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Volume 7

EINSTEIN'S RELATIVITY AND BEYOND

NEW SYMMETRY APPROACHES

Jong-Ping Hsu

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NEW SYMMETRY APPROACHES

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Preface

The purpose of this book is primarily to expound the idea that the first principle of relativity (i.e., the form of a physical law is the same in any inertial frame) taken by itself, implies a new theoretical framework which is consistent with the Lorentz and Poincaré group properties and with all previous experiments. The motivation of this work is *not* to show that special relativity is wrong in some way, but instead to show that special relativity is in some sense over specified and that removal of the over specification leads to a fresh view of the physical world with new concepts and results which are unobtainable through special relativity.

This new framework uses a four-dimensional symmetry formalism in which the three spatial variables as well as the temporal variable are all expressed in units of length. Furthermore, because it is based on only one postulate, it is the logically simplest theory which has Lorentz and Poincaré invariance. This is important because it is absolutely essential to insist that a fundamental physical theory should be derivable from the smallest possible set of basic principles. The 4-dimensional symmetry by itself dictates the kinematics of particles and fields in inertial frames in a manner which can also be extended to non-inertial frames through a limiting procedure.

Three aspects related to this simplest 4-dimensional symmetry framework are discussed:

(1) The first principle of relativity is shown to be the essence of relativity theory. All previous experimental results related to Einstein's theory of special relativity can be derived and understood based solely on this symmetry principle, without invoking an additional postulate regarding the universal constancy of the speed of light. We call such a theory "taiji relativity." As a result, we also find that the number of truly universal and fundamental constants in inertial frames is reduced to two, the atomic fine structure constant $\alpha_e=1/137.03604$ and a second constant $J=3.51773\times10^{-38}$ gram·cm. The speed of light (measured in cm/sec) and the Planck constant are found not to be truly fundamental.

(2) The renaissance of the common-sense concept of time $t=t'$ is shown to be possible. By embedding this "common time" in a 4-dimensional symmetry framework of the Lorentz and Poincaré groups, rather than the 3-dimensional framework of the Galilean group, we can construct a viable theory ("common

relativity") which is consistent with all known experiments. Such a theory has certain advantages over special relativity in that we can introduce the notion of the canonical evolution for a system of N particles and derive the invariant Liouville equation. This cannot be done, in principle, in special relativity because each particle has its own relativistic time, so that one cannot have a canonical evolution and can only derive N *one-particle* Liouville equations rather than one single invariant Liouville equation for N particles.

(3) The first principle of relativity can also lead to an 'absolute' theory of spacetime physics in non-inertial frames undergoing a constant linear acceleration or a uniform rotation. This is accomplished on the basis of limiting 4-dimensional symmetry, which requires that transformations for non-inertial frames must smoothly reduce to the familiar 4-dimensional transformation of relativity in the limit of zero acceleration. The relativity theory of spacetime for inertial frames is simply the limiting case of such a theory of spacetime ("taiji spacetime") for non-inertial frames. Particle dynamics and quantizations of fields in linearly accelerated frames are discussed. The universal constants in non-inertial frames are found to be the same as the ones in inertial frames mentioned above. Thus, α_e and J are the truly universal and fundamental constants in the physical world since almost all reference frames in the universe are, strictly speaking, non-inertial.

I want to express my gratitude to Bonnie Hsu for her patience and assistance in the preparation of this book. The writing of the book was supported in part by the Potz Science Fund and the Jing Shin Research Fund of the University of Massachusetts Dartmouth. I would like to thank the Academia Sinica and the Beijing Normal University in Beijing for their hospitality. I am indebted to Leslie Hsu, Wolfhard Kern, George Leung, John Dowd, Ed King and Kevin Smith for reading many chapters and improving the text. I would also like to thank Leonardo Hsu for his many valuable suggestions made during our discussions of the ideas presented here.

I have greatly benefitted from discussions with colleagues C. Chiu, L. Hsu, H. Margenau, T. Sherry and T. Y. Wu. Over the past decade, students in my classes (H. Lu, H. Stevens, Jr., Y. C. Wei, *et al*) have helped to clarify some ideas with their questions and comments. Lastly, I would like to express my sincere gratitude to B. Bertotti and the referees of my early papers on Common Relativity. Because of their understanding and open-mindedness, I was able to

publish my works on Common Relativity twenty-four years after I first conceived of the idea during my first year of college. The acceptance for publication strongly motivated me to further develop several new ideas which have engrossed my thought for years, and to publish a series of papers which have eventually led to the writing of this book.

Herein, I have sketched some of the questions which have arisen, in the hope that some readers may participate in the research of this subject of physics which is of fundamental importance.

As the author, I must take sole responsibility for any mistakes in the book and would be grateful to have my attention called to them by readers.

Institute of Theoretical Physics
Academia Sinica, Beijing
November, 1999

J. P. Hsu
(UMass Dartmouth)

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Overview

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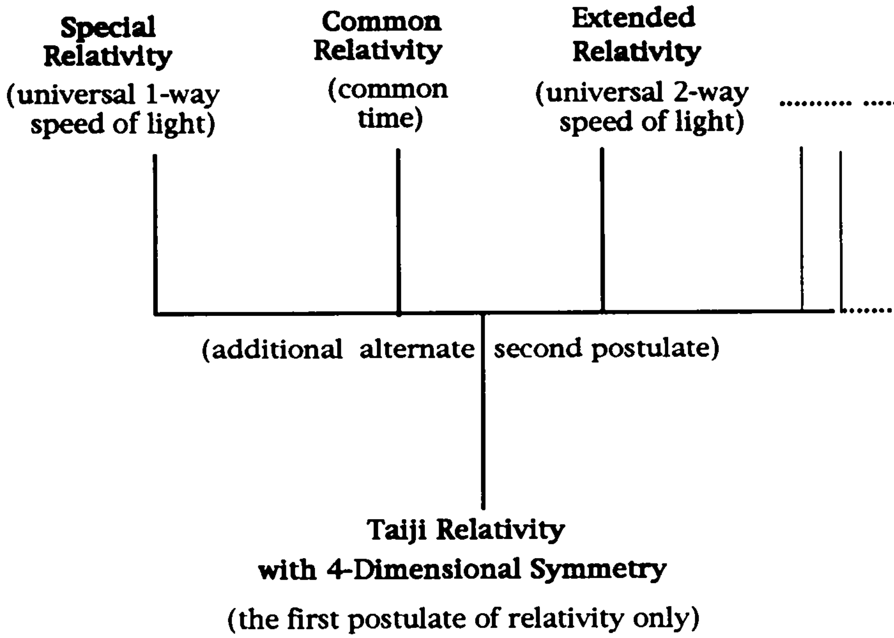

Logical Connections of Relativity Theories
with 4-Dimensional Symmetry

0.

Introduction

"Truth loves its limits, for there it meets the beautiful."

R. Tagore, *Fireflies*

"We are not saying that special relativity is wrong. Instead, ... (the new) relativity shows that relativistic time, or any particular time system for that matter, is not a necessary ingredient for a theory for it to correctly reproduce all known experimental results. The four-dimensional symmetry of the physical frmework is all that matters."

J. P. Hsu and Leonardo Hsu
Physics Letters A **196**, 1 (1994)

0a. Limitations of Special Relativity

There is no doubt that special relativity, the creation of a young Albert Einstein, is a beautiful theory. It is logically rigorous and wonderful in its agreement with experimental observations over a wide range of physical phenomena. In addition, its union with quantum mechanics has led to numerous significant results, including the celebrated Dirac equation, the stunning prediction of the existence of anti-particles, quantum field theories and the powerful Feynman rules. By now, all these successes of relativity are taken for granted.

But, however powerful and beautiful it may be, special relativity has its limitations. The web that physicists weave with concepts and laws often restricts their own thinking. There is little chance for making progress simply by repeatedly going over the successes of a physical theory. One must continually push at the boundaries of knowledge if the frontiers are to be extended. The reason that the limitations of a theory are so attractive to physicists is simply because they are the joyful beginning of a 'love affair' in research.

Four questions related to the limitations and possible extensions of special relativity and field theory are discussed in the following chapters. While answers will be provided for the first three, we can only speculate on the fourth because of its profound difficulty and the lack of any experimental hint at present. For now, let us briefly examine each of the four questions.

0b. Question #1: Can the theory of relativity be formulated solely on the basis of the first principle of relativity[1] (without assuming the constancy of the speed of light)?

In 1904, Poincaré delivered an important address, entitled "The Principles of Mathematical Physics," to the International Congress of Arts and Science in St. Louis.[2] During his talk, he discussed some hypotheses related to relativity and conjectured a whole new mechanics with the following features: (a) the principle of relativity, (b) the velocity of light as an unpassable limit, and (c) the contraction of moving bodies in their direction of motion. These were later stated in his lectures in Göttingen (1909) as the three hypotheses for a new mechanics.[3] In 1905, Einstein formulated his well-known special relativity based on two postulates: (A) the relativity principle and (B) the constancy of the speed of light.

When special relativity was proposed, some physicists felt that the second postulate concerning the constancy of the speed of light was too radical a change in physical concepts. They wondered:[1] Can one formulate a theory of relativity solely on the basis of the first postulate of special relativity, without assuming the universal constancy of the speed of light?

This historic question was discussed by Ritz (1908), Tolman (1910), Kunz (1910), Comstock (1910) and Pauli (1921, 1958), among others.[1] All concluded that to do so would be nearly impossible, that one would have to abandon Maxwell's equations for the vacuum and the whole of electrodynamics would have to be constructed from scratch. However, as we shall see, their conclusion turns out to be incorrect because they failed to recognize the existence of the **4-dimensional symmetry associated with the first postulate of relativity alone.** The 4-dimensional symmetry is the symmetry of the Lorentz and Poincaré groups, in which spatial and temporal (or evolution) variables in basic physical laws are treated on equal footing. We stress that all four variables must have the same dimension. In other words, the evolution variable **must** be expressed in units of length. The importance of this property will become clear later.

The new idea which leads us to the unexpectedly affirmative answer was stumbled upon in 1990 while re-analyzing the old question of whether the Lorentz transformation can be completely determined using only experimental results.[4] As it turns out, the answer is no, unless an additional theoretical assumption (such as the universality of the speed of light) is made. Purely on the basis of precision experiments, one can only obtain a Lorentz-like transformation which does not involve the constant speed of light c, but which turns out to have exactly the Lorentz group properties. This finding suggests that there exists a new "Lorentz and Poincaré invariance" which does not involve a constant corresponding to the speed of light at all and hence, that there is a new "4-dimensional symmetry" which can provide the necessary mathematical framework to give an affirmative answer to this long-standing question. In fact, this new Lorentz-like transformation is precisely the one which would be derived using *only* the first postulate of special relativity. Thus, this "4-dimensional symmetry" is associated only with the first postulate of special relativity and does not involve the constant speed of light c. It paves the way to a new and logically simplest theory of relativity, called taiji relativity. This result was published in a paper[5] in 1994.

The existence of taiji relativity illustrates more fully that the second postulate of the universality of the speed of light is truly a postulate, or *a truth by definition,* and a trap from which people can see only the particular four-dimensional symmetry of special relativity. The postulate of the universality of the speed of light is superfluous, blocking physicists' sight from an appreciation of the flexibility of the concept of time and from a true understanding of the universality of the speed of light as merely a free invention of the mind.

Many different theories of relativity with 4-dimensional symmetry can be constructed on the basis of alternate second postulates. Comparing Einstein's theory of relativity and Euclidean geometry, Einstein's second postulate (i.e., the universality of the speed of light) plays a role similar to Euclid's fifth postulate (i.e., the parallel postulate).[6]

0c. Question #2: Can one generalize the 4-dimensional transformation for inertial frames to non-inertial frames with a constant acceleration or rotation? In accelerated frames, the speed of light is no longer a universal constant; is the Planck constant still a universal constant?

This second question is closely related to an issue addressed by the young Einstein in his 1907 paper, two years after his successful application of the principle of relativity to inertial frames, namely "*Is it conceivable that an analogous principle of relativity holds for systems which are accelerated relative to each other?*" This question can be answered with the help of taiji relativity. A simple generalized transformation for frames with constant linear acceleration (or rotation) can be obtained by assuming that the transformation must reduce to the 4-dimensional transformation of relativity in the limit of zero acceleration. This assumption is called "limiting 4-dimensional symmetry," which is not only a natural assumption, but also absolutely necessary since *a non-inertial frame becomes an inertial frame in the limit of zero acceleration and an inertial frame must display the 4-dimensional symmetry of the Lorentz and Poincaré invariance.* As it turns out, the Planck constant, like the speed of light, is no longer a universal constant in non-inertial frames. Such accelerated transformations lead to a new and more general theory of space and time, implying a different, and truly universal constant.

0d. Question #3: Within the 4-dimensional symmetry framework of special relativity, it appears to be impossible, in principle, to generalize the classical Liouville equation for many-particle systems to a Lorentz invariant Liouville equation. Can we overcome this difficulty?

According to taiji relativity, it is possible to have infinitely many four-dimensional symmetry frameworks, each of which has a different concept of time. Within the four-dimensional symmetry framework of taiji relativity, relativistic time is but one of many possibilities which are all consistent with experimental results. For example, one could also make all clocks read the same time ("common time") for all observers without contradicting any experimental results. In the framework of taiji relativity, each particular concept of time corresponds to the assumption of a particular relation between the time t in F and time t' in F'. As we shall see in chapter 6, any given relationship between t and t' can always be physically realized by clock systems because the reading and the rate of ticking of any clock can be adjusted. Two of these are particularly simple. The first is the relativistic time of special relativity. The second, which is elaborated in chapter 12, is known as common relativity.

Poincaré first stated in 1898 that "**the simultaneity of two events or the order of their succession, as well as the equality of two time intervals, must be defined in such a way that the statements of the natural laws be as simple as possible.**"[7]

Our third question can be answered on the basis of this point of view. Common relativity has the advantage over special relativity in treating many particle systems. For N-particle systems, common relativity with a single time enables us to introduce the notion of the canonical evolution of a system and to derive the Liouville equation, a basic equation of the statistical physics. This cannot be done, in principle, in special relativity because each particle has its own relativistic time, so that one cannot have a canonical evolution and can only derive N one-particle Liouville equations which are not practically useful. Hidden here is a simplicity which will be explored and discussed in chapter 12.

0e. Question #4: In view of the profound divergence difficulties in quantum field theory, is the spacetime 4-dimensional symmetry exact at very large momenta or short distances?

This final question is extremely non-trivial and has no clear answer as yet. In view of the fundamental difficulties of the divergences in quantum field theory, the limitations of the conventional concepts of space and time become more clear. *With piercing insight, Schwinger noted that "a convergent theory [i.e., finite quantum electrodynamics] cannot be formulated consistently within the framework of present space-time concepts [special relativity]. To limit the magnitude of interactions while retaining the* *customary coordinate description is contradictory, since no mechanism is provided for precisely localized measurements.*"[8]

In his book, *Theory of Fundamental Processes*, Feynman analyzed possible modifications in quantum electrodynamics at very high energies. He said that "*relativity plus quantum mechanics seems to be exceedingly restrictive, but we are also undoubtedly adding unknown tacit assumptions (such as indefinitely short distances in space.)* "[9]

The profound difficulties of divergence in relativistic quantum field theories appear to suggest that the concept of locality has to be modified. The assumption of locality requires that the theory of quantum fields should be based on dynamical variables that are each localized at some point in spacetime.[10] However, locality is interlocked with the conventional description of space and time in special relativity, which has extensive experimental support.[11] Locality cannot be changed without modifying the traditional concept of space-time at short distances. Perhaps, a new dawn in the landscape of physics is not far off. Perhaps one may soon be able to see clearly the nonexistence of indefinitely short wavelengths and hence, an inherent fuzziness at short distances in space! It appears that common relativity could provide a more satisfactory foundation than special relativity for phenomena related to high energies, short distances and many-particle systems.[12]

In summary, taiji relativity has several interesting implications for various ideas which have been advanced during the past 90 years to extend or to change the concepts of time and simultaneity. These ideas were discussed by Reichenbach (1927), Edwards (1963), Winnie (1970), Mansouri and Sexl (1977), Hsu and Sherry (1976, 1980), Yuan Zhong Zhang, and others.[13] Unfortunately, the formulations of all these ideas, with the exception of those of Hsu and Sherry, did not have the explicit 4-dimensional symmetry of the Lorentz and Poincaré groups and, hence, are incompatible with quantum electrodynamics and other gauge field theories. However, taiji relativity indicates that such an incompatibility simply shows that these ideas were not properly formulated rather than incorrect. In addition, taiji relativity can provide a formulation for these various ideas which are compatible with the 4-dimensional symmetry of the Lorentz and Poincaré groups.[14]

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5. Jong-Ping Hsu and Leonardo Hsu, Phys. Letters A **196**, 1 (1994); Leonardo Hsu and Jong-Ping Hsu, Nuovo Cimento B, **111**, 1283 (1996); Jong-Ping Hsu and Leonardo Hsu, in *JingShin Physics Symposium in Memory of Professor Wolfgang Kroll* (World Scientific, Singapore. New Jersey, 1997), pp. 176-193.
6. The parallel postulate of Euclid asserts that, given any straight line L_1 and point P not on L_1, then in the plane determined by L_1 and P there is precisely one straight line through P and parallel to L_1. For about 2000 years, the parallel postulate was believed to be a necessary truth of a consistent geometry. But now we know that it is not necessary.
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1.

A Brief Review of Space and Time.

"What can be named is not the permanent name."

Lao-tzu (~600 B.C.) *Dauhder-jing*

1a. Space and Objects

The existence of space and time is intuitively clear and is taken for granted by all but the most hard-boiled of philosophers. In the most basic terms, space is the grand stage for all physical phenomena, while time is intimately related to our perception of the motion of objects and the changes in phenomena. As human beings, our perception of space and time are inherently different. An object in space can move left and right, forwards and backwards, up and down, or some combination of the three. This implies that the space we live in is three dimensional. However, at present, objects cannot move in time in the same way that they move in space, except in stories such as H. G. Wells' *The Time Machine.*[1] Although we can easily return to a previous location in space as many times as we wish, time appears to flow uniformly without being affected by anything.

Space itself is an "empty stage,"[2] without reference marks, so the only way to denote the position of an object (which we may idealize as a point particle for now) on a straight line is to specify its distance relative to another object with the help of some measuring device, such as a meter stick. There is absolutely no other way to define the concept of position. Such a method of measuring space is possible only because of the invariance in the form or size of solid bodies under certain transformations, namely spatial translations and rotations. As will become clear later, the invariance of solid bodies under spatial motion is closely related to the invariant forms of physical laws.

A convenient way to express the spatial positions of objects is to set up a coordinate system (e.g., three mutually perpendicular lines in the Cartesian case) relative to which any object's position can be measured. Thus, in the case of Cartesian coordinates, the position of any object at any instant can be completely defined by three numbers which are the shortest possible distances of the object from each of the axes.

Ignoring gravitational effects, the physical space we inhabit appears to be described by Euclidean geometry. The origin of the concepts of Euclidean geometry probably lies in the intuitions and world experiences of the ancients. As an independent branch of mathematics, however, Euclidean geometry now has a life of its own and often extends beyond any physical structure we might encounter. If gravity is not ignored, then our space is curved, according to general relativity, and described by non-Euclidean geometry, which was discovered independently by G. F. Gauss (1777-1855), J. Bolyai (1802-1860), and N. I. Lobachevsky (1793-1856). A more general non-Euclidean geometry (now called Riemannian geometry) was later constructed by G. F. B. Riemann (1826-1866).

As science progressed through the ages, the ability to make accurate and reproducible measurements of space and time became more and more important. The standards of measurement have been improved and modified. For example, the international definition of the meter has changed several times. The original standard meter bar was made of platinum in 1793. Its length at 0°C was supposedly one ten-millionth of the length of the Earth's meridian at sea level. As the techniques of spectroscopy became available, the definition of the meter was changed to be 1,553,164.13 times the wavelength of the red line of cadmium in air at standard temperature and pressure (1 atm and 0°C). In 1960, the meter was redefined as 1,650,763.73 wavelengths of the orange-red line of Krypton-86. Using interferometers, one could now make measurements of extremely high accuracy and reproducibility. Most recently, in 1983, the definition of the meter was once again altered, this time being identified with the distance traveled by light through vacuum in 1/299792458 of a second, where the speed of light is defined as 299792458 meters per second. This latest definition is not as hard to realize as one may think since there are atomic clocks which are capable of making such precise time measurements. For example, using light signals, the distance between the Earth and the moon can be measured to within a few meters accuracy.

1b. Time and Motion

As we have mentioned, if there were no motion of objects and no change in the universe, then it will be impossible to define time. Newtonian mechanics gave us the simplest intuitive concept of space and time: Space is absolute and is defined in terms of a reference coordinate system; time, on the other hand, is treated as a separate entity consistent with human intuition.

The measurement of time has also undergone many changes and improvements throughout the ages, but the one thing that all time pieces have had in common is that they all have used some regularly recurring events to mark the passage of time. In the early days, water clocks, hourglasses and sundials were used to measure time. In the early 1600's, Galileo discovered the constancy of the period of a swinging pendulum and for hundreds of years afterwards, this property has been used to construct all sorts of clocks. The Dutch scientist C. Huygens was probably the first to invent a pendulum clock around 1656 by applying Galileo's law of the pendulum. Electric clocks were made in the second half of the 19th century, but were not used extensively until after 1930. Their hands are driven by a synchronous electric motor controlled by an alternating current with a stable frequency. In 1929, the quartz clock was invented, which used the vibration of a quartz crystal to drive a synchronous motor at a very precise rate. The turning of a balance wheel and the oscillation of a quartz crystal have enabled the reduction in size of timepieces. For the most precise time measurements in modern research labs, atomic clocks, in which oscillations based on the frequency of radiation from atomic and molecular transitions, are used to mark time. Cesium atomic clocks are currently used as a standard of measurement. In a cesium clock, one second is defined as the time interval for 9,192,631,770 cycles of the oscillation with an accuracy of a few parts per billion.

1c. Inertial Frames of References

For the description of physical phenomena, we must have a frame of reference, which is a system of coordinates to indicate the spatial position of a particle, and clocks fixed in the frame to indicate the time. With the help of meter sticks and clocks in a frame, one can give operational definitions of space and time in that frame, provided that clocks at different places are synchronized by a certain procedure. For simplicity, we assume the use of identical meter sticks and identical clocks for all inertial frames. Roughly speaking, this means that the length of a meter stick on the ground (F frame) is the same as that of *another* meter stick in a train (F' frame) with constant velocity. Furthermore, the rate of ticking of a clock on the ground with respect to ground-observers is the same as that of *another* clock at rest in the train with respect to the train-observers. These assumed properties are consistent with the fact that all inertial frames are equivalent.

There exist frames of reference in which a particle moves with a constant velocity if the particle is not acted upon by external forces. These frames are called ***inertial frames.*** Clearly, all inertial frames move with constant velocities relative to one another. We may remark that why a particle has this inertial property, i.e., it remains in motion when it is moving under zero net force, is not known in physics. This inertial property of all matter was discovered by Galileo and is called the **principle of inertia.** His discovery marked a great advance in our understanding of motion.

In a given frame of reference, we can define spatial coordinates for particles and evolution variables to describe their motions. For simplicity, we usually assume that there are meter sticks, clocks *and observers* everywhere in a frame, so that space and time of a particle's motion can be recorded. We usually call the evolution variable 'time' and we define a unit for it such as the second. It is clear that the definition of a unit of time is an arbitrary convention for convenience. In other words, the change of the unit of time does not change physics. Suppose one uses seconds as the unit on Monday and minutes as the unit on Tuesday, etc. Physics does not change. Actually, if one uses a non-uniform time t^*, say, $t^* = t^3$, where t is the usual uniform time to describe motions, physics also does not change. The reason is that a non-uniform time amounts to a continuous change of the unit of time. In this case, we will still have the conservation of momentum, etc. and what will happen will happen. As an example, consider the collision of the Titanic and the iceberg. If one uses a non-uniform time t^* to measure the velocity of the iceberg, its velocity v^* will be different from the usual velocity v measured by using t. Will this change of time prevent the tragic collision from happening?

Furthermore, one may call the evolution variable w instead of t and define its unit to be, say, centimeters. Now the "velocity," dx/dw, of a particle measured by using w will be a pure number, without dimension. Will physics be changed?

In our discussions later, it is important to keep this flexibility of the evolution variable or "time" in mind. We shall demonstrate that the usual physical properties of time measured in seconds or minutes are actually not inherent properties of nature. They are human conventions imposed upon a physical theory for describing nature, and therefore, they are not essential for physics, as we shall see in chapter 7.

1d. Space and Time Transformations

We stress that just having a grid of identical clocks (synchronized according to a certain procedure) and meter sticks *in one inertial frame* does not completely define space and time in physics. One may ask an elementary question:

If a particle is at position (x,y,z) at time t [or w] as measured by observers in a particular inertial frame, what is the corresponding position (x',y',z') and time t' [or w'] for *the same particle* as measured by observers in a different inertial frame?

In order to answer this question, *there must be specific relations between the space and time coordinates of any two inertial frames. Only then are space and time completely defined.* Once we have such space and time transformations, the mathematical framework for describing physical phenomena is complete. One must merely follow logic to understand their meanings and implications. It is important to note that such transformations can only be expressed in terms of the Cartesian coordinate system and not in spherical coordinates or any other coordinate system. Why? If one attempts to use any other coordinate system to express space and time transformations for reference frames with uniform motion in a straight line, it simply does not work. The Cartesian coordinate provides the simplest equations which describe straight lines. In this sense, the Cartesian coordinate system is preferred for the transformations of inertial frames. In contrast, when one restricts oneself to a particular inertial frame, one can choose any coordinate system to describe physical phenomena and to solve physical problems.

The space and time transformations for inertial frames are of fundamental importance because they determine the basic space-time symmetry of the frame and restrict possible forms of the dynamics of interactions, but of course, cannot dictate the actual dynamics themselves. It is fair to say that the basic dynamics of particles and fields that is known so far is determined by several symmetry principles including local gauge symmetry, and by simplicity. For example, the interaction between a charged particle and the electromagnetic fields cannot be uniquely determined by the gauge symmetry principle; simplicity (e.g., an uncomplicated algebraic form for the electromagnetic coupling) also must be invoked.

In this connection, one may say that space and time in physics as a whole are not completely known because their properties in non-inertial frames have not yet been satisfactorily defined and tested experimentally. In chapters 21 and 22, we shall discuss space, time and physics in simple non-inertial frames with a constant linear acceleration based on symmetry principles.

1e. Absolute Time, Relative Time, Common Time and Taiji Time

Newton asserted absolute time and absolute space[3] in his *Principia* (1687) before stating his laws of motion. These ideas of absolute time and space were taken for granted for more than 200 years.

In the beginning of the 20th century, Einstein's special relativity introduced a revolutionary new concept of space and time based on two principles, (I) the invariance of physical laws under a constant velocity, and (II) a universal and constant speed of light. Consequently, time and space are relative and are interlocked, which differ drastically from the Newtonian concepts of absolute space and time.[3] In any inertial frame, one uses the 4-coordinate (ct,x,y,z) to express physical laws. *The relativity of space and time is completely described by the Lorentz transformations which connects the 4-coordinates of different inertial frames.* For example, one can have relativities of simultaneity, length contraction and time dilatation. The evolution variable of a physical system is the relativistic time t and is measured in, say, seconds. Such a time can be operationally defined by the invariant phase of an electromagnetic wave, $\exp\{i(\omega t-\mathbf{k}\cdot\mathbf{r})\}$, where $\omega=|\mathbf{k}|c$. In any inertial frame, clocks can be synchronized based on the second postulate, i.e., the universal constancy of the speed of light c = 299792458 m/sec.

In the 1970's, it was shown that one could formulate a 4-dimensional symmetry framework in which there is a common time for all observers in different inertial frames. Such a theory is called common relativity[4] and is consistent with experiments, as we shall see in chapter 12. Common time can be physically realized because the reading and the rate of ticking of clocks can be adjusted. It turns out that although common time does provide a universal meaning of simultaneity, it is not unique in the 4-dimensional symmetry framework and, therefore, it is not the same as the Newtonian concept of absolute time. (See section 12e in chapter 12.)

Near the end of the 20th century, it was realized that a physical theory of relativity could be formulated solely on the basis of the first principle (I) of relativity, namely, the invariance of physical laws under a constant velocity.[5] It is logically the simplest theory of relativity with 4-dimensional symmetry. The theory is called "taiji relativity" because taiji denotes, in ancient Chinese thought, the ultimate principle as it existed before the creation of the world.[6] It must be stressed that this does not mean that special relativity is wrong, but instead implies that special relativity is actually over specified by the second principle (II). In other words, it is not necessary to postulate the speed of light to be a universal constant. Taiji relativity dictates that the 4-coordinate (w,x,y,z) must be used to express physical laws which display the 4-dimensional symmetry. Furthermore, it also dictates that the evolution variable w, called taiji-time, must be measured in the same units as the spatial components x, y, and z, (say, centimeters,) although our intuition seems to allow us to give the evolution variable a different unit such as seconds, just as in classical mechanics. The taiji-time w (measured in centimeters) cannot be expressed in terms of the usual time t (measured in seconds) unless a second postulate is made. Nevertheless, the taiji-time w can be operationally defined through the invariant phase of electromagnetic waves, i.e., $\exp\{i(k_0 w - \mathbf{k}\cdot\mathbf{r})\}$, where $k_0=|\mathbf{k}|$. Physics can be formulated on the basis of taiji relativity for all inertial frames, just as in special relativity. The framework of taiji relativity provides a base for a truly unified standard[7] for 'frequency' k_0, taiji-time w and length because light always propagates with a derived constant 1 with no units, as we shall see later in chapter 7.

Furthermore, taiji relativity leads to a fresh view of the physical world and new results which are unobtainable through special relativity. For example, it shows that relativistic time, common time, or any particular time[8] measured in, say, seconds can be accommodated in a 4-dimensional symmetry framework and can correctly reproduce all known experimental results.

However, it also shows that none of them is a necessary ingredient of a theory for it to correctly reproduce all known experiments. Taiji relativity has the advantage of not being tied to any particular concept of time measured in seconds. It is also helpful to explore physics in non-inertial frames[9] in which the speed of light is not constant.

Since there are many possible times in 4-dimensional symmetry framework, which one should be used? The simplest criterion to follow is the one proposed by Poincaré. He believed that time "must be defined in such a way that the statement of the natural laws be as simple as possible."[10] For one-particle systems, common time, relativistic time and taiji time are more or less equal in making the natural laws as simple as possible. Nevertheless, for many-particle systems, common time has the unique and unequaled advantage of making the natural laws the simplest within the 4-dimensional symmetry framework, as we shall see in chapter 13.

Reference

1. H. G. Wells, *The Time Machine* (1895). There are some interesting passages in the book. The Time Traveler ardently explained to his guests: "The geometry, for instance, they taught you at school is founded on a misconception.... There are really four dimensions, three which we call the three planes of Space, and a fourth, Time. There is, however, a tendency to draw an unreal distinction between the former three dimensions and the latter, because it happens that our consciousness moves intermittently in one direction along the latter from the beginning to the end of our lives.... There is no difference between Time and any of the three dimensions of space, except that our consciousness moves along it." (pp. 1-2) Thus, it appears that long before Poincaré (1906) and Minkowski (1909) published their views that the Lorentz transformations could be interpreted as a 'rotation' of the coordinates system in a 4-dimensional spacetime, etc., Wells was the first to argue for a 4-dimensional vision of the universe. Of course, the four dimensions of spacetime are, strictly speaking, not *completely* equivalent, as explained in this paragraph. Sometime, people call it (3+1)-dimensional spacetime.
2. Space is usually considered as a set of points satisfying specified geometric postulates. Physically, this is true if and only if one ignores the effects of quantum fields. See section 4d and T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood Academic Publishers, New York; and Science Press, Beijing, 1981), pp. 378-405. This book gives a comprehensive discussion on fundamental aspects of particle physics and quantum fields related to the vacuum (i.e., the "aether" or the "ether", in old language). For a brief discussion of changing views of geometry and space, see O. Veblen and J. H. C. Whitehead, *The Foundations of Differential Geometry* (Cambridge Univ. Press, London, 1954), pp. 31-33.
3. Newton stated: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equally without relation to anything external..." "Absolute space, in its own nature, without relation to anything external, remains always similar and immovable..."
4. Jong-Ping Hsu, Nuovo Cimento B, **74**, 67 (1983); J. P. Hsu, Found. Phys. **8**, 371 (1978); **6**, 317 (1976); J. P. Hsu and T. N. Sherry, *ibid* **10**, 57 (1980); J. P. Hsu and C. Whan, Phys. Rev. **A 38**, 2248 (1988), Appendix. See chapter 12.

5. J. P. Hsu and L. Hsu, Phys. Letters A **196**, 1 (1994); (Erratum) ibid **217**, 359 (1996); Leonardo Hsu and Jong-Ping Hsu, Nuovo Cimento **111B**, 1283 (1996); Jong-Ping Hsu and Leonardo Hsu, in *JingShin Physics Symposium in Memory of Professor Wolfgang Kroll* (World Scientific, Singapore. New Jersey, 1997), pp. 176-193.
6. See, for example, *The New Lin Yutang Chinese-English Dictionary* (Ed. Lai Ming and Lin Tai-Yi, Panorama Press, Hong Kong, 1987), p. 90.
7. H. Hellwig, Metrologia **6**, 118 (1970).
8. It includes Reichenbach's time. H. Reichenbach, *The Philosophy of Space and Time* (Dover, New York, 1858), p. 127. See chapters 17 and 20 for more discussions.
9. Leonardo Hsu and Jong-Ping Hsu, Nuovo Cimento **112B**, 575 (1997) and 1147 (1997); Jong-Ping Hsu and Leonardo Hsu, Chinese J. Phys. **35**, 407 (1997). See also chapters 21-23.
10. H. Poincaré, Rev. Métaphys. Morale **6**, 1 (1898).

2.

The Nontrivial Pursuit of Earth's Absolute Motion

2a. Newton, Classical Mechanics and Invariant Laws of Motion

Once an inertial frame is set up with operational definitions of space, time, and observers, one can proceed to discuss mechanical phenomena. Mechanics is the study of the motions and interactions of objects and is probably the oldest branch of physics. The first person to undertake a systematic study of mechanics was Isaac Newton (1642–1727).[1] In his *Principia* (1687), Newton established his three now famous laws of motion and derived a large number of interesting and useful results. This work forms the foundation of what we now refer to as classical mechanics. Based on the method used to obtain the vast majority of his results, it would be fair to say that Newtonian mechanics is as complete and rigorous as Euclidean geometry.

Though not all of Newton's work has remained valid, Newtonian mechanics has two very important and fundamental properties which have had great significance in the later development of all branches of physics.

The first property is that Newton's laws of motion are **invariant** under Galilean transformations. What do we mean by invariant? Simply that the laws of mechanics (or their representative equations) have the same form for all observers whose relative motion is described by a constant velocity. This is equivalent to making the statement that, as far as the laws of mechanics are concerned, all inertial frames are completely equivalent. In classical mechanics, this invariance of laws is a basic symmetry property,[2] which is a simple generalization of the concept of symmetry as applied to geometry. The **Symmetry** of an object is, in the words of Herman Weyl, that "*one does something to it, and after one finishes doing it, the object looks the same.*" For example, a sphere rotated by any angle or displaced by any amount in space is still a sphere. This concept can be generalized in several ways—for example, the shortest distance between two points in space remains the same under any translation or rotation of the coordinate axes describing that space.

It is extremely fortunate that Newton's law of motion is the same in any inertial frame. If the law took a different form in different inertial frames, it would be much more complicated to understand the physical world. In this sense, the *invariance of physical laws is interlocked with the* ***simplicity*** *of physics.*

As an example, consider a set of Cartesian axes (x', y', z') fixed in an inertial frame F', say, a train which is moving with a constant velocity $\mathbf{V}=(V_x, V_y, V_z)$ relative to the inertial frame F (the ground) with Cartesian axes (x, y, z). Then

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t\,, \qquad t' = t\,, \tag{2.1}$$

give the coordinate transformation between the two frames, with the stipulation that the origins of F and F' coincide at time t=0 and that the two sets of axes are parallel to each other. This is the Galilean transformation. It is also explicitly assumed that there exists a unique time which is independent of any reference to special frames of coordinates. This is another important point that we will return to later. The ground or a laboratory on the rotating Earth may be considered to be inertial frame to a very good approximation during short time intervals, if gravity can be neglected.

Intuitively, one might expect that the laws of physics should remain the same when going from a particular coordinate system to one which is rotated or translated by a fixed amount from the original system. This is indeed true. This stands to reason as moving to a different place or tilting one's head should not change the outcome or characteristics of physical phenomena. This is a fundamental property of physics, as it cannot be explained in terms of something more basic. Let us see how one of Newton's laws holds up under a Galilean transformation.

Newton's law of motion can be expressed as

$$m\frac{d^2\mathbf{r}_b}{dt^2} = \mathbf{F}(\mathbf{r}_b - \mathbf{r}_a)\,, \tag{2.2}$$

where m is the mass of an object and $\mathbf{r}_b$ is its position. The vector $\mathbf{r}_a$ may be seen as the position of the source of the force on our object or as an equilibrium point for our system. Using equation (2.1) above, one can verify that Newton's law (2.2) is invariant under a Galilean transformation, i.e., that the equation looks exactly the same, except that all quantities are now measured from the primed frame:

$$m\frac{d^2\mathbf{r}'_b}{dt^2} = \mathbf{F}(\mathbf{r}'_b - \mathbf{r}'_a) , \tag{2.3}$$

where the mass m is a scalar, m=m'. This simple property, which holds for all of Newton's laws, implies that the F frame (ground) and the F' frame (the moving train) are physically equivalent. Since the mechanical laws of motion are the same in both inertial frames, one cannot use observations of mechanical phenomena to say which frame is "at rest" and which one is "moving." This is the *primitive principle of relativity for mechanics*, sometimes called the Galilean principle of relativity.

This particular symmetry property of mechanics had been known qualitatively to both Eastern and Western scientists for quite a while before Newton's time. In the western literature, Galileo (1564–1642) appeared to be the first one to discuss it in a famous passage from his book "Dialogue Concerning the Two Chief World Systems" (1632). He observed that if one was below decks on a large ship moving with constant velocity in smooth waters, one could not tell whether or not the ship was moving. In Chinese literature, there is a very similar observation in a book called "*Shang-Shu Weei*" (*An Appendix to the Book of History*), dating about a thousand and four hundreds years before Galileo's, which reads: "The earth (ground) always moves and never stops, and yet one does not know it; just like if one sits in a boat with windows closed, the boat moves and one does not feel it."[3] Unfortunately, this observation was not taken seriously by Chinese thinkers and did not have any impact on further scientific development in China.

Returning to our discussion of Newtonian mechanics, the second important property of Newton's laws is that he postulated an absolute time as well as an absolute space, corresponding to the frame of absolute rest. As far as his concept of time was concerned, it appeared so obvious and intuitive that it was taken for granted as correct for more than two hundred years until the time of Poincaré and Einstein. Newton called it "absolute, true and mathematical time", which "of itself and from its own nature, flows equally without relation to anything external." However, his idea of absolute space was criticized by Leibniz, Huygens and others. Leibniz argued that there is no philosophical need for any conception of space apart from the relations of material objects. Given that mechanical phenomena were not sufficient to determine a frame of absolute rest, Newton was probably aware of the logical weakness in his postulated absolute space, but he employed theological arguments to strengthen his idea of the absolute space, declaring that the absolute space was simply the space relative to God. About 150 years later, when the idea of a stationary aether permeating all space was popular, it was only natural to associate this aether with the frame of absolute rest.

2b. Maxwell's Suggestion for Finding Absolute Motion and Michelson's Interferometer

In the nineteenth century, the study of electric and magnetic phenomena was still a relatively new subject. When electromagnetic waves were found to be associated with the oscillations of fields, the similarity to mechanical oscillations was too great for people to believe that the waves could be transmitted through empty space. The theory of an aether that permeated all space and matter and provided a medium for the oscillations of the fields made perfect sense at that time and provided a convenient and pleasing analogy to the mechanical case. It may be somewhat puzzling for modern physicists to see why the idea of an aether was so popular among physicists in the nineteenth century. However, the very existence of the electromagnetic field was considered by them to be the unequivocal and firm evidence for the existence of the aether.

Given the aether and a frame of reference that could be associated with absolute rest, it was only natural to try to discover the velocity of the Earth through this medium. As mentioned before, mechanical phenomena do not provide meaningful tests because of Galilean invariance. However, when light was discovered to be a form of electromagnetic wave, optics seemed to present a possible solution. Initially, such experiments appeared to be impossible, considering the precision that would be required. Maxwell (1831-1879) wrote that "all methods . . . by which it is practicable to determine the velocity of light from terrestrial experiments depend on the measurement of the time required for the double journey from one station to the other and back again, and the increase of this time on account of a relative velocity of the ether equal to that of Earth would be only about one hundred millionth part of the whole time of transmission, and would therefore be quite insensible."

This apparently insurmountable difficulty was overcome by the genius of A. A. Michelson (1852–1931) in 1881 with his invention of the interferometer. The Michelson experiment split one monochromatic beam of light into two beams propagating in perpendicular directions for equal distances. When the two were rejoined, an interference pattern of light and dark rings could be observed due to the phase difference in the beams. According to the Galilean transformation, the velocity of light should not be isotropic in a terrestrial laboratory. Therefore, when the apparatus is rotated through 90°, changing the directions of the two equal arms, the interference bands should be observed to shift by an easily measurable amount dependent on the earth's velocity. As is well known, however, no shift was ever observed in this or any of numerous similar reliable experiments that followed it.[4]

The same experiment was repeated by Michelson together with Morley in 1887 resulting in the same null outcome to much better accuracy, and was published in the *American Journal of Science*.[5] The null result was a disappointment to Lorentz, Michelson and others who believed in aether. However, a positive reaction to this null result was received by the editor of *Science* in 1889, a very short paper with the title "The Ether and the Earth's Atmosphere" by G. F. FitzGerald:[6]

> "I have read with much interest Messrs. Michelson and Morley's wonderfully delicate experiment attempting to decide the important question as to how far the ether is carried along by the earth. Their result seems opposed to other experiments showing that the ether in the air can be carried along only to an inappreciable extent. I would suggest that almost the only hypothesis that can reconcile this opposition is that the length of material bodies changes, according as they are moving through the ether or across it, by an amount depending on the square of the ratio of their velocities to that of light. We know that electric forces are affected by the motion of the electrified bodies relative to the ether, and it seems a not improbable supposition that the molecular forces are affected by the motion, and that the size of a body alters consequently. It would be very important if secular experiments on electric attractions between permanently electrified bodies, such as in a very delicate quadrant electrometer, were instituted in some of the equatorial parts of

the earth to observe whether there is any diurnal and annual variation of attraction — diurnal due to the rotation of the earth being added and subtracted from its orbital velocity, and annual similarly for its orbital velocity and the motion of the solar system."

This was the first spark of theoretical thought, the FitzGerald contraction, that eventually caused a prairie fire in theoretical physics in the 20th century.

References

1. Newton is arguably the greatest scientist and genius. However, as a man in the eye of his own friends and contemporaries, "Newton was profoundly neurotic of a not unfamiliar type." "His deepest instincts were occult, esoteric, semantic — with profound shrinking from the world, a paralyzing fear of exposing his thoughts, his beliefs and his discoveries, in all nakedness to the inspection and criticism of the world. 'Of the most fearful, cautious and suspicious temper that I ever knew' said Whiston, his successor in the Lucasian Chair." "He parted with and published nothing except under the extreme pressure of friends. ... His peculiar gift was the power of holding continuously in his mind a purely mental problem until he had seen straight through it." "He regarded the Universe as a cryptogram set by the Almighty — just as he himself wraps the discovery of the calculus in a cryptogram when he communicated with Leibnitz. By pure thought, by concentration of mind, the riddle, he believed, would be revealed to the initiate." "He believed that the clues to the riddle of the universe were to be found partly in the evidence of the heavens and in the constitution of elements,... but also partly in certain papers and traditions handed down by the brethren in an unbroken chain back to the original cryptic revelation in Babylonia." "Very early in life Newton abandoned orthodox belief in the Trinity. ... He was persuaded that the revealed documents give no support to the Trinitarian doctrines which were due to late falsification. The revealed God was one God." This belief is presumably related to his idea of Absolute Space, being the one and only one that is relative to God. See "Newton, the Man" by J. M. Keynes, in *The World of Physics,* (Ed. J. H. Weaver, Simon and Schuster, New York, 1987), Vol. 1, PP. 537–547.
2. A. Zee, *Fearful Symmetry, The Search for Beauty in Modern Physics* (Macmillan, New York, 1986).
3. "*Shang-Shu Weei*" (*An Appendix to the Book of History*), published in the East Han Dynasty (AD 23–AD 221), the author was unknown.
4. A. A. Michelson, Am. J. Sci. **22**, 120 (1881). See also A. P. French, *Special Relativity* (W. W. Norton & Company, New York, 1968), pp. 51–58; A. Pais, *Subtle is the Lord..., The Science and the Life of Albert Einstein* (Oxford Univ. Press, Oxford, 1982), pp. 111–122.

5. A. A. Michelson and E. W. Morley, Am. J. Sci. 34, 333 (1887).

6. G. F. FitzGerald, Science 13, 349 (1889). FitzGerald was a tutor at Trinity College in Dublin in 1879 and became a professor in 1881. He was the first physicist to suggest a method of producing radio waves by an oscillating electric current, which was verified experimentally later by H. R. Hertz between 1885 and 1889. FitzGerald frankly told his friends that he was not in the least sensitive to making mistakes. His habit was to rush out with all sorts of crude notions in hope that they might set others to thinking and lead to some advance. He did precisely this in the short paper he submitted to *Science*.

3.

On the Right Track — Voigt, Lorentz and Larmor

3a. "Absolute Contraction of Length" and Lorentz's Heuristic Local Time

The Michelson-Morley experiment carried out in 1887 confirmed Michelson's original experimental null result of 1887 with a 20-fold increase in accuracy. In 1889, G. F. FitzGerald (1851-1901) proposed that the null result obtained could be explained if bodies in motion through the aether underwent a length contraction in the direction of motion "by an amount depending on the square of the ratio of their velocities to that of light," as mentioned in chapter 2. Of course, there was no other experimental evidence for this conclusion, so this was merely an ad hoc explanation. However, since the aether was generally believed to exist at that time, FitzGerald's idea was "natural" and attractive. This idea of FitzGerald influenced Larmor and Lorentz in obtaining the relativistic transformation of space and time.

The next development was carried out by H. A. Lorentz (1853-1928). He finished his doctoral dissertation in 1875, in which he refined Maxwell's electromagnetic theory so that it explained more satisfactorily the reflection and the refraction of light. He became a professor of mathematical physics at Leiden University in 1878 at the age of 25. His main effort was to complete Maxwell's theory to explain the relationship among electricity, magnetism and light by introducing the Lorentz force and by suggesting that the oscillations of charged particles inside the atom was the source of light. To test his idea on the source of light, he reasoned that a strong magnetic field should have an effect on the oscillations of these charged particles. P. Zeeman, a student of Lorentz, carried out experiments to confirm this idea in 1896. This phenomenon is now known as the Zeeman effect. For these experiments, both Zeeman and Lorentz were awarded the Nobel Prize in 1902.

In 1895, Lorentz was investigating the physical effects on electric and optical phenomena due to the Earth's motion through the aether, taking Maxwell's equations to be a description of such phenomena in the aether. This amounted to finding the proper transformation for Maxwell's equation from the aether frame F_a (the frame of absolute rest) to the Earth's moving frame, F'. At this time, it was known that while Newton's mechanical equations were invariant under the Galilean transformation, Maxwell's equations were not. In order to make them invariant from frame to frame to first order in V/c, Lorentz discovered that one had to introduce a new time t' in the moving frame F':

$$t' \approx t - Vx/c^2, \qquad x' \approx x - Vt, \qquad y' = y, \qquad z' = z, \qquad (3.1)$$

where t is the time measured in the aether frame F_e and the relative motion of the two frames is taken to be solely along the parallel x and x' axes. Lorentz was not the first to write down such a transformation, however. In 1887, Woldemar Voigt (1850–1919) had introduced a non-absolute time $t' = t - Vx/c^2$ while studying Doppler shifts.[1]

Lorentz considered this t' in (3.1) to be a local time which had no physical meaning since it contradicted the absolute time t' = t in the Galilean transformation (2.1), which was too intuitively obvious to be incorrect. He later remarked that

> "a transformation of the time was necessary, so I introduced the conception of local time which is different for different frames of reference which are in motion relative to each other. But I never thought that this had anything to do with real time. This real time for me was still represented by the older classical notion of an absolute time, which is independent of any reference to special frames of coordinates. There existed for me only one true time. I considered my time-transformation only as a heuristic working hypothesis, so the theory of relativity is really solely Einstein's work."[2]

On the other hand, Poincaré regarded Lorentz as the one who had conceived the principle of relativity for electromagnetic phenomena and was the first physicist to call the formulae obtained by Lorentz the "Lorentz transformation".[3] In one of his essays, "Space and Time," Poincaré wrote, prophetically: "Will not the principle of relativity, as conceived by Lorentz, impose upon us an entirely new conception of space and time and thus force us to abandon some conclusions which might have seemed established?"[4]

3b. Exact Transformations Discovered by Larmor and Lorentz

Sir Joseph Larmor (1857–1942) was educated at Belfast and Cambridge. He taught at Cambridge from 1885 to 1932 and was the Lucasian Professor of Mathematics in the University of Cambridge. Larmor was knighted in 1909. He did pioneering work in the rate of energy radiation from an accelerated electron and in explaining the splitting of spectrum lines by a magnetic field. Like many physicists at that time, he believed that matter consists entirely of electric particles moving in aether. Today, he is mainly known for his work on the wobbling motion of an atomic orbit when an atom is subjected to an external magnetic field, called "Larmor precession" and the rate of the precession is known as the "Larmor frequency." He also obtained "Larmor's formula" for the total power radiated by an accelerated charge. In retrospect, however, Larmor's most significant contribution is his discovery of the exact relativistic spacetime transformation which is now called the "Lorentz transformations." In the fierce competition of scientific research today, to be the first person to make a discovery is everything. To be the second is nothing. In light of this, one may be surprised that, before he died in 1942, Larmor never made any claim of being the first to conceive of this important exact spacetime transformation, which specifies the transformation of space and time between inertial frames and forms the basis of Einstein's special theory of relativity.

Larmor studied Lorentz's paper of 1895 which contained the first order transformation (3.1). He was able to improve it and obtained an exact transformation which can be written in the familiar form:

$$x' = \gamma(x - Vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{Vx}{c^2}\right); \quad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}, \quad (3.2)$$

after a change of variables. This important result was presented and discussed in his book *Aether and Matter*[5] which was completed in 1898 and published in 1900. Apparently, this work went largely unnoticed as this set of equations is now known as the Lorentz transformation. Larmor also derived what we now know as the relativistic length contraction from the transformation (3.2). At the time, it was thought that this was the contraction that FitzGerald had proposed. However, this is not the case since the FitzGerald contraction is an actual physical contraction of the length of an object due to its motion in absolute space, while the contraction implied by (3.2) is an effect due to the relative motion of the observer. Larmor did not discuss the physical meaning of t'. Presumably, his viewpoint on that account was the same as that of Lorentz.

In 1899, Lorentz wrote down the exact transformation with an additional factor K:

$$x' = K\gamma(x - Vt)\,, \quad y' = Ky\,, \quad z' = Kz\,, \quad t' = K\gamma(t - \frac{Vx}{c^2})\,. \tag{3.3}$$

It is unknown whether or not he was aware of Larmor's work at the time. He noted that the scale factor K could not be determined by the Michelson–Morley experiment and that it required "a deeper knowledge of the phenomena." Nevertheless, he stressed that the length contraction implied by the spatial components in (3.3) were precisely those which one had to assume in order to explain the Michelson–Morley experiment. In 1904, Lorentz wrote down the transformation

$$x' = \gamma(x - Vt)\,, \quad y' = y\,, \quad z' = z\,, \quad t' = \gamma(t - \frac{Vx}{c^2})\,. \tag{3.4}$$

He attempted to fix K in (3.3) to have the value 1 by considering the transformation properties of the equation of motion of an electron in an external field. However, because he made a mistake in the transformation equations for velocities, his proof was valid only to first order in V/c.[6] This mistake was corrected by Poincaré in 1905.

As mentioned earlier, Voigt[7] (1850–1919) had obtained a similar transformation as early as 1887 while studying Doppler shifts in his paper 'Über das Dopplersche Prinzip.'[1] His transformation was the same as that found in (3.3), but with the scale factor K set to $1/\gamma$:

$$x' = (x - Vt)\,, \quad y' = y/\gamma\,, \quad z' = z/\gamma\,, \quad t' = (t - \frac{Vx}{c^2})\,. \tag{3.5}$$

Reflecting for a moment, one sees that Voigt had actually introduced two revolutionary ideas into physics,

(a) the concept of non-absolute time t', which is almost the correct relativistic time, and

(b) the universal and constant speed c for the propagation of light in all inertial frames.

Furthermore, Voigt showed the invariance of the wave equation,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi = 0, \tag{3.6}$$

under his transformation (3.5). If people had been imaginative enough at that time, they might have recognized the potential of these ideas to open up a whole new field of physics. Of course, this never happened. In the time when the ideas of Newtonian absolute time and space dominated physics, people simply dismissed Voigt's ideas as nonsense. Later, in 1906, Lorentz said that regrettably, Voigt's transformations had escaped his notice all those years.[8]

It would be fair to say that Lorentz first conceived the *invariance* of the laws of the electromagnetic field and that his transformations embody the whole mathematical essence of the new concepts of space and time in special relativity. However, Lorentz believed in the aether.[9] Consequently, the relativity of space and time and the principle of relativity were not apparent to him before 1905. He did not believe in the non-absolute time found in equation (3.4) even though he discovered it himself and, moreover, he did not attempt to formulate a theory of relativity[10] for mechanics and electrodynamics as Poincaré and Einstein did. Lorentz arrived at the new concept of time by struggling with the old idea of absolute time. Unfortunately, the preconceptions of absolute time and aether remained the language of his thinking for the rest of his life.

References

1. W. Voigt, Nachr. Ges. Wiss. Göttingen. 41 (1887); see also Phys. Zeitschr. 9, 762 (1908). See also W. Pauli, *Theory of Relativity* (first appeared in 1921, translated from the German by G. Field, Pergamon Press, London, 1958), pp. 1–3.
2. W. Gv. Rosser, *An Introduction to the Theory of Relativity* (Butterworths London 1964), P. 64.
3. H. Poincaré, Compt. Rend. Acad. Sci. Paris **140**, 1504 (1905) and Rend. Circ. Mat. Palermo **21**, 129 (1906).
4. H. Poincaré, *Mathematics and Science: Last Essays* (Dover, N.Y. 1963; originally published in 1913).
5. J. Larmor, *Aether and Matter* (Cambridge University Press, Cambridge, 1900.), pp. 167–170 and pp. 173–175. This book has a long subtitle: "A development of the dynamical relations of the aether to material systems on the basis of the ATOMIC CONSTITUTION OF MATTER including a discussion of the influence of the earth's motion on optical phenomena." The notations of Larmor for his transformations are messy. For example, he wrote the following expression: $\varepsilon^{1/2}x'$, y', z', $\varepsilon^{-1/2}t'-(v/c^2)\varepsilon^{1/2}x'$, where $\varepsilon=(1-v^2/c^2)^{-1}$. One has to follow the notation used in the first order approximation to find the relations $t' = t$, $z' = z$, $y' = y$ and $x'=(x-vt)$ and to obtain the familiar expression: $\varepsilon^{1/2}(x-vt)$, y, z, $\varepsilon^{1/2}(t-vx/c^2)$. In chapter XI, the title reads "MOVING MATERIAL SYSTEM: APPROXIMATION CARRIED TO THE SECOND ORDER." This seems to suggest that, at that time, he was not aware of or did not regard his transformation to be correct and exact to all orders. Like most physicists at that time, Larmor apparently believed that the aether played a fundamental role in the physical universe. In section 123, he speculated on a hydrodynamic theory "which would construct an atom out of vortex rings" and "the circulation of the vortex is however in the dynamical theory an unalterable constant, so that the one system cannot be changed by natural processes into the other."
6. A. Pais, *Subtle is the Lord..., The Science and the Life of Albert Einstein* (Oxford Univ. Press, Oxford, 1982), p. 126.
7. W. Voigt was a colleague of Klein, Hilbert, Minkowski, Runge, Wiechert, Prandtl and Schwarzschild at Göttingen, which had been developed in to a world famous center of learning in sciences by Gauss. Voigt published a

book: *Magneto- und Elektrooptik*, Leipzig, 1908. He also gave lectures on optics and taught an advanced course in optical experiments. He was Max Born's teacher and used to investigate crystal properties with the help of symmetry considerations. Voigt appreciated his talented former student Born who later became his friend Minkowski's assistant. In 1909, after the death of Minkowski, young Born gave his first report on his work concerning the constant acceleration of an electron and its electromagnetic mass to the Mathematical Society. There were continual interruptions and attacks, so that Born was confused and got entangled in his formulae. Klein stopped the talk and declared that he had never listened to such a bad lecture in all his life. Born was completely broken and seriously considered giving up physics and entering one of the technical colleges in order to become an engineer. However, with the help of Runge and Hilbert, Born got a second chance and convinced Klein. After the second report, Voigt encouraged Born to stay and become a lecturer at Göttingen by saying that Born's paper seemed to him a very suitable thesis, and Born could count on his support in the faculty, offering Born a lectureship (Privatdozent). It appears that Voigt's encouragement to the young Born eventually made a bigger contribution to physics than his own research on crystals and magneto-optical phenomena.

8. Lorentz and Voigt corresponded about the Michelson-Morley experiment around 1888, and Lorentz knew some of Voigt's work, though not the Voigt transformation (3.5). In 1906, Lorentz said in his Columbia University lectures that Voigt had applied the transformation (3.5) to show the invariance of the wave equation $[\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - (1/c^2)\partial^2/\partial t^2]\phi=0$. He also commented: "The idea of the transformations....might therefore have been borrowed from Voigt and the proof that it does not alter the form of the equations for the free ether is contained in his paper.[1]" See H. A. Lorentz, *The Theory of Electrons* (ser. 169. Teubner, Leipzig, 1909; Dover, New York, 1952), p. 198, and, also, A. Pais, ref. 6, pp. 121–122.
9. Lorentz said in 1909 that he could not but regard the aether as endowed with a certain degree of substantiality, however different it may be from all ordinary matter. See H. A. Lorentz, ref. 8, p. 230.
10. The main difference between Lorentz's and Einstein's attitude toward relativity can be seen clearly from Lorentz's statement: "....the chief

difference being that Einstein simply postulates what we have deduced, with some difficulty and not altogether satisfactorily, from the fundamental equations of the electromagnetic field. By doing so, he may certainly take credit for making us see in the negative result of experiments like those of Michelson, Rayleigh and Brace, not a fortuitous compensation of opposing effects, but the manifestation of a general and fundamental principle." See H. A. Lorentz, ref. 8, p. 230.

Woldemar Voigt (1850-1919): The unsung hero of special relativity (Photo courtesy of Helmut Rohlfing, Göttingen University.)

4.

Poincaré's Contributions and the Aether (Past and Present)

4a. A Remarkable Insight of Physical Time

If one examines the writings of the great French mathematician and physicist Jules Henri Poincaré (1854–1912) at the turn of the 20th century, one would see a foreshadowing of most of the notions which would later become part of the special theory of relativity.[1] In 1895, he noted the impossibility of detecting earth's absolute motion and, in July of 1905, completed his theory of relativity based on *the postulate of relativity* and a definition—choosing the units of length and of time so that the speed of light was equal to unity. Many physicists consider Poincaré's main contributions to relativity to be of a mathematical nature. However, this does not adequately represent his work. In the literature, historical accounts of relativity theory regarding Poincaré's contributions have been controversial. For example, at one extreme, one author has claimed that Poincaré discovered the principle of relativity for all physical laws in 1904 and that the originators of the relativity theory are Lorentz and Poincaré rather than Einstein.[2] At the opposite end of the spectrum, Holton[3] asserts that Poincaré's principle of relativity is equivalent to the Galilean-Newtonian principle of relativity, ignoring his comprehensive papers on relativity and electrodynamics finished in June and July 1905. It seems fair to say that the truth is somewhere in between these two views.[4]

In 1895, Poincaré had already noted the impossibility of detecting the earth's absolute motion. "Experiment has revealed a multitude of facts which can be summed up in the following statement: It is impossible to detect the absolute motion of matter, or rather the relative motion of ponderable matter with respect to the ether; all that one can exhibit is the motion of ponderable matter with respect to ponderable matter."[5]

Around 1898, Poincaré was examining the concepts of absolute time and absolute simultaneity in response to the then often-debated question of the measurement of time intervals. He appears to have been the first physicist to discuss and analyze the concept of time from what people later named the "operational viewpoint." In an article entitled "La Mesure du Temps",[6] Poincaré stressed that "*we have no direct intuition about the equality of two time intervals.* People who believe they have this intuition are the dupes of an illusion" (the italics are Poincaré's). This is a very remarkable insight. Noting earlier definitions of simultaneity which he found unsatisfactory, he wrote that "it is difficult to separate the qualitative problems of simultaneity from the quantitative problem of the measurement of time; either one uses a chronometer, or one takes into account a transmission velocity such as the one of light, since one cannot measure such a velocity without measuring a time." He finally concluded that "**the simultaneity of two events or the order of their succession, as well as the equality of two time intervals, must be defined in such a way that the statement of the natural laws be as simple as possible.** In other words, all rules and definitions are but the result of an unconscious opportunism."

With this penetrating understanding of physical time, it was natural that Poincaré in 1900 showed a strong interest in and gave the first correct physical interpretation to the "local time" t' which was introduced by Lorentz in 1895. Lorentz investigated the influence of the Earth's motion on electric and optical phenomena. He realized that if the experiments carried out in a terrestrial laboratory could not reveal any effect on the motion, then the equations of the theory must have the same form when going from the absolute rest frame $F_{ar}(t,x)$ of the aether to the terrestrial frame F'(t',x'). To obtain this result to the first order in (V/c), Lorentz introduced approximate transformations of time $t' = t - Vx/c^2$ and space $x' = x - Vt$, where y, z and the second order terms in V/c were neglected.

Local time t' in a moving frame F' differs from the true time t in the frame of absolute rest F_{ar} by the amount Vx/c^2 for each point on the x axis in F. Lorentz considered the local time t' to be nothing but a convenient mathematical quantity to simplify Maxwell's equations in a moving frame F' and that it did not have any physical meaning at all. However, in a paper published in 1900,[7] Poincaré showed that the local time t' could be given a simple physical interpretation: Suppose observers at various points along the x' axis of the moving frame F' synchronize their respective clocks by exchanging light signals with an observer at the origin and that the speed of light is independent of the motion of its source. The difference between true time and local time would be accounted for by an observer at rest in F_{ar} by the amount each clock in F' had been thrown out of "true synchrony" by its translation during the exchange of signals. All these discussions hold only for the first order approximation in (V/c) because the exact Lorentz transformations had not yet been derived. Historically, this was the first approximate operational definition of time. The exact operational definition of time was discussed by Poincaré in 1904, as we shall see below.

4b. Poincaré's Innovative Principle of Relativity

The idea of relativity had already engrossed Poincaré's mind for about ten years when he proposed it as the "*principle of relativity*" in 1904 and reformulated physics in accordance with it in 1905. It was a long evolutionary process for him to finally grasp the principle based on an inductive approach. As mentioned before, in 1895, Poincaré had already noted the impossibility of detecting earth's absolute motion.[5] Five years later in 1900, he called it the "principle of relative motion."[7] In his book *Science and Hypothesis* published in 1902, Poincare's principle of relative motion is stated in chapter VII, "RELATIVE AND ABSOLUTE MOTION," as follows: "The movement of any system whatever ought to obey the same laws, whether it is referred to fixed axes or to the movable axes which are implied in uniform motion in a straight line."[7] Although Lorentz had initiated research in "relativity" as early as 1895 and attempted to show that Maxwell's equations are invariant under a new space and time transformation, he did not conceive the principle of relativity to be generally and rigorously valid and never believed in the relativistic time, even though he himself discovered it.

In his address to the Paris Congress of 1900, Poincaré discussed hypotheses in physics and the theories of modern physics.[8] It was a comprehensive and impressive survey of all branches of physics up to that time, 1900, the very beginning of the 20th century. He asked a burning question:

"Does the aether really exist?"

which was one of the central questions in physics at that time. The importance of this question can be seen by the fact that Einstein came back to this question in 1920 from the viewpoint of gravity, that Dirac wrote a paper with essentially the same title in 1951 based on relativistic quantum electrodynamics, and that particle physicists such as T. D. Lee, Weisskopf, Bjorken and others continued to discuss it in the 1980's and afterward based on modern gauge field theory, as we shall see in section 4d.

The essence of Poincaré's argument was that the aether exists but that there is a conspiracy of dynamical effects so that the velocity of an object moving through the aether cannot be detected by optical phenomena. He believed that the conspiracy causing the cancellations of the velocity-dependent terms should be rigorous and absolute, holding for all orders, rather than just for the first order terms.

In 1904, Poincaré delivered an important address to the International Congress of Arts and Sciences in St. Louis with the title "THE PRINCIPLES OF MATHEMATICAL PHYSICS," in which he listed and discussed six general principles of physics.[9] One of them is what he called the "**principle of relativity**, according to which the laws of physical phenomena should be the same, whether for an observer fixed, or for an observer carried along in a uniform movement of translation; so that we have not, and could not have any means of discerning whether or not we are carried along in such a motion." He explained it as follows: "Indeed, experience has taken on itself to ruin this interpretation of the principle of relativity; all attempts to measure the velocity of the earth in relation to the ether have led to negative results. This time experimental physics has been more faithful to the principle than mathematical physics; ... but experiment has been stubborn in confirming it. .....And finally Michelson has pushed precision to its limit: nothing has come of it." In this connection, it is worthwhile to note that Poincaré's principle of relativity is clearly a new one and is *not* the Galilean-Newtonian principle of relativity in classical mechanics. In fact, this principle is exactly the same as his "principle of relative motion" proposed in 1900,[7] as mentioned previously. We stress that *this is a pure symmetry principle because it asserts that the form of physical laws should be the same in any inertial frame.* In modern language, it asserts that physical laws must display 4-dimensional symmetry of the Lorentz and the Poincaré groups.

Taken alone, Poincaré's principle of relativity does not provide the foundation for special relativity. *However, with this principle of relativity, Poincaré has laid the foundation for a broad four-dimensional symmetry framework, of which special relativity is, in the restricted sense of adding an extra postulate, a particular case.* This is to be expounded in chapter 7.

Poincaré even went one step further by treating Lorentz's ingenious idea of a local time t' as a physical time and gave t' a rigorous operational definition based on his principle of relativity. Consider two observers (or the station A and the station B) in uniform relative motion who wish to synchronize their clocks by means of light signals. According to Poincaré, clocks synchronized in this manner do not mark 'true' time if the frame of reference has an absolute motion, *since the velocity of light is not isotropic in that frame.* However, this leads to no contradiction because all physical laws in the moving frame are the same as those in a rest frame. Thus, an observer in the moving frame has no means of detecting the anisotropy and thereby ascertaining a difference between his local time and "true" time. The difference "matters little since we have no means of perceiving it." "All phenomena which happen at A, for example, will be late, but all will be equally so, and the observer who ascertains them will not perceive it since his watch is slow; so as the principle of relativity would have it, he will have no means of knowing whether he is at rest or in absolute motion."

Here, in 1904, appeared for the first time the procedure for what is now called the operational definition of physical time. Poincaré continued: "Unhappily, that does not suffice, and complementary hypotheses are necessary; it is necessary to admit that bodies in motion undergo a uniform contraction in the sense of the motion.... Thus, the last little difference find themselves compensated. And then there is still the hypothesis about force. Force, whatever be their origin, gravity as well as elasticity, would be reduced in a certain proportion in a world animated by a uniform translation; or, rather, this would happen for the components perpendicular to the translation; the components parallel would not change."

Poincaré concluded his talk with a marvelous vision: "Perhaps we should construct a whole new mechanics, that we only succeed in catching a glimpse of, where inertia increasing with the velocity, the velocity of light would become an impassable limit."[9]

4c. Poincaré's Theory of Relativity Based on 1 Postulate and 1 Definition.

We have seen that, in 1904,[9] Poincaré discussed, among other principles, one *postulate* related to relativity, (a) the principle of relativity, and a closely related *property*, (b) "no velocity could surpass that of light" (which was stated as a character of an entirely new mechanics in the physics of the future). These obviously correspond, though are not identical, to the two postulates of Einstein's special relativity. Poincaré also mentioned two necessary complementary hypotheses right after he discussed the synchronization of clocks using light signals. They are (c) that bodies in motion suffer a uniform contraction in their direction of motions and (d) that the components of forces perpendicular to the translation would be reduced. Because his idea of a new mechanics based on three hypotheses (a), (b) and (c) was not stated until 1909 in his lectures in Göttingen,[10] and the lack of explicitly physical interpretations of space, time and relativity,[4] many physicists tend not to consider Poincaré as an originator of special relativity.

In June and July 1905, Poincaré completed two papers, both entitled "On the Dynamics of the Electron,"[1] regarding his principle of relativity, the Lorentz transformations and their physical implications. The first was a detailed summary of the second much longer paper, which was Poincaré's last major work on relativity and was written at about the same time as Einstein finished his first paper on special relativity. It is interesting to note that, roughly speaking, the mathematician Poincaré took a more physical approach to discussing his relativity theory, while the physicist Einstein took a more mathematical (axiomatic) approach in his formulation of the theory. In his work, Poincaré stayed close to experimental evidence, noting that neither the aberration of starlight and related phenomena nor the work of Michelson revealed any evidence for an absolute motion of the earth.

He said: "It seems that this impossibility to disclose experimentally the absolute motion of the earth is a general law of nature; we are led naturally to admit this law, which we shall call the *Postulate of Relativity*, and to admit it unrestrictedly. Although this postulate, which up till now agrees with experiment, must be confirmed or disproved by later more precise experiments, it is in any case of interest to see what consequences can flow from it." And before he wrote down Maxwell's equations and the Lorentz force, he chose the units of length and of time so that the speed of light equals unity. In other words, he defined $c=1$ by a suitable choice of units rather than by postulating the speed of light to be a universal constant, in sharp contrast with Einstein. Poincaré never mentioned an experimental test of the universality of the speed of light, although he did mention that the postulate of relativity must be tested experimentally. This indicates that he did not treat the invariance of physical laws and the constancy of the speed of light on equal footing. This has been considered a weakness in Poincaré's understanding of physics. However, it will be argued that this is actually a major strength of Poincaré's power of thinking. Conceptually, there appears to be a significant difference between Einstein's second postulate for special relativity and Poincaré's definition regarding the true nature of the constancy of the speed of light c. We shall come back to this important point later in section 4f and in chapters 7 and 17.

In any case, we have a clear-cut answer to the question: How many postulates did Poincaré base his theory of relativity in the Rendiconti paper[1]?

Answer: He made one postulate (the postulate of relativity) and one definition concerning the speed of light to formulate relativity theory. (Apart from these, Maxwell's equations and the Lorentz force must also be assumed for discussions of electromagnetic phenomena; and the Newtonian equation of motion must be assumed for discussions of particle dynamics. But these are not on the same level as the fundamental postulates in relativity theory.)

Poincaré wrote down the Maxwell equations, the continuity equation for the conservation of charge, the wave equations for the scalar and vector potentials and the Lorentz force. Then he stressed that "these equations admit a remarkable transformation discovered by Lorentz, which is of interest because it explains why no experiment is capable of making known to us the absolute motion in the universe." The transformations discovered by Lorentz in 1899 are given by (3.3), i.e.,

$$x'=K\gamma(x-\beta t), \quad y'=Kz, \quad z'=Kz, \quad t'=K\gamma(t-\beta x); \tag{4.1}$$

$$\gamma = 1/\sqrt{1-\beta^2}, \qquad \beta = V/c.$$

Poincaré called (4.1) the Lorentz transformations and rigorously proved the invariance of Maxwell's equations under these transformations. Poincaré derived the transformation equations for the force per unit charge and pointed out that they differ significantly from those found by Lorentz. He further discussed in detail that the transformations (4.1) formed a group which was generated by seven infinitesimal transformations (with $K \neq 1$.)[1]

When Poincaré discussed the Lorentz group, it was clear that his derivation of the Lorentz transformation (4.1) was no different from that used in many of today's textbooks (with K=1): He explained that every transformation of this group can be decomposed into a transformation of the form $x_1'=Kx_1$, $t'=Kt$ and a linear transformation that leaves invariant the quadratic form $\mathbf{x}^2 - t^2$, where $\mathbf{x}=(x_1,x_2,x_3)$. Perhaps because this simple derivation was not written down explicitly in terms of the invariance of physical laws and because his paper was not widely read, many people were not aware of Poincaré's derivation of the Lorentz transformations. Most physicists believed that he had simply assumed the transformation, as Lorentz and Larmor did. As a result, it is usually believed incorrectly that Einstein was the first one who derived the Lorentz transformations from first principles.

Poincaré did not discuss time and clock synchronization because they were discussed in his previous papers. He did discuss a procedure for making spatial measurements, writing "How do we make our measurements? By transporting to mutual juxtaposition objects considered as invariable solids, one would reply at first; but this is no longer true in the present theory if one admits Lorentzian contraction. In this theory two equal lengths are by definition two lengths which are traversed by light in equal times."

The elegance of Poincaré's style is most evident in his use of the principle of least action (or the Lagrangian formulation) for Maxwell's equations with sources in vacuum and the equations of motion for charged particles which were shown to be invariant under the Lorentz transformations. He was the first physicist to employ the Lorentz covariant Lagrangian formalism for discussions of physical theories. Today, the elegant Lorentz covariant Lagrangian formalism has become a standard method for treating quantum field theories and relativistic particle dynamics. *Poincaré also considered the inverse of the Lorentz transformations, replacing V with -V and rotating the coordinate axes by 180° around the y axis, to conclude that K=1 for any V consistent with the obvious condition that K=1 when V=0.* Then he used the Lorentz transformation to discuss the dynamics and the contraction of an extended electron in detail. The electromagnetic field due to the presence of the electron can react on the electron itself. The interaction energy between the electron and the electromagnetic field due to the presence of the electron itself is called the self-energy and is an inherent part of the electron. He discussed the self-energy of the electron and noted that an extended electron with negative charge cannot be stable.

In these discussions, Poincaré wrote down the complete Lorentz invariant action integrals of the electron, the electromagnetic fields and their interaction. For example, he discussed quasi-stationary and arbitrary motions of an electron and obtained the exact Lagrangian for the electron: $L = m(1-u^2)^{1/2} = m/\gamma$, $u = |\mathbf{u}|$, in which he called m the experimental mass (i,e, the rest mass) of the electron.[1] Once the Lagrangian for the electron is given, its kinetic properties and equations of motion are completely determined. The correct expressions for the relativistic energy ($L-\mathbf{u}\cdot[\partial L/\partial \mathbf{u}]$) and momentum ($\partial L/\partial u_i=-Du_i/u$, $D=-dL/du$), and the transverse and the longitudinal mass (D/u and dD/du) were given. Furthermore, the exact covariant equation of motion was given by $(d/dt)[m\mathbf{u}\gamma]=\int\rho(\mathbf{E}+\mathbf{u}\times\mathbf{B})d^3x$ (in modern notation), which reduces to the familiar form, $(d/dt)[m\mathbf{u}\gamma]=e(\mathbf{E}+\mathbf{u}\times\mathbf{B})$, for a point charge distribution. (See equations (64), (92), (93), (94), (95) and (95') in Schwartz's translation with modern notations.[1]) This covariant equation of motion for a charged particle is important because it shows for the first time that Newton's second law, $F=ma=d(mu)/dt$, in classical mechanics can be generalized to be consistent with the principle of relativity. This task is difficult to carry out if one does not use the Lorentz invariant Lagrangian and variational calculus. (In contrast, Einstein failed in deriving the exact covariant equation of motion for a charged particle in 1905, as we shall see in chapter 5 below.) All these show Poincaré's mastery of electrodynamics and mathematics and his difference from Einstein in style and taste.

Poincaré concluded that "if one admits this impossibility (of manifesting absolute motion) one must admit that electrons when in motion contract so as to become ellipsoid of revolution whose two axes remain constant; one must then admit ... the existence of a supplementary potential proportional to the volume of the electron." Thus, his principle of relativity led him to discover the "Poincaré stresses" to maintain the stability of an extended electron, a model discussed by Lorentz and others. This was the climax of his paper. However, the electron is now assumed to be a point-like object in quantum mechanics and quantum electrodynamics, so that Poincaré stresses do not play a significant role in modern physics, although they are frequently mentioned. We note that since the assumption of point-like particles is related to the profound divergence difficulty in quantum field theory, it is possible that a non-point-like particle might emerge in the physics of the future and that Poincaré stresses may again become important, unless the quantization of the electric charge and its stability can be understood on the basis of some unexpected new principles.

In this connection, it is interesting to note Poincaré's outlook of the physical world: When an extended electron is at rest, the equilibrium of the Coulomb force and the Poincaré stresses result in a spherical electron. However, when in motion, an electron would suffer a contraction such that the extended electron undergoes a deformation precisely according to the Lorentz transformation. In this sense, *he appeared to believe that his non-electromagnetic stresses gave a dynamical explanation to the contraction of the extended electron.*[1]

In the last part of his regular paper, he discussed a "relativistic theory of gravitation" which was a generalization of Newtonian gravity to be consistent with the principle of relativity. His generalized gravitational theory involved retarded action-at-a-distance interaction and the concept of gravitational waves which were "assumed to propagate with the velocity of light." It was the first such attempt and not satisfactory. However, the significant result is the 4-dimensional symmetry framework (with 4-vectors and Lorentz invariants, etc.) that he developed, anticipating Minkowski, as a powerful tool to study physics. After discussing general properties of the basic scalar equation of propagation and transformation properties of the gravitational force, Poincaré said: "In order to make further progress, it is necessary to look for the *invariants of the Lorentz group* (the italics are Poincaré's.) We know that the transformations of this group (taking K=1) are the linear transformations which do not change the quadratic form $x^2+y^2+z^2-t^2$. .... Let us consider x, y, z, $t\sqrt{-1}$; δx, δy, δz, $\delta t\sqrt{-1}$;.... as coordinates of ... points...in a space of four-dimensions. We see that the Lorentz transformation is just a rotation of this space about the origin, regarded as fixed. We shall have as invariants only the...distances between the points among themselves and the origin,... $x^2+y^2+z^2-t^2$, $x\delta x + y\delta y + z\delta z - t\delta t$, etc...."

Indeed, Poincare's statement concerning the "the *invariants of the Lorentz group*" is precisely a modern view of the theory: "Mathematically speaking, therefore, the special theory of relativity is the theory of invariants of the Lorentz group," as Pauli said in 1921 (when he was 21 years old) in the article on the theory of relativity written for the *Mathematical Encyclopedia*, which later became his book "*Theory of Relativity.*"[2]

The essence of Poincaré's argument for K=1 in (4.1) is the principle of relativity which implies that both frames, F and F', are physically completely equivalent. Otherwise, if one sets $K=\sqrt{1-V^2/c^2}=1/\gamma$, then the trans-formation (4.1) becomes

$$x' = x - Vt\,, \quad y' = \frac{y}{\gamma}\,, \quad z' = \frac{z}{\gamma}\,, \quad t' = t - \frac{Vx}{c^2}\,, \tag{4.2}$$

or

$$\left(x = \frac{x'+Vt'}{1-V^2/c^2}\,, \quad y = \frac{y'}{\sqrt{1-V^2/c^2}}\,, \quad z = \frac{z'}{\sqrt{1-V^2/c^2}}\,, \quad t = \frac{t'+Vx'/c^2}{1-V^2/c^2}\,. \right)$$

This particular transformation was first obtained by Voigt in 1887 in a work in which he discussed Doppler shifts and invariance of wave equations. Clearly, Voigt's transformation preserves the "four-dimensional intervals": $s(0)^2=s'(V)^2$, where

$$\begin{aligned} s(0)^2 &= c^2t^2 - r^2\,, && \text{for frame F (at rest)}\,, \\ s'(V)^2 &= \left(1 - \frac{V^2}{c^2}\right)^{-1}(c^2t'^2 - r'^2)\,, && \text{for frame F' (moving)}\,, \end{aligned} \tag{4.3}$$

or $s(V)^2 = s'(0)^2$, where

$$\begin{aligned} s(V)^2 &= \left(1 - \frac{V^2}{c^2}\right)(c^2t^2 - r^2)\,, && \text{in F (moving)}\,, \\ s'(0)^2 &= c^2t'^2 - r'^2\,, && \text{in F' (at rest)}\,. \end{aligned} \tag{4.4}$$

Note that the four dimensional intervals are not the ones one knows from special relativity but are multiplied by a velocity-dependent factor. This transformation implies that one of the frames is really moving with an "absolute velocity" V. Whether such an "absolute velocity" V can be detected turns out to be non-trivial and is still an open question,[11] as we shall see below in section 4e.

4d. The Concept of an "Aether" Never Fades Away

The concept of aether (or ether) is an old and cherished one which is intertwined with the physical properties of the vacuum as a whole. In the age of Newton and Maxwell, aether meant, among other things, the medium for the propagation of light and *was identified with the classical electromagnetic field.* Its existence was taken for granted, just as the existence of the electromagnetic field. Nowadays, it is simply called the "vacuum" in modern quantum field theory. The properties of the vacuum are so complicated and non-classical from the viewpoint of quantum fields that it is totally beyond the dreams of Faraday, Maxwell, Lorentz, Poincaré and Einstein.

In the 19th century, the universe was believed to be built of objects with mechanical properties. In particular, light was believed to be supported by a material substance, called aether, when it propagated in vacuum. Maxwell and other physicists tried to understand and visualize electromagnetic field as a mechanical stress in a material substance. Some physicists believed ordinary matter to be condensed aether. Others such as Lord Kelvin held that matter is only the locus of those points at which the aether is animated by vortex motions. Riemann believed it to be locus of those points at which aether is constantly destroyed. The properties and the names associated with the aether may change, but the essential idea has never faded away even after Einstein declared it to be superfluous based on his momentous theory of relativity. We note that in the early 20th century, the concept of the state of motion of the aether had to be given up because (a) it was unobservable and (b) it became superfluous to a physical formalism since the electromagnetic field was regarded as an independent physical reality.

Nevertheless, many well-known physicists continued to reexamine and discuss the aether from time to time. For example, in 1920 Einstein himself gave a talk on aether and general relativity. He identified aether with the gravitational field and said that "the aether of the general theory of relativity is a medium without mechanical and kinematic properties, but which codetermines mechanical and electromagnetic events."[12] However, there was no further investigation along this line of thought.

In 1951, Dirac published a paper in the journal Nature with the title "Is there an Aether?"—which is the same question that Poincaré asked in 1900. Dirac discussed the existence of an aether on the basis of modern physics:

"If one reexamines the question in the light of present-day knowledge, one finds that the aether is no longer ruled out by relativity, and good reasons can now be advanced for postulating an aether."[13]

After the significance of gauge theories[12,14] based on Yang-Mills fields was recognized, many theorists discussed the physical properties of the vacuum in the 1970s. More recently in 1981, T. D. Lee discussed vacuum, in his book *Particle Theory and Introduction to Field Theory*, as the source of asymmetry in quantum field theory (related to CP nonconservation and spontaneous symmetry breaking in the unified electroweak theory, etc.) and vacuum as a color dielectric medium for permanent quark confinement.[15] Lee said that

> "...since at that (Faraday's) time the nonrelativistic Newtonian mechanics was the only one available, the vacuum was thought to provide an absolute frame which could be distinguished from other moving frames by measuring the velocity of light.... We know now that vacuum is Lorentz-invariant, which means that just by our running around and changing the reference system we are not going to alter the vacuum."

Here, the vacuum is the aether in old language. Lee pointed out that from Dirac's hole theory we know that the vacuum is actually quite complicated, even though it is Lorentz-invariant and is the lowest energy state of the system. Then Lee asked that important and long-standing question:

"Could the vacuum be regarded as a physical medium?"

This is a question that has engrossed the minds of many physicists such as Huygens, Faraday, Maxwell, Michelson, Lorentz, Poincaré, Einstein, Dirac and others throughout the ages.[14] Lee's answer was as follows:

> "If under suitable conditions the properties of the vacuum, like those of any medium, can be altered physically, then the answer would be affirmative. Otherwise, it might degenerate into semantics."

He then proceeded to analyze and discuss the physical properties of the vacuum (or the aether) on the basis of two remarkable phenomena in modern physics, (i) missing symmetry and (ii) permanent quark confinement.

Who says the concept of the aether is dead ?

Some people may wonder whether Poincaré's acceptance of the principle of relativity is only provisional because he did not give up the concept of the aether. In contrast, Einstein put his faith in the two principles of relativity and abandoned the notion of an aether completely. In this connection, it should be stressed that from the viewpoint of modern quantum field theory, the existence of an aether is not incompatible with the principle of relativity or Lorentz and Poincaré invariance. In this sense, Poincaré's belief turns out to be not completely wrong from a modern field-theoretic viewpoint, and his idea of "absolute rest" appears to be only a question of semantics in the sense of T. D. Lee.

4e. Conformal Transformations for Inertial Frames with Absolute Velocity and "Conformal 4-Dimensional Symmetry" with the Constant Speed of Light

Theoretical physicists and mathematicians like to discuss general cases. In fact, Poincaré derived most of the equations in his relativity paper[1] based on the transformation (4.1) with the arbitrary constant $K(\beta)\neq 1$. He explained that every transformation in this group, which he called the Lorentz group, can be decomposed into a transformation of the form, $r' = Kr$, $t' = Kt$, and a linear transformation that leaves invariant the quadratic form $r^2 - t^2$. Thus, in modern language, Poincaré actually discussed the invariance of Maxwell's equations under conformal transformations. The conformal transformation is a larger class of coordinate transformations for which $c^2t'^2-r'^2$ is proportional, though not generally equal, to $c^2t^2-r^2$, and which therefore also leaves the speed of light universal and invariant, as shown in equations (4.2)-(4.4). Concerning the special case K=1, Poincaré said: "...we have to suppose that K is a function of β and it is a question of choosing this function in such a way that this group portion, which I shall call P, forms also a group." In this connection, it is worthwhile to note that the concept of conformal invariance was explicitly introduced into physics by Cunningham who showed that Maxwell's equations are invariant not only under the Lorentz group (10 parameters), but also under the conformal group C_o (15 parameters).[16]

In connection with the existence of an aether, let us briefly examine further possible physical implications of an absolute velocity (i.e., a velocity relative to a frame of absolute rest) based on the 4-dimensional framework with the conformal symmetry. Suppose we have three inertial frames $F_a = F(0)$, $F' = F'(V)$ and $F'' = F''(U)$, where V and U are the absolute velocities of F' and F'' relative to the absolute rest frame F_a=F(0) along the +x direction. For a general K(V), we assume the four-dimensional intervals as[9]

$$s^2 = c^2t^2 - r^2, \qquad \text{for } F(0),$$

$$s'^2 = K^{-2}(V)\,(c^2t'^2 - r'^2), \qquad \text{for } F'(V), \tag{4.5}$$

$$s''^2 = K^{-2}(U)\,(c^2t''^2 - r''^2), \qquad \text{for } F''(U).$$

If we insist that the four-dimensional interval be invariant, we obtain the following "absolute" transformations:

$$x' = K(V)\frac{x-Vt}{\sqrt{1-\beta_V^2}}, \quad y' = K(V)y, \quad z' = K(V)z, \quad t' = K(V)\frac{t-Vx/c^2}{\sqrt{1-\beta_V^2}}, \tag{4.6}$$

and

$$x'' = K(U)\frac{x-Ut}{\sqrt{1-\beta_U^2}}, \quad y'' = K(U)y, \quad z'' = K(U)z, \quad t'' = K(U)\frac{t-Ux/c^2}{\sqrt{1-\beta_U^2}}, \tag{4.7}$$

where $\beta_V = V/c$ and $\beta_U = U/c$.

It can be shown that the transformations (4.6) and (4.7) form a conformal group because, for example, the transformation between F'(V) and F''(U) has the same form:

$$x'' = \rho\frac{x'-V't'}{\sqrt{1-\beta'^2}}, \quad y'' = \rho y', \quad z'' = \rho z', \quad t'' = \rho\frac{t'-V'x'/c^2}{\sqrt{1-\beta'^2}}, \tag{4.8}$$

where

$$\rho = \frac{K(U)}{K(V)}, \quad V' = \frac{U-V}{1-UV/c^2}, \quad \beta' = \frac{V'}{c}. \tag{4.9}$$

Mathematically, the Voigt transformation (4.2) is clearly a special case of a conformal transformation (4.6). The physical relevance of the conformal transformations is in general not yet clear.[17]

One may interpret absolute transformations such as those shown above as a theory of "conformal four-dimensional symmetry" with an absolute motion and with a constant speed of light c. Thus, it is interesting to see whether these transformations can be excluded by modern high precision experiments of special relativity. Surprisingly enough, it is highly nontrivial to exclude experimentally this type of "absolute" transformation.[11] For example, let us consider experiments designed to measure the Doppler shift to high precision. The transformations of the contravariant wave 4-vector $k^\mu=(k^0,k^1,k^2,k^3)=(\omega/c,k_x,k_y,k_z)$ between two moving frames, $F'\equiv F'(V)$ and $F''\equiv F''(U)$, are given by

$$k''_x = \rho\frac{k'_x - \beta' k'^0}{\sqrt{1-\beta'^2}}, \quad k''_y = \rho k'_y, \quad k''_z = \rho k'_z, \quad k''^0 = \rho\frac{k'^0 - \beta' k'_x}{\sqrt{1-\beta'^2}}, \tag{4.10}$$

where $(k'_x,k'_y,k'_z) = (k'^1,k'^2,k'^3)$ and

$$\rho = \frac{K(U)}{K(V)}, \quad V' = \frac{U-V}{1-UV/c^2}, \quad \beta' = \frac{V'}{c}. \tag{4.11}$$

Suppose one performs the experiment in the $F'(V)$ frame and that the atoms are at rest in $F''(U)$ so that $k''_x = 2\pi/\lambda''$ and $k''^0 = 2\pi\nu''/c$. In practice, one cannot know the wavelength and frequency λ'' and ν'' of the light emitted by the atoms as measured by observers in the $F''(U)$ frame. One can only compare the shifted λ' and ν' with the unshifted quantities λ'_o and ν'_o associated with the same kind of atoms at rest in the laboratory frame $F'(V)$. Since $F'(V)$ and $F''(U)$ are not completely equivalent, as shown in (4.5), the wavelength and the frequency of light emitted by atoms at rest in F' and measured by observers in F' are not the same as those emitted by the same kind of atoms at rest in F" and measured in F". Thus, one does not have the usual relations $\lambda'' = \lambda'_o$ and $\nu'' = \nu_o$ of special relativity. Rather, one has in general

$$\frac{k''_x}{K(U)} = \frac{k'_{xo}}{K(V)} \quad \text{and} \quad \frac{(\omega''/c)}{K(U)} = \frac{(\omega'_o/c)}{K(V)} \tag{4.12}$$

for the contravariant wave 4-vector $k'^\mu = (k'^0,k'^1,k'^2,k'^3)$ because of the metric tensor in (4.5) for the two frames $F'(V)$ and $F''(U)$. Thus, one obtains

$$\frac{1}{\lambda'' K(U)} = \frac{1}{\lambda'_o K(V)} \quad \text{and} \quad \frac{\nu''}{K(U)} = \frac{\nu'_o}{K(V)}. \tag{4.13}$$

It follows from (4.10) and (4.13) that the observable Doppler shifts in the moving frame F'(V) are given by:[11]

$$\frac{1}{\lambda'_o} = \frac{1}{\lambda'}\frac{(1-\beta')}{\sqrt{1-\beta'^2}}, \qquad \nu'_o = \nu'\frac{(1-\beta')}{\sqrt{1-\beta'^2}}, \tag{4.14}$$

which are exactly the same as those in special relativity. Therefore, we see that the absolute velocities V and U cannot be determined individually by the Doppler shift experiment. The best one can do is to determine the "relative velocity" β' between F'(V) and F"(U), as shown by (4.14). Note that if one uses the covariant wave 4–vectors, $k'_\mu = K^{-2}(V)k'^\mu$ and $k''_\mu = K^{-2}(U)k''^\mu$, the relations corresponding to (4.10) and (4.12) will be different. Nevertheless, the final result (4.14) for Doppler shift remains the same. Furthermore, the Michelson–Morley experiment also cannot rule out the validity of the "absolute transformation" (4.6) and (4.7) because the speed of light is still the universal constant c in such a theory with "conformal 4-dimensional symmetry."

Incidentally, these results are consistent with Poincaré's view that the aether exists but that the absolute velocity of a moving frame cannot be detected. Nevertheless, there is a difference: Namely, there is *no conspiracy* of dynamical effects due to the interaction of aether and matter so that the velocity of an object moving through the aether cannot be detected by optical phenomena. Rather, the effects of the absolute velocities V and U are suppressed and disappear from the Doppler shift experiment and the Michelson–Morley experiment because of the *inherent properties of the conformal 4–dimensional symmetry.*

4f. Poincaré's Contributions to Relativity and Symmetry Principle

Poincaré's paper[1] gave a complete logical foundation for relativity theory, which includes the mathematical framework and the basic invariant equations of motion for both electromagnetic fields and charged particles. There is little doubt that Einstein formulated the theory of special relativity independently, even though conceptually he may have benefited from Poincaré's nontechnical articles and Lorentz's early papers, as we shall discuss in chapter 5. It is certain that there is no experimental difference between Poincaré's and Einstein's relativity theories because their equations are identical, provided Einstein's approximate and non-Lorentz invariant dynamical equations for charged particles are properly corrected. (See chapter 5.)

Both Poincaré and Einstein should be credited for the discovery and the establishment of the fundamental principle of relativity for physical laws. Thus we shall call this fundamental symmetry principle in physics the "Poincaré-Einstein Principle." In our discussions, the following statements may be taken to be equivalent:

(a) the Poincaré-Einstein principle,
(b) the first postulate of relativity,
(c) the 4-dimensional symmetry of the Lorentz group,
(d) the 4-dimensional symmetry,
(e) the invariance of physical laws,
(f) the equivalence of all inertial frames,
(g) the Lorentz and Poincaré invariance.

Henri Poincaré was generally acknowledged to be the most outstanding mathematician at the turn of the 20th century. He was a man with enormous interest and rare insight in all branches of mathematics, astronomy and theoretical physics. After Gauss, he was the last mathematical universalist—doing creative work of high quality in all four main divisions of mathematics, arithmetic, algebra, geometry and analysis.[18] In addition, Poincaré made significant contributions towards the theory of special relativity in the years from 1895 to 1905 through his strenuous effort.

The main reason that Poincaré did not achieve a complete grasp of the very essence of relativity (or "reciprocity") appears to be because of his fundamental viewpoint — namely that the aether exists and there is a conspiracy of *dynamical effects* due to interaction of aether and matter such that the velocity of an object moving through the aether cannot be detected. This outlook of the physical world urged him to search for a dynamical explanation for the length contraction of the electron, as he did in his Rendiconti paper.[1] He did not have a satisfactory answer to explain the conspiracy of dynamical effects. Apparently, a dynamical explanation for the length contraction was very natural and popular among physicists at that time. FitzGerald also speculated that "the molecular force are affected by the motion, and that the size of a body alters consequently."[19] Even as late as 1912, three months before his death, Poincaré delivered a lecture (to the French Society of Physics, April 11, 1912) entitled "The Relations Between Matter and Ether."[20] A new aether may actually exist according to modern theory of gauge fields, but it will be quite different from the aether that Lorentz and Poincaré had in mind. Furthermore, one probably cannot prove experimentally that his belief in a conspiracy of *dynamical effects* due to interaction of aether and matter is wrong. However, as far as relativity theory is concerned, it simply makes the theory more complicated conceptually and is not necessary. In this aspect, Einstein had a more profound understanding: Namely, **it does not need to be explained!!!**

We may remark that Poincaré was not alone in scientific research to have had an incorrect conceptual interpretation of his equations. Maxwell also had wrong ideas of molecular vortices and idle wheels for the electromagnetic field in his equations, but physicists still call them Maxwell's equations.

Why did this thing happen?

In an interesting discussion of innovation in physics, Dyson made a piercing observation:[21] "When the great innovation appears, it will almost certainly be in a muddled, incomplete and confusing form. To the discoverer himself it will be only half-understood; to everybody else it will be a mystery. For any speculation which does not at first glance look crazy, there is no hope." He explained that "the reason why new concepts in any branch of science are hard to grasp is always the same; contemporary scientists try to picture the new concept in terms of ideas which existed before. The discoverer himself suffers especially from this difficulty; he arrived at the new concept by struggling with the old ideas, and the old ideas remain the language of his thinking for a long time afterward." This makes it clear why a young scientist has an edge over a mature scientist in grasping a new concept. For example, one can see a difference between the young Einstein and the mature Poincaré regarding having a sound grip of the "crazy" properties of space, time and relativity related to the Lorentz transformations around 1905; or of the young Heisenberg versus the mature Einstein in understanding the "crazy" uncertainty principle in quantum mechanics around 1927.

It seems fair to say that Poincaré ushered symmetry principles onto the stage of twentieth century physics because of

(i) his 1904 proposal[9] of the symmetry principle of relativity for all laws of physics,

(ii) his recognition[1] that the Lorentz transformations of space and time preserve the invariant $x^2 + y^2 + z^2 - c^2t^2$ and form a symmetry group, and that the Lorentz group has 6 generators (three for spatial rotations and three for boosts),

(iii) his insight into the relation between the Lorentz transformations and a 'rotational symmetry' in a 4-dimensional space,[1]

(iv) his use of the *invariants of the Lorentz group* and the Lorentz covariant Lagrangian formalism for treating fields and particle dynamics, which is particularly powerful for treating a physical system with symmetry properties.

These insights show that his incomparable understanding of the symmetry properties of the physical world was deeper than anybody else's at the very beginning of the twentieth century. With the help of the enormous impact of special relativity and general relativity, symmetry principles are now generally believed to play a universal and fundamental role in revealing the simplicity of the physical world. In particular, the view that *symmetry dictates interactions* took root mainly through the works of H. Weyl and of C. N. Yang and R. L. Mills in modern quantum field theory and was stressed in particular by Yang.[22] The most spectacular results of the power of symmetry principles shows up in Dirac's prediction of the existence of anti-particles, in the establishment of the unified electroweak theory and quantum chromodynamics based on non-Abelian gauge fields discovered by Yang and Mills. Today, one hundred years after Poincaré proposed the symmetry principle of relative motion for all physical laws, symmetry principles in physics have transcended both kinetic and dynamic properties and gone right to the very heart of our understanding of the universe.

In this connection, we may remark that although Poincaré did not discuss the most general symmetry group in 4-dimensional spacetime (i.e., the inhomogeneous Lorentz group generated by four translations and six rotations), this group is nowadays called the "Poincaré group" by many physicists and mathematicians. Furthermore, "Lorentz and Poincaré invariance" is now a standard term in the 1997 ICSU/AB International Classification System for Physics, and in Physics and Astronomy Classification Scheme. These attributions appear to be in memory of his insight of the 4-dimensional symmetry of spacetime. The Poincaré group has important applications in particle physics, as shown by Wigner and others.[23]

To many physicists, it is apparent from Poincaré's reasoning that the speed of light is not really a universal constant in all inertial frames; it only appears to be a constant in those frames moving with respect to the aether as a result of compensating effects of motion on length and time.[24] Indeed, this may explain why he simply defined $c \equiv 1$ by specially choosing the units of length and of time, and why he did not call the constancy of the speed of light a postulate.[25] Furthermore, he did not mention the possibility of experimentally testing the properties of the speed of light, in contrast to the 'postulate of relativity.' In this sense, Poincaré's viewpoint turns out to be closer to the recent result of taiji relativity (in chapter 7) based *solely* on the Poincaré–Einstein principle of relativity.[26] From this recent vantage point, it was Poincaré's profound insight to treat "the invariance of physical laws" as a fundamental postulate and "the constancy of the speed of light" as a definition. This view has not been appreciated for nearly a century.

After special relativity was established, there was some work towards formulating a theory of relativity without introducing any postulates or definitions concerning the speed of light. From this point of view, Poincaré may be credited as a pioneer in the search toward logically the simplest theory of relativity theory based solely on one single postulate.

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2. Sir E. Whittaker, *A History of the Theories of Aether and Electricity: The Modern Theories 1900–1926* (Philosophical Library, New York, 1954), p. 40 and in *Biographical Memoirs of Fellows of the Royal Society* (London, 1955), p. 42.
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4. B. Hoffmann, Relativity, in *Dictionary of the History of Ideas*, vol. IV (Ed. P. P. Wiener, Charles Scribner's Sons, New York, 1973), pp. 74–92. Hoffmann observed that "along with the later fame of Einstein there grew a popular mythology correctly attributing the theory of relativity to him, but seriously slighting the work of Poincaré." In ref. 1, C. Cuvaj also said: "Frequent misunderstanding of the work of Poincaré and Einstein has resulted in controversy tending to obscure the main achievements of Poincaré." Nevertheless, physicists such as Pauli, Feynman, J. J. Sakurai, Pais and many others gave Poincaré credit for his endeavors towards relativity theory. See W. Pauli, *Theory of Relativity* (Pergamon Press, London, 1958), p. 1 and p. 21 ; R. P. Feynman, *The Character of Physical Law* (MIT Press, 1965), pp. 91–92; J. J. Sakurai, *Advanced Quantum Mechanics* (Benjamin/Cummings, 1984), p. 12 and Pais, in ref. 1. For different views of Poincaré's contributions, see also C. Scribner, Jr., Am. J. Phys. **32**, 672 (1964); C. Lanczos (one of Einstein's closest collaborators), *The Variational Principles of Mechanics* (4th ed. Univ. of Toronto Press, 1977), pp. 291–293; C. Cuvaj, ref. 1, and references therein.
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7. H. Poincaré, Arch. Néerl. [2], 5, 271 (1900). See also *Science and Hypothesis* (first published in 1902, Dover Publications, Inc. 1952), p.111. Some physicists interpreted Poincaré's principle of relative motion to be simply the principle of relativity for mechanical experiments. The interpretation is untenable: In his address to the Paris Congress in 1900, he discussed whether optical and electrical phenomena are influenced by the motion of the earth and said that "many experiments have been made on the influence of the motion of the earth. The results have always been negative."
8. The English translation of this address can be found in H. Poincaré, *Science and Hypothesis* (first appeared in 1902, Dover Publications, Inc. 1952), pp. 140–182.
9. H. Poincaré, Bull. Sci. Math. 28, 302 (1904). This address was translated into English by G. B. Halsted and published in The Monist, 15, 1 (1905) and also in chapters 7 to 9 of *The Value of Science* by Poincaré, (in *The Foundations of Science*, G. R. Halsted, Tran. Science Press, New York, 1913; or Dover, New York, 1958). To appreciate the vision of Poincaré in this talk, it is interesting to note that he even speculated about the future of physical law: "Physical law will then take an entirely new aspect; it will no longer be solely a differential equation, it will take the character of a statistical law." This prophecy was finally realized by the discoveries of Schrödinger's wave equation and Born's statistical interpretation of the wave function more than 20 years later.
10. A. Pais, in ref. 1, p. 150 and pp. 167–168. Pais discussed Poincaré's third postulate for a new mechanics in 1909 and reached a conclusion: "It is evident that as late as 1909 Poincaré did not know that the contraction of rods is a consequence of the two Einstein's postulates. Poincaré therefore did not understand one of the most basic traits of special relativity." However, this third hypothesis may also be viewed from a different angle: We note that Poincaré made a great effort (in his Rendiconti paper in ref. 1) to give a consistent dynamical theory of an extended electron: "It is necessary to assume a special force which explains simultaneously both the contraction and the constancy of two of the axes. I have sought to determine this force, and I have found that it can be represented by a

constant external pressure acting on the deformable and compressible electron, whose work is proportional to the change in volume of this electron." Furthermore, he also stated in his paper: "How do we make our measurements? By transporting to mutual juxtaposition objects considered as invariable solids, one would reply at first; but this is no longer true in the present theory if one admits Lorentzian contraction. In this theory two equal lengths are by definition two lengths which are traversed by light in equal times." Thus, he knew that the length contraction follows from the Lorentz transformations which can be obtained on the basis of the principle of relativity and the definition c=1 for the speed of light. In the last part of his Rendiconti paper, Poincaré explained: "We know that the transformations of this group (taking K=1) are the linear transformations which do not change the quadratic form $x^2+y^2+z^2-t^2$." Therefore, it appears to be more reasonable to interpret his third hypothesis as a manifestation of his views that the contraction of length is a dynamical result rather than simply a kinematic property of the Lorentz transformations. Since he did not have a satisfactory explanation for the conspiracy of dynamical effects in general, he cast it as a hypothesis. Of course, Poincaré's view differs from our present understanding of length contraction in special relativity as a kinematic property of the Lorentz transformations. However, the difference is perhaps only a semantic one in the sense that there is no experimental difference between his "dynamical contraction" and Einstein's kinematic contraction of lengths because the basic dynamical equations for particles and fields in Poincaré's relativity are identical to those in Einstein's relativity.

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the elementary particles. See also E. C. G. Sudarshan, "Some Problems of Natural Philosophy" (CPT preprint, UT at Austin, Presidential Address to the Symposium on Particle Physics, Univ. of Madras, Madras, India, January, 1971).

14. For example, theoretical physicists J. Bjorken gave a talk on "The New Ether" in the Niels Bohr Centennial Symposium (Nov. 1985, Boston), and V. Weisskopf on "The Vacuum in Quantum Field Theories" at Harvard University in Nov., 1985. More recently, in explaining his theory of QCD Lite, F. Wilczek stated: "Most of the mass of ordinary matter, for sure, is the pure energy of moving quarks and gluons. The remainder, a quantitatively small but qualitatively crucial remainder – it includes the mass of electrons – is all ascribed to the confounding influence of a pervasive medium, the Higgs field condensate." In other words, "what we call empty space, or vacuum, is filled with a condensate spawned by Higgs field." (See Physics Today, January 2000, page 13.)
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25. This appears to be consistent with conventionalism (H. Poincaré and Pierre Duhem) which attempted to do justice to the arbitrary elements in theory construction and held that all coherent scientific theories have equal validity and, therefore, one theory cannot be more true than another. For example, according to Poincaré, the geometrical axioms are "*neither synthetic a priori intuitions nor experimental facts.*" "They are conventions. Our choice among all possible conventions is *guided* by experimental facts; but it remains *free*, and is only limited by the necessity of avoiding every contradiction, and thus it is that postulates may remain rigorously true even when the experimental laws which have determined their adoption are only approximate. In other words, *the axioms of geometry* (I do not speak of those of arithmetic) *are only definitions in disguise.* What, then, are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true and if the old weights and measures are false; if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true than another; it can only be more convenient." (The italics are Poincaré's.) See H. Poincaré, *Science and Hypothesis* (first appeared in 1902, Dover Publications, Inc. 1952), p. 50.
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5.

Young Einstein's Novel Creation Based on 2 Postulates

"Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labeled as 'conceptual necessities,' '*a priori* situations,' etc. The road of scientific progress is frequently blocked for long periods by such errors."

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ALBERT EINSTEIN: Person of the Century
"He was the pre-eminent scientist in a century dominated by science. The touchstones of the era—the Bomb, the Big Bang, quantum physics and electronics—all bear his imprint."

Frederic Golden, TIME, Dec. 31, 1999, page 62

5a. The Power of a Young Mind

In contrast to Lorentz and Poincaré, Einstein was a young physicist at the turn of the century and hence had more fresh ideas and more flexibility in his thinking. Rather than being rooted in the ideas of the traditional physicists of his time, he was able to discard the idea of an aether without any reservations. His approach to the whole problem was quite novel, mostly aesthetic and logically deductive, rather than mathematical. Furthermore, the young Einstein showed a more profound understanding of the heart of the matter, namely, the physical properties of space and time.

Although Einstein did not refer to other physicists' earlier works in his first 1905 paper on relativity, it is certain that he had known of the work of Poincaré and Lorentz.[1] These days, if an unknown physicist were to submit a research paper without any references to previous work by other scientists, it is certain that the paper would be rejected by any of the well-known journals such as *Physical Review* or *Nuovo Cimento*. Presumably, this absence of references was the sign of a young rebel outside the academic circle. It certainly made it more difficult for later historians to trace the origin of special relativity and to assess the originality of his paper.

Around the time Einstein was formulating his ideas about relativity, his personal life was in turmoil. He was a young theoretical physicist who tried and failed for years to find an academic position after graduating. He was passionately in love with his classmate, Mileva Maric, and they had a baby girl, Lieserl, before their marriage but were forced to give the baby away. Moreover, his father passed away and his mother was vehemently opposed to Einstein's marriage. It was the worst time in Einstein's life, yet the best time for Einstein's physics. This curious circumstance is not unlike what has been known for a poet and his poems.

In his first 1905 paper on relativity,[2] Einstein derived all of the essential results contained in Poincaré's papers of 1905, except the exact covariant equations of motion for a charged particle. These results are now standard textbook material and, hence, will not be discussed in detail here. To read his paper is to see the exuberant new ideas and the beauty of nature. Although most people tend to think that Einstein created special relativity all by himself, it is more reasonable to consider special relativity as both an effect and a cause of scientific progress.

5b. Einstein's Formulation of Special Relativity with 2 Postulates

Einstein's approach to special relativity was as follows: He simply observed that "the phenomena of electrodynamics and mechanics possess no properties corresponding to the idea of absolute rest", and from this idea, proceeded to formulate special relativity entirely on the basis of two postulates or principles:

(PI) The form of a physical law is the same in any inertial frame.

(PII) In all inertial frames, the speed of light c is the same whether the light is emitted from a source at rest or in motion.

The first postulate was originally stated by Einstein as follows: "The laws in accordance with which the states of physical systems vary are not dependent on whether these changes of state are referred to one or to the other of two coordinate systems that are in uniform translational relative motion."[2] It is the same as Poincaré's 'principle of relative motion'[1] proposed in 1900 or his 'principle of relativity' as discussed in the1904 address in St.

Louis: ".... the laws of physical phenomena must be the same for a 'fixed' observer or for an observer who has a uniform motion of translation relative to him ..." This first postulate implies that all inertial frames are equivalent.

The second postulate is also known as the principle of the constancy of the speed of light. This particular property of the speed of light in vacuum (or of electromagnetic waves in general) was actually built into the Voigt transformation (3.5) in 1887 and the exact Lorentz transformations (3.4) obtained by Larmor (1898) and Lorentz (1899). From this historical viewpoint, neither of the two statements above postulated by Einstein was anything original. However, it took Einstein's extraordinary insight to understand the role they play, to combine these two postulates of physics together consistently, and to extract their revolutionary implications regarding space and time, which were so different from the widely accepted Newtonian ideas of space and time.

The second postulate was originally stated by Einstein in the beginning of his paper as follows: 'Every light ray moves in the "stationary" coordinate system with the fixed velocity c, independently of whether this ray is emitted by a stationary or a moving body.'[2] Schwartz has pointed out that Einstein used an alternative expression of his second postulate during the derivation of the coordinate transformations, namely: "The speed of light in vacuum has the same value in every inertial frame." Are these two versions of the second postulate the same?

Let us consider the first version: The speed of light c is independent of the motion of the light source. That is, we have

$$c(\text{source a}) = c(\text{source b}) \tag{5.1}$$

for any two light sources a and b. This statement can also be seen in Pauli's book, *Theory of Relativity*, and many other textbooks. Taken as it stands, it is not a complete statement of the second postulate of special relativity and can lead to misunderstandings. For example, suppose we have N light sources moving with different velocities. In general, the speed of light emitted by the source S_i, $i=1,2,3,...N$, can be denoted by c_i and c'_i in the frames F and F' respectively. Given this, the statement of the postulate (5.1) implies only that

$$\begin{aligned} &c_1 = c_2 = \dots\dots = c_N = c\,, && \text{in F frame}\,, \\ &c'_1 = c'_2 = \dots\dots = c'_N = c'\,, && \text{in F' frame}\,. \end{aligned} \tag{5.2}$$

Mathematically, (5.2) means that we can use one single equation

$$x^2 + y^2 + z^2 = c^2t^2 \tag{5.3}$$

in F to describe the propagation of a light signal emitted from any source S_i, $i=1,2,....,N$. Similarly, we can use the equation

$$x'^2 + y'^2 + z'^2 = c'^2t'^2 \tag{5.4}$$

in a different frame F' (moving at a constant speed relative to F) to describe the propagation of light emitted from any of the sources S_i, $i=1,2,......,N$. However, (5.1) does not say explicitly whether c and c', i.e., the speeds of the same light signals measured in F and F', are the same or not. We note that both equations (5.3) and (5.4) have the same form and, therefore, are consistent with the Poincaré-Einstein principle of relativity. In other words, the Poincaré-Einstein principle by itself cannot tell us whether the values c and c' are the same or not because it is concerned only with the form of physical laws and not with the values of physical quantities which may appear in them.

Einstein's second version of the second postulate has also been widely used in the literature. In light of the previous discussions, we see that this is also incomplete. A complete statement of the second postulate must address both the motion of the source and the observer, as it does in (PII).

Much discussion and confusion has surrounded the second postulate of relativity regarding the question of whether or not it can be tested experimentally. Some physicists say that it can be tested, while others say it cannot.[3] Both sides are partially right, but remain incomplete in that they consider only some of the physical properties implied in the universality of the speed of light. Let us now see what we mean by this.

The universality of the speed of light c consists of two distinct physical properties:

(a) The value of c is independent of the motion of the light source.

(b) The value of c is independent of the motion of the reference frame.

While (a) can be tested in any inertial frame, (b) cannot. The key to understanding the difference lies in a subtle point concerning the synchronization of clocks. Property (a) is independent of the synchronization of clocks. For example, to test (a), one may put two stable atomic clocks C_A and C_B at rest at different places A and B in the same frame, making sure that they tick with the same rate. To check whether they are ticking with the same frequency, it could be arranged for the clocks to emit light pulses with every tick and an observer standing halfway between the clocks could see whether or not he receives the light signals at the same rate. It is not necessary to synchronize the clocks to read the same time.

Now, we set things up so that a source S_1 moving with a velocity V_1 emits a light signal from A when $t_{A1}=0$ and an observer records its arrival time t_{B1} at B. We repeat the same procedure for another source S_2, where $V_2 \neq V_1$, and record the times $t_{A2}=0$ (by resetting clock A) and t_{B2}. If $t_{B1}=t_{B2}$, then one confirms the independence of the speed of light from the motion of its source without ever knowing an explicit value for c in different frames.

Property (b) however, is interlocked with the synchronization of clocks in different frames. To test (b), we must measure the value of the speed of the same light signal in two different frames, F and F', which are in relative motion. We stress that there is no substitute for measuring the (one-way) speed of light in this way.[4] Before we measure, we must synchronize the readings of *identical clocks* at different places in F and F', where "identical" means that the rate of ticking of the F-clocks measured in F is the same as that of the F'-clocks measured in F'. However, in order to synchronize their readings by the usual procedure we must assume that the (one-way) speeds of light in F and F' are the same because this is the basis for fixing the relation between t and t'. After synchronization, times t and t' are related by the Lorentz transformation. There is a vicious circle here: When we use these synchronized clocks in F and F' to measure the (one-way) speed of light, then we get the same values because that is precisely what we have assumed in order to adjust the readings of identical clocks in different frames in the first place. *Thus, property (b) is nothing but a truth by definition. A definition cannot be proved wrong. As Einstein said, it is a free invention of mind.*

In this connection, it is interesting to note that, in his book *Science and Hypothesis,*[5] Poincaré examined with the utmost care the role of hypothesis: "There are several kinds of hypothesis; that some are verifiable, and when once confirmed by experiment become truths of great fertility; that others may be useful to us in fixing our ideas; and finally, that others are hypothesis only in appearance, and reduce to definitions or to conventions in disguise.

The latter are to be met with especially in mathematics and in the sciences to which it is applied. From them, indeed, the sciences derive their rigour; such conventions are the result of the unrestricted activity of the mind, which in this domain recognises no obstacle. For here the mind may affirm because it lays down its own laws; but let us clearly understand that while these laws are imposed on our science, which otherwise could not exist, they are not imposed on Nature." It is clear that property (b) of the second postulate of special relativity , i.e., that the value of c is independent of the motion of the reference frame, is of the latter sort.

Sometimes, following statements were made:[6]

(i) "From the principle of relativity it follows in particular that the velocity of propagation of interaction is the same in all inertial systems of reference. Thus the velocity of propagation of interactions is a universal constant. This constant (.....) is also the velocity of light"

(ii) "The consequence of the Maxwell-Lorentz equations that in a vacuum light is propagated with the velocity c, at least with respect to a definite inertial system K, must therefore be regarded as proved. According to the principle of special relativity, we must also assume the truth of this principle for every other inertial system."

These statements are not completely satisfactory because *the principle of relativity, in the sense that all inertial frames are equivalent, implies only that the form of physical laws should be invariant but not that the values of parameters in equations such as the speed of light should be invariant.* We shall show a counter example to these statements later: The existence of Taiji Relativity solely based on the principle of relativity clearly shows that these view points are incorrect. (See chapter 7.)

5c. The Derivation of the Lorentz Transformations

Einstein showed that the coordinate transformation between F and F' can be derived from the two postulates. He started from the form invariance of the law for the propagation of light, as shown in (5.3) and (5.4) with c'=c and then obtained the transformation equations in (3.3) where K is an arbitrary scale factor depending on V only. Based on the arguments that

(a) the product of this transformation (2.11) and its inverse should yield the identity, so that $K(V)K(-V)=1$, and

(b) the transformations on y and z should not change if $V \to -V$, thus $K(V)=K(-V)$,

it follows that $K(V)=1$ for any V, because $K(0)=+1$ for $V=0$. The crucial point in these arguments is that the inverse transformation must have the same form as the original transformation, according to the principle of relativity. (Note that eq.(4.1) corresponds to (3.3) with $K(V)=\sqrt{1-V^2/c^2}=1/\gamma$. In this case, the transformation and its inverse do not have the same form.)

These days the Lorentz transformation is usually derived from the invariance of the four-dimensional space-time interval,

$$s^2 = c^2 t^2 - \mathbf{r}^2 = c^2 t'^2 - \mathbf{r}'^2 = s'^2, \tag{5.5}$$

rather than using (5.3) and (5.4) with $c'=c$. In this way one immediately obtains the Lorentz transformation without having to find K(V). Since mathematically, (5.5) is a definition of the four-dimensional space-time interval, but the Lorentz transformation is supposed to be derived from the form invariance of a physical law, one may ask:

To what physical law does the equation in (5.5) correspond?

The answer depends on the values of s^2. When $s^2>0$, (5.5) is equivalent to the equation of motion of a non-interacting particle with a mass $m>0$ moving with a constant velocity $\mathbf{v}=\mathbf{r}/t$ in an inertial frame:

$$(E/c)^2 - \mathbf{p}^2 = m^2 c^2, \tag{5.6}$$

$$E=\frac{mc^2}{\sqrt{1-\mathbf{v}^2/c^2}}, \qquad \mathbf{p}=\frac{m\mathbf{v}}{\sqrt{1-\mathbf{v}^2/c^2}}, \qquad \mathbf{v}=\frac{\mathbf{r}}{t}, \qquad m>0. \tag{5.6a}$$

When $s^2=0$, (5.5) is the equation of motion of a massless non-interacting particle such as a photon. In this case, we still have the relation, $(E/c)^2 - \mathbf{p}^2 = 0$, however, (5.6a) is no longer applicable. When $s^2<0$, (5.5) has no physical meaning, i.e. it is not a law of motion for any known physical object. (One might say that (5.5) with $s^2<0$ describes the motion of a "tachyon" with $v>c$ and $m^2<0$, but tachyons have never been detected experimentally.)

A rigorous proof of the equivalence of the spacetime interval (5.5) and the physical laws (5.6) can be done with the help of a covariant variational calculus.[7]

5d. Novel Relative Properties of Space and Time

The new properties of space and time are the most difficult concepts to understand in special relativity. The first clear and complete explanations were given by the young Einstein with little historic burden of preconceptions in his 1905 paper. Let us consider the novel properties of space and time in the Lorentz transformations (3.2), i.e.,

$$x'=\gamma(x-Vt), \quad y'=y, \quad z'=z, \quad t'=\gamma(t-Vx/c^2); \quad \gamma=\frac{1}{\sqrt{1-(V/c)^2}}. \tag{5.7}$$

(A) Relativity of length.

The length of a rod does not have an absolute meaning because its length depends on who measures it. Suppose there is a rod at rest in the F' frame, parallel to the x'-axis. Its length measured by observers in its rest frame is called the *proper length of a rod*:

$$L'_0 = x'_2 - x'_1, \tag{5.8}$$

where x'_2 and x'_1 are the coordinates of the two ends of the rod in this frame F'. What is its length as measured in another frame F? To determine its length L in F, one must find the coordinates x_2 and x_1 for the two ends of the rod at the same time $t_2=t_1=t$ in F. According to the Lorentz transformation (5.7), one finds

$$x'_2=\frac{x_2-Vt}{\sqrt{1-V^2/c^2}}, \qquad x'_1=\frac{x_1-Vt}{\sqrt{1-V^2/c^2}}, \tag{5.9}$$

which are the two ends of the rod as measured at the same time t in F. Thus, one obtains the relation

$$L=L'_0\sqrt{1-V^2/c^2}, \qquad L=x_2-x_1, \qquad t=t_2=t_1, \tag{5.10}$$

which shows that a moving rod with a length L is contracted by a factor $\sqrt{1-V^2/c^2}$ in its direction of motion. We have seen that in the frame in which the rod is at rest, it has its greatest length (which is also called the proper length of the rod). If the rod is parallel to the y- or z-axis, its length will not be changed due to motion along the x-axis, but its thickness will be.

The contraction of length in (5.10) is called the Lorentz contraction. We note that the original FitzGerald–Lorentz contraction is absolute rather than relative, because it is assumed to be a real physical contraction when the rod has absolute motion, i.e. motion relative to the aether. However, in special relativity, there is no absolute motion and, hence, no absolute contraction. If one puts a rod L_0 at rest in F and parallel to the x-axis, the length L' of this rod as measured by observers in F' at a certain time t' will be

$$L' = L_0 \sqrt{1-V^2/c^2} , \qquad (t' = t_2' = t_1') . \tag{5.11}$$

Comparing (5.10) and (5.11), we see the relativity of length contraction. (This is intimately related to the relativity of simultaneity, i.e. $t_2'-t_1'=0$ and $t_2-t_1=0$ cannot be both true when the relative velocity V between F and F' frames is not zero, $V \neq 0$.)

Now suppose there is a meter stick at rest in F and another meter stick at rest in F'. One may wonder: if one compares them, which meter stick is really contracted? This question cannot be answered until the conditions for their comparison are defined in terms of measurement of length. Different results for length contraction such as (5.10) and (5.11) are obtained under different conditions of measurement. Nevertheless, if one does not use the Lorentz transformations with a certain simultaneity condition to compare them, can one know whether the length of a meter stick at rest in F has the same length as another meter stick at rest in F'? The answer is affirmative, according to the Poincaré–Einstein principle of relativity, which implies the complete equivalence of the two inertial frames F and F'.

Why? If one is not satisfied with this answer, there is really no deeper reason than the Poincaré–Einstein principle of relativity. As mentioned before in chapter 4, sec. 4f, the symmetry principles, as we understand them at the present state of knowledge of physics, have transcended both kinetic and dynamic properties and gone right into the very heart of our understanding of the universe. However, one may explain it in a different way: Suppose two identical meter sticks are made by the same factory in F, and one moves a meter stick to another inertial frame F'. The rigidity of a meter stick depends on the stability of atomic structure which, in turn, depends on fundamental constants and the Dirac equation. (Note that there are truly universal and fundamental constants and there are 'fundamental constants' that appear right now to be universal. This point will be elaborated later in chapter 10.) If the fundamental constants and the Dirac equation are the same in F and F', then the Bohr radius of a hydrogen atom at rest in F must be the same as that of another hydrogen atom at rest in F'. Therefore, the length of a meter stick at rest in F has the same length as meter stick at rest in F':

$$L_0'(\text{at rest in F'}) = L_0(\text{at rest in F}) . \tag{5.12}$$

(B) Relativity of time.

The time interval between two events does not have an absolute meaning because it, too, depends on the conditions of measurement. To discuss this, we must first set up a clock system in each frame. Suppose one has clocks with identical mechanisms of ticking, some are at rest in F and others are at rest in F'. Einstein's procedure of synchronizing them is as follows:

A light signal is sent out from a point a at time t_{1a} (according to the clock at a) to another point b, arriving at time t_b (according to the clock at b), and reflected back by a mirror to point a at time t_{2a} (again, according to the clock at a). We adjust the reading of the clock at b to satisfy the condition

$$t_b = \frac{1}{2}(t_{2a} - t_{1a}) + t_{1a} . \tag{5.13}$$

This is called *Einstein's synchronization of clocks in an inertial reference frame*. By this procedure, one can synchronize all clocks in any given frame. As a result, the times t' and t in F' and F respectively will be related by the last relation in the Lorentz transformation (5.7). Einstein's synchronization procedure is the operational definition of relativistic time. It gives the precise physical meaning of time in the Lorentz transformation. Furthermore, it helps us to see that Newtonian absolute time cannot be realized by an operational procedure and, hence, that absolute and unique time for all observers is unphysical.

Now suppose a clock is at rest in the F frame at the point $\mathbf{r}=(x,y,z)$. If a ball hits this point $\mathbf{r}$ at time t_1 and another ball arrives at the same point $\mathbf{r}$ at time t_2, the time interval for these two events is

$$\Delta t = t_2 - t_1 , \tag{5.14}$$

according to the clock at the point **r**. This is known as the proper time between the two events. Now what is the time which elapses between these two events in another frame F'? From the transformation (5.7), we find

$$t_1' = \frac{t_1 - Vx/c^2}{\sqrt{1-V^2/c^2}} , \qquad t_2' = \frac{t_2 - Vx/c^2}{\sqrt{1-V^2/c^2}} . \tag{5.15}$$

It follows that

$$\Delta t' = t_2' - t_1' = \frac{\Delta t}{\sqrt{1-V^2/c^2}} , \quad (\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}) , \tag{5.16}$$

as observed by people in the F' frame. To observers in F', the clock at **r** in F is moving and the result (5.16) shows time dilation, that is, a moving clock appears to slow down: $\Delta t' > \Delta t$.

We stress again, the time dilation is always relative in special relativity. To compare the rate of ticking of clocks in two frames, we require two or more clocks in one frame and one clock in the other frame. The clocks that slows down is always the one which is being compared with different clocks in the other system.[8] (See Appendix D.)

5e. Physical Implications of Einstein's Special Relativity

After Einstein derived the Lorentz transformations, noted that they form a group and proved the invariance of Maxwell's equations under the Lorentz transformations, he proceeded to derive and discuss many interesting physical results. Apart from discussions of physical properties of space and time, he obtained the law of velocity addition, the law of aberration, the Doppler shift for any angle between a monochromatic light ray and the x-axis, and the relativistic kinetic energy $mc^2(\gamma - 1)$. All these are exact results.

However, when he discussed the equation of motion of a charged particle in an external electromagnetic field, he obtained only approximate results which hold for the first order terms in v/c:

$$m\gamma^3\frac{d^2x}{dt^2} = eE_x, \quad m\gamma\frac{d^2y}{dt^2} = e(E_y - vH_z/c), \quad m\gamma\frac{d^2z}{dt^2} = e(E_z + vH_y/c), \quad (5.17)$$

where the velocity v is in the x direction. This unsatisfactory situation was noted and resolved by Max Planck in 1906 by using the correct relativistic momentum $\mathbf{p} = mv/\sqrt{1-v^2/c^2}$ and its transformation laws, apparently without being aware of Poincaré's previous work published in 1906.[9]

We shall not go into details of Einstein's 1905 paper since all physical results in special relativity are included in our following discussions in this book as a special case (i.e., by setting w = ct and w' = ct' in taiji relativity) or a limiting case when accelerations are involved.

5f. Einstein and Poincaré

Let us briefly compare Einstein's and Poincaré's work up to 1905. Poincaré finished his last major paper on relativity at the peak of his fame and intellectual power. The young Einstein wrote his first paper on special relativity and started vigorously to pursue relativity and its generalization with all his ingenuity and with many more fruitful results later. Both were influenced by the pioneer work of Lorentz. These three physicists made the fundamental contributions to the theory of special relativity.

It is fair to say that both Einstein and Poincaré realized that "the only possible way in which a person moving and a person standing still could measure the speed (of light) to be the same was that their sense of time and their sense of space are not the same, that the clocks inside the space ship are ticking at a different speed from those on the ground, and so forth."[10]

Einstein constructed the logical foundation of special relativity based on two postulates and gave the clear and complete physical interpretation of space and time in his theory, including the definition of simultaneity, the relativity of length and time, and the physical meaning of the equations obtained with respect to moving bodies and moving clocks. This physical interpretation was the most difficult part of the theory and turned out to be the greatest strength of Einstein's paper. In contrast, Poincaré's formulation was based on one postulate, the principle of relativity, and one definition $c \equiv 1$ by choosing suitable units of length and time. He did not give a clear and complete physical interpretation of space and time. Although there are differences in their concepts and interpretations, the two formulations lead to exactly the same set of basic equations for field and particle dynamics, provided Einstein's approximate dynamical equations for charged particles are corrected. Thus, there is no experimental difference between them.

Einstein used about six pages to derive the Lorentz transformation based on the invariance of the law of the propagation of light $c^2t^2-x^2-y^2-z^2=0$, the operational definitions of space and time, and the universal constancy of the speed of light. In contrast, Poincaré used one sentence to explain that the Lorentz transformation can be derived as the result of "a linear transformation that leaves invariant the quadratic form $c^2t^2-x^2-y^2-z^2$."[11]

As a great mathematician, Poincaré understood the Lorentz group completely as a group with six parameters, three for spatial rotations and three for constant linear motions, while Einstein showed it as a group with one parameter for the constant motion in a given direction. (See chapter 9, section 9b, after equation (9.18).) The understanding of the group property is crucial because, mathematically, the special theory of relativity is the theory of invariants of the Lorentz group and, physically, symmetry of the theory completely depends on the group property.

Poincaré believed in the existence of an as yet undetected aether, while Einstein did not believe in the aether. It was widely believed by most people that Einstein was right and Poincaré was wrong. However, this belief is no longer tenable from the viewpoint of modern gauge field theory and particle physics. Based on the unified electroweak theory and quantum chromodynamics, the physical vacuum is quite complicated, contrary to Einstein's belief. (See chapter 4, sec. 4d.) Nevertheless, it was very important to separate Maxwell's equations from mechanical model of the aether, and this was accomplished by Einstein.

The basic formulation of relativistic electrodynamics consists of two parts: the invariant equations of the electromagnetic fields and the invariant equations of motion of the charged particles. Being a grandmaster of mathematics, Poincaré gave a complete formulation of electrodynamics, which included the mathematical framework and the Lorentz invariant fundamental equations of motion for charged particles and electromagnetic fields in his relativity theory. He did all this elegantly by using the principle of least action (i.e., the Lagrangian formulation) and by choosing units so that $c\equiv1$.[12] In contrast, Einstein's 1905 formulation of the covariant electrodynamics is mathematically less elegant and not completely satisfactory: His result for the transverse mass $m/(1-v^2/c^2)$ differs from the correct expression $m/\sqrt{1-v^2/c^2}$ obtained by Lorentz and Poincaré.[9] Moreover, his resultant equations of motion for charged particles are not Lorentz invariant because they are only approximations which hold for small velocities and accelerations.

It appears that the physicist Einstein and the mathematician Poincaré had quite different styles and tastes, as shown by the notations and emphases in their papers. In particular, Einstein believed in the speed of light as a truly universal constant. His paper was more appealing to physicists and had a great impact on the development of physics. He derived results which could be directly tested experimentally. In contrast, Poincaré's Rendiconti paper published in a mathematical journal was not widely read by physicists and, hence, did not have much influence in the early development of relativity.[13] He treated the constant speed of light as nothing more than a definition, which was consistent with his conventionalism.[14] However, his paper showed elegance and generality based on the Lorentz covariant Lagrangian formulation. The difference in their views concerning the constancy of the speed of light is conceptually significant, and we shall return to this point in chapters 7 and 17.

Having said all this, Einstein is generally considered to have a more profound understanding of physical space, time and relativity. In the early years, Lorentz was probably in the best position to appreciate and assess the works of Poincaré and Einstein. He was particularly impressed by "a remarkable reciprocity that has been pointed out by Einstein" and credited Einstein for "making us see in the negative result of experiments like those of Michelson, Rayleigh and Brace, not a fortuitous compensation of opposing effects, but the manifestation of a general and fundamental principle."[15]

When Pauli discussed Einstein and the development of physics, he said: 'Nowadays we speak with some justification of the "Lorentz Group"; but as a matter of history it was precisely the group property of his transformations that *Lorentz* failed to recogniz; this was reserved for Poincaré and Einstein independently. It is regrettable that a certain amount of dispute about priority has arisen over this.'[16]

References

1. See, for example, A. Pais, *Subtle is the Lord...., The Science and the Life of Albert Einstein* (Oxford Univ. Press, Oxford, 1982), pp. 133–134 and pp. 163–166. Pais stresses that, in 1902, "Einstein and his friends did much more than just browse through Poincaré's writings. Solovine has left us a detailed list of books which the Akademic members read together. Of these, he singles out one and only one, *La Science et l'Hypothese*, for the following comment: '[This] book profoundly impressed us and kept us breathless for weeks on end'! " Poincaré's book *La Science et l'Hypothese* was published in 1902. Pais therefore believes that, "prior to his own first paper on relativity, Einstein knew the Paris address in which Poincaré suggested that the lack of any evidence for motion relative to the aether should hold generally to all orders in v/c and that 'the cancellation of the [velocity-dependent] terms will be rigorous and absolute.' But there is more. In *La Science et l'Hypothese*, there is a chapter on classical mechanics in which Poincaré writes, 'There is no absolute time; to say that two durations are equal is an assertion which has by itself no meaning and which can acquire one only by convention.... Not only have we no direct intuition of the equality of two durations, but we have not even direct intuition of the simultaneity of two events occuring in different places; this I have explained in an article entitled "La Mesure du Temps".' " Furthermore, in chapter VII, RELATIVE AND ABSOLUTE MOTION, of this book, Poincaré discussed the 'principle of relative motion': "The movement of any system whatsoever ought to obey the same laws, whether it is referred to fixed axes or to the movable axes which are implied in uniform motion in a straight line." This is precisely the principle of relativity that Poincaré proposed two years later in his 1904 address in St. Louis. On page 166, Pais made some comments: "It cannot be said, however, that the content of Einstein's June 1905 paper depends in any technical sense on these important remarks by Poincaré. Others in Einstein's position might perhaps have chosen to mention Poincaré at the earliest opportunity. However, it does not seem to me that Einstein had compelling reasons to do so in 1905." Nevertheless, based on today's guidance for referees to accept papers submitted to, say, Physical

Review or Nuovo Cimento, it is very unlikely that such an omission of any reference to relevant works will be allowed for a paper to be published.

2. **A. Einstein, Ann. Phys. (Leipzig)** ***17*****, 891 (1905)**; translated by H. M. Schwartz, in Am. J. Phys. **45**, 18 (1977).
3. See, for example, J. D. Jackson, *Classical Electrodynamics* (John Wiley & sons, New York, 2nd ed.), p. 514; K. Brecher, Phys. Rev. Lett. **39**, 1051 (1977); A. A. Tyapkin, Lett. Nuovo Cimento **7**, 760 (1973).
4. One may think that there are other ways to test the universality of the speed of light in different frames, e.g., measuring the frequency ω of an electromagnetic wave with a known wavelength in different frames. This is untenable because such a measurement is based on the invariant phase $(\omega t - \mathbf{k}\cdot\mathbf{r})$ in which the constancy of the speed of light has already been assumed. One may ask: What happens to the invariant phase if the speed of light is not assumed to be a constant? The answer is that the invariant phase can only be written as $[k_0 w - \mathbf{k}\cdot\mathbf{r}]$, where $(k_0,\mathbf{k})$ and $(w,\mathbf{r})$ are 4-vectors, so that time (measured in seconds) does not appear and, hence, cannot be defined via the oscillation of electromagnetic waves. For more detailed discussions, see chapter 10, between eqs. (10.18) and (10.19). For some experimental tests of special relativity, see M. P. Haugen and C. M. Will, Phys. Today, May 1987 and Yuan-zhong Zhang, *Special Relativity and Its Experimental Foundations* (World Scientific, Singapore, 1997).
5. H. Poincaré, *Science and Hypothesis* (Dover, New York, 1952; first appeared in 1902), pp. xxii–xxiii; see also pp. 152–153.
6. L. Landau and E. Lifshitz, "*The Classical Theory of Fields*" (Addison-Wesley press, Cambridge, 1951), p. 2; A Einstein, *The Meaning of Relativity* (5th ed., Princeton University Press, 1956), p. 27.
7. This equivalence can be shown by using a covariant variational calculus, in which one treats the second end point as variable and admits only actual paths. The usual variational calculus fixed both end points. See for example L. Landau and E. Lifshitz, "*The Classical Theory of Fields*" (Addison-Wesley press, Cambridge, 1951), pp. 28–29. If $ds^2=dw^2-dr^2$, we have the usual action S for a free particle with a mass m,

$$S = -m\int_a ds, \qquad \text{(R5.1)}$$

where the first end point a is fixed. The variational principle $\delta S = 0$ under the stated conditions will lead to

$$p^\mu = -\partial S/\partial x_\mu = m dx^\mu/ds , \qquad \mu=0,1,2,3 . \tag{R5.2}$$

Therefore, one has the energy-momentum relation for a free particle

$$p^\mu p_\mu = m^2 . \tag{R5.3}$$

If (R5.3) holds, then using (R5.2), one can obtain $ds^2=dw^2-dr^2$. For a photon with mass m=0, the above calculations cannot be applied. Rather, one can still show that $dw^2-dr^2=0$ is not an independent assumption. It can be derived from Maxwell's equations. The solution of a free plane wave in the vacuum is given by

$$E(w,\mathbf{r}) = E_0 \sin(k_0 w - \mathbf{k}\cdot\mathbf{r}) , \tag{R5.4}$$

where $k_0^2 - \mathbf{k}^2 = 0$. To calculate the velocity of the propagation of this wave, one simply looks at the motion of the node, $E(w,\mathbf{r})=0$, or $k_0 w - \mathbf{k}\cdot\mathbf{r} = k_0 w - kr\cos\theta = 0$. Along the direction of the wave propagation $\mathbf{k}$ (i.e., $\theta=0$), one has $dr/dw = k_0/k = 1$, i.e.,

$$dw^2 - dr^2 = 0 . \tag{R5.5}$$

Clearly, these discussions hold, if one sets w=ct and $k_0=\omega/c$ for the case of special relativity.

8. One interesting question is as follows: Suppose the rate of ticking of clocks in two inertial frames are not compared by the above method or by using the Lorentz transformations. Suppose we just consider one clock in F ticking with respect to the F-observers and another clock in F' ticking with respect to the F'-observers, what is the relation for these two rates of tickings? The answer is obvious if one considers the equivalence of all inertial frames. (See Appendix D for a detailed discussion.)
9. M. Planck, Verh. Deutsch. Phys. Ges. 4, 136 (1906). Planck also used the covariant Lagrangian formalism to obtain the correct equations of

motion for a charged particle. It is worthwhile to note that not all of the kinematic properties inherent in the Lorentz transformation were immediately made clear by Einstein's 1905 paper. For example, the increase in the mass of a particle as a function of its velocity is one such kinematical property which was not discovered until later. As late as 1909, Max Born, though familiar with special relativity, still took the time to calculate the electromagnetic mass of the electron to find its velocity dependence, similar to what Lorentz had done previously. In light of this, one may wonder: Who was the referee of Einstein's original paper on relativity? It is not known with certainty who the referee was, but in view of its style and content, very likely it was Planck. Planck was not only immediately interested in Einstein's work on relativity, he also asked von Laue to discuss it in a colloquium and sent him to Bern to meet with Einstein soon after the colloquium. These events are well documented. Helmut Recehnberg, private e-mail correspondence, Sept. 1, 1999.

10. R. Feynman, *The Character of Physical Law* (MIT Press, Cambridge, MA. 1965), p. 92.
11. Some people believed that Einstein derived the Lorentz transformations, while Poincaré assumed them in order to get the invariance of the electromagnetic equations. This appears to be untenable. In the last part of his Rendiconti paper, Poincaré considered (x, y, z, $t\sqrt{-1}$) to be the coordinates of a point in a space of four-dimensions and stated that the Lorentz transformation is just a rotation of this space about the origin, regarded as fixed. Poincaré explained: "We know that the transformations of this group (taking K=1) are the linear transformations which do not change the quadratic form $x^2+y^2+z^2-t^2$." As a mathematician of a very high caliber, this explanation clearly showed that he knew that the Lorentz transformation could be derived based on the invariant $t^2 - x^2 - y^2 - z^2$. (See also ref. 7.) It was understood that the calculations were simple and need not have been written explicitly in the paper. Poincaré's method of derivation has been used by many authors of textbooks to obtain the Lorentz transformations, see for example, V. Barger and M. Olsson, *Classical Mechanics: A Modern Perspective* (2nd ed., McGraw-Hill, Nw York, 1973), pp. 347-348.

12. Today, quantum field theorists all use the Lorentz covariant Lagrangian formulation and choose units in which $c = \hbar = 1$ in their formulations and investigations, just as Poincaré did in 1905.
13. A very important and useful reference book of relativity for decades, DAS RELATIVITÄTSPRINZIP, which collected the important papers of Lorentz, Einstein, Minkowski and Weyl, appeared in a series on monographs edited by Otto Blementhal (based on the suggestion of Sommerfeld) and was first published in German by Teubner, Leipzig, 1913. In this book, Sommerfeld stated in his NOTES that "Whereas Minkowski's ideas on the vector of the first kind, or four-vector, were in part anticipated by Poincaré (Rend. Circ. Mat. Palermo, 21, 1906), the introduction of the six-vector is new" and that "Minkowski's relativistic form of Newton's law for the special case of zero acceleration mentioned in the text included in the more general form proposed by Poincaré (loc. cit)." Its English translation, THE PRINCIPLE OF RELATIVITY, was first published in 1923. It is regrettable that Poincaré's original and comprehensive paper was included in neither of them. It was translated into English around 1970 by C. W. Kilmister (*Special Theory of Relativity* [Pergamon, New York, 1970]) and by H. M. Schwartz, Am. J. Phys. **39**, 1287 (1971), **40**, 862 (1972), **40**, 1282 (1972)).
14. See ref. 20 in chapter 4.
15. Lorentz and Pauli were familiar with and understood the works of both Poincaré and Einstein. It is somewhat odd that Pauli credited Poincaré for the principle of relativity, while Lorentz credited Einstein for it. They discussed relativity theory and regarded it as Einstein's work. See, for example, H. A. Lorentz, *The Theory of Electrons* (first appeared in 1909, Dover, New York, 1952), pp. 223–224, 226–228 and 321–325. In his book, Lorentz discussed only the "Poincaré stress" for the equilibrium of an extended electron and not the formulation of relativity theory for the electromagnetic fields and the dynamics of charged particles based on the principle of relativity (and the definition $c=1$.)
16. W. Pauli, *Writings on Physics and Philosophy* (Edited by C. P. Enz and Karl von Meyenn, translated by R. Schlapp, Springer-Verlag, Berlin, 1994), p. 118; *Neue Zurcher Zeitung*, 12. Januar 1958.

6.

Minkowski's 4-Dimensional Spacetime, Adjustable Clocks and Flexibility in the Concept of Time

6a. The Completion of Special Relativity by Minkowski's Idea of 4-Dimensional Spacetime

The physical theory of special relativity is usually considered to have been completed in 1905. The 4-dimensional symmetry framework of Minkowski and Poincaré is generally viewed to have been a purely mathematical development. However, in view of the later development of general relativity, covariant quantum electrodynamics, unified electroweak theory and quantum chromodynamics, both the new physical ideas in special relativity and the explicit 4-dimensional symmetry framework are necessary for calculating physical results. Thus, it appears to be more reasonable to consider the idea of the 4-dimensional spacetime as an integral and necessary part of the physics of special relativity. We know that quantum mechanics is incomplete without the probabilistic interpretation of the wave function. Similarly, we stress that special relativity is not really complete without having the 4-dimensional interpretation of the Lorentz transformation and of the form of physical laws.

The radical idea of a 4-dimensional spacetime for physical laws was first conceived by Poincaré in 1905, as discussed in section 4c. In his Rendiconti paper, Poincaré considered (x, y, z, it), (with c=1), as coordinates of a point in a space of 4-dimensions and stated that the Lorentz transformation is just a rotation of this space about the origin, regarded as fixed. In this way, one has an invariant distance between the point and the origin,

$$x^2 + y^2 + z^2 - t^2 = x^2 + y^2 + z^2 + (it)^2, \qquad i = \sqrt{-1}\ .$$

Thus, the geometry of such a 4-dimensional space (x, y, z, it) is closely related to Euclidean geometry. Poincaré further explained: "We know that the transformations of this group (with K=1) are the linear transformations which do not change the quadratic form $x^2+y^2+z^2-t^2$."

Unfortunately, Poincaré's idea remained almost completely unnoticed until 1908. The same idea, its elegance and important applications were again suggested and expounded by the Russian mathematician Hermann Minkowski (1864–1909) at Göttingen.[1] The impact of Minkowski's work on 4–dimensional spacetime has been enormous. His essential idea was that the invariant theory of the Lorentz group can be represented geometrically, so that it is a natural generalization of the tensor calculus for a 4–dimensional space. The presence of tensor calculus forever changed the landscape of theoretical physics. It greatly simplified calculations and proofs in theoretical physics. Einstein used nearly six pages to prove the invariance of Maxwell's equations with sources in his original paper.[2] The same proof can be done in about six lines by using tensor notation. Furthermore, a few years later the ideas of 4–dimensional spacetime and tensor calculus paved the way for Einstein's creation of general relativity — a theory of gravity based on a 4–dimensional curved spacetime with Riemann geometry.

We know that, in principle, the electrodynamics of moving bodies was solved in 1905 by Einstein and Poincaré. Nevertheless, Minkowski further discussed phenomenological electrodynamics of moving bodies in 1908. When the structure of matter is not completely known, the prediction of macroscopic phenomena of moving bodies is not trivial even if phenomena associated with bodies at rest are assumed to be known experimentally. Minkowski showed that within the framework of special relativity, the equations and the boundary conditions for phenomenological electrodynamics of moving bodies can be derived from Maxwell's equations and the boundary conditions for bodies at rest.[3]

It turns out that there is a flexibility in the concept of time within the 4–dimensional symmetry framework. The first specific example was discussed by Reichenbach in 1928.[4] He proposed a more general procedure for clock synchronization using light signals, which includes Einstein's clock synchronization as a special case. Although his idea was logically sound, for many years it could not be implemented to derive a spacetime transformation which is consistent with the 4–dimensional symmetry of the Lorentz and Poincaré groups.[5] Seventy years later, it was shown that it is indeed possible for Reichenbach's general concept of time to be implemented in transformations compatible with the 4–dimensional symmetry of the Lorentz and Poincaré groups.[6]

6b. The Collision of the Titanic and Haywire Clocks

To demonstrate the flexibility we have in defining a physical time, let us consider an example. In the physical world, what happens will happen, regardless of how we measure space and time. In our discussions, *time usually denotes the evolution variable measured in, for example, seconds.* As an example, let us consider the disastrous collision between the Titanic and an iceberg. Suppose the Titanic is at location T at time t_1 and moves with a constant velocity $\mathbf{V}=(dx/dt,0,0)=(V,0,0)$ to location C at time t_2 where it collides with an iceberg, which also happens to move to the position C at time t_2. Thus, the Titanic moves a distance D during the time interval t_2-t_1

$$D=\int_{t_1}^{t_2} Vdt = V(t_2-t_1) = \int_{x_1}^{x_2} \frac{dx}{dt}\, dt = x_2 - x_1 , \tag{6.1}$$

according to the stable, uniform clocks we are used to. This relation is the root cause of the disastrous collision. Now suppose the clocks which measure the velocity of the Titanic suddenly go haywire, ticking with different rates and reading different times t' at different positions. The velocity V' measured by these haywire clocks will now be some complicated function of t' rather than a constant. Will the relation (6.1) be changed so that the collision would not occur? Of course not. In other words, we can represent t' by any given complicated function A(x,t), (assuming that the length of meter sticks does not change),

$$t' = A(x,t) , \tag{6.2}$$

so that the velocity V' measured by t',

$$V' = \frac{dx}{dt'} = \frac{dx}{dt}\frac{dt}{dt'} = V/(\frac{\partial A}{\partial x} V + \frac{\partial A}{\partial t}) , \tag{6.3}$$

is now a complicated function of space x and time t due to the factor dt/dt', in contrast to the constant velocity dx/dt=V. However, during the time interval $t'_1=A(x_1,t_1)$ and $t'_2=A(x_2,t_2)$ (corresponding to t_1 and t_2) the Titanic will still move a distance

$$D = \int_{t_1'}^{t_2'} V'(t')dt' = \int \frac{dx}{dt'}dt' = \int_{x_1}^{x_2} dx = x_2 - x_1, \tag{6.4}$$

which is precisely the same as the distance D in (6.1), as we expected.

6c. The Primacy of the 4–Dimensional Symmetry

The same considerations can be applied to the light beams in the Michelson–Morley experiment instead of the Titanic. In this case, if we use some crazy set of clocks, the speed of light may not be a constant, but the experiment will still produce a null result.

Why, then, is the relativistic time so important and crucial in physics?

The answer to this question is that *the important and crucial property for understanding physical phenomena is the four-dimensional symmetry of the Poincaré–Einstein principle rather than the relativistic time.* In other words, *the synchronization of clocks to realize relativistic time is a matter of convention rather than something inherent in nature or essential in physics.* This viewpoint was stressed both by Poincaré[7] and Reichenbach.[4] From a physical viewpoint, the essence of relativity is the principle of relativity of Poincaré and Einstein for the *form* of physical laws rather than the relativistic time or the universal value for the speed of light c. This will be discussed and explicitly demonstrated in chapter 7, 8 and 12.

6d. A Flexible Concept of Time

First, however, we would like to make it absolutely clear that no matter how complicated the function $A(x,t)$ may be, the new time t' in (6.2) can always be physically realized once $A(x,t)$ is given, because with today's computer technology, it is always possible to design a timepiece with the desired rate of ticking $\partial A/\partial t$ which will show the desired time $A(x,t) = t'$.[8]

Secondly, if one wants to use the usual stable and uniform time in an inertial frame, the synchronization of clocks *in a given inertial frame*, say, F, can always be carried out, in principle, by anything with a constant velocity — light, sound, a bullet, etc. by the method described in the previous chapter.

In this way, the time t in F is completely defined. While this is intuitively clear, the nontrivial part is how one should set up clocks in another frame F' which is moving with a constant velocity **V**=(V,0,0). *It must be stressed that it is only when time in other inertial frames is specified do we have a complete definition of time in physics.* The synchronized clock system in F' makes all the difference in our concept of time. Each F' clock has a position **r**=(x,y,z) as observed in F. For each of these F'-clocks at a particular position **r** at time t, we can arrange for its time t' to read

$$t' = A(x,t)\,, \qquad \mathbf{V} = (V,0,0)\,, \tag{6.5}$$

where A(x,t) is some given function. A specific choice of A(x,t) corresponds to a particular concept of time. For example, with the choice

$$t' = A(x,t) = \gamma(t - \frac{Vx}{c^2})\,, \tag{6.6}$$

one has chosen to use "relativistic time" in one's universe. On the other hand, if one chooses

$$t' = A(x,t) = t\,, \tag{6.7}$$

one has "common time," that is, all observers in all frames of reference use the same time. However, within the four-dimensional symmetry framework, "common time" is not absolute in the sense of Newton and leads to different predictions of physical results. (See chapter 12.)

By adjusting the reading and the rate of ticking of clocks in different frames of reference, we have great flexibility in picking our particular concept of time. It turns out that, within the four-dimensional symmetry framework based on the Poincaré-Einstein principle, (6.6) and (6.7) are the two algebraically simplest concepts of time. As we shall demonstrate later, *the four-dimensional symmetry, and not relativistic time, is the crux!* In other words, the physical world can be understood based on the four-dimensional symmetry without ever specifying the function A(x,t) in (6.2) and the time t in F to be uniform.[8]

References

1. H. Minkowski (1864–1909), F. Klein and D. Hilbert were considered three "prophets" in the mecca of German mathematics at Göttingen in the early 20th century. Minkowski and Hilbert were friends since their schooldays in Königsberg. Around 1882, the French Academy proposed the problem of the representation of a number as the sum of five squares. Minkowski's investigation led him far beyond the stated problem. In the spring of 1883, the 18 year old Minkowski and a well-known English mathematician H. Smith were awarded jointly the French Academy Grand Prize. The young Hilbert was attracted to become friends with the shy, kind-hearted and gifted Minkowski. They often took long walks in the woods. They engrossed themselves in the problems of mathematics and exchanged their newly acquired understandings and thoughts on every subject. In this way, they formed a friendship for life. In 1902, when Hilbert was offered a very honored position in Berlin, he grabbed this opportunity and proposed that a new professorship be created expressly for Minkowski at Göttingen. As a result, these two great mathematicians were able to continue their walks in the woods at Göttingen. The joy of Klein, the head of the group, and mathematics students knew no limit. "To me, he was a gift from heaven" wrote Hilbert. Unfortunately, Minkowski died suddenly after an operation for appendicitis in 1909. Apart from his well-known 4-dimensional spacetime as the mathematical framework for special relativity, he also made important contributions on "Minkowskian geometry" by modifying Euclid's congruence axioms (i.e., the axiom of the free mobility of rigid point systems) and on the "geometry of numbers." In the latter, his theorem on lattice points in convex regions provides insight into apparently unrelated subjects such as algebraic number fields, finite groups, and the arithmetic theory of quadratic forms.
2. H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, *The Principle of Relativity* (translated by W. Perrett and G. B. Jeffery, Methuen and Company, 1923), pp. 51-55 and 59–61.
3. W. Pauli, *Theory of Relativity* (Pergamon Press, London, 1958), pp. 99–111.
4. H. Reichenbach, *The Philosophy of Space and Time* (first published in 1928; translated by M. Reichenbach and J. Freud; Dover, New York, 1958), p. 127. Einstein's definition of simultaneity, $t_2 = t_1+(1/2)(t_3-t_1)$, is essential

for special relativity. But Reichenbach believed that "it is not epistemologically necessary." Suppose there are two identical clocks, clock 1 located at the origin of the F frame and clock 2 at point x on the x-axis. A light signal starts from the origin (event 1) at time t_1, it reaches clock 2 (event 2) at time t_2 and returns to the origin (event 3) at time t_3. Reichenbach's concept of time can be realized by synchronizing clock 2 to read t_2 by the relation $t_2 = t_1 + \varepsilon[t_3 - t_1]$, where ε is restricted by $0< \varepsilon <1$, so that causality is preserved, i.e., t_2 cannot be earlier than t_1. It can be shown that indeed Reichenbach's more general definition of simultaneity for time t can be implemented in an "extended relativity" without violating 4-dimensional symmetry of the Lorentz group. See also A. Grünbaum, *Philosophical Problems of Space and Time* (Reidel Publishing Co., Boston, 1973), chapter 12. For a detailed discussion of "extended relativity", see chapters 17-20.

5. W. F. Edwards, Am. J. Phys. 31, 482–489 (1963); J. A. Winnie, Phil Sci. **37**, 81–99 and 223–238 (1970);
6. Leonardo Hsu, Jong–Ping Hsu and Dominik A. Schneble, Nuovo Cimento **111B**, 1299 (1996); Jong–Ping Hsu and Leonardo Hsu, *ibid.* **112B**, 575 (1997); and Leonardo Hsu and Jong–Ping Hsu, *ibid.* **112B**, 1147 (1997).
7. See chapter 4. Note that this view did not escape Einstein when he explained the idea of time in physics: "....That light requires the same time to traverse the path A→M as for the path B→M is in reality neither a *supposition nor a hypothesis* about the physical nature of light, but a *stipulation* which I can make of my own freewill in order to arrive at a definition of simultaneity." (M is the mid–point between A and B.) See A. Einstein, *Relativity, The Special and General Theory* (trans. R. W. Lawson, Crown Publishers, INC., New York, 1961), p. 23.
8. J. P. Hsu and L. Hsu, Phys. Lett. A **196**, 1 (1994); Leonardo Hsu and Jong–Ping Hsu, Nuovo Cimento B, **111**, 1283 (1996); Jong–Ping Hsu and Leonardo Hsu, in *JingShin Physics Symposium in Memory of Professor Wolfgang Kroll* (World Scientific, Singapore. New Jersey, 1997), pp. 176–193.

7.

Taiji Relativity Based Solely on 1 Principle – the First Principle of Relativity

"Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labeled as 'conceptual necessities,' '*a priori* situations,' etc. The road of scientific progress is frequently blocked for long periods by such errors. It is therefore not just an idle game to exercise our ability to analyze familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little, from the data of experience. In this way they are deprived of their excessive authority."

Einstein (Phys. Zeitschr. 17,101 (1916))

7a. Refreshingly Innocent Questions

It is absolutely essential to insist that a fundamental physical theory should be derivable from the simplest possible set of basic principles. With this in mind, we turn our attention to the following questions: Can a theory of relativity be formulated solely on the basis of the first principle of relativity, without assuming the universal constancy of the speed of light? Is it necessary for clock systems to be adjusted to satisfy a specific relation for t' and t?

It turns out that, indeed, a general theoretical framework for space and 'time' can be explicitly constructed solely on the basis of the Poincaré–Einstein principle of relativity. We term such a theory "taiji relativity,"[1] which has the ultimate logical simplicity with regard to its foundations. The 4-dimensional symmetry of the Poincaré–Einstein principle of relativity dictates that the four spacetime variables (w,x,y,z), where w is the evolution variable, must be treated on equal footing and must all have the same units of length. As a result, the basic dimensions of physics are reduced from the usual three (i.e., time, length and mass) to two (i.e., length and mass.) Also, the usual three fundamental and universal constants (the speed of light c, the Planck constant and the electric charge) are reduced to two.[1]

We note that *one fundamental difference of taiji relativity from special relativity is that the evolution variable (i.e., the taiji-time w) must be measured in units of length, just like the spatial components, x, y and z.* The first principle of relativity and Maxwell's equations can determine the relation between one centimeter of w and one second of time t, as defined conventionally. The usual constant speed of light c=299792458m/sec is neither necessary nor fundamental to physics. Physical laws expressed in terms of the 4-coordinate $x^\mu=(w,x,y,z)$ display four-dimensional symmetry and are consistent with all known experiments such as the Michelson-Morley experiment, the Doppler shifts of wavelength and frequency, the energy-momentum relations for a particle, and the decay-length dilatation.

Such a new theory of relativity suggests that the four-dimensional symmetry itself *is inherent and truly fundamental* in the sense that it is a fundamental and necessary component of any theory if it is to correctly explain and predict phenomena in the physical world. In contrast, the universal constancy of the speed of light c is neither necessary nor fundamental in the physical world *but instead, a human convention imposed upon physical theories for describing nature.* Thus it seems appropriate to term such a theory of relativity "taiji relativity" because the word "taiji" denotes, in ancient Chinese thought, the ultimate principles or the condition which existed before the creation of the world.[2]

The purpose of our new formalism of four-dimensional taiji relativity *is not* to show that special relativity is wrong in some way, *but instead to show that special relativity is in some sense overspecified and that removal of the overspecification leads to a fresh view of the physical world and new concepts and results which are unobtainable through conventional special relativity.*

7b. 4-Dimensional Taiji Transformations

Based on four-dimensional symmetry considerations, we describe an "event" with the coordinates

$$(w, x, y, z) = x^\mu, \quad \text{and} \quad (w', x', y', z') = x'^\mu, \quad \mu = 0,1,2,3, \tag{7.1}$$

in two inertial frames F(w,x,y,z) and F'(w',x',y',z') respectively. The variable w or w' is interpreted as the evolution variable of a physical system.

We derive the four-dimensional transformation between these coordinates. For simplicity, we start with the convention that F and F' frames have relative motion along parallel x and x' axes, that the origins of F and F' coincide at the taiji-time w=w'=0, and that the transformation is linear (i.e., $w'=a_1w+a_2x$, $x'=b_1x+b_2w$, $y'=y$, $z'=z$). The principle of relativity implies that the four-dimensional interval s^2 is invariant:

$$s^2 = w'^2 - r'^2 = w^2 - r^2 = x_\mu x^\mu ;$$
$$r^2 = \mathbf{r}^2, \qquad x_\mu = (w, -x, -y, -z) = (w, -\mathbf{r}), \qquad \mu = 0,1,2,3. \tag{7.2}$$

Its differential form is $ds^2=dw^2-dr^2$. We note that this is not a separate assumption because the relation (7.2) for the 4-dimensional interval is actually the law of "energy-momentum" for the motion of a free particle with mass $m \geq 0$ (if $s^2 \geq 0$),[3]

$$p_0^2 - \mathbf{p}^2 = m^2, \tag{7.3}$$

where $p^\mu=mdx^\mu/ds$, $\mu=0,1,2,3$. The relative motion between F and F' satisfies a condition that F' moves with a dimensionless velocity B, measured in terms of w, i.e., when $dx'/dw'=0$, $dx/dw=B$. Using (7.2) and this condition, we can determine all four unknown constants in terms of a single parameter $B(=-b_2/b_1)$. We obtain the taiji transformations of coordinates:

$$w' = \gamma(w - Bx), \quad x' = \gamma(x - Bw), \quad y' = y, \quad z' = z, \tag{7.4}$$

$$\gamma = \frac{1}{\sqrt{1-B^2}}. \tag{7.5}$$

Its inverse transformation can be derived from (7.4) and (7.5):

$$w = \gamma(w' + Bx'), \quad x = \gamma(x' + Bw'), \quad y = y', \quad z = z'. \tag{7.6}$$

The set of four-dimensional coordinate transformations (7.4) forms precisely the Lorentz group. The Lorentz group properties of (7.4) can be demonstrated by the usual method. As usual, these transformations can also be generalized to form the Poincaré group by including constant translations $b^{\mu}=(b^0,b^1,b^2,b^3)$ along the four axes. (See eq. (9.30) in chapter 9.)

The propagation of light is described by setting $s^2=0$ in (7.2) or $ds^2=dw^2-dr^2=0$. This is not a separate and independent assumption because it can be derived from the free wave solution of the Maxwell equations, as shown in ref. 5 of chapter 5. We can now derive the taiji-speed of light β_L implied by (7.2) as

$$\beta_L = |dr/dw| = 1, \qquad ds^2 = dw^2 - dr^2 = 0, \tag{7.7}$$

where we have used the differential form of (7.2). This holds in all inertial frames since ds^2 is invariant and also holds for lights emitted from any source. Although it may look as if we are setting c=1, as we shall see later, this is not the same as simply setting c=1 in special relativity.

7c. Taiji Time and Clock Systems

Now let us consider physical devices called "taiji-clocks" which show taiji-time w. First, the four-dimensional symmetry naturally dictates that all four variables w, x, y and z have the same dimension of length, the same units (say, cm) and are homogeneous in any inertial frame. The usual clocks, whose readings of t have the unit of seconds, can be modified to become "taiji clocks" in such a way that their readings of w have the unit of centimeters. According to the four-dimensional symmetry or eq. (7.7), $\beta_L=1$, the units of taiji clocks must be such that during the passage of $\Delta w=1$cm, a light signal travels a spatial distance $|\Delta r| = 1$cm along any direction in vacuum. In other words, the duration of the usual time interval $\Delta t=1$ second corresponds to the taiji-time interval $\Delta w=29979245800$cm. We note that this numerical value 29979245800 has no fundamental significance because it completely depends on our definitions of the speed of light. (At present, one centimeter for lengths is defined in terms of the defined value of the speed of light c and the frequency of the atomic clock.) Since taiji relativity implies the invariant and absolute taiji-speed (7.7) for light signals, $\beta_L=1$, for all frames and all directions, we can set up a taiji-clock system in each frame, and use light signals to synchronize these clocks as follows:

Suppose there are taiji clocks located at positions A and B in a frame. The taiji-time interval required for a light signal to go from A to B must be equal to the taiji-time interval required for a light signal to go from B to A, according to the law (7.7). The taiji-time interval must also be equal to the spatial distance between A and B, $|\mathbf{r}_A-\mathbf{r}_B|$, because $\beta_L=1$. Thus, when a light signal starts from A at w_1, is reflected back by a mirror at B, and returns to A at taiji-time w_2, then the taiji-time at which the signal hits B is

$$w_1 + \frac{1}{2}(w_2 - w_1) = w_1 + |\mathbf{r}_A - \mathbf{r}_B| . \tag{7.8}$$

This synchronization procedure resembles that in special relativity, *except that (7.8) is not based on an additional second principle.* Rather, it is dictated by the law (7.7) which is implied by Maxwell's equations and *the first principle of relativity.* Note that any change of the relation (7.8) for taiji-time w by, say, including an extra parameter similar to that suggested by Reichenbach will violate the 4-dimensional symmetry of taiji relativity.[4]

The taiji-time w can also be understood as an *optical path length,* because, for a finite interval, the law for the propagation of a light signal, starting from the origin r=0 at w=0, is described by the law:

$$w^2 - r^2 = 0, \qquad \text{or} \qquad w = r > 0, \tag{7.9}$$

where r is the distance traveled by the light signal during the taiji-time interval $\Delta w=w$. Evidently, this law (7.9) or $\beta_L \equiv r/w=1$ is "taiji invariant", i.e., unchanged under the taiji transformation (7.4). All other particles with masses $m>0$ have a smaller taiji velocity, $\beta<1$, as implied by the transformation (7.4).

7d. Taiji Velocity Transformations

From equations (7.4)-(7.5), we can derive the transformations of taiji-velocities,

$$\beta'_x = \frac{(\beta_x - B)}{(1 - \beta_x B)}, \quad \beta'_y = \frac{\beta_y}{\gamma(1 - \beta_x B)}, \quad \beta'_z = \frac{\beta_z}{\gamma(1 - \beta_x B)}; \tag{7.10}$$

$$\boldsymbol{\beta}' = \left(\frac{dx'}{dw'}, \frac{dy'}{dw'}, \frac{dz'}{dw'}\right), \qquad \boldsymbol{\beta} = \left(\frac{dx}{dw}, \frac{dy}{dw}, \frac{dz}{dw}\right).$$

The form of (7.10) is exactly the same as that of the velocity transformation in special relativity because both theories have the 4-dimensional symmetry of the Lorentz group. The only difference is that the constant speed of light c=29979245800cm/sec is not involved in (7.10) at all.

7e. Comparisons with Special Relativity

In order to avoid confusion, one must be very careful in comparing taiji relativity and special relativity because of their conceptual differences, their formal similarity due to the four-dimensional symmetry, and one's preconception of space and time. For purposes of comparison with special relativity, we *formally* introduce a time variable measured in seconds for each frame and describe events with the coordinates

$$x^\mu = (w, x, y, z) = (bt, \mathbf{r}),$$

$$x'^\mu = (w', x', y', z') = (b't', \mathbf{r}'), \qquad \mu = 0,1,2,3, \tag{7.11}$$

in F and F', respectively. It must be emphasized that, *in taiji relativity, the usual concept of time is undefined*, i.e., time variables t, t', t".... in F, F', F",....respectively are not related by specific functions. Logically, nothing can prevent one from defining t by setting up a specific clock system in one frame, say, F. However, the relation between t and t', for example, *can never be specified* within the framework of taiji relativity due to the lack of a second postulate. If one specifies a relation for t and t', *then one has made a second postulate so that the time t is now defined for all frames* (due to the group properties of the coordinate transformations); one will then have a theory of relativity which differs from taiji relativity. In (7.11) we include b and b' as (unknown) variables with the dimensions of velocity, so that each coordinate has the dimension of length. We cannot identify b and b' with a constant c as is done in special relativity since we have no postulate regarding the universality of the speed of light. ***Although t' and b' are separately undefined in terms of b and t, their product b't'≡w' is well-defined in terms of bt≡w in taiji relativity.***

To further simplify the comparison with special relativity, we may also set things up so that the speed of light is constant and isotropic in the F frame by synchronizing the clocks in the F frame by the usual method in special relativity. ***It is important to note that we make this definition in one frame only and that this could have been any frame, not just F, since all frames are equivalent, according to the first postulate of relativity. Thus, we have not picked out any preferred frame.*** Since we have abandoned Einstein's second postulate, we have no information as to what the speed of light is in any other frame, and thus, do not know how to synchronize the clocks (which read time t or t') in any other frame by using light signals. As we see, Einstein's second postulate serves to fix a relation between the clocks in different inertial frames. One might object here that we have simply introduced a different postulate by requiring the speed of light to be constant and isotropic in one frame. However, the existence of such a frame is not critical to our discussion, as we have shown previously. The theory would stand just as well without it, except that the comparison with special relativity and further explanations would be made unnecessarily complicated mathematically.

Thus, keeping the conventional interpretation that s^2 should equal 0 when discussing the propagation of light, we see that we can now identify b in (7.11) with the constant c, which we use to denote the constant and isotropic speed of light in the F frame only. We can now write the taiji transformations (7.4) and (7.5) as

$$b't' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z, \tag{7.12}$$

where $$\beta = V/c, \qquad \gamma = 1/\sqrt{1-\beta^2}. \tag{7.13}$$

Note that, with the definition b=c in the F frame, V in (7.13) is the usual constant velocity of the F' frame as measured by the F-observers.

Now let us characterize a formal relationship between the times t and t' as follows:

$$t' = A(\mathbf{V}\cdot \mathbf{r},t) = A(x,t), \qquad \mathbf{V} = (V,0,0), \tag{7.14}$$

where A(x,t) is some unspecified function. Again, we stress that t' is just a symbol and has no physical meaning because its value is unknown or unspecified in taiji relativity. *Choosing a specific* A(x,t) *is equivalent to making a second postulate and one will obtain a different theory of relativity, as mentioned previously.* In this case, the F' clocks can then be synchronized by a new method without relying on knowing the speed of a light signal. *This is possible because the reading and the rate of ticking of any clock are adjustable.* For example, our "clocks" in F' could be some kind of computerized machines which have the capability to measure (or to accept information concerning) their positions in the F frame, obtain t from the nearest F clock, and then compute and display t' using A(x,t). Such a clock may seem absurd compared to our usual concept of time, but the conventional time system we use is itself merely the result of a postulate (namely the second postulate of special relativity regarding the universal speed of light) which we have grown used to. Any specific choice of A(x,t) for time t' must be regarded as a valid and physical time of the F' frame, as long as it can be physically realized by clocks and is consistent with known experimental results. *It must be emphasized that, according to taiji relativity, the inherent evolution variable is the taiji-time w with the dimensions of length, rather than the usual concept of time.* Furthermore, taiji relativity gives no relation between the times t and t' in (7.12). *A concept of time is well defined if and only if the time relation between any two inertial frames is explicitly stated.*

From equations (7.12)-(7.14), we derive the transformations of velocities,

$$c' = \frac{\gamma(c - \beta v_x)}{B_u}, \qquad v'_x = \frac{\gamma(v_x - \beta c)}{B_u}, \qquad v'_y = \frac{v_y}{B_u}, \qquad v'_z = \frac{v_z}{B_u}; \tag{7.15}$$

$$v' = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}\right), \qquad v = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),$$

$$B_u = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial t}, \qquad A = A(x,t),$$

where B_u is an unknown function dependent on A(x,t) and we have again made the definition

$$c' = \frac{d(b't')}{dt'}, \tag{7.16}$$

which is consistent with the law for the propagation of light: $ds^2=d(b't')^2-dr'^2=c'^2dt'^2-dr'^2=0$. Equation (7.15) gives us a formal relation between c and c', where $c'=c'(v_x)$ is actually undefined because it contains an unknown function B_u. It is essential to note that c' and v' *are separately unspecified.* However, *their ratio* v'/c' in (7.15) *is independent of B_u and, hence, well-defined.*

7f. Einstein's Time, Common Time, Reichenbach's Time and Unspecified Time

Many different additional second postulates can be made to obtain different theories of relativity. Although there are infinitely many possibilities, let us consider three simple cases:

(a) If one makes the additional postulate that the taiji-times w and w' are related to our usual concept of time t and t' by:

$$w = ct, \qquad w' = ct', \tag{7.17}$$

then all of the above equations reduce to those of special relativity. In this case, one has Einstein's relativistic time.

(b) If one makes the additional postulate that the taiji-times w and w' are related to our usual concept of time t and t' by:

$$w = ct, \qquad w' = b't, \qquad \text{(or equivalently } w = bt', \ w' = ct'), \tag{7.18}$$

then one has what we term "common time."[5] In this case, one has a 4-dimensional theory with a common time for all observers. Such a theory is called "common relativity" which will be discussed in detail in chapter 12. We may remark that (7.18) can also be written as w=bt' and w'=ct', since all inertial frames are equivalent. Also, common time is not unique as shown in (7.18) and, hence, not absolute. It differs from Newtonian absolute time which is embedded in a 3-dimensional symmetry framework.

(c) If one makes the additional postulate that the two-way speed of light (i.e., the round-trip distance divided by the round-trip time interval) in any frame is a universal constant, then one has "extended relativity".[6] In this case one might have

$$w = ct, \qquad w' = b't', \tag{7.19}$$

where t and t' are an "extended relativistic time" which are related by

$$t' = \gamma[(1 - \beta q')t - (\beta - q')x/c], \qquad \beta = V/c, \tag{7.20}$$

which involves the parameter q'. One can verify that the special case $q'=0$ corresponds to special relativity.[6]

In taiji relativity, one may formally write $w=bt$ and $w'=b't'$ if one wishes. However, we stress that b, t, b' and t' are separately undefined and unknown because of the lack of a second postulate. Since there is no relation between t and t', the concept of time measured in seconds is undefined in taiji relativity.

An infinite number of second postulates which are consistent with experimental results are possible. In this sense, taiji relativity can generate all kinds of different relativity theories with different concepts of time (i.e., different relations between t and t') and different properties of the speed of light (i.e., different relations between c and c').

7g. Discussions and Remarks

Following are some commonly asked questions about taiji relativity and their answers.

(A) Since the time t' in F' is, in general, non-uniform and a function of $\mathbf{r}'$, the velocity of F is also, in general, a function of $\mathbf{r}'$, as shown in (7.15). How then, can one define the concept of an "inertial frame" in taiji relativity?

An "inertial frame" in taiji relativity cannot be defined by the usual criterion of constant velocity V' because the "velocity" in the sense of $d\mathbf{r}'/dt$ is undefined. However, we define the taiji-velocity $\mathbf{B}'=d\mathbf{r}'/dw'$ which is the ratio between V' and c' because $d\mathbf{r}'/dw' = [d\mathbf{r}'/dt']/[dw'/dt'] = \mathbf{V}'/c'$. An "inertial frame" is thus one which has a constant taiji-velocity.

(B) Is taiji relativity really based on only one single postulate? According to (7.12), there is a second postulate that the speed of light is constant in one frame F.

The relation w = ct for taiji-time w in the F frame is not necessary for the formulation of taiji relativity. It is a technical 'definition' made solely to simplify the comparison with special relativity. And we do not need to assume the existence of F. The reason is that there are infinitely many frames, F, F', F",... in taiji relativity, even if the F frame with w=ct does not exist, the physics will not be changed.

(C) Isn't taiji relativity just a change of variables, $ct_E \rightarrow w=bt$ and $ct'_E \rightarrow w'=b't'$, of special relativity with relativistic times t_E and t'_E ?

The answer is no because the physical results of taiji relativity cannot be obtained from the corresponding expressions in special relativity merely by a change of variables. Taiji relativity and special relativity are two different and distinct theories. Special relativity has c, which is a universal constant in all inertial frames. However, in taiji relativity, there is no meaningful constant c related to the speed of light in all inertial frames. If one compares these two theories, their differences involve more than a change of variables in special relativity. The reasons are as follows: In special relativity, t_E and t'_E have a definite relationship (i.e., time t_E is defined) and the speed of light is a universal constant in all inertial frames. *However, in taiji relativity, t and t' have no relation whatsoever (i.e., time t is undefined) and, moreover, speed of light, measured in terms of time, is undefined and unknown.* To be more specific, the basic laws such as Maxwell equations (see eq.(10.20) in chapter 10) and the Dirac equation (see eq.(10.12)) are not related to their corresponding equations in special relativity by a mere change of variables. The fundamental difference between the two theories is that taiji relativity is based solely on the first principle of relativity, while special relativity is based on two principles. As a result, conventional QED involves three fundamental and universal constants, while QED based on taiji relativity has only two fundamental and universal constants.[7]

(D) Isn't it true that a formalism of space and time (as measured in seconds) must have two postulates? How can taiji relativity be based only on one postulate?

It is true that a formalism of space and time must involve two postulates. However, taiji relativity is not a formalism of space and time t because the usual concept of time as measured in seconds is undefined. Rather, it is a formalism of space and taiji-time w, which is measured in units of length.

(E) It seems that the difference between the taiji transformation (7.4) and the Lorentz transformation is superfluous. Doesn't (7.4) indicate that one can have $w=ct_E$ and $w'=ct'_E$?

No, (7.4) does not indicate that. If one wants to relate w to a 'time variable' t with the dimension of seconds in a general frame without making an explicit second postulate, one *can only write* $w=bt$, where b is an undefined variable which has the dimension of velocity. In taiji relativity, there are not enough postulates to specify b separately from t in a physically meaningful way. In a general frame, the time t is, therefore, also arbitrary and cannot by itself be regarded as the fourth dimension. That role is taken by the evolution variable w which we call the "taiji-time." *We stress that the arbitrariness of the time relation $w=bt$ is not the motivation of our investigation. Rather, it is merely a consequence of eliminating the second postulate of special relativity.* If and only if one makes a second postulate (the universal constancy of the speed of light), does one have $w=ct_E$ and $w'=ct'_E$ and hence, special relativity. As we shall show in the next chapters however, the loss of the usual concept of time does not in any way prevent taiji relativity from being a physically useful theory. It can still predict experimental results and is in agreement with all of the experimental results which have been interpreted as confirmations of special relativity. These experiments include the Michelson-Morley experiment, the Fizeau experiment, the aberration of light, experiments related to covariant Maxwell equations and Lorentz force, and so on. *Taiji relativity thus implies that relativistic time or any particular time system, for that matter, is not a necessary component of a theory based solely upon the first principle of relativity.*

References

1. Jong-Ping Hsu and Leonardo Hsu, Phys. Letters A **196**, 1 (1994); Leonardo Hsu and Jong-Ping Hsu, Nuovo Cimento B, **111**, 1283 (1996); Jong-Ping Hsu and Leonardo Hsu, in *JingShin Physics Symposium in Memory of Professor Wolfgang Kroll* (World Scientific, Singapore. New Jersey, 1997), pp. 176–193.
2. See, for example, *The New Lin Yutang Chinese-English Dictionary* (Ed. Lai Ming and Lin Tai-Yi, Panorama Press, Hong Kong, 1987), p. 90.
3. For a free particle, $r^2/w^2=\beta^2$=constant. Eq. (7.2) can be written as $m^2w^2/s^2-m^2r^2/s^2=m^2$, which is exactly the same as the 'energy-momentum" relation in eq. (7.3) because p_o amd **p** are given by equations (10.3) and (10.4) in chapter 10. For a rigorous proof based on covariant variational calculus, see ref. 5 in chapter 5.
4. Such a "time" will also be inconsistent with Poincaré's criterion of physical time discussed in section 4a of chapter 4.
5. Jong-Ping Hsu, Phys. Lett. A97, 137 (1983); Nuovo Cimento B74, 67 (1983); J. P. Hsu and C. Whan, Phys. Rev. A3**8**, 2248 (1988), Appendix. These papers discuss a four-dimensional symmetry framework based on the usual first postulate and a different second postulate, namely, a common time $t'=t=t_c$ for all observers in different inertial frames.
6. Leonardo Hsu, Jong-Ping Hsu and Dominik Schneble, Nuovo Cimento **111B**, 1299 (1996).
7. See chapter 10 for a detailed discussion.

8.

The Arbitrary Speed of Light in Taiji Relativity and the Michelson-Morley Experiment

8a. Does the Michelson-Morley Experiment Imply a Constant and Isotropic Speed of Light?

The Michelson-Morley experiment was first carried out by A. A. Michelson in 1881, more than twenty years before the birth of special relativity. It had been suggested by Maxwell in 1878 that such an experiment could reveal the absolute velocity of the Earth moving through the aether. However, the experiment turned out to have a null result which indicated that the absolute motion of the Earth could not be detected by optical phenomena. This "alarming result" was the first experiment which stimulated the search for the theory of relativity and it played a central role in the earlier work (1886–1905) of Lorentz and Poincaré, although not in Einstein's work. Lorentz was deeply concerned about the null result of the Michelson-Morley experiment for a long time and corresponded with W. Voigt around 1888.

The null result of the Michelson-Morley experiment has been interpreted in many different ways at different times by different people:

Originally, it was interpreted as the contraction of the absolute length of a rod along the direction of its motion through the aether by FitzGerald and Lorentz. Nowadays, it is widely seen as a confirmation of the universal constancy of the speed of light. Some physicists, however, interpret it only as support for the universal constancy of the two-way speed of light[1] rather than that of the one-way speed of light in special relativity. None of these are completely correct, as we shall explain below.

What we shall now show is that the Michelson-Morley experiment does not imply *any* specific property concerning the speed of light (measured in traditional units such as cm/sec.) *In fact, the null result can be shown to be consistent with an arbitrary speed of light, as long as the propagation of light is described by a law which displays the 4-dimensional symmetry of the Poincaré-Einstein principle (or of taiji relativity).*

Let us consider in detail the arbitrariness of the speed of light and the great flexibility of the usual concept of time t within taiji relativity. Within the conceptual framework of taiji relativity, both the time t and the speed of light c are separately undefined and, hence, arbitrary. Nevertheless, we can show that taiji relativity, with its 4-dimensional symmetry of Lorentz and Poincaré groups, is still consistent with experiments.

We start with the equation $ds^2=dw^2-dr^2=0$, which describes the propagation of a light signal using the taiji-time w in the frame F. This equation is not an independent assumption for a light signal in vacuum because it can be derived from Maxwell's equations, which are postulated to describe electromagnetic fields.[2] The equation for the propagation of this light signal in F' must be $ds^2=dw'^2-dr'^2=0$. In taiji relativity, although we cannot say anything definite about the time t or the speed of light c when taken separately, the Michelson-Morley experiment can still be understood in terms of the optical path, where the optical path is defined as the distance traveled by a light signal measured in units of the wavelengths of light. For example, a distance of one meter in a vacuum would be equivalent to an optical path of 2×10^6 for green light with a wavelength of 500 nm. Thus the optical path depends on the wavelength of the light under consideration, the distance traveled by the light signal, and the index of refraction n of the medium through which the light travels. This number dictates the phase of a wave at a certain point in space.

Taiji-time w in the present theory is precisely the same as the optical path in vacuum (n=1) because the propagation of light satisfies the invariant law

$$w'^2 - r'^2 = w^2 - r^2 = 0\,, \tag{8.1}$$

for a finite interval of taiji-time w.

Let us consider the case in which the apparatus of the Michelson-Morley experiment[3,4] is at rest in the F' frame. For purposes of the following discussion, we consider the simplified diagram showing a Michelson-Morley apparatus in figure 8.1. Light from a source reaches an inclined half-silvered mirror P which splits the light beam into two. One beam travels to mirror A and is reflected back to P. A fraction of this first beam is then reflected into a telescope. The second beam goes to a mirror B before being reflected back to P.

A fraction of this second beam interferes with the first beam and travels with it to the viewing device. If the light is monochromatic, an interference pattern of light and dark bands or rings will be seen in the telescope. This apparatus is known as a Michelson interferometer.

Suppose the angle between the direction of travel of the light signal going out along one of the interferometer arms and the x'-axis is θ' in F', as shown in Fig. 8.1 below.

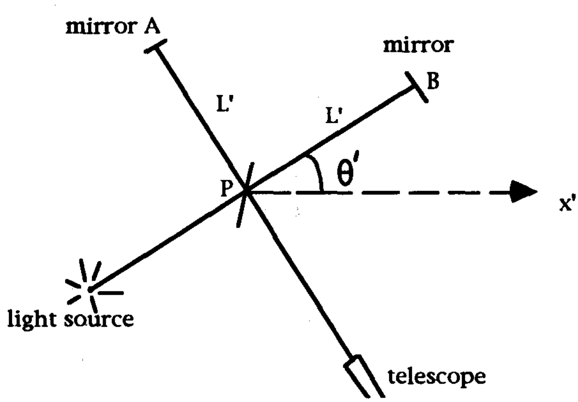

Fig. 8.1 A Michelson-Morley apparatus in an arbitrary orientation in the F' frame.

The corresponding angle as measured by observers in F is θ. During the return of the light signal along the arm PB, the angle is $\theta'_r = \pi + \theta'$ in F'. Putting $(\Delta x'/\Delta w')_{\theta'} = (\cos\theta', 0, 0)$ and $(\Delta x'/\Delta w')_{\theta'_r} = (-\cos\theta', 0, 0)$ separately in (7.6), we have

$$\Delta w(\theta) = \gamma \Delta w'(\theta')(1 + B\cos\theta') ,$$

$$\Delta w(\theta_r) = \gamma \Delta w'(\theta'_r)(1 + B\cos\theta'_r) = \gamma \Delta w'(\theta')(1 - B\cos\theta') , \qquad (8.2)$$

for a light signal going out and back along one of the arms of the interferometer. The optical path Δw_{rt} for such a round-trip of light, observed in the F frame, turns out to be independent of the angle θ,

$$\Delta w_{rt} = \Delta w(\theta) + \Delta w(\theta_r) = 2\gamma L', \qquad L' = \Delta w'(\theta'), \tag{8.3}$$

where L' is the length of the interferometer arm. The relation $L'=\Delta w'(\theta')$ is a consequence of the invariant law (8.1) for the F' frame. Since Δw_{rt} is independent of the angle θ, its value for light signals traveling along each of the two arms of the Michelson-Morley interferometer is identical. As a consequence, when the interferometer is rotated through 90^o, we would not expect to see any change in the interference pattern. Thus, taiji relativity, in which the speed of light is not postulated to be a universal constant, can lead to the null result for the Michelson-Morley experiment in any inertial frame.

8b. The Michelson-Morley Experiment Supports the First Postulate of Relativity

We have shown that, contrary to the usual belief, the null result of the Michelson-Morley experiment does not depend on postulating a universal speed of light or having a particular transformation for time. All that is required is that the law of propagation of light displays 4-dimensional symmetry and satisfies the first postulate of relativity. In other words, only the optical path needs to be well-defined, as given in equation (8.1). This suffices for the understanding of the Michelson-Morley experiment.

In order to make this point clear, we will now show that although the speed of light c' measured by F' observers using arbitrarily running clocks may be a complicated and arbitrary function of space and time, the *form* of the law for the propagation of light in F' turns out to be as simple as that in special relativity:

$$(b't')^2 - \mathbf{r}'^2 = 0, \tag{8.4}$$

$$\frac{d(b't')}{dt'} = c' \quad \text{or} \quad b't' = \int_0 c'dt' = \int_0 d(b't') .$$

We stress that the fact that the quantities t', b' and c' are not well-defined when taken separately does not upset the simplicity of the form of a physical law such as (8.4). This reveals the power of the 4-dimensional symmetry inherent in the Poincaré–Einstein principle. Equation (8.4) is merely (8.1) with the well-defined taiji–time w' written in terms of the product of two arbitrary functions c' and dt' (i.e., dw'=c'dt').

Let us try to use an arbitrary time t' and a correspondingly arbitrary and anisotropic speed of light c' in the F' frame to discuss the same Michelson–Morley experiment described previously. This example enables us to see the real roles played by t' and c' and why their separate and individual complications do not affect observable results. We again take the lengths of the two arms of the interferometer to be L', as shown in Fig. 8.1. In this case, it is clearly not physically meaningful to calculate time intervals Δt' for the propagation of light signals. We can, however, calculate the optical path length c'dt' with the help of the well-defined taiji–time dw'=c'dt'. The reason for this is as follows: it is the optical path that actually defines the number of wavelengths through which the two beams of light travel and this number is what determines the phases of the two beams when they recombine at the half–silvered mirror P. The phase relationship of these two beams then determines the interference pattern which is seen in the telescope.

Let us use w'_1 to denote the taiji–time interval required for a light signal to go from the beam splitter P to the far end of one of the interferometer arms say, mirror B, and w'_2 to denote the taiji–time interval for the same light signal to make the return trip back to the beam splitting point P in Fig. 8.1. The optical path lengths for the horizontal and vertical arms (L'_h and L'_v) are given by

$$L'_h = \int_0^{w'_1} c'dt' + \int_{w'_1}^{w'_2} c'dt' = 2L', \qquad c'dt' = d(b't') = dw', \tag{8.5}$$

$$L'_v = \int_0^{w'_2} c'dt' = 2L'. \tag{8.6}$$

Since w'=b't' is defined by the transformation (7.4), we can express the limits of integration, i.e., w'_1 and w'_2 in (8.5) and (8.6), in terms of the arm length,

$$w'_1 = L',$$
$$w'_2 - w'_1 = L', \quad (8.7)$$

where we have used (8.4).

Note that the results (8.5) and (8.6) obtained in the F' frame in terms of the taiji–time (or optical path length) appear to be as simple as those in special relativity because all inertial frames are equivalent. In special relativity, it is usually stressed that the nature of relativistic time is the key to understanding physical phenomena in different frames. *However, within the framework of taiji relativity, it is the invariance of physical laws which is crucial.* In general, one might think that physical phenomena involving the propagation of light cannot be discussed in terms of the time t' itself in taiji relativity because t' by itself is undefined. What we want to stress is that none of the previously established experimental results related to special relativity ever actually depended on the properties of time alone. We note that, in the Michelson-Morley experiment, it is the phase difference that matters, and the phase of the wave depends on the optical path it has travelled. That is why taiji relativity works and why the second postulate of special relativity is not needed.

In fact, in the vast majority of such experiments, such as the Michelson–Morley experiment, the measured quantities are actually lengths, which are well–defined quantities in taiji relativity, and can be expressed in terms of the product of c' and dt', (i.e., dw', or the optical path).

As we noted before, the two optical path lengths are identical for both arms of the interferometer in the Michelson–Morley experiment, so that we will obtain the null result in any inertial frame. From operational viewpoint, the interference fringes shown in the telescope in this experiment are directly determined by the phase difference $\mathbf{k}'\cdot\mathbf{r}'$ between the two beams, which in turn is determined by the optical path lengths of the interferometer arms. Thus, the time t', the speed of light and its frequency are not directly or separately involved in the Michelson–Morley experiment. This is the reason why the Michelson–Morley experiment cannot reveal any physical properties of the speed of light.

In taiji relativity, both length and wavelength have well–defined 4–dimensional transformation properties, while the time t' and the speed of light are undefined. In this connection, we note that the frequency ν' of light is also undefined because it is given by the relation $\nu'=c'/\lambda'$, where the wavelength λ' *is* well-defined, since it has a definite transformation property. The speed of light c' however, is undefined. (See equations (10.15)-(10.17) in chapter 10.)

To summarize, we have seen that the null result of the Michelson-Morley experiment does not unambiguously show that the speed of light, or even the two-way speed of light, is a universal constant. Even with an arbitrary and anisotropic speed of light within the four-dimensional symmetry framework of taiji relativity, a null result is still predicted for the Michelson-Morley experiment in all inertial frames as long as physical laws display 4-dimensional symmetry. *Therefore, we conclude that the null result in the Michelson-Morley experiment is actually experimental evidence for the first postulate of relativity rather than evidence for the second postulate concerning the universality of the speed of light.*

8c. Do Any Experiments Really Show the Universal Constancy of the Speed of Light c?

Given the previous discussion, *why are all experiments related to the speed of light interpreted to indicate that the speed of light is a universal constant?* This is related to the fact that the web that physicists weave with concepts and laws often restricts their own thinking. The Michelson-Morley and the Kennedy-Thorndike experiments are considered to be observational evidence in support of Einstein's second postulate, i.e., the speed of light in vacuum always has the same value c. However, in the preceding section, we have shown that the Michelson-Morley experiment does not actually give us any information regarding the speed of light. A similar analysis based on optical path lengths can be carried out for all previous optical experiments involving interference, including the Kennedy-Thorndike experiment, the two masers experiment, the two laser experiment, the Fizeau experiment, etc.,[5] to show that they are all consistent with taiji relativity.

In this connection, we note that in order to explain Fizeau experiment in taiji relativity, one should calculate Fresnel's drag coefficient for light in a medium with a refractive index n by using the transformation of taiji-velocities (7.10). First, we note that the taiji speed of light in the medium (at rest in frame F') slows from $\beta_L'=1$ in vacuum to $\beta_L'=1/n$, where $n>1$. Suppose a medium is at rest in the F' frame which moves with a taiji-velocity B parallel to the direction of light, as observed in F. The taiji-speed of light in such a moving medium observed by a stationary observer in F can be calculated from the inverse of transformation (7.10):

$$\beta=\frac{\beta_L' + B}{1 + B\beta_L'} \approx \frac{1}{n} + B\left(1 - \frac{1}{n^2}\right). \tag{8.8}$$

We see from this that Fresnel's drag coefficient, $(1-n^{-2})$, in taiji relativity is exactly the same as that obtained in special relativity. Its value does not depend on the speed of light c. Thus, the optical path involved in the Fizeau experiment[3] can be still calculated using taiji-velocities (8.8) and taiji-time. Suppose each tube has the length L and the taiji-speed of the water in Fizeau's experiment is B. The difference in the optical path length is $\Delta w = 2L/[n^{-1} - B(1-n^{-2})] - 2L/[n^{-1} + B(1-n^{-2})] \approx 2LBn^2(1-n^{-2})$. This result is exactly the same as that obtained in special relativity because the taiji-speed B for the water has the same numerical value as the usual expression v/c, (which can be seen by using equation (8.11) below.) The optical path difference Δw observed in the Fizeau experiment was given an ad hoc explanation by Fresnel in terms of a partial dragging of the light by medium. Einstein explained it on the basis of two postulates of special relativity. Here, again, we stress that it can be explained solely on the basis of the first postulate of special relativity.

Let us consider the following high-energy experiment: Some physicists have claimed that the existence of an ultimate velocity,[6] $c \approx 3\times10^8$ m/sec, in the universe has been established experimentally by the result that the measured velocity v of an electron is always smaller than the speed of light, $v < c$, even though its energy can be increased to about 15MeV, about 30 times larger than its rest energy of 0.5 MeV. The ultimate velocity of this experiment is consistent with the speed of light c=299792458m/sec. This experiment also confirms the relativistic relation

$$E=\frac{mc^2}{\sqrt{1-(v/c)^2}}, \tag{8.9}$$

which is the zeroth component of the energy-momentum 4-vector $p^\mu=(p^0,\mathbf{p})= (E/c^2,\mathbf{p})$ in special relativity.

At SLAC (the Stanford Linear Accelerator Center in Stanford, CA), electrons can be accelerated to an energy of about 50 GeV, which is 100,000 times larger than its rest energy, to support relation (8.9).[7] Furthermore, the relation (8.9) holds in many different inertial frames (i.e., because the earth revolves on its own axis and orbits the sun, an earth laboratory is co-moving with many different inertial frames for a short time interval during the course of a year), so this experiment is often interpreted as confirmation that the electron velocity v can only approach, but never exceed, c in any inertial frame. Since all properties of physical experiments based on such a high-energy electron beam are consistent with special relativity, the relation (8.9) with E=50 GeV could be interpreted as a strong confirmation that the electron velocity v can only approach, but never exceed, c in any inertial frame.

Indeed, within the available energy, the relation (9) is confirmed, so that it appears to be impossible for the electron to move faster than light. However, they do not unambiguously or objectively establish the existence of a constant ultimate velocity in nature with the numerical value of 299792458 m/sec. The reason is that such experiments are also consistent with taiji relativity, in which there is no well-defined universal and ultimate speed of light c.

In taiji relativity, the "energy-momentum" 4-vector $p^\mu = m\,dx^\mu/ds$, $\mu=0,1,2,3$, satisfies the 4-dimensional symmetry relation $(p^0)^2 - \mathbf{p}^2 = m^2$, as shown in equation (7.3). The zeroth component of p^μ is given by

$$p^0 = \frac{m}{\sqrt{1-(dr/dw)^2}} = \frac{m}{\sqrt{1-\beta^2}}, \qquad \mathbf{r} = (x,y,z)\,. \tag{8.10}$$

From the framework of taiji relativity, we can then interpret the previously described high-energy experiment as follows: The electron velocity has been accelerated to a very large taiji-velocity $\beta=0.99999998$, which corresponds to the ratio $p^0/m \approx 20{,}000$. Thus, the relation (10) of taiji relativity is also consistent with "high-energy" experiments and, therefore, it appears that β can only approach, but never exceed, 1.

In order to compare (8.9) and (8.10) directly, one may introduce a variable t which corresponds to the relativistic time. One can then express β in (8.10) in the following form

$$\beta = \frac{dr}{dw} = \frac{dr/dt}{dw/dt} = \frac{dr/dt^*}{dw/dt^*}. \tag{8.11}$$

It must be stressed that only w and r have a definite transformation and that t and t* can be arbitrary functions without affecting (8.10). Only when Einstein's second postulate is made do we have a well-defined relation w=ct in the F frame and w'=ct' in another frame F', so that β=dr/dw=v/c and (8.10) reduces to (8.9). On the other hand, if t* denotes Reichenbach's more general time, then β is the ratio v*/c* of two separate well-defined functions v* and c* and one still does not have a constant speed of light. (See section 17d in chapter 17 for details.) We have seen that the "energy" of a particle is given by (8.10), and is independent of any particular time t or t*, as shown in (8.11). This is dictated by the 4-dimensional symmetry of the Poincaré-Einstein principle of relativity.

There is a class of high-energy experiments, in which the speed of light emitted from a moving source is shown to be the same as that emitted from a source at rest.[3,5] Although this supports the claim that the speed of light is independent of any motion of its source, it does not prove the universality of the speed of light (i.e., that the same value c of the speed of the light signal will be measured by observers in different frames). (See discussions in section 5b in chapter 5.) In this connection, it is worthwhile to note that the taiji-speed of light is also independent of any motion of the source because relation (7.7) does not refer to any specific source.

8d. Physical Quantities Measured by Using Taiji Time w

In any inertial frame, we can set up a synchronized clock system which reads the taiji-time w, as discussed in chapter 7. If one compares the two lengths $\Delta x = x_2 - x_1$ and $\Delta x' = x'_2 - x'_1$ at the same taiji-time $w_2 = w_1$ we will obtain the usual length contraction result

$$\Delta x' = \gamma \Delta x, \qquad \Delta w = 0, \tag{8.12}$$

from (7.4). However, if they are compared at the same taiji-time $w'_2 = w'_1$, we obtain the reciprocal relation:

$$\Delta x' = \gamma^{-1}\Delta x\,, \tag{8.13}$$

showing the relativity between the F and F' frames, as in special relativity.

Let us also consider the measurement of "frequency" in terms of taiji-time w. In the F frame with synchronized taiji-clocks, a plane wave is described by

$$A\exp[-ik^\mu x_\mu] = A\exp[-i(k_o w - \mathbf{k}\cdot\mathbf{r})]\,, \tag{8.14}$$

where $k^\mu=(k^o,\mathbf{k})=(k_o,k)$ is the wave four-vector and $x_\mu=(w,-\mathbf{r})$. In the F' frame, this plane wave is described by

$$A\exp[-ik'_\mu x'^\mu] = Ae^{+i\Phi'}\,, \tag{8.15}$$

where $k'_\mu=(k'_o,-\mathbf{k}')$ and $x'^\mu=(w',x',y',z')$. In the framework of taiji relativity, k^μ and k'^μ are related by a 4-dimensional transformation (see eq. (10.15) in chapter 10), where k^o and k'^o are called the "frequency" measured by taiji-times w and w', respectively. As with velocities and times, the frequency ω' measured in terms of the time t' in F' has no definite transformation to relate it to ω, since the time t is undefined when taken alone. As we shall see later, however, the quantity $k'^o=\omega'/c'$ *is* well-defined and plays an important role in the emission and absorption of photons in the taiji framework. (See eq.(10.14) in chapter 10.)

We have now seen that, in taiji relativity, all physical quantities which are measured using the time t' have no definite transformation to relate them to their corresponding quantities as measured from other inertial frames. However, the laws of physics themselves are not affected since they are dictated solely by the first principle of relativity, i.e., invariance of physical laws under constant velocity. Thus, the Poincaré-Einstein principle or the four-dimensional symmetry in taiji relativity dictates that the most simple and natural "time" is the taiji-time with the dimensions of length.

References

1. The 2-way speed of light was introduced and discussed in physics long time ago. For example, Maxwell noted in 1878 that "all methods....by which it is practicable to determine the velocity of light from terrestrial experiments depends on the measurement of the time required for the double journey from one station to the other and back again." (See ref. 3 in chapter 2.) Let us consider two inertial frames F and F', where F' is moving along the x-axis. Suppose there is a clock 1 located at the origin of the F frame and a mirror at point x=L on the x-axis. A light signal starts from the origin at time t_1, it reaches the mirror and returns to the origin at time t_3. The two-way speed c_{2w} of light moving along the x-axis and returning to the starting point in F is defined by $c_{2w}=2L/(t_3-t_1)$. The round-trip trajectory of this light signal forms a "closed loop" in F. However, from the viewpoint of observers in F', the trajectory of this light signal forms an "open-loop" because the returning point differs from the starting point in the F' frame.
2. Note that the law ds=0 for the propagation of light is a consequence of the plane wave solution of Maxwell's equations (given by eq. (10.20) in chapter 10.)
3. For discussions of the Michelson-Morley experiment and the Fizeau experiment in special relativity see, for example, A. P. French, *Special Relativity* (Norton & Company, New York, 1968), pp. 51-57, pp. 46-48 and pp. 131-132.
4. For a discussion of the Michelson-Morley experiment in "extended relativity" based on Reichenbach's concept of time (see ref. 2 in chapter 6) and Edwards' universal 2-way speed of light, see Leonardo Hsu, Jong-Ping Hsu and Dominik A. Schneble, Nuovo Cimento **111B**, 1305 and 1310 (1996).
5. See, for example, Yuan Zhong Zhang, *Special Relativity and Its Experimental Foundations* (World Scientific, 1997), chapter 8.
6. W. Bertozzi, Am. J. Phys., 32, 551 (1964).
7. Particle Data Group, *Review of Particle Physics*, The European Physical Journal C, vol. 3, 1998, p. 142. Maximum electron beam energy in high-energy collider: E=50 GeV at SLAC (SLC, 1989) and E=92 GeV (100 GeV forseen) at CERN (LEP, 1997).

9.

Lorentz and Poincaré Invariance Without Involving a Constant Corresponding to the Speed of Light

9a. Group Properties of Taiji Transformations

Lorentz and Poincaré invariance is closely intertwined with the group properties of the 4-dimensional coordinate transformations. At first glance, one sees that there is an obvious one-to-one correspondance between the 4-dimensional taiji transformation (7.4) and the Lorentz transformation (5.7), even though there is no universal constant corresponding to the speed of light in the taiji transformations. Thus, one would expect the set of 4-dimensional taiji transformations (7.4) to form a Lorentz group (see below), just as the Lorentz transformations do. This observation seems mathematically trivial. However, the new concepts behind the taiji transformations (i.e., that the second postulate regarding a constant speed of light is superfluous, and that there are infinitely many possibilities for specifying the time t through the relations w=bt and w'=b't') are highly non-trivial.

In this chapter, we shall show that the taiji transformations, which are derived solely from the first postulate of special relativity, (i.e., the Poincaré-Einstein principle) have precisely the properties of the Lorentz group and the Poincaré group. This is crucial to answering the historic question discussed by Ritz, Tolman, Comstock and Pauli in chapter 0. Furthermore, it is also crucial[1] for the formulation of quantum field theories and particle physics on the basis of taiji relativity. Once the 4-dimensional group properties of the taiji transformations are established, it follows that, regardless of which concept of time one uses, (e.g., Reichenbach's more general time, Edwards' universality of the 2-way speed of light, or common time) one always has a valid 4-dimensional symmetry framework with which to understand and investigate phenomena in the physical world.

Mathematically, a group consists of a set $G^{\circ}=\{g_0, g_1, ... g_k, ..\}$ of elements (or operators) g_k and an operation (also known as a "multiplication rule" and denoted by a •) such that:

(a) Combining any two elements g_i and g_k using the operation leads to another elements g_n in the set G°,

$$g_i \bullet g_k = g_n .$$

(b) There exists a unit element g_o in G° such that for any element g_i, one has the relation

$$g_o \bullet g_i = g_i \bullet g_o = g_i .$$

(c) For any element g_k, there exists an inverse element $g_f \equiv g_k^{-1}$ such that

$$g_k \bullet g_k^{-1} = g_k^{-1} \bullet g_k = g_o .$$

(d) The operation obeys the associate rule,

$$(g_i \bullet g_k) \bullet g_n = g_i \bullet (g_k \bullet g_n) .$$

The most important groups are those related to geometrical or physical transformations.[2] For the Lorentz group, these elements are the transformation matrices in (9.11) below. Thus, the Lorentz group can be defined as the set of all 4x4 real matrices that leave the 4–dimensional interval $w^2-x^2-y^2-z^2$ invariant. These matrices are singled–valued and continuous functions of six parameters, three angles for rotations and three velocities for motion in 3–dimensional space. These parameters can change continuously, so that the Lorentz group is a continuous group or Lie group.

Let us consider the group properties of the taiji transformations which relate two inertial frames, F and F', which have their axes oriented in the same directions. First, we consider the special case where the relative motion between the F and F' frames is characterized by a dimensionless constant taiji–velocity B, measured in terms of taiji–time w, along parallel x and x' axes. The origins of F(w,x,y,z) and F'(w',x',y',z') coincide at the taiji–time w=w'=0. In addition to these two standard reference frames, let us also consider a third frame F"(w",x",y",z"), with axes similarly oriented, which moves with a constant taiji–velocity $\boldsymbol{\beta}=(\text{B}_1,0,0)$ as measured in terms of the taiji–time w by observers in the F frame. For simplicity, we shall call $\boldsymbol{\beta}$ or B_1 the velocity. The constant velocity of F" measured in F' is then denoted by $\beta' = (B_1', 0, 0)$ and is related to β by (7.10):

$$\beta'_x = B_1' = \frac{B_1 - B}{1 - BB_1} . \tag{9.1}$$

In analogy with equation (7.4), the 4-dimensional transformations between F and F" are given by

$$w'' = \gamma_1(w - B_1 x) , \quad x'' = \gamma_1(x - B_1 w) , \quad y'' = y , \quad z'' = z ; \tag{9.2}$$

$$\gamma_1 = \frac{1}{\sqrt{1 - B_1^2}} .$$

The inverse transformations can be easily obtained and are:

$$w = \gamma(w' + Bx') , \quad x = \gamma(x' - Bw') , \quad y = y' , \quad z = z' . \tag{9.3}$$

Based on (9.2), (9.3) and (7.10), one can show that the taiji transformations between F' and F" are

$$w'' = \gamma_1'(w' - B_1' x') , \quad x'' = \gamma_1'(x' - B_1' w') , \quad y'' = y' , \quad z'' = z' ; \tag{9.4}$$

$$B_1' = \frac{B_1 - B}{1 - B_1 B} , \quad \gamma_1' = \gamma\gamma_1(1 - B_1 B) = \frac{1}{\sqrt{1 - B_1'^2}} ,$$

which have the same form as the taiji transformations between F and F', or F and F".

Other group properties, such as the existence of an identity transformation [e.g., B=0 in (9.3)] , the existence of an inverse transformation [e.g., (9.3) is the inverse transformation of (7.4)] and the fact that the transformations obey the associative rule, can be verified. These properties and result (9.4) show that the set of 4-dimensional taiji transformations forms precisely the Lorentz group.[3] These 4-dimensional group properties are essential for taiji relativity to be consistent with the results of all known experiments.

Thus, we have shown that the Lorentz group can actually accommodate a wide class of different concepts of the time t and that postulating the speed of light to be a universal constant is not a requirement for transformations to form such a group.

9b. The Lorentz Group Without Involving the Constant Speed of Light

In the previous section, we discussed group properties of the taiji transformations for inertial frames with relative motion solely along the x and x' axes. In general, the taiji transformations relate quantities in inertial frames with relative motion along x, y, and z directions that leave the 4–dimensional interval $w^2-x^2-y^2-z^2$ invariant. Consequently, the contents of the Lorentz group are much richer than what we have previously discussed. There are many different transformations which leave $w^2-x^2-y^2-z^2$ invariant. For example, there is a subset involving three discrete transformations: a spatial inversion ($w \to w$, $\mathbf{r} \to -\mathbf{r}$), a time inversion ($w \to -w$, $\mathbf{r} \to \mathbf{r}$) and a spacetime inversion ($w \to -w$, $\mathbf{r} \to -\mathbf{r}$). Another subset is a six-parameter (three angles for rotations and three constant velocities for boosts) continuous group, which was first noted and discussed by Poincaré in 1905.

As stated before, the Lorentz group is defined as the set of all 4×4 real matrices that leave the interval s^2 invariant, where

$$s^2 = w^2 - x^2 - y^2 - z^2 = g_{\mu\nu}x^\mu x^\nu = x_\nu x^\nu ; \qquad (9.5)$$

$$g_{00} = 1 , \qquad g_{11} = g_{22} = g_{33} = -1 , \qquad g_{\mu\nu} = 0 \;\; \text{for} \;\; \mu\neq\nu , \qquad (9.6)$$

$$x^\mu = (w, \mathbf{r}) , \quad x_\mu = g_{\mu\nu}x^\nu = (w, -\mathbf{r}) , \qquad \mu=0,1,2,3 .$$

The metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ satisfy the relation $g_{\mu\nu}g^{\mu\rho}=\delta_\nu{}^\rho$.

Suppose x_1 and x_2 denote two coordinate 4–vectors (related by the taiji transformations), then we have the following three types of invariant intervals in the 4–dimensional spacetime:

$$\Delta s^2 = (x_{2\nu} - x_{1\nu})(x_2{}^\nu - x_1{}^\nu) > 0 : \qquad \text{a "time-like" interval,} \qquad (9.7)$$

$$\Delta s^2 = (x_{2\nu} - x_{1\nu})(x_2^\nu - x_1^\nu) = 0: \quad \text{a "light-like" interval,} \tag{9.8}$$

$$\Delta s^2 = (x_{2\nu} - x_{1\nu})(x_2^\nu - x_1^\nu) < 0: \quad \text{a "space-like" interval.} \tag{9.9}$$

Physically, these invariant expressions describe the invariant laws of motion for three types of particles: (See ref. 7 in chapter 5.)

(9.7) for massive particles, $m^2 > 0$,
(9.8) for massless particles, $m^2 = 0$,
(9.9) for tachyons (moving faster than the taiji-speed of light $\beta_L=1$), $m^2 < 0$.

The first two laws have been experimentally demonstrated. However, it is not known why the third law (9.9) for "tachyons" is not physically realized in nature[4] or why tachyons, if they "exist" at all, have no interaction with ordinary particles.

Let us consider the continuous Lorentz group with six parameters. With the help of tensor notation,[3] the general 4-dimensional taiji transformations can be parametrized as follows,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{9.10}$$

where we have employed the summation convention for repeated indices. Any quantity Q^μ with four components which satisfy the transformation (9.10), $Q'^\mu=\Lambda^\mu{}_\nu Q^\nu$, is called a contravariant 4-vector. The transformation tensor $\Lambda^\mu{}_\nu$ and the metric tensor $g_{\alpha\beta}$ satisfy the relation

$$g_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu g_{\alpha\beta}. \tag{9.11}$$

The metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ can be used to raise and lower the indices of any tensor. (For example, a covariant 4-vector Q_μ is, by definition, related to the corresponding contravariant vector Q^ν by $Q_\mu=g_{\mu\nu}Q^\nu$, and in general one has $A^{\cdots\mu\rho\cdots\alpha}=g^{\mu\nu}A^{\cdots}{}_\nu{}^{\rho\cdots\alpha}$.) The metric of the Lorentz group, given in (9.6), is $g_{\mu\nu}=(1,-1,-1,-1)$ which has three negative signs and one positive sign. Thus, the Lorentz group is denoted by O(3,1).

The 4-dimensional taiji transformation (9.10) can be written in matrix form and $(\Lambda^\mu{}_\nu)$ can be considered a 4x4 matrix. (See (9.15) below.) The matrix form of (9.11) is $g=\Lambda^T g \Lambda$, where the superscript T denotes the transpose of a matrix. The set of matrices $\{\Lambda\}=\{\Lambda,\Lambda',\Lambda'',...\}$ leaves the interval s^2 in (9.5) invariant and can be considered as elements of the Lorentz group. Indeed, one can verify that $\{\Lambda\}$ satisfies the group properties mentioned previously in section 9a.

The anti-symmetric operator $L^{\mu\nu}$ of the Lorentz group was first discussed by Poincaré in his Rendiconti paper. It is defined as follows:

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) , \qquad \partial^\mu = \partial/\partial x_\mu . \tag{9.12}$$

It can be verified that $L^{\mu\nu}$ generates the algebra of the Lorentz group,

$$[L^{\mu\nu}, L^{\mu\nu}] = i(g^{\mu\beta}L^{\nu\alpha} + g^{\nu\alpha}L^{\mu\beta} - g^{\mu\alpha}L^{\nu\beta} - g^{\nu\beta}L^{\mu\alpha}) . \tag{9.13}$$

We can parametrize the matrix Λ of the taiji transformations (9.10) in terms of ϕ by using equation (7.4):

$$w' = \gamma(w - Bx) = w \cosh\phi - x \sinh\phi ,$$

$$x' = \gamma(x - Bw) = x \cosh\phi - w \sinh\phi , \tag{9.14}$$

$$y' = y , \quad z' = z ;$$

$$\gamma = \frac{1}{\sqrt{1-B^2}} = \cosh\phi , \qquad \gamma B = \sinh\phi .$$

The Lorentz group is non-compact because the range of the parameter B does not include the endpoint 1. If the the metric tensor is $g_{\mu\nu}=(1,1,1,1)$ [i.e., $s'^2=g_{\mu\nu}x^\mu x^\nu=w^2+x^2+y^2+z^2$] instead of that in (9.5)), then the group would be the 4-dimensional orthogonal group O(4) rather than O(3,1). Since the coordinates (w,x,y,z) are real numbers, this O(4) is actually the 4-dimensional rotational group. The parameters of the group O(4) have finite ranges which include the endpoints and, hence, is compact. The 4-dimensional space associated with O(4) is mathematically the perfect generalization of the ordinary 3-dimensional space with the distance $x^2+y^2+z^2$. However, such a perfect generalization is not perfect from the physical viewpoint. The reason is that the invariant interval $s'^2=g_{\mu\nu}x^\mu x^\nu=w^2+x^2+y^2+z^2$ has only a geometric meaning and has nothing to do with laws of physics. This is in sharp contrast with the invariant interval (9.5) or (9.7)–(9.9), which is also applicable to the laws of physics.

The taiji transformation (9.14) can also be written in matrix form

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \equiv \Lambda(10,\phi) \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \tag{9.15}$$

where the transformation matrix $\Lambda(10,\phi)$ denotes a rotation in the wx (or x^0x^1) plane in the 4-dimensional spacetime. For an infinitesimal transformation, Λ, in (9.15) takes the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon\lambda^\mu{}_\nu, \qquad \lambda^{\mu\nu} = -\lambda^{\nu\mu}. \tag{9.16}$$

The infinitesimal generator m^{10} for the rotation in (9.15) is defined as

$$m^{10} = -i\frac{d}{d\phi}\,\Lambda(10,\phi)|_{\phi=0} = -i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9.17}$$

Similarly, for rotations in the x^0x^2 and x^0x^3 planes, the infinitesimal generators are given by

$$m^{20} = -i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad m^{30} = -i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{9.18}$$

In equations (9.1)–(9.4), we see that two boosts in the x direction generate another boost. However, a boost in the x direction followed by a boost in the y direction does not generate another boost. Instead, it generates a spatial rotation. In fact, this is the physical origin of the Thomas precession.[5] Therefore, we must also introduce three generators for spatial rotations in order to have a closed algebra. This important property was recognized by Poincaré, but not by Einstein, in their original works on relativity in 1905. The infinitesimal generators for spatial rotations are

$$m^{12} = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad m^{23} = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$m^{31} = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{9.19}$$

In general, the infinitesimal generator $m^{\mu\nu}$ is defined to satisfy the relation $m^{\mu\nu}=-m^{\nu\mu}$. An arbitrary infinitesimal taiji transformation can be written as

$$\Lambda(\omega) = 1 + \frac{i}{2}\,\omega_{\mu\nu}m^{\mu\nu}, \qquad \omega_{\mu\nu} = -\,\omega_{\nu\mu}. \tag{9.20}$$

A finite rotation in the $\mu\nu$ plane (in the sense μ to ν) is given by exponentiation

$$\Lambda(\mu\nu, \phi) = \exp(i\phi m^{\mu\nu})\ . \tag{9.21}$$

It can be shown that the generators $m^{\mu\nu}$ satisfy the commutation relation

$$[m^{\mu\nu}, m^{\mu\nu}] = i(g^{\mu\beta}m^{\nu\alpha} + g^{\nu\alpha}m^{\mu\beta} - g^{\mu\alpha}m^{\nu\beta} - g^{\nu\beta}m^{\mu\alpha})\ . \tag{9.22}$$

Let us define the spatial-rotational generators J^i and boost-generators K^i as follows:

$$J^i = \frac{1}{2}\,\varepsilon^{ijk}\, m^{jk}, \qquad K^i = m^{0i}. \tag{9.23}$$

One can verify the following commutation relations:

$$[J^i, J^j] = i\,\varepsilon^{ijk}J^k, \tag{9.24}$$

$$[K^i, K^j] = -\,i\,\varepsilon^{ijk}J^k, \tag{9.25}$$

$$[J^i, K^j] = i\,\varepsilon^{ijk}K^k. \tag{9.26}$$

If one makes the following linear combinations of these rotational and boost generators,

$$M^i = \frac{1}{2}(J^i + iK^i)\,, \tag{9.27}$$

$$N^i = \frac{1}{2}(J^i - iK^i)\,, \tag{9.28}$$

one can show that M^i and N^k commute,

$$[M^i\,, N^k] = 0\,, \tag{9.29}$$

so that the algebra of the Lorentz group has been split up into two pieces. Each piece generates a separate group called SU(2). One can use the irreducible representations of SU(2) to construct representations of the Lorentz group.[6]

9c. The Poincaré Group with Ten Generators and Without Involving the Constant Speed of Light

In order to have the most general transformations in flat 4–dimensional spacetime, one can generalize (9.10) to include translations such as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + b^{\nu}\,, \tag{9.30}$$

where b^{ν} is a constant and real 4–vector. These are known as the Poincaré transformations. Note that (9.30) does not leave the 4–dimensional interval (9.5) invariant. Nevertheless, the interval Δs^2 in (9.7)–(9.9) are still invariant. The set of transformations (9.30) forms a Poincaré group with 10 generators (3 angles for rotations, 3 constant velocities for boosts and 4 constants in b^{ν} for translations.) Thus, in addition to the six generators $L_{\mu\nu}$ given by (9.12), there are four translation generators $P_{\mu} = i\partial_{\mu}$ (with a suitable choice of units, J=1). These generators satisfy the following commutation relations:

$$[L^{\mu\nu}, L^{\mu\nu}] = i(g^{\mu\beta}L^{\nu\alpha} + g^{\nu\alpha}L^{\mu\beta} - g^{\mu\alpha}L^{\nu\beta} - g^{\nu\beta}L^{\mu\alpha})\,, \tag{9.31}$$

$$[L_{\mu\nu}, P_{\alpha}] = i(g_{\nu\alpha}P_{\mu} - g_{\mu\alpha}P_{\nu})\,, \tag{9.32}$$

$$[P_\mu, P_\nu] = 0 . \tag{9.33}$$

This is the Lie algebra of the Poincaré group (or the Poincaré algebra.)

Based on these generators, one can construct two invariant (or Casimir) operators:

$$P_\mu P^\mu \quad \text{and} \quad W_\mu W^\mu , \tag{9.34}$$

where

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} P_\nu L_{\alpha\beta} . \tag{9.35}$$

The tensor $\varepsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric unit 4–tensor of the fourth rank and satisfies $\varepsilon^{0123}=1$. Its components change sign upon interchange of any two indices, so that the components different from zero are equal to ±1.

The two invariant operators in (9.34), $P_\mu P^\mu$ and $W_\mu W^\mu$, commute with all generators of the Poincaré group. The operator $P_\mu P^\mu$ has a clear physical interpretation, namely, it is the square of the mass of a particle. It is easier to see the meaning of $W_\mu W^\mu$, if one makes a taiji transformation to the rest frame of the particle. In this case, the eigenvalues of P_μ are given by (m,0,0,0) and we have

$$W^o = 0, \qquad W^i = \frac{1}{2} \varepsilon^{ijk0} m L_{jk} , \tag{9.36}$$

where W^i/m, i=1,2,3 are simply the usual rotational matrices in the three spatial dimensions which obey the angular momentum commutation relations. Since the particle is at rest, W^i is just the spin operator. The eigenvalues of the square of $\mathbf{W}=(W^i)$ are therefore given by

$$\mathbf{W}^2 = m^2 s(s+1) , \qquad m>0, \tag{9.37}$$

where s is the spin eigenvalue of a particle, s = 0, 1/2, 1, 3/2, 2,... . Thus, if m > 0, a particle with spin s has 2s+1 components (i.e., 2s+1 independent states for a given momentum 4–vector). However, if m = 0, a particle with a spin s can have only 2 components. For example, the photon has spin s=1, but only two independent polarization states. This property of the photon is a purely kinematic property dictated by the 4-dimensional symmetry of Lorentz and Poincaré groups.

It is interesting that all particles in the physical world can be classified according to the eigenvalues of the two invariant operators in (9.34):

$$P_\mu P^\mu > 0, \qquad s = 0,1,2,3,\dots \text{ (bosons)}, \qquad s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \text{ (fermions)} \tag{9.38}$$

$$P_\mu P^\mu = 0, \qquad s = \pm s, \tag{9.39}$$

$$P_\mu P^\mu = 0, \qquad \text{continuous } s, \tag{9.40}$$

$$P_\mu P^\mu < 0, \qquad \text{tachyons}. \tag{9.41}$$

Particles with the properties in (9.40) and (9.41) have not been detected experimentally. Nevertheless, it is of great significance to the Poincaré group that all particles (or fields) can be classified by using the eigenvalues of the two invariant operators, one related to the mass of the particle and the other to its spin.[7] In contrast, the Lorentz group does not have the translation operator P_μ (or $P_\mu P^\mu$), and so is inadequate for the classification of elementary particles. Note that mass and spin are two fundamental properties of elementary particles. The fact that particles have a discrete spin can be understood on the basis of spacetime symmetry.[8] However, the fact that elementary particles (such as quarks and leptons) also have discrete masses is far from being understood.[9]

References

1. Although the form of transformations with 4-dimensional symmetry is simple, its significance to the formulation of particle physics and quantum field theory has not been fully appreciated by some authors who have attempted to modify the synchronization procedure for clocks and/or to test a different ideas of time or the speed of light. For example, Winnie and Edwards attempted to obtain new transformations of space and time. However, the resultant transformations they obtained are untenable because the 4-dimensional symmetry of the Lorentz and Poincaré groups were lost. See discussions in chapter 17.
2. See, for example, P. Roman, *Theory of Elementary Particles* (2nd. ed., North-Holland, Amsterdam, 1961), pp. 8-51; Wu-Ki Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985), pp. 173-211.
3. See, for example, J. D. Jackson, *Classical Electrodynamics* (2nd ed., John Wiley & Sons, New York, 1975), pp. 532-541.
4. For a discussion of tachyons, see G. Feinberg, Phys. Rev. 159, 1089 (1967); E. C. G. Sudarshan, in *The Encycloopedia of Physics* (edited by R.M. Besancon, Van Nostrand Reinhold, New York, 1974), p. 309 and references therein. Equations (9.7)-(9.9) appears to suggest that the absence of physical particles with $m^2<0$ is an asymmetry in Nature. To restore the symmetry and to interpret the apparent asymmetry, it was proposed that the Finsler geometry should be used to desccribe the 4-dimensional spacetime. See S. L. Cao, in *The Proceedings of International Workshop on Gravitation and Astrophysics*, 1999 (Ed. L. Liu, et al, to be published by World Scientific in 2000) and references therein.
5. J. D. Jackson, ref. 2, pp. 541-547.
6. S. Weinberg, *The Quantum Theory of Fields, I. Foundations* (Cambridge Univ. Press, New York, 1995), pp. 49-74; M. Kaku, *Quantum Field Theory, A Modern Introduction* (Oxford Univ. Press, New York, 1993), pp. 50-56.
7. The physical content of the Poincaré group is richer than that of the Lorentz group and has not been fully explored. See E. P. Wigner, Ann. Math. **40**, 149 (1939); E. Inönü and E. P. Wigner, Nuovo Cimento IX, 705 (1952); V. Bargmann, Ann. Math. **59**, 1 (1954); S. Weinberg, ref. 5, pp. 55-81
8. In the world of elementary particles and quantum fields, there is a necessary and fundamental connection between spin and statistics. This

can be understood with the help of the 4–dimensional symmetry of the Lorentz group together with other requirements such as local fields, unique ground states, and microscopic causality. See J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw–Hill, New York, 1965), pp. 170–172.

9. This suggests that symmetry principles alone are inadequate for understanding the phenomenon of mass and that perhaps, interactions of particles should be taken into consideration. However, it is very puzzling that the electron and the muon appear to have exactly the same interactions (except for the "extremely" weak gravitational interaction), and yet the muon mass is about two hundred times larger than the electron mass. For a recent discussion on the origin of the mass of ordinary matter, see ref. 14 in chapter 4.

10.

Truly Universal Constants and Physical Laws Based on Taiji Relativity

10a. Truly Universal Constants and Invariant Actions

One of the burning questions of taiji relativity is the following: *If the speed of light is not a truly universal constant, why do all experiments that measure the speed of light in different inertial frames give the same definite value c=29979245800cm/sec?* (An earth laboratory at different times is co-moving with different inertial frames for short time intervals.)

To answer this question, let us first consider physical laws involving the momentum four-vector and the wave four-vector of a particle. In the framework of taiji relativity, the speed of light c' in F' is undefined, so that the invariant action for a free particle *cannot* be written in the usual form $-\int mc'ds'$, where $ds'^2=dx'_\mu dx'^\mu$. Nevertheless, mds'=mds is invariant because the interval ds (=ds') and the mass m are scalars. Thus, in order for the action to be consistent with 4-dimensional symmetry principles, we assume that the action S for a charged particle in an electromagnetic field in a general frame F has the form:

$$S = \int(-mds - \bar{e}a_\mu dx^\mu) = \int L dw, \tag{10.1}$$

$$L = -m\frac{ds}{dw} - \bar{e}a_\mu \frac{dx^\mu}{dw} = -m\sqrt{1-\beta^2} - \bar{e}a_0 + \bar{e}\mathbf{a}\cdot\boldsymbol{\beta}, \qquad \boldsymbol{\beta} = \frac{d\mathbf{r}}{dw},$$

$$ds = \sqrt{dw^2 - d\mathbf{r}^2} = dw\sqrt{1-\beta^2},$$

$$\bar{e} = -1.6021891\times10^{-20}\sqrt{4\pi}\ (\text{g}\cdot\text{cm})^{1/2},$$

and is given in terms of quantities measured in an inertial frame F with taiji-time w. The universal constant $\bar{e}$ is in Heaviside–Lorentz units. Comparing this action with the corresponding action S_{sr} in special relativity for the F frame, we see that $S=S_{sr}/c$, $\bar{e}=e/c$ (e in esu) and $a_\mu=a_\mu(w,\mathbf{r})=A_\mu(w,\mathbf{r})/c$, where A_μ is the usual electromagnetic vector potential in special relativity. The "canonical momentum" of the particle is defined as

$$\mathbf{P} = \frac{\partial L}{\partial \boldsymbol{\beta}} = \mathbf{p} + \bar{e}\mathbf{a}, \qquad \mathbf{p} = \frac{m\boldsymbol{\beta}}{\sqrt{1-\beta^2}}. \tag{10.2}$$

The Hamiltonian H is now defined as

$$H = [(\partial L/\partial \boldsymbol{\beta}) \cdot \boldsymbol{\beta} - L] = p_0 + \bar{e}a_0, \qquad p_0 = \frac{m}{\sqrt{1-\boldsymbol{\beta}^2}}. \tag{10.3}$$

We note that the quantities in both (10.2) and (10.3) have the dimensions of mass. They can be interpreted as the "momentum" and the "energy" associated with a charged particle moving in an electromagnetic field. From (10.2), (10.3) and $H = P_0$ we have the canonical momentum 4-vector $(P_0, \mathbf{P})$ and the invariant relation

$$(P_0 - \bar{e}a_0)^2 - (\mathbf{P} - \bar{e}\mathbf{a})^2 = m^2. \tag{10.4}$$

We also have the usual "kinetic energy" p_0 and momentum $\mathbf{p}$ in (10.3) and (10.2) for a free particle. They are the components of $p^\mu=(p_0,\mathbf{p})=m dx^\mu/ds=\left(m/\sqrt{1-\beta^2}, m\boldsymbol{\beta}/\sqrt{1-\beta^2}\right)$ which satisfy the 4-dimensional taiji transformation and hence, form the momentum four-vector in taiji relativity,

$$p_0' = \gamma(p_0 - Bp_x), \quad p_x' = \gamma(p_x - Bp_0), \quad p_y' = p_y, \quad p_z' = p_z; \tag{10.5}$$

$$\gamma = \frac{1}{\sqrt{1-B^2}},$$

where $p_0=p^0$ and $\mathbf{p}=(p_x,p_y,p_z)=(p^1,p^2,p^3)$ have the dimensions of mass. As a result, the concept of mass m and that of "energy" p_0 are the same thing in taiji relativity. In contrast, they are two related concepts in special relativity. In this connection, it is worthwhile to note that quantities such as

$$\frac{mv'}{\sqrt{1-v'^2/c'^2}} \quad \text{and} \quad \frac{mc'}{\sqrt{1-v'^2/c'^2}}$$

have no physical meaning and are not conserved quantities in taiji relativity.

According to quantum mechanics, the wave four-vector k^μ must be proportional to the four-momentum p^μ. Thus, in general we have

$$p'^\mu = J'k'^\mu, \qquad \text{in F' frame ;} \qquad (10.6)$$

$$p^\mu = Jk^\mu, \qquad \text{in F frame .} \qquad (10.7)$$

Based on symmetry considerations, the same proportional relation should hold in any inertial frame such that J'=J. Therefore, the proportional constant J must be a universal constant.[1] Its value can be obtained by comparing (10.7) in F with the usual relation in special relativity (SR) $p^\mu_{SR}=\hbar k^\mu$, where $p^\mu_{SR}/c \Leftrightarrow p^\mu$, $k^0_{SR}=\omega/c \Leftrightarrow k^0$ and $\mathbf{k}_{SR}=\mathbf{k}$ in F. Note that the last relation must be satisfied because the spatial components $\mathbf{r}$ in taiji relativity is the same as the usual $\mathbf{r}_{SR}$. As a result, we can deduce that the value of J must be $\hbar/c=h/(2\pi c)$:[1]

$$J = 3.5177293 \times 10^{-38}\ \text{g·cm}. \qquad (10.8)$$

It must be emphasized that J is a truly universal constant because it is independenvsvolution variable w in the 4-dimensional symmetry framework. This is in sharp contrast to the conventional universal constants c and ħ, whose universality depends on the second postulate of special relativity, which dictates a certain specific transformation of the time t. The truly universal constant J in taiji relativity plays the role of the Planck constant in the convenient theory. For example, we have expressions such as $\mathbf{p}=iJ\nabla$, $\exp(ip_\mu x^\mu/J)$, $d^3r d^3p/(2\pi J)^3$ and $[J^2(\partial^2/\partial w^2)-J^2\nabla^2-m^2]\Phi(x^\mu)=0$.

For a charged particle moving in the electromagnetic 4-potential $a^\mu(w,\mathbf{r})$, we have the relation (10.4):

$$(p_0 - \bar{e}a_0)^2 - (\mathbf{p} - \bar{e}\mathbf{a})^2 = m^2, \qquad \text{in F},$$
$$(p'_0 - \bar{e}a'_0)^2 - (\mathbf{p}' - \bar{e}\mathbf{a}')^2 = m^2, \qquad \text{in F'}. \qquad (10.9)$$

If one compares equation (10.9) with the corresponding special relativistic relation in the F frame, one has $\bar{e}=e/c$ (where e is measured in electrostatic units, esu.) and $a^\mu(w,\mathbf{r}) \Leftrightarrow A^\mu(ct,\mathbf{r})/c$; the latter is necessary, so that the action for the free electromagnetic field

$$-(1/4)\int f_{\mu\nu}f^{\mu\nu}d^3r dw \ , \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \, ,$$

has the same dimensions as S_f in (10.1), as is required for consistency. Thus, the universal constant $\bar{e}$ in the taiji relativity is

$$\bar{e} = -1.6021891\times 10^{-20}\sqrt{4\pi}\ (\text{g}\cdot\text{cm})^{1/2} \, . \tag{10.10}$$

That is, only the electric charge measured in electromagnetic units (emu) is a universal constant in taiji relativity. The electric charge e measured in electrostatic units (esu) is not a universal constant in taiji relativity. It can be verified that the dimensionless electromagnetic coupling strength α_e has the usual dimensionless value of

$$\alpha_e = \bar{e}^2/(4\pi J) = 1/137.0359895 \, .$$

Note that there are three fundamental and universal constants $\hbar$, e and c in quantum electrodynamics based on special relativity. However, from the viewpoint of taiji relativity, there are only two such constants, J and $\bar{e}$. It must be stressed that J and $\bar{e}$ *are truly universal constants because they are independent of the specific transformation of the time t.* In other words, they are independent of how time t is defined, or whether it is defined at all. In taiji relativity, the speed of light is not defined and, hence, unknown.

There are other 4-dimensional formulations of relativity based on different transformations of the time t. For example, if one looks at the speed of light in the framework of common relativity,[2] in which a common time for all observers is well-defined, one sees that the speed of light is also not a truly universal constant.

10b. Atomic Structures and Doppler Shifts

The covariant Dirac Hamiltonian for a hydrogen atom is $H_D = p_0$:

$$H_D = -\boldsymbol{\alpha}_D\cdot\mathbf{p} - \beta_D m - \frac{\bar{e}^2}{4\pi r} \, , \qquad \mathbf{p} = iJ\boldsymbol{\nabla} \, , \tag{10.11}$$

where α_D and β_D are constant Dirac matrices. Since H_D transforms as the zeroth component of p^μ, the covariant Dirac equation is given by

$$iJ\frac{\partial}{\partial w}\Psi = H_D\Psi , \tag{10.12}$$

where $\partial/\partial w$ and H_D have the same transformation properties. Since the taiji-time w plays the role of the evolution parameter for a physical system, H_D has the dimensions of mass and the Dirac equation leads to atomic *mass levels* rather than the usual "energy levels." Following the usual method, we find that Dirac's equation (10.12) gives the following atomic mass levels for a hydrogen atom:

$$M_n = \frac{m}{\sqrt{1+\dfrac{\alpha^2}{\{n' + [(j+\frac{1}{2})^2-\alpha^2]^{1/2}\}^2}}} , \qquad \alpha = \frac{\overline{e}^2}{(4\pi J)} . \tag{10.13}$$

We conclude that, in taiji relativity, an atomic system can only emit or absorb "mass quanta" with a "moving mass" Jk^0 determined by the transition between two mass levels M_k and M_n:

$$\begin{aligned} &Jk^o , \qquad \text{in } F , \qquad (M_k - M_n = Jk^o) , \\ &Jk'^o , \qquad \text{in } F' , \qquad (M'_k - M'_n = Jk'^o) . \end{aligned} \tag{10.14}$$

This concept is very important to the understanding of the experimental results of Doppler shifts within the framework of taiji relativity.

The Doppler shift is given by the four-dimensional transformation of the wave vectors k^μ and k'^μ (in F and F' frames):

$$k'^o = \gamma(k^o - Bk_x) , \quad k'_x = \gamma(k_x - Bk^o) , \quad k'_y = k_y , \quad k'_z = k_z . \tag{10.15}$$

This satisfies the invariance relation, $(k'^o)^2-\mathbf{k}'^2=(k^o)^2-\mathbf{k}^2=0$, for electro-magnetic waves. (It is the 4-dimensional form of the relations, $\nu'\lambda'=c'$, where $k'^o=2\pi\nu'/c'$ and $|\mathbf{k}'|=2\pi/\lambda'$. Note that ν' and c' are separately undefined in taiji relativity; only their ratio k'^o is well-defined.) Since $k_x=|\mathbf{k}|\cos\theta=k^o\cos\theta$, we have the exact Doppler effect for "taiji-frequency" k^o:

$$k'^0 = k^0 \left(\frac{1 - B\cos\theta}{\sqrt{1 - B^2}} \right), \tag{10.16}$$

which is consistent with experiments because of (10.14), (10.13) and (9.8).

The invariant phase, $ik'_\mu x'^\mu$, of waves written as (9.8) is important because in the F' frame, the four-dimensional symmetry of taiji relativity dictates that the mixture of two waves must be expressed in terms of (w',x',y',z') rather than (t',x',y',z'):

$$A_o \sin(k'_{01} w' - \mathbf{k}'_1 \cdot \mathbf{r}') + B_o \sin(k'_{02} w' - \mathbf{k}'_2 \cdot \mathbf{r}'), \tag{10.17}$$

in the F' frame. Also, suppose we set $k'_x = k'\cos\theta' = k'^0 \cos\theta'$ in (10.15). We then obtain the usual formula for the aberration of star light

$$\cos\theta' = \frac{\cos\theta - B}{1 - B\cos\theta}, \qquad k_x = k\cos\theta. \tag{10.18}$$

Thus, we see that precise experiments of Doppler shifts do not uniquely or unambiguously imply that the speed of light is a universal constant because they can also be understood from the point of view of taiji relativity, a theory in which the speed of light is undefined. In this sense, the experimental result of precise Doppler shifts is a confirmation of the four-dimensional symmetry of physical laws rather than of the universality of the speed of light (or of relativistic time.)

It is important to note that only when one makes the additional postulate of the universal constancy of the speed of light does one have w'=ct' and w=ct. In that case, (10.17) reduces to the usual expression $A_o \sin(\omega'_1 t' - \mathbf{k}'_1 \cdot \mathbf{r}') + B_o \sin(\omega'_2 t' - \mathbf{k}'_2 \cdot \mathbf{r}')$ in special relativity, because when one assumes a constant speed of light c in F', one also has the relation $k'^0 = \omega'/c$. Therefore, if an experiment involves using the conventional electromagnetic frequency to measure time, the result will always be consistent with a constant speed of light. *Such experiments are nothing more than a check of the self-consistency of a theory. Why is this not sufficient to confirm the object existence of an inherent constant speed of light in nature? The reason is that the additional postulate for the speed of light is not necessary for understanding such experiments.*

10c. Dirac's Conjecture of Truly Fundamental Constants vs. Taiji Relativity's Results, and the Origin of the "Universal Value" c = 29979245800cm/sec

In his 1963 article "The Evolution of the Physicist's Picture of Nature", Dirac gave an interesting account of how physical theory has developed in the past and how it can perhaps be expected to develop in the future. Within the conventional framework of physics based on special relativity and quantum mechanics, there are some fundamental constants in nature: the charge e, the Planck constant $\hbar$, and the speed of light c. Notes that a constant in physics must be universal in order to be fundamental. Using these constants, one can construct a dimensionless number $\alpha_e = e^2/(4\pi\hbar c)$, which turns out to be very close to 1/137. Why? Many physicists have proposed ideas to explain it, but none is satisfactory or widely accepted. This has been a mystery for a long time and no one knows why it should have this particular value. Dirac said that "there will be a physics in the future that works when $\hbar c/e^2$ has the value 137 and that will not work when it has any other value." In this connection we may remark that whether or not there is a factor of 4π in α_e is simply a matter of definition and not important since one can always choose suitable units (e.g., Gaussian units, as used by Dirac in the previous quotation) for the charge e such that α_e does not contain the factor of 4π.

Dirac gave an interesting outlook on the physics:

"The physics of the future, of course, cannot have the three quantities $\hbar$, e, and c all as fundamental quantities. Only two of them can be fundamental, and the third must be derived from those two. It is almost certain that c will be one of the two fundamental ones. The velocity of light, c, is so important in the four-dimensional picture, and it plays such a fundamental role in the special theory of relativity, correlating our units of space and time, that it has to be fundamental. Then we are faced with the fact that of the two quantities $\hbar$ and e, one will be fundamental and one will be derived. If $\hbar$ is fundamental, e will have to be explained in some way in terms of the square root of $\hbar$, and it seems most unlikely that any fundamental theory can give e in terms of a square root, since square roots do not occur in basic equations. It is much more likely that e will be the fundamental quantity and that $\hbar$ will be explained in terms of e^2. Then there will be no square root in the basic equations. I think one is on safe ground if one makes the guess that in the physical picture we shall have at some future stage e and c will be fundamental quantities and $\hbar$ will be derived."[3]

Compare Dirac's outlook in 1963 with our present understanding of relativity and the 4–dimensional symmetry of physical laws based solely on the first postulate of relativity, one sees a quite different perspective: The theory of taiji relativity shows that the constant, c=29979245800cm/sec, is not necessary to the formulation and understanding of physics. *Dirac's conjecture was based on the implicit assumption that the evolution (or temporal) variable must be measured with a dimension different from that of space variables.* This turns out not to be necessary. As we have discussed in chapter 7, the first postulate alone requires that the temporal variable w and the space variables x, y and z have the same dimension. These four variables (w,x,y,z) form the 4–dimensional symmetry framework for physical laws and may be called 4–dimensional taiji spacetime, to avoid confusion with the Minkowski's spacetime based on special relativity. In taiji spacetime, the quantity which correlates the units of 'time' w and space is simply 1, which turns out to be the dimensionless taiji–speed of light derived from Maxwell's equations in taiji relativity.

Now let us analyze the origin of the constant value, c=29979245800cm/sec, from the viewpoint of the Poincaré–Einstein principle alone, i.e., by the viewpoint of taiji relativity. Suppose a physicist would like to introduce the time t' as the evolution variable in the F' frame. It must be related to w' by w'=b't', as required by 4–dimensional symmetry. In taiji relativity, the invariant phase of an electromagnetic wave can be written as

$$k'_{\mu}x'^{\mu}=(\omega'/c')b't'-\mathbf{k}'\cdot\mathbf{r}', \qquad c'=d(b't')/dt',$$

where c' and b' are functions which are separatedly undefined. These two functions are different but related, because the speed of light c' is given by the invariant law of propagation (7.7) in the frame F' with w'=b't', $[d(b't')]^2-dr'^2\equiv(c'dt')^2-r'^2=0$, i.e., d(b't')=c'dt'. Now, suppose he wants to measure the time t' by using the electromagnetic frequency ω'. One may think that he does not need to make any assumption about the speed of light. However, this is not true. The requirement for measuring the time t' using the frequency ω' forces the invariant phase to have the form, $\omega't'-\mathbf{k}'\cdot\mathbf{r}'$ and the electromagnetic oscillation to have a form similar to the sin(ω't') in (10.17). Therefore, this requirement amounts to making the additional assumption c'=b'=c=constant in the invariant phase of electromagnetic waves in taiji relativity. Thus, we see that ***assuming that the time t can be measured using the electromagnetic frequency (in any inertial frame) within the four-dimensional symmetry framework is equivalent to assuming that the speed of light is universal.*** With the definitions of the length of a centimeter and the duration of one second, the speed of light turns out to have the value 29979245800cm/sec experimentally, when one measures frequency and wavelength. If the length of one centimeter or the duration of one second changes, the value of c also changes. As a result, there is no inherent significance to the units of length and time or the value of the speed of light c. Furthermore, the assumption of the constancy of the speed of light is not necessary. Therefore, previous experiments do not imply the existence of a universal constant inherent in Nature. This answers the question raised in the beginning of the chapter.

If one looks at the physical world from the simplest viewpoint, i.e., based solely on the 4-dimensional symmetry of the Poincaré-Einstein principle alone, electromagnetic waves have two types of periodic behavior, one is related to our perception of spatial length determined by $|\mathbf{k}|$ (or the wave length $\lambda=2\pi/|\mathbf{k}|$) and the other is related to our perception of "duration" determined by k_o, as shown in the expression (10.17). Since $|\mathbf{k}|=k_o$, the units for spatial and temporal intervals must be the same and these waves propagate with a dimensionless speed of 1. In other words, the relationship between units for the spatial interval $\Delta x=1$cm and the "time" interval $\Delta w=1$cm are completely dictated by the Maxwell equations and the Poincaré-Einstein principle. Since our ancestors did not know the Poincaré-Einstein principle and their intuitive sense of time was different from that of space, they used a quantity with a different dimension (seconds) to measure time intervals. As a result, we have a quantity called frequency ω measured in the unit of 1/second. It is purely by the artificial and accidental choice of the duration of $\Delta t=1$sec that it is equal to the "taiji-time" interval $\Delta w=29{,}979{,}245{,}800$cm. This is the reason why when one includes an additional second postulate in taiji relativity, which amount to setting $w=ct$, $w'=ct'$, etc., the ratio of these two units appears as the "universal value" in all inertial frames.

Presumably, in the future of particle physics, there will be a basic principle or symmetry which will dictate a relation between the units of mass and length.[4] The quantum constant J in (10.8) and the electric charge $\bar{e}$ in (10.10) will then be reduced to dimensionless constants, just like the taiji-speed of light $\beta_L=1$. However, the electromagnetic coupling strength (i.e., the fine structure constant) $\alpha_e = \bar{e}^2/J \approx 1/137$ will not be changed as it is already dimensionless. It is reasonable to make the guess that quantum mechanics of the future will be able to determine the value of the electromagnetic coupling strength α_e and, hence, to explain the quantization of the electric charge. The fundamental and universal constants, which are dimensionless, may be called "taiji universal constants" since they are independent of any units of measurement and inherent in nature.

10d. The Maxwell Equations Without the Constant Speed of Light c

The invariant action for a classical charged particle moving in the electromagnetic field is assumed to be

$$S_{tot} = \int\left(-mds - \bar{e}a_\mu dx^\mu\right) - \frac{1}{4}\int f^{\mu\nu}f_{\mu\nu}d^3r dw \quad , \tag{10.19}$$

where $f_{\mu\nu}=\partial_\mu a_\nu - \partial_\nu a_\mu$. The Lagrange equations of motion for a charged particle in an electromagnetic field can be derived from (10.19) or from the Lagrangian in (10.1), to be more specific. We then have

$$\frac{dp^\mu}{ds} = \bar{e}f^{\mu\nu}\frac{dx_\nu}{ds}, \tag{10.20}$$

$$p^\mu = (p^0,\mathbf{p}) = \left(\frac{m}{\sqrt{1-\beta^2}}, \frac{m\boldsymbol{\beta}}{\sqrt{1-\beta^2}}\right) , \qquad x_\mu = (w,-\mathbf{r}) .$$

This can be generalized to the case of a continuous charge distribution in space.[5] The second term in (10.19) becomes $-\int a_\mu j^\mu d^3r dw$. With the help of the delta function $\int\delta(\mathbf{r}-\mathbf{r}_0)d^3r=1$, one can formally replace $\bar{e}dx^\mu$ by

$$\int \bar{e}\delta(\mathbf{r}-\mathbf{r}_0)d^3r \,\frac{dx^\mu}{dw}\, dw = \int\left(\rho\frac{dx^\mu}{dw}\right)d^3r dw = \int j^\mu d^3r dw . \tag{10.21}$$

We have the following forms of the Maxwell equations in taiji relativity

$$\partial_\mu f^{\mu\nu} = j^\nu , \qquad \partial_\lambda f^{\mu\nu} + \partial_\mu f^{\nu\lambda} + \partial_\mu f^{\lambda\mu} = 0 ; \tag{10.22}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad x^\lambda = (w,\mathbf{r}).$$

One can write the field-strength tensor $f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu$ in matrix form[6]

$$f^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{10.23}$$

where the electric fields $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ are expressed in terms of the 4-potentials $a^\mu = (a^0, \mathbf{a})$ as

$$\mathbf{E} = \frac{\partial \mathbf{a}}{\partial w} - \nabla a^0, \qquad \mathbf{B} = \nabla \times \mathbf{a}. \tag{10.24}$$

Since $a^\mu(w,\mathbf{r}) \Leftrightarrow A^\mu(ct,\mathbf{r})/c$, the electromagnetic fields $\mathbf{E}(w,\mathbf{r})$ and $\mathbf{B}(w,\mathbf{r})$ are related to the usual fields as defined in special relativity $\mathbf{E}_{SR}(ct,\mathbf{r})$ and $\mathbf{B}_{SR}(ct,\mathbf{r})$ by:

$$\mathbf{E}(w,\mathbf{r}) \Leftrightarrow \mathbf{E}_{usu}(ct,\mathbf{r})/c, \qquad \mathbf{B}(w,\mathbf{r}) \Leftrightarrow \mathbf{B}_{usu}(ct,\mathbf{r})/c. \tag{10.25}$$

In terms of $\mathbf{E}$, $\mathbf{B}$, and the 4-current $J^\mu = (\rho, \mathbf{J})$, the first equation in (10.22) can be written as

$$\nabla \cdot \mathbf{E} = \rho, \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial w} = \mathbf{J}. \tag{10.26}$$

The second equations in (10.22) can be written as

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial w} = 0. \tag{10.27}$$

We see that the speed of light does not explicitly appear in the new form of the Maxwell equations in a general inertial frame. The Maxwell equations can be written in the familiar form

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \text{etc.}, \tag{10.28}$$

if and only if the speed of light is assumed to be a constant c. Thus, the invariance of Maxwell's equations alone does not logically imply the universality of the speed of light c, contrary to many physicists' beliefs. Nevertheless, when $\rho=0$ and $J=0$, the wave equations (10.26) and (10.27) can be written in the form,

$$\frac{\partial^2 \mathbf{f}}{\partial w^2} - \nabla^2 \mathbf{f} = 0 , \qquad \mathbf{f} = \mathbf{E} \text{ or } \mathbf{B} . \tag{10.29}$$

This implies that there is a constant dimensionless taiji speed of light, $\beta_L=1$, measured in terms of taiji time w. This result of $\beta_L=1$ is, of course, consistent with equation (7.7) and with equations (R5.4) and (R5.5) in reference 5 of chapter 5.

References

1. A better way to see the universality of the new quantum constant J is from the invariant action in quantum electrodynamics. See, for example, Jong-Ping Hsu and Leonardo Hsu, Phys. Letters A **196**, 1 (1994).
2. Jong-Ping Hsu, Phys. Lett. A **97**, 137 (1983); Nuovo Cimento B **74**, 67 (1983); J. P. Hsu and C. Whan, Phys. Rev. A **38**, 2248 (1988), Appendix. These papers discuss a four-dimensional symmetry framework based on the usual first postulate and a different second postulate, namely, a common time t'=t for all observers in different inertial frames.
3. P. A. M. Dirac, Sci. Am. **208**, 48 (1963).
4. For example, in quantum mechanics or field theory of the future, the fundamental commutation relations, which involve J, are forced to take a certain dimensionless value. Equivalently, the momentum and the inverse wavelength associated with a particle are required to have the same dimension and to be measured in the same units.
5. L. Landau and E. Lifshitz, *The Classical Theory of Fields* (translated by M. Hamermesh, Addison-Wesley, Cambridge, Mass. 1951), pp. 70–73.
6. See, for example, J. D. Jackson, *Classical Electrodynamics* (2nd Edition, John Wiley & Sons, New York,1975), pp. 549–551.

11.

Quantum Electrodynamics Based on Taiji Relativity and Dilatation of Lifetimes and Decay-Lengths

11a. Quantum Electrodynamics Based on Taiji Relativity

In quantum electrodynamics (QED), the invariant action S_q, involving Dirac's electron field ψ, the photon field a_μ and the new quantum constant J, is assumed to take the usual form,[1]

$$S_q = \int L d^4x, \quad L = \overline{\psi}[\gamma^\mu(iJ\partial_\mu - \overline{e}a_\mu) - m]\psi - (1/4)f_{\mu\nu}f^{\mu\nu}, \tag{11.1}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu},$$

$$J = 3.5177293\times10^{-38}\ \text{g·cm}, \quad \overline{e} = -1.6021891\times10^{-20}\sqrt{4\pi}\ (\text{g·cm})^{1/2}, \tag{11.2}$$

where $d^4x=dwd^3r$ and the electromagnetic coupling strength is $\alpha_e = \overline{e}^2/(4\pi J) \approx 1/137$, as it should.[2] Each term in the Lagrangian density L has the dimension of mass/(length)^4.

For quantization of the Dirac fields, the "canonical momentum" π_b conjugate to ψ_b and the Hamiltonian density for a free electron are defined as

$$\pi_b = \frac{\partial L_\psi}{\partial(\partial_0\psi_b)}, \qquad L_\psi = \overline{\psi}[\gamma^\mu iJ\partial_\mu - m]\psi,$$

$$H = \pi\partial_0\psi - L_\psi . \tag{11.3}$$

For free photon (a_μ) and electron (ψ) fields, we have

$$a_\mu(w,\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p};\alpha}\sqrt{J/(2p_0)}\,[a(\mathbf{p},\alpha)\ \varepsilon_\mu(\alpha)\exp(-ip\cdot x/J)$$
$$+ a^\dagger(\mathbf{p},\alpha)\ \varepsilon_\mu(\alpha)\exp(ip\cdot x/J)], \tag{11.4}$$

$$\psi(w,\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p};s}\sqrt{m/p_0}\,[b(\mathbf{p},s)u(\mathbf{p},s)\exp(-ip\cdot x/J)$$
$$+ d^\dagger(\mathbf{p},s)v(\mathbf{p},s)\exp(ip\cdot x/J)]\,, \qquad p\cdot x = p_\mu x^\mu\,, \qquad (11.5)$$

where

$$[a(\mathbf{p},\alpha),\, a^\dagger(\mathbf{p}',\alpha')] = \delta_{\mathbf{pp}'}\delta_{\alpha\alpha'}\,,$$

$$[b(\mathbf{p},s),\, b^\dagger(\mathbf{p}',s')] = \delta_{\mathbf{pp}'}\delta_{ss'}\,, \qquad [d(\mathbf{p},s),\, d^\dagger(\mathbf{p}',s')] = \delta_{\mathbf{pp}'}\delta_{ss'}\,, \qquad (11.6)$$

and all other commutators such as $[a(\mathbf{p},\alpha),\, a(\mathbf{p}',\alpha')]$ and $[a^\dagger(\mathbf{p},\alpha),\, a^\dagger(\mathbf{p}',\alpha')]$ vanish. Of course, commutators for quantized fields $\psi(w,\mathbf{r})$ and $a_\mu(w,\mathbf{r})$ can be derived from (11.4)–(11.6). The Dirac equation based on taiji relativity can be derived from (11.1),

$$iJ\frac{\partial\psi}{\partial w} = [\boldsymbol{\alpha}_D\cdot(-iJ\nabla - \bar{e}\mathbf{a}) + \beta_D m + \bar{e}a_0]\psi\,, \quad \alpha_D^k = \gamma^0\gamma^k\,, \quad \beta_D = \gamma^0\,, \qquad (11.7)$$

where $\boldsymbol{\alpha}_D = (\alpha_D^1, \alpha_D^2, \alpha_D^3)$ and β_D are the usual constant Dirac matrices.[1]

In view of the equations of motion (11.7), we must use the taiji-time w in a general frame as the evolution variable for a state $\Phi^{(S)}(w)$ in the Schrödinger representation:

$$iJ\frac{\partial\Phi^{(S)}(w)}{\partial w} = H^{(S)}(w)\Phi^{(S)}(w)\,, \qquad H^{(S)} = H_o^{(S)} + H_I^{(S)}\,, \qquad (11.8)$$

because the evolution of a physical system is assumed to be described by a Hamiltonian operator which has the same transformation property as the taiji-time w or $\partial/\partial w$.

The usual covariant formalism of perturbation theory can also be applied to such a QED.[3] To illustrate this, let us briefly consider the interaction representation and the S-matrix based on taiji relativity. The transformations of the state vector $\Phi(w)$ and operator O from the Schrödinger representation to the interaction representation are

$$\Phi(w) \equiv \Phi^{(I)}(w) = \exp[iH_o^{(S)}w/J]\Phi^{(S)}(w)\,, \qquad (11.9)$$

$$O(w) \equiv O^{(I)}(w) = \exp[iH_o^{(S)}w/J]O^{(S)}\exp[-iH_o^{(S)}w/J] \,. \tag{11.10}$$

Because $O^{(S)}$ and O(w) are the same for w=0, we have

$$iJ\frac{\partial\Phi(w)}{\partial w} = H_I(w)\Phi(w)\,, \qquad H_I = \exp[iH_o^{(S)}w/J]H_I^{(S)}\exp[-iH_o^{(S)}w/J]\,, \tag{11.11}$$

$$O(w) = \exp[iH_o^{(S)}w/J]O(0)\exp[-iH_o^{(S)}w/J] \,. \tag{11.12}$$

The U-matrix can be defined in terms of the taiji-time w: $\Phi(w)=U(w,w_0)\Phi(w_0)$, $U(w_0,w_0)=1$. It follows from (11.11) and (11.12) that

$$iJ\frac{\partial U(w,w_0)}{\partial w} = H_I(w)U(w,w_0) \,. \tag{11.13}$$

If a physical system is in the initial state Φ_i at taiji-time w_0, the probability of finding it in the final state Φ_f at a later taiji-time w is given by

$$|\langle\Phi_f|U(w,w_0)\Phi_i\rangle|^2 = |U_{fi}(w,w_0)|^2. \tag{11.14}$$

The average transition probability per unit taiji-time for $\Phi_f \rightarrow \Phi_i$ is

$$\frac{|U_{fi}(w,w_0) - \delta_{fi}|^2}{(w - w_0)} \,. \tag{11.15}$$

As usual, we can express the S-matrix in terms of the U-matrix, i.e. $S=U(\infty,-\infty)$ and obtain the following form

$$S = 1 - \frac{i}{J}\int_{-\infty}^{\infty} H_I(w)dw + \left(\frac{-i}{J}\right)^2 \int_{-\infty}^{\infty} dw \int_{-\infty}^{w} H_I(w)H_I(w')dw' + \dots\dots \,. \tag{11.16}$$

For w-dependent operators, one can introduce a w-product W* (corresponding to the usual chronological product), so that (11.16) can be written in an exponential form:

$$S = W^*\{\exp[-\frac{i}{J}\int_{-\infty}^{\infty} H_I(x^\mu)dwd^3r]\}, \qquad \int_{-\infty}^{\infty} H_I(x^\mu)d^3r = H_I(w). \tag{11.17}$$

For simplicity, one may set J = 1 (the "natural unit" in taiji QED) in the following discussions, so that one has the relation of dimensions

$$[L^{1/4}] = [a_\mu] = [\psi^{2/3}] = [\text{mass}] = [1/\text{length}]. \tag{11.18}$$

Consequently, the classical electron radius r_e and electron Compton wavelength λ_e, for example, are given by $r_e = \alpha_e J/m_e = \alpha_e/m_e$ and $\lambda_e = J/m_e = 1/m_e$ respectively, and have the usual values.

To obtain the rules for Feynman diagrams in taiji QED, we follow the usual quantization procedure and define L_{TQED} by adding a gauge fixing term in the Lagrangian (10.30),

$$L_{TQED} = L - \frac{1}{2\rho}(\partial^\mu a_\mu)^2, \qquad J = 1. \tag{11.19}$$

where ρ is a gauge parameter. As usual, we define the *M*-matrix as follows:

$$S_{if} = \delta_{if} - i(2\pi)^4\delta^4(p_f^{(tot)} - p_i^{(tot)})[\Pi_{ext\ par}(n_j/V)]^{1/2}M_{if}, \tag{11.20}$$

where "ext par" denotes external particles and $n_j = m_j/p_{0j}$ for spin 1/2 fermions and $1/2p_{0j}$ for bosons.[3] Because of the 4-dimensional symmetry in (11.19) and (11.20), the rules for writing M_{if} are formally the same as those in the usual QED, except that certain quantities (e.g., w, J, p_μ and $\bar{e}$) have different dimensions from the corresponding quantities in conventional QED. To wit,

(a) the covariant photon propagator is now given by

$$\frac{-i[g_{\mu\nu} - (1-\rho)k_\mu k_\nu/(k^2 + i\varepsilon)]}{(k^2 + i\varepsilon)}, \qquad k^2 = k^\sigma k_\sigma, \tag{11.21}$$

(b) the electron propagator is

$$\frac{-i}{(\gamma^\mu p_\mu - m + i\varepsilon)}, \tag{11.22}$$

(c) the electron-photon vertex is

$$-i\ \bar{e}\gamma^{\mu}, \tag{11.23}$$

and (d) each external photon line has an additional factor ε_{μ}. Also, each external electron line has u(s,p) for the absorption of an electron and $\bar{u}$(s,p) for the emission of an electron, etc.

Other rules such as taking the trace with a factor of -1 for each closed electron loop, integration with $d^4k/(2\pi)^4$ over a momentum k_{μ} not fixed by the conservation of four-momentum at each vertex, etc. are the same as usual.

Thus, if one calculates scattering cross sections and decay rates (with respect to the taiji-time w) of a physical process, one will get formally the same result as that in conventional QED.[3] Let us consider an example. For a scattering process $1+2\rightarrow 3+4+....+N$, the differential cross section $d\sigma$, which has the dimension of area, is given by

$$d\sigma = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - (m_1 m_2)^2}}\ |M_{if}|^2 \left[\prod_{ext\ fer}(2m_{fer})\right] \frac{d^3p_3}{(2\pi)^3 2p_{03}} \ .....$$

$$...... \frac{d^3p_N}{(2\pi)^3 2p_{0N}} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 .... - p_N) S_o\ , \tag{11.24}$$

where $p_0=\sqrt{\mathbf{p}^2+m^2}$ and S_o denotes a factor 1/(n!) for each kind of (n) identical particles in the final state. If the initial particles are unpolarized, one takes the average over initial spin states. When there is no external fermion in a process, then $[\Pi_{ext\ fer}\,(2m_{fer})]$ in (11.24) is replaced by 1.

11b. Experimental Measurements of Dilatation for Decay-Lengths and Decay-Lifetimes

Now, let us consider the mean lifetime of an unstable particle. Since the time t is undefined in taiji relativity, a burning question is:

How can taiji relativity explain the well-established experimental results of the "lifetime dilatation" of unstable particles?

The answer is that experiments which purport to measure the mean lifetime of unstable particles in flight actually measure the mean decay-lengths rather than the lifetimes, where the decay length is the distance traveled by an unstable particle before decaying. Furthermore, the decay-length dilatation can be calculated and it can be shown that this length is dilated by a γ factor in quantum field theory based on taiji relativity. Let us consider the decay rate $\Gamma(1\to 2+3+...+N)$ for a physical process $1\to 2+3+...+N$. It is given by

$$\Gamma(1\to 2+3+...+N) = \lim_{w\to\infty} \int \frac{|\langle f|S|i\rangle|^2}{w} \frac{d^3x_2 d^3p_2}{(2\pi J)^3} \cdots \frac{d^3x_N d^3p_N}{(2\pi J)^3}, \tag{11.25}$$

which has the dimensions of inverse length in taiji relativity. Since a particle's lifetime is measured in terms of taiji-time, which has the units of length, this decay rate can be interpreted as the inverse of the particle's "decay-length" or the mean distance which a group of such particles would travel before their number is decreased by a factor of $1/e=1/2.71828$ due to decay. The decay length D is given by

$$D = 1/\Gamma(1\to 2+3+...+N). \tag{11.26}$$

Thus, in taiji relativity, one has the "rest decay length" D_0 for a particle decay at rest, corresponding to the "rest lifetime" in the conventional theory. Also, instead of the dilatation of the lifetime of a particle in flight, we have dilatation of the distance it travels before decaying. Such a dilatation is physically correct because it is equal to the experimentally determined distance travelled by the particles.

Let us consider a simple specific example, i.e., the muon decay $\mu^-(p_1) \to e^-(p_2)+\nu_\mu(p_3)+ \bar{\nu}_e(p_4)$ with the usual V–A coupling. The decay S-matrix in the momentum space is given by

$$\langle f|S|i\rangle \propto (p_{01}p_{02}p_{03}p_{04})^{-1}\, \delta^4(p_1-p_2-p_3-p_4)\, M_{sc},$$

$$M_{sc} = (G/\sqrt{2})[\,\bar{\nu}_\mu(p_3)\gamma^\lambda(1-\gamma_5)\,\mu(p_1)]\,[\,\bar{e}(p_2)\gamma_\lambda(1-\gamma_5)\,\nu_e(p_4)]\,. \tag{11.27}$$

The decay length, defined by $D=1/\Gamma(1\to 2+3+...+N)$, can be calculated and is found to be

$$\frac{1}{D} \propto \frac{1}{p_{01}} \int \frac{d^3p_2}{p_{02}} \frac{d^3p_3}{p_{03}} \frac{d^3p_4}{p_{04}} \delta^4(p_1-p_2-p_3-p_4) \sum_{spin} |M_{sc}|^2 . \tag{11.28}$$

Everything to the right of $1/p_{01}$ in (11.28) is invariant under a taiji four-dimensional transformation so that the decay length D is indeed proportional to $\sqrt{p_1{}^2+m_1{}^2} = p_{01}$.[4] We note that, when a particle at rest decays, its decay-length is not zero and can be expressed in terms of taiji-time w. It should be stressed that result (11.28) is sufficient to understand all previous experiments of the "lifetime dilatation" of unstable particles because it is the decay-length of an unstable particle in flight which is the quantity that is directly measured in these experiments.

In order to see this, let us consider the "lifetime-dilatation experiment" involving the muons produced by collisions of cosmic rays and air molecules in the upper atmosphere, one counts the number of muons at the top of a mountain of known height and then counts the number of muons again at sea level to find out how many have lived long enough to reach the detectors there. Thus it is the mean decay distance that is directly measured in the cosmic-ray experiment.[5]

Similarly, the experimental setup in laboratories for measuring "lifetime dilatation" is roughly as follows:

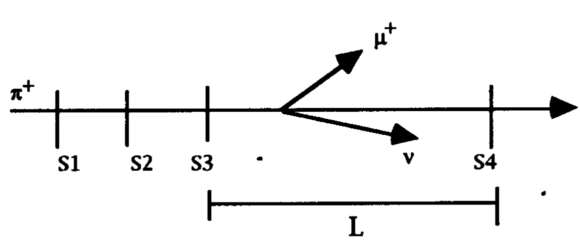

Fig. 11.1 Experimental arrangement for measuring the mean life of particle decay in flight

A narrow beam of unstable particles, for example, pions, passes through three detectors, S_1, S_2 and S_3. Some of the pions decay between detectors S_3 and S_4, according to the reaction $\pi^+ \to \mu^+ + \nu_\mu$. If the distance L between S_3 and S_4 is sufficiently large, the probability of a μ^+ entering the detector S_4 is negligible because the μ^+'s produced in the decay move out in all directions. The coincidence of signals in S_1, S_2 and S_3 indicates a high-speed charged particle passing all three detectors, i.e., moving in the horizontal beam direction. By counting these coincident signals, one has the total number of charged pions N_0 in the beam before decay. Similarly, the coincidence of signals in S_1, S_2, S_3 and S_4 gives the number of pions, N' remaining after traveling a length L in the horizontal direction of the beam. In taiji relativity, the relationship between N_0 and N' can be given in terms of taiji-time as,

$$N' = N_0 \exp[-\Gamma w'] . \qquad \Gamma = \Gamma(\pi^+ \to \mu^+ + \nu_\mu) , \tag{11.29}$$

where w' is the proper taiji-time of the pion (at rest in the F' frame, i.e., $\Delta x'=0$) moving from S_3 to S_4. This relationship can also be expressed in terms of the taiji-time w of the laboratory frame F,

$$w' = w\sqrt{1-\beta^2} , \tag{11.30}$$

where $\beta=dx/dw$ is the known taiji-velocity of the pion. Thus, using $w=L/\beta$ equation (11.29) can be written as

$$N' = N_0 \exp[-\Gamma w\sqrt{1-\beta^2}\,] = N_0 \exp[-\Gamma\sqrt{1-\beta^2}\, L/\beta] . \tag{11.31}$$

Experimentally, one changes L and measures N' and N_0 for each L and then plots a graph of $\ln(N'/N_0)$ versus L to get a straight line with a slope of $-\Gamma\sqrt{1-\beta^2}/\beta$. From the slope and the known taiji-velocity β, one can then obtain the mean life of a positive pion,

$$w(\pi^+) = D(\pi^+) = \frac{1}{\Gamma} = 780.45 \text{ cm} . \tag{11.32}$$

Since the time interval $\Delta t=1$ second corresponds to $\Delta w=299792458$ meters, as discussed in section 7c, chapter 7, the result (11.32) corresponds to the conventional pion mean life $\tau(\pi^+)=2.603\times10^{-8}$ seconds.

Within the framework of special relativity with a universal speed of light c, the decay-length D can be converted to the decay lifetime τ by

$$\tau = \frac{D}{c} = \frac{1}{c\Gamma} \tag{11.33}$$

in any frame. Thus, in special relativity, one has the dilatation of both the decay-length and the lifetime. However, within the four-dimensional symmetry framework of taiji relativity, one can only talk about the decay-length dilatation in general. Such a decay-length dilatation is completely relative within the framework of taiji relativity and special relativity. (See Appendix D for a more detailed discussion of the relativity of "lifetimes" or decay-length and the "twin paradox.")

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3. Since the 4-dimensional symmetry of the Lorentz group is shared by both special relativity and taiji relativity, the usual formalism of quantum fields can be applied to field theory based on taiji relativity. J. J. Sakurai, *Advanced Quantum Mechanics* (Addison–Wesley, Reading, MA, 1967), pp. 171–172 and pp. 181–188; S. Weinberg, *The Quantum Theory of Fields,* vol. I, Foundations (Cambridge University Press, New York, N.Y., 1995), pp. 134–147.
4. See, for example, Particle Data Group, *Particle Physics Booklet* (July, 1996, American Institute of Physics), p. 11. A muon's mean life τ is $2.197\text{x}10^{-6}$ seconds and its decay-length is given by $c\tau$=658.654 meters. See also A. P. French, *Special Relativity* (W. W. Norton & Company, New York, 1968), pp. 98–101. Recently, the lifetime (or the decay-length) dilatation predicted by the 4-dimensional symmetry of the Lorentz group has been tested and confirmed by measuring the decay lifetime of k_s^0 in flight at several hundreds of GeV (i.e.,$\gamma \sim 10^3$.) See N. Grossman, K. Heller, C. James, et al, Phys. Rev. Lett. **59**, 18 (1987).
5. See, for example, A. P. French, *Special Relativity* (W. W. Norton & Company, New York, 1968), pp. 98–103.

12.

Common Relativity: A Common Time for all Observers

12a. Why Common Time?

The formulation of physical concepts should provide a comprehensive conceptual framework for the whole of physics, including both single particle and many-particle systems. It is therefore necessary that the 4-dimensional symmetry framework be based on concepts and equations that can be unified with those used in statistical systems with many particles. We have so far talked about taiji relativity with the 4-dimensional symmetry of the Lorentz and Poincaré groups and shown that it is consistent with all known experiments even though the transformation property of time t is arbitrary. However, in order to deal with many-particle systems, it is better to have a specific and simple transformation for time t so that the canonical evolution of a statical system is possible. Thus, we will explore one particularly simpe transformation of t which is called common time. The 4-dimensional theoretical framework with common time can be obtained from taiji relativity by making an additional postulate of common time for all inertial frames, without upsetting the 4-dimensional Lorentz and Poincaré invariance. Such a 4-dimensional theory of relativity is called "common relativity."

The treatment of common relativity and its physical implications given in chapters 12-16 are based on two postulates: the Poincaré-Einstein principle and a common time for all observers. They are developed as far as possible and involve a considerable departure from the conventional treatment.

The motivations for introducing common time into the 4-dimensional symmetry framework[1,2] are as follows:

(i) There appears to be an inborn psychological and physiological tendency and limitation to the human perception of nature with one single evolution variable independent of space. This was shown in the early formulations of classical physics. From the pedagogical viewpoint, a theory of relativity with common time could serve as an innovative approach to explaining 4-dimensional physics to students and laymen, in harmony with the intuition of daily life.

(ii) The expansion of the universe as a whole suggests a single cosmic time as the proper evolution variable.

(iii) Common time appears to be the only concept which enables us to introduce the canonical evolution, so that we can describe enormously complex macroscopic systems of N-particles by reducing N one-particle Liouville equations to a single invariant Liouville equation. This is a significant advantage of common time over relativistic time, taiji-time or any of the other possible times allowed in the 4-dimensional symmetry framework.[3,4]

To avoid confusion, it should be stressed that common time in the 4-dimensional symmetry framework is not unique and, hence, not absolute. Common time is not the same as the Newtonian absolute time in 3-dimensional symmetry framework, even though it does provide a concept of *universal simultaneity* for observers in all inertial frames. Common relativity still preserves the 4-dimensional symmetry of physical laws and is consistent with all known experiments. It also reveals the truly universal and fundamental constants in nature, J and $\bar{e}$.[1,2]

12b. Two Basic Postulates of Common Relativity

Common relativity is entirely based on two postulates, which are analogous to the two postualtes of Einstein's special relativity:

(I). The form of a physical law is the same in any inertial frame.

(II). Physical time t_c is the same in any inertial frame.

The first postulate is identical to the principle of relativity of Poincaré and Einstein. It is satisfied by any one of the inclusive four-dimensional symmetry frameworks; the second postulate selects a particular four-dimensional framework with a common time t_c for all observers,

$$t'_c = t_c . \tag{12.1}$$

Thus, common time is a scalar, i.e., an invariant quantity.

It should be stressed that this postulate of common time in (12.1) is really a definition rather than a postulate, just as the universality of the speed of light in special relativity. It is a truth by definition which makes the theory self-consistent and therefore, can never be proven wrong. This may sound strange because we know that the same "definition" (12.1) in the Galilean transformation leads to incorrect physical results. However, in this case the blame lies with the 3-dimensional symmetry of the Galilean transformations rather than the definition (12.1). When relation (12.1) is implemented within a 4-dimensional symmetry framework, one gets correct physical results consistent with experiments, as we shall see below.

In a general inertial frame, the 4-coordinate of an event is denoted by $x^\mu=(w,x,y,z)$, where w is the product of a function b and the invariant common time t_c, $w=bt_c$. I will call the function b the "ligh" so that w or bt_c will be called the "lightime." In common relativity, (t_c,x,y,z) is no longer a 4-vector, the reason being that the common time t_c is an invariant scalar and is not proportional to the zeroth component of a coordinate 4-vector.

Of course, for common time to make sense, one must be able to physically synchronize clocks which read the common time for all observers. Some have argued that it would take signals with infinite speed to synchronize clocks in such a way as to satisfy (12.1) and, hence, that it cannot be realized physically. However, as we have discussed in chapter 7, there are, in fact, ways to realize clock systems in different inertial frames in accordance with equation (7.14). Common time can be physically realized through clock systems in F and F' because the reading and the rate of ticking of clocks in both frames can be adjusted to read the same time. We shall return to this point in sections (12c) and (12e) below.

12c. The Space-Lightime Transformations and Physical Clocks

Let us consider two inertial frames, F and F', which have relative motion along a common x/x' axis. Suppose F' moves with a constant velocity V=(V,0,0), as measured by observers in F. Since F and F' are equivalent, we may start with, say, the F frame to synchronize F-clocks by assuming that the speed of light is constant and isotropic in F. (Of course, one may choose the F' frame to synchronize F'-clocks. This will be discussed in section 12e.) One may think that this assumption constitutes a third postulate in common relativity. However, the existence of this F frame is not necessary to common relativity. One could imagine a physical world of frames F, F', F", etc. from which F suddenly disappeared. No physics in common relativity would change. This is just a technical assumption for convenience of our discussions.

Now the coordinate 4-vectors of an event as recorded by observers in F and F', and their derivatives with respect to the invariant common time can be denoted by

$$x^\mu = (ct_c, x, y, z) \quad \text{and} \quad x'^\mu = (w', x', y', z'), \qquad w' = b't_c,$$
$$\frac{dx^\mu}{dt_c} = (c, \mathbf{v}) \quad \text{and} \quad \frac{dx'^\mu}{dt_c} = (c', \mathbf{v}'), \qquad \frac{dw'}{dt_c} = c', \tag{12.2}$$

in F and F' respectively, where the fourth (or the zeroth) dimension is lightime $w'=b't_c$ in F' (or ct in F). And c' denotes speeds of light measured in the F' frame.

The Poincaré–Einstein principle of relativity implies the symmetry between F and F' in the following forms

$$x' = Ax - Bct_c = \gamma(x - \beta ct_c),$$
$$x = Ax' + Bb't_c = \gamma(x' + \beta b't_c), \tag{12.3}$$

and $y'=y$ and $z'=z$. If an object is at rest in F', $d\mathbf{r}'/dt_c=0$, then its velocity as measured in F must be $\mathbf{V}=(V,0,0)$, because F' is moving with the velocity $(V,0,0)$ related to F. This condition leads to $B/A=V/c=\beta$, where we have used the first relation in (12.3). The invariant laws for the propagation of light in F and F' are, as usual, assumed to be given by $ds^2=ds'^2=0$, i.e., $dx^2 - [d(ct_c)]^2 = dx^2-c^2dt_c^2 = 0$ and $dx'^2-[d(b't_c)]^2=dx'^2-c'^2dt_c^2 = 0$, where $c'=d(b't_c)/dt_c$ and the "ligh" function b' are in general not a constant. Evidently, these invariant laws for the propagation of light lead to $dx/dt_c=c$ and $dx'/dt_c=c'$ for light moving along the +x direction. It follows from (12.3) that $A^2-B^2=\gamma^2(1-\beta^2)=1$. This result and $\beta=V/c$ lead to the relation $\gamma=1/\sqrt{1-(V/c)^2}$. Thus, the 4-dimensional space–lightime transformation is found to be

$$b't_c = \gamma(ct_c - \beta x), \quad x' = \gamma(x - \beta ct_c), \quad y' = y, \quad z' = z; \tag{12.4}$$
$$\gamma = 1/\sqrt{1-\beta^2}, \quad \beta = V/c.$$

This is precisely the result obtained by setting the coordinates $x^\mu=(ct_c,x,y,z)$ and $x'^\mu=(b't_c,x'y'z')$ in the 4-dimensional taiji transformations (7.4) with B replaced by $\beta=V/c$, as it should.

In a general frame, we denote the coordinate x^μ of an event and the distance squared $(S_{12})^2$ between two events as

$$x^\mu = (w,x,y,z) , \qquad w = bt_c ,$$

and

$$(s_{12})^2 = (w_1 - w_2)^2 - (\mathbf{r}_1 - \mathbf{r}_2)^2 \tag{12.5}$$

$$= \begin{cases} (ct_{c1}-ct_{c2})^2 - (\mathbf{r}_1-\mathbf{r}_2)^2 ; & \text{in F} \quad w_1=ct_{c1} , \quad w_2=ct_{c2} , \\ (b'_1t_{c1}-b'_2t_{c2})^2 - (\mathbf{r}'_1-\mathbf{r}'_2)^2 ; & \text{in F'} \quad w'_1=b'_1t_{c1} , \quad w'_2=b'_2t_{c2} . \end{cases}$$

One can verify that $(s_{12})^2$ is invariant under the space-lightime transformations (12.4). The ligh function b' in the F' frame is completely determined by (12.4). Since the common time t_c does not transform in (12.4), the ligh function b' is necessary in order to keep the four-dimensional interval invariant in common relativity. We note that for two events to occur at the same time $t_{c2}-t_{c1}=\Delta t_c=0$ but at different places $x_2-x_1=\Delta x\neq 0$, we have

$$(b_2 - b_1)t_{c1} = -\gamma\beta(x_2 - x_1) \tag{12.6}$$

from (12.4). Also, when $t \to 0$ and $x\neq 0$, (12.4) leads to

$$b't_c = -\gamma\beta x , \qquad t_c \to 0 . \tag{12.7}$$

That is, the ligh function b' tends to approach infinity such that $b't_c\neq 0$ in the limit $t_c\to 0$. This mathematical property of b' does not lead to any physical problems in common relativity.

In inertial frames, we must have clocks which read common time for all observers. How to realize such common time for observers in different frames? A simple method is as follows: we choose any one of them, say, F, to synchronize clocks by defining the speed of light is constant (or isotropic) in F and only in F. In other words, we define that the one-way speed of light is the same as the two-way speed of light (which can be measured by using just one stable atomic clock without synchronization) in the F frame. Then, the reading and the rate of ticking of any F'-clock can be adjusted to read the same time as the synchronized F-clock nearby. In this way, observers in F and F' have a common time. Or it is even simpler by requiring the F'-observers to use a nearly F-clock to record time. In this way, one does not have to set up a grid of clocks in the F' frame and other inertial frames.

12d. Relativity of the Speed of Light Measured by Using Common Time

With a common time t_c for all observers, the velocities of a particle measured in F and F' are defined respectively by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt_c}, \qquad \mathbf{v}' = \frac{d\mathbf{r}'}{dt_c}. \tag{12.8}$$

These velocities transform like the spatial components of a coordinate 4-vector in (12.4) because the common time t is a scalar. Thus, the four-dimensional transformations of velocities in common relativity are given by

$$c' = \gamma(c - \beta v_x) \equiv c'(v_x), \quad v'_x = \gamma(v_x - \beta c), \quad v'_y = v_y, \quad v'_z = v_z; \tag{12.9}$$

$$c' = \frac{d(b' t_c)}{dt_c}.$$

This can also be derived from the space-lightime transformation (12.4). As we shall see below, if v is the velocity of light, i.e., $|\mathbf{v}|=c$, then c' in (12.9) is the one-way speed of this light signal as measured by F' observers. However, if **v** is the velocity of any other object, $0 \le |\mathbf{v}| < c$, then c' is a general two-way speed of light. This will be amplified in section 12f below.

We note that there is an "apparent asymmetry" between the F and F' frames. This asymmetry appears in the space-lightime transformation (12.4) and the velocity transformation (12.9):

(a) If observers in F and F' compare the length of a rod, they find that

$$\Delta x' = \gamma \Delta x = \frac{\Delta x}{\sqrt{1 - V^2/c^2}}, \qquad \Delta t_c = 0, \tag{12.10}$$

regardless of whether the rod is at rest in F or at rest in F'. It seems that the relativity of length contraction is lost.

(b) the speed of light is isotropic in F and anisotropic in F':

$$c' = \frac{c \mp V}{\sqrt{1-V^2/c^2}}, \qquad \text{for } v_x = \pm c, \qquad (12.11)$$

where c=299792458 m/s.

However, none of these results implies an inherent asymmetry between F and F', as we shall see below.

12e. The Symmetry Between Any Two Frames F and F'

It must be stressed that the apparent asymmetry between F and F' (as shown in (12.10) and (12.11)) is completely due to the fact that we started with the F frame to synchronize clocks by assuming that the speed of light is isotropic in F. According to the Poincaré-Einstein principle of relativity, there can be no preferred inertial frame. Thus, if one wishes, this clock system could be discarded and another one set up starting with the F' frame. The F' clocks could be synchronized by using light signals which are assumed to be isotropic and to have the constant speed c'=299792458 m/s in the F' frame. We could then require all observers to use this grid of clocks in F' to record the time of events. In this way, we also have a common time t'_c for all observers in different frames. *Here, we see clearly that common time within the four-dimensional symmetry framework is universal but not unique and, therefore, not absolute in the sense of the Newtonian time.* In the preceding case, we would have the following space-lightime transformations between (c't',**r**') in F' and (bt', **r**) in F, where t' is a common time:

$$bt'_c = \gamma'(c't'_c - \beta'x'), \quad x = \gamma'(x' - \beta'c't'_c), \quad y = y', \quad z = z'; \qquad (12.12)$$

$$\gamma' = \frac{1}{\sqrt{1-\beta'^2}}, \quad \beta' = \frac{V'}{c'},$$

where V' < 0 is the constant velocity of F as measured in F' and the "ligh" b is a function in the transformation. We would obtain the following results:

(a') If observers in F and F' compare the length of a rod, they find that

$$\Delta x = \gamma' \Delta x', \qquad \text{for} \quad \Delta t'_c = 0, \tag{12.13}$$

regardless of whether the rod is at rest in F or at rest in F'.

(b') The speed of light is isotropic in F' and anisotropic in F:

$$\frac{dx}{dt'_c} = c = \gamma'(c' - V'), \qquad \text{for} \quad \frac{dx'}{dt'_c} = c' = \text{const.} \tag{12.14}$$

Thus, there is no inherent asymmetry between any two inertial frames F and F'.

12f. The Two-Way Speed of Light

Based on the transformation of velocities in (12.9), we can now discuss properties of the two-way speed of light in the F' frame, in which the (one-way) speed of light is not isotropic. Suppose a light signal, starting from the origin $\mathbf{r}'_A=0$ at time $t_c=0$, travels outward and reaches an arbitrary point $\mathbf{r}'_B=(x',y',0)$ at $t_c=t_{cB}$ and then returns to $\mathbf{r}'_A$ at time $t_c=t_{cA}$. The average speed (i.e. the two-way speed) of light c'_{av} in F' satisfies

$$c'_{av}\, t_{cA} = 2L', \qquad L' = \sqrt{x'^2+y'^2}, \tag{12.15}$$

where

$$t_{cA} = \frac{L'}{c'(A\to B)} + \frac{L'}{c'(B\to A)} .$$

Since the propagation of light satisfies the invariant law $s^2=0$, we have the relation $c'^2-v'^2_x-v'^2_y-v'^2_z=0$ in F'. From (12.9) with $v_x=c\cos\theta_1$, and $v_x=-c\cos\theta_2$ corresponding to the light signals propagating from A to B and from B to A, we have

$$\begin{aligned}
&c'(A\to B) = c'(c\cos\theta_1) = \gamma c(1-\beta\cos\theta_1), \qquad c\cos\theta_1 = \gamma^{-1}(c'_{AB}\cos\theta'+\gamma\beta c)\\
&c'(B\to A) = c'(-c\cos\theta_2) = \gamma c(1+\beta\cos\theta_2), \qquad -c\cos\theta_2 = \gamma^{-1}(-c'_{BA}\cos\theta'+\gamma\beta c)\\
&v_x' = c'_{AB}\cos\theta' = c'(A\to B)\cos\theta', \qquad \theta'=\theta'_1=\theta'_2.
\end{aligned} \tag{12.16}$$

$$c'(A\rightarrow B)\cos\theta' = c'(A\rightarrow B)\cos\theta'_1 = -c'_{BA}\cos\theta'_2.$$

It follows that

$$c'(A\rightarrow B) = \frac{c\sqrt{1-\beta^2}}{1+\beta\cos\theta'}, \qquad c'(B\rightarrow A) = \frac{c\sqrt{1-\beta^2}}{1-\beta\cos\theta'}. \tag{12.17}$$

Thus, the two-way speed of light is found to be

$$c'_{av} = \frac{2L'}{t_{CA}} = c\sqrt{1-\beta^2}, \tag{12.18}$$

which is indeed isotropic in the F' frame.

This isotropy of the two-way speed of light in (12.18) guarantees that the Michelson-Morley experiment performed with an apparatus at rest in the F' frame will obtain a null result, even though the one-way speed of light is not isotropic. This differs from the conventional explanation of the Michelson-Morley experiment.

Next, let us consider the Kennedy-Thorndike experiment. It is an important variation of the Michelson-Morley experiment, in which the difference in length of the two arms in the Michelson interferometer is kept large on purpose. Also, the apparatus is kept fixed in the laboratory and the interference fringes are observed over a period of months. Nevertheless, the velocity-dependence of the two-way speed of light in (12.18) cannot be detected in the Kennedy-Thorndike experiment because the fringe shift is related to the quantity δ

$$\delta = \frac{c'_+\,\Delta t_+}{\lambda'_+} + \frac{c'_-\,\Delta t_-}{\lambda'_-} = \frac{2|L'_2 - L'_1|}{\lambda'}, \tag{12.19}$$

where $|L'_2 - L'_1|$ is the difference in the lengths of the two arms. Since there is no relative motion between the light source and the apparatus in F', we have the relation $\lambda'_+ = \lambda'_- = \lambda'$. Thus the fringe shift turns out to be independent of the motion of the F' frame and common relativity is also consistent with the Kennedy-Thorndike experiment: During the course of one day or half a year, no fringe shift will be observed.

These results show the powerful and far-reaching consequence of the

Poincaré–Einstein Principle of relativity, independent of a particular concept of time being used in the 4–dimensional framework.

12g. The Inverse Transformations and the Lorentz Group

The inverse transformation of (12.4) in common relativity is

$$ct_c = \gamma(b't_c + \beta x'), \quad x = \gamma(x' + \beta b't_c), \quad y = y', \quad z = z'. \tag{12.20}$$

One could also express β and $\gamma = 1/\sqrt{1-\beta^2}$ in terms of quantities measured in F'. Suppose an object is at rest in F, i.e. $v = dr/dt_c = 0$. Its velocity $v' = (v', 0, 0)$ measured in F' is given by (12.9):

$$V' = v_x' = -\gamma\beta c = -\gamma V, \quad v_y' = y_z' = 0, \quad c'(0) = \gamma c. \tag{12.21}$$

Thus, we can write (12.20) in the following form,

$$ct_c = \gamma'(b't_c + \beta' x'), \quad x = \gamma'(x' - \beta' b't_c), \quad y = y', \quad z = z'; \tag{12.22}$$

$$\beta' = \left| \frac{V'}{c'(0)} \right| = \beta, \qquad \gamma' = \frac{1}{\sqrt{1-\beta'^2}} = \gamma.$$

Note that the apparent asymmetric result in (12.21),

$$V' = -\gamma V, \tag{12.23}$$

is purely due to our convention that the speed of light is isotropic in the F frame. If we instead assume that the speed of light is isotropic in the F' frame, as discussed previously in section (12e) concerning the symmetry between F and F', we will have the reciprocal relation

$$V = -\gamma V', \tag{12.24}$$

which can be derived from (12.12).

Apart from the inverse transformations, other group properties, such as the 4–dimensional transformations having identity, associativity and obeying the law of composition, can be verified. These properties show that *the set of 4-dimensional transformations in common relativity forms the Lorentz group.* In general, the Lorentz and Poincaré groups discussed in chapter 9 for taiji relativity hold also for common relativity. This 4-dimensional symmetry of the Lorentz and Poincaré invariance is essential for common relativity to be consistent with all previous experiments.

12h. 4-Dimensional Maxwell Equations and Lorentz Force with Scalar Physical Time

In common relativity, the invariant action S_{com} for a classical charged particle moving in the electromagnetic field is assumed to have the same form as that in (10.19),

$$S_{com} = \int\{-mds - \bar{e}a_\mu dx^\mu\} - \frac{1}{4}\int f^{\mu\nu}f_{\mu\nu}d^3r dw \ , \tag{12.25}$$

$$x^\mu = (w,\mathbf{r}) = (bt_c,\mathbf{r}) \ , \quad x_\mu = (w,-\mathbf{r}) = (bt_c,-\mathbf{r}) \ , \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \ ,$$

$$ds^2 = dx^\mu dx_\mu = [d(bt_c)]^2 - [d\mathbf{r}]^2 \ ,$$

where x^μ is the coordinate 4-vector, t_c is common time and ds^2 is the differential form of (12.5) in common relativity. The universal constant $\bar{e}<0$ is in Heaviside-Lorentz units, as given in (10.1). The invariant action for a charged particle interacting with the electromagnetic field is given by

$$S_{cf} = \int\{-mds - \bar{e}a_\mu dx^\mu\} = \int L_c dt_c \ , \qquad a_\mu = (a_0, -\mathbf{a}) \ , \tag{12.26}$$

$$L_c = -m\sqrt{C^2-\mathbf{v}^2} - \bar{e}a_0 C + \bar{e}\mathbf{a}\cdot\mathbf{v} \ , \qquad \frac{ds}{dt_c} = \sqrt{C^2-\mathbf{v}^2} \ , \quad C = \frac{d(bt_c)}{dt_c} \ .$$

The canonical momentum **P** and the Hamiltonian H_c for a charged particle are defined, as usual, by

$$\mathbf{P} = \frac{\partial L_C}{\partial \mathbf{v}} = \mathbf{p} + \bar{e}\mathbf{a},$$

$$H_C = [(\partial L_C/\partial \mathbf{v})\cdot\mathbf{v} - L_C] = Cp_0 + \bar{e}a_0 C, \tag{12.27}$$

$$p_0 = \frac{m}{\sqrt{1-v^2/C^2}}, \qquad \mathbf{p} = \frac{m\mathbf{v}/C}{\sqrt{1-v^2/C^2}}, \qquad p^\mu p_\mu = m^2,$$

where $p^\mu=(p^0,\mathbf{p})$ and $p_\mu=(p_0,p_1,p_2,p_3)=(p^0,-\mathbf{p})$ are the 4–momenta of the particle. Note that the canonical momentum $(P^0,\mathbf{P})$, where $P^0=H_C/C$, also forms a 4–vector. The Lagrange equation of motion for a charged particle in the electromagnetic field can be derived from (12.26) by the variational calculus. One has

$$\frac{dp^\mu}{ds} = \bar{e}\, f^{\mu\nu}\frac{dx_\nu}{ds}, \tag{12.28}$$

$$p^\mu = m\frac{dx^\mu}{ds} = \left(\frac{m}{\sqrt{1-v^2/C^2}}, \frac{m\mathbf{v}/C}{\sqrt{1-v^2/C^2}}\right), \qquad C = \frac{d(bt_C)}{dt_C},$$

where $p^\mu=(p^0,\mathbf{p})$ is the usual momentum for a particle. Multiplying both sides of (12.28) by ds/dt_C, one obtains the equation of motion expressed in terms of common time t_C,

$$\frac{dp^\mu}{dt_C} = \bar{e}f^{\mu\nu}\frac{dx_\nu}{dt_C}, \tag{12.29}$$

i.e., $\dfrac{dp^0}{dt_C} = \bar{e}\mathbf{E}\cdot\mathbf{v}$, $\qquad \dfrac{d\mathbf{p}}{dt_C} = \bar{e}(\mathbf{E}C + \mathbf{v}\times\mathbf{B})$,

where p^0 and $\mathbf{p}$ are given in (12.28) and $\bar{e}(\mathbf{E}C+\mathbf{v}\times\mathbf{B})$ is the Lorentz force in common relativity.

For a continuous charge distribution in space, the second term in (10.25) becomes $-\int a_\mu j^\mu d^3r dw$, where $w=bt_C$, similar to that in (10.19). Maxwell's equations in a general inertial frame in common relativity are thus

$$\partial_\mu f^{\mu\nu} = j^\nu, \quad \partial_\lambda f^{\mu\nu} + \partial_\mu f^{\nu\lambda} + \partial_\mu f^{\lambda\mu} = 0; \tag{12.30}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad x^\lambda = (bt_c, \mathbf{r}), \quad f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu .$$

One can write the field-strength tensor $f^{\mu\nu}$ in matrix form

$$f^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{12.31}$$

$$\mathbf{E} = \frac{\partial \mathbf{a}}{\partial (bt)} - \nabla a_0, \quad \mathbf{B} = \nabla \times \mathbf{a} .$$

Since $a^\mu(w,\mathbf{r}) \Leftrightarrow A^\mu(ct,\mathbf{r})/c$, the fields $\mathbf{E}(w,\mathbf{r})$ and $\mathbf{B}(w,\mathbf{r})$ are related to the usual $\mathbf{E}_{usu}(ct,\mathbf{r})$ and $\mathbf{B}_{usu}(ct,\mathbf{r})$ by the correspondences in (10.25).

The transformations of the field tensor $f^{\mu\nu}$ are given by

$$f'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} f^{\alpha\beta}, \tag{12.32}$$

where $\partial x'^\mu/\partial x^\alpha$ can be calculated from the coordinate transformation (12.4). Since the fields $\mathbf{E}$ and $\mathbf{B}$ are elements of the field tensor $f^{\alpha\beta}$, we have the explicit transformation for the electromagnetic fields :

$$\begin{aligned} E'_x &= E_x, & E'_y &= \gamma(E_y - \beta B_z), & E'_z &= \gamma(E_z + \beta B_y), \\ B'_x &= B_x, & B'_y &= \gamma(B_y + \beta E_z), & B'_z &= \gamma(B_z - \beta E_y). \end{aligned} \tag{12.33}$$

Note that one can use transformations (12.9) and (12.33) for velocities and electromagnetic fields to show that $\bar{e}\mathbf{E}\cdot\mathbf{v}$ and the Lorentz force $\bar{e}(\mathbf{E}C + \mathbf{v} \times \mathbf{B})$ in (12.29) form a 4-vector $(\bar{e}\mathbf{E}\cdot\mathbf{v}, \bar{e}(\mathbf{E}C + \mathbf{v} \times \mathbf{B}))$, i.e., they transform like a 4-vector $(dp^0/dt_c, d\mathbf{p}/dt_c)$ or $(p^0, \mathbf{p})$.

In terms of $\mathbf{E}$, $\mathbf{B}$, and the 4-current $J^\mu = (\rho, \mathbf{J})$, the first equation in (12.31) can be written as

$$\nabla\cdot\mathbf{E} = \rho, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial (bt_c)} = \mathbf{J} . \tag{12.34}$$

The second equation in (12.31) can be written as

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial(bt_c)} = 0. \tag{12.35}$$

In common relativity, the speed of light does not explicitly appear in the new form of the Maxwell equations in a general inertial frame. Note that the Maxwell equations can be written in the familiar form

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} - \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t_c} = 0, \quad \text{etc.}, \tag{12.36}$$

if and only if the speed of light is assumed to be a constant c. Thus, even if the time measured in seconds is explicitly introduced in a theory, the 4–dimensional invariance of Maxwell equations do not necessarily imply the universality of the speed of light. Nevertheless, when $\rho = 0$ and $\mathbf{J} = 0$, wave equations (12.34) and (12.35) can be written in the 4–dimensional form,

$$\frac{\partial^2 g}{\partial w^2} - \nabla^2 g = 0, \qquad w = bt_c, \tag{12.37}$$

where the function $g=g(bt_c,\mathbf{r})$ stands for any component of E or $\mathbf{B}$. Let us consider the solution of (12.37) with y=z=0. Equation (12.37) with $g=g(bt_c,x)$ leads to the solution in a general inertial frame,

$$g(bt_c,x) = g_1(x+bt_c) + g_2(x-bt_c),$$

where g_1 and g_2 are two arbitrary functions. Since b is a function in general, it is not the speed of the wave. To obtain the speed of the wave, we may consider the amplitude g_2 which satisfies the equation $x - bt_c$ = constant. Differentiating with respect to the common time t, we find the speed of this wave to be

$$\frac{dx}{dt_c} = \frac{d(bt_c)}{dt_c} = C, \tag{12.38}$$

where the speed of light C as measured by the common time t_c in a general inertial frame is not isotropic, as shown in equation (12.11). Only in the inertial frame $F(ct_c,x,y,z)$ does one have a constant speed of light because b=C=c.

12i. Quantum Electrodynamics Based on Common Relativity

In common relativity, the action S_Q for quantum electrodynamics involves Dirac's electron field ψ, the photon field a_μ and the new quantum constant J. The invariant action S_Q is assumed to take the usual form,

$$S_Q = \int L\, d^4x, \qquad L = \overline{\psi}[\gamma^\mu(iJ\partial_\mu - \overline{e}a_\mu) - m]\psi - \frac{1}{4}f_{\mu\nu}f^{\mu\nu}, \qquad (12.39)$$

$$J = 3.5177293 \times 10^{-38}\ \text{g·cm}, \qquad \overline{e} = -1.6021891 \times 10^{-20}\sqrt{4\pi}\ (\text{g·cm})^{1/2},$$

where $d^4x = d(bt_c)d^3r$. The electromagnetic coupling strength in common relativity is the same as that in taiji relativity, $\alpha_e = \overline{e}^2/(4\pi J) \approx 1/137$, as it should. Also, $J\partial_\mu$ and $\overline{e}a_\mu$ in (12.39) have the dimension of mass.

Following the steps from (10.32) to (10.48), we have the S-matrix and the M-matrix for quantum electrodynamics as follows:

$$S_{if} = \delta_{if} - i(2\pi)^4\delta^4(p_f^{(tot)} - p_i^{(tot)})[\Pi_{ext\ par}(n_j/V)]^{1/2}M_{if}, \qquad (12.40)$$

where "ext par" denotes external particles, $n_j = m_j/p_{0j}$ for spin 1/2 fermions and $n_j = 1/(2p_{0j})$ for bosons.

It is important to note that quantum electrodynamics based on common relativity possesses the 4-dimensional symmetry, i.e., the Lorentz and Poincaré invariance, as shown in the action (12.39). The Lorentz and Poincaré invariance dictates that the evolution variable in basic physical laws must be the lightime bt_c, as one can see in the Maxwell equations (12.34) and (12.35). Therefore, the Feynman rules for writing M_{if} are formally the same as those in the usual QED, except that certain quantities (e.g., $w=bt_c$, J, p_μ and $\overline{e}$) have different dimensions from the corresponding quantities in conventional QED.

The Feynman rules for QED in common relativity are as follows:

(a) the covariant photon propagator is now given by

$$\frac{-i[g_{\mu\nu} - (1-\rho)k_\mu k_\nu/(k^2 + i\varepsilon)]}{(k^2 + i\varepsilon)}, \qquad k^2 = k_\mu k^\mu, \qquad (12.41)$$

(b) the electron propagator is

$$\frac{-i}{(\gamma^{\mu}p_{\mu} - m + i\varepsilon)}, \tag{12.42}$$

(c) the electron-photon vertex is

$$-i\,\bar{e}\gamma^{\mu}. \tag{12.43}$$

Also, each external photon line has an additional factor ε_{μ}, each external electron line has u(s,p) for the absorption of an electron and $\bar{u}(s,p)$ for the emission of an electron, etc. Other rules such as taking the trace with a factor -1 for each closed electron loop, integration with $d^4k/(2\pi)^4$ over a momentum k_{μ} not fixed by the conservation of four-momentum at each vertex, etc. are the same as the usual.

Thus, if one calculates scattering cross sections and decay rates (with respect to the lightime $w=bt_c$) of a physical process, one will get formally the same result as that in conventional QED. For example, let us consider the decay rate $\Gamma(1\to 2+3+...+N)$ for a physical process $1\to 2+3+...+N$. It is given by

$$\Gamma(1\to 2+3+...+N) = \lim_{w\to\infty} \int \frac{|\langle f|S|i\rangle|^2}{w} \frac{d^3x_2d^3p_2}{(2\pi J)^3} \cdots \frac{d^3x_Nd^3p_N}{(2\pi J)^3}, \tag{12.44}$$

$$w=bt_c.$$

The quantity $\Gamma(1\to 2+3+...+N)$ has the dimension of inverse length. Its inverse is particle's "lifetime" measured in terms of lightime bt_c which has the dimension of length and, hence, it may be called "decay-length." The decay-length D is given by

$$D = 1/\Gamma(1\to 2+3+...+N). \tag{12.45}$$

Thus, in common relativity, one has the "rest decay-length" D_0 for a particle decay at rest, corresponding to the "rest lifetime" times the speed of light in the conventional theory. The results in equations (12.44) and (12.45) imply that common relativity also has the dilatation of the decay-length consistent with experiment.

12j. New Properties in Common Relativity

In common relativity, there are new properties which special relativity simply cannot have. They are

1. invariant common time t_c for all inertial frames,

2. invariant "genergy" $G = \frac{m}{\sqrt{C^2-v^2}} = \frac{P_0}{C}, \quad C = \frac{d(bt_c)}{dt_c}$, (12.46)

3. invariant volume $V_I = \int \frac{d^4x}{c_0}\, \delta(t_c - \frac{x^0}{b}) = \int \frac{|b|}{c_0}\, dxdydz$, (12.47)

where $d^4x = dx^0dx^1dx^2dx^3$ and $c_0 = 29979245800$cm/sec is the scaling constant for the invariant volume V_I to have the same dimension as the usual volume. The last two invariant quantities will be discussed at length in the next chapter.

These new physical properties have new and important consequences, especially for many-particle systems. For example, in common relativity if we generalize the equation of motion (12.29) to a system of charged particles such as a plasma, we have

$$\frac{dp_i^\mu}{dt_c} = \bar{e} F^{\mu\nu}_{(i)} \frac{dx_{\nu i}}{dt_c}, \qquad i=1,2,...N, \tag{12.48}$$

where t_c is the common time for all particles and $F^{\mu\nu}_{(i)}$ is the electromagnetic field tensor produced by all particles except the ith particle. On the other hand, in special relativity, one must use the proper time τ_i of the ith particle to express the covariant equations of motion,

$$\frac{dp_i^\mu}{d\tau_i} = \frac{e}{c} F^{\mu\nu}_{(i)} \frac{dx_{\nu i}}{d\tau_i}, \qquad i=1,2,...N. \tag{12.49}$$

A proper time τ_i, i=1,2,...N, for each particle or the lack of a single evolution variable for a system of particles appears to be incompatible with the notion of canonical evolution of a many-particle system and with the evolution of a stochastic process. This will be discussed in chapter 13.

To summarize the new features of common relativity:

1. Common relativity offers a new simple picture of the physical world, incorporating the four-dimensional symmetry of physical laws while retaining our intuitive picture of the physical world at one instant of time.

2. Within the four-dimensional framework, common time t_c resembles Newtonian absolute time because it is a universal time for all observers. However, it is not unique as shown in equations (12.4) and (12.12) and, therefore, not absolute in the sense of the Newtonian concept of time.

3. Common time is essentially the same as the intuitive common-sense time which is taken for granted by people in daily life. If one uses this common time in the three-dimensional symmetry framework of the Galilean transformation, one gets experimentally incorrect results. However, common time within a four-dimensional symmetry framework is consistent with all known experiments.

4. When we perform experiments and observations, the external world appears three-dimensional to our consciousness at each instant of time. Such intuition about the physical world is lost when one employs the conventional four-dimensional framework with the relativistic time to many-particle systems. In particular, the concept of a Hamiltonian system with many degrees of freedom breaks down, and one can no longer have the notion of canonical evolution of a many-particle system because of the lack of a single time for all particles or all frames of reference.[4] As the result, one cannot derive the invariant Liouville equation in special relativity. However, one can derive the invariant Liouville equation in common relativity.

5. Common time allows new concepts to be naturally introduced in common relativity that is impossible to do in special relativity. In particular, common relativity has a new invariant quantity which we call the "genergy" G which is closely related to the energy of a particle. It allows us to have fuzziness at short distances and to define an invariant temperature. In particular, we can have an invariant Planck law for black-body radiation, in contrast to a non-invariant Planck law in special relativity. Thus, one may have to discuss the small anisotropy in the cosmic back-ground radiation based on reasons other than the 'absolute' motion of our solar system in the cosmic radiation.[5]

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13.

Common Time and Many-Particle Systems in a 4-Dimensional Symmetry Framework

13a. Problems of Relative Simultaneity for Many-Particle Systems

In classical mechanics, the motion of a many-particle system can be described in terms of the position of the center of mass $\mathbf{r}_C$, which has a well-defined Galilean transformation property. Yet, in the usual relativistic mechanics, one does not have a well-defined concept of the center of mass for a system of particles because the moving mass is not a constant. The center of energy $\mathbf{R}_{CE}$ defined by

$$\mathbf{R}_{CE} = \sum_i \frac{E_i \mathbf{r}_i}{E_{tot}}, \qquad E_{tot} = \sum_k E_k ,$$

in special relativity corresponds to the center of mass $\mathbf{r}_C$ in classical physics, where E_i and $\mathbf{r}_i$ are the relativistic energy and position vector of the ith particle, respectively. Clearly, $\mathbf{R}_{CE}$ reduces to $\mathbf{r}_C$ in the nonrelativistic limit because E_i becomes the mass m_i. However, the components of $\mathbf{R}_{CE}$ do not transform as components of a 4–vector under the Lorentz transformations. A similar situation occurs with the total momentum

$$P_\mu = \sum_i p_{i\mu} ,$$

which is not a 4–vector in general. The reason is that the sums P_μ are, in fact, sums of the values of $p_{i\mu}$ at a given instant in the given frame. However, in a different frame these values are no longer the momenta of each of the individual particles at the same instant. Therefore, when one transforms the total momentum P_μ to a new frame F', one has not only to form

$$\sum_i p'_{i\mu} ,$$

according to the 4–vector rule, but also must recalculate these quantities appropriately to the simultaneity of the F' frame.[1] Only for noninteracting particles is the sum P_μ a 4-vector because each $p_{i\mu}$ is constant.

Apart from these problems, there are other difficulties within the framework of special relativity:[2]

(i) The concept of a Hamiltonian system with many degrees of freedom breaks down because of the lack of a single time for a system of N particles. The relativistic time appears to be incompatible with the notion of the canonical evolution of a many-particle system and with the evolution of a stochastic process.

(ii) Since the notion of a 3-dimensional spatial volume is not a covariant concept, the uniform density in configuration space is not normalizable.

These properties lead to several difficulties in treating many-particle systems within the conceptual framework of special relativity because, in sharp contrast to Newtonian mechanics, it does not have a single time which is applicable to all particles or to all inertial frames of references.[2,3]

In contrast, common relativity does not have these difficulties. *It is the natural and unique four-dimensional generalization of the classical three-dimensional framework with one single time.* As demonstrated in chapter 12, common time is universal but not unique and, hence, differs from the Newtonian absolute time. However, it provides a universal simultaneity, just like the absolute time in classical mechanics. As a result, common time embedded in a 4-dimensional symmetry framework enables us to formulate the Hamiltonian dynamics of a many-particle system and to define an effective and invariant 6-dimensional μ-space and 6N-dimensional phase space for a covariant statistical mechanics, in sharp contrast to relativistic time.[3]

We stress that the assumption of common time in space-lightime necessitates the existence of the ligh function b in a general inertial frame, $b = x_0/t_c = w/t_c$, which transforms as the zeroth component of a 4-vector because the common time t_c is a scalar. This is important because the ligh function b enables us to define an invariant volume V_I in a general inertial frame:

$$V_I = \int \frac{1}{c_0} d^4x\delta(t_c - \frac{x^0}{b}) = \int \frac{|b|}{c_0} dxdydz, \qquad c_0 \equiv c, \tag{13.1}$$

where $d^4x = dx^0dx^1dx^2dx^3 = dx^0dxdydz$ and the *scaling constant* $c_0 \equiv c$ is introduced to preserve the usual dimension of the volume v_I. Note that there is no restriction on the zeroth component x^0 in $d^4x=dx^0dxdydz$, just as $x^0\equiv w$ in equations (7.1) and (7.4) within taiji relativity. The δ-function in (13.1) imposes on x^0 a restriction such that one has common time for all inertial frames. Since both b and x^0 transform as the zeroth component of the coordinate 4-vector, the condition imposed by the δ-function, i.e., $t_c-x^0/b=0$, is invariant. This can be seen as follows:

$$t_c - \frac{x^0}{b} = t_c - \frac{x'^0M}{b'N} = 0, \qquad M = \left(1 + \frac{\beta x'}{x'^0}\right), \qquad N = \left(1 + \frac{\beta x'}{b't_c}\right). \tag{13.2}$$

From the expression for N and the constraint $[t_c - x'^0M/(b'N)] = 0$, one can obtain the relation between M and N, $N=1/[1-\beta x'/(x'^0M)]$. Thus one has

$$t_c - \frac{x'^0M}{b'N} = t_c - \frac{x'^0M[1 - \beta x'/(x'^0M)]}{b'}$$

$$= t_c - \frac{[x'^0(1 + \beta x'/x'^0) - \beta x']}{b'} = t_c - \frac{x'^0}{b'}. \tag{13.3}$$

It follows from (13.2) and (13.3), that the volume element $dV_I = c_0^{-1}d^4x\delta(t_c - x^0/b)$ $=(|b|/c_0)dxdydz$ is invariant under the 4-dimensional coordinate transformation in common relativity. The numerical value of c_0 in a general inertial frame is defined to be identical to the speed of light in F, $c_0=c=b$, so that V_I is the same as the usual volume in the frame F. Even if the speed of light is not isotropic in an inertial frame F', one can still make this definition without violating the 4-dimensional symmetry of common relativity because c_0 is just a scaling constant. In the F'-frame, an invariant integral involving $dV_I=(|b'|/c)dx'dy'dz'$ is, in general, not simple for carrying out calculations because of the ligh function b'. Nevertheless, we can always transform to the F-frame, in which b=c and $dV_I=dxdydz$, in order to calculate invariant integrals.

One can also understand the new invariant volume dV_I in (13.3) as the volume of the parallelepiped in the space-lightime defined by four vectors a_μ, b_ν, c_α, and d_β:

$$dV_1 = \varepsilon^{\mu\nu\alpha\beta} a_\mu b_\nu c_\alpha d_\beta, \tag{13.4}$$

where $a_\mu=(b/c_0,0,0,0)$, $b_\nu=(0,dx,0,0)$, $c_\alpha=(0,0,dy,0)$ and $d_\beta=(0,0,0,dz)$. It is the same as the invariant expression (13.4) provided a_μ, b_ν, c_α, d_β are vectors lying in the direction of the coordinate axes and of length dw, dx, dy, dz, respectively. And the hypersurface $d\Sigma^\mu$ in the 4-dimensional space-lightime,

$$d\Sigma^\mu = (d\Sigma^0, d\Sigma^1, d\Sigma^2, d\Sigma^3), \tag{13.5}$$

transforms like a 4-vector, where $d\Sigma^0 = dxdydz$ is equal to the three dimensional volume element. Geometrically, the 4-vector $d\Sigma^\mu$ is equal in absolute magnitude to the "area" of the element of hypersurface and is in the direction perpendicular to all lines lying in this element.

The existence of the invariant volume V_I is important to four-dimensional thermodynamics and statistical mechanics. It enables us to define the notion of a box (or a 3-dimensional spatial volume) and to normalize the uniform density in configuration space.

13b. Invariant Hamiltonian Dynamics and Phase Space

In order to see what new quantities are covariant under the common relativity transformation and properties of the invariant 6N-dimensional phase space, let us briefly consider kinematics and the invariant Hamiltonian dynamics. The invariant "action function" for a free particle is assumed to be

$$S = -\int m ds = -\int m\sqrt{v^\mu v_\mu}\, dt \equiv \int L dt_C,$$

$$L = -m\sqrt{v^\mu v_\mu} = -m\sqrt{C^2 - v^2}, \tag{13.6}$$

where $ds^2 = dw^2 - dr^2 = (C^2 - v^2)dt_C^2$ and $C = dw/dt_C$ in a general inertial frame. The 4-momentum p^μ of a free particle is defined as

$$p^\mu = -\frac{\partial L}{\partial v_\mu} = \frac{mv^\mu/C}{\sqrt{1 - v^2/C^2}} = Gv^\mu, \qquad v_\mu = (C, -\mathbf{v}), \quad v^\mu = (C, \mathbf{v}), \tag{13.7}$$

where p^μ has the dimension of mass. Note that (13.7) is consistent with the usual definition of momentum $\mathbf{p} = \partial L/\partial \mathbf{v}$. Also, although mv^μ is a 4-vector, but it is not the physical momentum. The quantity G,

$$G=\frac{m}{\sqrt{C^2-v^2}}=\frac{\sqrt{\mathbf{p}^2+m^2}}{C}=\frac{p^0}{C}, \tag{13.8}$$

is an invariant because it is the ratio of corresponding components of two 4-vectors, p^μ and v^μ, with $p_x/v_x=p^0/C$. Its invariance can also be seen from (13.7) because p^μ and v^μ are 4-vectors with the same transformation properties in common relativity. We call G "genergy" because it is equivalent (except for a multiplicative constant) to the conserved energy p^0 in the F-frame, in which C=c so that the conservation of energy implies the conservation of the invariant genergy. Since special relativity does not have a four-velocity measured in terms of a scalar common time, it does not have an invariant corresponding to the genergy (13.8).

With the help of the invariant genergy G, we can define the four-coordinate R^μ_c of the "center of mass" for an N-particle system in common relativity,

$$R^\mu_c=\frac{Q^{\mu *}}{\sum_k G_k}, \qquad Q^{\mu *}=\sum_{i=1}^{N} G_i x^\mu_i . \tag{13.9}$$

The spatial components $\mathbf{R}_c$ can be satisfactorily identified with the relativistic center of mass for a system of particles in common relativity. It can be verified that $\mathbf{R}_c$ reduces to the classical center of mass when the velocities of the particles are small. Similarly, the total momentum

$$P_\mu=\sum_i p_{i\mu}$$

in common relativity is a 4-vector because the common time gives us a universal definition of simultaneity which is applicable to all observers or all inertial frames of reference.

Let us consider N charged particles with mass m_a, a=1,2,..., N, within the framework of the common-relativistic Hamiltonian dynamics.[3] We begin with 8N-dimensional "extended phase space" with the basic variables $x^a{}_\nu$, $p^a{}_\nu$, to define the Poisson brackets and the generators of the Poincaré group.

$$\{x^\mu{}_a, x^\nu{}_b\}=0, \qquad \{x^\mu{}_a, p^\nu{}_b\}=g^{\mu\nu}\delta_{ab}, \qquad \{p^\mu{}_a, p^\nu{}_b\}=0, \tag{13.10}$$

where

$$\{A,B\} \equiv \frac{\partial A}{\partial x_a^\lambda}\frac{\partial B}{\partial p_{a\lambda}} - \frac{\partial A}{\partial p_{a\lambda}}\frac{\partial B}{\partial x_a^\lambda}, \qquad x_a^\lambda = (w_a, x_a, y_a, z_a).$$

We impose 2N constraints of the form

$$K_a \equiv (P_a^\mu - \bar{e}\, a_a^\mu)^2 - m_a^2 = 0, \qquad Z_a(x_a^0, t_c) = x_a^0 - b_a t_c = 0, \tag{13.11}$$

where P_a^μ is the canonical momentum 4-vector, $P_a^\mu = p_a^\mu - \bar{e} a_a^\mu$, and a_a^μ is the electromagnetic potential acting on the particle a. Thus, we end up with 6N basic independent variables. The Dirac Hamiltonian is

$$H_D = W_a K_a, \tag{13.12}$$

where W_a is determined by the constraints $Z_a=0$ which must hold for all time t_c, i.e. $dZ/dt_c=0$.

The Hamiltonian equations of motion with common time t_c are then[3]

$$\frac{dx_a^\mu}{dt_c} = \{x_a^\mu, H_D\} = \frac{\partial H_D}{\partial p_{a\mu}}, \qquad \frac{dp_a^\mu}{dt_c} = \{p_a^\mu, H_D\} = -\frac{\partial H_D}{\partial x_{a\mu}}, \tag{13.13}$$

where

$$W_a = -A_{ab}\frac{\partial Z_b}{\partial t_c}, \qquad A_{ab}\{Z_b, K_c\} = \delta_{ac}.$$

One can verify that Hamilton's equations of motion (13.13) with the Dirac Hamiltonian H_D in (13.12) and the constraints (13.11) lead to the equation of motion for a charged particle,

$$\frac{dp_a^\mu}{dt_c} = \bar{e}\, f_a^{\mu\nu} v_{a\nu}, \qquad a=1,2,...N, \tag{13.14}$$

$$v_{a\nu} = \frac{C_a(P_{a\nu} - \bar{e} a_{a\nu})}{(P_a^0 - \bar{e} a_a^0)},$$

$$dZ_a(x_a^0,t_c) = dx_a^0 - \frac{d(b_a t_c)}{dt_c}\,dt_c = dx_a^0 - C_a dt_c\,, \qquad \frac{\partial Z_a}{\partial t_c} = -C_a\,,$$

$$A_{ac} = \frac{\delta_{ac}}{2(P_a^0 - \bar{e}a_a^0)}\,,$$

$$W_a = \frac{C_a}{2(P_a^0 - \bar{e}a_a^0)}\,,$$

where a is not summed. The resultant equation of motion (13.14) is the same as that in (12.29), as it should. Thus, the evolution of the N-particle system in terms of the common time t_c is completely determined by Hamilton's equations (13.13) and the initial data, i.e., the initial momenta and positions of the particles.

A basic unsolved problem in statistical mechanics based on special relativity is the phase space. In general inertial frames, it appears impossible to have a meaningful and Lorentz-invariant phase space related to initial data at a given time for all observers in different frames, in contrast with the classical case based on Newtonian laws. In Newtonian physics, observers can, in principle, perform measurements at a given absolute time. Thus one can specify a system of N particles by the vectors $(\mathbf{q}_1,...,\mathbf{q}_N,\mathbf{p}_1,...,\mathbf{p}_N)$ which span the phase space of the system.

In common relativity, suppose that there is a clock and an observer at each point in space in every inertial frame. The F-frame observers can perform instantaneous measurements of $\mathbf{r}$ and $\mathbf{v}$ for each particle at a given common time t_c using the nearest clock so as to remove the need to account for the finiteness of the speed of light. Thus, we have the data of coordinates and momenta of particles $(\mathbf{r}_a,\mathbf{p}_a)$, a=1,2,..., N, at common time t_c. These data and dynamical equations completely determine the evolution of the system. Of course, the data can also be transformed to another frame F' using the space-lightime transformations which relate

$$(ct, \mathbf{r}_a) \quad \text{and} \quad (c, \mathbf{v}_a) \qquad \text{in F}, \qquad a=1,2,...,N,$$

and

$$(b'_a t, \mathbf{r}'_a) \quad \text{and} \quad (c'_a, \mathbf{v}'_a) \qquad \text{in F'}, \qquad a=1,2,...,N.$$

The initial positions and velocities of the particles in F can also be directly obtained by the F' observers by measuring $\mathbf{r}'_a, \mathbf{v}'_a$ and computing b'_a and c'_a with the help of eq. (12.4). In this way, we have global knowledge of the whole system at a given instant of common time in statistical mechanics based on common relativity.

With the help of (13.3) and (13.11), we can define in μ-space an effective six-dimensional volume element (for particles with $m \geq 0$)

$$d\mu = d^4x\, \delta(\frac{x^0}{b} - t_c)\, d^4p\, \delta(p_\lambda{}^2 - m^2)\, \theta(p^0) 2G\,, \tag{13.15}$$

which is an invariant because the common time t_c, the rest mass m, the genergy G of a particle and the ratio x_0/b are all scalars in common relativity. In the particular frame F, one can see that eq. (13.15) reduces to $d^3x d^3p$ by carrying out the integrations over x^0 and p^0. The 6N-dimensional phase space in common relativity has the invariant volume element

$$d\Gamma = \prod_{a=1}^{N} d\mu_a\,. \tag{13.16}$$

The physical and independent variables in phase space are the only really important ones. We can eliminate the nonphysical variables from the theory by using the constraint equations. *However, the elimination may be awkward and may spoil the four-dimensional feature in the equations; therefore, we shall retain the possibility of constraint equations in general theory.* That is, we treat the N-particle phase space as an invariant 8N-dimensional extended phase space with 2N invariant constraints (13.11). In this way, we have, effectively, a physical 6N-dimensional phase space which is invariant under space-lightime transformations.

13c. The Invariant Kinetic Theory of Gases

Let us consider kinetic theory based on a one-particle distribution function $D_1(x^A, t_c)$ and a kinetic equation for $D_1(x^A, t_c)$ within the framework of common relativity. In order to show explicitly the 4-dimensional symmetry of the theory of gases, we use $x^A = (x^\mu, p^\mu)$, where p^μ is the 4-momentum of the particle in (13.7), as the coordinates of the extended μ-space, i.e. the one–

particle phase space. The reason for this is that the 4-dimensional transformations of coordinates can only be expressed in terms of the Cartesian coordinates x^μ. The invariant time-dependent distribution function $D_1(x^A,t_C)$ in the μ-space is normalized by the expression

$$\int D_1(x^A,t_C)d\mu = \int D^* d^4x d^4p = N ,$$

$$D^* \equiv D_1(x^A,t)\, \delta(t_C - \frac{x^0}{b})\, \delta(p_\lambda{}^2 - m^2)\, \theta(p^0) 2G , \tag{13.17}$$

where $d\mu$ is given by (13.15) and $N(t_C)$ is the number of particles in the gas at time t_C. The function $D^*d^4xd^4p/N$ can be formally interpreted as the probability of finding the particle at common time t_C in the volume element d^4xd^4p around the point x^A in the 8-dimensional space. If one carries out the integration over x^0 and p^0 from $-\infty$ to $+\infty$, then $[(1/N)\int D^*dx^0dp^0]d^3xd^3p$ is the probability of finding the particle at time t_C in the spatial volume element d^3x around the point $\mathbf{r}$ and has values of its momentum in the region d^3p around the point $\mathbf{p}$.

This is in sharp contrast with the usual relativistic kinetic theory in which the distribution D' is normalized on a six-dimensional manifold through a current[4]

$$\int D' 2p_\mu \theta(p^0)\delta(p_\lambda{}^2 - m^2) d^4p d\Sigma_\mu = N , \tag{13.18}$$

because of the lack of a single evolution variable for a system of particles. In other words, the usual relativistic kinetic theory is a statistics of curves rather than a statistics of points. In the conventional theory, such a six-dimensional manifold is, in general, not meaningful because of the arbitrariness of Σ (except the case when the current is conserved) and that the distribution D' is actually not the probability density.[2]

The invariant distribution $D_1(x^A,t_C)$ is a function in the eight-dimensional space with the property (13.17). The common-relativistic one-particle Liouville equation for the invariant distribution $D_1(x^A,t_C)$ can be derived by considering the eight-vectors $x^A(t_C)$ of the particle coordinates, the corresponding velocities

$$\dot{x}^A(t_C) = \frac{dx^A(t_C)}{dt_C} = (\dot{x}^\mu(t_C), \dot{p}^\mu(t_C)) = (v^\mu, F^\mu) , \tag{13.19}$$

and the current density $j^A(x^A,t_c)$,

$$j^A(x^A,t_c) = D_1(x^A,t_c)\dot{x}^A(t_c). \tag{13.20}$$

The current density $j^A(x^A,t_c)$ satisfies the continuity equation in the eight-dimensional space

$$\frac{\partial}{\partial t_c} D_1(x^A,t_c) + \frac{\partial}{\partial x^A}[D_1(x^A,t_c)\,\dot{x}^A(t_c)] = 0\,. \tag{13.21}$$

Using Hamilton's equations of motion (13.13), we have

$$\frac{\partial \dot{x}^A}{\partial x^A} = \frac{\partial}{\partial x^\mu}\frac{\partial H_D}{\partial p_\mu} + \frac{\partial}{\partial p_\mu}\left(-\frac{\partial H_D}{\partial x^\mu}\right) = 0, \tag{13.22}$$

and, therefore, we can write (13.21) as

$$\frac{\partial}{\partial t_c} D_1(x^A,t_c) + \left[v^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}\right] D_1(x^A,t_c) = \frac{d}{dt_c} D_1(x^A,t_c) = 0\,. \tag{13.23}$$

This is a simple example of the Liouville theorem, to be discussed in section 13d below. Clearly, if the one-particle distribution does not depend on t_c explicitly, i.e., $D_1 = D_1(x^A)$, then we have an invariant distribution $D_1(x^A)$ which obeys equation (13.23) without having the first term.

In order to see its connection to the nonrelativistic case, we write the invariant distribution $D_1(x^A,t_c)$ in the following form

$$D_1(x^A,t_c) \equiv \overline{D}\big(x^0(t_c),\mathbf{r},p^0(t_c),\mathbf{p},t_c\big)\delta\big(x^0 - x^0(t_c)\big)\delta\big(p^0 - p^0(t_c)\big)\,, \tag{13.24}$$

$$x^0(t_c) = bt_c\,, \quad p^0(t_c) = \sqrt{\mathbf{p}^2(t_c)+m^2}\,.$$

It follows from (13.23) and (13.24) that

$$\frac{\partial \overline{D}}{\partial t_c} + \mathbf{v}\cdot\frac{\partial \overline{D}}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial \overline{D}}{\partial \mathbf{p}} = 0, \tag{13.25}$$

because $(\partial/\partial t_c)\,\delta(x^0 - x^0(t_c))\,\delta(p^0 - p^0(t_c)) = 0$ and

$$\bar{D}\left[v^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}\right]\delta(x^0 - x^0(t_c))\,\delta(p^0 - p^0(t_c))$$

$$= \bar{D}\left[\dot{x}^0 \frac{\partial}{\partial x^0} + \dot{p}\frac{\partial}{\partial p^0}\right]\,\delta(x^0 - x^0(t_c))\,\delta(p^0 - p^0(t_c))$$

$$= -\left\{\left[\dot{x}^0 \frac{\partial}{\partial x^0} + \dot{p}^0 \frac{\partial}{\partial p^0}\right]\bar{D}\right\}\delta(x^0 - x^0(t_c))\,\delta(p^0 - p^0(t_c))\,, \qquad (13.26)$$

In the nonrelativistic case, we have $|\mathbf{p}(t_c)| \approx |m\mathbf{v}| << p^0(t_c) \approx m$ and $C \approx b \approx c$ in the frame F, so that

$$\int_{[x^0 p^0]} d\mu = d^3x d^3p\,,$$

$$\int D_1(x^A, t_c)\delta\big(x^0 - x^0(t_c)\big)\delta\big(p^0 - p^0(t_c)\big)dx^0 dp^0 \qquad (13.27)$$

$$= D_1(x^A, t_c)|_{x^0 = bt_c;\ p^0 = p^0(t_c)} = \bar{D}(\mathbf{r}, \mathbf{p}, t_c)\,.$$

The last expression in (13.27) is the one-particle distribution in the classical six–dimensional μ–space $(\mathbf{r},\mathbf{p})$ and it satisfies eq. (13.25). The non-relativistic distribution $\bar{D}(\mathbf{r},\mathbf{p},t_c)$ is the probability density of finding the particle at common time t_c around the point $(\mathbf{r},\mathbf{p})$ in the F frame in which C=b=c.

A general 4-dimensional kinetic equation for a non-equilibrium gas is given by

$$\left(\frac{\partial}{\partial t_c} + v^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}\right)D_1 = P(D_1)\,, \qquad D_1 = D_1(x^A, t_c)\,, \qquad F^\mu = \frac{dp^\mu}{dt_c}\,, \qquad (13.28)$$

where $P(D_1)$ is given by Boltzmann's collision postulate (i.e. the variation of D_1 per unit time due to collisions, see eq. (13.78) below) and it should fulfill certain requirements.[5]

Once the solution of the kinetic equation (13.28) is obtained, one can compute the energy–momentum tensor of the relativistic fluid in the theory:

$$T^{\mu\nu}(x) = \int d^4p 2\theta(p^0)\delta(p_\lambda{}^2-m^2)p^\mu p^\nu D_1(x^A)\,, \quad p^\mu = \frac{m dx^\mu}{ds} = Gv^\mu\,. \tag{13.29}$$

The explicit knowledge of this tensor implies the equations of relativistic hydrodynamics based on common relativity.

The average value of a measurable quantity $Q(x,p)$ is defined as

$$<Q(x,p)> = \int Q(x,p)D_1(x,p)d\mu\,, \qquad x^A = (x,p)\,. \tag{13.30}$$

The particle current density $j^\mu(x)$ in the 4-dimensional spacetime can be defined in terms of $j^\mu(x,p)=D_1(x,p)v^\mu$ in (13.20) with A=0,1,2,3, as follows,

$$j^\mu(x) = \int d^4p 2G\theta(p^0)\delta(p_\lambda{}^2-m^2)D_1(x,p)v^\mu\,, \qquad Gv^\mu = p^\mu\,. \tag{13.31}$$

Also, the invariant density of a fluid can be defined as

$$\int D_1(x,p)\delta(p_\lambda{}^2-m^2)2G\theta(p^0)d^4p\;. \tag{13.32}$$

This can be verified in the simple case where $D_1(x,p) = A\rho(x)\exp(G/\tau)$.

In the presence of the external electromagnetic-field tensor $f^{\mu\nu}$, when $P(D_1)=0$ and $\partial D_1/\partial t_c=0$, (13.29), (13.28), (12.29) and $Gv^\mu = p^\mu$ lead to

$$\partial_\mu T^{\mu\nu} = j_\mu(x)f^{\mu\nu}\,, \tag{13.33}$$

where $j_\mu(x)$ is given by (13.31). Clearly, the result (13.33) reduces to the conservation law $\partial_\mu T^{\mu\nu}=0$ when $f^{\mu\nu}= 0$.

13d. The Invariant Liouville Equation

We consider the Gibbs ensemble of similar systems of N particles, i.e. identical in composition and macroscopic conditions, but in different states. (cf. section (13e) below) Such an ensemble is represented by an invariant N-particle density function $D_N(x_a^A, t_c) \equiv \rho_I(x_a, p_a, t_c)$, a=1,2,...N, which is normalized by the relation

$$\int \rho_I(x_a,p_a,t_c)d\Gamma = \int \rho^* \prod_{b=1}^{N} d^4x_b d^4p_b = N, \qquad (x_a,p_a) \equiv (x_a^\mu,p_a^\mu), \tag{13.34}$$

$$\rho^* \equiv \rho_I(x_a,p_a,t_c) \prod_{a=1}^{N} \delta(t_c - x_{0a}/b_a)\delta(p_a^2 - m_a^2)\theta(p_a^0)(2G_a), \tag{13.35}$$

where $d\Gamma$ is given by (13.16). The density ρ_I satisfies the continuity equation

$$\frac{\partial}{\partial t_c}\rho_I + \frac{\partial}{\partial x_a^\mu}(v_a^\mu \rho_I) + \frac{\partial}{\partial p_a^\mu}(\dot{p}_a^\mu \rho_I) = 0, \tag{13.36}$$

where both a=1,2,...,N and μ=0,1,2,3, are summed over. With the help of the Dirac Hamiltonian and eq. (13.13), we are able to obtain one single 4–dimensional invariant Liouville equation in common relativity,

$$\left(\frac{\partial}{\partial t_c} + v_a^\mu \frac{\partial}{\partial x_a^\mu} + F_a^\mu \frac{\partial}{\partial p_a^\mu}\right)\rho_I = \frac{d}{dt_c}\rho_I = 0, \tag{13.37}$$

for a system of N particles. We note that, in the framework of special relativity, one can only obtain N *one–particle* Liouville equations[2] rather than one single invariant Liouville equation:

$$\left(\frac{\partial}{\partial \tau_b} + \frac{dx_b^\mu}{d\tau_b}\frac{\partial}{\partial x_b^\mu} + \frac{dp_b^\mu}{d\tau_b}\frac{\partial}{\partial p_b^\mu}\right)\rho_1(x_b,p_b,\tau_b) = \frac{d}{d\tau_b}\rho_1(x_b,p_b,\tau_b) = 0, \tag{13.38}$$

b is not summed, b=1,2,...N,

where τ_b is the proper time of the particle b. The basic reason for this property is the absence of a universal (or common) time for all observers in special relativity. As a result, within the framework of special relativity, there is no notion of the canonical evolution of an N-particle system and the entire history of the system of N particles is represented by an N-dimensional manifold rather than a one-dimensional manifold.

With the help of (13.35) and the relation (13.26) for each particle, the Liouville equation (13.37) takes the more familiar form

$$\left(\frac{\partial}{\partial t_c} + \sum_a \mathbf{v}_a \cdot \frac{\partial}{\partial \mathbf{r}_a} + \sum_a \mathbf{F}_a \cdot \frac{\partial}{\partial \mathbf{p}_a}\right)\rho = 0, \tag{13.39}$$

$$\rho_I(x_a,p_a,t_c) \equiv \rho\left(x_a^0(t_c),\mathbf{r}_a,p_a^0(t_c),\mathbf{p}_a,t_c\right)\prod_{b=1}^{N}\delta\left(x_b^0 - x_b^0(t_c)\right)\delta\left(p_b^0 - p_b^0(t_c)\right),$$

$$x_a^0(t_c) = b_a t_c, \quad p_a^0(t_c) = \sqrt{\mathbf{p}_a^2(t_c)+m^2}, \quad G_a = \frac{p_a^0(t_c)}{C_a}, \quad \text{(a is not summed)},$$

where the explicit four-dimensional feature of (13.37) has been spoiled. However, the 3-dimensional form in (13.39) is the same as the usual Liouville equation in the nonrelativistic limit.

The ensemble average of a measurable property $f(x_a,p_a)$ of a system is defined by

$$\langle f(x_a,p_a)\rangle = \frac{\int f(x_a,p_a)\rho_I(x_a,p_a)d\Gamma}{\int \rho_I(x_a,p_a)d\Gamma}. \tag{13.40}$$

Relations $p_{0a}=\sqrt{\mathbf{p}_a^2+m_a^2}$ and $x_{0a}=b_a t_c$ are understood in the functions $\rho(x_a,p_a,t_c)$ and $f(x_a,p_a)$ in (13.40).

13e. Invariant Entropy, Temperature and Maxwell–Boltzmann Distribution

Let us consider the distribution of momenta of a relativistic gas at equilibrium in a box having the invariant volume (13.2). Suppose one divides all the quantum states of an individual particle of the gas into groups denoted by $i=1,2,3.....$ Each group contains neighbouring states (having neighbouring genergies.) Both the number of states w_i in group i and the number of particles n_i in these states are very large. Then the set of numbers n_i will completely describe the macroscopic state of the gas. Each particle in group i has an invariant genergy G_i. They satisfy the conditions

$$\sum_i n_i = N, \qquad \sum_i n_i G_i = G_{tot}. \tag{13.41}$$

Putting each of the n_i particles into one of the w_i states, one obtains $(w_i)^{n_i}$ possible distributions. Since all particles are identical, these possible distributions contain some identical ones which differ only by a permutation of the particles. The number of permutations of n_i particles is $n_i!$. Thus the statistical weight of the distribution of n_i particles over w_i states is $(w_i)^{n_i}/n_i!$. We define the entropy S_C of the gas as a whole to be

$$S_C = \sum_i \ln \Delta\Gamma_i , \qquad \Delta\Gamma_i = (w_i)^{n_i}/(n_i!) . \tag{13.42}$$

When the gas is in the equilibrium state, the entropy of the system must be a maximum. Suppose the average number of particles in each of the quantum states of the group i is denoted by $\langle n_i \rangle$, i.e., $\langle n_i \rangle = n_i/w_i$. One can find those $\langle n_i \rangle$, which give S_C in (13.42) its maximum possible value subject to the conditions in (13.41). By the usual methods of maximization of S_C and the Lagrange multipliers, we obtain the most probable distribution

$$\langle n_i \rangle = A \exp[-G_i/\tau] , \tag{13.43}$$

where A is a normalization constant and τ is a "scalar temperature." We stress that $\langle n_i \rangle$ is an invariant function and can be used to define invariant thermodynamic quantities.[3] (For a derivation of the form (13.43) with the genergy, see also equation (13.95) below.)

Similarly, we can obtain the invariant Maxwell-Boltzmann equilibrium distribution (normalized by (13.17))

$$D_I^{MB}(p) = BN \exp[-G/\tau] , \tag{13.44}$$

with

$$G = \frac{\sqrt{p^2+m^2}}{C} , \qquad B = \frac{1}{4} pm^2 V_I \tau C K_2\left(\frac{m}{c\tau}\right) ,$$

where $K_\nu(z)$ is the Bessel function of imaginary argument and can be expressed in the terms of the Hankel function $H_\nu^{(1)}(iz)$: $K_\nu(z) = (\pi i/2)\exp[-i\pi\nu/2]H^{(1)}(iz)$. In the nonrelativistic limit, we choose the frame F in which C=c and make the approximation of large mass m or $\sqrt{p^2+m^2} \approx m+mv^2/2c^2$. We have

$$D_I^{MB} \approx \frac{N\exp[-m/c\tau - p^2/2mc\tau]}{V_I\, 4\pi m^2 c\tau\sqrt{\pi c\tau/2m}\, \exp[-m/c\tau]}$$

$$= \frac{N/V_I}{(2\pi mc)^{3/2}} \exp[-mv^2/2c^3\tau]\,. \tag{13.45}$$

Comparing it with the usual Maxwell-Boltzmann distribution $\propto \exp[-mv^2/2k_BT]$, we see that the invariant temperature τ is related to the usual temperature T and the Boltzmann constant k_B by the relation (in the F frame):

$$c^3\tau = k_BT\,. \tag{13.46}$$

Note that the concept of invariant temperature τ is closely related to the fact that the genergy G of a particle is invariant and proportional to the energy in the frame F. In special relativity, the usual procedure is to introduce an inverse temperature 4-vector for the invariant form of the Maxwell-Boltzmann distribution. However, it is not at all clear that such a notion of a temperature 4-vector has any physical meaning.[2]

13f. The Invariant Boltzmann-Vlasov Equation

For a non-equilibrium gas, one may describe the system in terms of the coordinates x^μ and velocities v^μ of the particles which comprise it. Suppose the system is described by the distribution $f(x^\mu,v^\mu)$. The rate of change of the distribution $f(x^\mu,v^\mu)$ in a general inertial frame is given by the invariant equation

$$\frac{df}{dt_c} = \frac{dx^\mu}{dt_c}\frac{\partial f}{\partial x^\mu} + \frac{dv^\mu}{dt_c}\frac{\partial f}{\partial v^\mu} = v^\mu\frac{\partial f}{\partial x^\mu} + a^\mu\frac{\partial f}{\partial v^\mu}\,, \tag{13.47}$$

$$x^\mu = (bt_c,x,y,z)\,, \quad \frac{dx^\mu}{dt_c} = v^\mu = (C,\mathbf{v})\,, \quad \frac{dv^\mu}{dt_c} = a^\mu = (a^0,\mathbf{a})\,, \quad C = \frac{d(bt_c)}{dt_c}\,,$$

where the velocities v^μ and accelerations a^μ are 4-vectors in common relativity because physical common time t_c is a scalar. The physical meaning of the distribution $f(x^\mu,v^\mu)$ can be seen from the following invariant relations

$$\int f(x^\mu,v^\mu)d^4v\delta(\dot{s}^2-C^2+v^2) = n(x^\mu) = n(bt_c,x,y,z)\,, \tag{13.48}$$

$$\dot{s}^2 \equiv \left(\frac{ds}{dt_c}\right)^2 = \left(\frac{dw}{dt_c}\right)^2 - \left(\frac{d\mathbf{r}}{dt_c}\right)^2 = C^2-v^2\,, \qquad \text{and} \qquad \frac{dw}{dt_c}=\frac{d(bt_c)}{dt_c}=C\,,$$

where $n(bt_c,x,y,z)$ is the number of molecules per unit volume at the time t and position $\mathbf{r}=(x,y,z)$ in a general inertial frame. The function $\delta(\dot{s}^2-C^2+v^2)$ is an invariant constraint of the 4-dimensional velocity integration $\int d^4v$. This is the same as a mass-shell constraint, $\delta(p_\lambda{}^2-m^2)$, in the momentum integration $\int d^4p$. In the framework of common relativity, we have, by definition, the simple familiar form of the density function $n(t,x,y,z)$ of molecules because of the relation $b=C=c=$constant in the frame F. In a general inertial frame, by integration over invariant volume given by (13.1), one has the total number N of the molecules,

$$\int n(x^\mu)dV_I = N\,, \tag{13.49}$$

which is a constant, independent of the common time t_c and the reference frame.

If the molecules do not collide, then one has $df/dt_c = 0$, i.e.,

$$v^\mu\frac{\partial f}{\partial x^\mu} + a^\mu\frac{\partial f}{\partial v^\mu} = 0\,, \qquad\qquad f = f(x^\mu,v^\mu)\,. \tag{13.50}$$

This is a special case of the Liouville theorem. If the molecules collide with each other, then df/dt_c is not equal to zero and can be understood based on Boltzmann's collision postulate.[5]

For a high temperature plasma, let us consider the generalized 4-dimensional Boltzmann-Vlasov equation based on common relativity. Vlasov first used the equation for the plasma. The equation in a general inertial frame is assumed to have the form of (13.50) with the acceleration $a^\mu=(a^0,\mathbf{a})$ of charged particles expressed in terms of the electromagnetic fields:

$$C\frac{\partial f}{\partial(bt_c)} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + a^0(\mathbf{E},\mathbf{B},\mathbf{v})\frac{\partial f}{\partial C} + \mathbf{a}(\mathbf{E},\mathbf{B},\mathbf{v})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0\,, \tag{13.51}$$

where $f=f(bt_c,\mathbf{r},C,\mathbf{v})\equiv f_1$ is the one-particle distribution function. The relation between the 4-acceleration $a^\mu=(a^0,\mathbf{a})$ and the electromagnetic fields can be obtained from the equation of motion (10.20) with the 4-coordinate $x^\mu=(w,\mathbf{r})\equiv(bt_c,\mathbf{r})$ and the 4-dimensional transformations in common relativity. The equation of motion of a charged particle can be written as

$$\frac{dp^\mu}{ds} = \bar{e}F^{\mu\nu}\frac{dx_\nu}{ds}, \qquad \text{or} \qquad \frac{dp^\mu}{dt_c} = \bar{e}F^{\mu\nu}\frac{dx_\nu}{dt_c}, \tag{13.52}$$

where $x_\mu=(bt_c,-\mathbf{r})$ and

$$p^\mu = m\frac{dx^\mu}{ds} = (p^0,\mathbf{p}) = \left(\frac{m}{\sqrt{1-\beta^2}}, \frac{m\boldsymbol{\beta}}{\sqrt{1-\beta^2}}\right), \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{C}. \tag{13.53}$$

The fourth component of the equation of motion in (13.52) is not independent of the three spatial components. The three independent equation of motion can be written as

$$\frac{d\mathbf{p}}{dt_c} = \bar{e}(C\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{13.54}$$

Using the expression of $\mathbf{p}$ in (13.53) and $dv^\mu/dt_c=(dC/dt_c,d\mathbf{v}/dt_c)=(a^0,\mathbf{a})$, eq. (13.54) becomes

$$\frac{m\gamma}{C}\left[\mathbf{a} + \gamma^2\frac{\mathbf{v}}{C^2}(\mathbf{v}\cdot\mathbf{a})\right] - m\gamma^3\frac{\mathbf{v}a^0}{C^2} = \bar{e}(C\mathbf{E} + \mathbf{v}\times\mathbf{B}), \qquad \gamma = \frac{1}{\sqrt{1-v^2/C^2}}. \tag{13.55}$$

Now let us consider the Boltzmann-Vlasov equation in the F frame, in which the speed of light is constant, i.e., $a^0=\dot{C}=0$ or $C=c=$constant. One can solve for the acceleration $\mathbf{a}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ by letting $\mathbf{a}= Z_1(C\mathbf{E} + \mathbf{v}\times\mathbf{B}) + Z_2\mathbf{v}(\mathbf{v}\cdot\mathbf{E})$ and using (13.55) with $\dot{C} = 0$ to determine Z_1 and Z_2. We obtain

$$\mathbf{a} = \frac{\bar{e}c}{m\gamma}(c\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \frac{\bar{e}}{m\gamma}\mathbf{v}(\mathbf{v}\cdot\mathbf{E}), \qquad \dot{C}=0. \tag{13.56}$$

Thus, the generalized 4-dimensional Boltzmann-Vlasov equation based on common relativity can be written as

$$\frac{\partial f}{\partial t_c} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \left\{\frac{\bar{e}c}{m\gamma}(c\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \frac{\bar{e}}{m\gamma}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\right\}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{13.57}$$

in the F frame. If the second order terms in (v/c) are neglected, equation (13.57) reduces to the usual Boltzmann–Vlasov equation,

$$\frac{\partial f}{\partial t_c} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\bar{e}c^2}{m}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} = 0. \tag{13.58}$$

Let us consider another inertial frame F' in which the speed of light c' is not a constant, $\dot{C}=\dot{c}'\neq 0$. To be specific, (13.55) is written in terms of primed quantities measured by observers in the F' frame,

$$\frac{m\gamma'}{c'}\left[\mathbf{a}' + \gamma'^2\frac{\mathbf{v}'}{c'^2}(\mathbf{v}'\cdot\mathbf{a}')\right] - m\gamma'^3\frac{\mathbf{v}'a_0'}{c'^2} = \bar{e}(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}'). \tag{13.59}$$

The quantity $a_0' = \dot{c}'$ in F' can be solved by multiplying (13.59) with $\mathbf{v}'$. We obtain

$$a_0' = \frac{c'}{\beta'^2}\left\{\frac{\mathbf{v}'\cdot\mathbf{a}'}{c'^2} - \frac{\bar{e}}{m\gamma'^3}\mathbf{v}'\cdot\mathbf{E}'\right\}, \qquad \boldsymbol{\beta}' = \frac{\mathbf{v}'}{c'}, \qquad \gamma' = \frac{1}{\sqrt{1-v'^2/c'^2}}. \tag{13.60}$$

Substituting (13.60) into (13.59), one has

$$\frac{m\gamma'}{c'}\left[\mathbf{a}' + \frac{\mathbf{v}'}{\beta'^2c'^2}(\mathbf{v}'\cdot\mathbf{a}')\right] - \frac{\bar{e}\mathbf{v}'}{\beta'^2c'}(\mathbf{E}'\cdot\mathbf{v}') = \bar{e}(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}'). \tag{13.61}$$

One can solve for the acceleration $\mathbf{a}'$ in terms of $\mathbf{E}'$ and $\mathbf{B}'$ by letting $\mathbf{a}' = Y_1(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}') + Y_2\mathbf{v}'(\mathbf{v}'\cdot\mathbf{E}')$ and using (13.61) to determine Y_1 and Y_2. One finds that

$$a_0' = \left\{\frac{\bar{e}c'}{m\gamma'} + Y_2\right\}\mathbf{v}'\cdot\mathbf{E}', \qquad \text{and}$$

$$\mathbf{a}' = \frac{\bar{e}c'}{m\gamma'}\left\{c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}' + Y_2\frac{\mathbf{v}'}{c'}(\mathbf{E}'\cdot\mathbf{v}')\right\}, \tag{13.62}$$

where the function Y_2 cannot be determined because we have four unknown variables and there are only three independent equations of motion. Fortunately, the 4–acceleration a^μ in F and a'^μ in F' must be related by the following 4–dimensional transformation:

$$a_0 = \gamma(a'_0 + \beta a'_x), \quad a_x = \gamma(a'_x + \beta a'_0), \quad a_y = a'_y, \quad a_z = a'_z. \tag{13.63}$$

For simplicity, let us consider the y–component in (13.63). From (13.56) and (13.62), we have

$$\frac{\bar{e}c}{m\gamma}(c\mathbf{E} + \mathbf{v}\times\mathbf{B})_y + \frac{\bar{e}}{m\gamma}v_y(\mathbf{v}\cdot\mathbf{E})$$

$$= \frac{\bar{e}c'}{m\gamma'}\left\{(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}')_y + Y_2\frac{v'_y}{c'}(\mathbf{E}'\cdot\mathbf{v}')\right\}. \tag{13.64}$$

Based on the transformations for **E**, **B** and v^μ in equations (12.33) and (12.9), one can show that

$$\frac{\bar{e}c}{m\gamma} = \frac{\bar{e}}{m}\sqrt{c^2 - v^2} = \frac{\bar{e}}{m}\sqrt{c'^2 - v'^2} = \frac{\bar{e}c'}{m\gamma'},$$

$$(c\mathbf{E} + \mathbf{v}\times\mathbf{B})_y = (c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}')_y, \quad \text{and} \quad v_y = v'_y. \tag{13.65}$$

It follows from (13.64) and (13.65) that

$$Y_2 = -\frac{\bar{e}c}{m\gamma}\,\frac{(\mathbf{E}\cdot\mathbf{v})/c}{(\mathbf{E}'\cdot\mathbf{v}')/c'}. \tag{13.66}$$

If one wishes, one may express c/γ and $(\mathbf{E}.\mathbf{v})/c$ in (13.66) in terms of quantities measured by observers in the F' frame with the help of the transformation properties of $(\mathbf{E}\cdot\mathbf{v})/c$ (or the 4–vector in (13.73) below) and the invariant $\sqrt{c^2-v^2} = \sqrt{c'^2-v'^2}$. Note that this function Y_2 can also be obtained from other relations in the transformation (13.63) for the 4–acceleration a^μ in common relativity.

Thus, the 4–acceleration a'^μ is completely determined by (13.62) and (13.66),

$$a'_0 = \dot{c}' = \frac{\bar{e}c'}{m\gamma'}\left\{\mathbf{v}'\cdot\mathbf{E}' - c'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right\},$$

$$\mathbf{a}' = \frac{\bar{e}c'}{m\gamma'}\left\{(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}') - \mathbf{v}'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right\}. \tag{13.67}$$

Thus, the generalized Boltzmann–Vlasov equation (13.51) in the F' frame can be written as

$$\frac{\partial f'}{\partial t_c} + \mathbf{v}'\cdot\frac{\partial f'}{\partial \mathbf{r}'} + \frac{\bar{e}c'}{m\gamma'}\left\{\mathbf{v}'\cdot\mathbf{E}' - c'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right\}\frac{\partial f'}{\partial c'}$$

$$+\frac{\bar{e}c'}{m\gamma'}\left\{(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}') - \frac{\bar{e}}{m\gamma'}\mathbf{v}'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right\}\cdot\frac{\partial f'}{\partial \mathbf{v}'} = 0, \qquad (13.68)$$

where $f'=f(b't_c,\mathbf{r}',c',\mathbf{v}')$. Since $(\mathbf{v}\cdot\mathbf{E},c\mathbf{E}+\mathbf{v}\times\mathbf{B})$ and $(c,\mathbf{v})$ are 4–vectors, the quantity $(\mathbf{v}\cdot\mathbf{E})/c$ in (13.68) can be expressed in terms of the primed quantities $\mathbf{v}'$, $\mathbf{E}'$, $\mathbf{B}'$ and c' measured by observers in F' by using transformations (12.9) and (12.33). Nevertheless, it is convenient to leave the unprimed quantity in (13.67) and (13.68). Using the velocity transformation (12.9), one can also show that the accelerations $a^\mu=(0,\mathbf{a})$ in (13.56) and a'^μ in (13.67) are related by the 4–dimensional transformation.

Furthermore, (13.56) and (13.67) give

$$a_0^2 - \mathbf{a}^2 = -\left\{\frac{\bar{e}c}{m\gamma}(c\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \frac{\bar{e}}{m\gamma}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\right\}^2$$

$$= -\left(\frac{\bar{e}c}{m\gamma}\right)^2\left[(c\mathbf{E} + \mathbf{v}\times\mathbf{B})^2 - (\mathbf{v}\cdot\mathbf{E})^2 - \frac{(\mathbf{v}\cdot\mathbf{E})^2}{\gamma^2}\right\}, \qquad (13.69)$$

where $a_0=0$, $\gamma=1/\sqrt{1-v^2/c^2}$, and

$$a_0'^2 - \mathbf{a}'^2 = \left(\frac{\bar{e}c'}{m\gamma'}\right)^2\left\{\left[\mathbf{v}'\cdot\mathbf{E}' - c'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right]^2 - \left[(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}') - \mathbf{v}'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right]^2\right\},$$

$$= -\left(\frac{\bar{e}c}{m\gamma}\right)^2\left\{(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}')^2 - \frac{(\mathbf{v}\cdot\mathbf{E})^2}{\gamma^2} - (\mathbf{v}'\cdot\mathbf{E}')^2\right\}, \qquad (13.70)$$

where we have used the invariant relation $c/\gamma = c'/\gamma'$. Note that the unprimed terms $(\mathbf{v}\cdot\mathbf{E})^2/\gamma^2$ in (13.69) and (13.70) are the same. Thus, the invariant relation

$$a_0^2 - \mathbf{a}^2 = -\mathbf{a}^2 = a_0'^2 - \mathbf{a}'^2 \qquad (13.71)$$

is equivalent to the invariant relation

$$(cE + v \times B)^2 - (v \cdot E)^2 = (c'E' + v' \times B')^2 - (v' \cdot E')^2. \qquad (13.72)$$

This is not surprising because

$$\frac{dp^\mu}{dt_c} = \bar{e}F^{\mu\nu}\frac{dx_\nu}{dt_c} = \bar{e}\Big((v \cdot E), (cE + v \times B)\Big) \qquad (13.73)$$

is a 4-vector in common relativity. It is stressed that the acceleration is the quantity with the simplest physical properties in common relativity because of the presence of the common time t_c and the 4-dimensional symmetry.

13g. Boltzmann's Transport Equation with 4-Dimensional Symmetry

For the classical kinetic theory of dilute gases for a system of N molecules enclosed in a box of volume V_I, we are interested in the invariant distribution function f under a 4-dimensional transformation. Let us define the distribution f as a function of the coordinates and momenta, $(x^\mu, p^\nu) \equiv (x,p)$, instead of a function of the coordinates and velocities (x^μ, v^ν). Such a definition of f(x,p) will make the 4-dimensional symmetry more explicit, as we shall see below. In a general inertial frame, the invariant volume element in the μ-space given by (13.15) can be rewritten as

$$d\mu = d^4x\, \delta(\frac{x^0}{b} - t_c)\, d^4p\, \delta(p_\lambda{}^2 - m^2)\, \theta(p^0)(2G)$$

$$= d^3x dx^0\, \delta(\frac{x^0}{b} - t_c) d^3p dp^0\, \delta(p_\lambda{}^2 - m^2)\, \theta(p^0)\Big(\frac{2p^0}{C}\Big).$$

The distribution function f(x,p) is defined so that classically, the number of molecules dN in a volume element d^3x about **r** and within a momentum space volume element d^3p around **p** at common time t_c is given by

$$dN = \int_{[x^0 p^0]} f(x,p) d\mu = \int f(x,p) dV_I\, d^3p \frac{C_0}{C}, \qquad (13.74)$$

$$dV_I = \int_{[x^0]} \frac{d^4x}{C_0}\, \delta\Big(t_c - \frac{x^0}{b}\Big), \qquad (13.75)$$

$$d^3p\frac{c_0}{C} = \frac{d^3p}{2p^0}(2Gc_0), \qquad p^0 = \sqrt{p^2+m^2}, \tag{13.76}$$

where integrations over x^0 and p^0 have been carried out in (13.74) and c_0 is a scaling constant defined in (13.1) to give the correct dimensions for dV_I .

Note that both dV_I and $d^3p(c_0/C)$ are invariant in common relativity and that they are large enough so that dN is a very large number ($\sim 10^{10}$) of molecules. Nevertheless, the volume elements dV_I and $d^3p(c_0/C)$ are very much smaller than macroscopic dimensions. Also, if the distribution f(x,p) is independent of the position **r**, we have the usual relation, $\int f(x,p)d^3p = N/V$, in the inertial frame $F(ct_c,\mathbf{r})$ in which $c_0=c=C$ and $\int dV_I = V_I$, where V_I is the volume of a box. The distribution f(x,p) can be identified with D_1 in (13.28) when D_1 does not depend on t_c explicitly, $f(x,p)=D_1(x,p)$.

Let us consider equation (13.28) with the collision term P(D), where we use D to denote the one-particle distribution function for the following discussions,

$$\left(\frac{\partial}{\partial t_c} + v^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}\right)D = P(D), \qquad D = D(x,p,t_c), \tag{13.77}$$

$$v^\mu = \frac{dx^\mu}{dt_c} = (C,\mathbf{v}), \qquad F^\mu = \frac{dp^\mu}{dt_c} = \left(\frac{dp^0}{dt_c}, \frac{d\mathbf{p}}{dt_c}\right),$$

where the collision term $P(D)=(\partial D/\partial t_c)_{coll}$ must be specified for (13.77) to be meaningful. In order to have an invariant expression for the collision term P(D), we use quantum mechanics to treat the scattering process. Although the molecules are regarded as classical objects, "they see each other as plane waves of definite momenta rather than wave packets of well-defined positions"[6] in scattering processes. By considering the collision process, $p_1+p_2 \to p'_1+p'_2$,[6] one can express $(\partial D/\partial t_c)_{coll} = (\partial D_1/\partial t_c)_{coll}$ in the invariant form,

$$P(D_1) = \left(\frac{D_1}{t_c}\right)_{coll} = \int \frac{d^3p_2}{2p_{20}}\frac{d^3p'_1}{2p'_{1o}}\frac{d^3p'_2}{2p'_{2o}}\,\delta^4(p_1+p_2-p'_1-p'_2\ )$$

$$\times |M_{fi}|^2(D'_1 D'_2 - D_1 D_2), \tag{13.78}$$

where $D_i=D(x,p_i,t_c)$ and $D'_i=D(x,p'_i,t_c)$, i=1,2. From equations (13.77) and (13.78), we have the 4-dimensional Boltzmann transport equation based on common relativity:

$$\left(\frac{\partial}{\partial t_c} + C\frac{\partial}{\partial(bt_c)} + \mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{dp_1{}^0}{dt_c}\frac{\partial}{\partial p_1{}^0} + \frac{d\mathbf{p}_1}{dt_c}\cdot\frac{\partial}{\partial \mathbf{p}_1}\right)D_1$$

$$= \int \frac{d^3p_2}{2p_{20}}\frac{d^3p_1'}{2p_{10}'}\frac{d^3p_2'}{2p_{20}'}\,\delta^4(p_1+p_2-p_1'-p_2')|M_{fi}|^2(D_1'D_2' - D_1D_2)\,. \qquad (13.79)$$

In the inertial frame $F(ct_c,\mathbf{r})$ in which $b=c=C$, if the distribution function D_1 is a function of x and p only,[7] then the LHS of (13.79) reduces to a more familiar form

$$\frac{\partial D_1}{\partial t_c} + \mathbf{v}_1\cdot\frac{\partial D_1}{\partial \mathbf{r}} + \frac{dp_1{}^0}{dt_c}\frac{\partial D_1}{\partial p_1{}^0} + \frac{d\mathbf{p}_1}{dt_c}\cdot\frac{\partial D_1}{\partial \mathbf{p}_1}\,, \qquad D_1=D_1(x,p)\,. \qquad (13.80)$$

Note that the third and fourth terms are not independent because of the relation $p_1{}^0=\sqrt{\mathbf{p}_1{}^2+m^2}$. However, it is more convenient to leave Boltzmann's transport equation in the form (13.80) [or the LHS of (13.79)] to have an explicit 4-dimensional form. We also note that $(dp_1{}^0/dt_c)$ is smaller then $|d\mathbf{p}_1/dt_c|$ by a factor v/c, as one can see from (13.73). Thus, if the velocity $v=|\mathbf{v}|$ is much smaller than c, (13.80) reduces to the usual form:

$$\frac{\partial D_1}{\partial t_c} + \frac{\mathbf{p}_1}{m}\cdot\frac{\partial D_1}{\partial \mathbf{r}} + \frac{d\mathbf{p}_1}{dt_c}\cdot\frac{\partial D_1}{\partial \mathbf{p}_1}\,. \qquad (13.81)$$

In this connection, it is worthwhile to note that if the Boltzmann–Vlasov equation for high temperature plasma is expressed in terms of the distribution function $D_1(x,p)$, we have

$$\frac{\partial D_1}{\partial(bt_c)} + \frac{\mathbf{p}}{p_o}\cdot\frac{\partial D_1}{\partial \mathbf{r}} + \bar{e}\frac{\mathbf{E}\cdot\mathbf{p}}{p^0}\frac{\partial D_1}{\partial p^0} + \bar{e}\left(\mathbf{E} + \frac{\mathbf{p}\times\mathbf{B}}{p_o}\right)\cdot\frac{\partial D_1}{\partial \mathbf{p}} = 0, \qquad (13.82)$$

in a general inertial frame, where we have used the following relations

$$p_o = \sqrt{\mathbf{p}^2+m^2} = \frac{m}{\sqrt{1-v^2/C^2}}, \quad \mathbf{p} = \frac{m\mathbf{v}/C}{\sqrt{1-v^2/C^2}}, \quad \frac{\mathbf{v}}{C} = \frac{\mathbf{p}}{p_o}, \qquad (13.83)$$

$$\frac{1}{C}\frac{dp_0}{dt_c} = \frac{\bar{e}\mathbf{E}\cdot\mathbf{v}}{C} = \bar{e}\frac{\mathbf{E}\cdot\mathbf{p}}{p^0}, \quad \frac{1}{C}\frac{d\mathbf{p}}{dt_c} = \bar{e}\left(\mathbf{E} + \frac{\mathbf{p}\times\mathbf{B}}{p_0}\right). \tag{13.84}$$

13h. Boltzmann's H Theorem with 4-Dimensional Symmetry

In common relativity, the invariant Boltzmann functional $H(t_c)$ can be defined in terms of the invariant distribution $D(p^\mu,t_c)\equiv D(p,t_c)$,

$$H(t_c) = \int D(p_1,t_c)\ln D(p_1,t_c)\,\frac{d^3p_1}{2p_{10}}, \qquad p_{10} = \sqrt{\mathbf{p}_1{}^2+m^2}, \tag{13.85}$$

where the invariant distribution D is assumed to be a function of the 4-momentum $p_1{}^\mu$ and common time t_c and which satisfies[8]

$$\left(\frac{\partial D_1}{\partial t_c}\right) = \int \frac{d^3p_2}{2p_{20}}\,\frac{d^3p_1'}{2p_{10}'}\,\frac{d^3p_2'}{2p_{20}'}\,\delta^4(p_1+p_2-p_1'-p_2')$$

$$\times|M_{fi}|^2(D_1'D_2' - D_1D_2), \qquad D_1 = D(p_1,t_c). \tag{13.86}$$

Note that both the distribution D_1 and $d^3p_1/(2p_{10})$ are invariant. It follows from (13.85) and (13.86) that

$$\frac{dH(t_c)}{dt_c} = \int \frac{d^3p_1}{2p_{10}}\,\frac{d^3p_2}{2p_{20}}\,\frac{d^3p_1'}{2p_{10}'}\,\frac{d^3p_2'}{2p_{20}'}\,\delta^4(p_1+p_2-p_1'-p_2')$$

$$\times|M_{fi}|^2(D_1'D_2' - D_1D_2)(1 + \ln D_1). \tag{13.87}$$

Note that the transition matrix M_{fi} is invariant under the interchange of $\mathbf{p}_1$ and $\mathbf{p}_2$, so that the integrand in (13.87) is unchanged under $\mathbf{p}_1\leftrightarrow\mathbf{p}_2$. From (13.87) and the new expression obtained by $\mathbf{p}_1\leftrightarrow\mathbf{p}_2$, we have

$$\frac{dH(t_c)}{dt_c} = \int \frac{d^3p_1}{2p_{10}}\,\frac{d^3p_2}{2p_{20}}\,\frac{d^3p_1'}{2p_{10}'}\,\frac{d^3p_2'}{2p_{20}'}\,\delta^4(p_1+p_2-p_1'-p_2')$$

$$\times|M_{fi}|^2(D_1'D_2' - D_1D_2)\left[1 + \frac{1}{2}\ln(D_1D_2)\right]. \tag{13.88}$$

For every scattering there is an inverse scattering with the same transition matrix M_{fi}. Thus, the integral (13.88) is invariant under the interchange $\{\mathbf{p}_1,\mathbf{p}_2\} \leftrightarrow \{\mathbf{p}'_1,\mathbf{p}'_2\}$,

$$\frac{dH(t_c)}{dt_c} = -\int \frac{d^3p_1}{2p_{10}} \frac{d^3p_2}{2p_{20}} \frac{d^3p'_1}{2p'_{10}} \frac{d^3p'_2}{2p'_{20}} \delta^4(p_1+p_2-p'_1-p'_2)$$

$$\times|M_{fi}|^2(D'_1D'_2 - D_1D_2)[1 + \frac{1}{2} \ln (D'_1D'_2)] . \qquad (13.89)$$

It follows from (13.88) and (13.89) that

$$\frac{dH(t_c)}{dt_c} = \frac{1}{4}\int \frac{d^3p_1}{2p_{10}} \frac{d^3p_2}{2p_{20}} \frac{d^3p'_1}{2p'_{10}} \frac{d^3p'_2}{2p'_{20}} \delta^4(p_1+p_2-p'_1-p'_2)$$

$$\times|M_{fi}|^2(D'_1D'_2 - D_1D_2)[\ln (D_1D_2) - \ln (D'_1D'_2)] . \qquad (13.90)$$

Since the integrand of this equation is never negative, we have

$$\frac{dH(t_c)}{dt_c} \leq 0 \qquad (13.91)$$

in all inertial frames.

We observe that equation (13.91) vanishes, i.e., $dH(t_c)/dt_c=0$, if and only if the integrand of (13.90) vanishes identically. In other words, the condition $dH(t_c)/dt_c=0$ is the same as

$$D_0(\mathbf{p}'_2)D_0(\mathbf{p}'_1) = D_0(\mathbf{p}_2)D_0(\mathbf{p}_1) , \qquad (13.92)$$

where $D_0(\mathbf{p})$ denotes the equilibrium distribution. This has an interesting implication concerning the explicit form of the invariant equilibrium (or Maxwell-Boltzmann) distribution $D_0(\mathbf{p})$. Equation (13.92) leads to

$$\ln D_0(\mathbf{p}_1) + \ln D_0(\mathbf{p}_2) = \ln D_0(\mathbf{p}'_1) + \ln D_0(\mathbf{p}'_2) \qquad (13.93)$$

for any possible collision process, $p_1+p_2 \rightarrow p'_1+p'_2$. Thus ,the relation (13.93) can be interpreted as a conservation law. Since the function $\ln D_0(\mathbf{p})$ is a scalar in common relativity, the only quantity which is both a scalar and conserved is the genergy $G(\mathbf{p})$. In the F frame, in which C=c, we have $G(\mathbf{p})=p_0/c=\sqrt{\mathbf{p}^2+m^2}/c$.

Clearly, the conservation of the genergy,

$$G(\mathbf{p}_1) + G(\mathbf{p}_2) = G(\mathbf{p}'_1) + G(\mathbf{p}'_2), \tag{13.94}$$

is the same as the conservation of the energy p_0 in the F frame. Since the genergy $G(\mathbf{p}_1)=p_{10}/C_1$ is an invariant, the relation (13.94) holds for all inertial frames. Thus, the only solution of (13.93) is

$$\ln D_0(\mathbf{p}) = -AG(\mathbf{p}) + \ln B, \quad \text{or} \qquad D_0(\mathbf{p}) = B\exp[-AG(\mathbf{p})], \tag{13.95}$$

where A and B are arbitrary constants which can be determined by observing the physical properties of a system.

In the literature, most of the formalisms on relativistic statistical mechanics are not manifestly covariant. Some of them are, strictly speaking, not covariant because of the approximations involved. Some formalisms lack proof of their effective covariance.[2]

We have demonstrated some novel and interesting features of the 4-dimensional symmetry framework of common relativity. This framework has a considerable advantage over the usual spacetime framework with regard to the formulation of statistical mechanics within a 4-dimensional symmetry framework. The existence of a scalar common time t_c, the 4-coordinate (bt_c,x,y,z), and the genergy G in common relativity is the key to preserving the invariant phase space of initial particle positions and velocities, the concept of a Hamiltonian system with many degrees of freedom, the Liouville equation and the invariant Maxwell-Boltzmann distribution with a scalar temperature.

References

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7. In this case $D_1=D_1(x,p)$, the 4–dimensional transport equation (13.79) or (13.80) can be used to derive the conservation theorem associated with the distribution function. (See ref. 5.) For example, one can show that

$$\int \frac{d^3p}{2p_0}\, Z(x,p)\left(\frac{D}{t_c}\right)_{coll} = 0,$$

and

$$\int \frac{d^3p}{2p_0}\, Z(x,p)\left(v^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}\right) D(x,p) = 0,$$

where $(\partial D/\partial t_c)_{coll}$ is given by (13.78), and $Z(x,p)$ denotes the conserved quantity associated with the collision process $p_1+p_2 \rightarrow p_1'+p_2'$, i.e., $Z(x,p_1)+Z(x,p_2)=Z(x,p_1')+Z(x,p_2')$. These results are useful for deriving conservation laws of mass, momentum, etc. in hydrodynamics in a 4–dimensional symmetry framework.
8. This is equivalent to assuming that the distribution function $D(p_1,t_c)$ in a general inertial frame is independent of the 4–coordinate x^μ and satisfies Boltzmann's transport equation (13.28) without an external force, $F^\mu=0$, and with the collision term $P(D_1)$ given by (13.78).

14.

Common Relativity and Quantum Mechanics

14a. Fuzziness at Short Distances and the Invariant 'Genergy'

In quantum mechanics, there is a problem related to the non-square-integrable coordinate representation because such a representation is incompatible with the usual probabilistic interpretation. This problem suggests the need for introducing *fuzzy coordinates* to modify physics at short distances.[1] Furthermore, the introduction of fuzzy coordinates is also motivated by the problem of locality and the related ultraviolet divergence in quantum field theories. Fuzzy coordinates may be significant because an inherent fuzziness at short distances is characterized by a radical length R. The radical length R is related to the concept of a fundamental length R_D in quantum field theory which has been discussed long time ago by Heisenberg and by Dirac.[2] Such an inherent fuzziness is also related to Feynman's idea[3] of a basic width for a modified δ-function for interactions. In 1949, Dirac made the following comments:

> "Presentday atomic theories involve the assumption of localizability, which is sufficient but is very likely too stringent ··· A less drastic assumption may be adequate, e.g., that there is a fundamental length λ such that the Poisson bracket of two dynamical variables must vanish if they are localized at two points whose separation is space-like and greater than λ, but need not vanish if it is less than λ."

Schwinger,[4] Feynman[5] and Wigner[6] also had a similar view.

Common relativity has a new invariant $G(\mathbf{p})=p^0/C=\sqrt{\mathbf{p}^2+m^2}/C$, which is called the "genergy." In the F frame in which the speed of light is, by definition, a constant, i.e., C=c, the genergy $G(\mathbf{p})$ is essentially the same as the conserved quantity, energy, $p^0=\sqrt{\mathbf{p}^2+m^2}$. This invariant genergy $G(\mathbf{p})$ can help a theory to realize the inherent fuzziness at short distances without upsetting the 4-dimensional symmetry within the framework of common relativity. Note that the genergy $G(\mathbf{p})$ is clearly not an invariant quantity in special relativity.

Thus, the presence of the genergy G(**p**) allows a possible departure of the 4–dimensional symmetry of special relativity at short distances. This is but one of the examples which show that there may be real physical differences between common relativity and special relativity.

First, let us show the invariance of the genergy G by direct calculations based on the 4–dimensional space–lightime transformations:[1] The 4–momentum p^μ in (13.7) has the same transformation property as the 4–coordinate x^μ (or dx^μ) in (12.4) and, hence, satisfies the following transformations:

$$p'^0 = \gamma(p^0 - \beta p_x), \quad p'_x = \gamma(p_x - \beta p^0), \quad p'_y = p_y, \quad p'_z = p_z, \tag{14.1}$$

where the spatial component of the momentum is $\mathbf{p}=(p_x,p_y,p_z)=(p^1,p^2,p^3)$. It follows from (14.1) and the velocity transformations (12.9) that the ratio p^0/c is invariant under the 4–dimensional transformations:

$$\frac{p'^0}{c'} = \frac{p^0}{c} = G(\mathbf{p}), \quad p'^0 = \sqrt{\mathbf{p}'^2+m^2}, \qquad p^0 = \sqrt{\mathbf{p}^2+m^2}. \tag{14.2}$$

Note that the genergy G(**p**) involves the spatial momentum vector **p**. This invariance is quite interesting because it suggests an invariant function involving **p** and a new universal constant I which can be related to a radical (or fundamental) length. The constant I allows us to define a dimensionless function D(**p**) in terms of the scalar genergy G(**p**) associated with a physical particle with a mass m,

$$D(\mathbf{p}) = \frac{1}{1 + I^2G^2(\mathbf{p})}. \tag{14.3}$$

We have seen that the constant I has the dimension of length/(mass·time). In this way, D(**p**) is a pure number and may be related to a probability. The constant I can be related to a radical length R defined by the relation

$$R = \frac{IJ}{c_0}, \tag{14.4}$$

where c_0 is a scaling constant defined in (13.2). The presence of c_0 in (14.4) may appear to be artificial, but, there is a good physical reason for it:

The units of length and time in physics are arbitrary choices. However, the fundamental constant of physics should be independent of such an arbitrary choice of units. The electromagnetic coupling strength $\alpha_e \sim 1/137$ is a truly universal and fundamental constant which is a pure number and, hence, is independent of any choice of units. In contrast, the constant I in (14.3) has the dimension of length/(mass·time) which indicates that its value is implicitly affected by such arbitrary choices of units. The presence of c_0 in (14.4) is precisely to remove the unwanted effects due to such artificial and arbitrary choices of units for time and mass. Consequently, the radical length R in nature represents a basic physical quantity which is independent of the artificial choices of units for mass and time. Of course, the numerical value of R still depends on an artificial choice of units for length. However, it should be stressed that the radical length R, if exists, should be regarded as the unit of length *chosen by nature rather than by humans.* This has physical significances, as we shall see later. Note that special relativity cannot have the invariant relations (14.2), (14.3) and (14.4).

14b. Fuzzy Quantum Mechanics with an Inherent Fuzziness in the Position of a Point Particle

In quantum mechanics and quantum field theories, *basic physical objects are assumed to be point particles which may have a definite and precise position in space (or a definite momentum, but not both at the same time because of the uncertainty principle).* As a result, the Coulomb potential produced by a point electron takes the form $-e/(4\pi r)$, which diverges at r=0 and this is intimately related to the fundamental divergence difficulty in quantum electrodynamics and other field theories. Also, the point particle picture implies that a quantum particle can have a position eigenstate $|q\rangle$ of a position operator Q;

$$Q|q\rangle = q|q\rangle . \tag{14.5}$$

Therefore, in principle, the position q of a quantum particle can be measured precisely, i.e.,

$$\Delta q_{min} = 0 . \tag{14.6}$$

However, the properties (14.5) and (14.6) do not have operational meanings because they cannot be realized. Furthermore, the position eigenstate (14.5) is, strictly speaking, not physically meaningful because its wave function is not square-integrable and, hence, cannot be normalized to have an elementary probabilistic interpretation. Thus, in quantum mechanics, one gives up the fundamental idea of a probabilistic interpretation of a state for the sake of introducing certain mathematical idealizations of what can be realized. This is not in harmony with the requirement of operational meaning in physics which states that one should formulate a physical theory using observable quantities and realizable states.

On the other hand, if one attempts to avoid the above difficulties by assuming the particle to have a finite size, then one immediately encounters many other difficult problems related to non-local interactions, non-local wave functions and so on.

One way to depart from the conventional point particle picture without these difficulties is as follows: We assume that the particle by itself has no size and no structure, but that its position cannot be measured with unlimited accuracy:[1]

$$\Delta q_{min} = R > 0, \tag{14.7}$$

where R is a very small radical length. The postulate (14.7) enables us to avoid unknown tacit assumptions such as indefinitely short distances in space.[5] This suggests a picture of a fuzzy-point particle, which closely resembles a fuzzy point in the fuzzy set theory of Zadeh.[7]

According to postulate (14.7), we replace the improperly idealized state $\langle q|$ with zero width by a new base state $(q|$, which has the width R. To accomplish this, it is convenient to use Klauder's continuous representation[8] of Hilbert space to express the new base $(q|$:

$$(q| = \int_{-\infty}^{\infty} dp \langle p| D(p) e^{ipq/J} \frac{1}{\sqrt{2\pi J}}, \tag{14.8}$$

where $\langle p|$ is the usual momentum eigenstate. In order to preserve known properties of a physical state at low momenta, the new function D(p) must satisfy

$$D(p) \sim \begin{cases} 1 & |p| \ll J/R \; ; \\ 0 & |p| \gg J/R \; . \end{cases} \tag{14.9}$$

Similarly, we can modify the usual momentum eigenstate $\langle p|$ to have an inherent fuzziness in the momentum state $(p|$ which has the minimum uncertainty,

$$\Delta p_{min} = \frac{J}{S} > 0, \tag{14.10}$$

where p and S have the dimension of mass and length, respectively. To be consistent with known physical states at low momenta, the length scale S should be extremely large. Presumably, it is related to the size of the observable physical universe: $S \geq 10^{10}$ light years. For simplicity, we shall postulate that the physical effects of a finite S are too small to be detected experimentally. Thus, we shall take the limit that S approaches infinity so that

$$(p| = \langle p| , \tag{14.11}$$

and concentrate on the new physical properties related to the finiteness of the radical length $R>0$.

Let us consider fuzzy base states. In the limit $S \to \infty$, base momentum states $(p|$ and $|p)$ have the usual orthogonal and completeness relations and so on,

$$(p|p') = \delta(p - p') , \qquad \int_{-\infty}^{\infty} dp|p)(p| = 1 ,$$

$$(p|P = (p|p , \qquad (p|Q = iJ \frac{\partial}{\partial p} (p| , \tag{14.12}$$

where Q and P are Hermitian operators for positions and momenta. On the other hand, since $R>0$, the position vector $(q|$, its dual vector $|q)$ and the Hermitian operators Q and P satisfy

$$(q|q') = (q'|q)^* = (2\pi J)^{-1} \int_{-\infty}^{\infty} dp D^2(p)\, e^{i(q-q')p/J} = D^2(\partial)\, \delta(q-q') , \tag{14.13}$$

$$\partial \equiv -iJ\frac{\partial}{\partial q},$$

$$(q|P = -iJ\frac{\partial}{\partial q}(q| , \qquad P|q) = iJ\frac{\partial}{\partial q}|q) , \qquad \int_{-\infty}^{\infty} dq|q)(q| = D^2(P) , \tag{14.14}$$

$$\int_{-\infty}^{\infty} dq[D^{-1}(\partial)|q)][D^{-1}(\partial)(q|] = \int_{-\infty}^{\infty} dq|q)[D^{-2}(\partial)(q|] = 1 , \tag{14.15}$$

$$(q|Q = (q|\,[q - iJ\frac{\partial \ln D(P)}{\partial P}] , \qquad Q|q) = [q + iJ\frac{\partial \ln D(P)}{\partial P}]\,|q) , \tag{14.16}$$

where $D^2(\partial)$ and $D^{-1}(\partial)$ are integral operators given by[9]

$$f(\partial)\phi(q) = (2\pi J)^{-1}\int_{-\infty}^{\infty} dp e^{ipq/J} f(p) \int_{-\infty}^{\infty} dq' e^{-ipq'/J}\phi(q') . \tag{14.17}$$

We have seen that the new continuous base states $(q|$ defined in (14.8) satisfy minimum requirements of coherent states: $(q|$ is a strongly continuous function of q and satisfies the relation (14.15). Nevertheless, the most important physical property of $(q|$ is given by (14.16), (14.13), (14.7): Namely, the position operator Q has neither eigenstates nor eigenvalues. It has only a fuzzy value with an uncertainty $\Delta q \geq R$. In other words, a "point particle" (i.e., no size and no structure), if measured, will never be found at one and only one point q at a time. Thus, it appears more appropriate to term $(q|$ and Q *fuzzy states* and *fuzzy dynamical variables*, respectively. (In principle, both P and Q should be fuzzy dynamical variables if the length scale S is treated as a finite quantity.) We may remark that $(q|$ is formally an eigenstate of the operator

$$Q_0 = Q + iJ\frac{\partial \ln D(P)}{\partial P},$$

with the eigenvalue q. However, such an operator Q_0 has no physical meaning in the present formalism of fuzzy quantum mechanics.

The fuzziness of the base state $(q|$ is completely determined by the function D(p). Let us consider a plausible argument for the function D(p) to have the following form:

$$D(p) = \frac{1}{a^2p^2 + 1}, \qquad a^2 = \frac{2R^2}{J^2}. \tag{14.18}$$

Classically, one can determine a particle's position by confining the particle in a certain range Δx by an attractive square-well potential. If the potential is narrower and deeper, we can determine the position more accurately. In the limiting case, we have a δ-function potential, $-V_0\delta(x-x_0)$, and the classical particle is precisely located at the point x_0. However, a quantum particle is described by the Schrödinger (or Klein-Gordon) equation with the form

$$\left[-\frac{J^2}{2m}\frac{d^2}{dx^2} - V_0\delta(x-x_0)\right]\phi(x) = E\,\phi(x), \tag{14.19}$$

rather than the classical Newtonian equation. The solution to (14.19) has the form

$$\phi(x) = A\exp\left[-\frac{|x-x_0|}{\sqrt{2}\,R}\right], \qquad R = \frac{J^2}{\sqrt{2}\,mV_0}, \tag{14.20}$$

which shows that the position of the particle is fuzzy (with the uncertainty $\Delta x \sim R$). The Fourier transform of the solution (14.20) leads to

$$\int_{-\infty}^{\infty} A\exp\left[-\frac{|x-x_0|}{\sqrt{2}\,R}\right]\exp\left[-\frac{i(x-x_0)p}{J^2}\right]dx = \frac{2\sqrt{2}AR}{p^2R^2/J^2 + 1}, \tag{14.21}$$

which has the same form as that in (14.18). In light of this, it is reasonable to use D(p) in (14.18) to describe the fuzzy base state $(q|$ in (14.8) for a quantum particle. It appears that the value R can only be determined by future experiments. So far, there is no compelling reason to identify R with the Planck length $\sim 10^{-33}$ cm or any other known length in physics.

According to the "probability axiom" of quantum mechanics, the physical base states associated with the dynamical variables Q and P must have a probabilistic meaning. Thus, these states should satisfy Klauder's postulates of continuous representation rather than satisfy the usual eigenbras and eigenkets. One might think that this is nothing new because one can always transform the fuzzy state (q| in (14.8) to the eigenstate ⟨q| by a nonunitary transformation. However, this is not true: Under such a nonunitary transformation, which preserves $PQ-QP=-iJ$, the position operator Q and the states (q| and |q) become

$$Q \to Q' = D(P)QD^{-1}(P) = Q - iJ\frac{\partial \ln D(P)}{\partial P}, \tag{14.22}$$

$$(q| \to \{q| = (q|D^{-1}(P) = \langle q| , \qquad |q) \to |q\} = D(P)|q) \neq |q\rangle \ . \tag{14.23}$$

Thus we have seen that the expectation value of the coordinate Q is not changed

$$(q_1|Q|q_2) = \{q_1|Q'|q_2\} \neq \langle q_1|Q|q_2\rangle , \tag{14.24}$$

i.e., the fuzzy feature of coordinates is not changed by such a nonunitary transformation.

We postulate such a fuzziness as a fundamental and inherent property of a particle's position. That is, no matter how one improves the technique and the apparatus, there will be an uncertainty $\Delta q \geq R$ associated with each measurement, even if the particle by itself has no size and structure. There emerges a new strange picture of the fuzzy point particle: Namely, *a particle is, at a given instant of time, partially located at one point, partially elsewhere and can never be completely at one point.* Of course, this is in harmony with Zadeh's original idea of fuzzy set theory[7] and in sharp contrast with both the classical and the conventional quantum-mechanical concept of a point particle.

14c. A Fuzzy Point and Modified Coulomb Potential at Short Distances

We have seen that the Klauder representation in Hilbert space naturally allows a basic length scale R to characterize continuous states. As long as $R>0$, our results indicate that space is continuous but fuzzy at short distances. The length scale R characterizes the smallest width of a wave packet that can be physically realized. Evidently, the fuzzy base vectors (q| form a submanifold in Hilbert space.

Since the radical length R seems to be so small that its effects have remained undetected, one wonders what type of effects an experimentalist should look for in the future. The fuzziness of the electron coordinates implies that the electron must have an "R-inherent charge distribution" given by

$$\rho_R(r) = -\frac{\bar{e}}{(2\pi J)^3}\int_{-\infty}^{\infty} D^2(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{r}/J}d^3p$$

$$= -\frac{\bar{e}}{10\sqrt{2}R^3}\exp[-r/(\sqrt{2}\,R)]\,, \tag{14.25}$$

$$D(\mathbf{p}) = \frac{1}{a^2\mathbf{p}^2 + 1}, \qquad a^2 = \frac{2R^2}{J^2},$$

where the form of $D(\mathbf{p})$ is assumed to be given by (14.18). Note that $\rho_R(r) \to -\bar{e}\delta^3(r)$ when R approaches 0, as it should. This R-inherent charge distribution is consistent with the fuzzy point picture of a particle. The modified Coulomb potential $V_R(r)$ produced by such a fuzzy point electron is given by

$$\int_{-\infty}^{\infty} d^3r' \frac{\rho_R(r')}{4\pi|\mathbf{r} - \mathbf{r}'|},$$

i.e.,

$$V_R(r) = -\bar{e}\int \frac{d^3k}{(2\pi)^3}\frac{D^2(k)}{k^2}e^{-i\mathbf{k}\cdot\mathbf{r}}, \qquad \mathbf{k} = \mathbf{p}/J\,,$$

$$= -\frac{\bar{e}}{4\pi r}\left[1 - \left(1 + \frac{r}{2\sqrt{2}R}\right)\exp\{-r/(\sqrt{2}R)\}\right]$$

$$= \begin{cases} -\bar{e}/(4\pi r) & r \gg R\,; \\ -\bar{e}/(8\sqrt{2}\pi R) & r \ll R\,; \end{cases} \tag{14.26}$$

which is finite at r=0. This suggests that the electromagnetic interaction will be asymptotically free at very high energies (or $p \gg \hbar/R$). The results (14.26) and (14.25) indicate that the presence of R should appear in, say, the e^+e^- scattering at very high energies:

$$e^- + e^+ \rightarrow e^- + e^+ \qquad \text{or} \qquad e^- + e^+ \rightarrow \mu^- + \mu^+ . \tag{14.27}$$

If one measures their differential cross sections carefully, a deviation from the usual quantum electrodynamics results can be attributed to the radical length R, provided that the electron is not a composite particle. The details of the deviation are related to the specific form of the function $D(\mathbf{p})$.

Of course, there is also the possibility that the constant R in Klauder's continuous representation is so small that it will forever elude detection. In this case, the present formalism is still advantageous to ordinary quantum mechanics because this formalism enables us to apply the probabilistic interpretation to all states of all observables regardless of whether they are discrete or continuous.

It is likely that Klauder's continuous representations for coordinates and momenta are not merely a matter of mathematical purism, but rather an inherent property of the physical world at very small and very large distances. Such representations are particularly interesting because they suggest a connection between the microscopic physical world and the new mathematics related to fuzzy set theory. Also, fuzziness at short distances is probably related to the solution of the divergence problem in field theories.

Note that it is highly nontrivial to accommodate the fuzziness of the coordinates and the usual 4-dimensional symmetry of spacetime in special relativity. If the fuzziness of Q is really fundamental, then it would imply that the usual 4-dimensional symmetry of special relativity becomes exact only at low energies or in the limit $R \rightarrow 0$.

14d. Inherent Probability for Suppression of Large Momentum States

Let us assume that the inherent fuzziness of the coordinate variables (or quantum particle's position) is a fundamental property of physics and is independent of the 4-dimensional symmetry of spacetime. We would expect that, apart from the usual relativistic time dilatation (or the dilatation of decay length), the lifetime of an unstable particle decay in flight with the momentum p will be further dilated due to the existence of the radical length R when $p \geq J/R$. (The new effect may be called 'radical dilatation.') Otherwise, the concept of quantum-mechanical probability is probably not really fundamental and could be modified in the future.

The new effect 'radical dilatation' for lifetimes of particles is related to the radical length R in the following way:

The fuzzy coordinate base state (14.8) leads to

$$(\mathbf{q}'|\mathbf{q}) = (2\pi J)^{-3} \int_{-\infty}^{\infty} D^2(\mathbf{p})\ e^{i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{p}/J} d^3p = D^2(\partial')\delta^3(\mathbf{q}'-\mathbf{q}), \quad (14.28)$$

where

$$D^2(\partial')f(\mathbf{q}') = (2\pi J)^{-3} \int_{-\infty}^{\infty} d^3p\ e^{i\mathbf{p}\cdot\mathbf{q}'/J}\, D^2(\mathbf{p}) \int_{-\infty}^{\infty} d^3q\ e^{-i\mathbf{p}\cdot\mathbf{q}/J} f(\mathbf{q}), \quad (14.29)$$

$$D^2(\mathbf{p}) = \frac{1}{[2R^2\mathbf{p}^2/J^2 + 1]^2}\ .$$

This shows the suppression of the momentum **p** by an "inherent probability" $D^2(\mathbf{p})$. This property can be properly implemented only in quantum field theory. This will be discussed in chapter 15. However, we still can see its partial effect in a physical process through an effective density of states, without invoking quantized fields. We first note that, mathematically, the momentum p can take any value between $-\infty$ and $+\infty$. However, the region in which the momentum of a particle can be physically realized is effectively finite. Thus, the three-dimensional momentum space appears to be "non-Euclidean" because one may picture the momentum space as having a volume element $D^2(\mathbf{p})d^3p$, as shown in (14.28). As a result, the number of one particle states in the range **p** and **p**+d**p** is given by the following "effective density of states,"

$$\left[\int d^3q\right]\frac{D^2(\mathbf{p})d^3p}{(2\pi J)^3} = [V]\frac{D^2(\mathbf{p})d^3p}{(2\pi J)^3}, \quad (14.30)$$

for a spin-zero particle. This result shows roughly the effect of suppression of large momentum contribution due to the new restriction (14.6) on the Heisenberg uncertainty relation.

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15.

Common Relativity and Fuzzy Quantum Field Theory

15a. Fuzzy Quantum Field Theories

It has been observed that finite quantum electrodynamics cannot be formulated consistently within the framework of special relativity. The reason is that limiting the magnitude of interactions while retaining the customary coordinate description is contradictory, since no mechanism is provided for precisely localized measurements.[1] This observation naturally suggests that the physical properties of space at very short distances are much different from those at macroscopic and atomic distances. The fact that no mechanism is provided for precisely localized measurements indicates that there is an inherent fuzziness at short distances and that the concept of a point-like particle, which works very well in the atomic and nuclear domains, must be modified so that it resembles some sort of non-point-like object. Spatially extended particles, fuzzy particles and string-like objects have all been discussed in the literature.[2] It is very difficult to make extended particles and string-like objects consistent with the 4-dimensional symmetry of the Lorentz and the Poincaré groups. So far, people have found strings which fit a symmetry framework of spacetime with 26 dimensions or 10 dimensions, but this has little relevance to present experiments and observations. Instead of trying to explain why a 10 or 26 dimensional spacetime has only 4 observable dimensions, if one can find a string-like object (no matter how strange it may be) which can exist mathematically in the 4-dimensional spacetime, then it would be a really surprising and important discovery.

On the other hand, one may ask: Is the 4-dimensional symmetry of Lorentz and Poincaré invariance so sacred that it must be absolutely preserved at all costs? A physical principle is sacred only as long as it is supported by experiments. However, experiments have limitations: Newton's law of motion is known to be good for the macroscopic world and small speeds and the Dirac equation is good for spatial regions where $\Delta r > 10^{-17}$ cm and for energies smaller than roughly 10^3 GeV. Nevertheless, whether physical principles are valid for describing the early universe or for particles when their energy reaches 10^{30}

GeV cannot be assured by anything. Based on the evolution of physical theories and discoveries of physical laws in the past 100 years, it seems reasonable to conjecture that most currently established physical laws will not last for another 100 years without modification.

Dirac commented on the divergence difficulties in the quantum theory of fields: "The difficulties, being of a profound character, can be removed only by some drastic change in the foundations of the theory, probably a change as drastic as the passage from Bohr's orbit theory to the present quantum mechanics."[3]

It is reasonable to assume that the 4–dimensional symmetry of Lorentz and Poincaré groups is a good starting principle, which can also be realized in different conceptual frameworks, as discussed in previous chapters. We shall now explore the physical origin and implications of fuzzy particles, which can be accommodated in the 4–dimensional symmetry of common relativity (but not that of special relativity.)

It appears quite possible that the divergence difficulties originate from the naive analogy between field quantities ($\varphi(t,\mathbf{r})$, $\partial\varphi(t,\mathbf{r})/\partial t$) and generalized coordinates and velocities ($q_i(t)$, $dq_i(t)/dt$). For decades, it seemed quite certain that one had to follow this "exact analogy" in order to apply the canonical procedure for quantizing the fields. However, if one insists on preserving this "exact analogy," one is led to the customary coordinate description at short distances and encounters profound divergence difficulties. This leads to some doubts about how fundamental this analogy in physics really is. According to this analogy, one simply replaces the Fourier coefficients c_k and c_k^* of a free classical field φ by the corresponding annihilation and creation operators:

$$c_k \to \frac{a_k}{\sqrt{2\omega_k}}, \qquad c_k^* \to \frac{a_k^\dagger}{\sqrt{2\omega_k}}. \tag{15.1}$$

In this way, the usual Hamiltonian H_{usu} of the field φ is

$$H_{usu} = \sum_k \omega_k (a_k a_k^\dagger + \frac{1}{2}), \tag{15.2}$$

where the field φ is enclosed in a cubic box of side $L=V^{1/3}$ and satisfies the periodic boundary conditions. Result (15.2) shows the analogy with the harmonic oscillator.

In this section, I have attempted to modify the conventional quantum field which allows, in principle, the existence of a physical wave with an infinitesimal wavelength, so that one can measure spatial positions to an arbitrary accuracy. The basic new idea is that such a wave does not exist even in principle. Thus, according to quantum mechanics, spatial measurement cannot be as accurate as one wants, in principle. In this sense, space is "fuzzy at short distances." Such a fuzziness suggests that there exists a fundamental probability distribution P(k,m) which approaches zero as the wavelength or $2\pi/k$ approaches zero. We assume that this is inherent in each harmonic oscillator with the momentum **k** of a field with a mass m.[4] Namely, instead of (15.1), one should replace the Fourier coefficients of a free classical field by

$$c_k \to \frac{a_k D(k,m)}{\sqrt{2\omega_k}},$$

$$c_k^* \to \frac{a_k^\dagger D(k,m)}{\sqrt{2\omega_k}}; \qquad D(k,m) = \sqrt{P(k,m)}\,, \quad k = |\mathbf{k}|\,. \tag{15.3}$$

Thus, the Hamiltonian of the field φ becomes

$$H = \sum_{\mathbf{k}} \omega_k P(k,m)(a_k a_k^\dagger + \frac{1}{2})\,. \tag{15.4}$$

The specific form of the probability distribution P(k,m) can only be determined by future experiments. Nevertheless, based on known experiments at low energies, the probability distribution P(k,m) must satisfy the following limits:

$$P(k,m) \to \begin{cases} 1 & k \to 0\,; \\ 0 & k \to \infty\,. \end{cases} \tag{15.5}$$

Thus, the concept of an inherent probability distribution has the effect of suppressing the large momentum contributions in (15.4). In addition, field oscillators have a finite zero-point energy, in contrast to the usual expression (15.2) which has an infinite zero-point energy. To make our discussion concrete, we will assume the invariant function D(k,m) in common relativity to be given by (14.3) with **p**=J**k**,

$$P(k,m) = \left[\frac{1}{1 + I^2 G^2(\mathbf{k})}\right]^2, \qquad G(\mathbf{k}) = \frac{\sqrt{\mathbf{k}^2+m^2}}{C}, \tag{15.6}$$

where the quantity I is a new fundamental constant. Presumably, the value of I is very small so that P(k,m)≈1 for presently available energies in the laboratory.

We may picture a field as a set of harmonic oscillators where each harmonic oscillator is associated with an intrinsic probability determined by its momentum. In the limit I→0, the probability distribution P(k,m)→1 and we have the usual field in which all states of the harmonic oscillator of the field are equally probable.

The probability distribution P(k,m) in (15.5) must have the following two important properties:

(i) P(k,m) depends only on the spatial components of the four-momentum vector of an oscillator. This dependence is necessary for unitarity of quantum field theories, i.e., the probability for any physical process to occur must be non-negative.

(ii) P(k,m) must be an invariant function, as required by the 4-dimensional symmetry.

These two requirements can only be satisfied by field theories formulated within the framework of common relativity. In sharp contrast, special relativity does not have the *invariant* genergy (14.2), which is related only to the spatial components of the momentum four-vector. Thus, quantum field theories *based on special relativity* cannot satisfy (15.5) or possess the two properties described above. Furthermore, the experimental evidence for the presence of P(k,m)≠1 in the future would imply that the spacetime symmetry of special relativity is only approximate at "low energies" and is not exact at very high energies or short distances.

According to Maxwell's equations and the postulate of the universal probability distribution for field oscillators, we expand the photon field **A**(ω,**r**) as follows:

$$\mathbf{A}(\omega,\mathbf{r}) = \int \frac{d^3k}{\sqrt{2k_0(2\pi)^3}} \sum_{\lambda=1}^{2} \boldsymbol{\varepsilon}(k,\lambda)\sqrt{P(k,0)}\,[(a(k,\lambda)e^{-ik\cdot x} + a^\dagger(k,\lambda)e^{ik\cdot x})], \tag{15.7}$$

$$k_0 = |\mathbf{k}|, \quad \mathbf{k}\cdot\boldsymbol{\varepsilon}(k,\lambda)=0, \quad \boldsymbol{\varepsilon}(k,\lambda)\cdot\boldsymbol{\varepsilon}(k,\lambda') = \delta_{\lambda\lambda'},$$

in the radiation gauge. Since photons are massless, m=0, they are associated with the invariant probability distribution P(k,0) in (15.7). The operators $a(k,\lambda)$ and $a^\dagger(k,\lambda)$ are assumed to satisfy the commutation relations,

$$[a(k,\lambda), a^\dagger(k',\lambda')] = \delta^3(k - k')\,\delta_{\lambda\lambda'},$$
$$[a(k,\lambda), a(k',\lambda')] = [a^\dagger(k,\lambda), a^\dagger(k,\lambda')] = 0. \tag{15.8}$$

As usual, we assume the gauge invariant Lagrangian L_a for the photon field a_μ :

$$L_a = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu}; \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{15.9}$$

Note that $a^\mu = (a^0, \mathbf{a})$ in common relativity has the same dimensions as A^μ/c in the special relativity because $\int L_a d^4x$ and J have the same dimensions. The Hamiltonian H_a and the momentum Q_a of the photon field can be written in the form

$$H_a = \int d^3k|\mathbf{k}|P(k,0)\sum_{\lambda=1}^{2} a(k,\lambda)a^\dagger(k,\lambda),$$
$$Q_a = \int d^3k\mathbf{k}P(k,0)\sum_{\lambda=1}^{2} a(k,\lambda)a^\dagger(k,\lambda). \tag{15.10}$$

The modified photon propagator involving the constant I is now given by

$$iD^{tr}_{\mu\nu}(x,I) = \langle 0|a_\mu(x)a_\nu(0)\theta(w)|0\rangle + \langle 0|a_\nu(0)a_\mu(x)\theta(-w)|0\rangle$$
$$= \int \frac{d^4k}{(2\pi)^4}\,\frac{P(k,0)e^{-ik\cdot x}}{k_\lambda^2 + i\varepsilon}\sum_{\lambda=1}^{2}\varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda), \tag{15.11}$$

$$\varepsilon^\mu(k,\lambda) = (0, \varepsilon(k,1), \varepsilon(k,2), \mathbf{k}/|\mathbf{k}|) = (0, \boldsymbol{\varepsilon}(k,\lambda)).$$

This propagator reduces to the Feynman propagator $iD^{tr}_{F\mu\nu}(x)$ in the limit $I \to 0$,

$$iD^{tr}_{\mu\nu}(x, 0) = iD^{tr}_{F\mu\nu}(x) , \qquad (15.12)$$

as it should.

The Coulomb potential of a charged particle is now produced by a fuzzy source rather than a point source. The fuzzy source is determined by the probability function P(k,0) in (15.11).[5] The modified Coulomb potential $V(r)=a^0(r)$ is

$$V(r) = -\bar{e} \int \frac{d^3k}{(2\pi)^3} \frac{P(k,0)}{\mathbf{k}^2} e^{-i\mathbf{k}\cdot\mathbf{x}} , \qquad (15.13)$$

which is consistent with (14.26). It is important to remember that P(k,0) should not be understood or interpreted as the form factor due to charge distribution in the classical sense. Rather, it is a quantum mechanical property related to the *inherent probability distribution of photon field oscillators.* In other words, such a photon can be pictured as a "fuzzy particle" rather than a point particle. A "fuzzy photon" can produce an effect similar to that produced by a non-point charge distribution. In general, any departure from the point picture of particles in electrodynamics implies a modification of the Coulomb potential at short distances which can be tested by very high-energy experiments in the future.

15b. Fuzzy Quantum Electrodynamics Based on Common Relativity

In fuzzy quantum electrodynamics (fuzzy QED), the invariant action S_q, involving a fuzzy electron field ψ and a photon field a_μ, is assumed to have the usual form:

$$S_q = \int L \, d^4x , \qquad L = \bar{\psi}[\gamma^\mu(iJ\partial_\mu - \bar{e}a_\mu) - m]\psi - (1/4)f_{\mu\nu}f^{\mu\nu} ; \qquad (15.14)$$

$$J = 3.5177293 \times 10^{-38}\, \text{g}\cdot\text{cm} , \qquad \alpha_e = \bar{e}^2/(4\pi J) \approx 1/137 . \qquad (15.15)$$

The wave equation of a free Dirac fields can be derived from (15.14). It has the usual form:

$$[\gamma^\mu iJ\partial_\mu - m]\psi = 0 . \qquad (15.16)$$

Fuzzy quantum electrodynamics is based on a general postulate: There is an inherent probability distribution P(p,m) associated with a field oscillator with momentum **p** and mass m of all fields. This is a natural generalization of the idea of "fuzzy quantum field" discussed in section 15a.

Based on the 4–dimensional formalism with a covariant gauge condition, we have the following free photon and electron fields:

$$a_\mu(w,\mathbf{r}) = \sum_{\mathbf{p};\alpha}\sqrt{J/(2Vp_0)}\sqrt{P(\mathbf{p},0)}[a(\mathbf{p},\alpha)\varepsilon_\mu(\alpha)\exp(-ip\cdot x/J)$$

$$+ a^\dagger(\mathbf{p},\alpha)\,\varepsilon_\mu(\alpha)\exp(ip\cdot x/J)] , \qquad (15.17)$$

$$\psi(w,\mathbf{r}) = \sum_{\mathbf{p};s}\sqrt{(m/Vp_0)}\sqrt{P(\mathbf{p},m)}[b(\mathbf{p},s)u(\mathbf{p},s)\exp(-ip\cdot x/J)$$

$$+ d^\dagger(\mathbf{p},s)v(\mathbf{p},s)\exp(ip\cdot x/J)] , \qquad p\cdot x = p_\mu x^\mu , \qquad (15.18)$$

where

$$[a(\mathbf{p},\alpha), a^\dagger(\mathbf{p}',\alpha')] = \delta_{\mathbf{pp}'}\delta_{\alpha\alpha'} ,$$

$$[b(\mathbf{p},s), b^\dagger(\mathbf{p}',s')] = \delta_{\mathbf{pp}'}\delta_{ss'} , \qquad [d(\mathbf{p},s), d^\dagger(\mathbf{p}',s')] = \delta_{\mathbf{pp}'}\delta_{ss'} , \qquad (15.19)$$

and all other commutators vanish. Of course, commutators for quantized fields $\psi(w,\mathbf{r})$ and $a_\mu(w,\mathbf{r})$ can be derived from (15.17)–(15.19). In a general inertial frame, the Dirac equation in (15.16) can be written as

$$iJ\partial\psi/\partial w = [- iJ\boldsymbol{\alpha}_D\cdot\nabla + \beta_D m]\psi . \qquad (15.20)$$

Note that $w = bt_c$ denotes the lightime, t_c is common time, and $\boldsymbol{\alpha}_D$, β_D are the usual constant Dirac matrices.

Since the lightime w plays the role of evolution variable in the invariant equations of motion (15.20), it is also the evolution variable for a state $\Phi^{(S)}(w)$ in the Schrödinger representation:

$$iJ\frac{\partial\Phi^{(S)}(w)}{\partial w} = H^{(S)}(w)\Phi^{(S)}(w), \qquad H^{(S)} = H_o^{(S)} + H_I^{(S)}, \tag{15.21}$$

because the evolution of a physical system is assumed to be described by a Hamiltonian operator which has the same transformation property as the lightime w or $\partial/\partial w$.

The conventional covariant formalism of perturbation theory can also be applied to fuzzy QED. The steps from (10.38) to (10.46) in chapter 10 also hold for fuzzy QED based on common relativity. For simplicity, we set J=1, so that the dimensions of the Lagrangian density L, photon fields a_μ and the electron field ψ are given by the relation,

$$[L^{1/4}] = [a_\mu] = [\psi^{2/3}] = [mass] = [1/length], \qquad J = 1. \tag{15.22}$$

To obtain the modified rules for Feynman diagrams in fuzzy QED, we follow the usual quantization procedure and define the fuzzy QED Lagrangian L_{FQED} by adding a gauge fixing term in the Lagrangian (15.14),

$$L_{FQED} = L - (\partial^\mu a_\mu)^2/2\rho, \qquad J = 1, \tag{15.23}$$

where ρ is a gauge parameter. We define the M-matrix as follows:

$$S_{if} = \delta_{if} - i(2\pi)^4\delta^4(p_f(tot) - p_i(tot))\sqrt{\Pi_{ext\ par}(n_j/V)}\ M_{if}, \tag{15.24}$$

where "ext par" stands for external particles. The quantity n_j denotes m_j/p_{0j} for spin 1/2 fermions and $1/(2p_{0j})$ for bosons. Because of the 4-dimensional symmetry in (15.23) and (15.24), the rules for writing M_{if} are formally the same as those in the usual QED:

(a) The fuzzy photon propagator is now given by

$$\frac{-iP(k,0)[g_{\mu\nu} - (1-\rho)k_\mu k_\nu/k^2]}{(k^2 + i\varepsilon)}, \qquad k^2 = k_\mu k^\mu. \tag{15.25}$$

(b) The fuzzy electron propagator is

$$\frac{-iP(p,m)}{(\gamma^\mu p_\mu - m + i\varepsilon)}. \tag{15.26}$$

(c) The electron–photon vertex is

$$-i\bar{e}\gamma^\mu . \tag{15.27}$$

(d) Each external photon line has an additional factor $\sqrt{P(k,0)}\ \varepsilon_\mu$. Also, each external electron line has $\sqrt{P(p,m)}\ u(s,p)$ for the absorption of an electron and $\sqrt{P(p,m)}\ \bar{u}(s,p)$ for the emission of an electron, etc.

Other rules such as taking the trace with a factor (−1) for each closed electron loop, integration with $d^4k/(2\pi)^4$ over a momentum k_μ not fixed by the conservation of four–momentum at each vertex, etc. are the same as the usual.

Thus, calculating scattering cross sections and decay rates (with respect to the lightime w) of a physical process, one will get formally the same result as that in conventional QED. For example, let us consider again the decay rate $\Gamma(1\to2+3+...+N)$ for a physical process $1\to2+3+...+N$ given by

$$\Gamma(1\to2+3+...+N) = \lim_{w\to\infty}\int \frac{|\langle f|S|i\rangle|^2}{w}\frac{d^3x_2 d^3p_2}{(2\pi J)^3}\cdots\frac{d^3x_N d^3p_N}{(2\pi J)^3}, \tag{15.28}$$

which has the dimensions of inverse length. Its inverse is a particle's lifetime measured in terms of lightime, $w=bt_C$, which has the dimension of length and, hence, it may be called the "decay-length." The decay-length D is given by

$$D = 1/\Gamma(1\to2+3+...+N). \tag{15.29}$$

Thus, in common relativity, one has the "rest decay-length" D_0 for a particle decay at rest, corresponding to the "rest lifetime" in the conventional theory. Also, instead of the dilatation of the lifetime of a particle in flight, we have dilatation of the decay–length it travels before decaying. Such a dilatation is physically correct because it is equal to the experimentally measured distance traveled by a decaying particle. (See chapter 11.)

For a scattering process $1+2\to3+4+....+N$, the differential cross section $d\sigma$, which has the dimension of area, is given by

$$d\sigma = \frac{1}{4[(p_1 \cdot p_2)^2 - (m_1 m_2)^2]^{1/2}} |M_{if}|^2 \Big[\prod_{ext\ fer} (2m_{fer})\Big] \frac{d^3\mathbf{p}_3}{(2\pi)^3 2p_{03}} \cdots$$

$$\cdots\cdots \frac{d^3\mathbf{p}_N}{(2\pi)^3 2p_{0N}} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots - p_N) S_o , \qquad (15.30)$$

where $p_0 = (\mathbf{p}^2+m^2)^{1/2}$ and S_o denotes a factor $1/n!$ for each kind of (n) identical particles in the final state. If the initial particles are unpolarized, one must take the average over initial spin states. When there is no external fermion in a process, then $[\Pi_{ext\ fer}(2m_{fer})]$ in (15.30) is replaced by 1.

The formal expressions (15.28) and (15.30) are the same as the conventional ones. The new and different effects in fuzzy QED came from the *M*-matrix, which involves modified propagators related to an inherent fuzziness at short distances as shown in eqs. (15.25) through (15.27).

15c. Experimental Tests of Possible Approximate 4-Dimensional Symmetry of Special Relativity at Very High Energies and Short Distances

Let us elaborate what new physical effects may be obtained due to an inherent probability distribution P(k,m) for field oscillators. Suppose one calculates the differential cross-section of unpolarized electrons scattered by an external potential (or a point-like nucleus.) One obtains the result

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} (Z\alpha)^2 P(p_i - p_f, 0) P(p_i, m) P(p_f, m) \left(\frac{\omega_p}{|\mathbf{p}_i|^4 \sin^4(\theta/2)} \right)$$

$$\times\left(1 - \beta^2 \sin^2\left(\frac{\theta}{2}\right)\right), \qquad (15.31)$$

$$P(p_i - p_f, 0) = \frac{\sqrt{(\mathbf{p}-\mathbf{p}_f)^2}}{C_d}, \qquad P(p_i, m) = \frac{\sqrt{\mathbf{p}_i^2+m^2}}{C_i} \equiv \frac{\omega_p}{C_i}, \qquad (15.32)$$

$$P(p_f, m) = \frac{\sqrt{\mathbf{p}_f^2+m^2}}{C_f}, \qquad |\mathbf{p}_i - \mathbf{p}_f|^2 = 4|\mathbf{p}_i|^4 \sin^4\left(\frac{\theta}{2}\right), \quad \beta = \frac{|\mathbf{p}_i|}{\omega_p}. \qquad (15.33)$$

If the laboratory is chosen to be the F frame in which the speed of light is, by definition, isotropic, then one has $C_d=C_i= C_f=c=29979245800$cm/sec. (See section 12c in chapter 12.) The differential cross section for the scattering of unpolarized electrons (15.31) is suppressed at large momenta by the inherent probability distribution associated with the photon in the intermediate state and the electrons in external states. In this calculation, the point-like nucleus is assumed to be at rest and its inherent probability distribution is negligible.

Next, let us consider the differential cross section for Møller scattering, (after C. Møller who first discussed this process in 1931) it is the scattering of two electrons, $e^-(p_1)+e^-(p_2)\rightarrow e^-(p_1')+e^-(p_2')$. Since we are interested in the effect due to the inherent probability distribution, we calculate the differential cross section at "high energies," $p_{10} >> 0$. The result is relatively simple to calculate in the center-of-mass frame,

$$\frac{d\sigma}{d\Omega} \approx \frac{r_0^2}{8}\left(\frac{m}{p_{10}}\right)^2 P^4(p,m)\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}P_-^2 + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}P_+^2 + \frac{2}{\cos^2(\theta/2)\sin^2(\theta/2)}P_-P_+\right], \tag{15.34}$$

$$\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2, \qquad \mathbf{p}' = \mathbf{p}_1' = -\mathbf{p}_2', \qquad p = |\mathbf{p}| = |\mathbf{p}'|,$$

$$P_\pm = P(k,0), \qquad k = |\mathbf{p}\pm\mathbf{p}'|, \qquad r_0 = \frac{\alpha_e}{m} \approx 2.82\times10^{-13}\ \text{cm},$$

where one may choose the center-of-mass frame to be the frame F in which the speed of light is isotropic.

One may also consider the pair annihilation process, i.e., $e^-(p_-) + e^+(p_+) \rightarrow \gamma(k_1) + \gamma(k_2)$. In the laboratory frame, in which $p_-=(m,0,0,0)$, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2 k_{10}^2}{8}\left(\frac{1}{|\mathbf{p}_+|(m + p_{+0})}\right)P(p_-,m)P(p_+,m)P(k_1,0)P(k_2,0)$$
$$\times\left\{\left[\frac{k_{10}}{k_{20}} - \frac{2(\varepsilon_1\cdot k_2)^2}{k_{20}m}\right]P_a^2 + \left[\frac{k_{20}}{k_{10}} - \frac{2(\varepsilon_2\cdot k_1)^2}{k_{10}m}\right]P_b^2\right.$$
$$\left. + \left[2 - 4(\varepsilon_2\cdot\varepsilon_1)^2 + \frac{2(\varepsilon_2\cdot k_1)^2}{k_{10}m} + \frac{2(\varepsilon_1\cdot k_2)^2}{k_{20}m}\right]P_aP_b\right\}, \tag{15.35}$$

$$P_a = P(p_- - k_1, m)\,, \quad P_b = P(p_- - k_2, m)\,, \quad k_i^\mu = (k_{i0}, \mathbf{k}_i)\,, \quad i=1,2\,.$$

In the limit of zero radical length, $R \to 0$, (i.e., $I \to 0$ in (15.6)), all inherent probability distributions reduce to 1, and the terms involving $(\varepsilon_2 \cdot k_1)$ and $(\varepsilon_1 \cdot k_2)$ cancel, so that (15.35) reduces to the usual result.

The new effects in the scattering cross sections (15.31) to (15.35) due to fuzziness at short distances can all be tested by high-energy experiments in the future. These are also experimental tests of the possible approximate nature of the 4-dimensional symmetry of special relativity at very high energies and short distances.

At present, there is experimental data for the decay of unstable particles in flight at high energies (several hundreds of GeV), $K_S^o(p) \to \pi^+(p_+) + \pi^-(p_-)$. They can be used to estimate the upper limit of the radical length R. The decay rate of K_S^o in flight is roughly modified by the inherent probability distribution as follows:

$$\Gamma(K_S^o \to \pi^+ + \pi^-) \approx \frac{1}{p_0} P(p, m_K) P(p_+, m_{\pi+}) P(p_-, m_{\pi-}) \times \text{const}\,. \tag{15.36}$$

At several hundred GeV, the masses of these mesons can be neglected in the estimate. The departure of $P(p,m_K)P(p_+,m_{\pi+})P(p_-,m_{\pi-})$ from 1 must be within the experimental error. Based on the result of the decay rate of K_S^o in flight,[6] we estimate that

$$I < 2\times10^{-4} \frac{c}{m_p}, \qquad \text{or} \qquad R < 4\times10^{-18}\,\text{cm}\,, \tag{15.37}$$

where we have ignored the meson masses. Since $m_p \approx 1\text{GeV}/c^2$, (15.37) and (15.6) indicate that the inherent probability distribution $P(k,m)$ becomes important only at extremely large momenta ($p >> 10^4$ GeV/c^2).

In summary, if we treat the concept of probability in quantum mechanics as fundamental so that the realizable position states of a particle must have a probabilistic interpretation, then the physically realizable large momentum states will be suppressed by an inherent probability distribution $P(p,m)$. This implies that dilatation of decay-lengths should come from both the usual relativistic effect and the 'radical dilatation' due to $R \neq 0$ at very high energies.

Note that the inherent probability P(p,m) is not a scalar under the Lorentz transformations of special relativity. Therefore, the spacetime 4–dimensional symmetry of special relativity is only approximate at extremely large momentum if $R \neq 0$ or $P(p,m) \neq 1$. Only in the limit of low energies or $R \to 0$, does the spacetime symmetry of special relativity become exact. However, the inherent probability P(p,m) can be expressed in terms of the genergy G(p), which is an invariant in common relativity. Therefore, the 4–dimensional symmetry of common relativity is not violated by the presence of the inherent probability distribution $P(p,m) \neq 1$.

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16.

Common Relativity and the 3 K Cosmic Background Radiation

16a. Implications of Non-Invariant Planck's Law of Blackbody Radiation

Since the anisotropy of the 3 K radiation was experimentally established,[1] many physicists have concluded that it is now possible to talk about "absolute motion".[2] This conclusion was reached within the framework of special relativity, in which the Planck law for blackbody radiation is not invariant under the Lorentz transformation. If one is not aware that the 4-dimensional symmetry can be understood from a broader viewpoint than that of just special relativity, this conclusion appears to be unavoidable.[3] However, to have a non-invariant Planck's law of the blackbody radiation is unnatural and is not in harmony with the fundamental Poincaré-Einstein principle of relativity for physical laws.

As we have shown in previous chapters, nature can be viewed from the standpoint of a 4-dimensional theory of relativity with a common (but not absolute) time for all observers. Existing experiments cannot distinguish between common relativity and special relativity because they both have the 4-dimensional symmetry of the Lorentz and the Poincaré groups. Nevertheless, the conceptual framework of common relativity is not exactly the same as that of special relativity. For example, we can introduce the concepts of an invariant genergy $G=p_0/C$, as shown in (14.2), and the invariant volume of a box for a many-particle system. With the help of these concepts, we can formulate a new covariant thermodynamics which leads to an invariant Planck law and, hence, the impossibility of detecting the Earth's motion relative to the cosmic microwave background radiation.[4]

16b. The Invariant Partition Function

Let us consider systems with discrete states labelled by an index i, since the generalization to systems with continuous states is straightforward. Each state of the system corresponds to an invariant genergy $G=p^0/C > 0$. Because of its invariance, the genergy G corresponds more closely to the scalar energy in classical physics than the non-scalar energy p^0. We assume, as usual, that the interaction between molecules (or particles) is sufficiently weak so that there is an exchange of genergy between molecules but no change in the structure and properties of a molecule.

Consider a gas with N particles, where N»1. The μ space volume is divided into K cells with K»1. Each particle in the cell ω_i is in the state i and has genergy G_i. Suppose there are n_i particles in the cell ω_i. Let us assume that the *a priori* probability of the cell ω_i to be occupied is 1, we have the probability

$$W(n_1,n_2, ...) = N!/(\prod_i n_i!) \tag{16.1}$$

for n_1 particles in ω_1, n_2 particles in ω_2, etc. The equilibrium state of the gas corresponds to a state of maximum $W(n_1,n_2, ...)$ consistent with the constraints

$$N = \sum_i n_i , \tag{16.2}$$

$$G_t = \sum_i n_i G_i = N\overline{G} , \tag{16.3}$$

where $\overline{G}$ is the average genergy of the constituent particles. (For simplicity, we do not consider modifications due to the radical length.) In equilibrium, we have $n_i = \overline{n}_i$, where

$$\overline{n}_i = N \exp[- \overline{\beta}G_i]/Z , \qquad \overline{\beta} = \text{parameter} , \tag{16.4}$$

which is obtained by varying $W(n_1,n_2, ...)$ with constraints (16.2) and (16.3). The invariant partition function Z is defined by

$$Z = \sum_i \exp[- \overline{\beta}G_i] , \qquad \overline{\beta} > 0 , \tag{16.5}$$

provided the summation over the states i does not change the invariance of (16.5). From (16.3) and (16.5), one has $\overline{G} = -\partial \ln Z/\partial \overline{\beta}$.

Once the partition function Z is known, all other thermodynamic quantities can now be derived. We can define an invariant "free genergy" F_c so that

$$Z = \exp[-\bar{\beta}F_c]\,. \tag{16.6}$$

We note that (16.5) is the invariant partition function of the Gibbs ensemble, in which there is no exchange of momentum, within the framework of common relativity. This is to be compared with the conventional Lorentz-invariant partition function, in which one has to introduce a 4-vector of the inverse temperature whose operational meaning is in general not at all clear.

In a particular frame F, the partition function in (16.5) can be identified with the usual partition function of the Gibbs ensemble $\sum_i \exp[-E_i/K_B T]$, and the invariant parameter $\bar{\beta}$ in (16.5) can be related to the Boltzmann constant $k_B = 1.38\times10^{-16}$ erg/deg K and the usual absolute temperature T. Since $G_i = m/c + p_i^2/2mc$, $p_i = mv_i/c$, and $E = mv_i^2/2$ in the F frame, we have

$$1/\bar{\beta} = k_B T/c^3 = \tau\,. \tag{16.7}$$

We term the invariant quantity τ the "common temperature". Evidently, the natural unit of τ is genergy rather than energy. Note that the usual temperature T is closely related to the energy E_i and that both T and E_i are not invariants in common relativity.

16c. Covariant Thermodynamics

In conventional thermodynamics, the relation between the entropy S, the heat Q, and the absolute temperature T is given by

$$dS = \frac{(dQ)_{rev}}{T}\,. \tag{16.8}$$

This suggests that we relate the "common entropy" S_c and "common heat" Q_c by the relation,[4]

$$dS_c = \frac{(dQ_c)_{rev}}{\tau}\,, \tag{16.9}$$

in the covariant thermodynamics formulated on the basis of common relativity. Relations (16.8) and (16.9) are the same in the frame F in which the speed of light has been defined to be isotropic, provided that $S_c=S/k_B$ and $Q_c=Q/c^3$. In general, the definitions of S_c and Q_c must be consistent with (16.9) and the Boltzmann equation

$$S_c = \ln W(n_1,n_2, ...) = \frac{S}{k_B}. \tag{16.10}$$

Let us consider a change of the genergy of a closed system. It is given by the invariant equation

$$N\delta\overline{G} = \sum_i (\,\overline{n}_i\delta G^i + G^i\delta\overline{n}_i) , \tag{16.11}$$

where the first term is always considered to be the "work" and the second term to be the "heat". Thus we can define the common heat Q_c as

$$N(\delta Q_c)_{rev} = \sum_i G^i\delta\,\overline{n}_i , \tag{16.12}$$

where the reversibility of the change is achieved by letting $\delta\,\overline{n}_i$ correspond to a change in which the probability W defined in (16.1) remains maximum. We see that the common heat Q_c is the change of genergy of the whole system resulting from a change in the statistical arrangement of its components. On the other hand, "common work" is due to the change of genergy of every component of a system without changing their statistical arrangement.[5]

The reversible change of $\overline{n}_i$ in (16.12) can be defined as

$$\delta\overline{n}_i = -N\frac{\partial^2 \ln Z}{\partial z_i \partial z_j}\delta z_j , \tag{16.13}$$

because Z in (16.5) can be considered to be a function of $z_i = \overline{\beta}G_i$ and (16.4) can be written as

$$\delta\overline{n}_i = -N\frac{\partial \ln Z}{\partial z_i}. \tag{16.14}$$

It follows from (16.13) and (16.12) that

$$\frac{1}{\tau}(\delta Q_c)_{rev} = -\sum_{ij} z_i \delta z_j \frac{\partial^2 \ln Z}{\partial z_i \partial z_j}, \tag{16.15}$$

where $G_i = z_i/\bar{\beta} = \tau z_i$. Thus $(\delta Q_c)_{rev}/\tau$ can be represented by a Pfaffian in the variable z_i and it is nothing but $d\bar{S}_c$ according to (16.9). After integration, we find the average common entropy $\bar{S}_c$ of a particle in the ensemble:

$$\bar{S}_c = \frac{\bar{G}}{\tau} + \ln Z, \tag{16.16}$$

where the constant of integration is neglected. Using (16.2)-(16.5) and (16.16), one can verify that

$$S_c = N\bar{S}_c = \ln W = -\sum_i \frac{e^{[-\bar{\beta}G_i]}}{Z} \ln\left(\frac{e^{[-\bar{\beta}G_i]}}{Z}\right), \tag{16.17}$$

where we have used the Stirling formula: $\ln N! \approx N(\ln N - 1)$.

For a thermodynamic system, the invariant entropy S_c can be expressed as a function of N, $G_t = N\bar{G}$ and V_I. The equations for thermodynamic equilibrium are invariant:

$$\frac{1}{\tau} = \left(\frac{\partial S_c}{\partial G}\right)_{V,N}, \tag{16.18}$$

$$\frac{P_c}{\tau} = \left(\frac{\partial S_c}{\partial V}\right)_{G,N}, \tag{16.19}$$

$$\frac{\mu_c}{\tau} = \left(\frac{\partial S_c}{\partial N}\right)_{G,V}, \tag{16.20}$$

where $G \equiv G_t$ and $V \equiv V_I$. The invariant pressure P_c and chemical potential μ_c are related to the usual pressure P and chemical potential μ by the relations

$$P_c = \frac{P}{c^3}, \qquad \mu_c = \frac{\mu}{c^3}, \tag{16.21}$$

in the F frame. One can verify that, in the rest frame F of the thermodynamic system, the present formalism reduces to the usual thermodynamics at low energies.

16d. Canonical Distribution and Blackbody Radiation

Because of the existence of the new invariant genergy (14.2) and the invariant volume V_I in (13.1), we are able to define a meaningful invariant density for particles. One cannot do this within the framework of special relativity because of the lack of an invariant genergy. The number of states available to a particle in dV_I can be written as

$$d\Gamma = 2Gc_0 dV_I\, \delta(p_\lambda^2 - m^2)\theta(p^0)d^4p \frac{1}{(2\pi J)^3}, \tag{16.22}$$

$$G = \frac{\sqrt{p^2+m^2}}{C}, \qquad J = 3.5177\times 10^{-38}\, \text{g}\cdot\text{cm},$$

in an arbitrary frame, where p_λ and $m \geq 0$ are respectively the 4–momentum and the mass of the particle. The invariance of $d\Gamma$ in (16.22) ensures that the summation over the states of a system (see eq. (16.5) does not change the invariance of the expression. In the particular frame F, we have $C=c=c_0$ and hence

$$d\Gamma = \frac{d^3x d^3p}{(2\pi J)^3}, \qquad \text{(in the F frame)}, \tag{16.23}$$

as it should.

For an ideal gas containing N particles with the same mass $m \geq 0$, we have the invariant canonical (or Gibbs) distribution

$$d\rho = \frac{N(2\pi J)^3}{4\pi m^2 \tau V_I c_0 K_2(m/\tau c_0)} e^{[-G/\tau]} d\Gamma, \tag{16.24}$$

$$m^2 K_2\left(\frac{m}{\tau c_0}\right) = \frac{1}{\tau c_0}\int_0^\infty dp\ p^2 \exp\left[-\frac{\sqrt{p^2+m^2}}{\tau c_0}\right] \tag{16.25}$$

$$= \frac{1}{\tau c_0}\int_0^\infty \sqrt{p_0{}^2-m^2}\; p_0 dp_0 \exp[-p^0/\tau c_0]$$

$$\rightarrow\; 2(c_0\tau)^2 \qquad \text{as} \qquad m\rightarrow 0\,.$$

For blackbody radiation, we have the partition function Z

$$Z = \sum_n \exp[-(G_n/\tau)]\,, \qquad G_n = nG = \frac{nJk_0}{C}\,, \tag{16.26}$$

where the wave 4–vector k^μ=(k_0, **k**) is related to the momentum p^μ by

$$p^\mu = Jk^\mu\,, \qquad k^\mu = (\frac{\omega}{C}, \mathbf{k})\,, \qquad J = 3.5177\times 10^{-38} g{\cdot}cm\,. \tag{16.27}$$

Note that for a massless particle, G is still invariant under the space–lightime transformation:

$$\frac{G}{J} = \frac{k'_0}{c'} = \frac{\gamma(k_0 - \beta k_x)}{\gamma(c - \beta v_x)} = \frac{k_0}{c}\,, \tag{16.28}$$

because k_x=$k_0\cos\theta$ and v_x=c $\cos\theta$. It follows from (16.26) that

$$\langle n\rangle = \frac{1}{Z}\sum_n n\; e^{[-nG/\tau]} = \frac{1}{e^{[G/\tau]} - 1}\,, \tag{16.29}$$

which is the *invariant* Planck distribution for blackbody radiation in the framework of common relativity. Comparing equation (16.29) with the usual expression $\langle n\rangle = 1/(\exp[\hbar k_0 c/k_B T] - 1)$ in the F–frame, we obtain

$$\frac{Jk_0}{c\tau} = \frac{\hbar k_0 c}{k_B T}\,, \qquad \frac{\hbar}{c} = J\,, \tag{16.30}$$

which is consistent with (16.7).

In the present formalism, we also have an invariant grand–canonical distribution, the invariant "thermodynamic potential" Ω_C, and the invariant Fermi–Dirac distribution, etc.:

$$\Omega_C = -\tau \ln \{\sum_n \exp(\mu_C/\tau - G_n/\tau)\} ,$$

$$\langle n \rangle_{FD} = 1/\{\exp[G/\tau - \mu_C/\tau] + 1\} . \tag{16.31}$$

16e. Question on Earth's "Absolute" Motion in the 3 K Radiation

The cosmic 3 K radiation was detected by Penzias and Wilson in 1965. It is an important discovery for understanding the Universe and is believed to be the left-over radiation from the early epoch of the Universe in which matter and radiation were in thermal equilibrium. The cosmic 3 K radiation involves about 10^3 photons per cm^3 and this temperature is assumed to be the present temperature of the universe because the effect of hot stars is negligible when one averages their temperature over all space. The cosmic 3 K radiation appears to be isotropic, having the same radiation intensity in all directions, as observed on the Earth. However, a very small anisotropy in the cosmic blackbody radiation has been detected. One usually interprets it as caused by Earth's motion relative to the radiation background. The interpretation is made within the usual framework of special relativity with certain implicit assumptions: one must consider two frames F and F' (which is moving with the speed V along the x-axis of F) immersed in the blackbody radiation. According to special relativity, suppose one has the usual Planck distribution in the F frame,

$$\langle n \rangle = \frac{1}{\exp[h\nu/k_B T] - 1} . \tag{16.32}$$

Observers in the F' frame will detect a modified distribution,

$$\langle n' \rangle = \frac{1}{\exp\left[\frac{h\nu'(1-V\cos\theta'/c)}{k_B T\sqrt{1-V^2/c^2}}\right] - 1} , \tag{16.33}$$

which is derived from (16.32) with the usual relation of special relativity

$$\nu' = \frac{\nu(1-V\cos\theta/c)}{\sqrt{1-V^2/c^2}} , \qquad \cos\theta' = \frac{\cos\theta + V/c}{1+V\cos\theta/c} , \tag{16.34}$$

and the assumptions that the Boltzmann constant k_B and the temperature T are Lorentz scalars. It follows that the spectral distribution of the energy of the blackbody radiation is given by

$$dE(k_0) = \frac{2d^3x d^3k}{(2\pi)^3} \frac{h\nu}{\exp[h\nu/k_B T] - 1}, \qquad \text{in F}$$

(16.35)

$$dE'(k_0') = \frac{2d^3x' d^3k'}{(2\pi)^3} \frac{h\nu'}{\exp[h\nu'(1+V\cos\theta'/c)\gamma/k_B T] - 1}, \qquad \text{in F'},$$

according to special relativity. The photon energies $h\nu$ and $h\nu'$ in (16.35) are related by the relativistic Doppler effect (16.34). Thus, the cosmic microwave background plays the role of a new "ether" and the frame F is the privileged and absolute frame in which the law of the Planck distribution takes a particular simple form, as one can see from (16.35). According to this view, nature provides an absolute frame of reference: the cosmic 3 K radiation represents a fixed system of coordinates in the Universe. Most physicists were impressed by this and concluded that we now are justified in talking about absolute motion.

Nevertheless, one notes that this interpretation is not in harmony with the Poincaré-Einstein principle of relativity for physical laws. Furthermore, this conclusion is untenable from the viewpoint of the present covariant thermodynamics based on common relativity. According to the present framework, the Planck formula corresponding to (16.35) in F and F' can be obtained by combining the invariant density of states (16.22) with two polarizations (i.e. $2d\Gamma$), the invariant Planck distribution (16.29) and the covariant photon "energy" Jk_0 (which has the dimension of mass):

$$dM(k_0) = \frac{2d\Gamma J k_0}{\exp[G/\tau] - 1}, \qquad \text{in F},$$

(16.36)

$$dM'(k_0') = \frac{2d\Gamma J k_0'}{\exp[G/\tau] - 1}, \qquad \text{in F'},$$

where we have used the invariant genergy in (16.28). The photon "energy" Jk_0 and Jk_0' in (16.36) are related by the Doppler effect in common relativity:

$$k_0' = (k_0 - \beta k_x')\gamma, \quad k'_x = (k_x - \beta k_0)\gamma, \quad k'_y = k_y, \quad k'_z = k_z,$$

$$k^{\mu} = (\frac{\omega}{c}, \mathbf{k}), \quad k'^{\mu} = (\frac{\omega'}{c'}, \mathbf{k}') . \tag{16.37}$$

Since the density of states (16.22) and the Planck distribution $<n>=\{\exp[G/\tau-1\}^{-1}$ are invariant, the quantity $dM'(k_0')$ is covariant and transforms like k_0' in common relativity. Therefore, from this viewpoint, the observed anisotropy in the cosmic 3 K radiation should not be interpreted as the "absolute" motion of the Earth. This is in contrast with the corresponding result of special relativity, in which $dE'(k'_0)$ in (16.35) does not have a covariant form and does not transform like k_0' because the Planck distribution is not invariant and takes the simplest form only in the frame F. Note that the Doppler shift of $h\nu'$ in (16.35) and Jk'_0 in (16.36) cannot be distinguished by previous experiments. The reason for this can be seen clearly by considering the quantities involved in the Doppler shift experiment.

Note that the relation between the frequency $\nu=ck_0/2\pi$ and $\nu'=c'k'_0/2\pi$ in (16.37) is quite different from that given by (16.34) in special relativity. This is because we have used common time to measure the speed of light c and c' and to define the frequency of a wave in our formalism. However, this does not contradict laboratory experiments testing the Doppler effect involving shifts of atomic-level structure, lasers, etc. The reason has been discussed in chapter 10 [eqs. (10.13)-(10.18)].

Experimentally, the anisotropy of the cosmic 3 K radiation was established at the level of one part in 10^3 to 10^4. According to common relativity, it should not be interpreted as being related to the angular dependence of the Planck distribution due to the absolute motion of the Earth. This conclusion is more in harmony with the principle of relativity if one formulates thermodynamics in such a way that the Planck distribution is angular independent and, hence, has the same form in all inertial frames.

From the present viewpoint, the anisotropy of the 3 K radiation indicates a new phenomenon in the cosmological scale. For example, it may be related to a small nonsymmetric expansion of the Universe or large-scale irregularities in the distribution of energy in the Universe. This could be clarified in the future when one is able to measure the dipole anisotropy ($\sim 10^{-5}$) due to the Earth's orbital motion and the quadruple anisotropy.

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17.

Extended Relativity: A Weaker Postulate for the Speed of Light

17a. 4-Dimensional Symmetry as a Guiding Principle

Only at the beginning of the twentieth century after the creation of the 4-dimensional symmetry, was it recognized that the concept of symmetry played a fundamental role in physics.[1] The 4-dimensional symmetry is one of the most thoroughly tested symmetry principles in physics. In his Nobel Lecture, C. N. Yang made the following piercing observation:

> "*Nature seems to take advantage of the simple mathematical representations of the symmetry laws. When one pauses to consider the elegance and the beautiful perfection of the mathematical reasoning involved and contrast it with the complex and far-reaching physical consequences, a deep sense of respect for the power of the symmetry laws never fails to develop.*"[2]

This quotation summarizes the essence of symmetry in physics, which will be illustrated below by an analysis of different viewpoints of the physical world to show how 4-dimensional symmetry is critical to any theory.

Let us consider the viewpoint of Reichenbach's more general concept of time and Edwards' universal two-way speed of light. This includes relativistic time and the universal speed of light of special relativity as a special case. Unfortunately, Reichenbach's and Edwards' treatments were, in general, not consistent with the Lorentz and Poincaré symmetry groups. The lack of the 4-dimensional symmetry of the Lorentz group will cause many problems in the formulation of quantum field theories, especially the Feynman rules for calculations in quantum electrodynamics or chromodynamics.[3] For the purpose of comparison, we shall explain in some detail the application of 4-dimensional symmetry to the ideas of Reichenbach and Edwards. In this way, we will also show their implications and complete consistency with experiments.

We term such a 4-dimensional theory "extended relativity." As a matter of fact, extended relativity is just the theory of taiji relativity with an additional second postulate, namely, the universal 2-way speed of light.

Using 4-dimensional symmetry as a guiding principle for discussing physical laws, we first analyze Edwards' original attempt in 1963 to formulate a relativity theory based on a weaker postulate for the speed of light. Edwards postulated that "the two-way speed of light in a vacuum as measured in two coordinate systems moving with constant relative velocity is the same constant regardless of any assumptions concerning the one-way speed."[4] He derived space and time transformations involving infinitely many possible physical times which can be physically realized through Reichenbach's convention of clock synchronization and which include relativistic time as a special case. Edwards' transformations were shown to be consistent with many experiments related to the propagation of light. Nevertheless, we show that they do not form the Lorentz group *so that, in general, physical laws are not invariant under such transformations.* As a result, it leads to an incorrect expression for the relativistic energy-momentum of a particle in the Lagrangian formalism.[3]

In view of these results, one may conclude that assuming the universality of the 2-way speed of light and Reichenbach's concept of time is wrong. However, that is not the case. We show that, Reichenbach's general convention of time and Edwards' universal 2-way speed of light can be accommodated in a 4-dimensional formalism which is consistent with the relativistic energy-momentum of a particle and the Lorentz and Poincaré groups. Such a theory, extended relativity, includes special relativity as a special case.

These results are physically and pedagogically interesting for the following reason: Physics students are usually puzzled by discussions of Reichenbach's convention of time and the impossibility of the unambiguous measurement of the one-way speed of light in the literature.[4] They ask: Can Reichenbach's convention of time be consistent with the Lorentz group properties of transformation between two inertial frames? Are the ideas of Reichenbach and Edwards viable? Our results suggest that the key to answering these questions is the 4-dimensional symmetry of the Lorentz group. The physical theory of taiji relativity based solely on the first postulate of relativity, i.e., the principle of relativity for physical laws, has been discussed in chapter 7. We show that the correct formalism of Edwards' theory based on weaker postulate for the speed of light can be obtained by imposing the additional second postulate of universal 2-way speed of light to taiji relativity. Thus, extended relativity gives a more restricted view of the 4-dimensional physical world than that of taiji relativity. It also shows the power of the first postulate of relativity from the vantage point of 4-dimensional symmetry.

17b. Edwards' Transformations with Reichenbach's Time

Let us consider two inertial frames F and F', where F' is moving along the x-axis. Suppose there are two identical clocks, clock 1 located at the origin of the F frame and clock 2 at point x on the x-axis. A light signal starts from the origin (event 1) at time t_1, it reaches clock 2 (event 2) at time t_2 and returns to the origin (event 3) at time t_3. Reichenbach's concept of time can be realized by synchronizing clock 2 to read t_2 by the relation[4]

$$t_2 = t_1 + \varepsilon[t_3 - t_1]\,, \tag{17.1}$$

where ε is restricted by $0 < \varepsilon < 1$, so that causality is preserved, i.e., t_2 cannot be earlier than t_1. The same synchronization procedure can be performed on clocks in the F' frame:

$$t'_2 = t'_1 + \varepsilon'[t'_3 - t'_1]\,. \tag{17.2}$$

By definition, the two-way speed c_{2w} of light along the x-axis in F is given by

$$c_{2w} = 2L/[t_3 - t_1]\,, \tag{17.3}$$

which is independent of t_2 and Reichenbach's parameter ε. Similarly, in the F' frame, we also have a constant two-way speed of light,

$$c'_{2w} = 2L'/[t'_3 - t'_1]\,. \tag{17.4}$$

Following Edwards, we shall assume these two-way speeds of light to have the same value,

$$c'_{2w} = c_{2w} = c\,. \tag{17.5}$$

The synchronization of clocks in F and F' according to (17.1)–(17.5) defines the Reichenbach time which clearly includes relativistic time as a special case ($\varepsilon = \varepsilon' = 1/2$.)

In special relativity, an Einstein clock at x is synchronized according to $t_E = t_o + x/c$ (where t_o is the time of the clock at the origin of F). The corresponding Reichenbach clock at x reads $t_R = t_o + 2\varepsilon x/c_{2w} \equiv t$, which follows from (17.1), (17.4) and (17.5). Thus, t_E and t are related by

$$t_E = t - (2\varepsilon - 1)\frac{x}{c} = t - q\frac{x}{c}, \qquad q = 2\varepsilon - 1. \tag{17.6}$$

Similarly, in the F' frame, we have the relation

$$t'_E = t' - \frac{q'x'}{c}, \qquad q' = 2\varepsilon' - 1. \tag{17.7}$$

For simplicity and without loss of generality, we set q = 0 in (17.6) , so that physics in the F frame is identical to that in special relativity. We now concentrate on physical implications of (17.7) in the F' frame to see whether Edwards' transformation is in contradiction with experimental results. Using (17.6) with q = 0, (17.7) and the Lorentz transformation involving Einstein time t_E and t'_E, one obtains the Edwards transformation between inertial frames F and F',

$$t' = \gamma\left\{(1 - \beta q')t - (\beta - q')\frac{x}{c}\right\}, \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z; \tag{17.8}$$

$$\beta = \frac{V}{c}, \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}},$$

where V is the speed of F' as seen from F. This type of transformation with Reichenbach's time was first derived and discussed by Edward.[4]

It is important to note that

(A) The coordinate vector (ct',x',y',z') is no longer a four–vector under the transformation (17.8) because (17.8) does not have the explicit 4–dimensional symmetry of the Lorentz group in general.

(B) In the limit V→0, i.e., F and F' become the same inertial frame, but (17.8) does not reduce to the identity transformation:

$$t' = (t - q'x), \quad x' = x, \quad y' = y, \quad z' = z. \tag{17.9}$$

This shows that Edwards' transformations do not form the Lorentz group, except in the special case $q'=q=0$.

(C) Under Edwards' transformations, one has

$$c^2t'^2 + 2q'x'ct' - x'^2(1 - q'^2) - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2, \qquad q=0. \tag{17.10}$$

Note that if $q \neq 0$, then $c^2t^2-x^2-y^2-z^2$ will be replaced by $c^2t^2+2qxct-x^2(1-q^2)-y^2-z^2$. It can be shown that this quadratic form holds also for infinitesimal intervals.

17c. Difficulties of Edwards' Transformations

The Edwards transformation (17.8) has been shown to be consistent with many experiments related to the propagation of light.[4] Moreover, since Edwards' transformation appears to be obtainable by a "change of time variables", (17.6) and (17.7), in the Lorentz transformation, one might think that it is equivalent to the Lorentz transformation. However, we have shown that this is not the case because Edwards' transformation (17.8) does not have the Lorentz group properties (except when $q=q'=0$) due to the transformation of t.[3] Thus, Edwards' transformation (17.8), as it stands, violates the 4-dimensional symmetry of the Lorentz group. As a result, one can show explicitly (see (17.14) below) that the time t' leads to an incorrect expression for the relativistic energy-momentum of a particle in the Lagrange formalism. This contradicts experiments in general, except when $q'=q=0$.

To wit, let us consider Edwards' transformation (17.8), where we have chosen a frame F in which the Reichenbach time equals the Einstein time by setting $q=0$. We stress that this frame can be chosen arbitrarily. (If one wishes, one can set $q'=0$ instead of $q=0$. The following arguments still hold with F and F' interchanged.) For a free particle, we have the actions S and S' in F and F' respectively:

$$S = -\int mc\, ds = \int L\, dt, \quad ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2,$$

$$L = - mc^2\sqrt{1-v^2/c^2}\ , \qquad \text{in F (with } q = 0) , \qquad (17.11)$$

$$S' = - \int mc\, ds' = \int L'dt' , \quad ds'^2 = c^2dt'^2 + 2q'dx'cdt' - dx'^2(1-q'^2) - dy'^2 - dz'^2 ,$$

$$L'(E) = - mc^2\sqrt{(1-q'v'_x/c)^2-v'^2/c^2}\ , \qquad \text{in F' (with } 1 > q' > -1) . \qquad (17.12)$$

Note that S = S' because ds = ds' which can be verified by taking differentials of (17.8) . The constant c in (17.12) is the universal 2-way speed of light measured in F'. The Lagrangians L and L' lead to the following momenta in F and F' respectively,

$$p_x(E) = \frac{\partial L}{\partial v_x} = \frac{mv_x}{\sqrt{1-v^2/c^2}} \equiv p_x , \quad \text{etc.} \quad \text{in F} , \qquad (17.13)$$

$$p'_x(E) = \frac{\partial L'}{\partial v'_x} = \frac{(mcq' - mq'v'_x + mv'_x)}{\sqrt{(1-q'v'_x/c)^2-v'^2/c^2}} , \qquad \text{etc.} \quad \text{in F'}. \qquad (17.14)$$

We see clearly that the momentum $p'_x(E)$ in F' contradicts experimental results, except when q'=q=0. Similarly, one can show that the result for the kinetic energy, defined by $\mathbf{p}'(E)\cdot\mathbf{v}' - L'$, in F' is also incorrect.

Before dismissing the expression (17.14), one should check whether (17.14) is the same as the result of changing the time variable in the normal special relativistic momentum $p'_x(SR)$,

$$p'_x(SR) = \frac{mu'_x}{\sqrt{1-u'^2/c^2}} , \qquad u'_x = \frac{dx'}{dt'_E} , \quad \text{etc.} \quad \text{in F'}. \qquad (17.15)$$

Under a change of time variable given in (17.7), we have the relation $dx'/dt' = (dx'/dt'_E)(dt'_E/dt')$, etc., i.e.,

$$u' = \frac{v'}{(1 - q'v'_x/c)} . \qquad (17.16)$$

From equations (17.15) and (17.16), we obtain

$$p'_x(SR) = \frac{mv'_x}{\sqrt{(1-q'v'_x/c)^2-v'^2/c^2}} \neq p'_x(E) , \qquad (17.17)$$

which differs from the momentum $p'_x(E)$ in (17.14). *This example shows that the lack of 4-dimensional symmetry in Edwards' formalism based on transformation (17.8) with $1 > q' > -1$ make it untenable.* Of course, there are many other difficulties related to the lack of symmetry such as the non-invariance of the Maxwell equations, the Klein-Gordon equation and laws in quantum electrodynamics under the transformations (17.8).

In the next section, however, we show that if one uses 4-dimensional symmetry of taiji relativity as a guiding principle, all these difficulties can be resolved.

17d. Extended Relativity — A 4-dimensional Theory with Reichenbach's Time

The method for the construction of a 4-dimensional symmetry framework without the usual relativistic time has been discussed before.[3,5] The logically simplest case is the 4-dimensional symmetry of taiji relativity which is based solely on the first postulate of relativity. It can be applied to guide the construction of a 4-dimensional framework for the present case with Reichenbach's time. For simplicity and without loss of generality, we choose q=0 in synchronizing clocks in the F frame, so that we have the usual relativistic time $t=t_E$. In the F' frame, clocks are synchronized to read the Reichenbach time t'. An event is, as usual, denoted by (w,x,y,z), where w=ct, in F. However, following taiji relativity discussed in chapter 7, the same event must be denoted by (w',x',y',z'), with w'=b't', in F' within the 4-dimensional symmetry framework. We stress that *it is necessary to introduce the function b' so that $(w',x',y',z')=(w',\mathbf{r}')$ transforms like a four vector and the laws of physics can display 4-dimensional symmetry.* Otherwise, the laws of physics cannot be invariant under the transformation from F to F'. With clocks in F and F' synchronized as discussed previously, the times t and t' are related as in (17.8):

$$t' = \gamma\left[(1-\beta q')t - (\beta - q')\frac{x}{c}\right], \qquad q' = 2\varepsilon' - 1 . \tag{17.18}$$

This may be considered as an assumption. Indeed, from the viewpoint of taiji relativity, this is a second postulate which is the analogous to assuming the universality of the two-way speed of light over a closed path in any inertial frame, as discussed in section 17b. As usual, we start with the invariance of the 4-dimensional interval s^2,

$$s^2 = b'^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2 , \qquad (17.19)$$

to derive the 4-dimensional transformation. Note that relation (17.19) follows from the first postulate of relativity, i.e., the Poincaré-Einstein principle, because it is equivalent to the law of motion, $p_0^2 - p^2 = m^2$, for a free particle with mass $m>0$. (See ref. 7 in chapter 5.) By the usual method for deriving the Lorentz transformation or the taiji transformation (7.4), we can derive the extended 4-dimensional transformation involving Reichenbach's time,

$$w' \equiv b't' = \gamma(ct - \beta x) , \quad x' = \gamma(x - \beta ct) , \quad y' = y , \quad z' = z ; \qquad (17.20)$$

$$\beta = \frac{V}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} .$$

where t' is given by equation (17.18). This extended transformation completely determines the function b'

$$b' = \frac{(ct - \beta x)}{\left[(1 - \beta q')t - (\beta - q')\frac{x}{c}\right]} , \qquad (17.21)$$

i.e., if x and t are known, we can always calculate the value of b' in (17.20) and (17.21). Note that the *fourth dimension in F' is now $b't' \equiv w'$ rather than t' or ct'*, where $w' = b't'$ may be called "lightime". Although the function b' in (17.21) and the time t' in (17.18) separately have complicated transformation properties, we stress that b' and t' are separately well-defined and that the product $b't' \equiv w'$ transforms as the fourth component of the coordinate 4-vector. The property of b' is completely determined by (17.21) or the transformation (17.20). From now on we will use $w' \equiv b't'$ to display explicitly the 4-dimensional symmetry of physical laws.

The extended transformation of velocities can be derived from (17.20):

$$c' \equiv \frac{d(w')}{dt'} = \frac{dt}{dt'}\gamma(c - \beta v_x) ,$$

$$v'_x = \frac{dt}{dt'}\gamma(v_x - \beta c) , \tag{17.22}$$

$$v'_y = \frac{dt}{dt'} v_y , \qquad v'_z = \frac{dt}{dt'} v_z ,$$

where dt/dt' may be obtained from differentiation of (17.18),

$$\frac{1}{[dt/dt']} = \frac{dt'}{dt} = \gamma\left\{(1 - \beta q') - (\beta - q')\frac{dx}{cdt}\right\} . \tag{17.23}$$

We stress that the definition $c' \equiv d(w')/dt'$ is quite natural because $dw'^2 - dr'^2 = 0$ is to be interpreted as the law of light propagation, $c'^2 dt'^2 - dr'^2 = 0$. Note that the transformation of the ratios v'/c' are precisely the same as those in special relativity

$$\frac{v'_x}{c'} = \frac{(\frac{v_x}{c} - \beta)}{(1 - \frac{v_x\beta}{c})} , \quad \frac{v'_y}{c'} = \frac{\frac{v_y}{c}}{\gamma(1 - \frac{\beta v_x}{c})} , \quad \text{etc.} \tag{17.24}$$

Let us consider the property of c' in F'. Its value depends on its direction of propagation. Suppose a light signal propagates along an angle θ as measured in F and θ' as measured in F', we have

$$v_x = c\cos\theta , \qquad v'_x = c'\cos\theta' . \tag{17.25}$$

From the inverse transformation of (17.22), we obtain $c = (dt'/dt)[\gamma(c' + \beta v'_x)]$, i.e.,

$$c' = \frac{c}{(dt'/dt)[\gamma(1 + \beta\cos\theta')]} = c'(\theta') . \tag{17.26}$$

Evidently, the average speed of light over any closed path in F is a constant which is equal to c. Now we will prove that the average speed of a light signal over any closed path is also c, though the speed of light is no longer isotropic in F'.

Suppose a light signal travels along the vectors $\mathbf{r}'_i$, where i = 1,2,...N, which form a closed path on the x'–y' plane in F'. The total distance L_{tot} and the total time T'_{tot} are given by

$$L'_{tot} = \sum_{i=1}^{N} r'_i, \quad \text{and} \quad T'_{tot} = \sum_{i=1}^{N} \frac{r'_i}{c'(\theta'_i)}; \qquad N > 1. \tag{17.27}$$

The average speed c'_{av} of the light signal over this closed path is

$$c'_{av} = \frac{L'_{tot}}{T'_{tot}} = \frac{L'_{tot}}{\left\{\frac{L'_{tot}}{c} + \frac{q'}{c}\sum_{i=1}^{N} r'_i \cos\theta'_i\right\}} = c, \tag{17.28}$$

where we have used

$$v'_{xi} = c'(\theta'_i)\cos\theta'_i, \tag{17.29}$$

$$c'(\theta'_i) = \frac{c}{\{(dt'/dt)_i[\gamma(1+\ \beta\cos\theta'_i)]}, \tag{17.30}$$

$$(dt'/dt)_i = \frac{(1\ +\ q'\cos\theta'_i)}{[\gamma(1\ +\ b\cos\theta'_i)]}, \qquad i = 1,2,....,N, \tag{17.31}$$

$$\sum_{i=1}^{N} r'_i \cos\theta'_i = 0. \tag{17.32}$$

Equation (17.32) is a property of a closed path in F'. Thus, the average speed of light over an arbitrary closed path is a universal constant c in extended relativity.

We note that a closed path for a light signal as observed by F observers is, in general, not a closed path as observed by F' observers. Suppose a signal starting from O(or O'), i.e., the origin of F (F'), travels to a point A (A') on the x-axis and is reflected back to B = O (B'≠ O) as observed in F (F'). The two-way speed of this signal in F is c. One may ask however, what is the average speed of this light signal as measured in F'? From Reichenbach's time (17.18) and extended transformation (17.20), we derive the space and time coordinates of these events in F',

$$t'(O') = x'(O') = 0\,, \tag{17.33}$$

$$t'(A') = \gamma\left\{[1 - \beta q']t(A) - (\beta + q')\frac{x(A)}{c}\right\}, \quad x'(A') = \gamma\,(x(A) - \beta ct(A))\,, \tag{17.34}$$

$$t'(B') = \gamma\left\{[1 - \beta q']t(B) - (\beta + q')\frac{x(B)}{c}\right\}, \quad x'(B') = \gamma\,(x(B) - \beta\, ct(B))\,. \tag{17.35}$$

Since $t(O)=x(O)=x(B)=0$, $x(A)=ct(A)=L$ and $t(B)= 2L/c$, the equations in (17.33)–(17.35) lead to the "average speed" $c'_{av}(nc)$ for such a non–closed path in F':

$$c'_{av}(nc) = \frac{x'(A') + |x'(B') - x'(A')|}{t'(A') + |t'(B') - t'(A')|}$$

$$= \frac{c}{[1 - \beta q']}\,. \tag{17.36}$$

Result (17.36) can also be derived from (17.22) directly because this average speed of light can be considered as two events which satisfy $\Delta x = x(B) - x(O) = 0$. As far as constant velocities are concerned, $\Delta x = 0$ is equivalent to $dx = 0$ or $dx/dt = v_x = 0$. Now we use the relation (17.23) with the condition $v_x = 0$, we obtain

$$(dt'/dt) = \gamma(1 - \beta\, q')\,, \qquad v_x = 0\,. \tag{17.37}$$

Thus using the expression for c' in (17.22) for a non–closed path in F', we also have the relation

$$c'(v_x{=}o) = \frac{c}{(1 - \beta q')}\,, \tag{17.38}$$

which is precisely the result (17.36).

17e. Two Basic Postulates of Extended Relativity

The physical foundation of the extended transformation (17.20) is based on two postulates: The invariance of the form of physical laws and Reichenbach's time (17.18). The latter may also be regarded as a definition of time.

As we have discussed previously, relation (17.18) is directly related to clock synchronization. To be specific, we have used the relation between Reichenbach's and Einstein's times, together with the Lorentz transformation, to obtain (17.18).

One may ask: Is it possible to derive the extended transformations (17.20) and (17.18) without using Einstein's time and the Lorentz transformation as a crutch? The answer is yes. Instead of postulating (17.18), one can use its equivalent postulate: Namely, that the 2-way speed of light over a closed path in any frame is a universal constant and independent of the motion of the light sources. This (second) postulate was first made by Edwards for his formalism.[4]

Let us now derive the relation (17.18) based on the two basic postulates of extended relativity:

[1]. The principle of relativity for physical laws: The form of a physical law must be invariant under coordinate transformations. (In other words, physical laws must display 4-dimensional symmetry.)

[2]. The 2-way speed of light over a closed path is a universal constant and is independent of the motion of sources.

To derive (17.18) from these two postulates, we first observe that, starting from (17.19) and following the steps from (7.2) to (7.4), we obtain (17.20) but not (17.18). Since the F' frame moves along the x-axis, its time t' can only be a linear function of t and x,

$$t' = Pt + Qx. \tag{17.39}$$

where unknown coefficients P and Q are to be determined. From (17.22) and (17.39), we have

$$c'(+) = \left(\frac{dt}{dt'}\right)_1 c\gamma(1-\beta), \qquad \frac{dx}{dt} = +c, \tag{17.40}$$

$$c'(-) = \left(\frac{dt}{dt'}\right)_2 c\gamma(1+\beta), \qquad \frac{dx}{dt} = -c, \tag{17.41}$$

where

$$\left(\frac{dt}{dt'}\right)_1 = P + Qc, \tag{17.42}$$

$$\left(\frac{dt}{dt'}\right)_2 = P - Qc. \tag{17.43}$$

Postulate [2] implies that

$$\frac{L'}{c'(+)} + \frac{L'}{c'(-)} = \frac{2L'}{c'_{2w}} = \frac{2L'}{c}. \tag{17.44}$$

It follows from equations (17.40) -(17.44) that

$$P + \beta cQ = \frac{1}{\gamma}. \tag{17.45}$$

Without loss of generality, we may express Q in terms of another parameter q' such that (17.39) is more closely related to the form (17.18) in Edwards' transformation,

$$Q = -\frac{\gamma(\beta + q')}{c}. \tag{17.46}$$

Relations (17.39), (17.45) and (17.46) lead to

$$t' = \gamma\left[(1 - \beta q')t - (\beta + q')\frac{x}{c}\right], \tag{17.47}$$

which is precisely the basic transformation of Reichenbach's time (17.18) .

17f. Invariant Action for a Free Particle in Extended Relativity

Although extended relativity involves a class of different concepts of time, realized by Reichenbach's procedures for clock synchronization based on a universal 2-way speed of light (over a closed path), all physical laws have the 4-dimensional form which are identical to those in special relativity.

Let us first demonstrate that extended relativity leads to a correct expression for momentum, in contrast to Edwards' formalism. The invariant action for a free particle in F' is

$$S' = -\int mc\, ds' = \int L'dt', \qquad c = c'_{2w} = c_{2w}, \tag{17.48}$$

$$ds'^2 = c'^2dt'^2 - dx'^2 - dy'^2 - dz'^2, \qquad c' = \frac{d(w')}{dt'}.$$

Note that c in (17.48) is the two-way speed of light, c_{2w}, which is a universal constant. The Lagrangian L' in the F' frame takes the form

$$L' = -mcc'\sqrt{1-v'^2/c'^2}, \tag{17.49}$$

which leads to the momentum **p'**,

$$\mathbf{p}' = \frac{(mc\mathbf{v}'/c')}{\sqrt{1-v'^2/c'^2}}. \tag{17.50}$$

The "energy" p'_0 is defined as the zeroth component of the momentum by

$$p'_0 = \frac{(\mathbf{p}'\cdot\mathbf{v}' - L')}{c'} = \frac{mc}{\sqrt{1-v'^2/c'^2}}. \tag{17.51}$$

These form the momentum 4-vector $p'_\mu=(p'_0,-\mathbf{p}')$ which satisfies the 4-dimensional invariant relation

$$p'^{\,2}_0 - \mathbf{p}'^2 = m^2c^2. \tag{17.52}$$

By Noether's theorem, this is the conserved energy-momentum in extended relativity.

We see clearly that the momentum **p'** and the energy p'_0 in F' is consistent with that measured in high energy experiments. As a matter of fact, the momentum p'_x given by (17.50) is precisely the same as the momentum $p'_x(SR)$ in (17.15) for the F' frame in special relativity. The reason is that relations (17.16) and (17.7) and the invariant property that ds^2 in F' for special relativity is the same as ds'^2 given by (17.48), {i.e. $ds^2 = c^2dt'^2_E - dx'^2 - dy'^2 - dz'^2 = c'^2dt'^2 - dx'^2 - dy'^2 - dz'^2$}:

$$\mathbf{p}'(SR) = \frac{m\mathbf{v}'c}{(1 - q'v'_x/c)\sqrt{c^2 - u'^2}} = \mathbf{p}'. \tag{17.53}$$

This example shows that for Reichenbach's time to be viable it must be embedded within a 4–dimensional framework.

Of course, the Klein–Gordon equation takes the form

$$(\partial_\mu^2 - m^2)\,\varphi = 0\,. \tag{17.54}$$

It can be shown to be invariant under the extended transformation (17.20) as well because the differential operators

$$\partial_\mu = \frac{\partial}{\partial x^\mu}\,, \qquad \text{and} \qquad \partial'_\mu = \frac{\partial}{\partial x'^\mu}\,, \tag{17.55}$$

are 4–vectors, where $x^\mu=(ct,\mathbf{r})$, $x'^\mu=(w',\mathbf{r}')=(b't',\mathbf{r}')$. It is interesting to see from (17.54) that, although one has times t and t' in extended relativity, the evolution variable in basic 4-dimensional equation is the lightime w'=b't' rather than Reichenbach's time t' in the F' frame. This is dictated by the 4-dimensional symmetry of the Lorentz and Poincaré groups and is in harmony with the evolution variable in taiji relativity.

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18.

Extended Relativity with the Lorentz Group and Lifetime Dilatation

18a. A Comparison of Extended Relativity and Special Relativity

Although Edwards' original transformations (17.8) can be obtained by a "change" of the time variable (17.7) in the Lorentz transformations, this does not imply that the two transformations are completely equivalent *experimentally and theoretically.* The reason is that it is not simply a change of variable within the framework of special relativity. To be specific, such a change of variable violates the second postulate of special relativity, i.e. the universal constancy of the speed of light. As a result, after the "change" of the time variable in (17.7), one is no longer within the same conceptual framework because, strictly speaking, special relativity is a theory which possesses the 4-dimensional symmetry with the Einstein time t'_E which has the usual time dilatation property. However, after the change of time variable in (17.7), one has extended relativity which possesses the 4-dimensional symmetry with the Reichenbach time t' given by (17.47) which does not have in general the usual time dilatation property. Nevertheless, we show in section 18d below that the experimental results of lifetime dilatation of unstable particles' decay in flight can be understood within extended relativity in terms of the dilatation of decay length.

We have shown in (17.12) that Edwards' original transformations lead to an incorrect momentum in the Lagrangian formalism because of its lack of 4-dimensional symmetry, as shown in equations (17.13)-(17.17). *In this sense, the first principle of relativity for physical laws has not been really incorporated in Edwards' formalism of relativity. The basic reason is that his formalism does not have 4-dimensional symmetry.* In particular, (ct',x',y',z') is not a four-vector. In sharp contrast, the present formalism of extended relativity based on the new transformations (17.20) between the coordinate four-vectors (w,x,y,z) and (w',x',y',z'), where w=ct and w'=b't', is explicitly consistent with the 4-dimensional symmetry of the Lorentz and the Poincaré groups or the Poincaré-Einstein principle of relativity.

If one compares the space-lightime transformations (17.20) with the Lorentz transformations, one sees that

$$ct'_E = b't', \tag{18.1}$$

where t'_E and t' are Einstein's time and Reichenbach's time, respectively. We note that the relation (18.1) does not imply that extended relativity is *an ordinary change of the time variable within the framework of special relativity.* All that it implies is that if the fourth component ct'_E of a coordinate 4–vector in special relativity is replaced by $b't'$, where b' is a function, one will get another 4–dimensional symmetry framework without the universal constant of the speed of light c. Therefore, *(18.1) is simply the direct consequence of replacing Einstein's second postulate by Edwards' weaker second postulate.*

If one takes the viewpoint that extended relativity is just a change of time variables given by (18.1), one should be able to obtain all results in extended relativity from the corresponding result in special relativity by the relation (18.1). For instance, the momentum (17.50) indeed can be obtained from the usual relativistic momentum (17.15) by a simple change of variable (18.1). However, this is not always the case. For example, a plane wave in F' is described by a Lorentz invariant function,

$$\exp\{i(\omega' t'_E - \mathbf{k}'\cdot\mathbf{r}')\}, \tag{18.2}$$

in special relativity. By a simple change of time variable (18.1), one obtains

$$\exp\{i(b't'\frac{\omega'}{c} - \mathbf{r}'\cdot\mathbf{k}')\}, \tag{18.3}$$

which leads to an *incorrect* wave 4–vector $(\omega'/c, \mathbf{k}')$ in extended relativity. The correct wave 4–vector in the F' frame should be

$$(k'^0, \mathbf{k}') = (\frac{\omega'}{c'}, \mathbf{k}'). \tag{18.4}$$

This is the only wave 4–vector which is consistent with the relation

$$v'\lambda' = c', \tag{18.5}$$

defined in extended relativity. We note that if quantum mechanics is formulated on the basis of extended relativity, the 4–momentum of a photon should be proportional to the extended wave 4–vector (18.4) rather than (ω'/c, $\mathbf{k}'$).

As a matter of fact, based on equation (18.2) one can obtain the correct invariant phase, involving (18.5) and the 4–coordinate (w',x',y',z') by changing both time and quantities related to time (e.g., frequency). *In doing this, essentially one must first know the correct wave 4–vector in extended relativity, i.e., one must be able to formulate 4–dimensional extended relativity independent of special relativity.* In this sense, it is gratifying that we are able to formulate a 4–dimensional theory of extended relativity based on its own basic postulates, as shown in section 17e.

From previous discussions, we conclude that *extended relativity and special relativity are two logically different theories* — the first has universal one–way speed of light and the second has universal two–way speed of light (which includes the former as a special case.) Nevertheless, they both have 4–dimensional symmetry of the Lorentz group, so that physical laws have the same forms in all inertial frames. (See section 18c.)

We have examined all known experiments. It turns out that these experiments cannot distinguish extended relativity from special relativity. For any invariant law in special relativity, there is a corresponding law of the same form in extended relativity. *This suggests that extended relativity is experimentally equivalent to special relativity.* We believe this indicates that physical properties such as the concepts of time (e.g., Einstein's time or Reichenbach's time, etc.) and the corresponding speed of light in the 4–dimensional symmetry framework are human conventions rather than the inherent nature of the physical world. This conclusion was already indicated by the results of taiji relativity.

18b. An Unpassable Limit and Non–Constant Speed of Light

From the expressions for extended relativistic momentum and energy in (17.50) and (17.51) or the extended transformations of velocities (17.22), one can see an interesting property of the non–constant speed of light c'. Namely, although c' differs in different directions in F', if one compares speeds of various physical objects in a given direction in F', the speed of light is *the*

maximum speed in the universe. This holds for all values of the parameter q' in (17.22). We stress that this "maximum speed" is a more general property than the speed of light being a universal constant because the latter corresponds to the special case q'=0 in (17.22). It is interesting to note that this property was discussed as a "character" of an entirely new mechanics by Poincaré in his vision of a relativity theory in 1904:[1]

"No velocity could surpass that of light."

According to taiji relativity, this property of unpassable limit is true for any relation between t and t', as the 4-dimensional theory holds for their unspecified relation (7.14). If one replaces (7.14) by any specific relation such as (7.17), (7.18) or (17.18), one still has this "character," provided the law of propagation of light displays the 4-dimensional symmetry. In this sense, the character of an entirely new mechanics is logically implied by Poincaré's first postulate, i.e., the principle of relativity.

Furthermore, even though the velocities c' and v' measured by using Reichenbach's time depend on the parameter q', the experimental results (such as the Michelson-Morley experiment, the conservation laws of momentum **p'** and 'energy' p'^0, the dilatation of lifetimes, etc.) turn out to be independent of q'. For example, the momentum **p'** in (17.50) depends on the ratio v'/c'=(dr'/dt')/(dw'/dt')=dr'/dw' which is independent of time t' or q'. Thus, the restriction for q' in (17.12) is really not necessary from an experimental viewpoint. The basic reason for these properties is that, within the 4-dimensional symmetry framework, the inherent evolution variable in the laws of physics is the lightime w' rather than Reichenbach's time t', as already indicated by taiji relativity. (See chapter 7.)

18c. The Lorentz Group and the Space-Lightime Transformations

If an object is at rest in the F frame, i.e., v = (0,0,0), the extended velocity transformations (17.22) lead to

$$c' = \frac{c}{(1-\beta q')} \equiv c'(0)\,,$$

$$v'_x = -\frac{\beta c}{(1-\beta q')} \equiv v'_x(0)\,, \qquad v'_y = v'_z = 0. \tag{18.6}$$

The result (18.6) implies $v'_x(0) \neq -V$ $(= -\beta c)$, which shows that the speed of the F frame, measured from F', is different from the speed of F' measured from F and, moreover, the difference depends on the parameter q'. When one thinks about it, one may be puzzled because common sense tells us that the speed of F measured from F' should be $-V$. However, as can be seen from special relativity and quantum mechanics, common sense is often a poor guide in modern physics. Let us consider the ratio $v'_x(0)/c'(0)$ in (18.6). This ratio is a constant, independent of dt/dt', and satisfies

$$\beta' \equiv -\frac{v'_x(0)}{c'(0)} = +\beta. \tag{18.7}$$

Thus, using the ratios of velocities, β' and β, one has the symmetry of velocities between F and F'. The reason for the simplicity shown in (18.7) is simply that the real evolution variable in extended relativity is the lightime $w'=b't'$ rather than the Reichenbach time t' in F'. The inverse transformation of (17.20) can be obtained. It takes the form:

$$ct = \gamma(b't' + \beta'x') = \gamma'(b't' + \beta'x'),$$

$$x = \gamma'(x' + \beta'b't'), \quad y = y', \quad z = z'; \qquad \gamma' = \gamma. \tag{18.8}$$

Let us consider also another frame F" moving with a constant velocity $\mathbf{V_1} = (V_1,0,0)$, as measured from F and $\mathbf{V'_1}=(V'_1,0,0)$, as measured in F'. Note that the ratio V'_1/c' is related to V_1 by (17.24):

$$\frac{V'_1}{c'} = \frac{(V_1/c - \beta)}{(1 - \beta V_1/c)}. \tag{18.9}$$

From (18.6), (18.7) and (18.9), we see that instead of V or V', one should use the ratio V/c or V'/c' to characterize the relative motion between inertial frames in extended relativity, as *this ratio will always be constant and more importantly, is independent of the parameter q'.*

The 4–dimensional transformations between F and F" are given by

$$b''t'' = \gamma_1(ct' - \beta_1 x), \quad x'' = \gamma_1(x - \beta_1 ct), \quad y'' = y, \quad z'' = z, \tag{18.10}$$

where

$$t'' = \gamma_1[(1-\beta_1 q'')ct - (\beta_1 - q'')x]\,\frac{1}{c}\,. \tag{18.11}$$

From (17.20) and (18.10), we can obtain the transformations between F' and F",

$$b''t'' = \gamma'_1(b't' - \beta'_1 x'), \quad x'' = \gamma'_1(x' - \beta'_1 b't'), \quad y'' = y', \quad z'' = z', \tag{18.12}$$

where

$$\beta'_1 = \frac{(\beta_1 - \beta)}{(1 - \beta_1\beta)} = \frac{V'_1}{c'}, \qquad \beta_1 = \frac{V_1}{c}, \tag{18.13}$$

$$\gamma'_1 = \gamma_1\gamma(1 - \beta_1\beta) = \frac{1}{\sqrt{1-\beta_1'^2}}, \qquad \gamma_1 = \frac{1}{\sqrt{1-\beta_1'^2}}, \tag{18.14}$$

$$t'' = \gamma'_1\Big[(1 - \beta'_1 q'')b't' - (\beta'_1 - q'')x'\Big]\frac{1}{c}\,. \tag{18.15}$$

This result (18.12), together with other properties such as the existence of an inverse transformation and associativity, demonstrates that *the set of extended 4-dimensional transformations forms precisely the Lorentz group.* This shows explicitly that the Lorentz group can accommodate a weaker postulate for the speed of light. Lorentz group property is the core of the 4-dimensional symmetry and is *crucial for extended relativity to be consistent with experiments and to be applicable to quantum field theories.*

18d. The Decay Rate and "Lifetime Dilatation"

Fundamental wave equations such as (17.53) with (17.54) show that 4-dimensional symmetry dictates the evolution parameter to be the lightime w rather than t in a general inertial frame. If we consider the decay rate $\Gamma(1\rightarrow 2+3+...+N)$ for a physical process $1\rightarrow 2+3+...+N$, we can follow steps from (11.1) to (11.4) because we have 4-dimensional symmetry in extended relativity. We may remark that this cannot be done in Edwards' original formulation because of the lack of explicit 4-dimensional symmetry in his transformations. In a general inertial frame with (w,x,y,z), we obtain precisely the same form as (11.4) for the decay rate:

$$\Gamma(1\rightarrow 2+3+...+N) = \lim_{w\rightarrow\infty}\int \frac{c|\langle f|S|i\rangle|^2}{w}\,\frac{d^3x_2 d^3p_2}{(2p_{02}\hbar)^3}\cdots\frac{d^3x_N d^3p_N}{(2p_{0N}\hbar)^3}\,. \qquad (18.16)$$

This definition of decay rate has the dimension of inverse time because of the presence of c in (18.16). The decay lifetime τ is given by $\tau=1/\Gamma(1\rightarrow 2+3+...+N)$. To illustrate the calculation of a decay rate, let us consider a simple example, i.e., the muon decay $\mu^-(p_1)\rightarrow e^-(p_2) + \nu_\mu(p_2) + \bar{\nu}_e(p_4)$ with the usual V–A coupling. Following the steps in chapter 11, the muon lifetime τ can be calculated and the result is[2]

$$\frac{1}{\tau} = \Gamma(1\rightarrow 2+3+...+N)$$

$$\propto \frac{1}{p_{01}}\int\frac{d^3p_2}{p_{02}}\,\frac{d^3p_3}{p_{03}}\,\frac{d^3p_4}{p_{04}}\,\delta^4(p_1-p_2-p_3-p_4)\sum_{spin}|M_{sc}|^2\,. \qquad (18.17)$$

Everything to the right of $1/p_{01}$ in (18.17) is invariant under the extended transformation so that the "decay lifetime" τ is indeed dilated by the usual γ factor:

$$\tau \propto \sqrt{\mathbf{p}_1{}^2+m_1{}^2} = p_{01} \propto \gamma. \qquad (18.18)$$

One should keep in mind, however, that it is really the decay length τc which is measured in the laboratory. We stress that such a "decay time" as measured in the F' frame is directly related to lightime (w'=b't') rather than Reichenbach's time t', as one can see from (18.16).

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19.

Physical Implications of Extended Relativity

19a. 4-Dimensional Symmetry with a Universal 2-Way Speed of Light

Guided by the 4-dimensional symmetry of taiji relativity, Reichenbach's general convention of time (or, equivalently, the universal 2-way speed of light) can be used as the "second postulate" for the construction of a theory of extended relativity with new 4-dimensional coordinate transformations. A second postulate is necessary to factorize, say, w' in the F' frame into a well-defined function b' (i.e., 'ligh' function) and Reichenbach's time t', i.e., w'=b't' which is called 'lightime'. It turns out that the lightime w', rather than Reichenbach's time, plays the role of evolution variable in physical laws because the lightime w' transforms as the zeroth component of a 4-vector and Reichenbach's time does not. This property and w'=b't' are crucial and they make extended relativity consistent with established energy-momentum of a particle, the Lorentz group and so on, as shown in chapters 17 and 18. Furthermore, we shall show below that the covariant lightime embedded in the 4-dimensional symmetry is also crucial for the formulation of a covariant quantum electrodynamics (QED) based on extended relativity.

Using Reichenbach's procedure to synchronize clocks in every inertial frame amounts to imposing a second postulate[1,2] upon taiji relativity, so that we have a well-defined 'extended' time, which includes Einstein's time as a special case. For simplicity and without loss of generality, we choose $q = 0$ in the synchronization of clocks in the F frame, so that we have Einstein's time $t = t_E$ in F as one can see from the relation (17.6); while in the F' frame, clocks are synchronized to read Reichenbach's time t'. Thus, an event is denoted by (ct,x,y,z) in F. Following taiji relativity, the same event must be denoted by (b't',x',y',z') in F' which is moving along the +x axis with a constant velocity $\beta = V/c$, as measured in F. We stress that *it is necessary to introduce the "ligh function" b' so that (b't',x',y',z') = (w', r') transforms like a four vector and the laws of physics can display 4-dimensional symmetry.* It was shown by Edwards on the basis of the universal 2-way speed of light that times t and t' were related by

$$t' = \gamma\left[(1-\beta q')t - (\beta - q')\frac{x}{c}\right], \qquad q' = 2\varepsilon' - 1, \quad \beta = V/c, \tag{19.1}$$

which was the basic property of Reichenbach's time.[2,3] Assuming the validity of relation (19.1) is effectively the same as assuming the universality of the 2-way speed of light over a closed path in any inertial frame within a 4-dimensional framework, as shown in (17.1)-(17.8). In analogy to the derivation of transformations for taiji relativity in chapter 7, the 4-dimensional space-lightime transformations for (ct,x,y,z) and (b't',x',y',z') in extended relativity can be obtained as follows:

$$w' \equiv b't' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z; \tag{19.2}$$

$$\beta = \frac{V}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad b' = c - \frac{q'x'}{t'},$$

where the transformation of t' and b' are separately given by (19.1) and (17.21). Note that t' and b' in (19.1) and (19.2) separately are non-covariant and have complicated transformation properties. However, their product, w'=b't', is a covariant quantity and has a simple transformation property as the zeroth component of a 4-vector. The physical realization of a clock system for the lightime w' in the F' frame will be discussed later in section 19f. It should be noted that without having this "ligh function" b' one cannot formulate a physical theory of extended relativity with the 4-dimensional symmetry of the Lorentz and Poincaré groups.

The extended transformations of velocities and accelerations, $\mathbf{v} = d\mathbf{r}/dt$, $\mathbf{v}' = d\mathbf{r}'/dt'$, $\mathbf{a} = d\mathbf{v}/dt$ and $\mathbf{a}' = d\mathbf{v}'/dt'$, can be derived from (19.2). We obtain

$$c' \equiv \frac{d(b't')}{dt'} = \frac{dt}{dt'}\gamma(c - \beta v_x),$$

$$v'_x = \frac{dt}{dt'}\gamma(v_x - \beta c), \qquad v'_y = \frac{dt}{dt'}v_y, \qquad v'_z = \frac{dt}{dt'}v_z; \tag{19.3}$$

and

$$a'_t = \frac{dc'}{dt'} = \frac{d^2t}{dt'^2}\gamma(c - \beta v_x) - \left(\frac{dt}{dt'}\right)^2\gamma\beta a_x ,$$

$$a'_x = \frac{d^2t}{dt'^2}\gamma(v_x - \beta c) + \left(\frac{dt}{dt'}\right)^2\gamma a_x , \qquad (19.4)$$

$$a'_y = \frac{d^2t}{dt'^2} v_y + \left(\frac{dt}{dt'}\right)^2 a_y , \quad a'_z = \frac{d^2t}{dt'^2} v_z + \left(\frac{dt}{dt'}\right)^2 a_z ;$$

where dt/dt' may be obtained from (19.1),

$$\frac{1}{dt/dt'} = \frac{dt'}{dt} = \gamma\{[1-\beta q'] - [\beta - q']\frac{v_x}{c}\} , \qquad (19.5)$$

$$\frac{d^2t}{dt'^2} = \frac{dt}{dt'}Z , \qquad Z \equiv \frac{(\beta-q')a_x/c}{(1-\beta q')-(\beta-q')v_x/c} .$$

We stress that the definition $c' \equiv d(b't')/dt'$ is quite natural because the invariant equation $ds^2 = 0$, i.e., $[d(b't')]^2 - d\mathbf{r}'^2 = 0$, is the law of the propagation of light, $c'^2dt'^2 - d\mathbf{r}'^2 = 0$, in extended relativity. *Thus,we still have an invariant law for the propagation of light in all inertial frames, although the speed of light is not a constant and Reichenbach's time is not covariant.* Furthermore, the transformations of the velocity ratios $\mathbf{v}'/c'$ turn out to be precisely the same as those in special relativity and taiji relativity:

$$\frac{v'_x}{c'} = \frac{(v_x/c - \beta)}{(1-v_x\beta/c)} v, \qquad \frac{v'_y}{c'} = \frac{(v_y/c)}{\gamma(1-\beta v_x/c)} , \quad \text{etc.} \qquad (19.6)$$

These relations are necessary for the space-lightime coordinate transformations (19.2) to form the Lorentz group. For the ratios of accelerations, we have

$$\frac{a'_x}{a'_t} = \frac{Z(v_x - c\beta) + (dt/dt')\gamma a_x}{Z(v_x - c\beta) - (dt/dt')\gamma\beta a_x} , \qquad (19.7)$$

$$\frac{a'_y}{a'_t} = \frac{Zv_y + (dt/dt')a_y}{Z(v_x - c\beta) - (dt/dt')\gamma\beta a_x} , \quad \text{etc.},$$

which are more complicated than those in special relativity because of the parameter $q' \neq 0$ in the function Z given by (19.5).

19b. Some Experimental Implications of Extended Relativity

The momentum 4-vector $p^\mu = (p^0,\mathbf{p}) = (p_0,p_x,p_y,p_z)$ and wave 4-vector $k^\mu = (k_0,k_x,k_y,k_z)$ transform like a coordinate four-vector (19.2). For example, we have

$$k_0' = \gamma(k_0 - \beta k_x), \quad k'_x = \gamma(k_x - \beta k_0), \quad k'_y = k_y, \quad k'_z = k_z. \tag{19.8}$$

For a light wave, $\mathbf{k} = (k\cos\theta, k\sin\theta, 0)$ and $\mathbf{k}' = (k'\cos\theta', k'\sin\theta', 0)$, eq. (19.8) leads to the usual formula for the aberration of star light,

$$\cos\theta' = \frac{(\cos\theta - \beta)}{(1 - \beta\cos\theta)}, \tag{19.9}$$

$$\sin\theta' = \frac{\sin\theta}{\gamma(1 - \beta\cos\theta)}, \tag{19.10}$$

where we have used $k = |\mathbf{k}| = k_0$ for a light wave.

In the Fizeau experiment, the observed drag coefficient can be explained by the addition law (19.6) for velocity ratios with $v_x/c = 1/n$:

$$\frac{v'_x}{c'} = \frac{(1/n - \beta)}{(1 - \beta/n)}, \qquad \beta = \frac{V}{c}. \tag{19.11}$$

The speed of light relative to the medium (at rest in the F frame) is $v_x = c/n$ in F, where n is the refractive index of the medium. Now, if the medium is moving with speed $V = \beta c$ parallel to the direction of the light, the ratio v'_x/c' observed by a person at rest in F' is given by (19.11) as

$$\frac{v'_x}{c'} = \frac{1}{n'_-} = \frac{(1/n - \beta)}{(1 - \beta/n)} \cong \frac{1}{n} - \beta\left(1 - \frac{1}{n^2}\right), \tag{19.12}$$

where n'_- is the "effective refractive index" of the moving medium with velocity +V. Assuming each tube in the Fizeau experiment has the length L' and the speed of the water is $\pm V = \pm \beta c$, the optical path difference $\Delta L'$ of the two beams of light turns out to be consistent with experiment:

$$\Delta L' = 2L'n'_{-} - 2L'n'_{+} = 4L'n^2 \frac{V}{c}\left(1 - \frac{1}{n^2}\right), \tag{19.13}$$

$$\frac{1}{n'_{\pm}} = \frac{1}{n} \pm \frac{V}{c}\left(1 - \frac{1}{n^2}\right), \tag{19.14}$$

because the optical path is just the distance in vacuum equivalent to the actual path length traveled by each beam.

The usual formula for the aberration of star light has been confirmed by experiments. It can be obtained from the inverse transformations of (19.9) and (19.10),

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}. \tag{19.15}$$

This relation shows that both the deviation of light when transforming to a new reference frame and the relation between two angles θ and θ' are independent of whether or not the one-way speed of light is a universal constant.

Note that the invariant law for the propagation of light,

$$c'^2 dt'^2 - dr'^2 = 0, \tag{19.16}$$

does not refer to any specific source and, hence, it *holds for light emitted from any source.* From a microscopic viewpoint of light emission by atoms, the state of motion of a macroscopic source of light is actually irrelevant. The reason is that photons are emitted from atoms which may be in violent motion and whose electrons make transition from one state to another state without having a definite momentum or velocity because of the uncertainty principle.

The results (19.8)–(19.10) and (19.13)–(19.15) in extended relativity are the same as those in special relativity. The reason is that *they actually depend only on the 4-dimensional symmetry of physical laws and are independent of the constancy of the speed of light.*

19c. Doppler Shifts of Frequency and Atomic Energy Levels

The extended transformation of wave four-vectors $k^\mu=(\omega/c,\mathbf{k})$ and $k'^\mu=(\omega'/c', \mathbf{k}')$ is given by (19.8):

$$\frac{\omega'}{c'} = \gamma\left(\frac{\omega}{c} - \beta k_x\right), \quad k'_x = \gamma\left(k_x - \beta\frac{\omega}{c}\right), \quad k'_y = k_y, \quad k'_z = k_z, \tag{19.17}$$

This implies the Doppler shift of wavelength in extended relativity to be

$$\frac{1}{\lambda'} = \gamma\left(\frac{1}{\lambda} - \frac{\beta}{\lambda}\right) = \frac{1}{\lambda}\sqrt{\frac{(1-\beta)}{(1+\beta)}}, \tag{19.18}$$

where $k_x = 2\pi/\lambda$, $k_y = k_z = 0$, $k'_x = 2\pi/\lambda'$. This result is precisely the same as that in special relativity.

However, the consistency of the Doppler frequency shift in (19.17) with the result of laser experiments is more subtle because c' is not a constant in extended relativity. Since experiments which measure the frequency shift involve the absorption of photons by atoms, we must first re-examine the nature of atomic levels from the point of view of extended relativity.

In extended relativity, Dirac's Hamiltonian H_D for a hydrogen atom is given by

$$ic\hbar\frac{\partial\psi}{\partial w} = H_D\psi,$$

$$H_D = \alpha_D\cdot Pc + \beta_D mc^2 - \frac{e^2}{4\pi r}, \qquad \mathbf{P} = -i\hbar\nabla, \tag{19.19}$$

Using the usual method, it can be shown that (19.19) leads to atomic energy levels

$$E_n = \frac{mc^2}{\sqrt{1 + \dfrac{\alpha^2}{\left[n - |\kappa| + \sqrt{\kappa^2 - \alpha^2}\right]^2}}}, \tag{19.20}$$

$$|\kappa| = j + \frac{1}{2}, \qquad \alpha = e^2/(4\pi c\hbar).$$

Thus when an electron jumps from a state n_1 to another state n_2, it will emit or absorb an energy quantum $c\hbar k_o$:

$$E_{n_2} - E_{n_1} = c\hbar k_o = \hbar\omega , \qquad \text{in F}, \qquad (19.21)$$

$$E'_{n_2} - E'_{n_1} = c\hbar k'_o = c\hbar \frac{\omega'}{c'} , \qquad \text{in F}' . \qquad (19.22)$$

where c in (19.22) is the constant 2-way speed of light. Note that c' and c in (19.22) do not cancel in F', except for the special case, q' = 0. If two photons with "energy" $(\omega_o/c)c\hbar$ and $(\omega'_o/c')c\hbar$ are emitted from two hydrogen atoms at rest in F and F' respectively, then by the equivalence of F and F' frames we have

$$\frac{\omega_o}{c} c\hbar = \frac{\omega'_o}{c'} c\hbar . \qquad (19.23)$$

Here, the ratio ω'_o/c' is isotropic. However, if $(\omega/c)c\hbar$ and $(\omega'/c')c\hbar$ are the energies of the same photon measured from F and F' respectively, then they are related by the Doppler shift (19.17)

$$\frac{\omega'}{c'} = \gamma\frac{\omega}{c}(1 - \beta\cos\theta), \qquad k_x = k\cos\theta = \frac{\omega}{c}\cos\theta . \qquad (19.24)$$

Evidently, only when c'=c do we have the usual relation for the Doppler *frequency* shift. In general, we can only talk about an 'energy shift' or a 'k_o shift' in extended relativity because the energy of a quantum particle is not always proportional to the corresponding frequency in all frames. From the operational viewpoint, the energy[1] of a particle or k_o is the zeroth component of a 4-vector and observable in the Doppler effect, but frequency (defined on the basis of Reichenbach's time) in general is not.

Experimentally, one never measures the frequency directly unless constant one-way speed of light is presupposed (so that frequency becomes well-defined in all frames); instead, what one really measures is the shift of atomic levels in interaction with radiation in a general frame. Thus, (19.20)–(19.24) are consistent with experiments of precision Doppler shifts. The situation is similar to that in taiji relativity discussed in section 10b.

Since Reichenbach's time is used in the F' frame, the speed c' and the frequency ω' of a photon emitted from an atom at rest in F' is anisotropic in general; however, the ratio ω'/c' is isotropic. The 4-dimensional symmetry dictates the mixture of two waves in F' to be expressed in terms of (w',**r'**) and (ω'/c',**k'**) rather than (t',**r'**) and (ω',**k'**). Thus the superposition of two waves is in general given by

$$A_0\sin(k'_{01}w' - \mathbf{k}'_1\cdot\mathbf{r}') + B_0\sin(k'_{02}w' - \mathbf{k}'_2\cdot\mathbf{r}'), \qquad w' = b't' . \qquad (19.25)$$

Only in the F frame, where c is, by definition, isotropic and constant, does one have the usual expression $A_0\sin(\omega_1 t - \mathbf{k}_1\cdot\mathbf{r}) + B_0\sin(\omega_2 t - \mathbf{k}_2\cdot\mathbf{r})$.

19d. Classical Electrodynamics Based on Extended Relativity

Although the (one-way) speed of light in extended relativity is not a universal constant, one still has the universal 2-way speed of light c. Thus, the usual action for a free particle, $-\int mcds$, is an invariant in extended relativity. If a charged particle with mass m and charge e moves in the presence of an electromagnetic field in a general frame, the invariant action can be assumed to have the usual form[1]

$$S = -\int mcds - \frac{e}{c}\int A_\mu dx^\mu - \frac{1}{4c}\int F_{\mu\nu}F^{\mu\nu}d^3rdw , \qquad (19.26)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \qquad (19.27)$$

$$ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu} = dt\sqrt{C^2 - v^2} , \qquad C = \frac{dw}{dt}, \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}, \qquad (19.28)$$

$$\frac{e}{c} = -1.6021891 \times 10^{-20}\sqrt{4\pi}\ \sqrt{g{\cdot}cm} , \qquad (19.29)$$

where $g_{\mu\nu} = (1,-1,-1,-1)$, $x^\mu = (w,x,y,z)$, e/c is in Heaviside-Lorentz units and A_μ is the same as the usual electromagnetic vector potential in special relativity. However, there is an important difference from special relativity, namely, the speed of light C = dw/dt in (19.28) in a general inertial frame is not a constant in extended relativity.

For a charged particle moving in an electromagnetic potential field A_μ, the invariant action is given by S_{cp},

$$S_{cp} = \int L_{cp} dt, \qquad L_{cp} = -mcC\sqrt{1-v^2/C^2} - \frac{e}{c} A_\mu \frac{dx^\mu}{dt}, \tag{19.30}$$

where $dx^\mu/dt = (C, \mathbf{v})$, $A_\mu = (A_o, -\mathbf{A})$ and S_{cp} equals the first two terms in (19.26). The canonical momentum of a charged particle is now given by

$$\mathbf{P} = \frac{\partial L_{cp}}{\partial \mathbf{v}} = \mathbf{p} + \frac{e\mathbf{A}}{c}, \tag{19.31}$$

where

$$\mathbf{p} = \frac{(mc\mathbf{v}/C)}{\sqrt{1-v^2/C^2}}.$$

We have used the universal 2-way speed of light c to make $\mathbf{p}$ in (19.31) having the usual dimension of mass times velocity. In contrast, there is no universal speed of light in taiji relativity, so that the covariant momentum in (10.3) and (10.5) must have the dimension of mass.[4] The covariant Hamiltonian H is defined as

$$H \equiv \left(\frac{\partial L}{\partial \mathbf{v}} \cdot \mathbf{v} - L\right)\frac{c}{C} = cp_0 + eA_0, \tag{19.32}$$

$$p_0 = \frac{mc}{\sqrt{1-v^2/C^2}}.$$

Note that the factor c/C in the definition (19.32) is necessary for the expression H/c to transform as the zeroth component of the momentum 4-vector, $H/c \equiv P_0$. Otherwise, the Hamiltonian will not be meaningful. From (19.31), (19.33) and $H/c = P_0$ we have

$$\left(P_0 - \frac{e}{c} A_0\right)^2 - \left(\mathbf{P} - \frac{e}{c} \mathbf{A}\right)^2 = m^2c^2. \tag{19.33}$$

The Lagrange equation of motion for a charged particle in the electromagnetic field has the usual form

$$\frac{dp^\mu}{ds} = \frac{e}{c} F^{\mu\nu} \frac{dx_\nu}{ds}, \tag{19.34}$$

where $p^\mu = (p^0, \mathbf{p})$ and $x_\mu = (w, -\mathbf{r})$.

Making the substitutions $\mathbf{P} \to -i\hbar\nabla$ and $P_0 \to i\hbar\partial/\partial w$ based on symmetry considerations, we obtain the extended relativistic Klein-Gordon equation

$$[(i\hbar\frac{\partial}{\partial w} - \frac{e}{c}A_0)^2 - (-i\hbar\nabla - \frac{e}{c}\mathbf{A})^2]\,\Phi(w,\mathbf{r}) = m^2c^2\Phi(w,\mathbf{r})\,. \tag{19.35}$$

For a continuous charge distribution in space, the second term in (19.26) should be replaced by $-\int A_\mu J^\mu d^3r dw$. In this case, the variation of (19.26) leads to the invariant Maxwell equations in a general frame

$$\begin{aligned} &\partial_\nu F^{\mu\nu} = J^\mu\,, \\ &\partial_\lambda F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0\,, \end{aligned} \tag{19.36}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\partial_\mu = \partial/\partial x^\mu$ and $x^\lambda = (w, \mathbf{r})$. Thus, Maxwell's equations have the invariant form, even if the one-way speed of light is not universal. This is consistent with the result in taiji relativity.

One can write the field-strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ in matrix form

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{19.37}$$

where the fields $\mathbf{E}$ and $\mathbf{B}$ have the usual dimension. Note that they differ by a constant factor c from the corresponding fields in taiji relativity discussed in section 10c. They can be expressed in terms of the usual 4-vector potentials $A^\mu = (A_0, \mathbf{A})$ as

$$\mathbf{E} = \frac{\partial \mathbf{A}}{\partial w} - \nabla A_0\,, \qquad \mathbf{B} = \nabla \times \mathbf{A}\,. \tag{19.38}$$

In terms of $\mathbf{E}$, $\mathbf{B}$, and the 4-current $J^\mu = (\rho, \mathbf{J})$, the first equation in (19.36) can be written as

$$\nabla \cdot \mathbf{E} = \rho, \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial w} = \mathbf{J}. \tag{19.39}$$

The second equation in (19.36) can be written as

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial w} = 0. \tag{19.40}$$

We have seen that the universal 2-way speed of light c does not explicitly appear in the Maxwell equations in a general inertial frame. The situation resembles that of taiji relativity discussed in section 10c.

19e. Quantum Electrodynamics Based on Extended Relativity

We will demonstrate that an extended quantum electrodynamics (QED) can be formulated because extended relativity has the 4-dimensional symmetry with the lightime w as the evolution parameter in a general inertial frame. The formulation is formally the same as QED based on taiji relativity, as discussed in section 10e. It should be stressed that extended QED cannot be formulated covariantly if one uses Reichenbach's time t' in Edwards' original transformations[3] as the evolution parameter, because t' does not transform as the zeroth component of the coordinate 4-vector. Within extended quantum electrodynamics, the invariant action S_q involves an electron field ψ. The electromagnetic 4-potential A_μ (i.e., the photon field) is assumed to be given by the usual expression[5]

$$\begin{aligned} S_q &= \int L d^4x \, , \\ L &= \bar{\psi}[\gamma^\mu(i\hbar\partial_\mu - \frac{e}{c} A_\mu) - mc]\psi - \frac{1}{4c} F_{\mu\nu}F^{\mu\nu}, \end{aligned} \tag{19.41}$$

where e is given by (19.29) and $d^4x = dwd^3r$. We may remark that there are two differences between (19.41) and the corresponding Lagrangian density in conventional QED, namely, *the constant c in (19.41) is the universal 2-way speed of light* and *d^4x is not the same as $cdtd^3r$ in a general reference frame* (in which $x^\mu = (w,x,y,z) \neq (ct,x,y,z)$.) For quantization of the electron fields, for example, the "canonical momentum" π_b conjugate to ψ_b is defined as usual to be[5]

$$\pi_b = \frac{\partial L_\psi}{\partial(\partial_0 \psi_b)} , \tag{19.42}$$

$$L_\psi = \bar{\psi}\, [\gamma^\mu i\hbar\partial_\mu - mc]\psi , \qquad \partial_0 = \frac{\partial}{\partial w} ,$$

and the Hamiltonian density for a free electron is $H = \pi\partial_0\psi - L_\psi$. Suppose free photon fields are enclosed in a box taken to be a cube of side $V^{1/3}$ and the electron field is normalized in a box with volume V. We have

$$A_\mu(w,\mathbf{r}) = \frac{1}{V^{1/2}}\sum_{\mathbf{p};\alpha}(\hbar/2p_0)^{1/2}[A(\mathbf{p},\alpha)\ \varepsilon_\mu(\alpha)\exp(-ip\cdot x/\hbar)$$
$$+ A^\dagger(\mathbf{p},\alpha)\ \varepsilon_\mu(\alpha)\exp(ip\cdot x/\hbar)] , \tag{19.43}$$

$$\psi(w,\mathbf{r}) = \frac{1}{V^{1/2}}\sum_{\mathbf{p};s}(m/p_0)^{1/2}[b(\mathbf{p},s)u(\mathbf{p},s)\exp(-ip\cdot x/\hbar)$$
$$+ d^\dagger(\mathbf{p},s)v(\mathbf{p},s)\exp(ip\cdot x/\hbar)] , \qquad p\cdot x = p_\mu x^\mu , \tag{19.44}$$

where

$$[A(\mathbf{p},\alpha),\ A^\dagger(\mathbf{p}',\alpha')] = \delta_{\mathbf{p}'\mathbf{p}}\delta_{\alpha\alpha'} ,$$

$$[b(\mathbf{p},s),\ b^\dagger(\mathbf{p}',s')] = \delta_{\mathbf{p}'\mathbf{p}}\delta_{ss'} , \qquad [d(\mathbf{p},s),\ d^\dagger(\mathbf{p}',s')] = \delta_{\mathbf{p}'\mathbf{p}}\delta_{ss'} , \tag{19.45}$$

and all other commutators vanish.[5]

The Dirac equation in extended relativity can be derived from (19.41). We obtain

$$i\hbar\frac{\partial\psi}{\partial w} = [\ \boldsymbol{\alpha}_D\cdot(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}) + \beta_D mc + \frac{e}{c}A_0]\psi . \tag{19.46}$$

In view of the equations of motion (19.35) and (19.46), we must use the lightime w rather than Reichenbach's time t in a general inertial frame as the evolution variable for a state $\Phi^{(S)}(w)$ in the Schrödinger representation:

$$i\hbar\,\frac{\partial\Phi^{(S)}(w)}{\partial w} = H^{(S)}(w)\Phi^{(S)}(w) , \qquad H^{(S)} = H_o^{(S)} + H_I^{(S)} , \tag{19.47}$$

because the evolution of a physical system is assumed to be described by a Hamiltonian operator which has the same transformation property as that of w.

The usual covariant formalism of perturbation theory[5] can also be applied to quantum field theory based on extended relativity. To illustrate this point, let us briefly consider the interaction representation and the S-matrix based on extended relativity. The transformations of the state vector $\Phi(w)$ and operator O from the Schrödinger representation to the interaction representation are

$$\Phi(w) \equiv \Phi^{(I)}(w) = \exp[iH_o^{(S)}w/\hbar]\Phi^{(S)}(w) , \tag{19.48}$$

$$O(w) = O^{(I)}(w) = \exp[iH_o^{(S)}w/\hbar]O^{(S)}\exp[-iH_o^{(S)}w/\hbar] . \tag{19.49}$$

Because $O^{(S)}$ and $O(w)$ are the same for $w = 0$, we have

$$i\hbar \frac{\partial\Phi(w)}{\partial w} = H_I(w)\Phi(w) , \tag{19.50}$$

where

$$H_I(w) = \exp[iH_o^{(S)}w/\hbar]H_I^{(S)}\exp[-iH_o^{(S)}w/\hbar] ,$$

$$O(w) = \exp[iH_o^{(S)}w/\hbar]O(0)\exp[-iH_o^{(S)}w/\hbar] , \tag{19.51}$$

The U-matrix can be defined in terms of the lightime w: $\Phi(w) = U(w,w_0)\Phi(w_0)$, $U(w_0,w_0) = 1$. It follows from (19.50) and (19.51), that

$$i\hbar \frac{\partial U(w,w_0)}{\partial w} = H_I(w)U(w,w_0) , \tag{19.52}$$

If a physical system is in the initial state Φ_i at lightime w_0, the probability of finding it in the final state Φ_f at a later lightime w is

$$|\langle\Phi_f|U(w,w_0)\Phi_i\rangle|^2 = |U_{fi}(w,w_0)|^2 . \tag{19.53}$$

Evidently, the average transition probability per unit lightime for $\Phi_f \to \Phi_i$ is

$$\frac{|U_{fi}(w,w_0) - \delta_{fi}|^2}{(w - w_0)} . \qquad (19.54)$$

As usual, we can express the S-matrix in terms of the U-matrix, i.e. $S = U(\infty, -\infty)$ and obtain the following form

$$S = 1 - (i/\hbar) \int_{-\infty}^{\infty} H_I(w)dw + (-i/\hbar)^2 \int_{-\infty}^{\infty} H_I(w)dw \int_{-\infty}^{w} H_I(w')dw' + \ldots\ldots . \qquad (19.55)$$

For w-dependent operators, one can introduce a w-product W(corresponding to the usual chronological product), so that one can write (19.55) in an exponential form:

$$S = W\{\exp[- (i/\hbar) \int_{-\infty}^{\infty} \mathcal{H}_I(x^\mu)dwd^3r]\} , \qquad (19.56)$$

$$\int_{-\infty}^{\infty} \mathcal{H}_I(x^\mu)d^3r = H_I(w) . \qquad (19.57)$$

For simplicity, one may set $\hbar = c = 1$, where c is the 2-way speed of light. (These are the "natural units" in extended relativity, similar to those in the conventional field theory based on special relativity.)

For the QED Lagrangian (19.41), we can derive Feynman rules based on extended relativity. Let us summarize the Feynman rules for QED with the Lagrangian L in (19.41) with a gauge-fixing term $(\partial_\mu A^\mu)^2/(2\rho)$:

$$L_{QED} = L - \frac{1}{2\rho} (\partial_\mu A^\mu)^2 , \qquad \hbar = c = 1 , \qquad e < 0 . \qquad (19.58)$$

The covariant photon and electron propagators are

$$\frac{-i[g_{\mu\nu} - (1 - \rho)k_\mu k_\nu/(k^2 + i\varepsilon)]}{(k^2 + i\varepsilon)} , \qquad (19.59)$$

$$\frac{i}{(\gamma_\mu p^\mu - m + i\varepsilon)} . \qquad (19.60)$$

The vertex factor is

$$-ie\gamma^{\mu}. \tag{19.61}$$

There is a factor ε_{μ} for the polarization vector of each external photon line and a factor $u(\mathbf{p},s)$ for each absorbed electron, a factor −1 for each closed fermion loop, etc. These rules are identical to those in the conventional theory, if the natural units ($\hbar = c = 1$) are used.

Thus, if one calculates scattering cross sections and decay rates (with respect to the lightime w) of a physical process in a general frame, one will get the same result as that in special relativity,[5] as discussed in section 18d in chapter 18. This is not surprising because one has the same 4-dimensionally symmetric Lagrangian (19.41) as that in special relativity.

19f. A Clock System for Lightime, Lifetime Dilatation and the Maximum Speed of Physical Objects

One may ask: How does one realize the evolution variable w' in the extended coordinate transformation (19.2) by a physical device? Since the invariant phase of an electromagnetic wave in the F' frame is given by ($k'_0 w' - \mathbf{k'}\cdot\mathbf{r'}$), where $k'_0 = |\mathbf{k'}|$, we can define lightime w' in terms of k'_0, just as length can be defined, as usual, by wave length λ' or $|\mathbf{k'}|$. The "clocks", which show lightime in this theory, are the same as those in taiji relativity because they have exactly the same 4-dimensional transformation property. However, the taiji-time w' in F' *cannot* be factorized into two well-defined b' and t' because of the absence of a second postulate; while lightime w' in common relativity and extended relativity *can* be factorized into two well-defined functions b' and t', as shown in equations (12.4) with common time t'=t, and (19.2) with Reichenbach's time (19.1). Another method of setting up a "clock system" which reads lightime $w' = b't'$ is to use the expression for b' [i.e., $b' = c - q'x'/t'$ given in (19.2)] and the value t', which is known through Reichenbach's synchronization procedure for a grid of ordinary clocks. Any clock has two adjustable parameters, its rate of ticking and reading. We may base our "clock system" on computer chips. We can program any "clock" in F' to obtain a time reading t' from the nearest F' clock in the grid and, based on its position x' and given values c and q', compute the lightime w' it should display.

Our discussions show that it is important to be aware of what quantities are actually measured in the experiments and what effects the assumption of a universal speed of light may have had on the interpretation of the results. For example, we have seen in section 18d that the "lifetime dilatation" of unstable particle decay in flight has little to do with the *non-covariant* property of Reichenbach's time with a general parameter q or q', because the lifetime τ is defined as the decay length divided by the constant 2-way speed of light c. The basic reason is as follows: The 4-dimensional symmetry dictates that the decay rates in any quantum field theory based on extended relativity can only be defined in terms of the *covariant* lightime w or w'. The lightime has the dimension of length and transforms as the zeroth component of a coordinate 4-vector.

We stress that the constant 2-way speed of light in extended relativity is in general not the maximum speed of physical objects in the universe. Rather, *it is the one way speed of light in a given direction that is the maximum speed of objects in that direction, as shown in (19.3). This holds for any inertial frame.* It is worthwhile to note that this property of light, being the "maximum speed" of all physical objects in any given direction, is the logical consequence of the first postulate of relativity, as shown in taiji relativity in chapter 7.

We have examined a number of experimental tests of special relativity and the formulations of classical electrodynamics and QED. All of them are consistent with extended relativity. These discussions can be generalized to other field theories such as unified electroweak theory and quantum chromodynamics. As we have seen, only the 4-dimensional symmetry of physical laws in extended relativity is absolutely essential for understanding experiments and for the formulation of classical electrodynamics and QED; the universality of the one-way speed of light is physically unnecessary. Similarly, from the viewpoint of taiji relativity, the second postulate of the constancy of the 2-way speed of light is also physically unnecessary.

In this connection, we stress that according to taiji relativity, the universality of the one-way or the two-way speed of light is a convention rather than an inherent property of the physical world. In other words, all experimental results in taiji relativity or extended relativity can be derived by simply using the quantities (w,x,y,z) and (w',x',y',z') without ever mentioning time t (or t') and the speed of light c=29979245800 cm/sec.

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20.

Determination of the Parameters of General Linear Transformations by Precision Experiments*

20a. A General Parameterization of Linear Transformations

Can the Lorentz transformation of special relativity be deduced entirely from experiments?

This question has been discussed and answered by several physicists.[1] Postulating a general parameterization of linear coordinate transformations between two inertial frames, it has been concluded that the Lorentz transformation of special relativity can be deduced entirely from three precision experiments:[1] the Michelson-Morley,[2] Kennedy-Thorndike[3] and Ives-Stilwell[4] experiments.

If true, this would experimentally exclude Reichenbach's extended concept of time, Edwards' postulate of the universal two-way speed of light, common time and the concept of taiji-time within the 4-dimensional symmetry framework. Thus, it is important to look into the analysis of these three experiments carefully.

Guided by a general parameterization of linear 4-dimensional transformations, Leon Hsu, then an undergraduate, re-examined the implications of the three precision experiments in a term paper for a physics course.[5] He found that, using the Einstein method for clock synchronization, one obtains the Lorentz transformations; without it, the three precision experiments are insufficient to specify a relation between times t and t' in two inertial frames. Furthermore, he obtained an infinite set of 4-dimensional coordinate transformations which has precisely the Lorentz group properties. Later, it was realized that no known experiment can specify a definite relation for times t and t'.[6]

*Although this chapter lies somewhat out of this book's main line of development, it complements the reasoning. Historically, the analysis in this chapter stimulated the new ideas of taiji relativity discussed in chapter 7.

Suppose we have two inertial frames, a rest frame F and a frame F' which moves along the x–axis of F with velocity V in a positive sense. In a general flat 4–dimensional framework, we use four variables to describe the coordinate of an event in F or F'

$$(w, x, y, z) \equiv x^{\mu} \qquad \text{or} \qquad (w', x', y', z') \equiv x'^{\mu}, \tag{20.1}$$

where $w \equiv bt$ and $w' \equiv b't'$.

For simplicity and without loss of generality, we define the speed of light to be isotropic in one of the two frames, say, F.[6] Using light signals to synchronize the clocks in F (à la Einstein), we have

$$bt = ct, \quad \text{but} \quad b't' \neq ct', \tag{20.2}$$

since we know nothing about the speed of light or the time t' in F'. We write our parameterization of a most general linear transformation as follows:

$$b't' = D(Ect + Gx), \quad x' = D(Ax + Bct), \quad y' = Dy, \quad z' = Dz. \tag{20.3}$$

Because b' is as yet undetermined, we assume a linear relation between t and t',

$$t' = Mt + Nx, \tag{20.4}$$

for simplicity of the discussion. This linear relation is not necessary, but without it, the ensuing discussion would become more involved. This relation can be physically realized by clock systems for any choice of M and N (which may be constants or functions of V) since the reading and rate of ticking of a clock may be arbitrarily set.[6] The only thing implied by (20.4) is that a clock moving with velocity V at any position x shows time Mt + Nx, while a clock at rest at the same position x in F shows time t.

To determine our parameters, we first use the condition that an object at rest in F' has the velocity V when measured from F. Setting dx' to 0 and dx/dt to V in (20.3), we get

$$\frac{B}{A} = -\frac{V}{c}. \tag{20.5}$$

Other parameters can only be determined by experiments.

20b. Determinations of Parameters by Three Experiments

The result of the Michelson–Morley experiment implies that the speed of light, averaged over a round trip, is the same in all directions for a given V. Note that, strictly speaking, the Michelson–Morley experiment cannot tell us the one–way speed of light. Suppose we have the apparatus set up in F' so that one arm is in the direction of motion while the other is perpendicular to it. Both arms have length L' as seen from F'. Using (20.3), the averaged velocity of the perpendicular path as measured from F' is

$$c'_{\perp} = \frac{dy'}{dt'} = \frac{Dc\sqrt{1-\beta^2}}{(M + NV)}, \qquad \frac{dx'}{dt'} = \frac{dz'}{dt'} = 0, \tag{20.6}$$

where $\beta = V/c$, while the average velocity of the parallel path is

$$c'_{\parallel} = \frac{2}{\left[\frac{1}{(c'_+)} + \frac{1}{(-c'_-)}\right]} = \frac{(Dc/A)(A^2 - B^2)}{(M + NV)}, \qquad c'_- < 0; \tag{20.7}$$

$$c'_+ = \frac{dx'}{dt'} \quad \text{with} \quad \frac{dx}{dt} = c; \qquad c'_- = \frac{dx'}{dt'} \quad \text{with} \quad \frac{dx}{dt} = -c.$$

Setting these two average speeds equal to each other and using (20.5), we get

$$A = \gamma, \qquad B = -\beta\gamma, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{20.8}$$

The positive root has been taken for A in order to make x and x' parallel rather than antiparallel.

Now we use the results of Kennedy and Thorndike, first writing the metric in terms of primed and unprimed variables by using (20.3) and (20.8),

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 = [(E + \beta G)\gamma D]^{-2}\left\{(b't')^2 - x'^2 (E^2 - G^2) - 2b't'x'\gamma(G+E)\beta\right\} - D^{-2}(y'^2 + z'^2). \tag{20.9}$$

The unchanging interference pattern seen in the original experiment and the constant frequency of the cavity laser observed by Hils and Hall[1] indicate that the final phase of a light signal making a round trip is independent of the velocity of the observer. This implies that over a round trip, the quantity $b't' = f(\mathbf{r}')$ ($s^2=0$ for the propagation of light) is also independent of V, since $b't'=f(\mathbf{r}')$ is merely the distance traveled by light in time t', $\{b't' = \int_0 d(b't') = \int_0 c'dt'$ = light path, see (20.19) below$\}$ and the wavelength of that light does not depend on the relation between t and t'. Solving for $b't' \equiv w'$ from $s^2=0$ in (20.9) for the propagation of light, we obtain

$$w'(\theta') = (L'\cos\theta')\gamma(G + \beta E) + L'\gamma(E + \beta G) > 0\,, \quad \text{for all } \theta'\,, \quad s^2 = 0\,, \qquad (20.10)$$

where

$$x' = L'\cos\theta'\,, \qquad y'^2 + z'^2 = L'^2\sin^2\theta'\,. \qquad (20.11)$$

So over a round trip, we have the light path

$$w'(\theta') + w'(\pi+\theta') = 2L'\gamma(E + \beta G)\,, \qquad s = 0\,, \quad w' = b't'\,. \qquad (20.12)$$

From (20.3), we see that A, E, and D equal 1 while B and G equal 0 when V = 0, so

$$\gamma(E + \beta G)|_{\beta=0} = 1\,. \qquad (20.13)$$

Since the Kennedy-Thorndike results require the light path (20.12) to be independent of velocity, (20.13) must hold for all V. Using (20.13) and (20.8), we then have

$$\gamma E = 1 - \beta\gamma G\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,. \qquad (20.14)$$

Finally, we turn to the Ives-Stilwell experiment to try to determine the remaining parameters. In order to proceed, we note that, within the 4-dimensional symmetry framework, the transformation property of a wave 4-vector k_μ must be the same as that of the differential operator $\partial_\mu = \partial/\partial x^\mu$, where x^μ

= (w,r) and k_μ = (k_o, -**k**). Thus, (20.3) leads to the transformation of the wave four-vector

$$k_o = D(Ek_o' + \beta\gamma k_x'), \quad k_x = D(\gamma k_x' - Gk_o'), \quad k_y = Dk_y', \tag{20.15}$$

where, for simplicity, the wave is assumed to have no z-component.

In the Ives-Stilwell experiment (which was carried out in the frame F), the experimental equivalent to observing a moving light source (at rest in F') perpendicularly to its direction of motion was performed, in order to observe only the second order Doppler shift.[4] The calculational equivalent is to set ϑ= 90 degrees in F or $k_x = 0$.

Using the equation $k_o^2 - \mathbf{k}^2 = 0$ (or $k_o = k_y$ for $k_x = k_z = 0$) and the transformation equation for k_o, we derive two expressions for k_o. Setting these equal to each other and using (20.14), we obtain values for G and E ,

$$G = -\beta\gamma, \qquad E = \gamma, \tag{20.16}$$

where G is chosen to be negative so that t and t' always have the same sign.

Only three parameters, D, M, and N, remain undetermined at this point. Since $k_o = 2\pi/\lambda$, we take the equations in (20.15) (where $k_x = k_o\cos\vartheta$ and $\vartheta = \pi/2$) together with (20.16) to get

$$\lambda = \gamma\frac{\lambda'}{D}. \tag{20.17}$$

Here, $\lambda' = \lambda'(0)$ is the wavelength of the radiation emitted by the light source as seen by an F'-observer who is co-moving with it. This quantity cannot be measured directly in the laboratory frame F, but one can measure $\lambda(0)$, the wavelength of an identical emitter which is at rest in the laboratory frame F. If we assume that $\lambda'(0) = \lambda(0)$, then the results of Ives and Stilwell indicate that the second order wavelength shift $[\lambda - \lambda(0)]/\lambda(0)$ of a moving light source is related to its velocity by $V^2/2c^2$. For our proposed transformation to be consistent with these experimental results, we must have D = 1. Note, however, that *assuming that $\lambda(0) = \lambda'(0)$ implies the equivalence of the frames F and F' or the absence of any kind of "absolute" velocity. This equivalence is actually postulated by the*

principle of relativity for physical laws. So, in this sense, the result D = 1 is assumed rather than derived from these experiments.

Having culled as much information as possible out of the three experiments, we can now write the transformation equations (20.3) as

$$b't' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z. \tag{20.18}$$

It is interesting to note that the resultant 4-dimensional transformation (20.18) is precisely the taiji transformation that one would obtain by assuming only the principle of relativity for physical laws, without postulating the universality of the speed of light, as discussed in chapter 7.

20c. Flexibility of the Relation for t and t' in 4-dimensional Symmetry Framework

The expression (20.1) with w≡bt and w'≡b't' expressed the *concept of taiji-time* w as follows:

The taiji-times w and w' in F and F', respectively, have well defined values in the unit of length. Furthermore, if one wants to introduce the conventional time t and t', one can only have w=bt and w'=b't' where the relation for t and t' cannot be specified in taiji relativity due to the lack of a second postulate. In other words, taiji-time implied that the relation between t and t' is completely flexible and can be arbitrary. In this sense, *taiji relativity implies a new "time gauge symmetry,"* as long as physical laws display the 4-dimensional symmetry.

The 4-dimensional symmetry property of (20.18) can be seen through, for example, the invariance of the law for propagation of light emitted from an arbitrary source,

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0,$$

$$(d(b't'))^2 - dx'^2 - dy'^2 - dz'^2 = c'^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = 0, \tag{20.19}$$

where c' = d(b't')/dt' is in general unknown[6] but c'dt' = dw' is well-defined. The result (20.18) lead to the transformations of velocity ratios:

$$\frac{v'_x}{c'} = \frac{v_x/c - V/c}{1 - v_x V/c^2}, \quad \frac{v'_y}{c'} = \frac{v_y/c}{1 - v_x V/c^2}, \quad \frac{v'_z}{c'} = \frac{v_z/c}{1 - v_x V/c^2}. \qquad (20.20)$$

where $v'_x = dx'/dt'$, $v_x = dx/dt$, etc. Based on (20.20), we can show that the taiji transformation (20.18) forms precisely the Lorentz group, even though t' is actually unspecified and unknown because the relation between t and t' cannot be specified by experiments.[6] We stress that the velocity ratio $\mathbf{v}'/c'$ is independent of the unspecified time t' because $\mathbf{v}'/c'=(d\mathbf{r}'/dt')/(dw'/dt')=d\mathbf{r}'/dw'$, where $w'=b't'$ is well defined in (20.18).

These transformations are independent of M and N so that the infinite set of transformations given by (20.18) and (20.4) with unspecified M and N all have the Lorentz group properties.[7] The Lorentz transformation in special relativity is only a special case with

$$M = (1 - \beta^2)^{-1/2} \quad \text{and} \quad N = -(\beta/c)(1-\beta^2)^{-1/2}. \qquad (20.21)$$

We observe that there is another simple and interesting case, i.e.,[8]

$$t' = t, \quad (\text{or } M = 1 \text{ and } N = 0 \text{ in eq. } (20.4)). \qquad (20.22)$$

This choice leads to the 4-dimensional symmetry framework with a common time for all observers.

It is important to note that taiji relativity never specifies M and N. The specification of M and N is equivalent to making an additional postulate. For example, if one imposes the additional postulate (20.21) [(20.22)] in taiji relativity, one will get special relativity [common relativity] which is logically a different theory. We may remark that the postulate (20.21) is effectively the same as assuming $b'=b=c$.

Finally, notice that at no point did we ever postulate a method of clock synchronization for all inertial frames, as Hils and Hall did in order to set their parameter e.[1] As a result, it was impossible to separate b' from t' in our transformation, or equivalently, to determine M and N.[8,6] As we did, Hils and Hall defined the speed of light to be isotropic in the rest frame F. But, by using relativistic clock synchronization and thus assuming that the speed of light was the same in both directions along a line in the moving frame F', they have actually fixed the speed of light to be a universal constant.[1] Without this stipulation, we see that an infinite number of transformations are allowed, each with the Lorentz group properties and each fully consistent with the results of these three experiments. We stress that other experiments such as 'lifetime dilatation' etc. cannot give more information to restrict the function b' in (20.18).[6]

We conclude that these three precision experiments are consistent with taiji-time and that they do not exclude either Reichenbach's extended concept of time or Edwards' weaker postulate for the speed of light embedded in a 4-dimensional symmetry framework.

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21.

Generalized Lorentz Transformations for Non-Inertial Frames Based on the Limiting 4-Dimensional Symmetry

"Every generalization is a hypothesis.... It is clear that any fact can be generalized in an infinite number of ways, and it is a question of choice. The choice can only be guided by considerations of simplicity."

H. Poincaré, *Science and Hypothesis*

21a. An Answer to Young Einstein's Question and its Implications

Ever since the precise formalisms for mechanics and electrodynamics in inertial frames were discovered and understood, physicists have attempted to explore the exact laws of physics in non-inertial frames.

"Is it conceivable that the principle of relativity also holds for systems which are accelerated relative to each other?"

This was the question the young Einstein asked in his 1907 paper, two years after he created special relativity for inertial frames (or systems).[1] He said that this question must occur to everyone who has followed the applications of the relativity principle. He considered two reference frames Σ_1 and Σ_2. The frame Σ_1 is accelerated in the +x direction with a constant acceleration g, and the frame Σ_2 is at rest in a homogeneous gravitational field which gives all objects an acceleration of $-g$ in the x-direction. *Einstein extended the principle of relativity to the case of the "uniformly accelerated frame" by assuming the complete physical equivalence of a homogenenous gravitational field and the corresponding constant acceleration of a reference frame.*[1]

However, such an idea of extension does not seem to enable physicists to derive an exact generalization of the Lorentz transformation for frames with uniform accelerations or rotations. One might think that perhaps this is partially due to the fact that the essential 4-dimensional symmetry of the Lorentz transformation was not understood at that time. But even today, although special relativity can deal with accelerations of particles, the exact operational meanings of constant linear accelerations and uniform rotations of non-inertial frames are still not completely clear.[2]

There is a difficulty in the conventional notion of constant acceleration. Such a constant acceleration g implies a velocity $v_I = gt_I$ as measured in an inertial frame, in which the time t_I is not limited in any way. Therefore, this velocity v_I will eventually exceed the speed of light after a sufficiently long time t_I, however small g may be. This violates special relativity and experimental results of particle kinematics and, therefore, the exact operational meaning of "constant linear acceleration" (CLA) cannot be satisfactorily based on the conventional concept of acceleration, namely, $dv_I/dt_I =$ constant *in an inertial frame*. Nevertheless, by considering small gt_I, Einstein in 1907 was able to obtain two results related to time in accelerated frames:

(A) $$\tau(x) = \tau(0)(1 + \frac{gx}{c^2}) . \qquad (21.1)$$

(B) The Maxwell equations in a uniformly accelerated frame Σ_1 have the same form as in an inertial frame, but with the velocity of light c replaced by

$$c(1 + \frac{gx}{c^2}) . \qquad (21.2)$$

Based on the equivalence principle, result (A) was experimentally confirmed by the gravitational red shift. On the other hand, result (B) was not completely satisfactory.

Evidently, the answer to Einstein's question concerning the applicability of the relativity principle to accelerated frames depends on the relationship between 4-dimensional symmetry of spacetime and non-inertial frames. The *4-dimensional* spacetime symmetry of relativity was introduced by Poincaré and Minkowski,[3] and is now one of the most thoroughly tested symmetry principles of the 20th century. It is the mathematical manifestation of the first postulate of special relativity, i.e., the symmetry or invariance of physical laws. The 4-dimensional symmetry is fundamental and is extremely powerful in helping us to understand physics.[4]

Although there is a definite non-equivalence between an inertial and a non-inertial frame, we shall demonstrate in this chapter that the question raised by Einstein in 1907, namely, whether "the principle of relativity also holds for systems which are accelerated relative to each other" can be answered affirmatively to a large extent. The reasons are as follows:

(A) *Any accelerated frame $F(w,x,y,z)$ must smoothly reduce to an inertial frame $F_I(w_I,x_I,y_I,z_I)$ in the limit of zero acceleration.*

(B) *The principle of relativity states that the laws of physics are invariant under a coordinate transformation between inertial frames. Logically, this is equivalent to the statement that the laws of physics in inertial frames must display the four-dimensional symmetry of the Lorentz and the Poincaré group.*

Thus, it is natural and worthwhile to investigate accelerated transformations on the basis of the principle of "limiting 4–dimensional symmetry". *This principle states that the laws of physics in non-inertial frames must display the 4–dimensional symmetry of the Lorentz group in the limit of zero acceleration.* As a result, accelerations of reference frames can be investigated on the basis of a purely kinematic approach and independent of the gravitational field. And physical results can be obtained without using gravity as a crutch, in contrast to the conventional approach based on general relativity.

The "4–dimensional" spacetime (w,x,y,z) of non–inertial frames is, of course, no longer the same as the relative spacetime of inertial frames in taiji relativity. This is obvious because there is no relativity or reciprocity between an inertial frame and a non–inertial frame. In other words, such a spacetime (w,x,y,z) cannot have the 4–dimensional symmetry of the Lorentz and the Poincaré groups. It is a more general 4–dimensional spacetime which includes the 4–dimensional spacetime of inertial frames as a special case when the acceleration approaches zero. To avoid confusion, let us call such a more general spacetime and evolution variable w of non–inertial frames *"taiji spacetime"* and *"taiji time." Thus, taiji spacetime contains both non–relative spacetime and relative spacetime of taiji relativity.* As usual, we use *non–relativistic* to describe properties or quantities in Newtonian physics; but we shall use *non–relative* to denote properties or quantities associated with non–inertial frames. Moreover, taiji spacetime for non–inertial frames is not a general covariant theory (such as general relativity) because only a particular type of coordinates can be used in an accelerated transformation, just as that only the Cartesian coordinates can be used in the Lorentz transformation.

Based on this limiting 4–dimensional symmetry principle, the answer to Einstein's question is *affirmative* because we can obtain a "minimal generalization" of Lorentz transformations for non-inertial frames with constant–linear–accelerations (CLA). "Minimal generalization" means the resultant equations involve minimal departure from the simple case with zero acceleration.

As we have discussed in chapter 4, the 4–dimensional symmetry of the Lorentz and the Poincaré groups is directly implied by and is the essence of the Poincaré–Einstein principle of relativity. It must be stressed that *limiting 4–dimensional symmetry is simply the 4–dimensional symmetry of the Lorentz and the Poincaré groups applied to non-inertial frames in the limit of zero acceleration.* We note that it is only the first postulate, but not the second postulate, of relativity theory or the 4–dimensional symmetry principle that is applied to non-inertial frames. The reason is obvious because the speed of light cannot be a universal constant in non-inertial frames.

Since 1909, accelerated motions and transformations have been discussed by many people.[5–8] These results are not satisfactory because they do not have the limiting 4–dimensional symmetry. In sections 21d, 21e and 21f below, we show that the idea of limiting symmetry leads to a linearly accelerated transformation (with two parameters, the constant acceleration and the initial velocity,) which reduces to the 4–dimensional symmetry form of taiji or special relativity theory in the limit of zero acceleration.

It is interesting and gratifying that the principle of limiting 4–dimensional symmetry has further physical implications: It can also lead to transformations for rotational frames and reveal the truly universal constants for physics in both non-inertial and inertial frames.

Suppose $F_I(w_I,x_I,y_I,z_I)$ is an inertial laboratory frame and $F(w,x,y,z)$ is a second, non-inertial frame whose origin rotates with an 'angular velocity' Ω (measured in terms of w) on a circle of radius R, and whose y–axis always points towards the origin of the $F_I(x_I^\mu)$ frame. The principle of limiting 4–dimensional symmetry states that in the limit $R\to\infty$ and $\Omega\to 0$ such that $R\Omega=\beta_o\neq 0$, the rotational transformation relating those two frames must reduce smoothly to the 4–dimensional symmetry form of relativity theory. On the other hand, when R approaches zero, $R\to 0$, the general rotational transformation leads to an exact rotational transformation with a fixed origin. The result turns out to be approximately the same as the usual rotational transformation (with t'=t). The transformations for rotating frames will be discussed in chapter 25.

When one examines the formulations of physical theories in these non-inertial frames, one notes that the speed of light is not a universal constant. Consequently, the charge e measured in the *electrostatic units* and the Planck constant $\hbar$ are also no longer universal constants of physics in non-inertial frames. They are replaced by the charge $\bar{e} = -1.6021891\times10^{-20}(4\pi)^{1/2}(g\cdot cm)^{1/2}$ and the new quantum constant $J = 3.5177293\times10^{-38}$ g·cm, where $\bar{e}$ is the electric charge measured in *electromagnetic (Heavyside-Lorentz) units*. In this connection, we note that if we consider the unified electroweak theory instead of quantum electrodynamics, we have one more constant, i.e., the Weinberg (or weak) angle, which is a dimensionless and truly universal constant. The dimensionless electromagnetic coupling strength $\alpha_e = \bar{e}^2/(4\pi J) \approx 1/137$ is still a universal constant, as it should be. These truly universal constants, $\bar{e}$ and J, are exactly the same as those in 'taiji relativity' which is based solely on the first postulate of relativity.[9] They are truly universal constants because their universal constancy holds in both inertial and non-inertial frames.

21b. Physical Time and Clocks in Linearly Accelerated Frames

The limiting 4-dimensional symmetry naturally dictates the smooth connection between the physical time of inertial frames and that of accelerated frames in the limit of zero acceleration. Since lifetime dilatations of particles with constant velocities can be described by the physical time in inertial frames, the corresponding "physical time" in accelerated frames must be able to describe, say, the accelerated "lifetime dilatation" or "decay-length dilatation" of an accelerated particle which decays in flight. In this sense, "times" in both inertial and non-inertial frames should have equal *physical significance.*

Thus, it is important to find out first what is the "physical time" in accelerated frames and how one can realize it through the setup of a clock system. These are very difficult problems in a general non-inertial frame, especially in rotating frames, because the speed of light becomes very complicated, so that the usual synchronization procedure based on light signals cannot be applied. However, the principle of limiting 4-dimensional symmetry can dictate certain properties of the "physical time" in a non-inertial frame.

Since there is no constant speed of light c in an accelerated frame F from the operational viewpoint, one can only denote the 4–coordinate of an event by (w,x,y,z) in general, according to the principle of limiting 4–dimensional symmetry. Note that w has the dimension of length and plays the role of the evolution variable. We avoid introducing the constant c = 29979245800 cm/sec into the formalism of physics in non-inertial frames. Although it is not physically or logically wrong to do so, defining w=ct for non-inertial frames is unnecessary and can hide and confuse truly universal constants. For example, if one uses c, one would be led to the conclusion that, for example, the Planck constant $\hbar$ is universal. However, if one does not introduce c, one can see that the truly universal constant for quantum mechanics in non-inertial frames is J = $3.5177293 \times 10^{-38}$ g·cm rather than $\hbar$. This can also be demonstrated in the physical theory of relativity (i.e., taiji relativity) which is based solely on the first principle of relativity and does not make the second postulate concerning the universal speed of light.[9]

For simplicity, let us first consider the physical time in CLA frames. In the twentieth century, there have been a number of attempts to define a transformation between the times in an inertial frame and one with a constant linear acceleration. Through an ingenious trick of clock synchronization with the help of three reference frames, Einstein in 1907 was able to obtain an important result (21.1) for clocks at different positions in a CLA frame F with small accelerations in the x direction:[1]

$$t_X \sim t_O\left(1 + \frac{gx}{c^2}\right), \qquad F = F(ct,x,y,z). \tag{21.3}$$

But this approximate result is inadequate to reveal the relation between time in an inertial frame and that in a CLA frame.

In 1909, Born solved the relativistic equation of motion, $dp_I/dt_I = F$ using a constant force F=mg in an inertial frame. Using the relativistic momentum for p_I, he obtained

$$\left(x_I + \frac{c^2}{g}\right)^2 - (ct_I)^2 = \left(\frac{c^2}{g}\right)^2, \tag{21.4}$$

which is a hyperbola in the $x_I t_I$ plane. The above results still do not lead to an exact transformation between an inertial frame and a CLA frame.

In 1943, Møller[6] obtained a transformation for time (and space) between an inertial frame $F_I(ct_I, x_I, y_I, z_I)$ and a CLA frame $F(ct,x,y,z)$ moving with a constant acceleration "a" in the x_I (or x) direction:

$$ct_I = \frac{c^2}{a} \sinh \frac{at}{c} + x \sinh \frac{at}{c}, \qquad (21.5)$$

based on Einstein's vacuum equation $R_{\mu\nu}=0$ and a postulated time-independent metric tensor of the form, $g_{\mu\nu}=(g_{oo}(x),-1,-1,-1)$. His transformation involves only one parameter, i.e., the constant acceleration, and hence it does not reduce to the Lorentz transformation with a non-zero constant velocity in the limit of zero acceleration.

In 1972, Wu and Lee[7] used a kinematical approach to derive a uniformly accelerated transformation in the x-direction by assuming a time-independent metric tensor $g_{\mu\nu}(x) = (g_{oo}(x),-1,-1,-1)$ and local Lorentz contraction of length. Their transformations turned out to be identical to that of Møller.

Recently, Hsu and Kleff[10] used Wu-Lee's kinematical approach to obtain a generalized transformation of time from an inertial frame to a CLA frame $F(ct,x,y,z)$ with an arbitrary initial velocity v_o and a constant acceleration "a" in the x direction. We have defined w=ct in the CLA frame for easy comparison with Møller's and Wu-Lee's works. This may be called the generalized Møller-Wu-Lee (MWL) transformation. One has the following transformation of physical time between an inertial frame F_I and a CLA frame F,

$$ct_I = \left(x + \frac{c^2}{a\gamma_o}\right)\sinh\left(\frac{a\gamma_o t}{c} + \tanh^{-1}\beta_o\right) - \frac{\beta_o c^2}{a}, \qquad (21.6)$$

$$\gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \quad \beta_o = \frac{v_o}{c}, \quad \beta = \frac{v}{c} = \tanh\left(\frac{a\gamma_o t}{c} + \tanh^{-1}\beta_o\right). \qquad (21.7)$$

One can verify that eq. (21.6) reduces to (21.5) in the limit of zero initial velocity, $\beta_o \to 0$. Moreover, (21.6) reduces to the usual 4-dimensional form, $ct_I = \gamma_o[ct+\beta_o x]$, in the limit of zero acceleration, $a \to 0$.

However, based on the limiting 4-dimensional symmetry (see section 21f below), one can obtain yet another transformation between an inertial frame F_I and a CLA frame $F(w,x,y,z)$:[11]

$$ct_I = \gamma\beta\left(x + \frac{1}{\alpha\gamma_o^2}\right) - \frac{\beta_o}{\alpha\gamma_o},$$

$$x_I = \gamma\left(x + \frac{1}{\alpha\gamma_o^2}\right) - \frac{1}{\alpha\gamma_o}, \qquad y_I = y, \qquad z_I = z; \tag{21.8}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \qquad \beta = \alpha w + \beta_o .$

This is called the Wu transformations for CLA frames. Note that β in (21.8) has the usual form of a linear function of the evolution parameter w, $\beta = \alpha w + \beta_o$, in contrast to the more complicated function of $\beta = v/c = \tanh(gt/c + \tanh^{-1}\beta_o)$ in (21.5) and (21.6). In this sense, (21.8) is formally a minimal generalization of the classical transformations for accelerated frames.

We stress that all the above results (21.3), (21.5), (21.6) and (21.8) are consistent for small velocities. Einstein's result (21.3) is equivalent to the metric tensor $g_{00}\sim(1+2gx/c^2)$ for the space associated with an accelerated frame.[6] Indeed, (21.5), (21.6) and (21.8) all imply the same metric tensor in the case of zero initial velocity, $\beta_o=0$, and small velocity, $\beta<<1$. (cf. Equations (21.39), (21.40), (21.49) and (21.50) below.)

From the Wu transformation (21.8) one can solve for w,

$$w = \frac{(w_I + \beta_o/\alpha\gamma_o)}{(\alpha x_I + 1/\gamma_o)} - \frac{\beta_o}{\alpha}, \qquad w_I \equiv ct_I . \tag{21.9}$$

The F clocks can be synchronized without relying on knowledge of the speed of a light signal. This is possible because any "clock" (in the sense of a device which shows the time) can be adjusted to run at an arbitrary rate and show an arbitrary time.[9] For example, our F "clocks" could be some kind of computerized machines which have the capability to measure their position x_I in the F_I frame, obtain $w_I = ct_I$ from the nearest F_I clock, and then compute and display w using (21.9) on some readout. Such clocks may seem strange compared to our usual concept of time. However, the conventional time system in an inertial frame which we use is itself merely the result of a postulate (namely the second postulate of special relativity regarding the universal speed of light) which we have grown used to. As long as it can be physically realized by clocks and is consistent with known experimental results (see sec. 7.), the relation (21.9) for the "time" w must be regarded as a valid and physical time of the CLA frame F. Such a grid of computerized clocks will automatically become the more familiar Einstein clocks of inertial frames, provided $w \equiv ct$, when the acceleration in (21.9) becomes zero. Therefore, the coordinates (w,x,y,z) in (21.8) for a CLA frame play the same role and have a similar physical meaning to the Cartesian coordinates in the transformations for inertial frames. We may remark that these discussions of computerized clocks and their synchronizations can be applied to any other accelerated and inertial[9] frames and their transformations.

In the following discussions, we shall use

$$(w,x,y,z) \quad \text{and} \quad (w_I,x_I,y_I,z_I) \tag{21.10}$$

as coordinates of non-inertial and inertial frames respectively. For simplicity, w and w_I are called "time", and they are related to the "decay-length dilatation" experimentally.

21c. Møller's Gravitational Approach to Accelerated Transformations

It appears that physical phenomena in a constant-linear-acceleration (CLA) frame have not been thoroughly investigated. Although there were lifetime dilatation experiments for particles in uniform circular motion, these results cannot be applied to the case of CLA motion in which a particle's speed changes. Also, we still do not have a satisfactory transformation between an inertial frame and a CLA frame which can be smoothly and naturally connected to the Lorentz transformation when the accelerated frame becomes an inertial frame. Møller used Einstein's vacuum equations $R_{ik} = 0$ and postulated a time-independent g_{00} and $g_{11} = g_{22} = g_{33} = -1$, $ds^2 = g_{00}(x)dw^2 - dr^2$, to obtain $g_{00}(x) = (1+gx/c^2)^2$ and a transformation between an accelerated frame and an inertial frame.[6] But his accelerated transformation cannot be smoothly connected to the Lorentz transformation in the limit of zero acceleration.

In this section, Einstein's equation $R_{ik} = 0$ is used as a guiding principle to obtain CLA transformations. Although Einstein's covariant equation holds for any coordinates, the Lorentz transformation prefers the Cartesian coordinates. Thus, the natural assumption of smooth connection between a CLA frame and an inertial frame dictates the CLA transformation to be expressed in quasi-

Cartesian coordinates which become Cartesian in the limit of zero acceleration. The reason for using $R_{ik} = 0$ is suggested by a heuristic view that the 'inertial force' of accelerated frames and the 'gravitational force' may be considered as being "unified" by Einstein's equation. Nevertheless, these two forces satisfy different "boundary conditions": Namely, in contrast to the gravitational force, the inertial force does not vanish at spatial infinity, and the transformations for accelerated frames should reduce to the Lorentz transformation when inertial forces vanish.

Let us first consider an inertial frame $F_I(w_I,x_I,y_I,z_I)$ with $w_I = ct_I$, and a CLA frame $F(w,x,y,z)$ moving with a constant acceleration along the x-axis. Based on the preceding discussions, it is natural to assume that ds^2 takes the form

$$ds^2 = c^2dt_I^2 - dx_I^2 - dy_I^2 - dz_I^2$$

$$= g_{00}(x,w)dw^2 + g_{11}(x)dx^2 + g_{22}(x)dy^2 + g_{33}(x)dz^2, \tag{21.11}$$

where g_{00} is assumed to be a function of w and x in general. Time t_I is shown by the Einstein clocks in the inertial frame F_I. As usual, one may define $w = ct$ in the CLA frame F if one wishes. Although $R_{ik} = 0$ holds for arbitrary coordinates, we postulate the metric (21.11) so that ds^2 and the resultant transformations are consistent with both Einstein's vacuum equation $R_{ik}=0$ and the new "boundary conditions" when the acceleration vanishes. Since the CLA frame F moves along the x-axis, we look for axial symmetric solutions with

$$g_{22} = g_{33} = -Y^2(x), \tag{21.12}$$

and all metric tensors are functions of x, except that g_{00} may be a function of x and w.

This physical property of $g_{00}(x,w)$ is crucial for the new CLA transformation; and it will be determined later. Now we can calculate Christoffel symbols $G^i{}_{jk} = g^{im}(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk})$ and the Ricci tensor $R_{ik} = \partial_m G^m{}_{ik} + \partial_k G^m{}_{im} + G^n{}_{ik}G^m{}_{nm} - G^m{}_{in}G^n{}_{km}$. The equations $R_{ii} = 0$, $i=0,1,2$, lead to

$$\partial^2{}_x W - \partial_x W\left(\frac{\partial_x X}{X} - 2\frac{\partial_x Y}{Y}\right) = 0, \tag{21.13}$$

$$\frac{\partial^2{}_x W}{W} + 2\frac{\partial^2{}_x Y}{Y} - \left(\frac{\partial_x X}{X}\right)\left[2\frac{\partial_x Y}{Y} + \frac{\partial_x W}{W}\right] = 0, \tag{21.14}$$

$$\frac{\partial^2{}_x Y}{Y} - \frac{\partial_x Y \partial_x X}{YX} + \left(\frac{\partial_x Y}{Y}\right)^2 + \frac{\partial_x Y \partial_x W}{YW} = 0, \tag{21.15}$$

respectively, where Y is given by (21.12) and

$$W^2 = g_{00}(x,w)\,, \qquad X^2 = -\,g_{11}(x)\,; \qquad W = W_1(x)W_2(w)\,. \tag{21.16}$$

We note that $R_{33} = 0$ gives the same equation as (21.15) and other components of R_{ik} vanish identically.

If $\partial^2{}_x Y = 0$, equations (21.13)-(21.15) lead to an exact solution

$$W_1 = \frac{f_3}{\sqrt{f_1 x + f_0}}\,, \qquad X = f_2\sqrt{f_1 x + f_0}\,, \qquad Y = f_1 x + f_0\,, \tag{21.17}$$

where f_1, f_2 and f_3 are constants. We stress that $W_2(w)$ in (21.16) is arbitrary because it cannot be determined by Einstein's equation. Physically, one expects that the metric tensor g_{22} should satisfy $-g_{22} = Y^2 = 1$ rather than $Y^2 = (f_1 x + f_0)^2$, since there is no motion at all along the y-axis. Furthermore, the accelerated transformation based on this solution cannot be smoothly connected to the Lorentz transformation (i.e., it does not satisfy the "limiting four-dimensional symmetry" or integrability conditions in (21.23) below). Therefore, the solution (21.17) with $f_1 \neq 0$ is not physically meaningful. Note that the case $f_1 = 0$ in (21.17) is trivial and uninteresting because it is equivalent to that of zero acceleration.

Let us concentrate on the non-trivial case $\partial_x Y = 0$ and $\partial_x W = 0$. We have the solution

$$Y = 1\,, \tag{21.18}$$

which satisfies the boundary conditions $g_{22}(0) = g_{33}(0) = -1$ at the origin. From (21.13) and (21.18), we deduce a general relation between $W_1(x)$ and $X(x)$:

$$\frac{dW_1(x)}{dx} = fX(x)\,, \tag{21.19}$$

where f is a constant of integration. Note that $W_1(x)$ can be determined if $X(x)$ is given or postulated. Furthermore, the time-dependent part of g_{00}, i.e., $W_2(w)$, still cannot be determined by Einstein's equation, just like the previous case $\partial_x Y = 0$. Thus we have seen that Einstein's covariant equation by itself does not lead to a specific form for $X(x)$, $W_1(x)$ and $W_2(w)$. We may remark that one must have specific functional forms for $X(x)$ and W in order to have finite transformations between a linearly accelerated frame and an inertial frame. Møller postulated $W=W_1(x)$ and $X(x)=1$ in (21.16) (i.e., time-independent g_{00} and $g_{11} = g_{22} = g_{33} = -1$) to obtain $g_{00}(x) = (1+gx/c^2)^2$ and a transformation between an accelerated frame and an inertial frame.[6]

21d. A Kinematical Approach to Accelerated Transformations Based on the Limiting 4-Dimensional Symmetry

Let us consider now the implications of the Poincaré-Einstein principle of relativity for the transformations between a CLA frame F and an inertial frame F_I. Suppose that a CLA frame F moves along parallel x and x_I axes and that the origins of F and F_I coincide at the taiji-time $w = w_I = 0$. We postulate that the invariant infinitesimal intervals for a CLA frame F and an inertial frame F_I are given by

$$ds^2 = W^2(w,x)dw^2 - dx^2 - dy^2 - dz^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2. \qquad (21.20)$$

The principle of relativity dictates that $W(w,x)$ must approach 1 in the limit of zero acceleration. This is equivalent to satisfying the requirement of the limiting 4-dimensional symmetry. The presence of the function $W(w,x)\neq 1$ indicates the physical non-equivalence of F and F_I. The local relation between $F(x^\mu)$ and $F_I(x_I{}^\mu)$ may be written in the form,

$$dw_I = \gamma(Wdw + \beta\, dx),$$
$$dx_I = \gamma(dx + \beta\, Wdw), \qquad dy_I = dy, \qquad dz_I = dz, \qquad (21.21)$$

where γ and β are functions of the time w in general, and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \,. \tag{21.22}$$

The local transformation (21.21) preserves the invariance of ds^2 in (21.20). In order for F(w,x,y,z) to smoothly reduce to an inertial frame in the limit of zero acceleration, there must exist a *global* transformation related to (21.21). In other words, the limiting 4-dimensional symmetry also dictates that the two unknown function W(w,x) and $\beta(w)$ must satisfy the following two integrability conditions for the differential relations in (21.21):

$$\frac{\partial(\gamma W)}{\partial x} = \frac{\partial(\gamma\beta)}{\partial w}, \qquad \text{and} \qquad \frac{\partial\gamma}{\partial w} = \frac{\partial(\gamma\beta W)}{\partial x}. \tag{21.23}$$

By separation of variables, $W(w,x)=W_W W_X$, the two equations in (21.23) lead to the same relation

$$\frac{dW_X}{dx} = \frac{\gamma^2}{W_W}\frac{d\beta}{dw} = k_1 , \tag{21.24}$$

where k_1 is a constant. Thus we have two solutions

$$W_X = k_1 x + k_2 , \tag{21.25}$$

$$W_W = \frac{(d\beta/dw)}{[k_1(1 - \beta^2)]} \,. \tag{21.26}$$

The implication of the principle of limiting 4-dimensional symmetry is as follows: the solution of the differential equation (21.26) depends on the physical properties of either β or W_W. Namely, if the "velocity function" β is known, then W_W can be determined, and vice versa. In other words, the principle of limiting 4-dimensional symmetry by itself cannot uniquely determine the function W(w,x). In general, there are infinitely many solutions for (21.24). We observe that the present case resembles the situation in which gauge symmetry cannot uniquely determine the electromagnetic action,[12] so that one must further postulate a minimal electromagnetic coupling. In the present case, we also postulate the minimal generalization of the Lorentz transformation:

The minimal generalization of the Lorentz transformation is to give up the untenable relation $v_I = gt_I + v_{Io}$ in an inertial frame F_I (as discussed in section

21a), and to retain the velocity function β as the usual linear function of time w only in the CLA frame,

$$\beta = \alpha w + \beta_o . \tag{21.27}$$

It follows from (21.25) and (21.26) that we have a time-dependent metric tensor $g_{oo}(w,x)=W^2$, where

$$W = W_w W_x = \gamma^2(\gamma_o^{-2} + \alpha x) = \frac{(\gamma_o^{-2} + \alpha x)}{(1 - \beta^2)}, \tag{21.28}$$

where the constants k_2 and k_1 in the product $W_w W_x$ are determined by the limiting condition $W(w,x) \to 1$ as $\alpha \to 0$.

However, there is another relatively simple generalization of the Lorentz transformation that can be obtained by postulating a time-independent metric tensor g_{oo} , i.e.,

$$g_{oo} = W_x \qquad \text{or} \qquad W_w = 1 . \tag{21.29}$$

It then follows from (21.25) and (21.26) that,

$$W_x = k_1 x + k_2 , \tag{21.30}$$

$$\beta = \tanh(k_1 w + k_3) , \tag{21.31}$$

with parameters k_1, k_2, and k_3 determined as follows. The constant k_3 in (21.31) can be determined by the initial condition that $\beta = \beta_o$ when time $w = 0$, i.e., $k_3 = \tanh^{-1}\beta_o$. Furthermore, since β is equal to β_o when the acceleration approaches zero, k_1 in (21.31) should be proportional to the constant acceleration, although k_1 may also depend on β_o, as shown in eq. (21.35) below. Finally, since W_x must be 1 if the acceleration (and thus k_1) vanishes, we see that $k_2=1$. Relations (21.30) and (21.31) are now determined as follows:

$$W_x = 1 + k_1 x , \qquad \text{and} \qquad \beta = \tanh(k_1 w^* + \tanh^{-1}\beta_o) , \tag{21.32}$$

where we denote the time as w^* in order to distinguish it from w in (21.27). Note that in this case, the usual simple relation between the velocity and time variables in (21.27) is replaced by the more complicated relation in (21.32). In this sense, (21.32) is not the minimal generalization. Nevertheless, we will also discuss the solution (21.32) because the assumption of a time-independent metric tensor was also made by Møller and by Wu and Lee,[6,7]. Equation (21.32) leads to a generalization of the transformation obtained by them to the case of non-zero initial velocity, though it is based on entirely different considerations.[10]

21e. Generalized Møller-Wu-Lee Transformations Based on the Limiting 4-Dimensional Symmetry

Substituting (21.32) in (21.21), one can carry out the integration and obtain the relations

$$w_I = \left(x+\frac{1}{k_1}\right)\sinh(k_1 w^* + \tanh^{-1}\beta_o) + k_6 ,$$

$$x_I = \left(x+\frac{1}{k_1}\right)\cosh(k_1 w^* + \tanh^{-1}\beta_o) + k_7 , \tag{21.33}$$

$$y_I = y, \qquad z_I = z.$$

Using the usual initial condition that if $w^* = x = 0$, then $w_I = x_I = 0$, the constants of integration k_6 and k_7 in (21.33) can be determined in terms of k_1 as

$$k_6 = \frac{-\gamma_o\beta_o}{k_1}, \quad k_7 = \frac{-\gamma_o}{k_1} . \tag{21.34}$$

In order to determine the role of constant acceleration in (21.33) and to compare it with the transformation obtained by Møller and by Wu and Lee,[6,7] we follow Wu and Lee and impose the boundary condition that the velocity β is related to the constant acceleration α^* by the usual relation $\beta = \alpha^* w_I + \beta_o$ at $x_I = 0$.[7] From (21.32), (21.33) and (21.34), k_1 is found to be

$$k_1 = \gamma_o\alpha^*, \tag{21.35}$$

because

$$\beta = \frac{\sinh(k_1 w^* + \tanh^{-1}\beta_o)}{\cosh(k_1 w^* + \tanh^{-1}\beta_o)} = \frac{w_I + \gamma_o\beta_o/k_1}{x_I + \gamma_o/k_1} = \alpha^* w_I + \beta_o, \quad \text{for} \quad x_I = 0.$$

Thus, the generalized Møller-Wu-Lee (MWL) transformation between the frames F_I and F is given by

$$w_I = \left(x + \frac{1}{\gamma_o\alpha^*}\right)\sinh(\gamma_o\alpha^* w^* + \tanh^{-1}\beta_o) - \frac{\beta_o}{\alpha^*},$$

$$x_I = \left(x + \frac{1}{\gamma_o\alpha^*}\right)\cosh(\gamma_o\alpha^* w^* + \tanh^{-1}\beta_o) - \frac{1}{\alpha^*}, \tag{21.36}$$

$$y_I = y, \qquad z_I = z.$$

In the limit of zero acceleration $\alpha^* \to 0$, the transformation (21.36) indeed reduces to a form with the four-dimensional symmetry of the Lorentz group

$$w_I = \gamma_o(w^* + \beta_o x), \quad x_I = \gamma_o(x + \beta_o w^*), \quad y_I = y, \quad z_I = z, \tag{21.37}$$

where

$$\sinh(\tanh^{-1}\beta_o) = \beta_o\gamma_o, \qquad \text{and} \qquad \cosh(\tanh^{-1}\beta_o) = \gamma_o.$$

The resultant transformation (21.36) is obtained on the basis of the principle of limiting 4-dimensional symmetry and time-independent g_{oo} given in (21.29). Interestingly enough, it turns out to be the same transformation as (21.6) with $a/c^2 = \alpha^*$, $w = w^*$ and $ct_I = w_I$, obtained by using Wu-Lee's kinematic assumptions with non-zero initial velocity β_o.[10]

The inverse transformation of (21.36) is

$$w^* = \frac{1}{\gamma_o\alpha^*}\left\{\tanh^{-1}\left[\frac{(w_I + \beta_o/\alpha^*)}{(x_I + 1/\alpha^*)}\right] - \tanh^{-1}\beta_o\right\},$$

$$x = \sqrt{(x_I + 1/\alpha^*)^2 - (w_I + \beta_o/\alpha^*)^2} - \frac{1}{\gamma_o\alpha^*}, \tag{21.38}$$

$$y_I = y, \quad z_I = z.$$

Differentiation of (21.36) give

$$dw_I = \gamma(W_x dw^* + \beta dx), \quad dx_I = \gamma(dx + \beta W_x dw^*), \quad dy_I = dy, \quad dz_I = dz, \tag{21.39}$$

$$W_x = 1+\gamma_o\alpha^* x, \qquad \beta = \tanh(\gamma_o\alpha^* w^* + \tanh^{-1}\beta_o), \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

It follows from (21.39) that

$$ds^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2$$

$$= W_x^2 dw^{*2} - dx^2 - dy^2 - dz^2 \equiv g^*_{\mu\nu} dx^\mu dx^\nu, \tag{21.40}$$

where $W_x = 1+\gamma_o\alpha^* x$, diagonal elements of $g^*_{\mu\nu}$ are $g^*_{\mu\nu} = (W_x^2, -1, -1, -1)$ and μ, $\nu = 0,1,2,3$.

We stress that the coordinate $x^\mu = (w^*, x, y, z)$ with metric tensor $g^*_{\mu\nu}$ given by (21.40) is the preferred coordinate system for the generalized MWL transformation. Contrary to the usual covariant formalism, the coordinates x^μ cannot be arbitrary because the principle of limiting 4–dimensional symmetry dictates that the transformation (21.38) or (21.36) must be expressed in terms of such coordinates (w,x,y,z), so that they can smoothly connect to the Cartesian coordinates in the limit of zero acceleration. This is similar to the case that the coordinate system chosen for the Lorentz transformation should be Cartesian and cannot be arbitrary. The distance x is measured by the usual meter stick (or the Bohr radius of a hydrogen atom) at rest in the CLA frame. Similarly, w can be determined by a grid of computerized clocks in the CLA frame F, as discussed in section 21b.

The speed of a light signal in the CLA frame is described by ds=0, i.e.,

$$\left|\frac{dr}{dw^*}\right| = W_x = (1+\alpha^*\gamma_o x), \tag{21.41}$$

and is consistent with Einstein's result (21.2) when one sets $w^*=ct$ and $\beta_o=0$.

If a particle is at rest in F_I, using (21.39) and (21.40) with $dx_I=0$ and $y_I=z_I=0$, we obtain the relation of the proper time (or decay-length) w_{Ip} of a particle at the time w^*, $w_{Ip}=\int dw_I = \int ds$, i.e.,

$$w_{Ip} = \int_0^{w^*} dw^* \sqrt{W_x^2 - \left(\frac{dx}{dw^*}\right)^2} = \alpha^*\gamma_o \int_0^{w^*} dw^* \frac{1}{\cosh^2(\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o)}$$

$$= \left(\frac{1}{\alpha^*} + x_I\right)[\tanh(\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o) - \beta_o] , \qquad (21.42)$$

where x_I is the fixed position of the particle. On the other hand, if a particle is at rest at x in F, the proper time (or decay-length) is $w^*_p=\int dw^*=(1/W_x)\int ds$ $=(1/W_x)\int dw_I\,[1-(dx_I/dw_I)^2]^{1/2}$, where x is fixed, so that we have

$$w^*_p = \frac{1}{(1+\alpha^*\gamma_o x)} \int_0^{w^*} dw_I \sqrt{1-\beta^2}$$

$$= \frac{1}{\gamma_o\alpha^*}[\tanh^{-1}Z(x,w_I) - \tanh^{-1}Z(x,0)] , \qquad (21.43)$$

where

$$Z(x,w_I) = \frac{(w_I + \beta_o/\alpha^*)}{\sqrt{(x + 1/\gamma_o\alpha^*)^2 + (w_I + \beta_o/\alpha^*)^2}} ,$$

$$\frac{dx_I}{dw_I}\bigg|_{dx=0} = \beta = \tanh(\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o) = \frac{(w_I + \beta_o/\alpha^*)}{(x_I + 1/\alpha^*)} .$$

Moreover, the usual acceleration, $d^2x_I/dw_I^2=d\beta/dw_I$ of a fixed point **r** in F, as measured in F_I, is

$$\frac{d^2x_I}{dw_I^2} = \frac{(x + 1/\gamma_o\alpha^*)^2}{[(x + 1/\gamma_o\alpha^*)^2 + (w_I + \beta_o/\alpha^*)^2]^{3/2}} , \qquad (21.44)$$

which is clearly not a constant. However,

$$\frac{d}{dw_I}\frac{\beta}{\sqrt{1-\beta^2}} = \gamma^3\frac{d\beta}{dw_I} = \frac{\alpha^*\gamma_o}{(1+x\gamma_o\alpha^*)} \qquad (21.45)$$

is a constant because x is fixed. This is consistent with Born's result (21.4) [or dp_I/dt_I = constant.]

21f. Minimal Generalization of the Lorentz Transformations — the Wu Transformations

For the minimal generalization of the Lorentz transformation (first described in section 21d), we retain the usual relation $\beta=\alpha w+\beta_o$. The postulate of limiting 4–dimensional symmetry and such a minimal generalization lead to the interval (21.20) with W given by (21.28). Substituting (21.28) into (21.22) and integrating, one obtains the CLA transformation (21.8):

$$w_I = \gamma\beta\left(x + \frac{1}{\alpha\gamma_o^2}\right) - \frac{\beta_o}{\alpha\gamma_o},$$

$$x_I = \gamma\left(x + \frac{1}{\alpha\gamma_o^2}\right) - \frac{1}{\alpha\gamma_o}, \qquad y_I = y, \qquad z_I = z; \qquad w_I = ct_I, \tag{21.46}$$

where the constants of integration, $\beta_o/\alpha\gamma_o$ and $1/\alpha\gamma_o$, are determined by initial conditions, i.e., when $w_I=x_I=0$, one has $w=x=0$. This is called the Wu transformation.[8] One can verify that the Wu transformation (21.46) indeed reduces to the 4–dimensional symmetry form (21.37) in the limit of zero acceleration.

The inverse Wu transformation can be derived from (21.46),

$$w = \frac{(w_I + \beta_o/\alpha\gamma_o)}{(\alpha x_I + 1/\gamma_o)} - \frac{\beta_o}{\alpha},$$

$$x = \sqrt{(x_I + 1/\gamma_o\alpha)^2-(w_I + \beta_o/\alpha\gamma_o)^2} - \frac{1}{\alpha\gamma_o^2}, \qquad y = y_I, \qquad z = z_I. \tag{21.47}$$

Differentiation of (21.46) gives

$$dw_I = \gamma(Wdw + \beta dx), \quad dx_I = \gamma[dx + \beta Wdw)], \quad dy_I = dy, \quad dz_I = dz; \tag{21.48}$$

$$\beta = \alpha w + \beta_o, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad W = \gamma^2(\gamma_o^{-2}+\alpha x). \tag{21.49}$$

The invariant infinitesimal interval is thus

$$ds^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I = g_{\mu\nu}dx^\mu dx^\nu, \tag{21.50}$$

$$g_{\mu\nu} = (W^2,-1,-1,-1) = (\gamma^4[\gamma_o^{-2}+\alpha x]^2,-1,-1,-1) .$$

The factor W in the metric tensor (21.50) and the differential form of the Wu transformation (21.48) may be called the Wu factor. Again, we note that the coordinates (w,x,y,z) with the metric tensor $g_{\mu\nu}$ given by (21.50) are the preferred coordinates for the accelerated Wu transformations and that the time w in (21.46) and (21.47) differs from that in (21.36) and (21.38). We shall discuss physical implications of this important difference in chapter 23.

The propagation of light is described by ds=0 in (21.50), so that the speed of light measured in terms of w in a CLA frame is now given by the Wu factor W:

$$\left|\frac{dr}{dw}\right| = W = \gamma^2(\gamma_o^{-2} + \alpha x) , \tag{21.51}$$

which is consistent with Møller's result (21.41) and Einstein's result (21.2) for small velocities, $\beta_o \sim 0$ and $|\beta^2| << 1$. For large velocities, however, they have significant differences.

If a particle is at rest in F_I at the location $(x_I,0,0)$, (21.50) implies the proper time w_{Ip} at time w to be $w_{Ip} = \int dw_I = \int ds$, i.e.,

$$w_{Ip} = \int_0^w dw\sqrt{W^2 - (dx/dw)^2} = \left(\frac{1}{\gamma_o} + \alpha x_I\right)w . \tag{21.52}$$

On the other hand, if a particle is at rest in F, **r**=const., (21.50) gives the proper time w_p at time w_I :

$$w_p = \int_0^w dw = \int\frac{ds}{W} = \frac{1}{(\gamma_o^{-2} + \alpha x)}\int_0^{w_I} dw_I\ [1 - (dx_I/dw_I)^2]^{3/2}$$

$$= \frac{1}{\alpha}\left\{\frac{(w_I + \beta_o/\alpha\gamma_o)}{\sqrt{(x + 1/\gamma_o^2\alpha)^2 + (w_I + \beta_o/\alpha\gamma_o)^2}} - \frac{(\beta_o/\alpha\gamma_o)}{\sqrt{(x + 1/\gamma_o^2\alpha)^2 + (\beta_o/\alpha\gamma_o)^2}}\right\}. \tag{21.53}$$

This can also be written as

$$w_I = \left(x + \frac{1}{\gamma_o^2 \alpha}\right)[\gamma\beta - \gamma_o\beta_o], \qquad \beta = \alpha w + \beta_o. \tag{21.54}$$

These relations for proper times can be tested experimentally by measuring the decay-length (or "lifetime") dilatation of a particle with constant linear acceleration.

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22.

Dynamics of Classical and Quantum Particles in Non-Inertial Frames with the Limiting 4-Dimensional Symmetry

22a. Classical Electrodynamics in Constant-Linear-Acceleration Frames

Within the 4-dimensional framework of taiji relativity, there are only two truly universal and fundamental constants in quantum electrodynamics: $J = 3.5177293\times10^{-38}$g·cm and $\bar{e} = -1.6021891\times10^{-20}(4\pi)^{1/2}(\text{g·cm})^{1/2}$, instead of the usual three, c, ħ, e (in the electrostatic unit, esu.)[1] These results of truly universal constants still hold in the present formalism of physics in accelerated frames, in which the speed of light is not a constant.

To be specific, let us assume that CLA frames satisfy the Wu transformation (21.46). Since the speed of light in an accelerated frame F is no longer a universal constant, the invariant action for a charged particle moving in the electromagnetic potential $a_\mu(x)$ is assumed to be[2]

$$S = \int(-mds - \bar{e}a_\mu dx^\mu) - \frac{1}{4}\int f_{\mu\nu}f^{\mu\nu}Wd^4x$$

$$= \int Ldw - \frac{1}{4}\int f_{\mu\nu}f^{\mu\nu}Wd^4x\,, \qquad (22.1)$$

$$ds^2 = W^2dw^2 - dx^2 - dy^2 - dz^2\,, \quad \sqrt{-\det\, g_{\mu\nu}} = W = \gamma^2(\gamma_o^{-2} + \alpha x)\,, \qquad (22.2)$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu\,, \quad dx^\mu = (dw,d\mathbf{r})\,, \quad dx_\mu = g_{\mu\nu}dx^\nu = (W^2dw,-d\mathbf{r})\,, \qquad (22.3)$$

in a CLA frame F. The differential of the coordinate $dx^\mu=(dw,d\mathbf{r})$ is, by definition, a contravariant vector. The covariant coordinate dx_μ is given by (22.3). Note that the invariant action S for a CLA frame does not involve the constant c and that $Wd^4x = \sqrt{-\det\, g_{\mu\nu}}\, dwdxdydz$ is the invariant volume element in taiji spacetime. We use taiji-time w as the evolution variable, so that the Lagrangian L, defined in (22.1), takes the form,

$$L = -m\sqrt{W^2-\beta_x^2-\beta_y^2-\beta_z^2} - \bar{e}(a_0 + a_i\beta^i)\,, \quad i=1,2,3\,, \qquad (22.4)$$

$$\boldsymbol{\beta} = d\mathbf{r}/dw = (\beta^1,\beta^2,\beta^3) \equiv (\beta_x,\beta_y,\beta_z) \,.$$

Note that $\boldsymbol{\beta}$ in (22.4) is the 'velocity' of the particle measured in terms of Taiji time w. In the limit $\alpha \to 0$, F becomes an inertial frame and, hence, $\bar{e}$ and a_μ correspond to the charge e(in esu) and the usual electromagnetic potential $A_\mu(ct,\mathbf{r})$ by $\bar{e} = e/c$ and $a_\mu(w,\mathbf{r}) \Leftrightarrow A_\mu(ct,\mathbf{r})/c$ respectively. The canonical momentum P_i of a particle in the CLA frame F is defined by

$$P_i = -\frac{\partial L}{\partial \beta^i} = p_i + \bar{e}a_i \,; \qquad P_i = -P^i, \qquad i = 1,2,3 \,; \tag{22.5}$$

$$p_i = (-m\Gamma\beta_x/W, -m\Gamma\beta_y/W, -m\Gamma\beta_z/W) = g_{ik}p^k \,, \qquad \beta_x \equiv \beta^1 = -\beta_1 \,, \text{ etc.} \tag{22.6}$$

$$\Gamma = \frac{1}{\sqrt{1-\beta^2/W^2}} \,, \qquad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 \equiv -\beta_i\beta^i \,, \qquad \beta^i = \frac{dx^i}{dw} \,. \tag{22.7}$$

The "Hamiltonian" $H = P_0$, which has the same dimension as that of P_i, is defined by

$$P_0 \equiv \left(\frac{\partial L}{\partial \beta^i}\beta^i - L\right) = p_0 + \bar{e}a_0 = W\sqrt{(P_i - \bar{e}a_i)^2 + m^2} + \bar{e}a_0 \equiv H \,; \tag{22.8}$$

$$p_0 = m\Gamma W = g_{00}p^0 \,, \qquad p^0 \equiv m\frac{dx^0}{ds} = m\frac{dw}{ds} \,. \tag{22.9}$$

Note that the contravariant momentum p^μ and the covariant momentum p_μ are related by $p_\mu = g_{\mu\nu}p^\mu$, i.e., $p_\mu = (p_0,p_1,p_2,p_3) = (W^2p^0,-p^1,-p^2,-p^3) \equiv (W^2p^0,-\mathbf{p})$ and the function W is given in (22.2).

We observe that the covariant momentum $p_\mu = (p_0,p_1,p_2,p_3)$ in (22.6) and (22.9) can be written as $p_\mu = m dx_\mu/ds = m g_{\mu\nu}dx^\nu/ds$. Since m and ds are invariant, p_μ should transform as $g_{\mu\nu}dx^\nu$. From the transformations of dx^ν in (21.48) and $g_{\mu\nu}$ given in (21.50), we obtain the transformations of the covariant momentum

$$p_{I0} = \gamma\left(\frac{p_0}{W} - \beta p_1\right) \,, \qquad p_{I1} = \gamma\left(p_1 - \beta\frac{p_0}{W}\right) \,, \qquad p_{I2} = p_2 \,, \qquad p_{I3} = p_3 \,; \tag{22.10}$$

$$\beta = \alpha w + \beta_o = \frac{(w_I + \beta_o/\alpha\gamma_o)}{(x_I + 1/\alpha\gamma_o)} = \frac{\sqrt{(x_I + 1/\gamma_o\alpha)^2 - (x + 1/\alpha\gamma_o^2)^2}}{(x_I + 1/\alpha\gamma_o)} \,,$$

where we have used $dx_\mu = g_{\mu\nu} dx^\nu$, $dx_{I\mu} = \eta_{\mu\nu} dx_I^\nu$, $\eta_{\mu\nu} = (1,-1,-1,-1)$ and (21.47). Note that β in (22.10) is the velocity of the CLA frame F and differs from that in (22.4)–(22.9). This transformation allows us to see how the particle's energy p_{I0} increases as a function of distance it travels in an inertial laboratory. Suppose the particle is in the CLA frame at $x^i = (x_o, 0, 0) = \text{constant}$, so that $dx^i/dw = 0$, $\Gamma = 1$ and $p_0 = mW$ in (22.7) and (22.9). We have $p_{I0} = m\gamma = m/\sqrt{1-\beta^2}$, which leads to

$$\frac{dp_{I0}}{dx_I} = \frac{m\alpha}{(\gamma_o^{-2} + \alpha x_o)} = \text{constant}, \tag{22.11}$$

where we have used β in (22.10) with $x = x_o = \text{constant}$. This result gives the operational meaning of constant acceleration in an inertial laboratory such as the Stanford Linear Accelerator Center. To be more specific, if $x_o = 0$ and the initial velocity $\beta_o = 0$, the constant acceleration α of a charged particle with mass m can be obtained by measuring $(1/m)(dp_{I0}/dx_I)$, where (dp_{I0}/dx_I) is related to the potential gradient of the accelerator.

The Lagrange equation of motion of a charged particle can be derived from the invariant action (22.1). We obtain

$$m\frac{Du_\mu}{ds} = \bar{e} f_{\mu\nu} u^\nu, \tag{22.12}$$

$$\bar{e} = -1.6021891 \times 10^{-20} \sqrt{4\pi}\, \sqrt{g \cdot cm}\ ,$$

$$u^\nu = \frac{dx^\nu}{ds}, \qquad Du_\mu = u_{\mu;\nu} dx^\nu \equiv D_\nu u_\mu dx^\nu,$$

$$D_\nu u_\mu = \frac{\partial u_\mu}{\partial x^\nu} - \Gamma^\rho{}_{\mu\nu} u_\rho \equiv \partial_\nu u_\mu - \Gamma^\rho{}_{\mu\nu} u_\rho, \qquad D_\nu u^\mu = \partial_\nu u^\mu + \Gamma^\mu{}_{\nu\rho} u^\rho,$$

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\, g^{\rho\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) = \Gamma^\rho{}_{\nu\mu},$$

where $;\nu$ or D_ν denotes the partial covariant differentiation[3] with respect to x^ν. And the Christoffel symbols $\Gamma^\rho{}_{\mu\nu}$ can be calculated with the metric tensor given by (21.50) for the space of a CLA frame with the Wu transformation. On the other hand, if one works in the space of a frame with the generalized MWL transformation, one uses the metric tensor in (21.40), i.e., $g_{\mu\nu}=(W_x^2,-1,-1,-1)$ where $W_x=1+\gamma_o\alpha^* x$, to calculate the Christoffel symbols and so on.

For a continuous charge distribution in space, the invariant action for the electromagnetic fields and their interaction is assumed to be

$$S_{em} = - \int a_\mu j^\mu W d^4x - \frac{1}{4}\int f_{\mu\nu} f^{\mu\nu} W d^4x . \tag{22.13}$$

It leads to the following Maxwell's equations

$$g^{\mu\nu} D_\mu f_{\nu\alpha} = j_\alpha , \qquad \partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0 , \tag{22.14}$$

$$f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu ,$$

$$D_\alpha f_{\mu\nu} = \partial_\alpha f_{\mu\nu} - \Gamma^\rho{}_{\mu\alpha} f_{\rho\nu} - \Gamma^\rho{}_{\nu\alpha} f_{\mu\rho} .$$

These Maxwell's equations are invariant under the Wu transformations for CLA frames. Note that the coordinates in the Wu transformations (21.46) for a CLA frame F(w,x,y,z) is not arbitrary. Rather, they are a particular coordinates with the metric tensor given by (21.48) and (21.50).

22b. Quantum Particles and Dirac's Equation in a CLA Frame

Equation (22.10) implies the invariant relation $g^{\mu\nu} p_\mu p_\nu = m^2$, which can be written as $g^{\mu\nu}(P_\mu - \bar{e} a_\mu)(P_\nu - \bar{e} a_\nu) = m^2$ by using equations (22.5) through (22.9) and $P^\mu = g^{\mu\nu} P_\nu$. This resultant equation for a classical charged particle suggests that the generalized Klein-Gordon equation for a quantum charged particle in a CLA frame should have the form

$$\left[g^{\mu\nu}(iJD_\mu - \bar{e} a_\mu)(iJD_\nu - \bar{e} a_\nu) - m^2\right]\Phi = 0 , \quad g^{\mu\nu} = (W^{-2}, -1, -1, -1) ,$$

i.e., $$\Big[W^{-2}(iJD_0 - \bar{e} a_0)^2 - (iJD_1 - \bar{e} a_1)^2 - (iJD_2 - \bar{e} a_2)^2$$

$$- (iJD_3 - \bar{e} a_3)^2 - m^2\Big]\Phi = 0 . \tag{22.15}$$

Note that the canonical momentum $P_\mu/(iJ)$ has to be replaced by covariant derivatives D_μ as defined in Riemannian Geometry. Similarly, the generalized Dirac equation for a CLA frame F should have the form

$$[\Gamma^\mu(x)(P_\mu - \bar{e}a_\mu) - m]\psi = 0, \tag{22.16}$$

or $$[W^{-1}\gamma^0(P_0 - \bar{e}a_0) + \gamma^1(P_1 - \bar{e}a_1) + \gamma^2(P_2 - \bar{e}a_2) + \gamma^3(P_3 - \bar{e}a_3) - m]\psi = 0,$$

where

$$P_\mu \equiv (P_0,-\mathbf{P}), \qquad \psi = \psi(w,x,y,z), \qquad \Gamma^\mu(x) = (W^{-1}\gamma^0, \gamma^1, \gamma^2, \gamma^3),$$
$$\{\gamma^a,\gamma^b\} = \eta^{ab}, \qquad \eta^{ab} = (1,-1,-1,-1), \quad a,b = 0,1,2,3. \tag{22.17}$$

The explicit form of the operator P_μ for the Dirac spinor turns out to be different from that for a scalar function. (See equations (24.41) and (24.49) in chapter 24.) Evidently, the generalized Dirac equation (22.16) reduces to the usual equation in the limit of zero acceleration, $\alpha \to 0$ or $W \to 1$. Note that P_0 and the momentum operator **P** have the dimensions of mass and the universal constant J has the dimension of (mass·length).[1] If one wishes, one can relate $\Gamma^\mu(x)$ in (22.16) to constant Dirac matrices γ^a, a=0,1,2,3, by the relation $\Gamma^\mu(x) = e_a{}^\mu(x)\gamma^a$, where $e_a{}^\mu(x)$ is a tetrad, i.e., a unit tangent vector and satisfies the relations:

$$\sum_{a=0}^{3} e_a{}^\mu(x)e_a{}^\nu(x) = g^{\mu\nu} \qquad \text{and} \qquad \sum_{a=0}^{3} e_{a\mu}(x)e_{a\nu}(x) = g_{\mu\nu}. \tag{22.18}$$

Note that the subscript a has no significance of covariance.

It turns out that when $\bar{e}=0$, we have the "free" Dirac equation in a CLA frame involving a "gauge covariant differentiation":

$$(i\Gamma^\mu J\nabla_\mu - m)\psi = 0, \quad \nabla_\mu = \left(\partial_0 + \frac{1}{2}(\partial_k W)\gamma^0\gamma^k, \partial_1, \partial_2, \partial_3\right), \tag{22.19}$$

which will be discussed in chapter 24. In the absence of the electromagnetic potential $a_\mu(x)$, or when the potentials do not involve time explicitly, one can use the separation of variables in (22.19) to find the w-dependent part of ψ :

$$\psi = F_w(w)\psi_r(\mathbf{r}) , \tag{22.20}$$

$$F_w = F_{wo} \exp\left\{\frac{-i}{J\gamma_o{}^2}\int_0^w \gamma^2(w')Mdw'\right\}$$

$$= F_{wo} \exp\left\{\frac{-iM}{2\alpha J\gamma_o{}^2} \ln\left|\frac{(1 + \beta)(1 - \beta_o)}{(1 - \beta)(1 + \beta_o)}\right|\right\} , \tag{22.21}$$

where M is a constant, $\beta=\alpha w+\beta_o$ and we have combined the space-dependent terms involving the same Dirac matrices together (e.g., $\gamma^k W^{-1}\partial_k W$ and $\gamma^k\partial_k$, k=1,2,3.) We note that (22.21) reduces to the usual form $F_w=F_{wo}\exp(-iMw/J)$ in the limit of vanishing acceleration, $\alpha\to 0$.

22c. Stability of Atomic Levels Against Constant Accelerations

A physical system is described by a Hamiltonian (e.g., given P_o in (22.16)) and the Hamiltonian in our formalism for accelerated frames has the dimension of mass. Thus, let us now consider the "mass level" of a hydrogen atom at rest in the CLA frame F. First, we must find out what is the generalized Coulomb potential produced by a charged particle which is at rest in a CLA frame. Such a Coulomb potential can be obtained by solving generalized Maxwell's equation $g^{\mu\nu}D_\mu f_{\nu\alpha} = j_\alpha$, (or $g^{\mu\nu}[D_\mu D_\nu a_\alpha - D_\mu D_\alpha a_\nu] = j_\alpha$) given by (22.14) in a CLA frame with the 4-potential $a_\mu = (a_o,0,0,0)$ and the current density $j_\mu = (\bar{e}\delta(\mathbf{r}),0,0,0)$ in a CLA frame.

Based on the covariant differentiations in (22.12) and (22.14), and the metric tensor $g_{\mu\nu}$ in (21.50) and that in (21.40), one can verify that $D_\mu D_\alpha A_\nu = D_\alpha D_\mu A_\nu$ for an arbitrary vector A_ν. (This is related to the fact that the Riemann curvature tensor $R^\nu{}_{\sigma,\mu\alpha}$ vanishes, $(D_\mu D_\alpha - D_\alpha D_\mu)A^\nu = -R^\nu{}_{\sigma,\mu\alpha}A^\sigma = 0$, for the spaces of CLA frames with both the metric tensors given in (21.50) and in (21.40).) As usual, we can choose a gauge condition $D^\nu a_\nu = g^{\nu\mu}D_\mu a_\nu = 0$ to simplify Maxwell's equations. Thus, we have $g^{\mu\nu}D_\mu D_\nu a_\alpha = j_\alpha$. When $a_\mu(w,\mathbf{r})=(a_o,0,0,0)$, this equation leads to the following generalized Coulomb equations for CLA frames:

$$g^{\mu\nu}D_\mu D_\nu a_0 = g^{\mu\nu}\partial_\mu\partial_\nu a_0 - \frac{2\alpha\beta}{1-\beta^2} g^{00}\partial_0 a_0 - \frac{\alpha}{\gamma_0^{-2}+\alpha x}\partial_1 a_0 = j_0 ,$$

$$\text{for} \quad g^{\mu\nu}=(\gamma^{-4}(\gamma_0^{-2}+\alpha x)^{-2},-1,-1,-1) ,$$

$$g^{\mu\nu}D_\mu D_\nu a_0 = g^{\mu\nu}\partial_\mu\partial_\nu a_0 - \frac{\gamma_0\alpha^*}{1+\gamma_0\alpha^* x}\partial_1 a_0 = j_0 , \tag{22.22}$$

$$\text{for} \quad g^{\mu\nu}=((1+\gamma_0\alpha^* x)^{-2},-1,-1,-1) .$$

These equations in general give complicated potentials produced by a simple point charge. Let us consider the static case, $a_0 = a_0(\mathbf{r})$ and $j_0 = \bar{e}\delta(\mathbf{r})$, with the first order approximation, $\alpha/(\gamma_0^{-2}+\alpha x) \approx \alpha$ and so on. Under these conditions, both equations in (22.22) lead to the differential equation and the solution for the generalized Coulomb potential in CLA frame:

$$\left(\nabla^2 + \alpha\frac{\partial}{\partial x}\right)a_0 = -\bar{e}\delta(\mathbf{r}) ,$$

$$a_0 = \frac{\bar{e}}{4\pi r}\left(1 - \frac{\alpha x}{2}\right) . \tag{22.23}$$

Indeed, they approach the usual equation and the Coulomb potential in the limit of zero acceleration, $\alpha \to 0$.

The generalized Dirac equation for the electron in an accelerated hydrogen atom (i.e., at rest in a CLA frame) is given by (22.19) with Γ^μ given by (22.17) and with the usual replacement: $\nabla_\mu \to (\nabla_\mu - \bar{e}a_\mu)$:

$$\left\{\frac{iJ}{W}\partial_0 + \frac{iJ}{2W}(\partial_1 W)\alpha_{Dx} - \frac{1}{W}\frac{\bar{e}^2}{4\pi r}\left(1 - \frac{\alpha x}{2}\right)\right.$$

$$\left. + \alpha_{Dx} iJ\frac{\partial}{\partial x} + \alpha_{Dy} iJ\frac{\partial}{\partial y} + \alpha_{Dz} iJ\frac{\partial}{\partial z} - \beta m\right\}\psi = 0 , \tag{22.24}$$

where $\boldsymbol{\alpha}_D = (\alpha_{Dx},\alpha_{Dy},\alpha_{Dz}) = (\beta_D\gamma^1, \beta_D\gamma^2, \beta_D\gamma^3)$ and $\beta_D \equiv \gamma^0$.

If one uses the Wu factor W given by (21.49), i.e., $W = \gamma^2(\gamma_0^{-2}+\alpha x)$ associated with the transformation (21.48), the effective potential $\bar{e}a_0/W =$

$-(\bar{e}^2/4\pi r)(1+\alpha x/2)/W$ becomes time-dependent due to acceleration of the whole atom because γ involves w. In this case, the atom is unstable and will radiate during acceleration. The radiation of a charged particle under acceleration is not completely solved.

On the other hand, suppose one uses the time-independent W given by (21.40), i.e., $W = W_x = (1+\gamma_o\alpha^* x)$ which is related to the generalized MWL transformation. The effective Coulomb potential $\bar{e}a_o/W$ is also time-independent and one has a stable atom, just as in an inertial frame.

We note that the generalized Dirac equation in (22.24) is, in general, complicated and does not have the usual spherical symmetry because of the presence of the metric tensor $g_{\mu\nu}$ and the generalized Coulomb potential given by (22.23). However, the violation of spherical symmetry turns out to be extremely small and the Dirac equation for the hydrogen atom can be well approximated by the usual form in an inertial frame. The reason is that we are interested only in the atomic domain, $r \sim 10^{-8}$cm. We have at present linear accelerators in an inertial frame F_I (laboratory) with a maximum voltage gradient of about 70 MeV per meter. We estimate the acceleration of a hydrogen atom to be

$$\alpha^* \sim 0.05/m \qquad \text{for} \quad \beta_o \sim 0.1 . \tag{22.25}$$

Thus, the extra x-dependent part in $(1+\gamma_o\alpha^* x)$ and the generalized Coulomb potential given by (22.23) is extremely small in comparison with 1,

$$\alpha^* x \sim 10^{-11} , \tag{22.26}$$

where x is roughly the size of the atom, $x \sim 10^{-8}$ cm. Therefore, the violation of spherical symmetry in the generalized Dirac equation is negligible for all practical purposes. As a result, the generalized Dirac equation in (22.24) can be well approximated by the usual Dirac equation, so that the mass level of an accelerated hydrogen atom is also given by[1]

$$M(n) = \frac{m}{\sqrt{1 + \dfrac{\alpha_e^2}{(n - h_d + \sqrt{h_d^2 - \alpha_e^2}\,)^2}}} , \qquad h_d = j + \frac{1}{2}, \tag{22.27}$$

where $\alpha_e = \bar{e}^2/(4\pi J) \sim 1/137$. We have seen that the atomic mass level structure is extremely stable against linear accelerations in this case. When an electron jumps from a state n1 to another state n2, it will emit or absorb a 'mass quantum' Jk_o:

$$M_I(n2) - M_I(n1) = Jk_{Io}, \qquad \text{in } F_I, \tag{22.28}$$

$$M(n2) - M(n1) = Jk_o, \qquad \text{in } F. \tag{22.29}$$

If two photons with "moving masses," Jk_{Io} and Jk_o, are emitted from two hydrogen atoms at rest in F_I and F respectively and measured immediately, then the results (22.27) through (22.29) imply

$$k_o(\text{rest}) = k_{Io}(\text{rest}), \tag{22.30}$$

where the symbols (rest) associated with k_o and k_{Io} refer to sources of photons. The result in (22.30) is isotropic when the photon is emitted and immediately measured in F. On the other hand, if the measurement is delayed, then the acceleration of the F frame will cause a Doppler-type shift.

22d. Electromagnetic Fields Produced by a Charge with Constant Linear Acceleration

In 1909, M. Born first discussed the motion of a charge with a constant linear acceleration based on special relativity.[4] Suppose the motion of the charge is along the x_I-axis of an inertial frame $F_I(w_I,x_I,y_I,z_I)$, he obtained the hyperbolic motion (21.4), i.e.,

$$x_I = \sqrt{(c^2/g)^2 + (ct_I)^2} - \frac{c^2}{g}, \qquad v_I = \frac{c^2 t_I}{\sqrt{(c^2/g)^2 + (ct_I)^2}}. \tag{22.31}$$

This motion can be viewed as the motion of a particle under a constant external force, F_I^{ext}=const., along the x-axis,

$$\frac{d}{dt_I} p_I = F_I^{ext}, \qquad p_I = \frac{m v_I}{\sqrt{1 - v_I^2/c^2}}, \qquad v_I = (v_I, 0, 0), \tag{22.32}$$

in special relativity. Born also obtained the electromagnetic fields associated with this motion of a charged particle.

Let us compare the motion described by (22.31) with the motion implied by the Wu transformation (21.46) for a CLA frame F(w,x,y,z). Suppose a particle is at rest in the CLA frame F at $x=x_o$ and $y=z=0$. Its position $(x_I,0,0)$ and velocity dx_I/dw_I can be expressed in terms of the time w_I in the inertial laboratory frame $F_I(w_I,x_I,y_I,z_I)$,

$$x_I = \sqrt{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(w_I + \frac{\beta_o}{\alpha\gamma_o}\right)^2} - \frac{1}{\alpha\gamma_o}, \tag{22.33}$$

$$\frac{dx_I}{dw_I} = \alpha w + \beta_o = \frac{w_I + \dfrac{\beta_o}{\alpha\gamma_o}}{\sqrt{\left(x_o + \dfrac{1}{\alpha\gamma_o^2}\right)^2 + \left(w_I + \dfrac{\beta_o}{\alpha\gamma_o}\right)^2}}, \quad \text{for} \quad x = x_o, \tag{22.34}$$

where we have used the Wu transformation (21.46) and (21.47). We observe that there are non-trivial dependencies on the initial velocity β_o and position x_o in the CLA motion in (22.33) and (22.34). The initial position $x_I(0)$ at $w_I = 0$ can be obtained from (22.33),

$$x_I(0) = \sqrt{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(\frac{\beta_o}{\alpha\gamma_o}\right)^2} - \frac{1}{\alpha\gamma_o}, \qquad w_I = 0, \tag{22.35}$$

$$= x_o, \qquad \text{for} \quad \beta_o = 0, \qquad \gamma_o = 1.$$

From (22.31), (22.33) and (22.34), we see that (22.31) corresponds to the special case of (22.33) and (22.34) when both the initial position and the initial velocity of the F frame are zero, $x_o = 0$ and $\beta_o = 0$. However, there is an important difference: The acceleration of the hyperbolic motion (22.31) is $d^2v_I/dt_I^2 = (c^6/g^2)/[(c^2/g)^2+(ct_I)^2]^{3/2}$, so that in the limit of zero acceleration $g \to 0$ one has the result $x_I \to 0$. It does not reduce to a constant motion with non-zero velocity. Note that the constant acceleration of a particle in special relativity is defined in its instantaneous rest frame $v_I = 0$, i.e., $t_I = 0$ from (22.31) and, hence $d^2v_I/dt_I^2 = g$ = constant. On the contrary, the limit of zero acceleration of the motion implied by the Wu transformation has a well-defined constant motion: $x_I = (x_o+\gamma_o w_I)/\gamma_o$ as $\alpha \to 0$. In other words, in this zero acceleration limit, we obtain exactly the 4–dimensional transformation with the constant velocity β_o. The reason for this difference is that the CLA motion in (22.33) and (22.34) satisfies the limiting 4–dimensional symmetry, while the hyperbolic motion (22.31) in general does not.

In the CLA motion, the acceleration of this particle can be derived from (22.34):

$$\frac{d^2x_I}{dw_I^2} = \frac{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2}{\left[\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(w_I + \frac{\beta_o}{\alpha\gamma_o}\right)^2\right]^{3/2}}, \qquad \text{for} \quad x = x_o. \tag{22.36}$$

Thus, the Wu transformation implies that the acceleration of this particle depends on its position x_o and the initial velocity β_o of the CLA frame. The relativistic equation of motion (22.32) cannot give this detailed information. As a result, if one uses (22.31) or (22.32) to derive transformations for reference frames with constant linear acceleration, one obtains the 4–dimensional group of conformal transformations in spacetime,[5] which turns out to be not physically meaningful and does not satisfy limiting 4–dimensional symmetry.

Now we have a more satisfactory Wu transformation which satisfies the limiting 4–dimensional symmetry and can provide more detailed information concerning accelerations. We would like to apply it to investigate electro–magnetic fields produced by an accelerated charge, classical radiation and energy conservation.

First, let us consider the electromagnetic fields produced by a charge $\bar{e}$, whose motion is described by (22.33) and (22.34). According to Maxwell's equations, the retarded 4–potentials $a_{I\mu}(f)$ at the field point f are the covariant Liénard-Wiechert potentials[6] as observed in an inertial frame F_I:

$$a_{I\mu}(f) = \frac{\bar{e}u_\mu(s)}{4\pi(x_I^f - x_I^s)^\mu u_\mu(s)}, \tag{22.37}$$

$$f \equiv (w_I^f, x_I^f, y_I^f, z_I^f), \qquad s \equiv (w_I^s, x_I^s, y_I^s, z_I^s),$$

$$u_\mu(s) = \eta_{\mu\nu}u^\nu(s), \qquad \eta_{\mu\nu} = (1,-1,-1,-1),$$

where s denotes the source point and $u^\nu(s)$ the 4–velocity of the charged particle (source)

$$u^\nu(s) = \left(\frac{dw_I^s}{ds}, \frac{dx_I^s}{ds}, 0, 0\right),$$

$$\frac{dw_I^s}{ds} = \frac{1}{\sqrt{1-\beta^2}} = \frac{\sqrt{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(w_I^s + \frac{\beta_o}{\alpha\gamma_o}\right)^2}}{x_o + \frac{1}{\alpha\gamma_o^2}}, \tag{22.38}$$

$$\frac{dx_I^s}{ds} = \frac{dx_I^s/dw_I^s}{\sqrt{1-\beta^2}} = \frac{w_I^s + \frac{\beta_o}{\alpha\gamma_o}}{x_o + \frac{\beta_o}{\alpha\gamma_o^2}},$$

where we have used (22.33) and (22.34). Note that the charged particle is assumed to be at rest at the position $(x_o,0,0)$ in the CLA frame $F(w,x,y,z)$, i.e., $\mathbf{r}^s = (x_I^s,0,0)$ in the inertial frame F_I.

It is important to note that the field time w_I^f is related to the source (or emission) time w_I^s by the relation

$$w_I^f - w_I^s = |\mathbf{r}_I^f - \mathbf{r}_I^s| > 0. \tag{22.39}$$

This is the causality condition by which the Liénard–Wiechert potential (22.37) must be retarded in order to be consistent with experiments and observations.

To find the potential $a_{I\mu}(f)$ at the field point $f \equiv (w_I^f, x_I^f, y_I^f, z_I^f)$, we must express the source coordinate $(w_I^s, x_I^s, y_I^s, z_I^s)$ in terms of the field point f. This can be done because the square of the causality condition (22.39) enables us to solve w_I^s in terms of $(w_I^f, x_I^f, y_I^f, z_I^f)$,

$$(w_I^f - w_I^s)^2 = (\mathbf{r}_I^f - \mathbf{r}_I^s)^2$$

$$= \left[x_I^f - \sqrt{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(w_I^s + \frac{\beta_o}{\alpha\gamma_o}\right)^2} + \frac{1}{\alpha\gamma_o}\right]^2 + \rho^2, \tag{22.40}$$

$$\rho^2 = (y_I^f - y_I^s)^2 + (z_I^f - z_I^s)^2 = (y_I^f)^2 + (z_I^f)^2.$$

For calculations, it is convenient to set

$$w_f = w_I^f + \frac{\beta_o}{\alpha\gamma_o}, \quad w_s = w_I^s + \frac{\beta_o}{\alpha\gamma_o}, \quad x_f = x_I^f + \frac{1}{\alpha\gamma_o}, \quad k = x_o + \frac{1}{\alpha\gamma_o^2}.$$

By straightforward calculations, we obtain

$$w_I^s + \frac{\beta_o}{\alpha\gamma_o} = \frac{(w_I^f + \frac{\beta_o}{\alpha\gamma_o})A^* - (x_I^f + \frac{1}{\alpha\gamma_o})B^*}{2\left[(x_I^f + \frac{1}{\alpha\gamma_o})^2 - (w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2\right]}, \tag{22.41}$$

$$A^* = (x_I^f + \frac{1}{\alpha\gamma_o})^2 + \rho^2 + k^2 - (w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2,$$

$$B^* = \sqrt{\left[(w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2 - (x_I^f + \frac{1}{\alpha\gamma_o})^2 - \rho^2 + k^2\right]^2 + 4k^2\rho^2}.$$

Similarly, we can also express the coordinate x_I^s of the charged particle in terms of the field point f,

$$x_I^s + \frac{1}{\alpha\gamma_o} = \sqrt{k^2 + (w_I^s + \frac{\beta_o}{\alpha\gamma_o})^2}$$

$$= \frac{(x_I^f + \frac{1}{\alpha\gamma_o})A^* - (w_I^f + \frac{\beta_o}{\alpha\gamma_o})B^*}{2\left[(x_I^f + \frac{1}{\alpha\gamma_o})^2 - (w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2\right]}, \tag{22.42}$$

where we have used (22.33), (22.34) and (22.41). It follows from (22.38), (22.41) and (22.42) that

$$(x_I^f - x_I^s)^\mu u_\mu(s) = \frac{B^*}{2\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2}. \tag{22.43}$$

Thus, the retarded potentials $a_{I\mu}(f)$ in (22.37) produced by a charged particle at rest at a point $(x_o,0,0)$ in the CLA frame F are given by

$$a_{Io}(f) = \frac{\bar{e}}{4\pi} \frac{(x_I^f + \frac{1}{\alpha\gamma_o})A^* - (w_I^f + \frac{\beta_o}{\alpha\gamma_o})B^*}{B^*\left[(x_I^f + \frac{1}{\alpha\gamma_o})^2 - (w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2\right]}, \tag{22.44}$$

$$a_{I1}(f) = - a_I^1(f) = - \frac{\bar{e}}{4\pi} \frac{(w_I^f + \frac{\beta_o}{\alpha\gamma_o})A^* - (x_I^f + \frac{1}{\alpha\gamma_o})B^*}{B^*\left[(x_I^f + \frac{1}{\alpha\gamma_o})^2 - (w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2\right]}, \tag{22.45}$$

$$a_{I2}(f) = a_{I3}(f) = 0 . \tag{22.46}$$

The electromagnetic field **E**(f) can be calculated from equations (22.44) through (22.46). We have

$$E_{I1}(f) = \frac{-\bar{e}}{\pi B^{*3}} (x_o + \frac{1}{\alpha\gamma_o{}^2})^2 \left[(x_o + \frac{1}{\alpha\gamma_o{}^2})^2 - (x_I^f + \frac{1}{\alpha\gamma_o})^2 + (w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2 + \rho^2\right],$$

$$E_{I2}(f) = \frac{2\bar{e}}{\pi B^{*3}} (x_o + \frac{1}{\alpha\gamma_o{}^2})^2 y_I^f (x_I^f + \frac{1}{\alpha\gamma_o}), \tag{22.47}$$

$$E_{I3}(f) = \frac{2\bar{e}}{\pi B^{*3}} (x_o + \frac{1}{\alpha\gamma_o{}^2})^2 z_I^f (x_I^f + \frac{1}{\alpha\gamma_o}) .$$

Similarly, the magnetic induction **B**(f) is found to be

$$B_{I1}(f) = 0 ,$$

$$B_{I2}(f) = - \frac{\bar{e}}{\pi B^{*3}} (x_o + \frac{1}{\alpha\gamma_o{}^2})^2 z_I^f (w_I^f + \frac{\beta_o}{\alpha\gamma_o}), \tag{22.48}$$

$$B_{I3}(f) = \frac{\bar{e}}{\pi B^{*3}} (x_o + \frac{1}{\alpha\gamma_o{}^2})^2 z_I^f (w_I^f + \frac{\beta_o}{\alpha\gamma_o}) .$$

In these calculations, we have used

$$E_I(f) = \left[-\frac{\partial a_I}{\partial w_I} - \frac{\partial a_{I0}}{\partial r_I}\right]_f, \qquad B_I(f) = \left[\nabla \times a_I\right]_f, \tag{22.49}$$

where $[\ ]_f$ denotes that the quantities in the bracket [ ] refer to the field point f. The function B^* is given by (22.41) and the coordinate of the charged particle in F_I is $(x_I^s, y_I^s, z_I^s) = (x_I^s, 0, 0)$. One can verify that Born's results correspond to the results (22.44)-(22.48) in the special case $x_o = 0$ and $\beta_o = 0$. We may remark that although these expressions involve $1/\alpha$, they do not diverge in the limit of zero acceleration $\alpha\to 0$. As a matter of fact, in the limit $\alpha\to 0$, the retarded potentials in (22.44)–(22.46) reduce to the usual ones in standard textbooks.[6] Moreover, all these electromagnetic fields in equations (22.47)–(22.48) vanish at the "black wall" $x_o = -1/\alpha\gamma_o^2$, the singularity wall of the CLA frame.

The causality condition (22.39) can be expressed in terms of the field point $f \equiv (w_I^f, x_I^f, y_I^f, z_I^f)$,

$$w_I^f - w_I^s = \frac{1}{2\left[(x_I^f + \frac{1}{\alpha\gamma_o})^2 - (w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2\right]} \times$$

$$\times (x_I^f + \frac{1}{\alpha\gamma_o}) B^* (w_I^f + \frac{\beta_o}{\alpha\gamma_o}) \left[(w_I^f + \frac{\beta_o}{\alpha\gamma_o})^2 - (x_I^f + \frac{1}{\alpha\gamma_o})^2 + \rho^2 + k^2\right] > 0. \quad (22.50)$$

When $\alpha\to 0$, it reduces to

$$w_I^f - w_I^s = \gamma_o^2\left[(x_I^f - \frac{x_o}{\gamma_o})\beta_o - \beta_o^2 w_I^f\right]$$

$$+ \gamma_o^2\sqrt{(x_I^f - x_o/\gamma_o - \beta_o w_I^f)^2 + \rho^2} > 0. \quad (22.51)$$

Thus, we have seen that the restriction on $(w_I^f, x_I^f, y_I^f, z_I^f)$ to satisfy the causality condition is not simple even in the limit $\alpha\to 0$.

According to classical electrodynamics, an accelerated charge emits electro-magnetic radiation. The radiation rate for a charge with arbitrary motion can be calculated by using the Liénard-Wiechert potentials (22.37):

$$R_{rad} = \frac{2}{3}\bar{e}^2 \frac{du_\mu(s)}{ds}\frac{du^\mu(s)}{ds}. \quad (22.52)$$

This holds generally for any source point s with the retarded velocity 4-vector $u^\mu(s) = (dw_I^s/ds, dx_I^s/ds, 0, 0)$ of any source point s. For the CLA motion of a charge, the acceleration 4-vector $du^\mu(s)/ds$ can be obtained from (22.38). We have

$$\frac{du^0(s)}{ds} = \frac{w_I^s + \frac{\beta_o}{\alpha\gamma_o}}{(x_o + \frac{\beta_o}{\alpha\gamma_o^2})^2}, \quad (22.53)$$

$$\frac{du^1(s)}{ds} = \frac{\sqrt{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(w_I^s + \frac{\beta_o}{\alpha\gamma_o}\right)^2}}{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2}, \tag{22.54}$$

$$\frac{du^2(s)}{ds} = \frac{du^3(s)}{ds} = 0.$$

They lead to the relation

$$\frac{du_\mu(s)}{ds}\frac{du^\mu(s)}{ds} = -\frac{1}{\left(x_o + \frac{\beta_o}{\alpha\gamma_o^2}\right)^2}. \tag{22.55}$$

Thus, the radiation rate for a charge with a CLA motion is a constant

$$R_{rad} = -\frac{2}{3}\frac{\bar{e}^2}{\left(x_o + \frac{\beta_o}{\alpha\gamma_o^2}\right)^2}. \tag{22.56}$$

The radiation rate R_{rad} has the dimension of mass/length because $\bar{e}$ has the dimension $(\text{mass}\times\text{length})^{1/2}$, as given by (22.12) and x_o has the dimension of length.

Let us estimate the radiation rate for an electron at a high energy laboratory, say, the Stanford Linear Accelerator with w_I=ct and a potential gradient of 60 MeV per meter. Let us set x_o=0, β_o=0 and use $\bar{e}$ in (22.12), the result (22.11) with (dp_{Io}/dx_I)=60MeV/c^2 per meter and the electron mass $m \approx 0.5\text{MeV}/c^2 \approx 10^{-30}$ Kg, we obtain

$$R_{rad} \approx -\bar{e}^2\alpha^2 \approx -10^{-40}\ \text{Kg/m}, \tag{22.57}$$

which corresponds to about 10^{-23} watts in conventional units. This is extremely small in comparison with the energy loss in circular motion. For example, in the 10 GeV Cornell electron synchrotron with an orbital radius of 100 meters, the loss of electron energy per turn is about 9 MeV,[6] which corresponds to about 10^{-2} MeV/m or 10^{-6} Watts. This is a significant loss of the electron energy. Thus, r-f power must be supplied to maintain electrons at a constant energy as they circulate. For the electron with a CLA motion, the loss of energy is completely negligible in the presently available energies.

22e. Covariant Radiative Reaction Force in Special Relativity and Common Relativity, and Conservation Laws for Radiations

A complete satisfactory treatment of the radiative reaction forces and their effects exist neither in classical electrodynamics nor in quantum electrodynamics. This is a profound difficulty which is intimately related to the conceptual framework of spacetime and elementary particles. Whenever a charged particle is accelerated by an external force, it emits radiation which carries away linear momentum and angular momentum in general. Thus, the emitted radiation must influence the particle's subsequent motion.

However, the situation in special relativity is not satisfactory because the covariant radiative reaction force vanishes for a charged particle which moves with a constant linear acceleration and radiates. In contrast, common relativity does not have this problem and is more satisfactory.

In the following discussions in this section, all coordinates and momenta refer to the charged particle (i.e., the source of electromagnetic emission) in a general inertial frame. So the superscript s for source and the subscript ı for inertial frame will be suppressed in this section. Only the constant $(x_o,0,0)$ refer to the position of the particle in a CLA frame.

First, let us consider the covariant radiative reaction force in special relativity. The usual form of the equation of motion such as (10.20) or (22.12) is not completely satisfactory because they do not take the radiative reaction force into account. Once the radiative reaction force appears, the system becomes non-holonomic because such a force cannot be derived from a potential in the Lagrangian formalism. The Abraham-Lorentz model of finite size (extended) electron was studied in detail and it leads to the Abraham radiative reaction 4-vector.[7] Dirac's formalism of Lorentz covariant classical electrodynamics[8] also had the same result:

$$\frac{dp^{\mu}}{ds} = F^{\mu}_{ext} + F^{\mu}_{rad}, \qquad c \equiv 1, \tag{22.58}$$

$$F^{\mu}_{ext} = \bar{e}F^{\mu\nu}u_{\nu}, \tag{22.59}$$

$$F^{\mu}_{rad} = \frac{2\bar{e}^2}{3m}\left(\frac{d^2p^{\mu}}{ds^2} + \frac{p^{\mu}}{m^2}\frac{dp_{\alpha}}{ds}\frac{dp^{\alpha}}{ds}\right). \tag{22.60}$$

Let us consider the CLA motion described by (22.33) and (22.34), which include hyperbolic motion as a special case. Since $p^{\mu} = m dx^{\mu}/ds$, the differentiation of (22.38) leads to

$$\frac{1}{m}\frac{d^2p^{\mu}}{ds^2} = \left(\frac{\sqrt{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(w + \frac{\beta_o}{\alpha\gamma_o}\right)^2}}{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^3}, \frac{w + \frac{\beta_o}{\alpha\gamma_o}}{\left(x_o + \frac{\beta_o}{\alpha\gamma_o^2}\right)^3}, 0, 0\right)$$

$$= \frac{p^{\mu}}{m\left(x_o + \frac{\beta_o}{\alpha\gamma_o^2}\right)^2}. \tag{22.61}$$

This second order derivative of momentum may be called the "jerk" 4–vector, since the time derivative of the accéleration is known as jerk in mechanics. Although this quantity "jerk" never appears in the fundamental laws of nature (i.e., elementary particle physics or quantum field theories), it is needed for the designs in robotics and in tracking systems for fast moving objects. From (22.54), (22.60), (22.61) and $p^{\mu}=mu^{\mu}$ we have the following result based on special relativity:

$$F^{\mu}_{rad} = \frac{2\bar{e}^2}{3m}\left[\frac{p^{\mu}}{\left(x_o + \frac{\beta_o}{\alpha\gamma_o^2}\right)^2} + \frac{(-p^{\mu})}{\left(x_o + \frac{\beta_o}{\alpha\gamma_o^2}\right)^2}\right] = 0, \qquad c \equiv 1. \tag{22.62}$$

This implies that, for CLA motions, the force of radiative reaction vanishes, so that equation (22.58) turns out to be the same as the conventional equation $dp^{\mu}/ds = \bar{e}F^{\mu\nu}u_{\nu}$, even though the charge emits electromagnetic radiation. This is indeed a very strange result. In the literature, Pauli and von Laue said that charges with CLA motion (i.e., the hyperbolic motion) do not radiate.[9] If they were right, the result (22.62) would be satisfactory. However, their statement is incorrect.[5] Other physicists such as Langevin, Poincaré and Heitler concluded that there is radiation.[4] The result (22.60) for the radiative reaction force of Abraham and Dirac appears to be unique because any force must satisfy the relation $p^\mu F_\mu=0$ which follows logically from the basic 4-dimensional law $ds^2 = dx_\mu dx^\mu$, $x^\mu = (ct,x,y,z)$, and the definitions $p^\mu = mu^\mu$ and $c = 1$ in special relativity. Furthermore, the 4-dimensional symmetry of the constant linear acceleration requires that the jerk 4-vector is proportional to the velocity 4-vector, as shown in (22.61). So the situation appears to be quite hopeless because 4-dimensional symmetry of special relativity seems to be exceedingly restrictive.

Fortunately, there is a different and distinct 4-dimensional symmetry of common relativity, as discussed in chapter 12. Within the framework of common relativity, one has the basic law $ds^2 = dx_\mu dx^\mu$ with $x^\mu = (bt_c,x,y,z)$, where b is a function, but common time t_c is an invariant quantity, just like ds. Consequently, one also has $p^\mu p_\mu = m^2$, where $p^\mu = mdx^\mu/ds$. (See equations (12.25), (12.27) and (12.28).) By differentiating $p^\mu p_\mu=m^2$ with respect to common time t_c twice, one obtains $(dp^\mu/dt_c)p_\mu = 0$ and $(d^2p^\mu/dt_c^2)p_\mu+(dp^\mu/dt_c)(dp_\mu/dt_c) = 0$. These relations imply that the covariant radiative reaction force has the form

$$\frac{d^2p^\mu}{dt_c^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{dt_c}\frac{dp^\alpha}{dt_c} . \tag{22.63}$$

Note that in the absence of the radiative reaction force F^μ_{rad} in (22.58), the equation of motion, $mdu^\mu/ds = \bar{e}f^{\mu\nu}dx_\nu/ds$, can be written in terms of invariant common time t_c, $dp^\mu/dt_c = \bar{e}f^{\mu\nu}dx_\nu/dt_c$, as shown in eq. (12.29) in chapter 12. Thus, based on common relativity, we can postulate the basic equation of motion for a charged particle emitting radiation to be

$$\frac{dp^\mu}{dt_c} = F^\mu_{ext} + F^\mu_{rad}, \tag{22.64}$$

$$F^\mu_{ext} = \bar{e}\, f^{\mu\nu}\frac{dx_\nu}{dt_c}, \tag{22.65}$$

$$F^\mu_{rad} = \frac{2\bar{e}^2}{3m}\frac{dt_c}{ds}\left(\frac{d^2p^\mu}{dt_c^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{dt_c}\frac{dp^\alpha}{dt_c}\right) . \tag{22.66}$$

The invariant factor dt_c/ds in (22.66) is necessary for the dimension to be correct.

Let us now consider the conservation of "energy" p^0 in the general equation of motion (22.64). The zeroth component F^0_{ext} and F^0_{rad} in (22.64) are respectively the rate of work done by the external force (i.e., dW_{ext}/dt_c) and by the radiative reaction force, as measured in common time t_c. Since d/dt_c = $(ds/dt_c)(d/ds) = \sqrt{C^2-v^2}\,(d/ds)$, where $C = d(bt_c)/dt_c$, we have

$$\frac{d^2p^\mu}{dt_c^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{dt_c}\frac{dp^\alpha}{dt_c}$$

$$= \sqrt{C^2-v^2}\;\left(\frac{d}{ds}\sqrt{C^2-v^2}\,\right)\frac{dp^\mu}{ds} + (C^2-v^2)\left(\frac{d^2p^\mu}{ds^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{ds}\frac{dp^\alpha}{ds}\right)$$

$$= v^\alpha\frac{dv_\alpha}{ds}\frac{dp^\mu}{ds}, \qquad v^\mu = \frac{dx^\mu}{dt_c} = (C, \mathbf{v}) \,. \tag{22.67}$$

where we have used (22.61). The result in (22.38) gives

$$v^\alpha = \frac{dx^\alpha}{dt_c} = \left(\frac{dw}{dt_c}\,,\,\frac{dx}{dt_c}\,,0\,,0\right)$$

$$= \left(C\,, \frac{\left(w + \frac{\beta_o}{\alpha\gamma_o}\right)C}{\sqrt{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(w + \frac{\beta_o}{\alpha\gamma_o}\right)^2}}\,,0\,,0\right) . \tag{22.68}$$

From equations (22.66) through (22.68), we obtain a new non-zero radiative reaction force in common relativity,

$$F^0_{rad} = \frac{2\bar{e}^2}{3m}\frac{dt_c}{ds}v^\alpha\frac{dv_\alpha}{ds}\frac{dp^\mu}{ds}$$

$$= -\frac{2\bar{e}^2}{3m}\;\frac{C\left(w + \frac{\beta_o}{\alpha\gamma_o}\right)^2}{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2\sqrt{\left(x_o + \frac{1}{\alpha\gamma_o^2}\right)^2 + \left(w + \frac{\beta_o}{\alpha\gamma_o}\right)^2}}\;, \tag{22.69}$$

in contrast to the result (22.62) in special relativity. We have used $dC/dt_c = 0$ by choosing the inertial frame to be the one in which the speed of light is constant. This can be done because $v^\alpha(dv_\alpha/ds)$ is invariant in common relativity. (See equation (12.9).) Note that F^0_{rad} is never positive, as it should.

In general, the zeroth component of (22.64) can be written as the conservation law of "energy" (which has the dimension of mass) for an arbitrary motion,

$$\frac{dp^0}{dt_c} = \frac{dW_{ext}}{dt_c} - \frac{dW_{rad}}{dt_c} , \tag{22.70}$$

$$\frac{dW_{rad}}{dt_c} = -\left(\frac{\gamma}{C}\frac{d}{dt_c}\left(\frac{2\bar{e}^2}{3}\frac{du^0}{dt_c}\right) + CR_{rad}\right) , \qquad u^0 = \frac{p^0}{m} , \tag{22.71}$$

where R_{rad} is a scalar quantity given by (22.52). Its physical meaning is that the rate of change in the kinetic energy of a particle equals the rate of work done by the external force minus the rate of work done by the radiative reaction force. The work done by the radiative reaction force is always negative and involves two parts:

(i) One part is related to the rate of change in $(2\bar{e}^2/3)(du^0/dt_c)$ which may be called "acceleration charge energy." The idea of "acceleration energy" was discussed by Schott.[4] This "acceleration charge energy" is independent of the sign of the charge and depends only on the rate of change of $u^0 = \gamma$, just like the energy p^0 of a particle.

(ii) Another part is related to the radiation rate R_{rad} which is never positive, as one can see from (22.56).

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But he also said parenthetically that radiation does occur when two uniform, rectilinear motions are connected by a "portion" of hyperbolic motion.

23.

Experimental Tests of Generalized Lorentz Transformations for Constant-Linear-Acceleration Frames

23a. Tests of Physical Time in Non-Inertial Frames

In the discussions of chapter 21, we obtained two simple transformations for constant-linear-acceleration (CLA) frames on the basis of the principle of limiting 4-dimensional symmetry. First are the Wu transformations (21.46) which are obtained by assuming minimal generalization of the Lorentz transformations, i.e., to preserve the usual linear expression for acceleration and velocity, $\beta = \alpha w + \beta_o$, in accelerated frames. Second are the generalized Møller-Wu-Lee (MWL) transformations (21.36), which are obtained by assuming the metric tensor $(g_{oo},-1,-1,-1)$ to be time-independent and has a more complicated and unusual relation for acceleration and velocity, $\beta = \tanh(\gamma_o\alpha^* w^* + \tanh^{-1}\beta_o)$. Both of the evolution variables w and w* in these two transformations can be considered as a generalization of the 'time' in inertial frames. Although the physical 'time' in CLA frames cannot be uniquely determined by the limiting 4-dimensional symmetry principle, neither is it arbitrary. Let us summarize the generalized MWL transformations and the Wu transformations based on the limiting 4-dimensional symmetry principle with an additional assumption:

(A) Generalized MWL transformations:

Additional assumption: $g_{oo}(x^\mu) = g_{oo}(x) \qquad \Rightarrow$

Result: $g_{oo}(x) = [1+\alpha^*\gamma_o x]^2, \quad \left(\frac{dx_I}{dw_I}\right)_x = \tanh(\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o)\,;$ (23.1)

(B) Wu transformations:

Additional assumption: $\left(\frac{dx_I}{dw_I}\right)_x = \alpha w + \beta_o, \qquad \Rightarrow$

Result: $g_{oo}(w,x) = [\gamma^2(\gamma_o^{-2}+\alpha x)]^2.$ (23.2)

It is interesting to observe that time w^* in the generalized MWL transformation (21.36) and time w in the Wu transformation (21.46) are related by the following relation:[1]

$$w = \frac{1}{\alpha}\left[\tanh(\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o) - \beta_o\right], \quad \alpha^* = \alpha\gamma_o . \tag{23.3}$$

This relation between w and w^* can be seen by comparing $(dx_I/dw_I)_x$ in (23.1) and (23.2). It follows that

$$W_x dw^* = (1+\alpha^*\gamma_o x)dw[\tanh^{-1}(\alpha w + \beta_o)] = W(w,x)dw\,, \tag{23.4}$$

and

$$ds^2 = dw_I^2 - dr_I^2 = W_x^2 dw^{*2} - dr^2 = W(w,x)^2 dw^2 - dr^2\,. \tag{23.5}$$

Although both times w^* in (21.36) and w in (21.46) reduce to the same "taiji time" (measured in, say, centimeters rather than seconds) for inertial frames in the limit of zero acceleration, a burning question is:

"*What is the physical time in constant-linear-acceleration frames?*"

Since the physical time in inertial frames is the one which is directly related to the dilatation of "lifetime" or the decay-length of an unstable particle decaying in flight, the same property must hold also for the physical time in CLA frames because of the requirement of the limiting 4-dimensional symmetry principle. Therefore, the question of physical time for accelerated frames can only be settled by experiments. One way to answer this question is to measure the decay-length dilatation to test the predictions of the Wu transformations and the generalized MWL transformations.

In order to conduct an experimental test of the predictions for these two different CLA transformations, there are two crucial questions to be answered: How can one measure the lifetime or decay-length of a particle at rest in a CLA frame? How do we measure the constant linear acceleration α^* or α? We will show that, according to limiting 4-dimensional symmetry, a "constant linear acceleration" means a constant change in the kinetic energy p_0 of an object per unit length, as measured in an inertial laboratory. To see this, let us consider the invariant action S_f for a "free particle" in a CLA frame (which may be associated with the generalized MWL transformations with time w^* or the Wu transformations with time w):

$$S_f = -\int_a^b m ds = \int_{w_a}^{w_b} L_w dw \ , \tag{23.6}$$

$$ds^2 = W^2 dw^2 - dx^2 - dy^2 - dz^2 \ , \qquad g_{\mu\nu} = (W^2, -1, -1, -1) \ ,$$

$$L_w = -m\sqrt{g_{\mu\nu} u^\mu u^\nu} \ , \qquad u^\mu \equiv \frac{dx^\mu}{dw} = (1, \beta^i) \ .$$

The function W in (23.6) may be either $W_x = (1+\alpha^*\gamma_o x)$ with w replaced by w* or $W(w,x) = \gamma^2(\gamma_o^{-2}+\alpha x)$.

As usual, the covariant momentum p_i and the corresponding "energy" p_0 (or the Hamiltonian) with the dimensions of mass are given in (22.5) and (22.8) with $\bar{e} = 0$. Thus, the covariant momentum 4–vector $p_\mu = (p_0, p_1, p_2, p_3) = (p_0, -\mathbf{p})$ transforms like the covariant coordinate $dx_\mu = g_{\mu\nu} dx^\nu$:

$$p_{I0} = \gamma\left(\frac{p_0}{W} - \beta p_1\right) , \quad p_{I1} = \gamma\left(p_1 - \beta\frac{p_0}{W}\right) , \quad p_{I2} = p_2 , \quad p_{I3} = p_3 \ . \tag{23.7}$$

One can verify the invariant relation

$$p_{I0}^2 - p_I^2 = \left(\frac{p_0}{W}\right)^2 - p^2 = m^2 \ . \tag{23.8}$$

Now suppose a particle is at rest in the CLA frame, $\mathbf{p} = 0$. We have $p_0 = mW$ and $p_{I0} = m\gamma$ from (23.7) and (23.8). Thus, when $\mathbf{r}$ is fixed, we obtain

$$\left(\frac{dp_{I0}}{dx_I}\right)_x = m\gamma^3 \frac{d\beta}{dx_I} \tag{23.9}$$

$$= \begin{cases} m\alpha/(\gamma_o^{-2}+\alpha x) \ ; & \beta = \alpha w + \beta_o \ ; \\ m\gamma_o\alpha^*/(1+\alpha^*\gamma_o x) \ ; & \beta = \tanh[\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o] \ . \end{cases}$$

The last relation involving α^* is consistent with constant acceleration given in (21.45).

We stress that the result (23.9) is intimately related to the "limiting four–

dimensional symmetry." The concept of linear and uniform acceleration of a particle can be defined in the sense of (23.9), i.e., constant change of a particle's "energy" p_{I0} per unit length, as measured in an inertial frame F_I. It is gratifying to see that this is precisely what is used in high energy laboratories these days. Therefore, the theories with limiting 4–dimensional symmetry already have partial experimental support.

23b. Experiments of Accelerated Decay-Length Dilatation and the Limiting 4–Dimensional Symmetry

For particle decay in flight with constant velocity, one can carry out calculations of the transition probability in quantum field theory (with the help of the Feynman rules) or use the Lorentz transformation to obtain the dilatation of the decay-length (or the lifetime) in any inertial frame. Strictly speaking, a particle's decay-length depends on the dynamics of its interactions.[2] We note that one cannot use the conventional quantum field theory based on perturbation calculations to calculate directly the accelerated dilatation of the decay-length *in CLA frames* because the conventional S-matrix (or the Feynman rules) are applicable only for free external particles. In inertial frames, free particles move with constant velocities in the initial and the final states of a physical process and have simple wave functions. However, in CLA frames the wave functions for these "free particles" are not simple. Fortunately, the *dilatation* of an accelerated particle's decay-length is a purely kinematic effect, in sharp contrast to the decay-length itself. This is true for measurements carried out in both inertial and CLA frames. Therefore, we can use the generalized MWL transformations or the Wu transformations to obtain the decay-length dilatation of particle decay in flight with constant acceleration.[1,3]

Let us consider a positive pion π^+ at rest at the origin $\mathbf{r} = 0$ in the CLA frame F. Suppose its 'lifetime' measured in terms of w (or decay-length) in F is $D_o = w_o$. According to the Wu transformation (21.46), its decay length $D_I = w_I$ measured in the inertial laboratory F_I is given by

$$D_I = \gamma\beta/\alpha\gamma_o^2 - \beta_o/\alpha\gamma_o\,, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \qquad \beta = \alpha D_o + \beta_o\,. \tag{23.10}$$

To see the new effect of acceleration α on the decay-length dilatation, let us expand D_I in terms of D_o and small α and β_o. Equation (23.10) leads to

$$D_I \approx D_o\gamma_o[1 + 1.5\alpha D_o\beta_o + 0.5\alpha^2 D_o^2]; \qquad (23.11)$$

$$\gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \qquad \beta = \alpha D_o + \beta_o \ll 1 .$$

This result predicts that the first order correction due to the acceleration α will make the lifetime more dilated when α and the velocity β_o are parallel to each other.

Suppose we have a linear accelerator in the inertial frame F_I with a voltage gradient of 10 MeV per meter. (Currently available voltage gradients are about 67 MeV per meter.) It is feasible to test the prediction (23.11) by measuring the decay lifetime of, say, a positive pion π^+. Using the relations (22.9) in the inertial laboratory,

$$p_{Io} - p_{Ioo} = m_\pi(\gamma - \gamma_o), \qquad m_\pi \approx 140 \text{ MeV}/c^2, \qquad (23.12)$$

$$p_{Io} - p_{Ioo} = m_\pi \alpha \gamma_o^2 \Delta x_I, \qquad \Delta x_I = 1 \text{ m}, \qquad r = 0, \qquad (23.13)$$

we estimate the acceleration α to be

$$\alpha \approx 0.063\frac{1}{m}, \quad \text{for } \beta_o = 0.1; \text{ and } \quad \alpha \approx 0.0025\frac{1}{m}, \quad \text{for } \beta_o = 0.943; \qquad (23.14)$$

for $p_{Io} - p_{Ioo} = 10$ MeV/c^2. Based on the rough approximation (23.11) with $D_o \approx$ 7.8m for π^+, we obtain its decay-length D_I in flight under acceleration of a voltage[4] gradient 10 MeV/m in the inertial laboratory frame F_I,

$$D_I \approx 7.8[1 + 0.07 + 0.12]\,\text{m}, \qquad \beta_o \approx 0.1, \qquad (23.15)$$

$$D_I \approx 7.8 \times 3[1 + 0.03 + 0.0002]\,\text{m}, \qquad \beta_o \approx 0.94, \qquad (23.16)$$

where the approximation is not good for large $\beta_o \approx 0.94$. If we use the exact relation (23.10), we obtain

$$D_I = 9.92m = 7.8[1+0.27]\,m, \qquad \text{for } \beta_o \approx 0.1, \tag{23.17}$$

and
$$D_I = 30.93m = 7.8\times3[1+0.32]m, \qquad \text{for } \beta_o \approx 0.943. \tag{23.18}$$

Here, for a simple illustration of the feasibility of this experimental test, we have used $D_o \approx 7.8m$, i.e., the decay-length of a particle at rest in the CLA frame F and measured by F-observers is approximately equal to that of the same kind of particle at rest in the inertial frame F_I and measured by F_I-observers. (Of course, this is only a crude approximation because there is no relativity or equivalence between an inertial frame F_I and a CLA frame. In this approximation of D_o, the 'rest decay-length' $D_o \approx 7.8m$ is numerically the same as $\tau_o c$, where τ_o is pion's rest lifetime in special relativity.) However, the decay-length D_o in (23.11) can be measured if one has data for several different values of α and β_o.

We can also consider the decay of the same positive pion π^+ (at rest at the origin $\mathbf{r} = 0$ in the CLA frame F) based on the generalized MWL transformations. Suppose its 'lifetime' measured in terms of w (or decay-length) in F is $D_o = w_o$. According to the generalized MWL transformation (21.36), its decay length $D_I = w_I$ measured in the inertial laboratory F_I is given by

$$L_I\,(MWL) = \frac{1}{\alpha^*}\left[\sinh(\alpha^*\gamma_o L_o)+\beta_o(\cosh(\alpha^*\gamma_o L_o) - 1)\right], \quad L_I = w_I. \tag{23.19}$$

Note that both (23.10) and (23.19) reduce to the usual relativistic decay length dilatation in the limit of zero acceleration,

$$L_I(MWL) = L_I(Wu) = \gamma_o L_o, \qquad \alpha^* = \alpha = 0, \tag{23.20}$$

as they should.

So far, the decay-length dilatation of *linearly* accelerated particles have not been directly measured. Our results (23.15)–(23.18) show that it is feasible to test them in linear accelerators. It is important to test the new effect of decay-length dilatation due to constant accelerations predicted by the Wu transformations and by the generalized MWL transformations. If such a new effect is confirmed, it could be useful for increasing the beam intensity of unstable particles in linear accelerators under some circumstances.

23c. Experiments on Wu–Doppler Effects of Waves Emitted from Accelerated Atoms

The new accelerated transformations (21.36) and (21.46) can also be experimentally tested by measuring a Doppler-type shift of wavelength of light emitted from an accelerated source.[5] From eq. (23.7) one obtains the Wu transformation of the covariant wave 4-vector $k_\mu = p_\mu/J$ between an inertial frame F_I and a CLA frame F:

$$k_{I0} = \gamma\left(\frac{k_0}{W} - \beta k_1\right), \quad k_{I1} = \gamma\left(k_1 - \beta\frac{k_0}{W}\right), \quad k_2 = k_3 = 0. \tag{23.21}$$

where $k_{I\mu} = (k_{I0}, -\mathbf{k}_I)$ and $k_\mu = (k_0, -\mathbf{k})$. Note that Jk_{I0} and Jk_0 are moving masses of the same photon measured from F_I and F respectively. Suppose the radiation source is at rest at the origin of the F frame, $\mathbf{r} = 0$. Experimentally, it is difficult to measure $k_0(\text{rest})$ and $k_1(\text{rest})$ for such a radiation source by observers in the accelerated frame F. Thus we have to express them in terms of quantities measured in the inertial frame (or laboratory) F_I. Using the approximate relation (22.30), i.e. $k_0(\text{rest}) = k_{I0}(\text{rest})$, and the Wu factor $W(w,0) = \gamma^2\gamma_o^{-2}$, we obtain a Doppler-type shift of k_{I0} (related to photon's moving mass or atomic mass level) and wavelength λ_I

$$k_{I0} = k_{I0}(\text{rest})\,\frac{(1+\beta)}{\gamma\gamma_o^{-2}}\,.$$

$$\frac{1}{\lambda_I} = \frac{1}{\lambda_I(\text{rest})}\,\frac{(1+\beta)}{\gamma\gamma_o^{-2}}\,, \qquad \beta = \alpha w + \beta_o, \tag{23.22}$$

for waves emitted from a CLA radiation source. These new results predicted by the Wu transformation were termed the Wu–Doppler shift.[5]

For an experimental test, it is more convenient to express 'velocities' in (23.22) in terms of distances. Suppose the radiation source (at $\mathbf{r} = 0$) enters the accelerated potential at $x_I = 0$ in F_I with an initial velocity β_o and emits radiation when it was accelerated to the point $x_I = L$ with the velocity β_I. We have

$$\beta_I = \left(\frac{dx_I}{dw_I}\right)_{x=0} = \beta = \sqrt{1 - \gamma_o^{-2}(1 + L\alpha\gamma_o)^{-2}}\,, \tag{23.23}$$

$$\gamma_I = \frac{1}{\sqrt{1 - \beta_I^2}} = \gamma_o(1 + \alpha L \gamma_o) . \tag{23.24}$$

It follows from (23.22)-(23.24) that the wavelength shift is given by

$$\delta\lambda_I = \lambda_I - \lambda_I(\text{rest}) \cong \lambda_I(\text{rest})\left[\alpha L - \sqrt{\beta_o^2 + 2\alpha L} - \frac{1}{2}\beta_o^2\right] . \tag{23.25}$$

Also, according to (22.28) and (23.22), the shift of mass levels of a CLA atom is

$$\delta M_I = Jk_{I0} - Jk_{I0}(\text{rest}) \cong Jk_{I0}(\text{rest})\left[\sqrt{\beta_o^2 + 2\alpha L} + \frac{1}{2}\beta_o^2 - \alpha L\right] . \tag{23.26}$$

The predictions (23.25) and (23.26) can be measured in the laboratory frame F_I by using a method similar to that of Ives–Stilwell.[6]

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24.

Quantizations of Scalar, Spinor and Electromagnetic Fields in Constant-Linear-Acceleration Frames

24a. Scalar Fields in Constant-Linear-Acceleration Frames

In constant-linear-acceleration (CLA) frames, physical equations and calculations are generally more complicated than those in inertial frames. The reason for this is that the differential time-axis dw in CLA frames is altered by the Wu factor $W(w,x)=\gamma^2(\gamma_o^{-2}+\alpha x) > 0$, as shown in (22.2), so that the physical space is bounded by a "black wall" at $x=-1/\alpha\gamma_o^2$. It is a wall-singularity where the speed of any physical signal or particle vanishes. Nevertheless, the CLA frame with a constant-linear-acceleration is the simplest non-inertial frame, so it offers the best opportunity to explore physical properties of quantum fields beyond inertial frames or the framework of relativity theory.

The investigation of quantum field theory in CLA frames may shed light on more complicated non-inertial frames and, perhaps, quantum gravity. Also, in quantum field theory, especially when considering the S-matrix and the Feynman rules, we must use a plane wave (for a free particle) which is very simple in inertial frames, but complicated in CLA frames. As a result, many calculations in CLA frames are most easily carried out by a change of variables. Such a change of variables effectively transforms calculations from a CLA frame to an inertial frame. Of course, there are non-trivial problems which need to be solved in CLA frames. For example, finding the Coulomb potential generated by a charged particle at rest in a CLA frame and solving "energy states" for such a potential, as discussed in chapter 22.

Since we have coordinate transformations between inertial frames and CLA frames, we can explore their implications for possible new physics of particles and fields in non-inertial frames. The equation of motion of a classical particle in CLA frames has been discussed in chapter 22. Even the simplest motion, i.e., constant velocity or momentum of a particle in an inertial frame, will have more complicated properties from the viewpoint of a CLA frame, as shown in the transformations of momentum (22.10).

A similar complication occurs in field theory: The Klein–Gordon equation for a "free" (i.e., non-interacting) particle in a CLA frame can be obtained from (22.15) by setting $\bar{e}$=0:

$$[J^2 g^{\mu\nu} D_\mu D_\nu + m^2]\Phi(w,\mathbf{x}) = 0 , \tag{24.1}$$

where **x** denotes the spatial vector and D_μ is the partial covariant derivative as defined in (22.12) with the fundamental metric tensor given by (21.40),

$$g^*_{\mu\nu} \equiv (W_x^2, -1, -1, -1) = ([1+\gamma_o\alpha^* x]^2, -1, -1, -1) , \tag{24.2}$$

based on the generalized MWL transformations (21.36), or given by (2.50)

$$g_{\mu\nu} \equiv (W^2, -1, -1, -1) = (\gamma^4[\gamma_o^{-2}+\alpha x]^2, -1, -1, -1) , \tag{24.3}$$

based on the Wu transformations (21.46). In the following discussion, the form of physical equations (or laws) involving W hold for both metric tensors in (24.2) and in (24.3). That is, the function W stands for either $[1 + \gamma_o\alpha^* x] > 0$ or $\gamma^2[\gamma_o^{-2} + \alpha x] > 0$.

Using the covariant expression for d'Alembert's operator $g^{\mu\nu}D_\mu D_\nu$, we find that

$$g^{\mu\nu}D_\mu D_\nu\Phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\; g^{\mu\nu}\partial_\nu\Phi\right)$$

$$= \{W^{-2}\partial_0^{\,2} - \nabla^2 - W^{-3}(\partial_0 W)\partial_0 - W^{-1}(\partial_1 W)\partial_1\}\Phi , \tag{24.4}$$

$$\sqrt{-g} = \sqrt{-\det g_{\sigma\rho}} = W > 0 ,$$

where

$$\partial_\mu = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial/\partial w, \partial/\partial \mathbf{x}) , \qquad \mathbf{x} = (x^1, x^2, x^3) . \tag{24.5}$$

The partial differential operators (24.5) are related to those in an inertial frame $\partial_{I\mu}=(\partial_{I0}, \partial_{I1}, \partial_{I2}, \partial_{I3})=(\partial/\partial w_I, \partial/\partial x_I^1, \partial/\partial x_I^2, \partial/\partial x_I^3)$ by the following relations,

$$\partial_{I0} = \gamma(W^{-1}\partial_0 - \beta\partial_1) , \quad \partial_{I1} = \gamma(\partial_1 - \beta W^{-1}\partial_0) , \quad \partial_{I2} = \partial_2 , \quad \partial_{I3} = \partial_3 , \tag{24.6}$$

which can be derived from the transformations in (21.46) (or (21.36) with W in (24.6) replaced by W_x.) They transform in the same way as the covariant momentum 4-vector p_μ in (22.10), as they should. It follows from (24.6) that

$$\partial_{I0}^2 - \nabla_I^2 = W^{-2}\partial_0^2 - \nabla^2 - W^{-3}(\partial_0 W)\partial_0 - W^{-1}(\partial_1 W)\partial_1 , \qquad (24.7)$$

which is consistent with the invariant expression for d'Alembert's operator $g^{\mu\nu}D_\mu D_\nu$ in (24.4).

For the CLA frame with the Wu transformations, W is given by (24.3), and the free Klein-Gordon equation (24.1) takes the form,

$$\left\{\frac{J^2}{W^2}\partial_0^2 - J^2\nabla^2 - \frac{2\alpha\beta\gamma^2 J^2}{W^2}\partial_0 - \frac{\alpha J^2}{(\gamma_o^{-2} + \alpha x)}\partial_1 + m^2\right\}\Phi = 0 . \qquad (24.8)$$

The solution of this "free" Klein-Gordon equation in a CLA frame takes the form

$$\Phi(x) = \Phi_0 \exp\left\{\frac{-iP(x)}{J}\right\}, \qquad x \equiv x^\mu , \qquad (24.9)$$

$$P(x) = p_{I0}w_I(w,x^1) + p_{I1}x_I^1(w,x^1) + p_{I2}x_I^2 + p_{I3}x_I^3$$

$$= p_{I0}[\gamma\beta(x^1 + \frac{1}{\alpha\gamma_o^2}) - \frac{\beta_o}{\alpha\gamma_o}] + p_{I1}[\gamma(x^1 + \frac{1}{\alpha\gamma_o^2}) - \frac{1}{\alpha\gamma_o}]$$

$$+ p_{I2}x^2 + p_{I3}x^3$$

$$= \frac{\gamma p_o}{W}[\gamma\beta(x^1 + \frac{1}{\alpha\gamma_o^2}) - \frac{\beta_o}{\alpha\gamma_o}] - \gamma\beta p_1[\gamma\beta(x^1 + \frac{1}{\alpha\gamma_o^2}) - \frac{\beta_o}{\alpha\gamma_o}]$$

$$+ \gamma p_1[\gamma(x^1 + \frac{1}{\alpha\gamma_o^2}) - \frac{1}{\alpha\gamma_o}] - \frac{\gamma\beta p_o}{W}[\gamma(x^1 + \frac{1}{\alpha\gamma_o^2}) - \frac{1}{\alpha\gamma_o}]$$

$$+ p_2x^2 + p_3x^3 , \qquad W = W(w,x^1) ,$$

where we have used (22.10). The phase P(x) reduces to the usual expression $(p_o w + p_1x^1 + p_2x^2 + p_3x^3)$ in the limit of zero acceleration $\alpha \to 0$. Note that p_{I0} and p_{I1} are constant momenta measured in F_I and that they are related to non-constant momenta p_μ in the CLA frame F by the transformation shown in (22.10). This "free" wave function (24.9) with the complicated phase P(x) in a CLA frame is actually a simple plane wave in an inertial frame F_I. If one substitutes the solution (24.9) with the phase P(x) in (24.8), one obtains

$$g^{\mu\nu}p_\mu p_\nu - m^2 = (W^2 p_0{}^2 - \mathbf{p}^2 - m^2) = 0 . \qquad (24.10)$$

This is the invariant relation for the momentum 4–vector, which takes the Lorentz invariant form $p_I{}^\nu p_{I\nu} - m^2 = 0$ in an inertial frame.

24b. Quantization of Scalar Fields in CLA Frames

The canonical quantization of fields mimics the dynamics in quantum mechanics. As a matter of fact, it is a generalization of the quantum mechanics of a finite system to an infinite system. In the following discussion, we shall closely follow the canonical quantization of fields in inertial frames. For simplicity, we will set $J=3.5177293\times 10^{-38}$g·cm≡1. The invariant action S for a neutral scalar field in CLA frames is assumed to take the form:

$$S = \int L_s d^4x , \qquad L_s = \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m^2\Phi^2) , \qquad (24.11)$$

$$g = \det g_{\sigma\rho} , \qquad J \equiv 1 ,$$

where the metric tensors are given in (24.2) or (24.3). The extra function of spacetime $\sqrt{-g}$ in the scalar Lagrangian density L_s comes from the invariant volume element $\sqrt{-g}d^4x = Wd^4x$. In the conventional field theory formulated in inertial frames, the Lagrangian density is required to be a functional only of the fields and their first derivatives. It is also required that the Lagrangian density does not have explicit dependence on the spacetime coordinates.[1] However, when one generalizes field theory from inertial frames to non-inertial frames, these requirements become too stringent and must be relaxed. This is because the fundamental metric tensor $g_{\mu\nu}(x)$ will show up in the invariant action. As a result, the Lagrangian density will effectively involve explicit spacetime dependence through $g_{\mu\nu}$ which stems from linear accelerations and is not a physical field in our formalism. We stress that the geometrical property of spacetime in a CLA frame is completely described by the metric tensor $g_{\mu\nu}$ and that the Riemann curvature tensor can be shown to be zero, so that the spacetime is flat and there is no gravity.[2]

By varying the scalar field Φ, one obtains the free Klein–Gordon equation for CLA frames

$$\frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\; g^{\mu\nu}\partial_\nu\Phi) + m^2\Phi = 0\,, \tag{24.12}$$

which is the same as (24.1). The energy–momentum tensor $T_{\mu\nu}$ for the scalar field is defined by

$$T^\mu{}_\nu = \frac{\partial L_s}{\partial(\partial_\mu\Phi)}\,\partial_\nu\Phi - \delta^\mu{}_\nu L_s$$

$$= \sqrt{-g}\;\partial^\mu\Phi\partial_\nu\Phi - \frac{1}{2}\sqrt{-g}\;\delta^\mu{}_\nu(g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - m^2\Phi^2)\,. \tag{24.13}$$

The momentum $\pi(w,\mathbf{x})$ conjugate to $\Phi(w,\mathbf{x})$ is defined by

$$\pi(w,\mathbf{x}) = \frac{\partial L_s}{\partial(\partial_0\Phi)} = \sqrt{-g}\; g^{00}\partial_0\Phi = \frac{1}{W}\,\partial_0\Phi\,, \qquad \mathbf{x} = (x^1, x^2, x^3)\,, \tag{24.14}$$

where the function W appears in (24.14) due to the acceleration of the reference frame. The Hamiltonian density H_s and the momentum density P_i of the scalar field are defined by

$$H_s = \pi\partial_0\Phi - L_s = \frac{1}{2}\sqrt{-g}(g^{00}\pi^2 + |\nabla\Phi|^2 + m^2\Phi^2) = T^0{}_0\,, \tag{24.15}$$

$$P_i = T^0{}_i = \pi(w,\mathbf{x})\partial_i\Phi\,.$$

To quantize the neutral scalar field, we postulate the equal–time commutation relation

$$[\Phi(w,\mathbf{x}),\pi(w,\mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}')\,, \tag{24.16}$$

$$[\Phi(w,\mathbf{x}),\Phi(w,\mathbf{x}')] = [\pi(w,\mathbf{x}),\pi(w,\mathbf{x}')] = 0\,.$$

As usual, we may express the scalar field operator $\Phi(w,x)$ in terms of an operator-valued amplitude $A(k)$, $k_\mu = p_\mu/J = p_\mu$, and its Hermitian conjugate $A^\dagger(k)$:

$$\Phi(w,x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^4k}{W}\, \delta(k_\mu k^\mu - m^2)\theta(k_0)[A(k)e^{-iP} + A^\dagger(k)e^{iP}] , \qquad (24.17)$$

$$\delta(k_\mu k^\mu - m^2) = \delta(k_0 k_0/W^2 - \mathbf{k}^2 - m^2)$$

$$= \frac{\delta(k_0 - W\omega_k)}{2k_0/W^2} + \frac{\delta(k_0 + W\omega_k)}{2|k_0|/W^2} , \quad \omega_k = \sqrt{\mathbf{k}^2 + m^2} , \qquad (24.18)$$

$$P = P(x) = (\gamma \frac{k_0}{W} - \gamma\beta k_1)\Big[\gamma\beta(x^1 + \frac{1}{\alpha\gamma_o^2}) - \frac{\beta_o}{\alpha\gamma_o}\Big]$$

$$+ (\gamma k_1 - \gamma\beta \frac{k_0}{W})\Big[\gamma(x^1 + \frac{1}{\alpha\gamma_o^2}) - \frac{1}{\alpha\gamma_o}\Big] + k_2 x^2 + k_3 x^3 , \qquad (24.19)$$

where $W = W(w, x^1)$ and $P(x) = P(x^\mu)$. We observe that the validity of the commutation relations (24.16) for canonical quantization and the Fourier transform (24.17) for the field $\Phi(w,x)$ are independent of the detailed form of the Lagrangian density. This is the reason why one can carry out canonical quantization of fields in non-inertial frames. Before quantization, $A^\dagger(k)$ in (24.17) is the complex conjugate amplitude of $A(k)$. After the quantization postulate (24.16) is made, one has quantum fields and the amplitude becomes an operator with $A^\dagger(k)$ the Hermitian conjugate to $A(k)$.

The relation (24.18) follows from the formula[3]

$$\delta^{(\kappa)}(f(x)) = \Big(\frac{d}{dx}\Big)^\kappa \delta(f(x)) = \sum_n \frac{1}{|f'(x_n)|} \Big(\frac{1}{f'(x)}\frac{d}{dx}\Big)^\kappa \delta(x - x_n) , \qquad (24.20)$$

$$f(x_n) = 0 , \qquad n=1,2,3... \qquad f'(x) = df(x)/dx ,$$

with $\kappa=0$ and $n=2$. It is important to note that the quantities $(\gamma k_0/W - \gamma\beta k_1)$ and $(\gamma k_1 - \gamma\beta k_0/W)$ in (24.19) are both constants associated with "plane waves,"

$$(\gamma k_0/W - \gamma\beta k_1) = \text{constant} \quad \text{and} \quad (\gamma k_1 - \gamma\beta k_0/W) = \text{constant} , \qquad (24.21)$$

even though β, γ, k_0, k_1 and W are separately non-constants in a CLA frame, as shown in (22.10). The invariant volume element $d^4k/\sqrt{g_{00}} = d^4k/W$ in the momentum space of a CLA frame can be obtained from d^4k_I in an inertial frame by the momentum transformation (22.10) with the covariant momentum 4-vector p_μ replaced by k_μ:

$$d^4k_I = J(k_{I\lambda}/k_\lambda)d^4k = d^4k/W, \tag{24.22}$$

where $J(k_{I\lambda}/k_\lambda)=1/W$ is the Jacobian.

With the help of (24.18), we can write the Fourier decomposition (24.17) for $\Phi(w,\mathbf{x})$ and (24.14) for $\pi(w,\mathbf{x})$ in the form

$$\Phi(w,\mathbf{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} [a(\mathbf{k})e^{-iP} + a^\dagger(\mathbf{k})e^{iP}], \quad a(\mathbf{k}) = \frac{A(k)}{\sqrt{2\omega_k}}, \tag{24.23}$$

$$\pi(w,\mathbf{x}) = \int \frac{d^3k \ i\omega_k}{\sqrt{(2\pi)^3 2\omega_k}} [-a(\mathbf{k})e^{-iP} + a^\dagger(\mathbf{k})e^{iP}], \quad \omega_k = \sqrt{k^2+m^2}. \tag{24.24}$$

where k_0 in the phase P is equal to $W\omega_k$, i.e., $k_0=W\sqrt{k^2+m^2}$. We can also express the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$, which are functions of the spatial momentum vector $\mathbf{k}$, in terms of the scalar field operator Φ:

$$a(\mathbf{k}) = i\int \frac{d^3x}{W\sqrt{(2\pi)^3 2\omega_k}} [e^{iP}\partial_0\Phi - (\partial_0 e^{iP})\Phi], \tag{24.25}$$

$$a^\dagger(\mathbf{k}) = -i\int \frac{d^3x}{W\sqrt{(2\pi)^3 2\omega_k}} [e^{-iP}\partial_0\Phi - (\partial_0 e^{-iP})\Phi]. \tag{24.26}$$

One can verify that the commutation relations

$$\begin{aligned} &[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k} - \mathbf{k}'), \\ &[a(\mathbf{k}), a(\mathbf{k}')] = [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0, \end{aligned} \tag{24.27}$$

are consistent with the equal-time commutators in (24.16). To see this, let us substitute (24.23) and (24.24) into the equal-time commutator (24.16). We have

$$[\Phi(w,\mathbf{x}),\pi(w,\mathbf{x}')] = \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\int\frac{d^3k'\ i\omega_{k'}}{\sqrt{(2\pi)^3 2\omega_{k'}}}$$

$$\times[a(\mathbf{k})e^{-iP(x)} + a^\dagger(\mathbf{k})e^{iP(x)}, -a(\mathbf{k})e^{-iP'(x')} + a^\dagger(\mathbf{k})e^{iP'(x')}]|_{w=w'}$$

$$= \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\int\frac{d^3k'\ i\omega_{k'}}{\sqrt{(2\pi)^3 2\omega_{k'}}}\ \delta^3(\mathbf{k}-\mathbf{k}')[e^{-iP+iP'} + e^{iP-iP'}]|_{w=w'}$$

$$= \int\frac{id^3k}{2(2\pi)^3}[e^{-iP+iP'} + e^{iP-iP'}]|_{w=w',\ \mathbf{k}=\mathbf{k}'}, \tag{24.28}$$

where the phase functions P and P' are given by (24.19),

$$P = P(x) = P(x,k)\ , \quad P'= P(x\to x', k\to k')\ , \quad x \equiv x^\mu , \quad k \equiv k^\mu , \quad \text{etc.} \tag{24.29}$$

Let us calculate the term $e^{-iP+iP'}$ in (24.28). We obtain

$$e^{-iP+iP'}|_{w=w',\ k=k'} = \exp\Big[-i\ \{(\gamma\frac{k_0}{W}-\gamma\beta k_1)[\gamma\beta(x^1+\frac{1}{\alpha\gamma_0{}^2})-\frac{\beta_0}{\alpha\gamma_0}]$$

$$+(\gamma k_1-\gamma\beta\frac{k_0}{W})[\gamma(x^1+\frac{1}{\alpha\gamma_0{}^2})-\frac{1}{\alpha\gamma_0}] + k_2x^2 + k_3x^3\}$$

$$+ i\{(\gamma\frac{k_0}{W}-\gamma\beta k_1)[\gamma\beta(x'^1+\frac{1}{\alpha\gamma_0{}^2})-\frac{\beta_0}{\alpha\gamma_0}]$$

$$+(\gamma k_1-\gamma\beta\frac{k_0}{W})[\gamma(x'^1+\frac{1}{\alpha\gamma_0{}^2})-\frac{1}{\alpha\gamma_0}] + k_2x'^2 + k_3x'^3\}\Big]$$

$$= \exp\Big[-i\frac{\gamma^2\beta k_0}{W}(x^1-x'^1) + i\gamma^2\beta^2k_1(x^1-x'^1) - i\gamma^2k_1(x^1-x'^1)$$

$$+ i\frac{\gamma^2\beta k_0}{W}(x^1-x'^1) - ik_2(x^2-x'^2) - ik_3(x^3-x'^3)\Big]$$

$$= \exp[-ik\cdot(x-x')]\ . \tag{24.30}$$

The other term $e^{iP-iP'}$ is just the complex conjugate of (24.30). Thus, equation (24.28) is the same as the equal-time commutator (24.16).

The Hamiltonian H of the scalar field,

$$H = \int H_s d^3x = \int d^3x \sqrt{-g}\ \frac{1}{2}[g^{00}\pi^2 + |\nabla\Phi|^2 + m^2\Phi^2] , \tag{24.31}$$

cannot in general be expressed in terms of the operators a(k) and $a^\dagger(k)$, because of the presence of the spacetime-dependent function $\sqrt{-g}$ = W in the integrand. Nevertheless, one still can derive physical results from the theory. The Hamiltonian H in (24.31) can be expressed in terms of the operators a(**k**) and $a^\dagger(\mathbf{k})$ only in the limit of zero acceleration, i.e.,W→1,. This suggests that one can define H_ω by the relation

$$H_\omega \equiv \int d^3x \frac{H_s}{\sqrt{-g}} = \int \frac{1}{2} d^3k\ \omega_k [a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})] . \tag{24.32}$$

This H_ω is not exactly the "Hamiltonian" of the scalar field defined in (24.31). It corresponds to p_0/W and is related to an "instantaneous conservation of energy" in a collision process. (See chapter 25.) Nevertheless, the momentum P_i of the scalar field can be expressed in terms of the operators a(**k**) and $a^\dagger(\mathbf{k})$:

$$P_i = \int P_i d^3x = \int \frac{1}{2} d^3k\ k_i [a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})] , \tag{24.33}$$

where a(**k**) and $a^\dagger(\mathbf{k})$ are respectively the annihilation and the creation operators. We observe that the momentum P_i is not constant:

$$[H, P_i] \neq 0 , \tag{24.34}$$

in contrast to the corresponding P_i in quantum field theory in inertial frames. The reason is obvious: a constant momentum in an inertial frame can no longer be a constant when it is measured in a CLA frame. The Hamiltonian H in (24.31) contains $\sqrt{-g}$ = W which is an explicit function of space and time. Only in the limit of zero acceleration does (24.34) vanish as it must.

One can verify that the operators $P_\mu=(P_0,P_i)=(H,P_i)$ and $M_{\mu\nu}$ generate translations and rotations in CLA frames:

$$i[P_\mu,\Phi] = \partial_\mu \Phi , \tag{24.35}$$

$$i[M_{\mu\nu},\Phi] = (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi , \tag{24.36}$$

where $M^{\mu\nu}$ is given by

$$M^{\mu\nu} = \int d^3x\sqrt{-g}\,[T^{0\nu}x^\mu - T^{0\mu}x^\nu] . \tag{24.37}$$

The Heisenberg equations of motion are given by

$$\frac{\partial\Phi}{\partial w} = \frac{1}{i}(\Phi H - H\Phi) = \frac{1}{i}[\Phi,H] , \tag{24.38}$$

$$\frac{\partial\pi}{\partial w} = \frac{1}{i}[\pi,H] , \qquad J = 1 . \tag{24.39}$$

Equations (24.14), (24.16) and (24.39) reproduce the field equation (24.12), as they should for consistency of quantization.

24c. Quantization of Spinor Fields in CLA Frames

The previous discussion of canonical quantization of scalar fields in non-inertial frames with constant linear acceleration can be applied to other fields. In this section, we will give a brief derivation of the main results for spinor fields.

The invariant action S_ψ for a free electron field ψ in CLA frames is assumed to be given by

$$S_\psi = \int L_\psi d^4x , \quad L_\psi = \frac{1}{2}\sqrt{-g}\;\overline{\psi}\Gamma^\mu i\partial_\mu\psi - \frac{1}{2}\sqrt{-g}\;(\partial_\mu\overline{\psi})\Gamma^\mu i\psi - \sqrt{-g}\;m\overline{\psi}\psi , \tag{24.40}$$

$$\{\Gamma^\mu,\Gamma^\nu\} = 2g^{\mu\nu}(x) , \quad \sqrt{-g} = W > 0 , \quad \Gamma^\mu = (W^{-1}\gamma^0, \gamma^1, \gamma^2, \gamma^3) ,$$

where $\gamma^\nu=(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are the constant Dirac matrices. The presence of the metric tensor $g^{\mu\nu}(x)$ is a new feature in the formalism of quantum field theory in CLA frames. The fermion Lagrangian must be symmetrized because Γ_μ is now a function of spacetime rather than a constant. Also, the non-universal speed of light does not appear in the Lagrangian density L_ψ. Thus, with the help of integration by parts, the Lagrangian in (24.40) can be written in the usual form in terms of a "gauge covariant differentiation" ∇_μ,

$$L_\psi = \sqrt{-g}\,\bar{\psi}\Gamma^\mu i\partial_\mu\psi + \frac{1}{2}\bar{\psi}[i(\partial_\mu\sqrt{-g}\,\Gamma^\mu)]\psi - \sqrt{-g}m\bar{\psi}\psi$$

$$= W\bar{\psi}\Gamma^\mu i\nabla_\mu\psi - Wm\bar{\psi}\psi, \qquad (24.41)$$

$$\nabla_\mu = (\partial_0 + \frac{1}{2}(\partial_k W)\gamma^0\gamma^k, \partial_1, \partial_2, \partial_3) .$$

The presence of the gauge covariant derivative can also be seen explicitly if one introduces vierbeins (or tetrads) to express the matrices Γ^μ. Four mutually orthogonal unit vectors, denoted by $e^a{}_\nu$, a=0,1,2,3, form an orthonormal tetrad, where ν is the covariant tensor index and the Latin suffix "a" is a label distinguishing the particular vector. The contravariant components of the same tetrad can be obtained by using $g^{\mu\nu}$ to raise the index ν, $e^{a\mu} = g^{\mu\nu}e^a{}_\nu$. The labels on the vectors have no tensorial meaning, but for convenience we shall raise and lower them by using $\eta_{ab} = (1, -1, -1, -1)$. We have

$$\Gamma^\mu = \gamma^a e_a{}^\mu, \quad e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}, \quad e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab}, \quad e^a{}_\mu = g_{\mu\nu}e^{a\nu}, \qquad (24.42)$$

where γ^a are the usual constant Dirac matrices, and Latin suffixes a, b, c,... may be called Lorentz indices because they are related to the Lorentz group. We have the following expressions for the non-zero diagonal elements:

$$g_{\mu\nu} = (W^2, -1, -1, -1), \quad \eta^{ab} = (1, -1, -1, -1), \quad e^a{}_\mu = (W, 1, 1, 1), \qquad (24.43)$$

$$e_{a\mu} = (W, -1, -1, -1), \quad e_a{}^\mu = (W^{-1}, 1, 1, 1), \quad e^{a\mu} = (W^{-1}, -1, -1, -1).$$

These vierbeins in (24.41) correspond to "scale gauge fields" (which differ from Yang-Mills fields or gauge fields) in the Poincaré gauge-invariant Lagrangian involving fermions.[4]

One can introduce the local Lorentz transformation of the spinor

$$\psi \to \exp[i\varepsilon^{ab}(x)\sigma_{ab}]\psi, \qquad \sigma_{ab} = (i/2)[\gamma_a\gamma_b - \gamma_b\gamma_a] . \qquad (24.44)$$

Corresponding to this local transformation, one has the gauge covariant partial derivative

$$\nabla_\mu \psi = [\partial_\mu - \frac{i}{4}\omega_\mu{}^{ab}\sigma_{ab}]\,\psi\,. \tag{24.45}$$

The connection $\omega_\mu{}^{ab}(x)$ is introduced so that one can *gauge the Lorentz group.* It is given by[5]

$$\omega_\mu{}^{ab}(x) = \frac{1}{2}e^{\lambda a}(\partial_\mu e_\lambda{}^b - \partial_\lambda e_\mu{}^b) + \frac{1}{4}e^{\rho a}e^{\lambda b}(\partial_\lambda e_\rho{}^c - \partial_\rho e_\lambda{}^c)e_{\mu c} - (a\leftrightarrow b)$$

$$= - e^{\lambda a}\partial_\lambda W\delta^{bo}\delta_{\mu 0} + e^{\lambda b}\partial_\lambda W\delta^{ao}\delta_{\mu 0}\,. \tag{24.46}$$

There are only two non-vanishing components for $\omega_\mu{}^{ab}(x)$:

$$\omega_0{}^{01} = -\,\omega_0{}^{10} = -\,\partial_1 W\,. \tag{24.47}$$

It follows from (24.45) and (24.47) that

$$\nabla_0\psi = [\partial_0 + \frac{1}{2}\gamma^0\gamma^1\partial_1 W]\,\psi\,, \tag{24.48}$$

$$\nabla_k\psi = \partial_k\psi\,, \qquad \text{for} \qquad k = 1,2,3\,.$$

One can verify that the results in (24.48) are precisely the same as $\nabla_\mu\psi$, with $\nabla_\mu = (\partial_0 + \frac{1}{2}(\partial_k W)\gamma^0\gamma^k, \partial_1, \partial_2, \partial_3)$, in (24.41) because the metric tensor $g_{\mu\nu}$ or the Wu factor W does not depend on y and z, i.e. $\partial_2 W = \partial_3 W = 0$.

From the Lagrangian (24.41), we obtain generalized Dirac equations for CLA frames:

$$(i\Gamma^\mu\nabla_\mu - m)\psi = 0\,, \qquad \bar{\psi}(-\overleftarrow{\nabla}_\mu i\Gamma^\mu - m) = 0\,. \tag{24.49}$$

For covariant quantization of the spinor fields in such CLA frames, we use the gauge covariant derivative (24.45) or (24.41) consistently and follow the canonical quantization as closely as possible. The "canonical momentum" π_b conjugate to ψ_b is defined as

$$\pi_b = \frac{\partial L_\psi}{\partial(\nabla_0 \psi_b)} = \frac{i}{W}\psi^\dagger , \qquad L_\psi = W\bar{\psi}\Gamma^\mu i\nabla_\mu \psi - Wm\bar{\psi}\psi , \qquad (24.50)$$

where the presence of W(w,x) in the Lagrangian, the canonical momentum (24.50) and the field equations (24.49) are the only source of complication of field quantization in CLA frames. The Hamiltonian density H_ψ for a free electron is defined as

$$H_\psi = \pi\nabla_0\psi - L_\psi = -\sqrt{-g}\,\bar{\psi}\gamma^k i\partial_k\psi + \sqrt{-g}\,m\bar{\psi}\psi , \qquad (24.51)$$

where we have used Γ^k, k=1,2,3, in (24.40) and ∇_k in (24.41). The energy–momentum tensor $T_{\mu\nu}$ for the spinor field is defined by

$$T^\mu{}_\nu = \frac{\partial L_\psi}{\partial(\nabla_\mu \psi)}\nabla_\nu\psi - \delta^\mu{}_\nu L_\psi . \qquad (24.52)$$

Thus, we have the energy H and momentum P_k of the spinor field:

$$H = \int d^3x T^0{}_0 = \int d^3x\sqrt{-g}\,(-i\bar{\psi}\,\gamma^k\partial_k\psi + m\bar{\psi}\psi) , \qquad T^0{}_0 = H_\psi , \qquad (24.53)$$

$$P_k = \int d^3x T^0{}_k = \int d^3x\sqrt{-g}\,i\bar{\psi}\,\gamma^0\nabla_k\psi = \int d^3x \frac{i}{W}\psi^\dagger\partial_k\psi . \qquad (24.54)$$

The fundamental equal–time anticommutation relations for spinor fields are postulated to be

$$\{\psi_\alpha(w, \mathbf{x}), \pi_\beta(w, \mathbf{y})\} = i\delta^3(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta} , \qquad (24.55)$$

$$\{\psi_\alpha(w, \mathbf{x}), \psi_\beta(w, \mathbf{y})\} = 0 , \quad \{\pi_\alpha(w, \mathbf{x}), \pi_\beta(w, \mathbf{y})\} = 0 ,$$

where $\pi_\alpha(w,y)$ is given by (24.50) and W must be treated as an ordinary function rather than an operator. From equations (24.53)–(24.55), one can verify that

$$i[P_\mu, \psi(x)] = \nabla_\mu\psi(x) , \qquad P_\mu = (H, P_k) . \qquad (24.56)$$

In the calculation of $[P_k, \psi(x)]$, the extra function $1/W$ in (24.54) disappears after the anticommutator (24.55) is used. Similar cancellations of W's occur in $[H, \psi(x)]$ when (24.55) and the free equation $i\Gamma^k\nabla_k\psi - m\psi = -i\Gamma^0\nabla_0\psi$ in (24.49) are used.

In order to carry out quantization in momentum space in terms of the creation and annihilation operators, we must first find the solution to the generalized free Dirac equation (24.49) in linearly accelerated frames. Because of the presence of the gauge covariant derivative ∇_μ, the time and space dependence of the wave function ψ is not just the phase $iP(x)$ for a free fermion. There is an additional gauge factor $G(w)$, which changes the magnitude of the "plane wave function," due to the acceleration of reference frames:

$$\psi = u(k)\exp[-iP(x) - G(w)] , \qquad P(x) = P(x^\mu) = P(w,x,y,z) , \tag{24.57}$$

$$(\Gamma^\mu k_\mu - m)u(k) = 0 , \tag{24.58}$$

where $u(k)$ is a 4–component spinor. The phase factor $P(x)$ is given by (24.19) and the time–dependent gauge factor $G(w)$ is found to be

$$G(w) = \int^w dw \frac{1}{2}\gamma^0\gamma^1(\partial_1 W) = \frac{1}{2}\gamma^0\gamma^1 \tanh^{-1}(\alpha w + \beta_o) , \quad \text{for} \quad W = \gamma^2[\gamma_o^{-2}+\alpha x] ,$$

$$G(w) = \frac{1}{2}\gamma^0\gamma^1 \alpha^* w , \qquad \text{for} \quad W = [1+\gamma_o\alpha^* x] , \tag{24.59}$$

where $\gamma = 1/\sqrt{1-(\alpha w+\beta_o)^2}$ and $\gamma_o = 1/\sqrt{1-\beta_o^2}$ and $\gamma^0\gamma^1$ satisfies the relation $(\gamma^0\gamma^1)^2 = 1$ because of the anti–commutation in (24.40).

For a free spinor field ψ, the Fourier expansion of a solution (24.57) for a "free" electron takes the form

$$\psi(w,\mathbf{x}) = \int \sqrt{\frac{m}{\omega_k}} \frac{d^3k}{\sqrt{(2\pi)^3}} \sum_{s=1}^{2} [b(\mathbf{k},s)u(\mathbf{k},s)\exp(-iP - G)$$

$$+ d^\dagger(\mathbf{k},s)v(\mathbf{k},s)\exp(iP - G)] , \tag{24.60}$$

$$k_0 = W\omega_k , \qquad \omega_k = \sqrt{\mathbf{k}^2+m^2} ,$$

where the phase P=P(x) is given by (24.19) and the gauge factor G=G(w) is given by (24.58)–(24.59). According to (24.58), the spinors $u(\mathbf{k},s)$ and $v(\mathbf{k},s)$ satisfy the equations

$$(\Gamma^\mu k_\mu - m)u(\mathbf{k},s) = 0\,, \qquad \bar{u}(\mathbf{k},s)(\Gamma^\mu k_\mu - m) = 0\,,$$
$$(\Gamma^\mu k_\mu + m)v(\mathbf{k},s) = 0\,, \qquad \bar{v}(\mathbf{k},s)(\Gamma^\mu k_\mu + m) = 0\,, \tag{24.61}$$

and the orthonormality relations

$$\bar{u}(\mathbf{k},s)u(\mathbf{k},s') = -\bar{v}(\mathbf{k},s)v(\mathbf{k},s') = \delta_{ss'}\,,$$
$$u^\dagger(\mathbf{k},s)u(\mathbf{k},s') = v^\dagger(\mathbf{k},s)v(\mathbf{k},s') = \frac{k_0}{m}\delta_{ss'}\,, \quad k_0 = W\sqrt{\mathbf{k}^2+m^2}\,. \tag{24.62}$$
$$\bar{v}(\mathbf{k},s)u(\mathbf{k},s') = \bar{u}(\mathbf{k},s)v(\mathbf{k},s') = 0\,.$$

We normalize the spinors so that $u^\dagger(\mathbf{k},s)u(\mathbf{k},s')=(k_0/m)\delta_{ss'}$. This is a covariant relation since both sides transform like the zeroth component of a 4-vector under accelerated Wu (or MWL) transformations. We also have the following completeness relations

$$\sum_{s=1}^{2} [u_\alpha(\mathbf{k},s)\bar{u}_\beta(\mathbf{k},s) - v_\alpha(\mathbf{k},s)\bar{v}_\beta(\mathbf{k},s)] = \delta_{\alpha\beta}\,,$$
$$\sum_{s=1}^{2} u_\alpha(\mathbf{k},s)\bar{u}_\beta(\mathbf{k},s) = \left(\frac{m + \Gamma^\mu k_\mu}{2m}\right)_{\alpha\beta}, \tag{24.63}$$
$$-\sum_{s=1}^{2} v_\alpha(\mathbf{k},s)\bar{v}_\beta(\mathbf{k},s) = \left(\frac{m - \Gamma^\mu k_\mu}{2m}\right)_{\alpha\beta}.$$

Upon second quantization of the spinor field, the Fourier expansion coefficients $b(\mathbf{k},s)$, $b^\dagger(\mathbf{k}',s')$, $d(\mathbf{k},s)$ and $d^\dagger(\mathbf{k}',s')$ become operators which annihilate and create particles. They are assumed to satisfy the anticommutation relations,

$$\{b(\mathbf{k},s),\ b^\dagger(\mathbf{k}',s')\} = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'},$$
$$\{d(\mathbf{k},s),\ d^\dagger(\mathbf{k}',s')\} = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \tag{24.64}$$

and all other anticommutators vanish. We may remark that anticommutators for the quantized fields $\psi(w,\mathbf{x})$ and $\psi^\dagger(w,\mathbf{x})$ in (24.55) can be shown to be consistent with (24.60) and (24.64). In the calculation of these anticommutators, the gauge factor G(w) in ψ and $\psi^\dagger$ cancels because of the relation $\gamma^0\gamma^1=-\gamma^1\gamma^0$, or

$$e^{-G(w)}\gamma^0 = \gamma^0 e^{G(w)}, \qquad G(w) = \int dw\ \frac{1}{2}\gamma^0\gamma^1(\partial_1 W). \tag{24.65}$$

Finally, we may remark that the Hamiltonian operator $H=\int d^3x T^0{}_0$ cannot be expressed in terms of the operators b, $b^\dagger$, d and $d^\dagger$ because of the presence of W in the integrand. This restriction occurs because the Hamiltonian H transforms like k_0 and $k_0=W\omega_k=W\sqrt{\mathbf{k}^2+m^2}$. However, we have

$$H_\omega = \int d^3x T^0{}_0/W = \int d^3k\omega_k[b^\dagger(\mathbf{k})b(\mathbf{k}) + d(\mathbf{k})d^\dagger(\mathbf{k})], \tag{24.66}$$

$$P_i = \int d^3x T^0{}_i = \int d^3k\ k_i\ [b^\dagger(\mathbf{k})b(\mathbf{k}) + d(\mathbf{k})d^\dagger(\mathbf{k})]. \tag{24.67}$$

The quantity H_ω in (24.66) is not exactly the Hamiltonian, in contrast to H in (24.53). Instead, it corresponds to p_0/W which is useful for treating scattering processes in field theory.

24d. Quantization of the Electromagnetic Field in CLA Frames

The action of the electromagnetic field is assumed to be that of the usual invariant form with a gauge fixing term involving a gauge parameter ρ,

$$S = \int d^4x L_{em}, \qquad L_{em} = -\frac{1}{4}\sqrt{-g}\ f_{\mu\nu}f^{\mu\nu} - \frac{1}{2\rho}\sqrt{-g}\ (D^\mu a_\mu)^2, \tag{24.68}$$

$$f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu, \tag{24.69}$$

$$D^\mu a_\mu = D_\mu a^\mu = D_\mu(a_\alpha g^{\mu\alpha}) = \partial_\mu(a_\alpha g^{\mu\alpha}) - a_\alpha g^{\mu\alpha}\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}$$

$$= g^{\mu\alpha}\partial_\mu a_\alpha - a_\alpha \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\alpha}\sqrt{-g}\,) \equiv \chi\,. \tag{24.70}$$

Varying the action S, we obtain the free electromagnetic equation

$$\partial_\mu(\sqrt{-g}\, f^{\mu\nu}) - \frac{1}{\rho}g^{\mu\nu}\sqrt{-g}\,\partial_\mu\chi = 0\,. \tag{24.71}$$

Since $f^{\mu\nu}$ is antisymmetric, the scalar field $\chi \equiv D^\mu a_\mu$ satisfies the free equation in CLA frames

$$\frac{1}{\sqrt{-g}}\,\partial_\nu(g^{\mu\nu}\sqrt{-g}\,\partial_\mu\chi) = D^\mu D_\mu\chi = 0\,, \tag{24.72}$$

which corresponds to the free equation $\partial^\mu\partial_\mu\chi=0$ in inertial frames.

The electromagnetic 4–potential can be written in the form

$$a_\mu(w,\mathbf{x}) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\sum_{\alpha=0}^{3}\,[a(\mathbf{k},\alpha)\,\varepsilon_\mu(\alpha)\exp(-iP(x))$$

$$+\,a^\dagger(\mathbf{k},\alpha)\,\varepsilon_\mu(\alpha)\exp(iP(x))]\,, \qquad k_0 = W\omega_k = W|\mathbf{k}|\,. \tag{24.73}$$

$$\sum_{\alpha=0}^{3}\,\varepsilon_\mu(\alpha)\,\varepsilon_\nu(\alpha) = -g_{\mu\nu} + (1-\rho)\frac{k_\mu k_\nu}{k^\rho k_\rho}\,, \tag{24.74}$$

$$[a(\mathbf{k},\alpha),\, a^\dagger(\mathbf{k}',\alpha')] = -\,\delta^3(\mathbf{k}-\mathbf{k}')g_{\alpha\alpha'}\,, \tag{24.75}$$

$$[a(\mathbf{k},\alpha),\, a(\mathbf{k}',\alpha')] = 0\,, \qquad [a^\dagger(\mathbf{k},\alpha),\, a^\dagger(\mathbf{k}',\alpha')] = 0\,.$$

The sum over α in (24.73) for four linearly independent unit polarization vectors $\varepsilon_\mu(\alpha)$, $\alpha=0,1,2,3$, is an ordinary sum and not a scalar product in 4–dimensional space. It can be quantized covariantly by using the Gupta–Bleuler method generalized for non–inertial frames.

Equation (24.72) is important for covariant quantization of the electromagnetic field by the Gupta–Bleuler method which involves both physical photons (i.e., the two transverse components) and unphysical photons (i.e., longitudinal and time–like photons). In the Gupta–Bleuler method, equation (24.72) guarantees that the unphysical photons do not interact and, hence, can be easily excluded from physical states and do not contribute to physical amplitudes.[6] In other words, it assures that when the condition for physical states $|\Psi>$,

$$k^\mu a_\mu(\mathbf{k})|\Psi> = 0\,, \quad \text{for all } \mathbf{k}, \quad k_0 = |\mathbf{k}|\,, \quad a_\mu(\mathbf{k}) = \sum_{\alpha=0}^{3} a(\mathbf{k},\alpha)\varepsilon_\mu(\alpha)\,, \qquad (24.76)$$

is imposed at, say, time w=0, it will hold for all times.

References

1. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields.* Vol. 2 (McGraw–Hill Book Company, New York, 1965), p. 14. C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954), see note added in proof on p. 193. The standard canonical formalism in quantum mechanics is not applicable to the quantization of a Lagrangian with explicit time dependence.
2. The Wu transformation does not change the curvature of spacetime, so it implies that the spacetime of CLA frames is flat (i.e., the Riemann curvature tensor vanishes), just like the spacetime of inertial frames. C. Møller, *The Theory of Relativity* (Oxford Univ. Press, London, 1952), p. 285.
3. I. M. Gel'fand and G.E. Shilov, *Generalized Functions* Vol. 1 (transl. by E. Saletan, Academic Press, New York, 1964), p. 185.
4. J. P. Hsu, Phys. Letters. **119B**, 328 (1982). (See appendix C.) It is shown that a gauge-invariant fermion Lagrangian with external spacetime symmetry groups (e.g., Lorentz and Poincaré groups) dictates the presence of two distinct gauge fields: the conventional Yang-Mills fields $b_\mu^{\hat{A}} = (b_\mu^{i}, b_\mu^{jk})$ and the new scale gauge fields e_A^μ.
5. The commutator of two gauge covariant derivatives ∇_μ gives a gauge curvature tensor $R_{\mu\nu}{}^{ab}$: $[\nabla_\mu, \nabla_\nu]\psi = (-i/4)R_{\mu\nu}{}^{ab}\sigma_{ab}\psi$, where $R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_\nu{}^{db}\eta_{cd} - \omega_\nu{}^{ac}\omega_\mu{}^{db}\eta_{cd}$. (This gauge curvature tensor $R_{\mu\nu}{}^{ab}$ resembles the Riemann curvature tensor $R^\alpha{}_{\beta\mu\nu}$ for curved spaces. The spacetime of CLA frames with the metric tensor $g_{\mu\nu}$ given by (24.2) and (24.3) is flat because the Riemann curvature tensor vanishes.) The connection $\omega_\mu{}^{ab}$ can be expressed in terms of the vierbein by imposing a constraint $\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \Gamma^\lambda{}_{\mu\nu} e_\lambda{}^a + \omega_\mu{}^{ac} e_\nu{}^b \eta_{cb} = 0$, where $\Gamma^\lambda{}_{\mu\nu}$ is the Christoffel symbol given in (22.12).
6. In contrast, the unphysical bosons (ghosts particles) in Yang–Mills fields do not satisfy a free equation and can interact with other particles to give unphysical contributions which upset the unitarity and gauge invariance of scattering amplitudes. It is much more involved to remove these unphysical contributions in gauge theory. See, for example, J. P. Hsu and J. A. Underwood, Phys. Rev. D12, 620 (1975); J. P. Hsu and E. C. G. Sudarshan, Phys. Rev. D9, 1678 (1974); M. Kaku, *Quantum Field Theory, A Modern Introduction* (Oxford Univ. Press, New York, 1993), pp. 298-306.

25.

Taiji Rotational Transformations with the Limiting 4-Dimensional Symmetry

25a. A Smooth connection of Rotational and Inertial frames

The 4-dimensional symmetry framework is the foundation of relativity theory and is perhaps the most thoroughly tested symmetry principle in the 20th century. It is the mathematical manifestation of the Poincaré-Einstein principle of relativity, i.e., the 4-dimensional symmetry (or invariance) of physical laws, which is extremely powerful in helping us to understand physics.[1] The power of 4-dimensional symmetry will be demonstrated by an analysis for a novel viewpoint of rotating non-inertial frames.

> "*Is it conceivable that the principle of relativity also holds for systems which are accelerated relative to each other* ?"

This was the question young Einstein asked in 1907.[2] The answer to Einstein's question is *affirmative* in a limiting sense because any non-inertial frame must reduce to an inertial frame in the limit of zero acceleration. This indicates that the 4-dimensional symmetry of physical laws must hold in the limit of zero acceleration for all non-inertial frames. We call this the principle of limiting 4-dimensional symmetry. Indeed, we are able to obtain a "minimal generalization" of the Lorentz transformation for noninertial frames with a constant-linear-acceleration, as discussed in chapter 21. "Minimal generalization" means that the resultant equations involve a minimal departure from the limiting case of zero acceleration. The resultant transformation for frames with constant-linear-acceleration gives a satisfactory explanation for the "constant-linear acceleration": Namely, it is better defined as a constant change in a particle's energy per unit distance travelled rather than the traditional definition of the rate of change of velocity with time in an inertial frame or laboratory.

In this chapter, we show that the affirmative answer to Einstein's question also holds in a limiting sense for rotational frames.[3] The reason is that the principle of "limiting 4-dimensional symmetry" also enables us to obtain a satisfactory minimal extension of the conventional classical rotational transformation to a rotating frame whose origin moves in a circle of non-zero radius around a fixed point in an inertial frame. Our results agree with high energy experiments involving unstable particles in a circular storage ring and are smoothly connected to the 4-dimensional transformations for inertial frames in the limit of zero acceleration. Furthermore, they turn out to support Pellegrini-Swift's analysis of the Wilson experiment,[4] in which they pointed out that rotational transformations cannot be locally replaced by Lorentz transformations.

25b. Taiji Rotational Transformations with the Limiting 4-Dimensional Symmetry

Suppose $F_I(w_I,x_I,y_I,z_I)$ is an inertial laboratory frame and $F(w,x,y,z)$ is a non-inertial frame whose origin moves in a circle of radius R around the origin of F_I with an 'angular velocity' Ω in such a way that the y-axis of the F frame always points to the origin of $F_I(x_I^\mu)$. Physical quantities without the subscript "I" are those measured by observers in the rotating frame $F(w,x,y,z)$.

In our discussions, we use w_I and w, respectively, as the evolution variables in F_I and F. This 'time' w can be physically realized in a rotating frame without relying on light signals. To avoid confusion, let us call w the "taiji time" which will reduce to Einstein time in the limit of zero acceleration. An operational meaning for w will be explained later.

Based on limiting 4-dimensional symmetry considerations and the classical rotational transformation

$$w_I = w\,, \qquad x_I = x\cos(\Omega w) - y\sin(\Omega w)\,,$$

$$y_I = x\sin(\Omega w) + y\cos(\Omega w)\,, \qquad z_I = z\,; \tag{25.1}$$

we write the general rotational transformation for $F_I(x_I^\mu)$ and $F(x^\mu)$ in the form:

$$w_I = Aw + B\boldsymbol{\rho}\cdot\boldsymbol{\beta}\,, \qquad x_I = Gx\cos(F\Omega w) + E(y-R)\sin(F\Omega w)\,,$$

$$y_I = Ix\sin(F\Omega w) + H(y-R)\cos(F\Omega w)\,, \qquad z_I = z\,; \tag{25.2}$$

$$\boldsymbol{\rho} = (x,y)\,, \qquad \mathbf{S} = (x, y-R)\,, \qquad \boldsymbol{\beta} = \boldsymbol{\Omega}\times\mathbf{S}\,, \qquad \boldsymbol{\Omega} = (0,0,\Omega)\,.$$

Note that the constant 'angular velocity' Ω is measured in terms of w by observers in F:

$$\Omega = d\phi/dw . \tag{25.3}$$

Thus, Ω has the dimension of inverse length and, hence, the 'velocity' $\boldsymbol{\beta} = \boldsymbol{\Omega}\times\mathbf{S}$ and Ωw are dimensionless. The functions A, B, E, F, G, H and I may depend on the coordinates x^μ in general, and will be determined by the limiting 4-dimensional symmetry principle.

In the limit $R\to 0$ and small Ω (or $|\boldsymbol{\Omega}\times\mathbf{S}|=\beta<<1$), the transformation (25.1) should reduce to the form of a classical rotational transformation (25.1). Thus, we have

$$-E \approx G \approx H \approx I \approx 1 , \qquad F \approx 1 , \qquad \text{for small } \Omega . \tag{25.4}$$

Furthermore, when R=0, the rotational transformation (25.1) must exhibit an x-y symmetry. This implies that

$$-E = G = H = I ; \qquad \text{for } R = 0 . \tag{25.5}$$

On the other hand, in the limit where $R\to\infty$ and $\Omega\to 0$ such that their product $R\Omega = \beta_o$ is a non-zero constant velocity along the x_I-axis, (25.1) must reduce to the 4-dimensional form

$$w_I = \gamma_o(w + \beta_o x) , \quad x_I = \gamma_o(x + \beta_o w) , \quad y_I = -\infty , \quad z_I = z ; \tag{25.6}$$

$$\gamma_o = \frac{1}{\sqrt{1 - \beta_o^2}} , \qquad \beta_o = R\Omega .$$

Note that we have $y_I = -\infty$ instead of $y_I = y$ because $R\to\infty$ in this limit.

Finally, the existence of the limiting transformation should hold for *both finite and differential forms* of the transformation. Thus, we also have the well-defined differential form,

$$dw_I = \gamma_o(dw + \beta_o dx) , \quad dx_I = \gamma_o[dx + \beta_o dw)] , \quad dy_I = dy , \quad dz_I = dz . \tag{25.7}$$

The limiting requirements in (25.6) and (25.7) lead respectively to

$$A = B = G = -EF = \gamma_o , \tag{25.8}$$

and $\quad H = 1 .$ (25.9)

These limiting 4-dimensional symmetry relations in (25.5), (25.8) and (25.9) do not lead to a unique solution for the unknown functions. The situation resembles the fact that gauge symmetry does not uniquely determine the electromagnetic action,[5] and one must postulate a minimal electromagnetic coupling. Here, we postulate the minimal generalization of the classical rotational transformation (25.1): Based on the relations in (25.5), (25.8) and (25.9), it is natural to have the following two properties: (i) The functions A, B, G, I and – E are simply extended from $\gamma_o=(1-\beta_o{}^2)^{-1/2}$ to $\gamma=(1-\beta^2)^{-1/2}$, where $\beta=S\Omega$, for non-vanishing 'angular velocity' Ω and finite R. (ii) Only the function H depends on R, i.e., $H = \gamma$ for $R = 0$ and $H = 1$ for $R\to\infty$. Thus, the limiting relations (25.5), (25.8), (25.9) together with the minimal generalization of (25.1) lead to the simplest and the most natural solution for the unknown functions in (25.1):

$$A = B = G = I = \gamma = \frac{1}{\sqrt{1-\beta^2}} , \qquad E = -\gamma ,$$
$$F = 1 , \qquad H = \frac{(\gamma + R/R_o)}{(1 + R/R_o)} , \tag{25.10}$$

where $\beta = |\Omega \times S| < 1$ and H is also obtained by requiring it to be the simplest function involving only the first power of γ. As we shall see below, this simplest solution (25.10) turns out to be consistent with all previous experiments of particles' energy-momentum in high energy accelerators involving rotational motion. The exact general rotational transformation is thus given by (25.1) and (25.10):

$$w_I = \gamma(w + \boldsymbol{\rho}\cdot\boldsymbol{\beta}) = \gamma(w + xR\Omega) , \quad x_I = \gamma[x\cos(\Omega w) - (y-R)\sin(\Omega w)] ,$$
$$y_I = \gamma[x\sin(\Omega w) + (H/\gamma)(y-R)\cos(\Omega w)] , \qquad z_I = z . \tag{25.11}$$

We stress that it is necessary for A, B, G, I and -E in (25.2) to approach γ in the limit $R \to \infty$ so that the limiting 4-dimensional symmetry is satisfied. To avoid confusion, let us call (25.11) the "taiji rotational transformation" in which w_I and w are evolution variables. Let us call $(w,\mathbf{r})$ the '*limiting Cartesian coordinates*' to distinguish it from the usual Cartesian coordinates $(w_I,\mathbf{r}_I)$. [The metric tensor $g_{\mu\nu}$ for the space of a rotating frame is given in (25.19) below.] The presence of the γ-factor and the 'scaling factor' H in (25.11) are new properties implied by the limiting 4-dimensional symmetry.

It is tedious, but straightforward, to show that the set of taiji rotational transformations (25.11) form a "taiji rotational group" which includes the Lorentz group as a special case.

25c. Physical Properties of Taiji Rotational Transformations

Presumably, the physical effects related to the terms R/R_o in (25.10) could be observed only in rotational motion with a large R. When the term R/R_o is small and negligible, $H/\gamma \approx 1$. In this case, the taiji rotational transformation (25.11) become very simple and closely resembles the classical form (25.1). In fact, all previous rotational experiments were performed under the condition $R \approx 0$. It appears that this approximation is sufficient for experiments in Earth laboratories. From now on, we shall ignore R/R_o so that (25.11) becomes

$$w_I = \gamma(w+\boldsymbol{\rho}\cdot\boldsymbol{\beta}) = \gamma(w+xR\Omega)\,, \qquad x_I = \gamma[x\cos(\Omega w)-(y-R)\sin(\Omega w)]\,,$$

$$y_I = \gamma[x\sin(\Omega w)+(y-R)\cos(\Omega w)]\,, \qquad z_I = z. \tag{25.12}$$

The inverse transformation of (25.12) is then

$$w = (w_I - \gamma xR\Omega)/\gamma, \quad x = [x_I\cos(\Omega w) + y_I\sin(\Omega w)]/\gamma,$$

$$y - R = [-x_I\sin(\Omega w) + y_I\cos(\Omega w)]/\gamma, \qquad z = z_I. \tag{25.13}$$

We stress that the exact taiji rotational transformation (25.11) can only be expressed by using the *limiting Cartesian coordinate* for the rotating frame F and cannot be written in terms of, say, the cylindrical coordinate, in sharp contrast to the conventional classical rotational transformation. This property is dictated by the limiting 4–dimensional symmetry and is similar to the fact that the Cartesian coordinate is preferred in the Lorentz transformation.

For discussions of experiments, it suffices to concentrate on the special case R=0 of (25.11):

$$w_I = \gamma w, \qquad x_I = \gamma[x\cos(\Omega w) - y\sin(\Omega w)],$$

$$y_I = \gamma[x\sin(\Omega w) + y\cos(\Omega w)], \qquad z_I = z; \tag{25.14}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \rho\Omega.$$

Let us compare the taiji rotational transformation (25.14) and the classical rotational transformation (25.1):

(A) The taiji rotational transformation and the Lorentz transformation are both exact and in harmony with the four–dimensional symmetry. In contrast, the classical rotational transformation (25.1) and the Galilean transformation are only approximately true to the first order in 'velocity' β.

(B) It is interesting that the taiji rotational transformation (25.14) predicts *the length of a rotating radius to be contracted by a γ factor*:

$$x_I^{\,2} + y_I^{\,2} = \gamma^2[x^2 + y^2], \tag{25.15}$$

independent of the variables w and w_I. If one replaces a rotational transformation at a very short time interval by a Lorentz transformation, one would be led to a completely different conclusion, namely, that a rotating radius does not contract because it is always perpendicular to the direction of motion.

(C) For a fixed γ, (25.14) gives $\Delta t_I = \gamma\Delta t$ which is independent of the spatial distance between two events, if we set $w = ct$ and $w_I = ct_I$. That is, the time t of clocks at rest in a rotating frame and located at a distance $\rho = (x^2 + y^2)^{1/2}$ from the center of rotation slows down by a factor of $\gamma = (1 - \beta^2)^{-1/2}$ in comparison with the time t_I of clocks in the inertial frame F_I. We stress that this is an absolute effect in that observers in both F_I and F agree that it is the accelerated clocks which are slowed. This effect is implied by the limiting four–dimensional symmetry.

Certain properties of the taiji rotational transformation (25.14) have been confirmed by high-energy particle kinematics experiments and "lifetime dilatation",[6] as we shall see later. In a sense, the unexpected relation (25.15) has been indirectly verified because the confirmed results [see equations in (25.31) and (25.32) below] are intimately related to the presence of the γ factors in (14) and, hence, (15).

25d. The Metric Tensors for Rotating Frames

The transformation of contravariant 4–vectors $dx_I^\mu=(dw_I,dx_I,dy_I,dz_I)$ and $dx^\mu=(dw,dx,dy,dz)$ can be derived from (25.11):

$$
\begin{aligned}
dw_I &= \gamma[dw + (\gamma^2\Omega^2 wx)dx + (\gamma^2\Omega^2 wy)dy] ,\\
dx_I &= \gamma[\{\cos(\Omega w) + \gamma^2\Omega^2 x^2\cos(\Omega w) - \gamma^2\Omega^2 xy \sin(\Omega w)\}dx\\
&\quad - \{\sin(\Omega w) + \gamma^2\Omega^2 y^2 - \gamma^2\Omega^2 xy\}dy - \{\Omega x \sin(\Omega w) + \Omega y\cos(\Omega w)\}dw] ,\\
dy_I &= \gamma[\{\sin(\Omega w) + \gamma^2\Omega^2 x^2\sin(\Omega w) + \gamma^2\Omega^2 xy\cos(\Omega w)\}dx \qquad (25.16)\\
&\quad + \{\cos(\Omega w) + \gamma^2\Omega^2 y^2\cos(\Omega w) + \gamma^2\Omega^2 xy\sin(\Omega w)\}dy\\
&\quad + \{\Omega x \cos(\Omega w) - \Omega y\sin(\Omega w)\}dw] ,\\
dz_I &= dz .
\end{aligned}
$$

To find the metric tensor $g_{\mu\nu}$ in the rotating frame F with R = 0, it is convenient to use (25.11) to write $ds^2= dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2$ in the following form first:

$$
\begin{aligned}
ds^2 &= d(\gamma w)^2 - (x^2+ y^2)\gamma^2\Omega^2 dw^2 - d(\gamma x)^2 - d(\gamma y)^2 - dz^2\\
&\quad + 2\gamma\Omega y d(\gamma x)dw - 2\Omega\gamma x d(\gamma y)dw . \qquad (25.17)
\end{aligned}
$$

Then with the help of the relation $d\gamma = \gamma^3\Omega^2(xdx+ydy)$, eq.(25.17) can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu , \qquad \mu, \nu = 0, 1, 2, 3 , \tag{25.18}$$

where the non-vanishing components of $g_{\mu\nu}$ are given by

$$\begin{aligned}
&g_{00} = 1 , \qquad g_{33} = -1 , \\
&g_{11} = -\gamma^2[1 + 2\gamma^2\Omega^2x^2 - \gamma^4\Omega^4x^2 (w^2 - x^2 - y^2)] , \\
&g_{22} = -\gamma^2[1 + 2\gamma^2\Omega^2y^2 - \gamma^4\Omega^4y^2 (w^2 - x^2 - y^2)] , \\
&g_{01} = \gamma^2[\Omega y + \gamma^2\Omega^2wx] , \qquad g_{02} = \gamma^2[-\Omega x + \gamma^2\Omega^2wy] , \\
&g_{12} = -\gamma^4\Omega^2xy[2 - \gamma^2\Omega^2(w^2 - x^2 - y^2)] .
\end{aligned} \tag{25.19}$$

The contravariant metric tensor $g^{\mu\nu}$ is found to be

$$\begin{aligned}
&g^{00} = \gamma^{-2}[1 - \Omega^4w^2(x^2 + y^2)] , \qquad g^{33} = -1 , \\
&g^{11} = -\gamma^{-2}[\gamma^{-2}(1 - \Omega^2x^2) - 2\gamma^{-2}\Omega^3wxy + \Omega^6w^2y^2(x^2 + y^2)] , \\
&g^{22} = -\gamma^{-2}[\gamma^{-2}(1 - \Omega^2y^2) + 2\gamma^{-2}\Omega^3wxy + \Omega^6w^2x^2(x^2 + y^2)] , \\
&g^{01} = -\gamma^{-2}[-\Omega y - \gamma^{-2}\Omega^2wx + \Omega^5w^2y(x^2 + y^2)] , \\
&g^{02} = -\gamma^{-2}[\Omega x - \gamma^{-2}\Omega^2wy - \Omega^5w^2x(x^2 + y^2)] , \\
&g^{12} = \gamma^{-2}[\gamma^{-2}\Omega^2xy - \gamma^{-2}\Omega^3w(x^2 - y^2) + \Omega^6w^2xy(x^2 + y^2)] .
\end{aligned} \tag{25.20}$$

(Its components can be obtained by using the momentum transformation (25.29) below and the invariant relation, $p_{I\nu}p_I{}^\nu = g^{\mu\nu}p_\mu p_\nu$.) All other components in (25.19) and (25.20) vanish. Indeed, we have also verified $g_{\mu\lambda}g^{\lambda\nu} = \delta^\nu{}_\mu$ based on (25.19) and (25.20).

25e. The Invariant Action for Electromagnetic Fields and Charged Particles in Rotating Frames and Truly Universal Constants

Now we are able to write the invariant action S_{em} in a rotating frame for a charged particle with mass m and charge $\bar{e}$ moving in the 4–potential a_μ:

$$S_{em} = -\int m ds - \bar{e} \int a_\mu dx^\mu - \frac{1}{4} \int f_{\mu\nu} f^{\mu\nu} \sqrt{-\det g_{\mu\nu}}\, d^4x, \tag{25.21}$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \qquad \partial_\lambda = \partial/\partial x^\lambda. \tag{25.22}$$

where ds is given by (25.18) and (25.19). Each term in (25.21) must have the same dimension (mass·length). In other words, $\bar{e} a_\mu$ must have the dimension of mass. This can be satisfied if $\bar{e}$ and a_μ are related to the usual charge e measured in electrostatic units (esu) and the usual 4–potential A_μ in special relativity (SR) by the relation

$$\bar{e} = e(\text{in esu})/c = -\, 1.6021891 \times 10^{-20} \sqrt{4\pi}\, \sqrt{g \cdot cm}\ , \tag{25.23}$$

$$a_\mu(w,\mathbf{r}) \sim [A_\mu(t,\mathbf{r})/c]_{SR},$$

where the truly universal constant $\bar{e}$ is in Heavyside–Lorentz units and the symbol ~ denotes correspondence. If one directly uses w as evolution variable and does not set w=ct in (25.21), then the *truly universal* charge[7] for rotating frames is $\bar{e}$ rather than the usual e(measured in esu), where $\bar{e}$ turns out to be the same as the usual charge measured in electromagnetic unit (emu). Also, the potential a_μ has the dimension $(g/cm)^{1/2}$, so that $\bar{e} a_\mu$ has the correct dimension of mass.

The Lagrange equation of motion of a charged particle can be derived from (25.21). We obtain

$$m\frac{Du_\mu}{ds} = \bar{e} f_{\mu\nu} u^\nu, \qquad u^\nu = dx^\nu/ds, \qquad u_\mu = g_{\mu\nu} u^\nu, \tag{25.24}$$

where $Du_\mu = u_{\mu;\nu} dx^\nu$, and ;ν denotes covariant differentiation with respect to x^ν.[8]

Starting with the invariant action (25.21) and replacing the second term with

$$-\int a_\mu j^\mu \sqrt{-\det\, g_{\mu\nu}}\, d^4x \qquad (25.25)$$

for a continuous charge distribution in space, we obtain the invariant Maxwell equations in a rotating frame:

$$f^{\mu\nu}{}_{;\nu} = j^\mu, \qquad \partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0\,. \qquad (25.26)$$

Based on gauge invariance and the taiji rotational invariance of the action (25.21), the electromagnetic potential must be a covariant vector, a_μ, in noninertial frames. Since the force **F** and the fields E, **B** are naturally related to a change of the potential a_μ with respect to a change in coordinate (i.e., x^μ, by definition), the fields **E** and **B** are naturally identified with components of the covariant tensor $f_{\mu\nu}$ as given by (25.22) in noninertial frames.

25f. The 4–Momentum and 'Lifetime Dilatation' of a Particle at Rest in a Rotating Frame

Although the speed of light is not a universal constant in a rotating frame F, we still can have a covariant four-momentum $p_\mu = (p_o, p_i)$ by using w, with units of length, rather than the more traditional t, with units of time, as the evolution variable in the Lagrangian formalism. From the invariant action $S_f = -\int m ds = \int L dw$ for a 'non-interacting particle' with mass m in the rotating frame F, we have the spatial components of the covariant four-momentum,

$$p_i = -\frac{\partial L}{\partial v^i} = m g_{\nu i}\frac{dx^\nu}{ds} \equiv g_{\nu i} p^\nu, \quad i = 1,2,3\,; \qquad (25.27)$$

$$L = -m\sqrt{g_{\mu\nu} v^\mu v^\nu}\ ,$$

where L and p_i both have the dimension of mass, and $v^\mu \equiv dx^\mu/dw = (1, v^i)$. The zeroth component p_0 (or the Hamiltonian) with the dimension of mass is defined as usual:

$$p_0 = \frac{\partial L}{\partial v^i} v^i - L = m g_{\nu 0} \frac{dx^\nu}{ds} = g_{\nu 0} p^\nu . \qquad (25.28)$$

The taiji rotational transformation of the differential operators $\partial/\partial x_I^\mu$ and $\partial/\partial x^\mu$ can be calculated from the inverse of (25.16). The covariant momentum p_μ has the same transformation properties as the covariant differential operator $\partial/\partial x^\mu$. This can also be seen from the quantum-mechanical relation $p_\mu \propto \partial/\partial x^\mu$. Thus, we have

$$p_{I0} = \gamma^{-1}(p_0 + \Omega y p_1 - \Omega x p_2) ,$$

$$p_{I1} = [-\gamma^{-2}\Omega^2 w x_I] p_0 + \gamma^{-2}[\gamma \cos(\Omega w) - \Omega^2 x_I x - \Omega^2 w x_I y] p_1$$

$$+ \gamma^{-2}[-\gamma \sin(\Omega w) - \Omega^2 x_I y + \Omega^3 w x_I x] p_2 ,$$

$$p_{I2} = [-\gamma^{-2}\Omega^2 w y_I] p_0 + \gamma^{-2}[\gamma \sin(\Omega w) - \Omega^2 y_I x - \Omega^3 w y_I y] p_1 \qquad (25.29)$$

$$+ \gamma^{-2}[\gamma \cos(\Omega w) - \Omega^2 y_I y + \Omega^3 w y_I x] p_2 ,$$

$$p_{I3} = p_3 .$$

Let us consider the kinematic properties of a particle. Suppose a particle is at rest in the rotating frame F, so that $dx^i = 0$ and, hence, $ds = dw$. Based on $p^\nu = m dx^\nu/ds$ in (25.27), the contravariant momenta are $p^i = 0$, $i = 1,2,3$, and $p^0 = m$. In this case the covariant momenta of this particle in F are

$$p_0 = m , \qquad p_1 = m\gamma^2(\Omega y + \gamma^2\Omega^2 w x) ,$$

$$p_2 = m\gamma^2(-\Omega x + \gamma^2\Omega^2 w y) , \qquad p_3 = 0 . \qquad (25.30)$$

This difference between p_μ and p^μ is due to the presence of the metric tensor components g_{01} and g_{02}. Its covariant momenta, as measured in an inertial frame F_I, are given by (25.29)–(25.30):

$$p_{I0} = \gamma m , \quad p_{I1} = m\gamma[\Omega x \sin(\Omega w) + \Omega y \cos(\Omega w)] ,$$

$$p_{I2} = - m\gamma[\Omega x \cos(\Omega w) - \Omega y \sin(\Omega w)] , \quad p_{I3} = 0 . \tag{25.31}$$

It is gratifying that the expression for the energy of a rotating particle, p_{I0} in (25.31), agrees with the results of high energy experiments performed in an inertial laboratory frame F_I. If one uses the classical rotational transformation (25.1), one will obtain $p_{I0} = m(1+ \rho^2\Omega^2)$ for the energy of such a particle at rest in F, which is clearly inconsistent with high energy experiments.

For the decay of a particle in flight in a storage ring, the particle can be considered to be at rest in the rotating frame F. The taiji rotational transformation (25.14) gives

$$\Delta w_I = \gamma \Delta w , \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}} , \qquad \beta = \rho\Omega . \tag{25.32}$$

Since w and w_I have the dimension of length, (25.32) can be understood as the dilatation of the decay length which is the quantity that is directly measured in experiments measuring decays of muons in flight in a storage ring.[6] We stress that this is an absolute effect in that observers in both F_I and F agree that it is the accelerated muons whose mean decay length is dilated. If one replaces a rotational transformation at a very short time interval by a Lorentz transformation, one obtains the same expression for the time interval $\Delta t_I = \gamma \Delta t'_I$ or, $\Delta w_I = \gamma \Delta w'_I$ with $w_I = ct_I$ and $w'_I = ct'_I$, by imposing a spatial relation $\Delta x'_I = 0$ for two events. One notes that, however, the time dilatation effect implied by the Lorentz transformations is relative rather than absolute. In this connection, we may remark that the relation $w_I = w$ for any $\beta = \rho\Omega \neq 0$ in the classical rotational transformation (25.1) is inconsistent with the muon experiment in a storage ring.

The new transformation (25.11) has several advantages for purposes of analyzing physical phenomena in rotating reference frames. One advantage relative to the Lorentz transformation is that since we derive our transformation from first principles, there are no conceptual difficulties arising from the improper use of a transformation designed to treat only inertial reference frames. Secondly, the coordinates involved in our transformation are physically meaningful. In contrast, the arbitrary coordinate systems devised for non-inertial frames based on general relativity in general, have no correspondence with physically measurable distances and times.[9]

References

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2. A. Einstein, Jahrb. Rad. Elektr. 4, 411 (1907).
3. Jong-Ping Hsu and Leonardo Hsu, in *JingShin Theoretical Physics Symposium in Honor of Professor Ta-You Wu* (Ed. J. P. Hsu and L. Hsu, World Scientific, Singapore. New Jersey), pp. 393-412; see also Nuovo Cimento B **112**, 575 (1997); and Leonardo Hsu and Jong-Ping Hsu, Nuovo Cimento B **112**, 1147 (1997).
4. G. N. Pellegrini and A. R. Swift, Am. J. Phys. **63**, 694 (1995) and references therein.
5. J. J. Sakurai, *Invariance Principles and Elementary Particles*, Princeton Univ. Press, 1964), p. v, pp. 3-5, and p. 10.
6. F. J. M Farley, J. Bailey and E. Picasso, Nature **217**, 17 (1968).
7. Jong-Ping Hsu and Leonardo Hsu, Phys. Letters A **196**, 1 (1994). See section 10a.
8. We note that $f_{\mu\nu}$ in (25.22) can also be written in terms of partial covariant derivatives, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu = a_{\nu;\mu} - a_{\mu;\nu}$. Also, the metric tensor $g_{\mu\nu}$ behaves like a constant under covariant differentation, $g_{\mu\nu;\mu} = 0$.
9. We quote from Møller: "We note that in the usual covariance formulation, it is admissible to use coordinate clocks of an arbitrary rate, provided that the time t defined by these coordinate clocks gives a reasonable chronological ordering of events. Also, for non-inertial frames, the space and time coordinates are arbitrary and lose every physical significance in the covariance formulation." (C. Møller, *The Theory of Relativity* (Oxford, London, 1969), p. 226.)

26.

Epilogue

We end this account of relativity theory by returning to the four main questions posed in chapter zero and briefly summarizing their answers.

> **Question #1**: Can the theory of relativity be formulated solely on the basis of the first principle of relativity (without assuming the constancy of the speed of light)?

The Poincaré-Einstein principle of relativity (i.e., the form of a physical law is the same in any inertial frame) is a fundamental and necessary component of any theory if it is to correctly explain and predict phenomena in the physical world. We have shown that the theory of taiji relativity can be formulated solely on the basis of the first principle of relativity, i.e., the Poincaré-Einstein principle, without assuming the constancy of the speed of light.

Einstein's second postulate of special relativity, namely that the speed of light is a universal constant c, is not a necessary component of a correct theory, according to the analysis of taiji relativity. Although the constant c has been described as "the foundation stone of Einstein's special theory of relativity"[1] and "so important in the four-dimensional picture, [playing] such a fundamental role in the special theory of relativity...that it has to be fundamental,"[2] *the universality of the speed of light has not and can not be unambiguously established by experiments.*[3]

> **Question #2:** Can one generalize the 4-dimensional transformation for inertial frames to non-inertial frames with a constant acceleration or rotation? In accelerated frames, the speed of light is no longer a universal constant; is the Planck constant still a universal constant?

Wu's *kinematic* approach to the investigation of spacetime transformations for non-inertial frames appears to be a viable one. This is supported by the analysis of transformations for reference frames with a constant-linear-acceleration or **constant rotational motion, on the basis of limiting 4-dimensional symmetry.** The results indicate that the physical taiji spacetime of non-inertial frames is more closely related to the limiting 4-dimensional symmetry rather than to the gravitational field equation. This result is important because it indicates that the spacetime coordinates (w,x,y,z) for both inertial and non-inertial frames must correspond to physically measurable distances and 'times.' This is in sharp contrast to the arbitrary coordinate systems devised for non-inertial frames based on general relativity, as discussed in section 25f.

Truly fundamental constants in physics should be universal in both inertial and non-inertial frames of references. According to this criterion, the constants J and $\bar{e}$ (or $\alpha_e = \bar{e}^2/J \approx 1/137$) are truly fundamental,[4] while $\hbar$, e (in esu), and c are not. This result has been substantiated by our generalization of the 4-dimensional transformation for inertial frames to non-inertial frames with a constant acceleration or rotation, and by the analysis of physics in non-inertial frames.

> **Question #3:** Within the 4-dimensional symmetry framework of special relativity, it appears to be impossible, in principle, to generalize the classical Liouville equation for many-particle systems to a Lorentz invariant Liouville equation. Can we overcome this difficulty?

The difficulty in this question can be overcome by using common relativity, a 4-dimensional theory of relativity with common time for all observers, to describe many-particle systems, as expounded in chapter 13.

> **Question #4:** In view of the profound divergence difficulties in quantum field theory, is the spacetime 4-dimensional symmetry exact at very large momenta or short distances?

The divergence difficulties have been a long-standing problem in particle physics and quantum field theories. So far, we can only conjecture that there might be an inherent fuzziness at short distances and discuss some of its implications. This fuzziness is intimately related to the property that an elementary particle should be pictured as a fuzzy point (in the sense of fuzzy set theory) rather than as a conventional geometrical point. Furthermore, it suggests a modification of the uncertainty principle as discussed in equation (14.7) and, hence, a greater departure from classical physics than quantum mechanics. The concept of a fuzzy point particle also suggests a modification of the concept of locality in quantum field theory, as shown in equations (14.26), (15.13), and section 15b. Such a picture of a fuzzy particle deserves to be theoretically and experimentally investigated further. According to the discussions in chapters 14 and 15, the 4-dimensional symmetry framework of common relativity appears to have advantages over special relativity for addressing this problem.

References

1. D. Halliday and R. Resnick, *Fundamentals of Physics* (3rd. edition, John Wiley, 1988), p. 543.
2. P. A. M. Dirac, Sci. Am. **208**, 48 (1963).
3. Experiments have confirmed that the speed of light is independent of the motion of the source. However, whether or not the speed of light is independent of the motion of the observer is a property which can not be tested by any experiment — past, present, or future, as discussed in Chapter 5, section 5b.
4. In the physics of the future, one may discover a new and compelling relation between the units of mass and length, which will further reduce the number of fundamental constants in physics. This could happen if, for example, the vacuum expectation value (which dictates the masses of gauge bosons) in the electroweak theory is also found to play the role of a fundamental length. Another possibility is the discovery of some new physical principle concerning phase space.

Appendix A.

Noether's Theorem in Both Linearly Accelerated and Inertial Frames

Aa. Noether's Theorem

In physics, Noether's theorem is of fundamental importance because it reveals an intimate relation between conservation laws or conserved quantities and symmetries of a physical system. It was first proved by Emmy Noether (1882–1935) in 1918.[1] It is usually employed in inertial frames. There was little discussion of its implications in non-inertial frames because the symmetry properties of a physical system viewed from non-inertial frames are not obvious. Furthermore, the conventional Noether's theorem based on symmetry in spacetime is generalized to the case with symmetry in the *phase space.*

In classical mechanics, Noether's Theorem can be stated as follows: Suppose a physical system is described by the Lagrangian equation with the Lagrangian $L = L(q_i,\dot{q}_i,w)$, where $\dot{q}_i = dq_i/dw$. Let

$$\begin{aligned} w' &= w + \varepsilon T(q_i,\dot{q}_i,w) , \\ q'_i &= q_i + \varepsilon Q_i(q_i,\dot{q}_i,w) , \end{aligned} \tag{A.1}$$

be a set of infinitesimal transformations with a continuous parameter ε. If the Lagrangian $L(q'_i,\dot{q}'_i,w')$, where $\dot{q}'_i = dq'_i/dw'$, satisfies

$$\left\{\frac{\partial}{\partial\varepsilon}\left[\frac{dw'}{dw}L(q'_i,\dot{q}'_i,w')\right]\right\}_{\varepsilon=0} = \frac{df}{dw} , \tag{A.2}$$

where f is some function, $f = f(q_i,\dot{q}_i,w)$, then

$$\sum_i p_i Q_i - HT - f \tag{A.3}$$

is a constant of motion, where

$$p_i = \partial L/\partial \dot{q}_i \quad \text{and} \quad H = \sum_i p_i \dot{q}_i - L. \tag{A.4}$$

To see the result (A.2) it suffices to calculate $(dw'/dw)L(q'_i,\dot{q}'_i,w')$ in (A.2) to the first order in ε. Since

$$\dot{q}'_i = \frac{dq_i'}{dw'} = \dot{q}_i + \varepsilon\frac{d\dot{Q}_i}{dw} - \varepsilon\dot{q}_i\frac{d\Gamma}{dw}, \tag{A.5}$$

we have

$$\left\{\frac{\partial}{\partial\varepsilon}\left[\frac{dw'}{dw}L(q'_i,\dot{q}'_i,w')\right]\right\}_{\varepsilon=0}$$

$$= \left\{\frac{\partial}{\partial\varepsilon}\left[\left(1+\varepsilon\frac{d\Gamma}{dw}\right)L\left(q_i+\varepsilon Q_i,\, \dot{q}_i+\varepsilon\frac{dQ_i}{dw} - \varepsilon\dot{q}_i\frac{d\Gamma}{dw},\, w+\varepsilon T\right)\right]\right\}_{\varepsilon=0}$$

$$= \left\{\frac{\partial}{\partial\varepsilon}\left[\left(1+\varepsilon\frac{d\Gamma}{dw}\right)L(q_i,\dot{q}_i,w) + \frac{\partial L}{\partial q_i}\varepsilon Q_i + \varepsilon(\partial L/\partial\dot{q}_i)\left(\frac{dQ_i}{dw} - \dot{q}_i\frac{d\Gamma}{dw}\right) + \frac{\partial L}{\partial w}\varepsilon T\right]\right\}_{\varepsilon=0}$$

$$= L\frac{d\Gamma}{dw} + \frac{\partial L}{\partial q_i}Q_i + (\partial L/\partial\dot{q}_i)\left(\frac{dQ_i}{dw} - \dot{q}_i\frac{d\Gamma}{dw}\right) + \frac{\partial L}{\partial w}T = \frac{df}{dw}. \tag{A.6}$$

Using (A.4), one has

$$T\frac{dH}{dw} = T\frac{d}{dw}\left([\partial L/\partial\dot{q}_i]\,\dot{q}_i\right) - T\frac{\partial L}{\partial q_i}\dot{q}_i - T\,[\partial L/\partial\dot{q}_i]\frac{d\dot{q}_i}{dw} - T\frac{\partial L}{\partial w}. \tag{A.7}$$

Substituting $(\partial L/\partial w)T$ in (A.6) into (A.7) and using (A.4), one obtains

$$\frac{d}{dw}\left(\sum_i p_i Q_i - HT - f\right) = (Q_i - T\dot{q}_i)\left(\frac{d}{dw}[\partial L/\partial\dot{q}_i] - \frac{\partial L}{\partial q_i}\right) = 0, \tag{A.8}$$

which vanishes because of the Lagrangian equation

$$\frac{d}{dw}[\partial L/\partial\dot{q}_i] - \frac{\partial L}{\partial q_i} = 0. \tag{A.9}$$

The result (A.8) implies that the quantity in (A.3) is a constant of motion.

To see the intimate relation among Noether's theorem and symmetry of a physical system, we observe that, if df/dw = 0, the condition (A.2) is equivalent to the property that the action of a physical system is invariant under the transformation (A.1) because

$$\left\{\frac{\partial}{\partial\varepsilon}\left[\frac{dw'}{dw}L(q'_i,\dot{q}'_i,w')\right]\right\}_{\varepsilon=0}$$

$$= \left\{\frac{\partial}{\partial\varepsilon}\left[L(q'_i,\dot{q}'_i,w')dw' - L(q_i,\dot{q}_i,w)dw\right]\frac{1}{dw}\right\}_{\varepsilon=0} = 0. \qquad (A.10)$$

In general, even if the difference, $[L(q'_i,\dot{q}'_i,w')dw' - L(q_i,\dot{q}_i,w)dw]$, is a total differential of some function $f(q_i,\dot{q}_i,w)$ rather than zero, we still have a conservation law, according to Noether's theorem. This is related to the fact that a total differential in the action of a physical system does not affect the equation of motion.

Now, let us consider a few specific cases to illustrate Noether's theorem.

Ab. Symmetry of Time and Space Translations in Inertial Frames

Let us consider a simple action,

$$S = \int_a^b -mds = \int_{w_{Ia}}^{w_{Ib}} L\ dw_I, \qquad (A.11)$$

$$L = L(\mathbf{r}_I,\dot{\mathbf{r}}_I,w_I) = -m\frac{ds}{dw_I} = -m\sqrt{1 - \dot{x}_I^2 - \dot{y}_I^2 - \dot{z}_I^2},$$

where $\mathbf{r}_I = (x_I, y_I, z_I)$ is the Cartesian coordinate and the velocity is $\dot{\mathbf{r}}_I = d\mathbf{r}_I/dw_I$. This Lagrangian L depends only on the velocities and does not depend on space and time. Suppose we consider infinitesimal translations in time and space

$$w_I \to w'_I = w_I + \varepsilon T_o,$$
$$x_I \to x'_I = x_I + \varepsilon Q_o, \qquad y_I \to y'_I = y_I, \qquad z_I \to z'_I = z_I, \qquad (A.12)$$

where T_o and Q_o are constant. One can verify that df/dw in (A.2) vanishes,

$$\left\{\frac{\partial}{\partial\varepsilon}\left[\frac{dw'_I}{dw_I} L(\mathbf{r}'_I,\dot{\mathbf{r}}'_I,w'_I))\right]\right\}_{\varepsilon=0}$$

$$= \left\{\frac{\partial}{\partial\varepsilon}\left[-m\sqrt{1 - \dot{x}_I^2 - \dot{y}_I^2 - \dot{z}_I^2}\right]\right\}_{\varepsilon=0} = 0, \tag{A.13}$$

so that f is simply a constant. Therefore, the result (A.3) of Noether's theorem implies

$$p_x Q_o - H T_o = \text{constant}. \tag{A.14}$$

If one considers only the time translation, i.e., $T_o = 1$ and $Q_o = 0$ in (A.12), the result (A.14) leads to the conservation of energy,

$$H = \text{constant}, \tag{A.15}$$

where H = E. Similarly, the symmetry of the Lagrangian (A.11) under space translations along the x-axis, i.e., $T_o = 0$ and $Q_o = 1$ in (A.12), implies by Noether's theorem the conservation of momentum, p_x = constant.

Ac. Symmetry of the Lorentz Group for Inertial Frames

We know that the action (A.11) is invariant under the infinitesimal 4-dimensional transformation

$$w'_I = w_I - \beta x_I, \qquad x'_I = x_I - \beta w_I, \qquad y'_I = y_I, \qquad z'_I = z_I. \tag{A.16}$$

Indeed, it follows from (A.11) and (A.16) that

$$\left\{\frac{\partial}{\partial\beta}\left[\frac{dw'_I}{dw_I} L(\mathbf{r}'_I,\dot{\mathbf{r}}'_I,w'_I))\right]\right\}_{\beta=0}$$

$$= \left\{\frac{\partial}{\partial\beta}\left(-m(1 - \beta\dot{x}_I)\left[(1 - \dot{\mathbf{r}}_I^2)(1 - 2\beta\dot{x}_I)\right]^{1/2}\right)\right\}_{\beta=0} = 0. \tag{A.17}$$

Thus, f in (A.2) is a constant. Noether's theorem (A.3) implies

$$p_{Ix}w_I - Hx_I = \text{constant}, \tag{A.18}$$

where

$$p_{Ix} = \partial L/\partial \dot{x}_I = m(d\dot{x}_I/dw_I)/[1-\dot{r}_I^2]^{1/2}, \qquad H = \mathbf{p}_I \cdot \dot{\mathbf{r}}_I - L = m/[1-\dot{r}_I^2]^{1/2}.$$

The invariance of the quantity (A.18) can also be directly verified by using the 4-dimensional transformations of coordinates and momenta, (7.4) and (10.5). The form (A.18) resembles that of angular momentum because, mathematically, it is related to the property that the Lorentz transformation (A.16) can be viewed as a rotation of w_I and x_I axes in a 4-dimensional space with y_I and z_I fixed.

If one pauses and reflects for a moment concerning the physical implications of (A.18), one cannot refrain from wondering: Why a splendid and useful Lorentz invariance turns out to be associated with such a dull and useless conservation law (A.18)?!

Ad. Generalized Translational Symmetry of the Wu Group in Accelerated Frames

In a non-inertial frame such as F(w,x,y,z) with a constant-linear-acceleration along the x axis, the energy p_0 and momentum p_x of a "free" (or non-interacting) particle are clearly not constant, as one can see from (22.6) and (22.9). Evidently, this is intimately related to the fact that the action (22.1) in F for a "free" particle, i.e., $a_\mu=0$, does not have the usual symmetries of translations in time w and space in the x direction, in contrast to the action in (A.11) in F_I.

However, there are more sophisticated symmetries associated with the action (22.1) with $a_\mu=0$ in non-inertial frames. They are the generalizations of translations (A.12) from an inertial frame F_I to an non-inertial frame F. Let us substitute the infinitesimal translations in (A.12) into the inverse Wu transformation (21.47)

$$w' = \frac{(w'_I + \beta_o/\alpha\gamma_o)}{(\alpha x'_I + 1/\gamma_o)} - \frac{\beta_o}{\alpha},$$

$$x' = \sqrt{(x'_I + 1/\gamma_o\alpha)^2 - (w'_I + \beta_o/\alpha\gamma_o)^2} - \frac{1}{\alpha\gamma_o^2}, \quad y = y_I, \quad z = z_I. \tag{A.19}$$

Using (A.19) and (A.12), we have the generalized translations in a CLA frame F(w,x,y,z),

$$w' = w + \varepsilon T,$$
$$x' = x + \varepsilon Q, \qquad y' = y, \qquad z' = z, \tag{A.20}$$

where ε is an infinitesimal parameter and the functions T and Q are given by

$$T = \frac{T_o}{\alpha(x_I + 1/\alpha\gamma_o)} - \frac{Q_o(w_I + \beta_o/\alpha\gamma_o)}{\alpha(x_I + 1/\alpha\gamma_o)^2} = T_o\left(\frac{\gamma}{W}\right) - Q_o\left(\frac{\beta\gamma}{W}\right), \tag{A.21}$$

$$Q = \frac{Q_o(x_I + 1/\alpha\gamma_o) - T_o(w_I + \beta_o/\alpha\gamma_o)}{\sqrt{(x_I + 1/\gamma_o\alpha)^2 - (w_I + \beta_o/\alpha\gamma_o)^2}} = \gamma Q_o - \gamma\beta T_o. \tag{A.22}$$

Let us evaluate (A.2) with the Lagrangian (22.4) with $\bar{e}$=0 and primed variables. We find that df/dw = 0,

$$\left\{\frac{\partial}{\partial\varepsilon}\left[\frac{dw'}{dw} L(\mathbf{r}',\dot{\mathbf{r}}',w')\right]\right\}_{\varepsilon=0}$$

$$= \left\{\frac{1}{dw}\frac{\partial}{\partial\varepsilon}\left(\sqrt{W'^2dw'^2 - dr'^2}\right\}_{\varepsilon=0}\right.$$

$$= \frac{Wdw^2(2\alpha\beta\gamma^2WT + \alpha\gamma^2Q) + W^2dwdT - dxdQ}{dw\sqrt{W^2dw^2 - dr^2}} = 0, \tag{A.23}$$

where we have used

$$\left\{\frac{\partial}{\partial\varepsilon}W\right\}_{\varepsilon=0} = 2W\alpha\beta\gamma^2T + \alpha\gamma^2Q, \tag{A.24}$$

$$\left\{\frac{\partial}{\partial\varepsilon}dw'^2\right\}_{\varepsilon=0} = 2dwdT, \qquad \left\{\frac{\partial}{\partial\varepsilon}dr'^2\right\}_{\varepsilon=0} = 2dxdQ, \tag{A.25}$$

$$dT = -\frac{\alpha\beta\gamma^3T_o\,dw}{W} - \frac{\alpha\gamma^3T_o\,dx}{W^2} - \frac{Q_o}{W}[\alpha\gamma - \alpha\beta^2\gamma^3]dw + \frac{Q_o}{W^2}\alpha\beta\gamma^3dx, \tag{A.26}$$

$$dQ = \alpha\beta\gamma^3Q_o\,dw - \alpha\gamma^3T_o\,dw. \tag{A.27}$$

Thus, Noether's theorem implies

$$p_x(\gamma Q_o - \gamma\beta T_o) - H\left(\frac{T_o\gamma}{W} - \frac{Q_o\beta\gamma}{W}\right) = \text{constant}, \tag{A.28}$$

in CLA frames. This is the generalization of the conservation (A.14) of inertial frames. For the generalized time translational symmetry, i.e., $T_o=1$, $Q_o=0$, the result (A.28) leads to

$$p_x(-\gamma\beta) - H\left(\frac{\gamma}{W}\right) = -\gamma\left(\frac{p_o}{W} + \beta p_x\right) = \text{constant}, \tag{A.29}$$

where p_o, p_x and W are functions of space **r** and time w. Similarly, the generalized space translational symmetry (i.e., $T_o=0$, $Q_o=1$) implies

$$p_x(\gamma) - H\left(-\frac{\beta\gamma}{W}\right) = \gamma\left(p_x + \beta\frac{p_o}{W}\right) = \text{constant}. \tag{A.30}$$

Equations (A.29) and (A.30) are consistent with the constant energy p_{I0} and momentum $p_{I1}=-p_I{}^1=-p_{Ix}$ in an inertial frame, as shown in the 4-momentum transformation (22.10).

Ae. Classical and Quantum Rings (or Closed Strings) in a Central Force Field

Some conservation laws for the motion of physical objects in a potential field are not related to Noether's theorem. For example, a string-like object, which has been extensively discussed in recent years, may have a different type of symmetry from that of ordinary particles. Let us consider a simple string which is closed and moves in a potential field with a constant radius. Based on a formal analogy between the equation of the Nambu string[2] and the massless Klein-Gordon equation with cyclic radial momentum p_r and cyclic angular momentum p_θ, it has been suggested that a closed quantum string with a constant radius could be described by a Hamiltonian with cyclic radial momenta p_r and p_θ.[3] The Lagrangian L_r for such a "ring" moving in a Coulomb-like potential V(r) was assumed to have the form

$$L_r(r,\theta,\phi,\dot{\phi}) = -m(1-r^2\dot{\phi}^2\sin^2\theta)^{1/2} - V(r) , \tag{A.31}$$

which has cyclic velocities $\dot{r}=dr/dw$ and $\dot{\theta}=d\theta/dw$. Since the generalized coordinates are $q_i=(q_1,q_2,q_3)=(r,\theta,\phi)$, their conjugate momenta are given by

$$p_r = \frac{\partial L_r}{\partial \dot{r}} = 0, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = 0, \quad p_\phi = \frac{\partial L_r}{\partial \dot{\phi}} = \frac{mr^2\dot{\phi}\sin^2\theta}{(1-r^2\dot{\phi}^2\sin^2\theta)^{1/2}}. \tag{A.32}$$

Thus, one cannot obtain a relation for the velocity $\dot{r}$ (or $\dot{\theta}$) in terms of p_r (or p_θ) and coordinates. In some cases, the lack of these relations will render the Legendre transformation and, hence, the Hamiltonian undefined.[4] However, in this case one can follow Routh's procedure for treating cyclic variables[5] and define a new Hamiltonian H_r for the ring

$$H_r(r,\theta,\phi,p_\phi) = p_\phi \dot{\phi} - L_r = \sqrt{m^2+p_\phi^2(r^2\sin^2\theta)} + V(r) , \tag{A.33}$$

where we have used the equations in (A.32). The usual Hamiltonian equations for ϕ and p_ϕ can be obtained. The momenta p_r and p_θ are cyclic in H_r. We also have the following equations (in which H_r plays the role of the Lagrangian) for the ring's motion:

$$\frac{\partial H_r}{\partial r} = \frac{d}{dw}(\partial H_r/\partial \dot{r}) = 0, \qquad \frac{\partial H_r}{\partial \theta} = \frac{d}{dw}(\partial H_r/\partial \dot{\theta}) = 0. \tag{A.34}$$

These equations determine the constant values of r and θ,

$$r = \alpha_r , \qquad \theta = \alpha_\theta . \tag{A.35}$$

For an arbitrary central potential V(r), the second equation in (A.34) leads to $\alpha_\theta=\pi/2$, while the numerical value of r depends on the specific form of the potential V(r) in the model.

Classically, the Hamiltonian (A.33) describes a particle with a mass m moving in a circular orbit. Since a rotating ring with a constant radius can be pictured as a collection of N particles moving in the same orbit, we can also interpret (A.33) as the Hamiltonian for a rotating ring with a total rest mass m.

The Hamiltonian (A.33) has the cyclic momenta p_r and p_θ which can be used to construct a "ring model of quarks" with permanent confinement of the quantum ring.[3]

Af. A Generalized Noether's Theorem for Symmetry in Phase Space

The constants of motion in (A.35) for inertial frames are not covered by the conventional Noether's theorem based on symmetry in spacetime. However, one can generalize Noether's theorem to imply (A.35) as a special case by considering symmetry in the *phase space.*

Let us introduce a new function called "Jingsian" J_s which symmetrizes the p and q variables explicitly and plays a double role of the Lagrangian and the Hamiltonian:

$$J_s(q_1,..q_n,p_1,..,p_s,\dot{q}_{s+1},..\dot{q}_n,t) \equiv L(q_i,\dot{q}_i,t) - \sum_{i=1}^{s}\frac{1}{2}(p_i\,\dot{q}_i - q_i\,\dot{p}_i)$$

$$-\frac{d}{dt}\sum_{i=1}^{s}\frac{1}{2}(p_i\,q_i), \qquad 0\le s\le n. \tag{A.36}$$

This form enables us to treat p_i and q_i on a more equal footing in performing variational calculations than the usual Hamiltonian form. That is, the Jingsian J_s makes the p-q symmetry more explicit and does not depend on $\dot{p}_i$, so that one can deal with symmetry related to the momenta $p_1, p_2,....,p_s$ and the coordinates $q_1, q_2,....,q_s$.

When s=0, the Jingsian J_0 is just the Lagrangian L, which is defined in a configuration space formed from the n generalized coordinates. And when s=n, J_n reduces to the negative Hamiltonian -H(q,p,t) defined in the phase space. The general Jingsian J_s, $1 \le s \le n-1$, is defined in a combined phase space $(p_1,..,p_s, q_1,..,q_s)$ and configuration space $(q_{s+1}, q_{s+2},....,q_n)$, which may be called "Jingsian space" for short. The Hamilton's principle $\delta S \equiv \delta\int_1^2 L dt = 0$ indicates that one can have the modified Hamilton's principle in the Jingsian space by expressing L in terms of the Jingsian J_s in (A.36). We have

$$\delta S = \delta \int_1^2 \Big\{ \sum_{i=1}^{s} \Big(\frac{1}{2} p_i \frac{dq_i}{dt} - \frac{1}{2} q_i \frac{dp_i}{dt}\Big) + \frac{d}{dt} \sum_{i=1}^{s} \frac{1}{2}(p_i q_i)$$

$$+ J_s(q_1, ... q_n, p_1, ..., p_s, \dot{q}_{s+1}, ... \dot{q}_n, t) \Big\} dt ,$$

$$= \int_1^2 \Big\{ \sum_{i=1}^{s} (\dot{q}_i \delta p_i - \dot{p}_i \delta q_i) + \sum_{i=1}^{s} \Big[\frac{\partial J_s}{\partial q_i} \delta q_i + \frac{\partial J_s}{\partial p_i} \delta p_i\Big]$$

$$+ \sum_{i=s+1}^{n} \Big[\frac{\partial J_s}{\partial q_i} - \frac{d}{dt} \frac{\partial J_s}{\partial q_i^*}\Big] \delta q_i \Big\} dt = 0 , \qquad q_i^* = \dot{q}_i , \tag{A.37}$$

where we have used the following conditions in the Jingsian space,

$$\delta q_i(t_1) = \delta q_i(t_2) = 0, \quad i = 1,2,...n ;$$
$$\delta p_i(t_1) = \delta p_i(t_2) = 0, \quad i = 1,2,...s . \tag{A.38}$$

We obtain the desired equations of motion

$$\frac{dp_k}{dt} = \frac{\partial J_s}{\partial q_k}, \qquad \frac{dq_k}{dt} = -\frac{\partial J_s}{\partial p_k}, \qquad k=1,2,...,s . \tag{A.39}$$

Equations (A.39) involving the coordinates $q_1,...,q_s$ and the momenta $p_1,...,p_s$ are in the form of Hamilton's equations with the negative Jingsian, $-J_s$, as the Hamiltonian. However, the (n-s) coordinates and velocities obey the Lagrange equations[2]

$$\frac{d}{dt}(\partial J_s/\partial \dot{q}_i) - (\partial J_s/\partial q_i) = 0 , \qquad i = s+1,....,n . \tag{A.40}$$

Suppose a physical system is described by the equations of motion (A.39) and (A.40). The last term involving a total differential in (A.37) does not contribute to the equations of motion and it can be ignored in the action functional without affecting physics. Thus, the action functional S of a physical system can be written in the following symmetrized form:

$$S = \int_1^2 \Big\{ \sum_{i=1}^{s} \left(\frac{1}{2}p_i\dot{q}_i - \frac{1}{2}q_i\dot{p}_i\right) + J_s(q_1,...q_n,p_1,...,p_s,\dot{q}_{s+1},.. \dot{q}_n,t)\Big\} dt . \qquad (A.41)$$

Suppose this action S is invariant under the following infinitesimal transformations in the Jingsian space,

$$p'_k = p_k + \varepsilon P_k^*, \qquad k=1,...s ;$$

$$q'_i = q_i + \varepsilon Q_i^*, \qquad i=1,...,n ; \qquad (A.42)$$

$$t' = t + \varepsilon T .$$

That is,

$$\Big\{\frac{\partial}{\partial\varepsilon}\Big[\Big(\sum_{i=1}^{s}\left(\frac{1}{2}p'_i\,\dot{q}'_i - \frac{1}{2}q'_i\,\dot{p}'_i\right) + J_s(q'_1,...q'_n;p'_1,...,p'_s,\dot{q}'_{s+1},....\dot{q}'_n,t')\Big)\frac{dt'}{dt}\Big]\Big\}_{\varepsilon=0}$$

$$= \frac{df}{dt}, \qquad \dot{p}'_k \equiv \frac{dp'_k}{dt'}, \qquad (A.43)$$

where P_k^*, Q_i^*, T and f are functions of $(q_1,..q_n,p_1,.,p_s,\dot{q}_{s+1},..,\dot{q}_n,t)$. Following similar steps from (A.5) to (A.9), we have the generalized Noether's theorem for the case with symmetry in the phase space:

$$\frac{d}{dt}\Big[HT + f - \sum_{i=1}^{n} p_i Q_i + \sum_{k=1}^{s} q_k P_k\Big] = 0 , \qquad (A.44)$$

where $P_i = (\frac{1}{2}P_1^*,..,\frac{1}{2}P_s^*)$ and $Q_i = (\frac{1}{2}Q_1^*,..,\frac{1}{2}Q_s^*,Q_{s+1}^*,...,Q_n^*)$.

References

1. She was born in Erlangen, Germany. Due to the strong prejudice existing in the early 20th century against women being professors, even the great Mathematician Hilbert could not gain her a permanent position at Göttingen University. His repeated attempts were frustrated for years until 1922 when she was named an "unofficial associate professor."
2. Y. Nambu, Lecture at the Copenhagen Summer Symposium, 1970 and *Proceedings of International Conference on Symmetries and Quark Model*, Wayne State University, 1969 (Gordon and Breach, New York, 1970), p. 269.
3. J. P. Hsu, J. Math. Phys. **30**, 2682 (1989). Within the 4-dimensional symmetry framework, the time variables related to conservation laws are flexible and not unique, as discussed in chapters 7 and 12 and 17.
4. When there is a cyclic velocity in a Lagrangian, the Legendre transformation has a problem because it cannot be defined in the usual sense. See J. Kevorkian, *Partial Difference Equations: Analytic Solution Techniques* (Wadsworth & Brooks/Cole 1990; Pacific Grove, CA), pp. 332–333. But in certain cases, one still can define a Hamiltonian for a quantum ring model in the absence of two velocities $\dot{r}$ and $\dot{\theta}$. See, for example, J. P. Hsu, J. Math. Phys. in ref. 3.
5. H. Goldstein, *Classical Mechanics* (2nd. Ed. Addison-Wesley, Reading MA., 1981), pp. 351–356.

Appendix B.

Quantum Electrodynamics in Both Linearly Accelerated and Inertial Frames

Ba. Quantum Electrodynamics of Bosons in CLA and Inertial Frames

Quantum scalar field operators obey equation (24.35) in CLA frames. This suggests that we use the taiji-time w in a general frame as the evolution variable for a state $\Phi^{(S)}(w)$ in the Schrödinger representation:

$$i\frac{\partial\Phi^{(S)}(w)}{\partial w} = H^{(S)}(w)\Phi^{(S)}(w) , \qquad H^{(S)} = H_o^{(S)} + H_I^{(S)}, \qquad J = 1 . \tag{B.1}$$

The reason is that the evolution of a physical system is assumed to be described by a Hamiltonian operator which has the same transformation property as $\partial/\partial w$. A covariant partial derivative is the same as an ordinary partial derivative, $D_\mu = \partial_\mu$, when they operate on scalar functions. We may remark that the form (B.1) is no longer true if the Hamiltonian involves spinor fields; in this case, the time derivative ∂_0 has to be replaced by the gauge covariant derivative ∇_0, according to equation (24.56).

It is natural to assume that the usual covariant formalism of perturbation theory[1] can also be applied to QED of scalar bosons in CLA frames, which are smoothly connected to inertial frames in the limit of zero acceleration. Let us briefly consider the interaction representation and the S-matrix in CLA frames. The transformations of the state vector $\Phi(w)$ and operator $O(w)$ from the Schrödinger representation to the interaction representation are defined as

$$\Phi(w) \equiv \Phi^{(I)}(w) = \exp[iH_o^{(S)}w]\Phi^{(S)}(w) , \tag{B.2}$$

$$O(w) \equiv O^{(I)}(w) = \exp[iH_o^{(S)}w]O^{(S)}\exp[-iH_o^{(S)}w] . \tag{B.3}$$

Because $O^{(S)}$ and $O(w)$ are the same for $w = 0$, we have

$$i\frac{\partial\Phi(w)}{\partial w} = H_I(w)\Phi(w), \quad H_I = \exp[iH_o^{(S)}w]H_I^{(S)}\exp[-iH_o^{(S)}w] , \tag{B.4}$$

$$O(w) = \exp[iH_o^{(S)}w]O(0)\exp[-iH_o^{(S)}w] . \tag{B.5}$$

The U-matrix can be defined in terms of the time w: $\Phi(w) = U(w,w_o)\Phi(w_o)$, $U(w_o,w_o) = 1$. It follows from (B.4) and (B.5) that

$$i\frac{\partial U(w,w_o)}{\partial w} = H_I(w)U(w,w_o) . \tag{B.6}$$

If a physical system is in the initial state Φ_i at time w_o, the probability of finding it in the final state Φ_f at a later time w is

$$|\langle\Phi_f|U(w,w_o)\,\Phi_i\rangle|^2 = |U_{fi}(w,w_o)|^2 . \tag{B.7}$$

Evidently, the average transition probability per unit time for $\Phi_f \to \Phi_i$ is

$$|U_{fi}(w,w_o) - \delta_{fi}|^2/(w - w_o) . \tag{B.8}$$

As usual, we can express the S-matrix in terms of the U-matrix, i.e. $S = U(\infty, -\infty)$ and obtain the following form

$$S = 1 - i\int_{-\infty}^{\infty} H_I(w)dw + (-i)^2\int_{-\infty}^{\infty} H_I(w)dw\int_{-\infty}^{w} H_I(w')dw' + \ldots\ldots . \tag{B.9}$$

For w-dependent operators, one can introduce a w-product W^* (corresponding to the usual chronological product), so that (B.9) can be written in an exponential form:

$$S = W^*\{\exp[-i\int_{-\infty}^{\infty} H_I(x^\mu)dwd^3r]\} , \qquad \int_{-\infty}^{\infty} H_I(x^\mu)d^3r = H_I(w) . \tag{B.10}$$

Since J is a truly universal constant, we can have the "natural units" J = 1 in both CLA and inertial frames. Thus, we have the relation of dimensions

$$[a_\mu] = [\psi^{2/3}] = [\text{mass}] = [1/\text{length}] , \tag{B.11}$$

in CLA and inertial frames.

To obtain the rules for Feynman diagrams for scalar QED in CLA frames, we follow the usual procedure[1] and assume L_{SQED} to be

$$L_{SQED} = L_{sp} - \sqrt{-g}\,(\partial^\mu a_\mu)^2/(2\rho)\,, \tag{B.12}$$

$$L_{sp} = \sqrt{-g}\,\{g^{\mu\nu}[(i\partial_\mu - \bar{e}a_\mu)\Phi^*][(i\partial_\nu - \bar{e}a_\nu)\Phi] - m^2\Phi^*\Phi\} - \frac{1}{4}\sqrt{-g}\,f_{\mu\nu}f^{\mu\nu}\,,$$

$$f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu\,, \quad \bar{e} = -1.6021891\times10^{-20}\sqrt{4\pi}\,(g\cdot cm)^{1/2}\,,$$

where ρ is a gauge parameter. The Lagrangian L_{sp} is gauge invariant and observable results are independent of the gauge parameter ρ.

To see that there is a "conservation" of 4-momentum at each vertex of the Feynman diagram in CLA frames, let us consider the wave function $\Phi(w,\mathbf{x}) = \Phi(x)$ for a "free particle" given by (24.23) with the phase P given by (24.19) and the condition (24.21) for a plane wave. In CLA frames, one can verify that

$$\frac{\partial}{\partial w} P = k_{I0}(x^1 + 1/\alpha\gamma_o^2)\gamma^3\alpha + k_{I1}(x^1 + 1/\alpha\gamma_o^2)\gamma^3\alpha\beta$$

$$= (\gamma k_{I0} + \gamma\beta k_{I1})W = k_0\,, \tag{B.13}$$

$$\frac{\partial}{\partial x^1} P = k_{I0}\gamma\beta + k_{I1}\gamma = k_1\,, \quad \frac{\partial}{\partial x^2} P = k_2\,, \quad \frac{\partial}{\partial x^3} P = k_3\,,$$

where we have used $(\partial/\partial w)\gamma\beta = \gamma^3\alpha$ and $(\partial/\partial w)\gamma = \gamma^3\alpha\beta$. Thus, the relation

$$i\frac{\partial}{\partial x^\mu}\, e^{-iP} = k_\mu\, e^{-iP} \tag{B.14}$$

holds for a "free wave" in CLA frames, just as in inertial frames.

However, the zeroth component p_0 (or k_0) of the covariant 4-momentum depends on the Wu alteration function W(w,x), as shown in (22.9), and is not conserved in a particle collision process. Fortuately, the conservation of "momentum" in a collision process, e.g., a+b→c+d, as observed in CLA frames can be expressed in terms of

$$(p_0/W, p_1, p_2, p_3) \equiv \bar{p}_\mu \equiv (\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3), \qquad \text{(B.15)}$$

$$g^{\mu\nu}p_\mu p_\nu = (p_0/W)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = \eta^{\mu\nu}\bar{p}_\mu\bar{p}_\nu, \quad \eta^{\mu\nu} = (1,-1,-1,-1).$$

This $\bar{p}_\mu$ may be termed "alteration momentum" which is not exactly a 4-vector under the Wu or the MWL transformation. But the momentum space of $\bar{p}_\mu$ is formally closer to the space of the 4-momentum $p_{I\mu}$ in inertial frames than that of the true 4-momentum p_μ as far as the S-matrix and Feynman rules are concerned. For the scattering process a+b→c+d, we have the following relations for momentum:

$$\bar{p}_{0a} + \bar{p}_{0b} = \gamma_B(p_{I0a} + p_{I0b} - \beta_B[p_{I1a} + p_{I1b}]),$$

$$\bar{p}_{1a} + \bar{p}_{1b} = \gamma_B(p_{I1a} + p_{I1b} - \beta_B[p_{I0a} + p_{I0b}]),$$

$$p_{2a} + p_{2b} = p_{I2a} + p_{I2b}, \qquad p_{3a} + p_{3b} = p_{I3a} + p_{I3b}; \qquad \text{(B.16)}$$

$$\bar{p}_{0c} + \bar{p}_{0d} = \gamma_A(p_{I0c} + p_{I0d} - \beta_A[p_{I1c} + p_{I1d}]),$$

$$\bar{p}_{1c} + \bar{p}_{1d} = \gamma_A(p_{I1c} + p_{I1d} - \beta_A[p_{I0c} + p_{I0d}]),$$

$$p_{2c} + p_{2d} = p_{I2c} + p_{I2d}, \qquad p_{3c} + p_{3d} = p_{I3c} + p_{I3d}; \qquad \bar{\mathbf{p}} = \mathbf{p},$$

which can be derived from the inverse transformation of (22.10) and (B.15). Since the 4-monentum is conserved in the inertial frame F_I, i.e., $p_{I0c} + p_{I0d} = p_{I0a} + p_{I0b}$ = constant and $\mathbf{p}_{Ic} + \mathbf{p}_{Id} = \mathbf{p}_{Ia} + \mathbf{p}_{Ib}$ = constant, we have the conservation of the alteration momentum in CLA frames:

$$\bar{p}_{0c} + \bar{p}_{0d} = \bar{p}_{0a} + \bar{p}_{0b} \qquad \text{and} \qquad \bar{\mathbf{p}}_c + \bar{\mathbf{p}}_d = \bar{\mathbf{p}}_a + \bar{\mathbf{p}}_b, \qquad \text{(B.17)}$$

at the "instant of collision," so to speak. The reason for their conservation is that, although both sides are not constant as shown in (B.16), they must be the same when $\beta_B = \beta_A$ which is realized at the instant of collision.

Based on the Wu transformation for coordinates and 4-momenta, we have

$$\int d^4x_I \ \exp(-ip_{I\mu}x_I^{\mu}) = (2\pi)^4\delta^4(p_I) = \int\sqrt{-g}d^4x\ e^{-iP(x)}$$

$$= (2\pi)^4\delta(\gamma\bar{p}_0 - \gamma\beta\bar{p}_1)\delta(\gamma\bar{p}_1 - \gamma\beta\bar{p}_0)\delta(\bar{p}_2)\delta(\bar{p}_3) \tag{B.18}$$

$$= (2\pi)^4\delta(\bar{p}_0)\delta(\bar{p}_1)\delta(\bar{p}_2)\delta(\bar{p}_3) = (2\pi)^4\delta^4(\bar{p}) = (2\pi)^4W\delta^4(p)\ ,$$

$$\delta(p_{I0})\delta(p_{I1}) = \delta(\gamma\bar{p}_0 - \gamma\beta\bar{p}_1)\delta(\gamma\bar{p}_1 - \gamma\beta\bar{p}_0) = \frac{\delta(\bar{p}_0)\delta(\bar{p}_1)}{J(p_{I\lambda}/\bar{p}_\lambda)} = \delta(\bar{p}_0)\delta(\bar{p}_1)\ ,$$

where we have used the 'free-wave' (24.9), the momentum transformation (22.10) and the Wu transformation (21.46). In the last equation, $J(p_{I\lambda}/\bar{p}_\lambda)$ is the Jacobian of the $p_{I\lambda}$ with respect to the $\bar{p}_\lambda$ which can be calculated by using (22.10). This result is the 2-dimensional generalization of the 1-dimensional case given by (24.20) with $\kappa = 0$.[2] In a CLA frame, the integral of a "plane wave" over the "whole spacetime" is limited and complicated by the presence of a "black wall" (i.e., a wall singularity) at $x = -1/(\alpha\gamma_o^2)$. The integration can be carried out by a change of variables and this amounts to using variables in an inertial frame as a crutch to obtain the result. The relation (B.17) or (B.18) implies that we have a conservation of momentum at a vertex in the generalized Feynman rules in CLA frames. Those properties in (B.13)–(B.18) are convenient for writing down the generalized Feynman rules for quantum electrodynamics in CLA frames.

As usual, if there are no identical particles in the final state, we define the relationship between the *M*- and S-matrices for initial (i) and final (f) states as follows:

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta^4(\bar{p}_f^{(tot)} - \bar{p}_i^{(tot)})[\Pi_{ext\ par}(n_j/V)]^{1/2}M_{fi}\ , \tag{B.19}$$

where "ext par" denotes external particles, $n_j = m_j/\omega_{kj}$ for spin 1/2 fermions and $n_j = 1/(2\omega_{kj})$ for bosons. Note that the S-matrix elements for physical processes which are observed and measured in CLA frames are defined only for those cases in which the momenta of the initial and final states are constant in an inertial frame.

Because of the 4-dimensional symmetry in (B.12) and (B.19), the generalized Feynman rules for writing the amplitude M_{fi} are formally the same as those in the usual QED, except that certain quantities (e.g., w, J and $\tilde{e}$) have different dimensions from the corresponding quantities in conventional QED.

One can use a more intuitive method of Feynman to obtain the generalized rules for Feynman diagrams in CLA frames. For example, the scalar boson propagator can be obtained from the scalar wave equations (24.1) and the relation (B.14) for a "free boson" with the momentum p_μ. The vertex of interaction can be obtained from the interaction Lagrangian density L_{SQED} in (B.12).[3]

The generalized Feynman rules for the amplitude M_{fi} in both constant-linear-acceleration frames and inertial frames are as follows:

(a) The covariant photon propagator is given by

$$\frac{-i\left\{g_{\mu\nu} - (1-\rho)\dfrac{k_\mu k_\nu}{(k^\lambda k_\lambda + i\varepsilon)}\right\}}{(k^\sigma k_\sigma + i\varepsilon)}, \tag{B.20}$$

where $\rho = 1$ is the Feynman gauge, and $\rho = 0$ the Landau gauge.

(b) The scalar boson propagator is

$$\frac{i}{(p^\mu p_\mu - m^2 + i\varepsilon)}. \tag{B.21}$$

(c) The vertex $\Phi(p)+\gamma(k,\mu) \to \Phi(p')$ is

$$-i\,\tilde{e}(p_\mu + p'_\mu), \tag{B.22}$$

where $\gamma(k,\mu)$ is an incoming photon line toward the vertex with the momentum k_λ and a polarization index μ.

(d) The vertex $\Phi+\gamma(\mu) \to \Phi+\gamma(\nu)$ has the factor

$$2i\,\tilde{e}^2 g_{\mu\nu}. \tag{B.23}$$

(e) Each external photon line with an index μ has a polarization vector ε_μ.

(f) A factor 1/2 for each closed loop containing only two photon lines, e.g., $\Phi+\Phi \to \gamma(\mu)+\gamma(\nu) \to \Phi+\Phi$.

Other rules such as a integration with $W^{-1}d^4k/(2\pi)^4 = d^4\bar{k}/(2\pi)^4$ over a momentum k_μ not fixed by the "conservation" of momentum at each vertex are the same as the usual rules in inertial frames.

Bb. Feynman Rules for QED in both CLA and Inertial Frames

To obtain the rules for Feynman diagrams of spinor QED in CLA frames, we have to replace the time derivative $\partial_0 = \partial/\partial w$ by the gauge covariant time derivative ∇_0, according to equation (24.56). We follow the usual quantization procedure and define L_{TQED} by adding a gauge fixing term in the Lagrangian density,

$$L_{TQED} = L - \sqrt{-g}\,(D^\mu a_\mu)^2/2\rho\,, \tag{B.24}$$

$$L = \sqrt{-g}\;\bar{\psi}\Gamma^\mu(i\nabla_\mu - \bar{e}a_\mu)\psi - \sqrt{-g}\;m\bar{\psi}\psi\,, \tag{B.25}$$

$$\nabla_\mu = (\partial_0 + \frac{1}{2}(\partial_k W)\gamma^0\gamma^k, \partial_1, \partial_2, \partial_3)\,,$$

where ρ is a gauge parameter. As usual, the M-matrix is defined in (B.19). One can verify that the plane-wave solution (24.57) of a free fermion satisfies

$$\nabla_\mu e^{(-iP(x)\,-\,G(w))} = k_\mu e^{(-iP(x)\,-\,G(w))} \tag{B.26}$$

in CLA frames.

The generalized Feynman rules for the amplitude M_{fi} of QED in both CLA frames and inertial frames are as follows:

(a) The covariant photon propagator is given by (B.20).

(b) The electron propagator is

$$\frac{-i}{(\Gamma^\mu p_\mu\; - \;m\; + \;i\varepsilon)}\,. \tag{B.27}$$

(c) The electron–photon vertex is

$$-\,i\bar{e}\Gamma^\mu\,. \tag{B.28}$$

(d) Each external photon line has an additional factor ε_μ. Each external electron line has u(s,p) for the annihilation of an electron and $\bar{u}$(s,p) for the creation of an electron. Each external positron line has v(s,p) for the annihilation of a positron and $\bar{v}$(s,p) for the creation of a positron.

Other rules such as taking the trace with a factor (-1) for each closed electron loop, integration with $d^4k/[W(2\pi)^4]$ over a momentum k_μ not fixed by the conservation of alteration momentum at each vertex are the same as the usual.

Thus, if one calculates scattering cross sections and decay rates (with respect to the taiji-time w) of a physical process, one will get formally a similar result as that in conventional QED. For example, let us consider the decay rate $\Gamma(1\to 2+3+...+N)$ for a physical process $1\to 2+3+...+N$. It is given by the expression

$$\Gamma(1\to 2+3+...+N) = \lim_{w\to\infty}\int \frac{|\langle f|S|i\rangle|^2}{w}\frac{d^3x_2 d^3p_2}{(2\pi)^3}\frac{d^3x_3 d^3p_3}{(2\pi)^3}\cdots\frac{d^3x_N d^3p_N}{(2\pi)^3},$$

$$= \int \frac{1}{2\omega_{p_1}} |M_{fi}|^2\Big[\prod_{ext\ fer}(2m_{fer})\Big]\frac{d^3p_2}{(2\pi)^3 2\omega_{p_2}}\cdots\cdots\cdots$$

$$\times\frac{d^3p_N}{(2\pi)^3 2\omega_{p_N}}(2\pi)^4\delta^4(\bar{p}_1+\bar{p}_2-\bar{p}_3-\bar{p}_4\cdots-\bar{p}_N)S_o, \tag{B.29}$$

where $\langle f|S|i\rangle = S_{fi}$ is the S-matrix element between the initial state i and the final state f given by (B.19). The decay rate $\Gamma(1\to 2+3+...+N)$ has the dimension of inverse length and S_o denotes a factor 1/n! for each kind of (n) identical particles in the final state. When there is no external fermion in a process, then $[\Pi_{ext\ fer}(2m_{fer})]$ in (B.29) is replaced by 1. The decay length D is given by $D = 1/\Gamma(1\to 2+3+...+N)$.

For a scattering process $1+2\to 3+4+....+N$, the differential cross section $d\sigma$, which has the dimension of area, is given by

$$d\sigma = \frac{1}{4[(p_1\cdot p_2)^2-(m_1m_2)^2]^{1/2}}|M_{fi}|^2\Big[\prod_{ext\ fer}(2m_{fer})\Big]\frac{d^3p_3}{(2\pi)^3 2\omega_{p_3}}\cdots\cdots$$

$$\cdots\cdots\frac{d^3p_N}{(2\pi)^3 2\omega_{p_N}}(2\pi)^4\delta^4(\bar{p}_1+\bar{p}_2-\bar{p}_3-\bar{p}_4\cdots-\bar{p}_N)S_o, \tag{B.30}$$

where $p_i = \bar{p}_i$, i = 1, 2, 3 and $\bar{p}_0 = \omega_p = (\mathbf{p}^2+m^2)^{1/2}$. If the initial particles are unpolarized, one takes the average over initial spin states. When there is no external fermion in a process, then $[\Pi_{ext\ fer}(2m_{fer})]$ in (B.30) is replaced by 1.

Bc. Some QED Results in Both CLA and Inertial Frames

Let us consider some well-known physical processes in QED[4] to illustrate the generalized Feynman rules in both CLA frames and inertial frames, and see how the conventional results in inertial frames are modified if they are measured in a laboratory with constant-linear-acceleration.

A. Electron Scattering from a Point-Like Proton

According to the generalized Feynman rules, the amplitude M_{fi} for such an electron scattering from a point-like proton, $e(p_i + p(P_i) \to e(p_f) + p(P_f)$ with the exchange of a photon $\gamma(q)$, where $q_\mu = p_{f\mu} - p_{i\mu}$, is given by

$$M_{fi} = \bar{u}(s_f,p_f)[-i\ \bar{e}\Gamma^\mu]u(s_i,p_i)\left[\frac{-ig_{\mu\nu}}{(q_\sigma q^\sigma+i\varepsilon)}\right]\bar{u}(S_f,P_f)[-i\ \bar{e}\Gamma^\mu]u(S_i,P_i)\ , \tag{B.31}$$

where we have used the Feynman gauge for the photon propagator ($\rho = 1$). The S-matrix element S_{fi} in (B.19) takes the form

$$S_{fi} = -i(2\pi)^4\delta^4(\bar{P}_f + \bar{p}_f - \bar{P}_i - \bar{p}_i)\left[\frac{m}{\omega_{pf}}\frac{m}{\omega_{pi}}\frac{M}{\omega_{Pf}}\frac{M}{\omega_{Pi}}\frac{1}{V^4}\right]^{1/2} M_{fi}\ , \tag{B.32}$$

where m and M are, respectively, masses for the electron and the proton. The differential cross section is given by (B.30),

$$d\sigma = \frac{mM}{[(\bar{p}_i\cdot\bar{P}_i)^2-(mM)^2]^{1/2}}|M_{fi}|^2\frac{md^3\bar{p}_f}{(2\pi)^3\omega_{pf}}\frac{Md^3\bar{P}_f}{(2\pi)^3\omega_{Pf}}$$

$$\times(2\pi)^4\delta^4(\bar{p}_f + \bar{P}_f - \bar{p}_i - \bar{P}_i)\ , \tag{B.33}$$

where $\bar{p}_0 = \omega_{pf} = (\mathbf{p}_f^2+m^2)^{1/2}$, $\omega_{Pf} = (\mathbf{P}_f^2+M^2)^{1/2}$ and $|M_{fi}|^2$ is given by

$$|M_{fi}|^2 = \frac{\bar{e}^4}{2m^2M^2\bar{q}^4}\,[\bar{P}_f\cdot\bar{p}_f\bar{P}_i\cdot\bar{p}_i + \bar{P}_f\cdot\bar{p}_i\bar{P}_i\cdot\bar{p}_f - m^2\bar{P}_f\cdot\bar{P}_i$$

$$- M^2\bar{p}_f\cdot\bar{p}_i + 2M^2m^2]\,, \tag{B.34}$$

Since $\bar{P}_f\cdot\bar{p}_f = \bar{P}_{f\mu}\bar{p}_f^{\mu}$, etc. we see that the differential cross section is formally the same as the usual one in an inertial frame, except that each individual momentum is not constant in CLA frames,

$$d\sigma(\text{CLA frame}) = d\sigma(\text{inertial frame})\,. \tag{B.35}$$

The origin of this identical result is the limiting 4–dimensional symmetry which dictates the invariance of the action or the S–matrix.

After integration, (B.35) gives the total cross section which can be pictured as the effective size of the target particle, i.e., proton. This effective size of the proton depends of the strength of the interaction. For the electromagnetic interaction, the coupling strength is $\alpha_e \sim 1/137 \sim 10^{-2}$, the weak interaction coupling strength is about 10^{-12}. The size (or cross section) of a proton is about 10^{-24} cm^2 from the 'viewpoint' of the electron. But from the 'viewpoint' of a neutrino, which has only weak interactions with the proton, the size of a proton is extremely small, about 10^{-44} cm^2.

B. Compton Scattering

The S–matrix element for the Compton scattering process, $\gamma(k) + e(p_i) \to \gamma(k') + e(p_f)$, is given by

$$S_{fi} = -i(2\pi)^4\delta^4(\bar{k}' + \bar{p}_f - \bar{k} - \bar{p}_i)\left[\frac{m}{\omega_{pf}}\frac{m}{\omega_{pi}}\frac{1}{2\omega_k}\frac{1}{2\omega_{k'}}\frac{1}{V^4}\right]^{1/2} M_{fi}\,. \tag{B.36}$$

where $\omega_k = |\bar{\mathbf{k}}| = |\mathbf{k}|$, $\omega_{pf} = \sqrt{\bar{p}_f^2 + m^2}$ and the *M*–matrix element is given by

$$M_{fi} = \bar{u}(s_f,p_f)\Big\{[-i\bar{e}\Gamma^{\alpha}\varepsilon'_{\alpha}]\left[\frac{-i}{\gamma^{\mu}(\bar{p}_{i\mu}+\bar{k}_{\mu})-m+i\varepsilon}\right][-i\bar{e}\Gamma^{\nu}\varepsilon_{\nu}]$$

$$+ [-i\bar{e}\Gamma^{\alpha}\varepsilon_{\alpha}]\left[\frac{-i}{\gamma^{\mu}(\bar{p}_{i\mu}-\bar{k}'_{\mu})-m+i\varepsilon}\right][-i\bar{e}\Gamma^{\nu}\varepsilon'_{\nu}]\Big\}u(s_i,p_i)\,, \tag{B.37}$$

according to the generalized Feynman rules.

We have seen that the differential cross section for the Compton scattering is also the same as that in an inertial frame,

$$d\sigma_{Compton}(\text{CLA frame}) = d\sigma_{Compton}(\text{inertial frame}) . \tag{B.38}$$

C. Self-Mass of the Electron

The self-mass of the electron is given by the expression

$$\delta m = \int \frac{d^4k}{W(2\pi)^4} \frac{-ig_{\mu\nu}}{[k_\sigma k^\sigma + i\varepsilon]} [-i\bar{e}\Gamma^\mu] \frac{-i}{[\Gamma^\rho(p_\rho - k_\rho) - m + i\varepsilon]} [-i\,\bar{e}\Gamma^\nu]$$

$$= \delta m(\text{inertial frame}) , \tag{B.39}$$

where we have set $\rho=1$ for simplicity and used (24.43) and (B.15). This is consistent with the fact that the (rest) mass of the electron in an inertial frame is the same as that of the electron in CLA frames, as shown in (23.8).

In all these calculations, the particles must move with constant velocities as measured in an inertial frame. This is the condition imposed in defining the S-matrix and for obtaining the generalized Feynman rules in both linearly accelerated and inertial frames. These discussons for QED can also be applied to non-inertial frames with a constant rotational motion, since we have the taiji rotational transformations (25.11) with limiting 4-dimensional symmetry.

References

1. See, for example, J. J. Sakurai, *Advanced Quantum Mechanics* (Addison–Wesley, Reading, MA, 1967), pp. 171–172 and pp. 181–188; S. Weinberg, *The Quantum Theory of Fields,* vol. I, Foundations (Cambridge University Press, New York, N.Y., 1995), pp. 134–147.
2. I. M. Gel'fand and G.E. Shilov, *Generalized Functions* Vol. 1 (transl. by E. Saletan, Academic Press, New York, 1964), p. 185.
3. As usual, one can use the interaction terms in iL_{SQED} and replace the field operators by appropriate 'free particle' wave functions. Omit $\exp(\pm iP(x))$ and the factors for external lines. The remainder is the vertex factor. The S-matrix expansion can be formulated by using the Lagrangian. See N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (transl. by G. M. Volkoff, Interscience Publishers, New York, 1959), pp. 206-226.
4. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), chapter 7.

Appendix C.

De Sitter and Poincaré Gauge-Invariant Fermion Lagrangians and Gravity*

We present a new fermion lagrangian which possesses exact symmetry under the local de Sitter group. The lagrangian involves new "scale gauge fields" related to the newtonian force and the usual Yang-Mills "phase gauge fields" related to a new "gravitational spin force" between two fermions. Generalization of the usual gauge theory for *external* symmetry groups is also discussed.

It has been suggested that gravity is related to gauge fields of four-dimensional symmetry such as the de Sitter group [1,2]. The idea is quite interesting because the de Sitter group possesses the maximum four-dimensional symmetry [3] and is the unique generalization of the Poincaré group. It also suggests the existence of a new "gravitational spin force" between objects with nonzero net spin densities. The de Sitter group is a rotational group in de Sitter space, which is the hypersurface of a four-dimensional sphere of a hyperbolic character in one direction, embedded in a five-dimensional space. The radius of the sphere is denoted by L. The de Sitter group reduces to the Poincaré group in the flat space limit $L \to \infty$.

One important ingredient in a realistic gauge theory of gravity is the fermion field — a source of the gravitational field. But in previous discussions [1,2] one either ignored the fermion field or discussed a fermion lagrangian which has only approximate symmetry under local de Sitter gauge transformations. It appears that one cannot get a fermion lagrangian with exact external gauge symmetry if one just employs the usual Yang-Mills fields, i.e. "phase gauge fields" [4].

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In this paper, we present a new fermion lagrangian, which has exact symmetry under the local de Sitter group. It is necessary that the lagrangian involves new "scale gauge fields" in addition to the usual Yang-Mills "phase gauge fields". They have different transformation property and, therefore, must be treated as different and independent fields.

Let us consider the generalization of $\gamma^\mu \partial_\mu \psi$ in the form for a non-abelian external symmetry group:

$$\Gamma^\mu D_\mu \psi \tag{C.1}$$

where Γ^μ involves both the Dirac matrices and scale gauge fields e^μ_A and the gauge-covariant derivative D_μ contains phase gauge fields $b^A_\mu = (b^i_\mu, b^a_\mu)$:

$$\Gamma^\mu \equiv e^\mu_A \gamma^A = e^\mu_k \gamma^k + e^\mu_{jk}\, i(\gamma^j\gamma^k - \gamma^k\gamma^j)/4L \equiv E^\mu_A Z^A, \tag{C.2}$$

$$D_\mu \equiv \partial_\mu - igb_\mu, \qquad b_\mu \equiv b^A_\mu Z_A, \tag{C.3}$$

$$Z_A = (Z_i, Z_a) = (\gamma_i/2L, i(\gamma_j\gamma_k - \gamma_k\gamma_j)/4), \qquad a \equiv jk,$$

$$\{\gamma_j, \gamma_k\} = 2\eta_{jk}, \qquad \eta_{jk} = (1, -1, -1, -1),$$

$$E^\mu_A = (2Le^\mu_i, e^\mu_{jk}/L).$$

Tile quantity Z_A is the matrix representation of the SO(3,2) de Sitter group generators:

$$[Z_B, Z_C] = if^A_{BC} Z_A, \qquad A = i, jk; \quad \text{etc.} \tag{C.4}$$

The local de Sitter gauge transformations are given by

$$\psi \to \psi' = \xi_d \psi,$$

$$b_\mu \to b'_\mu \equiv b_\mu + (\partial_\mu \xi_d)\, \xi_d^{-1}/(ig), \tag{C.5}$$

$$\Gamma^\mu \to \Gamma'^\mu = \xi_d \Gamma^\mu \xi_d^{-1},$$

where

$$\xi_d = \exp[i\omega^A(x)Z_A] .$$

The gauge functions $\omega^A(x)=(\omega^i(x), \omega^a(x))$ are real and arbitrary.

It can be shown that $\bar{\psi}(\Gamma^\mu D\mu+m)\psi$ is invariant under the local de Sitter gauge transformations (C.5):

$$\bar{\psi}'(\Gamma'^\mu D'_\mu+m)\psi' = \bar{\psi}(\Gamma^\mu D_\mu+m)\psi . \qquad \text{(C.6)}$$

We stress that this symmetry property holds if and only if both e^μ_i and e^μ_{jk} are introduced.

Note that the field e^μ_A is dimensionless and is related to a change in the scale rather than a change in the phase [4], so that e^μ_A may be termed a "scale gauge field". In view of the presence of this new scale gauge field, the present gauge theory is a generalization of the Yang-Mills theory. Such a generalization appears to be necessary because the de Sitter group is an external symmetry group, in which the generator Z_A does not commute with γ_k, in contrast to the case of an internal symmetry group.

The phase field strength $F^A_{\mu\nu}$ is given by

$$(D_\mu D_\nu - D_\nu D_\mu)\psi = igF^A_{\mu\nu}Z_A \psi , \qquad \text{(C.7)}$$

i.e.,

$$F^A_{\mu\nu} = \partial_\mu b^A_\nu - \partial_\nu b^A_\mu + gif^A_{BC} b^B_\mu b^C_\nu . \qquad \text{(C.8)}$$

One can verify that $F_{\mu\nu} \equiv F^A_{\mu\nu}Z_A$ is gauge covariant and transforms as follows:

$$F_{\mu\nu} \to F'_{\mu\nu} = \xi_d F_{\mu\nu} \xi_d^{-1} . \qquad \text{(C.9)}$$

Thus, $Tr(F_{\mu\nu}F_{\alpha\beta})$ is a gauge-invariant quantity:

$$Tr(F'_{\mu\nu}F'_{\alpha\beta}) = Tr(F_{\mu\nu}F_{\alpha\beta}) , \qquad \text{(C.10)}$$

which is usually used as the lagrangian for the phase field b^A_ν.

We observe that $Tr(\Gamma^\mu \Gamma^\nu)$ is gauge invariant:

$$Tr(\Gamma'^\mu \Gamma'^\nu) = Tr(\Gamma^\mu \Gamma^\nu) \, . \tag{C.11}$$

Thus, we expect that e^μ_A enters the lagrangian for the scale field through the combination

$$\begin{aligned} \bar{g}^{\mu\nu} &\equiv Tr(\Gamma^\mu \Gamma^\nu /4) \\ &= \eta^{ik} e^\mu_i \, e^\nu_k + e^\mu_{ij} \, e^\nu_{kn} \eta^{ik} \eta^{jn}/(2L) \\ &\to g^{\mu\nu} \quad \text{as} \quad L \to \infty . \end{aligned} \tag{C.12}$$

$$g^{\mu\nu} \equiv \eta^{ik} e^\mu_i e^\nu_k \, .$$

For large L, $\bar{g}^{\mu\nu}$ is approximately the same as $g^{\mu\nu}$. In the limit $L \to \infty$, it is natural to interpret e^μ_i as the vierbein component and $g^{\mu\nu}$ as the spacetime metric. Thus we can interpret $\bar{g}^{\mu\nu}$ as the spacetime metric in the present theory with the de Sitter gauge group. We are able to define the affine connection $\bar{\Gamma}^\alpha_{\mu\nu}$ and the Riemann curvature tensor $\bar{R}^\alpha{}_{\lambda\mu\nu}$ in terms of the new gauge-invariant metric $\bar{g}^{\mu\nu}$ by the usual relations:

$$\bar{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2} \, \bar{g}^{\lambda\alpha}(\partial_\nu \bar{g}_{\mu\lambda} + \partial_\mu \bar{g}_{\lambda\nu} - \partial_\lambda \bar{g}_{\nu\mu}) \, , \qquad \bar{g}^{\mu\lambda} \, \bar{g}_{\lambda\nu} = \delta^\mu{}_\nu \, ,$$

$$\bar{R}^\alpha{}_{\lambda\mu\nu} = \partial_\nu \bar{\Gamma}^\alpha_{\lambda\mu} - \partial_\mu \bar{\Gamma}^\alpha_{\lambda\nu} + \bar{\Gamma}^\beta_{\lambda\mu} \, \bar{\Gamma}^\alpha_{\beta\nu} - \bar{\Gamma}^\beta_{\lambda\nu} \, \bar{\Gamma}^\alpha_{\beta\mu} \, . \tag{C.13}$$

In this way, $\bar{\Gamma}^\alpha_{\mu\nu}$, $\bar{R}^\alpha{}_{\lambda\mu\nu}$ and $\bar{g}^{\mu\lambda}$ are all invariant under the local de Sitter gauge transformations (C.5). The invariant lagrangian for these fields is uniquely determined by the principle of gauge invariance and the principle of general covariance:

$$\int d^4x (\det \bar{g}_{\mu\nu})^{1/2} \Big[\tfrac{1}{2}(i\bar{\psi}\Gamma^\mu D_\mu \psi + \text{h.c.}) - \bar{\psi} m \psi - \tfrac{1}{8} Tr(F_{\mu\nu} F^{\mu\nu}) + \frac{1}{8\pi G}\bar{R}\Big] \, , \tag{C.14}$$

where G is a constant and $\bar{R} = \bar{R}^{\alpha}{}_{\mu\nu\alpha}\bar{g}^{\mu\nu}$. Field equations for e_i^μ, b_μ^A and ψ can be determined from the lagrangian (C.14).

When we take the limit $L \to \infty$, some components of gauge fields, i.e. b_μ^i and e_{ij}^μ disappear from the theory and we obtain the following lagrangian *L*:

$$L = (\det \bar{g}_{\mu\nu})^{1/2}\left[L_\psi - \frac{1}{8}\mathrm{Tr}(F_{\mu\nu}^a Z_a F_{\alpha\beta}^b Z_b)g^{\mu\alpha}g^{\nu\beta} + \frac{1}{8\pi G}R\right] , \qquad (C.15)$$

$$R = R^{\alpha}{}_{\mu\nu\alpha}g^{\mu\nu} = (\partial_\alpha \Gamma_{\mu\nu}^\alpha - \partial_\nu \Gamma_{\mu\alpha}^\alpha + \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\alpha}^\alpha - \Gamma_{\mu\alpha}^\lambda \Gamma_{\lambda\nu}^\alpha)g^{\mu\nu} ,$$

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\lambda\alpha}(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\nu\mu}) ,$$

$$g^{\mu\nu} = e_i^\mu e_k^\nu \eta^{ik}, \; g_{\mu\nu} = e_\mu^i e_\nu^k \eta_{ik} ,$$

$$e_\mu^i e_i^\nu = \delta^\nu{}_\mu , \qquad e_\mu^i e_k^\mu = \delta^i{}_k ,$$

$$\mathrm{Tr}(F_{\mu\nu}^a Z_a F_{\alpha\beta}^b Z_b) = 2F_{\mu\nu}^{ik} F_{\alpha\beta}^{mn} \eta_{im}\eta_{kn} , \qquad a=ik , \; b=nm ,$$

$$L_\psi = \frac{1}{2} i\bar{\psi}\Gamma^\mu (\partial_\mu - igb_\mu)\psi + \frac{1}{2} i\bar{\psi}(\overleftarrow{\partial}_\mu - igb_\mu)\Gamma^\mu \psi - \bar{\psi}m\psi ,$$

where the last term in (C.15) is identical with Einstein's lagrangian. The lagrangian (C.15) is invariant under local Poincaré gauge transformations (i.e. the transformations (C.5) with $L \to \infty$) and general coordinate transformations (with the spacetime metric tensor $g^{\mu\alpha}$). The Poincaré gauge-invariant lagrangian (C.15) differs from that discussed by Kibble [5]. From the viewpoint of symmetry, the lagrangian (C.15) is more satisfactory than Kibble's lagrangian involving fermions.

Physically, the radius L of the de Sitter space is probably very large. In this case, physical effects of b_μ^i and e_{ij}^μ are negligible. Experimentally, the difference between the de Sitter gauge-invariant lagrangian (C.14) and the Poincaré gauge-invariant lagrangian (C.15) cannot be distinguished in the near future. So the *important fields* are b_μ^a and e_i^μ, which are generated by the spin density and mass density, respectively. This can be seen from the field equations derived from (C.15). For example, the source of b_μ^a in the approximation of static and weak fields is

$$b_{\mu}^{ik} = g\varepsilon^{ikj}\bar{U}(r)\sigma_j U(r)\,, \qquad \mu = 0\,,$$

$$= 0\,, \qquad \mu \neq 0\,, \qquad \text{(C.16)}$$

where U(r) and σ_j are respectively the positive-energy Pauli spinor and the Pauli matrices. Of course, the result (C.16) can also be derived from the de Sitter gauge-invariant lagrangian (C.14) with the approximation of very large L.

These gauge fields are interpreted as follows: *Gravity is related to scale gauge fields rather than the usual Yang-Mills gauge fields* because the scale gauge field is generated by the mass density, according to gauge-invariant lagrangians. The massless Yang-Mills field b_μ is generated by the spin density of fermions and corresponds to a new long-range force between two bodies with nonzero spin densities. The strength of this new force is determined by a new dimensionless coupling constant g^2, which is independent of the newtonian gravitational constant G. These hold for both finite G. These hold for both finite L (i.e. the de Sitter group) and infinite L (i.e. the Poincaré group).

Our interpretation of gauge fields for external symmetry groups differs from previous interpretations by Kibble and others [5-7] (see also refs. [2,8]). We may remark that a "contact spin force" between fermions has been discussed by Kibble based on a non-gauge-invariant fermion lagrangian. Since the external four-dimensional symmetry group is a fundamental symmetry of nature, the prediction of the new long-range gravitational spin force should be taken seriously.

We conclude that gauge field theory based on external four-dimensional symmetry groups dictates the presence of a new "scale gauge field", which differs from the Yang-Mills "phase gauge field" [9]. Furthermore, the theory predicts a new long-range gravitational spin force between fermions. It appears that the quantization of these fields cannot be accomplished by a straightforward application of the usual quantization procedure for Yang-Mills fields. This needs further investigation.

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Appendix D.

The Relativity of Lifetime Dilatation and an Experimental Test of "Twin Particles" Involving Linear Accelerations

Da. Three Relations of t' and t ($\Delta t'=\gamma\Delta t$, $\Delta t'=\Delta t/\gamma$, $\Delta t'_{20}=\Delta t_{10}$) for "Twin Particles" Under Different Conditions of Measurements in Special Relativity

Let us consider the *relativity* of the lifetime dilatation in special relativity (or, equivalently, that of decay-length dilatation in taiji relativity) and experimental tests of "twin particles" involving linear accelerations. The discussion can also illustrate some interesting and puzzling aspects of problems related to the so-called "clock paradox" or "twin paradox."[1,2] Some physicists[3] insist that the effect of acceleration on the twin is very important and must be taken into account, in sharp contrast to the conventional interpretation, which will be discussed below. In view of different and incompatible views in the literature, it is highly desirable that the matter can be resolved by a direct and unambiguous experimental test with linear accelerators. This can be done in the near future.

When one talks about the lifetime of unstable particles such as pions, it is understood that one is talking about the *mean lifetime* which is measured by observing the decays of many pions. The basic reason for this is that the decay of a single unstable pion is dictated by quantum-mechanical laws of probability and does not have a single fixed value of lifetime for all pions. Nevertheless, *the physical time should have the same property as the mean lifetime of unstable particles.*

In order to observe relativistic motions and effects, clocks and twins must be accelerated to speeds comparable to that of light, but since they are macroscopic objects, the task is difficult. However, it is useful to note that as far as "twins" are concerned, no pair of human identical twins are more twin-like than two identical unstable particles. At the present time, the decay lifetime dilatation of unstable particles in flight has been experimentally established beyond a reasonable doubt. Furthermore, *if the acceleration is neglected in numerical calculations*, then the "twin-particle paradox" can be treated and calculated completely within special relativity.

Within the conceptual framework of special relativity, some people have used the experimental results of the lifetime dilatation of unstable particles to support the conventional interpretation of the twin paradox.[4] This goes as follows: The traveler twin's rocket lifts off and reaches a constant velocity V in a negligibly short time. After traveling for a very long distance L_o as measured by the stay-at-home twin on the earth, the rocket reverses its velocity, comes back to the earth and stops. Reversal and stopping again occur in a negligibly short time. The stay-at-home twin records an elapsed time $T_o=2L_o/V$. However, the traveler twin will have recorded an elapsed time of $2T_o\sqrt{1-V^2/c^2}$, and will be younger than his stay-at-home twin.[5] This result agrees with the experimental evidence for the lifetime dilatation of particles decaying in flight.[4]

However, this argument is not completely satisfactory because the relation for the lifetime dilatation involving *constant linear velocity* is completely relative and is symmetric (or reciprocal) between the twins or any two inertial frames. Thus, it cannot be used to conclude that the stay-at-home twin is *absolutely* younger than his traveler twin within the framework of special relativity.

To see the flaw in the preceding line of reasoning, let us consider the "twin-particle paradox" in detail within the framework of special relativity. Suppose an unstable pion π_1 is at rest in an inertial frame F and another pion π_2 is at rest in a second inertial frame F' which moves relative to F with a constant velocity V along the +x direction. Let us consider the pion π_1. Its mean lifetime is Δt_{10} as measured by observers in F and $\Delta t_1'$ as measured by observers in F'. These two time intervals are related by

$$\Delta t_1' = \frac{\Delta t_{10}}{\sqrt{1-V^2/c^2}}, \qquad \Delta x=0, \qquad (D.1)$$

because π_1 is at rest in F (i.e. $\Delta x=0$ in the Lorentz transformations (5.7)) and Δt_{10} is the "proper lifetime." Similarly, for the second pion π_2, its lifetimes as measured by observers in F and F' (i.e. Δt_2 and $\Delta t_{20}'$) satisfy the relation

$$\Delta t_2 = \frac{\Delta t_{20}'}{\sqrt{1-V^2/c^2}}, \qquad \Delta x'=0, \qquad (D.2)$$

because π_2 is at rest in F' (i.e. $\Delta x'=0$). It should be stressed that the result (D.1) [or (D.2)] is a relationship between the lifetime of a single pion as measured by two different observers and thus is not yet related to the experimental result, in which a single observer compares the lifetimes of two different pions π_1 and π_2, one at rest and one in motion. For example, if F is the laboratory frame, experiments show that

$$\Delta t_2 = \frac{\Delta t_{10}}{\sqrt{1-V^2/c^2}}. \tag{D.3}$$

i.e., the lifetime of π_2 decaying in flight is longer than that of π_1, which is at rest in the laboratory frame F. It follows from relations (D.1)–(D.3) that

$$\Delta t_{10} = \Delta t'_{20}\,, \tag{D.4}$$

and

$$\Delta t'_1 = \Delta t_2\,. \tag{D.5}$$

Result (D.4) implies that the lifetime Δt_{10} of π_1 at rest in F as measured by observers at rest in F is the same as the lifetime $\Delta t'_{20}$ of π_2 at rest in F' as measured by observers at rest in F'. The physical reason for the equality in (D.4) is exactly the same as that for the equality of meter sticks in equation (5.12). This is completely in harmony with the equivalence of the two inertial frames F and F' in special relativity. *The two time intervals,* Δt_{10} *and* $\Delta t'_{20}$, *in (D.4) are not related by the Lorentz transformations.*

The results in (D.1), (D.2), (D.4) and (D.5) bring out the most puzzling aspect of relativity theory.

Logically, (D.4) is *directly* implied by the first principle of relativity. One should say that the relation (D.4) together with the relation (D.2) derived from the Lorentz transformations, leads to the experimental prediction (D.3). Now suppose the decay process, the clock ticking, and the aging process are the same. Then it could be argued that the result (D.4) suggests that the aging of the twin brothers in F(earth) and F'(rocket ship) are the same, when they are brought back together both have the same age, *provided that the time intervals of accelerations are negligible and the ages are measured according to the conditions related in (D.4).* For example, the twins may express their ages in terms of the mean lifetime of the particles decaying at rest relative to them. This appears to be the qualitative argument of Dingle.[6] It must be stressed that the equality in the relation (D.4), i.e., the twins have the "same age," cannot be observed by a single observer, until they are brought back together with negligible effects due to acceleration.

Clearly, the results (D.1) and (D.2) are just another way of saying that two observers in different frames comparing two time intervals will arrive at different conclusions depending on how the intervals are measured. This is one of the most basic traits of relativity theory. One cannot use the lifetime dilatation experiment, which gives a relationship between measurements made by a single observer, to rule out result (D.4) because *it refers to measurements made by two different observers.* Similarly, to design an experiment to rule out (D.4) and confirm (D.2) (or vice versa) is impossible. If one reflects for a moment, one can see that both (D.2) and (D.4) are correct for different conditions of measurement within the conceptual framework of special relativity. This is the so-called "paradox"—the heart of the problem which is the source of a long controversy.[6] So far, all experiments support the first postulate of relativity that two inertial frames F and F' are equivalent and symmetrical as long as their relative velocity is constant. However, in the final analysis, it can be asserted unequivocally that, logically, there is absolutely no paradox in relativity theory.

As a result of this analysis, it appears reasonable to conclude that (i) the relativity (or the reciprocal relation of the two particles' lifetimes) can only be broken by taking into account the acceleration[7] of one of the particles and (ii) the numerical difference between the two lifetimes must be determined by taking the effects of linear accelerations into account. All physicists appear to agree with the conclusion (i), but not (ii).[8] Thus, a direct experimental test of these different views is warranted.

Db. A Direct Experiment on the interpretations of the "Twin Paradox" by Using Twin Particles

Since special relativity has been tested by hundreds of experiments, one might think that there is no point in doing one more experiment to test it. However, this experiment involving constant-linear-acceleration is necessary. The reason is that it really does not test special relativity. Rather, it tests various interpretations of the "twin paradox." Furthermore, it tests the transformations for linearly accelerated frames and gives clues to the understanding of physics in non-inertial frames, as discussed in chapter 23.

An idealized experimental setup for testing the "paradox" of the twins by using identical particles is as follows:

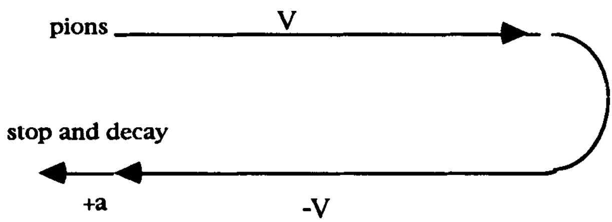

Fig. D.1 An idealized experimental arrangement to illustrate the test of the "twin paradox"

To be specific for comparisons, let us first consider an idealized trip (as shown in Fig. D.1). We have two twin brothers, T_1 and T_2, who were born at the same time in the earth laboratory and have the same life expectancy. (The following discussions hold also for two identical clocks with the same 'life expectancy.') Let us omit the initial acceleration and suppose T_1 (travelling twin) moves with a constant velocity V over a very long distance L_0. He then turns 180^o and travels with the same speed V over the same distance. Then twin T_1 decelerates and stops within a certain time interval. There are two ways to check whether he is absolutely younger: One is to measure his age when he returns and one is to measure his remaining lifetime, and to compare with his stay-at-home brother T_2.

Now suppose one replaces the two twin brothers by two "identical groups" of identical pions in an inertial laboratory. One has the following clear analogous experimental situation:

A bunch of identical pions are created in a high-energy laboratory. They are separated into two groups, denoted by $G(\pi_1)$ and $G(\pi_2)$, and both groups move with a constant velocity V. The first group $G(\pi_1)$ (representing the travelling twin) moves with a constant velocity V over a suitable distance L_0, as measured from the earth laboratory frame F. Then it undergoes an acceleration which reverses its velocity, so that it returns with a constant velocity $V' = -V$. After it travels a distance L_0, it is decelerated (by a field) within a certain time to zero velocity. When the group $G(\pi_1)$ stops, one counts the number of pions left in the group and deduces the mean lifetime $\tau(\pi_1)$.

On the other hand, the second group $G(\pi_2)$ is allowed to move with the constant velocity without any acceleration or deceleration. One can measure its mean lifetime decaying in flight with the velocity V. As discussed in section 11b, after the effect due to motion is taken into account, the result can lead to the "rest lifetime" $\tau_0(\pi_2)$ which is the same as the lifetime of the pions produced at rest in the laboratory. Therefore, $\tau_0(\pi_2)$ corresponds to the lifetime of the stay-at-home twin because these pions in the second group $G(\pi_2)$ are neither accelerated nor decelerated. With the help of $\tau_0(\pi_2)$, one can calculate the number of pions left in the $G(\pi_2)$, if it is produced at rest, at the time when the group $G(\pi_1)$ stops.

Note that, in this experiment, one compares the lifetimes of both groups, $G(\pi_1)$ and $G(\pi_2)$, when they are at rest in the earth laboratory, after the traveler group of pions has returned. This measurement is free from the reciprocal relations of the lifetime dilatation when they have constant relative motion.

One can perform experiments with different distance L_o and/or the acceleration. The result can test the conventional relation

$$\tau(\pi_1) - \tau_o(\pi_2) = \frac{2L_o}{V}(1 - \sqrt{1-V^2/c^2}) > 0 . \tag{D.6}$$

provided twin's lifetime (or clocks' time) and particles' mean lifetime have the same physical property.[9]

In order to test the "twin paradox", one should choose particles with suitable lifetimes, vary the distance L_o of the particle moving with a constant velocity and L_o should be sufficiently large, so that the difference in (D.6) can be detected. In view of these considerations, the muon with a longer lifetime ($c\tau \approx 6.6\times10^4$ cm) is more suitable than the pion ($c\tau \approx 780$ cm) for such an experiment.

Actually, it is not necessary for the angle between **V** and **V'** to be 180°. All that is needed is for the traveler group to have the experience of acceleration. This property could simplify the experiment. A much more simplified version of the "twin paradox" experiment is to do just half of the round-trip as follows:

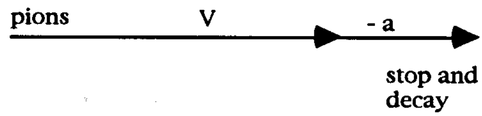

Fig. D.2 An idealized half-trip test of the "twin paradox" by using twin particles

The traveler twin can be represented the pions in Fig. D.2 because they travel a certain distance L_o and are decelerated by a field to stop and decay. In this case their age difference will be just half of that in (D.6):

$$\tau(\pi_1) - \tau_o(\pi_2) = \frac{L_o}{V}\left(1 - \sqrt{1-V^2/c^2}\right) > 0 . \qquad \text{(D.7)}$$

This is the simplest experiment to test the "twin paradox."[10]

The effect on lifetime due to the acceleration "a" of charged particles in a potential field should be investigated. The experimental results can also test another two views:

$$\text{"Naive view":} \quad \tau(\pi_1) - \tau_o(\pi_2) = 0, \qquad \text{(D.8)}$$

$$\text{"Noninertial view":} \quad \tau(\pi_1) - \tau_o(\pi_2) = f(L_o,V,a). \qquad \text{(D.9)}$$

The function $f(L_o,V,a)$ can be calculated if one has accelerated transformations, as discussed in chapter 23.

It must be stressed that this type of experiment tests only various interpretations of the theory of special relativity regarding the "twin paradox" or the "clock paradox," but not the theory itself. In other words, even if, say, (D.9) is confirmed by future experiments, this does not invalidate Einstein's theory of special relativity because it involves assumptions related to accelerations of reference frames.

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3. For example, Einstein (in 1916), Møller and Ta-You Wu, see ref. 8 below.
4. The time dilatation experiments have been carried out by using unstable particles in linear and in circular motions (e.g., muon experiment in a storage ring). Some authors use experimental results involving circular motion to support the conventional interpretation of the "twin paradox". This is not satisfactory because there is clearly no relativity (and, hence, no paradox) between a twin at rest and a twin in circular motion. (The reasonis that a frame in circular motion involves acceleration and cannot be described by special relativity, as discussed in chapter 25.) See, for example, D. Halliday and R. Resnick, *Fundamentals of Physics* (third edition, John Wiley, New York, 1988) pp. 960-961. Some physicists used the time dilatation in constant linear motion to support the conventional interpretation. See, for example, F. S. Crawford, Jr., Nature, 179, 35 (1957).
5. The crucial asymmetry of the twin's ages is usually explained as follows: The traveler twin feels an acceleration, whereas his twin on the earth does not. Logically, since the asymmetry of the twin is solely caused by the acceleration, one must calculate explicitly the asymmetry in their age due to accelerations. Most people did not do this, except Møller, Wu, and Lee, who derived a transformation for linearly accelerated frame and did the calculations (ref. 8).
6. There was a raging controversy concerning the "twin paradox" around 1958: Some people, e.g., H. Dingle, believed that the travelling twin will have the same age as his stay-at-home twin after he returns to the earth. (See Selected Reprints by American Institute of Physics in ref. 2.) If one takes the "twin-particle paradox" as an example, one could argue that the two particles have the same mean life (i.e., the relation in eq. (D.4)) when measured by two different observers at rest relative to each particle. However, this equality (D.4) cannot be shown by using the Lorentz

transformation. Rather, it follows directly from the first postulate of special relativity, namely, that all inertial frames are equivalent. In this sense, the result (D.4) is itself a basic postulate of special relativity and has been confirmed indirectly by many previous experiments which support relativity.

In this connection, we note that the "twin-particle paradox" differs from the "twin paradox" [or "clock paradox"] in two aspects: (i) There is no acceleration which reverses linear velocity involved in the measurement that was done in all previous experiment of lifetime dilatation involving LINEAR MOTION. (ii) the experimental asymmetry of their lifetimes [or the asymmetry of clocks' times in the Lorentz transformation] is completely due to the asymmetric conditions of measurements (i.e., $\Delta x=0$ or $\Delta x'=0$) and has nothing to do with the acceleration of particles or clocks for a certain period of time, as discussed in section 11b. If travelling twin's acceleration which reverses his velocity and the condition $\Delta x'=0$ can be shown to be equivalent, then the controversy would be clarified.

7. One might think that the previous experience of acceleration of one inertial frame F could break the symmetry between two inertial frames F and F'. If this were the case, then we would expect that the spectrum of a hydrogen atom (which has previously been accelerated by a_1) until at rest in a laboratory will be different from that of another hydrogen atom (which has a different previous acceleration $a_2 \neq a_1$) until at rest in the laboratory. But so far there seems to be no such difference being detected in any laboratory.
8. For a discussion of the "clock paradox" with the effect of acceleration taken into account (beyond the framework of special relativity) and calculated exactly on the basis of Møller's accelerated transformation, see Ta-You Wu and Y. C. Lee, Intern. J. Theor. Phys. 5, 307 (1972); Ta-You Wu, *Theoretical Physics*, vol. 4, *Theory of Relativity* (in Chinese, Lian Jing Publishing Co., Taipei, 1978) pp. 172–175; C. Møller, Dan. Mat. Fys. Medd. 20, No. 19 (1943), and *The Theory of Relativity* (Oxford, London, 1969) pp. 258–263. The correctness of the Wu-Lee's precise result can be tested experimentally by measuring a particle's mean decay lifetime under constant acceleration or deceleration. It is interesting to note that in his 1905 paper, Einstein first mentioned the following thought experiment: Suppose two synchronized clocks A and B are initially at the same position. Suppose clock A leaves and

moves along a closed path and returns. When clock A stops at the original position, it falls behind relative to clock B. However, Einstein changed his mind in 1916: He said that the logic of special relativity does not suffice for the explanation of this phenomenon since non-inertial frames are involved. See, for example, A. Pais, *Subtle is the Lord..., The Science and the Life of Albert Einstein* (Oxford Univ. Press, Oxford, 1982), p. 145. Today, at the dawn of the 21st century, most physicists believe the effects of accelerations can be neglected in the *quantitative* calculations in the "twin paradox" problem and follow Einstein's conclusion in 1905 rather than that in 1916. Such a difference can only be settled by experiments.

9. For the twins, one can measure their remaining lifetime τ(R, traveler) after the traveler twin returns. However, this cannot be done with the twin particles. When the traveler group $G(\pi_1)$ returns, one can only measure their mean lifetime, according to quantum mechanics and field theory.

10. Actually, this type of half-trip experiment has been carried out in many experiments in the early days to measure the mean lifetimes of particles decaying at rest. However, they were designed neither to test the "twin paradox" nor to detect effects due to acceleration, so that possible relevant effects were not explicitly investigated.

Name Index

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EINSTEIN'S RELATIVITY AND BEYOND
NEW SYMMETRY APPROACHES

by ***J P Hsu*** *(University of Massachusetts Dartmouth)*

The purposes of this book are (1) to explore and expound relativity physics and 4-dimensional symmetry from the logically simplest viewpoint by making one single postulate instead of two; and (2) to indicate the simplest generalization of the Lorentz transformation in order to cope with frames with constant linear accelerations. *The fundamentally new ideas of the first purpose are developed on the basis of* the term paper of a Harvard physics undergraduate. They lead to an unexpected affirmative answer to the long-standing question of whether it is possible to construct a relativity theory without postulating the constancy of the speed of light and retaining only the first postulate of special relativity. This question was discussed in the early years following the discovery of special relativity by many physicists, including Ritz, Tolman, Kunz, Comstock and Pauli, all of whom obtained negative answers. Furthermore, the new theory of relativity indicates the truly universal and fundamental constants in physics, and provides a broad view of relativistic physics beyond special relativity. It substantiates the view and sheds light on the understanding that the 4-dimensional symmetry framework can accommodate many different concepts of physical time, including common time and Reichenbach's general concept of time. This logically simplest viewpoint of relativity allows a natural extension of the physics of particles and fields from inertial frames to noninertial frames in which the speed of light is not constant. New predictions in physics resulting from this new viewpoint are discussed. The book is based on papers by the author and his collaborators in *Physics Letters A*, *Nuovo Cimento B*, and *Physical Review A* and *D*.

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