DE GRUYTER

Kecai Cao, YangQuan Chen

FRACTIONAL ORDER CROWD DYNAMICS

CYBER-HUMAN SYSTEM MODELING AND CONTROL

FRACTIONAL CALCULUS IN APPLIED SCIENCES AND ENGINEERING 4

Kecai Cao, YangQuan Chen
Fractional Order Crowd Dynamics

Fractional Calculus in Applied Sciences and Engineering

Volume 4

Kecai Cao, YangQuan Chen

Fractional Order Crowd Dynamics

Cyber-Human System Modeling and Control

DE GRUYTER

Mathematics Subject Classification 2010
Primary: 93-02, 93A30, 93C99; Secondary: 70-02, 70B99

Authors
Dr Kecai Cao
Nanjing University of Posts
and Telecommunications
School of Automation
No. 66 Xinmofan Road
210003 Nanjing
P.R. China
caokecai@gmail.com

Prof. Dr YangQuan Chen
University of California
Merced
MESA Lab
School of Engineering
5200 N. Lake Road
Merced, CA 95343
USA
yqchen@ieee.org

ISBN 978-3-11-047281-3
e-ISBN (PDF) 978-3-11-047398-8
e-ISBN (EPUB) 978-3-11-047283-7
ISSN 2509-7210

Library of Congress Cataloging-in-Publication Data
Names: Cao, Kecai, author. | Chen, YangQuan, 1966- author.
Title: Fractional order crowd dynamics : cyber-human system modeling and control / Kecai Cao, YangQuan Chen.
Description: 1 Edition. | Boston/Berlin : De Gruyter, 2018. | Series: Fractional calculus in applied sciences and engineering, ISSN 2509-7210 ; 4
Identifiers: LCCN 2018008284| ISBN 9783110472813 (hardback : alk. paper) | ISBN 9783110473988 (PDF) | ISBN 9783110472837 (EPUB)
Subjects: LCSH: Pedestrian traffic flow--Mathematical models. | Fractional calculus. | BISAC: MATHEMATICS / Applied. | MATHEMATICS / Calculus. | MATHEMATICS / Differential Equations. | MATHEMATICS / Mathematical Analysis.
Classification: LCC HE336.P43 C36 2018 | DDC 363.32/30151583--dc23 LC record available at https://lccn.loc.gov/2018008284

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

Typesetting: VTeX UAB, Lithuania
Printing and binding: CPI books GmbH, Leck
Cover image: naddi/iStock/thinkstock

www.degruyter.com

To our colleagues, friends, and families

Preface

Cyber-physical systems (CPSs), which have been defined as “computational thinking and integration of computation around the physical dynamic systems where sensing, decision, actuation, computation, networking, and physical processes are mixed,” have received a lot of attention in the past ten years. With the development of CPSs, human-centric cyber-physical systems (HCPSs) or cyber-human systems (CHSs) have drawn much attention of researchers. The CHS has evolved from human–machine symbiosis, i. e., humans acting as operators of complex engineering systems, humans as agents in multi-agent systems, humans as elements in controlled systems, and so on. Although we can distinguish the research relevant to CHSs according to the topics mentioned above, CHS research lies in a three-dimensional space comprising humans, computers, and the environment. Not only individuals but also a collection of people and even the whole society can be included in the human dimension, where the people’s capability to be enhanced and the people’s needs to be satisfied have to be considered. Computing devices or mobile devices and even computational systems of sensors and visual/audio devices that are embedded in the surrounding physical environment are categorized in the computer dimension. The environment dimension is composed of discrete physical computational devices, immersive virtual environments, and some mixed reality systems in between. The interplay of humans with computer and environment needs deep multidisciplinary joint research efforts to be made.

With this monograph, we wish to provide the reader with a comprehensive understanding of CHSs with an emphasis on our past experience in the topic, especially as regards crowds of pedestrians. In the past five years, we have worked on the topic of modeling and control of crowds of pedestrians. Fractional calculus has been introduced in the modeling and control of crowds of pedestrians, respectively.

This book is divided into two parts: the modeling part and the control part. We first point out some of the problems that existed in the research of modeling and control of crowds of pedestrians. Then modeling of the crowds of pedestrians is studied at the micro-scale, the macro-scale, and the meso-scale, respectively, with the introduction of fractional calculus. Limitations of the existing model of crowds has been reduced when the modeling problem is considered within the framework of fractional calculus. Fractional control of the microscopic model and the macroscopic model are further considered based on the modeling results obtained in the first part. Fractional protocols or controllers are constructed to better address the cluster consensus problem or evacuation problem of crowds at micro-scale and macro-scale, respectively. Finally, for further comprehension of CHSs, an intelligent evacuation system (IES) is also presented in the last chapter of this book, where a simulation platform and an experimental platform are presented.

This monograph is a result of five years of study under the name of FCCD (fractional calculus and crowd dynamics) with a group website which also serves as the

https://doi.org/10.1515/9783110473988-201

service website (http://mechatronics.ucmerced.edu/fccd) for this brief monograph. The first author's one-year visit to the second author at Utah State University under the financial support of China Scholarship Council (CSC) has been fruitful and productive, leading to new ideas in research. The first author acknowledges the financial support of the National Natural Science Foundation of China (Grant Nos. 61374055, 61503194), Key University Science Research Project of Jiangsu Province (Grant No. 17KJA120003), Natural Science Foundation of Jiangsu Province (Grant No. BK20161520), China Postdoctoral Science Foundation (Grant No. 2013M541663), Jiangsu Planned Projects for Postdoctoral Research Funds (Grant No. 1202015C), Scientific Research Foundation of the Ministry of Education of China for Returned Scholars (Grant No. BJ213022). The second author acknowledges the financial support of NIDDR (National Institute on Disability and Rehabilitation Research) under the project title "Experimental Research on Pedestrian and Evacuation Behaviors of Individuals with Disabilities; Theory Development Necessary to Characterize Individual-Based Models." We acknowledge the following researchers for collaboration in work on the project: Dr Keith Christensen (PI), Dr Anthony Chen, Dr Yong Kim, Dr M. S. Sharifi, and Dr Daniel Stuart. Dr Caibin Zeng has also contributed to the modeling discussions involving fractional calculus. We wish to thank Prof. Changpin Li, the series editor-in-chief of "Fractional Calculus in Applied Sciences and Engineering" for this book project invitation. Last but not least, our thanks go to Nadja Schedensack, project editor Mathematics and Physics Science & Technology of De Gruyter. We sincerely hope this monograph is both motivating and stimulating.

Nanjing, Jiangsu, China — Ke-Cai Cao
Merced, California, USA — YangQuan Chen

Contents

Acronyms

CPS	Cyber-physical systems
CHS	Cyber-human system
DDM	Drift diffusion model
DIFF-MAS2D	Diffusion with networked movable actuators and sensors in 2D
DMM	Decision-making model
FO-DIFF-MAS2D	Fractional diffusion with networked movable actuators and sensors in 2D
FOLTI	Fractional-order linear time-invariant
FPE	Fokker–Planck equation
FVM	Finite-volume method
GIS	Geographic information system
GPS	Global positioning system
GUI	Graphic user interface
HCPS	Human-centric cyber-physical system
HJB	Hamilton–Jacobi–Bellman
IES	Intelligent evacuation system
LRD	Long-range interaction
LWR	Lighthill–Whitham–Richards
MFG	Mean field game
ODE	Ordinary differential equation
PDE	Partial differential equation
PW	Payne–Whitham
TAFC	Two-alternative forced choice

https://doi.org/10.1515/9783110473988-202

1 Introduction

Abstract: Problems in modeling and control of crowds of pedestrians are firstly presented. Short comments on previous research and the organization of this book are also presented in this chapter.

1.1 Motivation

Humans are the most socially complex animals on this planet. It is not surprising that research related to crowds of pedestrian has received a lot of attention in recent years. A lot of work has been conducted for particles, vehicles, robots, animals, and even human beings from the perspectives of behavior, psychology, cognitive, and network theory (see [8, 9, 11, 23, 20]). Among the previous work, problems related to crowds of pedestrians are the most challenging due to difficulties in modeling of human beings, as there are no universal tools to characterize the complex temporal and spatial features of crowds.

On the other side, more and more tragedies due to people's stampede have been reported in recent years. The most tragic stampede occurred in Mecca in 1990, where 1426 pilgrims were trampled to death or suffocated. In evacuation, people got injured or lost their lives due to panic motion or running in every direction without aim. The catastrophic events have demonstrated the need to reanalyze and reexamine current evacuation policies and procedures for crowds of pedestrians. Thus, policy makers urgently need better crowd management or evacuation strategies.

The problems confronted in research of crowds of pedestrians can be listed as follows:

(1) How to characterize or obtain a satisfactory social-dynamic model for crowds of pedestrians that is much closer to reality compared to the previous model.

(2) How to enforce and stabilize the desired pattern formation of crowds of pedestrians and how to avoid some rare or dangerous formation pattern in evacuation of crowds.

1.2 Current status of research

1.2.1 Modeling of crowds

According to the differences in scales, the models for crowds of pedestrians can be categorized by microscopic model,macroscopic model and mesoscopic model (see [10, 3, 14, 2, 7]).

https://doi.org/10.1515/9783110473988-001

(1) The Newton principle is a powerful tool to describe the motion of particles at microscopic scale but the heavy burden of computation will not make it a better choice with the increase of the number of particles.
(2) Conservation of mass and momentum is the basic principle employed in obtaining the macroscopic model. Although the computation burden has been reduced greatly, the individual character of each pedestrian has been ignored when using this kind of method and the heterogeneity of different pedestrians can't be easily characterized at the macro-scale.
(3) For a mesoscopic model, not only the computation burden has been reduced, but also the heterogeneity of different pedestrians can be guaranteed. However, qualitative and quantitative results are not easy to obtain for integral-differential equations obtained at this scale.

Besides the problem of multiple scales in modeling crowds of pedestrians, there are a lot of other effects that influence the pattern of motion of crowds, such as imitating behavior of neighbors, following external signals, psychological unity, emotional intensity, and level of violence, as shown in [5]. The present status is that there is no common agreement on which model is the best one in describing this kind of complex social dynamics. A new methodology and theory are required for better approximating and characterizing this complex social-dynamic system.

1.2.2 Control of crowds

Compared to the modeling of crowds mentioned above, control of crowds is much more challenging as shown in recent work; see [14, 4, 10, 12, 6, 24].

- A lot of evacuation procedures or policies have been designed using computer simulations. Considering the adaptability and robustness to the environment, the method of simulation is not a good choice as the obtained modeling results or evacuation policies are not effective anymore in different buildings or different scenarios.
- In some of the previous research based on the mathematical model, pedestrians have been treated as particles and some of the characteristics of human beings have been neglected in control of the crowds, such as short-range and long-range interactions, effects of memory, and statistical characters at the temporal or spatial scale.
- In large crowds of pedestrians, self-organization or cooperative movement have been observed a long time ago. But there is little research on how to realize the desired formation patterns and prevent dangerous patterns so that a stampeding tragedy can be avoided.

As there is no perfect model for all kinds of scenarios, there are no universal controllers that can solve all evacuation problems. Based on models that are much closer to reality, many efforts have been made as regards control of crowds for the purpose of better management and efficient evacuation of crowds.

1.2.3 Comments

Some manuscripts have been published in recent years concerning the modeling and control problems of crowds of pedestrians, such as [22, 13, 25, 19, 1, 18, 21, 26, 17, 16]. But the authors found that some important characteristics of the crowds have been neglected in previous research and their effects should be reconsidered and reexamined in both the modeling and the control stages for crowds of pedestrians.

(1) *Integer order versus fractional order at temporal scale.*
The movements of each pedestrian are the results of a complex interaction between physical and psychological issues. Inter-event time has been proved to play an important role in analyzing the movement of crowds, as shown in [27]. Contrary to the fact that the distribution of inter-event time satisfies a power law distribution in most cases, an exponential law distribution has been assumed in most of the previous research within the framework of calculus of integer order. The calculus of fractional order has been introduced at the temporal scale as a remedy for this gap in this book.

(2) *Integer order versus fractional order in spatial scale.*
Another important thing should be pointed out: the spatial scale is assumed to be uniform and the dimensions of space are restricted to one dimension, two dimensions, and three dimensions in the previous research. But these assumptions are only reasonable if the crowds of pedestrians can fill space like particles of gases or fluids, while this is not the case as is clear from observations. Theoretically, only a normal diffusive process has been considered in the previous research and few results have been reported for sub-diffusive or super-diffusive processes for modeling of crowds.

(3) *Short-range interactions versus long-range interactions*
Short-range interactions have been extensively considered in the schooling of fish and flocking of birds and in the control of multi-agent systems, while long-range interactions dominating a system's phase transition only has received attention recently. Based on the results obtained in [15], long-range interactions at the micro-scale have been proved to be closely connected to the dynamic model of fractional order at the macro-scale. Not only short-range interactions but also long-range interactions can easily be manipulated using the framework of the calculus of fractional order.

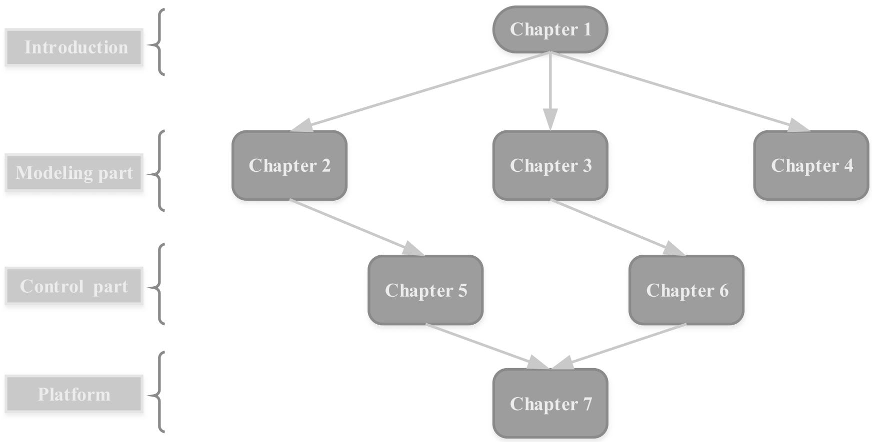

Figure 1.1: Organization of this book.

1.3 Organization of this book

In the first part of this book, a dynamic model of fractional order for crowds of pedestrians is studied at the micro-scale, macro-scale, and meso-scale. Ordinary differential equations (ODEs) of fractional order, partial differential equations (PDEs) of fractional order and coupled ODE-PDEs of fractional order have been obtained for modeling of crowds where the characteristics of temporal, spatial, and long-range interactions mentioned above have been embedded. Based on the obtained models, control or evacuation of crowds is considered in the second part of this book. An intelligent evacuation system based on FO-Diff-MAS2D is also introduced to illustrate or show the effectiveness of the theoretical results. The organization of this book is shown in Figure 1.1.

References

[1] N. Bellomo. *Modeling Complex Living Systems: A Kinetic Theory and Stochastic Game Approach*. Birkhäuser, Boston, 2008.

[2] N. Bellomo and C. Dogbe. On the modeling of traffic and crowds a survey of models, speculations, and perspectives. *SIAM Review*, 53(3):409–463, 2011.

[3] N. Bellomo, B. Piccoli, and A. Tosin. Modeling crowd dynamics from a complex system viewpoint. *Mathematical Models and Methods in Applied Sciences*, 22:1–29, 2012.

[4] N. Bellomo, C. Bianca, and V. Coscia. On the modeling of crowd dynamics: an overview and research perspectives. *S⃗eMA Journal*, 54(1):25–46, 2013.

[5] A. E. Berlonghi. Understanding and planning for different spectator crowds. *Safety Science*, 18(4):239–247, 1995.

[6] P. Bogdan and R. Marculescu. A fractional calculus approach to modeling fractal dynamic games. In *Proceedings of the IEEE Conference on Decision and Control and European Control Conference*, pages 255–260, 2011.

[7] D. Christian. On the modelling of crowd dynamics by generalized kinetic models. *Journal of Mathematical Analysis and Applications*, 387(2):512–532, 2012.

[8] I. D. Couzin. Collective cognition in animal groups. *Trends in Cognitive Sciences*, 13(1):36–43, 2008.

[9] I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin. Effective leadership and decision-making in animal groups on the move. *Nature*, 433:513–516, 2005.

[10] E. Cristiani, B. Piccoli, and A. Tosin. Multiscale modeling of granular flows with application to crowd dynamics. *Multiscale Modelling and Simulation*, 9(1):155–182, 2011.

[11] A. Czirok. Collective motion of self-propelled particles kinetic phase transition in one dimension. *Physical Review Letters*, 82(1):209–212, 1999.

[12] C. Dogbe. Applicable thermostatted models to crowd dynamics: Comment on "thermostatted kinetic equations as models for complex systems in physics and life sciences" by Carlo Bianca. *Physics of Life Reviews*, 9(4):410–412, 2012.

[13] H. Haken. *Information and Self-Organization A Macroscopic Approach to Complex Systems*. Springer, Berlin, Heidelberg, 2006.

[14] D. Helbing, L. Buzna, A. Johansson, and T. Werner. Self-organized pedestrian crowd dynamics: Experiments, simulations, and design solutions. *Transportation Science*, 39(1):1–24, 2005.

[15] R. Ishiwata and Y. Sugiyama. Relationships between power-law long-range interactions and fractional mechanics. *Physica A*, 391(23):5827–5838, 2012.

[16] B. Jaume. *Fundamentals of Traffic Simulation*. Springer Science and Business Media, Berlin, 2010.

[17] P. Kachroo. *Pedestrian Dynamics: Mathematical Theory and Evacuation Control*. CRC Press, Taylor & Francis Group, London, 2009.

[18] P. Kachroo, S. J. Al-nasur, S. A. Wadoo, and A. Shende. *Pedestrian Dynamics Feedback Control of Crowd Evacuation*. Springer-Verlag, Berlin, Heidelberg, 2008.

[19] W. W. F. Klingsch, C. Rogsch, A. Schadschneider, and M. Schreckenberg. *Pedestrian and Evacuation Dynamics*. Springer-Verlag, Berlin, Heidelberg, 2010.

[20] M. Moussaid, D. Helbing, and G. Theraulaz. How simple rules determine pedestrian behavior and crowd disasters. *Proceedings of the National Academy of Sciences of the United States of America*, 108(17):6884–6888, 2011.

[21] N. Pelechano, J. M Allbeck, and N. I. Badler. *Virtual Crowds Methods, Simulation, and Control*. Morgan & Claypool, Williston, 2008.

[22] A. Quarteroni and A. Veneziani. Analysis of a geometrical multiscale model based on the coupling of ODE and PDE for blood flow simulations. *Multiscale Modeling and Simulation*, 1:173–195, 2003.

[23] K. Spieser and D. E. Davison. A cooperative multi-agent approach for stabilizing the psychological dynamics of a two-dimensional crowd. In *Proceedings of the American Control Conference*, pages 5737–5742, 2009.

[24] D. Stuart, K. Christensen, A. Chen, K.-C. Cao, C. Zeng, and Y. Q. Chen. A framework for modeling and managing mass pedestrian evacuations involving individuals with disabilities: Networked segways as mobile sensors & actuators. In *Proceedings of the ASME 2013 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, DETC2013-12652, 2013.

[25] D. Thalmann and S. Raupp Musse. *Crowd Simulation*. Springer, Berlin, 2007.

[26] H. Timmermans. *Pedestrian Behavior: Models, Data Collection and Applications*. Emerald Group Publishing Limited, Bingley, 2009.

[27] B. J. West, M. Turalska, and P. Grigolini. *Networks of Echoes Imitation, Innovation and Invisible Leaders, volume Computatio*. Springer International Publishing, Switzerland, 2014.

Part I: **Fractional modeling of large crowds of pedestrians**

First problem confronted in research of crowds of pedestrians is how to model the crowds of pedestrians or which kind of model should be used to describe it. Different to previous research on crowds of pedestrians where some simulation softwares have been employed to simulate or predict the movement of crowds of pedestrians, explicit mathematical models will be given in this part according to different scales so that much more scientific policies can be derived based on the obtained mathematical models. Compared with dynamic models reported in the previous publications, another difference lies in that time and spatial characteristics of crowds of pedestrians have been considered in the modeling process. Calculus of fractional order is introduced into modeling the movement of crowds of pedestrians where some of the crowd's characteristics in time scale and spatial scale have been well described by fractional calculus. The work reported in this part provides a lot of interesting directions and hope the readers can get some inspirations from the content of this part.

2 Microscopic model of fractional order for evacuation of crowds

Abstract: In modeling of crowds of pedestrians using calculus of integer order, the distribution of time between interchanging opinions is assumed to satisfy a Gaussian distribution, while this assumption is not always satisfied in reality. Burst phenomenon which is commonly observed in decision-making process of human is also hard to be described using calculus of integer order. In the previous research, the dynamic decision-making process has been seldom considered. Modeling of the two-alternative decision-making process for evacuation of crowds of pedestrians from a bounded room has been firstly considered in this chapter using the framework of fractional calculus. Dynamic decision-making models of fractional order have been presented for symmetric interactions and asymmetric interactions, respectively. A stability analysis for asymmetric interactions has been studied using the technique of linearization. Finally, a collective decision-making process for asymmetric interactions is proved to depend on the initial distribution, while this is not the case for symmetric interactions. Future work based on the results of this chapter is also discussed.

2.1 Introduction

Two-alternative forced choice (TAFC) is a method of psycho-physics developed by Fechner [14] where each subject is forced to choose the correct option from two alternatives, as shown in Figure 2.1. As a lot of behaviors have been observed consistently using TAFC, many different kinds of models have been proposed in recent years to describe the dynamics of TAFC in the previous research.

The following assumptions, firstly presented in [5], have been widely assumed in the previous research.

Assumption 2.1.
(1) Evidence favoring each alternative is integrated over time.
(2) The dynamic process is subject to random fluctuations.
(3) The decision is made when sufficient evidence has accumulated favoring one alternative over the other.

Under Assumption 2.1, extensive research has been conducted to understand the complex dynamic process for animals or human beings from different aspects such as modeling work in [20, 1, 2], system analysis in [21, 10, 11], and even controller design in [25, 26, 23].

https://doi.org/10.1515/9783110473988-002

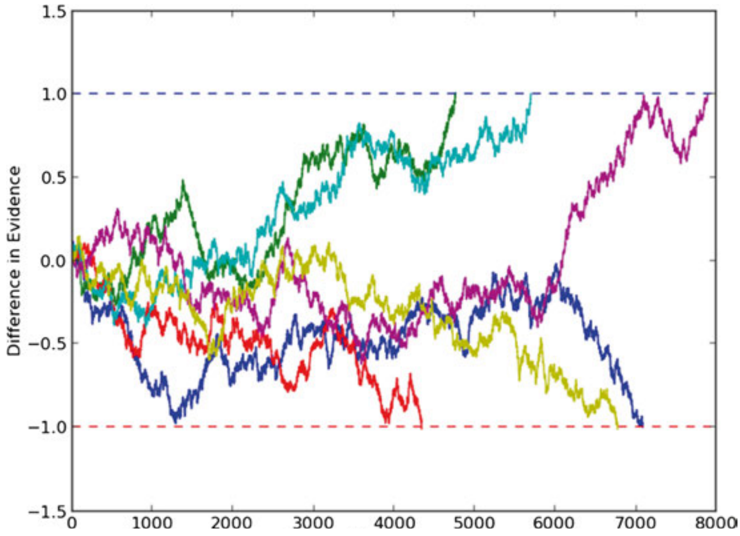

Figure 2.1: Example of six evidence accumulation sequences for unbiased TAFC from Wikipedia.

2.1.1 Isolated decision-making model for TAFC

In the isolated decision-making model (DMM) for TAFC in the previous research, the following well-known *Weber–Fechner law* has been widely used, based on which a drift–diffusion model (DDM) has been derived and proved to be a better choice for describing the dynamics of TAFC, as shown in the review paper [5].

Definition 2.1 (Weber–Fechner law). The following equation describes the relationship between the smallest change in stimuli dS that can be perceived and the current intensity of stimulus S:

$$\frac{dS}{S} = \text{constant}.$$

The relationship between changing decisions and evidence accumulating can be described similarly.

The following DDM has been treated as standard model in modeling of the TAFC problem, as shown in [40, 27, 36, 28]:

$$dx = Adt + cdW(t), \tag{2.1}$$

where dx denotes the change in x over a small time interval dt, $A \in R$ and $c \in R$ are, respectively, the drift rate and the diffusion rate, and $W(t)$ is the standard one-dimensional Wiener process. The pure diffusion model (2.1) has been extended to the following Ornstein–Uhlenbeck model in [9]:

$$dx = (\lambda x + A)dt + cdW(t), \tag{2.2}$$

where the change rate of x is assumed to depend on its current value. Considering the accumulating of different evidence that is favoring different alternatives, the Race model

$$\begin{cases} dx_1 = I_1 dt + cdW_1, \\ dx_2 = I_2 dt + cdW_2 \end{cases} \tag{2.3}$$

has also been proposed where a decision will be made if the accumulation of evidence reaches some predetermined thresholds. Based on the similarity between the decision-making process and Brownian motion in one-dimensional space, the Race model (2.3) is further generalized in [38] to describe two-dimensional diffusion process on a disk. Also based on the Race model (2.3), mutual interactions between these two evidence-accumulating processes are introduced in [42], where the inhibitory effect of one evidence-accumulating process imposed on the other evidence-accumulating process has been studied, and the following mutual inhibition model has been proposed:

$$\begin{cases} dx_1 = (-kx_1 - \omega x_2 + I_1)dt + cdW_1, \\ dx_2 = (-kx_2 - \omega x_1 + I_2)dt + cdW_2, \end{cases} \tag{2.4}$$

where k is the decay of the accumulators and ω is the rate of mutual inhibition.

Succinctly stated, model (2.1) has become the basic model for TAFC in the isolated case. In the work of [4], the pure drift–diffusion model (2.1) has been proved to be equivalent to the well-known Bayesian model. Furthermore, the drift–diffusion model (2.1) is also the optimal DMM for TAFC tasks in statistics, as shown in [5]. Recently, the motion of a noise-driven, over-damped particle was employed in [37] to describe the TAFC process and the following non-linear diffusion model:

$$\partial_T X = -\frac{dE(X)}{dX} + \sigma\xi^3,$$

where $E(X) = -\eta\Delta vX - \mu\bar{v}\frac{X^2}{2} \mp \frac{X^4}{4}$ has been used to solve the curse of dimension that is encountered in the following diffusion models for decision making of multiple neurons:

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, \ldots, x_n; I + I_1), \\ \dot{x}_2 &= f(x_2, x_1, \ldots, x_n; I + I_2), \\ \dot{x}_3 &= f(x_1 + x_2, x_1 \cdots, x_n; I_3), \\ &\vdots \\ \dot{x}_n &= f(x_1 + x_2, x_1 \cdots, x_n; I_n). \end{aligned}$$

2.1.2 Networked decision-making model for TAFC

Compared to research on isolated decision-making processes, a lot of efforts have been made by biologists and engineers on modeling and analysis of the collective decision-making process so that we can understand and control the collective behavior of animals, man-made systems, and even crowds of human beings.

The previous isolated DMM for TAFC has been extended to the following networked DMM in [33]:

$$dx_k = \left[\beta + \sum_{j=1}^{n} a_{ij}(x_j - x_k)\right]dt + \sigma dW_k \tag{2.5}$$

or in matrix form

$$dx = [b - Lx]dt + CdW, \tag{2.6}$$

where $a_{ij} > 0$ means the weight of difference between agent k and agent j, while $a_{ij} = 0$ implies that agent k and agent j do not communicate, $x := (x_1, x_2, \ldots, x_n)^T$, $b := \beta 1^n$, $C := \sigma I$, $dW = \mathrm{col}(dW_1, dW_2, \ldots, dW_n)$, and L is the graph Laplacian matrix defined as

$$l_{kj} := \begin{cases} \sum_{j=1, j\neq k}^{n} a_{kj}, & k = j, \\ -a_{kj}, & k \neq j. \end{cases}$$

The effects of different communication topologies on collective decision-making processes (2.5) or (2.6) have been analyzed in [33] and [34]. Obtained results of [33] have been further enhanced in [35] to characterize the information centrality and ordering of each node, which is very useful in selecting leaders in the TAFC problem. The networked drift–diffusion model (2.5) under network interactions was also adopted in [39] to study the trade-off between speed and accuracy of the evidence accumulating process where the reduced DDM has been used to approximate the original DDM. A weight factor u has been introduced in [15] to characterize the balance between holding of personal opinion and following his (her) neighbor's opinion where the following uninformed and informed models (α is the preference term) have been proposed:

$$dx_k = -d_i x_i + \sum_{j=1}^{n} uS(x_j)dt, \tag{2.7}$$

$$dx_k = -d_i x_i + \sum_{j=1}^{n} uS(x_j)dt + \alpha_i, \tag{2.8}$$

and rich phenomena such as bifurcations under different values of u have been studied and implemented to validate observed behaviors in crowds.

Similar to the Race model (2.3) and the inhibition model (2.4) where the evidence accumulating process is explicitly described, the following DMM for TAFC has also been presented in [41] to describe the cooperative interactions among individuals:

$$\begin{cases} \dot{p}_{i1}(t) = -g_{12}(t)p_{i1}(t) + g_{21}(t)p_{i2}(t), \\ \dot{p}_{i2}(t) = g_{12}(t)p_{i1}(t) - g_{21}(t)p_{i2}(t), \end{cases} \tag{2.9}$$

where $g_{ij}(t)$ is the transition probability from choosing one decision to choosing the other decision and bifurcation phenomenon that is depending on the strength of interaction is also explicitly studied in [41]. This is based on the fact that the sequence of time duration for isolated decision makers is a renewal Poisson process, while the time series generated by a networked or self-organized cluster is not a Poisson process. The DMM (2.9) of integer order for TAFC has been further generalized to the following fractional Langevin equation for an infinite number of individuals in [45] using techniques of the inverse Laplace transform:

$$\partial_t^{\mu-1}\phi(t) = -\lambda^{\mu-1}\phi(t) + \varepsilon(t), \tag{2.10}$$

where the inverse power law distribution of waiting time has been included for all-to-all network and the stochastic fluctuations in the network has been described using $\varepsilon(t)$. Based on the important role of leaders in collective decision making as observed in animal systems, the influence of the number of informed agents (leaders) and the location of these leaders in TAFC have also been considered in [13] and [15], respectively.

2.1.3 TAFC in evacuation of crowds

For efficient evacuation of crowds of pedestrians, the decision-making process has been incorporated in the study of the evacuation problem. The social-force model proposed by [20] has been enhanced using information as regards pedestrians at the exit and information of followers in [46] where the decision-making process has been embedded in the evacuation problem of pedestrians. The effects of the strength of interactions, the distribution of initial leaders, and even the memory of human beings have been considered in the social contagion model in [19], where cascading propagation of information has been studied. Choosing between two asymmetric exits for pedestrians in one corridor was also conducted in [22] using an experimental study and it seems reasonable to allocate people with similar physical abilities to use the same egress routes for realization of cooperative and efficient evacuation. The influence of different interactions on evacuation patterns and evacuation time has been considered in [29] using non-cooperative game theory. Game theory was also utilized in [6] to prove that mutual cooperation will contribute to efficient evacuation, and mean-field games have been employed in [8] to model the evacuating of crowds where a macroscopic optimal control problem has been solved under non-linear mobilities of pedestrians.

Although there are some results on the evacuation of crowds under consideration of the decision-making process, these results are only conducted using the framework of calculus of integer order. Some elements, such as the effects of memory and the distributions of time and space, have been neglected in the previous study. Much more appropriate models can be obtained based on characterizing these "human nature" effects using appropriate tools.

2.1.4 Comments on the previous research

Research efforts on isolated decision making or networked decision making were so far based on integer-order calculus. Specifically, the evidence aggregation process for the TAFC problem has been extensively considered within the framework of calculus of integer order. But more and more evidence has shown that it is much more appropriate to model the dynamics of a decision-making process using calculus of fractional order rather than calculus of integer order, such as memory effects in [12, 18], inverse power law distribution on temporal and spatial scale for movement of human beings, and even long-range interactions in [3] for group of animals.

Memory: Collective memory has been proved to play an important role in the generation of different formation patterns in [12]. The effects of memory on decision time for opinion dynamics are discussed in [18]. Memory effects are also included in the design of interactions in modeling particle's motion in [16] for one-dimensional space and in [43] for two-dimensional space.

Statistic character of time: As pointed out in [44], the exponential distribution of the waiting time for an isolated individual should be replaced with an inverse power law distribution due to interactions among nodes of the network. This kind of non-Poisson intermittency that has been observed in transition from uncoordinated random decisions to organized decisions is closely connected with the fractional calculus.

Statistic character of space: The work of [7] has shown that the traveling of human beings on geographical scales can be described as a super-diffusive process, while the distribution of traveling distance satisfies an inverse power law and the probability of lasting time satisfies a distribution with heavy tail.

Long-range interactions: The roles of long-range interactions and short-range interactions have been studied early for protein molecules in [17] and it was shown that the long-range interactions are essential for cooperative behavior, while the short-range interactions are responsible for accelerating the transitions. Based on flocking data of European starlings (*Sturnus vulgaris*), the theory of maximum entropy has been introduced in [3] in modeling the flocking behavior where both the presence of long-range and scale-free correlations among pairs of birds have been confirmed.

2.1.5 Overview of this chapter

Based on the above statements, dynamics of the decision-making process for evacuation of crowds has been considered in this chapter using the calculus of fractional order. The dynamic model of fractional order for choosing between two exits has been constructed with symmetric interactions and asymmetric interactions, respectively. As a generalization of the DMM of integer order that has been proposed in [44], a stability analysis of the obtained dynamic model has been performed using linearization around desired equilibrium. The relationship with the previous model and discussion of future work for the obtained models are also given for completeness.

This chapter is organized as follows. In Section 2.2, some preliminaries such as the definition of fractional calculus adopted and lemmas used are firstly presented; Section 2.3 is devoted to showing the model of fractional order with symmetric and asymmetric interactions where a local stability analysis is also presented. Finally, in Section 2.4, comparisons between our model of fractional order and the previous model of integer order are discussed and future work based on the work of this chapter is also included in this section.

2.2 Problem formulation

2.2.1 Definitions and lemmas

Definition 2.2 (TAFC [From Wikipedia]). Two-alternative forced choice[1] (TAFC) is a method for measuring the subjective experience of a person or animal through their pattern of choices and response times. The subject is presented with two alternative options, only one of which contains the target stimulus, and is forced to choose which one is the correct option.

For fractional calculus, as shown in [32], there are mainly two widely used fractional operators: Caputo and Riemann–Liouville (RL) fractional operators, where the traditional definitions of the integral and derivative of a function are generalized from integer orders to real or complex orders. The following Caputo definition for a fractional derivative is adopted in this book because the Laplace transform of the Caputo

1 https://en.wikipedia.org/wiki/Two-alternative_forced_choice derivative allows for utilization of initial values of classical integer-order derivatives with known physical interpretations.

Definition 2.3 (Caputo's fractional derivative). The Caputo fractional-order differentiation for $f(t)$ with order α is defined by

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, $$

where n is an integer satisfying $n - 1 < \alpha < n$.

For a fractional-order linear time-invariant (FOLTI) system in the pseudo-state-space model

$$ \begin{cases} {}_{0}^{C}D_t^{q} x(t) = Ax(t) + bu(t), \\ \quad y(t) = Cx(t), \end{cases} \tag{2.11} $$

the stability of the commensurate system (2.11) is guaranteed by the following lemma.

Lemma 2.1 (Stability of FOLTI system [31]). *System* (2.11) *is stable if the following condition is satisfied:*

$$ \left|\arg(\mathrm{eig}(A))\right| > q\frac{\pi}{2}, $$

where $0 < q < 2$ *and* $\mathrm{eig}(A)$ *represents the eigenvalue of matrix* A.

2.2.2 Problem considered in this chapter

How do evacuees find their way to escape from a dangerous room? How should one describe the dynamic process of decision making of each pedestrian? These are the problems to be considered in this chapter.

We analyze the evacuation problem in a parallelogram room with two symmetrical exits, as shown in Figure 2.2. The final choice of each pedestrian is the result of interacting with his/her neighbors and balancing his/her preference of these two exits. For the pedestrian in yellow color, shown in Figure 2.2, the choice between the left exit and right exit has been made based on the distribution of red ones and blue ones. For easy of analysis,each pedestrian has been assumed to be homogeneous in the decision-making process in this chapter.

2.3 Fractional decision making model

Due to the evidence mentioned in Section 2.1.4, which is related to calculus of fractional order, the following coupled master equation of fractional order is firstly pro-

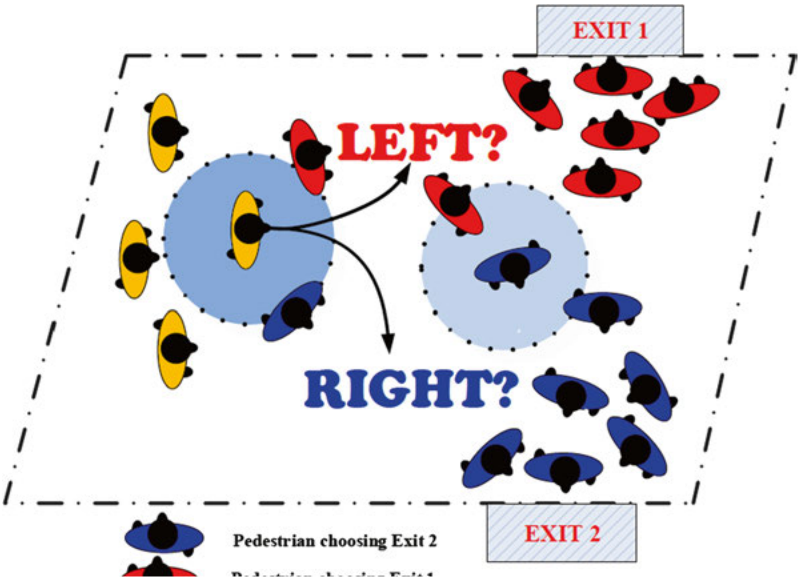

Figure 2.2: Evacuation of crowds of pedestrians in a room with two exits.

posed for TAFC:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} p_{i1}(t) = -g_{12}(t)p_{i1}(t) + g_{21}(t)p_{i2}(t), \\ {}_{t_0}^{C}D_t^{\alpha} p_{i2}(t) = g_{12}(t)p_{i1}(t) - g_{21}(t)p_{i2}(t), \end{cases} \tag{2.12}$$

where $\alpha \in (0, 2)$ and $p_{ij}(t) \in [0, 1]$ is the probability of the ith pedestrian being in the state of choosing the jth exit ($j = 1, 2$). The transition probability from choosing exit m to choosing exit n is denoted by $g_{mn}(t)$, which is dependent on his/her neighbor's choice

$$g_{mn}(t) = (\pi_m(t) - \pi_n(t))^2,$$

where $m \neq n = 1, 2$ and

$$\pi_s(t) = \frac{N_s(t)}{N},$$

where N is the total number of neighbors for the ith pedestrian and $N_s(t)$ is the number of the ith pedestrian's neighbors who choose the sth exit at time t ($s = 1, 2$). Similarly, we denote the following probability in the following sections:

$$\Sigma_s(t) = \frac{\Omega_s(t)}{\Omega},$$

where Ω is the total number of the sth nodes in the network and $\Omega_s(t)$ is the number of pedestrians in the entire network who choose the sth exit at time t ($s = 1, 2$).

2.3.1 Symmetric interaction

Similar to the relationship between action and reaction of forces, $g_{12}(t) = g_{21}(t)$ is assumed under symmetric interactions. We denote the following $\xi(t)$ as one global variable:

$$\xi(t) = \Sigma_1(t) - \Sigma_2(t) = \frac{1}{\Omega}[\Omega_1(t) - \Omega_2(t)], \tag{2.13}$$

which concerns the differences of choosing different alternatives in TAFC. In an all-to-all network, $\pi_s(t) = \Sigma_s(t)$ is satisfied, which means that $\Sigma_s(t)$ will be a reasonable approximation of the average opinion of the entire network no matter the number of pedestrians goes to infinity or not.

Thus, based on the fractional DMM in (2.12), the opinion of the whole network can be described as

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}\frac{\sum p_{i1}(t)}{N} = -g_{12}(t)\frac{\sum p_{i1}(t)}{N} + g_{21}(t)\frac{\sum p_{i2}(t)}{N}, \\ {}_{t_0}^{C}D_t^{\alpha}\frac{\sum p_{i2}(t)}{N} = g_{12}(t)\frac{\sum p_{i1}(t)}{N} - g_{21}(t)\frac{\sum p_{i2}(t)}{N}. \end{cases} \tag{2.14}$$

Subtracting the second line from the first line of system (2.12), the dynamic equation of the global variable $\xi(t)$ can be described as

$${}_{t_0}^{C}D_t^{\alpha}\xi(t) = -2g_{12}(t)\frac{\sum p_{i1}(t)}{N} + 2g_{21}(t)\frac{\sum p_{i2}(t)}{N}$$

or

$${}_{t_0}^{C}D_t^{\alpha}\xi(t) = -2g_{12}(t)\xi(t). \tag{2.15}$$

Theorem 2.1. *Under the assumption that the network is all-to-all, the DMM* (2.15) *that is proposed for the evacuation problem with two exits is asymptotically stable under symmetric interactions, where* $\alpha \in (0, 2)$.

Proof. The proof of asymptotical stability of equilibrium can be found in references [32, 31] and is omitted here. □

Remark 2.1. As the probability of choosing one exit is equal to the probability of choosing the other exit with symmetric interactions, there is no difference between the motion of particles and the motion of pedestrians. Thus the number of pedestrians that evacuated from different exits is equal to each other.

2.3.2 Asymmetric interaction

Similar to derivations in the previous subsection, the dynamics of the opinion of the whole network under asymmetric interactions can also be obtained based on the fractional microscopic model (2.14). The dynamic description of the whole network's opinion can be computed as

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}\xi(t) &= -g_{12}(t)\frac{\sum p_{i1}(t)}{N} + g_{21}(t)\frac{\sum p_{i2}(t)}{N} \\
&\quad - g_{12}(t)\frac{\sum(1-p_{i2}(t))}{N} + g_{21}(t)\frac{\sum(1-p_{i1}(t))}{N} \\
&= -g_{12}(t)\frac{\sum p_{i1}(t)}{N} + g_{21}(t)\frac{\sum p_{i2}(t)}{N} \\
&\quad - g_{12}(t)\left[1 - \frac{\sum p_{i2}(t)}{N}\right] + g_{21}(t)\left[1 - \frac{\sum p_{i1}(t)}{N}\right] \\
&= -(g_{12}(t) + g_{21}(t))\left(\frac{\sum p_{i1}(t)}{N} - \frac{\sum p_{i2}(t)}{N}\right) \\
&\quad + (g_{21}(t) - g_{12}(t)).
\end{aligned}$$

Then the generalized fractional dynamic model for TAFC can be described as

$$ {}_{t_0}^{C}D_t^{\alpha}\xi(t) = -(g_{12}(t) + g_{21}(t))\xi(t) + (g_{21}(t) - g_{12}(t)), \tag{2.16}$$

where the transition probability between different choices that is proposed in [44],

$$ g_{ij}(t) = g_0 \exp[K\{\pi_j(x,t) - \pi_i(x,t)\}] \tag{2.17}$$

is used ($i \neq j = 1, 2$). Then under the assumption of all-to-all network and an infinite number of pedestrians, the transition probability $g_{ij}(t)$ in (2.17) can also be written

$$\begin{cases} g_{12} = g_0 \exp(-K\xi(t)), \\ g_{21} = g_0 \exp(k\xi(t)), \end{cases} \tag{2.18}$$

where preference in choosing one of the two exits has been greatly amplified or suppressed. Thus system (2.16) can also be written as

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}\xi(t) &= -g_0(\exp(k\xi(t)) + \exp(-k\xi(t)))\xi(t) \\
&\quad + g_0(\exp(k\xi(t)) - \exp(-k\xi(t)))
\end{aligned}$$

or

$$ {}_{t_0}^{C}D_t^{\alpha}\xi(t) = -2g_0\cosh(k\xi(t))\xi(t) + 2g_0\sinh(k\xi(t)). \tag{2.19}$$

For stability analysis of (2.19), the potential function

$$ V(\xi) = \frac{2g_0}{K}\left[\frac{K+1}{K}\cosh(k\xi) - \xi\sinh(K\xi)\right]$$

is constructed firstly. Contrary to Figure 3.8 for $V(\xi)$ that is shown in [44], the figures for the potential function and its equilibria are redrawn in Figure 2.3 to provide an overview for stability analysis.

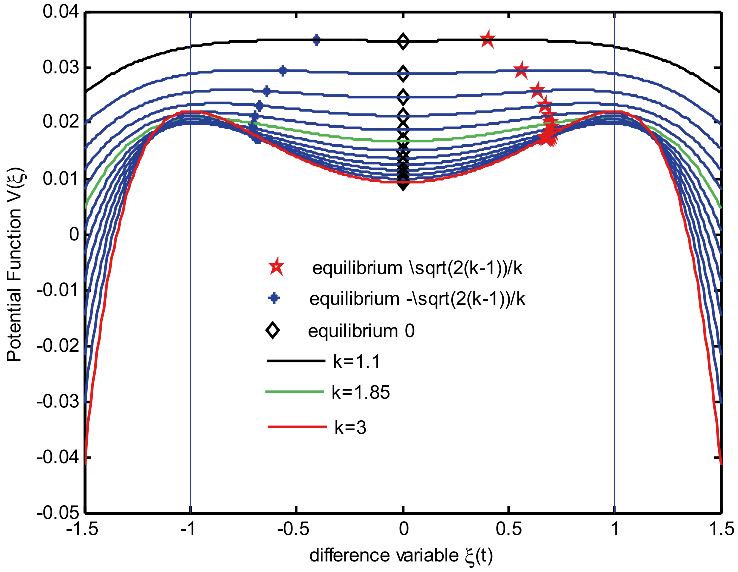

Figure 2.3: Potential function $V(\xi)$ as a function of ξ for $g_0 = 0.01$ and different K.

All the equilibria ξ_{eq} of the system (2.19) can be computed as

$$\xi_{\text{eq}} = 0 \quad \text{and} \quad \xi_{\text{eq}} = \pm \frac{\sqrt{2(K-1)}}{K}$$

after setting the right side of (2.19) to zero and expanding the hyperbolic function using a Taylor series, as shown in Figure 2.3. The stability of each equilibrium is shown in the following theorem.

Theorem 2.2. *Under the assumption of all-to-all network, each equilibrium of the DMM* (2.19) *with* $\alpha \in (0, 2)$ *is locally and asymptotically stable even under asymmetric interaction rules* (2.18).

Proof. (1) *Stability analysis of* $\xi_{\text{eq}} = 0$.

When the initial value of global difference variable $\xi(t)$ is close to zero, the asymmetric interaction is similar to a symmetric interaction. Due to the following facts

$$\begin{cases} g_{12} = g_0 \exp(-K\xi(t)) \to g_0, \\ g_{21} = g_0 \exp(k\xi(t)) \to g_0. \end{cases}$$

Thus system (2.16) can be reduced to

$$ {}_{t_0}^{C}D_t^{\alpha}\xi(t) = -2g_0\xi(t). \tag{2.20}$$

It is easy to see that system (2.20) is locally asymptotically stable based on Lemma 2.1 for $\alpha \in (0, 2)$.

(2) *Stability analysis of* $\xi_{\text{eq}} = \pm\frac{\sqrt{2(k-1)}}{k}$.

After substituting the following linear approximation of a hyperbolic function:

$$\begin{cases}\cosh K\xi = \cosh K\xi_{\text{eq}} + \sinh K\xi_{\text{eq}}(\xi - \xi_{\text{eq}}),\\ \sinh K\xi = \sinh K\xi_{\text{eq}} + \cosh K\xi_{\text{eq}}(\xi - \xi_{\text{eq}})\end{cases}$$

into system (2.19), system (2.19) can be rewritten as

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}\xi(t) &= -2g_0(\cosh K\xi_{\text{eq}} + \sinh K\xi_{\text{eq}}\cdot(\xi - \xi_{\text{eq}}))\xi(t)\\ &\quad + 2g_0(\sinh K\xi_{\text{eq}} + \cosh K\xi_{\text{eq}}\cdot(\xi - \xi_{\text{eq}}))\\ &= -2g_0(\cosh K\xi_{\text{eq}} + \sinh K\xi_{\text{eq}}\cdot(\xi - \xi_{\text{eq}}))\xi(t)\\ &\quad + 2g_0\cosh K\xi_{\text{eq}}\cdot\xi(t)\\ &= -2g_0\sinh K\xi_{\text{eq}}\cdot(\xi - \xi_{\text{eq}})\xi(t),\end{aligned}$$

where $\sinh K\xi_{\text{eq}} = \xi_{\text{eq}}\cosh K\xi_{\text{eq}}$ is used due to the definition of equilibrium. Denoting $\eta(t) = \xi(t) - \xi_{\text{eq}}$, then system (2.19) can be rewritten as

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}\eta(t) &= -2g_0\sinh K\xi_{\text{eq}}\eta(t)(\eta(t) + \xi_{\text{eq}})\\ &= -2g_0\xi_{\text{eq}}\sinh K\xi_{\text{eq}}\eta(t) - 2g_0\sinh K\xi_{\text{eq}}\eta^2(t).\end{aligned}$$

Thus, in neighborhood of each equilibrium $\xi_{\text{eq}} = \pm\frac{\sqrt{2(k-1)}}{k}$, system (2.19) can be approximated by

$${}_{t_0}^{C}D_t^{\alpha}\eta(t) = -2g_0\xi_{\text{eq}}\sinh K\xi_{\text{eq}}\eta(t), \tag{2.21}$$

where $\xi_{\text{eq}}\sinh K\xi_{\text{eq}} > 0$ is satisfied if $\xi_{\text{eq}} \neq 0$. Based on the stability of the linear time-invariant system of fractional order shown in Lemma 2.1, it is easy to see that these two equilibria $\xi_{\text{eq}} = \pm\frac{\sqrt{2(k-1)}}{k}$ are also locally and asymptotically stable. □

Remark 2.2. Based on the results obtained in Theorem 2.2, it can be seen that the final value of the global difference variable is dependent on the initial distribution of the crowd's choice.

- An asymmetric interaction can be reduced to a symmetric interaction if the initial distribution of the whole crowd's choice is close to zero and this balance will be kept as time goes on. In other words, if half of the crowd chooses to evacuate from exit 1 and the other half of the crowd chooses to evacuate from exit 2, this situation will be kept under symmetric interaction.
- It is often the case that the number of people choosing one exit is not equal to that choosing the other exit. This can be validated using the results presented in Theorem 2.2 with symmetric interaction. If one of the exits is much preferred for some of the crowds, this kind of preference can be amplified, while the choices of the other exit will be suppressed using the interaction rules proposed in (2.18).

This kind of situation can be approved by the observation from a real evacuation process that a lot of people try to exit from one exit, while there are not so many people in the other exit.
- In a real evacuation of crowds of pedestrians, it is much preferred to avoid the phenomenon of following the majority. Then how to reallocate the pedestrians in different exits is an interesting problem and will be helpful for better management of crowds.

2.4 Discussions

2.4.1 Relationship to the previous models

(1) Due to the characteristics mentioned above for decision-making processes, there are many limitations in previous research where the aggregation and diffusion of evidence are only considered under the framework of calculus of integer order.
(2) Advantages of using calculus of fractional order in modeling of the drift–diffusion process lie in that more human-nature characteristics can be easily described or modeled such as memory effects, statistic characteristics on temporal and spatial scale, and short-range/long-range interactions among pedestrians.

As generalization of previous pedestrian models of integer order, the obtained model of fractional order has provided us with much more freedom in characterizing and understanding the complexity of the decision-making process.

2.4.2 Future topics

Related topics for future work are listed as follows.
(1) Homogeneity in the decision-making process has been assumed for each pedestrian in this chapter. Due to the heterogeneity in the aggregation and dissemination of evidence in the decision-making process, the dynamic model of fractional variable order or the dynamic model of fractional distributed order that has been proposed in [30, 24] will be useful in solving the challenges caused by heterogeneity.
(2) As all-to-all framework and infinite number of pedestrians have been assumed in this paper, it is more reasonable to consider the decision making of crowds for finite number of pedestrians and more general communication topologies. The following DMM of fractional order

$$ {}_{t_0}^{C}D_t^{\alpha}\xi(t) = -(g_{12}(t) + g_{21}(t))\xi(t) + (g_{21}(t) - g_{12}(t)) + \varepsilon(t) $$

may be a better choice for the decision making process of this case.

(3) The stability of multiple equilibria of the obtained fractional DMM has been analyzed locally without external forces. How to realize the transition from one equilibrium to another equilibrium is an interesting problem that is worthy of putting much more more efforts.

2.5 Conclusion

For the two-alternative choosing problem in evacuation of crowds, fractional calculus has been employed to describe the dynamic decision-making process of each pedestrian which makes the model obtained much closer to reality compared to that of integer order. Stability analysis of the obtained DMM of fractional order has been considered under symmetric and asymmetric interactions, respectively. It was found that the final opinion of a whole crowd depends on the initial distribution of all pedestrians' choices under asymmetric interactions, while this is not the case for symmetric interactions. Based on the DMM of fractional order that is proposed in this chapter, the effects of different communication topologies on TAFC are addressed by the authors now.

References

[1] N. Bellomo and C. Dogbe. On the modeling of traffic and crowds a survey of models, speculations, and perspectives. *SIAM Review*, 53(3):409–463, 2011.

[2] N. Bellomo, B. Piccoli, and A. Tosin. Modeling crowd dynamics from a complex system viewpoint. *Mathematical Models and Methods in Applied Sciences*, 22:1–29, 2012.

[3] W. Bialek, A. Cavagna, I. Giardina, T. Mora, E. Silvestri, M. Viale, and A. M. Walczak. Statistical mechanics for natural flocks of birds. *Proceedings of the National Academy of Sciences of the United States of America*, 109(13):4786–4791, 2011.

[4] S. Bitzer, H. Park, F. Blankenburg, and S. J. Kiebel. Perceptual decision making: drift–diffusion model is equivalent to a Bayesian model. *Frontiers in Human Neuroscience*, 8:102, 2014.

[5] R. Bogacz, E. Brown, J. Moehlis, P. Holmes, and J. D. Cohen. The physics of optimal decision making: A formal analysis of models of performance in two-alternative forced-choice tasks. *Psychological Review*, 113(4):700–765, 2006.

[6] S. Bouzat and M. N. Kuperman. Game theory in models of pedestrian room evacuation. *Physical Review E*, 89(3):32806, 2014.

[7] D. Brockmann, L. Hufnagel, and T. Geisel. The scaling laws of human travel. *Nature*, 439(7075):462–465, 2006.

[8] M. Burger, M. Di Francesco, P. A. Markowich, and M. T. Wolfram. Mean field games with nonlinear mobilities in pedestrian dynamics. *Physics*, 19(5):1311–1333, 2013.

[9] J. R. Busemeyer and J. T. Townsend. Decision field theory: a dynamic-cognitive approach to decision making in an uncertain environment. *Psychological Review*, 100(3):432–459, 1993.

[10] K. Cao, Y. Chen, D. Stuart, and D. Yue. Cyber-physical modeling and control of crowd of pedestrians: a review and new framework. *IEEE/CAA Journal of Automatica Sinica*, 2(3):334–344, 2015.

[11] K. Cao, Y. Q. Chen, and D. Stuart. A fractional micromacro model for crowds of pedestrians based on fractional mean field games. *IEEE/CAA Journal of Automatica Sinica*, 3(3):261–270, 2016.
[12] I. D. Couzin, J. Krause, R. James, G. D. Ruxton, and N. R. Franks. Collective memory and spatial sorting in animal groups. *Journal of Theoretical Biology*, 218:1–11, 2002.
[13] I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin. The effective leadership and decision-making in animal groups on the move. *Nature*, 433:513–516, 2005.
[14] G. T. Fechner. *Elemente der Psychophysik, Volume 2*. Breitkopf & Hartel, Leipzig, 1889.
[15] A. Franci, V. Srivastava, and N. E. Leonard. A realization theory for bio-inspired collective decision-making. *Mathematics*, http://arxiv.org/abs/1503.08526v1, 2015.
[16] A. Galante and D. Levy. Modeling selective local interactions with memory. *Physica D: Nonlinear Phenomena*, 260:176–190, 2013.
[17] N. Go and H. Taketomi. Respective roles of short- and long-range interactions in protein folding. *Proceedings of the National Academy of Sciences of the United States of America*, 75:559–563, 1978.
[18] L. Guzman-Vargas and R. Hernandez-Peerez. Small-world topology and memory effects on decision time in opinion dynamics. *Physica A: Statistical Mechanics and its Applications*, 372(2):326–332, 2006.
[19] S. Hasan and S. V. Ukkusuri. A threshold model of social contagion process for evacuation decision making. *Transportation Research. Part B: Methodological*, 45(10):1590–1605, 2011.
[20] D. Helbing and P. Molnar. Social force model for pedestrian dynamics. *Physical Review E*, 51(5):4282–4286, 1995.
[21] D. Helbing, I. Farkas, and T. Vicsek. Simulating dynamical features of escape panic. *Nature*, 407(28):487–490, 2000.
[22] S. Heliovaara, J.-M. Kuusinen, T. Rinne, T. Korhonen, and H. Ehtamo. Pedestrian behavior and exit selection in evacuation of a corridor – an experimental study. *Safety Science*, 50(2):221–227, 2012.
[23] V. G. Ivancevic and T. T. Ivancevic. Control and synchronization in complex systems and human crowds. In *New Trends in Control Theory*, volume 19 of *Series on Stability, Vibration and Control of Systems, Series A*, pages 445–574. World Scientific, Singapore, 2012.
[24] Z. Jiao, Y. Q. Chen, and I. Podlubny. *Distributed-Order Dynamic Systems Stability, Simulation, Applications and Perspectives*. Springer, Berlin, 2012.
[25] P. Kachroo. *Optimal and Feedback Control for Hyperbolic Conservation Laws*. PhD thesis, Virginia Polytechnic Institute and State University, 2007.
[26] P. Kachroo, S. J. Al-nasur, S. Amin Wadoo, and A. Shende. *Pedestrian Dynamics Feedback Control of Crowd Evacuation*. Springer-Verlag, Berlin, Heidelberg, 2008.
[27] D. Laming. Information theory of choice-reaction times. In *Proceedings of IEEE Intelligent Transportation Systems*, pages 334–339, 1968.
[28] D. Laming. Choice reaction performance following an error. *Acta Psychologica*, 43(3):199–224, 1979.
[29] S. M. Lo, H. C. Huang, P. Wang, and K. K. Yuen. A game theory based exit selection model for evacuation. *Fire Safety Journal*, 41(5):364–369, 2006.
[30] F. Mainardi, A. Mura, G. Pagnini, and R. Gorenflo. Time-fractional diffusion of distributed order. *Journal of Vibration and Control*, 14(9–10):1267–1290, 2008.
[31] I. Petras. *Fractional-Order Nonlinear Systems Modeling, Analysis and Simulation*. Higher Education Press/Springer-Verlag, Beijing/Berlin, Heidelberg, 2011.
[32] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[33] I. Poulakakis, L. Scardovi, and N. E. Leonard. Coupled stochastic differential equations and collective decision making in the two-alternative forced-choice task. In *Proceedings of 2010 American Control Conference*, pages 69–74, 2010.
[34] I. Poulakakis, L. Scardovi, and N. E. Leonard. Node classification in networks of stochastic evidence accumulators. http://arxiv.org/abs/1210.4235, 2012.
[35] I. Poulakakis, G. F. Young, L. Scardovi, and N. E. Leonard. Information centrality and ordering of nodes for accuracy in noisy decision-making networks. *IEEE Transactions on Automatic Control*, 61(4):1040–1045, 2016.
[36] R. Ratcliff. A theory and of memory and retrieval. *Psychological Review*, 8(2):59–108, 1978.
[37] A. Roxin and A. Ledberg. Neurobiological models of two-choice decision making can be reduced to a one-dimensional nonlinear diffusion equation. *PLOS Computational Biology*, 4(3):1–13, 2008.
[38] P. L. Smith. Diffusion theory of decision making in continuous report. *Psychological Review*, 123(4):425–451, 2016.
[39] V. Srivastava and N. E. Leonard. Collective decision-making in ideal networks: The speed-accuracy tradeoff. *IEEE Transactions on Control of Network Systems*, 1(1):121–132, 2014.
[40] M. Stone. Models for choice-reaction time. *Psychometrika*, 25(3):251–260, 1960.
[41] M. Turalska, M. Lukovic, B. J. West, and P. Grigolini. Complexity and synchronization. *Physical Review E*, 80(2), 2009.
[42] M. Usher and J. L. McClelland. The time course of perceptual choice: The leaky, competing accumulator model. *Psychological Review*, 108(3):550–592, 2001.
[43] D. Weinberg and D. Levy. Modeling selective local interactions with memory: Motion on a 2D lattice. *Physica D: Nonlinear Phenomena*, 278–279:13–30, 2014.
[44] B. J. West, M. Turalska, and P. Grigolini. *Networks of Echoes Imitation, Innovation and Invisible Leaders*. Springer International Publishing, Switzerland, 2014.
[45] B. J. West, M. Turalska, and P. Grigolini. Fractional calculus ties the microscopic and macroscopic scales of complex network dynamics. *New Journal of Physics*, 17(4):1–13, 2015.
[46] Z. Zainuddin and M. Shuaib. Modification of the decision-making capability in the social force model for the evacuation process. *Transport Theory and Statistical Physics*, 39(1):47–70, 2010.

3 Macroscopic model of fractional order for crowds of pedestrians

Abstract: The diffusion process of integer order has been extensively used for modeling of crowds in macro-scale in the previous research. Observations have revealed that the classical diffusion process is not sufficient in characterizing the movement of crowds in different scenarios. A macroscopic model of integer order has been further generalized in this chapter using the fractional conservation law of mass/momentum, so that the fractional time and fractional space can be effectively characterized using calculus of fractional order. For completeness, models for multiple types of pedestrians under attractive and repulsive forces are also considered and macroscopic equivalent models of fractional order such as the sub-diffusion, super-diffusion, and porous phenomena are also included.

3.1 Introduction

It is common to see gathering of people in real life due to some public activities such as sports, religion, exhibition, traveling, or celebrating. More and more tragedies due to people's stampedes were reported in recent years, such as the stampede in Nigeria and the stampede in Shanghai as shown in Figure 3.1. The most tragic stampede occurred in Mecca in 1990, where 1426 pilgrims were trampled to death or suffocated. Incomplete statistics show that more than 4000 people died and more than 9000 were injured from 2000 to 2010. From the long list of tragedies of recent years, shown in Table 3.1, one concludes that there is an urgent need to reanalyze and reexamine the current evacuation policies and procedures for evacuation of crowds. In order to fully understand and control the motion of crowds, some work has been done in recent years on the modeling and control of crowds of pedestrians, such as [42, 24, 48, 33, 2, 32, 41, 49, 31, 1]. Some manuscripts have also been published on this topic in recent years, such as [42, 24, 48, 33, 2, 32, 41, 49, 31, 1, 27].

There are a lot of characteristics that should be considered in modeling and control of crowds, such as self-organization, following leaders, common motivations, psychological unity, emotional intensity, and level of violence, as shown in [4]. Thus research related to crowds of pedestrians has received a lot of attention in recent years. A lot of work has been conducted from the perspectives of behavior, psychology, cognition, and network sciences to analyze the motion problems related to particles, vehicles, robots, animals, and even human beings (see [36, 16, 45, 12, 14]). According to the work of [26, 3, 15, 17, 6, 46], modeling of crowds has been categorized as microscopic models, macroscopic models, and mesoscopic models.

https://doi.org/10.1515/9783110473988-003

Figure 3.1: Stampede in Nigeria (from www.chinadaily.com.cn) and stampede in Shanghai (from www.chinanews.com).

Table 3.1: Catastrophic events occurred around the world in recent years.

Time	Location	Place	Injuries and deaths
Oct. 13, 2013	Bond tia, the central of India	Bridge	115 deaths, more than 100 injuries
Jan. 1, 2011	State of Kerala, the southern Indian	Road	104 deaths, 50 injuries
Nov. 22, 2010	Diamond island, Phnom penh of Cambodia	Bridge	339 deaths, hundreds of people were injured
Jul. 25, 2010	Duisburg, Germany	Tunnel entrance	19 deaths, more than 100 injuries
Dec. 7, 2009	Yucai middle school, Xiangtan city of HuNan province	Stair	8 deaths, 26 injuries
Mar. 29, 2009	Abidjan, Cote d'Ivoire	Gym	19 deaths, 132 injuries
Feb. 4, 2006	Manila area, Philippines	Gym	74 deaths, 342 injuries
Feb. 12, 2006	Mecca, Saudi Arabia	Mosque	362 deaths, 300 injuries
Aug. 31, 2005	Baghdad, Iraq	Mosque	965 deaths, 815 injuries
May 9, 2001	Arak, the capital of Ghana	Gym	126 deaths
Jun. 2, 1990	Mecca, Saudi Arabia	Undergrand passage	1426 deaths

Fascinated by the phenomena observed, such as schooling of fish and flocking of birds, a lot efforts have been made to find connections between microscopic interactions and macroscopic patterns. It has been found in [13] that different formation patterns can be generated for swarming of fish through simply tuning the area of attraction and repulsion. Relationship between repulsion and attraction has been quantitatively analyzed in [35]. Contrary to the local interactions used in [13] and in [35], the role of long-range interactions was firstly considered in [50], where transition phenomena from disordered states to ordered states have been observed under long-range interactions. A phase transition from chaos to turbulence has also been reported in [53], where long-range interactions (described by $\frac{1}{l^{1+\alpha}}$, where l is the distance between oscillators and α is the tuning parameter) have been added for synchronization of nonlinear oscillators. Transitions from a purely diffusive regime to flocking patterns have been also realized in [28] through tuning the size of the range and strength of the interactions under local, non-local, or configuration-dependent interactions.

Another thing that should never be neglected is the effect of memory. It has been shown in [13] that the collective memory plays an important role in generating different collective behaviors such as alignment, swarm, and torus behavior. On the hysteresis phenomenon we read the following. (From Wikipedia: *Hysteresis* is the dependence of the output of a system not only on its *current input*, but also on *its history of past inputs*.) It has been explicitly studied in [12], where a non-linear relationship between collective behaviors and the range of interactions has been explicitly shown in Figure 3.2, from which it is obvious that the change of a group's behavior not only depends on the current control input but also depends on the history of individual

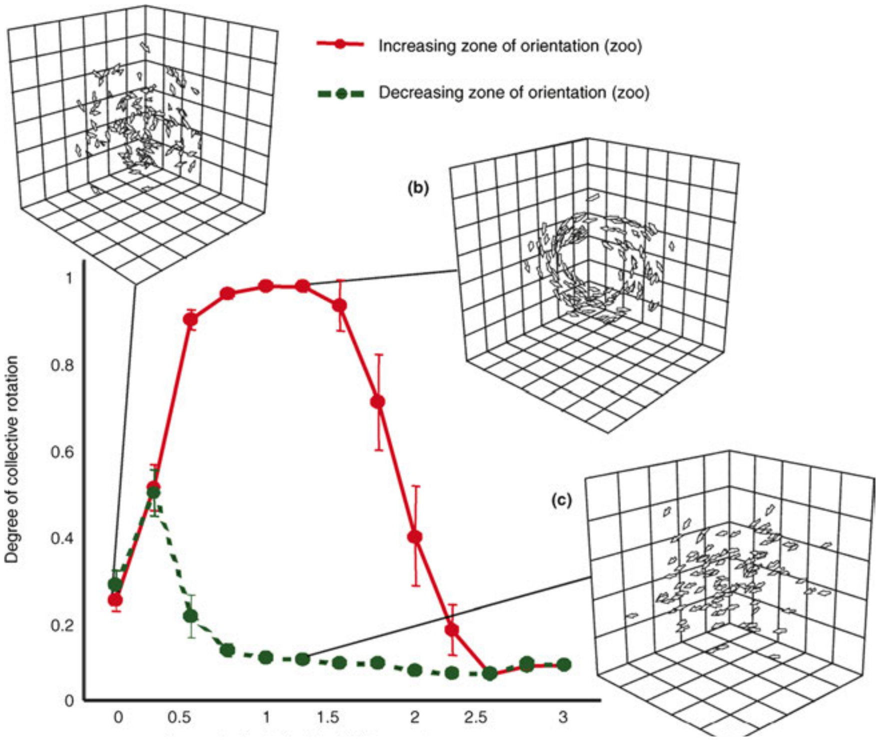

Figure 3.2: Hysteresis phenomenon borrowed from [12].

behavior and the previous shape of the group. The hysteresis phenomena or effects of memory are so common that they can be found in acceleration, deceleration, and equilibrium of traffic flows as explored in [54]. The fractional Fokker–Planck equation has been derived in [47] for particles with memory if the correlating function of probability density is in the form of a power law. Considering the memory of crowds, a nonnegative kernel function $\eta(\cdot)$ has been introduced in [10] to generalize the Lighthill–Whitham–Richards (LWR) model using the following convolution:

$$\rho \star \eta = \int_{R^2} \eta(x-\xi)\rho(t,\xi)d\xi,$$

where the preferred path can be found and regions of high density can be avoided.

Phenomena such as an individual's memory and long-range interactions are not easily characterized within the framework of calculus of integer order. Recent advances in fractional calculus have shown that these characteristics are fully or partially connected with the calculus of fractional order. As a generalization of calculus of integer order, the models derived with the help of fractional calculus are much closer to reality.

3.2 Mathematical model for crowds of pedestrians

3.2.1 Microscopic model for pedestrians

When the density of crowds is low, each pedestrian can move freely and interactions among pedestrians can be modeled using the framework of social forces. A lot of research has also been conducted using Newton's laws, as shown in the following microscopic model:

$$m_i \frac{dv_i}{dt} = f_i^S + \sum_{j=1}^{n} f_{ij}^N + \sum f_k^W, \tag{3.1}$$

where f_i^S is the self-driven force towards some desired velocity, f_{ij}^N is the interaction between agent i and its neighbor j, and f_k^W represents the interactions with the environment, such as walls or corridors.

In current research, only external effects have been considered in most of the previous modeling work at micro-scale. For example, interactions with the external environment such as the spatial–temporal markings have been described using a potential function and have been added to the right side of (3.1), as done in [25]. An external potential function is also employed in [39], where the relationship between formation patterns and different types of interactions has been studied. It has been proved that neither internal forces nor external forces alone can perform well in the evacuation of crowds of pedestrians and a practical and efficient method is introducing both of them in the modeling of crowds in (3.1), where appropriate f_{ij}^N and f_k^W have been constructed for panic crowds. Similarly, it has been shown in [34] that not only external forces, such as interactions with neighbors and environments, but also some internal forces, such as the forces due to personal will and personal emotion, should be considered in the modeling process.

Most of the simulation results are also conducted at this scale, as shown in [41], because the methods are simple and animations are realistic. In order to realize simulations with high realism and real-time animation, the idea of mapping the desired behaviors to a stable solutions of classical non-linear dynamic system such as the Van der Pol oscillator or fixed-point attractor has been introduced in [22] and [37]. With the help of the non-linear transformation introduced in [37], connecting the pedestrian's periodic or aperiodic motion with one structurally stable system, real-time animation has been realized in low-dimensional space without losing details.

Remark 3.1. Comments on microscopic model:
- The main advantages of microscopic model lie in the heterogeneity of pedestrians, which can be considered explicitly, and the simulation results obtained are highly realistic. But the microscopic model is not a good choice if the number of pedestrians becomes very high as some unnecessary interactions or effects have been included.
- Most of the previous work has treated each pedestrian as a single physical particle and little work has been conducted considering the effects of pedestrians' memory or other internal effects.
- Local interacting rules have received a lot of attention in the previous research, while long-range interacting rules are not so popular.

3.2.2 Macroscopic model for large crowds

Based on the methods of modeling of fluids, great progress has been made in constructing macroscopic models for traffic systems. Due to the similarity between traffic systems and crowds of pedestrians at macro-scale, modeling of macroscopic crowds has benefited a lot from the macroscopic model for traffic systems, where the well-known LWR model (3.2) and Payne–Whitham (PW) model (3.3) have been proposed in [31]. The macroscopic model proposed for a traffic system in one-dimensional space and two-dimensional space are described by

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}f(t,x) = 0 \tag{3.2}$$

and

$$\begin{cases} \frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\rho(t,x)v(t,x) = 0, \\ \frac{\partial}{\partial t}v(t,x) + v(t,x)\frac{\partial}{\partial x}v(t,x) = \frac{V(\rho) - v}{\tau} - \frac{A(\rho)_x}{\rho} + \mu\frac{v_{xx}}{\rho}, \end{cases} \tag{3.3}$$

where ρ denotes the density of the crowd and $f(t,x)$ is the flux of the crowd, $V(\rho)$ is the equilibrium speed, $\frac{V(\rho)-v}{\tau}$ is the relaxation term, $\frac{A(\rho)_x}{\rho}$ is the anticipation term, and $\mu\frac{v_{xx}}{\rho}$ is the viscosity term. The relationship between flux and density has attracted a lot of interest, as shown in the following Greenshield model:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}v_f\rho\left(1 - \frac{\rho}{\rho_m}\right) = 0,$$

where $v(\rho) = v_f(1 - \frac{\rho}{\rho_m})$ is assumed, or the following Greenberg model:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}v_f\rho\ln\left(\frac{\rho_m}{\rho}\right) = 0,$$

where $v(\rho) = v_f\ln(\frac{\rho_m}{\rho})$ is assumed. Also based on (3.3), the following diffusion model can be similarly obtained:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\left[v_f\left(1 - \frac{\rho}{\rho_m}\right) - \frac{D}{\rho}\frac{\partial\rho}{\partial x}\right] = 0,$$

where $v(\rho) = [v_f(1 - \frac{\rho}{\rho_m}) - \frac{D}{\rho}\frac{\partial\rho}{\partial x}]$ is assumed, v_f is the velocity of free flow, ρ is the maximum density, and D is the diffusion coefficient. Compared with the classical LWR model, a much more complex relationship between flux and density has been constructed in [55] where both a high-order LWR model and a low-order LWR model have been given. A new relationship between speed and density is also discussed in [11], where the Lighthill–Whitham model in panic scenarios has been derived. Based on the fundamental conservation law of mass and momentum, the macroscopic model (3.2) has been further generalized to the following two-dimensional case in [29]:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho y) = 0, \tag{3.4}$$

where different types of pedestrians with different walking habits have been considered.

The interactions among pedestrians have been assumed to be the same at the macro-scale since it is not easy to incorporate the heterogeneity of each pedestrian. In order to characterize the crowds of pedestrians more precisely, mesoscopic models are much preferred. A modeling procedure based on the following time-varying measures has been proposed in [15] for better characterization of the crowd–pedestrian system:

$$\mu_t = \theta m_t + (1 - \theta)M_t, \tag{3.5}$$

where $m_t = \sum_{j=1}^{N} \delta P_j(t)$ and $dM_t(x) = \rho(t,x)dx$ are the microscopic and macroscopic mass, respectively. Both topological interactions such as the Braess paradox (obstacles may contribute to improving the flow of people in some situations) characterized at macro-scale and anisotropic interactions such as the granular role of some pedestrians described at micro-scale can be considered within this framework using the time-varying measure (3.5). Similar models such as that are composed of an agent-based microscopic model and a flow-based macroscopic model have been shown in [52] and [38], where the accuracy of the microscopic model and the efficiency of the macroscopic model were combined using the techniques of aggregation and dis-aggregation. Details of the mesoscopic model for crowds of pedestrians are left to the next chapters.

Remark 3.2. Comments on macroscopic model:
- The computation time has been greatly decreased as pedestrians have been treated as same particles with same characteristics at macro-scale.
- The main disadvantage of macroscopic models lies in the heterogeneity of pedestrians: interactions or mobilities cannot be characterized or considered in detail at this scale.

3.3 Macroscopic model of fractional order based on conservation law

3.3.1 Definitions and lemmas

The following definitions of the fractal derivative and some lemmas to be used are first presented.

Definition 3.1 ([40]). For a set $F \subset R$ and a subdivision $P_{[a,b]}$, $a < b$, the mass function $\gamma^{\alpha}(F,a,b)$ is given by

$$\gamma^{\alpha}(F,a,b) = \lim_{\delta\to 0} \inf_{\{P_{[a,b]}:|P|<\delta\}} \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^{\alpha}}{\Gamma(\alpha+1)} \theta(F,[x_i,x_{i+1}]),$$

where

$$\theta(F,[x_i,x_{i+1}]) = 1 \quad \text{if } F \cap [x_i,x_{i+1}]$$

is non-empty and zero otherwise, $P_{[a,b]}$ is a subdivision of the interval $[a,b]$, and

$$|P| = \max_{0\le i\le n-1} (x_{i+1}-x_i),$$

the infimum being taken over all subdivisions P of $[a,b]$ such that $|P| < \delta$.

Definition 3.2 ([40]). Let a_0 be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F, a_0, x) & \text{if } x \geq a_0, \\ -\gamma^\alpha(F, x, a_0) & \text{otherwise.} \end{cases}$$

Definition 3.3 ([40]). The fractal derivative for F^α-derivative of f at x is

$$\mathcal{D}_F^\alpha(f(x)) = \text{F-}\lim_{y \to x} \frac{f(y) - f(x)}{S_F^\alpha(y) - S_F^\alpha(x)} \tag{3.6}$$

if the limit exists.

Definition 3.4 ([40]). A point x is a point of change of a function f, if f is not constant over any open interval (c, d) containing x. The set of all points of change of f is called the set of change of f and is denoted by $\text{Sch}(f)$.

Remark 3.3. From Definitions 3.1 to 3.3 listed above, it is easy to see that the definition of integer order can be treated as a special case of fractal derivative when $\alpha = 1$. Thus the fractal calculus offers us much more freedom in modeling dynamic behaviors where ordinary differential equations (ODEs) and methods of calculus of integer order may be inadequate.

Lemma 3.1 ([40]). *Let $f : R \to R$ be a continuous function and F^α a differentiable function such that $\text{Sch}(f)$ is contained in an α perfect set F and let $h : R \to R$ be F-continuous, such that*

$$h(x)\chi_F(x) = \mathcal{D}_F^\alpha(f(x)).$$

Then

$$\int_a^b h(x) d_F^\alpha x = f(b) - f(a). \tag{3.7}$$

3.3.2 Conservation law of mass/momentum

Denote by $\rho(t, x)$ the density of the fluid in the one-dimensional space that is shown in Figure 3.3. The fluid enters at the left edge x_1 and leaves from the right edge x_2 and the

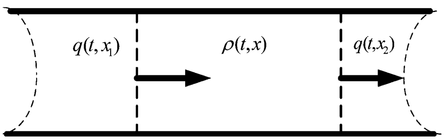

Figure 3.3: One-dimensional flow.

changes of the density are only caused by the fluxes at the boundary x_1 and x_2, which are defined as

$$q(t, x_1) = \rho(t, x_1)v(t, x_1),$$
$$q(t, x_2) = \rho(t, x_2)v(t, x_2),$$

where $v(t, x)$ is the velocity of the fluid.
The following mathematical model can be derived in parallel using conservation law of mass/momentum.

(1) Conservation law in integral form
It is assumed that there are no exits or entrance ramps in Figure 3.3. Then the number of pedestrians within $[x_1, x_2]$ at a given time t is given by

$$N = \int_{x_1}^{x_2} \rho(x, t)dx. \tag{3.8}$$

The change of pedestrians within $[x_1, x_2]$ is caused by pedestrians entering or leaving at the boundaries x_1 and x_2. Assuming that no pedestrians are created or destroyed, the following conservation law in differential form can be derived:

$$\frac{dN}{dt} = f_{x_1}(\rho, v) - f_{x_2}(\rho, v), \tag{3.9}$$

where $f(\rho, v) = \rho(t, x)v(t, x)$ is the flow of pedestrians.

After substituting (3.8) into (3.9), it is easy to obtain

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x, t)dx = f_{x_1}(\rho, v) - f_{x_2}(\rho, v). \tag{3.10}$$

If the endpoints x_1 and x_2 are independent and not fixed with time, then the above full derivative can be replaced by a partial derivative and the following conservation law in integral form can be obtained:

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(x, t)dx = f_{x_1}(\rho, v) - f_{x_2}(\rho, v). \tag{3.11}$$

(2) Conservation law in differential form
If $\rho(t, x)$ and $v(t, x)$ are differentiable functions, the change of the flux at the boundaries can be written as

$$\begin{aligned} f_{x_1}(\rho, v) - f_{x_2}(\rho, v) &= \rho(t, x_1)v(t, x_1) - \rho(t, x_2)v(t, x_2) \\ &= -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(\rho, v)dx. \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11), we have

$$\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}f(\rho,v)\right]dx=0. \tag{3.13}$$

Since (3.13) is satisfied for all independent x_1 and x_2, the following conservation equation in differential form can be obtained:

$$\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}f(\rho,v)=0. \tag{3.14}$$

Considering the integral on $[x_1,x_2]$ for both sides of equation (3.14), it is easy to obtain

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\rho(x,t)dx+f\big(\rho(x_2,t),v(x_2,t)\big)-f\big(\rho(x_1,t),v(x_1,t)\big)=0. \tag{3.15}$$

It is easy to see from (3.15) that the changes of crowds in the bounded area are caused by the net influx at the boundaries. Thus (3.14) is the conservation law in differential form.

(3) General conservation law

- *Case 1: Two-dimensional space*
  Consider the conservation law in differential form in a two-dimensional space,

$$\frac{\partial}{\partial t}\rho(x,y,t)+\frac{\partial}{\partial x}f_1(\rho,v)+\frac{\partial}{\partial y}f_2(\rho,v)=0, \tag{3.16}$$

  where $f_1(\rho,v)$ and $f_2(\rho,v)$ mean the flux along x direction and y direction, defined as

$$f_1(\rho,v)=\rho(x,y,t)u(x,y,t) \quad\text{and}\quad f_2(\rho,v)=\rho(x,y,t)v(x,y,t),$$

  where $u(x,y,t)$ and $v(x,y,t)$ are the velocities at (x,y,t) along x direction and y direction, respectively. The corresponding conservation law in integral form is

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\rho(x,y,t)dxdy=\int_{y_1}^{y_2}\rho(x_1,y,t)dy+\int_{x_1}^{x_2}\rho(x,y_1,t)dy-\int_{y_1}^{y_2}\rho(x_2,y,t)dy$$
$$-\int_{x_1}^{x_2}\rho(x,y_2,t)dy.$$

- *Case 2: n-dimensional space*
  For modeling in n dimensional space, the differential conservation law

$$\frac{\partial\rho}{\partial t}+\nabla\cdot F(u)=0$$

and the integral conservation law

$$\frac{\partial}{\partial t}\int_{V_F} udV + \oint_S F_n(u)dS = 0$$

have been proposed in [23], where $\nabla \cdot F(u)$ means the divergence of the vector field $F(u)$.

3.3.3 Fractional-order modeling based on conservation law

Fractional calculus has shown great potential in different applications, such as particles in fluid, plasma physics, quantum optics and many other fields. Some phenomena, such as self-similarity, non-stationary and spiky phenomena, short or long memory, and long-range interactions, are closely related to fractional calculus.

The authors have noticed that potential applications of fractional calculus in modeling and control of crowds have been mentioned or implied in the previous research [6, 30, 43, 5, 44]. Based on the previous research, some intrinsic characteristics at the temporal scale and spatial scale have been added in the modeling of crowds. Thus, the mathematical models proposed in the previous work, such as that in [31], are further generalized and can be considered as a special case of the work of this chapter.

Consider the following mass of pedestrians from $x = x_1$ to $x = x_2$ at time t, as shown in Figure 3.3:

$$\text{mass in } [x_1, x_2] \text{ at time } t = \int_{x_1}^{x_2} \rho(x,t)dx^{\alpha}. \tag{3.17}$$

The total mass of pedestrians that enters the section from the edge at $x = x_1$ is given by

$$\text{inflow at } x_1 \text{from time } t_1 \text{ to } t_2 = \int_{t_1}^{t_2} \rho(x_1,t)v(t,x_1)dt^{\beta}. \tag{3.18}$$

Similarly, the total mass of pedestrians that leaves the section from the edge at $x = x_2$ is given by

$$\text{outflow at } x_2 \text{ from time } t_1 \text{ to } t_2 = \int_{t_1}^{t_2} \rho(x_2,t)v(t,x_2)dt^{\beta}. \tag{3.19}$$

Applying the conservation law of mass, the change of mass in space $[x_1, x_2]$ on time interval $[t_1, t_2]$ is equal to the mass that enters at x_1 minus the mass that exits at

x_2 during the time interval $[t_1, t_2]$, which can be described by

$$\int_{x_1}^{x_2} \rho(t_2, x) dx^\alpha - \int_{x_1}^{x_2} \rho(t_1, x) dx^\alpha = \int_{t_1}^{t_2} \rho(x_1, t) v(t, x_1) dt^\beta - \int_{t_1}^{t_2} \rho(x_2, t) v(t, x_2) dt^\beta.$$

The above equation can also be written as the following double integral form:

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \frac{\partial}{\partial t^\beta} \rho(t, x) + \frac{\partial}{\partial x^\alpha} [\rho(t, x) v(t, x)] dt^\beta dx^\alpha = 0. \tag{3.20}$$

Since equation (3.20) is satisfied for every t and x, the following macroscopic model of fractional order can be derived for the one-dimensional case:

$$\frac{\partial}{\partial t^\beta} \rho(t, x) + \frac{\partial}{\partial x^\alpha} [\rho(t, x) v(t, x)] = 0. \tag{3.21}$$

Remark 3.4. Compared to dynamic models of integer order in the previous research, some advantages and disadvantages of the obtained model of fractional order are listed:
- *In time domain*: Only a normal diffusive process has been considered in the previous study due to the limitations of calculus of integer order. Besides the normal diffusive process, sub-diffusive process or super-diffusive process can also be utilized to describe the crowds of pedestrians using calculus of fractional order.
- *In spatial domain*: The dimensions of space are only limited to 1, 2, or 3 in the previous research, while a fractional dimension can also be included within the framework of fractional calculus.

Remark 3.5. A traffic model of fractional order has also been proposed in [51] with $\alpha = \beta$, where the dimensions of fractal time and fractal space are assumed to be equal to each other. Thus the model proposed in [51] can also be treated as a special case of our work.

Remark 3.6. We do not want to prove that models of integer order are not effective anymore in reality; we merely want to show that calculus of fractional order as a generalization of calculus of integer order has provided us with much more freedom in characterizing and understanding the complexities of crowds of pedestrians. The authors admit that there is a lot of challenging work left to do for the obtained model of fractional order, such as crowd controller design, stability analysis, and performance evaluation.

3.4 Extensions of macroscopic model

3.4.1 Equivalent model for multiple types of pedestrians

Partial differential equations for crowds composed by one single type of pedestrians have been proposed in the previous section, while it is obvious that the whole crowds are always composed of several types of pedestrians, who have different walking habits, destinations, and mobilities. Thus it is reasonable to study the macroscopic description for crowds composed of different types of pedestrians.

Similar to the fractional model in one-dimensional space in (3.21), each type of pedestrians in two dimensional space can be described by

$$-\frac{\partial \rho_i}{\partial t^\beta} + \frac{\partial}{\partial x^\alpha}(\rho_i u_i(x,y,t)) + \frac{\partial}{\partial y^\alpha}(\rho_i v_i(x,y,t)) = 0,$$

where $u_i(x,y,t)$ and $v_i(x,y,t)$ are the flow velocity of the ith type of pedestrians along the x and y directions without considering the effect of other pedestrian flows. Denote ϕ_i as the potential that each type of pedestrian has in reaching his destination. Then $u_i(x,y,t)$ and $v_i(x,y,t)$ can also be written as

$$u_i(x,y,t) = w_i(\rho)\frac{-\frac{\partial \phi_i}{\partial x}}{\sqrt{(\frac{\partial \phi_i}{\partial x})^2 + (\frac{\partial \phi_i}{\partial y})^2}}, \quad v_i(x,y,t) = w_i(\rho)\frac{-\frac{\partial \phi_i}{\partial y}}{\sqrt{(\frac{\partial \phi_i}{\partial x})^2 + (\frac{\partial \phi_i}{\partial y})^2}},$$

where the velocity function $w_i(\rho)$ is a velocity function that depends on the density of the entire crowds. Based on the inspiration of the classical Greenshield model, the following velocity function is selected in two-dimensional space:

$$w_i(\rho) = v_i^f\left(1 - \frac{\rho}{\rho_m}\right), \tag{3.22}$$

where v_i^f is the free flow speed, ρ is the total density of the whole crowd, which is different from ρ_i used in Greenshield's model, and ρ_m is the maximum density of the whole crowd. Then each type of pedestrian can be described as

$$-\frac{\partial \rho_i}{\partial t^\beta} + \frac{\partial}{\partial x^\alpha}\left[\rho_i v_i^f\left(1 - \frac{\rho}{\rho_m}\right)\frac{-\frac{\partial \phi_i}{\partial x}}{\sqrt{(\frac{\partial \phi_i}{\partial x})^2 + (\frac{\partial \phi_i}{\partial y})^2}}\right]$$
$$+ \frac{\partial}{\partial y^\alpha}\left[\rho_i v_i^f\left(1 - \frac{\rho}{\rho_m}\right)\frac{-\frac{\partial \phi_i}{\partial y}}{\sqrt{(\frac{\partial \phi_i}{\partial x})^2 + (\frac{\partial \phi_i}{\partial y})^2}}\right] = 0.$$

For crowds composed of N types of pedestrians, the steady flow per unit width of each type is given by

$$
\begin{aligned}
q_1 &= \rho_1 w_1(\rho), \\
q_2 &= \rho_2 w_2(\rho), \\
&\vdots \\
q_N &= \rho_N w_N(\rho).
\end{aligned}
\tag{3.23}
$$

Dividing each q_i in (3.23) by v_{if} and adding yields the effective flow velocity,

$$
q_{eff} = \sum_{i=1}^{N} \frac{q_i}{v_{if}} = \sum_{i=1}^{N} \rho_i \left(1 - \frac{\rho}{\rho_m}\right) = \left(1 - \frac{\rho}{\rho_m}\right) \sum_{i=1}^{N} \rho_i = \rho \left(1 - \frac{\rho}{\rho_m}\right).
$$

As the density ρ in Greenshield's model (3.22) is the total velocity, defined as

$$
\rho = \sum_{i=1}^{N} \rho_i,
$$

the equivalent flux of the crowds composed of N types of pedestrians is given by

$$
\rho^2 \left(1 - \frac{\rho}{\rho_m}\right).
$$

Thus the equivalent model for multiple types of pedestrians can be described by

$$
-\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^\alpha} \left[\rho^2 \left(1 - \frac{\rho}{\rho_m}\right) \cos \alpha\right] + \frac{\partial}{\partial y^\alpha} \left[\rho^2 \left(1 - \frac{\rho}{\rho_m}\right) \sin \alpha\right] = 0,
$$

where

$$
\cos \alpha = \left\langle \frac{-\frac{\partial \phi_i}{\partial x}}{\sqrt{(\frac{\partial \phi_i}{\partial x})^2 + (\frac{\partial \phi_i}{\partial y})^2}} \right\rangle, \quad \sin \alpha = \left\langle \frac{-\frac{\partial \phi_i}{\partial y}}{\sqrt{(\frac{\partial \phi_i}{\partial x})^2 + (\frac{\partial \phi_i}{\partial y})^2}} \right\rangle,
$$

and $\langle \cdot \rangle$ means the average direction of these N types of pedestrians.

3.4.2 Macroscopic model with attraction and repulsion

Different from the linear speed–density relationship, which is widely used in (3.22), the speed $w_i(\rho)$ can be further generalized to the following non-local form:

$$
\Gamma(\rho) = G(K * \rho) = G\left(\int_{-\infty}^{\infty} K(x - \xi)\rho(\xi, t) d\xi \right).
\tag{3.24}
$$

Table 3.2: Mutual interaction forces used in the previous literature.

References	Attraction	Repulsion	Comments
[19]	$\frac{1}{2\pi m_r^2} e^{-(s-s_r)^2/(2m_r^2)}$	$\frac{1}{2\pi m_a^2} e^{-(s-s_a)/(2m_a^2)}$	Gaussian interactions
[7, 35]	$e^{\frac{\lvert s \rvert}{s_a}}$	$e^{\frac{\lvert s \rvert}{s_r}}$	Morse interactions
[8, 9]	$\frac{z^\alpha}{\alpha}$	$\frac{z^r}{r}$	Power law interactions
[21]	r	$\frac{a}{r^n}$	Hybrid interactions
⋮	⋮	⋮	⋮

Based on the non-local interactions in (3.24), the fractional macroscopic model (3.21) can be rewritten as

$$\frac{\partial}{\partial t^\beta}\rho(t,x) + \frac{\partial}{\partial x^\alpha}\left[\rho(t,x)G\left(\int_{-\infty}^{\infty} K(x-\xi)\rho(\xi,t)d\xi\right)\right] = 0, \tag{3.25}$$

where the kernel $K(\cdot)$ means the difference between repulsive and attractive forces and is defined by

$$K(s) = \operatorname{sgn}(s)\big(l_r K_r(s) - l_a K_a(s)\big)$$

as in [18] or in [35, 20], where K_r is the repulsive function and K_a is the attractive function. Commonly used repulsive functions and attractive functions in the previous research have been summarized in Table 3.2.

3.4.3 Macroscopic model with diffusion and attraction

In order to counteract the effects of a cohesive potential function (attraction forces), the effect of motility (repulsion forces) has been added to prevent the density ρ from converging to a delta function (complete collapse of crowds) in [30]. Similar to the work of [30], diffusion terms have been added into the fractional model (3.25), resulting in

$$\frac{\partial}{\partial t^\beta}\rho(t,x) = \mathcal{D}\rho(t,x) - \frac{\partial}{\partial x^\alpha}\left[\rho(t,x)G\left(\int_{-\infty}^{\infty} K(x-\xi)\rho(\xi,t)d\xi\right)\right],$$

to counteract the action of attraction forces, where $\mathcal{D}$ is an operator describing the diffusion of crowds. In order to satisfy the conservation law of mass, the following condition is assumed:

$$\int_{-\infty}^{\infty} \mathcal{D}\rho(t,x)dx = 0,$$

which can be obtained through integration of (3.25). Denote $P(t,x) = \frac{\rho(t,x)}{C}$, where C is selected to ensure that $P(t,x)$ is a probability density function. Then the macroscopic model of fractional order with competition between diffusive and attractive forces can be written as

$$\frac{\partial}{\partial t^{\beta}} P(t,x) = \mathcal{D}P(t,x) - C\frac{\partial}{\partial x^{\alpha}}\left[P(t,x)G\left(\int_{-\infty}^{\infty} K(x-\xi)P(\xi,t)d\xi\right)\right]. \tag{3.26}$$

Possible choices of the above linear operator $\mathcal{D}$ are listed in the following based on the work of [30]:

(a) Second-order differential operator

$$\mathcal{D} = D\frac{\partial^2}{\partial x^2} - v\frac{\partial}{\partial x}.$$

(b) Linear pseudo-differential operator on space variable x

$$\mathcal{F}\{Dp(x,t); x \to q\} = -a(q)\tilde{p}(q,u).$$

(c) Linear pseudo-differential operator on space variable x and time variable t

$$\mathcal{F}\{Dp(x,t); x \to q, t \to u\} = -b(q,u)p^{*}(q,u),$$

where

$$\mathcal{F}\{g(x); x \to q\} = \int_{-\infty}^{\infty} e^{iqx}g(x)dx$$

is the Fourier transformation.

3.5 Conclusion

Macroscopic models for crowds of pedestrians have been considered in this chapter within the framework of conservation law of mass. Compared with the previous work on modeling of crowds of pedestrians, some characteristics at temporal scale and spatial scale have been included in the modeling of crowds of pedestrians with the help of calculus of fractional order. A macroscopic model of fractional order was constructed using the conservation law where models proposed in the previous research can be considered as a special case. Macroscopic model for multiple types of pedestrians and macroscopic model with competition between attractive forces and repulsive forces have also been presented in this chapter.

References

[1] J. Barcelo. *Fundamentals of Traffic Simulation*. Springer Science and Business Media, Berlin, 2010.

[2] N. Bellomo. *Modeling Complex Living Systems: A Kinetic Theory and Stochastic Game Approach*. Birkhäuser, Boston, 2008.

[3] N. Bellomo, C. Bianca, and V. Coscia. On the modeling of crowd dynamics: an overview and research perspectives. *SeMA Journal*, 54(1):25–46, 2013.

[4] A. E. Berlonghi. Understanding and planning for different spectator crowds. *Safety Science*, 18(4):239–247, 1995.

[5] P. Bogdan and R. Marculescu. Cyberphysical systems workload modeling and design optimization. *IEEE Design & Test of Computers*, 28(4):78–87, 2011.

[6] P. Bogdan and R. Marculescu. A fractional calculus approach to modeling fractal dynamic games. In *Proceedings of the IEEE Conference on Decision and Control and European Control Conference*, pages 255–260, 2011.

[7] J. A. Carrillo, S. Martin, and V. Panferov. A new interaction potential for swarming models. *Physica D: Nonlinear Phenomena*, 260:112–126, 2013.

[8] J. A. Carrillo, M. Chipot, and Y. Huang. On global minimizers of repulsive–attractive power–law interaction energies. *Philosophical Transactions, A: Mathematical, Physical and Engineering Sciences*, 372:20130399, 2014.

[9] R. Choksi, R. C. Fetecau, and I. Topaloglu. On minimizers of interaction functionals with competing attractive and repulsive potentials. *Annales de L'Institut Henri Poincaré*, 32(6):1283–1305, 2015.

[10] R. M. Colombo and L.-M. Magali. Nonlocal crowd dynamics models for several populations. *Acta Mathematica Scientia*, 32(1):177–196, 2012.

[11] R. M. Colombo, P. Goatin, and M. D. Rosini. A macroscopic model for pedestrian flows in panic situations. *International Series Mathematical Sciences and Applications*, 32:255–272, 2010.

[12] I. D. Couzin. Collective cognition in animal groups. *Trends in Cognitive Sciences*, 13(1):36–43, 2008.

[13] I. D. Couzin, J. Krause, R. James, G. D. Ruxton, and N. R. Franks. Collective memory and spatial sorting in animal groups. *Journal of Theoretical Biology*, 218:1–11, 2002.

[14] I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin. Effective leadership and decision-making in animal groups on the move. *Nature*, 433:513–516, 2005.

[15] E. Cristiani, B. Piccoli, and A. Tosin. Multiscale modeling of granular flows with application to crowd dynamics. *Multiscale Modelling and Simulation*, 9(1):155–182, 2011.

[16] A. Czirok. Collective motion of self-propelled particles kinetic phase transition in one dimension. *Physical Review Letters*, 82(1):209–212, 1999.

[17] C. Dogbe. Applicable thermostatted models to crowd dynamics: Comment on "thermostatted kinetic equations as models for complex systems in physics and life sciences" by Carlo Bianca. *Physics of Life Reviews*, 9(4):410–412, 2012.

[18] R. Eftimie. Hyperbolic and kinetic models for self-organized biological aggregations and movement: a brief review. *Journal of Mathematical Biology*, 65(1):35–75, 2012.

[19] R. Eftimie, G. De Vries, M. A. Lewis, and F. Lutscher. Modeling group formation and activity patterns in self-organizing collectives of individuals. *Bulletin of Mathematical Biology*, 69(5):1537, 2007.

[20] R. C. Fetecau and R. Eftimie. An investigation of a nonlocal hyperbolic model for self-organization of biological groups. *Journal of Mathematical Biology*, 61(4):545–579, 2010.

[21] R. C. Fetecau, Y. Huang, and T. Kolokolnikov. Swarm dynamics and equilibria for a nonlocal aggregation model. *Nonlinearity*, 24(10):2681–2716, 2011.

[22] M. A. Giese, A. Mukovskiy, A.-N. Park, L. Omlor, and J.-J. E. Slotine. Real-time synthesis of body movements based on learned primitives. In *Visual Motion Analysis*, volume 5604 in *Lecture Notes in Computer Science*, pages 107–127, 2009.

[23] J. Glimm, X. L. Li, Y. Liu, and N. Zhao. Conservative front tracking and level set algorithms. *Proceedings of the National Academy of Sciences*, 98(25):14198–14201, 2001.

[24] H. Haken. *Information and Self-Organization A Macroscopic Approach to Complex Systems*. Springer, Berlin, 2006.

[25] D. Helbing. Active walker model for the formation of human and animal trail systems. *Physical Review E*, 56(3):2527–2539, 1998.

[26] D. Helbing, L. Buzna, A. Johansson, and T. Werner. Self-organized pedestrian crowd dynamics: Experiments, simulations, and design solutions. *Transportation Science*, 39(1):1–24, 2005.

[27] R. Hilfer. *Applications of Fractional Calculus in Physics*. World Scientific Publishing Company, Singapore, 2000.

[28] M.-O. Hongler, R. Filliger, and O. Gallay. Local versus nonlocal barycentric interactions in 1D agent dynamics. *Mathematical Biosciences and Engineering*, 11(2):303–315, 2014.

[29] R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research. Part B: Methodological*, 36(6):507–535, 2002.

[30] B. D. Hughes and K. Fellner. Continuum models of cohesive stochastic swarms: The effect of motility on aggregation patterns. *Physica D: Nonlinear Phenomena*, 260:26–48, 2013.

[31] P. Kachroo. *Pedestrian Dynamics: Mathematical Theory and Evacuation Control*. CRC Press/Taylor & Francis Group, Boca Raton/London, 2009.

[32] P. Kachroo, S. J. Al-nasur, S. Amin Wadoo, and A. Shende. *Pedestrian Dynamics Feedback Control of Crowd Evacuation*. Springer-Verlag, Berlin, Heidelberg, 2008.

[33] W. W. F. Klingsch, C. Rogsch, A. Schadschneider, and M. Schreckenberg. *Pedestrian and Evacuation Dynamics*. Springer-Verlag, Berlin, Heidelberg, 2010.

[34] R. Lohner. On the modeling of pedestrian motion. *Applied Mathematical Modelling*, 34(2):366–382, 2010.

[35] A. Mogilner, L. Edelstein-Keshet, L. Bent, and A. Spiros. Mutual interactions, potentials, and individual distance in a social aggregation. *Journal of Mathematical Biology*, 47(4):353–389, 2003.

[36] M. Moussaid, D. Helbing, and G. Theraulaz. How simple rules determine pedestrian behavior and crowd disasters. *Proceedings of the National Academy of Sciences of the United States of America*, 108(17):6884–6888, 2011.

[37] A. Mukovskiy, J.-J. E. Slotine, and M. A. Giese. Dynamically stable control of articulated crowds. *Journal of Computational Science*, 4(4):304–310, 2013.

[38] X. Muzhou, M. Lees, C. Wentong, Z. Suiping, and L. M. Yoke Hean. Hybrid modelling of crowd simulation. *Procedia Computer Science*, 1(1):57–65, 2010.

[39] M. R. D. Orsogna, Y. L. Chuang, A. L. Bertozzi, and L. S. Chayes. Self-propelled particles with soft-core interactions patterns, stability, and collapse. *Physical Review Letters*, 96(10):104302, 2006.

[40] A. Parvate and A. D. Gangal. Fractal differential equations and fractal-time dynamical systems. *PRAMANA Indian Academy of Sciences*, 64(3):389–409, 2005.

[41] N. Pelechano, J. M. Allbeck, and N. I. Badler *Virtual Crowds Methods, Simulation, and Control*. Morgan & Claypool Publishers, Williston, 2008.

[42] A. Quarteroni and A. Veneziani. Analysis of a geometrical multiscale model based on the coupling of ODE and PDE for blood flow simulations. *Multiscale Modeling and Simulation*, 1:173–195, 2003.

[43] M. Romanovas, L. Klingbeil, M. Traechtler, and Y. Manoli. On fractional models for human

motion tracking. In *Proceedings of Symposium on Fractional Differentiation and Its Applications*, 2012.
[44] M. Romanovas, M. Traechtler, L. Klingbeil, and Y. Manoli. On fractional models for human motion tracking. *Journal of Vibration and Control*, 20(7):986–997, 2014.
[45] K. Spieser and D. E. Davison. Multi-agent stabilisation of the psychological dynamics of one-dimensional crowds. *Automatica*, 45(3):657–664, 2009.
[46] D. Stuart, K. Christensen, A. Chen, K.-C. Cao, C. Zeng, and Y. Q. Chen. A framework for modeling and managing mass pedestrian evacuations involving individuals with disabilities: Networked segways as mobile sensors & actuators. In *Proceedings of the ASME 2013 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, DETC2013-12652, 2013.
[47] V. E. Tarasov and G. M. Zaslavsky. Fokker–Planck equation with fractional coordinate derivatives. *Physica A: Statistical Mechanics and its Applications*, 387(26):6505–6512, 2008.
[48] D. Thalmann and S. Raupp Musse. *Crowd Simulation*. Springer,Berlin, 2007.
[49] H. Timmermans. *Pedestrian Behavior: Models, Data Collection and Applications*. Emerald Group Publishing Limited, Bingley, 2009.
[50] J. Toner and T. Yuhai. Long-range order in a two-dimensional dynamical *xy* model: How birds fly together. *Physical Review Letters*, 75:4326–4329, 1995.
[51] L.-F. Wang, X.-J. Yang, D. Baleanu, C. Cattani, and Y. Zhao. Fractal dynamical model of vehicular traffic flow within the local fractional conservation laws. *Abstract and Applied Analysis*, pages 1–5, 2014.
[52] M. Xiong, W. Cai, S. Zhou, M. Yoke-Hean Low, F. Tian, D. Chen, O. N. G. Daren Wee Sze, and B. D. Hamilton. A case study of multi-resolution modeling for crowd simulation. In *Proceedings of Simulation Multiconference, San Diego, California, USA*, 2009.
[53] G. M. Zaslavsky, M. Edelman, and V. E. Tarasov. Dynamics of the chain of forced oscillators with long-range interaction: From synchronization to chaos. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 17(4):043124, 2007.
[54] H. M. Zhang. A mathematical theory of traffic hysteresis. *Transportation Research. Part B: Methodological*, 33(1):1–23, 1999.
[55] H. M. Zhang. New perspectives on continuum traffic flow models. *Networks and Spatial Economics*, 1:9–33, 2001.

4 Mesoscopic model of fractional order for crowds of pedestrians

Abstract: Based on the microscopic model and the macroscopic model proposed in the previous two chapters, mesoscopic models of fractional order for crowds of pedestrians are further considered in this chapter. In order to characterize the competitive and cooperative interactions among pedestrians, fractional mean-field games are utilized when the number of pedestrians goes to infinity. Mesoscopic models of fractional order that are composed of fractional backward and fractional forward equations are constructed for crowds of pedestrians in this chapter.

4.1 Introduction

Helbing's framework of social forces has received a lot of attention in modeling crowds of pedestrians due to its similarity with the framework of Newton's principle and it is easy to understand. Another reason for the widespread use of this framework lies in the heterogeneity of the pedestrians, such as individual mobilities or reactions, which can be considered explicitly using Helbing's framework. Thus not only theoretical work but also simulation results have gained a lot of attention, as shown in [5, 10, 9, 29, 28].

At the same time, macroscopic models were also studied by a lot of researchers, as in [20, 15, 17, 18], to relieve the burden of computation when the number of pedestrians goes to infinity. Although the computational burden in macroscopic model has been reduced greatly compared to that in microscopic model, pedestrians have been treated as same particles in the previous research. Thus, the main disadvantage of macroscopic model is that individual characters or heterogeneity of each pedestrian has been neglected in macroscopic model.

For crowds with large numbers of pedestrians, it is impossible and not necessary to consider all interactions one by one. Methods based on mean field have been used to approximate the effects of interactions, whose basic idea is replacing the total interactions with an averaged interaction that is in "mean-field" form to relieve the burden of computation. Methods based on mean field have been applied to control of multi-agent systems in [25, 26, 27], where decentralized consensus protocols and decentralized optimizing algorithms were considered. In recent years, this method was also introduced in the modeling problem for crowds of pedestrians. Coupled dynamic models composed of backward Hamilton–Jacobi–Bellman (HJB) equations and forward Fokker–Planck equations have been constructed for crowds of pedestrians in [13] using a mean-field limit approach. In [21], the method of mean-field games (MFGs) is utilized to model the congestion and aversion phenomenon that is observed in inter-

https://doi.org/10.1515/9783110473988-004

actions of two types of pedestrians, and coupled dynamic models composed of backward (HJB) equations and forward Fokker–Planck equations have been constructed. The theory of MFGs has also been used in modeling traffic systems at meso-scale, as shown in [8], where a micro–macro model has been proposed. Other applications of the mean-field methods are referred to physical systems, financial systems, and social-dynamic systems as shown in [1, 7, 14, 13].

Table 4.1: Comparisons of models for crowds in different scales.

	Heterogeneity	Interactions	Computation ($N \to \infty$)
Microscopic model	Good	Good	Not good
Macroscopic model	Not good	Not good	Good
Mesoscopic model	Acceptable	Acceptable	Acceptable

Remark 4.1. Comments on microscopic, macroscopic, and mesoscopic model for crowds of pedestrians are summarized in Table 4.1.
- The main advantages of the microscopic model lie in modeling heterogeneity of each pedestrian, which can be considered explicitly, but the burden of computation and the curse of dimensions are the main disadvantages for microscopic model when the number of pedestrians goes to infinity.
- The computation time has been greatly decreased in macroscopic models but some of the heterogeneity of pedestrians are lost in macroscopic model.
- As a negotiation between microscopic model and macroscopicmo del, these two advantages mentioned above have been combined in mesoscopic model.

As shown by [6], the mesoscopic scale has been selected as the most appropriate scale to describe the complex characteristics of human crowds. Mesoscopic models of fractional order are considered in this chapter for crowds of pedestrians. Based on the previous two chapters, fractional MFG theory is introduced in this chapter to model the competitive and cooperative interactions among pedestrians. For completeness, the mesoscopic model of integer order and the mesoscopic model of fractional order are both included in this chapter.

4.2 Mesoscopic model of integer order

4.2.1 Mesoscopic model based on kinetic theory

Boltzmann's transport equation is an effective and powerful tool in describing the statistical behavior of thermodynamic systems since the nineteenth century. It has been widely used in astrophysics, engineering, social science, and even biology. Although Boltzmann's equation described by

$$\frac{\partial}{\partial t} f(t,x,\xi) + \xi \cdot \nabla_x f(t,x,\xi) = 0 \tag{4.1}$$

looks very simple, great success and many generalizations of this equation have been made in previous research. As both variables in macro-scale and variables in micro-scale are connected in equation (4.1), a lot of mesoscopic model for crowds of pedestrians have been reported based on the Boltzmann equation (4.1).

The following mesoscopic model has been firstly proposed in [15] based on the conservation laws of mass and momentum at macro-scale,

$$\frac{\partial}{\partial t}\rho_\mu + \frac{\partial}{\partial x}(\rho_\mu v_\mu) = \int m_\mu q_\mu dv_\mu + \sum_\mu \left[\frac{m_\mu}{m_\nu}\rho_\nu \chi_\mu^{\mu\nu}(1) - \rho_\mu \chi_\mu^{\mu\nu}(1) \right], \tag{4.2}$$

where ρ_μ is the density, the first term on the right is caused by pedestrians entering or leaving the areas of interest, and the second term describes the effects caused by internal motivations and external influences. The results of [15] were further generalized to crowds in two-dimensional space in [16],

$$\partial_t \rho + [\partial_{x_1}(\rho v_1) + \partial_{x_2}(\rho v_2)] + [\partial_{v_1}(\rho A_1) + \partial_{v_2}(\rho A_2)] = (\partial_t \rho)^+_{\text{event}} - (\partial_t \rho)^-_{\text{event}},$$

where $\partial_{x_1}(\rho v_1) + \partial_{x_2}(\rho v_2)$ means the changes of density due to convection, $\partial_{v_1}(\rho A_1) + \partial_{v_2}(\rho A_2)$ are terms describing the acceleration and deceleration of crowds, and $(\partial_t \rho)^+_{\text{event}} - (\partial_t \rho)^-_{\text{event}}$ describes the influence of events. Nonlinear mesoscopic models have been reported in [2] to describe the evolution of the crowds' distribution under competitions and interactions for different types of crowds,

$$\frac{\partial f_i}{\partial t}(t,u) = J[f_i](t,u) + \gamma_i(t,u),$$

where f_i is the distribution of the ith type of crowds at time t, $J[f_i](t,u)$ describes the gain and loss of the distribution f_i, and $\gamma_i(t,u)$ describes the production and migration of the ith type of crowds due to artificial inlet. A general mesoscopic model, proposed in [4], is

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} + F(t,x)\frac{\partial f}{\partial V} = Q(f,u), \tag{4.3}$$

where $f(\cdot)$ is the distribution function, $F(\cdot)$ is the acceleration applied to it by environment, and $Q(\cdot)$ is derived by phenomenological arguments that is depending on f and on local gross quantities u. Not only binary interactions, but also mean-field interactions can be described using the general mesoscopic model (4.3).

More and more general interactions have been considered in mesoscopic models so that the obtained models are much closer to real crowds of pedestrians. For example, Enskog-like interactions and stochastic interactions have been considered in [12], nonlinear interactions instead of linear interactions have been studied in [3], and both short-range and long-range interactions have been reported in [11].

4.2.2 Mesoscopic model based on mean-field games

MFGs have been recently introduced by Lasry and Lions in [22, 23, 24] and seem to be very useful in modeling complex interactions in large group of "intelligent individuals." The general framework for MFGs can be shown by the differential games among N players whose microscopic models are described by

$$\dot{x}_i(t) = v_i(t)dt + \sigma d\mathcal{B}_i, \tag{4.4}$$

where $i = 1, \dots, N$, $v_i(t)$ is the control input or velocity of each pedestrian, and $\mathcal{B}_i$ is an independent Brownian motion. Each pedestrian tries to minimize the following cost function:

$$E_{x_i}\left[\int_t^T \left[\frac{1}{2}|v_i(s)|^2 + L\left(x_s, m\left(s, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j(s)}\right)\right)\right]ds + G\left(X_T, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j(T)}\right)\right], \tag{4.5}$$

where the terms $\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j(s)}$ and $\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j(T)}$ means the influence of other pedestrians on the ith pedestrian. Due to the ability of anticipating the evolution of crowds, some terms depending on the final state can also be added

$$\mathcal{E}\left[\int_0^T L(X_t^x, u_t)dt + V(X_t^x)dt + \Psi(X_T^x)\right],$$

where $L(X_t^x, u_t)$ is the control and position cost, $V(X_t^x)$ is the state cost, and $\Psi(X_T^x)$ are terms depending on the final state of the crowds. The solution of the above problem has been reduced to find a solution to the following MFG system:

$$\begin{cases} \partial_t m - \dfrac{\sigma^2}{2}\Delta m + \operatorname{div}(m)\partial_p H(x, \nabla v) = 0, \\ \qquad \partial_t v + \dfrac{\sigma^2}{2}\Delta v + H(x, \nabla v) = V(m), \end{cases} \tag{4.6}$$

where H is the Legendre transform of L as in [13].

(1) Deterministic mean-field games

For simplicity, all pedestrians in large crowds of pedestrians are assumed to have identical motivations and the movement of each pedestrian is assumed to be described by

$$\dot{x}_i(t) = v(t, x_i(t)).$$

Denote $m(t, x)$ as the density with respect to the Lebesgue measure on R^d. Then the following approximation

$$\frac{1}{N}\sum_{i=1}^{N} F(x_i(t)) \approx \int_{R^d} m(t, x)F(x)dx \tag{4.7}$$

is satisfied for any smooth, compactly supported functions $F(x)$. After differentiating both sides of (4.7), the following advection equation can be obtained:

$$\partial_t m(t,x) + \nabla \cdot (mv)(t,x) = 0, \tag{4.8}$$

which describes the evolution of crowds without random noise.

Contrary to the value function in the classical optimization problem, the following value function has been used in deterministic MFGs:

$$\inf \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s, m(s,X_s)) \right] ds + G(X_T, m(T,X_T)), \tag{4.9}$$

where $m(t,x)$ is the non-negative density function that satisfies $\int_{R^d} m(s,x)dx = 1$ for all $s \in [0,T]$. Parallel to the above analysis in classical optimization problems, the following HJB equation can be obtained:

$$-\partial_t J(t,X) + \frac{1}{2}\left|\nabla_x J(t,X)\right|^2 = L(X,m). \tag{4.10}$$

Thus the mesoscopic model for large crowds without noise can be described by the following coupled equations composed of the backward HJB equation and the forward advection equation:

$$\begin{cases} -\partial_t J(t,X) + \frac{1}{2}\left|\nabla_x J(t,X)\right|^2 = L(X,m), \\ \partial_t m(t,x) + \nabla \cdot (mv)(t,x) = 0. \end{cases}$$

(2) Stochastic mean-field games

Different from the problem considered in deterministic case, now each pedestrian is described by

$$\dot{x}_i(t) = v(t, x_i(t))dt + \sigma d\mathcal{B}_t,$$

where $\mathcal{B}_t$ is the standard Brownian motion. Based on the approximation of

$$\frac{1}{N}\sum_{i=1}^{N} F(x_i(t) + v(t,x_i(t))dt + \sigma d\mathcal{B}_t) \approx \int_{R^d} m(t+dt,x)F(x)dx, \tag{4.11}$$

the following Fokker–Planck equation can be derived to describe the evolution of crowd's distribution at macro-scale

$$\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \nabla \cdot (mv)(t,x) = 0. \tag{4.12}$$

At micro-scale, each pedestrian tries to minimize the following value function:

$$E\left[\int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s, m(s,X_s)) \right] ds + G(X_T, m(T,X_T)) \right].$$

After Taylor expansion, the HJB equation

$$-\partial_t J(t,X) - \frac{\sigma^2}{2}\Delta J + \frac{1}{2}\left|\nabla_x J(t,X)\right|^2 = L(X,m) \tag{4.13}$$

can be derived with the help of Ito's formula. Thus the mesoscopic model for large crowds with noise can be described by the following coupled backward HJB equation and forward Fokker–Planck equation:

$$\begin{cases} -\partial_t J(t,X) + \frac{1}{2}\left|\nabla_x J(t,X)\right|^2 = L(X,m), \\ \partial_t m(t,x) + \nabla \cdot (mv)(t,x) = 0. \end{cases}$$

Remark 4.2. In real crowds of pedestrians, the movement and decision making of each pedestrian depend on his/her neighbors, which may be stochastically changing according to time, space, or other stochastic effects. Different kinds of MFGs have been proposed according to different couplings at micro-scale and different pay-off functions introduced in decision-making, which have been summarized in Table 4.2.

Table 4.2: Categories of MFGs.

	Movement	**Payoff**
Weak MFG	(4.14)	(4.15)
Weak MFG	(4.16)	(4.17)
Weak MFG	(4.18)	(4.17)
Strong MFG	(4.18)	(4.19)

$$dx_{j,t} = f_{jt}(x_{jt}, u_{jt})dt + \sigma_{jt}(x_{jt}, u_{jt})d\mathcal{B}_{jt}, \tag{4.14}$$

$$V_{jt} = \frac{1}{N}\sum_{i=1}^{N} \bar{V}(x_{jt}, u_{jt}, x_{it}), \tag{4.15}$$

$$dx_{j,t} = f_{jt}(x_{jt}, u_{jt}, x_{-jt}, u_{-jt})dt + \sigma_{jt}(x_{jt}, u_{jt})d\mathcal{B}_{jt}, \tag{4.16}$$

$$V_{jt} = \frac{1}{N}\sum_{i=1}^{N} \bar{V}(x_{jt}, u_{jt}), \tag{4.17}$$

$$dx_{j,t} = \sum_{i\in N_j} \omega_{ij}(t) f_{jt}(x_{jt}, u_{jt}, x_{it}, u_{it})dt + \sum_{i\in N_j} \omega_{ij}(t)\sigma_{jt}(x_{jt}, u_{jt}, x_{it}, u_{it})d\mathcal{B}_{jt}, \tag{4.18}$$

$$V_{jt} = \sum_{i\in N_j} \omega_{ij}(t)\bar{V}(x_{jt}, u_{jt}, x_{it}, u_{it}). \tag{4.19}$$

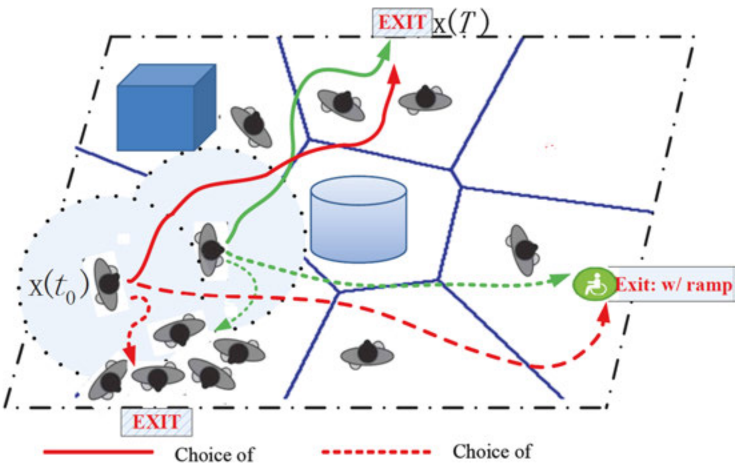

Figure 4.1: Movement of pedestrians based on fractional mean-field games.

4.3 Mesoscopic model of fractional order

4.3.1 Hamilton–Jacobi–Bellman equation of fractional order

(1) Deterministic case

For each pedestrian i, we assume the following cost function to be minimized in his/her movement from the initial starting point $x(t_0) = x_0$ to the desired location $x(T)$, as shown in Figure 4.1:

$$J(t_0, x_0) = \inf_{v(\cdot)} \int_{t_0}^{T} f(t, x(t), v(t))dt^{\alpha} + h(T, x(T)), \tag{4.20}$$

where the terminal cost is described using the convex function $h(T, x(T))$ and the convex even function $f(t, x(t), v(t))$ means the running cost during the movement.

Similar to the derivation of the HJB equation of integer order in Section 4.2.2, the fractional HJB equation will be discussed first and then the optimal velocity will be proposed for each pedestrian.

Denote the cost function from (t_0, x_0) to (T, x_T) by $J(t_0, x_0)$. Then the pedestrian will arrive at a new place $x_0 + vdt^{\alpha}$ after an infinitesimal time interval dt^{α}. The running cost on the time interval $(t_0, t_0 + dt^{\alpha})$ can be described by

$$f(v)dt^{\alpha}.$$

For pedestrians at the new position $(t_0 + dt^{\alpha}, x_0 + vdt^{\alpha})$, the cost function for the remaining journey is described by $J(t_0 + dt^{\alpha}, x_0 + vdt^{\alpha})$. It is easy to obtain the following relationship between $J(t_0, x_0)$ and $J(t_0 + dt^\alpha, x_0 + vdt^\alpha)$:

$$J(t_0, x_0) = J(t_0 + dt^\alpha, x_0 + vdt^\alpha) + f(v)dt^\alpha. \tag{4.21}$$

Based on a Taylor expansion, equation (4.21) can be rewritten as

$$J(t_0, x_0) = J(t_0, x_0) + dt^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} J(t_0, x_0) + v \cdot \frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0) + f(v) \right], \tag{4.22}$$

and the optimal problem (4.20) is now transformed into finding a proper v minimizing

$$v \cdot \frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0) + f(v), \tag{4.23}$$

where minimum v will be unique and will be some functions of $\frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0)$ due to the strict convexity condition. The Legendre transformation $H : R^d \to R$ of $f : R^d \to R$ with

$$H(p) := \sup_{v(\cdot)} [v \cdot p - f(v)] \tag{4.24}$$

is introduced in solving the minimum problem (4.23). Then the minimization value of equation (4.23) is just $-H(\frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0))$, since $f(\cdot)$ is an even function. Then substituting the minimum value $-H(\frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0))$ into equation (4.22),

$$J(t_0, x_0) = J(t_0, x_0) + dt^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} J(t_0, x_0) - H\left(\frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0) \right) \right]$$

will be satisfied for any t_0 and any x_0. Then the fractional HJB equation is derived as

$$-\frac{\partial^\alpha}{\partial t^\alpha} J(t_0, x_0) + H\left(\frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0) \right) = 0. \tag{4.25}$$

From the above discussions, we know that there are some v minimizing the following expression:

$$v \cdot \frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0) + f(v),$$

and $\widetilde{v} = -v$ maximizes the following:

$$v \cdot \frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0) - f(v).$$

As seen from (4.24), $\widetilde{v}$ as a function of p should satisfy

$$\frac{\partial}{\partial \widetilde{v}} (\widetilde{v} \cdot p - f(\widetilde{v})) = 0.$$

Thus the derivative of $H(p)$ can be obtained using the chain rule as follows:

$$\begin{aligned}\frac{d}{dp}H(p) &= \frac{\partial H}{\partial \widetilde{v}}\frac{\partial \widetilde{v}}{\partial p} + \frac{\partial H}{\partial p} \\ &= \frac{\partial}{\partial \widetilde{v}}(\widetilde{v}\cdot p - f(\widetilde{v}))\frac{\partial \widetilde{v}}{\partial p} + \frac{\partial H}{\partial p} \\ &= \widetilde{v},\end{aligned}$$

and the following velocity for each pedestrian to move in the next step is derived:

$$v = -H'\left(\frac{\partial^\beta}{\partial x^\beta}J(t_0, x_0)\right).$$

(2) Stochastic case

When the pedestrian's movement evolves according to the controlled stochastic differential equation shown in (4.4), the behavior of the pedestrians can now be evaluated by the following stochastic cost function:

$$J(t_0, x_0) = E\left[\int_{t_0}^{T} f(t, x(t), v(t))dt^\alpha + h(T, x(T))\big|_{x_s = x_0}\right]. \tag{4.26}$$

After Taylor expansion using Ito's formula, the cost function (4.26) can be written as

$$J(t_0, x_0) = J(t_0, x_0) + dt^\alpha\left[\frac{\partial^\alpha}{\partial t^\alpha}J(t_0, x_0) + v\cdot\frac{\partial^\beta}{\partial x^\beta}J(t_0, x_0) + \frac{\sigma^2}{2}\Delta J(t_0, x_0) + L(v)\right],$$

where the derivation has been omitted as it is similar to that in the previous section. Finally, the dynamic equation of fractional order

$$\frac{\partial^\alpha}{\partial t^\alpha}J(t_0, x_0) - H\left(\frac{\partial^\beta}{\partial x^\beta}J(t_0, x_0)\right) + \frac{\sigma^2}{2}\Delta J(t_0, x_0) = 0 \tag{4.27}$$

can be constructed to describe how the pedestrians evaluate their behavior under Brownian motion noise. Compared with the deterministic case in (4.25), the only difference is that an additional term of $\frac{\sigma^2}{2}\Delta J(t_0, x_0)$ has been added in equation (4.27).

Remark 4.3. A typical quadratic cost function that does not depend on the position of pedestrians can be selected as $\frac{1}{2}|v|^2$ to penalize pedestrians that are moving too fast. Much more generalized running cost functions that are depending on time, position and velocity, can also be used as shown in the following section.

4.3.2 Mesoscopic model of fractional order with interactions

In constructing the mesoscopic model, a decomposition of the motion space is firstly conducted using the Voronoi diagram technique. Based on the decomposition of motion space, the microscopic model of fractional order and the macroscopic model of fractional order are coupled on the Voronoi diagram using techniques of aggregation and dis-aggregation, as shown in Figure 4.2. Within this framework, not only local interactions, which have received a lot of attention in the previous study, but also non-local interactions, which are seldom considered, can be mixed together in the mesoscopic models of this chapter.

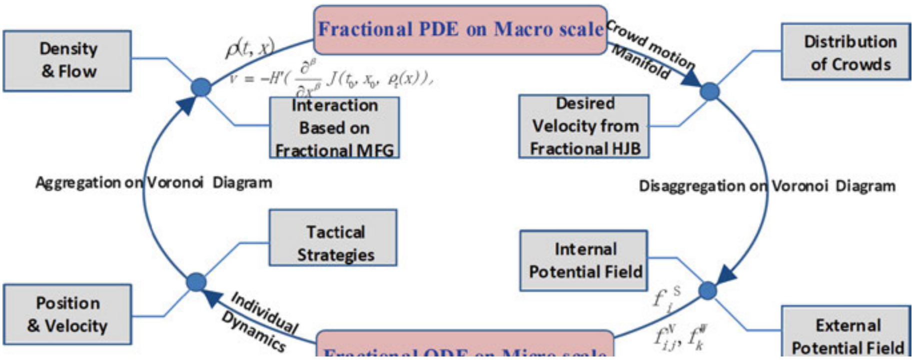

Figure 4.2: Fractional micro–macro model of crowds of pedestrians.

(1) MFGs model with local interactions

We describe the movement of each pedestrian by

$$\frac{\partial x_i^2(t)}{\partial t^2} = \sum_{j=1}^{n} f_{ij} + F(x_i(t)), \tag{4.28}$$

where $x_i(t)$ is the displacement from the exit, $F(x_i(t))$ is the external force that the ith pedestrian has received, and the term $\sum_{j=1}^{n} f_{ij}$ means the interaction between the ith pedestrian and all his/her neighbors, which is defined as

$$\sum_{j=1}^{n} f_{ij} = \sum_{j=1}^{n} J(i,j)x_j, \tag{4.29}$$

where $J(i,j)$ means the strength of interactions between the ith pedestrian and the jth pedestrian. The following $J(i,j)$ has been selected for local interactions:

$$J(i,j) = \delta_{i+1,j} - 2\delta_{i,j} + \delta_{i-1,j},$$

where $\delta_{i,j}$ is the Kronecker symbol and

$$\sum_{j=1}^{n} f_{ij} = (x_{i+1}(t) - x_i(t)) - (x_i(t) - x_{i-1}(t)).$$

It has been proved in [30] that the continuous macroscopic equation when the number of pedestrians goes to infinity can be described as

$$\frac{\partial^2}{\partial t^2}\rho(t,x) = G\frac{\partial^2}{\partial x^2}\rho(t,x) + F(u). \tag{4.30}$$

Since both the differentiations with respect to time and space in (4.30) are of integer order, the HJB equation selected in this case is the integer version of (4.25), which is described by

$$-\frac{\partial}{\partial t}J(t_0,x_0,\rho_t(x)) + H\left(\frac{\partial}{\partial x}J(t_0,x_0,\rho_t(x))\right) = 0.$$

Thus, the mesoscopic model based on MFG can be obtained as follows:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\rho(t,x) - G\dfrac{\partial^2}{\partial x^2}\rho(t,x) - F(u) = 0, \\ -\dfrac{\partial}{\partial t}J(t_0,x_0,\rho_t(x)) + H\left(\dfrac{\partial}{\partial x}J(t_0,x_0,\rho_t(x))\right) = 0. \end{cases}$$

(2) MFGs model with non-local interactions

For non-local interactions, the long-range interaction, defined in [19],

$$J(n) = |n|^{-(\beta+1)}$$

will be used in constructing the mesoscopic model based on MFGs, where β is a positive non-integer number. The dynamic equation with Riesz derivatives of fractional order

$$\frac{\partial^2}{\partial t^2}\rho(t,x) - \frac{\partial^\beta}{\partial |x|^\beta}\rho(t,x) - F(u) = 0$$

can be derived to model the distribution of crowds in this case, where $0 < \beta < 1$ comes from the order for long-range interactions. The mesoscopic model for crowds of pedestrians in this case can be described by

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\rho(t,x) - \dfrac{\partial^\beta}{\partial |x|^\beta}\rho(t,x) - F(u) = 0, \\ -\dfrac{\partial}{\partial t}J(t_0,x_0,\rho_t(x)) + H\left(\dfrac{\partial^\beta}{\partial |x|^\beta}J(t_0,x_0,\rho_t(x))\right) = 0. \end{cases}$$

(3) Generalized mesoscopic model

Based on the recent advances in motion of random particles, it is well known that the classical diffusion model,

$$\frac{\partial W}{\partial t} = K\frac{\partial^2}{\partial x^2}W(x,t),$$

can be used to describe the movement of a particle's diffusion when the mean waiting time and the jump length variance are finite. Using calculus of fractional order, the classical diffusion can be further generalized as follows.

- *The power law waiting time and jump length variance are finite.*
  If the probability distribution function (PDF) of the waiting time satisfies the following power law:

$$\psi(t) \backsim t^{-(1+\alpha)} \quad (0 < \alpha < 1) \tag{4.31}$$

  and the variance of jump length is finite, then the movement of these particles can be described by the following sub-diffusion equation:

$$\frac{\partial^\alpha W}{\partial t^\alpha} = K \frac{\partial^2}{\partial x^2} W(x,t),$$

  where the Caputo fractional derivative is used.
- *The power law jump length variance and waiting time are finite.*
  If the probability distribution function (PDF) of the jump length variance satisfies the power law

$$\eta(x) \backsim x^{-(1+\beta)} \quad (0 < \beta < 2) \tag{4.32}$$

  and the mean waiting time is finite, then the movement of these particles can be described by the following super-diffusion equation:

$$\frac{\partial W(x,t)}{\partial t} = K \frac{\partial^\beta}{\partial x^\beta} W(x,t),$$

  where the spatial Riesz fractional derivative is used and $W(x,t)$ is the probability distribution function (PDF) in x at time t.
- *Power law waiting time and power law jump length variance.*
  Based on the above statement, the generalized dynamic model of fractional order can be obtained:

$$\frac{\partial^\alpha W}{\partial t^\alpha} = K \frac{\partial^\beta}{\partial x^\beta} W(x,t)$$

  when both of the two assumptions (4.31) and (4.32) are satisfied.

Based on [8] addressing the traffic system, we assume the following utility function or velocity-choosing scheme for the ith pedestrian:

$$f_i^N(x_i, v_i) = v_i\left(1 - F\left(\frac{1}{N}\sum \omega(x_j - x_i)\right)\right),$$

where the first term v_i means that the ith pedestrian tries to arrive at his destination as fast as possible and the second term means that the ith pedestrian tunes his/her velocity according to pedestrians around him. The bounded non-negative anticipating function $\omega(\cdot)$ has been introduced to weight different impacts from different neighbors. Thus for the ith pedestrian, cooperative and competitive interactions with other pedestrians are manifested through choosing the velocity for the next step.

Denote by N the number of interacting pedestrians, by $\rho_t(y)$ the number of pedestrians in interval $[x, x + dx^\beta]$, and by $\omega(\cdot)$ the smooth anticipating function that is compactly supported functions on the space. Then the following expression can be satisfied:

$$\lim_{N\to\infty} \frac{1}{N} \sum \omega(x_j - x_i) \to \int_0^\infty \rho_t(y)\omega(y - x)dy^\beta. \tag{4.33}$$

Using the Lebesgue–Stieltjes integral, (4.33) can also be written as

$$\frac{1}{N} \sum \omega(x_j - x_i) \approx \int_0^\infty \omega(y - x)d\Gamma_t^N(y),$$

where $\Gamma_t^N(x)$ is the empirical distribution function for the crowds. If there is a nondecreasing right-continuous function $\Gamma_t(x)$ such that the following expression is satisfied:

$$\int_0^\infty \omega(y - x)d\Gamma_t^N(y) \to \int_0^\infty \omega(y - x)d\Gamma_t(y) \quad (N \to \infty),$$

then the following limits

$$\frac{1}{N} \sum \omega(x_j - x_i) \to \int_0^\infty \rho_t(y)\omega(y - x)dy^\beta \quad (N \to \infty),$$

$$v_i\left(1 - F\left(\frac{1}{N} \sum \omega(x_j - x_i)\right)\right) \to v_i\left(1 - F\left(\int_0^\infty \rho_t(y)\omega(y - x)dy^\beta\right)\right)$$

will be satisfied when the number of pedestrians N goes to infinity. Then the macroscopic model in terms of density and velocity and fractional HJB equation can be obtained using similar derivations as shown in Section 4.3.1. Thus the generalized model

$$\begin{cases} -\dfrac{\partial^\alpha}{\partial t^\alpha} J(t_0, x_0, \rho_t(x)) + H\left(\dfrac{\partial^\beta}{\partial x^\beta} J(t_0, x_0, \rho_t(x))\right) = 0, \\ \dfrac{\partial}{\partial t^\alpha}\rho(t, x) + \dfrac{\partial}{\partial x^\beta}[\rho(t, x)v(t, x)] = 0, \end{cases} \tag{4.34}$$

which is based on MFGs, can be proposed for crowds of pedestrians at meso-scale.

Remark 4.4. (1) Due to the coupling relationship between the microscopic model and macroscopic model in (4.34), the optimal velocity v for the next step can be solved from the first line of equation (4.34) under initial conditions of $J(T, X_T)$ and $\rho_0(x)$, while the density of crowds $\rho(t, x)$ can be solved from the second line of equation (4.34) under the initial condition of $\rho_0(x)$ and the following v:

$$v = -H'\left(\frac{\partial^\beta}{\partial x^\beta} J(t_0, x_0, \rho_t(x))\right).$$

(2) Comparison with a recent study on the mesoscopic model.
- Only functions of Dirac type and exponential type for $\omega(x_j - x_i)$ have been considered in [8] using the framework of calculus of integer order. Some other generalized anticipating function such as the following one of inverse power law

$$f_i^N(x_i, v_i) = v_i\left[1 - F\left(\frac{1}{N}\sum(|x_j - x_i| + 1)^{-2}\right)\right],$$

can be easily considered within the framework of calculus of fractional order.
- MFG theory has been utilized in [13] for modeling crowds of pedestrians at meso-scale. But work of [13] was mainly conducted within the framework of calculus of integer order and some statistical features at temporal scale and spatial scale have not been considered, such as power law in the distribution of crowds, power law in the distribution of inter-event time, and long-range interactions among pedestrians.

4.4 Conclusion

Due to the complexity of crowds of pedestrians, a mesoscopic model of fractional order has been proposed based on the microscopic model and the macroscopic model in the previous two chapters. Fractional MFGs has also been utilized to describe the mesoscopic model when the number of pedestrians goes to infinity. Fractional backward–forward partial differential equations have been presented in the end. As only some theoretical work has been obtained in this chapter, a lot of work has been left for future research as regards this topic, such as the existence and uniqueness of solution, the rate of convergence, and the stability of equilibria.

References

[1] Y. Achdou, F. Camilli, and I. Capuzzo-Dolcetta. Mean field games numerical methods for the planning problem. *SIAM Journal on Control and Optimization*, 50(1):77–109, 2012.
[2] L. Arlotti, N. Bellomo, and M. Lachowicz. Kinetic equations modelling population dynamics. *Transport Theory and Statistical Physics*, 29(1–2):125–139, 2000.
[3] L. Arlotti, E. De Angelis, L. Fermo, M. Lachowicz, and N. Bellomo. On a class of integro-differential equations modeling complex systems with nonlinear interactions. *Applied Mathematics Letters*, 25(3):490–495, 2012.

[4] N. Bellomo, M. Delitala, and V. Coscia. On the mathematical theory of vehicular traffic flow I. Fluid dynamic and kinetic modelling. *Mathematical Models and Methods in Applied Sciences*, 12(12): 1801–1843, 2002.
[5] N. Bellomo, C. Bianca, and V. Coscia. On the modeling of crowd dynamics: an overview and research perspectives. *SeMA Journal*, 54(1):25–46, 2013.
[6] N. Bellomo, D. Clarke, L. Gibelli, P. Townsend, and B. J. Vreugdenhil. Human behaviours in evacuation crowd dynamics: From modelling to "big data" toward crisis management. *Physics of Life Reviews*, 18:1–21, 2016.
[7] P. E. Caines. Mean field stochastic control. Technical report, Proceedings of the 48th Conference on Decision and Control, 2009.
[8] G. Chevalier, J. LeNy, and R. Malhame. A micro–macro traffic model based on mean-field games. In *Proceedings of the American Control Conference*, pages 1983–1988, 2015.
[9] I. D. Couzin. Collective cognition in animal groups. *Trends in Cognitive Sciences*, 13(1):36–43, 2008.
[10] I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin. Effective leadership and decision-making in animal groups on the move. *Nature*, 433:513–516, 2005.
[11] S. De Lillo, M. C. Salvatori, and N. Bellomo. Mathematical tools of the kinetic theory of active particles with some reasoning on the modelling progression and heterogeneity. *Mathematical and Computer Modelling*, 45(5–6):564–578, 2007.
[12] M. Delitala. Nonlinear models of vehicular traffic flow-new frameworks of the mathematical kinetic theory. *Comptes Rendus. Mécanique*, 331(12):817–822, 2003.
[13] C. Dogbe. Modeling crowd dynamics by the mean-field limit approach. *Mathematical and Computer Modelling*, 52(9–10):1506–1520, 2010.
[14] O. Gueant. A reference case for mean field games models. *Journal de Mathématiques Pures et Appliquées*, 92(3):276–294, 2009.
[15] D. Helbing. A fluid dynamic model for the movement of pedestrians. *Complex Systems*, 6(5):391–415, 1992.
[16] S. P. Hoogendoorn and P. H. L. Bovy. Gas-kinetic modelling and simulation of pedestrian flows. *Journal of the Transportation Research Board*, 1710(1):28–36, 2007.
[17] R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research. Part B: Methodological*, 36(6):507–535, 2002.
[18] R. L. Hughes. The flow of human crowds. *Annual Review of Fluid Mechanics*, 35:169–182, 2003.
[19] R. Ishiwata and Y. Sugiyama. Relationships between power-law long-range interactions and fractional mechanics. *Physica A*, 391(23):5827–5838, 2012.
[20] P. Kachroo. *Pedestrian Dynamics: Mathematical Theory and Evacuation Control*. CRC Press/Taylor & Francis Group, Boca Raton/London, 2009.
[21] A. Lachapelle and M.-T. Wolfram. On a mean field game approach modeling congestion and aversion in pedestrian crowds. *Transportation Research. Part B: Methodological*, 45(10):1572–1589, 2011.
[22] J.-M. Lasry and P.-L. Lions. Mean field games I: The stationary case. *Comptes Rendus. Mathématique*, 343(9):619–625, 2006.
[23] J.-M. Lasry and P.-L. Lions. Mean field games II: Finite horizon and optimal control. *Comptes Rendus. Mathématique*, 343(10):676–684, 2006.
[24] J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, 2007.
[25] M. Nourian, R. P. Malhame, Huang M., and P. E. Caines. Mean field (NCE) formulation of estimation based leader-follower collective dynamics. *International Journal of Robotics & Automation*, 26(1):120–129, 2011.

[26] M. Nourian, P. E. Caines, R. P. Malhame, and M. Huang. Mean field lqg control in leader-follower stochastic multi-agent systems likelihood ratio based adaptation. *IEEE Transactions on Automatic Control*, 57(11):2801–2816, 2012.

[27] M. Nourian, P. E. Caines, R. P. Malhame, and M. Huang. Nash, social and centralized solutions to consensus problems via mean field control theory. *IEEE Transactions on Automatic Control*, 58(3):639–653, 2013.

[28] N. Shiwakoti, M. Sarvi, G. Rose, and M. Burd. Animal dynamics based approach for modeling pedestrian crowd egress under panic conditions. *Transportation Research. Part B*, 45(9):1433–1449, 2011.

[29] W. Song, X. Xu, B.-H. Wang, and S. Ni. Simulation of evacuation processes using a multi-grid model for pedestrian dynamics. *Physica A*, 363:492–500, 2006.

[30] V. E. Tarasov. *Fractional Dynamics Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Springer, Berlin, 2011.

Part II: **Fractional control of large crowds of pedestrians**

Controlling flow of pedestrians has been recognized to be a very important research area especially in scenarios of emergencies. Feedback controllers can not only improve the performance of flow of pedestrians but also enhance the robustness to internal or external perturbations. Compared with the previous study on feedback control of flow of pedestrians, dynamic model of fractional order has been considered in this part where appropriate controllers has been proposed. Information obtained through using different sensors is also introduced to guarantee realization of smooth and efficient evacuation of crowds of pedestrians. As only some simple control problems have been considered for the mathematical model of fractional order, there is a lot of interesting work unexplored along this direction. The authors hope that this part can provide some inspirations for other researchers who are interested in control of crowds of pedestrians.

5 Cluster consensus for crowds of pedestrians at micro-scale

Abstract: As the phenomenon of cluster consensus is commonly observed in movement of crowds, the cluster consensus problem is considered in this chapter. Different to previous work on cluster consensus problems where a consensus protocol of integer order has been studied for systems of integer order, the cluster consensus problem for a system of fractional order is studied in this chapter. One sufficient condition has been obtained for the cluster consensus of system of fractional order. The influence of the fractional order on the cluster consensus is also analyzed using simulations. Results obtained in this chapter can provide some reference for choosing leaders in evacuation control of crowds of pedestrians.

5.1 Introduction

Due to the similarity to Newton's principle, the microscopic model for crowds of pedestrians in the social-force framework has received a lot of attention in different scenarios, such as panic crowds [12], consensus of crowds with leaders [9], interactions among pedestrians [6, 5], or interactions among subgroups [25]. Some other modeling methods at micro-scale have also been reported in recent years, such as the agent-based model shown in [11, 24], or cellular automaton models in [33]. For a short review and some comments on the microscopic model, please refer to Chapter 2 of this book or the recent reviews in [33] and [4].

Compared to the modeling problem for crowds of pedestrians, control of crowds of pedestrians is recognized to be another very important problem, especially during emergencies. How to generate desired motion patterns or avoid undesired motion patterns has received a lot of attention. The results obtained in the previous research can be roughly categorized as control of crowds with leaders and control of crowds without leaders.

5.1.1 Control of microscopic pedestrians using leaders

Based on inspirations from swarming of ants, schooling of fish, and flocking of birds, how to formulate or control the collective behaviors has interested a lot of researchers from control, communication, and computer science. One of the main reasons is that the fascinating phenomena mentioned above can be generated from simple or basic interacting rules and control of this kind of collective behaviors can be realized through controlling just a small part of the group.

https://doi.org/10.1515/9783110473988-005

The consensus problem has firstly been proposed in [15], where the average-consensus problem has been realized through appropriately choosing the moving direction of each agent. As far as the evacuation of crowds is concerned, the result obtained in [2] has shown that the evacuation rate can be greatly improved by adding some leaders with global knowledge and it is always good for the evacuation process if we scatter some leaders in the crowds of pedestrians. Influence of number of added leaders and their topology on the consensus problem has been further discussed in [10] using experiments and it has been proved that a small number of leaders are enough for driving a large number of uninformed individuals to reach consensus without explicit communications.

5.1.2 Control of microscopic pedestrian without leaders

At the same time, there is also some work on control of crowds without using leaders.

(1) Decentralized control of microscopic pedestrians
Using the methodology of decomposition, control problems for one complex system can be transformed into control problems for several simpler subsystems. A lot of decentralized controllers have been constructed using the framework of decomposition in the previous research. Decentralized controllers for a large number of stochastic agents have been given in [18] where not only the evolution at temporal scale but also the evolution at spatial scale has been considered. Decentralized controllers have also been constructed using the framework of decomposition in [13], where a complex linear–quadratic–Gaussian (LQG) games problem has been reduced to one game problem of two players.

The advantages of the decentralized framework are that the burden of computation has been greatly reduced and the obtained results can be easily extended to systems of large numbers. Compared to distributed controllers, drawback of this mehod lies in that neighbor's information has not been used in design of decentralized controllers.

(2) Distributed control of microscopic pedestrians
Since most of the consensus protocols constructed for a multi-agent system lie in this framework, we will not state them one by one and the interested reader may refer to the review papers in [3, 19] (on multi-agent systems), [28, 30, 29] (on control of crowds), and the references therein.

One big challenge for distributed control is the burden of computation especially when the number of pedestrians goes to infinity. Some previous research has adopted the mean-field methods to estimate the influence of neighboring agents to relieve the burden of computation. Mean-field methods have been used in [20] in controller design for agents with linear stochastic dynamics. Work of [31] has used mean-field methods to estimate effects of neighboring pedestrians where distributed controllers for random agents have been constructed. LQG controllers based on mean field have been proposed in [14] for socially optimal control problems where cost functions of each agent are coupled with each other. Similar results can also be found in [21, 22], where control of stochastic multi-agent systems is studied and statistical information of neighboring agents are obtained using mean-field methods.

In order to study interactions among agents, game theory has also been combined with mean-field method in control of agents of large numbers, which we will call mean-field games in the following. Based on the mean-field games introduced in [17], dynamics of humans' decision-making process has been considered in [8], where mean-field game theory has been employed in the backward Hamilton–Jacobi–Bellman (HJB) equation and forward Fokker–Planck equation (FPE). The forward–backward equations have also been utilized in [16] to describe aversion and congestion phenomena at macroscopic scale. Something that should be pointed out is that the information used in controllers based on the mean-field method is just initial distribution of crowds. The widely used topology condition such as connected graphs or jointly connected graphs in multi-agent systems is no longer needed. Thus this method is much preferred in control of crowds.

5.1.3 Motivation for work of this chapter

The phenomenon of cluster consensus is commonly observed in the movement of animals, as shown in Figure 5.1, the movement of pedestrians, as shown in Figure 5.2, and even in the distribution of people in a stadium, as is shown in Figure 5.3. Fascinated by these phenomena, the focus of this chapter lies on the cluster consensus problem of microscopic pedestrians. A microscopic model of fractional order is firstly proposed in this chapter; then cluster consensus of this model under a consensus protocol of integer order is studied in this chapter. Simulation results using Matlab are employed to illustrate that the consensus protocol of integer order is still effective for the clus-

Figure 5.1: Cluster phenomena observed in groups of animals. Source: www.vcg.com.

Figure 5.2: Cluster phenomena observed in evacuation of people. Source: www.news.cn and www.119.cn.

Figure 5.3: Cluster phenomena observed in groups of people. Source: www.vcg.com.

ter consensus problem of a fractional integrator when $\alpha \in (0, 1)$. Our conclusion and future topics are also included in this chapter.

5.2 Microscopic model of fractional order

5.2.1 Nice properties of fractional calculus [7, 23]

It is well known that one integral process $\phi(t)$ can be described by

$$\frac{d\phi}{dt} = -\int_0^t K(t-\tau)\phi(\tau)d\tau, \tag{5.1}$$

where the memory kernel $K(t)$ has the form of a power law $t^{\alpha-1}$ and one differential process $\eta(t)$ can be described by

$$\frac{d^\alpha \eta}{dt^\alpha} = B\eta(t), \tag{5.2}$$

where B is an operator.

Remark 5.1. Contrary to kernel of exponential form where no memory has been considered, the kernel of power-law form is much closer to reality as the influence of past experience on movement of each pedestrian can be easily included.

5.2.2 Microscopic model of fractional order

The following dynamic model of integer order has been extensively used in previous research:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m_i \dfrac{dv_i}{dt} = f_i^S + \sum_{j=1}^{n} f_{ij}^N + \sum f_k^W, \end{cases} \tag{5.3}$$

under one common assumption that the movement of each pedestrian is continuous and differentiable everywhere, where x_i is the position of each pedestrian and v_i is the velocity of each pedestrian. The assumption of a smooth trajectory can be satisfied if we observe the movement of each pedestrian at large scale. That is not the case if the trajectory of each pedestrian is zoomed in and described at a very small scale. Then zigzag phenomenon will be observed in the trajectory of each pedestrian, as shown in Figure 5.4. The zigzag phenomenon has been also observed in traffic control systems and studied in [7] from the viewpoint of fractional calculus. Similarly, the trajectory of each pedestrian is not smooth due to the interactions with neighbors and thus fractional calculus can also be introduced in modeling of pedestrian at micro-scale.

Concerning the dynamics of each pedestrian, the transformed velocity $d^\alpha x/dt^\alpha$ has been utilized to describe the dynamics of each pedestrians as follows:

$$\begin{cases} \dfrac{d^\alpha x_i}{dt^\alpha} = v_i, \\ m_i \dfrac{d^\beta v_i}{dt^\beta} = f_i^S + \sum_{j=1}^{n} f_{ij}^N + \sum f_k^W, \end{cases} \tag{5.4}$$

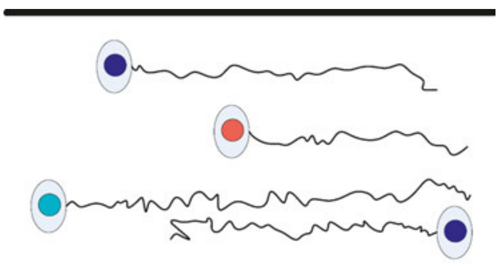

Figure 5.4: Zigzag phenomenon of each pedestrian.

where f_i^S is the self-driven force towards some desired velocity, f_{ij}^N is the interaction between agent i and its neighbor j, and f_k^W represents the interactions with the environment, such as walls or corridors.

5.3 Consensus of microscopic pedestrians of fractional order

5.3.1 Consensus algorithm for integer-order system

For consensus of a multi-agent system composed of N nodes that are described by

$$\frac{dx_i}{dt} = u_i,$$

it is well known that consensus will be realized using the following protocol:

$$u_i(t) = \sum_{j \in N_i} a_{ij}[x_j(t) - x_i(t)], \tag{5.5}$$

where $a_{ij} \neq 0$ if node i and node j are connected.

5.3.2 Consensus algorithm for fractional-order system

Considering the position of fractional order in a closed system, there are three kinds of consensus algorithms, as shown by [26].

Case 1: *Fractional-order system with integer-order consensus protocol.* The dynamics for a fractional integrator under the integer-order consensus protocol of (5.5) can be described as

$$\frac{d^\alpha x_i}{dt^\alpha} = \sum_{j \in N_i} a_{ij}[x_j(t) - x_i(t)].$$

Case 2: *Integer-order system with fractional-order consensus protocol.* For an integrator of integer order, the following fractional dynamics can be obtained under fractional consensus protocol as follows:

$$\frac{dx_i}{dt} = \sum_{j \in N_i} a_{ij}[{}_0^C D_t^\beta x_j(t) - {}_0^C D_t^\beta x_i(t)].$$

Case 3: *Fractional-order system with fractional-order consensus protocol.* The following general fractional dynamics can be easily derived for an integrator of fractional order under fractional consensus protocol

$$\frac{d^\alpha x_i}{dt^\alpha} = \sum_{j \in N_i} a_{ij}[{}_0^C D_t^\beta x_j(t) - {}_0^C D_t^\beta x_i(t)].$$

For the Caputo definition of fractional derivative, case 2 and case 3 can be transformed to case 1 with the help of the Caputo integral operator ${}_0^C D_t^{-\beta}$. Thus case 1 can be considered as a generalized form of the fractional consensus dynamics.

5.3.3 Analysis of fractional consensus

The fractional consensus of case 1 will be analyzed first in this section to show that consensus of a fractional integrator can be realized using a consensus protocol of integer order.

Theorem 5.1. *For the fractional integrator system described by*

$$\frac{d^\alpha x_i}{dt^\alpha} = u_i, \tag{5.6}$$

where $i = 1, 2, \ldots, N$ is the index of each integrator, under the following consensus protocol:

$$u_i = \sum_{j \in N_i} a_{ij}[x_j(t) - x_i(t)], \tag{5.7}$$

consensus will be reached if the Laplacian matrix $\mathcal{L}_N$ has a simple zero eigenvalue. Furthermore, if $\mathcal{L}_N$ has a simple zero eigenvalue and $v = [v_1, v_2, \ldots, v_n]^T \geq 0$ satisfying $1_N^T v = 1$ and $\mathcal{L}_N^T v = 0$, then $E_{\alpha,1}(-\mathcal{L}_N t) \to 1_N v^T$, as $t \to \infty$, where $E_{\alpha,1}(\cdot)$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Proof. Denote $A = -\mathcal{L}_N$ and let J be the Jordan form associated with A, i. e., $A = SJS^{-1}$. Then, based on the definition of the Mittag-Leffler function, we have

$$E_{\alpha,1}(At^\alpha) = SE_{\alpha,1}(Jt^\alpha)S^{-1}$$

and $E_{\alpha,1}(-\lambda t^\alpha) \to 0$ as $t \to \infty$ for $\forall \lambda > 0$. Thus we see that $E_{\alpha,1}(Jt^\alpha)$ converges to the following matrix:

$$E_{\alpha,1}(Jt^\alpha) \to \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{as } (t \to \infty).$$

Since A has only one zero eigenvalue and all the other eigenvalues lie in the left-hand side of the plane, there is only one non-zero element in the above matrix.

Notice that $AS = SJ$ and the first column of matrix S is the right eigenvector w_r of A with respect to the eigenvalue 0. Similarly, the first row of matrix S^{-1} is the left eigenvector w_l of matrix A with respect to the eigenvalue 0 as $S^{-1}A = JS^{-1}$ is satisfied. Straightforward calculation shows that

$$E_{\alpha,1}(At^{\alpha}) \to S \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} S^{-1} = w_r w_l^T \quad \text{as } (t \to \infty).$$

When the communication graph is an indirect and connected graph or a strongly connected digraph, the sum of each row of Laplacian matrix A equals zero and zero is the only eigenvalue of A with the right eigenvector $w_r = [1\,1 \cdots 1]^T$. After substituting the right eigenvector w_r, we have

$$E_{\alpha,1}(At^{\alpha}) \to \mathbf{1}w_l^T \quad \text{as } (t \to \infty).$$

Thus the average-consensus problem is realized for the fractional system (5.6) under the consensus protocol (5.7). □

5.4 Cluster consensus algorithm for fractional-order system

For a complex network that is composed of $n+m$ pedestrians, we use a graph $G = (x, \varepsilon)$ of $n + m$ nodes to describe the topology of these pedestrians. Denote by $G_1(x^1, \varepsilon^1)$ and $G_2(x^2, \varepsilon^2)$ the sub-networks that are formed by n and m pedestrians with $x^1 = (x_1, x_2, \ldots, x_n)^T$ and $x^2 = (x_{n+1}, x_{n+2}, \ldots, x_{n+m})^T$. Hence, all the pedestrians are divided into two groups and the pedestrians in each group build up a sub-network $G_s(s = 1, 2)$.

Definition 5.1 (Cluster consensus). For integrators of fractional order, the cluster consensus problem is to find some protocols to asymptotically solve the average-consensus problem for each subgroup, which can also be described by

$$(1) \quad \lim_{t\to\infty} x_i(t) = \frac{1}{n}\sum_{j=1}^{n} x_j(0) \qquad \forall i \in \mathfrak{p}_1,$$

$$(2) \quad \lim_{t\to\infty} x_i(t) = \frac{1}{m}\sum_{j=n+1}^{n+m} x_j(0) \quad \forall i \in \mathfrak{p}_2,$$

where $\mathfrak{p}_1 = \{1, 2, \ldots, n\}$, $\mathfrak{p}_2 = \{n + 1, n + 2, \ldots, n + m\}$.

Assumption 5.1. Both of $G_1(x^1, \varepsilon^1)$ and $G_2(x^2, \varepsilon^2)$ are strongly connected balanced graphs, where a graph is called a balance graph if the following two conditions are satisfied:

(1) $\sum_{j=n+1}^{n+m} a_{ij} = 0, \forall i \in \mathfrak{p}_1$;
(2) $\sum_{j=1}^{n} a_{ij} = 0, \forall i \in \mathfrak{p}_2$.

Different from the average-consensus problem of one group that has been proposed in previous research, the following consensus protocol has been proposed for the cluster consensus problem:

$$u_i = \begin{cases} \sum_{j \in N_{1i}} a_{ij}[x_j(t) - x_i(t)] + \sum_{j \in N_{2i}} a_{ij} x_j(t), & \forall i \in \mathfrak{p}_1, \\ \sum_{j \in N_{1i}} a_{ij} x_j(t) + \sum_{j \in N_{2i}} a_{ij}[x_j(t) - x_i(t)], & \forall i \in \mathfrak{p}_2, \end{cases} \tag{5.8}$$

where coupling terms between these two subgroups have been added. Under the cluster consensus protocol (5.8), the closed-loop system can be described by

$$\frac{d^\alpha x_i}{dt^\alpha} = \begin{cases} \sum_{j \in N_{1i}} a_{ij}[x_j(t) - x_i(t)] + \sum_{j \in N_{2i}} a_{ij} x_j(t), & \forall i \in \mathfrak{p}_1, \\ \sum_{j \in N_{1i}} a_{ij} x_j(t) + \sum_{j \in N_{2i}} a_{ij}[x_j(t) - x_i(t)], & \forall i \in \mathfrak{p}_2, \end{cases}$$

or in the following matrix form:

$${}_0^C D_t^\alpha x(t) = -Lx(t), \tag{5.9}$$

where $L = [l_{ij}]$ and l_{ij} is defined by

$$l_{ij} = \begin{cases} -a_{ij}, & j \neq i, \\ \sum_{j \in N_{1i}} a_{ij}, & j = i. \end{cases}$$

Denote $e(t) = [e^1(t), e^2(t)]^T$ as the state error, where $e^1(t) = [e_1(t), e_2(t), \ldots, e_n(t)]^T$, $e^2(t) = [e_{n+1}(t), e_{n+2}(t), \ldots, e_{n+m}(t)]^T$, and $e_i(t)$ is defined by

$$e_i(t) = \begin{cases} x_i(t) - \frac{1}{n} \sum_{j=1}^{n} x_j(0), & \forall i \in \mathfrak{p}_1, \\ x_i(t) - \frac{1}{m} \sum_{j=n+1}^{n+m} x_j(0), & \forall i \in \mathfrak{p}_2. \end{cases} \tag{5.10}$$

Based on the closed-loop system (5.9), the dynamics of the state error (5.10) can be written as

$${}_0^C D_t^\alpha e(t) = -Le(t). \tag{5.11}$$

Thus the consensus protocol (5.8) can solve the cluster consensus problem if and only if the zero solution of (5.11) is asymptotically stable. One sufficient condition for cluster consensus is presented in Theorem 5.2 based on the Lyapunov stability on fractional systems.

Lemma 5.1 ([1]). *Let $x(t) \in R^n$ be a continuous and differentiable function. Then, for any time instant $t \geq t_0$, the following inequality is satisfied:*

$$\frac{1}{2} {}_0^C D_t^\alpha (x^T(t) x(t)) \leq x^T(t) {}_0^C D_t^\alpha x(t), \qquad \forall \alpha \in (0, 1).$$

Theorem 5.2. *Under the assumption of* (5.1)*, the cluster consensus problem of system* (5.6) *can be realized using the protocol* (5.8) *if there exist matrices such that*

$$\begin{bmatrix} -2\lambda(L_{11}) + M_1 + X_1 & W \\ W^T & -2\lambda(L_{22}) + M_2 + X_2 \end{bmatrix} < 0,$$

where

$$M_1 = \Theta_{21}^T Z_2 \Theta_{21}, \quad M_2 = \Theta_{12}^T Z_1 \Theta_{12}, \quad W = (Y_1 - I)\Theta_{12} + \Theta_{12}^T (Y_2^T - I),$$

and Θ_{12}, Θ_{21} *are defined by*

$$\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}^T L \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Theta_{11} & 0 & \Theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & \Theta_{21} & 0 & \Theta_{22} \end{bmatrix},$$

where U_1 *and* U_2 *are orthogonal matrices.*

Proof. For the analysis of the stability of system (5.11), the following Lyapunov function is constructed:

$$V(e) = \frac{1}{2}\|e^1\|^2 + \frac{1}{2}\|e^2\|^2.$$

Due to Lemma 5.1, the fractional differential of $V(e)$ can be computed as

$${}_0^C D_t^\alpha V(e) \le -(e^1)^T L_{11} e^1 - (e^1)^T L_{12} e^2 - (e^2)^T L_{21} e^1 - (e^2)^T L_{22} e^2. \tag{5.12}$$

after replacing u_i with the following consensus protocol:

$$u_i = \sum_{j \in N_{1i}} a_{ij} x_j \quad \forall i \in \mathfrak{p}_1.$$

Since $G_1(x^1, \varepsilon^1)$ is a strongly connected balanced graph from Assumption 5.1, the inequality

$$\dot{V}_1(e) = -(e^1)^T L_{11} e^1 \le -\lambda(L_{11})\|e^1\|^2$$

can easily be checked for $V_1(e) = \frac{1}{2}(e^1)^T e^1$, where $\lambda(L_{11})$ is called the algebraic connectivity of the graph L_{11}. Similarly, the inequality

$$-(e^2)^T L_{22} e^2 \le -\lambda(L_{22})\|e^2\|^2$$

can also be obtained for $G_2(x^2, \varepsilon^2)$.

Denote

$$\widetilde{X}_i = \begin{bmatrix} 0 & 0 \\ 0 & \bar{X}_i \end{bmatrix}, \quad \widetilde{Y}_i = \begin{bmatrix} 0 & 0 \\ 0 & \bar{Y}_i \end{bmatrix}, \quad \widetilde{Z}_i = \begin{bmatrix} 0 & 0 \\ 0 & \bar{Z}_i \end{bmatrix},$$

where $i = 1, 2$ and $\bar{X}_i$, $\bar{Y}_i$, $\bar{Z}_i$ satisfy

$$\begin{bmatrix} \bar{X}_i & \bar{Y}_i \\ \bar{Y}_i & \bar{Z}_i \end{bmatrix} > 0.$$

Then, similar to the derivation for group consensus of the integer-order system in [32], ${}_0^C D_t^\alpha V(t) < 0$ in (5.12) is guaranteed if

$$\begin{bmatrix} -2\lambda(L_{11}) + \widetilde{M_1} + \widetilde{X_1} & \widetilde{W} \\ \widetilde{W}^T & -2\lambda(L_{22}) + \widetilde{M_2} + \widetilde{X_2} \end{bmatrix} < 0,$$

where $\widetilde{M_1} = L_{21}^T \widetilde{Z_2} L_{21}$, $\widetilde{M_2} = L_{12}^T \widetilde{Z_1} L_{12}$, and $\widetilde{W} = (\widetilde{Y_1} - I)L_{12} + L_{12}^T(\widetilde{Y}_2^T - I)$. Thus, after linear transformation of orthogonal matrices, the cluster consensus of system (5.6) can be realized if the following inequality is satisfied:

$$\begin{bmatrix} -2\lambda(L_{11}) + M_1 + X_1 & W \\ W^T & -2\lambda(L_{22}) + M_2 + X_2 \end{bmatrix} < 0. \qquad \square$$

Remark 5.2. Based on the results of Theorem 5.1, the cluster consensus of the fractional-order integrator (5.6) can also be realized using the consensus protocol (5.7) of integer order where $\alpha \in (0, 1)$. The assumption of balanced graph in Assumption 5.1 has played an important role where the interactions between different subgroups have been balanced with each other and the cluster consensus problems have been simplified to the consensus problems of two subgroups. Unfortunately, the assumption of a balanced graph is hard to satisfy and testify in the evacuation of crowds of pedestrians.

5.5 Simulation results

Considering unexpected or dangerous events in a real-life experiment, only simulation results are conducted to show the difference between models of fractional order and models of integer order in this chapter. In the following simulations, evacuation problems of six pedestrians are simulated using models of integer order and fractional order, respectively. Since it is reasonable that not everyone can get the information of the desired position during evacuation, we have assumed that there are two conflicting opinions that are insisted by one informed leader in each subgroup. The communication topology is presented in Figure 5.5.

5.5.1 Cluster consensus for pedestrians of integer order

The order of the dynamic model for each pedestrian is assumed to be 1 in this case and the simulation results are shown in Figure 5.6 and Figure 5.7. Consensus of X coordinates and Y coordinates is separated into two subgroups, as shown in Figure 5.6; the phenomenon of cluster consensus can be easily observed from Figure 5.7 where the trajectory of each pedestrian of integer order has been plotted.

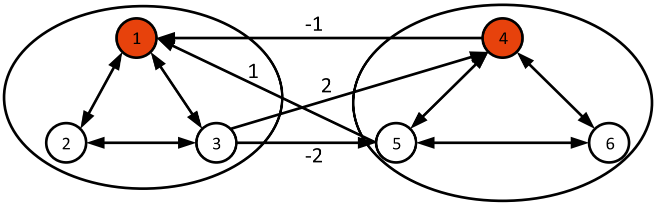

Figure 5.5: Balanced communication graph for six agents of fractional order with two leaders.

5.5.2 Cluster consensus for pedestrians of fractional order

(1) Simulations with the same fractional order $\alpha \in (0, 1)$.
For the analysis of the influence of the fractional order on the cluster consensus problem, two different scenarios with $\alpha = 0.95$ and $\alpha = 0.85$ have been selected in the following simulations.

α = 0.95: The order of the dynamic model for each pedestrian is assumed to be 0.95 in this case and the simulation results are shown in Figure 5.8 and Figure 5.9. The consensus of X coordinates and Y coordinates is separated into two subgroups, as shown in Figure 5.8, and the phenomenon of cluster consensus can be easily observed from Figure 5.9.

α = 0.85: The order of the dynamic model for each pedestrian is assumed to be 0.85 in this case and the simulation results are shown in Figure 5.10 and Figure 5.11. Consensus of X coordinates and Y coordinates are also separated into two subgroups as shown in Figure 5.10 and the phenomenon of cluster consensus can also be observed from Figure 5.11. Compared to the simulation results obtained when $\alpha = 0.95$, the distance between the final value of cluster consensus is smaller in this case.

(2) Simulations with mixed fractional order $\alpha \in (0, 1)$.
From the communication topology shown in Figure 5.5, we see that only the first pedestrian in the first subgroup can receive information from the second subgroup. Similarly only the fourth pedestrian in the second subgroup can receive information from the first subgroup. For further understanding of the influence of the fractional order and the communication topology, two kinds of scenarios with mixed fractional order [0.95 0.65 0.65 0.95 0.65 0.65] (Case 1) and [0.75 0.65 0.65 0.75 0.65 0.65] (Case 2) have been considered in the following simulations.

Case 1: Compared to the fractional order of uninformed agents, the fractional order of the informed agent is selected very close to 1 in this case. As shown in Figure 5.12 and Figure 5.13, cluster consensus can be observed using the same consensus protocol.

Case 2: The fractional order of the informed agent has been changed from 0.95 to 0.75 in this case, while the fractional orders of the uninformed agents are the same as

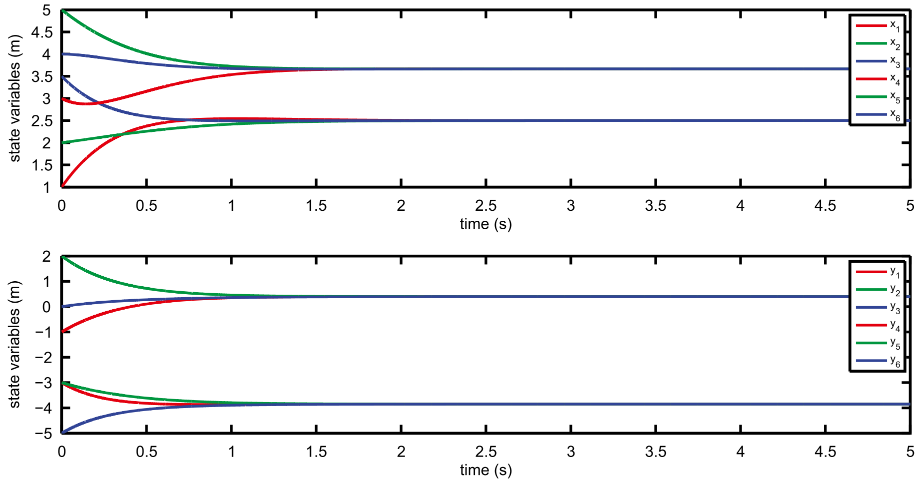

Figure 5.6: Cluster consensus of dynamic pedestrians (5.3) with integer order.

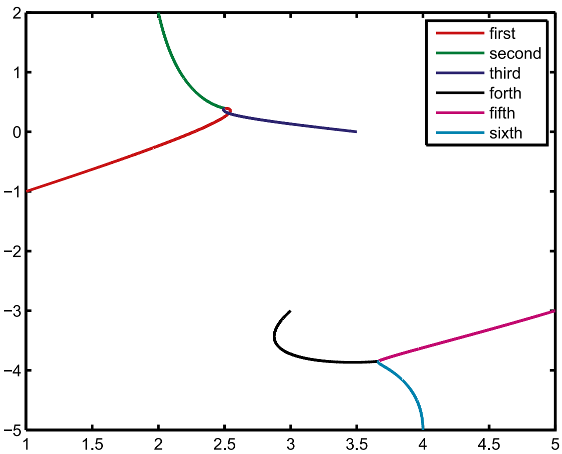

Figure 5.7: Trajectories of the movement of pedestrians of integer order.

those in Case 1. From the results shown in Figure 5.14 and Figure 5.15, we see that the distance between the two clusters has been greatly decreased in this case.

Remark 5.3. The assumption of balanced communication topology has played an important role in the cluster consensus problem. Besides the conclusion that the consensus protocol of integer order is still effective for the cluster consensus of the fractional agents, simulation results has shown that the closer to 1 of the informed agent's

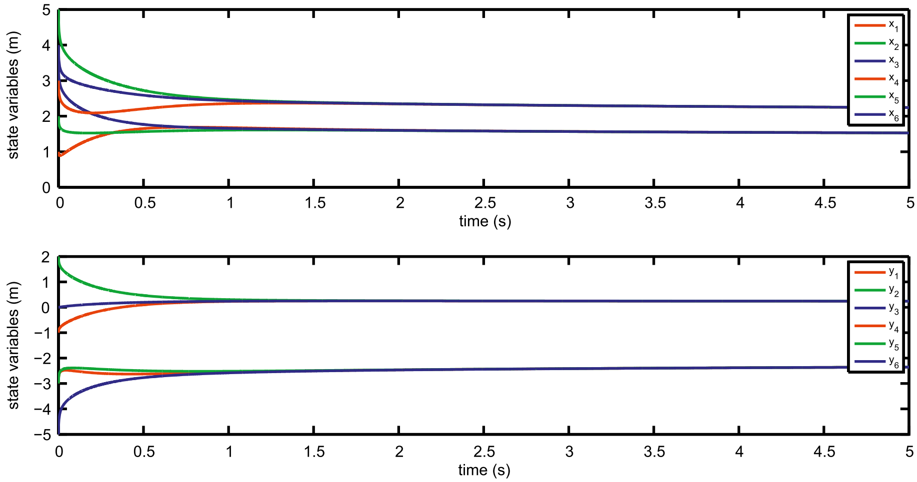

Figure 5.8: Cluster consensus of dynamic pedestrians of (5.4) with fractional order ($\alpha = 0.95$).

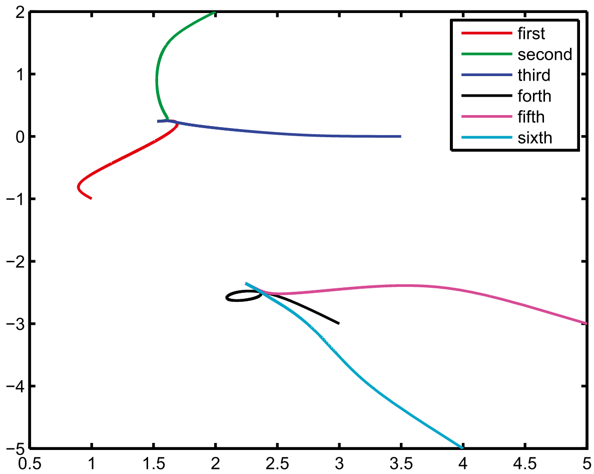

Figure 5.9: Trajectories of the movement of pedestrians of with fractional order ($\alpha = 0.95$).

fractional order, the easier to realize cluster consensus. Therefore, the selection of appropriate leader is of great importance in evacuation control of large crowds.

5.6 Conclusions and further work

Cluster consensus for fractional pedestrians has been considered in this chapter at micro-scale. Based on the previous research on the consensus of integrator systems, sufficient conditions for cluster consensus of the fractional system have been dis-

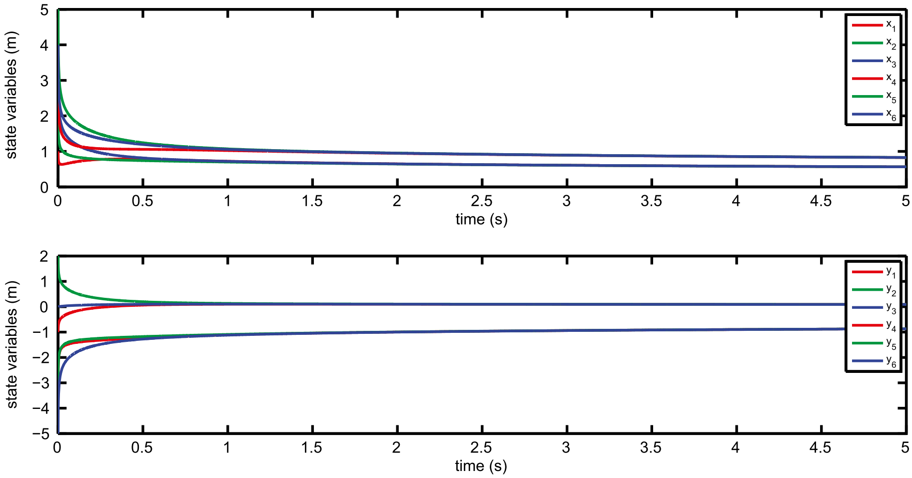

Figure 5.10: Cluster consensus of dynamic pedestrians of (5.4) with fractional order ($\alpha = 0.85$).

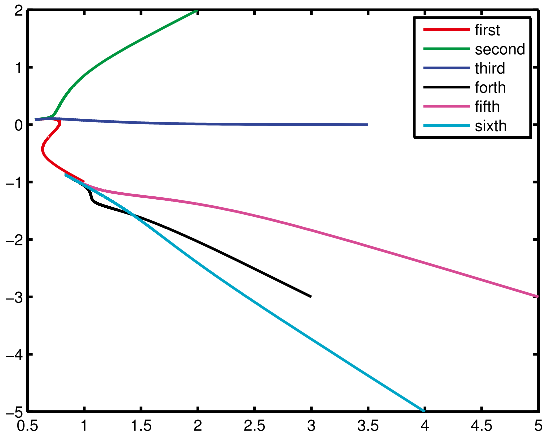

Figure 5.11: Trajectories of the movement of pedestrians with fractional order ($\alpha = 0.85$).

cussed in this chapter. Simulation results using Matlab are presented to illustrate the effectiveness of the results obtained.

Based on the results obtained in this chapter, there are also some interesting topics that are worthy of further investigation within the framework of fractional calculus.

More general communication topology: A static topology has been assumed in the cluster consensus problem in this chapter. It is not always the case, especially in the evacuation of crowds of pedestrians. Dynamic or switching topologies may be very reasonable assumptions.

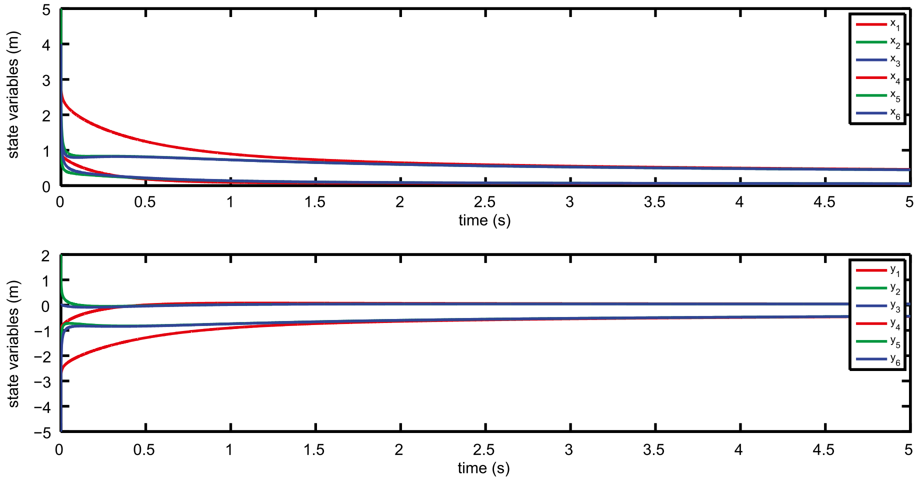

Figure 5.12: Cluster consensus of dynamic pedestrians of (5.4) with fractional order [0.95 0.65 0.65 0.95 0.65 0.65].

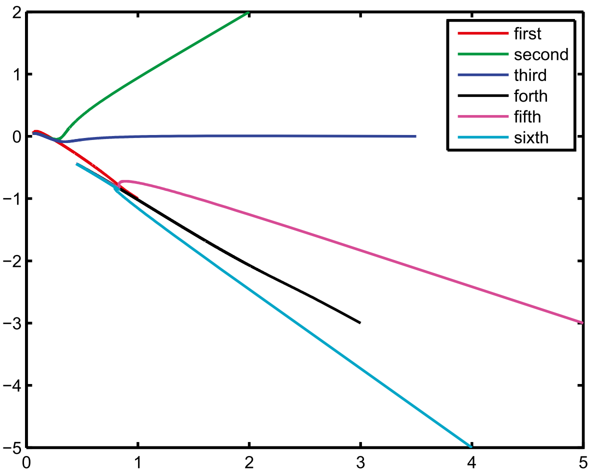

Figure 5.13: Trajectories of the movement of pedestrians with fractional order [0.95 0.65 0.65 0.95 0.65 0.65].

The balanced graph has played an important role in the cluster consensus problem in this chapter. The balanced assumption has made the cluster consensus problem much easier as the interactions between different subgroups have been canceled or balanced. In real evacuations of crowds, the balanced assumption is also hard to satisfy.

Noise is essential: All systems are prone to be affected by noise or perturbations. It should be pointed out that not all of the noise is bad for the system and some of the

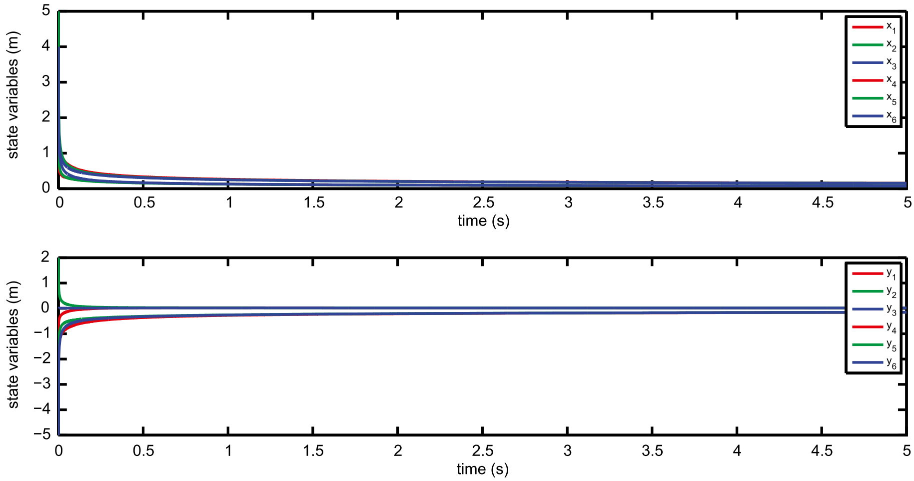

Figure 5.14: Cluster consensus of dynamic pedestrians of (5.4) with fractional order [0.75 0.65 0.65 0.75 0.65 0.65].

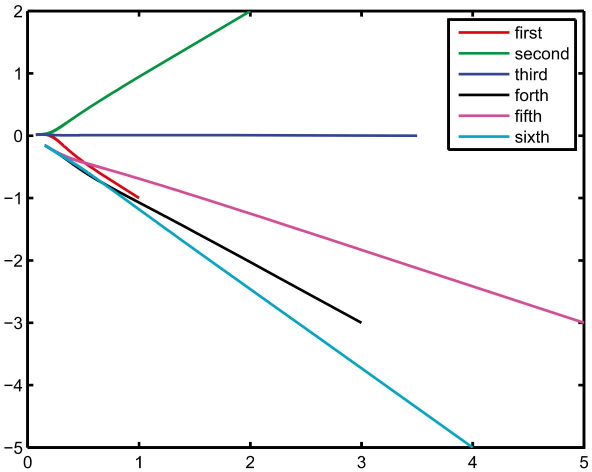

Figure 5.15: Trajectories of the movement of pedestrians with fractional order [0.75 0.65 0.65 0.75 0.65 0.65].

noise can facilitate ordering in the collective behavior of self-propelled particles, as shown in [27], where noise or perturbations can drive the particles from an inefficient regime into a much more efficient one with more interactions. As the role of noise in the evacuation of crowds of fractional order is not clear now, it is still one open problem now.

Role of leadership: In control of crowds of pedestrians using leaders, there are also some problems unclear or unsolved now, such as the role of the leadership in collective behavior, what kind of leaders should be chosen, the relationship between optimal placement of leaders, and the efficient evacuation of crowds. For efficient and safe evacuation of crowds, all these problems are worth to put more efforts.

References

[1] N. Aguila-Camacho, M. A. Duarte-Mermoud, and J. A. Gallegos. Lyapunov functions for fractional order systems. *Communications in Nonlinear Science and Numerical Simulation*, 19(9):2951–2957, 2014.

[2] F. Aube and R. Shield. Modeling the effect of leadership on crowd flow dynamics. *Lecture Notes in Computer Science: Cellular Automata*, 3305:601–611, 2004.

[3] Y. Cao, W. Yu, W. Ren, and G. Chen. An overview of recent progress in the study of distributed multi-agent coordination. *IEEE Transactions on Industrial Informatics*, 9(1):427–438, 2013.

[4] K. Cao, Y. Chen, D. Stuart, and D. Yue. Cyber-physical modeling and control of crowd of pedestrians: a review and new framework. *IEEE/CAA Journal of Automatica Sinica*, 2(3):334–344, 2015.

[5] I. D. Couzin. Collective cognition in animal groups. *Trends in Cognitive Sciences*, 13(1):36–43, 2008.

[6] I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin. Effective leadership and decision-making in animal groups on the move. *Nature*, 433(7025):513–516, 2005.

[7] S. Das. *Functional Fractional Calculus*. Springer-Verlag, Berlin, Heidelberg, 2011.

[8] C. Dogbe. Modeling crowd dynamics by the mean-field limit approach. *Mathematical and Computer Modelling*, 52(9–10):1506–1520, 2010.

[9] J. R. G. Dyer, C. C. Ioannou, L. J. Morrell, D. P. Croft, I. D. Couzin, D. A. Waters, and J. Krause. Consensus decision making in human crowds. *Animal Behaviour*, 75(2):461–470, 2008.

[10] J. R. G. Dyer, A. Johansson, D. Helbing, I. D. Couzin, and J. Krause. Leadership, consensus decision making and collective behaviour in humans. *Philosophical Transactions of the Royal Society B*, 364(1518):781–789, 2009.

[11] R. L. Goldstone and M. A. Janssen. Computational models of collective behavior. *Trends in Cognitive Sciences*, 9(9):424–430, 2005.

[12] D. Helbing, I. Farkas, and T. Vicsek. Simulating dynamical features of escape panic. *Nature*, 407(28):487–490, 2000.

[13] M. Huang. Large-population LQG games involving a major player: The Nash certainty equivalence principle. *SIAM Journal on Control and Optimization*, 48(5):3318–3353, 2010.

[14] M. Huang, P. E. Caines, and R. P. Malhame. Social optima in mean field LQG control: Centralized and decentralized strategies. *IEEE Transactions on Automatic Control*, 57(7):1736–1751, 2012.

[15] A. Jadbabaie, J. Lin, and A. Stephen Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[16] A. Lachapelle and M.-T. Wolfram. On a mean field game approach modeling congestion and aversion in pedestrian crowds. *Transportation Research. Part B: Methodological*, 45(10):1572–1589, 2011.

[17] J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, 2007.

[18] T. Li and J.-F. Zhang. Asymptotically optimal decentralized control for large population stochastic multiagent systems. *IEEE Transactions on Automatic Control*, 53(7):1643–1660, 2008.

[19] H.-B. Min, Y. Liu, S.-C. Wang, and F.-C. Sun. An overview on coordination control problem of multi-agent system. *Acta Automatica Sinica*, 38(10):1557–1570, 2012.
[20] M. Nourian, R. P. Malhame, M. Huang, and P. E. Caines Mean field (NCE) formulation of estimation based leader-follower collective dynamics. *International Journal of Robotics & Automation*, 26(1):120–129, 2011.
[21] M. Nourian, P. E. Caines, R. P. Malhame, and M. Huang. Mean field LQG control in leader-follower stochastic multi-agent systems likelihood ratio based adaptation. *IEEE Transactions on Automatic Control*, 57(11):2801–2816, 2012.
[22] M. Nourian, P. E. Caines, R. P. Malhame, and M. Huang. Nash, social and centralized solutions to consensus problems via mean field control theory. *IEEE Transactions on Automatic Control*, 58(3):639–653, 2013.
[23] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1999.
[24] H. Rahmandad and J. Sterman. Heterogeneity and network structure in the dynamics of diffusion comparing agent-based and differential equation models. *Management Science*, 54(5):998–1014, 2008.
[25] H. Singh, R. Arter, L. Dodd, P. Langston, E. Lester, and J. Drury. Modelling subgroup behaviour in crowd dynamics DEM simulation. *Applied Mathematical Modelling*, 33(12):4408–4423, 2009.
[26] W. Sun, Y. Li, C. Li, and Y. Q. Chen. Convergence speed of a fractional order consensus algorithm over undirected scale-free networks. Asian Journal of Control, 13(6):936–946, 2011.
[27] T. Vicsek and A. Zafeiris. Collective motion. *Physics Reports*, 517(3–4):71–140, 2012.
[28] S. A. Wadoo and P. Kachroo. Feedback control design and stability analysis of one dimensional evacuation system. In *Proceedings of IEEE Intelligent Transportation Systems Conference*, pages 618–623, 2006.
[29] S. A. Wadoo and P. Kachroo. Feedback control of crowd evacuation in one dimension. *IEEE Transactions on Intelligent Transportation Systems*, 11(1):182–193, 2010.
[30] S. A. Wadoo, S. Al-nasur, and P. Kachroo. Feedback control of macroscopic crowd dynamic models. In *Proceedings of American Control Conference*, pages 2558–2563, 2008.
[31] B.-C. Wang and J.-F. Zhang. Distributed control of multi-agent systems with random parameters and a major agent. *Automatica*, 48(9):2093–2106, 2012.
[32] J. Yu and L. Wang. Group consensus of multi-agent systems with directed information exchange. *International Journal of Systems Science*, 43(2):334–348, 2012.
[33] X. Zheng, T. Zhong, and M. Liu. Modeling crowd evacuation of a building based on seven methodological approaches. *Building and Environment*, 44(3):437–445, 2009.

6 Feedback control of crowds of pedestrians at macro-scale

Abstract: The evacuation of crowds of pedestrians is considered at macro-scale in this chapter. Based on the inspiration of diffusion processes and anomalous diffusion processes of fluids, distributed controllers have been constructed within the framework of calculus of fractional order. Compared to the previous work conducted within the framework of calculus of integer order, much more freedom has been provided in control of crowds at macro-scale using the controllers obtained. Simulation results in two-dimensional space are also presented to illustrate the effectiveness of the fractional and distributed controllers proposed in this chapter.

6.1 Introduction

Tragedies due to people's crushing or trampling have often been reported in public gatherings such as sports, meetings, exhibitions, or transportation in recent years. In order to prevent the occurrence of these tragedies and understand the reason behind these accidents, a lot of research has been conducted in modeling, predicting, and even controlling the behavior of crowds of pedestrians. Both simulation-based evacuation policies and model-based evacuation controllers have been reported in the previous research. Due to its simplicity and eye-catching features, the simulation-based method has been widely used in modeling or predicting the evolution of crowds. On the other hand, some explicit and analytic controllers have also been constructed where the performance of transient or stable process is greatly improved with the help of feedback.

(1) Simulation-based evacuation policies

Numerical simulations based on Monte Carlo have been used in [2] to validate mesoscopic models of crowds of particles. A simulator based on a co-existing microscopic model and macroscopic model has been proposed in [18]. The improvement of efficiency and accuracy using this simulator has been shown on a multi-resolution model for crowds. Simulation-based evacuation policies have also been used in [8] to navigate the crowds in real-time and predict the flow of crowds in [4]. Autonomous mobile robots have been introduced to improve the performance of evacuation in [6] where computer simulations are employed to show the effectiveness without constructing explicit controllers. In order to decrease the discrepancy between simulations and real crowds, a micro-spatial environment such as the geographic information system has been introduced into the simulation study of crowds in [11], where a much more realistic flow of crowds has been simulated. Many more references on the simulation of crowds can also be found in the book of [12] and in [7].

https://doi.org/10.1515/9783110473988-006

(2) Model-based evacuation controllers

In the evacuation of crowds of pedestrians, the moving direction and the moving speed are two important indices for smooth and efficient evacuation of crowds. Thus the velocity of crowds has been selected as control input in [14] and [15], where distributed feedback controllers have been constructed at macro-scale for the control of crowds in one- and two-dimensional space. The main idea of [14] and [15] is to approximate an infinite dimensional system using a finite dimensional system, so that the theory of non-linear control can be used in the design of feedback controllers. One of the main problems of this framework is that the original system may still be unstable, even if the obtained controllers work very well on the reduced system.

Without resorting to the technique of approximation, distributed controllers have been directly constructed for the macroscopic partial differential equations in [17] and [16], where diffusion-based controllers, advection-based controllers, and advective-diffusion-based controllers have been presented. A comparison of these distributed controllers is further considered in [3] where one interesting phenomenon has been reported that some problems may arise if only diffusion-feedback controller is adopted in control of crowds, because during the evacuation of crowds there is no preferred direction to follow for each pedestrian. Work of [3] has also shown that a faster evacuation of crowds is possible if diffusion–advection state feedback controllers are used, since the direction of evacuation is provided by the advection term in the diffusion-advection feedback controllers.

In order to avoid undesirable congestion and blockages in the evacuation of crowds, optimal feedback controllers have been considered in [9] and [10] by adjusting their velocities. Robust controllers were also considered for crowds of pedestrians in recent years. For example, Lyapunov techniques have been utilized in [1] to construct velocity controllers for automated highway systems where not only position and time but also the effects of lanes, drivers, and destinations have been included in the macroscopic partial differential equation (PDE) model; diffusion-based feedback controllers that were proposed in [3] have been further generalized using the methods of Lyapunov redesign in [19] to deal with the disturbance attenuation problem in the evacuation of crowds; robust controllers based on sliding-model control have been constructed in [13] for the control of macroscopic crowds with matched uncertainties and unmatched uncertainties.

Although some contribution has been achieved for the control of crowds in the previous study, main work has been conducted within the framework of calculus of integer order and there are also some conservation in the controller obtained. Feedback evacuation controllers for a macroscopic model of fractional order are considered in this chapter, where the techniques of feedback linearization have been utilized. Extension of obtained controllers such as diffusion-based controllers, sub-diffusion-based controllers, and super-diffusion-based controllers, have also been presented.

The organization of this chapter is as follows. Mathematical models of integer order and fractional order in one-dimensional space and in two-dimensional space are firstly shown in Section 6.2. The controller design for crowds of pedestrians is given in Section 6.3. Simulations of the evacuation process in closed and squared area with or without exits are shown in Section 6.4, where an approximating process using the Lax–Friedrichs scheme has been included.

6.2 Fractional macroscopic model

Mathematical models of integer order and fractional order for crowds of pedestrians are firstly presented based on the principle of conservation law. PDEs of integer order and fractional order have been obtained for crowds of pedestrians in one-dimensional space as shown in Figure 6.1 and in two-dimensional space as shown in Figure 6.2.

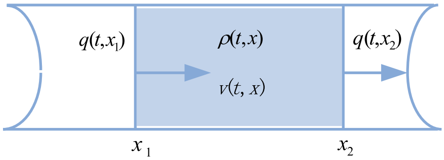

Figure 6.1: Conservation law of mass.

6.2.1 Macroscopic model of integer order in one-dimensional corridor

Similar to Chapter 3, the following model can be obtained for the evacuation of crowds in a one-dimensional corridor:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}q(x,t) = 0,$$

with the following initial and boundary conditions:

$$\rho(x,t_0) = \rho_0(x),$$
$$\rho(0,t_0) = 0, \qquad \rho(L,t) = 0,$$

where $\rho(x,t)$ denotes the density of people, which is a function of position x and time t and $q(x,t) = \rho(x,t)v(x,t)$ is the flow of crowds at a given x and at time t. With the help of the conservation law of momentum, the evolution of flux of crowds can also be described by

$$\frac{\partial}{\partial t}(\rho(x,t)v(x,t)) + \frac{\partial}{\partial x}(\rho(x,t)v(x,t)^2) = -\frac{\partial p(x,t)}{\partial x},$$

where $p(x,t)$ means the pressure imposed on the crowds. Thus the dynamic model for crowds in a one-dimensional corridor can be written as

$$\begin{cases} \frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}q(x,t) = 0, \\ \frac{\partial}{\partial t}(\rho(x,t)v(x,t)) + \frac{\partial}{\partial x}(\rho(x,t)v(x,t)^2) = -\frac{\partial p(x,t)}{\partial x}, \end{cases}$$

with the following initial condition:

$$\rho(x,t_0) = \rho_0(x), \quad q(x,t_0) = q_0(x)$$

and the boundary condition

$$\rho(0,t_0) = \rho(L,t) = 0, \quad q(0,t_0) = q(L,t) = 0.$$

6.2.2 Macroscopic model of fractional order in one-dimensional corridor

Parallel to the dynamic model of integer order, the conservation law of mass also plays an important role in constructing dynamic model of fractional order. Based on the inspiration of diffusion, sub-diffusion, and super-diffusion of fluids, viscoelasticity of fluids has been introduced in modeling the movement of pedestrians at macro-scale. For simplicity, only the fractional order at the time scale has been considered in this section.

Similar to the modeling process that is shown in Chapter 3, the macroscopic model of integer order

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}[\rho(t,x)v(t,x)] = 0 \tag{6.1}$$

can be generalized to the following macroscopic model of fractional order:

$$\frac{\partial^\alpha}{\partial t^\alpha}\rho(t,x) + \frac{\partial}{\partial x}[\rho(t,x)v(t,x)] = 0, \tag{6.2}$$

with the following initial condition and boundary conditions:

$$\begin{aligned} &\rho(x,t_0) = \rho_0(x), \\ &\rho(0,t_0) = 0, \qquad \rho(L,t) = 0, \end{aligned}$$

where the viscoelasticity of fluids has been described using the fractional order α ($\alpha \in (0,1)$), $\rho(x,t)$ denotes the density of people, which is a function of position x and time t, and $q(x,t) = \rho(x,t)v(x,t)$ is the flow of crowds at a given x and at time t.

Thus for crowds in a one-dimensional space corridor, the dynamic model with viscoelasticity can be described by

$$\begin{cases} \frac{\partial}{\partial t^\alpha}\rho(t,x) + \frac{\partial}{\partial x}[\rho(t,x)v(t,x)] = 0, \\ \frac{\partial}{\partial t}(\rho(x,t)v(x,t)) + \frac{\partial}{\partial x}(\rho(x,t)v(x,t)^2) = -\frac{\partial p(x,t)}{\partial x}, \end{cases}$$

with the initial conditions:

$$\rho(x,t_0) = \rho_0(x), \quad q(x,t_0) = q_0(x)$$

and the boundary conditions

$$\rho(0,t_0) = \rho(L,t) = 0, \quad q(0,t_0) = q(L,t) = 0.$$

6.2.3 Macroscopic model of fractional order in two-dimensional space

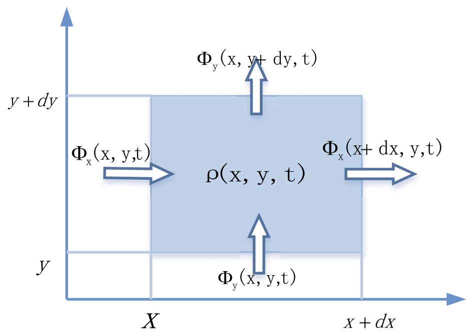

Figure 6.2: Two-dimensional flow of crowds.

Similarly, for the evacuation problem of crowds in two-dimensional space, as shown in Figure 6.2, the macroscopic model of fractional order can be described by

$$\frac{\partial}{\partial t^\alpha}\rho(x,y,t) + \frac{\partial}{\partial x}[\rho(x,y,t)u(x,y,t)] + \frac{\partial}{\partial y}[\rho(x,y,t)v(x,y,t)] = 0,$$

with the following initial condition and boundary conditions:

$$\begin{aligned}&\rho(x,y,t_0) = \rho_0(x,y),\\ &\rho(0,y,t_0) = \rho(x,0,t_0) = \rho(L,y,t_0) = \rho(x,L,t) = 0,\end{aligned}$$

where the viscoelasticity of the crowd flow has been described using the fractional order α ($\alpha \in (0,1)$). Considering the pressure imposed on the crowds, the dynamic model for crowds of pedestrians in two-dimensional space can be described by

$$\begin{cases}\dfrac{\partial}{\partial t^\alpha}\rho(x,y,t) + \dfrac{\partial}{\partial x}[\rho(x,y,t)u(x,y,t)] + \dfrac{\partial}{\partial y}[\rho(x,y,t)v(x,y,t)] = 0,\\ \dfrac{\partial}{\partial t}(v(x,y,t)) + v(x,y,t)\dfrac{\partial}{\partial x}(v(x,y,t)) + u(x,y,t)\dfrac{\partial}{\partial y}(v(x,y,t)) + \dfrac{C_0}{\rho}\rho_x\\ \quad = -\dfrac{w_i(\rho) - v(x,y,t)}{\tau},\\ \dfrac{\partial}{\partial t}(u(x,y,t)) + v(x,y,t)\dfrac{\partial}{\partial x}(u(x,y,t)) + u(x,y,t)\dfrac{\partial}{\partial y}(u(x,y,t)) + \dfrac{C_0}{\rho}\rho_y\\ \quad = -\dfrac{w_i(\rho) - u(x,y,t)}{\tau},\end{cases}$$

where $v(x,y,t)$ and $u(x,y,t)$ are velocity of crowds along the x direction and the y direction, $w_i(\rho)$ is the Greenshield relation between the velocity and density defined in (3.22), and τ is the relaxation time.

Remark 6.1. For simplicity, only the viscoelasticity at temporal scale has been considered for crowds at macro-scale, where the viscoelasticity has been assumed to just depend on time in this chapter. In real evacuations of crowds, the viscoelasticity of crowds may be changing with time and may be different at different positions. Thus it is interesting to consider the modeling problem using fractional calculus not only at the temporal scale but also at the spatial scale.

6.3 Controller design for crowds of pedestrians

6.3.1 Controller of integer order

Normal diffusive process

Based on the Greenshield relation between velocity and density defined in (3.22), the macroscopic model of integer order (6.1) can be written as

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\left[u(x,t)\left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right)\rho(x,t)\right] = 0, \tag{6.3}$$

where the free flow speed $v_f(x,t)$ in (3.22) has been assigned the role of control input, which is denoted by $u(x,t)$. Then, under the feedback controller

$$u(x,t) = -\left[\left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right)\rho(x,t)\right]^{-1} D\frac{\partial\rho(x,t)}{\partial x}, \tag{6.4}$$

the evolution of the closed-loop system can be described using the following classical Fick diffusion equation:

$$\frac{\partial}{\partial t}\rho(x,t) = D\frac{\partial^2}{\partial x^2}\rho(x,t).$$

Besides Fick diffusion, some other diffusion processes, such as sub-diffusion and super-diffusion, have been selected as the objective of control in controller design in the following section so that some other scenarios of the evacuation of crowds can be realized.

6.3.2 Controller of fractional order

Fick's diffusion is an ideal process and cannot be used to describe some phenomena observed in real crowds, such as the phenomenon of stampedes and the phenomenon of fast evacuation of crowds in different scenarios. Thus, the anomalous diffusion of fluids is introduced in the controller design in this section so that different evacuation processes can be united into the framework of fractional calculus.

(1) Sub-diffusion process

– Under the controller (6.4), the closed-loop system for macroscopic model (6.2) is

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\rho(x,t) = D\frac{\partial^{2}}{\partial x^{2}}\rho(x,t), \tag{6.5}$$

which is a sub-diffusion process for the evacuation of crowds with viscoelasticity depending on time.

– Under the controller

$$u(x,t) = -\left[\left(1-\frac{\rho(x,t)}{\rho_{\max}}\right)\rho(x,t)\right]^{-1} D\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\frac{\partial}{\partial x}\rho(x,t), \tag{6.6}$$

where $0 < \alpha < 1$, the closed-loop system for the macroscopic model (6.3) is also the sub-diffusion process (6.5).

Remark 6.2. Sub-diffusion evacuation can be realized not only in a macroscopic model of fractional order but also in a macroscopic model of integer order. Compared to Fick diffusion, a longer time may be needed in realizing the evacuation of crowds, which is caused by the presence of viscoelasticity in (6.2) or by improper evacuation policies in (6.6).

(2) Super-diffusion process

– Under the controller of spatial fractional order which is described by

$$u(x,t) = -\left[\left(1-\frac{\rho(x,t)}{\rho_{\max}}\right)\rho(x,t)\right]^{-1} D\frac{\partial^{\beta}}{\partial x^{\beta}}\rho(x,t), \tag{6.7}$$

where $1 < \beta < 2$, the closed-loop system of the macroscopic model (6.1) can be described by the following super-diffusion process:

$$\frac{\partial}{\partial t}\rho(x,t) = D\frac{\partial^{\beta}}{\partial x^{\beta}}\rho(x,t). \tag{6.8}$$

– Under the controller

$$u(x,t) = -\left[\left(1-\frac{\rho(x,t)}{\rho_{\max}}\right)\rho(x,t)\right]^{-1} D\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\frac{\partial}{\partial x}\rho(x,t), \tag{6.9}$$

where the information of the history of the density's gradient has been used, the closed-loop system of the macroscopic model (6.3) can also be described by the following super-diffusive process in the time domain:

$$\frac{\partial^{2-\alpha}}{\partial t^{2-\alpha}}\rho(x,t) = D\frac{\partial^2}{\partial x^2}\rho(x,t). \tag{6.10}$$

Remark 6.3. Compared to the sub-diffusive process (6.5), the super-diffusive process (6.10) will accelerate the crowd's evacuation, and stampede accidents may be avoided using the controller proposed in (6.7) or (6.9).

(3) General-diffusive process

Substituting the controller (6.4) into the generalized fractional dynamic model

$$\frac{\partial}{\partial t^{\alpha}}\rho(t,x) + \frac{\partial}{\partial x^{\beta}}[\rho(t,x)v(t,x)] = 0, \tag{6.11}$$

then the closed-loop system can be described by the following general-diffusive process:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\rho(x,t) = D\frac{\partial^{\beta}}{\partial x^{\beta}}\rho(x,t), \tag{6.12}$$

where $\alpha < 1$ and $1 < \beta < 2$.

Remark 6.4. In the general diffusion process, the final motion pattern is determined by the competition between sub-diffusive processes and super-diffusive processes in equation (6.12). Thus much more freedom has been provided in the modeling and control problems for the crowds of pedestrians.

6.4 Simulation results

In order to avoid searching the solution of PDEs, finite-volume methods (FVMs) have been used to divide the space into grid cells or finite volumes, as shown in Figure 6.3.

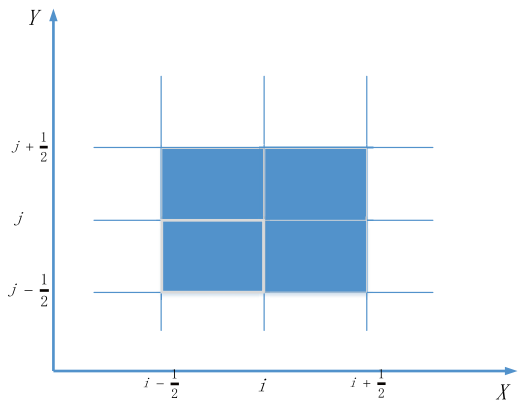

Figure 6.3: Lax–Friedrichs scheme in two-dimensional space.

The partial derivatives of a function $f(x,y,t)$ along the x axis and y axis have been approximated using

$$\frac{\partial}{\partial x}f(x_i,y_j,t)+\frac{\partial}{\partial y}f(x_i,y_j,t)=\frac{1}{\Delta x}\left[f(x_{i+\frac{1}{2}},y_j,t)-f(x_{i-\frac{1}{2}},y_j,t)\right]+\frac{1}{\Delta y}\left[f(x_i,y_{j+\frac{1}{2}},t)-f(x_i,y_{j-\frac{1}{2}},t)\right]$$

in the following simulations.

6.4.1 Simulation in closed and squared area without exits

Simulation results on fractional macroscopic model in two-dimensional space are firstly conducted where $\beta = 1$ is imposed for simplicity. The Lax–Friedrichs scheme has been used to approximate the spatial derivatives in solving the non-linear PDEs due to its efficiency in computation. Based on the Lax–Friedrichs scheme, the PDE in two-dimensional space

$$\frac{\partial}{\partial t^{\alpha}}\rho(t,x,y)+\frac{\partial}{\partial x}[\rho(t,x,y)v(t,x,y)]+\frac{\partial}{\partial y}[\rho(t,x,y)v(t,x,y)]=0$$

has been transformed into

$$\frac{\partial}{\partial t^{\alpha}}\rho(t,x,y)+\frac{1}{2Dx}[\rho(t,x+1,y)v(t,x+1,y)-\rho(t,x-1,y)v(t,x-1,y)]+\frac{1}{2Dy}[\rho(t,x,y+1)v(t,x,y+1)-\rho(t,x,y-1)v(t,x,y-1)]=0$$

in the following simulations.

We have used the following initial Gaussian distribution in simulations:

$$\rho(x,y,0)=C\exp(-(x-a)^2-(y-b)^2),$$

where $C = 1$ is the density value and (a,b) determines the center of initial density distribution. The average speed of free flow has been chosen to be $v_x = v_y = 1.36\,\mathrm{m\,s^{-1}}$ for each pedestrian, as done in many previous studies. Pedestrians are also assumed to move freely within a square area with no obstacles and no exits.

Simulation results for $\alpha = 0.6$ and $\alpha = 1$ are shown in Figure 6.4 to Figure 6.5 and Figure 6.6 to Figure 6.7, respectively. From Figure 6.4 and Figure 6.6, it can be concluded that pedestrians described by a fractional model are much more scattered in the closed square area than that is described by a model of integer order. The same conclusions can be obtained from comparisons between Figure 6.5 and Figure 6.7.

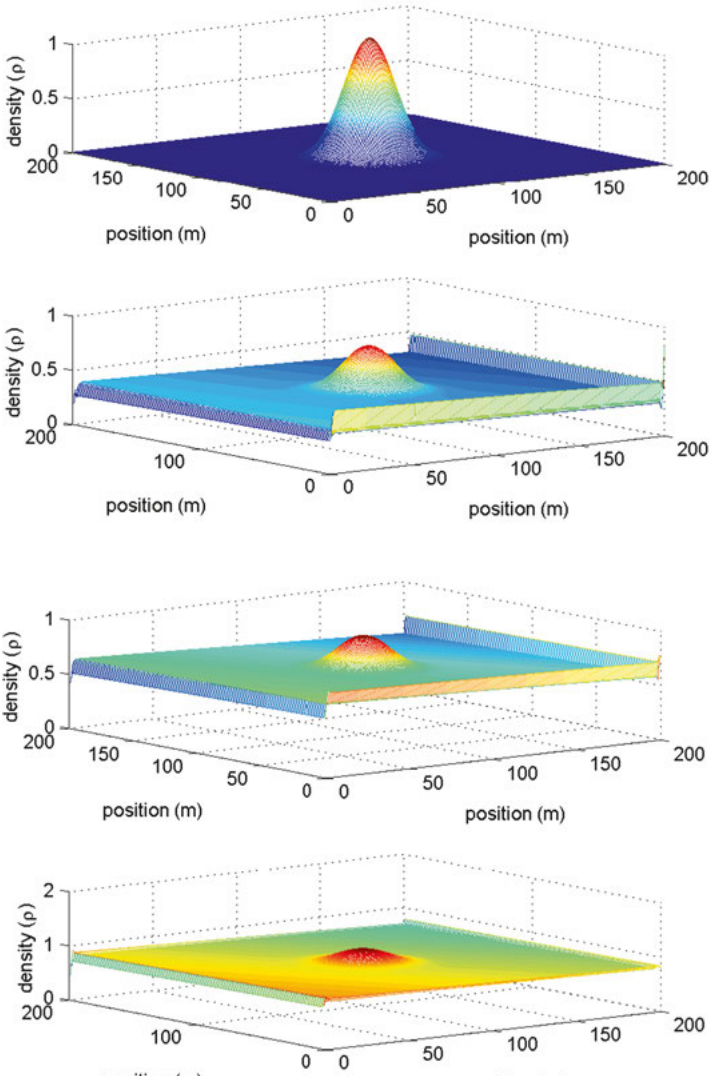

Figure 6.4: Density response for crowds of pedestrians with $\alpha = 0.6$ using the Lax–Friedrichs scheme.

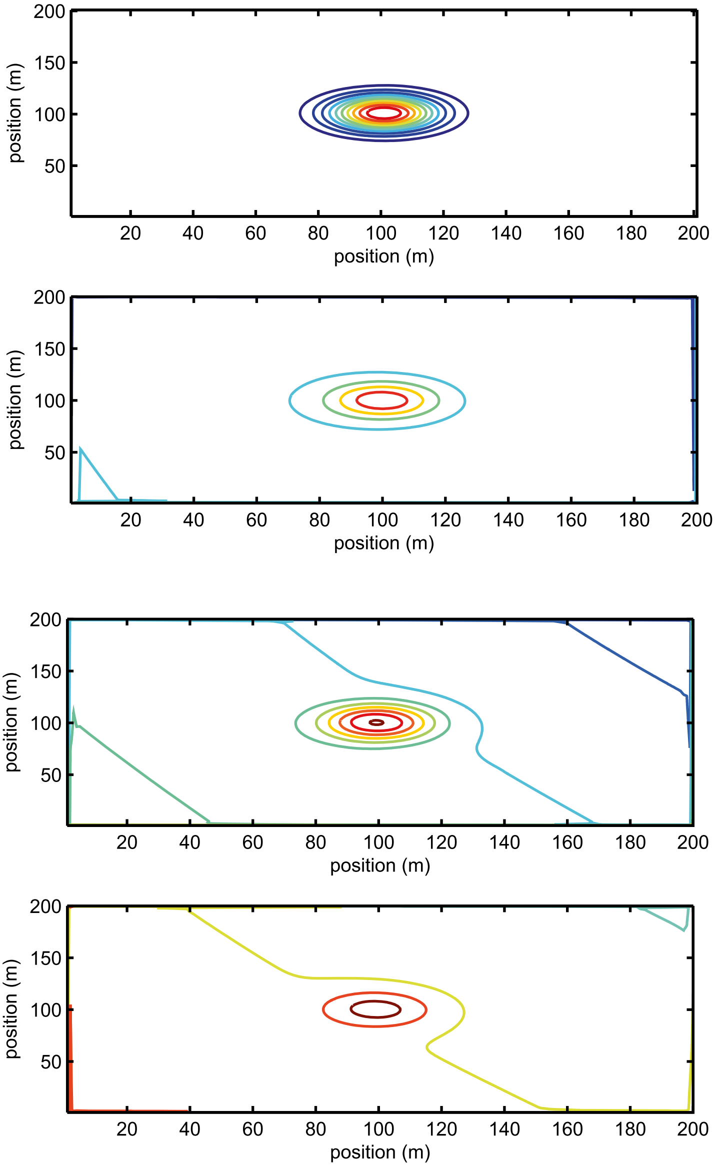

Figure 6.5: Contour of the density response for crowds of pedestrians with $\alpha = 0.6$ using the Lax–Friedrichs scheme.

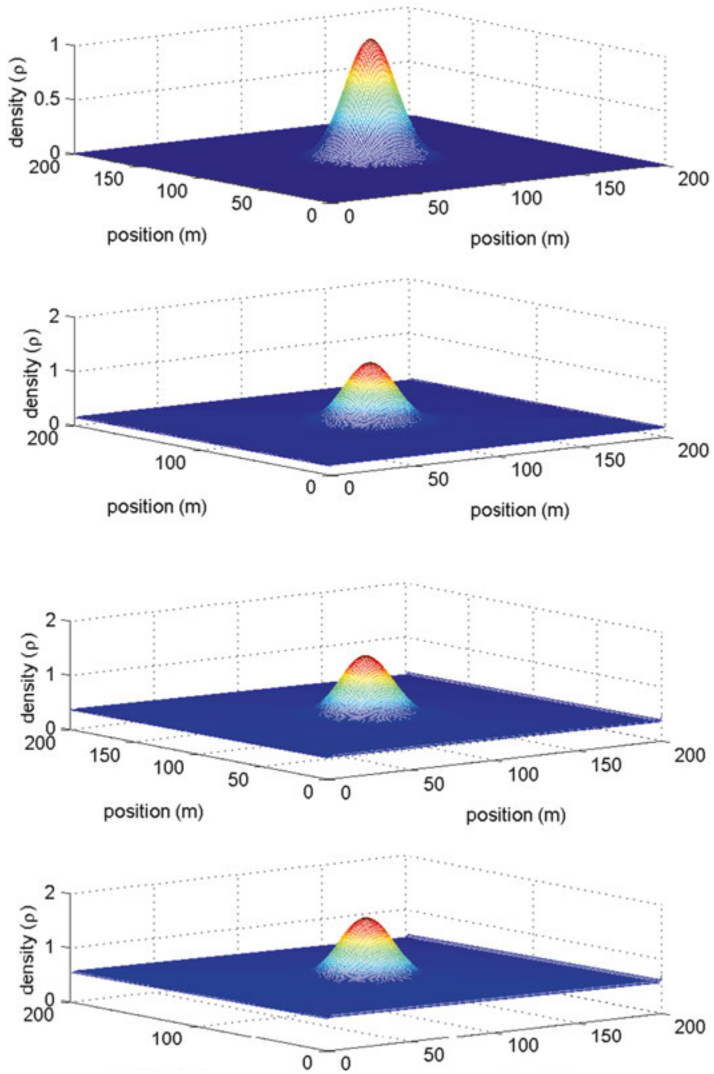

Figure 6.6: Density response for crowds of pedestrians of integer order using the Lax–Friedrichs scheme.

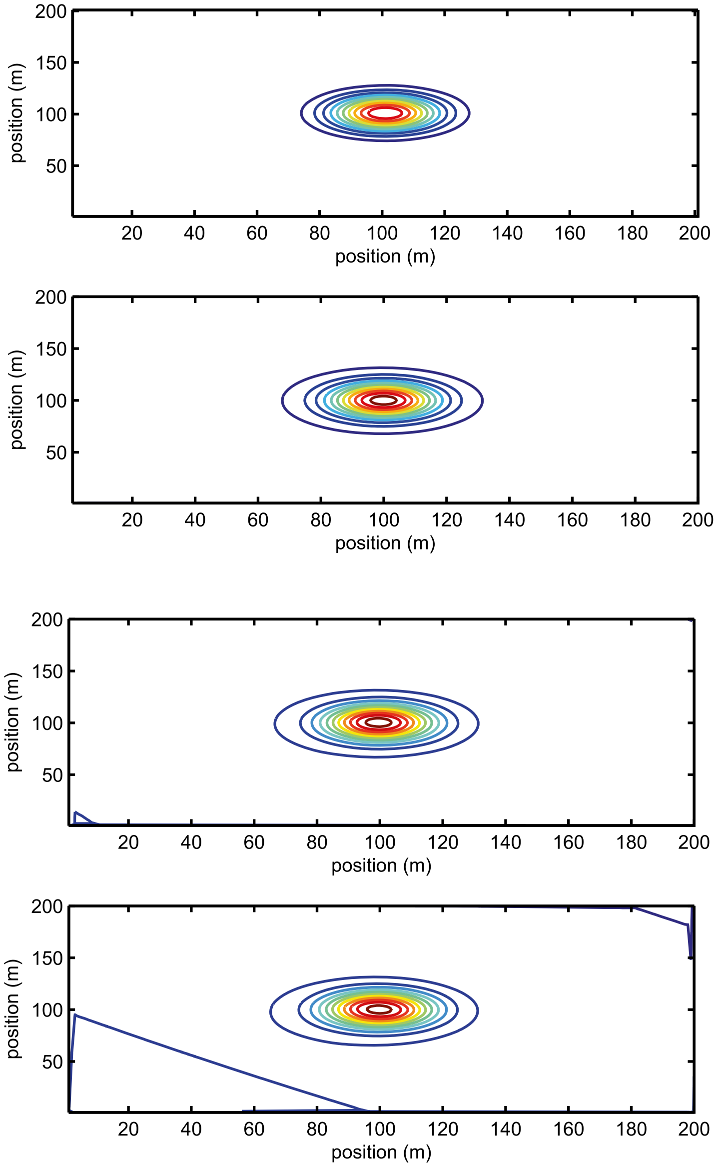

Figure 6.7: Contour of the density response for crowds of pedestrians of integer order using the Lax–Friedrichs scheme.

6.4.2 Simulation in closed and squared area with one exit

Based on the results obtained in Section 6.4.1, the following dynamic model has been simulated for evacuations of crowds in closed and squared corridor with one exit:

$$\begin{cases} \frac{\partial}{\partial t^\alpha}\rho(t,x,y) + \frac{\partial}{\partial x}[\rho(t,x,y)v(t,x,y)] + \frac{\partial}{\partial y}[\rho(t,x,y)v(t,x,y)] = 0, \\ v_t + vv_x = \frac{V - v}{\tau} - \frac{C_0^2}{\rho}\rho_x, \\ u_t + uu_y = \frac{U - u}{\tau} - \frac{C_0^2}{\rho}\rho_y, \end{cases}$$

where $C_0 = 0.8$ is the anticipation term which describes the response of each pedestrian to the density of people in his/her neighborhood and V and U are the desired velocity along x direction and y direction, which are taken to be

$$\begin{cases} V = V(\rho)\frac{x_e - x_i}{\sqrt{(x_e - x_i)^2 + (y_e - y_i)^2}}, \\ U = U(\rho)\frac{y_e - y_i}{\sqrt{(x_e - x_i)^2 + (y_e - y_i)^2}}, \end{cases}$$

where $V(\rho)$ and $U(\rho)$ are the flux–density relationships in Greenshield's model in [5].

The simulation results are shown in Figure 6.8 and Figure 6.9, where dynamic models with fractional order 0.85 and 1 have been used. Form the results obtained, it is easy to see that the density of pedestrians around the exit is much lower for the fractional model than that obtained using a model of integer order.

Remark 6.5. The final distribution of crowds will be different if a different fractional order is selected in the design of feedback controllers in the simulation study. Thus, how to choose the fractional order of feedback controllers to generate the desired distribution of crowds is an interesting problem that is worthy of further consideration in future research.

6.5 Conclusion

Macroscopic models of fractional order for crowds of pedestrians are firstly presented in this chapter where only the viscoelasticity at the temporal scale has been considered for simplicity of modeling and controller design. Based on the macroscopic model of fractional order, distributed feedback controllers have been presented where not only classical diffusion process but also anomalous diffusion process has been realized for evacuation of crowds using the controllers presented in this chapter. Simula-

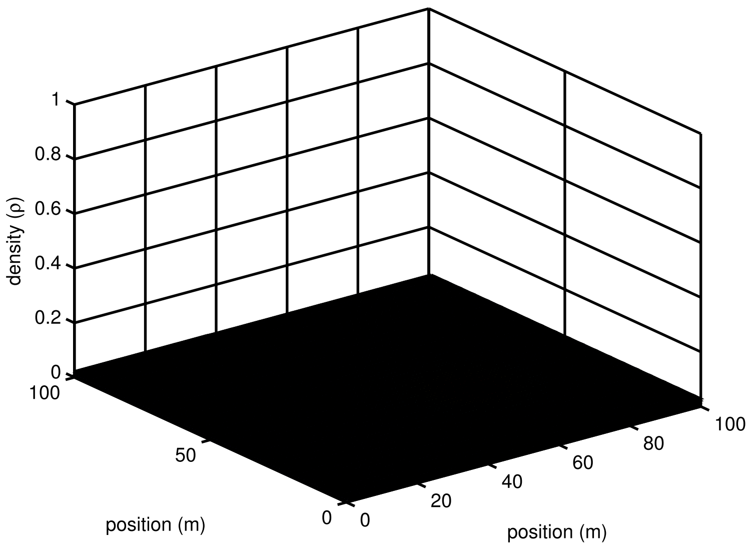

Figure 6.8: Density response for crowds of pedestrians with $\alpha = 0.85$ using the Lax–Friedrichs scheme.

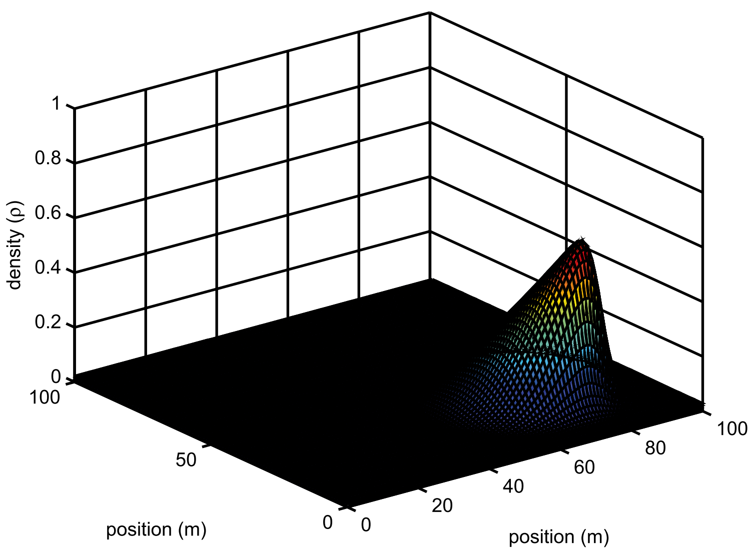

Figure 6.9: Contour of the density response for crowds of pedestrians with $\alpha = 1$ using the Lax–Friedrichs scheme.

tion results based on the Lax–Friedrichs scheme have also been presented to show the diversity and effectiveness of the modeling and control techniques that are proposed within the framework of fractional calculus.

Although some theoretical results and some initial simulations are presented in this paper, there is much more work unexplored as regards this topic, such as stability analysis of classical or anomalous diffusion process, performance evaluation of obtained distributed controllers.

References

[1] L. Avarez, R. Horowitz, and P. Li. Traffic flow control in automated highway systems. *Control Engineering Practice*, 7(9):1071–1078, 1999.

[2] N. Bellomo and L. Gibelli. Behavioral crowds: Modeling and Monte Carlo simulations toward validation. *Advances in Fluid–Structure Interaction*, 141:13–21, 2016.

[3] H. Dong, X. Yang, Y. Chen, and Q. Wang. Pedestrian evacuation in two-dimension via state feedback control. In *Proceedings of American Control Conference*, pages 302–306, 2013.

[4] A. Hanisch, J. Tolujew, K. Richter, and T. Schulze. Online simulation of pedestrian flow in public buildings. In *Proceedings of the 2003 Winter Simulation Conference, volume 2*, pages 1635–1641, 2003.

[5] P. Kachroo, S. J. Al-nasur, S. A. Wadoo, and A. Shende. *Pedestrian Dynamics Feedback Control of Crowd Evacuation*. Springer-Verlag, Berlin, Heidelberg, 2008.

[6] J. A. Kirkland and A. A. Maciejewski. A simulation of attempts to influence crowd dynamics. In *Proceedings of IEEE International Conference on Systems, Man and Cybernetics*, pages 4328–4333. IEEE, New York 2003.

[7] N. Pelechano, J. M. Allbeck, and N. I. Badler. *Virtual Crowds Methods, Simulation, and Control*. Morgan & Claypool Publishers, Williston, 2008.

[8] J. Pettre, P. de H. Ciechomski, J. Maim, B. Yersin, J.-P. Laumond, and D. Thalmann. Real-time navigating crowds scalable simulation and rendering. *Computer Animation and Virtual Worlds*, 17(3–4):445–455, 2006.

[9] A. Shende, M. P. Singh, and P. Kachroo. Optimization-based feedback control for pedestrian evacuation from an exit corridor. *IEEE Transactions on Intelligent Transportation Systems*, 12(4):1167–1176, 2011.

[10] A. Shende, M. P. Singh, and P. Kachroo. Optimal feedback flow rates for pedestrian evacuation in a network of corridors. *IEEE Transactions on Intelligent Transportation Systems*, 14(3):1053–1066, 2013.

[11] Y. Song, J. Gong, L. Niu, Y. Li, Y. Jiang, W. Zhang, and T. Cui. A grid-based spatial data model for the simulation and analysis of individual behaviours in micro-spatial environments. *Simulation Modelling Practice and Theory*, 38:58–68, 2013.

[12] D. Thalmann and S. Raupp Musse. *Crowd Simulation*. Springer, Berlin, 2007.

[13] S. A. Wadoo. Sliding mode control of crowd dynamics. *IEEE Transactions on Control Systems Technology*, 21(3):1008–1015, 2013.

[14] S. A. Wadoo and P. Kachroo. Feedback control design and stability analysis of one dimensional evacuation system. In *Proceedings of IEEE Intelligent Transportation Systems Conference*, pages 618–623, 2006.

[15] S. A. Wadoo and P. Kachroo. Feedback control design and stability analysis of two dimensional evacuation system. In *Proceedings of IEEE Intelligent Transportation Systems Conference*, pages 1108–1113, 2006.

[16] S. A. Wadoo and P. Kachroo. Feedback control of crowd evacuation in one dimension. *IEEE Transactions on Intelligent Transportation Systems*, 11(1):182–193, 2010.

[17] S. A. Wadoo, S. Al-nasur, and P. Kachroo. Feedback control of macroscopic crowd dynamic models. In *Proceedings of American Control Conference*, pages 2558–2563, 2008.
[18] M. Xiong, W. Cai, S. Zhou, M. Yoke-Hean Low, F. Tian, D. Chen, D. W. S. Ong, and B. D. Hamilton. A case study of multi-resolution modeling for crowd simulation. In *Proceedings of Simulation Multiconference, San Diego, California, USA*, 2009.
[19] X. Yang, H. Dong, Y. Chen, and Q. Wang. Pedestrian evacuation in two-dimension via robust feedback control. In *Proceedings of IEEE International Conference on Control and Automation*, pages 1087–1091, 2013.

7 Intelligent evacuation systems for crowds of pedestrians

Abstract: Based on the results obtained for fractional modeling and fractional control of crowds, intelligent evacuation systems for modeling of crowds and control of evacuation are considered in this chapter. Both simulation platform and an experiment platform are studied where control, information, communication, and computation techniques have been embedded to enhance crowd safety and management without changing the physical structure of the facility. We hope that this chapter can provide some reference for the design of intelligent evacuation systems.

7.1 Introduction

Catastrophic events around the world have demonstrated the need to reanalyze and redesign evacuation policies and procedures. The dynamic and uncertain nature of disasters also leads to the need for changing backup contingency plans and adaptation to current evacuation needs. For solving this problem, micro- or macro-simulation models are employed to study the complex evacuation problem of crowds and understand the influences of different kinds of elements such as stochastic perturbations and the built public environment. Based on these simulation and experimental studies, the response behavior of crowds to hazardous events or public facilities will be examined and visualized for the given mathematical model, regulation method, and evacuation policy, proposed in previous research. Also based on these simulation and experiment results, the performance of the crowds as a whole in some life-saving tasks, such as preventing, preparing for, and recovering from hazardous events can be further optimized and improved.

With the dynamic model of fractional order and distributed controllers for evacuation control developed in the previous chapters, the next step is to implement the evacuation of crowds in simulation and in real emergency evacuations. There are some simulation platforms and experiment platforms, such as VISSIM, EXODUS, Simulex, PSCrowd, PEDSIM, and VISWALK, that have been proposed to study the evacuation problem of crowds, and a lot of simulation results have been obtained on these platforms. The main problem existing in these platforms is that the model adopted is very simple and the application of the obtained controllers or evacuation policies is also limited for these simple models. Another issue that should be pointed out is that the platforms mentioned above are not effective anymore if the modeling and control problem are conducted within the framework of fractional calculus.

https://doi.org/10.1515/9783110473988-007

A big map that is used in simulation platforms and experiment platforms is firstly presented in Section 7.2. The main parts of the simulation platform and experiment platform are shown in Section 7.3 and Section 7.4, respectively.

7.2 Big map for intelligent evacuation systems

In order to study the complex crowd–pedestrian system and provide some references for control of the crowd–pedestrian system, a general framework, called cyber-human systems (CHSs), is proposed in this chapter, as shown in Figure 7.1.

The CHS is composed of two coupling parts, called the "PHYSICAL PART" and the "CYBER PART" in Figure 7.1. Transferring of information between these two parts has been implemented through networked Segways, with on-board emergency response personnel, and facility sensing and actuation. In the cyber part, ordinary differential equations (ODEs), partial differential equations (PDEs), and integral-differential equations are employed to describe the crowds of pedestrians using calculus of fractional order in micro-scale, macro-scale, and meso-scale, respectively. Interesting information such as speed, density, flux, and even formation patterns that are obtained through CCTV, Segways, cell phones, and some other sensors can be used in calculating the models obtained, controlling the crowds, and even predicting the tragedy of a stampede that is going to occur. From the viewpoint of a closed-loop system, the flow of information and the major points of each part can easily be seen from Figure 7.1.

7.2.1 Modeling crowds of pedestrians using fractional calculus

Calculus of fractional order is as old as calculus of integer order. After more than 300 years' development, it is well known that more and more systems and phenomena can be better described or approximated using calculus of fractional order (or systems

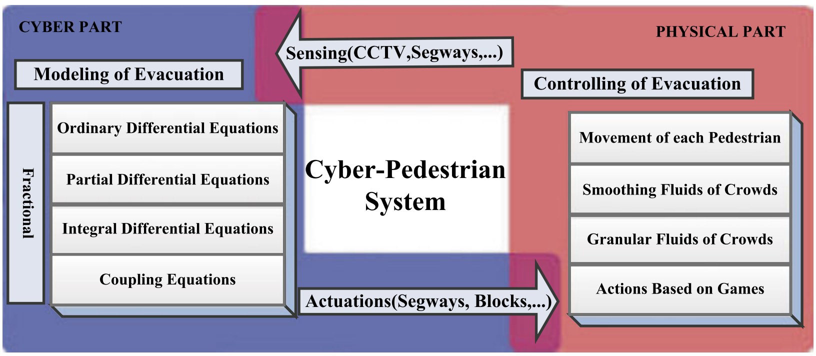

Figure 7.1: Cyber-human systems for modeling, control, and management of crowds.

containing fractional derivatives and integrals) such as self-similarity ("burstiness") and fractal dimensionality, which has been found in Internet traffic data [22], scale-invariance and power-laws in empirical data, as shown in [10] and [2]. More references can be found in [9, 24, 39, 27, 15].

Based on the connection between fractional calculus and the observed phenomena in crowds of pedestrians such as fractal distributions, fractional dynamic games, and fractional evolutions of distributions, modeling of crowds has been firstly considered within the framework of fractional calculus. According to different scales, modeling of crowds has been categorized into three kinds of models.

– At micro-scale level, the behavior of each pedestrian can be described by the ODEs based on some widely used methods such as the social force model (3.1) or an agent based model. A lot of work has been done to find the appropriate parameters in microscopic models using empirical or observed data from different crowds so that the heterogeneity of the pedestrians can be properly described. Another effective way to model the heterogeneity of crowds is using a model of fractional distributed order, as shown in Figure 7.2, especially when the effects of memory and motion habits are included at micro-scale.

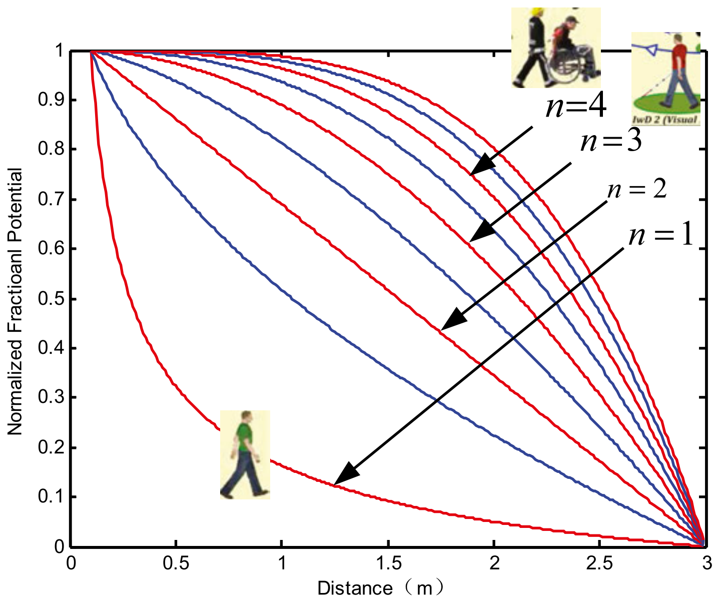

Figure 7.2: Normalized effect of activity variables on potential fields.

– At macro-scale level, the density of crowds is so high that the motion of all pedestrians can be modeled as continuum fluids where PDEs can be derived using conservation laws of mass or momentum. The main differences between crowds of pedestrians and smoothing fluids are that different patterns of motion, such as crossing or intersecting, are allowed in the crowd–pedestrian system due to the freedom of choosing different routes. We believe that not only the characteristics at temporal scale but also the characteristics at spatial scale should be considered in modeling of crowds of pedestrians, as shown by the following dynamic model of fractional order:

$$\frac{\partial}{\partial t^{\alpha}}\rho(t,x) + \frac{\partial}{\partial x^{\beta}}\left[\rho(t,x)v(t,x)\right] = 0, \tag{7.1}$$

which was firstly proposed in [5].
- At meso-scale level with medium density, the dynamics of an evacuation or egress process is similar to the diffusion process of active particles in many aspects such as the porous or granular patterns in smoothing fluids. Fractional convection–diffusion equations are useful tools to model crowds of pedestrians in this case because the phenomenon of porosity that is observed at this scale is connected to fractional calculus. Heterogeneous pedestrians can be modeled using mobile potential fields indexed by activity variables, as shown in Figure 7.2, to guarantee the heterogeneity on this level. Interactions between the microscopic model and the macroscopic model can also be realized through mean-field games to increase the validation of the models obtained and relieve the burden of computation.

Remark 7.1. Comments on modeling of crowds using fractional calculus:
- As there is no general or universal method for modeling crowds of pedestrians for all kinds of scenarios, it is reasonable to choose the most appropriate model for different problems. Since the microscopic model is powerful in describing the heterogeneity of pedestrians, we choose to use the microscopic model when the density is low. With increasing of the density, granular flows with the porosity phenomenon can be observed in large crowds. This kind of heterogeneity is modeled using different mobile potential fields, which can easily be added to the right-hand side of equation (4.3) to describe their influences at micro-scale. The porosity or granular phenomenon disappears when the density becomes very high. Then the macroscopic model can be constructed using a generalized conservation law of mass or momentum as done in the previous research.
- The macroscopic model is responsible for generating a homogenizing effect with desirable smoothness at macro-scale, and the microscopic model is responsible for characterizing heterogeneity and interactions from the macro-scale. For the mesoscopic model, not only the heterogeneity and the porous patterns of crowds can be explicitly characterized, but also the interactions between micro-scale and macro-scale can be included in the integral-differential equations.
- Although we describe the modeling of crowds at different scales according to their densities, the models obtained are not independent of each other, as shown in Figure 7.3. The macro-scale variables such as the density or flow come from an aggre-

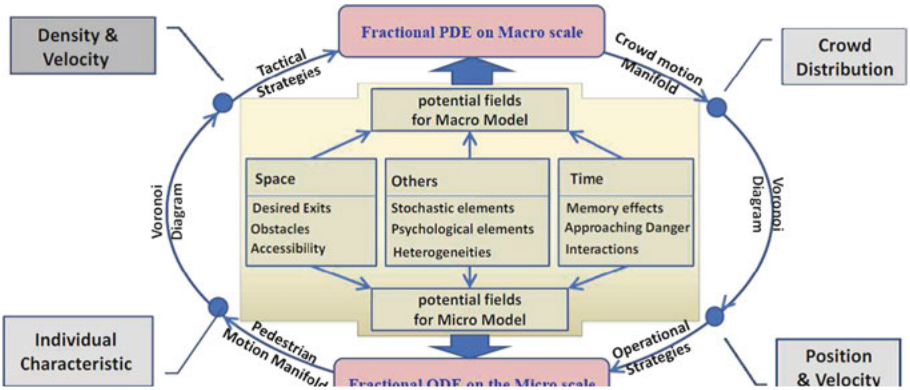

Figure 7.3: Fractional model at micro-scale and macro-scale.

gation of micro-scale data of each pedestrian and the motion of each pedestrian is also affected or constrained by the people around him/her.

7.2.2 Control of crowds of fractional order

Finding an appropriate method to model the complex crowd–pedestrian system is very important. But the more important thing is: what should be done after the modeling stage and how can that be done with the obtained model? Control of the crowd–pedestrian system is much more important in order to prevent a stampede tragedy in gathering of large crowds and may provide some suggestions for planning and design of infrastructure facilities at railway stations, stadiums, and airports, as well as management of pedestrian flows in such facilities. The reader is referred to [18, 17, 19, 20, 21, 43] for preliminary work on control of the crowd–pedestrian system.

Similar to the modeling of crowds of pedestrians, control of the crowd–pedestrian system is also categorized into three different kinds.

- At the microscopic level, the control of each pedestrian is focused on analyzing the relationship between different interactions and collective patterns, where not only interactions based on topology but also short-range and long-range interactions will play an important role.
  - The dynamics of a crowd–pedestrian system with long-range interactions is closely related to fractional calculus and has a significant effect in generating different collective patterns.
  - Mobile Segways will be employed to tune the range of interactions to generate some desired collective patterns.

- At the macroscopic level, fractional controllers based on fractional convection, diffusion, or both of them can be constructed for control of crowds of pedestrians in different scenarios.
  - Due to the high densities in this scale, it is not easy to inject control agents into the fluids. In our framework, mobile Segways with emergency personnel will be dispatched to control the inflow and outflow of crowds from outside based on the theory of boundary control to guarantee the smooth evacuation of high-density crowds without breakdowns.
  - Feedback controllers based on diffusion process or on diffusion–convection process will compensate the feedback controller constructed within the framework of calculus of integer order. Thus more scenarios of evacuation can be realized using diffusion-based controllers or diffusion-convection-based controllers.
- At the mesoscopic level, a coupled equation composed of forward fractional diffusion–convection equations and backward fractional Hamilton–Jacobi–Bellman (HJB) equations are used to model crowds of pedestrians, where the forward part describes the evolution of crowds and the backward part describes the evolution of the decision-making process.
  - Mean-field games can be used to estimate a neighbor's influence and reduce the burden of communication and computation in the coupled mesoscopic model.
  - Mobile Segways with global instructions can be used to guide or drive the crowds through broadcasting or changing the structure of the environment to control the velocity or flow of crowds.

Remark 7.2. Comments on the above control framework:
- For crowds with low density, mobile Segways can easily be added to the crowds as informed leaders and control of the crowd–pedestrian system can be realized through coordination between uninformed pedestrians and informed Segways.
- For crowds of high density, it is not wise to add mobile Segways into the crowds due to the high density of pedestrians. The movement of each pedestrian is totally determined by his/her neighbors. Due to the tight connection among pedestrians, evacuation control of crowds can be realized by Segways placed on the boundary of crowds.
- For crowds with medium density, interactions between microscopic model and macroscopic model are the main challenges in the control of the mesoscopic model. Mean-field-based methods such as mean-field games can be used to simplify the interaction between the microscopic model and the macroscopic model.

7.3 Simulation platform for modeling and evacuation of crowds

7.3.1 DIFF-MAS2D [25]

Due to the requirements of data processing and computation ability in modeling and control of crowds of pedestrians, use of Matlab is much preferred in simulation research. We try to study the modeling and control problem using the platform of diffusion with networked movable actuators and sensors in two-dimensional domain (Diff-MAS2D). Diff-MAS2D is a simulation software package for the control of the diffusion process using moving actuators and moving sensors and it has been firstly developed in [25] for simulating the measurements and control of diffusion processes using Matlab script and Simulink.

The main reason to develop this software was that no currently available software package (Matlab, Maple, Mathematica, MathCAD, Ansys, Nastran, FEMLAB) is able to solve the problem that Diff-MAS2D is able to solve. Diff-MAS2D is written completely in Matlab script and Simulink.

Diff-MAS2D is able to solve the following problem numerically:

$$\frac{\partial u(x,y,t)}{\partial t} = k\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) + f_c(x,y,t) + f_d(x,y,t),$$

where $0 \le x \le 1$ and $0 \le y \le 1$ is the spatial domain, $t \ge 0$ is the time domain, $u(x,y,t)$ is the variable we want to control, k is a positive real constant related to the system parameters, $f_c(x,y,t)$ is the control from the actuators, and $f_d(x,y,t)$ is the disturbance.

An arbitrary combination of the following two types of boundary conditions can be used as the boundary condition for each boundary ($x = 0$, $x = 1$, $y = 0$, or $y = 1$):

- Dirichlet boundary condition:

  $$u = C,$$

  where C is a real constant,
- Neumann boundary condition:

  $$\frac{\partial u}{\partial n}u = C_1 + C_2 u,$$

  where C_1 and C_2 are two real constants and n is the outward direction normal to the boundary.

We are using a number of moving sensors to measure $u(x,y,t)$ and moving actuators as controllers. The control effect of each actuator is assumed to be concentrated rather than actually distributed. The error generated by this assumption can be neglected if, at any time instant, the area affected by each controller is very small compared to the whole area $0 \le x \le 1$ and $0 \le y \le 1$.

Diff-MAS2D is able to simulate the above problem for the following cases:

- Any number of sensors and actuators.
- Sensors and actuators can be collocated or non-collocated.
- Disturbances can be movable.
- Movement of sensors and actuators can be open-loop (designed by the user as functions of time only) or closed-loop (designed by the user as functions of time, sensor data, sensor position/velocity, and actuator position/velocity).
- Arbitrary control algorithms designed by the user.

7.3.2 FO-Diff-MAS2D [6, 7]

FO-Diff-MAS2D is a platform for measurement and control of fractional diffusion model with mobile sensors and mobile actuators whose dynamic process is described by the time fractional differential equation:

$$ {}_C D_{0,t}^{\alpha} u(x,y,t) = k_\alpha \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right) + f_c(u,x,y,t) + f_d(\tilde{u},x,y,t) \tag{7.2} $$

and by the space fractional differential equation

$$ \frac{\partial u(x,y,t)}{\partial t} = k_\beta \left(\frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta} + \frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta} \right) + f_c(u,x,y,t) + f_d(\tilde{u},x,y,t), \tag{7.3} $$

where $u(x,y,t)$ is the density to be controlled, k_α and k_β are positive constants representing the diffusion rate, $f_d(u,x,y,t)$ is the source, $\tilde{u}(x,y,t)$ is the measured data of $u(x,y,t)$ from the sensors, $f_c(u,x,y,t)$ is the control input by mobile actuators to neutralize the controlled density, and its exact form depends on the closed-loop control law designed by the user based on a certain control performance requirement. The term ${}_C D_{0,t}^{\alpha} u(x,y,t)$ is the Caputo fractional derivative of order α $(0 < \alpha \le 1)$ defined by

$$ {}_C D_{0,t}^{\alpha} u(x,y,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u(x,y,\tau) d\tau, & 0 < \alpha < 1, \\ \frac{\partial u(x,y,t)}{\partial t}, & \alpha = 1, \end{cases} $$

and the operators $\frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta}$ are Riesz fractional derivatives defined by

$$ \frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta} = \begin{cases} -C_\beta ({}_{RL}D_{a,x}^{\beta} u + {}_{RL}D_{x,b}^{\beta} u), & 1 < \beta < 2, \\ \frac{\partial^2 u}{\partial x^2}, & \beta = 2, \end{cases} $$

where ${}_{RL}D_{a,x}^{\beta} u$ and $D_{x,b}^{\beta} u$ are left/right Riemann–Liouville derivatives for variable x defined in [31]. A similar definition for $\frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta}$ is omitted here.

Similar to the platform of DIFF-MAS2D, FO-Diff-MAS2D also uses the finite-difference method to discretize the spatial derivative in (7.3), and it uses the fractional central difference to approximate the space fractional derivative in (7.3). Then it leaves the time domain integration to Matlab/Simulink. Specifically, FO-Diff-MAS2D is used to solve a two-dimensional fractional diffusion equation. As an extension of Diff-MAS2D in [25], the main features of FO-Diff-MAS2D are listed as follows:

- Sensors and actuators can be collocated or non-collocated.
- Disturbances can be movable and time-varying.
- The mobility platform dynamics of sensors and actuators can be modeled as either first-order or second-order.
- Movement of sensors and actuators can be open-loop or closed-loop.
- Arbitrary control algorithms can be applied in $f_c(u, x, y, t)$.

The Oustaloup algorithms proposed in [30] can be used to realize the fractional differentiation of an unknown function. The continuous filter

$$G_f(s) = K \cdot \prod_{k=-N}^{k=N} \frac{(s + \omega_k')}{(s + \omega_k)} \tag{7.4}$$

has been constructed in the approximation on frequency band (ω_b, ω_h), where

$$\omega_k' = \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}(1-\gamma)}{2N+1}}, \quad \omega_k = \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}(1+\gamma)}{2N+1}},$$

$$K = \left(\frac{\omega_h}{\omega_b}\right)^{-\frac{\gamma}{2}} \prod_{k=-N}^{k=N} \frac{\omega_k}{\omega_k'}.$$

Based on the Mason formula for the transfer function, (7.4) can be constructed using Simulink of Matlab, as shown in Figure 7.4. Then, based on the state space realization of equation (7.4), the fractional-integrator part that is shown in Figure 7.5 can be successfully realized. The initial lay-outs and snapshot of the running of this simulation platform are shown in Figure 7.6 and Figure 7.7, respectively.

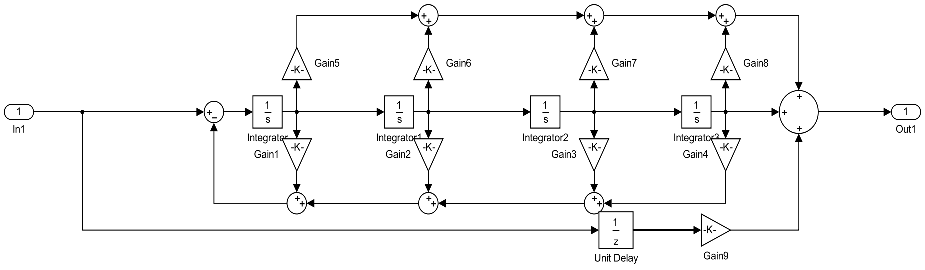

Figure 7.4: Block diagram of continuous filter using Simulink.

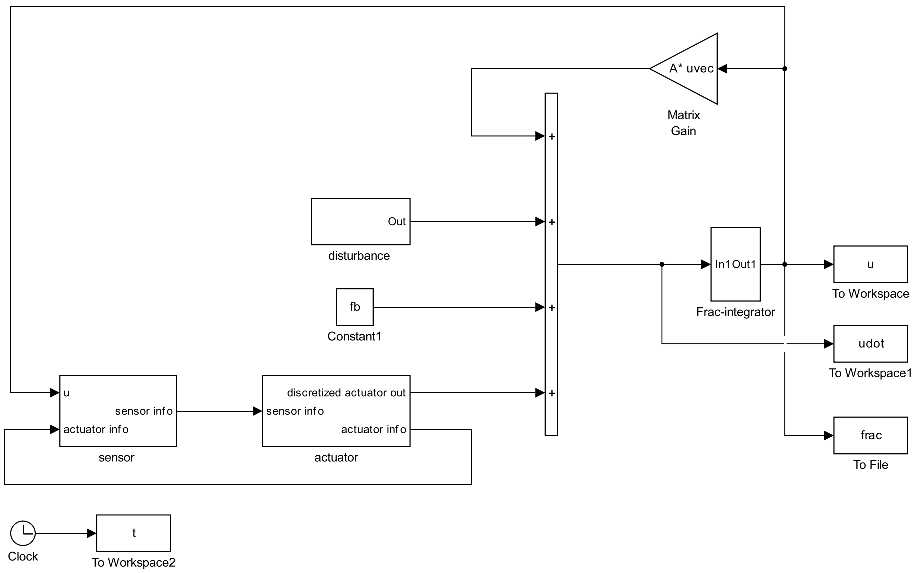

Figure 7.5: Main framework of FO-DIFF-MAS2D.

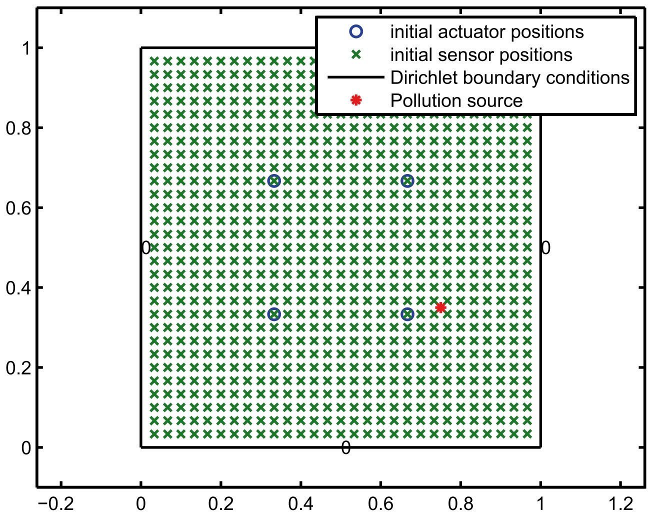

Figure 7.6: Initial lay-outs of actuators, sensors, and obstacle.

Remark 7.3. Comments on platform of FO-Diff-MAS2D.

To the best of our knowledge, Diff-MAS2D is the only available software package capable of simulating control of the diffusion process using movable sensors and actuators. FO-Diff-MAS2D is also the only software package for simulation of the time fractional diffusion process or the space fractional diffusion process controlled using

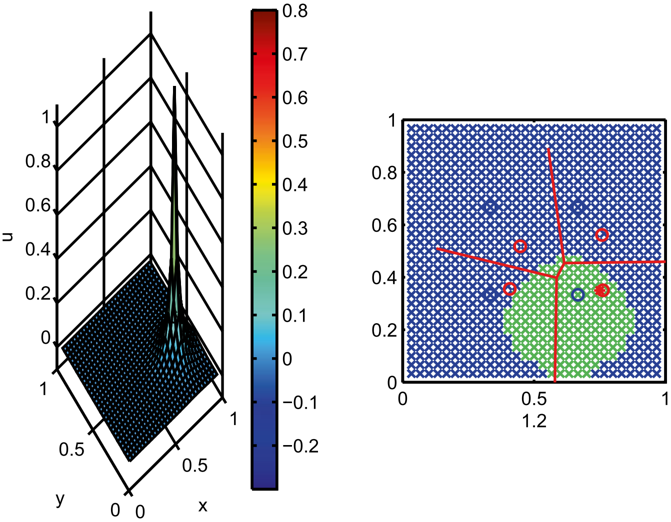

Figure 7.7: Running of fractional diffusion simulation.

moving actuators and moving sensors. Thus this platform is a good choice for the simulation task of modeling and controlling crowds of pedestrians. With Diff-MAS2D and FO-Diff-MAS2D, some hard questions might be answered, such as the following:
- Given large crowds of pedestrians, what are the minimal number of sensors and the minimal number of actuators required?
- What are the advantages (disadvantages) of the collocated scheme or the non-collocated scheme in evacuation control of crowds of pedestrians?
- Are there any better controllers or better schemes for modeling and controlling the evacuation of crowds?

7.4 Experiment platform for modeling and evacuation of crowds

7.4.1 Control architecture

A four-layer hierarchical control architecture that is similar to the one proposed in [21] has been used in the experiment platform as shown in Figure 7.8. The four-layer architecture is composed of a network layer, link layer, planning layer, and regulation layer. The main purposes and functions of each layer are listed as follows:

Network layer: The task in this layer is to assign an escape route for each evacuee in the system so that the evacuation of crowds can be finished in optimal time. Due to changing of the environment and the influence of stochastic noise, escape routes based on feedback controllers are much more preferred compared to open-loop controllers.

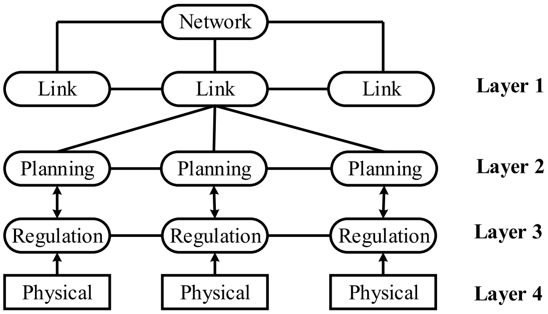

Figure 7.8: Four-layer architecture of the experiment platform.

Link layer: For efficient evacuation of crowds in an emergency scenario, the whole crowd is firstly decomposed into different sections according to the structure of buildings, stadiums, or stations using the techniques of Voronoi diagrams. Then different evacuation speeds and exits will be assigned by different link layers. In order to avoid jamming or a stampeding tragedy, the evacuation speed will be derived from the macroscopic models (PDEs in Chapter 3) using the density information on each cell of the Voronoi diagram.

Planning layer: Based on the escape route received from the network layer and the evacuation speed received from the link layer, the main task of the planning layer is to produce a plan so that each evacuee in different links can move along the assigned path. Another task of this layer is cooperating with other planning layers to avoid conflicts and collisions among the cells of the Voronoi diagram.

Regulation layer: The main task of the regulation layer is to satisfy requests from the planning layer using feedback controllers based on the current position and velocity of each evacuee. After various controllers in this layer have been implemented using hardwares in the physical layer, next action in time will be conducted to realize the evacuation of whole crowds.

Based on the four-layer hierarchical control architecture, the infrastructure of the intelligent evacuation system (IES) can be described by Figure 7.9, which is composed of the "physical" part and the "cyber" part as shown in Figure 7.1. The transfer of informa-

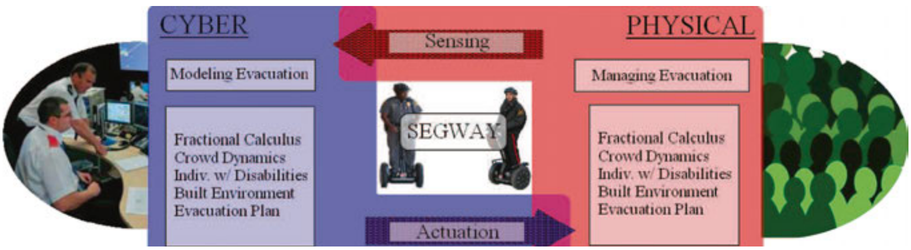

Figure 7.9: Mass pedestrian evacuation system.

tion between these two parts is implemented through the use of networked Segways with on-board emergency response personnel and facility sensing and actuation.

7.4.2 Data acquisition

Originally, collecting data of each pedestrian has been finished by counts and surveys such as questionnaires as done in [35], where the "manually" collected data have been used in analysis of behavior of pedestrians. Now with the vast development of the computer, communication, and computation, there are many different appliances and methods for collecting the data of crowds such as cell phones, GPS, and video tracking [41].

Cell phones: The benefits of mobile technology have made mobile phones a convenient method for data collection. It has been used in [36] for mobility detection and in [23] for analysis of the way-finding behavior of pedestrians. Due to some problems, such as data storage and data uploading, this method still needs to be further explored.

GPS: The global positioning system (GPS) has been used in [1, 26] and [34] for analysis of the pedestrian's spatial behavior such as calibrating activity and determining the location. However, application of this method is restricted to outdoor environments due to the accuracy problem and loss of GPS signal in a building environment.

Video: Different from the accuracy problem existing in cell phones and GPS, the technique of video image process has gained a lot of attention in modeling and control of crowds in recent years. Counts of pedestrians and trajectories of the crowds have been collected using the technique of video image processing to calibrate and validate the mathematical models obtained.

- Automatic counting of pedestrians has been studied in [32] and [33], where some algorithms or filters have been proposed for automatic counting of pedestrians. Similarly, an automatic counting method for bi-directional pedestrians has also been studied in [8]. Although the methods proposed for automatic counting are useful for obtaining density or flux information of the crowds, they cannot be used in motion planing problems and trajectory tracking problems; this is important for the evacuation of crowds where the actual trajectory of each pedestrian is desired, as shown in Figure 7.10 and Figure 7.11.
- With the development of video image processing techniques, this method has been introduced into the research of modeling and control of crowds. Parameters for crowds, such as free speed, moving direction, and density of people, have been obtained in [13, 12, 11, 40] using the technique of video image processing. Pedestrian tracking methods are also realized using the technique of video image processing in [40, 38, 28, 37]. Considering the performance

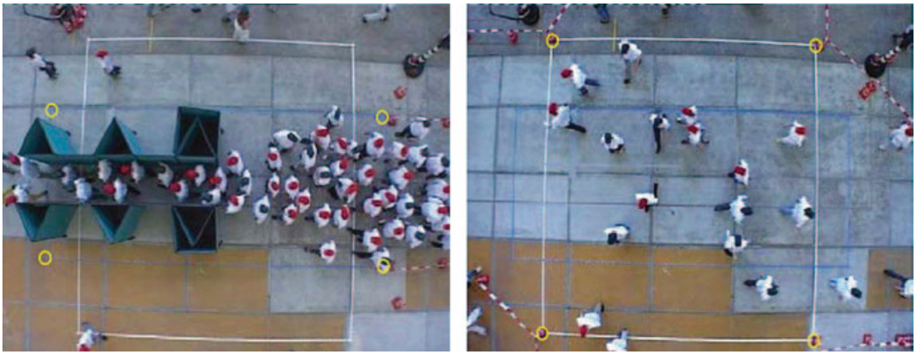

Figure 7.10: Extracting microscopic pedestrian characteristics from video data in [16].

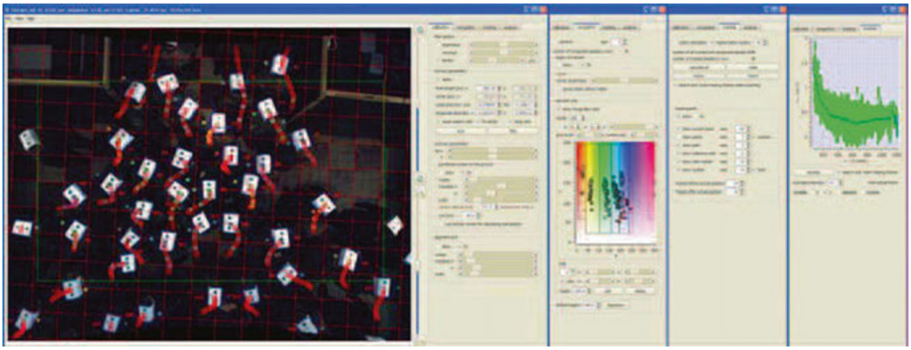

Figure 7.11: PEtrack: Automatic extraction of pedestrian trajectories from video recordings in [3].

requirements in video tracking and data reduction, there are many issues unexplored in concrete applications of these methods, such as what kind of sensors should be used and where these sensors should be arranged in a concrete scenario. The interested reader is referred to [42, 14, 4] for control using mobile sensors and actuators in distributed parameter systems.

Remark 7.4. Different from static sensing receiving a lot attention in previous studies, it is much preferred to use mobile sensing in modeling and control of crowd–pedestrian systems. Besides the use of facility cameras, emergency sensors, cell phones, and GPS, security personnel on Segways are also introduced as sensors and actuators in our framework as shown in Figure 7.12 and Figure 7.13. Previous work on the simulation platform DIFF-MASS2D has provided some reference for architecture design and realization of distributed sensing and control of crowds.

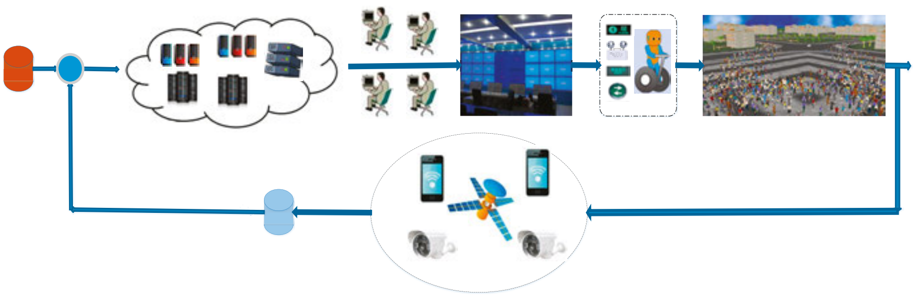

Figure 7.12: The framework for evacuation experiment of crowds of pedestrians.

Figure 7.13: Center of Monitor and Control in Nanjing University of Posts and Telecommunications.

7.4.3 Data extraction

The main task of the data analysis is extracting useful information from each pedestrian's movement to calibrate the microscopic model or the macroscopic model, predict the future behavior of crowds, and even control the movement of crowds so that efficient evacuation can be realized without stampeding.

Due to the powerful numeric engine and friendly programming environment with interactive tools, MATLAB and related data analysis products have provided very convenient ways in statistical analysis, image processing, signal processing, and some other domains [29]. To facilitate the analyzing of various aspects of the data obtained in experiments of crowds, a graphic user interface (GUI) has been created using Matlab. Although there are some other forms of programming that can be used, Matlab is selected due to its ease of use by several different programmers and also its ability to display graphs and manipulate data quickly. The complete GUI can be found in Figure 7.14, which is fast in constructing a prototype model and analyzing a system's performance.

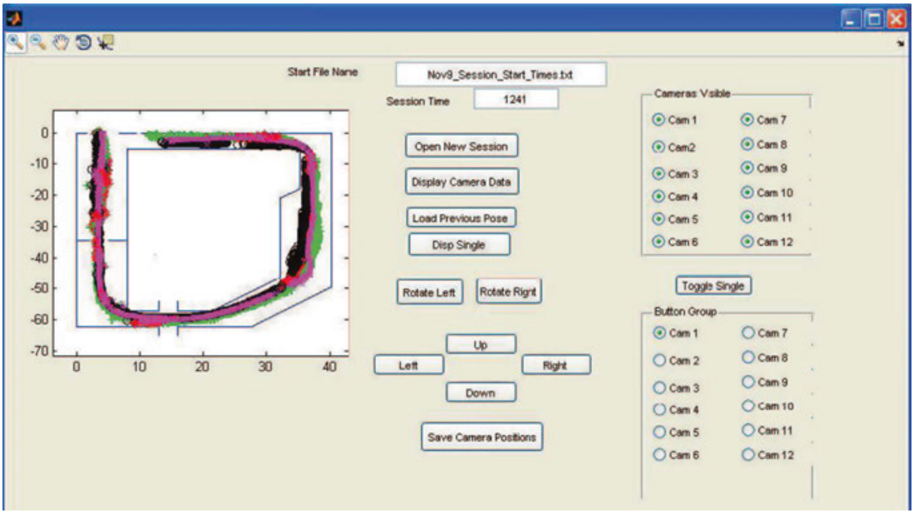

Figure 7.14: Matlab GUI created for data extraction and analysis in [37].

The data acquisition and data extraction mentioned above have been conducted in a series of large-scale crowd experiments in 2012, to study heterogeneous combinations of individuals with disabilities within a crowd. To track pedestrians, each individual wore a graduation cap with a marker axed that can be tracked via a series of cameras. The experiments took place in a gym of Utah State University, where a circuit was built containing a built environment possessing common facility structures such as a doorway, a bottleneck, corners, an oblique corner, and varying hallway widths, as shown in [37]. All structures are Americans with Disabilities Act Accessibility Guidelines-compliant. The circuit of the environment, the overview of the crowd movement, and the extracted information using the Matlab-GUI are shown in Figure 7.15.

7.5 Conclusion

An intelligent evacuation system based on technology of control, communication, and computation has been proposed in this chapter. Main components of the simulation

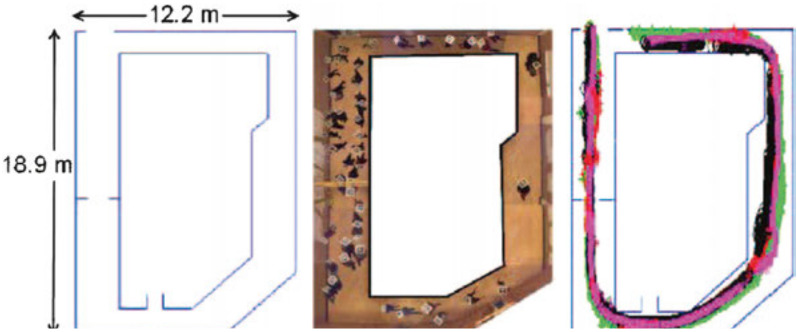

Figure 7.15: Experiment circuit of crowds.

platform and experiment platform have been introduced for modeling and control of crowds of pedestrians. Potential topics related to the simulation and experiment studies on these platforms are also mentioned for future research.

References

[1] D. Ashbrook and T. Starner. Using GPS to learn significant locations and predict movement across multiple users. *Personal and Ubiquitous Computing*, 7(5):275–286, 2003.

[2] P. Bogdan and R. Marculescu. Towards a science of cyber-physical systems design. In *Proceedings of IEEE/ACM International Conference on Cyber-Physical Systems*, pages 99–108, 2011.

[3] M. Boltes, A. Seyfried, B. Steffen, and A. Schadschneider. Automatic extraction of pedestrian trajectories from video recordings. In *Pedestrian and Evacuation Dynamics 2008*, pages 43–54. Springer, Berlin, Heidelberg, 2010.

[4] A. G. Butkovskiy and L. M. Pustyl'nikov. *Mobile Control of Distributed Parameter Systems*. John Wiley & Sons, New York, 1987.

[5] K.-C. Cao, C. Zeng, D. Stuart, and Y. Q. Chen. Fractional order dynamic modeling of crowd pedestrians. In *Proceedings of the Fifth Symposium on Fractional Differentiation and Its Applications*, 2012.

[6] J. Cao, C. Li, and Y. Q. Chen. Compact difference method for solving the fractional reaction–subdiffusion equation with Neumann boundary value condition. *International Journal of Computer Mathematics*, 92(1):167–180, 2014.

[7] J. Cao, Y. Q. Chen, and C. Li. Multi-UAV-based optimal crop-dusting of anomalously diffusing infestation of crops. In *Proceedings of American Control Conference*, pages 1278–1283, 2015.

[8] T. H. Chen. An automatic bi-directional passing-people counting method based on color image processing. In *Proceedings of IEEE International Carnahan Conference on Security Technology*, pages 200–207, 2003.

[9] Y. Q. Chen, D. Xue, and H. Dou. Fractional calculus and biomimetic control. In *Proceedings of IEEE International Conference on Robotics and Biomimetics*, pages 901–906, 2004.

[10] A. Clauset, C. R. Shalizi, and M. E. J. Newman. Power-law distributions in empirical data. *SIAM Review*, 51(4):661–703, 2009.

[11] W. Daamen. Modeling passenger flows in public transport facilities. *Free Flow Speeds*, 2004.

[12] W. Daamen and S. Hoogendoorn. Controlled experiments to derive walking behaviour. *European Journal of Transport and Infrastructure Research*, 3:39–59, 2003.
[13] W. Daamen and S. P. Hoogendoorn. Experimental research of pedestrian walking behavior. *Transportation Research Record*, 1828:20–30, 2003.
[14] A. El Jai and A. J. Pritchard *Sensors and Controls in The Analysis of Distributed Systems*. John Wiley & Sons, New York, 1988.
[15] R. Hilfer. *Applications of Fractional Calculus in Physics*. World Scientific Publishing Company, Singapore, 2000.
[16] S. P. Hoogendoorn, W. Daamen, and P. H. L. Bovy. Extracting microscopic pedestrian characteristics from video data. In *Proceedings of Annual Meeting of Transportation Research Part B*, pages 1–15, 2003.
[17] P. Kachroo. *Nonlinear Control Strategies and Vehicle Traction Control*. PhD thesis, University of California at Berkeley, 1993.
[18] P. Kachroo. *Pedestrian Dynamics: Mathematical Theory and Evacuation Control*. CRC Press/Taylor & Francis GroupBoca Raton/London, 2009.
[19] P. Kachroo and K. Ozbay. Solution to the user equilibrium dynamic traffic routing problem using feedback linearization. *Transportation Research. Part B*, 32(5):343–360, 1998.
[20] P. Kachroo, S. A. Shedied, J. S. Bay, and H. Vanlandingham. Dynamic programming solution for a class of pursuit evasion problems the herding problem. *IEEE Transactions on Systems, Man and Cybernetics, Part C*, 31(1):35–41, 2001.
[21] P. Kachroo, S. J. Al-nasur, S. A. Wadoo, and A. Shende. *Pedestrian Dynamics Feedback Control of Crowd Evacuation*. Springer-Verlag, Berlin, Heidelberg, 2008.
[22] W. E. Lelland On the self-similar nature of Ethernet traffic. *IEEE/ACM Transactions on Networking*, 2(1):1–15, 1994.
[23] C. Li. User preferences, information transactions and location-based services: A study of urban pedestrian wayfinding. *Computers, Environment and Urban Systems*, 30(6):726–740, 2006.
[24] Y. Li, H. Sheng, and Y. Q. Chen. On distributed order integrator differentiator. *Signal Processing*, 91(5):1079–1084, 2010.
[25] J. Liang and Y. Q. Chen. Diff-MAS2-User's Manual. Technical report, Center for Self-Organizing and Intelligent Systems (CSOIS), Department of Electrical and Computer Engineering College of Engineering, Utah State University, 2004.
[26] L. Lin, D. Fox, and H. Kautz. Learning and inferring transportation routines. In *National Conference on Artificial Intelligence*, pages 348–353, 2004.
[27] C. F. Lorenzo and T. T. Hartley. Variable order and distributed order fractional operators. *Nonlinear Dynamics*, 29:57–98, 2002.
[28] O. Masoud and N. P. Papanikolopoulos. Robust pedestrian tracking using a model-based approach. In *Proceedings of IEEE Intelligent Transportation System*, pages 338–343, 1997.
[29] Matlab. The Mathworks Inc. www.mathworks.com, 2016.
[30] A. Oustaloup, F. Levron, B. Mathieu, and F. M. Nanot. Frequency-band complex noninteger differentiator: characterization and synthesis. *IEEE Transactions on Circuits and Systems. I, Fundamental Theory and Applications*, 47(1):25–39, 2000.
[31] I. Podlubny. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, volume 198. Academic Press, San Diego, 1999.
[32] G. Sexton and X. Zhang. Automatic human head location for pedestrian counting. In *Proceedings of International Conference on Image Processing & Its Applications, volume 2*, pages 535–540, 2002.
[33] G. Sexton, X. Zhang, G. Redpath, and D. Greaves. Advances in automated pedestrian counting. In *European Convention on Security and Detection*, pages 106–110, 1995.

[34] N. Shoval and M. Isaacson. Application of tracking technologies to the study of pedestrian spatial behavior. *The Professional Geographer*, 58(2):172–183, 2006.

[35] V. P. Sisiopiku and D. Akin. Pedestrian behaviors at and perceptions towards various pedestrian facilities: an examination based on observation and survey data. *Transportation Research. Part F*, 6(4):249–274, 2003.

[36] T. Sohn, A. Varshavsky, A. Lamarca, M. Y. Chen, T. Choudhury, I. Smith, S. Consolvo, J. Hightower, W. G. Griswold, and E. De Lara. Mobility detection using everyday GSM traces. In P. Dourish, A. Friday (eds) *Ubiquitous Computing*, *Lecture Notes in Computer Science, volume 4206*, pages 212–224, Springer, Berlin, Heidelberg, 2010.

[37] D. S. Stuart. *Microscopic Modeling of Crowds Involving Individuals with Physical Disability: Exploring Social Force Interaction*. PhD thesis, Utah State University, 2015.

[38] M. J. Sullivan, C. A. Richards, C. E. Smith, and O. Masoud. Pedestrian tracking from a stationary camera using active deformable models. In *Proceedings of the Intelligent Vehicles Symposium*, pages 90–95, 1995.

[39] H. G. Sun, W. Chen, and Y. Chen. Variable-order differential operator in anomalous diffusion modeling. *Physica A*, 388:4586–4592, 2009.

[40] K. Teknomo. *Microscopic Pedestrian Flow Characteristics: Development of an Image Processing Data Collection and Simulation Model*. Tohoku University Japan, Sendai, 2002.

[41] H. Timmermans. *Pedestrian Behavior: Models, Data Collection and Applications*. Emerald Group Publishing Limited, Bingley, 2009.

[42] C. Tricaud and Y. Q. Chen. *Optimal Mobile Sensing and Actuation Policies in Cyber-Physical Systems*. Springer-Verlag London Limited, London, 2012.

[43] S. A. Wadoo, Sliding Mode Control of Crowd Dynamics. *IEEE Transactions on Control Systems Technology*, 21(3):1008–1015, 2013.

Index

Fractional Calculus in Applied Sciences and Engineering

Volume 3
Michal Fečkan, JinRong Wang, Michal Pospíšil
Fractional-Order Equations and Inclusions, 2017
ISBN 978-3-11-052138-2, e-ISBN (PDF) 978-3-11-052207-5,
e-ISBN (EPUB) 978-3-11-052155-9

Volume 2
Bruce J. West
Nature's Patterns and the Fractional Calculus, 2017
ISBN 978-3-11-053411-5, e-ISBN (PDF) 978-3-11-053513-6,
e-ISBN (EPUB) 978-3-11-053427-6

Volume 1
Dingyü Xue
Fractional-Order Control Systems.
Fundamentals and Numerical Implementations, 2017
ISBN 978-3-11-049999-5, e-ISBN (PDF) 978-3-11-049797-7,
e-ISBN (EPUB) 978-3-11-049719-9