Exotic Option Pricing and Advanced Lévy Models

Edited by

Andreas E. Kyprianou, Wim Schoutens and Paul Wilmott

John Wiley & Sons, Ltd

Exotic Option Pricing and Advanced Lévy Models

Exotic Option Pricing and Advanced Lévy Models

Edited by

Andreas E. Kyprianou, Wim Schoutens and Paul Wilmott

John Wiley & Sons, Ltd

Copyright © 2005 John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex PO19 8SQ, England

Telephone (+44) 1243 779777

Email (for orders and customer service enquiries): cs-books@wiley.co.uk
Visit our Home Page on www.wiley.com

Other Wiley Editorial Offices

John Wiley & Sons Inc., 111 River Street, Hoboken, NJ 07030, USA

Jossey-Bass, 989 Market Street, San Francisco, CA 94103-1741, USA

Wiley-VCH Verlag GmbH, Boschstr. 12, D-69469 Weinheim, Germany

John Wiley & Sons Australia Ltd, 42 McDougall Street, Milton, Queensland 4064, Australia

John Wiley & Sons (Asia) Pte Ltd, 2 Clementi Loop #02-01, Jin Xing Distripark, Singapore 129809

John Wiley & Sons Canada Ltd, 22 Worcester Road, Etobicoke, Ontario, Canada M9W 1L1

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic books.

British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library

ISBN-13 978-0-470-01684-8
ISBN-10 0-470-01684-1

Typeset in 10/12pt Times by Laserwords Private Limited, Chennai, India
Printed and bound in Great Britain by Antony Rowe Ltd, Chippenham, Wiltshire
This book is printed on acid-free paper responsibly manufactured from sustainable forestry in which at least two trees are planted for each one used for paper production.

Contents

Contributors

Hansjörg Albrecher
Department of Mathematics, Graz University of Technology, Steyrergasse 30, A-8010 Graz, Austria

Ariel Almendral
Norwegian Computing Center, Gaustadalleen 23, Postbox 114, Blindern, N-0314 Oslo, Norway

Pauline Barrieu
Statistics Department, London School of Economics, Houghton Street, London, WC2A 2AE, UK

Nadine Bellamy
Equipe d'Analyse et Probabilités, Université d'Evry Val d'Essonne, Rue du Père Jarlan, 91025 Evry Cedex, France

Peter Carr
Bloomberg LP, 731 Lexington Avenue, New York, NY 10022, USA

Terence Chan
School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh, EH14 4AS, Scotland, UK

José Manuel Corcuera
Facultat de Matematiques, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, E-08007 Barcelona, Spain

Ernst Eberlein
Department of Mathematical Stochastics, University of Freiburg, Eckerstraße 1, D-79104, Freiburg, Germany

Pavel V. Gapeev
Institute of Control Sciences, Russian Academy of Sciences, Profsoyuznaya Str. 65, 117997 Moscow, Russia

Hélyette Geman
University of Paris Dauphine, Paris, France and ESSEC-Finance Department, 95021 Cergy-Pontoise, France

Ali Hirsa
Caspian Capital Management, LLC, 745 Fifth Avenue, 28th Floor, New York, NY 10151, USA

Jan Kallsen
HVB-Institute for Mathematical Finance, Munich University of Technology, D-85747 Garching, Germany

Christoph Kuehn
Johann Wolfgang Goethe-Universität, Fachbereich Mathematik (Fach 187), D-60054 Frankfurt am Main, Germany

Andreas E. Kyprianou
School of Mathematical and Computer Sciences, Heriot Watt University, Edinburgh, EH14 4AS, Scotland, UK

R. Loeffen
Department of Mathematics, University of Utrecht, PO Box 80.010, 3508 TA Utrecht, The Netherlands

Dilip B. Madan
Department of Finance, Robert H. School of Business, Van Munching Hall, University of Maryland, College Park, MD 20742, USA

David Nualart
Facultat de Matematiques, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, E-08007 Barcelona, Spain

Antonis Papapantoleon
Department of Mathematical Stochastics, University of Freiburg, Eckerstraße 1, D-79104, Freiburg, Germany

Goran Peskir
Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000, Aarhus, Denmark

Wim Schoutens
Katholieke Universiteit Leuven - U.C.S, W. de Croylaan 54, B-3001 Leuven, Belgium

Irwin Simons
ING SWE, Financial Modelling, Marnixlaan 24, B-1000 Brussels, Belgium

Jurgen Tistaert
ING SWE, Financial Modelling, Marnixlaan 24, B-1000 Brussels, Belgium

Nadia Uys
Programme for Advanced Mathematics of Finance, School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Witwatersrand 2050, South Africa

Nick Webber
Warwick Business School, University of Warwick, Coventry CV4 7AL, UK

Preface

Since around the turn of the millennium there has been a general acceptance that one of the more practical improvements one may make in the light of the shortfalls of the classical Black–Scholes model is to replace the underlying source of randomness, a Brownian motion, by a Lévy process. Working with Lévy processes allows one to capture distributional characteristics in the stock returns such as semi-heavy tails and asymmetry, as well as allowing for jumps in the price process with the interpretation as market shocks and effects due to trading taking place in business time rather than real time. In addition, Lévy processes in general, as well as having the same properties as Brownian motion in the form of stationary independent increments, have many well understood probabilistic and analytical properties which make them attractive as mathematical tools.

At the same time, exotic derivatives are gaining increasing importance as financial instruments and are traded nowadays in large quantities in over the counter markets. The consequence of working with markets driven by Lévy processes forces a number of new mathematical challenges with respect to exotic derivatives. Many exotic options are based on the evolving historical path of the underlying. In terms of pricing and hedging, this requires an understanding of fluctuation theory, stochastic calculus and distributional decompositions associated with Lévy processes. This current volume is a compendium of articles, each of which consists of a discursive review and recent research on the topic of *Exotic Option Pricing and Advanced Lévy Models* written by leading scientists in this field.

This text is organized as follows. The first two chapters can be seen as an introduction to Lévy processes and their applications. The first chapter, by A. E. Kyprianou and R. Loeffen, gives a brief introduction to Lévy processes, providing several examples which are commonly used in finance, as well as examining in more detail some of their fine and coarse path properties. To apply Lévy processes in practice one needs good numerics. In Chapter 2, N. Webber discusses recent progress in the development of simulation methods suitable for most of the widely used Lévy processes. Speed-up methods, bridge algorithms and stratified sampling are some of the many ingredients. These techniques are applied in the context of the valuation of different kinds of exotic options.

In the second part, one can see Lévy-driven equity models at work. In Chapter 3, H. Geman and D. Madan use pure jump models, in particular from the CGMY class, for the evolution of stock prices and investigate in this setting the relationship between the statistical and risk-neutral densities. Statistical estimation is conducted on different world indexes. Their conclusions depart from the standard applications of utility theory to asset pricing which assume a representative agent who is long the market. They argue that one must have at a minimum a two-agent model in which some weight is given to an agent who is short the market. In Chapter 4, W. Schoutens, E. Simons and J. Tistaert calibrate different Lévy-based stochastic volatility models to a real market option surface and price by Monte Carlo techniques a range of exotics options. Although the different models discussed can all be nicely calibrated to the option surface – leading to almost identical vanilla prices – exotic option prices under the different models discussed can differ considerably. This investigation is pushed further by looking at the prices of moment derivatives, a new kind of derivative paying out realized higher moments. Even more pronounced differences are reported in this case. The study reveals that there is a clear issue of model risk and warns of blind use of fancy models in the realm of exotic options.

The third part is devoted to pricing, hedging and general theory of different exotics options of a European nature. In Chapter 5, E. Eberlein and A. Papapantoleon consider time-inhomogeneous Lévy processes (or additive processes) to give a better explanation of the so-called 'volatility smile', as well as the 'term structure of smiles'. They derive different kinds of symmetry relations for various exotic options. Their contribution also contains an extensive review of current literature on exotics driven in Lévy markets. In Chapter 6, H. Albrecher and W. Schoutens present a simple static super-hedging strategy for the Asian option, based on stop-loss transforms and comonotonic theory. A numerical implementation is given in detail and the hedging performance is illustrated for several stochastic volatility models. Real options form the main theme of Chapter 7, authored by P. Barrieu and N. Bellamy. There, the impact of market crises on investment decisions is analysed through real options under a jump-diffusion model, where the jumps characterize the crisis effects. In Chapter 8, J.M. Corcuera, D. Nualart and W. Schoutens show how moment derivatives can complete Lévy-type markets in the sense that, by allowing trade in these derivatives, any contingent claim can be perfectly hedged by a dynamic portfolio in terms of bonds, stocks and moment-derivative related products.

In the fourth part, exotics of an American nature are considered. Optimal stopping problems are central here. Chapter 9 is a contribution at the special request of the editors. This consists of T. Chan's original unpublished manuscript dating back to early 2000, in which many important features of the perpetual American put pricing problem are observed for the case of a Lévy-driven stock which has no positive jumps. G. Peskir and N. Uys work in Chapter 10 under the traditional Black–Scholes market but consider a new type of Asian option where the holder may exercise at any time up to the expiry of the option. Using recent techniques developed by Peskir concerning local time–space calculus, they are able to give an integral equation characterizing uniquely the optimal exercise boundary. Solving this integral equation numerically brings forward stability issues connected with the Hartman–Watson distribution. In Chapter 11, P. Carr and A. Hirsa give forward equations for the value of an American put in a Lévy market. A numerical scheme for the VG case for very fast pricing of an American put is given in its Appendix. In the same spirit, A. Almendral discusses the numerical valuation of American options under the CGMY model. A numerical solution scheme for the Partial-Integro-Differential Equation is provided; computations are accelerated by the Fast-Fourier Transform. Pricing American options and their early exercise boundaries can be carried out within seconds.

The final part considers game options. In Chapter 13, C. Kühn and J. Kallsen give a review of the very recent literature concerning game-type options, that is, options in which both holder and writer have the right to exercise. Game-type options are very closely related to convertible bonds and Kühn and Kallsen also bring this point forward in their contribution.

Last, but by far not least, P. Gapeev gives a concrete example of a new game-type option within the Black–Scholes market for which an explicit representation can be obtained.

We should like to thank all contributors for working hard to keep to the tempo that has allowed us to compile this text within a reasonable period of time. We would also like to heartily thank the referees, all of whom responded gracefully to the firm request to produce their reports within a shorter than normal period of time and without compromising their integrity.

This book grew out of the 2004 Workshop, *Exotic Option Pricing under Advanced Lévy Models*, hosted at EURANDOM in The Netherlands. In addition to the excellent managerial and organizational support offered by EURANDOM, it was generously supported by grants from Nederlands Organizatie voor Wetenschappelijk Onderzoek (The Dutch Organization for Scientific Research), Koninklijke Nederlandse Akademie van Wetenschappen (The Royal Dutch Academy of Science) and *The Journal of Applied Econometrics*. Special thanks goes to Jef Teugels and Lucienne Coolen. Thanks also to `wilmott.com` and `mathfinance.de` for publicizing the event.

A. E. Kyprianou, Edinburgh, UK
W. Schoutens, Leuven, Belgium
P. Wilmott, London, UK

About the Editors

Andreas E. Kyprianou
Address: School of Mathematical and Computer Sciences, Heriot Watt University, Edinburgh, EH14 4AS, Scotland, UK
E-mail: kyprianou@ma.hw.ac.uk
Affiliation: Heriot Watt University, Scotland, UK

Andreas Kyprianou has a degree in mathematics from Oxford University and a PhD in probability theory from Sheffield University. He has held academic positions in the Mathematics and/or Statistics Departments at The London School of Economics, Edinburgh University, Utrecht University and, currently, Heriot Watt University. He has also worked for nearly two years as a research mathematician with Shell International Exploration and Production. His research interests are focused on pure and applied probability with recent focus on Lévy processes. He has taught a range of courses on probability theory, stochastic analysis, financial stochastics and Lévy processes on the Amsterdam–Utrecht Masters programme in Stochastics and Financial Mathematics and the MSc programme in Financial Mathematics at Edinburgh University.

Wim Schoutens
Address: Katholieke Universiteit Leuven – UCS, W. De Croylaan 54, B-3001 Leuven, Belgium
E-mail: Wim.Schoutens@wis.kuleuven.be
Affiliation: Katholieke Universiteit Leuven, Belgium

Wim Schoutens has a degree in Computer Science and a PhD in Science (Mathematics). He is a research professor at the Department of Mathematics at the Catholic University of Leuven (Katholieke Universiteit Leuven), Belgium. He has been a consultant to the banking industry and is the author of the Wiley book *Lévy Processes in Finance – Pricing Financial Derivatives*.

His research interests are focused on financial mathematics and stochastic processes. He currently teaches several courses related to financial engineering in different Master programmes.

Paul Wilmott
Address: 'Wherever I lay my hat'
E-mail: paul@wilmott.com
Affiliation: Various

Paul Wilmott has undergraduate and DPhil degrees in mathematics. He has written over 100 articles on mathematical modeling and finance, as well as internationally acclaimed books including *Paul Wilmott on Quantitative Finance*, published by Wiley. Paul has extensive consulting experience in quantitative finance with leading US and European financial institutions. He has founded a university degree course and the popular Certificate in Quantitative Finance. Paul also manages wilmott.com.

About the Contributors

Hansjoerg Albrecher is Associate Professor of Applied Mathematics at the Graz University of Technology. He studied Mathematics in Graz, Limerick and Baltimore, receiving his doctorate in 2001. He held visiting appointments at the Katholieke Universiteit Leuven and the University of Aarhus. Research interests include ruin theory, stochastic simulation and quantitative finance.

Ariel Almendral will take up a research position at the Norwegian Computing Center, starting in August 2005. In 2004, he obtained his PhD from the University of Oslo, Norway. In his thesis he focused on numerical methods for financial derivatives in the presence of jump processes, from a differential equation perspective. Parts of his PhD research were carried out at Delft University of Technology, The Netherlands, where he held a postdoctoral position for a year.

Pauline Barrieu has been a lecturer in the Department of Statistics at the London School of Economics since 2002, after obtaining a PhD in finance (doctorate HEC, France) and a PhD in Mathematics (University of Paris 6, France). Her research interests are mainly problems at the interface of insurance and finance, in particular, optimal design of new types of derivatives and securitization. She also works on quantitative methods for assessing financial and non-financial risks, on stochastic optimization and environmental economics.

Nadine Bellamy is Associate Professor in Mathematics at the University of Evry, France. Her PhD thesis (University of Evry, 1999) deals with hedging and pricing in markets driven by discontinuous processes and her current research interests are related to optimization and real options problems.

Dr Peter Carr heads Quantitative Research at Bloomberg LP. He also directs the Masters in Mathematical Finance program at NYU's Courant Institute. Formerly, Dr Carr was a finance professor for eight years at Cornell University. Since receiving his PhD in Finance from UCLA in 1989, he has published extensively in both academic and industry-oriented journals. He has recently won awards from *Wilmott Magazine* for Cutting Edge Research and from *Risk Magazine* for Quant of the Year.

Terence Chan completed his PhD at Cambridge University UK after which he obtained his current position at Heriot-Watt University, Edinburgh. Among his research interests are Lévy processes but he only occasionally dabbles in financial mathematics to maintain the illusion that he is doing something of practical use!

José Manuel Corcuera is an associate professor since 1997 at the Faculty of Mathematics of the University of Barcelona. His main research interest is in the theoretical aspects of statistics and quantitative finance.

Ernst Eberlein is Professor of Stochastics and Mathematical Finance at the University of Freiburg. He is a co-founder of the Freiburg Center for Data Analysis and Modeling (FDM), an elected member of the International Statistical Institute and at present Executive Secretary of the Bachelier Finance Society. His current research focuses on statistical analysis and realistic modeling of financial markets, risk management, as well as pricing of derivatives.

Pavel Gapeev was born in Moscow in 1976. He studied and obtained his PhD in Stochastics at Moscow State University in 2001. He is now working as a Senior Researcher at the Institute of Control Sciences, Russian Academy of Sciences in Moscow. He has held a visiting appointment at Humboldt University, Berlin (2001/2002) in addition to some short term research visits to Aarhus, Bochum, Copenhagen, Frankfurt, Helsinki, and Zurich. His main field of research is stochastic analysis and its applications into financial mathematics, optimal control, optimal stopping, and quickest detection. Apart from mathematics he is interested in arts, sports and travelling, and enjoys playing the violin.

Hélyette Geman is a Professor of Finance at the University Paris Dauphine and ESSEC Graduate Business School. She is a graduate of Ecole Normale Superieure in Mathematics, holds a Masters degree in theoretical physics and a PhD in mathematics from the University Pierre et Marie Curie and a PhD in Finance from the University Pantheon Sorbonne. Professor Geman has published more than 60 papers in major finance journals including the Journal of Finance, Mathematical Finance, Journal of Financial Economics, Journal of Banking and Finance and Journal of Business. Professor Geman's research includes asset price modelling using jump-diffusions and Lévy processes, commodity forward curve modelling and exotic option pricing for which she won the first prize of the Merrill Lynch Awards. She has written a book entitled *Commodities and Commodity Derivatives* (John Wiley & Sons Ltd, 2005).

Ali Hirsa joined Caspian Capital Management as the Head of Analytical Trading Strategy in April 2004. At CCM his responsibilities include design and testing of new trading strategies. Prior to his current position, Ali worked at Morgan Stanley for four years. Ali is also an adjunct professor at Columbia University and New York University where he teaches in the mathematics of finance program. Ali received his PhD in applied mathematics from University of Maryland at College Park under the supervision of Dilip B. Madan.

Jan Kallsen is a Professor of Mathematical Finance at Munich University of Technology. His research interests include pricing and hedging in incomplete markets and the general theory of stochastic processes.

Christoph Kühn is Junior Professor at the Frankfurt MathFinance Institute. He holds a diploma in Mathematical Economics from the University of Marburg and a PhD in mathematics from Munich University of Technology. His main research interests are pricing and hedging of derivatives in incomplete markets and the microstruture of financial markets.

Ronnie Loeffen was born in 1981 in the Netherlands and has recently received a Master's degree in Mathematics at the University Utrecht. The subject of his Master's thesis was American options on a jump-diffusion model.

Dilip B. Madan is Professor of Finance at the Robert H. Smith School of Business, University of Maryland. He is co-editor of Mathematical Finance and served as President of the Bachelier Finance Society 2002–2003. He has been a consultant to Morgan Stanley since 1996. He now also consults for Bloomberg and Caspian Capital. His primary research focus is on stochastic processes as they are applied to the management and valuation of financial risks.

David Nualart is Professor at the Faculty of Mathematics of the University of Barcelona. His research interests include a variety of topics in stochastic analysis, with emphasis on stochastic partial differential equations, Malliavin calculus and fractional Brownian motion. He is the author of the monograph *Malliavin Calculus and Related Topics*.

Antonis Papapantoleon is a research assistant at the Department for Mathematical Stochastics, University of Freiburg. He received a Diploma in Mathematics from the University of Patras (2000) and an MSc in Financial Mathematics from the University of Warwick (2001). From January to August 2002 he worked at the FX Quantitative Research group of Commerzbank in Frankfurt.

Goran Peskir is the Chair in Probability at the School of Mathematics, University of Manchester. In the period 1996–2005 he was an Associate Professor at the Department of Mathematical Sciences, University of Aarhus in Denmark. He is an internationally leading expert in the field of Optimal Stopping and author to over sixty papers dealing with various problems in the field of probability and its applications (optimal stopping, stochastic calculus, option pricing). Together with Albert Shiryaev he has co-authored the book *Optimal Stopping and Free-Boundary Problems*.

Erwin Simons works in Quantitative Modeling at ING Brussels. After 3 years of front-office experience in Equity derivatives pricing, over the last year he switched to Interest-Rate derivatives modeling. He holds a PhD in Applied Mathematics from the Catholic University Leuven, von Karman Institute for Fluid Dynamics on the subject of large-scale computing of incompressible turbulent flows.

Jurgen Tistaert joined the Credit Risk Management Department of ING Brussels at the end of 1996 where he developed several rating, exposure and risk/performance models. He moved to Financial Markets in 2001, where the team is focusing on the R&D of pricing models for a broad range of derivative products. Before joining ING, he was a research assistant at the Quantitative Methods Group of K.U. Leuven Applied Economics Faculty, where he currently is appointed as a Fellow.

Nadia Uys completed her Bachelors in Economic Science, majoring in Mathematical Statistics and Actuarial Science, at the University of the Witwatersrand in 2000, followed with Honours in Advanced Mathematics of Finance in 2001. Her MSc dissertation entitled 'Optimal Stopping Problems and American Options' was completed under the supervision of Professor G. Peskir (University of Aarhus) and Mr H. Hulley (Sydney Polytechnic) and received a distinction in 2005. She is currently teaching in the Programme in Advanced Mathematic of Finance at the University of the Witwatersrand and engaging in research toward a PhD under the supervision of Professor F. Lombard (University of Johannesburg).

Nick Webber is Director of the Financial Options Research Centre, University of Warwick. Formerly Professor of Computational Finance at Cass Business School, he is interested not only in theoretical financial mathematics, but also in methods for the fast evaluation of options prices under a variety of assumptions for returns distributions. As well as work with Lévy processes and numerical methods he has also worked on copulas, credit models and interest rates.

1
Lévy Processes in Finance Distinguished by their Coarse and Fine Path Properties

Andreas E. Kyprianou
Heriot Watt University, Edinburgh, UK

and

R. Loeffen
University of Utrecht, The Netherlands

Abstract

We give a brief introduction to Lévy processes and indicate the diversity of this class of stochastic processes by quoting a number of complete characterizations of coarse and fine path properties. The theory is exemplified by distinguishing such properties for Lévy processes which are currently used extensively in financial models. Specifically, we treat jump-diffusion models (including Merton and Kou models), spectrally one-sided processes, truncated stable processes (including CGMY and Variance Gamma models), Meixner processes and generalized hyperbolic processes (including hyperbolic and normal inverse Gaussian processes).

1.1 INTRODUCTION

The main purpose of this text is to provide an *entrée* to the compilation *Exotic Options and Advanced Lévy Models*. Since path fluctuations of Lévy processes play an inevitable role in the computations which lead to the pricing of exotic options, we have chosen to give a review of what subtleties may be encountered there. In addition to giving a brief introduction to the general structure of Lévy processes, path variation and its manifestation in the Lévy–Khintchine formula, we shall introduce classifications of drifting and oscillation, regularity of the half line, the ability to visit fixed points and creeping. The theory is exemplified by distinguishing such properties for Lévy processes which are currently used extensively in financial models. Specifically, we treat jump-diffusion models (including Merton and Kou models), spectrally one-sided processes, truncated stable processes (including CGMY and variance gamma models), Meixner processes and generalized hyperbolic processes (including hyperbolic and normal inverse Gaussian processes).

To support the presentation of more advanced path properties and for the sake of completeness, a number of known facts and properties concerning these processes are reproduced from the literature. We have relied heavily upon the texts by Schoutens (2003) and Cont and Tankov (2004) for inspiration. Another useful text in this respect is that of Boyarchenko and Levendorskii (2002).

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

The job of exhibiting the more theoretical facts concerning path properties have been greatly eased by the existence of the two indispensable monographs on Lévy processes, namely Bertoin (1996) and Sato (1999); see, in addition, the more recent monograph of Applebaum (2004) which also contains a section on mathematical finance. In the course of this text, we shall also briefly indicate the relevance of the path properties considered to a number of exotic options. In some cases, the links to exotics is rather vague due to the fact that the understanding of pricing exotics and advanced Lévy models is still a 'developing market', so to speak. Nonetheless, we believe that these issues will in due course become of significance as research progresses.

1.2 LÉVY PROCESSES

We start with the definition of a real valued Lévy process followed by the Lévy–Khintchine characterization.

Definition 1 *A Lévy process* $X = \{X_t : t \geq 0\}$ *is a stochastic process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *which satisfies the following properties:*

(i) The paths of X *are right continuous with left limits almost surely.*
(ii) $X_0 = 0$ *almost surely.*
(iii) X *has independent increments; for* $0 \leq s \leq t$, $X_t - X_s$ *is independent of* $\sigma(X_u : u \leq s)$.
(iv) X *has stationary increments; for* $0 \leq s \leq t$, $X_t - X_s$ *is equal in distribution to* X_{t-s}.

It turns out that there is an intimate relationship between Lévy processes and a class of distributions known as *infinitely divisible* distributions which gives a precise impression of how varied the class of Lévy processes really is. To this end, let us devote a little time to discussing infinitely divisible distributions.

Definition 2 *We say that a real valued random variable* Θ *has an infinitely divisible distribution if for each* $n = 1, 2, \ldots$ *. there exists a sequence of iid random variables* $\Theta_1, \ldots, \Theta_n$ *such that*

$$\Theta \stackrel{d}{=} \Theta_{1,n} + \cdots + \Theta_{n,n}$$

where $\stackrel{d}{=}$ *is equality in distribution. Alternatively, we could have expressed this relation in terms of probability laws. That is to say, the law* μ *of a real valued random variable is infinitely divisible if for each* $n = 1, 2, \ldots$ *there exists another law* μ_n *of a real valued random variable such that* $\mu = \mu_n^{*n}$*, the* n*-fold convolution of* μ_n.

The full extent to which we may characterize infinitely divisible distributions is carried out via their characteristic function (or Fourier transform of their law) and an expression known as the Lévy–Khintchine formula.

Theorem 3 (Lévy–Khintchine formula) *A probability law* μ *of a real valued random variable is infinitely divisible with characteristic exponent* Ψ,

$$\int_{\mathbb{R}} e^{iux} \mu(dx) = e^{-\Psi(u)} \text{ for } u \in \mathbb{R},$$

if and only if there exists a triple (γ, σ, Π)*, where* $\gamma \in \mathbb{R}$*,* $\sigma \geq 0$ *and* Π *is a measure supported on* $\mathbb{R}\backslash\{0\}$ *satisfying* $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \Pi(dx) < \infty$*, such that*

$$\Psi(u) = i\gamma u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (1 - e^{iux} + iux\mathbf{1}_{(|x|<1)})\Pi(dx)$$

for every $u \in \mathbb{R}$*.*

Definition 4 *The measure* Π *is called the Lévy (characteristic) measure and the triple* (γ, σ, Π) *are called the Lévy triple.*

Note that the requirement that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$ necessarily implies that the tails of Π are finite. On the other hand, should Π be an infinite measure due to unbounded mass in the neighbourhood of the origin, then it must at least integrate locally against x^2 for small values of x.

Let us now make firm the relationship between Lévy processes and infinitely divisible distributions. From the definition of a Lévy process we see that for any $t > 0$, X_t is a random variable whose law belongs to the class of infinitely divisible distributions. This follows from the fact that for any $n = 1, 2, \ldots$

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}) \tag{1.1}$$

together with the fact that X has stationary independent increments. Suppose now that we define for all $u \in \mathbb{R}$, $t \geq 0$

$$\Psi_t(u) = -\log \mathbb{E}\left(e^{iuX_t}\right)$$

then by using equation (1.1) twice we have for any two positive integers m, n that

$$m\Psi_1(u) = \Psi_m(u) = n\Psi_{m/n}(u)$$

and hence for any rational $t > 0$

$$\Psi_t(u) = t\Psi_1(u). \tag{1.2}$$

If t is an irrational number, then we can choose a decreasing sequence of rationals $\{t_n : n \geq 1\}$ such that $t_n \downarrow t$ as n tends to infinity. Almost sure right continuity of X implies right continuity of $\exp\{-\Psi_t(u)\}$ (by dominated convergence) and hence equation (1.2) holds for all $t \geq 0$.

In conclusion, any Lévy process has the property that

$$\mathbb{E}\left(e^{iuX_t}\right) = e^{-t\Psi(u)}$$

where $\Psi(u) := \Psi_1(u)$ is the characteristic exponent of X_1 which has an infinitely divisible distribution.

Definition 5 *In the sequel we shall also refer to* $\Psi(u)$ *as the characteristic exponent of the Lévy process.*

Note that the law of a Lévy process is uniquely determined by its characteristic exponent. This is because the latter characterizes uniquely all one-dimensional distributions of X. From the property of stationary independent increments, it thus follows that the characteristic exponent characterizes uniquely all finite dimensional distributions which themselves uniquely characterize the law of X.

It is now clear that each Lévy process can be associated with an infinitely divisible distribution. What is not clear is whether given an infinitely divisible distribution, one may construct a Lévy process such that X_1 has that distribution. This latter issue is resolved by the following theorem which gives the Lévy–Khintchine formula for Lévy processes.

Theorem 6 *Suppose that* $\gamma \in \mathbb{R}$, $\sigma \geq 0$ *and* Π *is a measure on* $\mathbb{R}\backslash\{0\}$ *such that* $\int_{\mathbb{R}}(1 \wedge |x|^2)\Pi(dx) < \infty$. *From this triple define for each* $u \in \mathbb{R}$

$$\Psi(u) = i\gamma u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} [1 - e^{iux} + iux\mathbf{1}_{(|x|<1)}]\Pi(dx). \tag{1.3}$$

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *on which a Lévy process is defined having characteristic exponent* Ψ.

It is clear from the Lévy–Khintchine formula that a general Lévy process must necessarily take the form

$$-\gamma t + \sigma B_t + J_t, t \geq 0$$

where $B := \{B_t : t \geq 0\}$ is a standard Brownian motion (and thus has normally distributed increments contributing the term $\sigma^2 u^2/2$ to Ψ) and $J := \{J_t : t \geq 0\}$ is a process independent of B. It is the process J which essentially is responsible for the huge diversity in the class of Lévy processes and also for the discontinuities or jumps in the path of X which are typically present.

The proof of Theorem 6 is rather complicated but nonetheless very rewarding as it also reveals much more about the general structure of the process J. In Section 1.4.1 we shall give a brief outline of the main points of the proof and in particular how one additionally gets a precise classification of the path variation from it. We move first, however, to some examples of Lévy processes, in particular those which have become quite popular in financial modelling.

1.3 EXAMPLES OF LÉVY PROCESSES IN FINANCE

Appealing to the idea of stochastically perturbed multiplicative growth the classic Black–Scholes model proposes that the value of a risky asset should be modeled by an exponential Brownian motion with drift. It has long been known that this assumption drastically fails to match the reality of observed data. Cont (2001) exemplifies some of the more outstanding issues. The main problem being that log returns on real data exhibit (semi) heavy tails while log returns in the Black–Scholes model are normally distributed and hence light tailed. Among the many suggestions which were proposed to address this particular problem was the simple idea to replace the use of a Brownian motion with drift by a Lévy processes. That is to say, a risky asset is modeled by the process

$$se^{X_t}, t \geq 0$$

where $s > 0$ is the initial value of the asset and X is a Lévy process.

There are essentially four main classes of Lévy processes which feature heavily in current mainstream literature on market modeling with pure Lévy processes (we exclude from the discussion stochastic volatility models such as those of Barndorff–Nielsen and Shephard (2001)). These are the jump-diffusion processes (consisting of a Brownian motion with drift plus an independent compound Poisson process), the generalized tempered stable processes (which include more specific examples such as Variance Gamma processes and CGMY), Generalized Hyperbolic processes and Meixner processes. There is also a small minority of papers which have proposed to work with the arguably less realistic case of spectrally one-sided Lévy processes. Below, we shall give more details on all of the above key processes and their insertion into the literature.

1.3.1 Compound Poisson processes and jump-diffusions

Compound Poisson processes form the simplest class of Lévy processes in the sense of understanding their paths. Suppose that ξ is a random variable with honest distribution F supported on $\mathbb{R}$ but with no atom at 0. Let

$$X_t := \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0$$

where $\{\xi_i : i \geq 1\}$ are independent copies of ξ and $N := \{N_t : t \geq 0\}$ is an independent Poisson process with rate $\lambda > 0$. Then, $X = \{X_t : t \geq 0\}$ is a compound Poisson process. The fact that X is a Lévy process can easily be verified by computing the joint characteristic of the variables $X_t - X_s$ and $X_v - X_u$ for $0 \leq v \leq u \leq s \leq t < \infty$ and showing that it factorizes. Indeed, standard facts concerning the characteristic function of the Poisson distribution leads to the following expression for the characteristic exponent of X,

$$\Psi(u) = \lambda(1 - \widehat{F}(u)) = \int_{\mathbb{R}} (1 - e^{iux})\lambda F(dx)$$

where $\widehat{F}(u) = E(e^{iu\xi})$. Consequently, we can easily identify the Lévy triple via $\sigma = 0$ and $\gamma = -\int_R x\lambda F(dx)$ and $\Pi(dx) = \lambda F(dx)$. Note that Π has finite total mass. It is not difficult to reason that any Lévy process whose Lévy triple has this property must necessarily be a compound Poisson process. Since the jumps of the process X are spaced out by independent exponential distributions, the same is true of X and hence X is pathwise piecewise constant. Up to adding a linear drift, compound Poisson processes are the only Lévy processes which are piecewise linear.

The first model for risky assets in finance which had jumps was proposed by Merton (1976) and consisted of the log-price following an independent sum of a compound Poisson process, together with a Brownian motion with drift. That is,

$$X_t = -\gamma t + \sigma B_t + \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0$$

where $\gamma \in \mathbb{R}$, $\{B_t : t \geq 0\}$ is a Brownian motion and $\{\xi_i : i \geq 0\}$ are normally distributed. Kou (2002) assumed the above structure, the so called jump-diffusion model, but chose the jump distribution to be that of a two-sided exponential distribution. Kou's choice of jump distribution was heavily influenced by the fact that analysis of first passage problems become analytically tractable which itself is important for the valuation of American put options (see Chapter 11 below). Building on this idea, Asmussen *et al.* (2004) introduce a jump-diffusion model with two-sided phasetype distributed jumps. The latter form a class of distributions which generalize the two-sided exponential distribution and like Kou's model, have the desired property that first passage problems are analytically tractable.

1.3.2 Spectrally one-sided processes

Quite simply, spectrally one-sided processes are characterized by the property that the support of the Lévy measure is restricted to the upper or the lower half line. In the latter case, that is $\Pi(0, \infty) = 0$, one talks of spectrally negative Lévy processes. Without loss of generality we can and shall restrict our discussion to this case unless otherwise stated in the sequel.

Spectrally negative Lévy processes have not yet proved to be a convincing tool for modeling the evolution of a risky asset. The fact that the support of the Lévy measure is restricted to the lower half line does not necessarily imply that the distribution of the Lévy process itself is also restricted to the lower half line. Indeed, there are many examples of spectrally negative processes whose finite time distributions are supported on $\mathbb{R}$. One example, which has had its case argued for in a financial context by Carr and Wu (2003) and Cartea and Howison (2005), is a spectrally negative stable process of index $\alpha \in (1, 2)$. To be more precise, this is a process whose Lévy measure takes the form

$$\Pi(dx) = \mathbf{1}_{(x<0)} c|x|^{-1-\alpha} dx$$

for some constant $c > 0$ and whose parameter σ is identically zero. A lengthy calculation reveals that this process has the Lévy–Khintchine exponent

$$\Psi(u) = c|u|^{\alpha} \left(1 + i \tan \frac{\pi\alpha}{2} \operatorname{sign} u\right).$$

Chan (2000, 2004), Mordecki (1999, 2002) and Avram *et al*. (2002, 2004), have also worked with a general spectrally negative Lévy process for the purpose of pricing American put and Russian options. In their case, the choice of model was based purely on a degree of analytical tractability centred around the fact that when the path of a spectrally negative process passes from one point to another above it, it visits all other points between them.

1.3.3 Meixner processes

The Meixner process is defined through the Meixner distribution which has a density function given by

$$f_{\text{Meixner}}(x; \alpha, \beta, \delta, \mu) = \frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left(\frac{\beta(x-\mu)}{\alpha}\right) \left|\Gamma\left(\delta + \frac{\mathrm{i}(x-\mu)}{\alpha}\right)\right|^2$$

where $\alpha > 0, -\pi < \beta < \pi, \delta > 0, m \in \mathbb{R}$. The Meixner distribution is infinitely divisible with a characteristic exponent

$$\Psi_{\text{Meixner}}(u) = -\log\left(\left(\frac{\cos(\beta/2)}{\cosh(\alpha u - \mathrm{i}\beta)/2}\right)^{2\delta}\right) - \mathrm{i}\mu u,$$

and therefore there exists a Lévy process with the above characteristic exponent. The Lévy triplet (γ, σ, Π) is given by

$$\gamma = -\alpha\delta\tan(\beta/2) + 2\delta\int_1^{\infty} \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)}\,\mathrm{d}x - \mu,$$

$\sigma = 0$ and

$$\Pi(\mathrm{d}x) = \delta \frac{\exp(\beta x/\alpha)}{x \sinh(\pi x/\alpha)} \mathrm{d}x. \tag{1.4}$$

The Meixner process appeared as an example of a Lévy process having a particular martingale relation with respect to orthogonal polynomials (see Schoutens and Teugels (1998) and Schoutens (2000)). Grigelionis (1999) and Schoutens (2001, 2002) established the use of the Meixner process in mathematical finance. Relationships between Mexiner distributions and other infinitely divisible laws also appear in the paper of Pitman and Yor (2003).

1.3.4 Generalized tempered stable processes and subclasses

The generalized tempered stable process has Lévy density $\nu := \mathrm{d}\Pi/\mathrm{d}x$ given by

$$\nu(x) = \frac{c_p}{x^{1+\alpha_p}} \mathrm{e}^{-\lambda_p x} \mathbf{1}_{\{x>0\}} + \frac{c_n}{(-x)^{1+\alpha_n}} \mathrm{e}^{\lambda_n x} \mathbf{1}_{\{x<0\}},$$

with $\sigma = 0$, where $\alpha_p < 2$, $\alpha_n < 2$, $\lambda_p > 0$, $\lambda_n > 0$, $c_p > 0$ and $c_n > 0$.

These processes take their name from stable processes which have Lévy measures of the form

$$\Pi(dx) = \left(\frac{c_p}{x^{1+\alpha}} \mathbf{1}_{\{x>0\}} + \frac{c_n}{(-x)^{1+\alpha}} \mathbf{1}_{\{x<0\}} \right) \mathrm{d}x,$$

for $\alpha \in (0, 2)$ and $c_p, c_n > 0$. Stable processes with index $\alpha \in (0, 1]$ have no moments and when $\alpha \in (1, 2)$ only a first moment exists. Generalized tempered stable processes differ in that they have an exponential weighting in the Lévy measure. This guarantees the existence of all moments, thus making them suitable for financial modelling where a moment-generating function is necessary. Since the shape of the Lévy measure in the neighbourhood of the origin determines the occurrence of small jumps and hence the small time path behaviour, the exponential weighting also means that on small time scales stable processes and generalized tempered stable processes behave in a very similar manner.

Generalized tempered stable processes come under a number of different names. They are sometimes called KoBoL processes, named after the authors Koponen (1995) and Boyarchenko and Levendorskii (2002). Carr *et al.* (2002, 2003) have also studied this six-parameter family of processes and as a consequence of their work they are also referred to as generalized CGMY processes or, for reasons which will shortly become clear, CCGMYY processes. There seems to be no uniform terminology used for this class of processes at the moment and hence we have simply elected to follow the choice of Cont and Tankov (2004).

Since

$$\int_{\mathbb{R}\setminus(-1,1)} |x|\nu(x)dx < \infty$$

it turns out to be more convenient to express the Lévy–Khintchine formula in the form

$$\Psi(u) = \mathrm{i}u\gamma' + \int_{-\infty}^{\infty} (1 - \mathrm{e}^{\mathrm{i}ux} + \mathrm{i}ux)\nu(x)\mathrm{d}x \tag{1.5}$$

where $\gamma' = \gamma - \int_{\mathbb{R}\setminus(-1,1)} x\nu(x)dx < \infty$. In this case, the characteristic exponent is given by

$$\Psi(u) = \mathrm{i}u\gamma' - A_p - A_n, \text{ where}$$

$$A_p = \begin{cases} \mathrm{i}uc_p + c_p(\lambda_p - \mathrm{i}u)\log\left(1 - \frac{\mathrm{i}u}{\lambda_p}\right) & \text{if } \alpha_p = 1 \\ -c_p\left(\frac{\mathrm{i}u}{\lambda_p} + \log\left(1 - \frac{\mathrm{i}u}{\lambda_p}\right)\right) & \text{if } \alpha_p = 0 \\ \Gamma(-\alpha_p)\lambda_p^{\alpha_p}c_p\left(\left(1 - \frac{\mathrm{i}u}{\lambda_p}\right)^{\alpha_p} - 1 + \frac{\mathrm{i}u\alpha_p}{\lambda_p}\right) & \text{otherwise} \end{cases}$$

$$A_n = \begin{cases} -\mathrm{i}uc_n + c_n(\lambda_n + \mathrm{i}u)\log\left(1 + \frac{\mathrm{i}u}{\lambda_n}\right) & \text{if } \alpha_n = 1 \\ -c_n\left(-\frac{\mathrm{i}u}{\lambda_n} + \log\left(1 + \frac{\mathrm{i}u}{\lambda_n}\right)\right) & \text{if } \alpha_n = 0 \\ \Gamma(-\alpha_n)\lambda_n^{\alpha_n}c_n\left(\left(1 + \frac{\mathrm{i}u}{\lambda_n}\right)^{\alpha_n} - 1 - \frac{\mathrm{i}u\alpha_n}{\lambda_n}\right) & \text{otherwise} \end{cases}$$

(see Cont and Tankov (2004), p. 122).

When $\alpha_p = \alpha_n = Y$, $c_p = c_n = C$, $\lambda_p = M$ and $\lambda_n = G$, the generalized tempered stable process becomes the so called CGMY process, named after the authors who first introduced it, i.e. Carr *et al.* (2002). The characteristic exponent of the CGMY process for $Y \neq 0$ and $Y \neq 1$ is often written as

$$\Psi_{\text{CGMY}}(u) = -C\Gamma(-Y)[(M - \mathrm{i}u)^Y - M^Y + (G + \mathrm{i}u)^Y - G^Y)] - \mathrm{i}u\mu, \tag{1.6}$$

which is the case for an appropriate choice of γ', namely

$$\gamma' = C\Gamma(-Y)\left(\frac{YM^Y}{M} - \frac{YG^Y}{G}\right) + \mathrm{i}\mu.$$

The properties of the CGMY process can thus be inferred from the properties of the generalized tempered stable process. Note that in this light, generalized tempered stable processes are also referred to as CCGMYY.

As a limiting case of the CGMY process, but still within the class of generalized tempered stable processes, we have the variance gamma process. The latter was introduced as a predecessor to the CGMY process by Madan and Seneta (1987) and treated in a number of further papers by Madan and co-authors. The variance gamma process can be obtained by starting with the parameter choices for the CGMY but then taking the limit as Y tends to zero. This corresponds to a generalized tempered stable process with $\alpha_p = \alpha_n = 0$. Working with $\gamma' = -C/M + C/G + \mu$, we obtain the variance gamma process with the characteristic exponent

$$\Psi_{\text{VG}}(u) = C\left[\log\left(1 - \frac{\mathrm{i}u}{M}\right) + \log\left(1 + \frac{\mathrm{i}u}{G}\right)\right] - \mathrm{i}u\mu. \tag{1.7}$$

The characteristic exponent is usually written as

$$\Psi_{\text{VG}}(u) = \frac{1}{\kappa}\log\left(1 - \mathrm{i}\theta\kappa u + \frac{1}{2}\sigma^2\kappa u^2\right) - \mathrm{i}u\mu,$$

where

$$C = 1/\kappa, \quad M = \frac{\sqrt{\theta^2 + 2\frac{\sigma^2}{\kappa}} - \theta}{\sigma^2} \quad \text{and} \quad G = \frac{\sqrt{\theta^2 + 2\frac{\sigma^2}{\kappa}} + \theta}{\sigma^2}$$

for $\theta \in \mathbb{R}$ and $\kappa > 0$. Again, the properties of the variance gamma process can be derived from the properties of the generalized tempered stable process.

1.3.5 Generalized hyperbolic processes and subclasses

The density of a generalized hyperbolic distribution is given by

$$f_{GH}(x; \alpha, \beta, \lambda, \delta, \mu) = C(\delta^2 + (x-\mu)^2)^{\frac{\lambda}{2}-\frac{1}{4}} \mathrm{K}_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right) \mathrm{e}^{\beta(x-\mu)},$$

$$\text{where} \quad C = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda \mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}$$

and with $\alpha > 0$, $0 \leq |\beta| < \alpha$, $\lambda \in \mathbb{R}$, $\delta > 0$ and $\mu \in \mathbb{R}$. The function K_λ stands for the modified Bessel function of the third kind with index λ. This distribution turns out to be infinitely divisible with a characteristic exponent

$$\Psi_{GH}(u) = -\log\left(\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+\mathrm{i}u)^2}\right)^{\lambda/2} \frac{\mathrm{K}_\lambda(\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2})}{\mathrm{K}_\lambda(\delta\sqrt{\alpha^2-\beta^2})}\right) - \mathrm{i}\mu u.$$

These facts are non-trivial to prove–see Halgreen (1979) who gives the proofs. The corresponding Lévy measure is rather complicated, being expressed as integrals of special functions. We refrain from offering the Lévy density here on account of its complexity and since we shall not use it in the sequel.

Generalized hyperbolic processes were introduced within the context of mathematical finance by Barndorff-Nielsen (1995, 1998) and Erbelein and Prause (1998).

When $\lambda = 1$, we obtain the special case of a hyperbolic process and when $\lambda = -\frac{1}{2}$, the normal inverse Gaussian process is obtained. Because the modified Bessel function has a simple form when $\lambda = -\frac{1}{2}$, namely

$$\mathrm{K}_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}} z^{-\frac{1}{2}} \mathrm{e}^{-z},$$

the characteristic exponent can be simplified to

$$\Psi_{NIG}(u) = \delta\left(\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2} - \sqrt{\alpha^2-\beta^2}\right).$$

Eberlein and Hammerstein (2002) investigated some limiting cases of generalized hyperbolic distributions and processes. Because for $\lambda > 0$

$$\mathrm{K}_\lambda \sim \frac{1}{2}\Gamma(\lambda)\left(\frac{z}{2}\right)^{-\lambda} \quad \text{when} \quad z \to 0,$$

we have that

$$\Psi_{\mathrm{GH}}(u) \sim -\log\left(\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+\mathrm{i}u)^2}\right)^{\lambda/2}\left(\frac{2\delta\sqrt{\alpha^2-\beta^2}}{2\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2}}\right)^{\lambda}\right)$$

$$= \lambda\log\left(\frac{\alpha^2-(\beta+\mathrm{i}u)^2}{\alpha^2-\beta^2}\right) = \lambda\log\left(1+\frac{u^2}{\alpha^2-\beta^2}-\frac{2\beta\mathrm{i}u}{\alpha^2-\beta^2}\right)$$

when $\delta \to 0$ and for $\mu = 0$. Here we write $f \sim g$ when $u \to \infty$ to mean that $\lim_{u\to\infty} f(u)/g(u) = 1$. So, we see that when $\delta \to 0$ and for $\mu = 0$, $\lambda = 1/\kappa$, $\beta = \theta/\sigma^2$ and $\alpha = \sqrt{\frac{(2/\kappa)+(\theta^2/\sigma^2)}{\sigma^2}}$, the characteristic exponent of the generalized hyperbolic process converges to the characteristic exponent of the variance gamma process. Because the variance gamma process is obtained by a limiting procedure, its path properties cannot be deduced directly from those of the generalized hyperbolic process. Indeed, we shall see they are fundamentally different processes.

1.4 PATH PROPERTIES

In the following sections, we shall discuss a number of coarse and fine path properties of general Lévy processes. These include path variation, hitting of points, creeping and regularity of the half line.

With the exception of the last property, none of the above have played a prominent role in mainstream literature on the modeling of financial markets. Initial concerns of Lévy-driven models were focused around the pricing of vanilla-type options, that is, options whose value depends on the distribution of the underlying Lévy process at a fixed point in time. Recently, more and more attention has been paid to exotic options which are typically path dependent. Fluctuation theory and path properties of Brownian motion being well understood has meant that many examples of exotic options under the assumptions of the classical Black–Scholes models can and have been worked out in the literature. We refer to objects such as American options, Russian options, Asian options, Bermudan options, lookback options, Parisian options, Israeli or game options, Mongolian options, and so on. However, dealing with exotic options in Lévy-driven markets has proved to be considerably more difficult as a consequence of the more complicated, and to some extent, incomplete nature of the theory of fluctuations of Lévy processes.

Nonetheless, it is clear that an understanding of course and fine path properties plays a role in the evaluation of exotics. In the analysis below, we shall indicate classes of exotics which are related to the described path property.

1.4.1 Path variation

Understanding the path variation for a Lévy process boils down to a better understanding of the Lévy–Khintchine formula. We therefore give a sketch proof of Theorem 6 which shows that for any given Lévy triple (γ, σ, Π) there exists a Lévy process whose characteristic exponent is given by the Lévy–Khintchine formula.

Reconsidering the formula for Ψ, note that we may write it in the form

$$\Psi(u) = \left[iu\gamma + \frac{1}{2}\sigma^2u^2\right] + \left[\int_{\mathbb{R}\setminus(-1,1)} (1-e^{iux})\Pi(dx)\right]$$
$$+ \left[\int_{0<|x|<1} (1-e^{iux}+ixu)\Pi(dx)\right]$$

and define the three terms in square brackets as $\Psi^{(1)}$, $\Psi^{(2)}$ and $\Psi^{(3)}$, respectively. As remarked upon earlier, the first of these terms, $\Psi^{(1)}$, can be identified as belonging to a Brownian motion with drift $\{\sigma B_t - \gamma t : t \geq 0\}$. From Section 1.3.1 we may also identify $\Psi^{(2)}$ as belonging to an independent compound Poisson process with intensity $\lambda = \Pi(\mathbb{R}\backslash(-1,1))$ and jump distribution $F(dx) = \mathbf{1}_{(|x|\geq 1)}\Pi(dx)/\lambda$. Note that this compound Poisson process has jump sizes of at least 1. The third term in the decomposition of the Lévy–Khintchine exponent above turns out to be the limit of a sequence of compound Poisson processes with a compensating drift, the reasoning behind which we shall now very briefly sketch.

For each $1 > \epsilon > 0$, consider the Lévy processes $X^{(3,\epsilon)}$ defined by

$$X_t^{(3,\epsilon)} = Y_t^{(\epsilon)} - t\int_{\epsilon<|x|<1} x\Pi(dx), \quad t \geq 0 \tag{1.8}$$

where $Y^{(\epsilon)} = \{Y_t^{(\epsilon)} : t \geq 0\}$ is a compound Poisson process with intensity $\lambda_\epsilon := \Pi(\{x : \epsilon < |x| < 1\})$ and jump distribution $\mathbf{1}_{(\epsilon<|x|<1)}\Pi(dx)/\lambda_\epsilon$. An easy calculation shows that $X^{(3,\epsilon)}$, which is also a compensated Poisson process, is also a martingale. It can also be shown with the help of the property $\int_{(-1,1)} x^2\Pi(dx) < \infty$ that it is a square integrable martingale. Again from Section 1.3.1, we see that the characteristic exponent of $X^{(3,\epsilon)}$ is given by

$$\Psi^{(3,\epsilon)}(u) = \int_{\epsilon<|x|<1} (1 - e^{iux} + iux)\Pi(dx).$$

For some fixed $T > 0$, we may now think of $\{\{X_t^{(3,\epsilon)} : t \geq [0,T]\} : 0 < \epsilon < 1\}$ as a sequence of right continuous square integrable martingales with respect to an appropriate filtration independent of ϵ. The latter space, when equipped with a suitable inner product, turns out to be a Hilbert space. It can also be shown, again with the help of the condition $\int_{(-1,1)} x^2\Pi(dx) < \infty$, that $\{\{X_t^{(3,\epsilon)} : t \geq [0,T]\} : 0 < \epsilon < 1\}$ is also a Cauchy sequence in this Hilbert space. One may show (in the right mathematical sense) that a limiting process $X^{(3)}$ exists which inherits from its approximating sequence the properties of stationary and independent increments and paths being right continuous with left limits. Its characteristic exponent is also given by

$$\lim_{\epsilon\downarrow 0} \Psi^{(3,\epsilon)} = \Psi^{(3)}.$$

Note that, in general, the sequence of compound Poisson processes $\{Y^{(\epsilon)} : 0 < \epsilon < 1\}$ does not converge without compensation. However, under the right condition $\{Y^{(\epsilon)} : 0 < \epsilon < 1\}$ does converge. This will be dealt with shortly. The decomposition of Ψ into $\Psi^{(1)}$, $\Psi^{(2)}$ and $\Psi^{(3)}$ thus corresponds to the decomposition of X into the independent sum of a Brownian motion with drift, a compound Poisson process of large jumps and a residual process of arbitrarily small compensated jumps. This decomposition is known as the *Lévy–Itô decomposition.*

Let us reconsider the limiting process $X^{(3)}$. From the analysis above, in particular from equation (1.8), it transpires that the sequence of compound Poisson processes $\{Y^{(\epsilon)} : 0 < \epsilon < 1\}$ has a limit, say Y, if, and only if, $\int_{(-1,1)} |x|\Pi(dx) < \infty$. In this case, it can be shown that the limiting process has a countable number of jumps and further, for each $t \geq 0$, $\sum_{0\leq s\leq t} |\Delta Y_s| < \infty$ almost surely. Hence, we conclude that a Lévy process has paths of bounded variation on each finite time interval, or more simply, has *bounded variation*, if, and only if,

$$\int_{\mathbb{R}} (1 \wedge |x|)\Pi(dx) < \infty \tag{1.9}$$

in which case we may always write the Lévy–Khintchine formula in the form

$$\Psi(u) = -iu\mathrm{d} + \int_{\mathbb{R}} (1 - e^{iux})\Pi(dx). \tag{1.10}$$

Note that we simply take $\mathrm{d} = \gamma - \int_{(-1,1)} x\Pi(dx)$ which is finite because of equation (1.9). The particular form of Ψ given above will turn out to be important in the following sections when describing other path properties. If within the class of bounded variation processes we have $\mathrm{d} > 0$ and supp $\Pi \subseteq (0, \infty)$, then X is an non-decreasing process (it drifts and jumps only upwards). In this case, it is called a *subordinator*.

If a process has unbounded variation on each finite time interval, then we shall say for simplicity that it has *unbounded variation*.

We conclude this section by remarking that we shall mention no specific links between processes of bounded and unbounded variation to particular exotic options. The division of Lévy processes according to path variation plays an important role in the further classification of forthcoming path properties. These properties have, in turn, links with features of exotic options and hence we make the association there.

1.4.2 Hitting points

We say that a Lévy process X can hit a point $x \in \mathbb{R}$ if

$$P(X_t = x \text{ for at least one } t > 0) > 0.$$

Let

$$C = \{x \in \mathbb{R} : P(X_t = x \text{ for at least one } t > 0) > 0\}$$

be the set of points that a Lévy process can hit. We say a Lévy process can hit points if $C \neq \emptyset$. Kesten (1969) and Bretagnolle (1971) give the following classification.

Theorem 7 *Suppose that X is not a compound Poisson process. Then X can hit points if and only if*

$$\int_{\mathbb{R}} \Re\left(\frac{1}{1+\Psi(u)}\right) du < \infty. \tag{1.11}$$

Moreover,

(i) If $\sigma > 0$, then X can hit points and $C = \mathbb{R}$.
(ii) If $\sigma = 0$, but X is of unbounded variation and X can hit points, then $C = \mathbb{R}$.
(iii) If X is of bounded variation, then X can hit points, if and only if, $\mathrm{d} \neq 0$ *where* d *is the drift in the representation (equation (1.10)) of its Lévy–Khintchine exponent Ψ. In this case, $C = \mathbb{R}$ unless X or $-X$ is a subordinator and then $C = (0, \infty)$ or $C = (-\infty, 0)$, respectively.*

The case of a compound Poisson process will be discussed in Section 1.5.1. Excluding the latter case, from the Lévy–Khintchine formula we have that

$$\Re(\Psi(u)) = \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}\backslash\{0\}} (1 - \cos(ux))\Pi(\mathrm{d}x)$$

and

$$\Im(\Psi(u)) = \gamma u + \int_{\mathbb{R}\backslash\{0\}} (-\sin(ux) + ux\mathbf{1}_{\{|x|<1\}})\Pi(\mathrm{d}x).$$

We see that for all $u \in \mathbb{R}$, we have $\Re(\Psi(u)) \geq 0$, $\Re(\Psi(u)) = \Re(\Psi(-u))$ and $\Im(\Psi(u)) = -\Im(\Psi(-u))$. So, because

$$\Re\left(\frac{1}{1+\Psi(u)}\right) = \frac{1 + \Re(\Psi(u))}{[1 + \Re(\Psi(u))]^2 + [\Im(\Psi(u))]^2},$$

we see that $\Re\left(\frac{1}{1+\Psi(u)}\right)$ as a function of u is always bigger than zero and is symmetric. It is also continuous, because the characteristic exponent is continuous. So, for all $p > 0$ we have

$$\int_{-p}^{p} \Re\left(\frac{1}{1+\Psi(u)}\right) \mathrm{d}u < \infty$$

and

$$\int_{-\infty}^{-p} \Re\left(\frac{1}{1+\Psi(u)}\right) \mathrm{d}u = \int_{p}^{\infty} \Re\left(\frac{1}{1+\Psi(u)}\right) \mathrm{d}u$$

and the question as to whether the integral (equation (1.11)) is finite or infinite depends on what happens when $u \to \infty$. If, for example, $\Re\left(\frac{1}{1+\Psi(u)}\right) \asymp g(u)$ when $u \to \infty$, then we can use g to deduce whether the integral (equation (1.11)) is finite or infinite. Note, we use the notation $f \asymp g$ to mean that there exists a $p > 0, a > 0$ and $b > 0$ such that $ag(u) \leq f(u) \leq bg(u)$ for all $u \geq p$, This technique will be used quite a lot in the examples we consider later on in the text.

An example of an exotic option which in principle makes use of the ability of a Lévy process to hit points is the so-called callable put option. This option belongs to a more general class of exotics called Game or Israeli options, described in Kifer (2000) (see also the review by Kühn and Kallsen (2005) in this volume). Roughly speaking, these options have the same structure as American-type options but for one significant difference. The writer also has the option to cancel the contract at any time before its expiry. The consequence of the writer cancelling the contract is that the holder is paid what they would have received had they exercised at that moment, plus an additional amount (considered as a penalty for the writer). When the claim of the holder is the same as that of the American put and the penalty of the writer is a constant, δ, then this option has been named a callable put in Kühn and Kyprianou (2005) (also an Israeli δ-penalty put option in Kyprianou (2004)). In the latter two papers, the value and optimal strategies of writer and holder of this exotic option have been calculated explicitly for the Black–Scholes market. It turns out there that the optimal strategy of the writer is to cancel the option when the value of the underlying asset hits precisely the strike price, providing that this happens early on enough in the contract. Clearly, this strategy takes advantage of case (i) of the above theorem. Suppose now for the same exotic option that instead of an exponential Brownian motion we work with an exponential Lévy process which cannot hit points. What would be the optimal strategies of the writer (and hence the holder)?

1.4.3 Creeping

Define for each $x \geq 0$ the first passage time

$$\tau_x^+ = \inf\{t > 0 : X_t > x\}.$$

Here, we work with the definitions $\inf \emptyset = \infty$ and if $\tau_x^+ = \infty$, then $X_{\tau_x^+} = \infty$. We say that a Lévy process X *creeps upwards* if for all $x \geq 0$

$$P(X_{\tau_x^+} = x) > 0$$

and that X *creeps downwards* if $-X$ creeps upwards. Creeping simply means that with positive probability, a path of a Lévy process continuously passes a fixed level instead of jumping over it.

A deep and yet enchanting aspect of Lévy processes, *excursion theory*, allows for the following non-trivial deduction concerning the range of $\{X_{\tau_x^+} : x \geq 0\}$. With probability one, the random set $\{X_{\tau_x^+} : x \geq 0\} \cap [0, \infty)$ corresponds precisely to the range of a certain subordinator, killed at an independent exponential time with parameter $q \geq 0$. The case that $q = 0$ should be understood to mean that there is no killing and hence that $\tau_x^+ < \infty$ almost surely for all $x \geq 0$. In the obvious way, by considering $-X$, we may draw the same conclusions for the range of $\{-X_{\tau_x^-} : x \geq 0\} \cap [0, \infty)$ where

$$\tau_x^- := \inf\{t > 0 : X_t < x\}.$$

Suppose that $\kappa(u)$ and $\widehat{\kappa}(u)$ are the characteristic exponents of the aforementioned subordinators for the ranges of the upward and downward first passage processes, respectively. Note, for example, that for $u \in \mathbb{R}$

$$\kappa(u) = q - i\mathrm{a}u + \int_{(0,\infty)} (1 - e^{iux})\pi(dx)$$

for some π satisfying $\int_{0\infty} (1 \wedge x)\pi(dx) < \infty$ and $\mathrm{a} \geq 0$ (recall that q is the killing rate). It is now clear from Theorem 7 that X creeps upwards, if and only if, $\mathrm{a} > 0$. The so-called Wiener–Hopf factorization tells us where these two exponents κ and $\widehat{\kappa}$ are to be found:

$$\Psi(u) = \kappa(u)\widehat{\kappa}(-u). \tag{1.12}$$

Unfortunately, there are very few examples of Lévy processes for which the factors κ and $\widehat{\kappa}$ are known. Nonetheless, the following complete characterization of upward creeping has been established.

Theorem 8 *The Lévy process X creeps upwards, if and only if, one of the following three situations occurs:*

(i) X has bounded variation and $\mathrm{d} > 0$ *where* d *is the drift in the representation (equation (1.10)) of its Lévy–Khintchine exponent* Ψ.

(ii) *X has a Gaussian component, ($\sigma > 0$).*
(iii) *X has unbounded variation, no Gaussian component and*

$$\int_0^1 \frac{x\Pi([x,\infty))}{\int_{-x}^0 \int_{-1}^y \Pi((-\infty,u])\mathrm{d}u\mathrm{d}y}\mathrm{d}x < \infty. \tag{1.13}$$

This theorem is the collective work of Miller (1973) and Rogers (1984), with the crowning conclusion in case (iii) being given recently by Vigon (2002).

As far as collective statements about creeping upwards and downwards are concerned, the situation is fairly straightforward to resolve with the help of the following easily proved lemma. (See Bertoin (1996), p. 16).

Lemma 9 *Let X be a Lévy process with characteristic exponent $\Psi(u)$.*

(i) *If X has finite variation then*

$$\lim_{u\uparrow\infty} \frac{\Psi(u)}{u} = -i\mathrm{d}$$

where d *is the drift appearing in the representation (equation (1.10)) of Ψ.*
(ii) *For a Gaussian coefficient $\sigma \geq 0$,*

$$\lim_{u\uparrow\infty} \frac{\Psi(u)}{u^2} = \frac{1}{2}\sigma^2.$$

From the above lemma we see, for example, that

$$\lim_{u\uparrow\infty} \frac{\kappa(u)}{u} \neq 0,$$

if and only if, X creeps upwards. Consequently, from the Wiener–Hopf factorization (equation (1.12)) the following well-established result holds (see Bertoin (1996), p. 175).

Lemma 10 *A Lévy process creeps both upwards and downwards, if and only if it has a Gaussian component.*

There is also a relation between hitting points and creeping. Clearly, a process which creeps can hit points. In the case of bounded variation we see that hitting points is equivalent to creeping upwards or downwards. However, in the case of unbounded variation, it can be that a process does not creep upwards or downwards, but still can hit points. We will see an example of this later on–see Remark 17. A process which hits a point but does not creep over it must therefore do so by jumping above and below that point an infinite number of times before hitting it.

When considering the relevance of creeping to exotic option pricing, one need only consider any kind of option involving first passage. This would include, for example, barrier options as well as Russian and American put options. Taking the latter case with infinite horizon, the optimal strategy is given by first passage below a fixed value of the underlying Lévy process. The value of this option may thus be split into two parts, namely, the premium for exercise by jumping clear of the boundary and the premium for creeping over the boundary. For the finite expiry case, it is known that the optimal strategy of the holder is to exercise when the underlying Lévy process crosses a time-varying barrier. In this case, a more general concept of creeping over moving boundaries may be introduced and it would be interesting to know whether the ability to creep over the optimal exercise boundary has any influence on the continuity or smoothness properties of the boundary as a function of time.

1.4.4 Regularity of the half line

For a Lévy process X (which starts at zero) we say that 0 *is regular for* $(0, \infty)$ (equiv. the upper half line) if X enters $(0, \infty)$ immediately. That is to say, if

$$\mathbb{P}(\tau^{(0,\infty)} = 0) = 1, \quad \text{where} \quad \tau^{(0,\infty)} = \inf\{t > 0 : X_t \in (0, \infty)\}.$$

Because of the Blumenthal 0–1 law, the probability $\mathbb{P}(\tau^{(0,\infty)} = 0)$ is necessarily zero or one. When this probability is zero, we say that 0 is irregular for $(0, \infty)$. We also say that 0 is regular for $(-\infty, 0)$ (equiv. the lower half line) if $-X$ is regular for the upper half line.

The following theorem is the conclusion of a number of works and gives a complete characterization of regularity for the upper half line (see Shtatland (1965), Rogozin (1968) and Bertoin (1997)).

Theorem 11 *For a Lévy process X, the point* 0 *is regular for* $(0, \infty)$*, if and only if, one of the following three situations occurs:*

(i) X is a process of unbounded variation.
(ii) X is a process of bounded variation and $\mathrm{d} > 0$ *where* d *is the drift in the representation (equation (1.10)) of its Lévy–Khintchine exponent* Ψ.
(iii) X is a process of bounded variation, $\mathrm{d} = 0$ *(with* d *as in (ii)) and*

$$\int_0^1 \frac{x\Pi(\mathrm{d}x)}{\int_0^x \Pi(-\infty, -y)\mathrm{d}y} = \infty. \tag{1.14}$$

Regularity of the lower half line has already proved to be of special interest to the pricing of American put options. In Alili and Kyprianou (2004). a perpetual American put is considered where the underlying market is driven by a general Lévy process. Building on the work of Mordecki (1999, 2002), Boyarchenko and Levendorskii (2002a) and Chan (2000, 2004), it is shown that the traditional condition of smooth pasting at the optimal exercise boundary may no longer be taken for granted. Indeed necessary and sufficient conditions are given for no smooth pasting. This condition is quite simply the regularity of $(-\infty, 0)$ for 0 (in other words the regularity of the upper half line for $-X$).

It was conjectured in Alili and Kyprianou (2005) that the very same condition would also characterize the appearance of smooth fit for the finite expiry American put where the boundary is time varying. Indeed, numerical simulations in Matache *et al.* (2003) and Almendral (2004) support this conjecture. A financial interpretation of a non-smooth fit condition has yet to be clarified.

1.5 EXAMPLES REVISITED

1.5.1 Compound Poisson processes and jump-diffusions

Suppose that X is a compound Poisson process. Clearly, X has paths of bounded variation, cannot creep upwards or downwards and is irregular for the upper and lower half lines. Since the Lévy measure is bounded, it is easy to reason that the real and imaginary part of its characteristic $\Psi(u)$, and hence $\Re\left(\frac{1}{1+\Psi(u)}\right)$, is bounded away from zero and that the integral (equation (1.11)) is infinite. Nonetheless, certain compound Poisson processes can hit points. Take the simple example of a Poisson process. This is a process which hits $\{0, 1, 2, \ldots .\}$. Other similar examples where the jump distribution is supported on a lattice are possible. This is the reason for the exclusion of compound Poisson processes from Theorem 7. However, it can be said that so long as the jump distribution F is diffuse, a compound Poisson process can hit no point other than 0, its initial holding point.

If X is a jump-diffusion then the above properties change drastically. In particular, if the Gaussian component is non-zero then this will dominate the paths of the process. This is because, until the first jump, which occurs at arbitrarily large times with positive probability, the process behaves as a Brownian motion with drift. It is clear that paths will be of unbounded variation, there will be regularity for the upper and lower half lines, the process may creep both upwards and downwards and any point can be hit with positive probability. Note the latter fact is a well-known property of Brownian motion and does not require Theorem 7.

1.5.2 Spectrally negative processes

By definition, spectrally negative processes creep upwards and hence can hit points. Therefore, unless there is a Gaussian component present, they cannot creep downwards. It is possible to have such processes of both bounded and unbounded variation according to the finiteness of the integral $\int_{-1}^{0} |x|\Pi(dx)$. Clearly, if it is a process of unbounded variation, then there is regularity for the upper and lower half lines.

If it is a process of bounded variation and not the negative of the subordinator, then by reconsidering equation (1.10) we see that necessarily the process must take the form of a strictly positive drift minus a subordinator. Consequently, from Theorem 11 in this case, there is regularity for the upper half line but not for the lower half line. This, in turn, implies that for spectrally negative Lévy processes, regularity of the lower half line coincides with having paths of infinite variation.

1.5.3 Meixner process

We begin with a known fact concerning path variation.

Proposition 12 *The Meixner process is of unbounded variation and hence is regular for the upper and lower half lines.*

Proof. Denote $\nu(x)$ as the density of the Lévy measure (equation (1.4)). We have to prove that $\int_{(-1,1)} |x|\nu(x)dx$ is infinite. For $x \in (0, 1)$ we have

$$|x|\nu(x) = \delta\frac{\exp(\beta x/\alpha)}{\sinh(\pi x/\alpha)} = 2\delta\frac{e^{(\beta+\pi)x/\alpha}}{e^{2\pi x/\alpha} - 1} \geq 2\delta\frac{1}{e^{2\pi x/\alpha} - 1}$$

and so

$$\int_0^1 |x|\nu(x)\mathrm{d}x \geq \int_0^1 2\delta \frac{1}{\mathrm{e}^{2\pi x/\alpha} - 1}\,\mathrm{d}x$$
$$= 2\delta \left[\frac{1}{2\pi/\alpha} \log\left(\mathrm{e}^{(2\pi/\alpha)x} - 1\right) - x \right]\Bigg|_0^1 = \infty$$

showing in particular that $\int_{(-1,1)} |x|\nu(x)dx = \infty$.

Proposition 13 *A Meixner process cannot hit points and therefore cannot creep.*

Proof. To see whether the Meixner process can hit points we have to employ the integral given in equation (1.11). In order to use this, we first split the characteristic exponent into its real and imaginary part. First note that

$$\cosh\left(\frac{\alpha u - \mathrm{i}\beta}{2}\right) = \cos\left(\frac{1}{2}\beta\right)\cosh\left(\frac{1}{2}\alpha u\right) - \sin\left(\frac{1}{2}\beta\right)\sinh\left(\frac{1}{2}\alpha u\right)\mathrm{i}.$$

Then with

$$r = \sqrt{\left[\cos\left(\frac{1}{2}\beta\right)\cosh\left(\frac{1}{2}\alpha u\right)\right]^2 + \left[\sin\left(\frac{1}{2}\beta\right)\sinh\left(\frac{1}{2}\alpha u\right)\right]^2} \quad \text{and}$$
$$q = \arctan\left(\frac{-\sin\left(\frac{1}{2}\beta\right)\sinh\left(\frac{1}{2}\alpha u\right)}{\cos\left(\frac{1}{2}\beta\right)\cosh\left(\frac{1}{2}\alpha u\right)}\right)$$

we have

$$\Psi(u) = -2\delta\log(\cos(\beta/2)) + 2\delta\log(r) + (2\delta q - mu)\mathrm{i}$$

and hence

$$\frac{1}{1+\Psi(u)} = \frac{1 + 2\delta\log\left(\frac{r}{\cos(\beta/2)}\right) - (2\delta q - mu)\mathrm{i}}{\left(1 + 2\delta\log\left(\frac{r}{\cos(\beta/2)}\right)\right)^2 + (2\delta q - mu)^2} \quad \text{and}$$
$$\Re\left(\frac{1}{1+\Psi(u)}\right) = \frac{1 + 2\delta\log\left(\frac{r}{\cos(\beta/2)}\right)}{\left(1 + 2\delta\log\left(\frac{r}{\cos(\beta/2)}\right)\right)^2 + (2\delta q - mu)^2}.$$

When $u \to \infty$, then $\cosh(\alpha u/2) \asymp \sinh(\alpha u/2) \asymp \mathrm{e}^{\alpha u/2}$, and so $r \asymp \mathrm{e}^{\alpha u/2}$ and $\log(r/\cos(\beta/2)) \asymp \frac{1}{2}\alpha u$ when $u \to \infty$. Further, $(2\delta q - mu)^2 \asymp m^2u^2$ when $u \to \infty$, because $\arctan(z) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ for all $z \in \mathbb{R}$. So

$$\Re\left(\frac{1}{1+\Psi(u)}\right) \asymp \frac{1+\delta\alpha u}{(1+\delta\alpha u)^2 + m^2u^2} \asymp u^{-1} \quad \text{when} \quad u \to \infty.$$

Because for all $p > 0$, $\int_p^\infty u^{-1}\mathrm{d}u = \infty$, we find that the integral (equation (1.11)) is infinite and therefore the Meixner process cannot hit points.

1.5.4 Generalized tempered stable process

We again begin with a known statement concerning path variation.

Proposition 14 *The generalized tempered stable process has bounded variation, if and only if,* $\alpha_p < 1$ *and* $\alpha_n < 1$.

Proof. We have to determine whether the integral given in equation (1.9) is finite or infinite, where this integral is given by

$$\int_{-1}^{1} |x|\nu(x)\mathrm{d}x = \int_0^1 \frac{c_p}{x^{\alpha_p}} \mathrm{e}^{-\lambda_p x} \mathrm{d}x + \int_{-1}^{0} \frac{c_n}{(-x)^{\alpha_n}} \mathrm{e}^{\lambda_n x} \mathrm{d}x.$$

It is clear, however, that this boils down to whether

$$\int_0^1 x^{-\alpha_p} \mathrm{d}x + \int_{-1}^{0} (-x)^{-\alpha_n} \mathrm{d}x$$

is finite or infinite and the above expression is only finite when $\alpha_p < 1$ and $\alpha_n < 1$.

Proposition 15 *In the case of unbounded variation, a generalized tempered stable process creeps upwards, if and only if,* $\alpha_p < \alpha_n$.

Proof. Because the integral given in equation (1.13) is independent of c_p and c_n, we can assume without loss of generality that $c_p = c_n = 1$. In the following calculations, $b_1, b_2, \ldots$ are constants in $\mathbb{R}$.

Recall from the previous proposition that we have unbounded variation if $\alpha_p \in [1, 2)$ or $\alpha_n \in [1, 2)$. We shall therefore prove the result under the additional assumption that there is unbounded variation because $\alpha_n \in [1, 2)$. Similar arguments then deliver the same conclusions when we assume that there is unbounded variation because $\alpha_p \in [1, 2)$.

For $u \in (0, 1]$ we have

$$\begin{aligned}
&\Pi([u,\infty)) \\
&\quad = \int_u^\infty \mathrm{e}^{-\lambda_p x} x^{-(1+\alpha_p)} \mathrm{d}x \\
&\quad \le \begin{cases}
\int_1^\infty \mathrm{e}^{-\lambda_p} x^{-(1+\alpha_p)} \mathrm{d}x + \int_u^1 \mathrm{e}^{-\lambda_p x} \mathrm{d}x = b_1 + b_2 \mathrm{e}^{-\lambda_p u} & \text{if } \alpha_p \le -1 \\
\int_1^\infty \mathrm{e}^{-\lambda_p x} \mathrm{d}x + \int_u^1 x^{-1} \mathrm{d}x = b_3 - \log(u) & \text{if } \alpha_p = 0 \\
\int_1^\infty \mathrm{e}^{-\lambda_p x} \mathrm{d}x + \int_u^1 x^{-(1+\alpha_p)} \mathrm{d}x = b_4 + b_5 u^{-\alpha_p} & \text{if } \alpha_p \in (-1, 2)\backslash\{0\}
\end{cases}
\end{aligned}$$

and for $u \in [-1, 0)$

$$\begin{aligned}
\Pi((-\infty, u]) &= \int_{-\infty}^{u} \mathrm{e}^{\lambda_n x} (-x)^{-(1+\alpha_n)} \mathrm{d}x \\
&\ge \int_{-1}^{u} \mathrm{e}^{-\lambda_n} (-x)^{-(1+\alpha_n)} \mathrm{d}x = b_6 (-u)^{-\alpha_n} + b_7
\end{aligned}$$

if $\alpha_n \in [1,2)$. Hence

$$\int_{-1}^{y} \Pi((-\infty,u])\,\mathrm{d}u \geq \begin{cases} b_8(-y)^{-\alpha_n+1} + b_9 + b_{10}(y+1) & \text{if } \alpha_n \in (1,2) \\ b_{11}\log(-y) + b_{12}(y+1) & \text{if } \alpha_n = 1 \end{cases}$$

and then

$$\int_{-x}^{0}\int_{-1}^{y} \Pi((-\infty,u])\,\mathrm{d}u\,\mathrm{d}y \geq \begin{cases} b_{13}x^{-\alpha_n+2} + b_{14}x^2 + b_{15}x & \text{if } \alpha_n \in (1,2) \\ b_{16}(x\log(x) - x) + b_{17}x^2 + b_{18}x & \text{if } \alpha_n = 1. \end{cases}$$

We then have for $\alpha_p \in (-1,2)\backslash\{0\}$ and $\alpha_n \in (1,2)$

$$\begin{aligned} \frac{x\Pi([x,\infty))}{\int_{-x}^{0}\int_{-1}^{y}\Pi((-\infty,u])\,\mathrm{d}u\,\mathrm{d}y} &\leq \frac{b_4x + b_5x^{-\alpha_p+1}}{b_{13}x^{-\alpha_n+2} + b_{14}x^2 + b_{15}x} \\ &= \frac{b_4 + b_5x^{-\alpha_p}}{b_{13}x^{-\alpha_n+1} + b_{14}x + b_{15}}, \end{aligned}$$

for $x \in (0,1]$. Define the right side of the above inequality $f(x)$ and note that it is continuous for all $x \in (0,1]$. When $x \to 0$, then

$$f \asymp x^{\alpha_n-1} \text{ for } \alpha_p < 0 \quad \text{and} \quad f \asymp x^{\alpha_n-1-\alpha_p} \quad \text{for } \alpha_p > 0.$$

Hence, $\int_0^1 f(x)\mathrm{d}x < \infty$ for $\alpha_p < \alpha_n$ and therefore the integral (equation (1.13)) is finite for these parameter values. When $\alpha_p \leq -1$ or $\alpha_p = 0$ or $\alpha_n = 1$, there is a similar upper bound for which the same conclusions can be drawn. We have thus so far shown that there is creeping upwards if $\alpha_p < \alpha_n$.

To prove the ‘only-if’ part, note that the Lévy density of $-X$ is $\nu(-x)$ and this density is the same as $\nu(x)$ except that the p-parameters and the n-parameters have switched places. So, we can immediately conclude from the previous analysis that X creeps downwards if $\alpha_p > \alpha_n$ and then from Lemma 10 we see that since there is no Gaussian component, X cannot creep upwards if $\alpha_p > \alpha_n$. Now only the case remains when $\alpha_p = \alpha_n \in [1,2)$. In this case, we can use for $u \in (0,1]$ the lower bound for $\Pi((-\infty,-u])$ as a lower bound for $\Pi([u,\infty))$ and the upper bound for $\Pi([u,\infty))$ as an upper bound for $\Pi((-\infty,-u])$ in order to create a lower bound for the integral (equation (1.13)) which turns out to be infinite.

Proposition 16 *In the case of unbounded variation, a generalized tempered stable process can hit points, unless $\alpha_p = \alpha_n = 1$ and $c_p = c_n$.*

Proof. Because this process creeps upwards or downwards when $\alpha_p \neq \alpha_n$, we only have to prove that the process can hit points when $\alpha_p = \alpha_n = \alpha \in [1,2)$. Let $r_p = \sqrt{1 + \frac{u^2}{\lambda_p^2}}$, $q_p = \arctan\left(-\frac{u}{\lambda_p}\right)$, $r_n = \sqrt{1 + \frac{u^2}{\lambda_n^2}}$ and $q_n = \arctan\left(\frac{u}{\lambda_n}\right)$. Then

$$\begin{aligned} A_p(u) &= \beta_p(r_p^\alpha\cos(\alpha q_p) - 1) + \mathrm{i}\beta_p\left(r_p^\alpha\sin(\alpha q_p) + \frac{\alpha u}{\lambda_p}\right) \\ A_n(u) &= \beta_n(r_n^\alpha\cos(\alpha q_n) - 1) + \mathrm{i}\beta_n\left(r_n^\alpha\sin(\alpha q_n) - \frac{\alpha u}{\lambda_n}\right), \end{aligned}$$

with $\beta_p = \Gamma(-\alpha)\lambda_p^\alpha c_p$ and $\beta_n = \Gamma(-\alpha)\lambda_n^\alpha c_n$. So, for $\alpha_p = \alpha_n \in (1,2)$

$$\Re\left(\frac{1}{1+\Psi(u)}\right) = \frac{1-\Re(A_p(u))-\Re(A_n(u))}{[1-\Re(A_p(u))-\Re(A_n(u))]^2+[u\gamma-\Im(A_p(u))-\Im(A_n(u))]^2}$$
$$\leq \frac{1-\Re(A_p(u))-\Re(A_n(u))}{[1-\Re(A_p(u))-\Re(A_n(u))]^2} = \frac{1}{1-\Re(A_p(u))-\Re(A_n(u))}.$$

We see that the above upper bound of $\Re\left(\frac{1}{1+\Psi(u)}\right)$ as a function of u is continuous and symmetric. So, the question whether the integral of this function from minus infinity to infinity is finite or infinite depends on how the function behaves when $u \to \infty$. When $u \to \infty$, then $r_p \asymp r_n \asymp u$, $q_p \to -\frac{1}{2}\pi$ and $q_n \to \frac{1}{2}\pi$. So, $\cos(\alpha q_p) = \cos(\alpha q_n) \to a$ for $u \to \infty$, where a is a constant smaller than zero. Because $\Gamma(-\alpha) > 0$ for $\alpha \in (1,2)$, we have that $1-\Re(A_p(u))-\Re(A_n(u)) \asymp u^\alpha$ when $u \to \infty$. Because for all $t > 0$ $\int_t^\infty u^{-\alpha}\mathrm{d}u < \infty$, the integral of the upper bound is finite and hence this process can hit points when $\alpha_p = \alpha_n \in (1,2)$.

Now, let $\alpha_p = \alpha_n = 1$. Then by using the same expressions for r_p, r_n, q_p and q_n as above,

$$A_p(u) = c_p(\lambda_p \log(r_p) + uq_p) + \mathrm{i}c_p(u + \lambda_p q_p - u\log(r_p))$$
$$A_n(u) = c_n(\lambda_n \log(r_n) - uq_n) + \mathrm{i}c_n(-u + \lambda_n q_n + u\log(r_n))$$

and then when $u \to \infty$,

$$\begin{array}{ll} 1-\Re(A_p(u))-\Re(A_n(u)) \asymp u & \\ u\gamma' - \Im(A_p(u)) - \Im(A_n(u)) \asymp u\log(u) & \text{if } c_p \neq c_n \\ u\gamma' - \Im(A_p(u)) - \Im(A_n(u)) \asymp u \text{ or } u\gamma' - \Im(A_p(u)) - \Im(A_n(u)) \asymp 1 & \text{if } c_p = c_n. \end{array}$$

So, when $c_p \neq c_n$, then $\Re\left(\frac{1}{1+\Psi(u)}\right) \asymp \frac{1}{u\log^2(u)}$ and because for large t,

$$\int_t^\infty \frac{1}{u\log^2(u)} = -\frac{1}{\log(u)}\bigg|_t^\infty < \infty,$$

the process can hit points. For the case where $c_p = c_n$, then $\Re\left(\frac{1}{1+\Psi(u)}\right) \asymp u^{-1}$ and the integral given in equation (1.11) is infinite.

Remark 17 *The last two propositions give us an example of a Lévy process which can hit points but cannot creep. Take the example of a CGMY process where the parameter* $Y \in (1,2)$. *To some extent this is not surprising. As noted earlier, the small time behaviour of generalized tempered stable processes should in principle be similar to the behaviour of stable processes due to the similarities in their Lévy measures in the neighbourhood of the origin. In this sense, the class of CGMY processes mentioned are closely related to a symmetric stable processes of unbounded variation and for this class it is well known that they can hit points but cannot creep. To see the latter fact, note from Lemma 10 that it is clear that a symmetric stable process (or indeed any Lévy process which is symmetric without a Gaussian component) cannot creep upwards nor downwards on account of symmetry. On the* *other hand, it is well known (cf. Chapter VIII in Bertoin (1996)) that for a symmetric stable process of index* α, $\Psi(u) = c|u|^{\alpha}$ *for some constant* $c > 0$, *and there is unbounded variation when* $\alpha \in (1, 2)$. *It is easily verified that with this choice of* Ψ, *the integral (equation (1.11)) is finite.*

In the case that a generalized tempered stable process has bounded variation, that is, when $\alpha_p < 1$ and $\alpha_n < 1$, we can use the drift d to determine whether the process can hit points, creeps or whether 0 is regular for $(0, \infty)$. However, then we have to know what the drift looks like. Comparing equations (1.2) and (1.6), we see that $\mathrm{d} = -\gamma' - \int_{-\infty}^{\infty} x\nu(x)\mathrm{d}x$. The latter integral can be computed explicitly; however, it is easier to use Lemma 9 (i) to identify the drift term since the Lévy–Khintchine formula is polynomial when there is bounded variation. Indeed, it is easy to see by inspection that

$$\mathrm{d} = -\gamma - d_p - d_n,$$

where

$$d_p = \begin{cases} -c_p\alpha_p\Gamma(-\alpha_p)\lambda_p^{\alpha_p-1} & \text{if } \alpha_p \in (-\infty, 1)\backslash\{0\} \\ \dfrac{c_p}{\lambda_p} & \text{if } \alpha_p = 0 \end{cases} \qquad \text{and}$$

$$d_n = \begin{cases} c_n\alpha_n\Gamma(-\alpha_n)\lambda_n^{\alpha_n-1} & \text{if } \alpha_n \in (-\infty, 1)\backslash\{0\} \\ -\dfrac{c_n}{\lambda_n} & \text{if } \alpha_n = 0. \end{cases}$$

Remark 18 *For the special cases of the CGMY and variance gamma processes, we note that the representation of the Lévy–Khintchine formula given in equations (1.6) and (1.7) yields in both cases that the drift term* $\mathrm{d} = \mu$.

Proposition 19 *When a generalized tempered stable process has bounded variation and drift equal to zero, then* 0 *is regular for* $(0, \infty)$, *if and only if,* $\alpha_p \geq \alpha_n$ *and at least one of these two parameters is not smaller than zero.*

Proof. We can use the integral (equation (1.14)) to determine whether 0 is regular for $(0, \infty)$. For $y \in (0, 1)$ we have

$$\Pi((-\infty, -y)) = \int_{-\infty}^{-y} \frac{c_n \mathrm{e}^{\lambda_n x}}{(-x)^{1+\alpha_n}}\mathrm{d}x \geq \int_{-1}^{-y} \frac{c_n \mathrm{e}^{\lambda_n x}}{(-x)^{1+\alpha_n}}\mathrm{d}x$$
$$\geq \int_{-1}^{-y} \frac{c_n \mathrm{e}^{\lambda_n}}{(-x)^{1+\alpha_n}}\mathrm{d}x = \begin{cases} b_1 y^{-\alpha_n} + b_2 & \text{if } \alpha_n \neq 0 \\ b_3 \log(y) & \text{if } \alpha_n = 0 \end{cases}$$

and

$$\Pi((-\infty, -y)) = \int_{-\infty}^{-1} \frac{c_n \mathrm{e}^{\lambda_n x}}{(-x)^{1+\alpha_n}}\mathrm{d}x + \int_{-1}^{-y} \frac{c_n \mathrm{e}^{\lambda_n x}}{(-x)^{1+\alpha_n}}\mathrm{d}x$$
$$\leq \int_{-\infty}^{-1} \frac{c_n \mathrm{e}^{\lambda_n x}}{(-x)^{1+\alpha_n}}\mathrm{d}x + \int_{-1}^{-y} \frac{c_n}{(-x)^{1+\alpha_n}}\mathrm{d}x$$
$$= \begin{cases} b_4 + b_5 y^{-\alpha_n} & \text{if } \alpha_n \neq 0 \\ b_6 + b_7 \log(y) & \text{if } \alpha_n = 0, \end{cases}$$

where $b_1, b_2, \dots$ are constants in $\mathbb{R}$. So,

$$\begin{cases} b_2 x + b_8 x^{-\alpha_n+1} \le \int_0^x \Pi((-\infty,-y))\mathrm{d}y \le b_4 x + b_9 x^{-\alpha_n+1} & \text{if } \alpha_n \neq 0 \\ b_3(x\log(x) - x) \le \int_0^x \Pi((-\infty,-y))\mathrm{d}y \le b_6 x + b_7(x\log(x) - x) & \text{if } \alpha_n = 0. \end{cases}$$

Note that the constants $b_1, b_2, \dots$ have values such that the above upper and lower boundaries are strictly positive for $x > 0$. Now, when $\alpha_p < 1, \alpha_n < 1$ and $\alpha_n \neq 0$, then

$$\frac{x\nu(x)}{\int_0^x \Pi((-\infty,-y))\mathrm{d}y} \le \frac{x c_p x^{-1-\alpha_p}\mathrm{e}^{-\lambda_p x}}{b_2 x + b_8 x^{-\alpha_n+1}} = \frac{c_p x^{-1-\alpha_p}\mathrm{e}^{-\lambda_p x}}{b_2 + b_8 x^{-\alpha_n}}.$$

Let $f(x)$ be the value of the right-hand side of above inequality for $x \in (0,1)$. Then, as $x \to 0$, then

$$f(x) \asymp x^{-1-\alpha_p} \quad \text{if } \alpha_n < 0 \quad \text{and} \quad f(x) \asymp x^{\alpha_n - \alpha_p - 1} \quad \text{if } \alpha_n > 0.$$

We conclude that the upper bound is finite if $\alpha_p < 0$ and $\alpha_n < 0$ and if $\alpha_p < \alpha_n$ and hence therefore 0 is irregular for $(0,\infty)$ in these cases. Because the lower bound on $\int_0^x \Pi((-\infty,-y))\mathrm{d}y$ has the same form as the upper bound, we can immediately conclude that in the other cases when $\alpha_n \neq 0$, 0 is regular for $(0,\infty)$. Now only the case remains when $\alpha_n = 0$. Here we have

$$\frac{x\nu(x)}{\int_0^x \Pi((-\infty,-y))\mathrm{d}y} \le \frac{x c_p x^{-1-\alpha_p}\mathrm{e}^{-\lambda_p x}}{b_3(x\log(x) - x)} = \frac{c_p x^{-1-\alpha_p}\mathrm{e}^{-\lambda_p x}}{b_3(\log(x) - 1)}.$$

Let $g(x)$ be the value of the right-hand side of above inequality for $x \in (0,1)$. Then, when $x \to 0$,

$$g(x) \asymp \frac{-1}{x^{1+\alpha_p}\log(x)}.$$

Because for all $t < 1$,

$$\int_0^t \frac{-1}{x^{1+\alpha_p}\log(x)}\mathrm{d}x < \infty,$$

if and only if, $\alpha_p < 0$, the upper bound is finite in this case and hence the integral (equation (1.14)) is finite. The lower bound has again the same form as the upper bound and so we conclude that this integral (equation (1.14)) is infinite when $\alpha_p \ge 0$ and $\alpha_n = 0$.

1.5.5 Generalized hyperbolic process

Because the Lévy measure of this process is very complicated, it is very difficult to use this measure to determine whether the process is of finite of infinite variation. However, this can also be determined by using the characteristic exponent with the help of Lemma 9. We follow the ideas in given Cont and Tankov (2004).

Proposition 20 *A generalized hyperbolic process is of unbounded variation and has no Gaussian component and hence* 0 *is regular for the upper and lower half lines.*

Proof. Using the properties of the logarithm we have

$$\Psi(u) = -\frac{\lambda}{2}\log(\alpha^2-\beta^2) + \frac{\lambda}{2}\log(\alpha^2-(\beta+\mathrm{i}u)^2) \\ -\log\left(\mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2}\right)\right) + \log\left(\mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)\right) - \mathrm{i}\mu u.$$

Let $r = \sqrt{(\alpha^2-\beta^2+u^2)^2+4\beta^2u^2}$ and $q = \arctan\left(\frac{-2\beta u}{\alpha^2-\beta^2+u^2}\right)$. Then

$$\Psi(u) = -\frac{\lambda}{2}\log(\alpha^2-\beta^2) + \frac{\lambda}{2}(\log(r)+\mathrm{i}q) - \Re\left(\log\left(\mathrm{K}_\lambda\left(\delta\sqrt{r}\mathrm{e}^{\frac{1}{2}q\mathrm{i}}\right)\right)\right) \\ -\Im\left(\log\left(\mathrm{K}_\lambda\left(\delta\sqrt{r}\mathrm{e}^{\frac{1}{2}q\mathrm{i}}\right)\right)\right) + \log\left(\mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)\right) - \mathrm{i}\mu u.$$

When $u \to \infty$, $r \sim u^2$ and $q \to 0$. The modified Bessel function, K_λ, has the following property:

$$\text{if } a \to \infty \text{ then } \mathrm{K}_\lambda(a+b\mathrm{i}) \sim \mathrm{e}^{-(a+b\mathrm{i})}\sqrt{\frac{\pi}{2(a+b\mathrm{i})}}.$$

So, $\mathrm{K}_\lambda\left(\delta\sqrt{r}\mathrm{e}^{\frac{1}{2}q\mathrm{i}}\right) \sim \mathrm{e}^{-\delta\sqrt{r}\mathrm{e}^{\frac{1}{2}q\mathrm{i}}}\sqrt{\frac{\pi}{2\delta\sqrt{r}\mathrm{e}^{\frac{1}{2}q\mathrm{i}}}}$ and therefore

$$\Re\left(\log\left(\mathrm{K}_\lambda\left(\delta\sqrt{r}\mathrm{e}^{\frac{1}{2}q\mathrm{i}}\right)\right)\right) \sim -\delta\sqrt{r}\cos\left(\frac{1}{2}q\right) + \frac{1}{2}\log(\pi) - \frac{1}{2}\log\left(2\delta\sqrt{r}\right)$$

and

$$\Im\left(\log\left(\mathrm{K}_\lambda\left(\delta\sqrt{r}\mathrm{e}^{\frac{1}{2}q\mathrm{i}}\right)\right)\right) \sim -\delta\sqrt{r}\sin\left(\frac{1}{2}q\right) - \frac{1}{4}q$$

when $u \to \infty$. So, $\Re(\Psi(u)) \sim \delta u$ and we conclude from Lemma 9 that the process is of infinite variation and has no Gaussian component.

Proposition 21 *A generalized hyperbolic process cannot hit points and hence cannot creep.*

Proof. We have seen from the proof above that $\Re(\Psi(u)) \sim \delta u$ and that $\Im(\Psi(u)) \sim \delta\sqrt{r}\sin\left(\frac{1}{2}q\right) + \mu u$, when $u \to \infty$. This implies that $\Re\left(\frac{1}{1+\Psi(u)}\right) \asymp u^{-1}$ when $u \to \infty$ and therefore the process cannot hit points.

1.6 CONCLUSIONS

Let us conclude with some tables with our findings for some of the more popular models we have mentioned. It will be useful to recall the notation

$$C = \{x \in \mathbb{R} : P(X_t = x \text{ for at least one } t > 0) > 0\}.$$

Meixner processes

$$\Psi_{\text{Meixner}}(u) = -\log\left(\left(\frac{\cos(\beta/2)}{\cosh((\alpha u - \mathrm{i}\beta)/2}\right)^{2\delta}\right) - \mathrm{i}\mu u.$$

Path variation:	Unbounded variation.
Hitting points:	$C = \emptyset$.
Creeping:	No creeping upwards or downwards.
Regularity:	Always regular for $(0, \infty)$ and $(-\infty, 0)$.

CGMY processes

$$\Psi_{\text{CGMY}}(u) = -C\Gamma(-Y)[(M - \mathrm{i}u)^Y - M^Y + (G + \mathrm{i}u)^Y - G^Y)] - iu\mu.$$

Path variation:	Unbounded variation $\Leftrightarrow Y \in [1, 2)$.
Hitting points:	$C = \emptyset \Leftrightarrow Y = 1$ or $Y \in (0, 1)$ and $\mu = 0$, otherwise $C = \mathbb{R}$.
Creeping:	Upwards creeping $\Leftrightarrow Y \in (0, 1)$ and $\mu > 0$. Downwards creeping $\Leftrightarrow Y \in (0, 1)$ and $\mu < 0$.
Regularity:	Irregular for $(0, \infty) \Leftrightarrow Y \in (0, 1)$ and $\mu < 0$. Irregular for $(-\infty, 0) \Leftrightarrow Y \in (0, 1)$ and $\mu > 0$.

Variance gamma processes

$$\Psi_{\text{VG}}(u) = C\left[\log\left(1 - \frac{\mathrm{i}u}{M}\right) + \log\left(1 + \frac{\mathrm{i}u}{G}\right)\right] - iu\mu.$$

Path variation:	Bounded variation
Hitting points:	$C = \emptyset \Leftrightarrow \mu = 0$, otherwise $C = \mathbb{R}$.
Creeping:	Upwards creeping $\Leftrightarrow \mu > 0$, Downwards creeping $\Leftrightarrow \mu < 0$
Regularity:	Regular for $(0, \infty) \Leftrightarrow \mu \geq 0$. Regular for $(-\infty, 0) \Leftrightarrow \mu \leq 0$.

Generalized hyperbolic processes

$$\Psi_{GH}(u) = -\log\left(\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + \mathrm{i}u)^2}\right)^{\lambda/2} \frac{\mathrm{K}_\lambda(\delta\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2})}{\mathrm{K}_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}\right) - \mathrm{i}\mu u.$$

Path variation:	Unbounded variation.
Hitting points:	$C = \emptyset$
Creeping:	No creeping upwards or downwards.
Regularity:	Always regular for $(0, \infty)$ and $(-\infty, 0)$.

ACKNOWLEDGEMENTS

We would like to express our thanks to Antonis Papapantoleon and a referee for careful reading of this manuscript.

REFERENCES

[1] Alili, L. and Kyprianou, A. (2005), "Some remarks on first passage of Lévy process, the American put and pasting principles", *Annals of Applied Probability*, to be published.

[2] Almendral, A. (2005), "Numerical valuation of American options under the CGMY process", in A. Kyprianou, W. Schoutens and P. Wilmott (Eds), *Exotic Option Pricing and Advanced Lévy Models*, Wiley, Chichester, UK, pp. 259–276.

[3] Applebaum, D. (2004), *Lévy Processes and Stochastic Calculus*, Cambridge Studies in Advanced Mathematics, Vol. 93, Cambridge University Press, Cambridge, UK.

[4] Asmussen, S., Avram, F. and Pistorius, M.R. (2004), "Russian and American put options under exponential phase-type Lévy models", *Stochastic Processes and their Applications*, **109**, 79–111.

[5] Avram F., Chan T. and Usabel M. (2002), "On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts", *Stochastic Processes and their Applications*, **100**, 75–107.

[6] Avram, F., Kyprianou, A.E. and Pistorius, M.R. (2004), "Exit problems for spectrally negative Lévy processes and applications to Russian, American and Canadized options", *Annals of Applied Probability*, **14**, 215–238.

[7] Barndorff-Nielsen, O.E. (1995), "Processes of normal inverse Gaussian type", *Finance and Stochastics*, **2**, 41–68.

[8] Barndorff-Nielsen, O.E. (1998), "Normal inverse Gaussian distributions and the modeling of stock returns", Research Report No. 300, Department of Theoretical Statistics, Aarhus University, Denmark.

[9] Barndorff-Nielsen, O.E. and Shephard, N. (2001), "Non-Gaussian Ornstein–Ulenbeck based models and some of their uses in financial econometrics", *Journal of the Royal Statistical Society, B*, **63**, 167–241.

[10] Bertoin, J. (1996), *Lévy Processes*, Cambridge University Press, Cambridge Tracts in Mathematics, Vol. 121, Cambridge, UK.

[11] Bertoin, J. (1997), "Regularity of the half-line for Lévy processes", *Bulletin des Sciences Mathématiques*, **121**, 345–354.

[12] Boyarchenko, S.I. and Levendorskii, S.Z. (2002), *Non-Gaussian Merton–Black–Scholes theory*, Advanced Series on Statistical Science and Applied Probability, Vol. 9, World Scientific, River Edge, NJ, USA.

[13] Boyarchenko, S.I and Levendorskii, S.Z. (2002a), "Perpetual American options under Lévy processes", *SIAM Journal on Control and Optimization*, **40**, 1663–1696.

[14] Bretagnolle, J. (1971), "Résultats de Kesten sur les processus à accroissements indépendants", in *Séminaire de Probabilités V*, (*Universitaire Strasbourg, Année Universitaire 1969–1970*), Lecture Notes in Mathematics, Vol. 191, Springer-Verlag, Berlin, Germany, pp. 21–36 (in French).

[15] Carr, P. and Wu, L. (2003), "The finite moment log stable process and option pricing", *Journal of Finance*, **58**, 753–778.

[16] Carr, P., Geman, H., Madan, D.B. and Yor, M. (2002), "The fine structure of asset returns: an empirical investigation", *Journal of Business*, **75**, 305–332.

[17] Carr, P., Geman, H., Madan, D.B. and Yor, M. (2003), "Stochastic volatility for Lévy processes", *Mathematical Finance*, **13**, 345–382.

[18] Cartea, A. and Howison, A. (2005), "Option pricing with Lévy-stable process", to be published.

[19] Chan, T. (2000), "American options driven spectrally by one sided Lévy processes", *Unpublished manuscript* (see also Chapter 9 in this volume).

[20] Chan, T. (2004), "Some applications of Lévy processes in insurance and finance", *Finance*, **25**, 71–94.

[21] Cont, R. (2001), “Empirical properties of asset returns: stylized facts and statistical issues”, *Quantitive Finance*, **1**, 223–236.

[22] Cont, R. and Tankov, P. (2004), *Financial Modeling with Jump Processes*, Chapman and Hall/CRC Financial Mathematics Series, Boca Raton, FL, USA.

[23] Eberlein, E. and van Hammerstein, E.A. (2004), “Generalized hyperbolic and inverse Gaussian distributions: limiting cases and approximation of processes”, *Seminar on Stochastic Analysis, Random Fields and Applications IV*, Progress in Probability, Vol. 58, Birkhäuser Verlag, Basel, Switzerland, pp. 221–264.

[24] Eberlein, E. and Prause, K. (2002), “The generalized hyperbolic model: financial derivatives and risk measures”, in H. Geman, D. Madan, S. Pliska and T. Vorst (Eds), *Mathematical Finance-Bachelier Congress 2000*, Springer-Verlag, Berlin, Germany, pp. 245–267.

[25] Grigelionis, B. (1999), “Processes of Meixner type”, *Lithuanian Mathematics Journal*, **39**, 33–41.

[26] Halgreen, C. (1979), “Self-decomposability of the generalized inverse Gaussian and hyperbolic distributions”, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **47**, 13–17.

[27] Kesten, H. (1969), “Hitting probabilities of single points for processes with stationary independent increments”, in *Memoirs of the American Mathematical Society*, Vol. 93, American Mathematical Society, Providence, RI, USA, pp. 1–129.

[28] Kifer, Y. (2000), “Game options”, *Finance and Stochastics*, **4**, 443–463.

[29] Koponen, I. (1995), “Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process”, *Physical Review, E*, **52**, 1197–1199.

[30] Kou, S.G. (2002), “A jump-diffusion model for option pricing”, *Management Science*, **48**, 1086–1101.

[31] Kühn, C. and Kyprianou, A.E. (2003), “Callable puts as composite exotic options”, to be published.

[32] Kühn, C. and Kallsen, J. (2005), “Convertible bonds: financial derivatives of game type”, in A. Kyprianou, W. Schoutens, and P. Wilmott (Eds), *Exotic Option Pricing and Advanced Lévy Models*, Wiley, Chichester, UK.

[33] Kyprianou, A.E. (2004), “Some calculations for Israeli options”, *Finance and Stochastics*, **8**, 73–86.

[34] Madan, D.P. and Seneta, E. (1987), “The VG for share market returns”, *Journal of Business*, **63**, 511–524.

[35] Matache, A.M. Nitsche, P.A. and Schwab, C. (2003), “Wavelet Galerkin pricing of American options on Lévy driven assets”, to be published.

[36] Merton, R. (1976), “Option pricing when underlying stock returns are discontinuous”, *Journal of Financial Economics*, **3**, 125–144.

[37] Miller, P.W. (1973), “Exit properties of stochastic processes with stationary independent increments”, *Transactions of the American Mathematical Society*, **178**, 459–479.

[38] Mordecki, E. (1999), “Optimal stopping for a diffusion with jumps”, *Finance and Stochastics*, **3**, 227–236.

[39] Mordecki, E. (2002), “Optimal stopping and perpetual options for Lévy processes”, *Finance and Stochastics*, **6**, 473–493.

[40] Pitman, J. and Yor, M. (2003), “Infinitely divisible laws associated with hyperbolic functions”, *Canadian Journal of Mathematics*, **55**, 292–330.

[41] Rogers, L.C.G. (1984), “A new identity for real Lévy processes”, *Annals of the Institute of Henri Poincaré*, **20**, 21–34.

[42] Rogozin, B.A. (1968), “The local behavior of processes with independent increments”, *Teoriga Veroyatnosteüi i ee Primeneniya* (*Theory of Probability and its Applications*), **13**, 507–512 (in Russian).

[43] Sato, K.-I. (1999), *Lévy Processes and Infinitely Divisible Distributions* (translated from the 1990 Japanese original, revised by the author), Cambridge Studies in Advanced Mathematics, Vol. 68, Cambridge University Press, Cambridge, UK.

[44] Schoutens, W. (2000), *Stochastic Processes and Orthogonal Polynomials*, Lecture Notes in Statistics, Vol. 146, Springer-Verlag, New York, NY, USA.

[45] Schoutens, W. (2001), “The Meixner process in finance”, EURANDOM Report 2001–2002, EURANDOM, Eindhoven, The Netherlands.

[46] Schoutens, W. (2002), “Meixner processes: theory and applications in finance”, EURANDOM report 2002–2004, EURANDOM, Eindhoven, The Netherlands.

[47] Schoutens, W. (2003), *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley, Chichester, UK.

[48] Schoutens, W. and Teugels, J.L. (1998), “Lévy processes, polynomials and martingales”, *Communications in Statistics: Stochastic Models*, **14**, 335–349.

[49] Shtatland, E.S. (1965), “On local properties of processes with independent increments”, *Teoriga Veroyatnosteüi i ee Primeneniya* (*Theory of Probability and its Applications*), **10**, 317–322 (in Russian).

[50] Vigon, V. (2002), “Votre Lévy rampe-t-il? (Does your Leṽy process creep?)”, *Journal of the London Mathematical Society, 2*, **65**(1), (2), 243–256 (in French).

2
Simulation Methods with Lévy Processes

Nick Webber
University of Warwick, Coventry, UK

Abstract

Lévy processes are increasingly important tools for modelling asset return's processes and interest rates. Although when Lévy processes are used, direct integration methods are sometimes available to price standard European options, other numerical techniques must generally be employed to price instruments whose pay-offs are either path-dependent or American.

This article discusses recent progress made in developing simulation methods suitable for some of the most widely used Lévy processes. Bridge algorithms are given for the VG and NIG processes and these algorithms are applied to valuing average rate options.

We consider the valuation of barrier options. It is shown how simulation bias can be reduced in this case. Once bias is absent, speed-up methods can be applied. Results are presented illustrating the bias reduction achieved for up-and-out and up-and-in barrier options.

2.1 INTRODUCTION

An important objective in financial mathematics is to find models of asset returns, interest rates and other financial processes in order to value and hedge derivative securities, and for VaR and risk management purposes. Once a model has been found it is hoped that the relevant values, hedge ratios, reserves and risk factors can be computed.

The standard Black–Scholes assumption, used in many applications and not just for simple options pricing, is that asset returns are normally distributed, and that joint returns distributions are normal.

Alas, this assumption is false. Marginal distributions are not normal and joint distributions are not jointly normal (so that, in particular, the joint distribution does not have a Gaussian copula). The side-effects of assuming joint normality are unfortunately not ignorable. Market option implied volatilities have distinct smiles and historical returns distributions are fat tailed and skewed – unlike those predicted by a joint normality assumption.

To overcome this problem, a number of different approaches are possible, some of which are discussed below. This article focuses on modelling univariate returns as Lévy processes and the application of simulation methods for option valuation. We are particularly concerned with finding effective numerical methods that can be used in practice to facilitate the calculation of option values when asset returns processes are Lévy.

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

The next section briefly discusses modelling returns as Lévy processes. In Section 2.3, we discuss an approach towards finding numerical methods based on the subordinator decomposition of a Lévy process. Sections 2.4 and 2.5 present theory and results related to fast simulation methods, while Section 2.6 applies the methods, together with bias reduction methods, to continuously reset barrier options. The final section provides a summary of our conclusions.

2.2 MODELLING PRICE AND RATE MOVEMENTS

Write S_t for the value at time $t \geq 0$ of an asset value, conditional upon S_0. Different approaches to modelling asset price movements are possible, a number of which have been explored in the literature.

1. A standard modelling technique in mathematical finance is to model $S = (S_t)_{t \geq 0}$ by specifying the SDE it satisfies. For example, by specifying SDEs for the asset process and for a stochastic volatility (for instance, Heston (1993) [15]).
2. By specifying the conditional distributions of S. For instance, by
   (a) giving the conditional distributions $F(S_t \mid S_0)$ themselves, or by
   (b) giving the densities $f(S_t \mid S_0)$, if they exist (for instance option pricing by log-normal mixtures (see Brigo and Mercurio (2001) [6])).
   (c) Giving the inverse distribution functions, $F^{-1}(S_t \mid S_0)$ (for instance, Corrado (2001) [9]).
3. As a time-changed Brownian motion (for instance, Geman *et al.* (2001) [13]).
4. As a Lévy process determined by its Lévy triple, (a, σ, υ). The process might have a Lévy density $k(x)$, where $\upsilon(\mathrm{d}x) = k(x)\,\mathrm{d}x$. (In practical applications one might approximate the Lévy process as a compound Poisson process.) Some of the many contributions here are cited in the next section.
5. By its time copula (see Bouyé *et al.* (2000) [4]).

Which ever way one models S, how might one calibrate to prices? It may be possible to use an implied pricing method to fit exactly to an implied volatility surface. More usually, a functional form is specified, either explicitly or implicitly, and parameter values in the functional form chosen to fit as closely as possible, by some criterion, to prices.

We choose here to assume that a Lévy Process is given, whose triple (a, σ, υ) is parameterized, and whose parameter values can be chosen to match to observed prices.

2.2.1 Modelling with Lévy processes

Consider an asset price process $S = (S_t)_{t \geq 0}$. Under the pricing measure with respect to the accumulator account numeraire suppose that

$$S_t = S_0 \exp(rt + L_t - \omega t) \tag{2.1}$$

where $L = (L_t)_{t \geq 0}$ is a Lévy process, r is the constant short rate and $\omega = \ln\left(\mathbb{E}\left[\exp(L_1)\right]\right)$ compensates for the drift in L, so that $\left(S_t \mathrm{e}^{-rt}\right)_{t \geq 0}$ is a martingale. Equation (2.1) is a standard assumption in the literature (Madan *et al.* (1998) [19], Barndorff-Nielsen and Shephard (2000) [2], Eberlein and Keller (1995) [11], etc).

Much work has been carried out, for various choices of L, to price options and to calibrate to empirical distributions. Important contributions include Carr *et al.* (2001) [7], Eberlein (2000) [10], Barndorff-Nielsen (2000) [2] and Rydberg (1999) [25].

There are also a number of interest rate models powered by Lévy processes in the literature (for instance, Eberlein and Raible (1999) [12]). One could consider a model for the short rate $r = (r_t)_{t\geq 0}$ in which under the pricing measure

$$\mathrm{d}r_t = \alpha\,(\mu\,(t) - r_t)\,\mathrm{d}t + \sigma \mathrm{d}L_t, \tag{2.2}$$

but we do not pursue interest rate models any further here.

The problem with equations (2.1) and (2.2) is that, in general, it is hard to price non-vanilla derivatives in models incorporating these processes. For instance, how would one value path-dependent options or Bermudan or American puts? Even for vanilla European options, finding a price may require a difficult numerical integration of a density function approaching the pathological.

In a 'Black–Scholes world', where L is a multiple of a Wiener process, even if analytical solutions are unavailable, PDE, Monte Carlo integration and lattice methods can generally be found that provide adequate numerical solutions to many pricing problems. Lattice methods are cheap, flexible and accurate, and can price American and Bermudan options. Monte Carlo methods can often be a long step and very good speed-up methods are available, so that path-dependent options are often quick and easy to price.

In a Lévy world, the behaviour of the Lévy measure over short time horizons can cause immense practical problems. Analytic solutions may involve difficult numerical integration (Madan *et al.* (1998) [19]). Fourier transform methods can work well for European options as long as the time horizon is not too short (Carr and Madan (1999) [8]). Monte Carlo methods can be used, either directly on the asset price process (equation (2.2) or indirectly through a representation of L as a subordinated Brownian motion (Rydberg (1997) [24]), a mean-variance mixture, or though an approximation as a compound Poisson process. However, sampling over a small time step is very hard if a Lévy density becomes unbounded near zero. PDE methods, such as the method of lines, can work for certain processes, but for a general process can prove very hard to use. Lattice methods need very high order branching and again a time step that is not too small.

The problem is that on the one hand there are too many small jumps, and on the other there are too many big jumps.

2.2.2 Lattice methods

Given the caveats noted above, can a lattice method work at all? The answer turns out to be yes, and hinges on one of the defining properties of Lévy processes: convergence in probability.

A process $X = (X_t)_{t\geq 0}$ converges in probability if

$$\forall \varepsilon > 0,\ \Pr\left[|X_t - X_s| > \varepsilon\right] \to 0 \text{ as } s \to t. \tag{2.3}$$

For a process which does not converge in probability, a lattice method might indeed be hard to construct. Consider the following example. Let $t \in \mathbb{R}^+$. Define $(X_t)_{t\geq 0}$ as

$$\begin{aligned} &\text{if } t \in \mathbb{Q} \text{ then} && \begin{cases} \Pr[X_t = 0] = \frac{1}{2}, \\ \Pr[X_t = 2] = \frac{1}{2}, \end{cases} \\ &\text{if } t \in \mathbb{R}\backslash\mathbb{Q} \text{ then} && \begin{cases} \Pr[X_t = 0] = \frac{1}{2}, \\ \Pr[X_t = 1] = \frac{1}{2}. \end{cases} \end{aligned} \tag{2.4}$$

This process clearly does not converge in probability. A computer only sees (essentially) rational values of t, a set of measure zero and a numerical method would not see the process taking values 1, even though this possibility occurs on a set of measure 1.

If a process does converge in probability then, by definition, as $\Delta t \to 0$, the probability of branching further away from a node at X_t than a fixed distance ΔX goes to zero, so that lattice methods cannot immediately be ruled out.

Kellezi and Webber (2003) [17] devised a lattice method and applied it to VG and NIG processes. They obtained discrete branching probabilities from several alternative representations of the Lévy processes L.

1. Directly from the density function of $L_{\Delta t}$. Essentially, this is equivalent to fitting to the characteristic function of the Lévy process.
2. From a representation as a subordinated Brownian motion, when the subordinator representation is known.
3. From the representation as a Lévy triple (a, σ, υ), via an approximation as a compound Poisson process.
4. From the moments of the process.

The last possibility can be rejected very rapidly. Even if the moments of the process are known, it is still possible to match only a finite number of them. In any case, moment matching is equivalent to fitting the characteristic function only at zero.

Kellezi and Webber (2004) [17] found it preferable to construct a lattice directly from the density function (known for the examples they give). Nevertheless, their lattice has very high order branching, is relatively slow, still runs into problems when attempting to price American options, and in any case cannot value path-dependent options.

Instead of investigating lattice methods any further, this article now turns its attention to simulation methods. Although these may not be usable with Bermudan or American options (although perhaps primal-dual methods could still work), they can value path-dependent options.

2.2.3 Simulation methods

It is often not possible to directly and accurately simulate a Lévy process; it may not be possible to sample directly from the distribution of L_t. When it is not possible to simulate directly from some distribution, either a terminal distribution in a long-step Monte Carlo method, or a distribution, exact or otherwise, representing a small time step Δt, several alternatives are possible. For instance:

1. Represent the distribution as a mean-variance mixture.

2. Express an underlying process as a time-changed Brownian motion.[1]
3. In the worst case, approximate the Lévy measure as compound Poisson.

Given two densities, $f(x \mid \alpha)$ and $g(\alpha)$, where $\alpha \in B$ and g is a density on B, their mixture distribution has the density

$$f_g(x) = \int_B f(x \mid y)\, g(y)\, \mathrm{d}y. \tag{2.5}$$

When f depends upon a single parameter α via its mean and variance, $f_g(x)$ is a mean-variance mixture.

Given such a representation, it may be possible to (i) draw from g to get a y, and then (ii) draw from $f(x \mid y)$ for x.

If a time-change representation exists, so that $X_t = w_{h(t)}$ for a Brownian motion w and a time change h (see below), then it may be possible to (i) sample from $h(t)$ to get a random time τ, and then (ii) sample from w_τ.

The time-change representation is used in the rest of this article.

2.3 A BASIS FOR A NUMERICAL APPROACH

We exploit the time-change representation of a Lévy process. Let $X = (X_t)_{t \geq 0}$ be a one-dimensional semi-martingale, and then (Monroe (1978) [20]) X is representable as a time-changed Brownian motion,

$$X_t = w_{h(t)}, \tag{2.6}$$

where w is a Brownian motion, with drift μ and volatility σ, say, and $h = (h_t)_{t \geq 0}$ is a stochastic time change. A Lévy process L is a semi-martingale and so this representation can be used. In general, h and w need not be independent. However, we assume that they are independent; this assumption is valid for all the processes we consider below.

We consider only the case when the time-change h is an increasing Lévy process. Then, X will also be a Lévy process. Since $w_t = \mu t + \sigma z_t$, for a Wiener process $(z_t)_{t \geq 0}$, we can write $L_t = \mu h_t + \sigma z_{h(t)}$.

Both the variance gamma (VG) and the normal inverse Gaussian (NIG) processes have time changes h, whose conditional distributions are taken from a set of generalized inverse Gaussian (GIG) distributions. Write $\delta_t = \delta t$. If $h_t \sim \mathrm{GIG}(\delta_t, \lambda, \gamma)$ has a GIG distribution, then the density f_t^{GIG} of h_t is

$$f_t^{\mathrm{GIG}}(h; \delta_t, \lambda, \gamma) = \left(\frac{\gamma}{\delta_t}\right)^\lambda \frac{1}{2K_\lambda(\delta_t \gamma)} h^{\lambda - 1} \exp\left(-\frac{1}{2}\left(\frac{\delta_t^2}{h} + \gamma^2 h\right)\right) \tag{2.7}$$

The set of GIG distributions is not closed under convolution.

[1] A time-change representation may yield a mean-variance representation.

The VG process. The gamma distribution is the limit of the GIG distribution as $\delta \to 0$ and $\lambda \to t/v$. If $h_t \sim \Gamma(t, tv)$ is a gamma variate, its density f_t^{Γ} conditional on $h_0 = 0$ is

$$f_t^{\Gamma}(x) = \frac{x^{\frac{t}{\nu}-1} \exp\left(-\frac{x}{v}\right)}{\nu^{\frac{t}{\nu}} \Gamma\left(\frac{t}{\nu}\right)}. \tag{2.8}$$

If L is VG we have

$$L_t \equiv L^{\mathrm{VG}}(t \mid \sigma, v, \mu) = \mu h_t + \sigma z_{h(t)} \tag{2.9}$$

where h_t has the density given by equation (2.8). For the VG process, the compensator ω in equation (2.1) is

$$\omega = -\frac{1}{\nu} \ln\left(1 - \mu\nu - \frac{1}{2}\sigma^2\nu\right). \tag{2.10}$$

The NIG process. The inverse Gaussian process is GIG with $\lambda = -\frac{1}{2}$. If $h_t \sim \mathrm{IG}(\delta_t, \gamma)$ is an inverse Gaussian process, then the density $f_t^{\mathrm{IG}}(x)$ of h_t is

$$f_t^{\mathrm{IG}}(x) = \frac{\delta_t}{\sqrt{2\pi}} x^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\frac{\gamma^2}{x}\left(x - \frac{\delta_t}{\gamma}\right)^2\right). \tag{2.11}$$

An NIG process L can be written

$$L_t \equiv L^{\mathrm{NIG}}(t \mid \mu, \theta, \delta, \gamma) = \mu t + \beta h_t + z_{h(t)} \tag{2.12}$$

where h_t has the density given by equation (2.11). For the NIG case, the compensator ω is

$$\omega = \mu + \delta\left(\gamma - \sqrt{\alpha^2 - (\beta+1)^2}\right) \tag{2.13}$$

where $\alpha^2 = \gamma^2 + \beta^2$.

2.3.1 The subordinator approach to simulation

Suppose we have a European derivative, with payoff H_T at time T and value c_t at times $t \leq T$, so that $c_T = H_T$. In the martingale valuation framework, we have

$$c_t = \mathbb{E}_t\left[\widehat{c}_T\right], \tag{2.14}$$

where $\widehat{c}_T = c_T \frac{p_t}{p_T}$ for a numeraire p_t, with expectations $\mathbb{E}_t[\bullet] \equiv \mathbb{E}[\bullet \mid \mathcal{F}_t]$ taken with respect to the martingale measure associated with p_t.

Suppose that H_T and p_t depend on a single-state variable S_t, which in turn depends upon a Lévy process L_t. In particular, suppose that, as in equation (2.1), under the pricing measure

$$S_t = S_0 \exp(rt + L_t - \omega t) \tag{2.15}$$

where r is a constant short rate.

Now, suppose that L has a subordinator representation $L_t = w_{h(t)}$. Since w_t and $h(t)$ are independent, the filtration $\mathcal{F}_t$ decomposes as $\mathcal{F}_t = \mathcal{F}_t^w \times \mathcal{F}_t^h$ and we can iterate the expectation:

$$c_t = \mathbb{E}\left[\widehat{c}_T \mid \mathcal{F}_t\right] \tag{2.16}$$

$$= \mathbb{E}\left[\widehat{c}_T \mid \mathcal{F}_t^w \times \mathcal{F}_t^h\right] \tag{2.17}$$

$$= \mathbb{E}\left[\mathbb{E}\left[\widehat{c}_T \mid \mathcal{F}_t^w \times \mathcal{F}_T^h\right] \mid \mathcal{F}_t^w \times \mathcal{F}_t^h\right]. \tag{2.18}$$

Informally, we can write

$$c_t = \mathbb{E}_t\left[\mathbb{E}\left[\widehat{c}_T \mid h\right]\right] \tag{2.19}$$

where $\mathbb{E}\left[\widehat{c}_T \mid h\right]$ represents the expected value of $\widehat{c}_T$, conditional upon knowing the path of h up to time T.

The inner and outer expectations can be simulated separately and so a possible procedure to value c_t by simulating L_t is to:

1. Simulate a path $\{h_{t_i}\}_{i=1,\dots,N}$ of h up to time T.
2. Given $\{h_{t_i}\}$, generate a path for w at times $\{h_{t_i}\}$ and set $L_{t_i} = w_{h(t_i)}$.
3. Compute $\widehat{c}_T$ from the path of L.
4. Repeat 10 000 times (say) and average.

This procedure can be used to value not just vanilla European options but also options that payoff at a directly determined stopping time, such as barrier options.[2]

2.3.2 Applying the subordinator approach

There are two expectations from equation (2.19). Each could, in principle, be computed by using either a lattice valuation method or a Monte Carlo valuation method.[3] Each possible pairing of valuation methods can be assessed for its appropriateness or inappropriateness for valuing path-dependent options (P or NP), for valuing options such as American or Bermudan type which may be exercised early (A or NA), and for its ease of calibration (C or NC). We obtain Table 2.1.

Using a Monte Carlo method for the inner expectation and a lattice method for the outer expectation results in the random lattice method (Kuan and Webber (2003) [18]).

Table 2.1 Valuation methods

Method		Outer	
		MC	Lattice
Inner	MC	P, NA, NC	NP, A, NC
	Lattice	NP, ∼A, C	NP, A, C

[2] However, not options whose stopping times are determined by optimality conditions, such as American options.
[3] It is not immediately clear how a PDE method might be used.

Rydberg (1997) [24] effectively used a Monte Carlo method for both the inner and outer expectation to simulate the NIG process. The procedure was later developed by Ribeiro and Webber (2002, 2003a, b) [21]–[23] who showed how to apply effective speed-up methods and bias-reduction methods. We expound their approach in the remainder of this article.

2.4 CONSTRUCTING BRIDGES FOR LÉVY PROCESSES

Plain Monte Carlo methods are very slow to achieve acceptable accuracy.[4] Various speed-up methods need to be employed to produce reasonable computation times. One efficient method is path regularization by stratified sampling. This ensures that sample points form a less clustered draw from the sample space Ω than a plain Monte Carlo method would produce. If an entire sample path is needed, rather than just a draw from the terminal time, then stratified sampling has to be used in conjunction with a bridge method.

2.4.1 Stratified sampling and bridge methods

We discuss here the background to stratified sampling and bridge methods. Good reviews can be found in Jäckel (2002) [16] and Glasserman (2003) [14].

Stratified sampling. It is easy to generate a stratified sample from the unit interval [0, 1]. Let $v_i \sim U[0,1]$, $i = 1, \ldots, Q$, be a sample of size Q from the uniform distribution $U[0,1]$, and then $u_i = \frac{i+v_i-1}{Q}$ is a stratified sample of size Q from $U[0,1]$. The set $\{u_i\}$ is guaranteed to have minimal clustering above the scale $1/Q$.

Stratified samples can be produced from other distributions by inverse transform. Suppose that $X \sim F_X$ is a random variate with distribution function F_X and that F_X^{-1} is computable. Let $u \sim U[0,1]$ be a uniform variate, and then $F_X^{-1}(u) \sim F_X$ has distribution F_X.

Given a stratified sample u_i, $i = 1, \ldots, Q$ from $U[0,1]$, the set $F_X^{-1}(u_i)$, $i = 1, \ldots, Q$, is a stratified sample from F_X.

Bridge sampling. Given a Lévy process L, where L_t has distribution F_t at time t (conditional on L_0), suppose that we have found a sample, $L_{i,N}$, $i = 1, \ldots, Q$, of L_{t_N} from F_{t_N}, possibly stratified. Given a value for L_0 at time $0 = t_0$, we would like to construct an entire sample path $L_0 = L_{i,0}, \ldots, L_{i,N}$ with the correct conditional distributions. This means being able to sample $L_{i,j}$, at time $0 < t_j < t_N$, conditional upon the values of $L_{i,0}$ and $L_{i,N}$.

In general, suppose that $X \sim F_X$, $Y \sim F_Y$ and $Z \sim F_Z$, with densities f_X, f_Y and f_Z, respectively, are random variates such that $Z = Y + X$. For instance, X, Y and Z could be increments in a Lévy process L.

Given a draw z of Z, we want to sample from the conditional distribution $X \mid Z$. Write $f_{X,Y}(x, y)$ for the joint density of X and Y. Then

$$f_{X|Z}(x) = \frac{f_{X,Y}(x, z-x)}{f_Z(z)}, \tag{2.20}$$

$$= \frac{f_X(x)\, f_Y(z-x)}{f_Z(z)} \quad \text{when } X, Y \text{ are independent.} \tag{2.21}$$

[4] Measured by the square root of the second moment of the Monte Carlo estimate. An internally generated estimate of this is the Monte Carlo standard error.

For a Lévy process L, take $X \sim F_{t_i}$, $Y \sim F_{t_j}$, $Z \sim F_{t_i+t_j}$, say, and then $f_{t_i|t_i+t_j}(x) \equiv f_{X|Z}(x)$ is the bridge density of L.

If the densities f_t are known, then the bridge density of L can be computed and perhaps a sampling method can be found.

2.4.2 Bridge sampling and the subordinator representation

When $L_t = w_{h(t)}$ has a subordinator representation we could construct a stratified bridge sample of L by first obtaining a stratified bridge sample of h, and then constructing a stratified bridge sample of w at the times given by our path for h. The bridge distribution of a Brownian motion is well known and it is easy to sample from it. We need only to know the bridge distribution of the subordinator h, and to be able to sample from it.

Ribeiro and Webber (2002, 2003*b*) [21], [23] find the bridge density and a sampling method for the bridge density, for both the gamma and the inverse Gaussian processes. We summarize their results here.

Let $0 = h_0 < \cdots < h_N$ be a series of values at increasing times for the subordinator process h. Given h_i and h_k at times $t_i < t_k$, we want to sample h_j at an intermediate time $t_i < t_j < t_k$.

Write $z = h_k - h_i$, $x = h_j - h_i$, and $y = h_k - h_j$, and set $\tau_z = t_k - t_i$, $\tau_x = t_j - t_i$, and $\tau_y = t_k - t_j$.

The gamma process. Given h_i and h_k, the bridge density of x from a gamma process with parameter ν is

$$f_{X|Z}(x) = \frac{1}{z}\frac{\Gamma\left(\frac{\tau_x}{\nu}+\frac{\tau_y}{\nu}\right)}{\Gamma\left(\frac{\tau_x}{\nu}\right)\Gamma\left(\frac{\tau_y}{\nu}\right)}\left(\frac{x}{z}\right)^{\frac{\tau_x}{\nu}-1}\left(1-\frac{x}{z}\right)^{\frac{\tau_y}{\nu}-1}. \tag{2.22}$$

An algorithm to sample from this density is

1. Generate $b_j \sim \mathrm{B}(\tau_x/\nu, \tau_y/\nu)$, where B is the beta distribution.
2. Set $h_j = h_i + b_j\,(h_k - h_i)$.

Existing algorithms to sample (by inverse transform) from the beta distribution are relatively slow. Even so, the numerical results in Section 2.5 below demonstrate the very great speed-ups possible with this sampling method.

The inverse Gaussian process. The bridge density of x from an inverse Gaussian process with parameter δ is

$$f_{X|Z}(x) = \frac{\delta}{\sqrt{2\pi}}\frac{\tau_x\tau_y}{\tau_z}\left(\frac{xy}{z}\right)^{-\frac{3}{2}}\exp\left(-\frac{1}{2}\delta^2\left(\frac{\tau_x^2}{x}+\frac{\tau_y^2}{y}-\frac{\tau_z^2}{z}\right)\right), \tag{2.23}$$

where $y = z - x$.

An algorithm to sample from this density is

1. Generate $q \sim \chi_1^2$.

2. Set $\lambda = \frac{\delta^2 \tau_x^2}{x}$ and $\mu = \tau_y/\tau_x$, and compute roots s_1 and s_2,

$$s_1 = \mu + \frac{\mu^2 q}{2\lambda} - \frac{\mu}{2\lambda}\sqrt{4\mu\lambda q + \mu^2 q^2}, \tag{2.24}$$

$$s_2 = \frac{\mu^2}{s_1}. \tag{2.25}$$

3. Set $s = s_1$ with probability p,

$$p = \frac{\mu(1+s_1)}{(1+\mu)(\mu+s_1)}, \tag{2.26}$$

   or else set $s = s_2$.
4. Finally, set $h_j = h_i + \frac{1}{1+s}(h_k - h_i)$.

Since sampling from a χ_1^2 distribution, by inverse transform or otherwise, is very fast, sampling from the inverse Gaussian distribution is also very quick.

Using the bridge. Given a method for sampling from the subordinator process h_t, we adopt the following algorithm.

1. Given $h_0 = 0$, construct a stratified sample $h_{i,N}$ of h_N.
2. Using the bridge distribution, construct $h_{i,N/2}$ conditional on h_0 and $h_{i,N}$.
3. Using binary chop, continue to generate a path $h_{i,j}$ for times t_j, $j = 0, \ldots, N$.
4. Generate $w_{h_{i,j}}$, and hence $S_{i,j}$ conditional on $h_{i,j}$.

At each intermediate time it is possible to continue to stratify the sample path. Generating a sample path requires a sequence of uniform variates from which samples with the desired distributions are obtained (by inverse transform). If n uniform variates are needed, this is equivalent to a draw from a unit hypercube of dimension n. To stratify $m \leq n$ of these draws, one performs a stratified sample on an m-dimensional hypercube, selecting the remaining $n - m$ draws without stratification. In practice, it is only possible to make a fully stratified draw from a hypercube of dimension 3 or so. To make a stratified draw from a hypercube of dimension $m > 3$, low discrepancy sampling is often used.

For the VG process it takes one uniform variate to generate (by inverse transform) a draw from the gamma bridge distribution for h_i and one more for each normal variate w_{h_i}; for the NIG process, it takes two uniform variates to draw for each h_i and one more for w_{h_i}. So, the VG process requires two uniform variates at each time step, and the NIG process requires three.

Even if sampling with low discrepancy sequences it may only be possible in practice to sample reliably from a unit hypercube of dimension at most a few dozen. A freely available downloadable code for generating Sobol sequence numbers[5] goes up to dimension 39. This means that (with binary chop) for the VG process it is possible to stratify at up to 16 times and for the NIG process one can stratify at up to 8 times. Draws for other times have to be made with ordinary non-stratified sampling. Even with this restriction, very good speed-ups are possible.

[5] See Bratley and Bennett (1988) [5]. Code is downloadable from www.netlib.org/toms/659.

2.5 VALUING DISCRETELY RESET PATH-DEPENDENT OPTIONS

A discretely reset path-dependent option is one in which the option payoff is computed from observations of the underlying asset value at certain discrete times (the reset times). In this section, we value discretely reset path-dependent options when returns to the underlying asset are either VG or NIG processes. We present convergence results (taken from Ribeiro and Webber (2002, 2003*b*) [21] [23]) and show how true standard errors are improved relative to plain Monte Carlo.

When a Monte Carlo method is used with stratified sampling, successive Monte Carlo estimates are correlated with one another. This means that the internally generated standard error measure is not a true reflection of the standard deviation of the Monte Carlo estimate. In the tables below, the true standard deviation is estimated by taking the sample standard deviation of the Monte Carlo estimates obtained from 100 replications of the Monte Carlo procedure.

To compare two Monte Carlo methods, we use the efficiency gain of one method over the other. Suppose we have two Monte Carlo procedures. Monte Carlo method i, $i = 1, 2$, gives an estimate with standard deviation σ_i in a time t_i. If the time taken t is proportional to the number of sample paths Q, and if σ is $O\left(-\frac{1}{2}\right)$ in Q, then the efficiency gain $E_{1,2}$, defined as

$$E_{1,2} = \frac{t_2\sigma_2^2}{t_1\sigma_1^2}, \tag{2.27}$$

is how many times faster method 1 is to achieve the same standard deviation as method 2.

In the following tables, K is the number of stratification times. $K = 0$ is plain Monte Carlo with $Q = 10^6$ paths. Bridge Monte Carlo uses $Q = 10^4$ paths. The benchmark is full low discrepancy with $Q = 10^6$.

The initial asset value is $S_0 = 100$, the short rate is $r = 0.1$ and the option strike is $X = 100$ with maturity time $T = 1$. The VG case uses parameter values $\sigma = 0.12136$, $v = 0.3$ and $\mu = -0.1436$ (based upon Madan *et al.* (1998) [19]). The NIG case uses parameters $\alpha = 75.49$, $\beta = -4.089$, $\delta = 3$ and $\mu = 0$ (based upon Rydberg (1997) [24]). Programmes were written in VBA 6.0 run on a 900 Mhz PC.

Tables 2.2 and 2.3 give values, standard deviations and computation times (in seconds) for a discretely reset average rate option when the underlying asset has either VG or NIG returns process. More illuminating are Tables 2.4 and 2.5, which give the efficiency gains in each case.

In the VG case, speed-ups of up to a factor of about 400 are possible and up to about 200 for the NIG case. Speed-ups tend to improve as both the number of reset dates and the number of stratification times increase, although this is not so evident in this example for the NIG case.

Here, we have given only a few of the results of Ribeiro and Webber. They investigate many more cases, including discrete barrier and lookback options, demonstrating in each case that worthwhile speed-ups are attainable.

Further speed-ups would seem to be possible. In both the VG and NIG cases, the number of stratification dates was constrained by the dimension of the low discrepancy sequence generated by the available software. There have also been criticisms of the quality of this generator (Jäckel (2002) [16]). With a better quality generator, capable of producing low discrepancy sequences of higher dimension, increased speed-ups would be possible.

Table 2.2 Values, standard deviations and computation times for average rate call options: comparison of plain and bridge Monte Carlo methods for the VG case (Ribeiro and Webber (2003b) [23])

K	4 resets	8 resets	16 resets	32 resets	64 resets	256 resets
0	6.7720 (0.0064) [85.6]	6.0666 (0.0058) [175.0]	5.7274 (0.0055) [335.4]	5.5497 (0.0053) [647.5]	5.4625 (0.0052) [1277]	5.4075 (0.0052) [5034]
1	6.7993 (0.029) [2.4]	6.0290 (0.025) [3.5]	5.7234 (0.023) [5.5]	5.5242 (0.025) [9.3]	5.4830 (0.029) [17.0]	5.3874 (0.024) [52.2]
2	6.7635 (0.011) [4.0]	6.0741 (0.012) [5.0]	5.7187 (0.014) [6.9]	5.5208 (0.014) [10.8]	5.4761 (0.013) [18.5]	5.376 (0.013) [54.9]
4	6.7594 (—) [13.0]	6.0656 (0.0065) [14.0]	5.7149 (0.0067) [16.0]	5.5510 (0.0072) [19.9]	5.4765 (0.0063) [27.6]	5.3934 (0.0064) [62.9]
8	—	6.0711 (—) [33.8]	5.7283 (0.0029) [35.8]	5.5465 (0.0033) [39.6]	5.4667 (0.0035) [47.4]	5.3997 (0.0035) [83.0]
16	—	—	5.7245 (—) [77.3]	5.5527 (0.0014) [80.7]	5.4627 (0.0017) [88.1]	5.4008 (0.0017) [123.1]
Benchmark	6.7626 (—) [1297]	6.0702 (—) [3370]	5.7250 (—) [7638]	—	—	—

Reproduced by permission of L.D. Donaldson.

In the VG case, the stratification algorithm for the gamma bridge distribution is constrained by the method used to generate stratified beta variates by inverse transform.[6] An improved algorithm for the inverse transform of the beta distribution would result in even greater speed-ups in this case.

2.6 VALUING CONTINUOUSLY RESET PATH-DEPENDENT OPTIONS

We would like to obtain for continuously reset barrier and lookback options the speed-ups that were found for discretely reset barrier and lookback options by Ribeiro and Webber (2002, 2003*b*) [21][23]. These same authors (2003a) [22] find an approximate method to achieve this result.

Particular problems arise when applying numerical methods to continuous barrier options. It turns out that the values of discretely reset barrier options converge only very slowly to values of corresponding continuously reset barrier options as the number of reset dates increases. Since numerical methods are set in discrete time, the values they find for continuously reset barrier options may converge only very slowly to the true value.

[6] The algorithm uses a root searching method.

Table 2.3 Values, standard deviations and computation times for average rate call options: comparison of plain and bridge Monte Carlo methods for the NIG case (Ribeiro and Webber (2002) [21])

K	4 resets	8 resets	16 resets	32 resets	64 resets	256 resets
0, $Q = 10^6$	8.5856 (0.0103) [71.6]	7.7892 (0.0094) [141.4]	7.4059 (0.0089) [278.6]	7.2312 (0.0087) [551.8]	7.1286 (0.0086) [1101.4]	7.0698 (0.0086) [4390.1]
1	8.6326 (0.044) [1.9]	7.8205 (0.042) [2.4]	7.3874 (0.048) [4.9]	7.2347 (0.041) [7.7]	7.0763 (0.047) [15.1]	7.0656 (0.044) [58.8]
2	8.5530 (0.021) [1.8]	7.7695 (0.022) [2.1]	7.4282 (0.022) [4.8]	7.2457 (0.026) [7.7]	7.0721 (0.026) [15.0]	7.0497 (0.021) [58.8]
4	8.5695 (—) [1.8]	7.7963 (0.010) [2.0]	7.4181 (0.011) [4.7]	7.2105 (0.011) [7.6]	7.1249 (0.011) [14.9]	7.0430 (0.012) [58.6]
8	—	7.7959 (—) [1.7]	7.4045 (0.0048) [4.3]	7.2121 (0.0059) [7.3]	7.1296 (0.0051) [14.6]	7.0519 (0.0059) [58.4]
Benchmark	8.5807 (—) [169]	7.8072 (—) [169]	—	—	—	—

Table 2.4 Efficiency gains for average rate call options: comparison of plain and bridge Monte Carlo methods for the VG case (Ribeiro and Webber (2003b) [23])

K	4 resets	8 resets	16 resets	32 resets	64 resets	256 resets
1	1.7	2.7	3.5	3.1	2.4	4.5
2	7.2	8.2	7.5	8.6	11	15
4	—	10	14	18	32	53
8	—	—	34	42	60	134
16	—	—	—	115	136	383

Reproduced by permission of L.D. Donaldson.

Table 2.5 Efficiency gains for average rate call options: comparison of plain and bridge Monte Carlo methods for the NIG case (Ribeiro and Webber (2002) [21])

K	4 resets	8 resets	16 resets	32 resets	64 resets	256 resets
1	2.1	3.0	2.0	3.2	2.4	2.9
2	9.8	12.4	9.9	8.2	8.2	12.2
4	—	58.8	42.3	42.8	45.7	40.1
8	—	—	219.4	167.1	216.2	157.0

For a Monte Carlo method, this feature is called 'simulation bias'. A discrete sample path for an underlying asset may not exceed a barrier level, but on the continuous path sampled by the discrete path, the barrier may have been hit in between times observed by the discrete sample path.

For lookback options, the maximum (minimum) found along a discrete path will always be less than (more than) the true maximum (minimum) achieved along the continuous path.

When the underlying asset has a geometric Brownian motion, simulation bias correction methods are available (Beaglehole *et al.* (1997) [3] and El Babsiri and Noel (1998) [1]). In this section, following Ribeiro and Webber, we show how these ideas can be extended to asset processes driven by Lévy processes. Only when bias has been removed, or at least significantly reduced, does it makes sense to apply speed-up methods such as the bridge methods discussed in previous sections.

2.6.1 Options on extreme values and simulation bias

Consider a continuously reset option maturing at time T, and let B be a barrier level. Given a (continuous) sample path $\{S_t\}_{t\in[0,T]}$ for at asset value, set

$$M_{0,T} = \max_{t\in[0,T]} \{S_t\} \tag{2.28}$$

$$m_{0,T} = \min_{t\in[0,T]} \{S_t\}, \tag{2.29}$$

An up-and-in barrier option with barrier level B has payoff $H_T(\omega)\,\mathrm{I}_{\{M_{0,T}\geq B\}}$, where $H_T(\omega)$ is the payoff at time T of the knocked-in option. Similarly, an up-and-out barrier option has payoff $H_T(\omega)\,\mathrm{I}_{\{M_{0,T}<B\}}$, where now $H_T(\omega)$ is the payoff to the option at time T if it is not knocked out.

A fixed strike lookback option has payoff $\left(M_{0,T} - X\right)_+$ and a floating strike lookback option has payoff $\left(S_T - m_{0,T}\right)_+$.

An ordinary Monte Carlo method, without bias correction would value a floating strike lookback as follows: Discretize time as $0 = t_0 < t_1 < \cdots < t_N = T$. Suppose the initial asset value at time t_0 is S_0.

1. For $j = 1, \ldots, Q$, generate the jth sample path $\{S_i^j\}_{i=0,\ldots,N}$, with $S_0^j = S_0$.
2. For each j set $m_j = \min_i\{S_i^j\}$ and $H_j = (S_N^j - m_j)_+$.
3. Having generated H_j, $j = 1, \ldots, Q$, set $\widehat{c}_0 = \mathrm{e}^{-rT}\frac{1}{Q}\sum_{j=1,\ldots,Q} H_j$.

where $\widehat{c}_0$ is the (ordinary) Monte Carlo estimate of the option value at time t_0. This is a very poor estimate of the true option value. For a floating strike lookback, $\widehat{c}_0$ will significantly underestimate the true option value. The problem here is that the extreme value will lie in between observations $\{S_i^j\}_{i=0,\ldots,N}$. The effect is very slow convergence, with very high bias.

The way to overcome the problem is to sample for $M_{0,T}$ and $m_{0,T}$ from their conditional distributions.

Suppose that the conditional extremes distribution $F_{0,T}^m(u) = \Pr\left[m_{0,T} \leq u \mid S_0, S_T\right]$ is (i) known and (ii) workable with, so that one can sample from it. Then, Beaglehole *et al.* (1997) [3] and El Babsiri and Noel (1998) [1] proposed the following corrected algorithm:

1. For $j = 1, \ldots, Q$, generate jth sample path $\{S_i^j\}_{i=0,\ldots,N}$, with $S_0^j = S_0$.
2. Now generate $m_{i,i+1}^j \mid S_i^j, S_{i+1}^j$ from the distribution $F_{t_i,t_{i+1}}^m(u)$ of $m_{t_i,t_{i+1}}$ and set $m_j = \min_i \left\{m_{i,i+1}^j\right\}$.
3. Set $H_j = \left(S_N^j - m_j\right)_+$ as before.
4. Finally set $\widehat{c_0} = \mathrm{e}^{-rT}\frac{1}{Q}\sum_{j=1,\ldots,Q} H_j$.

where $\widehat{c_0}$ is now an unbiased estimate of the true option value.

2.6.2 Bias correction for Lévy processes

$F_{0,T}^m(u)$ is known when $(S_t)_{t\geq 0}$ has geometric Brownian motion. In our case, S_t has geometric Lévy motion and $F_{0,T}^m(u)$ will not usually be known. However, we can find an approximation procedure that gives very good results.

Write $\Pr\left[\tau^X < t \mid X_0, X_T, B_t\right]$ for the probability that the hitting time τ^X of a process X to a barrier level B_t, conditional on its values X_0 and X_T at times 0 and T, is less than t.

Let $R = (R_t)_{t\geq 0}$ with $R_t = \ln(S_t/S_0)$, and set $\underline{B} = \ln(B/S_0)$. Let $M_{0,T}^S$, etc. denote the minimum of a process S in the interval $[0, T]$. Then, we have, for example, for an up-and-in barrier option,

$$\Pr\left[M_{0,T}^S \geq B \mid S_0, S_T\right] = \Pr\left[\tau^S \leq T \mid S_0, S_T, B\right] \tag{2.30}$$

$$= \Pr\left[\tau^R \leq T \mid R_0, R_T, \underline{B}\right] \tag{2.31}$$

$$= \Pr\left[\tau^{R-(r-\omega)t} \leq T \mid R_0, R_T - (r-\omega)T, \underline{B} - (r-\omega t)t\right] \tag{2.32}$$

$$= \Pr\left[\tau^L \leq T \mid L_0, L_T, \underline{B} - (r-\omega)t\right] \tag{2.33}$$

where $L_t = w(h_t)$ (we suppose w_t has volatility σ).

For a geometric Lévy motion, the bridge hitting time distribution is equivalent to a bridge hitting time distribution of a Lévy process to a non-constant barrier. In two special cases, it is possible to get rid of the non-constant term.

First, when L_t is a Brownian motion, so that $h_t = t$, the $(r-\omega)t$ term can be absorbed into the drift of w_t and S_t is just a geometric Brownian motion. In this case, $F_{0,T}^m(u)$ is known,

$$F_{0,T}^m(u) = \Pr\left[M_{0,T}^S \leq B \mid S_0, S_T\right] \tag{2.34}$$

$$= \Pr\left[M_{0,T}^R \leq \underline{B} \mid R_0, R_T\right] \tag{2.35}$$

$$= 1 - \exp\left(-\frac{2}{\sigma^2 T}\left(\underline{B} - R_0\right)\left(\underline{B} - R_T\right)\right). \tag{2.36}$$

It is straightforward to sample from this distribution. Let $u \sim U[0,1]$ be a uniform variate, then

$$\tilde{M} = \frac{1}{2}\left(R_0 + R_T + \sqrt{(R_0 - R_T)^2 - 2\sigma^2 T\ln(u)}\right) \tag{2.37}$$

is a draw from $M_{0,T}^R$.

Secondly, suppose $r = \omega$, and so $(r - \omega)\,t = 0$. If h_t is deterministic, then, immediately,

$$\Pr\left[M_{0,T}^S \le B \mid S_0, S_T\right] = \Pr\left[M_{0,T}^R \le \underline{B} \mid R_0, R_T\right] \tag{2.38}$$

$$= \Pr\left[M_{0,T}^L \le \underline{B} \mid L_0, L_T\right] \tag{2.39}$$

$$= \Pr\left[M_{h(0),h(T)}^w \le \underline{B} \mid w_{h(0)}, w_{h(T)}\right] \tag{2.40}$$

$$= 1 - \exp\left(-2\frac{\left(\underline{B} - w_{h(0)}\right)\left(\underline{B} - w_{h(T)}\right)}{\sigma^2\left(h\left(T\right) - h\left(0\right)\right)}\right). \tag{2.41}$$

If h_t is stochastic, then we can apply iterated expectation:

1. Simulate a path for h_t.
2. Conditional on this, sample for $M_{0,T}^S$.
3. At step (2), h_t is deterministic and so it is possible to use the result from the first case above.

Note that, in fact, it is unnecessary to sample an entire path of h_t, and so just sample h_T at the terminal time T.

In the general case when $r \neq \omega$, there is a non-constant barrier and it is necessary to make an approximation. This approximation is good as the time step goes to zero.

Note that $\widehat{B}_t = \underline{B} - (r - \omega)\,t$ is a linear barrier. As $T \to 0$, the maximum deviation $(r - \omega)\,T$ of $\widehat{B}_t$ from $\underline{B}$ goes to zero, and the proportional deviation $\frac{(r-\omega)T}{\underline{B}-L_0}$ also goes to zero. Hence, when T is small we make the approximation

$$\Pr\left[M_{0,T}^S \ge B \mid S_0, S_T\right] = \Pr\left[\tau^L \le T \mid L_0, L_T, \underline{B} - (r - \omega)\,t\right] \tag{2.42}$$

$$\sim \Pr\left[\tau^L \le T \mid L_0, L_T + (r - \omega)\,T, \underline{B}\right] \tag{2.43}$$

$$= \Pr\left[M_{0,T}^L \ge \underline{B} \mid L_0, L_T + (r - \omega)\,T\right]. \tag{2.44}$$

We have reduced the problem to the second case considered above. The algorithm becomes:

1. Discretize time between 0 and T, $0 = t_0, \ldots, t_N = T$, so that $\Delta t_i = t_{i+1} - t_i$ is sufficiently small.
2. Draw a sample path $0 = h_0, \ldots, h_N = h_T$ for h.
3. Sample $M_{t_i,t_{i+1}}^S$ over each sub-interval and apply the standard bias correction computation as in Section 2.6.1.

We see in the examples below, taken from Ribeiro and Webber (2003*a*) [22], that this procedure is very effective.

2.6.3 Variation: exceedence probabilities

The algorithm just described requires us to sample $M_{t_i,t_{i+1}}^S$ over each time step to determine if the barrier level has been breached. Sometimes, an alternative procedure is available that enables us to avoid this sampling, computing instead the probability of hitting the barrier over the entire sample path.

Write $f_{S,\tau}(S,\tau)$ for joint density of S_T and τ. Then, $f_{S,\tau}(S,\tau) = f_\tau(\tau \mid S)\, f_S(S)$, where f_S is the density of S_T and $f_\tau(\tau \mid S)$ is the conditional density of τ (on S_T).

The value c_0 of a barrier option, for instance, an up-and-in, is

$$c_0 = \mathrm{e}^{-rT} \iint H_T(S)\, \mathrm{I}_{\{\tau \le T\}} f_{S,\tau}(S,\tau)\, \mathrm{d}S\mathrm{d}\tau \tag{2.45}$$

$$= \mathrm{e}^{-rT} \int \left(\int H_T(S)\, \mathrm{I}_{\{\tau \le T\}} f_\tau(\tau \mid S)\, \mathrm{d}\tau \right) f_S(S)\, \mathrm{d}S \tag{2.46}$$

$$= \mathrm{e}^{-rT} \iint H_T(S)\, F_{0,T}^{S_0,S} f_S(S)\, \mathrm{d}S \tag{2.47}$$

where

$$F_{0,T}^{S_0,S} = \int \mathrm{I}_{\{\tau \le T\}} f_\tau(\tau \mid S)\, \mathrm{d}\tau = \Pr[\tau < T \mid S_0, S_T = S] \tag{2.48}$$

is the bridge hitting time distribution.

Now, instead of sampling from the maximum at each step, for each sample of S_T^j, compute $F_{0,T}^{S_0,S_T^j}$ and set $H_j = \left(S_T^j - X\right)_+ F_{0,T}^{S_0,S_T^j}$. If $F_{0,T}^{S_0,S_T^j}$ can only be approximated accurately for small values of T, generate a sample path $\left\{S_i^j\right\}_{i=0,..,N}$ for S and set

$$F_{0,T}^{S_0,S} = 1 - \prod_{i=0,\dots,N-1} \left(1 - F_{t_i,t_{i+1}}^{S_i^j,S_{i+1}^j}\right) \tag{2.49}$$

over the jth path.

The obvious analogous procedure can be followed for up-and-out barrier options.

Compared to the previous algorithm, this procedure has a much reduced standard error since it significantly reduces the amount of sampling required.

2.6.4 Application of the bias correction algorithm

We now quote results from Ribeiro and Webber (2003*a*) [22] showing the convergence of the bias correction algorithm when valuing up-and-out and up-and-in barrier options when the underlying asset is driven by, either a VG or an NIG process. Parameter values are the same as in Section 2.5. Ribeiro and Webber give additional results showing how the algorithm performs for various barrier and lookback options.

Figures 2.1 and 2.2 value an up-and-out barrier option in the VG and NIG cases, respectively. The barrier level is $B = 120$. These figures show convergence as the number of time steps increases of (i) the ordinary Monte Carlo methods, (ii) the sampling (random) algorithm of Section 2.6.2, and (iii) the exceedence algorithm of Section 2.6.3. The horizontal axis is denominated in computation time (in seconds), rather than the number of time steps, so that the three methods can be fairly compared.

We see that in each case the bias-corrected methods are converging to a very different level than the level reached by the ordinary method. In fact, the ordinary method is still

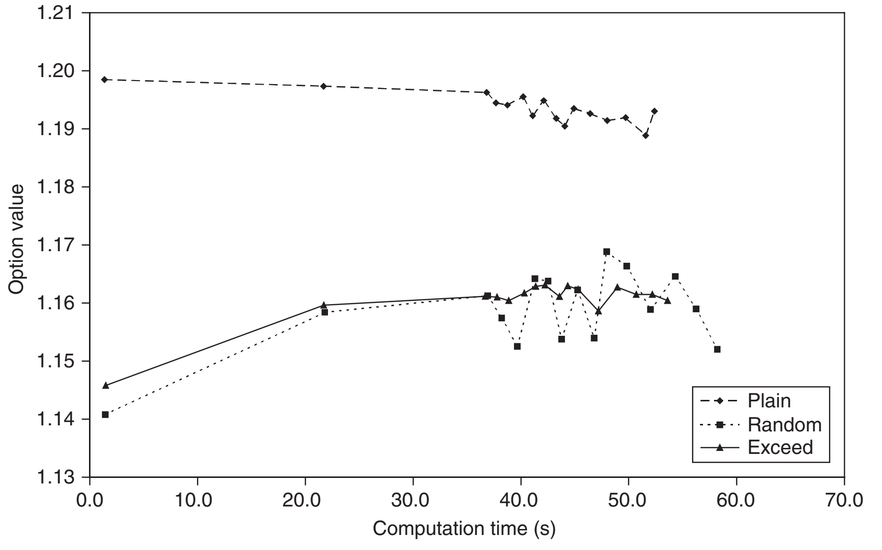

Figure 2.1 Convergence data for the up-and-out barrier option–VG case (Ribeiro and Webber (2000a) [22])

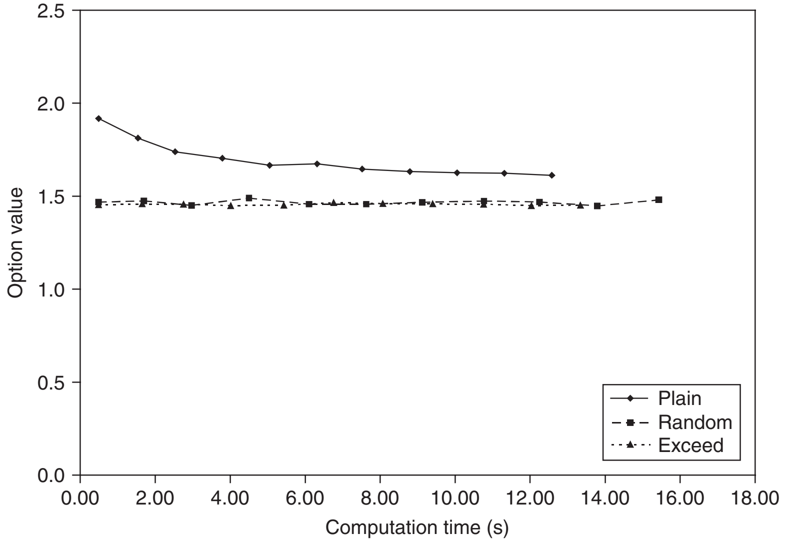

Figure 2.2 Convergence data for the up-and-out barrier option–NIG case (Ribeiro and Webber (2000a) [22])

converging and will only approach the true option value asymptotically. Note that the apparent variation in values given by the exceedence method is significantly less than that given by the sampling method.

Figures 2.3 and 2.4 show convergence data for up-and-in barrier options for the VG and NIG cases, respectively. Just like the up-and-out options, the ordinary Monte Carlo method is not achieving values anywhere near the bias-corrected values.

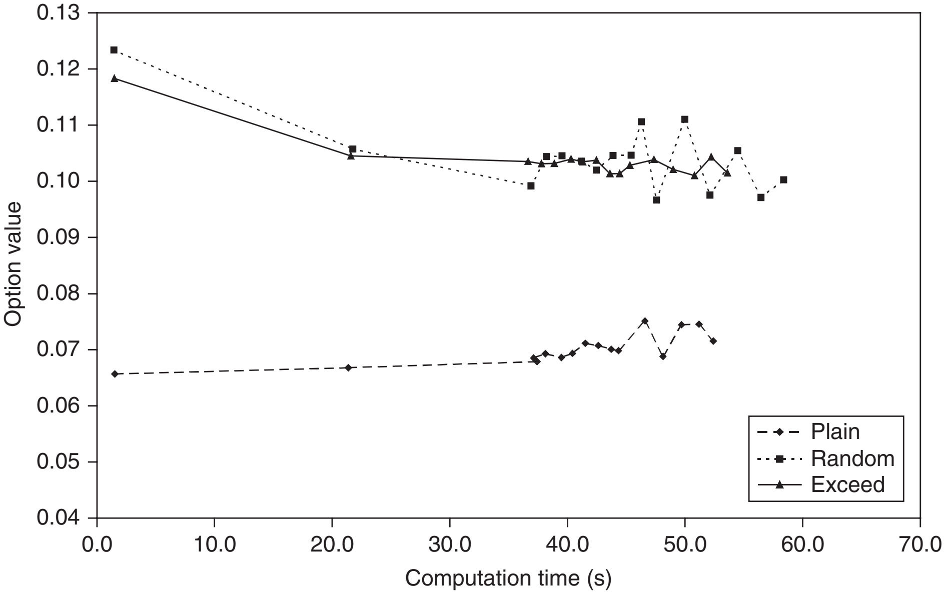

Figure 2.3 Convergence data for the up-and-in barrier option–VG case (Ribeiro and Webber (2000a) [22])

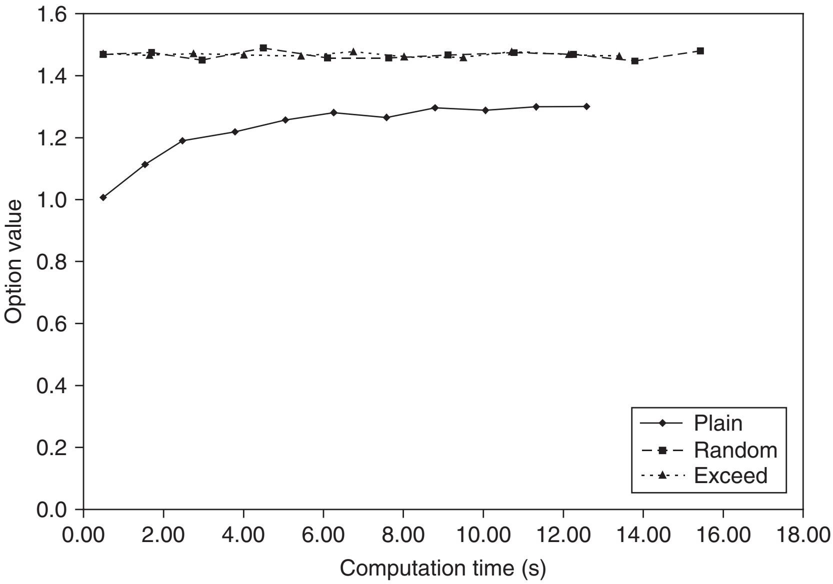

Figure 2.4 Convergence data for the up-and-in barrier option–NIG case (Ribeiro and Webber (2000a) [22])

Once bias has been removed, it now makes sense to apply the speed-ups developed in Section 2.4. Ribeiro and Webber report the result of doing this, finding very good speed-ups here as before.

2.7 CONCLUSIONS

To use Lévy processes, you need good numerics. Based on work carried out by Ribeiro and Webber, this article has looked at simulation methods for Lévy processes.

We have seen that very good speed-ups are possible when the subordinator representation is used as a starting point for a bridge distribution simulation method used in conjunction with stratified sampling. For the VG and NIG processes, the relevant bridge distributions are available and quick sampling methods have been developed.

The subordinator approach seems very fruitful, and no doubt can be developed further. This article has only discussed its use with the VG and NIG processes, but in principle there is no reason why this approach should not be applied to other classes of Lévy processes and other types of options.

REFERENCES

[1] El Babsiri, M. and Noel, G. (1998), "Simulating path dependent options: A new approach", *Journal of Derivatives*, Winter; 65–83.

[2] Barndorff-Nielsen, O.E. and Shephard, N. (2000), "Modelling by Lévy processes for financial econometrics", *Mimeo*, University of Aarhus, Denmark, pp. 1–38.

[3] Beaglehole, D.R., Dybvig, P.H. and Zhou, G. (1997), "Going to extremes: Correcting simulation bias in exotic option valuation", *Financial Analysts Journal*, January/February, 62–68.

[4] Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G. and Roncalli, T. (2000), "Copulas for finance: A reading guide and some applications", *Mimeo*, GRO, Crédit Lyonnais, Paris, France (July).

[5] Bratley, P. and Bennett, L.F. (1988), "Algorithm 659 implementing Sobol's quasirandom sequence generator", *Working Paper*, University of Montreal, Montreal, QU, Canada, pp. 88–100.

[6] Brigo, D. and Mercurio, F. (2001), *Interest Rate Models: Theory and Practice*, Springer Finance, Springer-Verlag, Berlin, Germany.

[7] Carr, P., Geman, H., Madan, D.B. and Yor, M. (2003), "Stochastic volatility for Lévy processes", *Mathematical Finance*, **13**, 345–382.

[8] Carr, P. and Madan, D.B. (1998), "Option valuation using the fast Fourier transform", *Mimeo, Journal of Computational Finance*, **2**, 61–73.

[9] Corrado, C.J. (2001), "Option pricing based on the generalized lambda distribution", *The Journal of Futures Markets*, **21**, 213–236.

[10] Eberlein, E. (2001), "Application of generalized hyperbolic Lévy motions to finance", in O.E. Barndorff-Nielsen, T. Mikosch and S. Resnich (Eds), *Lévy Processes: Theory and Applications*, Birkhäusar, Boston, MA, USA, pp. 319–336.

[11] Eberlein, E. and Keller, U. (1995), "Hyperbolic distributions in finance", *Bernoulli*, **1**, 281–299.

[12] Eberlein, E. and Raible, S. (1999), "Term structure models driven by general Lévy processes", *Mathematical Finance*, **9**, 31–35.

[13] Geman, H., Madan, D.B. and Yor, M. (2000), "Time changes hidden in Brownian subordination", *Working Paper*, University of Maryland, College Park, MD, USA.

[14] Glasserman, P. (2003), *Monte Carlo Methods in Financial Engineering*, Springer-Verlag, Berlin, Germany.

[15] Heston, S.L. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options", *Review of Financial Studies*, **6**, 327–343.

[16] Jackel, P. (2002), *Monte Carlo Methods in Finance*, Wiley Finance Series, Wiley, Chichester, UK.

[17] Kellezi, E. and Webber, N. (2004), "Valuing Bermudan options when asset returns are Lévy processes", *Quantitative Finance*, **4**, 87–100.

[18] Kuan, G. and Webber, N. (2003), "Calibrating interest rate models driven by Lévy processes with a random lattice", *Working Paper*, Cass Business School, City University, London, UK.

[19] Madan, D.B., Carr, P.P. and Chang, E.C. (1998), "The variance gamma process and option pricing", *European Finance Review*, **2**, 79–105.

[20] Monroe, I. (1978), "Processes that can be embedded in Brownian motion", *Annals of Probability*, **6**, 42–56.

[21] Ribeiro, C. and Webber, W. (2002), "A Monte Carlo method for the normal inverse Gaussian option valuation model using an inverse Gaussian bridge", *Working Paper*, Cass Business School, City University, London, UK.

[22] Ribeiro, C. and Webber, N. (2003a), "Correcting for simulation bias in Monte Carlo methods to value exotic options in models driven by Lévy processes", *Working Paper*, Cass Business School, City University, London, UK.

[23] Ribeiro, C. and Webber, N. (2003b), "Valuing path-dependent options in the variance-gamma model by Monte Carlo with a gamma bridge", *Journal of Computational Finance*, **7**, 81–100.

[24] Rydberg, T.H. (1997), "The normal inverse Gaussian Lévy process: Simulation and approximation", *Communications in Statistics: Stochastic Models*, **13**, 887–910.

[25] Rydberg, T.H. (1999), "Generalized hyperbolic diffusion processes with applications in finance", *Mathematical Finance*, **9**, 183–201.

3
Risks in Returns: A Pure Jump Perspective

Hélyette Geman
University of Paris Dauphine and ESSEC, France

and

Dilip B. Madan
Robert H. Smith School of Business, University of Maryland, USA

Abstract

From a financial perspective, two return densities are critical: They evaluate the likelihood and prices of events and are usually referred to as statistical and risk neutral densities. We use pure jump models for the evolution of stock prices and investigate in this setting the relationship between these two entities. We infer from this analysis that prices dominate likelihoods in both tails of the return distribution while the opposite is the case in the center. It is also observed that the domination is greater on the downside as compared to the upside. Consequently, the ratio of the two densities has an asymmetric U-shape. These results are inconsistent with the standard applications of utility theory to asset pricing which assume a representative agent who is long the market; in this case, the previous ratio is monotonically decreasing in returns and is not U-shaped. For a market consistent application of utility theory, it is required that, at a minimum, one admits a two-agent model in which some weight is given to an agent who is short the market. When prices exceed probabilities, the purchase of claims contingent on these events reflects a negative risk premium or a price paid to buy insurance. In the opposite case, we have a positive risk premium earned as compensation for risk bearing and the purchase is an investment. The purchase of stock is then seen as a portfolio combining investment and insurance.

3.1 INTRODUCTION

It is now well recognized in financial analysis that one may associate with each economic event two probability measures. The first assesses the likelihood of the event and is called the statistical, physical or real measure. A large part of traditional statistical and econometric analysis of economic data is singularly focused on the study of this measure and its description as a stochastic process in either discrete or continuous time. On the other hand, from the analysis of contingent claims in modern option pricing theory, it is known that futures prices (to avoid discounting considerations) of contracts paying a dollar on the realization of the event, define another probability measure that is called the risk neutral, martingale or pricing measure. A considerable part of empirical analysis in finance is devoted solely to this measure, partly motivated by the Breeden and Litzenberger (1978) observation on

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

how market option prices partially reveal this measure. The existence of the latter measure and its equivalence to the first is fundamentally tied to assumptions of market efficiency and the absence of arbitrage (Kreps (1981), Harrison and Pliska (1981) and Delbaen and Schachermayer (1994)). Understanding the transition from one to the other is critical in the management of financial risks. The following simple example illustrates the role played by both measures.

To focus attention, we consider at time t, for an equity index at the current level U_t, the event associated with this index reaching particular levels L at some prespecified time, $T > t$. Denote by $p_{t,T}(U_t, L)$ the statistical density of this event; the corresponding risk neutral density is $q_{t,T}(U_t, L)$. These two densities interact in the determination, for example, of the value at risk in a contingent claim with cash flow $c_T(L)$ at time T. The change in value over an interval of length h, of this cash flow on a marked to market basis is the random variable

$$\begin{aligned} & e^{-r(T-t+h)} \int_0^\infty c_T(L) q_{t+h,T}(U_{t+h}, L)\, dL - e^{-r(T-t)} \int_0^\infty c_T(L) q_{t,T}(U_t, L)\, dL \\ = \; & \Phi(U_t, U_{t+h}), \end{aligned}$$

where it is supposed that interest rates are constant at the continuously compounded rate of r. The risk in this position is assessed by the statistical density

$$p_{t,t+h}\,(U_t, U_{t+h})$$

and the value at risk at the 0.95 confidence interval defined as the corresponding quantile of the statistical distribution.

We note importantly that the contingent claim example is in fact quite general. It is now recognized explicitly that even bonds are claims contingent on the absence of counterparty default while equity is itself an option in the presence of outstanding bonds, or even otherwise when we take particular note of limited liability.

An equally important entity is the ratio of the two densities

$$y_{t,T}(U_t, L) = \frac{q_{t,T}(U_t, L)}{p_{t,T}(U_t, L)}$$

which is called the change of measure density (or the Radon–Nikodym derivative of the measure q with respect to the measure p). By incorporating the measure change into the valuations above, one may perform all expectations with respect to the statistical measure and write the change in value as

$$\begin{aligned} & e^{-r(T-t+h)} \int_0^\infty c_T(L) y_{t+h,T}(U_{t+h}, L) p_{t+h,T}(U_{t+h}, L)\, dL \\ & - e^{-r(T-t)} \int_0^\infty c_T(L) y_{t,T}(U_t, L) p_{t,T}(U_t, L)\, dL \\ = \; & \Phi(U_t, U_{t+h}) \end{aligned}$$

It is clear from these expressions that an understanding of the measure change $y_{t,T}(U_t, L)$ makes important contributions to risk management and investment decisions. The difficulty however, lies in making observations on the measure change. This is because, although one may extract $q_{t,T}$ from option prices using the methods of Breeden and Litzenberger (1978), this occurs at values of T reflecting traded option maturities $T - t$ at time t; these are typically at intervals of a month. In contrast, the statistical density is best estimated at the horizon of daily returns. This time discrepancy is difficult to overcome. There is little one may do about accessing risk neutral densities at maturities below the first liquid traded maturity. On the other hand, one may be tempted to construct monthly returns out of daily returns assuming independence and stationarity; however, the considerable evidence in support of correlated squared returns makes these assumptions problematic. For other recent approaches in this direction, the reader is referred to Jackwerth (2000) and Bliss and Panigirtzoglou (2002).

The approach we take here is to attempt to observe from options data and time series data the limiting densities as T approaches t. Furthermore, to recover the classical financial setting focusing on *log returns*, we first change variables to these magnitudes by making the transformation $L = U_t e^l$ and subsume the dependence on the current observed level U_t into the subscript t.

$$\widetilde{q}_{t,T}(l) \equiv q_{t,T}(U_t, U_t e^l) U_t e^l$$
$$\widetilde{p}_{t,T}(l) \equiv p_{t,T}(U_t, U_t e^l) U_t e^l$$

These limiting densities may be constructed on normalization by $(T - t)$ as follows

$$k_Q(l) = \lim_{T \to t} \frac{\widetilde{q}_{t,T}(l)}{T - t}$$
$$k_P(l) = \lim_{T \to t} \frac{\widetilde{p}_{t,T}(l)}{T - t}$$

We note importantly that the division by $(T - t)$ is necessary as the numerator in each case goes to zero for $l \neq 0$ and goes to infinity for $l = 0$ (as the limiting measures are Dirac measures at $l = 0$). Another key and different observation is the fact that, unfortunately, for continuous processes both limits remain zero for $l \neq 0$. For discontinuous processes in contrast, the situation is different; in this wide collection, we choose for tractability the class of purely discontinuous Lévy processes. For these pure jump processes, the above limits are well defined for all $l \neq 0$ and converge to the Lévy measures defined by $k_Q(l)$, and $k_P(l)$, respectively.

The statistical Lévy measure, $k_P(l)$, has the heuristic interpretation of the expected number of jumps of size l in log returns per unit time. Analogously, $k_Q(l)$ is the futures price of a contract paying at unit time the dollar number of jumps of size l that occur in this period. Apart from these horizon matching considerations, the use of Lévy processes in modeling asset returns, both statistically and risk neutrally, has a number of other well noted advantages. First, from the statistical perspective, it is well known that kurtosis levels in short period returns are substantially above 3, arguing for non-Gaussian distributions. Lévy processes easily accommodate a much richer structure of moments for short horizon returns, including negative skewness when needed. Risk neutrally, these processes easily capture short maturity skews that are prominent in options data. The transition from the statistical to the risk neutral probability is also less constrained, as, in principle, all moments may be altered, unlike the diffusion case where local volatilities must remain the same. For further details on the applications of Lévy processes in finance, we refer the reader to Schoutens (2003).

From the perspective of studying the ratio of the risk neutral to the statistical Lévy measure, it is useful to work with processes that have simple analytical forms for this entity and are capable of both providing a good fit to the data and of synthesizing the high activity levels observed in the markets. This leads us to 'infinite activity' Lévy processes (see Geman *et al.* (2001)) that have a sufficiently rich parametric structure to capture at least the first four moments of the local motions. A particularly attractive example is provided by the CGMY model (Carr *et al.* (2002)) with further properties described in the next section. Other candidates include the Normal Inverse Gaussian model of Barndorff-Nielsen (1998), the Meixner process studied by Schoutens and Teugels (1998), and the generalized hyperbolic model (Barndorff-Nielsen (1977), Eberlein and Prause (1998) and Prause (1999)).

The next section presents the details of the *CGMY* model employed in this current study. This section is followed by estimation details presented in Section 3.3, for both the statistical analysis and the inference of the risk neutral process. In comparing the two probabilities at the instantaneous level, we consider explicitly here the structure of returns on securities paying the market gap risk. These are securities that pay a dollar whenever there is a large up or down move of a prespecified size. Section 3.4 presents the results for five world equity indexes (USA, UK, Germany, Spain and Japan) showing that world-wide tail gap risk securities for both positive and negative moves are insurance-based with expected negative rates of return reflecting the presence of insurance premia, while the central part of the return distribution represents investment where positive rates of return reflect the expected risk compensation.

We anticipate that market participants taking long positions protect themselves by buying downside gap risk claims and pay the requisite insurance charge for this service. On the other hand, participants short the market protect themselves by buying upside gap risk claims and pay the insurance charge on this side. The relative strength of the long side to the short side is then reflected in the larger premia for downside gap risk claims, as compared to the comparable upside gap risk claims. We see in the structure of the change of measure density the ways in which investment risk and insurance protection complement each other in the financial markets of the world.

3.2 CGMY MODEL DETAILS

The general idea is to model the statistical and risk neutral log price relative over an interval, $X(t+h) - X(t) = \ln(S(t+h)/S(t))$, as the increment of a purely discontinuous Lévy process. Such processes have independent and identically distributed increments over non-overlapping intervals of equal length with infinitely divisible densities. They are characterized by the Lévy-Khintchine decomposition for their characteristic exponents, $\psi(u)$ by

$$E\left[\exp(iuX(t))\right] = \exp\left(-t\psi(u)\right) \tag{3.1}$$

$$\psi(u) = i\gamma u + \int_{-\infty}^{\infty} \left(1 + iux\mathbf{1}_{|x|\leq a} - e^{iux}\right) k(x)\, dx$$

where γ is called the drift coefficient and $k(x)$ is the Lévy density that integrates x^2 in a neighborhood of 0. The processes may, in general, have infinite variation in that the limiting sum of absolute changes in the log price over smaller and smaller time intervals tends to infinity. In the special case of a finite limit, we have finite variation and the characteristic exponent then has the representation

$$\psi(u) = i\gamma u + \int_{-\infty}^{\infty} \left(1 - e^{iux}\right) k(x)\, dx.$$

In the finite variation case, the process for $X(t)$ may be written as the difference of two increasing processes

$$X(t) = X_p(t) - X_n(t)$$

where the increasing processes $X_p(t)$, $X_n(t)$ have characteristic exponents, $\psi_p(u)$, $\psi_n(u)$

$$\begin{aligned}
\psi_p(u) &= i\gamma_p u + \int_0^{\infty} \left(1 - e^{iux}\right) k(x)\, dx \\
\psi_n(u) &= i\gamma_n u + \int_0^{\infty} \left(1 - e^{iux}\right) k(-x)\, dx \\
\gamma &= \gamma_p - \gamma_n
\end{aligned}$$

For the infinite variation case, we add to the difference of two increasing compound Poisson processes X_p^a, X_n^a with characteristic exponents

$$\begin{aligned}
\psi_p^a(u) &= i\gamma_p u + \lambda_p^a \int_a^{\infty} \left(1 - e^{iux}\right) f_p^a(x)\, dx \\
\psi_n^a(u) &= i\gamma_n u + \lambda_n^a \int_a^{\infty} \left(1 - e^{iux}\right) f_n^a(x)\, dx \\
\lambda_p^a &= \int_a^{\infty} k(x)\, dx \;;\; f_p^a(x) = \frac{k(x)}{\lambda_p^a} \;,\; x > a \\
\lambda_n^a &= \int_{-\infty}^{-a} k(x)\, dx \;;\; f_n^a(x) = \frac{k(-x)}{\lambda_n^a} \;,\; x > a \\
\gamma &= \gamma_p - \gamma_n
\end{aligned}$$

the limit as ε tends to zero, of the compensated jump compound Poisson martingale $X^{\varepsilon}(t)$

$$X^{\varepsilon}(t) = \sum_{s \le t} \Delta X_s \mathbf{1}_{\varepsilon < |\Delta X_s| < a} - t \int_{\varepsilon < |x| < a} x k(x)\, dx$$

that accommodates the behavior of the infinite variation small jump component with characteristic exponent

$$\psi^{\varepsilon}(u) = \int_{\varepsilon < |x| < a} \left(1 + iux \mathbf{1}_{\varepsilon < |x| < a} - e^{iux}\right) k(x)\, dx.$$

The precise model is specified on providing an explicit form for the Lévy measure $k(x)\, dx$.

For this choice, we adopt a synthesis of the known Lévy measures for the gamma process and the process with increments having the stable distribution. The result is a parametrically rich synthesis permitting control over local skewness and kurtosis, that accommodate long tail distributions yet has finite moments and permits both finite and infinite variation processes. This is the *CGMY* model with Lévy measure

$$k_{CGMY}(x) = C\frac{e^{-Mx}}{x^{1+Y}}\mathbf{1}_{x>0} + C\frac{e^{-G|x|}}{|x|^{1+Y}}\mathbf{1}_{x<0}. \tag{3.2}$$

For $G = M = 0$, we get the stable process and $Y = 0$ yields the variance gamma model that is also the difference of two gamma processes. The values of $Y \geq 1$ have infinite variation, while the values of $Y \geq 0$ have infinitely many jumps in any interval or represent infinite activity processes.

The characteristic function for the *CGMY* process is obtained by explicitly evaluating the integral in the exponent of the characteristic function (equation (3.1)) against the specific Lévy measure (equation (3.2)) (see Carr *et al.* (2002) for further details) to get that

$$\begin{aligned}\phi_{CGMY}(u) &= E[e^{iuX_{CGMY}(t)}] \\ &= \exp\left(tC\Gamma(-Y)\left[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right]\right)\end{aligned}$$

The stock price process associated with the *CGMY* model has the construction

$$S(t) = S(0)\exp\left(at + X_{CGMY}(t) - t\log\left(E\left[e^{X_{CGMY}(1)}\right]\right)\right)$$

and the characteristic function for the log stock price is given by

$$\phi_{\ln(S(t))}(u) = \exp\left(iu(\ln(S(0) + (a - \log\phi_{CGMY}(-i))t))\right)\phi_{CGMY}(u). \tag{3.3}$$

For the estimation of the statistical process we employ Fourier inversion methods to maximize the log likelihood of demeaned daily stock returns binned into cells at which the Fourier inversion computes the probability element. For the risk neutral process, we employ monthly option data and use Fourier inversion as described in Carr and Madan (1998) to estimate the risk neutral process.

The process for the measure change is then given explicitly as the stochastic exponential of the compensated jump martingale

$$m(t) = \int_0^t \int_{-\infty}^{\infty} (H(x) - 1)(\mu(dx, ds) - k_{CGMY}(x)\,dx\,ds)$$

where the counting measure $\mu(dx, ds)$ counts all of the jumps in the process $X_{CGMY}(t)$ at the level x, while the Lévy measure compensates these occurrences by their arrival rate under the model. The function $H(x)$ is the ratio of the two Lévy densities and, as noted in the introduction, it is also the limit of the ratio of the two probability elements over short horizons.

The stochastic exponential, $M(t)$, of $m(t)$ may be explicitly written out and is given by

$$M(t) = \exp\left(-t\int_{-\infty}^{\infty}(H(x) - 1)k_{CGMY}(x)\,dx\,ds\right)\prod_{s\leq t} H(\Delta X_{CGMY}(s))$$

and along any path one exaggerates the probability by the ratio of the limiting risk neutral probability to its statistical counterpart, renormalized to constitute a probability.

Once we have estimated the two Lévy measures we may explicitly compute the function $H(x)$. This function must satisfy certain integrability in the neighborhood of zero, but if we focus attention on the complement of a small interval near zero, then one may accommodate robust estimates of both the statistical and risk neutral processes. To maintain this robustness, we consider the function H in the complement of an interval around zero. This is also consistent with the broad concerns of risk management where the focus is on the effects of the larger moves.

The explicit expression for the function $H(x)$, written as the ratio of two *CGMY* Lévy densities, is not particularly instructive, but the function may be easily computed and it is the economic content of this entity that is of interest to us. This is best revealed by considering the short horizon ratio of the risk neutral and statistical probability elements in the space of log price relatives. For each size of move in the log price relative, here represented by x on the real line, the function $H(x)$ is the ratio of the price of a claim paying a dollar were this move to occur, to its probability. Hence, we may see the reciprocal $(H(x))^{-1}$ as the ratio of a security expected cash flow to its price, and so as a rate of return. Since the two probabilities integrate to unity, if these rates of return are equalized across all the assets represented by the jump size x, then this equalized rate of return is unity and the two probabilities are identical. Such an outcome rarely occurs, with rates of return being typically positive for values of x near zero, while they are negative for values of x away from zero on both sides. We also broadly understand securities with positive rates of return as risky investments with the return as a measure of risk compensation. For negative rates of return, these occur when a security is a hedge for market participants and the inflated price is inclusive of a risk premium. Essentially, securities associated with small jump sizes constitute risk that the investing public is willing to undertake as part of the nature of markets, but large moves on either side elicit premia from participants who are long the market, while the short side wishes to cover against the large positive moves. As a result, there is a hedging demand for large moves based on insurance considerations. We naturally expect this hedging demand to be considerably pronounced with respect to market indices and restrict attention here to world market equity indices. The objective then reduces to the estimation of the statistical and risk neutral Lévy processes in the *CGMY* parametric class, followed by a construction and comment on the explicitly observed country specific functions $H(x)$.

In contrast to the procedure followed in Carr *et al.* (2002), we shall here directly price securities paying a dollar when price moves occur in the tail of the distribution and directly compute the return on the securities. Negative returns we associate with securities having an insurance theoretic basis with investment securities reflecting positive returns. The basic pattern we observe is that investment occurs in the center of the distribution, while insurance claims relate to the two tails. Of course, market participants who are long the market and those who are short focus their insurance activities on opposite tails with the asymmetry arising from the relative strength of the long positions. The pricing details and return computations in the context of the *CGMY* model are described in detail in the next section.

3.3 ESTIMATION DETAILS

The *CGMY* model is analytical in its characteristic function (see equation (3.3)) and this makes estimation feasible via the fast Fourier transform for both the statistical density and the option prices using the methods of Carr and Madan (1998). This section briefly reviews the methods employed. We note up front the difficulty of comparing these two densities as the statistical process is best studied by modeling daily continuously compounded returns while the most liquid short maturity options are around a month in the time to maturity. As already noted, we hope to make the comparison via the implied instantaneous returns as they are captured in the Lévy measure for the two processes. Equally importantly, we do not impose restrictions on the relationship between the two measures as the foundations for such restrictions are questionable at best. Furthermore, such an imposition can lead to a poor quality of estimation in one or both of the two measures involved. We take up the details for the statistical estimation first, followed by a description of the risk neutral estimation.

3.3.1 Statistical estimation

The statistical density $f(x_t)$ for the daily continuously compounded return $x_t = \ln(S_t/S_{t-1})$ may be evaluated by

$$f(x_t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu(\ln(S_{t-1})+x_t)}\phi_{\ln(S_t)}(u)\,du.$$

One may then employ maximum likelihood estimation whereby we maximize, over the *CGMY* parameter set, the objective $\sum_{t=1}^{T}\ln(f(x_t))$. However, this is a computationally expensive procedure as it requires a separate Fourier inversion for each data point x_t. We choose instead to maximize the likelihood of the binned data. Specifically, we first construct a histogram with cell centers at the points a_i, for $i = 1, \ldots, N$. The histogram places the proportion p_i in the cell with center a_i. The centers are chosen to match the points at which the Fourier inversion computes the density. A single Fourier inversion then yields the values $f(a_i)$ for $i = 1, \ldots, N$ and the log likelihood of the binned data is given by

$$\sum_{i=1}^{N} p_i \ln(f(a_i)).$$

We maximize this binned likelihood over the parameter space of the *CGMY* model.

Our interest is in the Lévy measure of the process and this defines the martingale component of the statistical process. We therefore do not estimate a mean rate of return, and perform the statistical estimation on the demeaned sample of time series returns. From the perspective of the instantaneous component this is adequate as there is no instantaneous mean return. The mean return is always realized over an interval of time and we seek to compare martingale components at the level of the instant. The pure jump perspective is important in this endeavor as continuous processes disappear at the instantaneous level with both the mean and the variance vanishing at comparable speeds. Pure jump processes on the contrary maintain an instantaneous risk exposure that one may assess via an estimate of the Lévy measure.

The purpose of the statistical estimation is to get a good grasp, at least, or at a minimum, of the unconditional return density. The focus on the binned data is precise in this regard. There are many other aspects of the statistical process that are not being evaluated in such a procedure. The nature of conditioning in describing forward returns, for example, is a point of omission. Other matters include autocorrelation in squared returns or stochastic volatility, the form of long-range dependence, and the structure of joint densities or correlation across assets. Since our aim is to compare short horizon probability elements, statistical and risk neutral, we do this here at an unconditional univariate level.

We do, however, wish to assess the quality of this unconditional characterization. There are a number of metrics one may employ in this regard. The *p−value* of a chi-squared goodness of fit statistic is one such measure. Other alternatives include *qq−plots*, the Kolmogorov−Smirnov statistic and the cross entropic or Hellinger distances between the estimated and empirical densities. These magnitudes are instructive in the context of alternatives for comparative or benchmarking purposes. For these reasons, we include the well-understood case of the Gaussian distribution and the parametric special case of the *VG* model. Given that these alternatives are special cases within the *CGMY* class, we present estimated chi-squared statistics and probability levels for the binned data.

3.3.2 Risk neutral estimation

Risk neutral estimation is essentially the estimation of a pricing measure and all of the parameters of the risk neutral process can be viewed as one-to-one with the market prices of specific assets at a point of time. The question arises as to how the risk neutral process should be estimated. Clearly, the input for this exercise includes prices of options (and we focus on out-of-the-money options) of all strikes and maturities as they were traded across a period of calendar time.

The precise estimation strategy depends on the task at hand. For example, one may hold risk neutral parameters constant through the time period across all strikes and maturities if one wishes to assess the adequacy of a particular martingale measure choice through time, with respect to its ability to explain the observed prices. This is the approach taken in many studies and we cite here Bates (1996, 2000) and Duffie *et al.* (2000). Alternatively, one may recognize, as we do, that a one- or two-dimensional Markov process is inadequate as a candidate for describing the evolution of the volatility surface and we do not wish to evaluate the adequacy of a particularly simple (Lévy process) martingale measure choice through time. When we go across calendar time, there are substantial movements in both risk neutral volatilities and skewness. Furthermore, with respect to Lévy processes, we quote the results of Konikov and Madan (2002) suggesting that such a process is even inadequate with respect to explaining the prices of options across all maturities at a single point of time.

Given our focus on the instantaneous move structure, we consider the estimation of risk neutral parameters using just the options of the shortest liquid maturity. We therefore employ just options with maturities between one and two months. It is now well known that a variety of Lévy processes are adequate to this task and we employ the *CGMY* model for the purpose here. Forcing constancy of these parameters across calendar time is likely to lead to a loss of quality in describing the risk neutral measure on each day and therefore bias the comparison of the short maturity risk neutral and statistical probability elements we wish to make. We therefore follow the method employed in, for example, Bakshi *et al.* (1997) and conduct separate estimates for each day. For other nonparametric approaches, the reader is referred to Ait-Sahalia and Lo (1998).

With respect to the specific procedure, we estimate parameters by minimizing the root mean square error between the observed prices and the model prices. For computing the model prices we employ the Fourier inversion methods introduced by Carr and Madan (1998). Specifically, we note that the Fourier transform $\gamma(u)$ in log strike of modified call option prices $e^{\alpha k}c(k)$, where k is the logarithm of the strike and $\alpha > 0$, is defined by

$$\gamma(u) = \int_{-\infty}^{\infty} e^{\alpha k} c(k) dk.$$

Carr and Madan (1998) show that γ is analytic in the characteristic function of log prices introduced in equation (3.3) as $\phi_{\ln(S_t)}(u)$. Explicitly we have that

$$\pi\gamma(u) = \frac{e^{-rt}\phi_{\ln(S_t)}(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}$$

The option prices then follow on Fourier inversion and

$$c(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-iuk}\gamma(u) du.$$

3.3.3 Gap risk expectation and price

After we have estimated the statistical and risk neutral parameters in the *CGMY* class of processes, we then evaluate and present a comparison of the expected payout and market price of claims that pay a dollar for all down and up moves over a prespecified limit. This allows us to evaluate the arrival rate of market gaps as judged by the statistical Lévy process and simultaneously evaluate the up front price of claim that promises such payouts for a one-year term.

The details for this calculation require us to evaluate the integral of the one-sided Lévy measure for moves of size a with overall arrival rate c, exponential decay at rate β and stable process coefficient γ, which we denote as $w(a, c, \beta, \gamma)$. Specifically, we have

$$\begin{aligned} w(a, c, \beta, \gamma) &= \int_a^{\infty} c \frac{e^{-\beta x}}{x^{1+\gamma}} dx \qquad (3.4) \\ &= c\beta^{\gamma} \left[\begin{array}{c} \frac{(a\beta)^{-\gamma} e^{-a\beta}}{\gamma} + \frac{(a\beta)^{1-\gamma} e^{-a\beta}}{\gamma(1-\gamma)} - \\ \frac{\Gamma(2-\gamma)}{\gamma(1-\gamma)} (1 - gammainc\ (\beta a, 2-\gamma)) \end{array} \right] \end{aligned}$$

where $\Gamma(x)$ is the *gamma* function and *gammainc* is the incomplete gamma function

$$gammainc(x, \nu) = \frac{1}{\Gamma(\nu)} \int_0^x g^{\nu-1} e^{-g} dg.$$

3.4 ESTIMATION RESULTS

We selected the index markets of the USA, UK, Germany, Spain and Japan for a detailed analysis of the statistical and risk neutral densities and a comparison of the associated Lévy measures. The risk neutral estimations were conducted for data on option prices on these indexes for twelve days in the year 2002, taking a mid-month day for each of the twelve months. The specific days were January 9, February 13, March 13, April 10, May 8, June 12, July 10, August 10, September 11, October 9, November 13 and December 11. For the statistical estimation, we employed time series of daily continuously compounded returns from January 1 1998 to December 31 2002.

Table 3.1 Statistical estimation results

Parameter	SPX	DAX	FTSE	IBEX	NIKKEI
Volatility	0.1679	0.2569	0.1718	0.2222	0.2445
σ	0.1662	0.2545	0.1691	0.2202	0.2424
ν	0.0034	0.0031	0.0024	0.0022	0.0024
θ	−0.0447	−0.4548	−0.0765	−0.3502	0.0610
C	13.02	23.04	0.2927	2.79	5.11
G	94.64	65.24	51.99	63.08	68.57
M	100.2	78.10	56.37	75.60	66.16
Y	0.5348	0.4925	1.21	0.8963	0.7982
chisq Gauss	463.5	213.9	211.9	132.7	168.1
chisq VG	47.9	49.4	65.3	35.7	46.2
chisq CGMY	42.0	49.8	48.8	32.2	47.3
pval Gauss	0.0	0.0	0.0	0.0	0.0
pval VG	8.82%	41.7%	0.2%	81%	54.7%
pval CGMY	22.6%	40.3%	7.5%	90.6%	50.2%

3.4.1 Statistical estimation results

The statistical estimation was conducted for the whole period on demeaned data and we report estimates of the volatility, the parameters of the *VG* process as they appear in Carr *et al*. (1998), and the parameters of the *CGMY* model. The former two are included for comparative purposes. We also report chisquare goodness of fit statistics and associated probability values for the three models. These results are all presented in Table 3.1.

We observe that both the *VG* and *CGMY* Lévy processes constitute a substantial improvement over the Gaussian model, with the *CGMY* making a further improvement for the *SPX*, *FTSE* and the *IBEX*. There are also strong statistical skews in the data for *DAX* and *IBEX*, as is evidenced by large values for θ in the *VG* estimates and for the departure for G from M in the *CGMY* estimates. Except for the *FTSE*, all of the processes are estimated as being of finite variation.

3.4.2 Risk neutral estimation results

The risk neutral estimations were conducted for each of the twelve days for options on each of the five indexes. This gives us 60 sets of risk neutral parameter estimates. We first present the average percentage error (APE), defined as the ratio of the average absolute error relative to the average option price, and the number of option prices (NOP) used in the fit for each of the five indexes for the twelve months. Except for three days on the *FTSE* and one day on the *NIKKEI* the model successfully fits the option prices (Table 3.2).

We averaged the risk neutral parameter estimates over the twelve days and the results are reported in Table 3.3.

3.4.3 Results on gap risk expectation and price

The gap risk of markets pertains to sudden large moves and we evaluate here the statistical expectation and market price of such tail events. For moves of sizes ranging from *one*

Table 3.2 Risk neutral fit summaries[a]

Month	SPX		DAX		FTSE		IBEX		NIKKEI	
	APE	NOP	APE	NOP	APE	NOP	APE	NOP	APE	NOP
January	0.0069	18	0.0153	29	0.0207	29	0.0098	64	0.0077	6
February	0.0102	20	0.0097	33	0.0249	25	0.0076	49	0.0229	8
March	0.0122	20	0.0095	29	0.0105	24	0.0064	47	0.0248	10
April	0.0109	17	0.0132	28	0.0311	32	0.0111	43	0.0160	6
May	0.0111	24	0.0086	32	0.0072	30	0.0107	54	0.0297	7
June	0.0104	19	0.0094	28	0.0155	31	0.0081	51	0.0086	6
July	0.0095	17	0.0054	33	NA	35	0.0107	66	0.0229	6
August	0.0083	42	0.0109	43	0.0102	52	0.0083	77	0.0067	7
September	0.0069	21	0.0076	28	NA	41	0.0097	54	0.0111	6
October	0.0060	23	0.0059	28	0.0082	48	0.0076	70	0.0190	6
November	0.0069	29	0.0091	31	NA	37	0.0080	66	0.0107	7
December	0.0068	17	0.0069	24	0.0079	34	NA	27	0.0283	4

[a] NA, not available.

Table 3.3 Risk neutral parameter estimates

Market	C	G	M	Y
SPX	0.8689	6.9420	31.1907	0.8801
DAX	1.2594	5.7464	27.9887	0.9914
FTSE	0.2902	5.1308	41.7202	1.1902
IBEX	0.8757	8.2975	40.9873	0.9625
NIKKEI	3.6502	10.2038	28.5528	0.9228

percent to 5% in steps of *one* percent, in both directions up and down, we determine the statistical expectation of such a move and its market price. The former is determined by integrating the statistical Lévy density over the appropriate region, while the latter integrates the risk neutral Lévy density. The parameters for these on all of the equity indexes studied here have been reported in the last two subsections. The closed form formula for the price or the expectation of the tail event is proportional to the value given by equation (3.4), with the factor of proportionality being the time step. Table 3.4 reports these proportionality factors for these expectations and market prices.

We observe from Table 3.4 that price is at a premium relative to probability for moves in either direction. The premium increases with the size of the move, and is substantially higher for negative moves as opposed to positive ones. This pattern in maintained in all markets. For smaller moves like *one* percent, the up side security is an investment as opposed to an insurance contract for the SPX and IBEX. Otherwise, all of the moves represent insurance contracts displaying negative rates of return.

We also present in Figure 3.1 a graph of the rates of return on the digital claims paying a dollar contingent on returns at a given level for each of the five world indexes studied here. The region of insurance is clearly seen to be in the tail, with higher premiums for the left tail or the down side moves.

Table 3.4 Expectations and Prices of tail events

Size Expectation/ price		SPX	DAX	FTSE	IBEX	NIKKEI
1%	Up expectation	26.77	56.64	20.85	38.69	54.86
	Up price	24.47	59.34	24.09	28.70	128.75
	Down expectation	29.22	70.63	22.38	47.39	52.73
	Down price	42.94	97.70	50.00	57.16	194.15
2%	Up expectation	4.42	12.30	3.86	6.87	11.52
	Up price	7.59	18.04	5.43	7.47	40.43
	Down expectation	5.13	17.81	4.38	9.75	10.76
	Down price	19.36	42.64	19.35	23.94	80.46
3%	Up expectation	0.98	3.51	1.09	1.75	3.37
	Up price	3.24	7.72	1.83	2.75	17.58
	Down expectation	1.21	5.85	1.31	2.86	3.07
	Down price	11.54	24.97	10.67	13.56	44.85
4%	Up expectation	0.25	1.13	0.37	0.53	1.14
	Up price	1.59	3.83	0.74	1.18	8.81
	Down expectation	0.32	2.16	0.47	0.98	1.02
	Down price	7.73	16.60	6.82	8.73	28.36
5%	Up expectation	0.07	0.39	0.14	0.17	0.42
	Up price	0.84	2.07	0.33	0.56	4.78
	Down expectation	0.09	0.86	0.18	0.37	0.37
	Down price	5.54	11.86	4.73	6.04	19.26

These consistent negative returns are quite substantial on the down side but are lower for the up side. This probably reflects the relative strength of those who are long the market and seek to cover the down side tail. For the up moves, it is the relatively weaker short side that is buying tail insurance and hence the lower absolute magnitude of negative returns on these securities.

Option contracts may be broadly seen as offering long investors down side tail cover, while the opposite holds for investors taking a short position. The investment is primarily in the center of the return distribution, with both tails being covered by parties taking opposite positions with respect to the aggregate market. These results are broadly consistent with the earlier observations with respect to the USA market by Carr *et al*. (2002) and Jackwerth (2000).

3.5 CONCLUSIONS

We compare statistical and risk neutral probability elements at a common maturity with a view to learning the relationship between prices and likelihoods of events. There is an inherent difficulty in making such a comparison as the two probability elements are not available at a matching horizon. Typically, one has good data on daily returns but risk neutral information is captured in the prices of options at, for example, a monthly horizon.

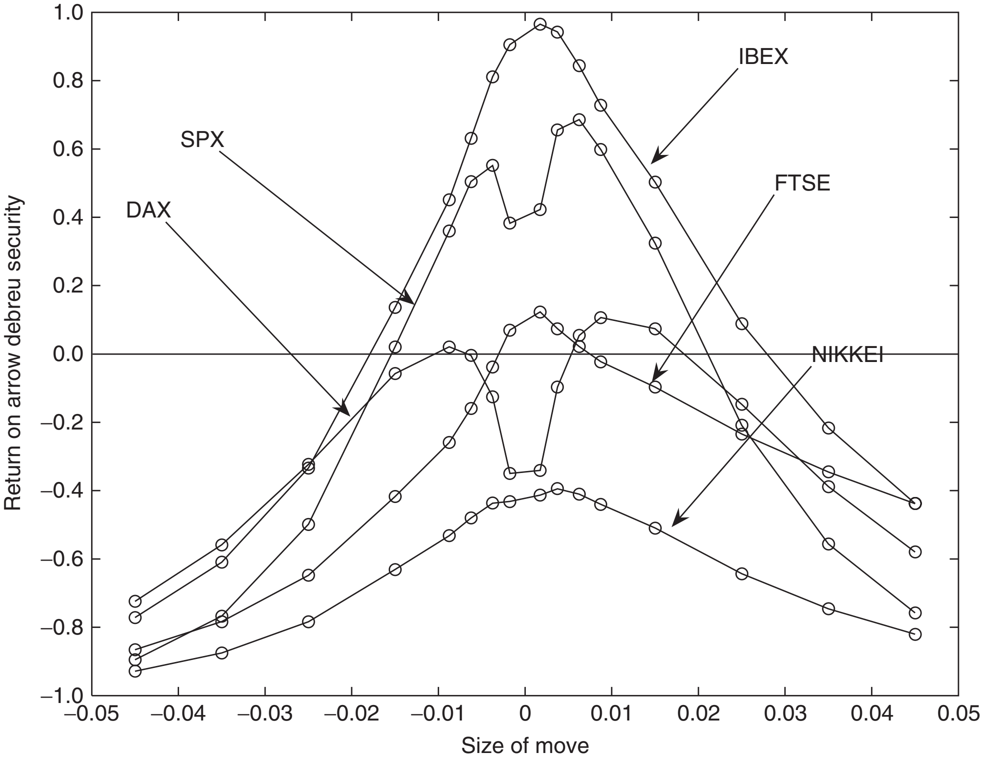

Figure 3.1 Investment and insurance by country index: rates of return on Arrow Debreu Securities contingent on the size of market moves

We work around this problem by considering both densities at the instantaneous level. The pure jump perspective of this paper is important in this regard as continuous processes collapse to delta functions at the instantaneous level, while pure jump Lévy processes converge to their Lévy densities.

Motivated by the desire to compare Lévy densities that are capable of capturing both the statistical return density and the prices of options, yet remain tractable for the evaluation of expected cash flows and market prices of claims contingent on market moves, we work with the *CGMY* model which includes the variance gamma, the stable and Brownian as special cases. This model has a Lévy density that is defined completely in terms of elementary functions and is therefore particularly well suited for the task at hand.

Statistical estimation is conducted on five world indexes, the *SPX*, *DAX*, *FTSE*, *IBEX* and the *NIKKEI*. Risk neutral estimation employs option prices through 2002. A comparison of the Lévy measures, via an evaluation of tail event probabilities and prices, reveals that tail prices substantially dominate probability in all markets and for moves on either side. The down side premia of price over probability is greater for negative moves as opposed to positive ones, and this is reflective of the insurance interests of the long side relative to the short side. The premia also rise with the size of the move on either side. These results support the view that tail event securities are insurance theoretic assets as opposed to investment assets, displaying the pattern of negative rates of return with prices inclusive of premia, while by contrast the center of the distribution represents the investment zone where positive rates of return reflect expected risk compensation.

Researchers interested in pricing issues in the context of access to the statistical density need to be aware of this two-sided feature of risk in markets. The pricing of risk has a two-sided feature that must balance insurance premia against investment-based compensations for risk taking. Both features are simultaneously present in worldwide equity index returns.

REFERENCES

[1] Ait-Sahalia, Y. and Lo, A. (1998), "Nonparametric estimation of state-price densities implicit in financial asset prices", *Journal of Finance*, **53**, 499–547.

[2] Bakshi, G., Cao, C. and Chen, Z. (1997), "Empirical performance of alternative option pricing models", *Journal of Finance*, **52**, 2003–2049.

[3] Barndorff-Nielsen, O.E. (1977), "Exponentially decreasing distributions for the logarithm of particle size", *Proceedings of the Royal Society of London, A*, **353**, 401–419.

[4] Barndorff-Nielsen, O.E. (1998), "Processes of normal inverse Gaussian type", *Finance and Stochastics*, **2**, 41–68.

[5] Bates, D. (1996), "Jumps and stochastic volatility: Exchange rate processes implicit in Deutschemark options", *Review of Financial Studies*, **9**, 69–108.

[6] Bates, D. (2000), "Post-'87 crash fears in S&P 500 futures options", *Journal of Econometrics*, **94**, 181–238.

[7] Bliss, R. and Panigirtzoglou, N. (2002), "Testing the stability of implied probability density functions", *Journal of Banking and Finance*, **26**, 381–422.

[8] Breeden, D. and Litzenberger, R.H. (1978), "Prices of state contingent claims implicit in options prices", *Journal of Business*, **51**, 621–651.

[9] Carr, P., Geman, H., Madan, D. and Yor, M. (2002), "The fine structure of asset returns: An empirical investigation", *Journal of Business*, **75**, 305–332.

[10] Carr, P. and Madan, D. (1998), "Option valuation using the fast Fourier transform", *Journal of Computational Finance*, **2**, 61–73.

[11] Delbaen, F. and Schachermayer, W. (1994), "A general version of the fundamental theorem of asset pricing", *Mathematische Annalen*, **300**, 463–520.

[12] Duffie, D., Pan, J. and Singleton, K. (2000), "Transform analysis and asset pricing for affine jump-diffusions", *Econometrica*, **68**, 1343–1376.

[13] Eberlein, E. and Prause, K. (1998), "The generalized hyperbolic model: financial derivatives and risk measures", FDM preprint, University of Freiburg, Freiburg, Germany.

[14] Geman, H., Madan, D. and Yor, M. (2001), "Time changes For Lévy processes", *Mathematical Finance*, **11**, 79–96.

[15] Harrison, J.M. and Pliska, S.R. (1981), "Martingales and stochastic integrals in the theory of continuous trading", *Stochastic Processes and their Applications*, **11**, 215–260.

[16] Jackwerth, J.C. (2000), "Recovering risk aversion from option prices and realized returns", *Review of Financial Studies*, **13**, 433–451.

[17] Konikov, M. and Madan, D. (2002), "Stochastic volatility via Markov chains", *Review of Derivatives Research*, **5**, 81–115.

[18] Kreps, D. (1981), "Arbitrage and equilibrium in economies with infinitely many commodities", *Journal of Mathematical Economics*, **8**, 15–35.

[19] Prause, K. (1999), “The generalized hyperbolic model: estimation, financial derivatives and risk measures”, *Ph.D. Thesis* University of Freiburg, Freiburg, Germany.

[20] Schoutens, W. (2003), *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley, Hoboken, NJ, USA.

[21] Schoutens, W. and Teugels, J.L. (1998), “Lévy processes, polynomials and martingales”, *Communications in Statistics: Stochastic Models*, **14**, 335–349.

4
Model Risk for Exotic and Moment Derivatives†

Wim Schoutens
Katholieke Universiteit Leuven – UCS, Leuven, Belgium

Erwin Simons
ING SWE, Financial Modelling, Brussels, Belgium

and

Jurgen Tistaert
ING SWE, Financial Modelling, Brussels, Belgium

Abstract

We show that several advanced equity option models incorporating stochastic volatility can be calibrated very nicely to a realistic implied volatility surface. More specifically, we focus on the Heston Stochastic Volatility model (with and without jumps in the stock price process), the Barndorff-Nielsen–Shephard model and Lévy models with stochastic time. All of these models are capable of accurately describing the marginal distribution of stock prices or indices and hence lead to almost identical European vanilla option prices. As such, we can hardly discriminate between the different processes on the basis of their smile-conform pricing characteristics. We therefore are tempted to apply them to a range of exotics. However, due to the different structure in path-behaviour between these models, we find that the resulting exotics prices can vary significantly. We subsequently introduce *moment derivatives*. These are derivatives that depend on the realized moments of (daily) log-returns. An already traded example of these derivatives is the Variance Swap. We show how to hedge these options and calculate their prices by Monte Carlo simulation. A comparison of these moment derivatives premiums demonstrates an even bigger discrepancy between the aforementioned models. This motivates a further study on how to model the fine stochastic behaviour of assets over time.

4.1 INTRODUCTION

Since the seminal publication [5] of the Black–Scholes model in 1973, we have witnessed a vast effort to relax a number of its restrictive assumptions. Empirical data show evidence for non-normal distributed log-returns together with the presence of stochastic volatility. Nowadays, a battery of models are available which capture non-normality and integrate

† This chapter is an extended version of Schoutens *et al*. [23]. The author(s) note that this paper does not necessarily reflect the views of their employer(s).

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

stochastic volatility. We focus on the following advanced models: the Heston Stochastic Volatility Model [14] and its generalization allowing for jumps in the stock price process (see e.g. [1]), the Barndorff-Nielsen–Shephard model introduced in [2] and Lévy models with stochastic time introduced by Carr *et al.* [8]. This class of models are build out of a Lévy process which is time-changed by a stochastic clock. The latter induces the desired stochastic volatility effect.

Our paper continues along the lines of Hull and Suo [16] and Hirsa *et al.* [15] and their study on the effect of model risk on the pricing of exotic options is extended in various aspects. This current paper is organized as follows. Section 4.2 elaborates on the technical details of the models and we state each of the closed-form characteristic functions. The latter are the necessary ingredients for a calibration procedure, which is tackled in Section 4.3. The pricing of the options in that framework is based on the analytical formula of Carr and Madan [7]. We will show that all of the above models can be calibrated very well to a representative set of European call options. Section 4.4 describes the simulation algorithms for the stochastic processes involved. Armed with good calibration results and powerful simulation tools, we will price a range of exotics. Section 4.5 presents the computational results for digital barriers, one-touch barriers, lookbacks and cliquet options under the different models. While the European vanilla option prices hardly differ across all models considered, we obtain significant differences in the prices of the exotics. These observations sparked our interest to push this study further in Section 4.6 by introducing and pricing moment derivatives. We extend a hedging formula as proposed by Carr and Lewis [6]. More precisely, we show how to hedge the realized kth moment swap by a dynamic trading strategy in bonds and stocks, a static position in vanilla options and a static position in the realized moment swaps of lower order. This paper concludes with a formal discussion and gives some directions for further research.

4.2 THE MODELS

We consider the risk-neutral dynamics of the different models. Let us shortly define some concepts and introduce their notation.

Let $S = \{S_t, 0 \leq t \leq T\}$ denote the stock price process and $\phi(u, t)$ the characteristic function of the random variable $\log S_t$, i.e.

$$\phi(u, t) = E[\exp(\mathrm{i}u \log(S_t))].$$

If for every integer n, $\phi(u, t)$ is also the nth power of a characteristic function, we say that the distribution is *infinitely divisible*. A Lévy process $X = \{X_t, t \geq 0\}$ is a stochastic process which starts at zero and has independent and stationary increments such that the distribution of the increment is an infinitely divisible distribution. A *subordinator* is a non-negative non-decreasing Lévy process. A general reference on Lévy processes is Bertoin [4]: for applications in finance see Schoutens [22].

The risk-free continuously compounded interest rate is assumed to be constant and denoted by r. The dividend yield is also assumed to be constant and denoted by q.

4.2.1 The Heston stochastic volatility model

The stock price process in the Heston Stochastic Volatility model (HEST) follows the Black–Scholes SDE in which the volatility is behaving stochastically over time:

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t, \quad S_0 \geq 0,$$

with the (squared) volatility following the classical Cox–Ingersoll–Ross (CIR) process:

$$\mathrm{d}\sigma_t^2 = \kappa(\eta - \sigma_t^2)\,\mathrm{d}t + \theta\sigma_t\,\mathrm{d}\tilde{W}_t, \quad \sigma_0 \geq 0,$$

where $W = \{W_t, t \geq 0\}$ and $\tilde{W} = \{\tilde{W}_t, t \geq 0\}$ are two correlated standard Brownian motions such that $\mathrm{Cov}[\mathrm{d}W_t\,\mathrm{d}\tilde{W}_t] = \rho\,\mathrm{d}t$.

The characteristic function $\phi(u, t)$ is in this case given by (see Heston [14] or Bakshi *et al.* [1]):

$$\begin{aligned}\phi(u,t) &= E[\exp(\mathrm{i}u\log(S_t))|S_0, \sigma_0^2]\\ &= \exp(\mathrm{i}u(\log S_0 + (r-q)t))\\ &\quad\times \exp(\eta\kappa\theta^{-2}((\kappa - \rho\theta u\mathrm{i} - d)t - 2\log((1 - g\mathrm{e}^{-dt})/(1-g))))\\ &\quad\times \exp(\sigma_0^2\theta^{-2}(\kappa - \rho\theta\mathrm{i}u - d)(1 - \mathrm{e}^{-dt})/(1 - g\mathrm{e}^{-dt})),\end{aligned}$$

where

$$d = ((\rho\theta u\mathrm{i} - \kappa)^2 - \theta^2(-\mathrm{i}u - u^2))^{1/2}, \tag{4.1}$$

$$g = (\kappa - \rho\theta u\mathrm{i} - d)/(\kappa - \rho\theta u\mathrm{i} + d). \tag{4.2}$$

4.2.2 The Heston stochastic volatility model with jumps

An extension of HEST introduces jumps in the asset price [1]. Jumps occur as a Poisson process and the percentage jump-sizes are lognormally distributed. An extension also allowing jumps in the volatility was described in Knudsen and Nguyen-Ngoc [18]. We opt to focus on the continuous version and the one with jumps in the stock price process only.

In the Heston Stochastic Volatility model with jumps (HESJ), the SDE of the stock price process is extended to yield:

$$\frac{\mathrm{d}S_t}{S_t} = (r - q - \lambda\mu_J)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t + J_t\,\mathrm{d}N_t, \quad S_0 \geq 0,$$

where $N = \{N_t, t \geq 0\}$ is an independent Poisson process with intensity parameter $\lambda > 0$, i.e. $E[N_t] = \lambda t$. J_t is the percentage jump size (conditional on a jump occurring) that is assumed to be lognormally, identically and independently distributed over time, with unconditional mean μ_J. The standard deviation of $\log(1+J_t)$ is σ_J:

$$\log(1+J_t) \sim \text{Normal}\left(\log(1+\mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2\right).$$

The SDE of the (squared) volatility process remains unchanged:

$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)\,dt + \theta\sigma_t\,d\tilde{W}_t, \quad \sigma_0 \geq 0,$$

where $W = \{W_t, t \geq 0\}$ and $\tilde{W} = \{\tilde{W}_t, t \geq 0\}$ are two correlated standard Brownian motions such that $\text{Cov}[dW_t\,d\tilde{W}_t] = \rho\,dt$. Finally, J_t and N are independent, as well as of W and of $\tilde{W}$.

The characteristic function $\phi(u,t)$ is in this case given by:

$$\begin{aligned}
\phi(u,t) &= E[\exp(\mathrm{i}u\log(S_t))|S_0, \sigma_0^2] \\
&= \exp(\mathrm{i}u(\log S_0 + (r-q)t)) \\
&\quad \times \exp(\eta\kappa\theta^{-2}((\kappa - \rho\theta u\mathrm{i} - d)t - 2\log((1-g\mathrm{e}^{-dt})/(1-g)))) \\
&\quad \times \exp(\sigma_0^2\theta^{-2}(\kappa - \rho\theta\mathrm{i}u - d)(1-\mathrm{e}^{-dt})/(1-g\mathrm{e}^{-dt})), \\
&\quad \times \exp(-\lambda\mu_J\mathrm{i}ut + \lambda t((1+\mu_J)^{\mathrm{i}u}\exp(\sigma_J^2(\mathrm{i}u/2)(\mathrm{i}u-1)) - 1)),
\end{aligned}$$

where d and g are as in equations (4.1) and (4.2).

4.2.3 The Barndorff-Nielsen–Shephard model

This class of models, denoted by BN–S, were introduced in Barndorff-Nielsen and Shephard [2] and have a comparable structure to HEST. The volatility is now modelled by an Ornstein Uhlenbeck (OU) process driven by a subordinator. We use the classical and tractable example of the Gamma-OU process. The marginal law of the volatility is Gamma-distributed. Volatility can only jump upwards and then it will decay exponentially. A co-movement effect between up-jumps in volatility and (down)-jumps in the stock price is also incorporated. The price of the asset will jump downwards when an up-jump in volatility takes place. In the absence of a jump, the asset price process moves continuously and the volatility decays also continuously. Other choices for OU-processes can be made: we mention especially the Inverse Gaussian OU process, leading also to a tractable model.

The squared volatility now follows a SDE of the form:

$$d\sigma_t^2 = -\lambda\sigma_t^2\,dt + dz_{\lambda t}, \tag{4.3}$$

where $\lambda > 0$ and $z = \{z_t, t \geq 0\}$ is a subordinator as introduced before.

The risk-neutral dynamics of the log-price $Z_t = \log S_t$ are given by:

$$dZ_t = (r - q - \lambda k(-\rho) - \sigma_t^2/2)\,dt + \sigma_t\,dW_t + \rho\,dz_{\lambda t}, \qquad Z_0 = \log S_0,$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of $z = \{z_t, t \geq 0\}$ and where $k(u) = \log E[\exp(-uz_1)]$ is the cumulant function of z_1. Note that the parameter ρ is introducing a co-movement effect between the volatility and the asset price process.

As stated above, we chose the Gamma-OU process. For this process, $z = \{z_t, t \geq 0\}$ is a compound-Poisson process:

$$z_t = \sum_{n=1}^{N_t} x_n, \tag{4.4}$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter a, i.e. $E[N_t] = at$ and $\{x_n, n = 1, 2, \dots\}$ is an independent and identically distributed sequence, and each x_n follows an exponential law with mean $1/b$. One can show that the process $\sigma^2 = \{\sigma_t^2, t \geq 0\}$ is a stationary process with a marginal law that follows a Gamma distribution with mean a and variance a/b. This means that if one starts the process with an initial value sampled from this Gamma distribution, at each future time point t, σ_t^2 is also following that Gamma distribution. Under this law, the cumulant function reduces to:

$$k(u) = \log E[\exp(-uz_1)] = -au(b+u)^{-1}.$$

In this case, one can write the characteristic function [3] of the log price in the form:

$$\begin{aligned}\phi(u,t) &= E[\exp(\mathrm{i}u \log S_t)|S_0, \sigma_0] \\ &= \exp\left(\mathrm{i}u(\log(S_0) + (r - q - a\lambda\rho(b-\rho)^{-1})t)\right) \\ &\quad \times \exp\left(-\lambda^{-1}(u^2 + \mathrm{i}u)(1 - \exp(-\lambda t))\sigma_0^2/2\right) \\ &\quad \times \exp\left(a(b - f_2)^{-1}\left(b \log\left(\frac{b - f_1}{b - \mathrm{i}u\rho}\right) + f_2\lambda t\right)\right),\end{aligned}$$

where

$$\begin{aligned}f_1 &= f_1(u) = \mathrm{i}u\rho - \lambda^{-1}(u^2 + \mathrm{i}u)(1 - \exp(-\lambda t))/2, \\ f_2 &= f_2(u) = \mathrm{i}u\rho - \lambda^{-1}(u^2 + \mathrm{i}u)/2.\end{aligned}$$

4.2.4 Lévy models with stochastic time

Another way to build in stochastic volatility effects is by making time stochastic. Periods with high volatility can be looked at as if time runs faster than in periods with low volatility. Applications of stochastic time change to asset pricing go back to Clark [9], who models the asset price as a geometric Brownian motion time-changed by an independent Lévy subordinator.

The Lévy models with stochastic time considered in this paper are built out of two independent stochastic processes. The first process is a Lévy process. The behaviour of the asset price will be modelled by the exponential of the Lévy process suitably time-changed. Typical examples are the Normal distribution, leading to the Brownian motion, the Normal Inverse Gaussian (NIG) distribution, the Variance Gamma (VG) distribution, the (generalized) hyperbolic distribution, the Meixner distribution, the CGMY distribution and many others. An overview can be found in Schoutens [22]. We opt to work with the VG and NIG processes for which simulation issues become quite standard.

The second process is a stochastic clock that builds in a stochastic volatility effect by making time stochastic. The above mentioned (first) Lévy process will be subordinated (or time-changed) by this stochastic clock. By definition of a subordinator, the time needs to increase and the process modelling the rate of time change $y = \{y_t, t \geq 0\}$ needs also to be positive. The economic time elapsed in t units of calendar time is then given by the integrated process $Y = \{Y_t, t \geq 0\}$ where

$$Y_t = \int_0^t y_s \, \mathrm{d}s.$$

Since y is a positive process, Y is an increasing process. We investigate two processes y which can serve for the rate of time change: the CIR process (continuous) and the Gamma-OU process (jump process).

We first discuss NIG and VG and subsequently introduce the stochastic clocks CIR and Gamma-OU. In order to model the stock price process as a time-changed Lévy process, one needs the link between the stochastic clock and the Lévy process. This role will be fulfilled by the characteristic function enclosing both independent processes as described at the end of this section.

NIG Lévy Process. A NIG process is based on the Normal Inverse Gaussian (NIG) distribution, NIG(α, β, δ), with parameters $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$. Its characteristic function is given by:

$$\phi_{NIG}(u; \alpha, \beta, \delta) = \exp\left(-\delta\left(\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right).$$

Since this is an infinitely divisible characteristic function, one can define the NIG-process $X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$, with $X_0^{(NIG)} = 0$, as the process having stationary and independent NIG distributed increments. So, an increment over the time interval $[s, s+t]$ follows a NIG$(\alpha, \beta, \delta t)$ law. A NIG-process is a pure jump process. One can relate the NIG-process to an Inverse Gaussian time-changed Brownian motion, which is particularly useful for simulation issues (see Section 4.4.1).

VG Lévy Process. The characteristic function of the VG(C, G, M), with parameters $C > 0$, $G > 0$ and $M > 0$, is given by:

$$\phi_{VG}(u; C, G, M) = \left(\frac{GM}{GM + (M - G)\mathrm{i}u + u^2}\right)^C.$$

This distribution is infinitely divisible and one can define the VG-process $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$ as the process which starts at zero, has independent and stationary increments and where the increment $X_{s+t}^{(VG)} - X_s^{(VG)}$ over the time interval $[s, s+t]$ follows a VG(Ct, G, M) law. In Madan *et al.* [19], it was shown that the VG-process may also be expressed as the difference of two independent Gamma processes, which is helpful for simulation issues (see Section 4.4.2).

CIR Stochastic Clock. Carr *et al.* [8] use as the rate of time change the CIR process that solves the SDE:

$$dy_t = \kappa(\eta - y_t)\,dt + \lambda y_t^{1/2}\,dW_t,$$

where $W = \{W_t, t \geq 0\}$ is a standard Brownian motion. The characteristic function of Y_t (given y_0) is explicitly known (see Cox *et al.* [12]):

$$\begin{aligned}\varphi_{CIR}(u, t; \kappa, \eta, \lambda, y_0) &= E[\exp(iuY_t)|y_0]\\ &= \frac{\exp(\kappa^2\eta t/\lambda^2)\exp(2y_0 iu/(\kappa + \gamma\coth(\gamma t/2)))}{(\cosh(\gamma t/2) + \kappa\sinh(\gamma t/2)/\gamma)^{2\kappa\eta/\lambda^2}},\end{aligned}$$

where

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}.$$

Gamma-OU Stochastic Clock. The rate of time change is now a solution of the SDE:

$$dy_t = -\lambda y_t\,dt + dz_{\lambda t}, \tag{4.5}$$

where the process $z = \{z_t, t \geq 0\}$ is as in equation (4.4) a compound Poisson process. In the Gamma-OU case, the characteristic function of Y_t (given y_0) can be given explicitly.

$$\begin{aligned}&\varphi_{\Gamma-OU}(u; t, \lambda, a, b, y_0) = E[\exp(iuY_t)|y_0]\\ &= \exp\left(iuy_0\lambda^{-1}(1 - e^{-\lambda t}) + \frac{\lambda a}{iu - \lambda b}\left(b\log\left(\frac{b}{b - iu\lambda^{-1}(1 - e^{-\lambda t})}\right) - iut\right)\right).\end{aligned}$$

Time-Changed Lévy Process. Let $Y = \{Y_t, t \geq 0\}$ be the process we choose to model our business time (remember that Y is the integrated process of y). Let us denote by $\varphi(u; t, y_0)$ the characteristic function of Y_t given y_0. The (risk-neutral) price process $S = \{S_t, t \geq 0\}$ is now modelled as follows:

$$S_t = S_0\frac{\exp((r - q)t)}{E[\exp(X_{Y_t})|y_0]}\exp(X_{Y_t}), \tag{4.6}$$

where $X = \{X_t, t \geq 0\}$ is a Lévy process. The factor $\exp((r-q)t)/E[\exp(X_{Y_t})|y_0]$ puts us immediately into the risk-neutral world by a mean-correcting argument. Basically, we model the stock price process as the ordinary exponential of a time-changed Lévy process. The process incorporates jumps (through the Lévy process X_t) and stochastic volatility (through the time change Y_t). The characteristic function $\phi(u, t)$ for the log of our stock price is given by:

$$\begin{aligned}\phi(u, t) &= E[\exp(iu\log(S_t))|S_0, y_0]\\ &= \exp(iu((r - q)t + \log S_0))\frac{\varphi(-i\psi_X(u); t, y_0)}{\varphi(-i\psi_X(-i); t, y_0)^{iu}},\end{aligned} \tag{4.7}$$

where

$$\psi_X(u) = \log E[\exp(\mathrm{i}uX_1)];$$

$\psi_X(u)$ is called the characteristic exponent of the Lévy process,

Since we consider two Lévy processes (VG and NIG) and two stochastic clocks (CIR and Gamma-OU), we will finally end up with four resulting models abbreviated as VG-CIR, VG-OUΓ, NIG-CIR and NIG-OUΓ. Because of (time)-scaling effects, one can set $y_0 = 1$, and scale the present rate of time change to one. More precisely, we have that the characteristic function $\phi(u, t)$ of equation (4.7) satisfies:

$$\begin{aligned}
\phi_{NIG\text{-}CIR}(u, t; \alpha, \beta, \delta, \kappa, \eta, \lambda, y_0) &= \phi_{NIG\text{-}CIR}(u, t; \alpha, \beta, \delta y_0, \kappa, \eta/y_0, \lambda/\sqrt{y_0}, 1), \\
\phi_{NIG-\Gamma OU}(u, t; \alpha, \beta, \delta, \lambda, a, b, y_0) &= \phi_{NIG-\Gamma OU}(u, t; \alpha, \beta, \delta y_0, \lambda, a, by_0, 1), \\
\phi_{VG\text{-}CIR}(u, t; C, G, M, \kappa, \eta, \lambda, y_0) &= \phi_{VG\text{-}CIR}(u, t; Cy_0, G, M, \kappa, \eta/y_0, \lambda/\sqrt{y_0}, 1), \\
\phi_{VG-\Gamma OU}(u, t; C, G, M, \lambda, a, b, y_0) &= \phi_{VG-\Gamma OU}(u, t; Cy_0, G, M, \lambda, a, by_0, 1).
\end{aligned}$$

Actually, this time-scaling effect lies at the heart of the idea of incorporating stochastic volatility through making time stochastic. Here, it comes down to the fact that instead of making the volatility parameter (of the Black–Scholes model) stochastic, we are making the parameter δ in the NIG case and the parameter C in the VG case stochastic (via the time). Note that this effect does not only influence the standard deviation (or volatility) of the processes; the skewness and the kurtosis are also now fluctuating stochastically.

4.3 CALIBRATION

Carr and Madan [7] developed pricing methods for the classical vanilla options which can be applied in general when the characteristic function of the risk-neutral stock price process is known.

Let α be a positive constant such that the αth moment of the stock price exists. For all stock price models encountered here, typically a value of $\alpha = 0.75$ will do fine. Carr and Madan then showed that the price $C(K, T)$ of a European call option with strike K and time to maturity T is given by:

$$C(K, T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-\mathrm{i}v \log(K))\varrho(v)\,\mathrm{d}v, \tag{4.8}$$

where

$$\varrho(v) = \frac{\exp(-rT)E[\exp(\mathrm{i}(v - (\alpha+1)\mathrm{i})\log(S_T))]}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v} \tag{4.9}$$

$$= \frac{\exp(-rT)\phi(v - (\alpha+1)\mathrm{i}, T)}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v}. \tag{4.10}$$

Using Fast Fourier Transforms, one can compute within a second the complete option surface on an ordinary computer. We apply the above calculation method in our calibration procedure and estimate the model parameters by minimizing the difference between market prices and model prices in a least-squares sense.

The data set consists of 144 plain vanilla call option prices with maturities ranging from less than one month up to 5.16 years. These prices are based on the implied volatility surface of the Eurostoxx 50 index, having a value of 2461.44 on October 7th, 2003. The volatilities can be found in Table 4.1. For the sake of simplicity and to focus on the essence of the stochastic behaviour of the asset, we set the risk-free interest rate equal to 3 percent and the dividend yield to zero.

Table 4.1 Implied volatility surface data (Eurostoxx 50 index; October 7th, 2003)

Strike	Maturity (year fraction)					
	0.0361	0.2000	1.1944	2.1916	4.2056	5.1639
1081.82			0.3804	0.3451	0.3150	0.3137
1212.12			0.3667	0.3350	0.3082	0.3073
1272.73			0.3603	0.3303	0.3050	0.3043
1514.24			0.3348	0.3116	0.2920	0.2921
1555.15			0.3305	0.3084	0.2899	0.2901
1870.30		0.3105	0.2973	0.2840	0.2730	0.2742
1900.00		0.3076	0.2946	0.2817	0.2714	0.2727
2000.00		0.2976	0.2858	0.2739	0.2660	0.2676
2100.00	0.3175	0.2877	0.2775	0.2672	0.2615	0.2634
2178.18	0.3030	0.2800	0.2709	0.2619	0.2580	0.2600
2200.00	0.2990	0.2778	0.2691	0.2604	0.2570	0.2591
2300.00	0.2800	0.2678	0.2608	0.2536	0.2525	0.2548
2400.00	0.2650	0.2580	0.2524	0.2468	0.2480	0.2505
2499.76	0.2472	0.2493	0.2446	0.2400	0.2435	0.2463
2500.00	0.2471	0.2493	0.2446	0.2400	0.2435	0.2463
2600.00		0.2405	0.2381	0.2358	0.2397	0.2426
2800.00			0.2251	0.2273	0.2322	0.2354
2822.73			0.2240	0.2263	0.2313	0.2346
2870.83			0.2213	0.2242	0.2295	0.2328
2900.00			0.2198	0.2230	0.2288	0.2321
3000.00			0.2148	0.2195	0.2263	0.2296
3153.64			0.2113	0.2141	0.2224	0.2258
3200.00			0.2103	0.2125	0.2212	0.2246
3360.00			0.2069	0.2065	0.2172	0.2206
3400.00			0.2060	0.2050	0.2162	0.2196
3600.00				0.1975	0.2112	0.2148
3626.79				0.1972	0.2105	0.2142
3700.00				0.1964	0.2086	0.2124
3800.00				0.1953	0.2059	0.2099
4000.00				0.1931	0.2006	0.2050
4070.00					0.1988	0.2032
4170.81					0.1961	0.2008
4714.83					0.1910	0.1957
4990.91					0.1904	0.1949
5000.00					0.1903	0.1949
5440.18						0.1938

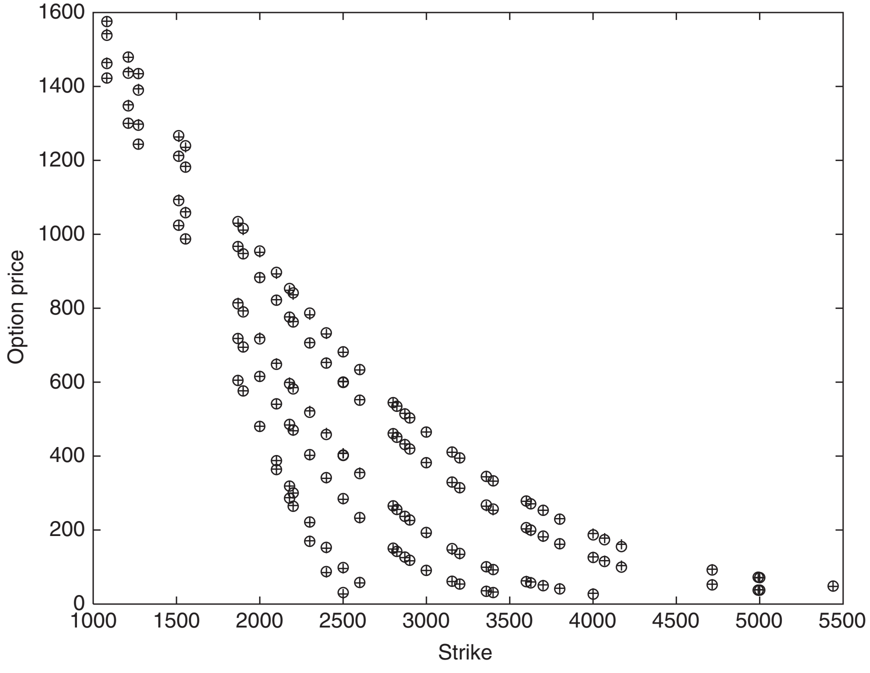

Figure 4.1 Calibration of the NIG-CIR model (Eurostoxx 50 index; October 7th, 2003)

Contrary to the approach described in Hirsa *et al.* [15], we search for a single global set of parameters per model which we fit (and which captures smile information) across the full range of maturities in the data set. This global parameter set can then be used to price path-dependent derivatives (e.g. payoffs at multiple points during its lifetime or moment derivatives; see Sections 4.6 and 4.7). This is in contrast with the parameter set resulting from a fitting procedure at a single maturity date, which can in principle only be used to price option payoffs occurring at that specific maturity.

The results of the global calibration are visualized in Figures 4.1 and 4.2 for the NIG-CIR and the BN−S model, respectively. The other models give rise to completely similar figures. Here, the circles are the market prices and the plus signs are the analytical prices (calculated via equation (4.8) using the respective characteristic functions and obtained parameters).

In Table 4.2, one finds the risk-neutral parameters for the different models. For comparative purposes, one computes several global measures of fit. We consider the root mean square error (*rmse*), the average absolute error as a percentage of the mean price (*ape*), the average absolute error (*aae*) and the average relative percentage error (*arpe*):

$$rmse = \sqrt{\sum_{options} \frac{(\text{Market price} - \text{Model price})^2}{\text{number of options}}}$$

$$ape = \frac{1}{\text{mean option price}} \sum_{options} \frac{|\text{Market price} - \text{Model price}|}{\text{number of options}}$$

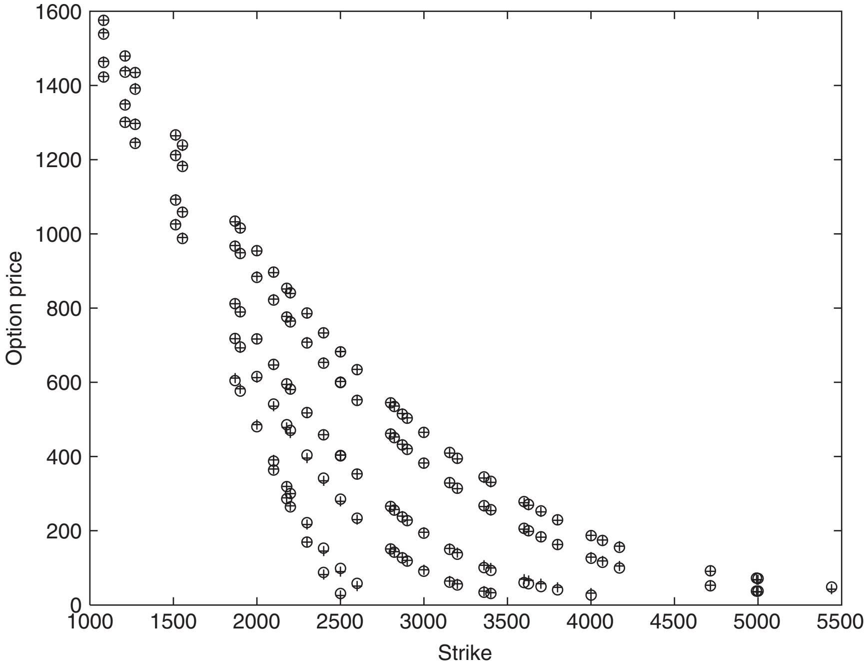

Figure 4.2 Calibration of the BN−S model (Eurostoxx 50 index; October 7th, 2003)

Table 4.2 Risk-neutral parameters for the different models

HEST
$\sigma_0^2 = 0.0654, \kappa = 0.6067, \eta = 0.0707, \theta = 0.2928, \rho = -0.7571$
HESJ
$\sigma_0^2 = 0.0576, \kappa = 0.4963, \eta = 0.0650, \theta = 0.2286, \rho = -0.9900, \mu_j = 0.1791,$ $\sigma_j = 0.1346, \lambda = 0.1382$
BN−S
$\rho = -4.6750, \lambda = 0.5474, b = 18.6075, a = 0.6069, \sigma_0^2 = 0.0433$
VG-CIR
$C = 18.0968, G = 20.0276, M = 26.3971, \kappa = 1.2145, \eta = 0.5501,$ $\lambda = 1.7913, y_0 = 1$
VG-OUΓ
$C = 6.1610, G = 9.6443, M = 16.0260, \lambda = 1.6790, a = 0.3484,$ $b = 0.7664, y_0 = 1$
NIG-CIR
$\alpha = 16.1975, \beta = -3.1804, \delta = 1.0867, \kappa = 1.2101, \eta = 0.5507,$ $\lambda = 1.7864, y_0 = 1$
NIG-OUΓ
$\alpha = 8.8914, \beta = -3.1634, \delta = 0.6728, \lambda = 1.7478, a = 0.3442,$ $b = 0.7628, y_0 = 1$

$$aae = \sum_{options} \frac{|\text{Market price} - \text{Model price}|}{\text{number of options}}$$

$$arpe = \frac{1}{\text{number of options}} \sum_{options} \frac{|\text{Market price} - \text{Model price}|}{\text{Market price}}$$

In Table 4.3, an overview of these measures of fit are given.

Table 4.3 Global fit error measures for the different models

Model	*rmse*	*ape*	*aae*	*arpe*
HEST	3.0281	0.0048	2.4264	0.0174
HESJ	2.8101	0.0045	2.2469	0.0126
BN–S	3.5156	0.0056	2.8194	0.0221
VG-CIR	2.3823	0.0038	1.9337	0.0106
VG-OUΓ	3.4351	0.0056	2.8238	0.0190
NIG-CIR	2.3485	0.0038	1.9194	0.0099
NIG-OUΓ	3.2737	0.0054	2.7385	0.0175

4.4 SIMULATION

In this section, we describe in detail how the particular processes presented in Section 4.2, can be implemented in practice in a Monte Carlo simulation pricing framework. For this, we first discuss the numerical implementation of the four building block processes which drive them. This will be followed by an explanation of how one assembles a time-changed Lévy process.

4.4.1 NIG Lévy process

To simulate a NIG process, we first describe how to simulate NIG(α, β, δ) random numbers. The latter can be obtained by mixing Inverse Gaussian (IG) random numbers and standard Normal numbers in the following manner. An IG(a, b) random variable X has a characteristic function given by:

$$E[\exp(\mathrm{i}uX)] = \exp(-a\sqrt{-2u\mathrm{i} + b^2} - b).$$

First, simulate IG$(1, \delta\sqrt{\alpha^2 - \beta^2})$ random numbers i_k, for example, by using the Inverse Gaussian generator of Michael, Schucany and Haas (see Devroye [13]). Then sample a sequence of standard Normal random variables u_k. NIG random numbers n_k are then obtained via:

$$n_k = \delta^2\beta\ i_k + \delta\sqrt{i_k}u_k.$$

Finally, the sample paths of a NIG(α, β, δ) process $X = \{X_t, t \geq 0\}$ in the time points $t_n = n\Delta t$, $n = 0, 1, 2, \ldots$ can be generated by using the independent NIG$(\alpha, \beta, \delta\Delta t)$ random numbers n_k as follows:

$$X_0 = 0, \qquad X_{t_k} = X_{t_{k-1}} + n_k, \quad k \geq 1.$$

4.4.2 VG Lévy process

Since a VG process can be viewed as the difference of two independent Gamma processes, the simulation of a VG process becomes straightforward. A Gamma process with parameters $a, b > 0$ is a Lévy process with Gamma(a, b) distributed increments, i.e. following a Gamma distribution with mean a/b and variance a/b^2. A VG process $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$ with parameters $C, G, M > 0$ can be decomposed as $X_t^{(VG)} = G_t^{(1)} - G_t^{(2)}$, where $G^{(1)} = \{G_t^{(1)}, t \geq 0\}$ is a Gamma process with parameters $a = C$ and $b = M$ and $G^{(2)} = \{G_t^{(2)}, t \geq 0\}$ is a Gamma process with parameters $a = C$ and $b = G$. The generation of Gamma numbers is quite standard. Possible generators are Johnk's gamma generator and Berman's gamma generator [13].

4.4.3 CIR stochastic clock

The simulation of a CIR process $y = \{y_t, t \geq 0\}$ is straightforward. Basically, we discretize the SDE:

$$\mathrm{d}y_t = \kappa(\eta - y_t)\,\mathrm{d}t + \lambda y_t^{1/2}\,\mathrm{d}W_t, \quad y_0 \geq 0,$$

where W_t is a standard Brownian motion. Using a first-order accurate explicit differencing scheme in time, the sample path of the CIR process $y = \{y_t, t \geq 0\}$ in the time points $t = n\Delta t$, $n = 0, 1, 2, \ldots$, is then given by:

$$y_{t_n} = y_{t_{n-1}} + \kappa(\eta - y_{t_{n-1}})\Delta t + \lambda y_{t_{n-1}}^{1/2}\sqrt{\Delta t}\; v_n,$$

where $\{v_n, n = 1, 2, \ldots\}$ is a series of independent standard Normally distributed random numbers. For other more involved simulation schemes, like the Milstein scheme, resulting in a higher-order discretization in time, we refer to Jäckel [17].

4.4.4 Gamma-OU stochastic clock

Recall that for the particular choice of an OU-Gamma process, the subordinator $z = \{z_t, t \geq 0\}$ in (equation (4.3)) is given by the compound Poisson process (equation (4.4)).

To simulate a Gamma(a, b)-OU process $y = \{y_t, t \geq 0\}$ in the time points $t_n = n\Delta t$, $n = 0, 1, 2, \ldots$, we first simulate in the same time points a Poisson process $N = \{N_t, t \geq 0\}$ with intensity parameter $a\lambda$. Then (with the convention that an empty sum equals zero)

$$y_{t_n} = (1 - \lambda\Delta t)y_{t_{n-1}} + \sum_{k=N_{t_{n-1}}+1}^{N_{t_n}} x_k \exp(-\lambda\Delta t\tilde{u}_k),$$

where $\tilde{u}_k$ is a series of independent uniformly distributed random numbers and x_k can be obtained from your preferred uniform random number generator via $x_k = -\log(u_k)/b$.

4.4.5 Path generation for time-changed Lévy process

The explanation of the building block processes above allow us next to assemble all the parts of the time-changed Lévy process simulation puzzle. For this one can proceed through the following five steps [22]:

(i) simulate the rate of time change process $y = \{y_t, 0 \leq t \leq T\}$;
(ii) calculate from (i) the time change $Y = \{Y_t = \int_0^t y_s \, ds, 0 \leq t \leq T\}$;
(iii) simulate the Lévy process $X = \{X_t, 0 \leq t \leq Y_T\}$;
(iv) calculate the time changed Lévy process X_{Y_t}, for $0 \leq t \leq T$;
(v) calculate the stock price process using equation (4.6). The mean correcting factor is calculated as:

$$\frac{\exp((r-q)t)}{E[\exp(X_{Y_t})|y_0]} = \frac{\exp((r-q)t)}{\varphi(-\mathrm{i}\psi_X(-\mathrm{i}); t, 1)}.$$

4.5 PRICING OF EXOTIC OPTIONS

As evidenced by the quality of the calibration on a set of European call options in Section 4.3, we can hardly discriminate between the different processes on the basis of their smile-conform pricing characteristics. We therefore put the models further to the test by applying them to a range of more exotic options. These range from digital barriers, one-touch barrier options, lookback options and finally cliquet options with local as well as global parameters. These first-generation exotics with path-dependent payoffs were selected since they shed more light on the dynamics of the stock processes. At the same time, the pricings of the cliquet options are highly sensitive to the forward smile characteristics induced by the models.

4.5.1 Exotic options

Let us consider contracts of duration T, and denote the maximum and minimum process, respectively, of a process $Y = \{Y_t, 0 \leq t \leq T\}$ as

$$M_t^Y = \sup\{Y_u; 0 \leq u \leq t\} \text{ and } m_t^Y = \inf\{Y_u; 0 \leq u \leq t\}, \quad 0 \leq t \leq T.$$

4.5.1.1 Digital barriers

We first consider digital barrier options. These options remain worthless unless the stock price hits some predefined barrier level $H > S_0$, in which case they pay at expiry a fixed amount D, normalized to 1 in the current settings. Using risk-neutral valuation, assuming no dividends and a constant interest rate r, the time $t = 0$ price is therefore given by:

$$\text{digital} = \mathrm{e}^{-rT} E_Q[1(M_T^S \geq H)],$$

where the expectation is taken under the risk-neutral measure Q.

Observe that with the current definition of digital barriers their pricing reflects exactly the chance of hitting the barrier prior to expiry. The behaviour of the stock after the barrier has been hit does not influence the result, in contrast with the classic barrier options defined below.

4.5.1.2 One-touch barrier options

For one-touch barrier call options, we focus on the following four types:

- The down-and-out barrier call is worthless unless its minimum remains above some 'low barrier' H, in which case it retains the structure of a European call with strike K. Its initial price is given by:

$$DOB = e^{-rT} E_Q[(S_T - K)^+ 1(m_T^S > H)]$$

- The down-and-in barrier is a normal European call with strike K, if its minimum went below some 'low barrier' H. If this barrier was never reached during the lifetime of the option, the option remains worthless. Its initial price is given by:

$$DIB = e^{-rT} E_Q[(S_T - K)^+ 1(m_T^S \leq H)]$$

- The up-and-in barrier is worthless unless its maximum crossed some 'high barrier' H, in which case it obtains the structure of a European call with strike K. Its price is given by:

$$UIB = e^{-rT} E_Q[(S_T - K)^+ 1(M_T^S \geq H)]$$

- The up-and-out barrier is worthless unless its maximum remains below some 'high barrier' H, in which case it retains the structure of a European call with strike K. Its price is given by:

$$UOB = e^{-rT} E_Q[(S_T - K)^+ 1(M_T^S < H)]$$

4.5.1.3 Lookback options

The payoff of a lookback call option corresponds to the difference between the stock price level at expiry S_T and the lowest level it has reached during its lifetime. The time $t = 0$ price of a lookback call option is therefore given by:

$$LC = e^{-rT} E_Q[S_T - m_T^S].$$

Clearly, of the three path-dependent options introduced so far, the lookback option depends the most on the precise path dynamics.

4.5.1.4 Cliquet options

Finally, we also test the proposed models on the pricing of cliquet options. These still are very popular options in the equity derivatives world which allow the investor to participate (partially) in the performance of an underlying over a series of consecutive time periods $[t_i, t_{i+1}]$ by 'clicking in' the sum of these local performances. The latter are measured relative to the stock level S_{t_i} attained at the start of each new subperiod, and each of the local performances is floored and/or capped to establish whatever desirable mix of positive and/or negative payoff combination. Generally, on the final sum an additional global floor (cap) is applied to guarantee a minimum (maximum) overall payoff. This can all be summarized through the following payoff formula:

$$\min\left(cap_{glob}, \max\left(floor_{glob}, \sum_{i=1}^{N} \min\left(cap_{loc}, \max\left(floor_{loc}, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)\right)\right)\right)$$

Observe that the local floor and cap parameters effectively border the relevant ‘local’ price ranges by centering them around the future, and therefore unknown, spot levels S_{t_i}. The pricing will therefore depend in a non-trivial subtle manner on the forward volatility smile dynamics of the respective models, further complicated by the global parameters of the contract. For an in-depth account of the related volatility issues, we refer to Wilmott [24].

4.5.2 Exotic option prices

We price all exotic options through Monte Carlo simulation. We consistently average over 1 000 000 simulated paths. All options have a lifetime of three years. In order to check the accuracy of our simulation algorithm, we simulated option prices for all European calls available in the calibration set. All algorithms gave a very satisfactory result, with pricing differences with respect to their analytic calibration values of less than 0.5 percent.

An important issue for the path-dependent lookback, barrier and digital barrier options above, is the frequency at which the stock price is observed for purposes of determining whether the barrier or its minimum level have been reached. In the numerical calculations below, we have assumed a discrete number of observations, namely at the close of each trading day. Moreover, we have assumed that a year consists of 250 trading days.

In Figure 4.3, we present simulation results with models for the digital barrier call option as a function of the barrier level (ranging from $1.05S_0$ to $1.5S_0$). As mentioned before, aside from the discounting factor e^{-rT}, the premiums can be interpreted as the chance of hitting the barrier during the option lifetime. In Figures 4.4–4.6, we show prices for all

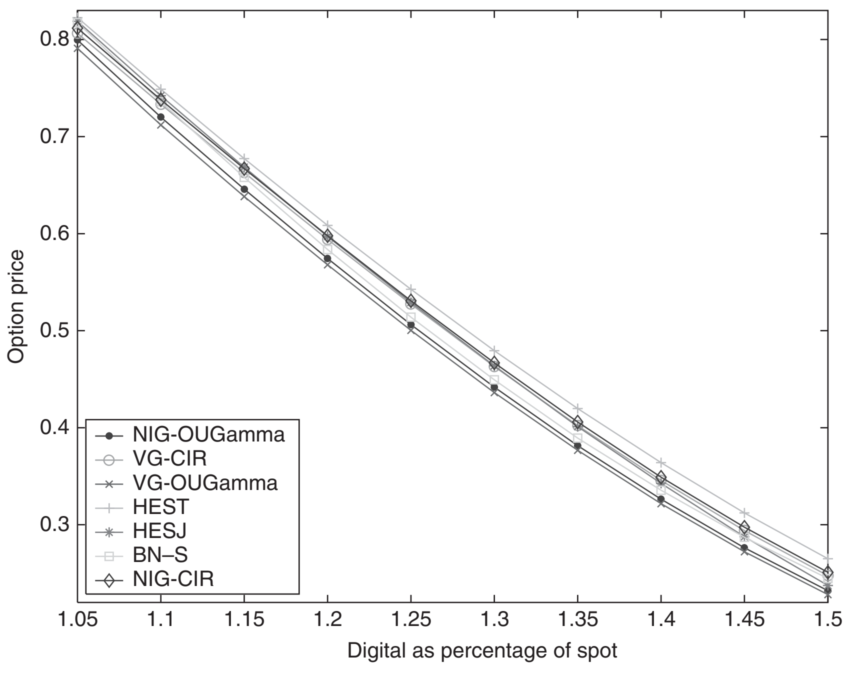

Figure 4.3 Digital barrier prices (Eurostoxx 50 index; October 7th, 2003)

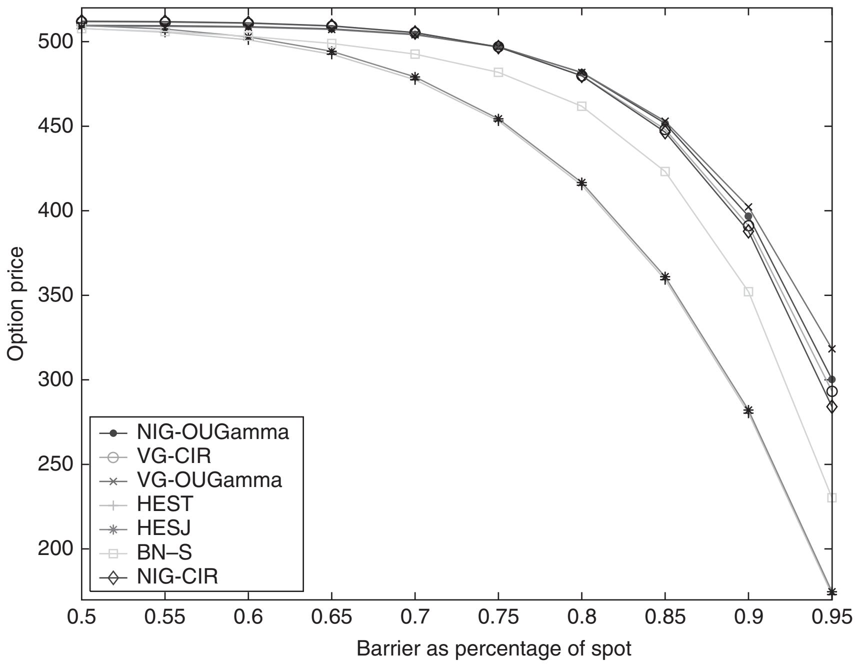

Figure 4.4 DOB prices (Eurostoxx 50 index; October 7th, 2003)

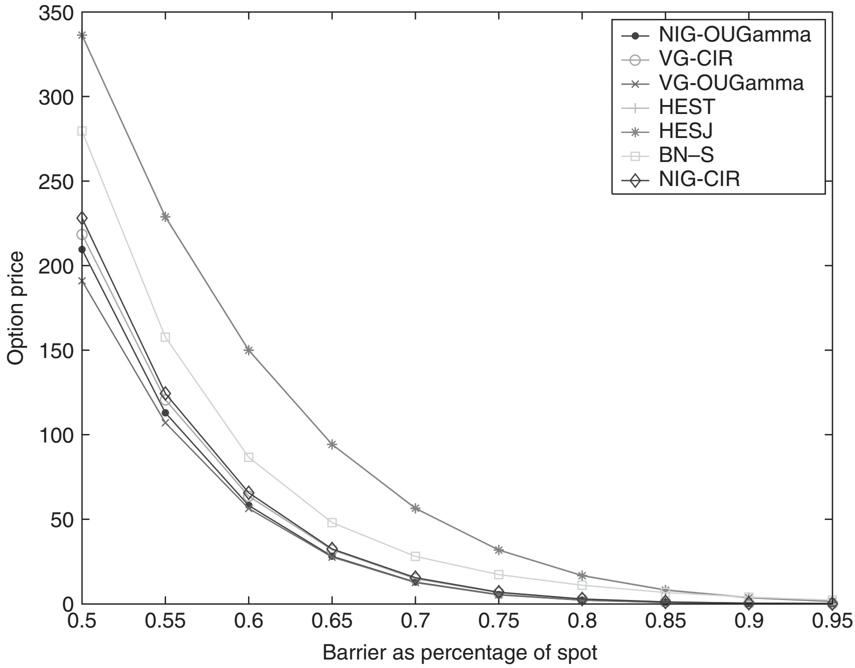

Figure 4.5 DIB prices (Eurostoxx 50 index; October 7th, 2003)

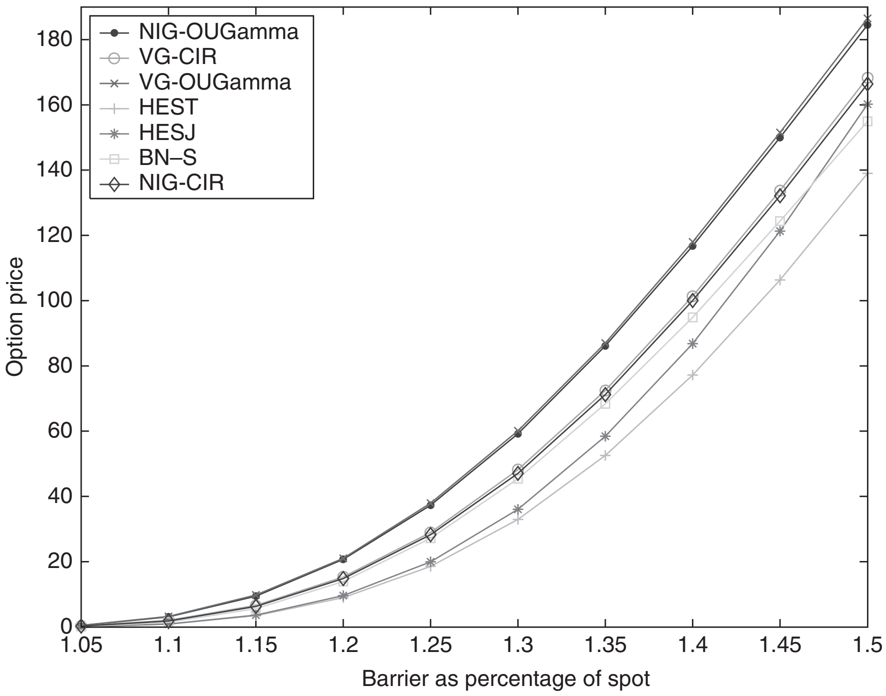

Figure 4.6 UOB prices (Eurostoxx 50 index; October 7th, 2003)

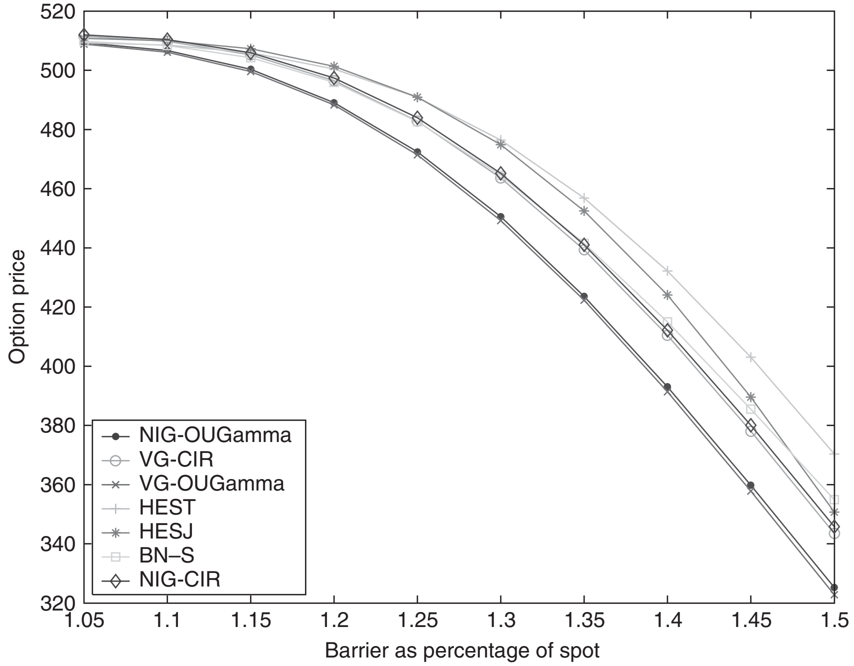

Figure 4.7 UIB prices (Eurostoxx 50 index; October 7th, 2003)

Table 4.4 Exotic option prices

	H/S_0	NIG-OUT	VG-CIR	VG-OUT	HEST	HESJ	BN–S	NIG-CIR
LC		722.34	724.80	713.49	844.51	845.18	771.28	730.84
Call		509.76	511.80	509.33	510.88	510.89	509.89	512.21
DOB	0.95	300.25	293.28	318.35	173.85	174.64	230.25	284.10
DOB	0.9	396.80	391.17	402.24	280.79	282.09	352.14	387.83
DOB	0.85	451.61	448.10	452.97	359.05	360.99	423.21	446.52
DOB	0.8	481.65	479.83	481.74	414.65	416.63	461.82	479.77
DOB	0.75	497.00	496.95	496.80	452.76	454.33	481.85	496.78
DOB	0.7	504.31	505.24	504.05	477.37	479.12	492.62	505.38
DOB	0.65	507.53	509.10	507.21	492.76	494.25	498.93	509.34
DOB	0.6	508.88	510.75	508.53	501.74	502.84	503.17	511.09
DOB	0.55	509.43	511.40	509.06	506.46	507.41	505.93	511.80
DOB	0.5	509.64	511.67	509.24	508.91	509.51	507.68	512.08
DIB	0.95	209.51	218.51	190.98	337.03	336.25	279.61	228.10
DIB	0.9	112.95	120.62	107.08	230.09	228.80	157.72	124.37
DIB	0.85	58.14	63.69	56.35	151.83	149.90	86.65	65.68
DIB	0.8	28.11	31.96	27.59	96.24	94.26	48.04	32.43
DIB	0.75	12.76	14.84	12.53	58.13	56.56	28.01	15.42
DIB	0.7	5.45	6.55	5.28	33.51	31.77	17.24	6.83
DIB	0.65	2.23	2.70	2.11	18.12	16.64	10.94	2.87
DIB	0.6	0.88	1.04	0.79	9.14	8.05	6.69	1.11
DIB	0.55	0.33	0.39	0.26	4.42	3.48	3.94	0.40
DIB	0.5	0.12	0.13	0.09	1.98	1.38	2.19	0.13
UIB	1.05	509.32	511.52	508.84	510.78	510.81	509.73	511.98
UIB	1.1	506.68	509.80	506.11	500.90	510.00	508.38	510.37
UIB	1.15	500.33	505.21	499.56	507.08	507.28	504.28	505.93
UIB	1.2	489.05	496.50	488.30	501.04	501.31	495.95	497.41
UIB	1.25	472.47	482.84	471.39	490.73	490.93	482.66	483.94
UIB	1.3	450.54	463.62	449.23	475.30	474.86	464.48	465.16
UIB	1.35	423.62	439.32	422.32	454.77	452.47	441.48	441.00
UIB	1.4	393.01	410.46	391.36	428.96	424.09	414.98	412.16
UIB	1.45	359.77	378.05	357.80	399.24	389.56	385.50	380.04
UIB	1.5	325.25	343.46	322.79	365.57	350.68	354.90	345.79
UOB	1.05	0.44	0.27	0.49	0.103	0.08	0.13	0.23
UOB	1.1	3.08	2.00	3.22	0.979	0.89	1.48	1.84
UOB	1.15	9.43	6.59	9.77	3.80	3.61	5.58	6.27
UOB	1.2	20.71	15.29	21.03	8.96	9.85	13.91	14.80
UOB	1.25	37.29	28.95	37.94	20.15	19.96	27.20	28.26
UOB	1.3	59.22	48.17	60.10	35.58	36.03	45.38	47.04
UOB	1.35	86.14	72.47	87.00	56.10	58.42	68.39	71.21
UOB	1.4	116.75	101.33	117.96	81.93	86.80	94.88	100.04
UOB	1.45	149.98	133.74	151.52	111.65	121.33	124.36	132.16
UOB	1.5	184.50	168.33	186.53	145.31	160.21	154.96	166.41
DIG	1.05	0.7995	0.8064	0.7909	0.8218	0.8189	0.8173	0.8118
DIG	1.1	0.7201	0.7334	0.7120	0.7478	0.7421	0.7360	0.7380

(*continued overleaf*)

Table 4.4 (*continued*)

	H/S_0	NIG-OUT	VG-CIR	VG-OUT	HEST	HESJ	BN−S	NIG-CIR
DIG	1.15	0.6458	0.6628	0.6382	0.6762	0.6685	0.6580	0.6670
DIG	1.2	0.5744	0.5940	0.5678	0.6069	0.5971	0.5836	0.5977
DIG	1.25	0.5062	0.5273	0.5003	0.5408	0.5290	0.5138	0.5308
DIG	1.3	0.4418	0.4630	0.4363	0.4769	0.4637	0.4493	0.4668
DIG	1.35	0.3816	0.4021	0.3767	0.4169	0.4012	0.3893	0.4059
DIG	1.4	0.3264	0.3456	0.3217	0.3603	0.3426	0.3355	0.3490
DIG	1.45	0.2763	0.2940	0.2722	0.3087	0.2877	0.2870	0.2975
DIG	1.5	0.2321	0.2474	0.2280	0.2610	0.2374	0.2446	0.2510

Table 4.5 Lookback option prices for the different models

HEST	HESJ	BN−S	VG-CIR	VG-OUΓ	NIG-CIR	NIG-OUΓ
844.51	845.19	771.28	724.80	713.49	730.84	722.34

one-touch barrier options (as a percentage of the spot). The strike K was always taken equal to the spot S_0. For reference, we summarize in Table 4.4 all option prices for the above discussed exotics. One can check that the barrier results agree well with the identity DIB + DOB = vanilla call = UIB + UOB, suggesting that the simulation results are well converged. Lookback prices are presented in Table 4.5. Consistently over all of the figures the Heston prices suggest that this model (for the current calibration) results in paths dynamics that are more *volatile*, breaching more frequently the imposed barriers. The results for the Lévy models with stochastic time change seem to move in pairs, with the choice of stochastic clock dominating over the details of the Lévy model upon which the stochastic time change is applied. The first couple, VG-Γ and NIG-Γ display very similar results, overall showing the least *volatile* path dynamics, whereas the VG-CIR and NIG-CIR prices consistently fall midway of the pack. Finally, the OU-Γ results without stochastic clock typically fall between the Heston and the VG-CIR and NIG-CIR prices.

Besides these qualitative observations, it is important to note the magnitude of the observed differences. Lookback prices vary over about 15 percent and the one-touch barriers over 200 percent, whereas for the digital barriers we found price differences of over 10 percent.

For the cliquet options, the prices are shown in Figures 4.8 and 4.9 for two different combinations. The numerical values can be found in Tables 4.6 and 4.7. These results are in-line with the previous observations. Variations of over 40 percent are noted.

4.6 PRICING OF MOMENT DERIVATIVES

These derivatives depend on the realized higher moments of the underlying. More precisely, their payoff is a function of powers of the (daily) log-returns and allows to cover different kinds of market *shocks*. Variance swaps were already created to cover changes in the volatility regime. Besides the latter, skewness and kurtosis also play an important role. To protect against a wrongly estimated skewness or kurtosis, moment derivatives of higher order can

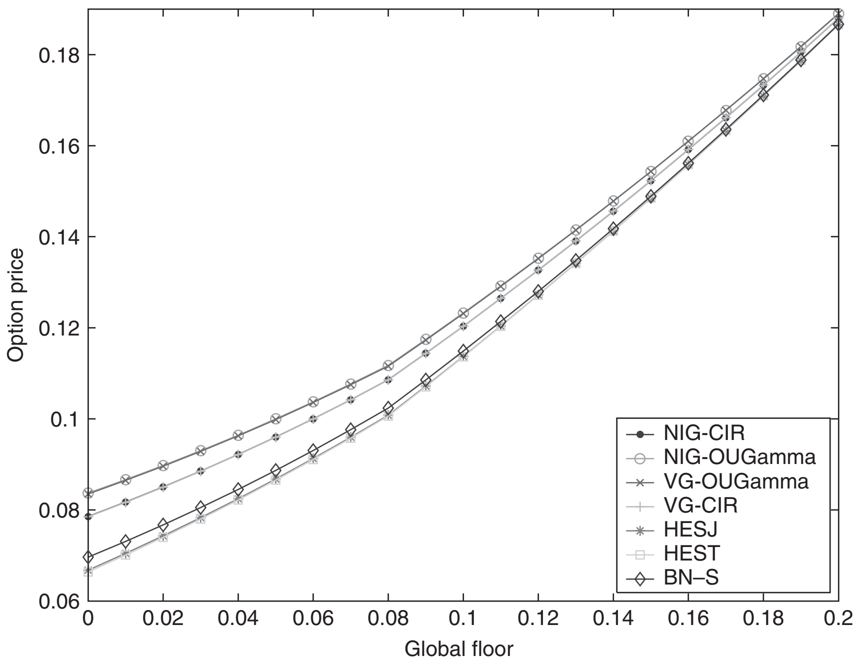

Figure 4.8 Cliquet prices: $cap_{loc} = 0.08$; $flo_{loc} = -0.08$; $cap_{glo} = +\infty$; $N = 3$; $t_1 = 1; t_2 = 3$ (Eurostoxx 50 index; October 7th, 2003)

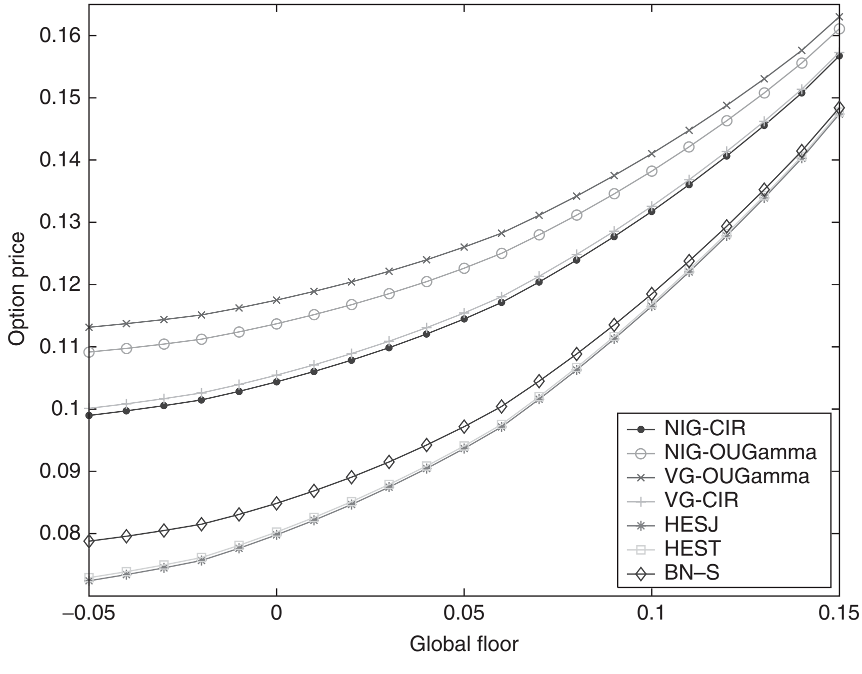

Figure 4.9 Cliquet Prices: $cap_{loc} = 0.05$; $flo_{loc} = -0.03$; $cap_{glo} = +\infty$; $T = 3$; $N = 6$; $t_i = i/2$ (Eurostoxx 50 index; October 7th, 2003)

Table 4.6 Cliquet prices: $cap_{loc} = 0.08$; $flo_{-loc} = 0.08$; $cap_{glo} = +\infty$; $flo_{glo} \in [0, 0.20]$; $N = 3$; $t_1 = 1$; $t_2 = 2$; $t_3 = 3$

flo_{glo}	NIG-CIR	NIG-OUT	VG-OUΓ	VG-CIR	HESJ	HEST	BN−S
0.00	0.0785	0.0837	0.0835	0.0785	0.0667	0.0683	0.0696
0.01	0.0817	0.0866	0.0865	0.0817	0.0704	0.0719	0.0731
0.02	0.0850	0.0897	0.0896	0.0850	0.0743	0.0757	0.0767
0.03	0.0885	0.0930	0.0928	0.0885	0.0783	0.0796	0.0805
0.04	0.0922	0.0964	0.0963	0.0921	0.0825	0.0837	0.0845
0.05	0.0960	0.1000	0.0998	0.0960	0.0868	0.0879	0.0887
0.06	0.1000	0.1037	0.1036	0.1000	0.0913	0.0923	0.0930
0.07	0.1042	0.1076	0.1075	0.1042	0.0959	0.0969	0.0976
0.08	0.1086	0.1117	0.1116	0.1085	0.1008	0.1017	0.1024
0.09	0.1144	0.1174	0.1173	0.1144	0.1072	0.1080	0.1085
0.10	0.1203	0.1232	0.1231	0.1203	0.1137	0.1145	0.1149
0.11	0.1264	0.1292	0.1291	0.1264	0.1204	0.1211	0.1214
0.12	0.1327	0.1353	0.1352	0.1327	0.1272	0.1279	0.1280
0.13	0.1391	0.1415	0.1414	0.1391	0.1342	0.1348	0.1348
0.14	0.1456	0.1478	0.1478	0.1456	0.1412	0.1418	0.1418
0.15	0.1523	0.1543	0.1543	0.1523	0.1485	0.1489	0.1489
0.16	0.1591	0.1610	0.1610	0.1591	0.1558	0.1562	0.1561
0.17	0.1661	0.1677	0.1678	0.1661	0.1633	0.1637	0.1635
0.18	0.1732	0.1747	0.1747	0.1733	0.1709	0.1712	0.1711
0.19	0.1805	0.1817	0.1818	0.1806	0.1787	0.1789	0.1788
0.20	0.1880	0.1889	0.1890	0.1880	0.1866	0.1868	0.1867

Table 4.7 Cliquet prices: $flo_{loc} = -0.03$; $cap_{loc} = 0.05$; $cap_{glo} = +\infty$; $T = 3$; $N = 6$; $t_i = i/2$

flo_{glo}	NIG-CIR	NIG-OUΓ	VG-OUΓ	VG-CIR	HESJ	HEST	BN−S
−0.05	0.0990	0.1092	0.1131	0.1001	0.0724	0.0762	0.0788
−0.04	0.0997	0.1098	0.1137	0.1008	0.0734	0.0771	0.0796
−0.03	0.1005	0.1104	0.1144	0.1017	0.0745	0.0781	0.0805
−0.02	0.1015	0.1112	0.1151	0.1026	0.0757	0.0762	0.0815
−0.01	0.1028	0.1124	0.1162	0.1039	0.0776	0.0811	0.0831
0.00	0.1044	0.1137	0.1175	0.1054	0.0798	0.0831	0.0849
0.01	0.1060	0.1152	0.1189	0.1071	0.0821	0.0853	0.0869
0.02	0.1079	0.1168	0.1204	0.1089	0.0847	0.0877	0.0891
0.03	0.1099	0.1185	0.1221	0.1109	0.0874	0.0904	0.0915
0.04	0.1121	0.1205	0.1240	0.1131	0.0904	0.0932	0.0942
0.05	0.1145	0.1226	0.1260	0.1154	0.0937	0.0963	0.0972
0.06	0.1171	0.1250	0.1283	0.1180	0.0971	0.0996	0.1004
0.07	0.1204	0.1280	0.1311	0.1213	0.1016	0.1039	0.1045
0.08	0.1239	0.1312	0.1342	0.1248	0.1063	0.1084	0.1088
0.09	0.1277	0.1346	0.1375	0.1286	0.1113	0.1132	0.1135
0.10	0.1317	0.1382	0.1410	0.1326	0.1165	0.1183	0.1185
0.11	0.1361	0.1421	0.1448	0.1368	0.1220	0.1237	0.1238
0.12	0.1406	0.1463	0.1488	0.1414	0.1278	0.1293	0.1294
0.13	0.1456	0.1508	0.1531	0.1462	0.1339	0.1352	0.1353
0.14	0.1508	0.1556	0.1576	0.1514	0.1403	0.1415	0.1415
0.15	0.1567	0.1611	0.1630	0.1573	0.1474	0.1484	0.1484

be useful. Recent studies by Nualart and Schoutens [20] [21] and Corcuera *et al.* [10] [11] suggest that functionals of powers of returns seem the natural choice to complete the market. It was shown that allowing trade in the power-assets of all orders in an incomplete Lévy market leads to a complete market. Power assets are strongly related to the realized higher moments and they mainly coincide in a discrete time framework [11].

4.6.1 Moment swaps

Consider a finite set of discrete times $\{t_0 = 0, t_1, \dots, t_n = T\}$ at which the path of the underlying is monitored. We denote the price of the underlying at these points, i.e. S_{t_i}, by S_i for simplicity. Typically, the t_i correspond to daily closing times and S_i is the closing price at day i. Note that then:

$$\log(S_i) - \log(S_{i-1}), \quad i = 1, \dots, n,$$

correspond to the daily log-returns. Next, we define the *moment swaps*. The kth-moment swap is a contract where the parties agree to exchange at maturity:

$$MOMS^{(k)} = N \times \left(\sum_{i=1}^{n} (\log(S_i) - \log(S_{i-1}))^k \right) = N \times \left(\sum_{i=1}^{n} \left(\log \left(\frac{S_i}{S_{i-1}} \right) \right)^k \right),$$

where N is the nominal amount.

A special case of these swaps is the second moment swap, better known as the Variance Swap. The non-centred payoff function in that case is given by:

$$VS = N \times \left(\sum_{i=1}^{n} (\log(S_i) - \log(S_{i-1}))^2 \right).$$

Basically, this contract swaps fixed (annualized) variance by the realized variance (second moment) and as such provides protection against unexpected or unfavourable changes in volatility. Higher moment swaps provide the same kind of protection. The $MOMS^{(3)}$ is related to realized skewness and provides protection against changes in the symmetry of the underlying distribution. $MOMS^{(4)}$ derivatives are linked to realized kurtosis and provide protection against the unexpected occurrences of very large jumps, or in other words, changes in the tail behaviour of the underlying distribution.

4.6.2 Moment options

Related to the above discussed swaps, we define the associated options on the realized kth moment. More precisely, a *moment option* of order k, pays out at maturity T:

$$\left(\sum_{i=1}^{n} (\log(S_i/S_{i-1}))^k - K \right)^+ .$$

The price of these options under risk-neutral valuation is given by:

$$MOMO^{(k)}(K, T) = \exp(-rT) E_Q \left[\left(\sum_{i=1}^{n} (\log(S_i/S_{i-1}))^k - K \right)^+ \right].$$

Note that since odd moments can be negative, the strike price for these options can range over the whole real line.

4.6.3 Hedging moment swaps

We focus on hedging the moment swaps which are written on the future price as underlying. The price process of the future is given by $F = \{F_t = \exp((r-q)(T-t))S_t\}$; we write $F_i = F_{ti}$.

In line with the results obtained by Carr and Lewis [6], first consider the following (Taylor-like) expansion of the kth power of the logarithmic function:

$$(\log(x))^k = k!\left(x - 1 - \log(x) - \frac{(\log(x))^2}{2!} - \frac{(\log(x))^3}{3!} - \cdots - \frac{(\log(x))^{k-1}}{(k-1)!} + \mathcal{O}((x-1)^{k+1})\right).$$

Substituting x by F_i/F_{i-1} leads to

$$(\log(F_i/F_{i-1}))^k = k!\left(\frac{\Delta F_i}{F_{i-1}} - \log(F_i/F_{i-1}) - \sum_{j=2}^{k-1}\frac{(\log(F_i/F_{i-1}))^j}{j!} + \mathcal{O}((\Delta F_i/F_{i-1})^{k+1})\right),$$

where $\Delta F_i = F_i - F_{i-1}$.

Summing over i gives a decomposition of the $MOMS^{(k)}$ (on a future) payoff:

$$\begin{aligned}
MOMS^{(k)} &= N \times \sum_{i=1}^{n} (\log(F_i/F_{i-1}))^k \\
&= Nk!\sum_{i=1}^{n}\left(\frac{\Delta F_i}{F_{i-1}} - \log(F_i/F_{i-1}) - \sum_{j=2}^{k-1}\frac{(\log(F_i/F_{i-1}))^j}{j!} + \mathcal{O}((\Delta F_i/F_{i-1})^{k+1})\right) \\
&= -Nk!(\log(F_T) - \log(F_0)) \\
&\quad + Nk!\sum_{i=1}^{n}\frac{\Delta F_i}{F_{i-1}} - N\sum_{j=2}^{k-1}\frac{k!}{j!}MOMS^{(j)} + \mathcal{O}\left(\sum_{i=1}^{n}(\Delta F_i/F_{i-1})^{k+1}\right)
\end{aligned} \tag{4.11}$$

Thus, up to $(k+1)$th-order terms the sum of the kth powered log-returns decomposes into the payouts from:

- $-k!$ log-contracts on the future with payoff $\log(F_T) - \log(F_0)$;
- a self-financing dynamic strategy $(k!\sum_{i=1}^{n}\frac{\Delta F_i}{F_{i-1}})$ in futures;
- a series of moment contracts of order strictly smaller than k.

The log-contract can be hedged by a dynamic trading strategy in combination with a static position in bonds, European vanilla call and put options maturing at time T. More precisely, first note that for any $L > 0$:

$$\log(F_T) - \log(F_0) = \frac{1}{L}(F_T - F_0) - u(F_T) + u(F_0), \tag{4.12}$$

for

$$u(x) = \left(\frac{x - L}{L} - \log(x) + \log(L)\right).$$

Moreover Carr and Lewis [6] show that:

$$u(F_T) = u(S_T) = \int_0^L \frac{1}{K^2}(K - S_T)^+ \, \mathrm{d}K + \int_L^{+\infty} \frac{1}{K^2}(S_T - K)^+ \, \mathrm{d}K. \tag{4.13}$$

Since $F_T - F_0 = \sum_{i=1}^n \Delta F_i$, substituting equation (4.13) into equations (4.12) and (4.11) implies:

$$MOMS^{(k)} \approx Nk!\left(\int_0^L \frac{1}{K^2}(K - S_T)^+ \, \mathrm{d}K + \int_L^{+\infty} \frac{1}{K^2}(S_T - K)^+ \, \mathrm{d}K.\right)$$
$$+N\left(k!\sum_{i=1}^{n}\left(\frac{1}{F_{i-1}} - \frac{1}{L}\right)\Delta F_i - u(F_0)\right) - N\sum_{j=2}^{k-1}\frac{k!}{j!}MOMS^{(j)}.$$

4.6.4 Pricing of moments swaps

We calculate under the different models, the risk-neutral expectation:

$$E_Q\left[MOMS^{(k)}\right].$$

We consistently average over 1 000 000 simulated paths. All options have a lifetime of 1 year. In Table 4.8, we clearly see how the price differences are even more pronounced as compared to the exotic option pricings discussed in Section 4.5.2.

Table 4.8 Moment swaps ($N = 10\ 000$) for the different models

Order	HEST	BN−S	VG-CIR	VG-OUΓ	NIG-CIR	NIG-OUΓ
$e^{-rT}E_Q\left[MOMS^{(2)}\right]$	623.89	804.60	557.55	628.85	557.75	641.71
$e^{-rT}E_Q\left[MOMS^{(3)}\right]$	−0.0807	−312.58	−21.03	−74.91	−21.69	−88.82
$e^{-rT}E_Q\left[MOMS^{(4)}\right]$	0.6366	322.40	7.8698	33.89	8.554	47.99

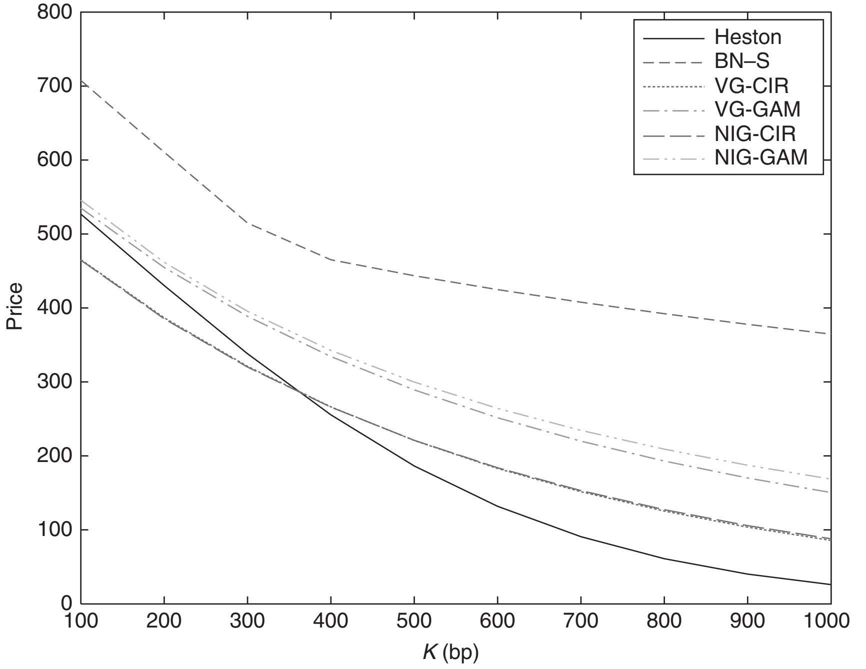

Figure 4.10 Moment option of order 2 ($N = 10\ 000$)

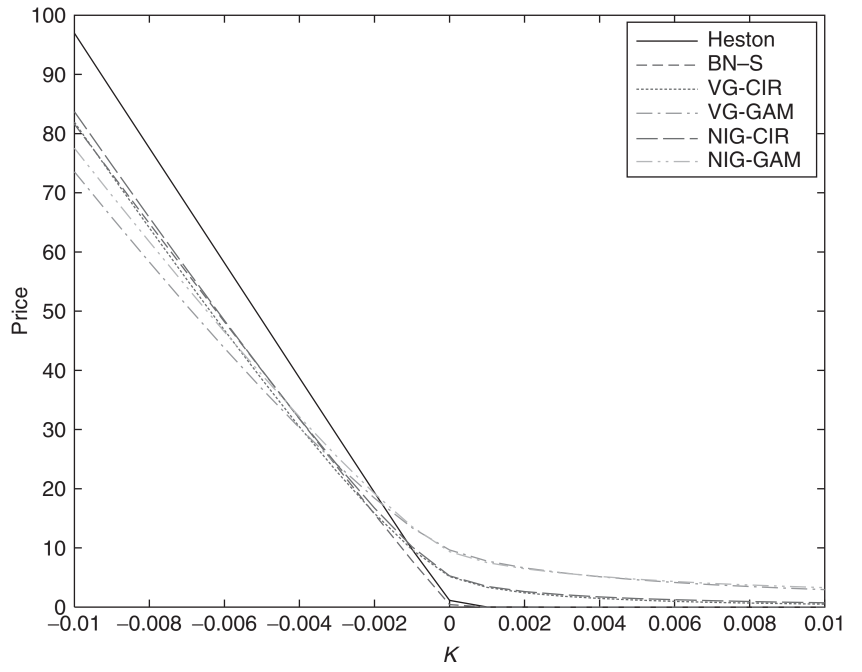

Figure 4.11 Moment option of order 3 ($N = 10\ 000$)

4.6.5 Pricing of moments options

Next, we calculate the prices of moment call option, $MOMO^{(k)}$, paying out at maturity T:

$$\left(\sum_{i=1}^{n}(\log(S_i/S_{i-1}))^k - K\right)^+,$$

where the price of these call moment options is by the risk-neutral valuation:

$$MOMO^{(k)}(K,T) = \exp(-rT)E_Q\left[\sum_{i=1}^{n}(\log(S_i/S_{i-1}))^k - K)^+\right].$$

We plot in Figures 4.10–4.11 the price for moment options of order 2 and 3; corresponding values for these options and fourth order moment option prices can be found in Tables 4.9–4.11.

The disparity between the models is amplified. The Lévy models with stochastic time-change seem again to move in the same pairs as in agreement with the results in Section 4.5.2, but now only up to the third-order moment option. The BN–S model has very pronounced second- and fourth-order moment option prices, while HEST drops (in absolute value) to very low values for the fourth-order moment option when compared to the other models.

4.7 CONCLUSIONS

We have looked at different models, all reflecting non-normal returns and stochastic volatility. Empirical work has generally supported the need for both ingredients.

We have demonstrated the clear ability of all proposed processes to produce a very convincing fit to a market-conform volatility surface. At the same time, we have shown that this calibration could be achieved in a timely manner by using a very fast computational

Table 4.9 Moment option data of order 2 ($N = 10\,000$) for the different models

K (bp)	HEST	BN–S	VG-CIR	VG-OUΓ	NIG-CIR	NIG-OUΓ
100	302.3301	491.4817	212.0101	183.3647	249.068	161.5099
200	219.1410	436.4667	152.6484	121.4477	186.381	100.5233
300	156.7058	394.0581	110.6050	83.4256	140.753	67.6916
400	109.5242	357.8177	80.4430	58.7503	106.257	47.7753
500	75.9747	326.6175	58.2142	42.1646	80.235	35.2828
600	52.4440	300.0791	42.2200	31.0541	60.917	26.9061
700	37.0312	277.1135	30.0486	23.5572	46.407	20.2450
800	25.9978	256.4670	21.2445	17.9608	36.175	15.0211
900	17.4472	238.1357	14.8481	14.1807	28.402	11.2782
1000	11.2276	221.1757	10.4386	11.1815	24.19	8.4718

Table 4.10 Moment option data of order 3 ($N = 10\,000$) for the different models

K	HEST	BN–S	VG-CIR	VG-OUΓ	NIG-CIR	NIG-OUΓ
−0.010	98.0459	79.9869	89.5862	84.6786	87.4962	82.2679
−0.009	88.2341	71.3740	79.9497	75.1442	78.1320	72.8796
−0.008	78.4223	62.8212	70.3448	65.6663	68.8121	63.5948
−0.007	68.6105	54.3490	60.8159	56.2683	59.5498	54.3998
−0.006	58.7987	46.0100	51.3447	46.9840	50.3831	45.3475
−0.005	48.9869	37.7781	41.9813	37.8554	41.3352	36.4821
−0.004	39.1751	29.6576	32.7639	28.9710	32.4824	27.8108
−0.003	29.3686	21.7306	23.7993	20.3739	23.8966	19.4166
−0.002	19.5673	14.0430	15.3273	12.3057	15.6840	11.4694
−0.001	9.8393	6.6657	7.5947	5.3112	8.1994	4.4442
0.000	0.7274	0.0997	1.9022	0.8162	2.4520	0.1462
0.001	0.0008	0	0.8012	0.2915	1.1520	0.0173
0.002	0	0	0.4866	0.1559	0.6438	0.0074
0.003	0	0	0.3267	0.0938	0.4213	0
0.004	0	0	0.2293	0.0486	0.2819	0
0.005	0	0	0.1614	0.0322	0.1998	0
0.006	0	0	0.1052	0.0224	0.1325	0
0.007	0	0	0.0648	0.0126	0.0873	0
0.008	0	0	0.0416	0.0028	0.0578	0
0.009	0	0	0.0268	0	0.0287	0
0.010	0	0	0.0170	0	0.0091	0

Table 4.11 Moment option data of order 4 ($N = 10\,000$) for the different models

K	HEST	BN–S	VG-CIR	VG-OUΓ	NIG-CIR	NIG-OUΓ
0.0001	0.0781	35.7465	1.9322	2.3416	5.2360	3.6095
0.0002	0.0259	35.4823	1.5977	2.0015	4.8754	3.3309
0.0003	0.0120	35.2471	1.3603	1.7655	4.6077	3.1274
0.0004	0.0065	35.0274	1.1821	1.5879	4.3995	2.9601
0.0005	0.0033	34.8220	1.0428	1.4542	4.2249	2.8158
0.0006	0.0011	34.6307	0.9281	1.3386	4.0755	2.6878
0.0007	0.0001	34.4525	0.8328	1.2403	3.9403	2.5750
0.0008	0	34.2810	0.7506	1.1552	3.8221	2.4746
0.0009	0	34.1127	0.6790	1.0805	3.7211	2.3819
0.0010	0	33.9479	0.6202	1.0160	3.6303	2.2979

procedure based on FFT. Note that an almost identical calibration means that at the time-points of the maturities of the calibration data set the marginal distribution is fitted accurately to the risk-neutral distribution implied by the market. If we have different models all leading to such almost perfect calibrations, all models have almost the same marginal distributions. It should, however, be clear that even if at all time-points $0 \leq t \leq T$ marginal distributions among different models coincide, this does not imply that exotic prices should also be the same. This can be seen from the following discrete-time example. Let $n \geq 2$ and $X = \{X_i, i = 1, \ldots, n\}$ be an iid sequence and let $\{u_i, i = 1, \ldots, n\}$ be an independent sequence which randomly varies between $u_i = 0$ and 1. We propose two discrete (be it unrealistic) stock price models, $S^{(1)}$ and $S^{(2)}$, with the same marginal distributions:

$$S_i^{(1)} = u_i X_1 + (1 - u_i) X_2 \text{ and } S_i^{(2)} = X_i.$$

The first process flips randomly between two states X_1 and X_2, both of which follow the distribution of the iid sequence, and so do all of the marginals at the time points $i = 1, \ldots, n$. The second process changes value in all time-points. The values are independent of each other and all follow again the same distribution of the iid sequence. In both cases, all of the marginal distributions (at every $i = 1, \ldots, n$) are the same (as the distribution underlying the sequence X). It is clear, however, that the maximum and minimum of both processes behave completely different. For the first process, the maximal $\max_{j \leq i} S_i^{(1)} = \max(X_1, X_2)$ and the minimal process $\min_{j \leq i} S_i^{(1)} = \min(X_1, X_2)$ for i being large enough, whereas for the second process there is much more variation possible and it clearly leads to other distributions. In summary, it should be clear that equal marginal distributions of a process do not at all imply equal marginal distributions of the associated minimal or maximal process. This explains why matching European call prices do not lead necessarily to matching exotic prices. It is the underlying fine-grain structure of the process that will have an important impact on the path-dependent option prices.

We have illustrated this by pricing exotics by Monte Carlo simulation, showing that price differences for one-touch barriers of over 200 percent are no exception. For lookback call options, a price range of more than 15 percent among the models was observed. A similar conclusion was valid for the digital barrier premiums. Even for cliquet options, which only depend on the stock realizations over a limited amount of time-points, prices vary substantially among the models. Moment derivatives amplify pricing disparity. At the same time, the presented details of the Monte Carlo implementation should allow the reader to embark on his/her own pricing experiments.

The conclusion is that great care should be taken when employing attractive ‘fancy-dancy’ models to price (or even more important, to evaluate hedge parameters for) exotics. As far as we know, no detailed study about the underlying path structure of assets has been carried out yet. Our study motivates such a deeper investigation.

ACKNOWLEDGEMENTS

The first author is a Postdoctoral Fellow of the Fund for Scientific Research, Flanders, Belgium (FWO – Vlaanderen). We thank Marc Jeannin for his devoted programming work.

REFERENCES

[1] Bakshi, G., Cao, C. and Chen, Z. (1997), “Empirical performance of alternative option pricing models”, *The Journal of Finance*, **LII**(5), 2003–2049.

[2] Barndorff-Nielsen, O.E. and Shephard, N. (2001), "Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics", *Journal of the Royal Statistical Society, B*, **63**, 167–241.

[3] Barndorff-Nielsen, O.E., Nicolata, E. and Shephard, N. (2002), "Some recent developments in stochastic volatility modelling", *Quantitative Finance*, **2**, 11–23.

[4] Bertoin, J. (1996), *Lévy Processes*, Cambridge Tracts in Mathematics, **121**, Cambridge University Press, Cambridge, UK.

[5] Black, F. and Scholes, M. (1973), "The pricing of options and corporate liabilities", *Journal of Political Economy*, **81**, 637–654.

[6] Carr, P. and Lewis, K. (2004), "Corridor variance swaps", *Risk Magazine*, **17**(2), 67–72.

[7] Carr, P. and Madan, D. (1998), "Option valuation using the fast fourier transform", *Journal of Computational Finance*, **2**, 61–73.

[8] Carr, P., Geman, H., Madan, D.H. and Yor, M. (2001), *Stochastic Volatility for Lévy Processes*, Prépublications du Laboratoire de Probabilités et Modèles Aléatoires, **645**, Universités de Paris 6 and Paris 7, Paris, France.

[9] Clark, P. (1973), "A subordinated stochastic process model with finite variance for speculative prices", *Econometrica*, **41**, 135–156.

[10] Corcuera, J.M., Nualart, D. and Schoutens, W. (2005a), "Completion of a Lévy market by power-jump assets", *Finance and Stochastics*, **9**, 109–127.

[11] Corcuera, J.M., Nualart, D. and Schoutens, W. (2005b), "Moment derivatives and Lévy-type market completion", in A.E. Kyprianou, W. Schoutens and P. Wilmott (Eds), *Exotic Option pricing and Advanced Lévy Models*, Wiley, Chichester, UK, pp. 169–193.

[12] Cox, J., Ingersoll, J. and Ross, S. (1985), "A Theory of the term structure of interest rates", *Econometrica*, **53**, 385–408.

[13] Devroye, L. (1986), *Non-Uniform Random Variate Generation*, Springer-Verlag, New York.

[14] Heston, S. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options", *Review of Financial Studies*, **6**, 327–343.

[15] Hirsa, A., Courtadon, G. and Madan, B.D. (2003), "The effect of model risk on the valuation of barrier options", *Journal of Risk Finance*, Winter, 1–8.

[16] Hull, J. and Suo, W. (2001), *A Methodology for Assessing Model Risk and its Application to the Implied Volatility Function Model*, Working Paper, University of Toronto, Toronto, ON, Canada, July.

[17] Jäckel, P. (2002), *Monte Carlo Methods in Finance*, Wiley, Chichester, UK.

[18] Knudsen, Th. and Nguyen-Ngoc, L. (2003), *Pricing European Options in a Stochastic Volatility-Jump-Diffusion Model*, DBQuant Working Paper, Deutsche Bank, London, UK; *Journal of Financial and Quantitative Analysis*, to be published.

[19] Madan, D.B., Carr, P. and Chang, E.C. (1998), "The variance gamma process and option pricing", *European Finance Review*, **2**, 79–105.

[20] Nualart, D. and Schoutens W. (2000), "Chaotic and predictable representations for Lévy processes", *Stochastic Processes and their Applications*, **90**, 109–122.

[21] Nualart, D. and Schoutens W. (2001), "Backwards stochastic differential equations and feynman–Kac formula for Lévy processes with applications in finance", *Bernoulli*, **7**, 761–776.

[22] Schoutens, W. (2003), *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley, Chichester, UK.

[23] Schoutens, W., Simons E. and Tistaert, J. (2004), "A perfect calibration! Now what?", *Wilmott Magazine*, March.

[24] Wilmott, P. (2002), "Cliquet options and volatility models", *Wilmott Magazine*, December.

5 Symmetries and Pricing of Exotic Options in Lévy Models

Ernst Eberlein and Antonis Papapantoleon
University of Freiburg, Freiburg, Germany

Abstract

Standard models fail to reproduce observed prices of vanilla options because implied volatilities exhibit a term structure of smiles. We consider time-inhomogeneous Lévy processes to overcome these limitations. Then the scope of this paper is two-fold. On the one hand, we apply measure changes in the spirit of Geman *et al*., to simplify the valuation problem for various options. On the other hand, we discuss a method for the valuation of European options and survey valuation methods for exotic options in Lévy models.

5.1 INTRODUCTION

The efforts to calibrate standard Gaussian models to the empirically observed volatility surfaces very often do not produce satisfactory results. This phenomenon is not restricted to data from equity markets, but it is observed in interest rate and foreign exchange markets as well. There are two basic aspects to which the classical models cannot respond appropriately: the underlying distribution is not flexible enough to capture the implied volatilities either across different strikes or across different maturities. The first phenomenon is the so-called *volatility smile* and the second one the *term structure of smiles*; together they lead to the *volatility surface*, a typical example of which can be seen in Figure 5.1. One way to improve the calibration results is to use stochastic volatility models; let us just mention Heston (1993) for a very popular model, among the various stochastic volatility approaches.

A fundamentally different approach is to replace the driving process. Lévy processes offer a large variety of distributions that are capable of fitting the return distributions in the real world and the volatility smiles in the risk-neutral world. Nevertheless, they cannot capture the term structure of smiles adequately. In order to take care of the change of the smile across maturities, one has to go a step further and consider time-inhomogeneous Lévy processes – also called *additive* processes – as the driving processes. For term structure models this approach was introduced in Eberlein *et al*. (2004) and further investigated in Eberlein and Kluge (2004), where cap and swaption volatilities were calibrated quite successfully. As far as plain vanilla options are concerned, a number of explicit pricing formulas is available for Lévy-driven models, one of which is also discussed in this article. The situation is much more difficult in the case of exotic options. The aim of this paper is to derive symmetries and to survey valuation methods for exotic options in Lévy models. By symmetries, we mean a relationship between pricing formulae for options of different

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

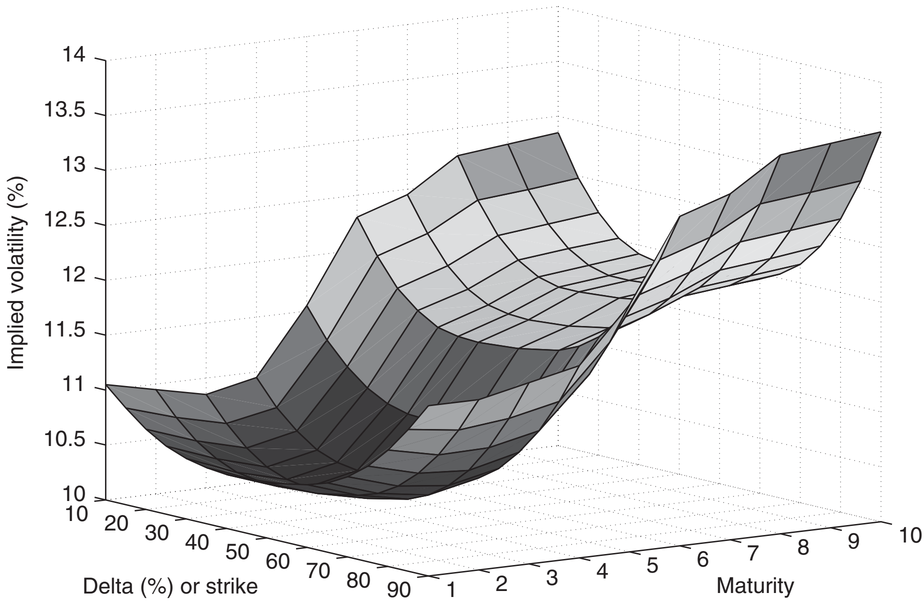

Figure 5.1 Implied volatilities of vanilla options on the Euro/Dollar rate: spot, 0.93; date, 5 November 2001. Data available at `http://www.Mathfinance.de/FF/sampleinputdata.txt`

type. Such a relation is of particular interest if it succeeds to derive the value of a complex payoff from that of a simpler one. A typical example is Theorem 5.1 (see below), where a floating strike Asian or lookback option can be priced via the formula for a fixed strike Asian or lookback option. Moreover, some symmetries are derived in situations where a put-call parity is not available.

The discussion here is rather general as far as the class of time-inhomogeneous Lévy processes is concerned. For implementation of these models, a very convenient class are the processes generated by the Generalized Hyperbolic distributions (cf. Eberlein and Prause (2002)).

The paper is organized as follows: in the next section, we present time-inhomogeneous Lévy processes, the asset price model and some useful results. In Section 5.3, we describe a method for exploring symmetries in option pricing. The next section contains symmetries and valuation methods for vanilla options while exotic options are tackled in the following section. Finally, in Section 5.6 we present symmetries for options depending on two assets.

5.2 MODEL AND ASSUMPTIONS

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete stochastic basis in the sense of Jacod and Shiryaev (2003, I.1.3). Let $\overline{T} \in \mathbb{R}_+$ be a fixed time horizon and assume that $\mathcal{F} = \mathcal{F}_{\overline{T}}$. We shall consider $T \in [0, \overline{T}]$. The class of uniformly integrable martingales is denoted by $\mathcal{M}$; for further notation, we refer the reader to Jacod and Shiryaev (2003). Let $D = \{x \in \mathbb{R}^d : |x| > 1\}$.

Following Eberlein *et al.* (2004), we use as driving process L a time-inhomogeneous Lévy process, more precisely, $L = (L^1, \ldots, L^d)$ is a *process* with *independent increments* and *absolutely continuous* characteristics, in the sequel abbreviated PIIAC. The law of L_t is described by the characteristic function

$$\mathbb{E}\left[e^{i\langle u,L_t\rangle}\right] = \exp\int_0^t \Big[i\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d}(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\lambda_s(dx)\Big]ds, \tag{5.2.1}$$

where $b_t \in \mathbb{R}^d$, c_t is a symmetric non-negative definite $d \times d$ matrix and λ_t is a Lévy measure on $\mathbb{R}^d$, i.e. it satisfies $\lambda_t(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\lambda_t(dx) < \infty$ for all $t \in [0, \overline{T}]$. The Euclidean scalar product on $\mathbb{R}^d$ is denoted by $\langle\cdot,\cdot\rangle$, the corresponding norm by $|\cdot|$ while $\|\cdot\|$ denotes a norm on the set of $d \times d$ matrices. The transpose of a matrix or vector v is denoted by $v^\top$ and $\mathbf{1}$ denotes the unit vector, i.e. $\mathbf{1} = (1, \dots, 1)^\top$. The process L has càdlàg paths and $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,\overline{T}]}$ is the filtration generated by L; moreover, L satisfies Assumptions ($\mathbb{AC}$) and ($\mathbb{EM}$) given below.

Assumption ($\mathbb{AC}$). Assume that the triplets (b_t, c_t, λ_t) satisfy

$$\int_0^{\overline{T}} \Big[|b_t| + \|c_t\| + \int_{\mathbb{R}^d}(1 \wedge |x|^2)\lambda_t(dx)\Big]dt < \infty.$$

Assumption ($\mathbb{EM}$). Assume there exists a constant $M > 1$, such that the Lévy measures λ_t satisfy

$$\int_0^{\overline{T}}\int_D \exp\langle u, x\rangle\lambda_t(dx)dt < \infty, \qquad \forall u \in [-M, M]^d.$$

Under these assumptions, L is a special semimartingale and its *triplet of semimartingale characteristics* (cf. Jacod and Shiryaev (2003, II.2.6)) is given by

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0, t] \times A) = \int_0^t\int_A \lambda_s(dx)ds, \tag{5.2.2}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$. The triplet of semimartingale characteristics (B, C, ν) completely characterizes the distribution of L. Additionally, L is *exponentially special* (cf. Kallsen and Shiryaev (2002) pp. 2.12–2.13).

We model the asset price process as an exponential PIIAC

$$S_t = S_0 \exp L_t \tag{5.2.3}$$

with $(S^1, \dots, S^d) = (S_0^1 e^{L^1}, \dots, S_0^d e^{L^d})$, where the superscript i refers to the i-th coordinate, $i \le d$. We assume that $\mathbb{P}$ is a risk neutral measure, i.e. the asset prices have mean rate of return $\mu^i \triangleq r - \delta^i$ and the auxiliary processes $\widehat{S_t^i} = e^{\delta^i t}S_t^i$, once discounted at the rate r, are $\mathbb{P}$-martingales. Here, r is the risk-free rate and δ^i is the dividend yield of the i-th asset. Notice that finiteness of $\mathbb{E}[\widehat{S_{\overline{T}}}]$ is ensured by Assumption ($\mathbb{EM}$).

The driving process L has the canonical decomposition (cf. Jacod and Shiryaev (2003), II.2.38 and Eberlein *et al.* (2004))

$$L_t = \int_0^t b_s \mathrm{d}s + \int_0^t c_s^{1/2} \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x) \tag{5.2.4}$$

where, $c_t^{1/2}$ is a measurable version of the square root of c_t, W a $\mathbb{P}$-standard Brownian motion on $\mathbb{R}^d$, μ^L the random measure of jumps of the process L and $\nu(\mathrm{d}t, \mathrm{d}x) = \lambda_t(\mathrm{d}x)\mathrm{d}t$ is the $\mathbb{P}$-compensator of the jump measure μ^L.

Because S is modelled under a risk neutral measure, the drift characteristic B is completely determined by the other two characteristics (C, ν) and the rate of return of the asset. Therefore, the i-th component of B_t has the form

$$B_t^i = \int_0^t (r - \delta^i)\mathrm{d}s - \frac{1}{2}\int_0^t (c_s\mathbf{1})^i \mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} (\mathrm{e}^{x^i} - 1 - x^i)\nu(\mathrm{d}s, \mathrm{d}x). \tag{5.2.5}$$

In a foreign exchange context, δ^i can be viewed as the foreign interest rate.

In general, markets modelled by exponential time-inhomogeneous Lévy processes are incomplete and there exists a large class of risk neutral (equivalent martingale) measures. An exception occurs in interest rate models driven by Lévy processes, where – in certain cases – there is a unique martingale measure; we refer to Theorem 6.4 in Eberlein *et al.* (2004). Eberlein and Jacod (1997) provide a characterization of the class of equivalent martingale measures for exponential Lévy models in the time-homogeneous case; this was later extended to general semimartingales in Gushchin and Mordecki (2002).

In this article, we do not dive into the theory of choosing a martingale measure; we rather assume that the choice has already taken place. We refer to Eberlein and Keller (1995), Kallsen and Shiryaev (2002) for the Esscher transform, Frittelli (2000), Fujiwara and Miyahara (2003) for the minimal entropy martingale measure and Bellini and Frittelli (2002) for minimax martingale measures, to mention just a small part of the literature on this subject. A unifying exposition – in terms of *f*-divergences – of the different methods for selecting an equivalent martingale measure can be found in Goll and Rüschendorf (2001).

Alternatively, one can consider the choice of the martingale measure as the result of a calibration to the *smile* of the vanilla options market. Hakala and Wystup (2002) describe the calibration procedure in detail; we refer to Cont and Tankov (2004) for a numerically stable calibration method for Lévy driven models.

Remark 2.1. In the above setting, we can easily incorporate dynamic interest rates and dividend yields (or foreign and domestic rates). Let D_t denote the domestic and F_t the foreign savings account, respectively; then, they can have the form

$$D_t = \exp\int_0^t r_s \mathrm{d}s \qquad \text{and} \qquad F_t = \exp\int_0^t \delta_s \mathrm{d}s$$

and equation (5.2.5) has a similar form, taking r_s and δ_s into account.

Remark 2.2. The PIIAC L is an *additive* process, i.e. a process with independent increments, which is stochastically continuous and satisfies $L_0 = 0$ a.s. (Sato (1999) Definition 1.6).

Remark 2.3. If the triplet (b_t, c_t, λ_t) is not time-dependent, then the PIIAC L becomes a (homogeneous) Lévy process, i.e. a *process* with *independent* and *stationary increments* (PIIS). In that case, the distribution of L is described by the *Lévy triplet* (b, c, λ), where λ is the Lévy measure and the compensator of μ^L becomes a product measure of the form $\nu = \lambda \otimes \lambda\!\!\lambda^1$, where $\lambda\!\!\lambda^1$ denotes the Lebesgue measure. In that case, equation (5.2.1) takes the form $\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp[t \cdot \psi(u)]$ where

$$\psi(u) = i\langle u, b\rangle - \frac{1}{2}\langle u, cu\rangle + \int_{\mathbb{R}^d} (\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\lambda(\mathrm{d}x) \tag{5.2.6}$$

which is called the *characteristic exponent* of L.

Lemma 2.4. *For fixed $t \in [0, \overline{T}]$, the distribution of L_t is infinitely divisible with Lévy triplet (b', c', λ'), given by*

$$b' := \int_0^t b_s\,\mathrm{d}s, \qquad c' := \int_0^t c_s\,\mathrm{d}s, \qquad \lambda'(\mathrm{d}x) := \int_0^t \lambda_s(\mathrm{d}x)\,\mathrm{d}s. \tag{5.2.7}$$

(The integrals should be understood componentwise.)

Proof. We refer to the proof of Lemma 1 in Eberlein and Kluge (2004).

Remark 2.5. The PIIACs $L^1, \ldots, L^d$ are independent, if and only if, the matrices C_t are diagonal and the Lévy measures λ_t are supported by the union of the coordinate axes; this follows directly from Exercise 12.10 in Sato (1999) or I.5.2 in Bertoin (1996) and Lemma 2.4. Describing the dependence is a more difficult task; we refer to Müller and Stoyan (2002) for a comprehensive exposition of various dependence concepts and their applications. We also refer to Kallsen and Tankov (2004), where a Lévy copula is used to describe the dependence of the components of multidimensional Lévy processes.

Remark 2.6. Assumption ($\mathbb{EM}$) is sufficient for all our considerations, but is in general too strong. In the sequel, we will replace ($\mathbb{EM}$), on occasion, by the *minimal* necessary assumptions. From a practical point of view though, it is not too restrictive to assume ($\mathbb{EM}$), since all examples of Lévy models we are interested in, e.g. the Generalized Hyperbolic model (cf. Eberlein and Prause (2002)), the CGMY model (cf. Carr *et al.* (2002)) or the Meixner model (cf. Schoutens (2002)), possess moments of all order.

We can relate the finiteness of the *g-moment* of L_t for a PIIAC L and a *submultiplicative* function g, with an integrability property of its compensator measure ν. For the notions of the g-moment and submultiplicative function, we refer to Definitions 25.1 and 25.2 in Sato (1999).

Lemma 2.7. (g-Moment). *Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$. Then the following statements are equivalent*

(1) $\int_0^{\overline{T}} \int_D g(x)\nu(\mathrm{d}t, \mathrm{d}x) < \infty$
(2) $\mathbb{E}\left[g(L_{\overline{T}})\right] < \infty.$

Proof. The result follows from Theorem 25.3 in Sato (1999) combined with Lemma 6 in Eberlein and Kluge (2004).

Now, since $g(x) = \exp\langle u, x\rangle$ is a submultiplicative function, we immediately get the following equivalence concerning Assumption ($\mathbb{EM}$).

Corollary 2.8. *Let $M > 1$ be a constant. Then the following statements are equivalent*

(1) $\int_0^{\overline{T}} \int_D \exp\langle u, x\rangle \nu(\mathrm{d}t, \mathrm{d}x) < \infty, \qquad \forall u \in [-M, M]^d$

(2) $\mathbb{E}\left[\exp\langle u, L_{\overline{T}}\rangle\right] < \infty, \qquad \forall u \in [-M, M]^d.$

We can describe the characteristic triplet of the *dual* of a one-dimensional PIIAC in terms of the characteristic triplet of the original process. First, we introduce some necessary notation and the next lemma provides the result.

Notation. We denote by $-\lambda_t$ the *Lévy measure* defined by

$$-\lambda_t([a, b]) := \lambda_t([-b, -a])$$

for $a, b \in \mathbb{R}$, $a < b$, $t \in \mathbb{R}_+$. Thus, $-\lambda_t$ is a non-negative measure and the mirror image of the original measure with respect to the vertical axis. For a compensator of the form $\nu(\mathrm{d}t, \mathrm{d}x) = \lambda_t(\mathrm{d}x)\mathrm{d}t$, we denote by $-\nu$ the (non-negative) measure, defined as

$$-\nu(\mathrm{d}t, \mathrm{d}x) := -\lambda_t(\mathrm{d}x)\mathrm{d}t.$$

Whenever we use the symbol "$-$" in front of a Lévy measure or a compensator, we will refer to measures defined as above.

Lemma 2.9 (dual characteristics)**.** *Let L be a PIIAC, as described above, with characteristic triplet (B, C, ν). Then $L^\star := -L$ is again a PIIAC with characteristic triplet $(B^\star, C^\star, \nu^\star)$, where $B^\star = -B$, $C^\star = C$ and $\nu^\star = -\nu$.*

Proof. From the Lévy-Khintchine representation we have that

$$\varphi_{L_t}(u) = \mathbb{E}\left[\mathrm{e}^{iuL_t}\right] = \exp \int_0^t \left[ib_s u - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux)\lambda_s(\mathrm{d}x)\right]\mathrm{d}s.$$

We get immediately

$$\begin{aligned}
\varphi_{-L_t}(u) &= \varphi_{L_t}(-u) \\
&= \exp \int_0^t \left[ib_s(-u) - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (\mathrm{e}^{i(-u)x} - 1 - i(-u)x)\lambda_s(\mathrm{d}x)\right]\mathrm{d}s \\
&= \exp \int_0^t \left[i(-b_s)u - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (\mathrm{e}^{iu(-x)} - 1 - iu(-x))\lambda_s(\mathrm{d}x)\right]\mathrm{d}s.
\end{aligned}$$

Then $b_t^\star = -b_t$, $c_t^\star = c_t$, and $\lambda_t^\star = -\lambda_t$ clearly satisfy Assumption ($\mathbb{AC}$). Hence, we can conclude that $L^\star$ is also a PIIAC and has characteristics $B_t^\star = \int_0^t b_s^\star \mathrm{d}s = -B_t$, $C_t^\star = \int_0^t c_s^\star \mathrm{d}s = C_t$ and $\nu^\star(\mathrm{d}t, \mathrm{d}x) = \lambda_t^\star(\mathrm{d}x)\mathrm{d}t = -\nu(\mathrm{d}t, \mathrm{d}x)$.

5.3 GENERAL DESCRIPTION OF THE METHOD

In this section, we give a brief and general description of the method we shall use to explore symmetries in option pricing. The method is based on the choice of a suitable numéraire and a subsequent change of the underlying probability measure; we refer to Geman *et al.* (1995) who pioneered this method.

The discounted asset price process, corrected for dividends, serves as the numéraire for a number of cases, in case the option payoff is homogeneous of degree one. Using the numéraire, evaluated at the time of maturity, as the Radon–Nikodym derivative, we form a new measure. Under this new measure, the numéraire asset is riskless while all other assets, including the savings account, are now risky. In case the payoff is homogeneous of higher degree, say $\alpha \geq 1$, we have to modify the asset price process so that it serves as the numéraire. As a result, the asset dynamics under the new measure will depend on α as well.

We consider three cases for the driving process L and the asset price process(es):

(P1): $L = L^1$ is a (1-d) PIIAC, $L^2 = k$ is constant and $S^1 = S_0^1 \exp L^1$, $S^2 = \exp L^2 = K$;
(P2): $L = L^1$ is a (1-d) PIIAC, $S^1 = S_0^1 \exp L^1$ and $S^2 = h(S^1)$ is a functional of S^1;
(P3): $L = (L^1, L^2)$ is a 2-dimensional PIIAC and $S^i = S_0^i \exp L^i$, $i = 1, 2$.

Consider a payoff function

$$f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \tag{5.3.1}$$

which is homogeneous of degree $\alpha \geq 1$, that is, for $\kappa, x, y \in \mathbb{R}_+^*$

$$f(\kappa x, \kappa y) = \kappa^\alpha f(x, y);$$

for simplicity we assume that $\alpha = 1$ and later – in the case of power options – we will treat the case of a more general α.

According to the general arbitrage pricing theory (Delbaen and Schachermayer (1994, 1998)), the value V of an option on assets S^1, S^2 with payoff f is equal to its discounted expected payoff under an equivalent martingale measure. Throughout this paper, we will assume that options start at time 0 and mature at T; therefore we have

$$V = \mathrm{e}^{-rT} \mathbb{E}\left[f\left(S_T^1, S_T^2\right)\right]. \tag{5.3.2}$$

We choose asset S^1 as the numéraire and express the value of the option in terms of this numéraire, which yields

$$\begin{aligned} \widetilde{V} = \frac{V}{S_0^1} &= \mathrm{e}^{-rT} \mathbb{E}\left[\frac{f\left(S_T^1, S_T^2\right)}{S_0^1}\right] \\ &= \mathrm{e}^{-\delta^1 T} \mathbb{E}\left[\frac{\mathrm{e}^{-rT} S_T^1}{\mathrm{e}^{-\delta^1 T} S_0^1} f\left(1, \frac{S_T^2}{S_T^1}\right)\right]. \end{aligned} \tag{5.3.3}$$

Define a new measure $\widetilde{\mathbb{P}}$ via the Radon–Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{-rT} S_T^1}{\mathrm{e}^{-\delta^1 T} S_0^1} = \eta_T. \tag{5.3.4}$$

After the change of measure, the valuation problem, under the measure $\widetilde{\mathbb{P}}$, becomes

$$\widetilde{V} = \mathrm{e}^{-\delta^1 T} \widetilde{\mathbb{E}} \left[f \left(1, S_T^{1,2} \right) \right] \tag{5.3.5}$$

where we define the process $S^{1,2} := \frac{S^2}{S^1}$.

The measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are related via the density process $\eta_t = \mathbb{E}[\eta_T | \mathcal{F}_t]$; therefore $\widetilde{\mathbb{P}} \overset{\text{loc}}{\sim} \mathbb{P}$ and we can apply Girsanov's theorem for semimartingales (cf. Jacod and Shiryaev (2003, III.3.24)); this will allow us to determine the dynamics of $S^{1,2}$ under $\widetilde{\mathbb{P}}$.

After some calculations, which depend on the particular choice of L^2 or S^2, we can transform the original valuation problem into a simpler one.

5.4 VANILLA OPTIONS

These results are motivated by Carr (1994), where a symmetry relationship between European call and put options in the Black and Scholes (1973) and Merton (1973) model was derived. This result was later extended by Carr and Chesney (1996) to American options for the Black–Scholes case and for general diffusion models; see also McDonald and Schroder (1998) and Detemple (2001).

This relationship has an intuitive interpretation in foreign exchange markets (cf. Wystup (2002)). Consider the Euro/Dollar market; then a call option on the Euro/Dollar exchange rate S_t with payoff $(S_T - K)^+$ has time-t value $V_c(S_t, K; r_d, r_e)$ in dollars and $V_c(S_t, K; r_d, r_e)/S_t$ in euros. This euro-call option can also be viewed as a dollar-put option on the Dollar/Euro rate with payoff $K(K^{-1} - S_T^{-1})^+$ and time-t value $K V_p(K^{-1}, S_T^{-1}; r_e, r_d)$ in euros. Since the processes S and S^{-1} have the same (Black–Scholes) volatility, by the absence of arbitrage opportunities, their prices must be equal.

5.4.1 Symmetry

For vanilla options, the setting is that of (**P1**): $L^1 = L$ is the driving $\mathbb{R}$-valued PIIAC with triplet (B, C, ν), $S^1 = \exp L^1 = S$ and $L^2 = k$, such that $S^2 = \mathrm{e}^k = K$, the strike price of the option.

In accordance with the standard notation, we will use σ_s^2 instead of c_s, which corresponds to the volatility in the Black–Scholes model. Therefore, the characteristic C in equation (5.2.2) has the form $C_t = \int_0^t \sigma_s^2 \mathrm{d}s$.

We will prove a more general version of Carr's symmetry, namely a symmetry relating *power options*; the payoff of the power call and put option, respectively, is

$$\left[(S_T - K)^+\right]^\alpha \qquad \text{and} \qquad \left[(K - S_T)^+\right]^\alpha$$

where $\alpha \in \mathbb{N}$ (more generally $\alpha \in \mathbb{R}$). We introduce the following notation for the value of a power call option with strike K and power index α

$$V_c(S_0, K, \alpha; r, \delta, C, \nu) = \mathrm{e}^{-rT} \mathbb{E}\left[(S_T - K)^+\right]^\alpha$$

where the asset is modelled as an exponential PIIAC according to equations (5.2.3)–(5.2.5) and $x^+ = \max\{x, 0\}$. Similarly, for a power put option we set

$$V_p(S_0, K, \alpha; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[(K - S_T)^+\right]^\alpha .$$

Of course, for $\alpha = 1$ we recover the European plain vanilla option and the power index α will be omitted from the notation.

Assumption ($\mathbb{EM}$) can be replaced by the following weaker assumption, which is the minimal condition necessary for the symmetry results to hold. Let $D_+ = D \cap \mathbb{R}_+$ and $D_- = D \cap \mathbb{R}_-$.

Assumption ($\mathbb{M}$)**.** The Lévy measures λ_t of the distribution of L_t satisfy

$$\int_0^{\overline{T}}\int_{D_-} |x|\lambda_t(\mathrm{d}x)\mathrm{d}t < \infty \qquad \text{and} \qquad \int_0^{\overline{T}}\int_{D_+} x\mathrm{e}^{\alpha x}\lambda_t(\mathrm{d}x)\mathrm{d}t < \infty.$$

Theorem 4.1. *Assume that* ($\mathbb{M}$) *is in force and the asset price evolves as an exponential PIIAC according to equations (5.2.3)–(5.2.5). We can relate the power call and put option via the following symmetry:*

$$V_c(S_0, K, \alpha; r, \delta, C, \nu) = K^\alpha S_0^\alpha \mathcal{C}_T \mathrm{e}^{\alpha\mathcal{C}_T^*} V_p(S_0^{-1}, \mathcal{K}, \alpha; \delta, r, C, -f\nu) \qquad (5.4.1)$$

where the constants $\mathcal{C}$ *and* $\mathcal{C}^*$ *are given by equations (5.4.3) and (5.4.10), respectively (see below),* $\mathcal{K} = K^{-1}\mathrm{e}^{-\mathcal{C}_T^*}$ *and* $f(x) = \mathrm{e}^{\alpha x}$.

Proof. First, we note that $[\mathrm{e}^{(\delta-r)t}S_t]^\alpha = S_0^\alpha \exp(\alpha(\delta - r)t + \alpha L_t)$ is not a $\mathbb{P}$-martingale; we denote by L^α the martingale part of the exponent; hence

$$L_t^\alpha = \int_0^t \alpha\sigma_s \mathrm{d}W_s + \int_0^t\int_{\mathbb{R}} \alpha x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x).$$

Since L^α is exponentially special, with Theorem 2.18 in Kallsen and Shiryaev (2002) we have that its exponential compensator, denoted CL^α, has the form

$$CL_t^\alpha = \frac{1}{2}\int_0^t \alpha^2\sigma_s^2 \mathrm{d}s + \int_0^t\int_{\mathbb{R}} (\mathrm{e}^{\alpha x} - 1 - \alpha x)\nu(\mathrm{d}s, \mathrm{d}x)$$

and $\exp(L^\alpha - CL^\alpha) \in \mathcal{M}$.

The price of the power call option expressed in units of the *numéraire* yields

$$\begin{aligned}\widetilde{V}_c := \frac{V_c}{S_0^\alpha} &= \frac{\mathrm{e}^{-rT}}{S_0^\alpha}\mathbb{E}\left[(S_T - K)^+\right]^\alpha \\ &= \mathrm{e}^{-\delta T}\mathbb{E}\left[\frac{\mathrm{e}^{-rT}S_T^\alpha K^\alpha}{\mathrm{e}^{-\delta T}S_0^\alpha}\left[(K^{-1} - S_T^{-1})^+\right]^\alpha\right]\end{aligned}$$

$$= e^{-\delta T} K^{\alpha} \mathbb{E}\Big[\exp\Big((\delta - r)T + \alpha \int_0^T b_s \mathrm{d}s + C L_T^{\alpha} \Big)$$

$$\times \exp\left(L_T^{\alpha} - C L_T^{\alpha} \right) \left[(K^{-1} - S_T^{-1})^+ \right]^{\alpha} \Big]$$

$$= e^{-\delta T} K^{\alpha} \mathcal{C}_T \mathbb{E}\Big[\exp\left(L_T^{\alpha} - C L_T^{\alpha} \right) \left[(K^{-1} - S_T^{-1})^+ \right]^{\alpha} \Big] \tag{5.4.2}$$

where, by using equations (5.2.5) and (5.2.2), we have that

$$\begin{aligned} \log \mathcal{C}_T &= (\delta - r)T + \alpha B_T + C L_T^{\alpha} \\ &= (\alpha - 1)(r - \delta)T + \frac{\alpha(\alpha - 1)}{2} \int_0^T \sigma_s^2 \mathrm{d}s \\ &\quad + \int_0^T \int_{\mathbb{R}} (e^{\alpha x} - \alpha e^x + \alpha - 1) \nu(\mathrm{d}s, \mathrm{d}x). \end{aligned} \tag{5.4.3}$$

Define a new measure $\widetilde{\mathbb{P}}$ via its Radon–Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(L_T^{\alpha} - C L_T^{\alpha} \right) = \eta_T \tag{5.4.4}$$

and the valuation problem (equation (5.4.2)) becomes

$$\widetilde{V}_c = e^{-\delta T} K^{\alpha} \mathcal{C}_T \widetilde{\mathbb{E}} \left[(\widetilde{K} - \widetilde{S}_T)^+ \right]^{\alpha} \tag{5.4.5}$$

where $\widetilde{K} = K^{-1}$ and $\widetilde{S}_t := S_t^{-1}$.

Since the measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are related via the density process (η_t), which is a positive martingale with $\eta_0 = 1$, we immediately deduce that $\widetilde{\mathbb{P}} \overset{\text{loc}}{\sim} \mathbb{P}$ and we can apply Girsanov's theorem for semimartingales (cf. Jacod and Shiryaev (2003) III.3.24). The density process can be represented in the usual form

$$\begin{aligned} \eta_t &= \mathbb{E}\left[\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \middle| \mathcal{F}_t \right] = \exp\left(L_t^{\alpha} - C L_t^{\alpha} \right) \\ &= \exp\Big[\int_0^t \alpha \sigma_s \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} \alpha x (\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x) \\ &\qquad - \frac{1}{2} \int_0^t \alpha^2 \sigma_s^2 \mathrm{d}s - \int_0^t \int_{\mathbb{R}} (e^{\alpha x} - 1 - \alpha x) \nu(\mathrm{d}s, \mathrm{d}x) \Big]. \end{aligned} \tag{5.4.6}$$

Consequently, we can identify the tuple (β, Y) of predictable processes

$$\beta(t) = \alpha \quad \text{and} \quad Y(t, x) = \exp(\alpha x)$$

which characterizes the change of measure.

From Girsanov's theorem, combined with Theorem II.4.15 in Jacod and Shiryaev (2003), we deduce that a PIIAC remains a PIIAC under the measure $\widetilde{\mathbb{P}}$, because the processes β and Y are deterministic and the resulting characteristics under $\widetilde{\mathbb{P}}$ satisfy Assumption $(\mathbb{AC})$.

As a consequence of Girsanov's theorem for semimartingales, we infer that $\widetilde{W}_t = W_t - \int_0^t \alpha\sigma_s \mathrm{d}s$ is a $\widetilde{\mathbb{P}}$-Brownian motion and $\widetilde{\nu} = Y\nu$ is the $\widetilde{\mathbb{P}}$ compensator of the jumps of L. Furthermore, as a corollary of Girsanov's theorem, we can calculate the canonical decomposition of L under $\widetilde{\mathbb{P}}$;

$$L_t = \int_0^t \widetilde{b}_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\widetilde{W}_s + \int_0^t \int_{\mathbb{R}} x(\mu^L - \widetilde{\nu})(\mathrm{d}s, \mathrm{d}x) \tag{5.4.7}$$

where

$$\begin{aligned}\widetilde{B}_t = \int_0^t \widetilde{b}_s \mathrm{d}s &= (r-\delta)t + \left(\alpha - \frac{1}{2}\right) \int_0^t \sigma_s^2 \mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}} (\mathrm{e}^{-\alpha x} - \mathrm{e}^{(1-\alpha)x} + x)\widetilde{\nu}(\mathrm{d}s, \mathrm{d}x)\end{aligned} \tag{5.4.8}$$

and hence, its triplet of characteristics is $(\widetilde{B}, C, \widetilde{\nu})$. Define its dual process, $L^\star := -L$ and by Lemma 2.9, we get that its triplet is $(B^\star, C^\star, \nu^\star) = (-\widetilde{B}, C, -\widetilde{\nu})$. The canonical decomposition of $L^\star$ is

$$L_t^\star = -\int_0^t \widetilde{b}_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s^\star + \int_0^t \int_{\mathbb{R}} x(\mu^{L^\star} - \nu^\star)(\mathrm{d}s, \mathrm{d}x) \tag{5.4.9}$$

and we can easily deduce that $\mathrm{e}^{(r-\delta)t} S_t^\star$ is not a $\widetilde{\mathbb{P}}$-martingale for $\alpha \neq 1$.

Adding the appropriate terms, we can re-write $L^\star$ as $L^\star := \mathcal{C}^* + \overline{L}$, where

$$\mathcal{C}^* = (1-\alpha) \int_0^\cdot \sigma_s^2 \mathrm{d}s - \int_0^\cdot \int_{\mathbb{R}} (\mathrm{e}^{-\alpha x} - \mathrm{e}^{(1-\alpha)x} + 1 - \mathrm{e}^{-x})\widetilde{\nu}(\mathrm{d}s, \mathrm{d}x) \tag{5.4.10}$$

and $\overline{L}$ is such that $\mathrm{e}^{(r-\delta)t}\overline{S}_t$ is a $\widetilde{\mathbb{P}}$-martingale. The characteristic triplet of $\overline{L}$ is $(B^\star - \mathcal{C}^*, C, \nu^\star)$ and $\overline{S}_t = S_0^{-1} \exp \overline{L}_t$.

Therefore, we can conclude the proof

$$\begin{aligned}\widetilde{V}_c &= \mathrm{e}^{-\delta T} K^\alpha \mathcal{C}_T \widetilde{\mathbb{E}}\left[(\widetilde{K} - \widetilde{S}_T)^+\right]^\alpha \\ &= \mathrm{e}^{-\delta T} K^\alpha \mathcal{C}_T \widetilde{\mathbb{E}}\left[(\widetilde{K} - \mathrm{e}^{\mathcal{C}_T^*}\overline{S}_T)^+\right]^\alpha \\ &= \mathrm{e}^{-\delta T} K^\alpha \mathcal{C}_T \mathrm{e}^{\alpha \mathcal{C}_T^*} \widetilde{\mathbb{E}}\left[(\mathcal{K} - \overline{S}_T)^+\right]^\alpha\end{aligned}$$

where $\mathcal{K} = \widetilde{K}\mathrm{e}^{-\mathcal{C}_T^*} = K^{-1}\mathrm{e}^{-\mathcal{C}_T^*}$.

Setting $\alpha = 1$ in the previous theorem, we immediately get a symmetry between European plain vanilla call and put options.

Corollary 4.2. *Assuming that* ($\mathbb{M}$) *is in force and the asset price evolves as an exponential PIIAC, we can relate the European call and put option via the following symmetry:*

$$V_c\left(S_0, K; r, \delta, C, \nu\right) = K S_0 V_p\left(S_0^{-1}, K^{-1}; \delta, r, C, -f\nu\right) \tag{5.4.11}$$

where $f(x) = \mathrm{e}^x$.

This symmetry relating European and also American plain vanilla call and put options, in exponential Lévy models, was proved independently in Fajardo and Mordecki (2003). Schroder (1999) proved similar results in a general semimartingale model; however, using a Lévy or PIIAC as the driving motion allows for the explicit calculation of the distribution under the new measure.

A different symmetry, again relating European and American call and put options, in the Black–Scholes model, was derived by Peskir and Shiryaev (2002), where they use the mathematical concept of *negative* volatility; their main result states that

$$V_c(S_T, K; \sigma) = V_p(-S_T, -K; -\sigma). \tag{5.4.12}$$

See also the discussion – and the corresponding cartoon – in Haug (2002).

In this framework, one can derive symmetry relationships between self-quanto and European plain vanilla options. This result is, of course, a special case of Theorem 6.4; nevertheless, we give a short proof since it simplifies considerably because the driving process is one-dimensional.

The payoff of the self-quanto call and put option is

$$S_T(S_T - K)^+ \qquad \text{and} \qquad S_T(K - S_T)^+,$$

respectively. Introduce the following notation for the value of the self-quanto call option

$$V_{qc}(S_0, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\big[S_T(S_T - K)^+\big]$$

and similarly, for the self-quanto put option we set

$$V_{qp}(S_0, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\big[S_T(K - S_T)^+\big].$$

Assumption ($\mathbb{EM}$) can be replaced by the following weaker assumption, which is the minimal condition necessary for the symmetry results to hold.

Assumption ($\mathbb{M}'$)**.** The Lévy measures λ_t of the distribution of L_t satisfy

$$\int_0^{\overline{T}}\int_{D_-}|x|\lambda_t(\mathrm{d}x)\mathrm{d}t < \infty \qquad \text{and} \qquad \int_0^{\overline{T}}\int_{D_+}\mathrm{e}^{2x}\lambda_t(\mathrm{d}x)\mathrm{d}t < \infty.$$

Theorem 4.3. *Assume that the asset price evolves as an exponential PIIAC and* ($\mathbb{M}'$) *is in force. We can relate the self-quanto and European plain vanilla call and put options via the following symmetry:*

$$V_{qc}(S_0, K; r, \delta, C, \nu) = S_0\mathrm{e}^{C_T^*}V_c(S_0, K^*; \delta, r, C, f\nu) \tag{5.4.13}$$

$$V_{qp}(S_0, K; r, \delta, C, \nu) = S_0\mathrm{e}^{C_T^*}V_p(S_0, K^*; \delta, r, C, f\nu) \tag{5.4.14}$$

where C^* *is given by equation (5.4.16) (see below),* $K^* = K\mathrm{e}^{-C_T^*}$ *and* $f(x) = \mathrm{e}^x$.

Proof. Expressing the value of the self-quanto call option in units of the *numéraire* as described in Section 5.3, we define a new measure $\widetilde{\mathbb{P}}$ via its Radon–Nikodym derivative given by equation (5.3.4) and the original valuation problem becomes

$$\widetilde{V}_{qc} = \mathrm{e}^{-\delta T} \widetilde{\mathbb{E}}\left[(S_T - K)^+\right]. \tag{5.4.15}$$

Now it suffices to calculate the characteristic triplet of L under $\widetilde{\mathbb{P}}$. Arguing as in the proof of Theorem 4.1, the density process η has the form of equation (5.4.6) for $\alpha = 1$; hence, the tuple (β, Y) of predictable processes that describes the change of measure is

$$\beta(t) = 1 \quad \text{and} \quad Y(t, x) = \exp(x).$$

Therefore, L has the canonical decomposition under $\widetilde{\mathbb{P}}$

$$L_t = \int_0^t \widetilde{b}_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\widetilde{W}_s + \int_0^t \int_{\mathbb{R}} x(\mu^L - \widetilde{\nu})(\mathrm{d}s, \mathrm{d}x)$$

where

$$\widetilde{b}_t = r - \delta + \frac{\sigma_t^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^{-x} - 1 + x)\mathrm{e}^x \lambda_t(\mathrm{d}x).$$

Notice that $\mathrm{e}^{(r-\delta)t}\mathrm{e}^{L_t}$ is not a $\widetilde{\mathbb{P}}$-martingale, but if we define L^* as

$$\begin{aligned} L_t^* := (\delta - r)t &+ \int_0^t \sigma_s \mathrm{d}\widetilde{W}_s + \int_0^t \int_{\mathbb{R}} x(\mu^L - \widetilde{\nu})(\mathrm{d}s, \mathrm{d}x) \\ &- \int_0^t \frac{\sigma_t^2}{2}\mathrm{d}s - \int_0^t \int_{\mathbb{R}} (\mathrm{e}^x - 1 - x)\mathrm{e}^x \nu(\mathrm{d}s, \mathrm{d}x) \end{aligned}$$

then $\mathrm{e}^{(r-\delta)t}\mathrm{e}^{L_t^*} \in \mathcal{M}$. Next, we re-express L as $L = L^* + \mathcal{C}^*$, where

$$\mathcal{C}_T^* = \exp\left[2(r-\delta)T + \int_0^T \sigma_s^2 \mathrm{d}s + \int_0^T \int_{\mathbb{R}} (\mathrm{e}^x + \mathrm{e}^{-x} - 2)\mathrm{e}^x \nu(\mathrm{d}s, \mathrm{d}x)\right]. \tag{5.4.16}$$

By re-arranging the terms in equation (5.4.15), the result follows.

5.4.2 Valuation of European options

We outline a method for the valuation of vanilla options, based on bilateral Laplace transforms, that was developed in the PhD thesis of Sebastian Raible (see Raible (2000) Chap. 3). The method is extremely fast and allows for the valuation not only of plain vanilla European derivatives, but also of more complex payoffs, such as digital, self-quanto and power options; in principle, every *European* payoff can be priced by using this method. Moreover, a large variety of driving processes can be handled, including Lévy and additive processes.

The main idea of Raible's method is to represent the option price as a *convolution* of two functions and consider its bilateral Laplace transform; then, by using the property that *the Laplace transform of a convolution equals the product of the Laplace transforms of the factors*, we arrive at two Laplace transforms that are easier to calculate analytically than the original one. Inverting this Laplace transform yields the option price.

A similar method, in Fourier space, can be found in Lewis (2001). See also Carr and Madan (1999) for some preliminary results that motivated this research. Lee (2004) unifies and generalizes the existing Fourier-space methods and develops error bounds for the discretized inverse transforms.

We first state the necessary assumptions regarding the distribution of the asset price process and the option payoff respectively.

(L1): Assume that $\varphi_{L_T}(z)$, the extended characteristic function of L_T, exists for all $z \in \mathbb{C}$ with $\Im z \in I_1 \supset [0, 1]$.
(L2): Assume that $\mathbb{P}_{L_T}$, the distribution of L_T, is absolutely continuous w.r.t. the Lebesgue measure λ^1 with density ρ.
(L3): Consider a European-style payoff function $f(S_T)$ that is integrable.
(L4): Assume that $x \mapsto \mathrm{e}^{-Rx}|f(\mathrm{e}^{-x})|$ is bounded and integrable for all $R \in I_2 \subset \mathbb{R}$.

In order to price a European option with payoff function $f(S_T)$, we proceed as follows.

$$\begin{aligned} V = \mathrm{e}^{-rT}\mathbb{E}[f(S_T)] &= \mathrm{e}^{-rT}\int_\Omega f(S_T)\mathrm{d}\mathbb{P} \\ &= \mathrm{e}^{-rT}\int_\mathbb{R} f(S_0\mathrm{e}^x)\mathrm{d}\mathbb{P}_{L_T}(x) \\ &= \mathrm{e}^{-rT}\int_\mathbb{R} f(S_0\mathrm{e}^x)\rho(x)\mathrm{d}x \end{aligned} \tag{5.4.17}$$

because of absolute continuity. Define $\zeta = -\log S_0$ and $g(x) = f(\mathrm{e}^{-x})$, and then

$$V = \mathrm{e}^{-rT}\int_\mathbb{R} g(\zeta - x)\rho(x)\mathrm{d}x = \mathrm{e}^{-rT}(g * \rho)(\zeta) \tag{5.4.18}$$

which is a convolution at point ζ. Applying bilateral Laplace transforms on both sides of equation (5.4.18) and using Theorem B.2 in Raible (2000), we get

$$\begin{aligned} \mathfrak{L}_V(z) &= \mathrm{e}^{-rT}\int_\mathbb{R} \mathrm{e}^{-zx}(g * \rho)(x)\mathrm{d}x \\ &= \mathrm{e}^{-rT}\int_\mathbb{R} \mathrm{e}^{-zx}g(x)\mathrm{d}x \int_\mathbb{R} \mathrm{e}^{-zx}\rho(x)\mathrm{d}x \\ &= \mathrm{e}^{-rT}\mathfrak{L}_g(z)\mathfrak{L}_\rho(z) \end{aligned} \tag{5.4.19}$$

where $\mathfrak{L}_h(z)$ denotes the bilateral Laplace transform of a function h at $z \in \mathbb{C}$, i.e. $\mathfrak{L}_h(z) := \int_\mathbb{R} \mathrm{e}^{-zx}h(x)\mathrm{d}x$. The Laplace transform of g is very easy to compute analytically and the Laplace transform of ρ can be expressed as the extended characteristic function φ_{L_T} of L_T. By numerically inverting this Laplace transform, we recover the option price.

The next theorem gives us an explicit expression for the price of an option with payoff function f and driving PIIAC L.

Theorem 4.4. *Assume that* (**L1**)–(**L4**) *are in force and let* $g(x) := f(\mathrm{e}^{-x})$ *denote the modified payoff function of an option with payoff* $f(x)$ *at time* T*. Assume that* $I_1 \cap I_2 \neq \emptyset$

and choose an $R \in I_1 \cap I_2$. *Letting* $V(\zeta)$ *denote the price of this option, as a function of* $\zeta := -\log S_0$, *we have*

$$V(\zeta) = \frac{e^{\zeta R - rT}}{2\pi} \int_{\mathbb{R}} e^{iu\zeta} \mathfrak{L}_g(R+iu)\varphi_{L_T}(iR-u)du, \tag{5.4.20}$$

whenever the integral on the right-hand side exists.

Proof. The claim can be proved by using the arguments of the proof of Theorem 3.2 in Raible (2000); there, no explicit statement is made about the driving process L; hence, it directly transfers to the case of a time-inhomogeneous Lévy process.

Remark 4.5. In order to apply this method, validity of the necessary assumptions has to be verified. (**L1**), (**L3**) and (**L4**) are easy to certify, while (**L2**) is the most demanding one. Let us mention that the distributions underlying the most popular Lévy processes, such as the Generalized Hyperbolic Lévy motion (cf. Eberlein and Prause (2002)), possess a known Lebesgue density.

Remark 4.6. The method of Raible for the valuation of European options can be applied to general driving processes that satisfy Assumptions (**L1**)–(**L4**). Therefore, it can also be applied to stochastic volatility models based on Lévy processes that have attracted much interest lately; we refer to Barndorff-Nielsen and Shephard (2001), Eberlein *et al.* (2003) and Carr *et al.* (2003) for an account of different models.

5.4.3 Valuation of American options

The method of Raible, presented in the previous section, can be used for pricing several types of European derivatives, but not path-dependent ones. The valuation of American options in Lévy-driven models is quite a hard task and no analytical solution exists for the finite horizon case.

For perpetual American options, i.e. options with infinite time horizon, Mordecki (2002) derived formulae in the general case in terms of the law of the extrema of the Lévy process, using a random walk approximation to the process. He also provides explicit solutions for the case of a jump-diffusion with exponential jumps. Alili and Kyprianou (2005) recapture the results of Mordecki by making use of excursion theory. Boyarchenko and Levendorskiĭ (2002c) obtained formulae for the price of the American put option in terms of the Wiener–Hopf factors and derive some more explicit formulae for these factors. Asmussen *et al.* (2004) find explicit expressions for the price of American put options for Lévy processes with two-sided phase-type jumps; the solution uses the Wiener–Hopf factorization and can also be applied to regime-switching Lévy processes with phase-type jumps.

For the valuation of finite time horizon American options, one has to resort to numerical methods. Denote by $x = \ln S$ the log price, $\tau = T - t$ the time to maturity and $v(\tau, x) = f(e^x, T-\tau)$ the time-t value of an option with payoff function $g(e^x) = \phi(x)$. One approach is to use numerical schemes for solving the corresponding partial integro-differential inequality (PIDI),

$$\frac{\partial v}{\partial \tau} - \mathcal{A}v + rv \geq 0 \qquad \text{in } (0,T)\times\mathbb{R} \tag{5.4.21}$$

subject to the conditions

$$\begin{cases} v(\tau, x) \geq \phi(x), & \text{a.e. in } [0, T] \times \mathbb{R} \\ (v(\tau, x) - \phi(x))\left(\frac{\partial v}{\partial \tau} - \mathcal{A}v + rv\right) = 0, & \text{in } (0, T) \times \mathbb{R} \\ v(0, x) = \phi(x) \end{cases} \tag{5.4.22}$$

where

$$\begin{aligned} \mathcal{A}v(x) = {} & \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \\ & + \int_{\mathbb{R}} \left(v(x+y) - v(x) - (\mathrm{e}^y - 1)\frac{\mathrm{d}v}{\mathrm{d}x}(x)\right)\lambda(\mathrm{d}y) \end{aligned} \tag{5.4.23}$$

is the infinitesimal generator of the transition semigroup of L; see Matache *et al.* (2003, 2005) for all of the details and numerical solution of the problem using wavelets. Almendral (2004) solves the problem numerically by using implicit–explicit methods in case the CGMY is the driving process. Equation (5.4.21) is a backward PIDE in spot and time *to* maturity; Carr and Hirsa (2003) develop a forward PIDE in strike and time *of* maturity and solve it by using finite-difference methods.

Another alternative is to employ Monte Carlo methods adapted for optimal stopping problems, such as the American option; we refer here to Rogers (2002) or Glasserman (2003). Këllezi and Webber (2004) constructed a lattice for Lévy-driven assets and applied it to the valuation of Bermudan options. Levendorskiĭ (2004) develops a non-Gaussian analog of the method of lines and uses Carr's randomization method in order to formulate an approximate algorithm for the valuation of American options. Chesney and Jeanblanc (2004) revisit the perpetual American problem and obtain formulae for the optimal boundary when jumps are either only positive or only negative. Using these results, they approximate the finite horizon problem in a fashion similar to Barone-Adesi and Whaley (1987). Empirical tests show that this approximation provides good results only when the process is continuous at the exercise boundary.

5.5 EXOTIC OPTIONS

The work on this topic follows along the lines of Henderson and Wojakowski (2002); they proved an equivalence between the price of floating and fixed strike Asian options in the Black–Scholes model. We also refer to Vanmaele *et al.* (2002) for a generalization of these results to forward-start options and discrete averaging in the Black–Scholes model.

5.5.1 Symmetry

For exotic options, the setting is that of (**P2**): $L^1 = L$ is the driving $\mathbb{R}$-valued PIIAC with triplet (B, C, ν), $S^1 = S_0^1 \exp L^1 = S$ and $S^2 = h(S)$ is a functional of S. The most prominent candidates for functionals are the maximum, the minimum and the (arithmetic) average; let $0 = t_1 < t_2 < \cdots < t_n = T$ be equidistant time points, and then the resulting processes, in case of discrete monitoring, are

$$M_T = \max_{0 \leq t_i \leq T} S_{t_i}, \quad N_T = \min_{0 \leq t_i \leq T} S_{t_i} \quad \text{and} \quad \Sigma_T = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}.$$

Table 5.1 Types of payoffs for Asian and lookback options

Option type	Asian payoff	Lookback payoff
Fixed strike call	$(\Sigma_T - K)^+$	$(M_T - K)^+$
Fixed strike put	$(K - \Sigma_T)^+$	$(K - N_T)^+$
Floating strike call	$(S_T - \Sigma_T)^+$	$(S_T - N_T)^+$
Floating strike put	$(\Sigma_T - S_T)^+$	$(M_T - S_T)^+$

Therefore, we can exploit symmetries between floating and fixed strike Asian and lookback options in this framework; the different types of payoffs of the Asian and lookback option are summarized in Table 5.1.

We introduce the following notation for the value of the floating strike call option, be it Asian or lookback

$$V_c(S_T, h(S); r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[(S_T - h(S)_T)^+\right]$$

and similarly, for the fixed strike put option we set

$$V_p(K, h(S); r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[(K - h(S)_T)^+\right];$$

a similar notation will be used for the other two cases.

Now we can state a result that relates the value of floating and fixed strike options. Notice that because stationarity of the increments plays an important role in the proof, the result is valid only for Lévy processes.

Theorem 5.1. *Assuming that the asset price evolves as an exponential Lévy process, we can relate the floating and fixed strike Asian or lookback option via the following symmetry:*

$$V_c\left(S_T, h(S); r, \delta, \sigma^2, \lambda\right) = V_p\left(S_0, h(S); \delta, r, \sigma^2, -f\lambda\right) \tag{5.5.1}$$

$$V_p\left(h(S), S_T; r, \delta, \sigma^2, \lambda\right) = V_c\left(h(S), S_0; \delta, r, \sigma^2, -f\lambda\right) \tag{5.5.2}$$

where $f(x) = \mathrm{e}^x$.

Proof. We refer to the proof of Theorems 3.1 and 4.1 in Eberlein and Papapantoleon (2005). The minimal assumptions necessary for the results to hold are also stated there.

Remark 5.2. These results also hold for *forward-start* Asian and lookback options, for *continuously* monitored options, for *partial* options and for Asian options on the *geometric* and *harmonic* average; see Eberlein and Papapantoleon (2005) for all of the details. Note that the equivalence result is not valid for *in-progress* Asian options.

5.5.2 Valuation of barrier and lookback options

The valuation of barrier and lookback options for assets driven by general Lévy processes is another hard mathematical problem. The difficulty stems from the fact that (a) the distribution of the supremum or infimum of a Lévy process is not known explicitly, and (b) the overshoot distribution associated with the passage of a Lévy process across a barrier is also not known explicitly.

Various authors have treated the problem in the case where the driving process is a spectrally positive/negative Lévy process; see, for example, Rogers (2000), Schürger (2002) and Avram *et al*. (2004). Kou and Wang (2003, 2004) have derived explicit formulas for the values of barrier and lookback options in a jump diffusion model where the jumps are double-exponentially distributed; they make use of a special property of the exponential distribution, namely the *memoryless* property, which allows them to explicitly calculate the overshoot distribution. Lipton (2002) derives similar formulas for the same model, making use of fluctuation theory.

Fluctuation theory and the Wiener–Hopf factorization of Lévy processes play a crucial role in every attempt to derive closed form solutions for the value of barrier and lookback options in Lévy-driven models. Introduce the notation

$$M_t = \sup_{0\le s\le t} L_s \qquad \text{and} \qquad N_t = \inf_{0\le s\le t} L_s$$

and let θ denote a random variable exponentially distributed with parameter q, independent of L. Then, the celebrated *Wiener–Hopf factorization* of the Lévy process L states that

$$\mathbb{E}[\exp(izL_\theta)] = \mathbb{E}[\exp(izM_\theta)] \cdot \mathbb{E}[\exp(izN_\theta)] \tag{5.5.3}$$

or equivalently

$$q(q - \psi(z))^{-1} = \varphi_q^+(z) \cdot \varphi_q^-(z), \qquad z \in \mathbb{R}, \tag{5.5.4}$$

where ψ denotes the characteristic exponent of L. The functions φ_q^+ and φ_q^- have the following representations

$$\varphi_q^+(z) = \exp\Big[\int_0^\infty t^{-1}\mathrm{e}^{-qt}\mathrm{d}t \int_0^\infty (\mathrm{e}^{izx} - 1)\mu^t(\mathrm{d}x)\Big] \tag{5.5.5}$$

$$\varphi_q^-(z) = \exp\Big[\int_0^\infty t^{-1}\mathrm{e}^{-qt}\mathrm{d}t \int_{-\infty}^0 (\mathrm{e}^{izx} - 1)\mu^t(\mathrm{d}x)\Big] \tag{5.5.6}$$

where $\mu^t(\mathrm{d}x) = \mathbb{P}_{L_t}(\mathrm{d}x)$ is the probability measure of L_t. These results were first proved for Lévy processes in Bingham (1975) – where an approximation of Lévy processes by random walks is employed – and subsequently by Greenwood and Pitman (1980) – where excursion theory is applied. See also the recent books by Sato (1999, Chapter 9) and Bertoin (1996, Chapter VI) respectively, for an account of these two methods.

Building upon these results, various authors have derived formulae for the valuation of barrier and lookback options; Boyarchenko and Levendorskiĭ (2002a) apply methods from potential theory and pseudodifferential operators to derive formulae for barrier and touch options, while Nguyen-Ngoc and Yor (2005) use a probabilistic approach based on excursion theory. Recently, Nguyen-Ngoc (2003) takes a similar probabilistic approach, motivated from Carr and Madan (1999) and derives quite simple formulae for the value of barrier and lookback options, which can be numerically evaluated with the use of Fourier inversion algorithms in two and three dimensions.

More specifically, let us denote by $V_c(M_T, K; T)$ the price of a fixed strike lookback option with payoff $(M_T - K)^+$, where $M_T = \max_{0\le t\le T} S_t$ and S is an exponential Lévy process. Choose $\gamma > 1$ and $\alpha > 0$ such that $\mathbb{E}[e^{2L_1}] < e^{r+\alpha}$ and set $V_c^{\alpha,\gamma}(M_T, K; T) = e^{-\alpha T - \gamma k} V_c(M_T, K; T)$ where $k = \log(K/S_0)$. Then, we have the following result.

Proposition 5.3. *If $k > 0$, then for all $q, u > 0$ we have:*

$$\int_0^\infty e^{-qT} dT \int_0^\infty e^{-uk} V_c^{\alpha,\gamma}(M_T, S_0 e^k; T) dk$$
$$= S_0 \frac{1}{q+r+\alpha} \frac{1}{z(z-1)} [\varphi^+_{q+r+\alpha}(i(z-1)) + (z-1)\varphi^+_{q+r+\alpha}(-i) - z] \qquad (5.5.7)$$

where $z = u + \gamma$.

Proof. We refer to the proof of Proposition 3.9 in Nguyen-Ngoc (2003).

The formula for the value of the floating strike lookback option is – as one could easily foresee – a lot more complicated than equation (5.5.7). Using the symmetry result of Theorem 5.1, this case can be dealt with via a change of the Lévy triplet and strike in the previous proposition.

The Wiener–Hopf factors are not known explicitly in the general case and numerical computation could be extremely time-consuming. Boyarchenko and Levendorskiĭ (2002b) provide some more efficient formulas for the Wiener–Hopf factors of – what they call – regular Lévy processes of exponential type (RLPE); for the definition refer to Section 1.2.2 in the above-mentioned reference. Given that L is an RLPE, $\varphi_q^+(z)$ has an analytic continuation on the half plane $\Im z > \omega$ and

$$\varphi_q^+(z) = \exp\Big[\frac{z}{2\pi i} \int_{-\infty+i\omega}^{+\infty+i\omega} \frac{\ln(q+\psi(u))}{u(z-u)} du\Big]. \qquad (5.5.8)$$

The family of RLPEs contains many popular – in mathematical finance – Lévy motions such as the Generalized Hyperbolic and Variance Gamma models (see Boyarchenko and Levendorskiĭ (2002b)).

Discretely monitored options have received much less attention in the literature than their continuous time counterparts. Borovkov and Novikov (2002) use Fourier methods and Spitzer's identity to derive formulae for fixed strike lookback options.

Various numerical methods have been applied for the valuation of barrier and lookback options in Lévy-driven models. Cont and Voltchkova (2005a, 2005b) study finite-difference methods for the solution of the corresponding PIDE (see also Matache *et al.* (2004)). Ribeiro and Webber (2003, 2004) have developed fast Monte Carlo methods for the valuation of exotic options in models driven by the Variance Gamma (VG) and Normal Inverse Gaussian (NIG) Lévy motions; their method is based on the construction of Gamma and Inverse Gaussian bridges, respectively, to speed up the Monte Carlo simulation. The recent book of Schoutens (2003) contains a detailed account of Monte Carlo methods for Lévy processes, also allowing for stochastic volatility.

5.5.3 Valuation of Asian and basket options

An explicit solution for the value of the arithmetic Asian or basket option is not known in the Black–Scholes model and, of course, the situation is similar for Lévy models. The difficulty is that the distribution of the arithmetic sum of log-normal random variables – more generally, random variables drawn from some log r infinitely divisible distribution – is not known in closed-form.

Večeř and Xu (2004) formulated a PIDE for all types of Asian options – including in-progress options – in a model driven by a process with independent increments (PII) or, more generally, a special semimartingale. Their derivation is based on the construction of a suitable self-financing trading strategy to replicate the average and then a change of numéraire – which is essentially the one we use – in order to reduce the number of variables in the equation. Their PIDE is relatively simple and can be solved by using numerical techniques such as finite-differences.

Albrecher and Predota (2002, 2004) use moment-matching methods to derive approximate formulae for the value of Asian options in some popular Lévy models such as the NIG and VG models; they also derive bounds for the option price in these models. See also the survey paper by Albrecher (2004) for a detailed account of the above mentioned results. Hartinger and Predota (2002) apply Quasi Monte-Carlo methods for the valuation of Asian options in the Hyperbolic model. Their method can be extended to the class of Generalized Hyperbolic Lévy motions, which contains the VG motion as a special case; see Eberlein and von Hammerstein (2004). Benhamou (2002), building upon the work of Carverhill and Clewlow (1992), uses the Fast Fourier transform and a transformation of dependent variables into independent ones, in order to value discretely monitored fixed strike Asian options. As he points out, this method can be applied when the return distribution is fat-tailed, with Lévy processes being prominent candidates.

Henderson *et al.* (2004) derive an upper bound for in-progress floating strike Asian options in the Black–Scholes model, using the symmetry result of Henderson and Wojakowski (2002) and valuation methods for fixed strike ones. Their pricing bound relies on a model-dependent symmetry result and a model-independent decomposition of the floating-strike Asian option into a fixed-strike one and a vanilla option. Therefore, given the symmetry result of Theorem 5.1, their general methodology can also be applied to Lévy models.

Albrecher *et al.* (2004) derive static super-hedging strategies for fixed strike Asian options in Lévy models; these results were extended to Lévy models with stochastic volatility in Albrecher and Schoutens (2005). The method is based on super-replicating the Asian payoff with a portfolio of plain vanilla calls, using the following upper bound

$$\left(\sum_{j=1}^{n} S_{t_j} - nK\right)^+ \leq \sum_{j=1}^{n}(S_{t_j} - nK_j)^+ \tag{5.5.9}$$

and then optimizing the hedge, i.e. the choice of K_js, using results from co-monotonicity theory.

Similar ideas appear in Hobson *et al.* (2004) for the static super-hedging of basket options. The payoff of the basket option is super-replicated by a portfolio of plain vanilla calls on each individual asset, using the upper bound

$$\left(\sum_{i=1}^{n} w_i S_T^i - K\right)^+ \leq \sum_{i=1}^{n}\left(w_i S_T^i - l_i K\right)^+ \tag{5.5.10}$$

where $l_i \geq 0$ and $\sum_{i=1}^{n} l_i = 1$; subsequently, the portfolio is optimized using co-monotonicity theory. Moreover, no distribution is assumed about the asset dynamics, since all of the information needed is the marginal distributions which can be deduced from the volatility smile; we refer to Breeden and Litzenberger (1978). This is also observed by Albrecher and Schoutens (2005).

5.6 MARGRABE-TYPE OPTIONS

In this section, we derive symmetry results between options involving two assets – such as Margrabe or Quanto options – and European plain vanilla options; therefore, we generalize results by Margrabe (1978) and Fajardo and Mordecki (2003) to the case of time-inhomogeneous Lévy processes. Schroder (1999) provides similar results for semimartingale models; the advantage of using a Lévy process or a PIIAC instead of a semimartingale as the driving motion, is that the distribution of the asset returns under the new measure can be deduced from the distribution of the returns of each individual asset under the risk-neutral measure.

For Margrabe-type options, the setting is that of (**P3**): $L = (L^1, L^2)$ is the driving $\mathbb{R}^2$-valued PIIAC with triplet (B, C, ν) and $S = (S^1, S^2)$ is the asset price process. For convenience, we set

$$S_t^i = S_0^i \exp\left[(r - \delta^i)t + L_t^i\right], \qquad i = 1, 2, \tag{5.6.1}$$

modifying the characteristic triplet (B, C, ν) accordingly.

With Theorem 25.17 in Sato (1999) and Lemma 2.4, Assumption ($\mathbb{EM}$) guarantees the existence of the moment generating function M_{L_t} of L_t for $u \in \mathbb{C}^d$ such that $\Re u \in [-M, M]^d$. Furthermore, for $u \in \mathbb{C}^d$ with $\Re u \in [-M, M]^d$, we have that

$$\begin{aligned} M_{L_t}(u) = \varphi_{L_t}(-iu) &= \mathbb{E}\left[\mathrm{e}^{\langle u, L_t\rangle}\right] \\ &= \exp \int_0^t \Big[\langle u, b_s\rangle + \frac{1}{2}\langle u, c_s u\rangle \\ &\qquad + \int_{\mathbb{R}^d} (\mathrm{e}^{\langle u, x\rangle} - 1 - \langle u, x\rangle)\lambda_s(\mathrm{d}x)\Big]\,\mathrm{d}s. \end{aligned} \tag{5.6.2}$$

The next result will allow us to calculate the characteristic triplet of a one-dimensional process, defined as a scalar product of a vector with the d-dimensional process L, from the characteristics of L under an equivalent change of probability measure.

Proposition 6.1. *Let L be a d-dimensional PIIAC with triplet (B, C, ν) under $\mathbb{P}$, let u, v be vectors in $\mathbb{R}^d$ and $v \in [-M, M]^d$. Moreover, let $\widetilde{\mathbb{P}} \overset{\text{loc}}{\sim} \mathbb{P}$, with density*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{\langle v, L_{\overline{T}}\rangle}}{\mathbb{E}[\mathrm{e}^{\langle v, L_{\overline{T}}\rangle}]}.$$

Then, the one-dimensional process $\widehat{L} := \langle u, L\rangle$ is a $\widetilde{\mathbb{P}}$-PIIAC and its characteristic triplet is $(\widehat{B}, \widehat{C}, \widehat{\nu})$ with

$$\begin{aligned} \widehat{b_s} &= \langle u, b_s\rangle + \frac{1}{2}\big(\langle u, c_s v\rangle + \langle v, c_s u\rangle\big) + \int_{\mathbb{R}^d} \langle u, x\rangle\big(\mathrm{e}^{\langle v, x\rangle} - 1\big)\lambda_s(\mathrm{d}x) \\ \widehat{c_s} &= \langle u, c_s u\rangle \\ \widehat{\lambda_s} &= \mathcal{T}(\kappa_s) \end{aligned}$$

where $\mathcal{T}$ is a mapping $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}$, such that $x \mapsto \mathcal{T}(x) = \langle u, x\rangle$, and κ_s is a measure defined by

$$\kappa_s(A) = \int_A \mathrm{e}^{\langle v,x\rangle}\lambda_s(\mathrm{d}x).$$

Proof. Because the density process (η_t) is given by $\eta_t = \mathrm{e}^{\langle v,L_t\rangle}\mathbb{E}[\mathrm{e}^{\langle v,L_t\rangle}]^{-1}$, by using equation (5.6.2) we get

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[\mathrm{e}^{z\langle u,L_t\rangle}\right] &= \mathbb{E}\left[\mathrm{e}^{z\langle u,L_t\rangle}\eta_t\right] \\
&= \mathbb{E}\left[\mathrm{e}^{z\langle u,L_t\rangle}\mathrm{e}^{\langle v,L_t\rangle}\mathbb{E}\left[\mathrm{e}^{\langle v,L_t\rangle}\right]^{-1}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{\langle zu+v,L_t\rangle}\right]\mathbb{E}\left[\mathrm{e}^{\langle v,L_t\rangle}\right]^{-1} \\
&= \exp\int_0^t \Big[\langle zu+v, b_s\rangle + \frac{1}{2}\langle zu+v, c_s(zu+v)\rangle \\
&\qquad + \int_{\mathbb{R}^d}(\mathrm{e}^{\langle zu+v,x\rangle} - 1 - \langle zu+v, x\rangle)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s \\
&\quad \times \exp\int_0^t -\Big[\langle v, b_s\rangle + \frac{1}{2}\langle v, c_s v\rangle \\
&\qquad + \int_{\mathbb{R}^d}(\mathrm{e}^{\langle v,x\rangle} - 1 - \langle v, x\rangle)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s \\
&= \exp\int_0^t \Big[z\Big\{\langle u, b_s\rangle + \frac{1}{2}\big(\langle u, c_s v\rangle + \langle v, c_s u\rangle\big) \\
&\qquad + \int_{\mathbb{R}^d}\langle u, x\rangle\big(\mathrm{e}^{\langle v,x\rangle} - 1\big)\lambda_s\mathrm{d}s\Big\} + \frac{1}{2}z^2\langle u, c_s u\rangle \\
&\qquad + \int_{\mathbb{R}^d}\big(\mathrm{e}^{z\langle u,x\rangle} - 1 - z\langle u, x\rangle\big)\mathrm{e}^{\langle v,x\rangle}\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s.
\end{aligned} \tag{5.6.3}$$

If we write κ_s for the measure on $\mathbb{R}^d$ given by

$$\kappa_s(A) = \int_A \mathrm{e}^{\langle v,x\rangle}\lambda_s(\mathrm{d}x), \tag{5.6.4}$$

$A \in \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{T}$ for the linear mapping $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}$ given by $\mathcal{T}(x) = \langle u, x\rangle$, then we get for the last term in the exponent of equation (5.6.3)

$$\int_{\mathbb{R}^d}\big(\mathrm{e}^{z\langle u,x\rangle} - 1 - z\langle u, x\rangle\big)\mathrm{e}^{\langle v,x\rangle}\lambda_s(\mathrm{d}x) = \int_{\mathbb{R}}\big(\mathrm{e}^{zy} - 1 - zy\big)\mathcal{T}(\kappa_s)(\mathrm{d}y)$$

by the change-of-variable formula. The resulting characteristics satisfy Assumption ($\mathbb{AC}$), and thus the result follows.

The valuation of options depending on two assets modelled by a two-dimensional PIIAC can now be simplified – using the technique described in Section 5.3 and Proposition 6.1 – to the valuation of an option on a one-dimensional asset. Subsequently, this option can be priced by using bilateral Laplace transforms, as described in Section 5.4.2.

The payoff of a Margrabe option, or option to exchange one asset for another, is

$$\left(S_T^1 - S_T^2\right)^+$$

and we denote its value by

$$V_m(S_0^1, S_0^2; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[\left(S_T^1 - S_T^2\right)^+\right]$$

where $\delta = (\delta^1, \delta^2)$. The payoff of the Quanto call and put option is

$$S_T^1\left(S_T^2 - K\right)^+ \qquad \text{and} \qquad S_T^1\left(K - S_T^2\right)^+,$$

respectively, and we will use the following notation for the value of the Quanto call option

$$V_{qc}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[S_T^1\left(S_T^2 - K\right)^+\right]$$

and similarly for the Quanto put option

$$V_{qp}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[S_T^1\left(K - S_T^2\right)^+\right].$$

The different variants of the Quanto option traded in Foreign Exchange markets are explained in Musiela and Rutkowski (1997). The payoff of a cash-or-nothing and a two-dimensional asset-or-nothing option is

$$1\!\!1_{\{S_T > K\}} \qquad \text{and} \qquad S_T^1 1\!\!1_{\{S_T^2 > K\}}.$$

The holder of a two-dimensional asset-or-nothing option receives one unit of asset S^1 at expiration, if asset S^2 ends up in the money; of course, this is a generalization of the (standard) asset-or-nothing option, where the holder receives one unit of the asset if it ends up in the money. We denote the value of the cash-or-nothing option by

$$V_{cn}(S_0, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[1\!\!1_{\{S_T > K\}}\right]$$

and the value of the two-dimensional asset-or-nothing option by

$$V_{an}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[S_T^1 1\!\!1_{\{S_T^2 > K\}}\right].$$

Notice that in the first case, r, δ, C and ν correspond to a one-dimensional driving process, while in the second case to a two-dimensional one.

Theorem 6.2. *Let Assumption* ($\mathbb{EM}$) *be in force and assume that the asset price evolves as an exponential PIIAC according to equations (5.2.3)–(5.2.5). We can relate the value of a Margrabe and a European plain vanilla option via the following symmetry:*

$$V_m(S_0^1, S_0^2; r, \delta, C, \nu) = \mathbb{E}[S_T^1]\mathrm{e}^{\widehat{\mathfrak{C}}_T} V_p\left(S_0^2/S_0^1, \mathcal{K}; \delta^1, r, \widehat{C}, \widehat{\nu}\right) \tag{5.6.5}$$

where $\mathcal{K} = \mathrm{e}^{-\widehat{\mathfrak{C}}_T}$, $\widehat{\mathfrak{C}}$ *is given by equation (5.6.9) (see below) and the characteristics* $(\widehat{C}, \widehat{\nu})$ *are given by Proposition 6.1 for* $v = (1, 0)$ *and* $u = (-1, 1)$.

Proof. Expressing the value of the Margrabe option in units of the *numéraire*, we get

$$\widetilde{V} := \frac{V_m}{S_0^1} = \frac{\mathrm{e}^{-rT}}{S_0^1}\mathbb{E}\left[\left(S_T^1 - S_T^2\right)^+\right]$$
$$= \mathrm{e}^{-\delta^1 T}\mathbb{E}\left[\frac{\mathrm{e}^{-rT}S_T^1}{\mathrm{e}^{-\delta^1 T}S_0^1}\frac{\eta_T^1}{\eta_T^1}\left(1 - \frac{S_T^2}{S_T^1}\right)^+\right]$$

where $\eta^1 = \mathbb{E}[\exp(L^1)] = \mathbb{E}[\exp\langle v, L\rangle]$, for $v = (1, 0)$, and by using equation (5.6.1) we get

$$= \mathrm{e}^{-\delta^1 T}\eta_T^1\mathbb{E}\left[\frac{\mathrm{e}^{L_T^1}}{\eta_T^1}\left(1 - \frac{S_T^2}{S_T^1}\right)^+\right]. \tag{5.6.6}$$

Define a new measure $\widetilde{\mathbb{P}}$ via its Radon–Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{L_T^1}}{\mathbb{E}[\mathrm{e}^{L_T^1}]}$$

and the valuation problem takes the form

$$\widetilde{V} = \mathrm{e}^{-\delta^1 T}\eta_T^1\widetilde{\mathbb{E}}\left[\left(1 - \widehat{S}_T\right)^+\right]$$

where, by using equation (5.6.1), we get

$$\widehat{S}_t := \frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1}\mathrm{e}^{(\delta^1-\delta^2)t+L_t^2-L_t^1} =: \widehat{S}_0\exp\left[(\delta^1 - \delta^2)t + \widehat{L}_t\right] \tag{5.6.7}$$

and $\widehat{L} := L^2 - L^1 = \langle u, L\rangle$ for $u = (-1, 1)$. The characteristic triplet of $\widehat{L}$, $(\widehat{B}, \widehat{C}, \widehat{\nu})$ under $\widetilde{\mathbb{P}}$, is given by Proposition 6.1 for $v = (1, 0)$ and $u = (-1, 1)$.

Observe that $\mathrm{e}^{(r-\delta^1)t}\widehat{S}_t$ is not a $\widetilde{\mathbb{P}}$-martingale. However, if we define

$$\overline{L}_t := (\delta^1 - r)t - \frac{1}{2}\int_0^t \widehat{c}_s\mathrm{d}s - \int_0^t\int_{\mathbb{R}}(\mathrm{e}^x - 1 - x)\widehat{\nu}(\mathrm{d}s, \mathrm{d}x)$$
$$+ \int_0^t \widehat{c}_s^{1/2}\mathrm{d}\widetilde{W}_s + \int_0^t\int_{\mathbb{R}} x(\mu^{\widehat{L}} - \widehat{\nu})(\mathrm{d}s, \mathrm{d}x) \tag{5.6.8}$$

where $\widetilde{W}$ is a $\widetilde{\mathbb{P}}$-standard Brownian motion and $\mu^{\widehat{L}}$ is the random measure of jumps of $\widehat{L}$, then $\mathrm{e}^{(r-\delta^1)t}\mathrm{e}^{\overline{L}_t} \in \mathcal{M}$. Therefore, we re-express the exponent of equation (5.6.7) as $\widehat{L}_t + (\delta^1 - \delta^2)t = \overline{L}_t + \widehat{\mathfrak{C}}_t$ where

$$\widehat{\mathfrak{C}}_t = (r - \delta^2)t + \int_0^t \widehat{b}_s\mathrm{d}s + \frac{1}{2}\int_0^t \widehat{c}_s\mathrm{d}s + \int_0^t\int_{\mathbb{R}}(\mathrm{e}^x - 1 - x)\widehat{\nu}(\mathrm{d}s, \mathrm{d}x) \tag{5.6.9}$$

and define $\overline{S}_t := \widehat{S}_0\exp\overline{L}_t$.

Now the result follows, because

$$\begin{aligned}\widetilde{V} &= \mathrm{e}^{-\delta^1 T}\eta_T^1 \widetilde{\mathbb{E}}\left[\left(1-\widehat{S}_T\right)^+\right]\\ &= \mathrm{e}^{-\delta^1 T}\eta_T^1 \widetilde{\mathbb{E}}\left[\left(1-\overline{S}_T \mathrm{e}^{\widehat{\mathfrak{C}}_T}\right)^+\right]\\ &= \mathrm{e}^{-\delta^1 T}\eta_T^1 \mathrm{e}^{\widehat{\mathfrak{C}}_T} \widetilde{\mathbb{E}}\left[\left(\mathrm{e}^{-\widehat{\mathfrak{C}}_T}-\overline{S}_T\right)^+\right].\end{aligned}$$

Theorem 6.3. *Let Assumption* ($\mathbb{EM}$) *be in force and assume that the asset price evolves as an exponential PIIAC according to equations (5.2.3)–(5.2.5). We can relate the value of a Quanto and a European plain vanilla call option via the following symmetry:*

$$V_{qc}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathbb{E}[S_T^1]\mathrm{e}^{\widehat{\mathfrak{C}}_T} V_p\big(S_0^2, \mathcal{K}; \delta^1, r, \widehat{C}, \widehat{\nu}\big) \tag{5.6.10}$$

where $\mathcal{K} = \mathrm{e}^{-\widehat{\mathfrak{C}}_T}$, *the constant* $\widehat{\mathfrak{C}}$ *is given by*

$$\widehat{\mathfrak{C}}_t = (2r - \delta^1 - \delta^2)t + \int_0^t \widehat{b}_s \mathrm{d}s + \frac{1}{2}\int_0^t \widehat{c}_s \mathrm{d}s + \int_0^t \int_{\mathbb{R}} (\mathrm{e}^x - 1 - x)\widehat{\nu}(\mathrm{d}s, \mathrm{d}x)$$

and the characteristics $(\widehat{C}, \widehat{\nu})$ *are given by Proposition 6.1 for* $v = (1, 0)$ *and* $u = (0, 1)$. *A similar relationship holds for the Quanto and European plain vanilla put options.*

Proof. The proof follows along the lines of that of Theorem 6.2.

Theorem 6.4. *Let Assumption* ($\mathbb{EM}$) *be in force and assume that the asset price evolves as an exponential PIIAC according to equations (5.2.3)–(5.2.5). We can relate the value of a two-dimensional asset-or-nothing and a cash-or-nothing option via the following symmetry:*

$$V_{an}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathbb{E}[S_T^1] V_{cn}\big(S_0^2, \mathcal{K}; \delta^1, r, \widehat{C}, \widehat{\nu}\big) \tag{5.6.11}$$

where $\mathcal{K} = K\mathrm{e}^{-\widehat{\mathfrak{C}}_T}$, *the constant* $\widehat{\mathfrak{C}}$ *is given by*

$$\widehat{\mathfrak{C}}_t = (2r - \delta^1 - \delta^2)t + \int_0^t \widehat{b}_s \mathrm{d}s + \frac{1}{2}\int_0^t \widehat{c}_s \mathrm{d}s + \int_0^t \int_{\mathbb{R}} (\mathrm{e}^x - 1 - x)\widehat{\nu}(\mathrm{d}s, \mathrm{d}x)$$

and the characteristics $(\widehat{C}, \widehat{\nu})$ *are given by Proposition 6.1 for* $v = (1, 0)$ *and* $u = (0, 1)$. *A similar relationship holds for the corresponding put options.*

Proof. The proof follows along the lines of that of Theorem 6.2.

Remark 6.5. Notice that the factor $\mathbb{E}[S_T^1]$ is the forward price of the asset S^1, the *numéraire* asset.

ACKNOWLEDGEMENTS

We thank Wolfgang Kluge for helpful discussions during the work on these topics. The second named author acknowledges the financial support provided through the European Community's Human Potential Programme under Contract HPRN-CT-2000-00100 DYN-STOCH.

REFERENCES

[1] Albrecher, H. (2004), "The valuation of Asian options for market models of exponential Lévy type", in M. Van Maele, A. De Schepper, J. Dhaene, H. Reynaerts, W. Schoutens and P. Van Goethem (Eds), *Handelingen Contact forum – 2nd Actuarial and Financial Mathematics Day*, Royal Flemish Academy of Belgium for Science and Arts, Brussels, Belgium, pp. 11–20.

[2] Albrecher, H., Dhaene, J., Goovaerts, M. and Schoutens, W. (2005), "Static hedging of Asian options under Lévy models: the comonotonicity approach", *The Journal of Derivatives*, **12**(3), 63–72.

[3] Albrecher, H. and Predota, M. (2002), "Bounds and approximations for discrete Asian options in a variance-gamma model", *Grazer Mathematische Berichte*, **345**, 35–57.

[4] Albrecher, H. and Predota, M. (2004), "On Asian option pricing for NIG Lévy processes", *Journal of Computational and Applied Mathematics* **172**, 153–168.

[5] Albrecher, H. and Schoutens, W. (2005), "Static hedging of Asian options under stochastic volatility models using Fast Fourier Transform", in A. Kyprianou, W. Schoutens and P. Wilmott (Eds), *Exotic Option Pricing and Advanced Lévy Models*, Wiley, Chichester, UK, pp. 129–147.

[6] Kyprianou A.E. and Alili, L. (2005), "Some remarks on first passage of Lévy process, the American put and pasting principles", *The Annals of Applied Probability*, to be published.

[7] Almendral, A. (2005), "Numerical valuation of American options under the CGMY process", in A. Kyprianou, W. Schoutens, and P. Wilmott (Eds), *Exotic Option Pricing and Advanced Lévy Models*, Wiley, Chichester, UK, pp. 259–276.

[8] Asmussen, S., Avram, F. and Pistorius, M.R. (2004), "Russian and American put options under exponential phase-type Lévy models", *Stochastic Processes and their Applications*, **109**, 79–111.

[9] Avram, F., Kyprianou, A. and Pistorius, M.R. (2004), "Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options", *Annals of Applied Probability*, **14**, 215–238.

[10] Barndorff-Nielsen, O.E. and Shephard, N. (2001), "Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics", *Journal of the Royal Statistical Society, Series B*, **63**, 167–241.

[11] Barone-Adesi, G. and Whaley, R.E. (1987), "Efficient analytic approximation of American option values", *Journal of Finance*, **42**, 301–320.

[12] Bellini, F. and Frittelli, M. (2002), "On the existence of minimax martingale measures", *Mathematical Finance*, **12**, 1–21.

[13] Benhamou, E. (2002), "Fast Fourier Transform for discrete Asian options", *Journal of Computational Finance*, **6**, 49–68.

[14] Bertoin, J. (1996), *Lévy Processes*, Cambridge University Press, Cambridge, UK.

[15] Bingham, N.H. (1975), "Fluctuation theory in continuous time", *Advances in Applied Probability*, **7**, 705–766.

[16] Black, F. and Scholes, M. (1973), "The pricing of options and corporate liabilities", *Journal of Political Economy*, **81**, 637–654.

[17] Borovkov, K. and Novikov, A. (2002), "On a new approach to calculating expectations for option pricing", *Journal of Applied Probability*, **39**, 889–895.

[18] Boyarchenko, S.I. and Levendorskiĭ, S.Z. (2002a), "Barrier options and touch-and-out options under regular Lévy processes of exponential type", *Annals of Applied Probability*, **12**, 1261–1298.

[19] Boyarchenko, S.I. and Levendorskiĭ, S.Z. (2002b), *Non-Gaussian Merton–Black–Scholes Theory*, World Scientific, Singapore.

[20] Boyarchenko, S.I. and Levendorskiĭ, S.Z. (2002c), "Perpetual American options under Lévy processes", *SIAM Journal on Control and Optimization*, **40**, 1663–1696.

[21] Breeden, D. and Litzenberger, R. (1978), "Prices of state-contingent claims implicit in option prices", *Journal of Business*, **51**, 621–651.

[22] Carr, P. (1994), "European put call symmetry", *Working Paper*, Cornell University, Ithaca, New York, NY, USA.

[23] Carr, P. and Chesney, M. (1996), "American put call symmetry", *Working Paper*, École des hautes études commerciales de Montreal (HEC), Montreal, QU, Canada.

[24] Carr, P., Geman, H. Madan, D.B. and Yor, M. (2002), "The fine structure of asset returns: an empirical investigation", *Journal of Business*, **75**, 305–332.

[25] Carr, P., Geman, H. Madan, D.B. and Yor, M. (2003), "Stochastic volatility for Lévy processes", *Mathematical Finance*, **13**, 345–382.

[26] Carr, P. and Hirsa, A. (2003), "Why be backward? Forward equations for American options", *Risk*, **16**(1), 103–107. Reprinted in A. Kyprianou, W. Schoutens and P. Wilmott (Eds) (2005), *Exotic Option Pricing and Advanced Lévy Models*, Wiley, Chichester, UK, pp. 237–257.

[27] Carr, P. and Madan, D.B. (1999), "Option valuation using the fast Fourier transform", *Journal of Computational Finance*, **2**, 61–73.

[28] Carverhill, A. and Clewlow, L. (1992), "Flexible convolution", in R. Tompkins (Ed.), *From Black–Scholes to Black Holes*, Risk Publications, London, UK, pp. 165–171.

[29] Chesney, M. and Jeanblanc, M. (2004), "Pricing American currency options in an exponential Lévy model", *Applied Mathematical Finance*, **11**, 207–225.

[30] Cont, R. and Tankov, P. (2004), "Nonparametric calibration of jump-diffusion option pricing models", *Journal of Computational Finance*, **7**(3), 1–49.

[31] Cont, R. and Voltchkova, E. (2005), "Finite difference methods for option pricing in jump-diffusion and exponential Lévy models", *SIAM Journal on Numerical Analysis*, to be published.

[32] Cont, R. and Voltchkova, E. (2005), "Integro-differential equations for option prices in exponential Lévy models", *Finance and Stochastics*, **9**, 299–325.

[33] Delbaen, F. and Schachermayer, W. (1994), "A general version of the fundamental theorem of asset pricing", *Mathematische Annalen*, **300**, 463–520.

[34] Delbaen, F. and Schachermayer, W. (1998), "The fundamental theorem of asset pricing for unbounded stochastic processes", *Mathematische Annalen*, **312**, 215–250.

[35] Detemple, J. (2001), "American options: symmetry properties", in J. Cvitanic, E. Jouini and M. Musiela (Eds), *Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, Cambridge, UK, pp. 67–104.

[36] Eberlein, E. and Jacod, J. (1997), "On the range of options prices", *Finance and Stochastics*, **1**, 131–140.

[37] Eberlein, E., Jacod, J. and Raible, S. (2005), "Lévy term structure models: no-arbitrage and completeness", *Finance and Stochastics*, **9**, 67–88.

[38] Eberlein, E., Kallsen, J. and Kristen, J. (2003), "Risk management based on stochastic volatility", *Journal of Risk*, **5**(2), 19–44.

[39] Eberlein, E. and Keller, U. (1995), "Hyperbolic distributions in finance", *Bernoulli*, **1**, 281–299.

[40] Eberlein, E. and Kluge, W. (2004), "Exact pricing formulae for caps and swaptions in a Lévy term structure model", *FDM Preprint 86*, University of Freiburg, Freiburg, Germany.

[41] Eberlein, E. and Papapantoleon, A. (2005), "Equivalence of floating and fixed strike Asian and lookback options", *Stochastic Processes and their Applications*, **105**, 31–40.

[42] Eberlein, E. and Prause, K. (2002), "The generalized hyperbolic model: financial derivatives and risk measures", in H. Geman, D. Madan, S. Pliska and T. Vorst (Eds), *Mathematical Finance-Bachelier Congress 2000*, Springer-Verlag, Berlin, Germany, pp. 245–267.

[43] Eberlein, E. and von Hammerstein, E.-A. (2004), "Generalized hyperbolic and inverse Gaussian distributions: limiting cases and approximation of processes", in R. Dalang, M. Dozzi and

F. Russo (Eds), *Seminar on Stochastic Analysis, Random Fields and Applications IV*, Progress in Probability, Vol. 58, Birkhäuser Verlag, Basel, Switzerland, pp. 221–264.

[44] Fajardo, J. and Mordecki, E. (2003), "Duality and derivative pricing with Lévy processes", Preprint No. 76, Centro de Matemática, Universidad de la República del Uruguay, Montevideo, Uruguay.

[45] Frittelli, M. (2000), "The minimal entropy martingale measure and the valuation problem in incomplete markets", *Mathematical Finance*, **10**, 39–52.

[46] Fujiwara, T. and Miyahara, Y. (2003), "The minimal entropy martingale measures for geometric Lévy processes", *Finance and Stochastics*, **7**, 509–531.

[47] Geman, H., El Karoui, N. and Rochet, J.-C. (1995), "Changes of numéraire, changes of probability measures and option pricing", *Journal of Applied Probability*, **32**, 443–458.

[48] Glasserman, P. (2003), *Monte Carlo Methods in Financial Engineering*, Springer-Verlag, New York, NY, USA.

[49] Goll, T. and Rüschendorf, L. (2001), "Minimax and minimal distance martingale measures and their relationship to portfolio optimization", *Finance and Stochastics*, **5**, 557–581.

[50] Greenwood, P. and Pitman, J. (1980), "Fluctuation identities for Lévy processes and splitting at the maximum", *Advances in Applied Probability*, **12**, 893–902.

[51] Gushchin, A.A. and Mordecki, E. (2002), "Bounds on option prices for semimartingale market models", *Proceedings of the Steklov Mathematical Institute* **237**, 73–113.

[52] Hakala, J. and Wystup, U. (2002), "Heston's stochastic volatility model applied to foreign exchange options", in J. Hakala and U. Wystup (Eds), *Foreign Exchange Risk*, Risk Publications, London, UK, pp. 267–282.

[53] Hartinger, J. and Predota, M. (2002), "Pricing Asian options in the hyperbolic model: a fast quasi Monte Carlo approach", *Grazer Mathematische Berichte*, **345**, 1–33.

[54] Haug, E.G. (2002), "A look in the antimatter mirror", *Wilmott Magazine*, September, 38–42.

[55] Henderson, V., Hobson, D., Shaw, W. and Wojakowski, R. (2004), "Bounds for in-progress floating-strike Asian options using symmetry", *Annals of Operational Research* (Special Issue on Financial Modelling), (to appear).

[56] Henderson, V. and Wojakowski, R. (2002), "On the equivalence of floating and fixed-strike Asian options", *Journal of Applied Probability*, **39**, 391–394.

[57] Heston, S.L. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options", *Review of Financial Studies*, **6**, 327–343.

[58] Hobson, D., Laurence, P. and Wang, T.-H. (2005), "Static-arbitrage upper bounds for the prices of basket options", *Quantitative Finance*, to be published.

[59] Jacod, J. and Shiryaev, A.N. (2003), *Limit Theorems for Stochastic Processes*, 2nd Edn, Springer-Verlag, Berlin, Germany.

[60] Kallsen, J. and Shiryaev, A.N. (2002), "The cumulant process and Esscher's change of measure", *Finance and Stochastics*, **6**, 397–428.

[61] Kallsen, J. and Tankov, P. (2004), "Characterization of dependence of multidimensional Lévy processes using Lévy copulas", *Working Paper*, École Polytechnique, Paris, France.

[62] Këllezi, E. and Webber, N. (2004), "Valuing Bermudan options when asset returns are Lévy processes", *Quantitative Finance*, **4**, 87–100.

[63] Kou, S.G. and Wang, H. (2003), "First passage times of a jump diffusion process", *Advances in Applied Probability*, **35**, 504–531.

[64] Kou, S.G. and Wang, H. (2005), "Option pricing under a double exponential jump diffusion model", *Management Science*, **50**, 1178–1192.

[65] Lee, R.W. (2004), "Option pricing by transform methods: extensions, unification, and error control", *Journal of Computational Finance*, **7**(3), 50–86.

[66] Levendorskiĭ, S.Z. (2004), "Pricing of the American put under Lévy processes", *International Journal of Theoretical and Applied Finance*, **7**, 303–335.

[67] Lewis, A. (2001), "A simple option formula for general jump-diffusion and other exponential Lévy processes", *Working Paper*, Optioncity.net.

[68] Lipton, A. (2002), "Assets with jumps", *Risk*, **15**(9), 149–153.

[69] Margrabe, W. (1978), "The value of an option to exchange one asset for another", *Journal of Finance*, **33**, 177–186.

[70] Matache, A.-M., Nitsche, P.-A. and Schwab, C. (2003), "Wavelet Galerkin pricing of American options on Lévy driven assets", *SAM Research Report 2003-06*, ETH, Zurich, Switzerland.

[71] Matache, A.-M., Schwab, C. and Wihler, T.P. (2005), "Fast numerical solution of parabolic integro-differential equations with applications in finance", *SIAM Journal on Scientific Computing*, to be published.

[72] Matache, A.-M., von Petersdorff, T. and Schwab, C. (2004), "Fast deterministic pricing of options on Lévy driven assets", *ESAIM Mathematical Modelling and Numerical Analysis*, **38**, 37–72.

[73] McDonald, R.L. and Schroder, M.D. (1998), "A parity result for American options", *Journal of Computational Finance*, **1**, 5–13.

[74] Merton, R.C. (1973), "Theory of rational option pricing", *Bell Journal of Economics and Management Science*, **4**, 141–183.

[75] Mordecki, E. (2002), "Optimal stopping and perpetual options for Lévy processes", *Finance and Stochastics*, **6**, 473–493.

[76] Müller, A. and Stoyan, D. (2002), *Comparison Methods for Stochastic Models and Risks*, Wiley, Chichester, UK.

[77] Musiela, M. and Rutkowski, M. (1997), *Martingale Methods in Financial Modelling*, Springer-Verlag, Berlin, Germany.

[78] Nguyen-Ngoc, L. (2003), "Exotic options in general exponential Lévy models", *Prépublication No 850*, Universités Paris 6, Paris, France.

[79] Nguyen-Ngoc, L. and Yor, M. (2005), "Lookback and barrier options under general Lévy processes", in Y. Ait-Sahalia and L.-P. Hansen (Eds), *Handbook of Financial Econometrics*, North-Holland, Amsterdam, The Netherlands, (to appear).

[80] Peskir, G. and Shiryaev, A.N. (2002), "A note on the call-put parity and a call-put duality", *Theory of Probability and its Applications*, **46**, 167–170.

[81] Raible, S. (2000), "*Lévy processes in finance: theory, numerics, and empirical facts*", *Ph.D. Thesis*, University of Freiburg, Freiburg, Germany.

[82] Ribeiro, C. and Webber, N. (2003), "A Monte Carlo method for the normal inverse Gaussian option valuation model using an inverse Gaussian bridge", *Preprint*, City University, London, UK.

[83] Ribeiro, C. and Webber, N. (2004), "Valuing path-dependent options in the variance-gamma model by Monte Carlo with a gamma bridge", *Journal of Computational Finance*, **7**, 81–100.

[84] Rogers, L.C.G. (2000), "Evaluating first-passage probabilities for spectrally one-sided Lévy processes", *Journal of Applied Probability*, **37**, 1173–1180.

[85] Rogers, L.C.G. (2002), "Monte Carlo valuation of American options", *Mathematical Finance*, **12**, 271–286.

[86] Sato, K.-I. (1999), *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, UK.

[87] Schoutens, W. (2002), “The Meixner process: theory and applications in finance”, in O.E. Barndorff-Nielsen (Ed.), *Mini-Proceedings of the 2nd MaPhySto Conference on Lévy processes*, Aarhus, Denmark, pp. 237–241.

[88] Schoutens, W. (2003), *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley, Chichester, UK.

[89] Schroder, M. (1999), “Changes of numeraire for pricing futures, forwards and options”, *Reviews of Financial Studies*, **12**, 1143–1163.

[90] Schürger, K. (2002), “Laplace transforms and suprema of stochastic processes”, in K. Sandmann and P.J. Schönbucher (Eds), *Advances in Finance and Stochastics*, Springer-Verlag, Berlin, Germany, pp. 285–294.

[91] Vanmaele, M., Dhaene, J., Deelstra, G., Liinev, J. and Goovaerts, M.J. (2005), “Bounds for the price of discrete arithmetic Asian options”, *Journal of Computational and Applied Mathematics*, to be published.

[92] Večeř, J. and Xu, M. (2004), “Pricing Asian options in a semimartingale model”, *Quantitative Finance*, **4**, 170–175.

[93] Wystup, U. (2002), “Vanilla options”, in J. Hakala and U. Wystup (Eds), *Foreign Exchange Risk*, Risk Publications, London, UK, pp. 3–14.

6
Static Hedging of Asian Options under Stochastic Volatility Models using Fast Fourier Transform

Hansjörg Albrecher
Graz University of Technology, Graz, Austria

and

Wim Schoutens
Katholieke Universiteit Leuven – UCS, Leuven, Belgium

Abstract

We present a simple static super-hedging strategy for the payoff of an arithmetic Asian option in terms of a portfolio of European options under various stochastic volatility models. Moreover, it is shown that the obtained hedge is optimal in some sense. The strategy is based on stop-loss transforms and comonotonicity theory. The numerical implementation is based on the Fast Fourier transform. We illustrate the hedging performance for several models calibrated to market data and compare the results with other (trivial) static super-hedging strategies.

6.1 INTRODUCTION

The design of efficient hedging strategies for exotic options is a challenging problem that has received a lot of interest during the last few years. In order to serve the needs of investors, increasingly complex financial products have been introduced in the market and the pricing and (in particular) hedging of these products is of great importance for assessing the involved risk when trading these instruments. However, many of the hedging techniques currently used in practice rely on market model assumptions that are clearly not sufficiently realistic (such as the Black–Scholes model). A common practice in the hedging of exotics is to calibrate the model to vanilla options traded in the market and then derive the corresponding hedging positions for the exotic option. If the model is then recalibrated to the market on the next day, say, then, in order to make the hedging strategy meaningful, the obtained parameter-set should be rather close to the one from the previous day so that only minor adjustments of the hedging portfolio are needed. That is, in addition to a good fit to historical market data, one crucial requirement for a sound market model is its stability in terms of hedging strategies. Empirical studies in that direction indicate that stochastic volatility models outperform classical models like Black-Scholes by far (see e.g. Bakshi *et al*. (1997) [7]).

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

Apart from that, proposed dynamic hedging strategies with continuously changing positions in the asset (such as delta-hedging) have various deficiencies (see e.g. Allen and Padovani (2002) [6]). These are typically based on assumptions like no limit on the frequency of rebalancing, zero transaction costs and full liquidity of the market. However, in practice these assumptions are usually not fulfilled and alternatives are asked for. The most favourable situation is the availability of a static hedging strategy for the exotic option, that is an initial hedge portfolio (in terms of the underlying and vanilla options), which will perfectly replicate the payoff at maturity without any portfolio adjustments during the lifetime of the option. For some exotic options (such as barrier, lookback and cliquet options), it is possible to derive semi-static hedging strategies, where portfolio adjustments are only needed at a finite (and typically small) number of times before maturity (see, for instance, Allen and Padovani (2002) [6] and Carr *et al.* (1998) [16]).

Another alternative is to look for a static super-hedging strategy, which is a portfolio of the underlying and vanilla options that will dominate the payoff of the exotic option without any adjustments during its lifetime. Such a strategy puts a floor on the maximum loss whatever the subsequent price path will look like and provides a simple way to hedge the product at the expense of a calculable additional cost (namely the difference of the cost of the hedge portfolio and the actual price of the option). At the same time, this strategy enjoys all the advantages of a static hedge: it is less sensitive to the assumption of zero transaction costs (both commissions and the cost of paying individuals to monitor the positions) and does not face the risk of dried-up liquidity when the market makes large moves (opposed to dynamic hedging (see e.g. Carr and Picron (1999) [19] and Carr and Wu (2002) [20]). Semi-static super-hedging strategies for barrier options are discussed in Brown *et al.* (2001) [14] and Neuberger and Hodges (2002) [32]).

This paper focuses on Asian options. Already, the pricing of these products is far from trivial, especially when leaving the Black–Scholes framework (see e.g. Albrecher and Predota (2002, 2004) [5, 4] and Večeř and Xu (2004) [41] or the recent survey by Albrecher (2004) [3]). Moreover, many of the available pricing techniques do not lead to an effective hedging strategy. A delta-hedging strategy for Asian options in a Black–Scholes model based on approximations was discussed in Jacques (1996) [28]. In Albrecher *et al.* (2003) [2], a simple static super-hedging strategy for arithmetic Asian call options consisting of a portfolio of European options has been derived and optimized using comonotonicity theory. The performance of the resulting strategy has been studied for models for asset price processes following an exponential Lévy model. In the present paper, we extend this approach to stochastic volatility models and investigate the performance of the resulting hedging strategy. As will be illustrated, the hedging error of this simple super-hedging strategy is very small if the option is in the money. For options at and out of the money, this strategy can be quite conservative, but the static nature of the hedge may compensate for parts of the gap.

The paper is structured as follows. In Section 6.2, several stochastic volatility models for the asset price process are introduced. In Section 6.3, we present the static super-hedging strategy in detail and illustrate how it can be optimized by comonotonicity techniques. The numerical implementation of the strategy for the various models on the basis of Fast Fourier transforms is discussed in Section 6.4. In Section 6.5, all of the models are calibrated to market data, namely to the same set of vanilla options on the S&P 500, and the performance of the corresponding hedging strategies is illustrated. Moreover, the issue of model risk is discussed.

Since it turns out that the developed hedging strategy only depends on the marginal risk-neutral densities of the asset price process at each averaging day of the Asian option, it can actually be implemented in a completely model-independent setup by estimating the marginal risk-neutral densities directly from the call option surface. This extension is discussed in Section 6.6.

6.2 STOCHASTIC VOLATILITY MODELS

In the sequel, we will briefly introduce various stochastic volatility models, all of which proved their smile-conform pricing abilities, and consider their risk-neutral dynamics.

Let $S = \{S_t, 0 \leq t \leq T\}$ denote the stock price process and $\phi(u, t)$ the characteristic function of the random variable $\log S_t$, i.e.

$$\phi(u, t) = E[\exp(\mathrm{i}u \log(S_t))].$$

We assume the stock pays out a continuous dividend; the dividend yield is assumed to be constant and denoted by q. We also have at our disposal a risk-free bank account, paying out a continuously compounded interest rate, which we assume to be constant and denote by r. The price process for the bank-account (bond) is thus given by $B = \{B_t = \exp(rt), t \geq 0\}$.

The stochastic dynamics of our stock price process will be driven by Lévy processes. A Lévy process $X = \{X_t, t \geq 0\}$ is a stochastic process which starts at zero and has independent and stationary increments such that the distribution of the increment is an *infinitely divisible distribution* (i.e. a distribution for which the characteristic function is also the nth power of another characteristic function, for every integer n). There is a one-to-one correspondence between Lévy processes and infinitely divisible distributions. A *subordinator* is a nonnegative nondecreasing Lévy process. A general reference for Lévy processes is Bertoin (1996) [12], while for applications in finance see Schoutens (2003) [38].

6.2.1 The Heston stochastic volatility model

In the Heston Stochastic Volatility model (HEST), the stock price process follows a Black–Scholes stochastic differential equation, in which the volatility behaves stochastically over time:

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\mathrm{d}t + \sigma_t \mathrm{d}W_t, \quad S_0 \geq 0,$$

where the (squared) volatility follows the classical Cox-Ingersoll-Ross (CIR) process:

$$\mathrm{d}\sigma_t^2 = \kappa(\eta - \sigma_t^2)\mathrm{d}t + \theta\sigma_t \mathrm{d}\tilde{W}_t, \quad \sigma_0 \geq 0.$$

Here, $W = \{W_t, t \geq 0\}$ and $\tilde{W} = \{\tilde{W}_t, t \geq 0\}$ are two correlated standard Brownian motions such that $\mathrm{Cov}[\mathrm{d}W_t, \mathrm{d}\tilde{W}_t] = \rho\, \mathrm{d}t$.

The characteristic function $\phi(u,t)$ is in this case given by (see Bakshi *et al.* [7] or Heston (1993) [27]):

$$\begin{aligned}\phi(u,t) &= E[\exp(\mathrm{i}u\log(S_t))|S_0,\sigma_0]\\ &= \exp(\mathrm{i}u(\log S_0+(r-q)t))\\ &\quad\times\exp(\eta\kappa\theta^{-2}((\kappa-\rho\theta u\mathrm{i}-d)t-2\log((1-g\mathrm{e}^{-dt})/(1-g))))\\ &\quad\times\exp(\sigma_0^2\theta^{-2}(\kappa-\rho\theta\mathrm{i}u-d)(1-\mathrm{e}^{-dt})/(1-g\mathrm{e}^{-dt})),\end{aligned}$$

where

$$d = ((\rho\theta u\mathrm{i}-\kappa)^2-\theta^2(-\mathrm{i}u-u^2))^{1/2}, \tag{6.1}$$

$$g = (\kappa-\rho\theta u\mathrm{i}-d)/(\kappa-\rho\theta u\mathrm{i}+d). \tag{6.2}$$

An extension of HEST introduces jumps in the asset price (Bakshi *et al.* [7]), while other extensions also allow for jumps in the volatility (see e.g. Knudsen and Nguyen-Ngoc (2003) [29]). Since for these extensions the characteristic function of the log stock price is also available, one can straightforwardly apply the methods described below for these models too.

6.2.2 The Barndorff-Nielsen–Shephard model

This class of models, denoted by BN–S, was introduced in Barndorff-Nielsen and Shephard (2000) [10] and has a structure similar to the Heston model. The difference is basically that here the volatility is modelled by an Ornstein–Uhlenbeck (OU) process driven by a subordinator. In this way, jumps are introduced into the volatility process. Volatility can only jump upwards and then will decay exponentially. A co-movement effect between up-jumps in volatility and (down)-jumps in the stock price is also incorporated. The squared volatility now follows a SDE of the form:

$$\mathrm{d}\sigma_t^2 = -\lambda\sigma_t^2\mathrm{d}t+\mathrm{d}z_{\lambda t}, \tag{6.3}$$

where $\lambda > 0$ and $z=\{z_t, t\ge 0\}$ is a subordinator.

The risk-neutral dynamics of the log-price $Z_t=\log S_t$ are given by

$$\mathrm{d}Z_t = (r-q-\lambda k(-\rho)-\sigma_t^2/2)\mathrm{d}t+\sigma_t\mathrm{d}W_t+\rho\mathrm{d}z_{\lambda t}, \qquad Z_0=\log S_0,$$

where $W=\{W_t, t\ge 0\}$ is a Brownian motion independent of $z=\{z_t, t\ge 0\}$ and where $k(u)=\log E[\exp(-uz_1)]$ is the cumulant function of z_1. Note that the parameter ρ introduces the co-movement effect between the volatility and the asset price process.

We use the classical and tractable example of the Gamma-OU process (other choices for OU-processes include the inverse Gaussian-OU process, which also leads to a tractable model [Schoutens (2003) [38], Section 7.2.1]). For a Gamma-OU process, $z=\{z_t, t\ge 0\}$ is a compound-Poisson process:

$$z_t = \sum_{n=1}^{N_t} x_n, \tag{6.4}$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter a, i.e. $E[N_t] = at$ and $\{x_n, n = 1, 2, \dots\}$ is an independent and identically distributed sequence of exponential random variables with mean $1/b$. One then has

$$\log E[\exp(-uz_1)] = -au(b+u)^{-1},$$

and it can be shown that σ^2 has a stationary marginal law that follows a Gamma distribution. The characteristic function of the log price can, in this case, be written in the form (cf. Barndorff-Nielson *et al.* (2002) [11])

$$\begin{aligned}\phi(u,t) &= E[\exp(\mathrm{i}u \log S_t)|S_0, \sigma_0] \\ &= \exp\left(\mathrm{i}u(\log(S_0) + (r - q - a\lambda\rho(b-\rho)^{-1})t)\right) \\ &\quad \times \exp\left(-\lambda^{-1}(u^2 + \mathrm{i}u)(1 - \exp(-\lambda t))\sigma_0^2/2\right) \\ &\quad \times \exp\left(a(b - f_2)^{-1}\left(b \log\left(\frac{b - f_1}{b - \mathrm{i}u\rho}\right) + f_2\lambda t\right)\right),\end{aligned}$$

where

$$\begin{aligned}f_1 &= f_1(u) = \mathrm{i}u\rho - \lambda^{-1}(u^2 + \mathrm{i}u)(1 - \exp(-\lambda t))/2, \\ f_2 &= f_2(u) = \mathrm{i}u\rho - \lambda^{-1}(u^2 + \mathrm{i}u)/2.\end{aligned}$$

6.2.3 Lévy models with stochastic time

Another way to incorporate stochastic volatility effects into the price process is by making time stochastic. Periods with high volatility can be interpreted as if time runs faster than in periods with low volatility. Applications of stochastic time change to asset pricing go back to Mandelbrot and Taylor (1967) [31] (see also Clark (1973) [21]). We consider the models introduced by Carr *et al.* (2003) [18].

The Lévy models with stochastic time considered in this paper are built out of two independent stochastic processes. The first process is a Lévy process. The behavior of the asset price will then be modelled by the exponential of the Lévy process, suitably time-changed. Typical examples for the generator of the Lévy process are the normal distribution (leading to Brownian motion), the Normal Inverse Gaussian (NIG) distribution (Barndorff-Nielsen (1995) [8] and Rydberg (1997) [34], the Variance Gamma (VG) distribution (Madan *et al.* (1998) [30]), the generalized hyperbolic distribution (Eberlein (1999) [25] and Rydberg (1999) [35], the Meixner distribution (Grigelionis (1999) [26], Schoutens and Teugels (1998) [36] and Schoutens (2002) [37]) and the CGMY distribution (Carr *et al.* (2002) [17]) (see Schoutens (2003) [38] for an overview). We opt to work with the VG and NIG processes for which simulation issues become quite standard.

The second process is a stochastic clock that builds in a stochastic volatility effect. The above mentioned (first) Lévy process will be subordinated (i.e. time-changed) by this stochastic clock. By definition of a subordinator, the time needs to increase and the process modelling the rate of time change $y = \{y_t, t \geq 0\}$ also needs to be positive. The economic time elapsed in t units of calendar time is then given by the integrated process $Y = \{Y_t, t \geq 0\}$ with

$$Y_t = \int_0^t y_s \mathrm{d}s. \tag{6.5}$$

Since y is a positive process, Y is an increasing process. We will consider two processes for the rate of time change y: the CIR process (which is continuous) and the Gamma-OU process (which is a jump process). We will first discuss NIG and VG and subsequently introduce the stochastic clocks CIR and Gamma-OU.

6.2.3.1 *NIG and VG processes*

The NIG(α, β, δ) distribution with parameters $\alpha > 0$, $|\beta| < \alpha$ and $\delta > 0$ has a characteristic function given by

$$\phi_{NIG}(u; \alpha, \beta, \delta) = \exp\left(-\delta\left(\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right)$$

and the VG(C, G, M) distribution with parameters $C > 0$, $G > 0$ and $M > 0$ has a characteristic function given by

$$\phi_{VG}(u; C, G, M) = \left(\frac{GM}{GM + (M - G)\mathrm{i}u + u^2}\right)^C.$$

Since both distributions are infinitely divisible, each of them generates a Lévy process $X = \{X_t, t \geq 0\}$ where the increment X_1 follows a NIG(α, β, δ) law (VG(C, G, M) law, respectively). The resulting process is called a NIG process (VG process, respectively). Due to convolution properties of these two distributions, increments of arbitrary length again follow the same law with just a change in parameters: An increment of the NIG-process over the time interval $[s, s + t]$ follows a NIG$(\alpha, \beta, \delta t)$ law and the increment of a VG-process over $[s, s + t]$ is VG(Ct, G, M)-distributed (see also Barndorff-Nielsen (1997) [9]).

6.2.3.2 *Stochastic clocks*

CIR Stochastic Clock:. Carr *et al.* (2003) [18] use as the rate of time change the CIR process that solves the SDE:

$$\mathrm{d}y_t = \kappa(\eta - y_t)\mathrm{d}t + \lambda y_t^{1/2}\mathrm{d}W_t,$$

where $W = \{W_t, t \geq 0\}$ is a standard Brownian motion. The characteristic function of Y_t (given y_0) is explicitly known (see Cox *et al.* (1985) [22]):

$$\begin{aligned}\varphi_{CIR}(u, t; \kappa, \eta, \lambda, y_0) &= E[\exp(\mathrm{i}uY_t)|y_0] \\ &= \frac{\exp(\kappa^2\eta t/\lambda^2)\exp(2y_0\mathrm{i}u/(\kappa + \gamma\coth(\gamma t/2)))}{(\cosh(\gamma t/2) + \kappa\sinh(\gamma t/2)/\gamma)^{2\kappa\eta/\lambda^2}},\end{aligned}$$

where

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2\mathrm{i}u}.$$

Gamma-OU Stochastic Clock:. Another choice for the rate of time change is the solution of the SDE:

$$dy_t = -\lambda y_t dt + dz_{\lambda t}, \tag{6.6}$$

where the process $z = \{z_t, t \geq 0\}$ is, as in equation (6.4), a compound Poisson process. In the Gamma-OU case, there is an explicit expression for the characteristic function of Y_t (given y_0):

$$\begin{aligned}
&\varphi_{\Gamma-OU}(u; t, \lambda, a, b, y_0)\\
&\quad = E[\exp(\mathrm{i}uY_t)|y_0]\\
&\quad = \exp\left(\mathrm{i}uy_0\lambda^{-1}(1-e^{-\lambda t}) + \frac{\lambda a}{\mathrm{i}u - \lambda b}\left(b\log\left(\frac{b}{b - \mathrm{i}u\lambda^{-1}(1-e^{-\lambda t})}\right) - \mathrm{i}ut\right)\right).
\end{aligned}$$

6.2.3.3 Time-changed Lévy process

Let $Y = \{Y_t, t \geq 0\}$ as defined in equation (6.5), be the process modelling our business time. The (risk-neutral) price process $S = \{S_t, t \geq 0\}$ is now modelled as follows:

$$S_t = S_0 \frac{\exp((r-q)t)}{E[\exp(X_{Y_t})|y_0]} \exp(X_{Y_t}), \tag{6.7}$$

where $X = \{X_t, t \geq 0\}$ is a Lévy process. The factor $\exp((r-q)t)/E[\exp(X_{Y_t})|y_0]$ puts us immediately into the risk-neutral world by a mean-correcting argument. Essentially, the stock price process is modelled as the ordinary exponential of a time-changed Lévy process. The process incorporates jumps (through the Lévy process X_t) and stochastic volatility (through the time change Y_t). The characteristic function $\phi(u, t)$ for the logarithm of our stock price is given by:

$$\begin{aligned}
\phi(u,t) &= E[\exp(\mathrm{i}u\log(S_t))|S_0, y_0]\\
&= \exp(\mathrm{i}u((r-q)t + \log S_0))\frac{\varphi(-\mathrm{i}\psi_X(u); t, y_0)}{\varphi(-\mathrm{i}\psi_X(-\mathrm{i}); t, y_0)^{\mathrm{i}u}},
\end{aligned} \tag{6.8}$$

where

$$\psi_X(u) = \log E[\exp(\mathrm{i}uX_1)]$$

and $\varphi(u; t, y_0)$ denotes the characteristic function of Y_t given y_0.

Since we consider two Lévy processes (VG and NIG) and two stochastic clocks (CIR and Gamma-OU), we will finally end up with four resulting models abbreviated as VG-CIR, VG-OUΓ, NIG-CIR and NIG-OUΓ. Due to (time-)scaling effects, one can without loss of generality scale the present rate of time change to 1 ($y_0 = 1$). For more details, see Carr *et al*. (2003) [18] or Schoutens (2003) [38].

6.3 STATIC HEDGING OF ASIAN OPTIONS

Consider now a European-style arithmetic average call option with strike price K, maturity T and n averaging days $0 \le t_1 < \cdots < t_n \le T$. Then, its price according to a risk-neutral pricing measure Q at time t is given by

$$AA_t = \frac{\exp(-r(T-t))}{n} E_Q\left[\left(\sum_{k=1}^{n} S_{t_k} - nK\right)^+ \Bigg| \mathcal{F}_t\right],$$

where $\{\mathcal{F}_t, 0 \le t \le T\}$ denotes the natural filtration of S.

In general, the distribution of the dependent sum $\sum_{k=1}^{n} S_{t_k}$ is not available, which makes pricing and hedging of these products difficult. However, for our super-hedging purposes it suffices to look for an upper bound of the above payoff. Assume for simplicity that $t = 0$ and that the averaging has not yet started. First note, that for any $K_1, \ldots, K_n \ge 0$ with $K = \sum_{k=1}^{n} K_k$, we have a.s.

$$\left(\sum_{k=1}^{n} S_{t_k} - nK\right)^+ = \Big((S_{t_1} - nK_1) + \cdots + (S_{t_n} - nK_n)\Big)^+ \le \sum_{k=1}^{n} \left(S_{t_k} - nK_k\right)^+ .$$

Hence

$$\begin{aligned} AA_0(K,T) &= \frac{\exp(-rT)}{n} E_Q\left[\left(\sum_{k=1}^{n} S_{t_k} - nK\right)^+ \Bigg| \mathcal{F}_0\right] \\ &\le \frac{\exp(-rT)}{n} \sum_{k=1}^{n} E_Q\left[\left(S_{t_k} - nK_k\right)^+ \Big| \mathcal{F}_0\right] \\ &= \frac{\exp(-rT)}{n} \sum_{k=1}^{n} \exp(rt_k) EC_0(\kappa_k, t_k), \end{aligned} \tag{6.9}$$

where $EC_0(\kappa_k, t_k)$ denotes the price of a European call option at time 0 with strike $\kappa_k = nK_k$ and maturity t_k.

In terms of hedging, this means that we have the following static super-hedging strategy: for each averaging day t_k, buy $\exp(-r(T - t_k))/n$ European call options at time $t = 0$ with strike κ_k and maturity t_k and hold these until their expiry. Then put their payoff on the bank account.

Since the upper bound (equation (6.9)) holds for all combinations of $\kappa_k \ge 0$ that satisfy $\sum_{k=1}^{n} \kappa_k = nK$, we still have the freedom to choose strike values that fit best to our purposes. The simplest choice is $\kappa_k = K$ $(k = 1, \ldots, n)$. If $q \le r$, we have $EC_0(K, t) \le EC_0(K, T)$ for every $K \ge 0$ and $0 \le t \le T$, and thus this trivial choice shows that the Asian option price is dominated by the price of a European option with the same strike and maturity, i.e.

$$AA_0(K,T) \le EC_0(K,T)$$

(this trivial hedging strategy of an Asian option in terms of the corresponding European option was already observed in Nielsen and Sandmann (2003) [33]). However, for our super-hedging purposes, we naturally look for that combination of κ_ks which minimizes the right-hand side of equation (6.9). In the Black–Scholes setting, this optimization problem was solved in Nielsen and Sandmann (2003) [33] by using Lagrange multipliers. In the general case of arbitrary arbitrage-free market models, this optimal combination can be determined by using stop-loss transforms and the theory of comonotonic risks (for a general introduction to comonotonicity techniques, see Dhaene *et al.* (2002a, b) [23]).

Let $F(x)$ be a distribution function of a non-negative random variable X; then its stop-loss transform $\Phi_F(m)$ is defined by

$$\Phi_F(m) = \int_m^{+\infty} (x-m)\mathrm{d}F(x) = E[(X-m)^+], \quad m \geq 0.$$

If we write

$$A_n = \sum_{k=1}^{n} S_{t_k}$$

and $F^t_{A_n}(x) = \mathbb{P}_Q(A_n \leq x|\mathcal{F}_t)$ for the distribution function under Q of A_n given the information $\mathcal{F}_t$, then we have

$$AA_t = \frac{\exp(-r(T-t))}{n}\,\Phi_{F^t_{A_n}}(nK). \tag{6.10}$$

In this way the problem of pricing an arithmetic average option is transformed to calculating the stop-loss transform of a sum of dependent risks. Concretely, we will look at bounds for stop-loss transforms based on comonotonic risks. A positive random vector $(X_1, \ldots, X_n)$ with marginal distribution functions $F_1(x_1), \ldots, F_n(x_n)$ is called *comonotone*, if for the joint distribution function $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \min\{F_1(x_1), \ldots, F_n(x_n)\}$ holds for every $x_1, \ldots, x_n \geq 0$. It immediately follows that the distribution of a comonotone random vector $(X_1, \ldots, X_n)$ with given marginal distributions $F_1(x_1), \ldots, F_n(x_n)$ is uniquely determined.

In Simon *et al.* (2000) [40], it was shown that an upper bound for the stop-loss transform of the sum of arbitrary dependent positive random variables $\sum_{k=1}^n X_k$ with marginal distributions $F_1(x_1), \ldots, F_n(x_n)$ is given by the stop-loss transform of the sum $S^c = \sum_{k=1}^n Y_k$, where $(Y_1, \ldots, Y_k)$ is the comonotone random vector with marginal distributions $F_1(x_1), \ldots, F_n(x_n)$. Let $F_{S^c}(x)$ denote the distribution function of $\sum_{k=1}^n Y_k$; then it follows from general comonotonicity results (see e.g. Dhaene *et al.* (2002a, b) [23]) that its inverse is given by

$$F^{-1}_{S^c}(x) = \sum_{k=1}^{n} F^{-1}_{X_k}(x), \quad x \geq 0. \tag{6.11}$$

The crucial result for our purposes is now that the stop-loss transform of a sum of comonotonic random variables can be obtained as a sum of the stop-loss transforms of the marginals evaluated at specified points (cf. Proposition 2 in Simon *et al.* (2000) [40]). More precisely,

$$\Phi_{F_{S^c}}(m) = \sum_{k=1}^{n} \Phi_{F_{X_k}}\left(F^{-1}_{X_k}(F_{S^c}(m))\right), \quad m \geq 0, \tag{6.12}$$

given that the marginal distribution functions involved are strictly increasing (which is always the case in our applications). At the same time,

$$\Phi_{F_{S^c}}(m) = E\left(\left(\sum_{k=1}^{n} Y_k - m\right)^+\right) \leq \sum_{k=1}^{n} E((Y_k - m_k)^+) = \sum_{k=1}^{n} \Phi_{F_{X_k}}(m_k) \tag{6.13}$$

whenever $\sum_{k=1}^{n} m_k = m$. Thus, the stop-loss transform of the comonotonic sum given by equation (6.12) represents the lowest possible bound in terms of a sum of stop-loss transforms of the marginal distributions.

This fact immediately translates to our setting of an arithmetic Asian option. Let $F(x_k; t_k)$ $(k = 1, \ldots, n)$ denote the conditional distribution of S_{t_k} under the risk-neutral measure Q (given the information available at time $t = 0$), i.e. for $x_k, t_k > 0$,

$$F(x_k; t_k) = \mathbb{P}_Q\left(S_{t_k} \leq x_k | \mathcal{F}_0\right). \tag{6.14}$$

Combining equations (6.9), (6.10), (6.12) and (6.13), the optimal combination of strike prices κ_k is given by

$$\kappa_k = F^{-1}\left(F_{S^c}(n\,K); t_k\right), \quad k = 1, \ldots, n. \tag{6.15}$$

In this way, we have obtained the optimal static super-hedge in terms of European call options with maturity dates equal to the averaging dates.

For the practical determination of the strike prices κ_k, the distribution function of the comonotone sum $F_{S^c}(x)$, as given by equation (6.11) has to be calculated and evaluated at nK. For this purpose, we need to approximate the risk-neutral marginal densities of the stock price at the averaging dates, which can be carried out efficiently by using Fast Fourier transforms (cf. Section 6.4.1 below). The κ_ks are then obtained by evaluating the inverse distribution function of $F(x; t_k)$.

6.4 NUMERICAL IMPLEMENTATION

6.4.1 Characteristic function inversion using FFT

For all of the above mentioned models, we have the characteristic function of the log-price process at our disposal. However, in order to determine the strike prices of our optimal hedge portfolio as described in Section 6.3, we need the corresponding density functions. Recall that the characteristic function, $\phi(u)$, is the Fourier-transform of the corresponding density function $f(x)$:

$$\phi(u) = \int_{-\infty}^{+\infty} \exp(iux) f(x) dx.$$

So, we need to apply an inverse Fourier-transformation. Next, we illustrate how this can be done fast and accurately by using the Fast Fourier Transform (FFT). The latter is an efficient algorithm for computing the following transformation of a vector $(\alpha_k, k = 1, \ldots, m)$ into a vector $(\beta_k, k = 1, \ldots, m)$:

$$\beta_k = \sum_{j=1}^{m} \exp(-i2\pi(j-1)(k-1)/N)\alpha_j.$$

Typically, m is a power of 2. The number of multiplications of the FFT algorithm is of the order $\mathcal{O}(m \log m)$ and this is in contrast to the straightforward evaluation of the above sums which give rise to $\mathcal{O}(m^2)$ multiplications.

We follow closely a technique described in Carr and Madan (1998) [15] in the context of option pricing. The classical inverse Fourier transform reads:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-\mathrm{i}ux)\phi(u)\mathrm{d}u.$$

Since f is real, we can write

$$f(x) = \frac{1}{\pi}\int_{0}^{+\infty} \exp(-\mathrm{i}ux)\phi(u)\mathrm{d}u.$$

Next, we are going to discretize the above integral and apply the trapezoid rule. We take a grid on the real line with grid-length $\Delta u > 0$:

$$u_j = (j-1)\Delta u, \qquad j = 1, \ldots, N.$$

One approximately has

$$\begin{aligned} f(x) &\approx \frac{\Delta u}{\pi}\left(\frac{1}{2}\exp(-\mathrm{i}u_1 x)\phi(u_1) + \sum_{j=2}^{N-1}\exp(-\mathrm{i}u_j x)\phi(u_j) + \frac{1}{2}\exp(-\mathrm{i}u_N x)\phi(u_N)\right) \\ &= \frac{\Delta u}{\pi}\sum_{j=1}^{N} w_j \exp(-\mathrm{i}x(j-1)\Delta u)\phi((j-1)\Delta u) \end{aligned}$$

where the weights w_j are given by

$$w_1 = \frac{1}{2}, \quad w_2 = 1, \quad w_3 = 1, \ldots, \quad w_{N-1} = 1, \quad w_N = \frac{1}{2}.$$

We will calculate the value of the density function f in the points

$$x_k = -b + \Delta x(k-1), \qquad k = 1, \ldots, N$$

where $\Delta x = 2b/(N-1)$, thus covering the interval $[-b, b]$ with an equally spaced grid. In these points we have

$$f(x_k) \approx \frac{\Delta u}{\pi}\sum_{j=1}^{N} w_j \exp(\mathrm{i}(j-1)b\Delta u)\exp(-\mathrm{i}(j-1)(k-1)\Delta u\Delta x)\phi((j-1)\Delta u).$$

If we choose the grid sizes such that

$$\Delta u \Delta x = \frac{2\pi}{N},$$

then

$$f(x_k) \approx \frac{\Delta u}{\pi}\sum_{j=1}^{N} w_j \exp(\mathrm{i}(j-1)\Delta u b)\exp(-\mathrm{i}(j-1)(k-1)2\pi/N)\phi((j-1)\Delta u).$$

This sum can be easily computed by FFT: $(f(x_k), k = 1, \ldots, N)$ is the FFT of the vector

$$\left(w_j\phi(u_j)\exp(\mathrm{i}u_j b),\ j = 1, \ldots, N\right).$$

Choosing N as a power of 2 allows very fast computation of the FFT.

6.4.2 Static hedging algorithm

In order to set up our hedge portfolio, we have to determine the inverse distribution function of the asset price at each averaging day (cf. equation (6.14)). This is carried out by numerically building up the distribution function from the approximated density function obtained in Section 6.4.1. The inverse is then determined by a bisection method from the corresponding table and linear interpolation between grid points is employed. In our implementation, we used 2^{14} points in the grid for both the densities and the inverse distribution functions, which turns out to be sufficient (in the sense that a further increase does not change the significant digits of the results). Next, the inverse of the distribution of the comonotone sum is built up according to equation (6.11) and then itself inverted in the above way. Finally, the strike prices κ_k of the European options are obtained by evaluating the inverse distribution functions of the marginals according to equation (6.15). This numerical procedure to obtain the strike prices for our hedging strategy is both accurate and very quick (the determination of the entire hedge portfolio takes less than a minute on a normal PC for each of the discussed stochastic volatility models).

6.5 NUMERICAL ILLUSTRATION

We give numerical results for an arithmetic Asian call option with a maturity of 1 year and averaging every month (i.e. 12 averaging days). First, the model parameters have to be determined from the market prices of vanilla options.

6.5.1 Calibration of the model parameters

Carr and Madan (1998) [15] developed pricing methods for the classical vanilla options which can be applied whenever the characteristic function of the risk-neutral stock price process is known. Using Fast Fourier transforms, one can compute within a second the complete option surface on an ordinary computer. In Schoutens (2003) [38], this method was used to calibrate the models (minimizing the difference between market prices and model prices in a least-squares sense) on a dataset of 77 option on the S&P 500 Index [Schoutens (2003) [38], Appendix C]. The results of the calibration are visualized in Figures 6.1 and 6.2 for the NIG-CIR and the Heston model, respectively. Here, the circles are the market prices and the plus signs are the model prices (calculated through the Carr–Madan formula by using the respective characteristic functions and obtained parameters). For details of the fit, see Schoutens (2003) [38]. The Heston model, which is not covered in Schoutens (2003) [38], gives rise to the following calibration errors:

$$ape = 1.31\%, \quad rmse = 1.0530, \quad aae = 0.8095, \quad arpe = 1.90\%.$$

Table 6.1 depicts the calibrated parameters for each of the six discussed stochastic volatility models, while Figure 6.3 shows the corresponding marginal density functions of $\log(S_t)$ for t ranging from 1 month up to 1 year for all six models obtained by Fast Fourier transform, as described in Section 6.4.1.

6.5.2 Performance of the hedging strategy

After the strike prices of the hedge portfolio are determined according to equation (6.15), the price of the hedging strategy is easily determined by using the Carr–Madan call option

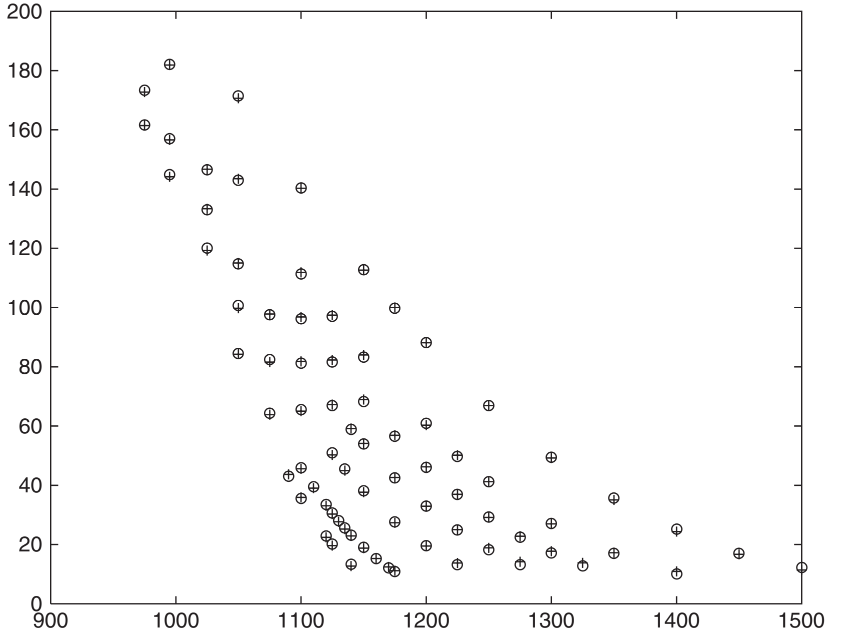

Figure 6.1 Calibration of the NIG-CIR Model

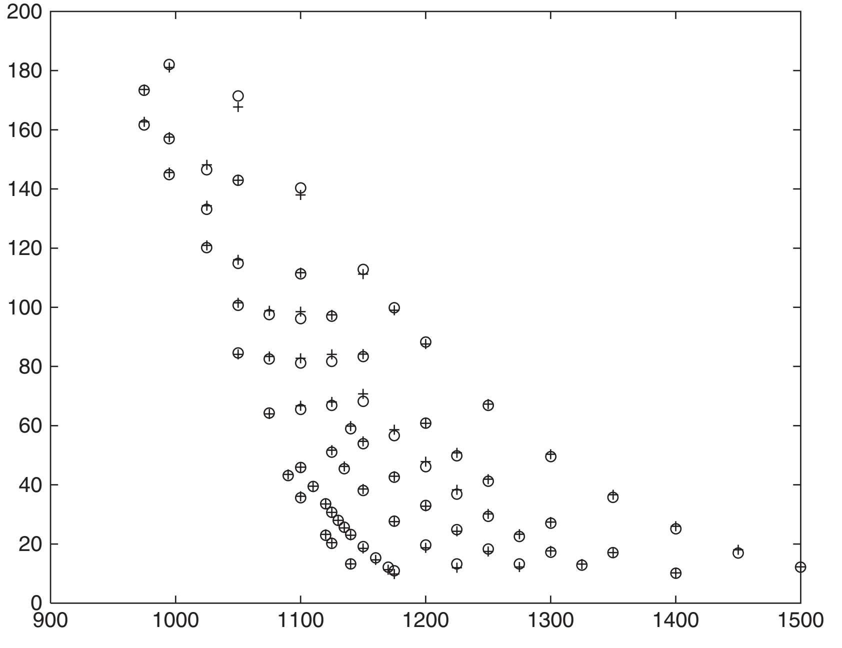

Figure 6.2 Calibration of Heston's Model

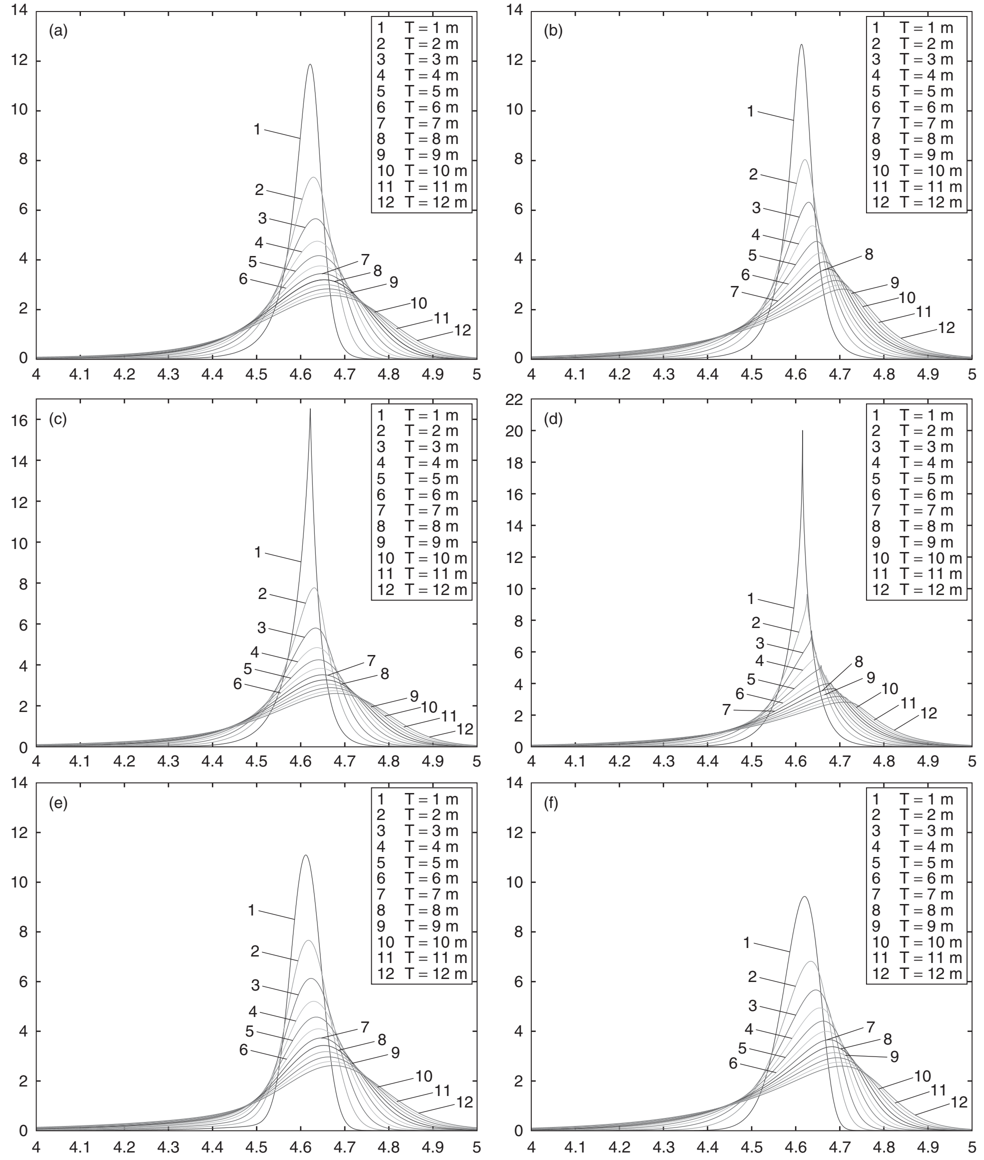

Figure 6.3 Marginal density functions for the various stochastic volatility models: (a) NIG-OUΓ; (b) NIG-CIR; (c) VG-OUΓ; (d) NIG-CIR; (e) BN−S; (f) Heston

pricing formula for European options and equation (6.9). Tables 6.2−6.7 compare the Monte Carlo simulated price of the Asian option AA_{MC} and the comonotonic super-hedge price AA_c, with the prices of two trivial super-hedging strategies, namely the trivial super-hedge using the European option price EC with identical strike and maturity (note that $q \leq r$) and the super-hedge equation (6.9) with all $\kappa_i = K$ with price AA_{tr}. The strike price is given as a percentage of the spot. For the Monte Carlo price, we used 1 million sample paths. The VG process was simulated as a difference of two Gamma processes (cf. Schoutens (2003)

Table 6.1 Risk-neutral parameters obtained by calibration to vanilla calls on S&P 500

HEST
$\sigma_0^2 = 0.0224, \kappa = 0.5144, \eta = 0.1094, \theta = 0.3354, \rho = -0.7392$
BN–S
$\rho = -1.2606, \lambda = 0.5783, b = 11.6641, a = 1.4338, \sigma_0^2 = 0.0145$
VG-CIR
$C = 11.9896, G = 25.8523, M = 35.5344, \kappa = 0.6020, \eta = 1.5560,$
$\lambda = 1.9992, y_0 = 1$
VG-OUΓ
$C = 11.4838, G = 23.2880, M = 40.1291, \lambda = 1.2517, a = 0.5841,$
$b = 0.6282, y_0 = 1$
NIG-CIR
$\alpha = 18.4815, \beta = -4.8412, \delta = 0.4685, \kappa = 0.5391, \eta = 1.5746,$
$\lambda = 1.8772, y_0 = 1$
NIG-OUΓ
$\alpha = 29.4722, \beta = -15.9048, \delta = 0.5071, \lambda = 0.6252, a = 0.4239,$
$b = 0.5962, y_0 = 1$

Table 6.2 Hedging performance in the BN–S model

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.6065	20.9648	21.1889	22.8511
90	11.7478	12.3153	12.4876	14.9462
100	4.5265	5.2411	5.2415	8.3470
110	0.9431	1.4128	1.6417	3.8643
120	0.1385	0.2972	0.5002	1.5736

Table 6.3 Hedging performance in Heston's model

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.2896	20.5088	20.7022	22.0898
90	11.3823	11.8872	12.0223	14.1997
100	4.3056	5.0132	5.0137	7.7280
110	0.6939	1.1328	1.3568	3.2476
120	0.0368	0.1193	0.2807	0.9834

Table 6.4 Hedging performance in the NIG-OUΓ model

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.3713	20.6307	20.7753	22.2822
90	11.4467	11.8830	11.9975	14.1826
100	4.4063	4.9562	4.9566	7.6203
110	0.8751	1.2170	1.4321	3.2497
120	0.0738	0.1566	0.3277	1.0465

Table 6.5 Hedging performance in the NIG-CIR model

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.2817	20.4979	20.6808	22.0975
90	11.4069	11.8418	11.9845	14.1909
100	4.4121	4.9588	4.9598	7.6878
110	0.9102	1.2704	1.4781	3.2162
120	0.1506	0.2864	0.4152	1.0910

Table 6.6 Hedging performance in the VG-OUΓ model

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.3528	20.5773	20.7447	22.2073
90	11.4380	11.8695	11.9896	14.1938
100	4.4083	4.9561	4.9567	7.6454
110	0.9070	1.2391	1.4559	3.2408
120	0.1061	0.1988	0.3506	1.0433

Table 6.7 Hedging performance in the VG-CIR model

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.3256	20.4907	20.6766	22.1156
90	11.4374	11.8395	11.9758	14.2022
100	4.4383	4.9605	4.9613	7.6906
110	0.9294	1.2723	1.4793	3.2159
120	0.1615	0.2883	0.4152	1.0898

[38], Section 8.4.2) while the NIG paths were obtained as described in Schoutens (2003) [38] (Section 8.4.5).

From Tables 6.2–6.7, we observe that the more in the money the Asian option is, the less is the difference between the option price and the comonotonic hedge. For an option with moneyness of 80% the difference is typically around 1.5%, whereas the classical hedge with the European call leads to a difference of almost 10%. For options out of the money, the difference increases, but is then substantially smaller than the differences for the other two trivial hedges. In view of the easy and cheap way in which this hedge can be implemented in practice, this static super-hedge approach seems to be competitive also in these cases.

As a by-product, we observe from the Monte Carlo estimates in Tables 6.2–6.7 that the model risk for Asian option prices can be quite substantial (note that all of the models are calibrated to the same set of vanilla option prices with a quite acceptable fit (the average percentage error of the fit is less than 2% for all the models (cf. Schoutens (2003) [38])), but the resulting marginal densities differ considerably (cf. Figure 6.3) and consequently the Asian option prices can differ quite a lot, especially if the option is out of the money). The issue of model risk for other exotic options has recently been discussed in Schoutens *et al.* (2004) [39].

6.6 A MODEL-INDEPENDENT STATIC SUPER-HEDGE

Since the hedging strategy introduced in this paper only depends on the risk-neutral marginal distribution functions on each averaging day of the Asian option, it can also be applied in a model-independent framework, if for all of the needed maturities t_k the European call prices $C(K, t_k)$ are available for every strike value K. In this case, the risk-neutral density function $f_{S_{t_k}}$ is given by the second derivative of $C(K, t_k)$ with respect to K (see e.g. Breeden and Litzenberger (1978) [13]):

$$f_{S_{t_k}}(K) = e^{r\,t_k} \frac{\partial^2 C(K, t_k)}{\partial K^2}.$$

In practice, call prices are available for a limited number of strike values K only, so that one has to use sophisticated statistical techniques to estimate $f(S_{t_k})$. For a recently developed efficient nonparametric estimation procedure utilizing shape restrictions due to no-arbitrage (such as monotonicity and convexity of the call price as a function of the strike), we refer to Aït-Sahalia and Duarte (2003) [1]. Once the density $f(S_{t_k})$ is available for all of the needed maturities t_k, the hedge portfolio can be determined in just the same way as described in the above sections.

6.7 CONCLUSIONS

We have shown that staticly hedging an Asian option in terms of a portfolio of European options is a simple and quick alternative to other strategies. Moreover, in contrast to most of the existing techniques, this approach is applicable in general market models whenever the risk-neutral density of the asset price distribution or an approximation of it is available. In particular, there is a fast algorithm to determine the hedge portfolio for various stochastic volatility models. Since the proposed hedging strategy is static, it is much less sensitive to the assumption of zero transaction costs and to the hedging performance in the presence of large market movements; no dynamic rebalancing is required. These advantages may sometimes compensate for the gap of the hedging price and the option price even for Asian options that are out of the money.

ACKNOWLEDGEMENTS

The authors would like to thank Peter Carr for fruitful discussions on the topic. H. Albrecher acknowledges support from the K.U. Leuven (Fellowship F/04/009) and the Austrian Science Foundation Project S-8308-MAT. W. Schoutens is a Postdoctoral Fellow of the Fund for Scientific Research, Flanders, Belgium (FWO – Vlaanderen).

REFERENCES

[1] Aït-Sahalia, Y. and Duarte, J. (2003), "Nonparametric option pricing under shape restrictions", *Journal of Econometrics*, **116**, 9–47.

[2] Albrecher, H., Dhaene, J., Goovaerts, M. and Schoutens, W. (2005), "Static hedging of Asian options under Lévy models", *Journal of Derivatives*, **12**(3), 63–72.

[3] Albrecher, H. (2004), "The valuation of Asian options for market models of exponential Lévy type", in M. Vanmaele, *et al.* (Eds), *Proceedings of the Second Day of Actuarial and Financial Mathematics*, Royal Flemish Academy of Arts and Sciences, Brussels, Belgium, pp. 11–20.

[4] Albrecher, H. and Predota, M. (2004), "On Asian option pricing for NIG Lévy processes", *Journal of Computational and Applied Mathematics*, **172**, 153–168.

[5] Albrecher, H. and Predota, M. (2002), "Bounds and approximations for discrete Asian options in a variance-gamma model", *Grazer Mathematische Berichte*, **345**, 35–57.

[6] Allen, S. and Padovani, O. (2002), "Risk management using quasi-static hedging", *Economic Notes*, **31**, 277–336.

[7] Bakshi, G., Cao, C. and Chen, Z. (1997), "Empirical performance of alternative option pricing models", *The Journal of Finance*, **LII**, 2003–2049.

[8] Barndorff-Nielsen, O.E. (1995), "Normal inverse Gaussian distributions and the modeling of stock returns", *Research Report No. 300*, Department of Theoretical Statistics, Aarhus University, Denmark.

[9] Barndorff-Nielsen, O.E. (1997), "Normal inverse Gaussian distributions and stochastic volatility models", *Scandinavian Journal of Statistics*, **24**, 1–13.

[10] Barndorff-Nielsen, O.E. and Shephard, N. (2000), "Modelling by Lévy processes for financial econometrics", in O.E. Barndorff-Nielsen, T. Mikosch and S. Resnick (Eds), *Lévy Processes – Theory and Applications*, Birkhäuser, Boston, MA, USA, pp. 283–318.

[11] Barndorff-Nielsen, O.E., Nicolata, E. and Shephard, N. (2002), "Some recent developments in stochastic volatility modelling", *Quantitative Finance*, **2**, 11–23.

[12] Bertoin, J. (1996), *Lévy Processes*, Cambridge Tracts in Mathematics, Vol. 121, Cambridge University Press, Cambridge, UK.

[13] Breeden, D. and Litzenberger, R. (1978), "Prices of state-contingent claims implicit in option prices", *Journal of Business*, **51**, 621–651.

[14] Brown, H., Hobson, D. and Rogers, L.C.G. (2001), "Robust hedging of barrier options", *Mathematical Finance*, **11**, 285–314.

[15] Carr, P. and Madan, D. (1998), "Option valuation using the fast Fourier transform", *Journal of Computational Finance*, **2**, 61–73.

[16] Carr, P., Ellis, K. and Gupta, V. (1998), "Static hedging of exotic options", *The Journal of Finance*, **53**, 1165–1190.

[17] Carr, P., Geman, H., Madan, D.H. and Yor, M. (2002), "The fine structure of asset returns: an empirical investigation", *Journal of Business*, **75**, 305–332.

[18] Carr, P., Geman, H., Madan, D.H. and Yor, M. (2003), "Stochastic volatility for Lévy processes", *Mathematical Finance*, **13**, 345–382.

[19] Carr, P. and Picron J. (1999), "Static hedging of timing risk", *Journal of Derivatives*, **6**, 57–70.

[20] Carr, P. and Wu, L. (2002), "Static hedging of standard options", Preprint.

[21] Clark, P. (1973), "A subordinated stochastic process model with finite variance for speculative prices", *Econometrica*, **41**, 135–156.

[22] Cox, J., Ingersoll, J. and Ross, S. (1985), "A theory of the term structure of interest rates", *Econometrica*, **53**, 385–408.

[23] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D. (2002a), "The concept of comonotonicity in actuarial science and finance: theory", *Insurance: Mathematics and Economics*, **31**, 3–33.

[24] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke D. (2002b), "The concept of comonotonicity in actuarial science and finance: applications", *Insurance: Mathematics and Economics*, **31**, 133–161.

[25] Eberlein, E. (1999), "Application of generalized hyperbolic Lévy motions to finance", in O.E. Barndorff-Nielsen T. Mikosch and S. Resnick (Eds), *Lévy Processes: Theory and Applications*, Birkhäuser, Boston, MA, USA, pp. 319–337.

[26] Grigelionis, B. (1999), "Processes of Meixner type", *Lithuanian Mathematics Journal*, **39**, 33–41.

[27] Heston, S. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options", *Review of Financial Studies*, **6**, 327–343.

[28] Jacques, M. (1996), "On the hedging portfolio of Asian options", *ASTIN Bulletin*, **26**, 165–183.

[29] Knudsen, Th. and Nguyen-Ngoc, L. (2003), "Pricing European options in a stochastic volatility-jump-diffusion model, DBQuant Working Paper, Deutsche Bank, London, UK; *Journal of Financial and Quantitative Analysis*, to be published.

[30] Madan, D.B., Carr, P. and Chang, E.C. (1998), "The variance gamma process and option pricing", *European Finance Review*, **2**, 79–105.

[31] Mandelbrot, B.B. and Taylor, H.M. (1967), "On the distribution of stock price differences", *Operations Research*, **15**, 1057–1062.

[32] Neuberger, A. and Hodges, S. (2002), "Rational bounds on the prices of exotic options", *IFA Working Paper 359*, London Business School, London, UK.

[33] Nielsen, J.A. and Sandmann, K. (2003), "Pricing bounds on Asian options", *Journal of Financial and Quantitative Analysis*, **38**, 449–473.

[34] Rydberg, T. (1997), "The normal inverse Gaussian Lévy process: simulation and approximation", *Communications in Statistics: Stochastic Models*, **13**, 887–910.

[35] Rydberg, T. (1999), "Generalized hyperbolic diffusions with applications in finance", *Mathematical Finance*, **9**, 183–201.

[36] Schoutens, W. and Teugels, J.L. (1998), "Lévy processes, polynomials and martingales", *Communications in Statistics: Stochastic Models*, **14**, 335–349.

[37] Schoutens, W. (2002), "The Meixner process: theory and applications in finance", *EURANDOM Report 2002-004*, EURANDOM, Eindhoven, The Netherlands.

[38] Schoutens, W. (2003), *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley, Chichester, UK.

[39] Schoutens, W., Simons, E. and Tistaert, J. (2004), "A perfect calibration! Now what?", *Wilmott Magazine*, March, 66–78.

[40] Simon S., Goovaerts, M. and Dhaene, J. (2000), "An easy computable upper bound for the price of an arithmetic Asian option", *Insurance: Mathematics and Economics*, **26**, 175–183.

[41] Večeř, J. and Xu, M. (2004), "Pricing Asian options in a semimartingale model", *Quantitative Finance*, **4**, 170–175.

7 Impact of Market Crises on Real Options

Pauline Barrieu
London School of Economics, London, UK

and

Nadine Bellamy
Université d'Evry Val d'Essonne, Evry, France

Abstract

We study the impact of market crises on investment decisions through real option theory. The framework we consider involves a Brownian motion and a Poisson process, with the jumps characterizing the crisis effects. We first analyze the consequences of different modelling choices. We then provide the real option characteristics and establish the existence of an optimal discount rate. We also characterize the optimal time to invest and derive some properties of its Laplace Transform (bounds, monotonicity, robustness). Finally, we specify the consequences of some wrong model specifications on the investment decision.

7.1 INTRODUCTION

Investment has always been a crucial question for firms. Should a given project be undertaken? In addition, if so, when is it the best time to invest? In order to answer these questions, the neo-classical criterion of Net Present Value (N.P.V.) is still widely used. It consists in investing, if and only if, the sum of the project discounted benefits is higher than the sum of its discounted costs. Such a criterion does, however, have several weaknesses. Among many others, the following facts are often mentioned:

- The N.P.V. method does not take into account potential uncertainty of future cash flows.
- It uses an explicit calculation for the cost of the risk.
- It focuses on present time: the investment decision can only be taken now or never.

However, reality is often more complex and flexible including, for instance, optional components for the project: a firm may have the opportunity (but not the obligation) to undertake the project, not only at a precise and given time, but during a whole period of time (or even without any time limit). In this sense, these characteristics may be related to that of an American call option, with the underlying asset being, for example, the ratio discounted benefits/discounted costs, and the strike level '1'. Therefore, the N.P.V. criterion implies that the American option has to be exercised as soon as it is in the money, which is obviously a sub-optimal strategy.

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

The use of a method based on option theory, such as the real option theory, would improve the optimality of the investment decision. Several articles have appeared as benchmarks in this field. The seminal studies of Brennan and Schwartz (1985), McDonald and Siegel (1986), Pindyck (1991) or Trigeorgis (1996) are often quoted as they present the fundamentals of this method, using particularly dynamic programming and arbitrage techniques. The literature on real options has been prolific from very technical papers to case studies and manuals for practitioners (see among many references, the book edited by Brennan and Trigeorgis (2000) or that of Schwartz and Trigeorgis (2001)). Such an approach better suits reality by taking into account project optional characteristics such as withdrawal, sequential investment, delocalization, crisis management, etc. In that sense, real option theory leads to a decision criterion that adapts to each particular project assessment.

However, real options have also some specific characteristics compared to 'classical' financial options. In particular, the 'risk-neutral' logic widely used in option pricing cannot apply here: the real options' underlying asset corresponds to the investment project flows and is generally not quoted on financial markets. Any replicating strategy of the option payoff is then impossible. So, the pricing is made under a prior probability measure (the historical probability measure or another measure chosen according to the investor's expectations and beliefs). Moreover, a specific feature of a real option framework is the key points of interest for the investor. More precisely, she is interested in:

- The cash flows generated by the project. They are represented by the 'price' of the real option. Note that the notion of 'price' is not so obvious in this framework. It corresponds rather to the value a particular investor gives to this project. However, for the sake of simplicity in the notations, we will use the terminology 'price' in the rest of this paper.
- But also, the optimal time to invest. This optimal time corresponds to the exercising time of the real option.

Therefore, it is important noticing that real options are above all a management tool for decision taking. Once the investment project has been well-specified, the major concern for the investor is indeed summarized in the following question: 'When is it optimal to invest in the project?'. In that sense, knowing the value of the option is less important than knowing its optimal exercising time. For that reason, in this paper, we focus especially on the properties of this optimal time. Moreover, real options studies are usually written in a continuous framework for the underlying dynamics. However, the existence of crises and shocks on investment markets generates discontinuities. The impact of these crises on the decision process is then an important feature to consider. This is especially relevant when some technical innovations may lead to instabilities in production fields.

For all of these reasons, this paper is dedicated to the analysis of the exercising time properties in an unstable framework. The modelling of the underlying dynamics involves a mixed-diffusion (made up of Brownian motion and Poisson process). The jumps are negative so as to represent troubles and difficulties occurring in the underlying market.

In the second section of this paper, we describe the framework of the study and analyze the consequences of different modelling choices. The crisis effect may be expressed via a Poisson process or the compensated martingale associated with it. Of course, there is an obvious relation between these models and they are equivalent from a static point of view. However, when studying the real option characteristics and their sensitivity towards the jump size, these models lead to various outcomes.

After analyzing the real option characteristics in the third section, we focus on the discount rate. We prove the existence of an optimal discount rate, considering the maximization of the Laplace transform of the optimal time to invest as a choice criterion. We also characterize the average waiting time.

In the fifth section we study the robustness of the element decision characteristics. We first specify the robustness of the optimal time to enter the project with respect to the jump size. We establish, in particular, that its Laplace transform is a decreasing function. Then, assuming that the investor only knows the expected value of the random jump size, we prove that this imperfect knowledge leads him/her to undertake the project too early.

In the last section, we focus on the impact of a wrong model specification, assuming that the investor believes in continuous underlying dynamics. In such a framework, we specify the error made in the optimal investment time.

All proofs are presented in the Appendix.

7.2 THE MODEL

7.2.1 Notation

In this paper, we consider a particular investor evolving in a universe, defined as a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. She has to decide whether to undertake a given investment project and, if so, when it is optimal to invest. We assume that the investor has no time limit to take this decision. Consequently, the time horizon we consider is infinite. The investment opportunity value at time $t = 0$ is then of the form

$$C_0 = \sup_{\tau \in \Upsilon} \mathbb{E}\left[\exp\left(-\mu\tau\right)\left(S_\tau - 1\right)^+\right]$$

where $\mathbb{E}$ denotes the expectation with respect to the prior probability measure $\mathbb{P}$, Υ is the set of the $(\mathcal{F}_t)$- stopping times and $(S_t, t \geq 0)$ is the process of the profits/costs ratio.

It is worthwhile noticing that the discount rate μ is usually different from the instantaneous risk-free rate. We will come back later to the real meaning of discount rate in such a framework and to the problem related to its choice.

The profits/costs ratio related to the investment project is characterized by the following dynamics

$$\begin{cases} dS_t = S_{t^-}\left[\alpha dt + \sigma dW_t + \varphi dM_t\right] \\ S_0 = s_0 \end{cases} \tag{A}$$

where $(W_t, t \geq 0)$ is a standard $(\mathbb{P}, (\mathcal{F}_t))$-Brownian motion and $(M_t, t \geq 0)$ is the compensated martingale associated with a $(\mathbb{P}, (\mathcal{F}_t))$-Poisson process N. The Poisson process is assumed to have a constant intensity λ and the considered filtration is defined by $\mathcal{F}_t = \sigma\left(W_s,\ M_s, 0 \leq s \leq t\right)$. Equivalently, the process $(S_t,\ t \geq 0)$ may be written in the form:

$$S_t = s_0 \exp(X_t)$$

where $(X_t,\ t \geq 0)$ is a Lévy process with the Lévy exponent Ψ

$$\mathbb{E}\left(\exp\left(\xi X_t\right)\right) = \exp\left(t\Psi\left(\xi\right)\right)$$

with

$$\Psi(\xi) = \xi^2\frac{\sigma^2}{2} + \xi\left(\alpha - \lambda\varphi - \frac{\sigma^2}{2}\right) - \lambda\left(1 - (1+\varphi)^{\xi}\right) \tag{7.2.1}$$

Hence, we have

$$\begin{aligned}\mathbb{E}\left(\exp\left(i\mathrm{X}_1\right)\right) &= \exp\left(i\xi\left(\alpha - \lambda\varphi - \frac{\sigma^2}{2}\right) - \xi^2\frac{\sigma^2}{2} + \lambda\left(e^{i\xi\ln(1+\varphi)} - 1\right)\right)\\ &= \exp\left(-\Phi(\xi)\right)\end{aligned}$$

Therefore, the Lévy measure associated with the characteristic exponent Φ is expressed in terms of the Dirac measure δ as:

$$\nu(dx) = \lambda\delta_{\ln(1+\varphi)}(dx)$$

Assumptions

In the rest of the paper, the following hypothesis (H) holds.

$$\begin{cases} (i) & 0 < s_0 < 1, \\ (ii) & \sigma > 0 \\ (iii) & 0 > \varphi > -1. \end{cases} \tag{H}$$

Assumption (i) states that s_0 is (strictly) less than 1: this is not a restrictive hypothesis, since the problem we study is a 'true' decision problem. In fact, delaying the project realization is only relevant in the case where the profits/costs ratio is less than one.

Assumption (iii) states that the jump size is negative as we study a crisis situation. The jump process allows us to take into account falls in the project business field. These negative jumps may be induced, for instance, by a brutal introduction of a direct substitute into the market, leading to a decrease in the potential sales. Moreover, we assume that the jump size is greater than -1. This hypothesis, together with the identity

$$S_t = s_0(1+\varphi)^{N_t} \times e^{(\alpha-\lambda\varphi)t} \times e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

ensure that the process S remains strictly positive.

We also impose the integrability condition

$$\mu > \sup(\alpha; 0) \tag{7.2.2}$$

There exists an optimal frontier L^*_φ such that

$$\sup_{\tau\in\Upsilon}\mathbb{E}(e^{-\mu\tau}(S_\tau - 1)^+) = \mathbb{E}(e^{-\mu\tau_{L^*_\varphi}}(S_{\tau_{L^*_\varphi}} - 1)^+)$$

where τ_L is the first hitting time of the boundary L by the process S, defined as

$$\tau_L = \inf\{t \geq 0; S_t \geq L\} \tag{7.2.3}$$

(For the proof, see, for instance, Darling *et al*. (1972) or Mordecki (1999).)

Before the profits/costs ratio S reaches the optimal boundary L^*_φ, it is optimal for the investor not to undertake the investment project and to wait. However, as soon as S goes beyond this threshold, it is optimal for her to invest.

7.2.2 Consequence of the modelling choice

In the framework previously described, we may work *a priori* with either of the two following models:

$$(A) \qquad \begin{cases} dS_t = S_{t^-}\,[\alpha dt + \sigma dW_t + \varphi dM_t] \\ S_0 = s_0 \end{cases}$$

$$(B) \qquad \begin{cases} dS_t = S_{t^-}\,[\overline{\alpha} dt + \sigma dW_t + \varphi dN_t] \\ S_0 = s_0 \end{cases}$$

In the case where all of the parameters are constant, these models are obviously equivalent and writing

$$\alpha = \overline{\alpha} + \lambda\varphi \tag{7.2.4}$$

is sufficient to see why. Note that the integrability condition for model (B) is expressed as

$$\mu > \max\,(\overline{\alpha} + \lambda\varphi; 0)$$

However, when studying the sensitivity of the different option characteristics with respect to the jump size, choosing (A) or (B) really matters. Indeed, monotonicity properties are significantly different in both frameworks, as underlined below.

- Let us first focus on the *optimal time to enter the project*, characterized by its Laplace transform defined as $\mathbb{E}(\exp(-\mu\tau_{L^*_\varphi}))$.

  Considering model (A), if the initial value of the profits/costs ratio is not 'too small', the Laplace transform of the optimal investment time is monotonic (this result is proved in Proposition 7.5.1). However, this monotonicity property does not hold any more for model (B) as is illustrated in Figure 7.1, which is done for the following set of parameters:

$$s_0 = 0.8; \quad \lambda = 0.1; \quad \overline{\alpha} = 0.05; \quad \mu = 0.15; \quad \sigma = 0.2.$$

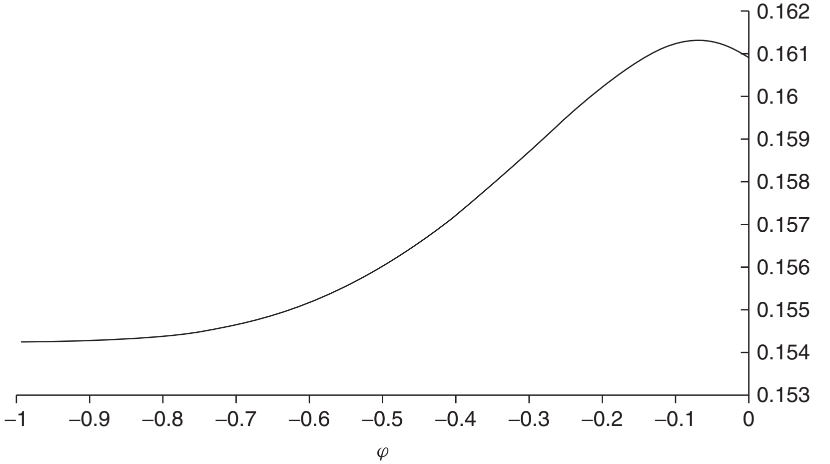

Figure 7.1 Laplace transform of the optimal time to invest (model (B))

- We now focus on the *investment opportunity value* C_0.

Proposition 7.2.1 *Let us consider model* (B). *Then, the investment opportunity value is an increasing function of the jump size.*

Figure 7.2 illustrates Proposition 7.2.1. It represents the variations of the investment value with respect to the jump size for different values of the jump intensity and for the following set of parameters:

$$s_0 = 0.8; \quad \overline{\alpha} = 0.05; \quad \mu = 0.15; \quad \sigma = 0.2.$$

However, this property of the investment opportunity value does not hold any more when considering model (A). Intuitively, the studied model leads to a double effect of the jump size on the underlying level: φ has a positive effect on the underlying by increasing the drift but it also has a negative effect on the underlying by acting on the Poisson process level:

$$dS_t = S_{t^-}\left((\alpha - \lambda\varphi)\,dt + \sigma dW_t + \varphi dN_t\right)$$

This double effect explains the differences between models (A) and (B), and in particular accounts for the following result: in setting (A), the maximum value of C_0 is not necessarily obtained for $\varphi = 0$.

As a conclusion, it cannot be said that one of these models is better or more relevant than the other one. From a static point of view (with respect to the parameter φ), both are mathematically equivalent. In particular, given the condition shown in equation (7.2.4), they lead to the same first and second moments for S. However, from a dynamic point of view with respect to the jump size, they are different.

In the setting (B), crisis is only detected as the spread between the level of S before and after a shock while on the other hand, in the setting (A), there is an additional effect of the shocks on the drift term of S. Economically speaking, both have their own interests and motivations. However, once a model is chosen, the consequences of this choice must be

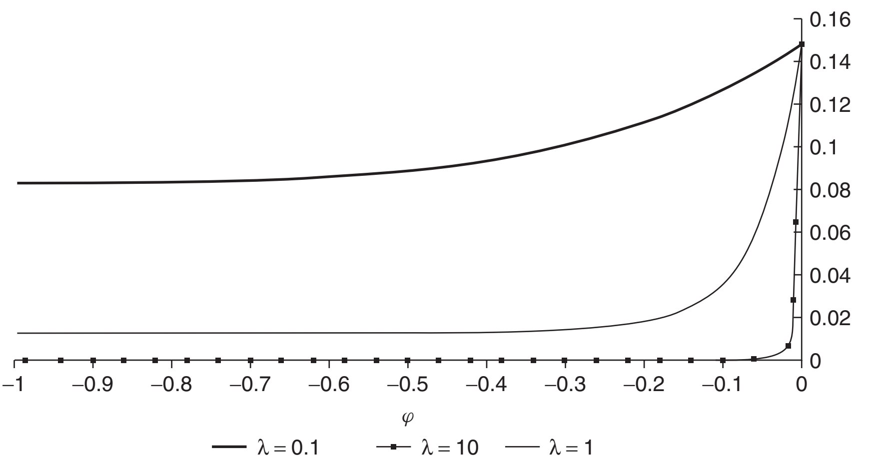

Figure 7.2 Investment opportunity value (model (B))

kept in mind, especially the implications for the monotonicity properties of the real option characteristics.

In this present study, since we are particularly interested in the optimal time to invest, we choose a martingale representation for the stochastic part of $\frac{dS_t}{S_{t-}}$; therefore, the model defined by (A) prevails in the following.

7.3 THE REAL OPTION CHARACTERISTICS

In this section, we first recall the classical formulae for the optimal time to invest and for the investment opportunity.

We denote by k_φ the unique real number defined in terms of the Lévy exponent Ψ defined in equation (7.2.1) since it satisfies:

$$k_\varphi > 1 \quad \text{and} \quad \Psi\left(k_\varphi\right) = \mu$$

Then the optimal profits/costs ratio L_φ^* satisfies:

$$L_\varphi^* = \frac{k_\varphi}{k_\varphi - 1}$$

The investment opportunity value at time 0 is given by:

$$C_0 = \left(\frac{s_0}{k_\varphi}\right)^{k_\varphi} \left(\frac{1}{k_\varphi - 1}\right)^{1-k_\varphi} \tag{7.3.1}$$

and the optimal investment time is characterized by its Laplace transform:

$$\mathbb{E}\left(\exp\left(-\mu \tau_{L_\varphi^*}\right)\right) = \left(\frac{s_0\left(k_\varphi - 1\right)}{k_\varphi}\right)^{k_\varphi} \tag{7.3.2}$$

(For detailed proofs, see, among others, Gerber and Shiu (1994), Bellamy (1999) and Mordecki (1999, 2002).)

It can be noticed that k_φ, as well as the optimal profits/costs ratio L_φ^*, depend on φ, λ and μ.

Remark 1. *In the framework we deal with, the so-called principle of smooth pasting is satisfied. Such a principle is always satisfied in a continuous framework but if the model is driven by discontinuous Lévy processes, this property can fail. In the model we consider, however, the smooth pasting principle still holds (see, for instance, Chan (2003, 2005), Boyarchenko and Levendorskii (2002), Alili and Kyprianou (2004) or Avram et al. (2004)).*

It is also easy to check that the optimal profits/costs ratio satisfies $L_\varphi^* > 1$. This underlines the interest of waiting before undertaking the project, as well as the gain in optimality obtained from considering a real option approach rather than the standard N.P.V. method (see, for instance, Dixit *et al.* (1993)).

Table 7.1 Values of the optimal benefits/costs ratio L^*_φ as a function of σ and φ

$\sigma\backslash\varphi$	−0.9	−0.5	−0.3	−0.1
0.1	16.25	6.59	4.39	3.29
0.2	17.13	6.93	4.74	3.69
0.4	18.24	8.19	6.08	5.12
0.6	20.23	10.26	8.19	7.29

The value of the optimal ratio may be much greater than the limit value '1'. This fact is at variance with the N.P.V. criterion and perfectly illustrates what McDonald and Siegel (1986) have called 'The value of waiting to invest'.

As an illustration, the optimal ratio L^*_φ is calculated in Table 7.1 for the following set of parameters:

$$\mu = 0.15; \quad \lambda = 1; \quad \alpha = 0.1; \quad s_0 = 0.8.$$

Note that high values for the volatility coefficient σ are also considered in this study. This is relevant since the underlying market related to the investment project may be more highly volatile than traditional financial markets (for instance, markets related to new technology).

7.4 OPTIMAL DISCOUNT RATE AND AVERAGE WAITING TIME

7.4.1 Optimal discount rate

We now focus on the discount rate μ and present some general comments about its choice, which is indeed crucial in this study. The rate μ does not correspond to the instantaneous risk-free rate, traditionally used in the pricing of standard financial options. In fact, in this real option framework, the rate μ characterizes the preference of the investor for the present or her aversion for the future. Choosing the 'right' μ is extremely difficult. Many different authors have been interested in this question (among many others, Weitzman (1998)). Some have also proved the existence of a specific relationship between discount rate and future growth rate (Gollier (2002), Gollier and Rochet (2002) and Kimball (1990)). The optimal choice criterion for the rate μ depends, however, on the considered framework. We present here a relevant criterion for this particular problem, corresponding to the maximization of the Laplace transform of the optimal investment time.

Proposition 7.4.1 *(i) There exists a unique real number $\widehat{\mu}$ strictly positive such that*

$$\mathbb{E}(\exp(-\widehat{\mu}\tau_{L^*_{\widehat{\mu}}})) = \max_{\mu} \mathbb{E}(\exp(-\mu\tau_{L^*_\mu}))$$

The real number $\widehat{\mu}$ agrees with an optimal choice of the discount rate μ.
(ii) The optimal discount rate $\widehat{\mu}$ increases with the jumps intensity and decreases with the jumps size.

This optimal discount rate is increasing with the absolute value of the jump size and with the intensity of the jumps. Such a behaviour seems rather logical as the occurrence and the frequency of negative jumps in the future make the value of the project decrease and represent an additional risk for the investor. The more important the jump intensity and size in absolute values are, the more the investor favours the present. Thus, she will choose a higher discount rate. Figure 7.3 shows the variations of the optimal rate $\widehat{\mu}$ with respect to φ for different values of λ and for the following set of parameters:

$$s_0 = 0.8; \quad \alpha = 0.1; \quad \sigma = 0.2.$$

Remark 2. *Other criteria may have been considered in order to choose an optimal rate. For instance, the maximization of C_0 could appear as an alternative. However, it is not a relevant criterion, since the function*

$$\mu \mapsto C_0 = \left(\frac{k_\varphi}{k_\varphi - 1} - 1\right)\left(\frac{s_0\left(k_\varphi - 1\right)}{k_\varphi}\right)^{k_\varphi}$$

is strictly decreasing.

7.4.2 Average waiting time

Another question relative to the best time to invest is, of course, that of the characterization of an average waiting time. If we denote this by T_c, it is defined as the unique element of $\mathbb{R}_+^*$ such that:

$$\mathbb{E}(\exp(-\widehat{\mu}\tau_{L^*_{\widehat{\mu}}})) = \exp(-\widehat{\mu}T_c)$$

Hence, T_c corresponds to the average waiting time. In fact, it is the certainty equivalent of τ_L when the utility criterion is exponential and the risk aversion coefficient is $\widehat{\mu}$. As previously seen, this rate $\widehat{\mu}$ can easily be interpreted as a future aversion coefficient (or a present preference coefficient) and T_c may be explicitly determined as:

$$T_c = -\frac{1}{\widehat{\mu}} \ln \mathbb{E}(\exp(-\widehat{\mu}\tau_{L^*_{\widehat{\mu}}}))$$

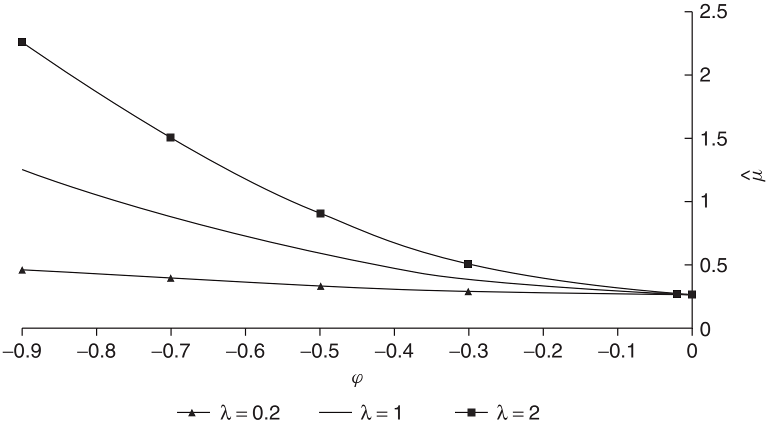

Figure 7.3 Optimal discount rate, $\hat{\mu}$

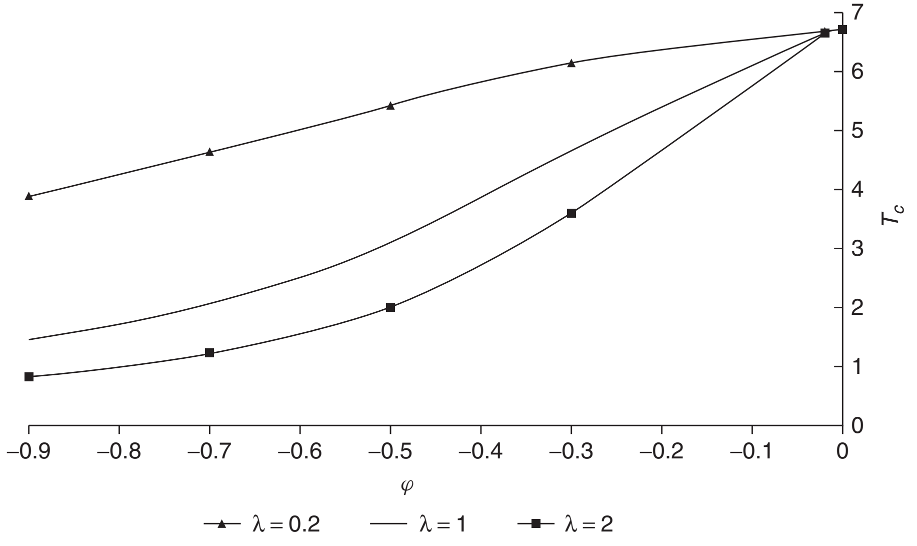

Figure 7.4 Average waiting time, T_c

From Proposition 7.4.1, we deduce that the average waiting time decreases with respect to the jump intensity, as well as to the absolute value of the jump size. This mathematical property can be economically understood as previously. In fact, jumps induce additional risks, increasing with previous jump intensity and the jump size absolute value.

The average waiting time can be related to an exponential utility criterion. Therefore, the investor we consider appears to be risk averse, with an exponential utility function and a risk aversion coefficient of $\widehat{\mu}$. So, in her decision process, she will take into account the expected profit as well as the associated risk. She will tend to reduce the risk induced by the business field by entering earlier in the project. Obviously, the more she waits, the greater the probability of jumps and then the risk are.

Figure 7.4 highlights this fact. It represents the variations of the average waiting time with respect to the jump size. The graphs are produced for different values of the jump intensity. All of these curves converge to the same point as the jump size tends to zero: this point corresponds to the average waiting time in the model without jump, or, in other words, in an universe without crisis. The following set of parameters has been used:

$$s_0 = 0.8; \quad \alpha = 0.1; \quad \sigma = 0.2.$$

7.5 ROBUSTNESS OF THE INVESTMENT DECISION CHARACTERISTICS

All of the different parameters of the model have to be estimated using historical data or strategic anticipations. Every estimation and calibration may lead to an error on the choice of the input parameters. Some stability (or robustness) of the results is an essential condition for a real practical use of a model.

7.5.1 Robustness of the optimal time to invest

As it has already been underlined, the optimal time to invest is the major concern of the investor. Hence, the robustness of its Laplace transform appears as a key point to be checked. We particularly focus on the study of the sensitivity of this quantity with respect to the jump size.

We study the behaviour of the Laplace transform of the optimal time to invest when the jump size is not perfectly known: the investor only knows that there exists $\underline{\varphi}$ and $\overline{\varphi}$ such that

$$-1 < \underline{\varphi} \leq \varphi \leq \overline{\varphi} < 0$$

We first provide a monotonicity result.

Proposition 7.5.1 *Let $\widehat{s_0}$ be the level defined as $\widehat{s_0} = \frac{k_0}{k_0-1}\exp(-\frac{1}{k_0-1})$. We assume that s_0 satisfies*

$$\widehat{s_0} < s_0 < 1 \tag{7.5.1}$$

Then, the Laplace transform of the optimal time to invest is an increasing function of the jump size.

Proposition 7.5.1 can be heuristically interpreted as follows: the more the jump size increases (hence decreases in absolute value), the more the investor delays entering the investment project. The maximum waiting time is attained in the lack of jump.

Remark 3. *The Assumption $\widehat{s_0} < s_0$ amounts to consider investment project only if the initial value is not 'too small'. From an economic point of view, such an assumption is not very restrictive. In fact, the investor will stop being interested in the project as soon as s_0 is below a given threshold. If, for example, we consider the following standard set of parameters*

$$\alpha = 0.10; \sigma = 0.20; \mu = 0.15,$$

then we get

$$\widehat{s_0} = 0.276.$$

Note that this level $\widehat{s_0}$ is far from the strike value 1.

Figure 7.5 shows the changes in the Laplace transform of the optimal time to invest with respect to φ for different values of λ. The following set of parameters is used:

$$s_0 = 0.8; \quad \alpha = 0.10; \quad \sigma = 0.20; \quad \mu = 0.15.$$

The robustness property of the Laplace transform is a straightforward consequence of Proposition 7.5.1.

Corollary 7.5.2 *We assume that the condition shown in equation (7.5.1) holds and*

$$-1 < \underline{\varphi} \leq \varphi \leq \overline{\varphi} < 0$$

Then, we have

$$\mathbb{E}(\exp(-\mu\tau_{L^*_{\underline{\varphi}}})) \leq \mathbb{E}(\exp(-\mu\tau_{L^*_{\varphi}})) \leq \mathbb{E}(\exp(-\mu\tau_{L^*_{\overline{\varphi}}})).$$

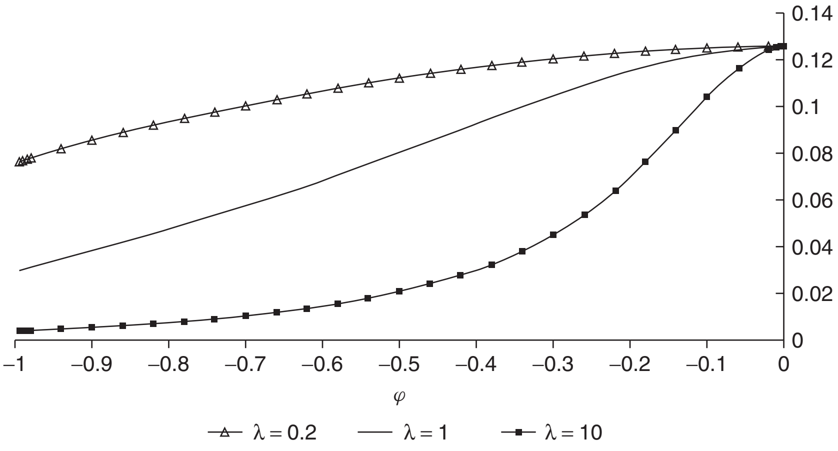

Figure 7.5 Laplace transform of the optimal time to invest for different values of λ

This result underlines the model robustness as far as the Laplace transform of the optimal time to invest is concerned. More precisely, if the investor does not know exactly the size of the jump, in other words the impact of the market crisis on the project, but knows, however, some boundaries for it, then she has an idea of the optimal time to enter the project. More precisely, the Laplace transform boundaries are expressed in terms of the boundaries for the market crisis impact. Equivalently, having some control or knowledge of the crisis impact enables the investor to have some control of her optimal time to invest.

7.5.2 Random jump size

We now consider the situation where the jump size is an unknown random variable Φ. We focus on the impact that this additional hazard may have on the investor decision.

Assuming that the investor estimates the jump size Φ by its expected value $\mathbb{E}(\Phi)$, we focus on the impact of such an error on her decision. Will she invest too early or too late? In order to answer this questions, we compare the 'true' Laplace transform of the optimal time to invest, with the Laplace transform estimated by means of $\mathbb{E}(\Phi)$.

The dynamics of the process of the project are now:

$$dS_t^{\Phi} = S_{t-}^{\Phi}\left(\alpha dt + \sigma dW_t + \Phi dM_t\right); \quad S_0^{\Phi} = s_0$$

and the investor builds her strategy from $S_t^{\mathbb{E}(\Phi)}$ where

$$dS_t^{\mathbb{E}(\Phi)} = S_{t-}^{\mathbb{E}(\Phi)}\left(\alpha dt + \sigma dW_t + \mathbb{E}(\Phi)\, dM_t\right); \quad S_0^{\mathbb{E}(\Phi)} = s_0$$

We assume that the random variable Φ is independent of the filtration generated by the Brownian motion and the Poisson process.

Let L_{Φ}^* be the true optimal benefit–cost ratio. If the investor only knows $\mathbb{E}(\Phi)$, she estimates this ratio by $L_{\mathbb{E}(\Phi)}^*$. The next proposition provides a comparison between these two quantities.

Proposition 7.5.3 *We assume that the condition shown in equation (7.5.1) holds. Then, the wrong specification in the model leads the investor to underestimate the optimal profits/costs ratio.*

Moreover we can précis the consequences of this error on the decision taking. We assume that the investor undertakes the project when the observed process of the benefits/costs ratio reaches what she supposes to be the optimal level. Therefore, her strategy is determined by the first hitting time of $L^*_{\mathbb{E}(\Phi)}$, instead of the first hitting time of L^*_{Φ} by process S. This proposition can be interpreted as follows: when the investor only knows $\mathbb{E}(\Phi)$, she tends to undertake the project too early.

7.6 CONTINUOUS MODEL VERSUS DISCONTINUOUS MODEL

In this section, we focus on the impact of a wrong model choice. This part extends the previous study of robustness. We suppose that the investor believes in a continuous underlying dynamics for S, while its true dynamics is given by (A). As a consequence, the investor governs her strategy according to the following process:

$$d\widetilde{S}_t = \widetilde{S}_t\left(\widetilde{\alpha}dt + \widetilde{\sigma}dW_t\right) \tag{$\widetilde{A}$}$$

where

$$\begin{cases} \widetilde{S}_0 &= s_0 \\ \widetilde{\alpha} &= \alpha \\ \widetilde{\sigma}^2 &= \sigma^2 + \lambda\varphi^2. \end{cases}$$

These equalities come directly from the calibration of both model (A) and $(\widetilde{A})$ on the same data set, leading to the same first and second moments for S and $\widetilde{S}$. The volatility parameter of the model without jump is different from that of the model with jumps: the absence of jump in the dynamics is indeed compensated by a higher volatility. In order to obtain the 'equivalent' volatility, the right brackets of S and $\widetilde{S}$ have to be equal. The process $\widetilde{S}$ is called 'equivalent process without jump'.

We now focus on the impact of such a wrong specification on the investment time. To this end, we first consider the error in the optimal profits/costs ratio.

7.6.1 Error in the optimal profit–cost ratio

We denote by $\widetilde{L}^*_{\varphi}$ the optimal profits/costs ratio in the model defined by $(\widetilde{A})$. More precisely, using the same arguments as presented in Section 7.3, $\widetilde{L}^*_{\varphi}$ is given by the following ratio

$$\widetilde{L}^*_{\varphi} = \frac{\widetilde{k}_{\varphi}}{\widetilde{k}_{\varphi} - 1}$$

where $\widetilde{k}_{\varphi}$ is the solution of

$$\widetilde{\psi}(k) = \frac{\sigma^2 + \lambda\varphi^2}{2}k^2 + \left(\alpha - \frac{\sigma^2 + \lambda\varphi^2}{2}\right)k = \mu \tag{7.6.1}$$

Note that this optimal ratio depends on the volatility parameter of the model, or equivalently, on both jump parameters φ and λ. For the sake of simplicity, as we are especially interested in the sensitivity with respect to the jump size, we use the notation $\widetilde{L}^*_\varphi$.

Proposition 7.6.1 *The previous wrong specification of the model leads the investor to underestimate the optimal profits/costs ratio, if and only if,*

$$\sigma^2 + \lambda\varphi^2 + 2\alpha \geq \mu \tag{7.6.2}$$

Note that for the usual values of the parameters, the inequality shown in equation (7.6.2) often holds. For instance, if we consider $\lambda = 1$, $\alpha = 0.1$, $\sigma = 0.2$ and $\mu = 0.15$, then $\sigma^2 + \lambda\varphi^2 + 2\alpha \geq \mu$ is true for all φ in $]-1, 0[$.

As an illustration, the relative error (expressed in percentage) on the optimal profits/costs ratio

$$RE(L^*, \varphi) = 100 \times \left(\frac{L^*_\varphi - \widetilde{L}^*_\varphi}{L^*_\varphi} \right)$$

is calculated in Table 7.2 for different values of the jump size φ and for the standard set of parameters:

$$s_0 = 0.8; \quad \alpha = 0.1; \quad \sigma = 0.20; \quad \mu = 0.15; \quad \lambda = 1.$$

Very naturally, the relative error becomes negligible as the jump size tends to zero. This error is still manageable when the jump size is not too large (up to -0.5). For larger values, however, the relative error becomes quite important to reach more than a third of the value of the ratio when the jump size is maximal.

Using the same argument as in the previous section, we can précis the consequences that this wrong specification has on the investor's strategy. The investor's waiting time is determined by $\widetilde{L}^*_\varphi$ instead of L^*_φ. So, if the condition shown in equation (7.6.2) holds, we can assert that the error in the model leads the investor to undertake the project too early.

This fact is brought to the fore by Figure 7.6. The optimal time to enter the project for a well-informed investor, as well as that of the previous investor, are respectively characterized by the Laplace transforms $\mathbb{E}(\exp(-\mu\tau_{L^*_\varphi}))$ and $\mathbb{E}(\exp(-\mu\tau_{\widetilde{L}^*_\varphi}))$.

Figure 7.6 represents the variations of these Laplace transforms with respect to the jump size φ. This is, carried out for the following values:

$$s_0 = 0.8; \quad \alpha = 0.1; \quad \sigma = 0.20; \quad \mu = 0.15; \quad \lambda = 1.$$

As another illustration, the relative error (expressed in percentage) on the Laplace transform of the optimal time to invest

$$RE(LT, \varphi) = 100 \times \left(\frac{LT - \widetilde{LT}}{LT} \right)$$

Table 7.2 Relative error on the optimal profits/costs ratio as a function of φ

φ	-0.995	-0.7	-0.5	-0.3	-0.1	-0.01
$RE(L^*, \varphi)$	38.30	15.81	7.11	1.87	0.08	0.01

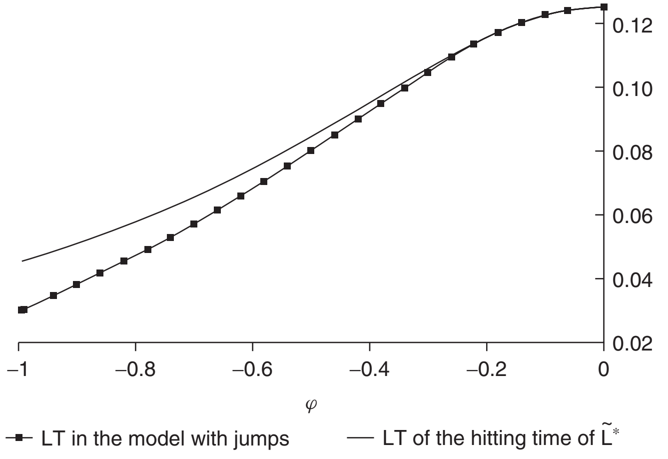

Figure 7.6 Laplace transform of the optimal time to invest in the model with jump and Laplace transform of the hitting time of the profit–cost ratio, $\tilde{L}^*$

Table 7.3 Relative error on the Laplace transform of the optimal time to invest as a function of φ

φ	−0.995	−0.7	−0.5	−0.3	−0.1	−0.02
$RE(LT, \varphi)$	−51.77	−14.68	−5.58	−1.26	−0.04	−0.01

is calculated in Table 7.3 for different values of the jump size φ and for the previous set of parameters.

The interpretation of these results is very similar to those associated with the relative error on the optimal profits/costs ratio. It can be noticed, however, that for large values of the jump size, the relative error becomes quite important to reach more than a half of the Laplace transform when the jump size is maximal. Hence, the impact of a wrong model specification could be important if the investor focuses on the optimal time to invest in the project.

7.6.2 Error in the investment opportunity value

In the 'true' model with jumps, the investment opportunity value is C_0. If we assume that the investor becomes involved in the project when the 'true' process S reaches the level $\widetilde{L}^*_\varphi$, then her investment opportunity value is

$$\widetilde{C}_0 = (\widetilde{L}^*_\varphi - 1)\mathbb{E}(\exp(-\mu\tau_{\widetilde{L}^*_\varphi}))$$

in which

$$\widetilde{C}_0 = (\widetilde{L}^*_\varphi - 1) \times \left(\frac{s_0}{\widetilde{L}^*_\varphi}\right)^{\widetilde{k}_\varphi}$$

where $\widetilde{k}_\varphi$ is the solution of equation (7.6.1).

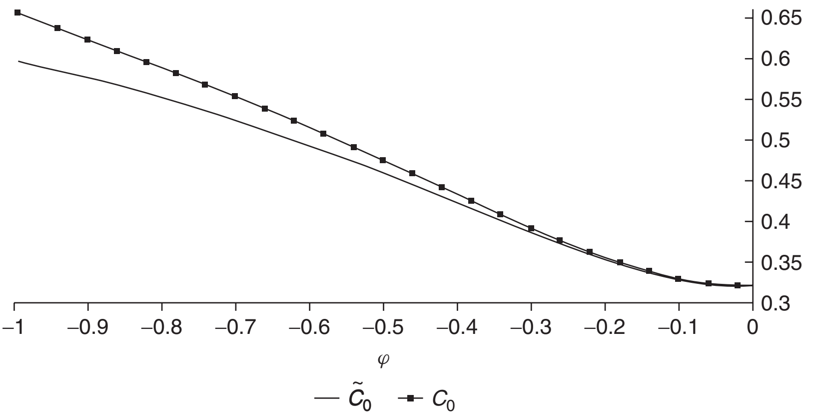

Figure 7.7 Investment value estimated with $\tilde{L}^*$ and the optimal investment value

Figure 7.7 represents the variations of C_0 and $\widetilde{C}_0$ with respect to the jump size φ. Of course, since $\widetilde{L}^*_\varphi$ differs from the optimal frontier L^*, we have for any φ,

$$\widetilde{C}_0 \leq C_0$$

and the loss $C_0 - \widetilde{C}_0$ comes from a wrong investment time. This loss tends to zero when the jump size tends to zero and this fact was expected as $\widetilde{L}^*_\varphi$ tends to the optimal frontier L^* when φ tends to 0.

The curves shown in Figure 7.7 are produced by using the following values:

$$s_0 = 0.8; \alpha = 0.1; \sigma = 0.20; \mu = 0.15; \lambda = 1.$$

As another illustration, the relative error (expressed in percentage) on the investment opportunity value

$$RE(C, \varphi) = 100 \times \left(\frac{C_0 - \widetilde{C}_0}{C_0} \right)$$

is calculated in Table 7.4 for different values of the jump size φ and for the previous set of parameters.

The relative error remains manageable even for large values of the jump size since it is always less than 10%. Therefore, the impact of a wrong model specification is relatively not so important if the investor focuses on the value of the investment opportunity.

Table 7.4 Relative error on the investment opportunity value as a function of φ

φ	−0.995	−0.7	−0.5	−0.3	−0.1	−0.02
$RE(C, \varphi)$	9.02	5.33	3.19	1.14	0.06	0.01

7.7 CONCLUSIONS

In this paper, we study the impact of market crises on investment decision via real option theory. The investment project, modelled by its profits/costs ratio, is characterized by a mixed diffusion process, whose jumps represent the consequences of crises on the investment field. After having analyzed the implications of different model choices, we study the real option associated with this investment project.

We establish the existence of an optimal discount rate, given a criterion based on this investment time and we characterize the average waiting time.

We study in detail the properties of the optimal investment time, through its Laplace transform, and focus, in particular, on its robustness when the underlying dynamics of the project are not well-known or are wrongly specified. We interpret the results in terms of the investment decision. More precisely, when the investor bases his/her decision on the expected value of the random jump size, he/she tends to undertake the project too early. The same property holds if he/she believes in a continuous dynamics for the underlying project.

In this paper, we focus on a single investor. The complexity of reality suggests, however, that different other aspects, in particular, strategic relationships between the economic agents, may play an important role. Investigating more general models involving strategic dimensions and game theory is a topic for future research.

APPENDIX

Proof of Proposition 7.2.1

Let S be defined by model (B). We define $\mathcal{C}(\varphi, L)$ as $\mathcal{C}(\varphi, L) = (L-1) \times \mathbb{E}\left(\exp\left(-\mu\tau_L^{\varphi}\right)\right)$ where $\tau_L^{\varphi} = \inf\{t \geq 0; S_t \geq L\}$. Hence

$$C_0(\varphi) = \mathcal{C}\left(\varphi, L_{\varphi}^*\right)$$

where L_{φ}^* is the optimal frontier, that is to say, the optimal benefit−cost ratio. Let φ_2 and φ_1 be such that $-1 < \varphi_2 < \varphi_1 < 0$. We have

$$C_0(\varphi_1) = \mathcal{C}\left(\varphi_1, L_{\varphi_1}^*\right) \geq \mathcal{C}\left(\varphi_1, L_{\varphi_2}^*\right)$$

Then inequality $\varphi_1 > \varphi_2$ leads to

$$\forall t \geq 0, S_t(\varphi_1) \geq S_t(\varphi_2)$$

and consequently

$$\mathbb{E}(\exp(-\mu\tau_{L_{\varphi_2}^*}^{\varphi_1})) \geq \mathbb{E}(\exp(-\mu\tau_{L_{\varphi_2}^*}^{\varphi_2}))$$

Finally, we get $C_0(\varphi_1) \geq \mathcal{C}(\varphi_1, L^*(\varphi_2)) \geq \mathcal{C}(\varphi_2, L^*(\varphi_2)) = C_0(\varphi_2)$.

Proof of Proposition 7.4.1

(i) The function $k \in]1, +\infty[\rightarrow \left(\frac{s_0(k-1)}{k}\right)^k$ admits a maximum for $k = \widehat{k}$, defined by:

$$\ln s_0 + \ln \frac{\widehat{k}-1}{\widehat{k}} + \frac{1}{\widehat{k}-1} = 0 \quad (7.A.1)$$

The study of the Lévy exponent Ψ leads to the existence of a unique value of μ, denoted by $\widehat{\mu}$, such that $\widehat{\mu} > \alpha$ and $k_{\varphi}^{(\widehat{\mu})} = \widehat{k}$. Moreover, $\widehat{\mu}$ satisfies:

$$\mathbb{E}(\exp(-\widehat{\mu}\tau_{L^*_{\widehat{\mu}}})) = \max_{\mu} \mathbb{E}(\exp(-\mu\tau_{L^*_{\mu}}))$$

Assertion (ii) comes from the definition of $\widehat{\mu}$ and the following properties of the Lévy exponent:

$$\forall k \in \,]1, \widehat{k}[\quad \forall \varphi \in \,]-1, 0[, \qquad \lambda \to \Psi(k)$$

is increasing and

$$\forall k \in \,]1, \widehat{k}[\quad \forall \lambda > 0, \qquad \varphi \to \Psi(k)$$

is decreasing.

Proof of Proposition 7.5.1

Let $\widehat{k}$ be defined by equation (7.A.1). We have

$$k_0 \leq \widehat{k} \Longleftrightarrow s_0 \geq \widehat{s_0}$$

where

$$\widehat{s_0} = \frac{k_0}{k_0 - 1} \exp\left(-\frac{1}{k_0 - 1}\right)$$

and where k_0 is the limit: $k_0 = \lim_{\varphi \to 0} k_{\varphi}$.

In order to get the conclusion, it suffices to prove that k_{φ} is strictly increasing with respect to the jump size φ.

Let $F :]-1; 0[\times]1; +\infty[\to \mathbb{R}$ be the function defined as: $F(\varphi, k) = \Psi(k) - \mu$ where Ψ is given by equation (7.2.1).

For any $(\varphi, k) \in \,]-1; 0[\times]1; +\infty[$ such that $F(\varphi, k) = 0$, we can easily check that $\Psi'(k) > 0$. Using the implicit function theorem, we get:

$$\frac{\partial k}{\partial \varphi} = -\frac{\frac{\partial F}{\partial \varphi}(\varphi, k)}{\frac{\partial F}{\partial k}(\varphi, k)}$$

and the inequality $\frac{\partial F}{\partial \varphi}(\varphi, k) < 0$ implies $\dfrac{\partial k}{\partial \varphi} > 0$. Hence the function $\varphi \mapsto k_{\varphi}$ is strictly increasing.

Proof of Proposition 7.5.3

We denote by Ψ_{Φ} and $\Psi_{\mathbb{E}(\Phi)}$ the Lévy exponents of the processes $\left(X_t^{\Phi}\right)_{t\geq 0}$ and $\left(X_t^{\mathbb{E}(\Phi)}\right)_{t\geq 0}$, where $X_t^{\Phi} = \ln\left(\dfrac{S_t^{\Phi}}{s_0}\right)$ and $X_t^{\mathbb{E}(\Phi)} = \ln\left(\dfrac{S_t^{\mathbb{E}(\Phi)}}{s_0}\right)$.

Let k_Φ (resp. $k_{\mathbb{E}(\Phi)}$) be the unique real number strictly greater than 1 such that $\Psi_\Phi(k_\Phi) = \mu$ (resp. $\Psi_{\mathbb{E}(\Phi)}\left(k_{\mathbb{E}(\Phi)}\right) = \mu$).

We have $\Psi_\Phi(k) = f(\Phi, k) + g(k)$ (resp. $\Psi_{\mathbb{E}(\Phi)}(k) = f(\mathbb{E}(\Phi), k) + g(k)$) where

$$f(\Phi, k) = \lambda(1+\Phi)^k - \lambda\Phi k \text{ and } g(k) = \frac{\sigma^2}{2}k^2 + \left(\alpha - \frac{\sigma^2}{2}\right)k - \lambda.$$

The convexity of the function $x \to f(x, k)$ for any $k > 1$, together with Jensen inequality, implies that

$$\forall k > 1, \ \Psi_{\mathbb{E}(\Phi)}(k) \leq \Psi_\Phi(k)$$

Hence

$$k_\Phi \leq k_{\mathbb{E}(\Phi)}$$

and from this last inequality, we conclude $L^*_\Phi \geq L^*_{\mathbb{E}(\Phi)}$.

Proof of Proposition 7.6.1

Let $\widetilde{\Psi}$ be the Lévy exponent of the process $\left(\widetilde{X}_t\right)_{t\geq 0}$ where $\widetilde{X}_t = \ln\left(\frac{\widetilde{S}_t}{s_0}\right)$ and $\widetilde{k}_\varphi$ be the unique real number such that

$$\widetilde{k}_\varphi > 1 \qquad \widetilde{\Psi}\left(\widetilde{k}_\varphi\right) = \mu.$$

Then, from the equalities

$$\Psi(0) = \widetilde{\Psi}(0) = 0 \text{ and } \Psi(2) = \widetilde{\Psi}(2) = \sigma^2 + \lambda\varphi^2 + 2\alpha,$$

we get

$$\widetilde{k}_\varphi \geq k_\varphi,$$

if and only if,

$$\sigma^2 + \lambda\varphi^2 + 2\alpha \geq \mu$$

and therefore we have

$$L^*_\varphi \geq \widetilde{L}^*_\varphi$$

REFERENCES

[1] Alili, L. and Kyprianou, A. (2004), "Some remarks on first passage of Lévy processes, the American put and pasting principles", *Annals of Applied Probability*, to be published.

[2] Avram, F., Kyprianou, A. and Pistorius, M. (2004), "Exit problems for spectrally negative Lévy processes and applications to Russian, American and Canadized options, *Annals of Applied Probability*, **14**, 215–238.

[3] Bellamy, N. (1999), *Hedging and Pricing in a Market Driven by Discontinuous Processes*, Ph. D. Thesis, Université d'Evry Val d'Essonne, Evry, France.

[4] Boyarchenko, S.I. and Levendorskii, S.Z. (2002), "Perpetual American options under Lévy processes", *SIAM Journal on Control and Optimization*, **40**, 1663–1696.

[5] Brennan, M.J. and Schwartz, E.S. (1985), "Evaluating natural resource investments", *Journal of Business*, **58**, 135–157.

[6] Brennan, M.J. and Trigeorgis, L. (Eds) (2000), *Project Flexibility, Agency, and Product Market Competition – New Developments in the Theory and Applications of Real Options*, Oxford University Press, New York, NY, USA.

[7] Chan, T. (2003), "Some applications of Lévy Processes in insurance and finance", *Finance*, **25**, 71–94.

[8] Chan, T. (2005), "Pricing perpetual American options driven by spectrally one-sided Lévy processes", in A.E. Kyprianou, W. Schoutens and P. Wilmott (Eds), *Exotic Option Pricing and Advanced Lévy Models*, Wiley, Chichester, UK, pp. 195–216.

[9] Darling, D.A., Ligget, T. and Taylor, H.M. (1972), "Optimal stopping for partial sums", *Annals of Mathematical Statistics*, **43**, 1363–1368.

[10] Dixit, A.K. and Pindyck, R. S. (1993). *Investment under Uncertainty*, Princeton University Press, Princeton, NJ, USA.

[11] Gerber, H.U. and Shiu, E.S. (1994), "Martingale approach to pricing perpetual American options", *ASTIN Bulletin*, **24**, 195–220.

[12] Gollier, C. (2002), "Time horizon and the discount rate", *Journal of Economic Theory*, **107**, 463–473.

[13] Gollier, C. and Rochet, J.C. (2002), "Discounting an uncertain future", *Journal of Public Economics*, **85**, 149–166.

[14] Kimball, M.S. (1990), "Precautionary saving in the small and in the large", *Econometrica*, **58**, 53–73.

[15] McDonald, R. and Siegel, D. (1986), "The value of waiting to invest", *Quarterly Journal of Economics*, **101**, 707–727.

[16] Mordecki, E. (1999), "Optimal stopping for a diffusion with jumps", *Finance and Stochastics*, **3**, 227–236.

[17] Mordecki, E. (2002), "Optimal stopping and perpetual options for Lévy processes", *Finance and Stochastics*, **6**, 473–493.

[18] Pindyck, R.S. (1991), "Irreversibility, uncertainty and investment", *Journal of Economic Literature*, **29**, 1110–1148.

[19] Schwartz, E.S. and Trigeorgis, L. (2001), *Real Options and Investment under Uncertainty: Classical Readings and Recent Contributions*, MIT Press, Cambridge, MA, USA.

[20] Trigeorgis, L. (1996), *Real Options – Managerial Flexibility and Strategy in Resource Allocation*, MIT Press, Cambridge, MA, USA.

[21] Weitzman, M.L. (1998), "Why the far-distant future should be discounted at its lowest possible rate", *Journal of Environmental Economics and Management,* **36**, 201–208.

8
Moment Derivatives and Lévy-type Market Completion

José Manuel Corcuera and David Nualart
University of Barcelona, Barcelona, Spain

and

Wim Schoutens
Katholieke Universiteit Leuven – UCS, Leuven, Belgium

Abstract

We show how moment derivatives can complete Lévy-type markets in the sense that by allowing trade in these derivatives any contingent claim can be perfectly hedged by a dynamic portfolio in terms of bonds, stocks and moment-derivative-related products.

Moment derivatives depend on the sum of the powered returns, i.e. the realized moments. Squared log-returns are the basis of the nowadays popular Variance Swaps. Higher-powered returns assess other kinds of important characteristics of the underlying distribution such as skewness and kurtosis.

We first work under a discrete time setting under which we assume that the returns of the stock price process are independent and identically distributed. Out of the Taylor expansion of the payoff function, we extract the positions one has to take in order to perfectly hedge the claim. We illustrate this by some illustrative examples such as the Trinomial tree model.

Next, we comment on the continuous time setting. In this case, a Martingale Representation Property lies at the heart of the completion on the market considered. Results in this exponential Lévy market were already obtained in previous work of these authors. A survey of the relevant results are given and the relation and similarities with the discrete setting are discussed.

8.1 INTRODUCTION

In this paper, we consider markets where the returns are independent and identically distributed (iid). Typically, these markets are incomplete, and the purpose of this work is to show a systematic way of completing these markets. We shall complete the market by introducing a series of assets related to the powers of the return process.

First we present the procedure in a discrete-time setting with discrete returns, while, secondly we consider more general returns, and finally we consider the continuous-time setting. In fact this latter case has been considered in Corcuera *et al*. (2004a) [10] and in such a case the new assets are based on the power-jump processes of the underlying Lévy process. In addition, these new assets can be related with options on the stock (see Balland (2002) [2])

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

and with contracts on realized variance (Carr and Madan (1998) [7] and Demeterfi *et al.* (1999) [15]) that have found their way into OTC markets and are now traded regularly. Higher order power-jump processes have a similar relationship with which one could call realized skewness and realized kurtosis processes. Contracts on these objects, however, are not common. Carr *et al.* (2002) [8] and Carr and Lewis (2004) [6] have studied contracts on the quadratic variation processes in a model driven by a so-called Sato process.

We give an explicit hedging portfolio for claims whose payoff function depends on the prices of the stock and the new assets at maturity. Then, if we introduce utility functions, we can obtain the optimal terminal wealth with respect to these utilities and by the completeness of the enlarged market we can obtain the optimal portfolio by duplicating the optimal wealth. This has been carried out by Corcuera *et al.* (2004b) [11], where we also analyze the case where the optimal portfolio consists only in stocks and bonds. This corresponds to complete the market with new assets in such a way that they are superfluous, that is, we do not improve the terminal expected utility by including these new assets in our portfolio. This is equivalent to choosing an appropriate risk-neutral or martingale measure (see Kallsen (2000) [18] and Schachermayer (2001) [28]). Moreover, this martingale measure is related to the *neutral derivative pricing* of Davis (1997) [13].

8.2 MARKET COMPLETION IN THE DISCRETE-TIME SETTING

We start by explaining the ideas in the most simple incomplete discrete market setting: the one-step trinomial market model. Next, we will consider a one-step market model, where the stock can attain m different values, then we will consider the same model but with n time-steps, and finally we will deal with a general multi-step market.

8.2.1 One-step trinomial market

In this model, we assume we have a risk-free bond paying out a constant interest rate $r > 0$, i.e. the bond has a deterministic value process: $B_0 = 1$ and $B_1 = 1 + r$. We have also a risky asset, a stock, which can move from its initial value $S_0 > 0$ to three different values at time 1. More precisely, we have $S_1 = S_0(1 + X_1)$, where X_1 can take the values $-1 < x_1 < x_2 < x_3$. It is a classical argument, that in order to avoid arbitrage one should have $x_1 < r < x_3$ (by investing in stocks you can lose more, but also gain more, than by investing in bonds).

This arbitrage-free market is one of the most simple cases of an incomplete market, in the sense that there exist contingent claims which cannot be hedged by positions in bonds and stocks. We will show that by introducing a moment option (a Variance-Swap-like derivative), the model can be completed. Moreover, we show that the position one has to take, in order to hedge any contingent claim, can just be read off from the Taylor expansion of the payoff function of the claim.

Indeed, suppose we allow also trade in a contingent claim, paying out X_1^2 at time 1. We will refer to this derivative as the $MOM^{(2)}$ derivative. Let us denote the price of this contingent claim at time zero by z_2.

In order to exclude arbitrage, there must be an equivalent martingale measure, making the discounted values of all traded securities martingales. Denoting the risk-neutral probabilities that X_1 attains the value x_i by $q_i > 0$, $i = 1, 2, 3$, we must have

$$q_1 + q_2 + q_3 = 1$$
$$q_1 x_1 + q_2 x_2 + q_3 x_3 = r$$
$$q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 = z_2(1 + r),$$

where the first equation is ensuring that we have a probability measure, the second equation makes the risk-neutral return on the stock equal to r and the third equation fixes the price of the $MOM^{(2)}$ derivative at z_2. This system of equations can be rewritten in matrix form as

$$\Xi \cdot q = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ r \\ z_2(1 + r) \end{bmatrix}.$$

Since Ξ is a Vandermonde matrix, $\det(\Xi) \neq 0$ and Ξ is invertable. So, the system has exactly one solution, namely:

$$q = \Xi^{-1} \cdot \begin{bmatrix} 1 \\ r \\ z_2(1 + r) \end{bmatrix}.$$

If this solution satisfies $q \in (0, 1)^3$, i.e. the q_is can be seen as probabilities, we have no arbitrage. Moreover, since the solution is unique, we have also, by the second fundamental theorem of asset pricing, that the market is complete. Note that to choose an arbitrage price z_2 is equivalent to choosing a risk-neutral probability q.

Consider now a general contingent claim, with payoff function $G(X_1)$ and develop this function into powers of X_1:

$$G(X_1) = a_0 + a_1 X_1 + a_2 X_1^2.$$

Since X_1 can take only three possible values, the series is cut off after the quadratic term. In order to hedge this claim one needs to carry out the following:

- Invest $(a_0 - a_1)(1 + r)^{-1}$ into bond.
- Buy a_1/S_0 units of stock, for a total price a_1.
- Buy a_2 units of $MOM^{(2)}$ derivatives, for a total price $a_2 z_2$.

At time $t = 1$, we have the following:

- The money invested in bond has grown to $a_0 - a_1$.
- We sell the a_1/S_0 stocks, giving us $a_1(1 + X_1)$ of money.
- The $MOM^{(2)}$ derivatives each pay out X_1^2. This leads to a total payout of $a_2 X_1^2$.

In total we thus end up with $a_0 + a_1 X_1 + a_2 X_1^2 = G(X_1)$ of money, exactly the payout of the contingent claim considered. In order to set up this strategy we needed

$$\frac{a_0 - a_1}{1 + r} + a_1 + a_2 z_2$$

of money, which in order to avoid arbitrage must be the initial price of the contingent claim with payoff function $G(X_1)$.

8.2.2 One-step finite markets

The above situation can easily be generalized to a (one-step) setting, where the random variable X_1 can take a finite number m of possible values $-1 < x_1 < \cdots < x_m$, with $x_1 < r < x_m$, to avoid arbitrage. For $m \geq 3$, the market is an incomplete market; there exist contingent claims which cannot be hedged by holding positions in bonds and stocks alone.

Assume trade is allowed into moment derivatives with payoff functions

$$MOM^{(k)} = X_1^k, \quad k = 2, \ldots, m-1.$$

So, besides investing in bonds and stocks, one can invest also into $m-2$ other derivatives, i.e. the $MOM^{(k)}$'s moment derivatives. Note that payoff functions and initial prices can be negative. For example, in the case of a negative payoff, the holder must pay the corresponding amount to the issuer. Let z_k, $k = 2, 3, \ldots, m-1$, be the initial price of the $MOM^{(k)}$ derivative.

In order to exclude arbitrage, there must be, as above, an equivalent martingale measure, making the discounted values of all traded securities martingales. Denoting the risk-neutral probabilities that X_1 attains the value x_i by $q_i > 0$, $i = 1, \ldots, m$, we must have

$$\begin{aligned} q_1 + \cdots + q_m &= 1 \qquad (8.1)\\ q_1 x_1 + \cdots + q_m x_m &= r \\ q_1 x_1^2 + \cdots + q_m x_m^2 &= z_2(1+r) \\ &\vdots \\ q_1 x_1^{m-1} + \cdots + q_m x_m^{m-1} &= z_{m-1}(1+r), \end{aligned}$$

where the first equation is ensuring that we have a probability measure, the second equation makes the risk-neutral return on the stock equal to r and the other equations fix the prices of the $MOM^{(k)}$ derivatives at z_k, $k = 2, \ldots, m-1$. With obvious notation (as above), the system has exactly one solution, namely:

$$q = \Xi^{-1} \cdot \begin{bmatrix} 1 \\ r \\ z_2(1+r) \\ \vdots \\ z_{m-1}(1+r) \end{bmatrix}. \qquad (8.2)$$

If this solution satisfies $q \in (0,1)^m$, i.e. the q_is can be seen as probabilities, we have no-arbitrage. Moreover, since the solution is unique, we have also that the market is complete.

Since X_1 can now take m possible values, the payoff of a contingent claim $G(X_1)$ can now be written into a Taylor expansion up to degree $m-1$:

$$G(X_1) = a_0 + a_1 X_1 + a_2 X_1^2 + \cdots + a_{m-1} X_1^{m-1}.$$

Completely analogous as in the trinomial setting, the hedging of this contingent claim can be carried out by performing the following:

- Invest $(a_0 - a_1)(1 + r)^{-1}$ into bond.
- Buy a_1/S_0 units of stock, for a total price a_1.
- For each $k = 2, 3, \ldots, m - 1$, buy a_k $MOM^{(k)}$ derivatives for a price $a_k z_k$.

At time $t = 1$, we have the following:

- The money invested in bond has grown to $a_0 - a_1$.
- We sell the a_1/S_0 stocks, giving us $a_1(1 + X_1)$ of money.
- For each $k = 2, 3, \ldots, m - 1$, each $MOM^{(k)}$ derivatives pays out X_1^k. This leads to a total payout of $\sum_{k=2}^{m-1} a_k X_1^k$.

In total, we thus end up with $a_0 + a_1 X_1 + \sum_{k=2}^{m-1} a_k X_1^k = G(X_1)$ of money, exactly the payout of the contingent claim considered. In order to set up this strategy, we needed

$$\frac{a_0 - a_1}{1 + r} + a_1 + \sum_{k=2}^{m-1} a_k z_k$$

of money, which in order to avoid arbitrage must be the initial price of the contingent claim with payoff function $G(X_1)$.

8.2.3 Multi-step finite markets

In this model, we consider a generalization of the above model, taking into account n time-steps. We assume that we have a risk-free bond paying out an interest rate r, i.e. the bond has a deterministic value process: $B_0 = 1$ and $B_i = (1 + r)^i$, $i = 1, \ldots, n$. We have also a risky asset, a stock, which has the following price process

$$S_0 > 0, \quad S_i = S_{i-1}(1 + X_i) = S_0(1 + X_1) \cdots (1 + X_i), \quad i = 1, \ldots, n.$$

We assume the X_is are defined on a stochastic basis $\{\Omega, \mathcal{F}, P, \mathbf{F}\}$, where $\mathbf{F} = \{\mathcal{F}_i\}_{i=1}^n$ is a filtration that describes how the information about the security prices is revealed to the investors. We will suppose that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_i = \sigma(S_1, \ldots, S_i)$, $i = 1, \ldots, n$ and $\mathcal{F} = \mathcal{F}_n$. In addition, we will assume that the X_is are iid and can attain m possible values $-1 < x_1 < \cdots < x_m$, with $x_1 < r < x_m$ to avoid arbitrage.

This arbitrage-free market is again an incomplete market. We will show that by introducing into the market, at each time-step, moment derivatives which mature one time-step later and payoff some power of the return the stock makes over that time-step, the model can be completed.

Assume at time $t = i - 1$, $i = 1, \ldots, n$ trade is allowed into, at this time newly introduced, moment derivatives ($MOM_i^{(k)}$) which mature at time $T = i$ and have a payoff function

$$MOM_i^{(k)} = X_i^k, \quad k = 2, \ldots, m - 1.$$

So, besides investing in bonds and stocks, one can invest at time zero also into the $MOM_1^{(k)}$, $k = 2, \ldots, m - 1$ derivatives. These derivatives mature at time $T = 1$. At this time, a set of $m - 2$ new derivatives are introduced into the market; these derivatives $MOM_2^{(k)}$, $k = 2, \ldots, m - 1$, mature at time $T = 2$, etc.

The X_is are iid with respect to P; in consequence, any possible path of the stock's prices has non-null P-probability. Then, if the system shown by equation (8.2) satisfies $q \in (0,1)^m$, we can find a risk-neutral probability Q such that the prices of the kth moment derivatives at their initiation are equal to say z_k, independently on the step time. This means that for each time $i = 1, \ldots, n$ we take $Q(X_i = x_j | \mathcal{F}_{i-1}) = q_j$. Note that under Q the X_is are also iid. Thus, for each $i = 1, \ldots, n$, we have

$$\text{price of } MOM_i^{(k)} \text{at time } i-1 \ = z_k, \qquad k = 2, \ldots, m-1.$$

By the model described in Section 8.2.2, any payoff function $G = G(X_1, X_2, \ldots, X_n)$ at time $t = n$ can be hedged by a portfolio built at $t = n-1$, having fixed the value of $(X_1, \ldots, X_{n-1}) = (x_1, \ldots, x_{n-1})$. In fact, we can write

$$G(x_1, \ldots, x_{n-1}, X_n) = \sum_{k=0}^{m-1} b_n^{(k)}(x_1, \ldots, x_{n-1}) X_n^k,$$

and the value of this portfolio at time $t = n-1$ will be

$$V_{n-1}(x_1, \ldots, x_{n-1}) = \frac{b_n^{(0)}(x_1, \ldots, x_{n-1}) - b_n^{(1)}(x_1, \ldots, x_{n-1})}{1+r} + b_n^{(1)}(x_1, \ldots, x_{n-1}) + \sum_{k=2}^{m-1} b_n^{(k)}(x_1, \ldots, x_{n-1}) z_k.$$

Then, we can replicate $G(x_1, \ldots, x_{n-2}, X_{n-1}, X_n)$ by a portfolio built at $t = n-2$, by duplicating $V_{n-1}(x_1, \ldots, x_{n-2}, X_{n-1})$. Finally, by backward induction we have that any contingent claim can be hedged by a self-financing portfolio.

8.2.4 Multi-step markets with general returns

With the same notation as in the previous case, let us assume that the Laplace transform of $(X_1, X_2, \ldots, X_n)$ is defined in an open neighborhood of the origin (under Q); then the polynomials are dense in $L^2(\mathcal{F}_n, Q)$. So, for any contingent claim, $G = G(X_1, X_2, \ldots, X_n) \in L^2(Q)$, if we are in the trading time $n-1$ with $(X_1, X_2, \ldots, X_{n-1}) = (x_1, x_2, \ldots, x_{n-1})$, we can write

$$G \stackrel{L^2}{=} \lim_{l \to \infty} \sum_{k=0}^{l} b_n^{(k,l)}(x_1, x_2, \ldots, x_{n-1}) X_n^k.$$

and by backward induction we can replicate G by a self-financing portfolio (see Corcuera *et al.* (2005) [10] for more details).

8.2.5 Power-return assets

Another way of completing the market, is by allowing trade in the so-called power-return assets. To simplify the exposition, we shall work under the finite market setting. We thus have a risk-free bond paying out an interest rate r, i.e. the bond has a deterministic value process: $B_0 = 1$ and $B_i = (1+r)^i$, $i = 1, \dots, n$. We have also a risky asset, a stock, which is the following price process

$$S_0 > 0, \quad S_i = S_{i-1}(1 + X_i) = S_0(1 + X_1) \cdots (1 + X_i), \quad i = 1, \dots, n,$$

and where the X_is are iid (with respect to P) and can attain m different values $-1 < x_1 < \cdots < x_m$, and $x_1 < r < x_m$. Assume now, that in this market $m-2$ new assets are introduced with price process

$$H_i^{(k)} = (1+r)^i \left(\sum_{j=1}^{i} X_j^k - \mu_k i \right), \quad k = 2, \dots, m-1,$$

where $\mu_k \in \mathbb{R}$. Let us make a few remarks on these assets. The asset with price process $H_i^{(k)}$ will be refereed to as the *kth order power-return asset*.

Remark 1 (Arbitrage) *To avoid arbitrage by the introduction of these power-return assets, some conditions are necessary on the constants μ_k. Classical theory says that to have an arbitrage-free market, there must exist an equivalent martingale measure, under which all of the discounted prices process of the assets are martingales. This condition translates into the existence of probabilities $0 < q_i < 1$, such that*

$$\begin{aligned} q_1 + \cdots + q_m &= 1 \qquad \text{(Condition H)} \\ q_1 x_1 + \cdots + q_m x_m &= r \\ q_1 x_1^2 + \cdots + q_m x_m^2 &= \mu_2 \\ &\vdots \\ q_1 x_1^{m-1} + \cdots + q_m x_m^{m-1} &= \mu_{m-1}. \end{aligned}$$

Remark 2 *The first condition forces the q_is to sum up to 1, as probabilities should do. The second condition forces the discounted stock price to be a martingale; the other ones force the discounted power-return asset prices to be martingales. These conditions are almost identical to the conditions in equation (8.1); just replace $z_k(1+r)$ by μ_k. In fact, if these condition are satisfied, it is straightforward to see that the $\{q_1, \dots, q_m\}$ are unique and hence the market is complete.*

Remark 3 (Relation with $MOM^{(k)}$ derivatives) *The two ways of completing the market are very related. To move from the one to the other, one should set $z_k(1+r) = \mu_k$ (as already noted in the previous remark). To exploit the relationship a bit more, we will briefly show how to set up a $MOM^{(k)}$ derivative by an investment strategy in power-return assets. Suppose, that we are at time $t = i-1$ and we want to generate at time i a payoff X_i^k, exactly like the $MOM_i^{(k)}$ derivative is doing. In order to achieve this, at time $i-1$ one should*

- *invest $-(1+r)^{-1} \left(\sum_{j=1}^{i-1} X_j^k - \mu_k i \right) = -(1+r)^{-i} H_{i-1}^{(k)} + (1+r)^{-1} \mu_k$ in bond;*
- *buy $(1+r)^{-i}$ power-return assets of order k, for the total price of $(1+r)^{-i} H_{i-1}^{(k)}$.*

In order to set up this portfolio, an amount (at time $i-1$*) of* $(1+r)^{-1}\mu_k = z_k$ *is needed, exactly the same amount as the time* $i-1$ *price of the* $MOM_i^{(k)}$ *derivative.*

Next, we will show, under the condition (H), that if trade is allowed in the power-return assets, the market is complete, in the sense that any contingent claim can be perfectly hedged by positions in bond, stock and the power-return assets. Let us consider a general contingent claim which can depend on the complete path followed by the underlying stock, i.e. the claim is characterized by a payoff function: $G(X_1, X_2, \ldots, X_n)$.

Write the discounted payoff function in the following form:

$$(1+r)^{-n}G(X_1, X_2, \ldots, X_n) = M_0 + \sum_{j=1}^{n} a_j(X_1, \ldots, X_{j-1})(X_j - r) \qquad (8.3)$$
$$+ \sum_{j=1}^{n}\sum_{k=2}^{m-1} a_j^{(k)}(X_1, \ldots, X_{j-1})(X_j^k - \mu_k).$$

Note that the functions a_j and $a_j^{(k)}$, $k = 2, \ldots, m-1$, only depend on $X_1, X_2, \ldots, X_{j-1}$ and are thus completely known at time $t = j-1$; in other words, the $a_j^{(k)}$s are $\mathcal{F}_{j-1}$ measurable or 'predictable'.

Then, let us consider the martingale

$$M_i = E_Q[(1+r)^{-n}G(X_1, X_2, \ldots, X_n)|\mathcal{F}_i], \quad i = 1, \ldots, n,$$

where Q is the risk-neutral probability defined by Condition (H). Since for $j = 1, \ldots, n$, $E_Q[X_j|\mathcal{F}_{j-1}] = r$ and $E_Q[X_j^k|\mathcal{F}_{j-1}] = \mu_k$, $k = 2, \ldots, m-1$, we have that

$$M_i = M_0 + \sum_{j=1}^{i} a_j(X_1, \ldots, X_{j-1})(X_j - r)$$
$$+ \sum_{j=1}^{i}\sum_{k=2}^{m-1} a_j^{(k)}(X_1, \ldots, X_{j-1})(X_j^k - \mu_k).$$

We know that the discounted value of any contingent claim is a Q-martingale. Then, $E_Q[(1+r)^{-n}G(X_1, X_2, \ldots, X_n)] = M_0$ is the initial price of the claim under consideration and $(1+r)^i M_i$ is the time $t = i$ price of this claim.

In order to hedge the claim, one should follow the following self-financing strategy. Just before the realization of S_i, $i = 1, \ldots, n$ take the following positions in, respectively, bonds, stocks, and kth order power-return assets, $k = 2, \ldots, m-1$:

- number of bonds $= \alpha_i = M_{i-1} - (1+r)^{-i+1}\beta_i S_{i-1} - (1+r)^{-i+1}\sum_{k=2}^{m-1}\beta_i^{(k)}H_{i-1}^{(k)}$,
- number of stocks $= \beta_i = (1+r)^i a_i(X_1, \ldots, X_{i-1})/S_{i-1}$,
- number of kth power-jump assets $= \beta_i^{(k)} = a_i^{(k)}(X_1, \ldots, X_{i-1})$, $k = 2, \ldots, m-1$.

Note the following

- The initial ($t = 0$) amount needed to set up the initial portfolio is:

$$\alpha_1 B_0 + \beta_1 S_0 + \sum_{k=2}^{m-1} \beta_1^{(k)} H_0^{(k)} = M_0.$$

- Just before the realization of S_i, the portfolio value is $(1+r)^{i-1} M_{i-1}$. By a straightforward calculation one can see that just after the realization of S_i (and before adjusting the portfolio again), the value is given by $(1+r)^i M_i$. This implies that the portfolio is self-financing. Moreover, since the value of the portfolio at time $t = n$ equals $(1+r)^n M_n = G(X_1, X_2, \dots, X_n)$, the portfolio is replicating the claim.

In conclusion, we have that the portfolio $(\alpha_i, \beta_i, \beta_i^{(2)}, \dots, \beta_i^{(m-1)}; i = 1, \dots, n)$ is the self-financing portfolio which replicates the claim $G(X_1, X_2, \dots, X_n)$ and has initial value M_0.

8.3 THE LÉVY MARKET

8.3.1 Lévy processes

Lévy processes are the natural continuous time analogs of the sums of iid random variables. Basically, they are processes with the same kind of structure in the increments: stationary and independent. However, not for any general distribution, one can define such a continuous time stochastic process, where the increments follow the given distribution. We have to restrict ourselves to so-called infinitely divisible distributions (see e.g. Bertoin (1996) [3] or Sato (2000) [27])

Given an infinitely divisible distribution with characteristic function $\phi(z)$, one can define a stochastic process (with càdlàg paths), $Z = \{Z_t, t \geq 0\}$, called a Lévy process, which starts at zero, has independent and stationary increments and such that the distribution of an increment over $[s, s+t]$, $s, t \geq 0$, i.e. $Z_{t+s} - Z_s$, has $(\phi(z))^t$ as the characteristic function. It is well known that Lévy processes are semimartingales.

The function $\psi(z) = \log \phi(z) = \log E[\exp(\mathrm{i}zZ_1)]$ is called the *characteristic exponent* and satisfies the following *Lévy–Khintchine formula* (see Bertoin (1996) [3]):

$$\psi(z) = \mathrm{i}\alpha z - \frac{c^2}{2} z^2 + \int_{-\infty}^{+\infty} (\exp(\mathrm{i}zx) - 1 - \mathrm{i}zx 1_{\{|x|<1\}}) \nu(\mathrm{d}x),$$

where $\alpha \in \mathcal{R}$, $c \geq 0$ and ν is a measure on $\mathcal{R}\backslash\{0\}$ with $\int_{-\infty}^{+\infty} (1 \wedge x^2)\nu(\mathrm{d}x) < \infty$. We say that our infinitely divisible distribution has a triplet of Lévy characteristics $[\alpha, c^2, \nu(\mathrm{d}x)]$. The measure $\nu(\mathrm{d}x)$ is called the *Lévy measure* of Z, while $\nu(\mathrm{d}x)$ dictates how the jumps occur. Jumps of sizes in the set A occur according to a Poisson process with parameter $\int_A \nu(\mathrm{d}x)$. If $c^2 = 0$ and $\int_{-1}^{+1} |x|\nu(\mathrm{d}x) < \infty$, it follows from standard Lévy process theory (see Bertoin (1996) [3] and Sato (2000) [27]) that the process is of finite variation (for applications of Lévy processes in finance, see Schoutens (2003) [29]).

From the Lévy–Khintchine formula, one can deduce that Z must be a linear combination of a standard Brownian motion $W = \{W_t, t \geq 0\}$ and a pure jump process $X = \{X_t, t \geq 0\}$:

$$Z_t = cW_t + X_t,$$

and where W is independent of X. Moreover

$$X_t = \int_{\{|x|<1\}} x(Q((0,t],\mathrm{d}x) - t\nu(\mathrm{d}x)) + \int_{\{|x|\geq 1\}} xQ((0,t],\mathrm{d}x) + \alpha t,$$

where $Q(\mathrm{d}t, \mathrm{d}x)$ is a Poisson random measure on $(0,+\infty) \times \mathcal{R}\backslash\{0\}$ with intensity $\mathrm{d}t \times \nu$, where ν is the Lévy measure of Z and $\mathrm{d}t$ denotes the Lebesgue measure.

8.3.2 The geometric Lévy model

The continuous analog (separating the deterministic trend) of the stock price model described in Section 8.2.3 model is the so-called (geometric) Lévy market model or (stochastic) exponential Lévy processes, under which we have initially to our disposal a bond, with price process $B = \{B_t = \exp(rt)\}$ and a stock. Under this model, the stock price process $S = \{S_t, t \geq 0\}$ is modelled by a Stochastic Differential Equation (SDE) driven by a general Lévy process $Z = \{Z_t, t \geq 0\}$:

$$\frac{\mathrm{d}S_t}{S_{t-}} = b\mathrm{d}t + \mathrm{d}Z_t, \qquad S_0 > 0. \tag{8.4}$$

Z is defined on a stochastic basis $\{\Omega, \mathcal{F}, P, \mathbf{F}\}$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}^T$ is the filtration that describes how the information about the security prices is revealed to the investors. We assume that $\mathcal{F}_t$ is $\sigma(S_s, 0 \leq s \leq t)$ completed with the P-null sets. The classical Black–Scholes model (Black and Scholes (1973) [5]) takes a Brownian motion for the Lévy process Z. We will allow more general Lévy processes (taking into account jumps).

For the purpose of our model, we require the process Z to satisfy certain conditions. We will suppose that the Lévy measure satisfies for some $\varepsilon > 0$, and $\lambda > 0$

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\nu(dx) < \infty. \tag{8.5}$$

This implies that

$$\int_{-\infty}^{+\infty} |x|^i \nu(\mathrm{d}x) < \infty, \qquad i \geq 2,$$

and that the characteristic function $E[\exp(\mathrm{i}uX_t)]$ is analytic in a neighbourhood of 0 and

$$E[\exp(-hZ_1)] < \infty \text{ for all } h \in (-h_1, h_2),$$

where $0 < h_1, h_2 \leq \infty$. So, all moments of Z_t (and X_t) exist.

8.3.3 Power-jump processes

Under our continuous-time setting, the role of the powered returns will be taken by power-jump processes.

These are built from the following transformations of $Z = \{Z_t, t \geq 0\}$. We set

$$Z_t^{(i)} = \sum_{0<s\leq t} (\Delta Z_s)^i, \qquad i \geq 2,$$

where $\Delta Z_s = Z_s - Z_{s-}$, and for convenience we put $Z_t^{(1)} = Z_t$. Note that $Z_t = \sum_{0<s\le t} \Delta Z_s$ is not necessarily true; it is only true in the bounded variation case (with c necessarily equal to zero). If we define $X_t^{(i)}$ in an analogous way, we have that $X_t^{(i)} = Z_t^{(i)}$, $i \ge 2$. The processes $X^{(i)} = \{X_t^{(i)}, t \ge 0\}$, $i \ge 2$, are again Lévy processes and are called the ith-power-jump processes (or the power-jump processes of order i). They jump at the same points as the original Lévy process, but the jumps sizes are the ith power of the jump size of the original Lévy process.

We have $E[X_t] = E[X_t^{(1)}] = tm_1 < \infty$ and (see Protter (1990), p. 29 [25])

$$E\left[X_t^{(i)}\right] = E\left[\sum_{0<s\le t} (\Delta X_s)^i\right] = t\int_{-\infty}^{\infty} x^i \nu(\mathrm{d}x) = m_i t < \infty, \qquad i \ge 2. \tag{8.6}$$

We denote by

$$Y_t^{(i)} = Z_t^{(i)} - E\left[Z_t^{(i)}\right] = Z_t^{(i)} - m_i t, \qquad i \ge 1,$$

the compensated processes.

Using Itô's formula (see Chan (1999) [9] or Protter (1990) [25]) for càdlàg semimartingales, one can show that equation (8.4) has an explicit solution

$$S_t = S_0 \exp\left(Z_t + \left(b - \frac{c^2}{2}\right)t\right) \prod_{0<s\le t} (1 + \Delta X_s)\exp(-\Delta X_s).$$

In order to ensure that $S_t > 0$ for all $t > 0$ almost surely, we need $\Delta X_t > -1$ for all t. We thus need that the Lévy measure ν is supported on a subset of $(-1, +\infty)$.

8.4 ENLARGING THE LÉVY MARKET MODEL

Suppose that we have an equivalent martingale measure Q under which Z remains a Lévy process. Under this measure, the discounted stock price process is a martingale and the process $\tilde{Z} = \{Z_t + (b - r)t, t \ge 0\}$ will be a Lévy process (with Lévy measure $\tilde{\nu}$); moreover, the process $\tilde{Z}$ is a martingale. Obviously, $\Delta\tilde{Z}_t = \Delta Z_t$ and $\tilde{Z}_t^{(i)} = Z_t^{(i)}$, $i \ge 2$. Let us consider (based on $\tilde{Z}$) the ith-power-jump processes $Y^{(i)} = \{Y_t^{(i)}, t \ge 0\}$. Note that for $i \ge 2$, $m_i = \int_{-\infty}^{+\infty} x^i \tilde{\nu}(\mathrm{d}x)$, and we will require $\tilde{\nu}$ to fulfil equation (8.5).

We will enlarge the Lévy market with what we will call *ith-power-jump assets.* More precisely, we will allow trade in assets with price process $H^{(i)} = \{H_t^{(i)}, t \ge 0\}$ where

$$H_t^{(i)} = \exp(rt)Y_t^{(i)}, \qquad i \ge 2.$$

By taking a suitable linear combination of the $Y^{(i)}$s, one obtains a set of pairwise strongly orthonormal martingales $\{T^{(i)}, i \ge 1\}$ (see Protter (1990) [25]). Each $T^{(i)}$ is a linear combination of the $Y^{(j)}$, $j = 1, 2, \ldots, i$:

$$T^{(i)} = c_{i,i}Y^{(i)} + c_{i,i-1}Y^{(i-1)} + \cdots + c_{i,1}Y^{(1)}, \qquad i \ge 1.$$

The constants $c_{i,j}$ can be calculated as described in Nualart and Schoutens (2000) [22]: they correspond to the coefficients of the orthonormalization of the polynomials $\{x^n, n \ge 0\}$

with respect to the measure $\mu(\mathrm{d}x) = x^2\nu(\mathrm{d}x) + c^2\delta_0(\mathrm{d}x)$. The resulting processes $T^{(i)} = \{T_t^{(i)}, t \geq 0\}$ are called the *orthonormalized ith-power-jump processes*. In addition, we will denote their orthonormalized version of $H_t^{(i)}$ by $\overline{H}^{(i)} = \{\overline{H}_t^{(i)}, t \geq 0\}$, where

$$\overline{H}_t^{(i)} = \exp(rt)T_t^{(i)}, \qquad i \geq 2.$$

Trade in the power-jump assets can be motivated as follows. Consider the 2nd-power-jump asset. This object in some sense measures the volatility of the stock, since it accounts for the squares of the jumps. If one believes that in the future there will be a more volatile environment than the current market's anticipation, trading the 2nd-power-jump asset can be of interest. In addition, if one would like to cover against periods of high (or low) volatility, they can be useful: Buying 2nd-power-jump assets can cover the possible losses due to such unfavourable periods. The same can be said for the higher order variation assets. Typically, the 3rd-power-jump assets is measuring a kind of asymmetry (cf. skewness) and the 4th-power-jump process is measuring extremal movements (cf. kurtosis). Trade in these assets can be of use if one likes to bet on the realized skewness or realized kurtosis of the stock: one believes that the market is not counting in asymmetry and possible extremal moves correctly. On the other hand, an insurance against a crash can be easily built from the 4th-power-jump (or ith-power jump, $i \geq 4$) assets.

Note, that clearly the discounted versions of the $H^{(i)}$ are the power-jump processes, and hence martingales:

$$E_Q[\exp(-rt)H_t^{(i)}|\mathcal{F}_s] = E_Q[Y_t^{(i)}|\mathcal{F}_s] = Y_s^{(i)}, \quad 0 \leq s \leq t.$$

Hence, the market allowing trade in the bond, the stock and the power-jump assets remains arbitrage-free.

8.4.1 Martingale representation property

Our Lévy process $Z = \{Z_t, t \geq 0\}$ has the Martingale Representation Property (MRP) in terms of the orthonormalized power-jump processes (see also Nualart and Schoutens (2000, 2001) [22] [23]) that is, every square-integrable martingale $M = \{M_t, t \geq 0\}$ can be represented as follows:

$$M_t = M_0 + \int_0^t h_s \mathrm{d}\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)}\, \mathrm{d}T_s^{(i)},$$

where h_s and $h_s^{(i)}$, $i \geq 2$ are predictable processes. such that

$$E\left[\int_0^t |h_s|^2\, \mathrm{d}s\right] < \infty$$

and

$$E\left[\int_0^t \sum_{i=2}^{\infty} |h_s^{(i)}|^2\, \mathrm{d}s\right] < \infty.$$

Note the similarity, except for the orthonormalization, between this MRP and equation (8.3).

The MRP implies that the market enlarged with the ith-power-jump assets is complete in the sense that for every square-integrable contingent claim X we can set up a sequence of self-financing portfolios whose values converge in $L^2(Q)$ to X. These portfolios will consist of finite number of bonds, stocks and ith-power-jump assets. We will say, for short, that X can be replicated. Note that this notion of completeness is equivalent to the notion of *approximately complete* of Björk and co-workers (given in Björk *et al.* (1997) [4]).

The details of the hedging strategy can be extracted out of the MRP. Consider a square-integrable contingent claim $X \in \mathcal{F}_T$ with maturity T. Let

$$M_t = E_Q[\exp(-rT)X|\mathcal{F}_t].$$

By the MRP given above, if we define

$$M_t^N := M_0 + \int_0^t h_s \mathrm{d}\tilde{Z}_s + \sum_{i=2}^{N} \int_0^t h_s^{(i)} \mathrm{d}T_s^{(i)}.$$

we have that

$$\lim_{N\to\infty} M_t^N = M_t,$$

in $L^2(Q)$. Define the sequence of portfolios (in terms of the orthonormalized ith-power-jump assets)

$$\phi^N = \{\phi_t^N = (\alpha_t^N, \beta_t, \beta_t^{(2)}, \beta_t^{(3)}, \ldots, \beta_t^{(N)}), t \geq 0\}, \quad N \geq 2$$

by

$$\alpha_t^N = M_{t-}^N - \beta_t S_{t-} e^{-rt} - e^{-rt} \sum_{i=2}^{N} \beta_t^{(i)} \overline{H}_{t-}^{(i)},$$

$$\beta_t = e^{rt} h_t \; S_{t-}^{-1},$$

$$\beta_t^{(i)} = h_t^{(i)}, \qquad i = 2, 3, \ldots, N.$$

Here, α_t^N corresponds to the number of bonds at time t, β_t is the number of stocks at that time and $\beta_t^{(i)}$ is the number of assets $\overline{H}^{(i)}$, $i = 2, 3, \ldots, N$, one needs to hold at time t. Then, it was shown in Corcuera *et al.* (2005) [10] that $\{\phi^N , N \geq 2\}$ is the sequence of self-financing portfolios which replicates X. In fact, the value V_t^N of ϕ^N at time t is given by

$$V_t^N = \alpha_t^N e^{rt} + \beta_t S_t + \sum_{i=2}^{N} \beta_t^{(i)} \overline{H}_t^{(i)} = e^{rt} M_t^N,$$

and so the sequence of portfolios $\{\phi^N , N \geq 2\}$ is replicating the claim.

Moreover, in the case of a contingent claim whose payoff is only a function of the value at maturity of the stock price, i.e. $X = f(S_T)$, one can compute explicitly the sequence of portfolios that replicates the contingent claim.

Note that the value of the contingent claim at time t is given by

$$F(t, S_t) = \exp(-r(T-t)) E_Q[f(S_T)|\mathcal{F}_t];$$

we call $F(t, x)$ the price function of X.

Denote by D_1 the differential operation with respect to the first variable, i.e. the time variable, and by D_2 the differential operator with respect to the space variable (the second variable – the stock price). Finally, denote by $\mathcal{D}$ the following integral operator:

$$\mathcal{D}F(t,x) = \int_{-\infty}^{+\infty} \left(F(t, x(1+y)) - F(t,x) - xyD_2F(t,x)\right)\tilde{\nu}(\mathrm{d}y).$$

If f is Lipschitz and under certain degeneracy conditions (see Chapter 12 in Cont and Tankov (2004) [12]), $F \in C^{1,2}$. In this case, we have that, in analogy with the Black–Scholes partial differential equation, in the Lévy market setting F will satisfy a Partial Differential Integral Equation (PDIE). More precisely, the price function (at time t) $F(t,x)$ satisfies (see Chan (1999) [9], Nualart and Schoutens (2001) [23] and Raible (2000) [26]):

$$D_1F(t,x) + rxD_2F(t,x) + \frac{1}{2}c^2x^2D_2^2F(t,x) + \mathcal{D}F(t,x) = rF(t,x). \tag{8.7}$$

with $F(T, S_T) = f(S_T)$.

The sequence of self-financing portfolios replicating a contingent claim X, with a payoff only depending on the stock price value at maturity and a price function $F(t,x) \in C^{1,\infty}$ which satisfies

$$\sup_{x<K, t\leq t_0} \sum_{n=2}^{\infty} |D_2^n F(t,x)| R^n < \infty, \tag{8.8}$$

for all $K, R > 0$, $t_0 > 0$, is given at time t by:

- number of bonds$= \alpha_t^N = B_t^{-1}F(t, S_{t-}) - S_{t-}D_2F(t, S_{t-}) - B_t^{-1}\sum_{i=2}^{N} \frac{S_{t-}^i D_2^i F(t, S_{t-})}{i!B_t} H_{t-}^{(i)}$ (8.9)
- number of stocks $= \beta_t = D_2F(t, S_{t-})$,
- number of ith-power-jump assets $= \beta_t^{(i)} = \frac{S_{t-}^i D_2^i F(t, S_{t-})}{i!B_t}$, $\quad i = 2, 3, \ldots, N$.

Remark 4 *In the Black–Scholes model, the risk-neutral dynamics of the stock price is given by the stochastic differential equation*

$$\frac{dS_t}{S_t} = \left(r - \frac{1}{2}\sigma^2\right)dt + dW_t, \qquad S_0 > 0,$$

where $W = \{W_t, t \geq 0\}$ is a standard Brownian motion. In this case, all processes $H^{(i)}$, $i \geq 1$ are equal to zero. Hence, it is clear that the market is already complete and that an enlargement is not necessary. Moreover the hedging portfolio is given by $\frac{F(s,S_s) - S_s D_2 F(s,S_s)}{B_s}$ number of bonds and $D_2F(s, S_s)$ number of stocks.

8.5 ARBITRAGE

We assume our market is already enlarged with the power-jump assets. So, we have chosen constants $a^{(i)}$, $i \geq 2$ and trade is allowed in the bond, the stock and the power-jump assets with price processes $H_t^{(i)} = \exp(rt)(X_t^{(i)} - a^{(i)}t)$, $i \geq 2$.

We investigate whether this enlargement leads to arbitrage or not. For instance, if we choose $a^{(i)}$ and r to equal zero this leads to arbitrage opportunities because all $H_t^{(i)}$ with even i are strictly increasing and starting at zero and trade is allowed in these objects. Actually, the choice of the constants $a^{(i)}$ may prevent arbitrage opportunities. We will discuss below how to make this choice, which is a delicate matter.

No arbitrage, in the usual sense and in our portfolios with a finite number of assets, is implied by the existence of an equivalent martingale measure under which all discounted assets in the market are martingales. This question is related to the moment problem and we will give sufficient conditions to ensure that there exists an equivalent martingale measure (and hence the market is arbitrage free): in continuous time the existence of an equivalent martingale measure is a sufficient but not a necessary condition to ensure no-arbitrage (see Delbaen and Schachermayer (1994) [14]). The problem in its full generality seems to be very hard and challenging.

8.5.1 Equivalent martingale measures

In this section, we will describe the measures, equivalent to the canonical (real world) measure under which the discounted stock price process is a martingale and under which Z remains a Lévy process. More precisely, we characterize all *structure preserving* P-equivalent martingale measures Q under which Z remains a Lévy process and the process $\tilde{S} = \{\tilde{S}_t = \exp(-rt)S_t, t \geq 0\}$ is an $\{\mathcal{F}_t\}$-martingale. Since we are considering a market with finite horizon T, locally equivalence will be the same as equivalence.

We have the following result (see Sato (2000), Theorem 33.1 [27]).

Theorem 5 *Let Z be a Lévy process with Lévy triplet $[\alpha, c^2, \nu(dx)]$ under some probability measure P. Then the following two conditions are equivalent.*

(a) There is a probability measure Q equivalent to P on $\mathcal{F}_t$ for any $t \geq 0$, such that Z is a Q-Lévy process with triplet $[\tilde{\alpha}, \tilde{c}^2, \tilde{\nu}(dx)]$.

(b) All of the following conditions hold:

(i) $\tilde{\nu}(dx) = H(x)\nu(dx)$ for some Borel function $H : \mathcal{R} \rightarrow (0, \infty)$.

(ii) $\tilde{\alpha} = \alpha + \int_{-\infty}^{+\infty} x\mathbf{1}_{\{|x|\leq 1\}}(H(x) - 1)\nu(dx) + Gc$ for some $G \in \mathcal{R}$.

(iii) $\tilde{c} = c$.

(iv) $\int_{-\infty}^{\infty}(1 - \sqrt{H(x)})^2\nu(dx) < \infty$.

The equivalent conditions in the previous theorem imply that the process $\tilde{W} = \{\tilde{W}_t, t \geq 0\}$ with

$$\tilde{W}_t = W_t - Gt$$

is a Brownian motion under Q and also, if ν and $\tilde{\nu}$ verify the condition shown in equation (8.5), the process X is a quadratic pure jump Lévy process with Doob–Meyer decomposition

$$X_t = \tilde{L}_t + \left(a + \int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x)\right) t,$$

where $\tilde{L} = \{\tilde{L}_t, t \geq 0\}$ is a Q-martingale and the Lévy measure is given by $\tilde{\nu}(\mathrm{d}x) = H(x)\nu(\mathrm{d}x)$.

We now want to find an equivalent martingale measure Q under which the discounted price process $\tilde{S}$ is a martingale. By the above theorem, under such a Q, X has the Doob–Meyer decomposition

$$X_t = \tilde{L}_t + \left(a + \int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x)\right) t,$$

where $\tilde{L} = \{\tilde{L}_t, t \geq 0\}$ is a Q-martingale. Noting that $\Delta L_t = \Delta \tilde{L}_t$, we have

$$\tilde{S}_t = S_0 \exp\left(c\tilde{W}_t + \tilde{L}_t + \left(a + b - r + cG - \frac{c^2}{2}\right) t\right)$$
$$\times \exp\left(t \int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x)\right) \prod_{0<s\leq t} (1 + \Delta\tilde{L}_s)\exp(-\Delta\tilde{L}_s).$$

Then, a necessary and sufficient condition for $\tilde{S}$ to be a Q-martingale is the existence of G and $H(x)$, with $\int_{-\infty}^{\infty}(1 - \sqrt{H(x)})^2\nu(\mathrm{d}x) < \infty$ such that

$$cG + a + b - r + \int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x) = 0. \tag{8.10}$$

Remark 6 *We remark (see e.g. Eberlein and Jacod (1997) [16]), that if there exists a (non-structure preserving) locally equivalent martingale measure Q_1 under which Z is not a Lévy process, there exists always a (structure preserving) locally equivalent martingale measure Q_2 under which Z is a Lévy process.*

A sufficient condition to guarantee that the enlarged market is free of arbitrage is the existence of an equivalent martingale measure Q making $\tilde{S}$ and the discounted $H^{(i)}$s martingales. If this measure is structure preserving, the condition that the discounted stock price must be a martingale comes down to the existence of G and $H(x)$ such that equation (8.10) holds. If we also want that the discounted $H^{(i)}$s, i.e. $X_t^{(i)} - a^{(i)}t$, be martingales for $i \geq 2$, using equation (8.6) together with the fact that the Lévy measure of X under Q is given by $H(x)\nu(\mathrm{d}x)$, this comes down to

$$\int_{-\infty}^{+\infty} x^i H(x)\nu(\mathrm{d}x) = a^{(i)}, \quad i \geq 2. \tag{8.11}$$

The question now is, do there exist G and $H(x)$ such that equations (8.10) and (8.11) hold simultaneously? This question is related to the moment problem: given a series of numbers $\{\mu_n\}$, find necessary and sufficient conditions for the existence of such a measure with μ_n as the nth moment. Another point is the uniqueness. A partial result is that if the moment problem has a solution with bounded support, then it will be unique (see Shohat and Tamarkin (1950) [30] or Ahiezer (1965) [1]). We have then the following proposition.

Proposition 7 *Suppose that $\nu(dx)$ has compact support: then, if there is a martingale measure in the market enlarged with the power-jump assets, the martingale measure is unique, structure preserving and the market is complete.*

Proof. If we have a martingale measure in the enlarged market, there exists, using the same arguments as in Eberlein and Jacod (1997) [16], an $H(x)$ verifying equations (8.10) and (8.11) with $H(x) > 0$. The measure $\mu(\mathrm{d}x) = x^2 H(x)\nu(\mathrm{d}x)$ is finite and has a bounded support. This implies that $H(x)$ is determined by the condition shown in equation (8.11). On the other hand, since the support is bounded, $H(x)\nu(\mathrm{d}x)$ verifies equation (8.5) and the model enlarged with the power-jump assets is complete. Finally, since the contingent claim $B_T \mathbf{1}_A$ with $A \in \mathcal{F}$ is replicable, the uniqueness of its initial arbitrage price, $E_Q(\mathbf{1}_A)$, implies the uniqueness of the martingale measure.

In general, uniqueness of the martingale measure implies completeness.

Proposition 8 *If the probability measure that makes the discounted stock price and the power-jump assets martingales is unique, that is, the martingale measure is unique, then the market is complete.*

Proof. Let Q be a martingale measure. We argue by contradiction. If the market is not complete, there exists a contingent claim $X \geq 0$, $X \in L^2(Q)$, not identically zero, which is orthogonal to any replicable contingent-claim. Define $Q^*(d\omega) = (1 + X)Q(d\omega)$. Then Q^* is a martingale measure different from Q. In fact, for any $s \leq t$, and $A \in \mathcal{F}_s$, we have

$$E_{Q^*}(\mathbf{1}_A(Y_t^{(i)} - Y_s^{(i)})) = E_Q(\mathbf{1}_A(Y_t^{(i)} - Y_s^{(i)})) + E_Q(X\mathbf{1}_A(Y_t^{(i)} - Y_s^{(i)})) = 0,$$

and $\{Y_t^{(i)}, t \geq 0\}$ are Q^*-martingales for all $i \geq 2$. Clearly, $\tilde{S}_t$ is also a Q^*-martingale.

8.5.2 Example: a Brownian motion plus a finite number of Poisson processes

Suppose

$$Z_t = cW_t + \sum_{j=1}^{n} c_j N_{j,t},$$

where $c \neq 0$, $W = \{W_t, t \geq 0\}$ a standard Brownian Motion and $N_j = \{N_{j,t}, t \geq 0\}$ are independent Poisson processes with intensity $a_j > 0$. The constants c_j, $j = 1, \ldots, n$ are assumed to be all different from each other and non-zero. Then, $X_t = \sum_{j=1}^{n} c_j N_{j,t}$ and $E[X_1] = \sum_{j=1}^{n} c_j a_j = a$, and

$$H_t^{(i)} = \exp(rt)\left(\sum_{j=1}^{n} c_j^i N_{j,t} - a^{(i)}t\right), \quad i = 2, 3, \ldots.$$

It is not that hard to see that $H_t^{(i)}$, for $i > n + 1$ can be written as a linear combination of the $H_t^{(i)}$, $i = 2, \ldots, n + 1$ (see Léon *et al.* (2002) [19]). In this case, we enlarge the market with only n objects, namely the assets following the price processes $H_t^{(i)}$, $i = 2, \ldots, n + 1$. In order that an equivalent martingale measure Q exists, we must have the existence of a G and H, such that

$$\int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x) = r - cG - a - b$$

$$\int_{-\infty}^{+\infty} x^i H(x)\nu(\mathrm{d}x) = a^{(i)}, \quad i = 2, \ldots, n + 1.$$

The support of H will now be the set $\{c_2, \ldots, c_{n+1}\}$ and the above equations reduce to

$$\sum_{j=1}^{n} c_j H(c_j) a_j = r - cG - b$$

$$\sum_{j=1}^{n} c_j^i H(c_j) a_j = a^{(i)}, \quad i = 2, \ldots, n+1.$$

There exists an equivalent martingale measure if the following system of equations for $H(c_j),\ j = 1, \ldots, n$ has a positive solution, i.e. $H(c_j) > 0,\ j = 1, \ldots, n$.

$$\begin{bmatrix} c_1^2 a_1 & c_2^2 a_2 & \ldots & c_n^2 a_n \\ c_1^3 a_1 & c_2^3 a_2 & \ldots & c_n^3 a_n \\ \ldots & \ldots & \ldots & \ldots \\ c_1^{n+1} a_1 & c_2^{n+1} a_2 & \ldots & c_n^{n+1} a_n \end{bmatrix} \times \begin{bmatrix} H(c_1) \\ H(c_2) \\ \ldots \\ H(c_n) \end{bmatrix} = \begin{bmatrix} a^{(2)} \\ a^{(3)} \\ \ldots \\ a^{(n+1)} \end{bmatrix}.$$

The existence (and uniqueness by Proposition 7) of a positive solution $H(c_j),\ j = 1, \ldots, n$ can be translated into the condition

$$C^{-1} \cdot a' > 0, \tag{8.12}$$

where C^{-1} is the inverse of the Vandermonde matrix

$$C = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ c_1 & c_2 & \ldots & c_n \\ \ldots & \ldots & \ldots & \ldots \\ c_1^{n-1} & c_2^{n-1} & \ldots & c_n^{n-1} \end{bmatrix}$$

and a' is the transpose of $[a^{(2)} \ldots a^{(n+1)}]$. Note that if all of the c_is are different from each other (as we assumed above), that $\det C \neq 0$.

For the calculation of the inverse of Vandermonde matrices, see Graybill (1983) [17] or Macon and Spitzbart (1958) [20], while for other applications of Vandermonde matrices in finance see Norberg (1999) [21].

8.6 OPTIMAL PORTFOLIOS

Definition 9 *A utility function is a mapping $U(x) : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ which is strictly increasing, continuous on $\{U > -\infty\}$, of class C^∞, strictly concave on the interior of $\{U > -\infty\}$ and satisfies*

$$U'(\infty) := \lim_{x \to \infty} U'(x) = 0.$$

Denoting by dom(U) *the interior of $\{U > -\infty\}$, we shall consider only two cases:*

Case 10 *dom$(U) = (0, \infty)$ in which case U satisfies*

$$U'(0) := \lim_{x \to 0^+} U'(x) = \infty.$$

Case 11 *dom*$(U) = \mathbb{R}$ *in which case* U *satisfies*

$$U'(-\infty) := \lim_{x \to -\infty} U'(x) = \infty.$$

Typical examples for Case 10 are the so-called HARA utilities, $U(x) = \frac{x^{1-p}}{1-p}$ for $p \in \mathbb{R}_+ \backslash \{0, 1\}$, and the logarithmic utility $U(x) = \log(x)$. A typical example for Case 11 is $U(x) = -\frac{1}{\alpha} e^{-\alpha x}$.

8.6.1 Optimal wealth

Given an initial wealth w_0 and an utility function U, we want to find the optimal terminal wealth $\mathcal{W}_T$, that is, the value of $\mathcal{W}_T$ that maximizes $E_P(U(\mathcal{W}_T))$. We will consider the optimization problem

$$\max \left\{ E_P(U(\mathcal{W}_T)) : E_Q \left(\frac{\mathcal{W}_T}{B_T} \right) = w_0 \right\}.$$

The corresponding Lagrangian is

$$E_P(U(\mathcal{W}_T)) - \lambda E_Q \left(\frac{\mathcal{W}_T}{B_T} - w_0 \right) = E_P \left(U(\mathcal{W}_T) - \lambda_T \left(\frac{\mathrm{d}Q_T}{\mathrm{d}P_T} \frac{\mathcal{W}_T}{B_T} - w_0 \right) \right).$$

Then, the optimal wealth is given by

$$\mathcal{W}_T = \left(U' \right)^{-1} \left(\frac{\lambda_T}{B_T} \frac{\mathrm{d}Q_T}{\mathrm{d}P_T} \right),$$

where λ_T is the solution of the equation

$$E_Q \left[\frac{1}{B_T} \left(U' \right)^{-1} \left(\frac{\lambda_T}{B_T} \frac{\mathrm{d}Q_T}{\mathrm{d}P_T} \right) \right] = w_0. \tag{8.13}$$

It is easy to check the existence and uniqueness of the optimal wealth from the conditions on U.

From equation (8.5) and under certain conditions on Q (see Corcuera *et al.* (2004b) [11]), we can write:

$$\mathcal{W}_T = \left(U' \right)^{-1} \left(m\,(T)\, S_T^{\frac{G}{c}} e^{V_T} \right),$$

where

$$m\,(t) := \frac{\lambda_t}{B_t} S_0^{-\frac{G}{c}} \exp \left(-\frac{1}{2} G^2 t - \frac{G}{c} \left(a + b - \frac{c^2}{2} \right) t \right.$$
$$\left. + t \int_{-\infty}^{+\infty} \left((\log H(x) - \frac{G}{c} \log(1+x)) H(x) - H(x) + 1 + \frac{G}{c} x \right) \nu(\mathrm{d}x) \right)$$

and

$$V_t = \int_{-\infty}^{+\infty} g(x)(Q((0,t], \mathrm{d}x) - tH(x)\nu(\mathrm{d}x)),$$

with

$$g(x) := \log H(x) - \frac{G}{c}\log(1+x).$$

It can be shown (see Corcuera *et al.* (2004b) [11]), that if we consider HARA and exponential utilities we have that the price process of the optimal portfolio is given by

$$E_Q\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = F(t, S_t, V_t)$$

with

$$F(t, x_1, x_2) := \phi(t,T)\left(U'\right)^{-1}\left(m(t)\, x_1^{\frac{G}{c}} e^{x_2}\right) + \chi(t,T). \tag{8.14}$$

We know that under an equivalent martingale measure Q, which is structure preserving, any random variable $\mathcal{W}_T \in L^2(\Omega, \mathcal{F}_T, Q)$ can be replicated and we have $w_0 = E_Q\left(\frac{\mathcal{W}_T}{B_T}\right)$. Now, by a generalization of equation (8.9) (see Theorem 4 in Corcuera *et al.* (2004b) [11]) we can find the composition of the portfolio with this price process. In fact, we have that the number of stocks and new assets are given, respectively, by

$$\beta_t = \frac{G\phi(t,T)m(t)\, S_{t-}^{\frac{G}{c}-1} e^{V_{t-}}}{cU''((U')^{-1}(m(t)\, S_{t-}^{\frac{G}{c}} e^{V_{t-}}))} = \frac{G\phi(t,T)U'(\mathcal{W}_{t-})}{cS_{t-}U''(\mathcal{W}_{t-})} \tag{8.15}$$

and

$$\beta_t^{(i)} = \frac{\phi(t,T)}{i!B_t}\frac{\partial^i}{\partial y^i}\left(U'\right)^{-1}\left(m(t)\, S_{t-}^{\frac{G}{c}} e^{V_{t-}} H(y)\right)\Bigg|_{y=0}, \quad i = 2, 3, \ldots \tag{8.16}$$

8.6.2 Examples

Example 12 *Consider* $U(x) = \log x$. *Then* $U'(x) = \left(U'\right)^{-1}(x) = \frac{1}{x}$. *Therefore, by solving equation (8.13), we have*

$$\mathcal{W}_T = w_0 B_T \frac{\mathrm{d}P_T}{\mathrm{d}Q_T} = \left(m(T)S_T^{\frac{G}{c}} e^{V_T}\right)^{-1}.$$

Therefore, we have that

$$E_Q\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = w_0 B_t E_Q\left[\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}|\mathcal{F}_t\right] = w_0 B_t \frac{\mathrm{d}P_t}{\mathrm{d}Q_t} = \mathcal{W}_t$$

and the price function of $\mathcal{W}_T$ *at time* t *is*

$$\mathcal{W}_t = \left(m(t)\, S_t^{\frac{G}{c}} e^{V_t}\right)^{-1},$$

that is, the wealth of the optimal portfolio at time t is the optimal terminal wealth for the period $[0,t]$; in other words, $\phi(t,T)=1$ and $\chi(t,T)=0$ in equation (8.14). Now, since $U''(x)=-\frac{1}{x^2}$, if we apply equation (8.15), we have that

$$\frac{\beta_t S_{t-}}{\mathcal{W}_{t-}}=-\frac{G}{c},$$

that is, the relative wealth invested in stocks is constant. From equation (8.16), the number of new assets is

$$\beta_t^{(i)}=\frac{\mathcal{W}_{t-}}{i!B_t}\frac{\partial^i}{\partial y^i}\frac{1}{H(y)}\bigg|_{y=0}, \quad i=2,3,\ldots$$

So, maximization with bonds and stocks corresponds to take

$$H(y)=\frac{1}{1-\frac{G}{c}y}.$$

where G verifies (see equation (8.10))

$$cG+a+b-r+\frac{G}{c}\int_{-\infty}^{+\infty}\frac{y^2}{1-\frac{G}{c}y}\nu(\mathrm{d}y)=0.$$

Example 13 *Consider $U(x)=\frac{x^{1-p}}{1-p}$ with $p\in\mathbb{R}_+\setminus\{0,1\}$. Then, $\left(U'\right)^{-1}(x)=x^{-\frac{1}{p}}$ and by solving equation (8.13) we have*

$$\mathcal{W}_T=w_0B_T\frac{\left(\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}\right)^{\frac{1}{p}}}{E_Q\left(\left(\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}\right)^{\frac{1}{p}}\right)}=\left(m(T)S_T^{\frac{G}{c}}e^{V_T}\right)^{-\frac{1}{p}}.$$

$\{\frac{\mathrm{d}P_t}{\mathrm{d}Q_t}, 0\le t\le T\}$ is a Q-exponential Lévy process (see Corcuera et al. (2005) [11]), and then

$$\begin{aligned}E_Q\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right]&=w_0B_t\frac{E_Q\left[\left(\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}\right)^{\frac{1}{p}}|\mathcal{F}_t\right]}{E_Q\left(\left(\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}\right)^{\frac{1}{p}}\right)}\\&=w_0B_t\frac{E_Q\left[\left(\frac{\mathrm{d}P_{T,t}}{\mathrm{d}Q_{T,t}}\right)^{\frac{1}{p}}\left(\frac{\mathrm{d}P_t}{\mathrm{d}Q_t}\right)^{\frac{1}{p}}|\mathcal{F}_t\right]}{E_Q\left(\left(\frac{\mathrm{d}P_{T,t}}{\mathrm{d}Q_{T,t}}\right)^{\frac{1}{p}}\left(\frac{\mathrm{d}P_t}{\mathrm{d}Q_t}\right)^{\frac{1}{p}}\right)}\\&=w_0B_t\frac{\left(\frac{\mathrm{d}P_t}{\mathrm{d}Q_t}\right)^{\frac{1}{p}}}{E_Q\left(\left(\frac{\mathrm{d}P_t}{\mathrm{d}Q_t}\right)^{\frac{1}{p}}\right)}=\mathcal{W}_t\end{aligned}$$

where $\frac{\mathrm{d}P_{T,t}}{\mathrm{d}Q_{T,t}} = \frac{\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}}{\frac{\mathrm{d}P_t}{\mathrm{d}Q_t}}$. *That is, again the wealth of the optimal portfolio at time t is the optimal terminal wealth for the period* $[0, t]$, *and* $\phi(t, T) = 1$ *and* $\chi(t, T) = 0$ *in equation (8.14). Now, since* $U'(x) = x^{-p}$ *and* $U''(x) = -px^{-p-1}$, *if we apply equation (8.15), we have that*

$$\frac{\beta_t S_{t-}}{\mathcal{W}_{t-}} = -\frac{G}{cp};$$

and by equation (8.16) the number of new assets is given by

$$\beta_t^{(i)} = \frac{\mathcal{W}_{t-}}{i!B_t} \frac{\partial^i}{\partial y^i} H(y)^{-\frac{1}{p}} \bigg|_{y=0}, \quad i = 2, 3, \ldots$$

So, we will have an optimal portfolio only based in bonds and stocks, if and only if,

$$H(y) = \frac{1}{\left(1 - \frac{G}{cp} y\right)^p},$$

where G verifies

$$cG + a + b - r + \int_{-\infty}^{\infty} y \left(\frac{1}{\left(1 - \frac{G}{cp} y\right)^p} - 1 \right) \nu(\mathrm{d}y) = 0.$$

Example 14 *Consider the exponential utility function*

$$U(x) = -\frac{1}{\alpha} e^{-\alpha x}$$

with $\alpha \in (0, \infty)$. *Then,* $\left(U'\right)^{-1}(x) = -\frac{1}{\alpha}\log x$ *and by solving equation (8.13) we have*

$$\mathcal{W}_T = w_0 B_T + \frac{1}{\alpha}\left(\log \frac{\mathrm{d}P_T}{\mathrm{d}Q_T} - E_Q\left(\log \frac{\mathrm{d}P_T}{\mathrm{d}Q_T}\right)\right) = -\frac{1}{\alpha}\log\left(m(T) S_T^{\frac{G}{c}} e^{V_T}\right)$$

Note, that in this case, $\mathcal{W}_{t-}$ *is not bounded by below and that there arises the problem of the admissibility of this optimal portfolio (see Kallsen (2000) [18]). In addition, we have that*

$$\begin{aligned}
E_Q\left[\frac{B_t}{B_T}\mathcal{W}_T | \mathcal{F}_t\right] &= w_0 B_t + \frac{B_t}{\alpha B_T}\left(E_Q\left[\log \frac{\mathrm{d}P_T}{\mathrm{d}Q_T} | \mathcal{F}_t\right] - E_Q\left(\log \frac{\mathrm{d}P_T}{\mathrm{d}Q_T}\right)\right) \\
&= w_0 B_t + \frac{B_t}{\alpha B_T}\left(E_Q\left[\log \frac{\mathrm{d}P_{T,t}}{\mathrm{d}Q_{T,t}} | \mathcal{F}_t\right] - E_Q\left(\log \frac{\mathrm{d}P_{T,t}}{\mathrm{d}Q_{T,t}}\right)\right. \\
&\quad \left. + \log \frac{\mathrm{d}P_t}{\mathrm{d}Q_t} - E_Q\left(\log \frac{\mathrm{d}P_t}{\mathrm{d}Q_t}\right)\right) \\
&= w_0 B_t + \frac{B_t}{\alpha B_T}\left(\log \frac{\mathrm{d}P_t}{\mathrm{d}Q_t} - E_Q\left(\log \frac{\mathrm{d}P_t}{\mathrm{d}Q_t}\right)\right) \\
&= \frac{B_t}{B_T}\mathcal{W}_t + w_0 B_t\left(1 - \frac{B_t}{B_T}\right).
\end{aligned}$$

Therefore, in this case $\phi(t,T) = \frac{B_t}{B_T}$ *and* $\chi(t,T) = w_0 B_t (1 - \frac{B_t}{B_T})$ *in equation (8.14). Now, since* $U'(x) = e^{-\alpha x}$ *and* $U''(x) = -\alpha e^{-\alpha x}$*, if we apply equation (8.15), we have that*

$$\frac{B_T}{B_t}\beta_t S_{t-} = -\frac{G}{c\alpha},$$

that is, the forward value of the wealth invested in stocks is constant. From equation (8.16), the number of new assets is constant:

$$\beta_t^{(i)} = -\frac{B_T}{i!\alpha}\,\frac{\partial^i}{\partial y^i}\log H(y)\bigg|_{y=0}, \quad i = 2, 3, \ldots$$

and we obtain the optimal portfolio based only in stocks and bonds by taking

$$H(y) = \exp\left(\frac{G}{c}y\right),$$

with G verifying

$$cG + a + b - r + \int_{-\infty}^{\infty} y\left(\exp\left(\frac{G}{c}y\right) - 1\right)\nu(\mathrm{d}y) = 0.$$

The corresponding martingale measure is then the Esscher measure (see Chan (1999) [9]).

Example 15 *Consider the quadratic utility*

$$U(x) = \gamma x - \frac{x^2}{2}, \quad x < \gamma$$

and then $(U')^{-1}(x) = \gamma - x$*, and by solving equation (8.13) we have*

$$\mathcal{W}_T = \gamma - (\gamma - w_0 B_T)\frac{\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}}{E_Q(\frac{\mathrm{d}P_T}{\mathrm{d}Q_T})} = \gamma - m(T) S_T^{\frac{G}{c}} e^{V_T}$$

and so

$$E_Q\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = \frac{B_t}{B_T}\gamma - \frac{B_t}{B_T}(\gamma - w_0 B_T)\frac{\frac{\mathrm{d}P_t}{\mathrm{d}Q_t}}{E_Q\left(\frac{\mathrm{d}P_t}{\mathrm{d}Q_t}\right)}$$

$$= \frac{B_t}{B_T}\gamma\left(1 - \frac{\gamma - w_0 B_T}{\gamma - w_0 B_t}\right) + \frac{B_t}{B_T}\frac{\gamma - w_0 B_T}{\gamma - w_0 B_t}\mathcal{W}_t$$

Therefore, in this case $\phi(t,T) = \frac{B_t}{B_T}\frac{\gamma - w_0 B_T}{\gamma - w_0 B_t}$ *and* $\chi(t,T) = \frac{B_t}{B_T}\gamma(1 - \frac{\gamma - w_0 B_T}{\gamma - w_0 B_t})$ *in equation (8.14). Now, since* $U'(x) = \gamma - x$ *and* $U''(x) = -1$ *if we apply equation (8.15), we have that*

$$\frac{B_T}{B_t}\frac{(\gamma - w_0 B_t)\beta_t S_{t-}}{(\gamma - w_0 B_T)(\gamma - \mathcal{W}_{t-})} = -\frac{G}{c},$$

the number of new assets is:

$$\beta_t^{(i)} = \frac{(\mathcal{W}_{t-} - \gamma)(\gamma - w_0 B_T)}{i! B_T(\gamma - w_0 B_t)} \frac{\partial^i}{\partial y^i} H(y)\bigg|_{y=0}, \ i = 2, 3, \ldots$$

and we obtain the optimal portfolio based only in stocks and bonds by taking

$$H(y) = 1 + \frac{G}{c} y.$$

with

$$G = \frac{(r - a - b)c}{c^2 + m_2}.$$

The corresponding martingale measure is the so-called minimal martingale measure *(see Chan (1999) [9]). Even though the quadratic utility is not a proper utility in the sense defined above, since this is decreasing for $x > \gamma$, it is interesting since the solution of the optimal problem with this utility is the same as that of the solution of* the mean-variance portfolio problem *if we choose*

$$\gamma = \frac{w_0((1+\rho)E_Q(\xi_T) - (1+r))}{E_Q(\xi_T) - 1}$$

where $\rho > r$ is a specified return (see Pliska (1997) [24]).

REFERENCES

[1] Ahiezer, N.I. (1965), *The Classical Moment Problem*, Hafner Publishing Company, New York, NY, USA.

[2] Balland, P. (2002), "Deterministic implied volatility models", *Quantitative Finance*, **2**, 31–44.

[3] Bertoin, J. (1996), *Lévy Processes*, Cambridge Tracts in Mathematics, **121**, Cambridge University Press, Cambridge, UK.

[4] Björk, T., Di Masi, G., Kabanov, Y. and Runggaldier, W. (1997), "Towards a general theory of bond markets", *Finance and Stochastics* **1**, 141–174.

[5] Black, F. and Scholes, M. (1973), "The pricing of options and corporate liabilities", *Journal of Political Economy*, **81**, 637–654.

[6] Carr, P. and Lewis, K. (2004), "Corridor variance swaps", *Risk*, **17**(2), 67–72.

[7] Carr, P. and Madan, D. (1998), "Towards a theory of volatility trading", in R. Jarrow (Ed.), *Volatility*, Risk Publications, London, UK, pp 417–427. Reprinted (2001) in M. Musiela, E. Jouini and J. Cvitanic (Eds), *Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, Cambridge, UK, pp. 458–476.

[8] Carr, P., Geman, H., Madan, D.H. and Yor, M. (2002), *Pricing Options on Realized Variance*, Prépublications du Laboratoire de Probabilités et Modèles Aléatoires, **768**, Universités de Paris 6 and Paris 7, Paris, France.

[9] Chan, T. (1999), "Pricing contingent claims on stocks driven by Lévy processes", *Annals of Applied Probability*, **9**, 504–528 (1999).

[10] Corcuera, J.M., Nualart, D. and Schoutens, W. (2005), "Completion of a Lévy market by power-jump assets", *Finance and Stochastics*, **9**, 109–127.

[11] Corcuera, J.M., Guerra J., Nualart, D. and Schoutens, W. (2004), "Optimal investment in a Lévy market", Preprint Institut de Matemática de la Universitat de Barcelona 249, Universitat de Barcelona, Barcelona, Spain.

[12] Cont, R. and Tankov, P. (2004), *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Press, Boca Raton, FL, USA.

[13] Davis, M. (1997), "Option pricing in incomplete markets", in M.A.H. Dempster and S.R. Pliska (Eds), *Mathematics of Derivative Securities*, Cambridge University Press, Cambridge, UK, pp. 227–254.

[14] Delbaen, F. and Schachermayer, W. (1994), "A general version of the fundamental theorem of asset pricing", *Mathematische Annalen*, **300**, 463–520.

[15] Demeterfi, K., Derman, E., Kamal, M. and Zhou, J. (1999), "A guide to volatility and variance swaps", *The Journal of Derivatives*, **6**(4), 9–32.

[16] Eberlein, E. and Jacod, J. (1997), "On the range of option prices", *Finance and Stochastics*, **1**, 131–140.

[17] Graybill, A. (1993), *Matrices with Applications to Statistics*, 2nd Edn, Wadsworth, Belmont, CA, USA.

[18] Kallsen, J. (2000), "Optimal portfolios for exponential Lévy processes", *Mathematical Methods of Operations Research*, **51**, 357–374.

[19] Léon, J.A., Vives, J., Utzet, F. and Solé, J.L. (2002), "On Lévy processes, Malliavin calculus and market models with jumps", *Finance and Stochastics*, **6**, 197–225.

[20] Macon, N. and Spitzbart, A. (1958), "Inverses of Vandermonde matrices", *American Mathematics Monthly*, **65**, 95–100.

[21] Norberg, R. (1999), On the Vandermonde matrix and its role in mathematical finance *Working paper*, Laboratory of Actuarial Mathematics (working paper), University of Copenhagen, Denmark.

[22] Nualart, D. and Schoutens W. (2000), "Chaotic and predictable representations for Lévy processes", *Stochastic Processes and their Applications*, **90**, 109–122.

[23] Nualart, D. and Schoutens W. (2001), "Backwards stochastic differential equations and Feynman–Kac formula for Lévy processes, with applications in finance", *Bernoulli*, **7**, 761–776.

[24] Pliska, S. (1997), *Introduction to Mathematical Finance*, Blackwell, Oxford, UK.

[25] Protter, Ph. (1990), *Stochastic Integration and Differential Equations*, Springer-Verlag, Berlin, Germany.

[26] Raible, S. (2000), "Lévy process in finance: theory, numerics, and empirical facts", *PhD Thesis*, University of Freiburg, Freiburg, Germany.

[27] Sato, K. (2000), *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, **68**, Cambridge University Press, Cambridge, UK.

[28] Schachermayer, W. (2001), "Optimal investment in incomplete financial markets", in H. Geman, D. Madan, S. Pliska and T. Vorst (Eds), *Mathematical Finance – Bachelier Congress 2000*, Springer-Verlag, Berlin, Germany, pp. 427–462.

[29] Schoutens, W. (2003), *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley, Chichester, UK.

[30] Shohat, J.A and Tamarkin, J.D. (1950), *The Problem of Moments*, Mathematical Surveys No. 1, American Mathematical Society, New York, NY, USA.

9

Pricing Perpetual American Options Driven by Spectrally One-sided Lévy Processes†

Terence Chan
Heriot-Watt University, Edinburgh, UK

Abstract

This paper considers the problem of pricing perpetual American put options on stocks whose price process is the exponential of a Lévy process (i.e. a process with stationary independent increments). When the price process has no negative jumps, the problem reduces to one of finding the law of a relevant first-passage time of the process, a problem which has already been well-studied. However, if the price process does have negative jumps, the problem is much more delicate as it involves finding the joint law of a first-passage time and position of the process at that time. This problem is only mathematically tractable under the assumption that the price process has no positive jumps. A renewal equation for the price is obtained in the case where the jump component of the Lévy process has finite variation. In the general case, a simple explicit formula is obtained for the optimal exercise boundary and a formula amenable to efficient numerical computation is obtained for the price of a perpetual put.

9.1 INTRODUCTION

Consider a (non dividend-paying) stock whose price at time t, S_t, is modelled as $S_t = S_0 \exp\{-Y_t\}$, where Y_t is a Lévy process (process with independent stationary increments) of the form

$$Y_t = \sigma B_t + X_t + ct, \quad Y_0 = 0, \tag{9.1.1}$$

where B_t is a standard Brownian motion and X_t is a jump process with stationary independent increments. We shall suppose that the market is already risk-neutral, so that for some discount factor $\delta > 0$, $e^{-\delta t} S_t$ is a martingale. Of course, such a model is incomplete: there are many equivalent martingale measures and contingent claims cannot be hedged perfectly. However, the purpose of this paper is not to address the problems associated with incompleteness of the market in this model; in particular, it does not deal with the question of how to choose a suitable equivalent martingale measure from the infinitely many available – this problem has been studied in, for example, Chan (1999) and the references cited there. Instead, the present article is concerned with the next step in pricing a contingent claim, namely, once an equivalent martingale measure has been chosen, how to calculate the expected payoff with respect to the chosen martingale measure. It is shown in Chan

† This paper was submitted at the special invitation of the editors. Please see Epilogue on page 215 for details.

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

(1999) that for this model, a Lévy process under the original measure remains a Lévy process under any equivalent martingale measure. We may therefore assume that $e^{-\delta t}S_t$ is already a martingale. (However, in the context of perpetual options considered here, it is perhaps worth bearing in mind that while an equivalent martingale measure is equivalent to the actual underlying probability measure over every finite time interval $[0, T]$, these two measures are mutually singular over $[0, \infty)$.)

In this paper, we consider the problem of pricing perpetual American options. These were first studied by Samuelson (1965), in relation to call options. In recent years, there has been renewed interest in this problem: some recent work includes Gerber and Shiu (1994, 1998) and Gerber and Landry (1998). While the implication of Samuelson (1965) seems to be that a few perpetual warrants (i.e. call options) once did exist, the main interest in perpetual put options in recent years arises from their relative mathematical tractability compared with finitely dated puts and the usefulness of the former in pricing the latter. The problem of pricing an American put option with a finite maturity presents many difficult mathematical problems because the optimal exercise level depends on the time to maturity and there are no reasonable models for which an explicit pricing formula is known. The article by Gerber and Shiu (1994) contains a long list of references to the literature on American options with finite maturity. The problem of pricing perpetual American puts is a much more mathematically tractable problem because the optimal exercise level is constant in time. The price of a perpetual American option with payoff function $\Pi(s)$ (under the equivalent martingale measure) is simply $\sup_\tau \mathbb{E}[e^{-\delta\tau}\Pi(S_\tau)]$ (in the case of payoffs depending only on the stock price at the time of exercise), where the supremum is taken over all stopping times τ.

The motivation for studying perpetual American options is three-fold. First, at the most obvious level, perpetual options can be thought of as approximations to options with a long time until expiry. Secondly, the mathematics associated with perpetual options have some applications to the ruin problem for certain generalizations of the classical model in ruin theory. Some recent work that have explored the connection between perpetual American options and ruin theory are Gerber and Landry (1998) and Gerber and Shiu (1997). The third and arguably most compelling motivation for studying perpetual American (put) options lies in their applications to numerical methods for pricing American puts with finite maturities (not necessarily long-dated). Various numerical approximation schemes developed in recent years involve evaluating perpetual options. MacMillan (1986) and Zhang (1995) consider the difference between the prices of a finite-maturity American option and a European option and show that under a suitable discretization, this difference can be interpreted as the price of a perpetual American option. Even more strikingly, Carr (1998) presents a recursive algorithm which involves calculating the prices of a sequence of American puts (with different payoffs) expiring at a sequence of independent exponential times. The key observation behind this idea of Carr is that the memoryless property of the exponential distribution reduces the problem with a random exponential time horizon to one with an infinite time horizon and an adjusted discount factor and payoff. To see this, let $T^{(\lambda)}$ be an exponential time with rate λ, independent of Y; then the price of an American option expiring at $T^{(\lambda)}$ is $\sup_\tau \mathbb{E}[e^{-\delta(\tau\wedge T^{(\lambda)})}\Pi(S_{\tau\wedge T^{(\lambda)}})]$ (where $x \wedge y = \min(x, y)$). Writing $\Pi(S_t) = \Pi(Y_t)$ for notational convenience, define $U(y) = \mathbb{E}^y[e^{-\delta T^{(\lambda)}}\Pi(Y_{T^{(\lambda)}})]$ (where $\mathbb{E}^y$ denotes expectation given $Y_0 = y$). The strong Markov property at stopping time τ says that $\tilde{Y}_s = Y_{s+\tau} - Y_\tau$ is independent of $\{Y_u : u \leq \tau\}$ and τ. Hence we may calculate as follows:

$$\mathbb{E}^y[e^{-\delta(\tau\wedge T^{(\lambda)})}\Pi(Y_{\tau\wedge T^{(\lambda)}})]$$

$$
\begin{aligned}
&= \mathbb{E}^y[e^{-\delta\tau}\Pi(Y_\tau)\mathbf{1}_{\{\tau<T^{(\lambda)}\}}] + \mathbb{E}^y[e^{-\delta T^{(\lambda)}}\Pi(Y_{T^{(\lambda)}})\mathbf{1}_{\{\tau\geq T^{(\lambda)}\}}] \\
&= \mathbb{E}^y[e^{-(\delta+\lambda)\tau}\Pi(Y_\tau)] + U(y) - \mathbb{E}^y\left[\int_\tau^\infty \lambda e^{-(\delta+\lambda)t}\Pi(Y_t)\,dt\right] \\
&= \mathbb{E}^y[e^{-(\delta+\lambda)\tau}\Pi(Y_\tau)] + U(y) - \mathbb{E}^y\left[\int_0^\infty \lambda e^{-(\delta+\lambda)(t+\tau)}\Pi(Y_{t+\tau})\,dt\right] \\
&= \mathbb{E}^y[e^{-(\delta+\lambda)\tau}\Pi(Y_\tau)] + U(y) - \mathbb{E}\left\{e^{-(\delta+\lambda)\tau}\mathbb{E}^{y+Y_\tau}\left[\int_0^\infty \lambda e^{-(\delta+\lambda)t}\Pi(\tilde{Y}_t)\,dt\right]\right\} \\
&= \mathbb{E}^y[e^{-(\delta+\lambda)\tau}\Pi(Y_\tau)] + U(y) - \mathbb{E}^y[e^{-(\delta+\lambda)\tau}U(Y_\tau)].
\end{aligned}
$$

Thus, the problem is equivalent to that of pricing a perpetual option with discount factor $\delta + \lambda$ and payoff $\Pi(y) - U(y)$.

We shall consider only options whose payoffs are bounded functions of the stock price at the time of exercise only (i.e. no path-dependent options). Consider first options of 'put' type, that is, those whose payoff $\Pi(s)$ is a decreasing function of the stock price s at the time of exercise. For such options, it is *a priori* clear that the optimal exercise time is of the form $\tau = \tau_L = \inf\{t \geq 0 : S_t \leq L\}$ for some L.

If Y_t has no positive jumps (so S_t has no downward jumps), we have $S_{\tau_L} = L$ and so finding the price of the option $\mathbb{E}[e^{-\delta\tau_L}\Pi(S_{\tau_L})] = \mathbb{E}[e^{-\delta\tau_L}\Pi(L)]$ reduces to finding the Laplace transform $\mathbb{E}[e^{-\delta\tau_L}]$ of the first-passage time $\tau_L = \inf\{t \geq 0 : Y_t \geq \log(S_0/L)\}$ of Y. Because $Y_{\tau_L} = \log(S_0/L)$, this problem is easily solved by an optional stopping argument on the martingale $\exp\{\lambda Y_t - \psi(\lambda)t\}$, where $\psi(\lambda) = \log \mathbb{E}[e^{\lambda Y_1}]$. A complete account of this problem is already given in Gerber and Shiu (1994) and we shall say no more on this situation in this paper.

On the other hand, if Y does have positive jumps, we no longer have $Y_{\tau_L} = \log(S_0/L)$ and so to find the option price $\mathbb{E}[e^{-\delta\tau_L}\Pi(S_{\tau_L})]$, we need the joint law of τ_L and Y_{τ_L}. This is a much more difficult problem and in general, no useful result can be obtained unless Y has no negative jumps. Gerber and Landry (1998) considered the case where the jump component X of Y is a compound Poisson process with only positive jumps. They derived a renewal equation for $V(S_0, L) = \mathbb{E}[e^{-\delta\tau_L}\Pi(S_{\tau_L})]$ and from this obtained a formula for the optimal exercise level L. Gerber and Shiu (1998) considered the same problem without the Brownian component and obtained similar results (although this is a special case of the situation considered in Gerber and Landry (1998), and a slightly different analysis is required).

The situation for options of 'call' type (those whose payoff $\Pi(s)$ is an increasing function of s) is the exact reverse: the case that S_t has no upward jumps can be handled easily by using an optional stopping argument, exactly as described in Gerber and Shiu (1994), while the case that S_t has no downward jumps can be handled using the methods presented here simply by reversing the sign of $\log S_t$: thus, $S_t = S_0 e^{Y_t}$, where Y_t is a Lévy process with no negative jumps, and the optimal time of exercise is a stopping time of the form $\tau_L = \inf\{t \geq 0 : S_t \geq L\}$. Since the mathematical problem of pricing calls on stocks whose price has no negative jumps is the same as that of pricing puts on stocks whose price has no positive jumps, and since in any case perpetual puts have more practical relevance, in this paper we shall consider only the latter problem: specifically, pricing perpetual options of 'put' type on stocks whose price $S_t = S_0 \exp\{-Y_t\}$ is driven by a

Lévy process Y of the form shown in equation (9.1.1), whose jump component X has no negative jumps. In the case that the jump component X has finite variation, we derive a renewal equation for $V(S_0, L) = \mathbb{E}[e^{-\delta\tau_L}\Pi(S_{\tau_L})]$, similar to that obtained by Gerber and Landry (1998) (see Section 9.4). While this is merely an easy extension of the results of Gerber and Landry (1998), the main contribution of this paper (see Section 9.5) consists in obtaining a simple explicit formula for the optimal exercise level and a formula amenable to efficient numerical computation for the price of a perpetual put option with payoff $\Pi(s) = (K - s)_+$, which applies to a general Lévy process Y_t with no negative jumps, whether or not the jump component has finite variation. This approach also yields as a by-product an explicit formula for the solution to the renewal equation for V obtained in Section 9.4.

9.2 FIRST-PASSAGE DISTRIBUTIONS AND OTHER RESULTS FOR SPECTRALLY POSITIVE LÉVY PROCESSES

In this section, we collect together some general results about Lévy processes which will be relied on heavily in subsequent sections.

We begin by quoting some basic formulae and properties of Lévy processes which will be referred to frequently in the sequel. The main reference here is Bertoin (1996) to which the reader is referred for further details.

A Lévy process Y_t is simply a process with stationary and independent increments: in other words, $Y_{s+t} - Y_s$ is independent of $\{Y_u : u \leq s\}$ and has the same distribution as $Y_t - Y_0$. Throughout this paper, we adopt the convention that all Lévy processes are right-continuous with left limits. We further suppose that $Y_0 = 0$ (if we require a process to start at any other point y, we shall use $y + Y_t$).

Since Y has stationary independent increments and $Y_0 = 0$, its characteristic function must take the form

$$\mathbb{E}[e^{-i\theta Y_t}] = e^{t\tilde{\psi}(\theta)} \tag{9.2.1}$$

for some function $\tilde{\psi}$, called the *Lévy exponent* of Y. The Lévy–Khintchine formula says that

$$\tilde{\psi}(\theta) = -\frac{\sigma^2\theta^2}{2} - i\tilde{b}\theta - \int_{\{|x|<1\}} (1 - e^{-i\theta x} - i\theta x)\,\nu(dx) - \int_{\{|x|\geq 1\}} (1 - e^{-i\theta x})\,\nu(dx) \tag{9.2.2}$$

for $\tilde{b},\ \sigma \in \mathbb{R}$ and for some σ-finite measure ν on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int \min(1, x^2)\,\nu(dx) < \infty. \tag{9.2.3}$$

The measure ν is called the *Lévy measure* of Y. The Lévy–Khintchine formula is intimately connected to the structure of the process Y itself; from the Lévy–Khintchine formula we can deduce that Y must be a linear combination of a Brownian motion and a jump process X with stationary independent increments which is independent of the Brownian motion. It is convenient to write $\tilde{b} = b + c$ and re-write equation (9.2.2) as

$$\tilde{\psi}(\theta) = -\frac{\sigma^2\theta^2}{2} - ib\theta - \int_{\{|x|<1\}} (1 - e^{-i\theta x} - i\theta x)\,\nu(dx) - \int_{\{|x|\geq 1\}} (1 - e^{-i\theta x})\,\nu(dx) - ic\theta$$

$$= -\frac{\sigma^2\theta^2}{2} + \tilde{\psi}_X(\theta) - ic\theta \tag{9.2.4}$$

where

$$\tilde{\psi}_X(\theta) = \log \mathbb{E}[e^{-i\theta X_1}] = -ib\theta - \int_{\{|x|<1\}} (1 - e^{-i\theta x} - i\theta x)\,\nu(dx) - \int_{\{|x|\geq 1\}} (1 - e^{-i\theta x})\,\nu(dx) \tag{9.2.5}$$

is the Lévy exponent of the jump component X of Y. Thus, from equation (9.2.4) we see that

$$Y_t = \sigma B_t + X_t + ct \tag{9.2.6}$$

where B is a standard Brownian motion and X is a Lévy process without Brownian component, independent of B. For the construction of such a process X, see Bertoin (1996). The reason we have chosen to consider a linear term $c\theta$ in equation (9.2.4) separately from the linear term $b\theta$ associated with the process X is because it is often convenient first to choose X – which would fix b – and then to choose c so as to make $e^{-\delta t - Y_t}$ a martingale.

In this paper, we consider only Lévy processes which have no negative jumps. A Lévy process with no negative (respectively positive) jumps is said to be *spectrally positive* (respectively *spectrally negative*). Note that for spectrally positive (respectively negative) processes, $\nu(-\infty, 0) = 0$ (respectively $\nu(0, \infty) = 0$). A process which is either spectrally positive or spectrally negative is said to be *spectrally one-sided* or *completely asymmetric*.

Henceforth, all Lévy processes Y will be assumed to satisfy the following conditions.

Assumption 9.2.1 *Y_t is a Lévy process which*

(i) is spectrally positive;
(ii) is not non-decreasing (i.e. not a subordinator).

In the sequel, we shall require $e^{-\delta t - Y_t}$ to be a martingale, in which case Y_t clearly cannot be non-decreasing, and so Assumption 9.2.1(ii) is no real restriction at all.

Because the jumps of a spectrally positive process are all positive, the Laplace transform $\mathbb{E}[e^{-\theta Y_t}]$ exists for all $\theta \geq 0$ and

$$\mathbb{E}[e^{-\theta Y_t}] = e^{t\psi(\theta)} \tag{9.2.7}$$

for some function ψ, which we shall refer to as the *Laplace exponent*. The Lévy–Khintchine formula for the Laplace exponent takes the form

$$\psi(\theta) = \frac{\sigma^2\theta^2}{2} - b\theta - \int_0^1 (1 - e^{-\theta x} - \theta x)\,\nu(dx) - \int_1^\infty (1 - e^{-\theta x})\,\nu(dx) - c\theta$$

$$= \frac{\sigma^2\theta^2}{2} + \psi_X(\theta) - c\theta \tag{9.2.8}$$

where

$$\psi_X(\theta) = \log \mathbb{E}[e^{-\theta X_1}] = -b\theta - \int_0^1 (1 - e^{-\theta x} - \theta x)\,\nu(dx) - \int_1^\infty (1 - e^{-\theta x})\,\nu(dx) \tag{9.2.9}$$

is the Laplace exponent of the jump component X of Y.

A Lévy process has finite variation, if and only if, it has no Brownian component (i.e. $\sigma = 0$) and

$$\int_0^1 x\,\nu(dx) < \infty. \tag{9.2.10}$$

In this case, we shall always choose $b = \int_0^1 x\,\nu(dx)$ in (9.2.9) and write the Laplace exponent of X as

$$\psi_X(\theta) = -\int_0^\infty (1 - e^{-\theta x})\,\nu(dx). \tag{9.2.11}$$

We next present some results concerning the first-passage time distribution of spectrally positive Lévy processes. The main reference here is Bingham (1975).

It is easy to see that the Laplace exponent ψ is convex and so the equation

$$\psi(\theta) = 0 \tag{9.2.12}$$

has at most two solutions. From equation (9.2.8), we see that $\theta = 0$ is always a solution: let ϕ_0 denote the largest solution to equation (9.2.12); let $\mu = \mathbb{E}(Y_1)$ – note that $-\infty < \mu \leq \infty$ – then, $\phi_0 = 0$ if $\mu \leq 0$ while $\phi_0 > 0$ if $\mu > 0$. Moreover, $\psi(\theta) \leq 0$ for $\theta \in [0, \phi_0]$ while ψ is increasing over the interval $[\phi_0, \infty)$; therefore, ψ has a unique continuous inverse $\phi(\theta) \geq \phi_0$ which is defined for $\theta \geq 0$ and satisfies $\psi(\phi(\theta)) = \theta$ for all θ and $\phi(\psi(\theta)) = \theta$ for $\theta \geq \phi_0$.

For $x \geq 0$, let

$$\tilde{T}_x = \inf\{t \geq 0 : Y_t > x\}. \tag{9.2.13}$$

The problem of pricing perpetual American options reduces to that of finding the joint law of $\tilde{T}_x$ and $Y(\tilde{T}_x)$. The main result we shall rely on is the following due to Bingham (1975) (Theorem 6b of that paper).

Theorem 9.2.1 *Suppose that Assumption 9.2.1 holds. Then the resolvent measure*

$$r_\theta(dx) = \int_0^\infty e^{-\theta t}\mathbb{P}(Y_t \in dx)\,dt$$

is absolutely continuous with respect to Lebesgue measure and the Laplace transform of its density function $r_\theta(x)$ is given by

$$\int_0^\infty e^{-\lambda x} r_\theta(x)\,dx = \frac{1}{\theta - \psi(\lambda)} - \frac{\phi'(\theta)}{\phi(\theta) - \lambda}. \tag{9.2.14}$$

The joint law of $\tilde{T}_x$ and $Y(\tilde{T}_x)$ is given by

$$\mathbb{E}[e^{-(\theta \tilde{T}_x + \eta Y(\tilde{T}_x))}] = \frac{\theta - \psi(\eta)}{\phi(\theta) - \eta} e^{-\eta x} r_\theta(x) + (\theta - \psi(\eta)) \int_x^\infty e^{-\eta z} r_\theta(z)\, dz \qquad (9.2.15)$$

for $\theta, \eta \geq 0$.

Finally, we shall need some results concerning the asymptotic behaviour of $\psi(\theta)$. While the results in Lemma 9.2.2 below are quite well-known in the Lévy processes literature and are by no means new, it is difficult to find a single self-contained reference for them. For the sake of completeness, we include a proof here.

Lemma 9.2.2 *Under Assumption 9.2.1,*

(i) if Y has finite variation then

$$\lim_{\theta \to \infty} \frac{\psi(\theta)}{\theta} = -c, \qquad (9.2.16)$$

while if Y has infinite variation then

$$\lim_{\theta \to \infty} \frac{\psi(\theta)}{\theta} = \infty; \qquad (9.2.17)$$

(ii) for any spectrally positive process Y,

$$\lim_{\theta \to \infty} \frac{\psi(\theta)}{\theta^2} = \frac{\sigma^2}{2}. \qquad (9.2.18)$$

Proof. (i) Suppose that Y has finite variation, so that equation (9.2.10) holds and $\sigma = 0$. Moreover, ψ_X is given by equation (9.2.11). An integration by parts shows that

$$\infty > \int_0^1 x\, \nu(dx) = \lim_{x \to 0} x\nu(x, 1] + \int_0^1 \nu(x, 1]\, dx, \qquad (9.2.19)$$

so that $\int_0^1 \nu(x, 1]\, dx < \infty$, which in turn implies that $\lim_{x \to 0} x\nu(x, 1] = 0$. Integrating by parts and noting that $1 - e^{-\theta x} \sim \theta x$ for small x gives

$$\begin{aligned} \psi(\theta) &= -\int_0^\infty (1 - e^{-\theta x})\, \nu(dx) - c\theta \\ &= -\theta \int_0^1 e^{-\theta x} \nu(x, 1]\, dx - \int_1^\infty (1 - e^{-\theta x})\, \nu(dx) - c\theta. \qquad (9.2.20) \end{aligned}$$

Since $\int_0^1 e^{-\theta x} \nu(x, 1]\, dx \to 0$ as $\theta \to \infty$ and $\int_1^\infty (1 - e^{-\theta x})\, \nu(dx)$ remains bounded as $\theta \to \infty$, the result shown in equation (9.2.16) follows.

On the other hand, if Y has infinite variation, then the integral in equation (9.2.19) diverges, and so either $\int_0^1 \nu(x,1]\,dx = \infty$, or $\lim_{x\to 0} x\nu(x,1] = \infty$, which also implies that $\int_0^1 \nu(x,1]\,dx = \infty$. Integrating by parts as before gives

$$\psi(\theta) = \frac{\sigma^2\theta^2}{2} - b\theta - \int_0^1 (1 - e^{-\theta x} - \theta x)\,\nu(dx) - \int_1^\infty (1 - e^{-\theta x})\,\nu(dx) - c\theta$$

$$= \frac{\sigma^2\theta^2}{2} - \theta \int_0^1 (e^{-\theta x} - 1)\nu(x,1]\,dx - \int_1^\infty (1 - e^{-\theta x})\,\nu(dx) + O(\theta). \quad (9.2.21)$$

The same argument as before, only this time $\int_0^1 (e^{-\theta x} - 1)\nu(x,1]\,dx \to \infty$ as $\theta \to \infty$, shows that equation (9.2.17) holds.
(ii) The same sort of argument as used for equation (9.2.19), but this time integrating by parts twice, shows that

$$\infty > \int_0^1 x^2\,\nu(dx) = 2\int_0^1 \int_x^1 \nu(y,1]\,dy\,dx.$$

Hence, integrating by parts once more at equation (9.2.21) gives

$$\psi(\theta) = \frac{\sigma^2\theta^2}{2} - \theta \int_0^1 (e^{-\theta x} - 1)\nu(x,1]\,dx - \int_1^\infty (1 - e^{-\theta x})\,\nu(dx) + O(\theta)$$

$$= \frac{\sigma^2\theta^2}{2} + \theta^2 \int_0^1 e^{-\theta x} \int_x^1 \nu(y,1]\,dy\,dx - \int_1^\infty (1 - e^{-\theta x})\,\nu(dx) + O(\theta). \quad (9.2.22)$$

Since $\int_0^1 e^{-\theta x} \int_x^1 \nu(y,1]\,dy\,dx \to 0$ as $\theta \to \infty$, the result (equation (9.2.18)) follows.

9.3 DESCRIPTION OF THE MODEL, BASIC DEFINITIONS AND NOTATIONS

The price process of a stock is given by $S_t = S_0 \exp\{-Y_t\} = \exp\{y - Y_t\}$, where

$$Y_t = \sigma B_t + X_t + ct, \quad Y_0 = 0, \quad (9.3.1)$$

is a Lévy process satisfying the basic Assumption 9.2.1. In addition, we assume that the market is risk-neutral – in other words, $e^{-\delta t} S_t = e^{y-\delta t - Y_t}$ is a martingale. Since $e^{-Y_t - \psi(1)t}$ is a martingale, this requires

$$\psi(1) = \delta, \quad \Longrightarrow \phi(\delta) = 1. \quad (9.3.2)$$

In order to achieve this, we require the drift c to be

$$c = -\delta + \frac{\sigma^2}{2} + \psi_X(1). \quad (9.3.3)$$

Note that if Y has finite variation, then $c < 0$.

We shall consider only options whose payoff is a bounded function of the stock price at the time exercise only. For notational convenience, we shall write the payoff of the option as a function of the logarithm of the price at the time of exercise: thus, if the price of the stock at time of exercise is s, the payoff is given by $\Pi(w)$, where $w = \log s$. The expected value of the discounted payoff obtained from exercising the option at a stopping time τ is given by

$$\mathbb{E}[e^{-\delta\tau}\Pi(y - Y_\tau)].$$

The price of such an option is then $\max_\tau \mathbb{E}[e^{-\delta\tau}\Pi(y - Y_\tau)]$. We shall consider only those options whose payoff $\Pi(w)$ is a decreasing function of w (i.e. options of 'put' type). For such options, it is clear from the form of the expected payoff and the fact that Y has stationary independent increments that the optimal time of exercise is a stopping time of the form

$$\tau_L = \inf\{t \geq 0 : S_t \leq L\}$$

for some L. It is equally clear that the optimal value of the exercise level L cannot depend on the initial stock price S_0. Writing $L = e^a$, we see that under our model,

$$\tau_L = T_{y-a} = \inf\{t \geq 0 : Y_t \geq y - a\}. \tag{9.3.4}$$

For a fixed choice of a, let $V(y, a)$ denote the expected value of the discounted payoff obtained from exercising the option at time T_{y-a}:

$$V(y, a) = \mathbb{E}[e^{-\delta T_{y-a}}\Pi(y - Y(T_{y-a}))]. \tag{9.3.5}$$

The price of such an option is then $\max_a V(y, a)$. The problem is then to find the optimal exercise level a which maximizes $V(y, a)$.

Notice that there is a small but important difference between the definition of T_x as shown in equation (9.3.4) and that of $\tilde{T}_x$ as in equation (9.2.13): $Y(T_x) \geq x$ whereas $Y(\tilde{T}_x) > x$. It is easy to see that $T_x = \tilde{T}_x$ almost surely *except* when $x = 0$ and Y has finite variation; in the latter case, $T_0 = 0$ by definition whereas, because $c < 0$ when Y has finite variation, Y will not become positive immediately (nor will it hit 0 again immediately), so that $\tilde{T}_0 > 0$ almost surely and by letting $x \downarrow 0$ in equation (9.2.15) we can obtain the joint law of $\tilde{T}_0$ and $Y(\tilde{T}_0)$. We have chosen the definition of T_x so as to ensure that $V(a, a) = \Pi(a)$, which is consistent with the fact that if the initial stock price $S_0 = e^y$ is at (or below) the chosen exercise level $L = e^a$, the option is exercised immediately at time 0, resulting in a payoff $\Pi(y)$. However, it must be emphasized that $V(a, a) = \Pi(a)$ is purely a consequence of the definition of T_{y-a}, which provides a neat way of expressing the expected payoff associated with a chosen exercise strategy – in particular, the statement that $V(a, a) = \Pi(a)$ is *not* the same as the *continuous junction* condition discussed in Gerber and Shiu (1998), which says that

$$V(a+, a) = \lim_{y \to a+} V(y, a) = \mathbb{E}[e^{-\delta\tilde{T}_0}\Pi(a - Y(\tilde{T}_0))] = \Pi(a), \tag{9.3.6}$$

From the definition of T_{y-a}, we already have $V(a-, a) = \lim_{y \to a-} V(y, a) = \Pi(a)$, and so if equation (9.3.6) holds, the function $y \mapsto V(y, a)$ would be continuous at $y = a$. If

Y has infinite variation, then since $T_0 = \tilde{T}_0 = 0$ almost surely, $V(y, a)$ is actually jointly continuous in y and a and the continuous junction condition (equation (9.3.6)) is always satisfied; on the other hand, if Y has finite variation, then in general $V(a+, a) \neq \Pi(a)$. However, as we shall see in Section 9.5, there is a unique a^* for which

$$\lim_{y \to a^*+} V(y, a^*) = \Pi(a^*)$$

and this a^* turns out to be the optimal exercise level (when Y has finite variation). The meaning of this special a^* is that, when the initial stock price is at the level $L^* = e^{a^*}$, the normal rule is to exercise the option immediately, resulting in a payoff $\Pi(a^*)$; however, we would get the same payoff if we waited until the stock price has actually fallen below L^* – the extra advantage in having $S(\tilde{T}_0) < L^*$ is exactly counter-balanced by the discount factor $e^{-d\tilde{T}}$.

9.4 A RENEWAL EQUATION APPROACH TO PRICING

Throughout this section, in addition to the basic Assumption 9.2.1, we make the following assumption.

Assumption 9.4.1 *The jump component X of Y has finite variation: thus, equation (9.2.10) holds.*

We derive a renewal equation for $V(y, a)$, the expected value of the payoff from exercising a perpetual option at level a. This is essentially the same renewal equation as obtained by Gerber and Landry (1998) in the case where X is a compound Poisson process; we show that there is an easy generalization to any jump process with finite variation.

Theorem 9.4.1 *Suppose Assumptions 9.2.1 and 9.4.1 hold. Let*

$$\beta = -\frac{2}{\sigma^2}(\psi_X(1) - \delta)$$

and let

$$h(s) = \frac{2}{\sigma^2} e^{-\beta s}, \quad \gamma(s) = e^s \int_s^\infty e^{-x}\, \nu(dx),$$

and

$$g(z) = \int_0^z h(z - s)\gamma(s)\, ds = \frac{2}{\sigma^2} e^z \int_0^z e^{-(\beta+1)(z-s)} \int_s^\infty e^{-x}\, \nu(dx)\, ds. \tag{9.4.1}$$

Then, $V(y, a)$, for $y > a$, satisfies

$$V(y, a) = \int_0^{y-a} V(y - z, a) g(z)\, dz + e^{-\beta(y-a)} \Pi(a) + \int_{y-a}^\infty \Pi(y - z) g(z)\, dz - e^{-\beta(y-a)} \int_0^\infty \Pi(a - z) g(z)\, dz. \tag{9.4.2}$$

Proof. We shall approximate X by a compound Poisson process and use the corresponding result of Gerber and Landry (1998). Thus, if ν is the Lévy measure of X, let $X_n(t)$ be a compound Poisson process with jump rate $\lambda_n = \nu(n^{-1}, \infty)$ and jump size distribution $P_n(dx) = \lambda_n^{-1}\mathbf{1}_{\{x>1/n\}}\,\nu(dx)$. Note that $\lambda_n P_n(dx) = \mathbf{1}_{\{x>1/n\}}\,\nu(dx)$. Let $Y_n(t) = \sigma B_t + X_n(t) + c_n t$, where the drift parameter c_n satisfies equation (9.3.3) with X replaced by X_n. Then, Y_n is precisely the process used by Gerber and Landry (1998) to model the logarithm of the stock price S_t. Let

$$\beta_n = -\frac{2}{\sigma^2}(\psi_{X_n}(1) - \delta)$$

and let

$$h_n(s) = \frac{2}{\sigma^2}e^{-\beta_n s}, \quad \gamma_n(s) = \lambda_n e^s \int_s^\infty e^{-x}\,P_n(dx),$$

and

$$g_n(z) = \int_0^z h_n(z-s)\gamma_n(s)\,ds = \frac{2\lambda_n}{\sigma^2}e^z \int_0^z e^{-(\beta_n+1)(z-s)} \int_s^\infty e^{-x}\,P_n(dx)\,ds. \tag{9.4.3}$$

Define $V_n(y,a)$ exactly as in equation (9.3.5) but with Y replaced by Y_n. Then Gerber and Landry (1998) showed that V_n satisfies

$$\begin{aligned} V_n(y,a) = {} & \int_0^{y-a} V_n(y-z,a)g_n(z)\,dz + e^{-\beta_n(y-a)}\Pi(a) \\ & + \int_{y-a}^\infty \Pi(y-z)g_n(z)\,dz - e^{-\beta_n(y-a)}\int_0^\infty \Pi(a-z)g_n(z)\,dz \end{aligned} \tag{9.4.4}$$

for $y > a$. To finish the proof, we only have to let $n \to \infty$. First, if ψ_n denotes the Laplace exponent of Y_n, it is easy to see that $\psi_n(\theta) \to \psi(\theta)$ for $\theta \geq 0$. This is equivalent to the weak convergence of Y_n to Y under a suitable topology, the J_1-topology of Skorohod (see Billingsley (1968)). Next, let $T_x(Y_n) = \inf\{t \geq 0 : Y_n(t) \geq x\}$. As a functional of Y_n, the first-passage time functional $T_x(\cdot)$ is continuous in the J_1-topology (see Whitt (1971)) and hence $T_x(Y_n)$ converges weakly under J_1 to $T_x(Y) = T_x$. These facts together imply that $V_n(y,a) \to V(y,a)$. In addition, $\beta_n \to \beta$ and $h_n(s) \to h(s)$, $\gamma_n(s) \to \gamma(s)$ pointwise. From the forms of h_n and γ_n, it is easy to see that h_n and γ_n – and hence g_n can be bounded by integrable functions and hence $g_n \to g$ by dominated convergence theorem. Finally, since g_n is bounded by an integrable function, letting $n \to \infty$ in equation (9.4.4) and using the dominated convergence theorem for the integrals on the right-hand side gives equation (9.4.2).

Note that if X has infinite variation, the integral in the definition of g diverges; this can be most readily seen if we interchange the order of integration and write

$$g(z) = \frac{2}{\sigma^2}e^{-\beta z}\int_0^\infty e^{-x}\left(\frac{e^{(\beta+1)\min(x,z)} - 1}{\beta+1}\right)\nu(dx).$$

There is a probabilistic explanation for why it is necessary to assume that X has finite variation in Theorem 9.4.1, which is related to the probabilistic interpretation of the function

g_n defined at equation (9.4.3). Let J_n denote the first time when Y_n attains a record high via a jump: formally

$$J_n = \inf\{t \geq 0 : Y_n(t) = S_n(t), Y_n(t-) \neq S_n(t-)\},$$

where $S_n(t) = \sup_{s \leq t} Y_n(s)$. Consider the joint density function $f_n(y, t)$ of $Y_n(J_n)$ and J_n. Then Gerber and Landry (1998) showed that

$$g_n(y) = \int_0^{\infty} e^{-\delta t} f_n(y, t)\, dt.$$

By conditioning on $Y_n(J_n)$, the level of the first record high in Y_n caused by a jump, Gerber and Landry (1998) derived the renewal equation shown by equation (9.4.4). The same probabilistic interpretations hold when we let $n \to \infty$: thus, if $J = \inf\{t \geq 0 : Y_t = S_t, Y_{t-} \neq S_{t-}\}$, then $g(y) = \int_0^{\infty} e^{-\delta t} f(y, t)\, dt$ where $f(y, t)$ is the joint density of $Y(J)$ and J. However, if X has infinite variation, it makes a jump which causes a new record immediately, and so $J = 0$ and $Y(J) = 0$ almost surely. Thus, there is a fundamental obstruction to using the conditioning argument of Gerber and Landry (1998) when X has infinite variation, rather than it being merely a question of certain expressions not behaving well in the limit.

To find the optimal exercise level a^*, we can use the *smooth pasting* condition, which says that the function $y \mapsto V(y, a)$ has a continuous first derivative at the optimal boundary a^*:

$$\lim_{y \to a^*+} \frac{\partial V(y, a^*)}{\partial y} = \Pi'(a^*).$$

By differentiating the right-hand side of equation (9.4.2), putting $y = a$ and equating this with $\Pi'(a)$, we obtain an equation for a which is identical to that obtained by Gerber and Landry (1998). In the case of an American put option, we have $\Pi(a) = (K - e^a)_+$ and the optimal exercise level a^* is given by

$$e^{a^*} = \frac{K\delta}{\sigma^2 - c - \int_0^{\infty} x e^{-x}\, \nu(dx)} \tag{9.4.5}$$

(where c is given by equation (9.3.3)), which is just the obvious extension of the formula obtained by Gerber and Landry (1998) to the present situation.

If $\sigma = 0$ (so Y has finite variation), then since $c < 0$, the time of the first record high caused by a jump is just $\tilde{T}_0$, the time of the first jump by Y above its initial level 0. In this case, the same method of approximation by compound Poisson processes, but this time using the results of Gerber and Shiu (1998), shows that V satisfies the following simpler renewal equation

$$V(y, a) = \int_0^{y-a} V(y - z, a)\tilde{g}(z)\, dz + \int_{y-a}^{\infty} \Pi(y - z)\tilde{g}(z)\, dz \tag{9.4.6}$$

where

$$\tilde{g}(z) = \frac{e^z}{c} \int_z^{\infty} e^{-x}\, \nu(dx).$$

The function $\tilde{g}$ has a similar probabilistic interpretation: $\tilde{g}(y) = \int_0^{\infty} e^{-\delta t} \tilde{f}(y, t)\, dt$ where $\tilde{f}(y, t)$ is the joint density of $Y(\tilde{T}_0)$ and $\tilde{T}_0$.

9.5 EXPLICIT PRICING FORMULAE FOR AMERICAN PUTS

In this section, we present explicit formulae for the optimal exercise level and value of a perpetual American put option, assuming only Assumption 9.2.1. Recall that the payoff function considered here is $\Pi(w) = (K - e^w)_+$ and that $r_\theta(x)$ denotes the resolvent density of Y; in other words, the density function of the measure $\int_0^\infty e^{-\theta t}\mathbb{P}(Y_t \in dx)\, dt$. The main result is encapsulated in the following theorem.

Theorem 9.5.1 *Suppose that Assumption 9.2.1 holds and consider a perpetual American put option with payoff* $\Pi(y - Y_T) = (K - e^{y-Y_T})_+$. *Then*

(i) the optimal exercise level is given by

$$L^* = e^{a^*} = \frac{K\delta}{\psi'(1)} \tag{9.5.1}$$

and for $y > a^*$, *the value of a perpetual put option is given by*

$$V(y, a^*) = K\delta \int_{y-a^*}^{\infty} r_\delta(z)\, dz; \tag{9.5.2}$$

(ii) if Y *has infinite variation, the optimal exercise level* a^* *is uniquely determined by the smooth pasting condition*

$$\lim_{y\to a^*+} \frac{\partial V(y, a^*)}{\partial y} = \Pi'(a^*) = -e^{a^*}; \tag{9.5.3}$$

(iii) if Y *has finite variation,* $V(y, a^*)$ *does not satisfy equation (9.5.3) – instead, the optimal exercise level* a^* *is uniquely determined by the continuity condition*

$$V(a^*+, a^*) = \lim_{y\to a^*+} V(y, a^*) = \Pi(a^*) = K - e^{a^*}. \tag{9.5.4}$$

Proof. (i) Suppose the option is exercised at level a at time T_{y-a} as described in Section 9.3 and we may assume that $a < \log K$. Then the value function is given by

$$V(y, a) = \mathbb{E}[e^{-\delta T}\Pi(y - Y_T)] = K\mathbb{E}[e^{-\delta T}] - e^y\mathbb{E}[e^{-\delta T - Y_T}],$$

where we have put $T = T_{y-a}$. For $y > a$, the terms on the right-hand side above are given by respectively putting $\theta = \delta$, $\eta = 0$ and $\theta = \delta$, $\eta = 1$ in equation (9.2.15) and noting the relationship shown in equation (9.3.2):

$$V(y, a) = K\delta\left[r_\delta(y - a) + \int_{y-a}^{\infty} r_\delta(z)\, dz\right] - \psi'(1)e^a r_\delta(y - a). \tag{9.5.5}$$

To find the optimal value of a so as to maximize $V(y, a)$, we differentiate equation (9.5.5) with respect to a to find

$$\frac{\partial V(y, a)}{\partial a} = (K\delta - \psi'(1)e^a)(r_\delta(y - a) - r'_\delta(y - a)) = 0. \tag{9.5.6}$$

The only solution to the above which does not depend on y is a^* as given by equation (9.5.1) and upon substitution into equation (9.5.5), we obtain equation (9.5.2). Finally, because the function $a \mapsto V(y, a)$ has a discontinuity at $a = y$ when Y has finite variation, we need to check that the optimal value of $V(y, a)$ is not in fact $\Pi(y)$ when $y > a^*$. To this end, we show that $V(y, y-) > \Pi(y)$, which implies that the optimal exercise level must be strictly less than y. Letting $a \to y-$ in equation (9.5.5) gives

$$V(y, y-) = K\delta \left(r_\delta(0+) + \int_0^\infty r_\delta(z)\, dz \right) - \psi'(1) e^y r_\delta(0+). \tag{9.5.7}$$

Putting $\lambda = 0$ into equation (9.2.14) and using equation (9.3.2) shows that

$$\int_0^\infty r_\delta(z)\, dz = \frac{1}{\delta} - \frac{1}{\psi'(1)}. \tag{9.5.8}$$

To find $r_\delta(0+)$, we use equation (9.2.14) together with equations (9.2.16) and (9.3.2) to obtain

$$\begin{aligned} r_\delta(0+) &= \lim_{\lambda \to \infty} \lambda \int_0^\infty e^{-\lambda x} r_\delta(x)\, dx = \lim_{\lambda \to \infty} \left[\frac{\lambda}{\delta - \psi(\lambda)} - \frac{\phi'(\delta)\lambda}{\phi(\delta) - \lambda} \right] \\ &= \frac{1}{c} + \phi'(\delta) = \frac{1}{c} + \frac{1}{\psi'(1)}. \end{aligned} \tag{9.5.9}$$

Substituting equations (9.5.9) and (9.5.8) into equation (9.5.7) gives

$$V(y, y-) = \frac{K\delta - \psi'(1) e^y}{c} + K - e^y. \tag{9.5.10}$$

Since $y > a^*$, $K\delta - \psi'(1)e^y < 0$ and recall that $c < 0$ when Y has finite variation. Hence, equation (9.5.10) shows that $V(y, y-) > \Pi(y) = (K - e^y)_+$.
(ii) Differentiating equation (9.5.2) with respect to y shows that

$$\frac{\partial V}{\partial y}(y, a^*) = -K\delta r_\delta(y - a^*),$$

and hence

$$\lim_{y \downarrow a^*} \frac{\partial V}{\partial y}(y, a^*) = -K\delta r_\delta(0+). \tag{9.5.11}$$

To find $r_\delta(0+)$, calculating as in equation (9.5.9) but this time using equation (9.2.17) gives

$$\begin{aligned} r_\delta(0+) &= \lim_{\lambda \to \infty} \lambda \int_0^\infty e^{-\lambda z} r_\delta(z)\, dz = \lim_{\lambda \to \infty} \left[\frac{\lambda}{\delta - \psi(\lambda)} - \frac{\psi_*'(\delta)\lambda}{\psi_*(\delta) - \lambda} \right] \\ &= \psi_*'(\delta) = \frac{1}{\psi'(1)}. \end{aligned} \tag{9.5.12}$$

Substituting this into equation (9.5.11) and using equation (9.5.1) immediately gives equation (9.5.3).

(iii) That a^* is uniquely determined by equation (9.5.4) follows immediately from equation (9.5.10). It is also immediately apparent upon substituting equation (9.5.9) into equation (9.5.11) that equation (9.5.3) does not hold.

We leave the reader to check that equation (9.5.1) agrees with equation (9.4.5) for the case where X has finite variation, and also that $V(a+, a) = \Pi(a)$ for all a when Y has infinite variation. Furthermore, note that the formula shown in equation (9.5.5) provides an explicit solution to the renewal equation (equation 9.4.2) when the jump component X has finite variation.

Of course, in order to actually evaluate equation (9.5.2), one still has to compute the resolvent density $r_\delta(x)$ for $x > 0$, for which there is rarely an explicit formula – unlike the simple formula $r_\theta(x) = \phi'(\theta)e^{\phi(\theta)x}$ for $x < 0$ (e.g. see Section 6 of Bingham (1975)). However, very often – and certainly for all the examples considered in the next section – $r_\delta(x)$ for $x > 0$ can be computed easily by Fourier inversion. First, observe that the Fourier transform of $r_\delta(x)$ is given by

$$\begin{aligned}\hat{r}_\delta(z) &= \int_{\mathbb{R}} e^{-izx} r_\delta(x)\,dx = \int_{\mathbb{R}} e^{-izx} \int_0^\infty e^{-\delta t}\mathbb{P}(Y_t \in dx)\,dt \\ &= \int_0^\infty e^{-\delta t}\mathbb{E}[e^{-izY_t}]\,dt = \int_0^\infty e^{-(\delta-\tilde{\psi}(z))t}\,dt = \frac{1}{\delta - \tilde{\psi}(z)}. \end{aligned} \tag{9.5.13}$$

Therefore, if

$$\int_{\mathbb{R}} \left| \frac{1}{\delta - \tilde{\psi}(z)} \right| dz < \infty, \tag{9.5.14}$$

$r_\delta(x)$ can be recovered by using the Fourier inversion formula

$$r_\delta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixz}\hat{r}_\delta(z)\,dz = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ixz}}{\delta - \tilde{\psi}(z)}\,dz, \tag{9.5.15}$$

which can be readily computed using a fast and efficient numerical algorithm like the Fast Fourier Transform (FFT). In particular, equation (9.5.14) is true if $\sigma \neq 0$; however, note that the latter equation implies that Y has infinite variation, since equation (9.2.16) implies $\int 1/|\delta - \tilde{\psi}(z)|\,dz = \infty$ when Y has finite variation.

Of course, one could also try to invert the Laplace transform at equation (9.2.14) but this is generally much harder computationally than Fourier inversion.

In principle, other payoff functions besides $\Pi(w) = (K - e^w)_+$ can be treated in the same way but in practice, this is much harder at the computational level as it entails the inversion of the Laplace transform (equation (9.2.15)).

9.6 SOME SPECIFIC EXAMPLES

We consider the following examples of Lévy processes for the jump component X.

Gamma process

A process is called a Gamma (α, β) process if its Lévy measure is

$$\nu(dx) = \alpha x^{-1}e^{-\beta x}\,dx, \quad x > 0. \tag{9.6.1}$$

Note that it has finite variation. It is well-known that X_t has the following Gamma distribution

$$\mathbb{P}(X_t \in dx) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} e^{-\beta t}\, dx.$$

The Lévy exponent is therefore given by

$$\tilde{\psi}_X(\theta) = -\alpha \log\left(1 + \frac{i\theta}{\beta}\right) \tag{9.6.2}$$

while its Laplace exponent is given by

$$\psi_X(\theta) = -\alpha \log\left(1 + \frac{\theta}{\beta}\right). \tag{9.6.3}$$

Stable process: index $\alpha \in (0, 1)$

A (spectrally positive) process is called stable of index α if its Lévy measure is

$$\nu(dx) = \beta x^{-\alpha-1}\, dx, \quad x > 0.$$

Thus, if $\alpha \in (0, 1)$, it has finite variation. Its Laplace exponent is given by

$$\psi_X(\theta) = -\beta \int_0^\infty (1 - e^{-\theta x}) x^{-\alpha-1}\, dx = -\frac{\beta\theta}{\alpha} \int_0^\infty e^{-\theta x} x^{-\alpha}\, dx = -\frac{\beta\Gamma(1-\alpha)}{\alpha}\theta^\alpha. \tag{9.6.4}$$

It is more convenient if we write the Lévy measure as

$$\nu(dx) = \frac{\beta\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1}\, dx. \tag{9.6.5}$$

The Laplace exponent is then

$$\psi_X(\theta) = -\beta\theta^\alpha. \tag{9.6.6}$$

To find the Lévy exponent, first note the following identity (see Erdélyi (1954), Section 2.3)

$$\int_0^\infty x^{-\alpha} \sin(\theta x)\, dx = \text{sgn}(\theta)|\theta|^{\alpha-1}\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}. \tag{9.6.7}$$

Hence, integrating by parts,

$$\begin{aligned}\int_0^\infty x^{-\alpha}(\cos(\theta x) - 1)\, dx &= \frac{\theta}{1-\alpha}\int_0^\infty x^{-(\alpha-1)}\sin(\theta x)\, dx \\ &= |\theta|^{\alpha-1}\frac{\Gamma(2-\alpha)}{1-\alpha}\cos\frac{\pi(\alpha-1)}{2} \\ &= |\theta|^{\alpha-1}\Gamma(1-\alpha)\sin\frac{\pi\alpha}{2}.\end{aligned} \tag{9.6.8}$$

To find the Lévy exponent, we have

$$\begin{aligned}\tilde{\psi}_X(\theta) &= \frac{\beta\alpha}{\Gamma(1-\alpha)}\int_0^\infty (e^{-i\theta x}-1)x^{-(\alpha+1)}\,dx\\ &= \frac{\beta\alpha}{\Gamma(1-\alpha)}\left(\int_0^\infty x^{-(\alpha+1)}(\cos(\theta x)-1)\,dx - i\int_0^\infty x^{-(\alpha+1)}\sin(\theta x)\,dx\right)\\ &= \frac{\beta\alpha}{\Gamma(1-\alpha)}\left(|\theta|^\alpha\Gamma(-\alpha)\sin\frac{\pi(\alpha+1)}{2} - i\ \mathrm{sgn}(\theta)|\theta|^\alpha\Gamma(-\alpha)\cos\frac{\pi(\alpha+1)}{2}\right)\\ &= -\beta|\theta|^\alpha\left(\cos\frac{\pi\alpha}{2} + i\ \mathrm{sgn}(\theta)\sin\frac{\pi\alpha}{2}\right). \qquad (9.6.9)\end{aligned}$$

(We have also used the identity $\Gamma(z+1) = z\Gamma(z)$ in the above calculation.)

Stable process: index $\alpha \in (1,2)$

This is arguably the most interesting of the examples considered here, as it is the only example where the jump component X has infinite variation. The Lévy measure here is given by

$$\nu(dx) = -\frac{\beta\alpha}{\Gamma(1-\alpha)}x^{-\alpha-1}\,dx, \quad x > 0. \qquad (9.6.10)$$

(Note that if $\alpha \in (1,2)$, $\Gamma(1-\alpha) < 0$.) Such stable processes are purely discontinuous martingales, which means that in equations (9.2.4) and (9.2.8)

$$b = -\int_1^\infty x\,\nu(dx)$$

and we write the Laplace exponent of X as

$$\begin{aligned}\psi_X(\theta) &= \frac{\beta\alpha}{\Gamma(1-\alpha)}\int_0^\infty (1-e^{-\theta x}-\theta x)x^{-(\alpha+1)}\,dx\\ &= \frac{\beta\theta}{\Gamma(1-\alpha)}\int_0^\infty (e^{-\theta x}-1)x^{-\alpha}\,dx = \beta\theta^\alpha, \qquad (9.6.11)\end{aligned}$$

where we have integrated by parts and used equation (9.6.4). Note the change in sign from the stable $0 < \alpha < 1$ case. To find the Lévy exponent, we integrate by parts and use equation (9.6.9):

$$\begin{aligned}\tilde{\psi}_X(\theta) &= \frac{\beta\alpha}{\Gamma(1-\alpha)}\int_0^\infty (1-e^{-i\theta x}-i\theta x)x^{-(\alpha+1)}\,dx\\ &= \frac{\beta(i\theta)}{\Gamma(1-\alpha)}\int_0^\infty (e^{-i\theta x}-1)x^{-\alpha}\,dx\\ &= \frac{\beta(i\theta)}{\Gamma(1-\alpha)}\left(|\theta|^{\alpha-1}\Gamma(1-\alpha)\sin\frac{\pi\alpha}{2} - i\ \mathrm{sgn}(\theta)|\theta|^{\alpha-1}\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}\right)\\ &= \beta|\theta|^\alpha\left(\cos\frac{\pi\alpha}{2} + i\ \mathrm{sgn}(\theta)\sin\frac{\pi\alpha}{2}\right). \qquad (9.6.12)\end{aligned}$$

Again, note the change in sign from the stable $0 < \alpha < 1$ case.

Finally, note that there is no such thing as a spectrally positive stable process with $\alpha = 1$ (other than a deterministic drift $t \mapsto \beta t$).

Numerical examples

The main purpose of this section is to illustrate the kind of calculations which can be carried out with these models, rather than to give a detailed comparison of the effects of using different Lévy processes. However, to achieve at least a certain degree of comparability among the different examples below, the parameters are chosen so that

$$\sigma^2 + \int_0^1 x^2 \, \nu(dx) = 4 \quad \text{and} \quad \nu[1, \infty) = 1 \tag{9.6.13}$$

in all of the examples. The first condition in equation (9.6.13) says that the contribution to the volatility (as measured by quadratic variation) coming from the Brownian fluctuations and the small jumps is the same in all of the examples, while the second condition in this equation (9.6.13) says that the rate at which large jumps occur is the same in all of the following examples.

Throughout the following examples, we take $S_0 = 1$ – so $y = 0$ – and $\delta = 0.1$. We then compute the prices of a perpetual American put with strikes $K = 0.75$, 1 and 1.25 using Theorem 9.5.1. In each case, the integrability condition (equation (9.5.14)) is satisfied and the resolvent density is computed by approximating the Fourier integral (equation (9.5.15)) with a suitable discrete Fourier sum which is then computed by an FFT algorithm. The details of how this is carried out are described in the Appendix. We consider the following models:

1. *Brownian motion plus Gamma (α,1/2) process*: here, $\alpha = 1.7864$, $\sigma^2 = 3.3555$ and according to equations (9.6.3) and (9.3.3), $c = -0.3848$ and from equation (9.2.8) $\psi'(1) = 2.5494$, and so the optimal exercise level is $L^* = e^{a^*} = 0.1K/2.5494$.
2. *Brownian motion plus stable* 1/2 *process*: we take $\beta = \sqrt{\pi}$, $\sigma^2 = 3.6667$ and according to equations (9.6.6) and (9.3.3), $c = -0.0391$ and $\psi'(1) = 2.8196$, and so $L^* = 0.1K/2.8196$.
3. *Brownian motion plus stable* 3/2 *process*: we take $\beta = 2\sqrt{\pi}$, $\sigma^2 = 1$ and from equations (9.6.11) and (9.3.3) we have $c = 3.9449$ and $\psi'(1) = 2.3725$, and so $L^* = 0.1K/2.3725$.

The results are summarized in the Table 9.1.

Table 9.1 Summary of the results obtained from the various models

Model	$K = 0.75$		$K = 1.00$		$K = 1.25$	
	Ex. Lev. L^*	Price	Ex. Lev. L^*	Price	Ex. Lev. L^*	Price
Brownian Motion + Gamma (α,1/2)	0.029	0.64	0.039	0.86	0.049	1.09
Brownian Motion + Stable (1/2)	0.027	0.30	0.035	0.40	0.044	0.51
Brownian Motion + Stable (3/2)	0.032	0.64	0.042	0.86	0.053	1.08

APPENDIX: USE OF FAST FOURIER TRANSFORM

Let $\{x(n)\}_{n=1}^{N}$ be a real sequence of length N. The discrete Fourier transform $\hat{x}$ of x, and its inverse, are given by the relations

$$\hat{x}(k) = \sum_{n=1}^{N} e^{-2\pi i(k-1)(n-1)/N} x(n) \tag{A.1}$$

$$x(n) = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i(k-1)(n-1)/N} \hat{x}(k) \tag{A.2}$$

The Fast Fourier Transform (FFT) is a fast and efficient numerical algorithm for computing equations (A.1) and (A.2). Although not strictly necessary, the FFT is at its most efficient if N is a power of 2.

The purpose of this appendix is not to describe the workings of the FFT – there are numerous texts written on the subject and the FFT is also implemented in many standard software packages – rather, the purpose is to describe how to recast the problem of computing the Fourier inversion formula (equation (9.5.15)) into a form equivalent to equation (A.2).

Let f be an integrable function: $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$. Its Fourier transform is given by $\hat{f}(z) = \int_{-\infty}^{\infty} e^{-izx} f(x)\, dx$, and so

$$\hat{f}(2\pi z) = \int_{-\infty}^{\infty} e^{-2\pi izx} f(x)\, dx. \tag{A.3}$$

Fix an integer M (ideally a power of 2) and let $N = M^2$ and $\Delta = M/N = 1/M$. We partition the interval $[-M/2, M/2]$ into steps of length Δ and approximate the Fourier integral (equation (A.3)) by a truncated discrete sum involving the values of $f(x)$, for $x = -M/2,\ -M/2 + \Delta \ldots,\ M/2 - \Delta$ as follows:

$$\hat{f}\big(2\pi[(k-1)\Delta - M/2]\big) = \sum_{n=1}^{N} e^{-2\pi i[(k-1)\Delta - M/2][(n-1)\Delta - M/2]} f\big((n-1)\Delta - M/2\big)\Delta. \tag{A.4}$$

giving an approximation for $\hat{f}(z)$ for $z = -M/2,\ -M/2 + \Delta,\ \ldots,\ M/2 - \Delta$. (For simplicity, we have chosen the crudest form of discrete approximation to equation (A.3); however, more sophisticated quadrature rules for the most part involve using weighted averages of sums of the form shown by equation (A.4) and so the method described below can be adapted to handle these more sophisticated approximations.) Rearranging equation (A.4) gives

$$\begin{aligned} &e^{-i\pi((k-1)-N/4)} \hat{f}\big(2\pi[(k-1)/M - M/2]\big) \\ &\qquad = \sum_{n=1}^{N} e^{-2\pi i(k-1)(n-1)/N} e^{i\pi((n-1)-N/4)} f\big((n-1)/M - M/2\big)/M. \end{aligned} \tag{A.5}$$

If $\hat{f}$ is also an integrable function, then the Fourier inversion formula holds:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{izx}\hat{f}(z)\,dz = \int_{-\infty}^{\infty} e^{2\pi iyx}\hat{f}(2\pi y)\,dy. \tag{A.6}$$

Approximating the above by a truncated sum over $[-M/2, M/2]$ as before gives

$$f\big((n-1)\Delta - M/2\big) = \sum_{k=1}^{N} e^{2\pi i[(k-1)\Delta - M/2][(n-1)\Delta - M/2]}\hat{f}\big(2\pi[(k-1)\Delta - M/2]\big)\Delta,$$

which when rearranged gives

$$\frac{e^{i\pi((n-1)-N/4)}f\big((n-1)/M - M/2\big)}{M} = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i(k-1)(n-1)/N}e^{-i\pi((k-1)-N/4)}\hat{f}\big(2\pi[(k-1)/M - M/2]\big). \tag{A.7}$$

Comparing equations (A.5) and (A.7) with equations (A.1) and (A.2), we see that defining

$$x(n) = M^{-1}e^{i\pi((n-1)-N/4)}f\big((n-1)/M - M/2\big) \tag{A.8}$$

$$\hat{x}(k) = e^{-i\pi((k-1)-N/4)}\hat{f}\big(2\pi[(k-1)/M - M/2]\big) \tag{A.9}$$

makes these equations identical.

For the purposes of Section 9.6, we have $f(x) = r_\delta(x)$, whose Fourier transform $\hat{r}_\delta(z)$ is given by equation (9.5.13). Defining $\hat{x}(k)$ as in equation (A.9), we can recover $x(n)$ from equation (A.2) (evaluated using FFT) and then equation (A.8) gives values of $r_\delta(x)$ for a discrete set of grid-points $x \in [-M/2, M/2]$, spaced $\Delta = 1/M$ apart. The values of $r_\delta(x)$ for the grid-points $x \in [y - a^*, M/2]$ are then used to approximate the integral in equation (9.5.2). The values shown in Table 9.1 are obtained by using $M = 2^9 = 512$.

ACKNOWLEDGEMENTS

I would like to thank the many people, including several anonymous referees and some of the editors of this volume, who have read and made suggestions for improving various earlier drafts of this article. I am also grateful to the editors for providing the opportunity for its publication.

REFERENCES

[1] Bertoin, J. (1996), *Lévy Processes*, Cambridge University Press, Cambridge, UK.

[2] Billingsley, P. (1968), *Convergence of Probability Measures*, Wiley, New York, NY, USA.

[3] Bingham, N.H. (1975), "Fluctuation theory in continuous time", *Advances in Applied Probability*, **7**, 705–766.

[4] Carr, P. (1998), "Randomization and the American put", *Review of Financial Studies*, **11**, 597–626.

[5] Chan, T. (1999), "Pricing contingent claims on stocks driven by Lévy processes", *Annals of Applied Probability*, **9**, 504–528.

[6] Erdélyi, A. (1954), *Tables of Integral Transforms*, Vol. 1, McGraw-Hill, New York, NY, USA.

[7] Gerber, H.U. and Landry, B. (1998), "On the discounted penalty at ruin in a jump-diffusion and the perpetual put option", *Insurance: Mathematics and Economics*, **22**, 263–276.

[8] Gerber, H.U. and Shiu, E.S.W. (1994), "Martingale approach to pricing perpetual American options", *ASTIN Bulletin*, **24**, 195–220.

[9] Gerber, H.U. and Shiu, E.S.W. (1997), "From ruin theory to option pricing", in *Joint Day Proceedings Volume of XXVIII International ASTIN Colloquium/7th International AFIR Colloquium*, pp. 157–176.

[10] Gerber, H.U. and Shiu, E.S.W. (1998), "Pricing perpetual options for jump processes", *North American Actuarial Journal*, **2**(3), 101–112.

[11] MacMillan, L. (1986), "Analytic approximation for the American put option", *Advance Futures Options Research*, **1**, 119–139.

[12] Samuelson, P.A. (1965), "Rational theory of warrant pricing", *Industrial Management Review*, **6**, 13–33.

[13] Whitt, W. (1971), "Weak convergence of first passage times", *Journal of Applied Probability*, **8**, 417–422.

[14] Zhang, X. (1995), "Formules quasi-explicites pour les options Américaines dans un modèle de diffusion avec sauts", *Mathematics and Computers in Simulation*, **38**, 151–161.

EPILOGUE†

This article is a revised (and hopefully improved) version of an original preprint first completed in early 2000. In the period between then and the publication of this present volume, there has been a number of new developments and papers which are related to the results presented here. The purely Brownian model considered in Carr (1998) has been extended by Avram *et al.* (2002) to models such as that presented here involving spectrally one-sided Lévy processes. Avram *et al.* (2004) further extend this method to Russian options. Many other authors have also contributed to the recent renewed interest in perpetual options of various kinds and below is a limited bibliography which also lists further papers whose themes are closely related to the present one. Much of this interest has centred on one-sided Lévy processes because of the relative ease in carrying out explicit computations; nevertheless, the more interesting case of two-sided Lévy processes has not been neglected and several authors (see, for example, Asmussen *et al*. (2004)) have succeeded in performing similar explicit computations in certain special cases.

FURTHER REFERENCES

1. Asmussen, S., Avram, F. and Pistorius, M.R. (2004), "Pricing American and Russian options under spectrally two-sided exponential Lévy models", *Stochastic Processes and their Applications*, **109**, 79–111.
2. Avram, F., Chan, T. and Usabel, M. (2002), "On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts", *Stochastic Processes and their Applications*, **100**, 75–107.

† Added at the request of the editors.

3. Avram, F. Kyprianou, A.E. and Pistorious, M.R. (2004), “Exit problems for spectrally negative Lévy processes and applications to Canadized Russian options”, *Annals of Applied Probability*, **14**, 215–238.
4. Boyarchenko, S.I. and Levendorskiǐ, S.Z. (2002a), “Perpetual American options under Lévy processes”, *SIAM Journal of Control Optimization*, **40**, 1663–1696.
5. Boyarchenko, S.I. and Levendorskiǐ, S.Z. (2002b), “Barrier options and touch-and-out options under regular Lévy processes of exponential type”, *Annals of Applied Probability*, **12**, 1261–1298.
6. Duistermaat, J.J., Kyprianou, A.E. and van Schaik, K. (2005), “Finite expiry Russian options”, *Stochastic Processes and their Applications*, **115**, 609–638.
7. Mordecki, E. (1999), “Optimal stopping for a diffusion with jumps”, *Finance and Stochastics*, **3**, 227–236.
8. Mordecki, E. (2002), “Optimal stopping and perpetual options for Lévy processes”, *Finance and Stochastics*, **6**, 473–493.

10
On Asian Options of American Type

Goran Peskir
University of Aarhus, Aarhus, Denmark†

and

Nadia Uys
University of the Witwatersrand, Witwatersrand, South Africa

Abstract

We show that the optimal stopping boundary for the early exercise Asian call option with floating strike can be characterized as the unique solution of a nonlinear integral equation arising from the early exercise premium representation (an explicit formula for the arbitrage-free price in terms of the optimal stopping boundary). The key argument in the proof relies upon a local time-space formula.

10.1 INTRODUCTION

According to financial theory (see, e.g. Karatzas and Shreve (1998) [7] or Shiryaev (1999) [18]), the arbitrage-free price of the *early* exercise *Asian call option* with *floating strike* is given as V in equation (10.2.1) below where I_τ/τ denotes the *arithmetic* average of the stock price S up to time τ. The problem was first studied by Hansen and Jørgensen (2000) [5] where approximations to the value function V and the optimal boundary b were derived. The main aim of this present paper is to derive exact expressions for V and b.

The optimal stopping problem (equation (10.2.1)) is three-dimensional. When a change-of-measure theorem is applied (as in Shepp and Shiryaev (1994) [16] and Kramkov and Mordecky (1994) [10]) the problem reduces to (equation (10.2.9)) and becomes two-dimensional. The problem (equation (10.2.9)) is more complicated than the well-known problems (Peskir (2005, 2003) [12] [13]) since the gain function depends on time in a nonlinear way. From the result of Theorem 3.1 below, it follows that the free-boundary problem (equations (10.2.10)–(10.2.14)) characterizes the value function V and the optimal stopping boundary b in a unique manner. Our main aim, however, is to follow the train of thought initiated by Kolodner (1956) [9] where V is initially expressed in terms of b, and b itself is then shown to satisfy a nonlinear integral equation. A particularly simple approach for achieving this goal in the case of the American put option has been suggested in Kim (1990) [8], Jacka (1991) [6] and Carr *et al*. (1992) [2] and we will take this up

† Centre for Analytical Finance (funded by the Danish Social Science Research Council) and Network in Mathematical Physics and Stochastics (funded by the Danish National Research Foundation).

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

in the present paper. We will moreover see (as in Peskir (2005, 2003) [12] [13]) that the nonlinear equation derived for b cannot have other solutions. The key argument in the proof relies upon a local time-space formula (see Peskir (2002) [11]).

The latter fact of uniqueness may be seen as the principal result of the paper. The same method of proof can also be used to show the uniqueness of the optimal stopping boundary solving nonlinear integral equations derived by Hansen and Jørgensen (2000) [5] and Wu *et al*. (1999) [19] where this question was not explicitly addressed. These equations arise from the early exercise Asian options (call or put) with floating strike based on *geometric* averaging. The early exercise Asian *put* option with floating strike can be dealt with analogously to the Asian call option treated here. For financial interpretations of the early exercise Asian options and other references on the topic, see Hansen and Jørgensen (2000) [5] and Wu *et al*. (1999) [19].

10.2 FORMULATION OF THE PROBLEM

The arbitrage-free price of the early exercise Asian call option with floating strike is given by the following expression:

$$V = \sup_{0<\tau\le T} \mathsf{E}\Big(e^{-r\tau}\Big(S_\tau - \frac{1}{\tau}\, I_\tau \Big)^+ \Big) \tag{10.2.1}$$

where τ is a stopping time of the geometric Brownian motion $S = (S_t)_{0\le t\le T}$ solving:

$$dS_t = rS_t\, dt + \sigma S_t\, dB_t \qquad (S_0 = s) \tag{10.2.2}$$

and $I = (I_t)_{0\le t\le T}$ is the integral process given by:

$$I_t = a + \int_0^t S_s\, ds \tag{10.2.3}$$

where $s > 0$ and $a \ge 0$ are given and fixed. (Throughout, $B = (B_t)_{t\ge 0}$ denotes a standard Brownian motion started at zero.) We recall that $T > 0$ is the expiration date (maturity), $r > 0$ is the interest rate and $\sigma > 0$ is the volatility coefficient.

By the change-of-measure theorem, it follows that:

$$V = \sup_{0<\tau\le T} \mathsf{E}\Big(e^{-r\tau} S_\tau \Big(1 - \frac{1}{\tau}\, X_\tau \Big)^+ \Big) = s \sup_{0<\tau\le T} \widetilde{\mathsf{E}}\Big(\Big(1 - \frac{1}{\tau}\, X_\tau \Big)^+ \Big) \tag{10.2.4}$$

where following Shepp and Shiryaev (1994) [16] and Kramkov and Mordecky (1994) [10], we set:

$$X_t = \frac{I_t}{S_t} \tag{10.2.5}$$

and $\widetilde{\mathsf{P}}$ is defined by $d\widetilde{\mathsf{P}} = \exp(\sigma B_T - (\sigma^2/2)T)\, d\mathsf{P}$ so that $\widetilde{B}_t = B_t - \sigma t$ is a standard Brownian motion under $\widetilde{\mathsf{P}}$ for $0 \le t \le T$. By Itô's formula, one finds that:

$$dX_t = (1 - rX_t)\, dt + \sigma X_t\, d\widehat{B}_t \qquad (X_0 = x) \tag{10.2.6}$$

under $\widetilde{\mathsf{P}}$ where $\widehat{B} = -\widetilde{B}$ is a standard Brownian motion and $x = a/s$. The infinitesimal generator of X is therefore given by:

$$\mathbb{L}_X = (1 - rx)\,\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\,x^2\frac{\partial^2}{\partial x^2}\,. \tag{10.2.7}$$

For further reference, recall that the strong solution of equation (10.2.2) is given by:

$$S_t = s\exp\left(\sigma B_t + \left(r - \frac{\sigma^2}{2}\right)t\right) = s\exp\left(\sigma \widetilde{B}_t + \left(r + \frac{\sigma^2}{2}\right)t\right) \tag{10.2.8}$$

for $0 \le t \le T$ under P and $\widetilde{\mathsf{P}}$, respectively. When dealing with the process X on its own, however, note that there is no restriction to assume that $s = 1$ and $a = x$ with $x \ge 0$.

Summarizing the preceding facts we see that the early exercise Asian call problem reduces to solving the following optimal stopping problem:

$$V(t, x) = \sup_{0<\tau\le T-t} \widetilde{\mathsf{E}}_{t,x}\left(\left(1 - \frac{1}{t+\tau}\,X_{t+\tau}\right)^+\right) \tag{10.2.9}$$

where τ is a stopping time of the diffusion process X solving equation (10.2.6) above and $X_t = x$ under $\widetilde{\mathsf{P}}_{t,x}$ with $(t, x) \in [0, T] \times [0, \infty\rangle$ given and fixed.

Standard Markovian arguments indicate that V from equation (10.2.9) solves the following free-boundary problem of parabolic type:

$$V_t + \mathbb{L}_X V = 0 \quad \text{in } C \tag{10.2.10}$$

$$V(t, x) = \left(1 - \frac{x}{t}\right)^+ \quad \text{for } x = b(t) \text{ or } t = T \tag{10.2.11}$$

$$V_x(t, x) = -\frac{1}{t} \quad \text{for } x = b(t) \text{ (smooth fit)} \tag{10.2.12}$$

$$V(t, x) > \left(1 - \frac{x}{t}\right)^+ \quad \text{in } C \tag{10.2.13}$$

$$V(t, x) = \left(1 - \frac{x}{t}\right)^+ \quad \text{in } D \tag{10.2.14}$$

where the continuation set C and the stopping set $S = \overline{D}$ are defined by:

$$C = \{\,(t, x) \in [0, T\rangle \times [0, \infty\rangle \mid x > b(t)\,\} \tag{10.2.15}$$

$$D = \{\,(t, x) \in [0, T\rangle \times [0, \infty\rangle \mid x < b(t)\,\} \tag{10.2.16}$$

and $b : [0, T] \to I\!R$ is the (unknown) optimal stopping boundary, i.e. the stopping time:

$$\tau_b = \inf\{\,0 \le s \le T - t \mid X_{t+s} \le b(t+s)\,\} \tag{10.2.17}$$

is optimal in equation (10.2.9) (i.e. the supremum is attained at this stopping time). It follows from the result of Theorem 3.1 below that the free-boundary problem (equations (10.2.10)–(10.2.14)) characterizes the value function V and the optimal stopping boundary b in a unique manner (proving also the existence of the latter).

10.3 THE RESULT AND PROOF

In this section, we adopt the setting and notation of the early exercise Asian call problem from the previous section. Below we will make use of the following functions:

$$F(t,x) = \widetilde{\mathsf{E}}_{0,x}\Big(\Big(1-\frac{X_t}{T}\Big)^+\Big) = \int_0^\infty \int_0^\infty \Big(1-\frac{x+a}{Ts}\Big)^+ f(t,s,a)\,ds\,da \tag{10.3.1}$$

$$G(t,x,y) = \widetilde{\mathsf{E}}_{0,x}\Big(X_t\, I\big(X_t \le y\big)\Big) = \int_0^\infty \int_0^\infty \Big(\frac{x+a}{s}\Big) I\Big(\frac{x+a}{s} \le y\Big) f(t,s,a)\,ds\,da \tag{10.3.2}$$

$$H(t,x,y) = \widetilde{\mathsf{P}}_{0,x}\,(X_t \le y) = \int_0^\infty \int_0^\infty I\Big(\frac{x+a}{s} \le y\Big) f(t,s,a)\,ds\,da \tag{10.3.3}$$

for $t > 0$ and $x, y \ge 0$, where $(s,a) \mapsto f(t,s,a)$ is the probability density function of (S_t, I_t) under $\widetilde{\mathsf{P}}$ with $S_0 = 1$ and $I_0 = 0$ given by:

$$\begin{aligned} f(t,s,a) = {} & \frac{2\sqrt{2}}{\pi^{3/2}\sigma^3}\,\frac{s^{r/\sigma^2}}{a^2\sqrt{t}}\,\exp\Big(\frac{2\pi^2}{\sigma^2 t} - \frac{(r+\sigma^2/2)^2}{2\sigma^2}\,t - \frac{2}{\sigma^2 a}\,(1+s)\Big) \\ & \times \int_0^\infty \exp\Big(-\frac{2z^2}{\sigma^2 t} - \frac{4\sqrt{s}}{\sigma^2 a}\cosh(z)\Big)\sinh(z)\,\sin\Big(\frac{4\pi z}{\sigma^2 t}\Big)\,dz \end{aligned} \tag{10.3.4}$$

for $s > 0$ and $a > 0$. For a derivation of the right-hand side in equation (10.3.4) see the appendix below.

The main result of the paper may be stated as follows.

Theorem 3.1
The optimal stopping boundary in the Asian call problem (equation (10.2.9)) can be characterized as the unique continuous increasing solution $b : [0,T] \to I\!R$ of the nonlinear integral equation:

$$\begin{aligned} 1 - \frac{b(t)}{t} = {} & F(T-t, b(t)) \\ & - \int_0^{T-t} \frac{1}{t+u}\Big(\Big(\frac{1}{t+u} + r\Big) G(u, b(t), b(t+u)) - H(u, b(t), b(t+u))\Big)\,du \end{aligned} \tag{10.3.5}$$

satisfying $0 < b(t) < t/(1+rt)$ for all $0 < t < T$. [The solution b satisfies $b(0+) = 0$ and $b(T-) = T/(1+rT)$, and the stopping time τ_b from equation (10.2.17) is optimal in equation (10.2.9).]

The arbitrage-free price of the Asian call option (equation (10.2.9)) admits the following 'early exercise premium' representation:

$$\begin{aligned} V(t,x) = {} & F(T-t, x) \\ & - \int_0^{T-t} \frac{1}{t+u}\Big(\Big(\frac{1}{t+u} + r\Big) G(u, x, b(t+u)) - H(u, x, b(t+u))\Big)\,du \end{aligned} \tag{10.3.6}$$

for all $(t,x) \in [0,T] \times [0,\infty\rangle$. [Further properties of V and b are exhibited in the proof below.]

Proof. The proof will be carried out in several steps. We begin by stating some general remarks which will be freely used below without further mentioning.

1. The reason that we take the supremum in equations (10.2.1) and (10.2.9) over $\tau > 0$ is that the ratio $1/(t+\tau)$ is not well defined for $\tau = 0$ when $t = 0$. Note, however, in equation (10.2.1) that $I_\tau/\tau \to \infty$ as $\tau \downarrow 0$ when $I_0 = a > 0$ and that $I_\tau/\tau \to s$ as $\tau \downarrow 0$ when $I_0 = a = 0$. Similarly, note in equation (10.2.9) that $X_\tau/\tau \to \infty$ as $\tau \downarrow 0$ when $X_0 = x > 0$ and $X_\tau/\tau \to 1$ as $\tau \downarrow 0$ when $X_0 = x = 0$. Thus, in both cases the gain process (the integrand in equations (10.2.1) and (10.2.9)) tends to 0 as $\tau \downarrow 0$. This shows that in either equation it is never optimal to stop at $t = 0$. To avoid similar (purely technical) complications in the proof to follow we will equivalently consider $V(t,x)$ only for $t > 0$ with the supremum taken over $\tau \geq 0$. The case of $t = 0$ will become evident (by continuity) at the end of the proof.

2. Recall that it is no restriction to assume that $s = 1$ and $a = x$ so that $X_t = (x + I_t)/S_t$ with $I_0 = 0$ and $S_0 = 1$. We will write X_t^x instead of X_t to indicate the dependence on x when needed. It follows that V admits the following representation:

$$V(t,x) = \sup_{0 \leq \tau \leq T-t} \widetilde{\mathsf{E}}\left(\left(1 - \frac{x + I_\tau}{(t+\tau)\, S_\tau}\right)^+\right) \tag{10.3.7}$$

for $(t,x) \in \langle 0, T] \times [0, \infty\rangle$. From equation (10.3.7) we immediately see that:

$$x \mapsto V(t,x) \text{ is decreasing and convex on } [0, \infty\rangle \tag{10.3.8}$$

for each $t > 0$ fixed.

3. We show that $V : \langle 0, T] \times [0, \infty\rangle \to I\!R$ is continuous. For this, using $\sup(f) - \sup(g) \leq \sup(f - g)$ and $(z-x)^+ - (z-y)^+ \leq (y-x)^+$ for $x, y, z \in I\!R$, we get:

$$\begin{aligned} V(t,x) - V(t,y) &\leq \sup_{0 \leq \tau \leq T-t} \left(\widetilde{\mathsf{E}}\left(\left(1 - \frac{x + I_\tau}{(t+\tau)\, S_\tau}\right)^+\right) - \widetilde{\mathsf{E}}\left(\left(1 - \frac{y + I_\tau}{(t+\tau)\, S_\tau}\right)^+\right)\right) \\ &\leq (y - x) \sup_{0 \leq \tau \leq T-t} \widetilde{\mathsf{E}}\left(\frac{1}{(t+\tau)\, S_\tau}\right) \leq \frac{1}{t}\,(y - x) \end{aligned} \tag{10.3.9}$$

for $0 \leq x \leq y$ and $t > 0$, where in the last inequality we used equation (10.2.8) to deduce that $1/S_t = \exp(\sigma \widehat{B}_t - (r + \sigma^2/2)t) \leq \exp(\sigma \widehat{B}_t - (\sigma^2/2)t)$ and the latter is a martingale under $\widetilde{\mathsf{P}}$. From equation (10.3.9) with equation (10.3.8) we see that $x \mapsto V(t,x)$ is continuous at x_0 uniformly over $t \in [t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$ (small enough) whenever $(t_0, x_0) \in \langle 0, T] \times [0, \infty\rangle$ is given and fixed. Thus, to prove that V is continuous on $\langle 0, T] \times [0, \infty\rangle$ it is enough to show that $t \mapsto V(t,x)$ is continuous on $\langle 0, T]$ for each $x \geq 0$ given and fixed. For this, take any $t_1 < t_2$ in $\langle 0, T]$ and $\varepsilon > 0$, and let τ_1^ε be a stopping time such that $\widetilde{\mathsf{E}}((1 - (X^x_{t_1+\tau_1^\varepsilon})/(t_1 + \tau_1^\varepsilon))^+) \geq V(t_1, x) - \varepsilon$. Setting $\tau_2^\varepsilon = \tau_1^\varepsilon \wedge (T - t_2)$ we see that $V(t_2, x) \geq \widetilde{\mathsf{E}}((1 - (X_{t_2+\tau_2^\varepsilon})/(t_2 + \tau_2^\varepsilon))^+)$. Hence we get:

$$\begin{aligned} V(t_1,x) - V(t_2,x) &\leq \widetilde{\mathsf{E}}\left(\left(1 - \frac{X^x_{t_1+\tau_1^\varepsilon}}{t_1 + \tau_1^\varepsilon}\right)^+\right) - \widetilde{\mathsf{E}}\left(\left(1 - \frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2 + \tau_2^\varepsilon}\right)^+\right) + \varepsilon \\ &\leq \widetilde{\mathsf{E}}\left(\left(\frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2 + \tau_2^\varepsilon} - \frac{X^x_{t_1+\tau_1^\varepsilon}}{t_1 + \tau_1^\varepsilon}\right)^+\right) + \varepsilon. \end{aligned} \tag{10.3.10}$$

Letting first $t_2 - t_1 \to 0$ using $\tau_1^\varepsilon - \tau_2^\varepsilon \to 0$ and then $\varepsilon \downarrow 0$ we see that $\limsup_{t_2 - t_1 \to 0}(V(t_1, x) - V(t_2, x)) \le 0$ by dominated convergence. On the other hand, let τ_2^ε be a stopping time such that $\widetilde{\mathsf{E}}((1 - (X^x_{t_2+\tau_2^\varepsilon})/(t_2 + \tau_2^\varepsilon))^+) \ge V(t_2, x) - \varepsilon$. Then we have:

$$V(t_1, x) - V(t_2, x) \ge \widetilde{\mathsf{E}}\Bigg(\bigg(1 - \frac{X^x_{t_1+\tau_2^\varepsilon}}{t_1 + \tau_2^\varepsilon}\bigg)^+\Bigg) - \widetilde{\mathsf{E}}\Bigg(\bigg(1 - \frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2 + \tau_2^\varepsilon}\bigg)^+\Bigg) - \varepsilon. \tag{10.3.11}$$

Letting first $t_2 - t_1 \to 0$ and then $\varepsilon \downarrow 0$ we see that $\liminf_{t_2 - t_1 \to 0}(V(t_1, x) - V(t_2, x)) \ge 0$. Combining the two inequalities we find that $t \mapsto V(t, x)$ is continuous on $\langle 0, T]$. This completes the proof of the initial claim.

4. Denote the gain function by $G(t, x) = (1 - x/t)^+$ for $(t, x) \in \langle 0, T] \times [0, \infty\rangle$ and introduce the continuation set $C = \{ (t, x) \in \langle 0, T\rangle \times [0, \infty\rangle \mid V(t, x) > G(t, x) \}$ and the stopping set $S = \{ (t, x) \in \langle 0, T\rangle \times [0, \infty\rangle \mid V(t, x) = G(t, x) \}$. Since V and G are continuous, we see that C is open and S is closed in $\langle 0, T\rangle \times [0, \infty\rangle$. Standard arguments based on the strong Markov property (cf. Shiryaev (1978) [17]) show that the first hitting time $\tau_S = \inf\{ 0 \le s \le T - t \mid (t + s, X_{t+s}) \in S \}$ is optimal in equation (10.2.9) as well as that V is $C^{1,2}$ on C and satisfies equation (10.2.10). In order to determine the structure of the optimal stopping time τ_S (i.e. the shape of the sets C and S), we will first examine basic properties of the diffusion process X solving equation (10.2.6) under $\widetilde{\mathsf{P}}$.

5. The state space of X equals $[0, \infty\rangle$ and it is clear from the representation (equation (10.2.5)) with equation (10.2.8) that 0 is an entrance boundary point. The drift of X is given by $\mu(x) = 1 - rx$ and the diffusion coefficient of X is given by $\sigma(x) = \sigma x$ for $x \ge 0$. Hence, we see that $\mu(x)$ is greater/less than 0, if and only if, x is less/greater than $1/r$. This shows that there is a permanent push (drift) of X towards the constant level $1/r$ (when X is above $1/r$ the push of X is downwards and when X is below $1/r$ the push of X is upwards). The scale function of X is given by $s(x) = \int_1^x y^{2r/\sigma^2} e^{2/\sigma^2 y}\, dy$ for $x > 0$, and the speed measure of X is given by $m(dx) = (2/\sigma^2)\, x^{-2(1+r/\sigma^2)}\, e^{-2/\sigma^2 x}\, dx$ on the Borel σ-algebra of $\langle 0, \infty\rangle$. Since $s(0) = -\infty$ and $s(\infty) = +\infty$, we see that X is recurrent. Moreover, since $\int_0^\infty m(dx) = (2/\sigma^2)^{-2r/\sigma^2}\, \Gamma(1 + 2r/\sigma^2)$ is finite we find that X has an invariant probability density function given by:

$$f(x) = \frac{(2/\sigma^2)^{1+2r/\sigma^2}}{\Gamma(1 + 2r/\sigma^2)} \, \frac{1}{x^{2(1+r/\sigma^2)}} \, e^{-2/\sigma^2 x} \tag{10.3.12}$$

for $x > 0$. In particular, it follows that $X_t/t \to 0$ $\widetilde{\mathsf{P}}$-a.s. as $t \to \infty$. This fact has an important consequence for the optimal stopping problem (equation (10.2.9)): if the horizon T is infinite, then it is never optimal to stop. Indeed, in this case letting $\tau \equiv t$ and passing to the limit for $t \to \infty$ we see that $V \equiv 1$ on $\langle 0, \infty\rangle \times [0, \infty\rangle$. This shows that the infinite horizon formulation of the problem (equation (10.2.9)) provides no useful information to the finite horizon formulation (such as in Peskir (2005, 2003) [12] [13], for example). To examine the latter beyond the trivial fact that all points (t, x) with $x \ge t$ belong to C (which is easily seen by considering the hitting times $\tau_\varepsilon = \inf\{ 0 \le s \le T - t \mid X_{t+s} \le (t + s) - \varepsilon \}$ and noting that $\widetilde{\mathsf{P}}_{t,x}(0 < \tau_\varepsilon < T - t) > 0$ if $x \ge t$ with $0 < t < T$), we will examine the gain process in the problem (equation (10.2.9)) using stochastic calculus as follows.

6. Setting $\alpha(t) = t$ for $0 \le t \le T$ to denote the diagonal in the state space and applying the local time–space formula (cf. Peskir (2002) [11]) under $\widetilde{\mathsf{P}}_{t,x}$ when $(t,x) \in \langle 0,T\rangle \times [0,\infty\rangle$ is given and fixed, we get:

$$
\begin{aligned}
G(t+s, X_{t+s}) &= G(t,x) + \int_0^s G_t(t+u, X_{t+u})\,du \\
&\quad + \int_0^s G_x(t+u, X_{t+u})\,dX_{t+u} + \frac{1}{2}\int_0^s G_{xx}(t+u, X_{t+u})\,d\langle X,X\rangle_{t+u} \\
&\quad + \frac{1}{2}\int_0^s \Big(G_x(t+u, \alpha(t+u)+) - G_x(t+u, \alpha(t+u)-)\Big)\,d\ell^{\alpha}_{t+u}(X) \\
&= G(t,x) + \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < \alpha(t+u)\big)\,du \\
&\quad - \sigma \int_0^s \frac{X_{t+u}}{t+u}\, I\big(X_{t+u} < \alpha(t+u)\big)\,d\widehat{B}_u + \frac{1}{2}\int_0^s \frac{d\ell^{\alpha}_{t+u}(X)}{t+u} \qquad (10.3.13)
\end{aligned}
$$

where $\ell^{\alpha}_{t+u}(X)$ is the local time of X on the curve α given by:

$$
\begin{aligned}
\ell^{\alpha}_{t+u}(X) &= \widetilde{\mathsf{P}}\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^u I\big(\alpha(t+v) - \varepsilon < X_{t+v} < \alpha(t+v) + \varepsilon\big)\,d\langle X,X\rangle_{t+v} \\
&= \widetilde{\mathsf{P}}\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^u I\big(\alpha(t+v) - \varepsilon < X_{t+v} < \alpha(t+v) + \varepsilon\big)\,\frac{\sigma^2}{2}\,X^2_{t+v}\,dv \qquad (10.3.14)
\end{aligned}
$$

and $d\ell^{\alpha}_{t+u}(X)$ refers to the integration with respect to the continuous increasing function $u \mapsto \ell^{\alpha}_{t+u}(X)$. From equation (10.3.13) we respectively read:

$$
G(t+s, X_{t+s}) = G(t,x) + A_s + M_s + L_s \qquad (10.3.15)
$$

where A and L are processes of bounded variation (L is increasing) and M is a continuous (local) martingale. We note, moreover, that $s \mapsto L_s$ is strictly increasing only when $X_s = \alpha(s)$ for $0 \le s \le T-t$, i.e. when X visits α. On the other hand, when X is below α then the integrand $a(t+u, X_{t+u})$ of A_s may be either positive or negative. To determine both regions exactly, we need to examine the sign of the expression $a(t,x) = x/t^2 - (1 - rx)/t$. It follows that $a(t,x)$ is larger/less than 0, if and only if, x is larger/less than $\gamma(t)$ where $\gamma(t) = t/(1+rt)$ for $0 \le t \le T$. By considering the exit times from small balls in $\langle 0,T\rangle \times [0,\infty\rangle$ with centre at (t,x) and making use of equation (10.3.13) with the optional sampling theorem (to get rid of the martingale part), upon observing that $\alpha(t) < \gamma(t)$ for all $0 < t \le T$ so that the local time part is zero, we see that all points (t,x) lying above the curve γ (i.e. $x > \gamma(t)$ for $0 < t < T$) belong to the continuation set C. Exactly the same arguments (based on the fact that the favourable regions above γ and on α are far away from X) show that for each $x < \gamma(T) = T/(1+rT)$ given and fixed, all points (t,x) belong to the stopping set S when t is close to T. Moreover, recalling equation (10.3.8) and the fact that $V(t,x) \ge G(t,x)$ for all $x \ge 0$ with $t \in \langle 0,T\rangle$ fixed, we see that for each $t \in \langle 0,T\rangle$ there is a point $b(t) \in [0,\gamma(t)]$ such that $V(t,x) > G(t,x)$ for $x > b(t)$ and $V(t,x) = G(t,x)$ for $x \in [0,b(t)]$. Combining it with the previous conclusion on S we find that $b(T-) = \gamma(T) = T/(1+rT)$. (Yet another argument for this identity will be given below. Note that this identity is different from the identity $b(T-) = T$ used in Hansen and Jørgensen (2000) [5, page 1126].) This establishes the existence of the non-trivial (non-zero) optimal stopping boundary b on a left-neighbourhood of T. We will now show that b extends (continuously and decreasingly) from the initial neighbourhood of T backward in time as long as it visits 0 at some time $t_0 \in [0, T\rangle$, and later in the second part of the proof below we will deduce that this t_0 is equal to 0. The key argument in the proof is provided by the following inequality. Notice that this inequality is not obvious *a priori* (unlike in Peskir (2005, 2003) [12] [13]) since $t \mapsto G(t, x)$ is increasing and the supremum in equation (10.2.9) is taken over a smaller class of stopping times $\tau \in [0, T-t]$ when t is larger.

7. We show that the inequality is satisfied:

$$V_t(t, x) \le G_t(t, x) \tag{10.3.16}$$

for all $(t, x) \in C$. (It may be noted from equation (10.2.10) that $V_t = -(1 - rx)V_x - (\sigma^2/2)x^2 V_{xx} \le (1 - rx)/t$ since $V_x \ge -1/t$ and $V_{xx} \ge 0$ by equation (10.3.8), so that $V_t \le G_t$ holds above γ because $(1 - rx)/t \le x/t^2$, if and only if, $x \ge t/(1 + rt)$. Hence, the main issue is to show that equation (10.3.16) holds below γ and above b. Any analytic proof of this fact seems difficult and we resort to probabilistic arguments.)

To prove equation (10.3.16), fix $0 < t < t + h < T$ and $x \ge 0$ so that $x \le \gamma(t)$. Let $\tau = \tau_S(t + h, x)$ be the optimal stopping time for $V(t + h, x)$. Since $\tau \in [0, T-t-h] \subseteq [0, T-t]$, we see that $V(t, x) \ge \widetilde{\mathsf{E}}_{t,x}((1 - X_{t+\tau}/(t + \tau))^+)$ so that using the inequality stated prior to equation (10.3.9) above (and the convenient refinement by an indicator function), we get:

$$\begin{aligned}
&V(t+h, x) - V(t, x) - \Big(G(t+h, x) - G(t, x)\Big) \\
&\le \widetilde{\mathsf{E}}\bigg(\Big(1 - \frac{x + I_\tau}{(t + h + \tau)\ S_\tau}\Big)^+\bigg) - \widetilde{\mathsf{E}}\bigg(\Big(1 - \frac{x + I_\tau}{(t + \tau)\, S_\tau}\Big)^+\bigg) - \Big(\frac{x}{t} - \frac{x}{t+h}\Big) \\
&\le \widetilde{\mathsf{E}}\bigg(\Big(\frac{x + I_\tau}{(t + \tau)\, S_\tau} - \frac{x + I_\tau}{(t + h + \tau)\, S_\tau}\Big) I\Big(\frac{x + I_\tau}{(t + h + \tau)\, S_\tau} \le 1\Big)\bigg) - \frac{xh}{t\,(t+h)} \\
&= \widetilde{\mathsf{E}}\bigg(\frac{x + I_\tau}{S_\tau}\Big(\frac{1}{t+\tau} - \frac{1}{t+h+\tau}\Big) I\Big(\frac{x + I_\tau}{(t + h + \tau)\, S_\tau} \le 1\Big)\bigg) - \frac{xh}{t\,(t+h)} \\
&= \widetilde{\mathsf{E}}\bigg(\frac{x + I_\tau}{(t + h + \tau)\, S_\tau}\ \frac{h}{t+\tau}\ I\Big(\frac{x + I_\tau}{(t + h + \tau)\, S_\tau} \le 1\Big)\bigg) - \frac{xh}{t\,(t+h)} \\
&\le \frac{h}{t}\ \widetilde{\mathsf{E}}\bigg(\frac{x + I_\tau}{(t + h + \tau)\, S_\tau} I\Big(\frac{x + I_\tau}{(t + h + \tau)\, S_\tau} \le 1\Big)\bigg) - \frac{xh}{t\,(t+h)} \le 0
\end{aligned} \tag{10.3.17}$$

where the final inequality follows from the fact that with $Z := (x + I_\tau)/((t + h + \tau)S_\tau)$ we have $V(t + h, x) = \widetilde{\mathsf{E}}((1-Z)^+) = \widetilde{\mathsf{E}}((1-Z)\, I(Z \le 1)) = \widetilde{\mathsf{P}}(Z \le 1) - \widetilde{\mathsf{E}}(Z\, I(Z \le 1)) \ge G(t + h, x) = 1 - x/(t + h)$ so that $\widetilde{\mathsf{E}}(Z\, I(Z \le 1)) \le \widetilde{\mathsf{P}}(Z \le 1) - 1 + x/(t + h) \le x/(t + h)$ as claimed. Dividing the initial expression in equation (10.3.17) by h and letting $h \downarrow 0$ we obtain equation (10.3.16) for all $(t, x) \in C$ such that $x \le \gamma(t)$. Since $V_t \le G_t$ above γ (as stated following equation (10.3.16) above) this completes the proof of equation (10.3.16).

8. We show that $t \mapsto b(t)$ is increasing on $\langle 0, T\rangle$. This is an immediate consequence of equation (10.3.17). Indeed, if (t, x) belongs to C and t_0 from $\langle 0, T\rangle$ satisfies $t_0 < t_1$, then by equation (10.3.17) we have that $V(t_0, x) - G(t_0, x) \geq V(t_1, x) - G(t_1, x) > 0$ so that (t_0, x) must belong to C. It follows that b cannot be strictly decreasing thus proving the claim.

9. We show that the smooth-fit condition equation (10.2.12) holds, i.e. that $x \mapsto V(t, x)$ is C^1 at $b(t)$. For this, fix a point $(t, x) \in \langle 0, T\rangle \times \langle 0, \infty\rangle$ lying at the boundary so that $x = b(t)$. Then $x \leq \gamma(t) < \alpha(t)$ and for all $\varepsilon > 0$ such that $x + \varepsilon < \alpha(t)$ we have:

$$\frac{V(t, x+\varepsilon) - V(t, x)}{\varepsilon} \geq \frac{G(t, x+\varepsilon) - G(t, x)}{\varepsilon} = -\frac{1}{t}. \tag{10.3.18}$$

Letting $\varepsilon \downarrow 0$ and using that the limit on the left-hand side exists (since $x \mapsto V(t, x)$ is convex), we get the inequality:

$$\frac{\partial^+ V}{\partial x}(t, x) \geq \frac{\partial G}{\partial x}(t, x) = -\frac{1}{t}. \tag{10.3.19}$$

To prove the converse inequality, fix $\varepsilon > 0$ such that $x + \varepsilon < \alpha(t)$, and consider the stopping times $\tau_\varepsilon = \tau_S(t, x+\varepsilon)$ being optimal for $V(t, x+\varepsilon)$. Then we have:

$$\begin{aligned} \frac{V(t, x+\varepsilon) - V(t, x)}{\varepsilon} &\leq \frac{1}{\varepsilon}\left(\widetilde{\mathsf{E}}\left(\left(1 - \frac{x+\varepsilon+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\, S_{\tau_\varepsilon}}\right)^+ - \left(1 - \frac{x+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\, S_{\tau_\varepsilon}}\right)^+\right)\right) \\ &\leq \frac{1}{\varepsilon}\, \widetilde{\mathsf{E}}\left(\frac{x+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\, S_{\tau_\varepsilon}} - \frac{x+\varepsilon+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\, S_{\tau_\varepsilon}}\right) = -\,\widetilde{\mathsf{E}}\left(\frac{1}{(t+\tau_\varepsilon)\, S_{\tau_\varepsilon}}\right). \end{aligned} \tag{10.3.20}$$

Since each point x in $\langle 0, \infty\rangle$ is regular for X, and the boundary b is increasing, it follows that $\tau_\varepsilon \downarrow 0$ $\widetilde{\mathsf{P}} - a.s.$ as $\varepsilon \downarrow 0$. Letting $\varepsilon \downarrow 0$ in equation (10.3.20) we get:

$$\frac{\partial^+ V}{\partial x}(t, x) \leq -\frac{1}{t} \tag{10.3.21}$$

by dominated convergence. It follows from equation (10.3.19) and (10.3.21) that $(\partial^+ V/\partial x)(t, x) = -1/t$ implying the claim.

10. We show that b is continuous. Note that the same proof also shows that $b(T-) = T/(1+rT)$ as already established above by a different method.

Let us first show that b is right-continuous. For this, fix $t \in \langle 0, T\rangle$ and consider a sequence $t_n \downarrow t$ as $n \to \infty$. Since b is increasing, the right-hand limit $b(t+)$ exists. Because $(t_n, b(t_n)) \in S$ for all $n \geq 1$, and S is closed, it follows that $(t, b(t+)) \in S$. Hence by equation (10.2.16) we see $b(t+) \leq b(t)$. Since the reverse inequality follows obviously from the fact that b is increasing, this completes the proof of the first claim.

Let us next show that b is left-continuous. Suppose that there exists $t \in \langle 0, T\rangle$ such that $b(t-) < b(t)$. Fix a point x in $\langle b(t-), b(t)]$ and note by equation (10.2.12) that for $s < t$ we have:

$$V(s, x) - G(s, x) = \int_{b(s)}^{x} \int_{b(s)}^{y} \big(V_{xx}(s, z) - G_{xx}(s, z)\big)\, dz\, dy \tag{10.3.22}$$

upon recalling that V is $C^{1,2}$ on C. Note that $G_{xx} = 0$ below α so that if $V_{xx} \geq c$ on $R = \{ (u, y) \in C \mid s \leq u < t \text{ and } b(u) < y \leq x \}$ for some $c > 0$ (for all $s < t$ close enough to t and some $x > b(t-)$ close enough to $b(t-)$) then by letting $s \uparrow t$ in equation (10.3.22) we get:

$$V(t,x) - G(t,x) \geq c \, \frac{\big(x - b(t)\big)^2}{2} > 0 \tag{10.3.23}$$

contradicting the fact that (t,x) belongs to $\overline{D}$ and thus is an optimal stopping point. Hence, the proof reduces to showing that $V_{xx} \geq c$ on small enough R for some $c > 0$.

To derive the latter fact we may first note from equation (10.2.10) upon using equation (10.3.16) that $V_{xx} = (2/(\sigma^2 x^2))(-V_t - (1-rx)V_x) \geq (2/(\sigma^2 x^2))(-x/t^2 - (1-rx)V_x)$. Suppose now that for each $\delta > 0$ there is $s < t$ close enough to t and there is $x > b(t-)$ close enough to $b(t-)$ such that $V_x(u,y) \leq -1/u + \delta$ for all $(u,y) \in R$ (where we recall that $-1/u = G_x(u,y)$ for all $(u,y) \in R$). Then from the previous inequality, we find that $V_{xx}(u,y) \geq (2/(\sigma^2 y^2))(-y/u^2 + (1-ry)(1/u - \delta)) = (2/(\sigma^2 y^2))((u - y(1+ru))/u^2 - \delta(1-ru)) \geq c > 0$ for $\delta > 0$ small enough since $y < u/(1+ru) = \gamma(u)$ and $y < 1/r$ for all $(u,y) \in R$. Hence, the proof reduces to showing that $V_x(u,y) \leq -1/u + \delta$ for all $(u,y) \in R$ with R small enough when $\delta > 0$ is given and fixed.

To derive the latter inequality we can make use of the estimate (equation (10.3.20)) to conclude that

$$\frac{V(u, y+\varepsilon) - V(u,y)}{\varepsilon} \leq -\widetilde{\mathsf{E}}\left(\frac{1}{(u+\sigma_\varepsilon)\, M_{\sigma_\varepsilon}}\right) \tag{10.3.24}$$

where $\sigma_\varepsilon = \inf\{\, 0 \leq v \leq T-u \mid X^{y+\varepsilon}_{u+v} = b(u) \,\}$ and $M_t = \sup_{0 \leq s \leq t} S_s$. A simple comparison argument (based on the fact that b is increasing) shows that the supremum over all $(u,y) \in R$ on the right-hand side of equation (10.3.24) is attained at $(s, x+\varepsilon)$. Letting $\varepsilon \downarrow 0$ in equation (10.3.24), we thus get:

$$V_x(u,y) \leq -\widetilde{\mathsf{E}}\left(\frac{1}{(u+\sigma)\, M_\sigma}\right) \tag{10.3.25}$$

for all $(u,y) \in R$ where $\sigma = \inf\{\, 0 \leq v \leq T-s \mid X^x_{s+v} = b(s) \,\}$. Since by regularity of X we find that $\sigma \downarrow 0$ $\widetilde{\mathsf{P}}$-a.s. as $s \uparrow t$ and $x \downarrow b(t-)$, it follows from equation (10.3.25) that:

$$V_x(u,y) \leq -\frac{1}{u} + \widetilde{\mathsf{E}}\left(\frac{(u+\sigma)\, M_\sigma - u}{u\,(u+\sigma)\, M_\sigma}\right) \leq -\frac{1}{u} + \delta \tag{10.3.26}$$

for all $s < t$ close enough to t and some $x > b(t-)$ close enough to $b(t-)$. This completes the proof of the second claim, and thus the initial claim is proved as well.

11. We show that V is given by the formula shown in equation (10.3.6) and that b solves equation (10.3.5). For this, note that V satisfies the following conditions:

$$V \text{ is } C^{1,2} \text{ on } C \cup D \tag{10.3.27}$$

$$V_t + \mathbb{L}_X V \text{ is locally bounded} \tag{10.3.28}$$

$$x \mapsto V(t,x) \text{ is convex} \tag{10.3.29}$$

$$t \mapsto V_x(t, b(t)\pm) \text{ is continuous.} \tag{10.3.30}$$

Indeed, the conditions (10.3.27) and (10.3.28) follow from the facts that V is $C^{1,2}$ on C and $V = G$ on D upon recalling that D lies below γ so that $G(t,x) = 1 - x/t$ for all

$(t,x) \in D$ and thus G is $C^{1,2}$ on D. [When we say in condition (10.3.28) that $V_t + \mathbb{L}_X V$ is locally bounded, we mean that $V_t + \mathbb{L}_X V$ is bounded on $K \cap (C \cup D)$ for each compact set K in $[0,T] \times I\!R_+$.] The condition (10.3.29) was established in equation (10.3.8) above. The condition (10.3.30) follows from equation (10.2.12) since according to the latter we have $V_x(t, b(t)\pm) = -1/t$ for $t > 0$.

Since conditions (10.3.27)–(10.3.30) are satisfied, we know that the local time-space formula (cf. Theorem 3.1 in Peskir (2002) [11]) can be applied. This gives:

$$\begin{aligned} V(t+s, X_{t+s}) &= V(t,x) + \int_0^s \big(V_t + \mathbb{L}_X V\big)(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, du \\ &\quad + \int_0^s \sigma\, X_{t+u}\, V_x(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, dB_u \\ &\quad + \frac{1}{2}\int_0^s \big(V_x(t+u, X_{t+u}+) - V_x(t+u, X_{t+u}-)\big)\, I\big(X_{t+u} = b(t+u)\big)\, d\ell^b_{t+u}(X) \\ &= \int_0^s \big(G_t + \mathbb{L}_X G\big)(t+u, X_{t+u})\, I\big(X_{t+u} < b(t+u)\big)\, du + M_s \end{aligned} \tag{10.3.31}$$

where the final equality follows by the smooth-fit condition (10.2.12) and $M_s = \int_0^s \sigma X_{t+u} V_x(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, dB_u$ is a continuous martingale for $0 \le s \le T-t$ with $t > 0$. Noting that $(G_t + \mathbb{L}_X G)(t,x) = x/t^2 - (1-rx)/t$ for $x < t$ we see that equation (10.3.31) yields:

$$V(t+s, X_{t+s}) = V(t,x) + \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < b(t+u)\big)\, du + M_s. \tag{10.3.32}$$

Setting $s = T-t$, using that $V(T,x) = G(T,x)$ for all $x \ge 0$, and taking the $\widetilde{\mathsf{P}}_{t,x}$-expectation in equation (10.3.32), we find by the optional sampling theorem that:

$$\begin{aligned} &\widetilde{\mathsf{E}}_{t,x}\left(\left(1 - \frac{X_T}{T}\right)^+\right) \\ &\quad = V(t,x) + \int_0^{T-t} \widetilde{\mathsf{E}}_{t,x}\left(\left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < b(t+u)\big)\right) du. \end{aligned} \tag{10.3.33}$$

Making use of equations (10.3.1)–(10.3.3) we see that equation (10.3.33) is the formula (10.3.6). Moreover, inserting $x = b(t)$ in equation (10.3.33) and using that $V(t, b(t)) = G(t, b(t)) = 1 - b(t)/t$, we see that b satisfies the equation (10.3.5) as claimed.

12. We show that $b(t) > 0$ for all $0 < t \le T$ and that $b(0+) = 0$. For this, suppose that $b(t_0) = 0$ for some $t_0 \in \langle 0, T\rangle$ and fix $t \in \langle 0, t_0\rangle$. Then, $(t,x) \in C$ for all $x > 0$ as small as desired. Taking any such $(t,x) \in C$ and denoting by $\tau_S = \tau_S(t,x)$ the first hitting time to S under $\widetilde{\mathsf{P}}_{t,x}$, we find by equation (10.3.32) that:

$$\begin{aligned} V(t+\tau_S, X_{t+\tau_S}) = G(t+\tau_S, X_{t+\tau_S}) &= \left(1 - \frac{X_{t+\tau_S}}{t+\tau_S}\right)^+ = V(t,x) + M_{t+\tau_S} \\ &= 1 - \frac{x}{t} + M_{t+\tau_S}. \end{aligned} \tag{10.3.34}$$

Taking the $\widetilde{\mathsf{P}}_{t,x}$-expectation and letting $x \downarrow 0$ we get:

$$\widetilde{\mathsf{E}}_{t,0}\left(1 - \frac{X_{t+\tau_S}}{t+\tau_S}\right)^+ = 1 \tag{10.3.35}$$

where $\tau_S = \tau_S(t,0)$. As clearly $\widetilde{\mathsf{P}}_{t,0}(X_{t+\tau_S} \geq T) > 0$ we see that the left-hand side of equation (10.3.35) is strictly smaller than 1, thus contradicting the identity. This shows that $b(t)$ must be strictly positive for all $0 < t \leq T$. Combining this conclusion with the known inequality $b(t) \leq \gamma(t)$, which is valid for all $0 < t \leq T$, we see that $b(0+) = 0$ as claimed.

13. We show that b is the unique solution of the nonlinear integral equation (10.3.5) in the class of continuous functions $c : \langle 0, T\rangle \to I\!\!R$ satisfying $0 < c(t) < t/(1+rt)$ for all $0 < t < T$. (Note that this class is larger than the class of functions having the established properties of b which is, moreover, known to be increasing.) The proof of the uniqueness will be presented in the final three steps of the main proof as follows.

14. Let $c : \langle 0, T] \to I\!\!R$ be a continuous solution of the equation (10.3.5) satisfying $0 < c(t) < t$ for all $0 < t < T$. We want to show that this c must then be equal to the optimal stopping boundary b.

Motivated by the derivation (10.3.31)–(10.3.33) which leads to the formula (10.3.6), let us consider the function $U^c : \langle 0, T] \times [0, \infty\rangle \to I\!\!R$ defined as follows:

$$\begin{aligned} U^c(t,x) = {} & \widetilde{\mathsf{E}}_{t,x}\left(\left(1 - \frac{X_T}{T}\right)^+\right) \\ & - \int_0^{T-t} \widetilde{\mathsf{E}}_{t,x}\left(\left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < c(t+u)\big)\right) du \end{aligned} \tag{10.3.36}$$

for $(t,x) \in \langle 0, T] \times [0, \infty\rangle$. In terms of equations (10.3.1)–(10.3.3), note that U^c is explicitly given by:

$$\begin{aligned} U^c(t,x) = {} & F(T-t, x) \\ & - \int_0^{T-t} \frac{1}{t+u}\left(\left(\frac{1}{t+u} + r\right) G\big(u, x, c(t+u)\big) - H\big(u, x, c(t+u)\big)\right) du \end{aligned} \tag{10.3.37}$$

for $(t,x) \in \langle 0, T] \times [0, \infty\rangle$. Observe that the fact that c solves equation (10.3.5) on $\langle 0, T\rangle$ means exactly that $U^c(t, c(t)) = G(t, c(t))$ for all $0 < t < T$. We will now, moreover, show that $U^c(t,x) = G(t,x)$ for all $x \in [0, c(t)]$ with $t \in \langle 0, T\rangle$. This is the key point in the proof (cf. Peskir (2005, 2003)[12] [13]) that can be derived using a martingale argument as follows.

If $X = (X_t)_{t \geq 0}$ is a Markov process (with values in a general state space) and we set $F(t,x) = \mathsf{E}_x(G(X_{T-t}))$ for a (bounded) measurable function G with $\mathsf{P}_x(X_0 = x) = 1$, then the Markov property of X implies that $F(t, X_t)$ is a martingale under P_x for $0 \leq t \leq T$. Similarly, if we set $F(t,x) = \mathsf{E}_x(\int_0^{T-t} H(X_u)\, du)$ for a (bounded) measurable function H with $\mathsf{P}_x(X_0 = x) = 1$, then the Markov property of X implies that $F(t, X_t) + \int_0^t H(X_u)\, du$ is a martingale under P_x for $0 \leq t \leq T$. Combining these two martingale facts applied to the time–space Markov process $(t+s, X_{t+s})$ instead of X_s, we find that:

$$U^c(t+s, X_{t+s}) - \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < c(t+u)\big)\, du \tag{10.3.38}$$

is a martingale under $\widetilde{\mathsf{P}}_{t,x}$ for $0 \le s \le T-t$. We may thus write:

$$U^c(t+s, X_{t+s}) - \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < c(t+u)\big)\, du = U^c(t,x) + N_s \tag{10.3.39}$$

where $(N_s)_{0 \le s \le T-t}$ is a martingale with $N_0 = 0$ under $\widetilde{\mathsf{P}}_{t,x}$.

On the other hand, we know from equation (10.3.13) that:

$$\begin{aligned} G(t+s, X_{t+s}) = G(t,x) + \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < \alpha(t+u)\big)\, du \\ + M_s + L_s \end{aligned} \tag{10.3.40}$$

where$M_s = -\sigma \int_0^s (X_{t+u}/(t+u))\, I(X_{t+u} < \alpha(t+u))\, d\widehat{B}_u$ is a continuous martingale under $\widetilde{\mathsf{P}}_{t,x}$ and $L_s = (1/2) \int_0^s d\ell^{\alpha}_{t+u}(X)/(t+u)$ is an increasing process for $0 \le s \le T-t$.

For $0 \le x \le c(t)$ with $t \in \langle 0, T \rangle$ given and fixed, consider the stopping time:

$$\sigma_c = \inf\{\, 0 \le s \le T-t \mid X_{t+s} \ge c(t+s) \,\}. \tag{10.3.41}$$

Using that $U^c(t, c(t)) = G(t, c(t))$ for all $0 < t < T$ (since c solves equation (10.3.5) as pointed out above) and that $U^c(T, x) = G(T, x)$ for all $x \ge 0$, we see that $U^c(t + \sigma_c, X_{t+\sigma_c}) = G(t+\sigma_c, X_{t+\sigma_c})$. Hence from equations (10.3.39) and (10.3.40) using the optional sampling theorem we find:

$$\begin{aligned} U^c(t,x) &= \widetilde{\mathsf{E}}_{t,x}\Big(U^c(t+\sigma_c, X_{t+\sigma_c})\Big) \\ &\quad - \widetilde{\mathsf{E}}_{t,x}\left(\int_0^{\sigma_c} \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < c(t+u)\big)\, du \right) \\ &= \widetilde{\mathsf{E}}_{t,x}\Big(G(t+\sigma_c, X_{t+\sigma_c})\Big) \\ &\quad - \widetilde{\mathsf{E}}_{t,x}\left(\int_0^{\sigma_c} \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < c(t+u)\big)\, du \right) \\ &= G(t,x) + \widetilde{\mathsf{E}}_{t,x}\left(\int_0^{\sigma_c} \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < \alpha(t+u)\big)\, du \right) \\ &\quad - \widetilde{\mathsf{E}}_{t,x}\left(\int_0^{\sigma_c} \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < c(t+u)\big)\, du \right) \\ &= G(t,x) \end{aligned} \tag{10.3.42}$$

since $X_{t+u} < \alpha(t+u)$ and $X_{t+u} < c(t+u)$ for all $0 \le u < \sigma_c$. This proves that $U^c(t,x) = G(t,x)$ for all $x \in [0, c(t)]$ with $t \in \langle 0, T \rangle$ as claimed.

15. We show that $U^c(t,x) \le V(t,x)$ for all $(t,x) \in \langle 0,T] \times [0,\infty\rangle$. For this, consider the stopping time:

$$\tau_c = \inf\{\, 0 \le s \le T-t \mid X_{t+s} \le c(t+s)\,\} \tag{10.3.43}$$

under $\widetilde{\mathsf{P}}_{t,x}$ with $(t,x) \in \langle 0,T] \times [0,\infty\rangle$ given and fixed. The same arguments as those given following equation (10.3.41) above show that $U^c(t+\tau_c, X_{t+\tau_c}) = G(t+\tau_c, X_{t+\tau_c})$. Inserting τ_c instead of s in equation (10.3.39) and using the optional sampling theorem we get:

$$U^c(t,x) = \widetilde{\mathsf{E}}_{t,x}\Big(U^c(t+\tau_c, X_{t+\tau_c})\Big) = \widetilde{\mathsf{E}}_{t,x}\Big(G(t+\tau_c, X_{t+\tau_c})\Big) \le V(t,x) \tag{10.3.44}$$

where the final inequality follows from the definition of V proving the claim.

16. We show that $c \ge b$ on $[0,T]$. For this, consider the stopping time:

$$\sigma_b = \inf\{\, 0 \le s \le T-t \mid X_{t+s} \ge b(t+s)\,\} \tag{10.3.45}$$

under $\widetilde{\mathsf{P}}_{t,x}$ where $(t,x) \in \langle 0,T\rangle \times [0,\infty\rangle$ such that $x < b(t) \wedge c(t)$. Inserting σ_b in place of s in equations (10.3.32) and (10.3.39) and using the optional sampling theorem we get:

$$\widetilde{\mathsf{E}}_{t,x}\Big(V(t+\sigma_b, X_{t+\sigma_b})\Big) = G(t,x) + \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_b}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg)\,du\bigg) \tag{10.3.46}$$

$$\begin{aligned}&\widetilde{\mathsf{E}}_{t,x}\Big(U^c(t+\sigma_b, X_{t+\sigma_b})\Big)\\ &\quad = G(t,x) + \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_b}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg) I\big(X_{t+u} < c(t+u)\big)\,du\bigg)\end{aligned} \tag{10.3.47}$$

where we also use that $V(t,x) = U^c(t,x) = G(t,x)$ for $x < b(t) \wedge c(t)$. Since $U^c \le V$ it follows from equations (10.3.46) and (10.3.47) that:

$$\widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_b}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg) I\big(X_{t+u} \ge c(t+u)\big)\,du\bigg) \ge 0. \tag{10.3.48}$$

Due to the fact that $b(t) < t/(1+rt)$ for all $0 < t < T$, we see that $X_{t+u}/(t+u)^2 - (1 - rX_{t+u})/(t+u) < 0$ in equation (10.3.48), so that by the continuity of b and c it follows that $c \ge b$ on $[0,T]$ as claimed.

17. We show that c must be equal to b. For this, let us assume that there is $t \in \langle 0,T\rangle$ such that $c(t) > b(t)$. Pick $x \in \langle b(t), c(t)\rangle$ and consider the stopping time τ_b from equation (10.2.17). Inserting τ_b instead of s in equations (10.3.32) and (10.3.39) and using the optional sampling theorem, we get:

$$\widetilde{\mathsf{E}}_{t,x}\Big(G(t+\tau_b, X_{t+\tau_b})\Big) = V(t,x) \tag{10.3.49}$$

$$\begin{aligned}&\widetilde{\mathsf{E}}_{t,x}\Big(G(t+\tau_b, X_{t+\tau_b})\Big)\\ &\quad = U^c(t,x) + \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\tau_b}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg) I\big(X_{t+u} < c(t+u)\big)\,du\bigg)\end{aligned} \tag{10.3.50}$$

where we also use that $V(t+\tau_b, X_{t+\tau_b}) = U^c(t+\tau_b, X_{t+\tau_b}) = G(t+\tau_b, X_{t+\tau_b})$ upon recalling that $c \geq b$ and $U^c = G$ either below c or at T. Since $U^c \leq V$ we see from equations (10.3.49) and (10.3.50) that:

$$\widetilde{\mathsf{E}}_{t,x}\left(\int_0^{\tau_b}\left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < c(t+u)\big)\, du\right) \geq 0. \tag{10.3.51}$$

Due to the fact that $c(t) < t/(1+rt)$ for all $0 < t < T$ by assumption, we see that $X_{t+u}/(t+u)^2 - (1-rX_{t+u})/(t+u) < 0$ in equation (10.3.51) so that by the continuity of b and c it follows that such a point (t,x) cannot exist. Thus c must be equal to b, and the proof is complete.

10.4 REMARKS ON NUMERICS

1. The following method can be used to calculate the optimal stopping boundary b numerically by means of the integral equation (10.3.5). Note that the formula (10.3.6) can be used to calculate the arbitrage-free price V when b is known.

Set $t_i = ih$ for $i = 0, 1, \ldots, n$ where $h = T/n$ and denote:

$$J(t, b(t)) = 1 - \frac{b(t)}{t} - F(T-t, b(t)) \tag{10.4.1}$$

$$K(t, b(t); t+u, b(t+u)) \tag{10.4.2}$$

$$= \frac{1}{t+u}\left(\left(\frac{1}{t+u} + r\right) G(u, b(t), b(t+u)) - H(u, b(t), b(t+u))\right). \tag{10.4.3}$$

Then, the following discrete approximation of the integral equation (10.3.5) is valid:

$$J(t_i, b(t_i)) = \sum_{j=i+1}^{n} K(t_i, b(t_i); t_j, b(t_j))\, h \tag{10.4.4}$$

for $i = 0, 1, \ldots, n-1$. Letting $i = n-1$ and $b(t_n) = T/(1+rT)$ we can solve equation (10.4.4) numerically and get a number $b(t_{n-1})$. Letting $i = n-2$ and using the values of $b(t_{n-1})$ and $b(t_n)$ we can solve equation (10.4.4) numerically and get a number $b(t_{n-2})$. Continuing the recursion, we obtain $b(t_n), b(t_{n-1}), \ldots, b(t_1), b(t_0)$ as an approximation of the optimal stopping boundary b at points $0, h, \ldots, T-h, T$.

It is an interesting numerical problem to show that the approximation converges to the true function b on $[0, T]$ as $h \downarrow 0$. Another interesting problem is to derive the rate of convergence.

2. To perform the previous recursion, we need to compute the functions F, G, H from equations (10.3.1)–(10.3.3) as efficiently as possible. Simply by observing the expressions (10.3.1)–(10.3.4) it is apparent that finding these functions numerically is not trivial. Moreover, the nature of the probability density function f in expression (10.3.4) presents a further numerical challenge. Part of this probability density function is the Hartman–Watson density discussed in Barrieu *et al.* (2003) [1]. As t tends to zero, the numerical estimate of the Hartman–Watson density oscillates, with the oscillations increasing rapidly in both amplitude and frequency as t gets closer to zero. Barrieu *et al.* (2003) [1] mention that this may be a consequence of the fact that $t \mapsto \exp(2\pi^2/\sigma^2 t)$ rapidly increases to infinity while $z \mapsto \sin(4\pi z/\sigma^2 t)$ oscillates more and more frequently. This rapid oscillation makes accurate estimation of $f(t, s, a)$ with t close to zero very difficult.

The problems when dealing with t close to zero are relevant to pricing the early exercise Asian call option. To find the optimal stopping boundary b as the solution to the implicit equation (10.4.4), it is necessary to work backward from T to 0. Thus, to get an accurate estimate for b when $b(T)$ is given, the next estimate of $b(u)$ must be found for some value of u close to T so that $t = T-u$ will be close to zero.

Even if we get an accurate estimate for f, to solve equations (10.3.1)–(10.3.3) we need to evaluate two nested integrals. This is slow computationally. A crude attempt has been made at storing values for f and using these to estimate F, G, H in equations (10.3.1)–(10.3.3) but this method has not produced reliable results.

3. Another approach to finding the functions F, G, H from equations (10.3.1)–(10.3.3) can be based on numerical solutions of partial differential equations. Two distinct methods are available.

Consider the transition probability density of the process X given by:

$$p(s, x; t, y) = \frac{d}{dy} \widetilde{\mathsf{P}}(X_t \le y \mid X_s = x) \tag{10.4.5}$$

where $0 \le s < t$ and $x, y \ge 0$. Since $p(s, x; t, y) = p(0, x; t-s, y)$, we see that there is no restriction to assume that $s = 0$ in the sequel.

4. The *forward equation* approach leads to the initial-value problem:

$$p_t = -((1-ry)\,p)_y + (D\,y\,p)_{yy} \quad (\,t > 0\,,\ y > 0\,) \tag{10.4.6}$$

$$p(0, x; 0+, y) = \delta(y-x) \quad (\,y \ge 0\,) \tag{10.4.7}$$

where $D = \sigma^2/2$ and $x \ge 0$ is given and fixed (recall that δ denotes the Dirac delta function). Standard results (cf. Feller (1952) [4]) imply that there is a unique non-negative solution $(t, y) \mapsto p(0, x; t, y)$ of equations (10.4.6) and (10.4.7). The solution p satisfies the following boundary conditions:

$$p(0, x; t, 0+) = 0 \quad (0 \text{ is entrance}) \tag{10.4.8}$$

$$p(0, x; t, \infty-) = 0 \quad (\infty \text{ is normal}). \tag{10.4.9}$$

The solution p satisfies the following integrability condition:

$$\int_0^\infty p(0, x; t, y)\, dy = 1 \tag{10.4.10}$$

for all $x \ge 0$ and all $t \ge 0$. Once the solution $(t, y) \mapsto p(0, x; t, y)$ of equations (10.4.6) and (10.4.7) has been found, the functions F, G, H from equations (10.3.1)–(10.3.3) can be computed using the general formula:

$$\widetilde{\mathsf{E}}_{0,x}(g(X_t)) = \int_0^\infty g(y)\, p(0, x; t, y)\, dy \tag{10.4.11}$$

upon choosing the appropriate function $g : I\!R_+ \to I\!R_+$.

5. The *backward equation* approach leads to the terminal-value problem:

$$q_t = (1-rx)\, q_x + D\, x^2\, q_{xx} \quad (\, t > 0\, ,\; x > 0\,) \tag{10.4.12}$$

$$q(T, x) = h(x) \quad (\, x \geq 0\,) \tag{10.4.13}$$

where $h : I\!R_+ \to I\!R_+$ is a given function. Standard results (cf. Feller (1952) [4]) imply that there is a unique non-negative solution $(t, x) \mapsto q(t, x)$ of equations (10.4.12) and (10.4.13). Taking $x \mapsto h(x)$ to be $x \mapsto (1-x/T)^+$ (with T fixed), $x \mapsto x\, I(x \leq y)$ (with y fixed), $x \mapsto I(x \leq y)$ (with y fixed), it follows that the unique non-negative solution q of equations (10.4.12) and (10.4.13) coincides with F, G, H from equations (10.3.1)–(10.3.3), respectively. (For numerical results of a similar approach, see Rogers and Shi (1995) [14].)

6. It is an interesting numerical problem to carry out either of the two methods described above and produce approximations to the optimal stopping boundary b by using equation (10.4.4). Another interesting problem is to derive the rate of convergence.

APPENDIX

In this section we derive the explicit expression for the probability density function f of (S_t, I_t) under $\widetilde{\mathsf{P}}$ with $S_0 = 1$ and $I_0 = 0$ given in equation (10.3.4) above.

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. With $t > 0$ and $\nu \in I\!R$ given and fixed, recall from Yor (1992, p. 527) [20] that the random variable $A_t^{(\nu)} = \int_0^t e^{2(B_s + \nu s)} ds$ has the conditional distribution:

$$\mathsf{P}\Big(A_t^{(\nu)} \in dy \;\Big|\; B_t + \nu t = x \Big) = a(t, x, y)\, dy \tag{10.A.1}$$

where the density function a for $y > 0$ is given by:

$$\begin{aligned} a(t, x, y) = {} & \frac{1}{\pi y^2} \exp\left(\frac{x^2 + \pi^2}{2t} + x - \frac{1}{2y}\Big(1 + e^{2x}\Big) \right) \\ & \times \int_0^\infty \exp\left(-\frac{z^2}{2t} - \frac{e^x}{y} \cosh(z) \right) \sinh(z)\, \sin\left(\frac{\pi z}{t} \right) dz. \end{aligned} \tag{10.A.2}$$

This implies that the random vector $\big(2(B_t + \nu t), A_t^{(\nu)}\big)$ has the distribution:

$$\mathsf{P}\Big(2(B_t + \nu t) \in dx,\, A_t^{(\nu)} \in dy \Big) = b(t, x, y)\, dx\, dy \tag{10.A.3}$$

where the density function b for $y > 0$ is given by:

$$\begin{aligned} b(t, x, y) &= a\left(t, \frac{x}{2}, y\right) \frac{1}{2\sqrt{t}}\, \varphi\left(\frac{x - 2\nu t}{2\sqrt{t}} \right) \\ &= \frac{1}{(2\pi)^{3/2} y^2 \sqrt{t}} \exp\left(\frac{\pi^2}{2t} + \Big(\frac{\nu + 1}{2}\Big) x - \frac{\nu^2}{2}\, t - \frac{1}{2y}\Big(1 + e^x\Big) \right) \\ &\quad \times \int_0^\infty \exp\left(-\frac{z^2}{2t} - \frac{e^{x/2}}{y} \cosh(z) \right) \sinh(z)\, \sin\left(\frac{\pi z}{t} \right) dz \end{aligned} \tag{10.A.4}$$

and we set $\varphi(z) = (1/\sqrt{2\pi})e^{-z^2/2}$ for $z \in I\!R$ (for related expressions in terms of Hermite functions, see Dufresne (2001) [3] and Schröder (2003) [15]).

Denoting $K_t = \alpha B_t + \beta t$ and $L_t = \int_0^t e^{\alpha B_s + \beta s} ds$ with $\alpha \neq 0$ and $\beta \in I\!R$ given and fixed, and using that the scaling property of B implies:

$$\mathsf{P}\left(\alpha B_t + \beta t \le x,\, \int_0^t e^{\alpha B_s + \beta s}\, ds \le y\right) = \mathsf{P}\left(2(B_{t'} + \nu t') \le x,\, \int_0^{t'} e^{2(B_s + \nu s)}\, ds \le \frac{\alpha^2}{4}\, y\right) \tag{10.A.5}$$

with $t' = \alpha^2 t/4$ and $\nu = 2\beta/\alpha^2$, it follows by applying equations (10.A.3) and (10.A.4) that the random vector (K_t, L_t) has the distribution:

$$\mathsf{P}\Big(K_t \in dx,\, L_t \in dy\Big) = c(t, x, y)\, dx\, dy \tag{10.A.6}$$

where the density function c for $y > 0$ is given by:

$$\begin{aligned} c(t, x, y) &= \frac{\alpha^2}{4}\, b\left(\frac{\alpha^2}{4}\, t,\, x,\, \frac{\alpha^2}{4}\, y\right) \\ &= \frac{2\sqrt{2}}{\pi^{3/2}\alpha^3}\, \frac{1}{y^2\sqrt{t}} \exp\left(\frac{2\pi^2}{\alpha^2 t} + \Big(\frac{\beta}{\alpha^2} + \frac{1}{2}\Big)x - \frac{\beta^2}{2\alpha^2}\, t - \frac{2}{\alpha^2 y}\Big(1 + e^x\Big)\right) \\ &\quad \times \int_0^\infty \exp\left(-\frac{2z^2}{\alpha^2 t} - \frac{4e^{x/2}}{\alpha^2 y} \cosh(z)\right) \sinh(z)\, \sin\left(\frac{4\pi z}{\alpha^2 t}\right) dz. \end{aligned} \tag{10.A.7}$$

From equations (10.2.8) and (10.2.3) we see that f satisfies:

$$f(t, s, a) = \frac{1}{s}\, c(t, \log(s), a) = \frac{1}{s}\, \frac{\alpha^2}{4}\, b\left(\frac{\alpha^2}{4}\, t,\, \log(s),\, \frac{\alpha^2}{4}\, a\right) \tag{10.A.8}$$

with $\alpha = \sigma$ and $\beta = r + \sigma^2/2$. Hence equation (10.3.4) follows by the final expression in equation (10.A.4).

REFERENCES

[1] Barrieu, P., Rouault, A. and Yor, M. (2003), "A study of the Hartman–Watson distribution motivated by numerical problems related to Asian options pricing", *Prépublication PMA* 813, Université Pierre et Marie Curie, Paris, France.

[2] Carr, P., Jarrow, R. and Myneni, R. (1992), "Alternative characterizations of American put options", *Mathematical Finance*, **2**, 78–106.

[3] Dufresne, D. (2001), "The integral of geometric Brownian motion", *Advances in Applied Probability*, **33**, 223–241.

[4] Feller, W. (1952), "The parabolic differential equations and the associated semi-groups of transformations", *Annals of Mathematics*, **55**, 468–519.

[5] Hansen, A.T. and Jørgensen, P.L. (2000), "Analytical valuation of American-style Asian options", *Management Science*, **46**, 1116–1136.

[6] Jacka, S.D. (1991), "Optimal stopping and the American put", *Mathematical Finance*, **1**, 1–14.

[7] Karatzas, I. and Shreve, S.E. (1998), *Methods of Mathematical Finance*, Springer-Verlag, Berlin, Germany.

[8] Kim, I.J. (1990), "The analytic valuation of American options", *Reviews in Financial Studies*, **3**, 547–572.

[9] Kolodner, I.I. (1956), "Free boundary problem for the heat equation with applications to problems of change of phase I. General method of solution", *Communications in Pure and Applied Mathematics*, **9**, 1–31.

[10] Kramkov, D.O. and Mordecky, E. (1994), "Integral option", *Theory of Probability and Applications*, **39**, 162–173.

[11] Peskir, G. (2002), "A change-of-variable formula with local time on curves", Research Report No. 428, Department of Theoretical Statistics, University of Aarhus, Denmark (30 pp.); *Journal of Theoretical Probability*, to be published.

[12] Peskir, G. (2005), "On the American option problem", Research Report No. 431, Department of Theoretical Statistics, University of Aarhus, Denmark (13 pp.); *Mathematical Finance*, **15**, 169–181.

[13] Peskir, G. (2003), "The Russian option: Finite horizon", Research Report No. 433, Department of Theoretical Statistics, University of Aarhus, Denmark (16 pp.); *Finance and Stochastics*, to be published.

[14] Rogers, L.C.G. and Shi, Z. (1995), "The value of an Asian option", *Journal of Applied Probability*, **32**, 1077–1088.

[15] Schröder, M. (2003), "On the integral of geometric Brownian motion", *Advances in Applied Probability*, **35**, 159–183.

[16] Shepp, L.A. and Shiryaev, A.N. (1994), "A new look at the Russian option", *Theory of Probability and Applications*, **39**, 103–119.

[17] Shiryaev, A.N. (1978), *Optimal Stopping Rules*, Springer-Verlag, Berlin, Germany.

[18] Shiryaev, A.N. (1999), *Essentials of Stochastic Finance (Facts, Models, Theory)*, World Scientific, Singapore.

[19] Wu, L., Kwok, Y.K. and Yu, H. (1999), "Asian options with the American early exercise feature", *International Journal of Theoretical and Applied Finance*, **1**, 101–111.

[20] Yor, M. (1992), "On some exponential functionals of Brownian motion", *Advances in Applied Probability*, **24**, 509–531.

11
Why be Backward? Forward Equations for American Options[†]

Peter Carr
Bloomberg LP, New York, NY, USA

and

Ali Hirsa
Caspian Capital Management, LLC, New York, NY, USA

Abstract

The purpose of this paper is to develop forward equations for standard American options. We assume that the returns on the underlying assets have stationary independent increments, or in other words, that the log price is a Lévy process. In all of these models, except for Black–Scholes, the existence of a jump component implies that the backward and forward equations contain an integral in addition to the usual partial derivatives. Despite the computational complications introduced by this term, we use finite differences to solve these fundamental partial integro-differential equations (PIDEs). Our approach to determining the forward equation for American options is to start with the well-known backward equation and then exploit the symmetries which essentially define Lévy processes. In the process of developing the forward equation, we also determine two hybrid equations of independent interest. To illustrate that our forward PIDE is a viable alternative to the traditional backward approach, we calculate American option values in the diffusion extended VG option pricing model.

11.1 INTRODUCTION

Valuing and hedging derivatives consistent with the volatility smile has been a major research focus for over a decade. A breakthrough occurred in the mid-1990s with the recognition that in certain models, European option values satisfied forward evolution equations in which the independent variables are the options' strike and maturity. More specifically, Dupire (1994) showed that under deterministic carrying costs and a diffusion process for the underlying price, no arbitrage implies that European option prices satisfy a certain partial differential equation (PDE), now called the Dupire equation. Assuming that one could observe European option prices of all strikes and maturities, then this forward PDE can be used to explicitly determine the underlying's instantaneous volatility as a function of the underlying's price

[†] The authors thank Dilip Madan and participants of the 2002 ICBI Barcelona conference and the 2002 Risk Boston conference. Errors are our own responsibility. The views expressed are those of the authors and do not necessarily reflect the position of Morgan Stanley. A shorter version of this paper was previously published in Risk, 16(1), pp. 103–107, 2003. The extended version is included here with permission of Incisive Media plc.

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

and time. Once this volatility function is known, the value function for European, American and many exotic options can be determined by a wide array of standard methods. As this value function relates theoretical prices of these instruments to the underlying's price and time, it can also be used to determine many greeks of interest as well.

Aside from their use in determining the volatility function, forward equations also serve a second useful purpose. Once one knows the volatility function, either by an explicit specification or by a prior calibration, the forward PDE can be numerically solved to efficiently value a collection of European options of different strikes and maturities, all written on the same underlying asset. Furthermore, as pointed out in Andreasen (1998), all the greeks of interest satisfy the same forward PDE and hence can also be efficiently determined in the same way.

Since the original development of forward equations for European options in continuous models, several extensions have been proposed. For example, Esser and Schlag (2002) develop forward equations for European options written on the forward price rather than the spot price. Forward equations for European options in jump diffusion models were developed in Andersen and Andreasen (1999) and extended by Andreasen and Carr (2002). It is straightforward to develop the relevant forward equations for European binary options or for European power options by differentiating or integrating the forward equation for standard European options. Buraschi and Dumas (2001) develop forward equations for compound options.* In contrast to the PDE's determined by others, their evolution equation is an ordinary differential equation whose sole independent variable is the intermediate maturity date.

Given the close relationship between compound options and American options, it seems plausible that there might be a forward equation for American options. The development of such an equation has important practical implications since all of the listed options on individual stocks are American-style. The Dupire equation cannot be used to infer the volatility function from market prices of American options, nor can it be used to efficiently value a collection of American options of differing strikes and maturities.

The purpose of this paper is to develop forward equations for standard American options. This problem is addressed for American calls on stocks paying discrete dividends in Buraschi and Dumas (2001) and it is also considered in a lattice setting in Chriss (1996). We direct our attention to the more difficult problem of pricing continuously exercisable American puts in continuous time models. To do so, we depart from the diffusive models which characterize most of the previous research on forward equations in continuous time. To capture the smile, we assume that prices jump rather than assuming that the instantaneous volatility is a function of stock price and time. Dumas *et al*. (1998) find little empirical support for the Dupire model whereas there is a long history of empirical support for jump-diffusion models.† In particular, we assume that the returns on the underlying asset have stationary independent increments, or in other words that the log price is a Lévy process. Besides the Black and Scholes (1973) model, our framework includes as special cases the variance gamma (VG) model of Madan *et al*. (1998), the CGMY model of Carr *et al*. (2002), the finite moment logstable model of Carr and Wu (2002), the Merton (1976) and Kou (2002) jump-diffusion models, and the hyperbolic models of Eberlein *et al*. (1998). In all of these models, except for Black–Scholes, the existence of a jump component implies that the backward and forward equations contain an integral in addition to the usual partial derivatives. Despite the

* However, their definition of a compound option is nonstandard in that the critical stock price is specified in the contract.

† For example, three recent papers documenting support for such models are Anderson *et al*. (2002), Carr *et al*. (2002) and Carr and Wu (2002).

computational complications introduced by this term, we use finite differences to solve both of these fundamental partial integro differential equations (PIDEs). To illustrate that our forward PIDE is a viable alternative to the traditional backward approach, we calculate American option values in the diffusion extended VG[‡] option pricing model and find very close agreement.

Our approach to determining the forward equation for American options is to start with the well-known backward equation and then exploit the symmetries which essentially define Lévy processes. In the process of developing the forward equation, we also determine two hybrid equations of independent interest. The advantage of these hybrid equations over the forward equation is that they hold in greater generality. Depending on the problem at hand, these hybrid equations can also have large computational advantages over the backward or forward equations when the model has already been calibrated. In particular, the advantage of these hybrid equations over the backward equation is that they are more computationally efficient when one is interested in the variation of prices or greeks across strike or maturity at a fixed time, e.g. market close.

The first of these hybrid equations has the stock price and maturity as independent variables. The numerical solution of this hybrid equation is an alternative to the backward equation in producing a spot slide, which shows how American option prices vary with the initial spot price of the underlying. If one is interested in understanding how this spot slide varies with maturity, then our hybrid equation is much more efficient than the backward equation.

Our second hybrid equation has the strike price and calendar time as independent variables. The numerical solution of this hybrid equation is an alternative to the forward equation in producing an implied volatility smile at a fixed maturity. If one is interested in understanding how the model predicts that this smile will change over time, then our hybrid equation is much more computationally efficient than the forward equation. This second hybrid equation also allows parameters to have a term structure, whereas our forward equation does not.[§] Hence, if one needs to efficiently value a collection of American options of different strikes in the time-dependent Black–Scholes model, then it is far more efficient to solve our hybrid equation than to use the standard backward equation.

The remainder of this paper is structured as follows. The next section introduces our setting and reviews the backward PIDE which governs American option values in this setting. The following section develops the first hybrid equation, while the subsequent section develops the second one. The penultimate section develops the forward equation for American options, while the final section summarizes and suggests further research.

11.2 REVIEW OF THE BACKWARD FREE BOUNDARY PROBLEM

Throughout this article, we focus on (standard) American puts on stocks leaving American calls and other underlyings as an exercise for the reader. We assume perfect capital markets, continuous trading, no arbitrage opportunities, continuous dividend payments and Markovian stock price dynamics under all martingale measures. We further assume that

[‡] For details on the use of finite differences for solving the backward PIDE for American options in the VG model, see Hirsa and Madan (2003).

[§] Note, however, that implied volatility can have a term or strike structure in our Lévy setting.

the spot interest rate and dividend yield are given by deterministic functions $r(t) > 0$ and $q(t) \geq 0$, respectively. Thus, we assume that under a risk-neutral measure Q, the stock price s_t satisfies the following stochastic differential equation:

$$ds_t = [r(t) - q(t)]s_{t-}dt + \sigma(s_{t-}, t)s_{t-}dW_t + \int_{-\infty}^{\infty} s_{t-}(e^x - 1)[\mu(dx, dt) - \nu(s_{t-}, x, t)\, dx\, dt], \tag{11.1}$$

for all $t \in [0, \overline{T}]$. Thus, the change in the stock price decomposes into three parts. The first part is the risk-neutral drift, comprised entirely of the dollar carrying cost of the stock. The second part is the diffusion part, expressed in terms of the instantaneous volatility function $\sigma(S, t)$. As usual, the term dW_t denotes increments of a standard Wiener process defined on the time set $[0, \overline{T}]$ and on a complete probability space $(\Omega, \mathcal{F}, Q)$. The third part is the jump part. The random measure $\mu(dx, dt)$ counts the number of jumps of size x in the log price at time t. The Hunt density $\{\nu(S, x, t), S > 0, x \in \Re, t \in [0, \overline{T}]\}$ is used to compensate the jump process $J_t \equiv \int_0^t \int_{-\infty}^{\infty} s_{t-}(e^x - 1)\mu(dx, ds)$, so that the last term in equation (11.1) is the increment of a Q jump martingale.¶ The jump martingale is specified in such a way that jumps to negative prices are impossible. Since the last two parts are both martingales, we have:

$$E^Q[s_t|s_0] = s_0 e^{\int_0^t [r(u)-q(u)]du},$$

where the initial stock price s_0 is positive.

Consider an American put option on the stock with a fixed strike price $K_0 > 0$ and a fixed maturity date $T_0 \in [0, \overline{T}]$. Let p_t denote the value of the American put at time $t \in [0, T_0]$. In this general setup, it is not yet known whether the American put value is monotone in S. Hence, we further assume whatever sufficient conditions on the coefficients that are needed so that the put value is monotone in S. Then, for each time $t \in [0, T_0]$, there exists a unique *critical stock price*, $\underline{s}(t)$, below which the American put should be exercised early, i.e.

$$\text{if } s_t \leq \underline{s}_t, \text{ then } p_t = \max[0, K_0 - s_t] \tag{11.2}$$

$$\text{and if } s_t > \underline{s}_t, \text{ then } p_t > \max[0, K_0 - s_t]. \tag{11.3}$$

The exercise boundary is the time path of critical stock prices, $\underline{s}_t$, $t \in [0, T_0]$. This boundary is independent of the current stock price s_0 and is bounded above by K_0. It is a smooth, nondecreasing function of time t whose terminal limit is:

$$\lim_{t \uparrow T_0} \underline{s}_t = K_0 \min\left[1, \frac{r(T_0)}{q(T_0)}\right].$$

Right at expiration, the critical stock price is the strike price, i.e. $\underline{s}_{T_0} = K_0$. Hence, when $q(T_0) > r(T_0)$, there is a discontinuity in the exercise boundary. Figure 11.1 plots the

¶ The function $\nu(S, x, t)$ must have the following properties:

$$\nu(S, 0, t) = 0, \quad \int_{-\infty}^{\infty} (x^2 \wedge 1)\nu(S, x, t)\, dx < \infty.$$

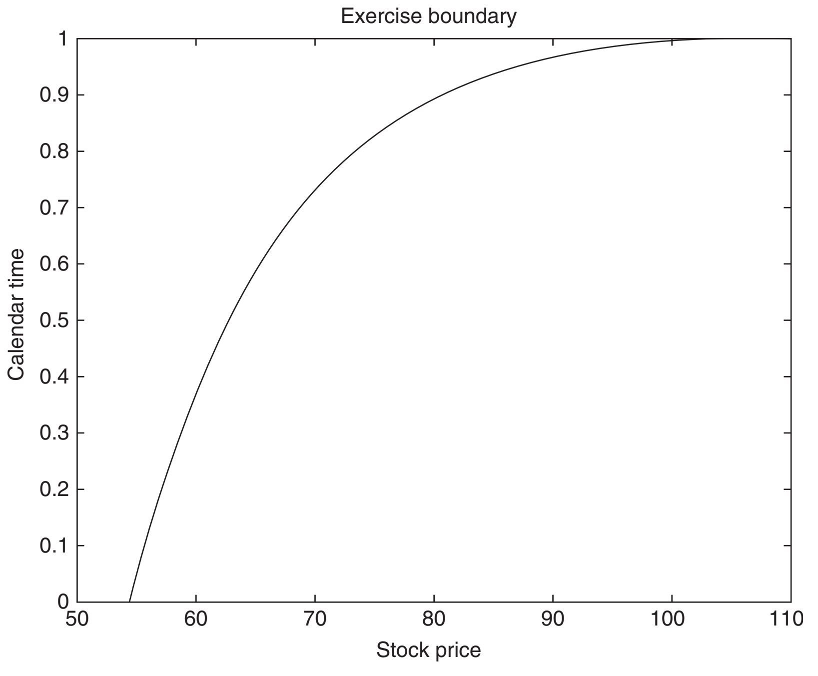

Figure 11.1 Exercise boundary in the DEVG model. Critical stock prices are computed from the DEVG model for the following inputs: $r = 0.06$; $q = 0.02$; $\sigma = 0.4$; $s = 0.3$; $v = 0.25$; $\theta = -0.3$; $K_0 = 110$; $T_0 = 1$. The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400 with initial price $S_0 = 100$

exercise boundary in the Diffusion Extended Variance Gamma (DEVG) model. This model extends the pure jump Variance Gamma model of Madan *et al.* (1998), by adding a diffusion component with constant volatility.

The American put value is also a function, denoted $p(s, t)$, mapping its domain $\mathcal{D} \equiv (s, t) \in [0, \infty) \times [0, T_0]$ into the nonnegative real line. The exercise boundary, $\underline{s}_t$, $t \in [0, T_0]$, divides this domain $\mathcal{D}$ into a *stopping region* $\mathcal{S} \equiv [0, \underline{s}_t] \times [0, T_0]$ and a *continuation region* $\mathcal{C} \equiv (\underline{s}_t, \infty) \times [0, T_0]$. Equation (11.2) indicates that in the stopping region, the put value function $p(s, t)$ equals its exercise value, $\max[0, K_0 - S]$. In contrast, the inequality expressed in equation (11.3) shows that in the continuation region, the put is worth more 'alive' than 'dead'. The transition between boundaries is smooth in the following sense:

$$\lim_{s \downarrow \underline{s}_t} p(s, t) = K_0 - \underline{s}_t, \qquad t \in [0, T_0] \tag{11.4}$$

$$\lim_{s \downarrow \underline{s}_t} \frac{\partial p(s,t)}{\partial s} = -1, \qquad t \in [0, T_0]. \tag{11.5}$$

The *value matching condition* (equation (11.4)) and equation (11.2) imply that the put value is continuous across the exercise boundary. Furthermore, the *high contact condition* (equation (11.5) and equation (11.2)) further imply that the put's *delta* is continuous. Equations (11.4) and (11.5) are jointly referred to as the *'smooth fit' conditions.*

The partial derivatives, $\frac{\partial p}{\partial t}$, $\frac{\partial p}{\partial s}$, and $\frac{\partial^2 p}{\partial s^2}$ exist and satisfy the following partial integro differential equation (PIDE):

$$\frac{\partial p(s,t)}{\partial t} + \frac{\sigma^2(s,t)s^2}{2}\frac{\partial^2 p(s,t)}{\partial s^2} + [r(t) - q(t)]s\frac{\partial p(s,t)}{\partial s} - r(t)p(s,t)$$
$$+ \int_{-\infty}^{\infty} \left[p(se^x, t) - p(s,t) - \frac{\partial}{\partial s}p(s,t)s(e^x - 1) \right] \nu(s,x,t)\,dx$$
$$+ 1(s < \underline{s}_t) \left\{ r(t)K_0 - q(t)s - \int_{\ln(\underline{s}_t/s)}^{\infty} [p(se^x,t) - (K_0 - se^x)]\nu(s,x,t)\,dx \right\} = 0. \tag{11.6}$$

The last term on the left-hand side (LHS) of equation (11.6) is the result of applying the integro-differential operator defined by the first two lines to the value $p(s,t) = K_0 - s$ holding in the stopping region.

The American put value function $p(s,t)$ and the exercise boundary $\underline{s}_t$ jointly solve a backward free boundary problem (FBP), consisting of the backward PIDE (equation (11.6)), the smooth fit conditions (equations (11.4) and (11.5)), and the following boundary conditions:

$$p(s, T_0) = \max[0, K_0 - s], \quad s > 0 \tag{11.7}$$

$$\lim_{s\uparrow\infty} p(s,t) = 0, \quad t \in [0, T_0] \tag{11.8}$$

$$\lim_{s\downarrow 0} p(s,t) = K_0, \quad t \in [0, T_0]. \tag{11.9}$$

These Dirichlet conditions force the American put value to its exercise value along the boundaries. As the efficient implementation of a finite difference scheme usually requires the use of positive finite spatial boundaries, our implementation replaces the last two conditions in the target problem by:

$$\lim_{s\uparrow\infty} p_{ss}(s,t) = 0, \quad t \in [0, T_0] \tag{11.10}$$

$$\lim_{s\downarrow 0} p_{ss}(s,t) = 0, \quad t \in [0, T_0]. \tag{11.11}$$

Hence, the put gamma is forced to zero along the spatial boundaries. Numerical experimentation suggests that imposition of the zero gamma condition on positive finite spatial boundaries tends to work better than imposing the Dirichlet conditions. The solution to this alternative specification is unique under the further condition that it be continuous along the entire boundary. Figure 11.2 plots American put values in the DEVG model against stock price and time.

11.3 STATIONARITY AND DOMAIN EXTENSION IN THE MATURITY DIRECTION

The last section assumed that the strike K and maturity T were fixed at K_0 and T_0, respectively. To derive a hybrid FBP for American put values, we first extend the domain of the problem to all $T \in [0, \overline{T}]$, keeping the strike price K fixed at K_0.

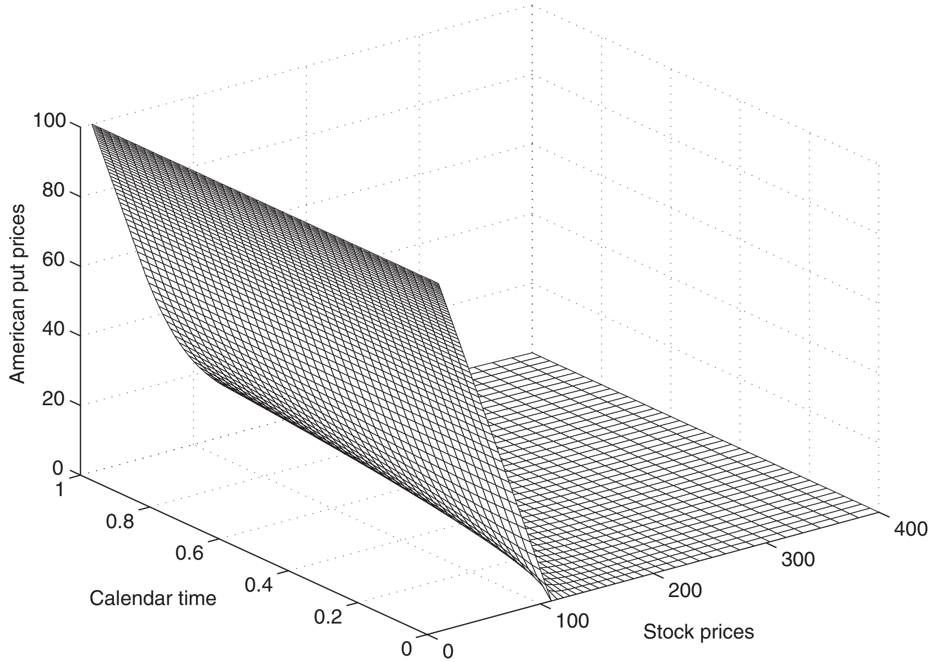

Figure 11.2 American put values in the DEVG model against stock price and calendar time. American put values are computed from the DEVG model for the following inputs: $r = 0.06$; $q = 0.02$; $\sigma = 0.4$; $s = 0.3$; $v = 0.25$; $\theta = -0.3$; $K_0 = 110$; $T_0 = 1$. The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400. The value of the American put at the initial stock price of $S_0 = 100$ is \$23.9875

Note that the exercise boundary depends on t, $r(t)$, $q(t)$, $\sigma(S,t)$, $\nu(S,x,t)$, T and K_0, but not on s. Suppressing the dependence on $r(t)$, $q(t)$, $\sigma(S,t)$, $\nu(S,x,t)$ and K_0, let $\underline{s}(t;T)$ be the function relating the exercise surface to t and T:

$$\underline{s}_t = \underline{s}(t;T), \quad t \in [0,T], T \in [0,\overline{T}].$$

The extended continuation region is a three-dimensional region denoted by Γ. This can be pictured as stacking the two-dimensional continuation regions up the Z-axis as T increases from 0. For each $T \in [0,\overline{T}]$, the union of the two-dimensional continuation region and the two-dimensional stopping region is the plane $S > 0, t \in [0,T]$. As T increases from zero, the area covered by this plane increases. Thus, the extended domain for the backward PIDE is the wedge $S > 0, t \in [0,T], T \in [0,\overline{T}]$. We note that the backward PIDE of the last section holds on this wedge with T_0 replaced by T. Let $\Pi(s,t;T)$ be the function solving this backward PIDE:

$$\frac{\partial \Pi(s,t;T)}{\partial t} + \frac{\sigma^2(s,t)s^2}{2}\frac{\partial^2 \Pi(s,t;T)}{\partial s^2} + [r(t)-q(t)]s\frac{\partial \Pi(s,t;T)}{\partial s} - r(t)\Pi(s,t;T)$$
$$+ \int_{-\infty}^{\infty}\left[\Pi(se^x,t;T) - \Pi(s,t;T) - \frac{\partial}{\partial s}\Pi(s,t;T)s(e^x-1)\right]\nu(s,x,t)\,dx$$

$$+ 1(s < \underline{s}(t;T)) \left\{ r(t)K_0 - q(t)s - \int_{\ln(\underline{s}(t;T)/s)}^{\infty} [\Pi(se^x, t; T) \right.$$
$$\left. - (K_0 - se^x)]\nu(s, x, t)\, dx \right\} = 0. \tag{11.12}$$

Now suppose stationarity, i.e. that $r(t)$, $q(t)$, $\sigma(S, t)$ and $\nu(S, x, t)$ are all independent of time t. It follows that the time derivative is just the negative of the maturity derivative:

$$\frac{\partial}{\partial t}\Pi(s, t; T) = -\frac{\partial}{\partial T}\Pi(s, t; T). \tag{11.13}$$

Dropping the dependence of $r(t)$, $q(t)$, $\sigma(S, t)$ and $\nu(S, x, t)$ on t and substituting equation (11.13) into equation (11.12) implies that the following relation holds in the extended domain:

$$-\frac{\partial \Pi(s, t; T)}{\partial T} + \frac{\sigma^2(s)s^2}{2}\frac{\partial^2 \Pi(s, t; T)}{\partial s^2} + (r - q)s\frac{\partial \Pi(s, t; T)}{\partial s} - r\Pi(s, t; T)$$
$$+ \int_{-\infty}^{\infty} \left[\Pi(se^x, t; T) - \Pi(s, t; T) - \frac{\partial}{\partial s}\Pi(s, t; T)s(e^x - 1) \right] \nu(s, x)\, dx$$
$$+ 1(s < \underline{s}(t;T)) \left\{ rK_0 - qs - \int_{\ln(\underline{s}(t;T)/s)}^{\infty} [\Pi(se^x, t; T) - (K_0 - se^x)]\nu(s, x)\, dx \right\} = 0. \tag{11.14}$$

We note that one can fix t at t_0 and just solve the above problem in the s, T plane if desired. In this case, the initial condition is:

$$\Pi(s, t_0; t_0) = \max[0, K_0 - s], \quad s > 0. \tag{11.15}$$

The Dirichlet boundary conditions are:

$$\lim_{s\uparrow\infty} \Pi(s, t_0; T) = 0, \quad T \in [t_0, \overline{T}] \tag{11.16}$$

$$\lim_{s\downarrow 0} \Pi(s, t_0; T) \sim K_0 - s, \quad T \in [t_0, \overline{T}]. \tag{11.17}$$

Alternatively, these Dirichlet conditions can be replaced by the following zero gamma conditions:

$$\lim_{s\uparrow\infty} \Pi_{ss}(s, t_0; T) = 0, \quad T \in [t_0, \overline{T}] \tag{11.18}$$

$$\lim_{s\downarrow 0} \Pi_{ss}(s, t_0; T) = 0, \quad T \in [t_0, \overline{T}]. \tag{11.19}$$

The smooth fit conditions are:

$$\lim_{s\downarrow \underline{s}(t_0;T)} \Pi(s, t_0, T) = K_0 - \underline{s}(t_0; T), \quad T \in [t_0, \overline{T}] \tag{11.20}$$

$$\lim_{s\downarrow \underline{s}(t_0;T)} \frac{\partial \Pi(s, t_0; T)}{\partial s} = -1, \quad T \in [t_0, \overline{T}]. \tag{11.21}$$

Figure 11.3 plots American put values in the DEVG model against stock price and maturity.

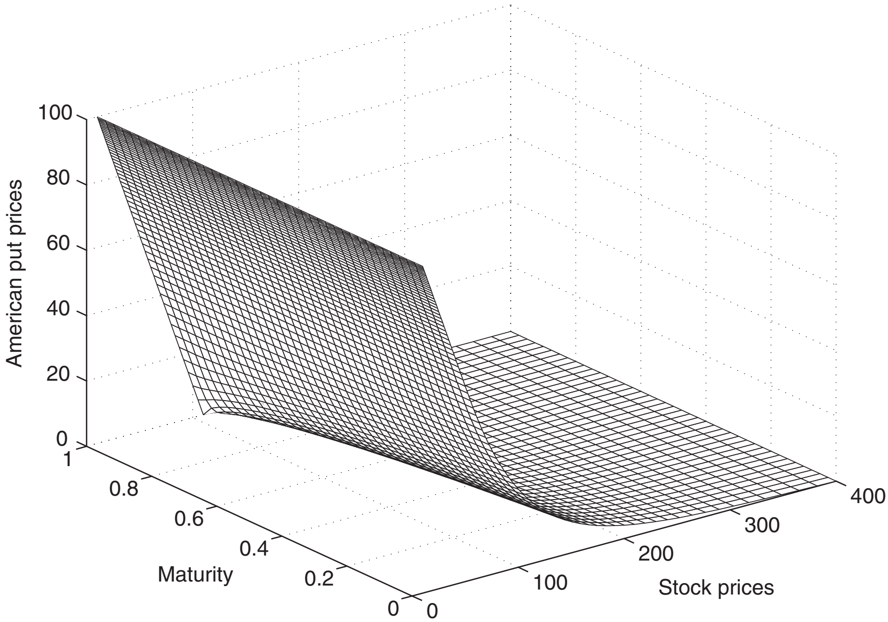

Figure 11.3 American put values in the DEVG model against stock price and maturity. American put values are computed from the DEVG model for the following inputs: $r = 0.06$; $q = 0.02$; $\sigma = 0.4$; $s = 0.3$; $\upsilon = 0.25$; $\theta = -0.3$; $K_0 = 110$; $T_0 = 1$. The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400

11.4 ADDITIVITY AND DOMAIN EXTENSION IN THE STRIKE DIRECTION

The last section assumed that the strike K was fixed at K_0 and that $r(t)$, $q(t)$, $\sigma(S, t)$ and $\nu(S, x, t)$ are all independent of time t. To derive a new hybrid PIDE for American put values, we further extend the domain of the problem to all $K > 0$. We also restore the dependence on t of $r(t)$, $q(t)$, $\sigma(S, t)$ and $\nu(S, x, t)$. On this larger domain, let $\underline{s}(t; T, K)$ be the function relating the exercise surface to t, T, and K:

$$\underline{s}_t = \underline{s}(t; T, K), \quad t \in [0, T], \quad T \in [0, \overline{T}], \quad K > 0.$$

We note that the backward PIDE (equation (11.12)) holding on the three-dimensional domain of the last section holds on the larger four-dimensional domain with K_0 replaced by all $K > 0$. Let $\Pi(s, t; K, T)$ be the function solving this backward PIDE on the extended four-dimensional domain:

$$\frac{\partial \Pi(s, t; K, T)}{\partial t} + \frac{\sigma^2(s, t)s^2}{2} \frac{\partial^2 \Pi(s, t; K, T)}{\partial s^2} + [r(t) - q(t)] \times s \frac{\partial \Pi(s, t; K, T)}{\partial s} - r(t)\Pi(s, t; K, T)$$

$$+\int_{-\infty}^{\infty}\left[\Pi(se^x,t;K,T)-\Pi(s,t;K,T)-\frac{\partial}{\partial s}\Pi(s,t;K,T)s(e^x-1)\right]\nu(s,x,t)\,dx$$

$$+\,1(s<\underline{s}(t;T,K))\left\{r(t)K-q(t)s-\int_{\ln(\underline{s}(t;T,K)/s)}^{\infty}[\Pi(se^x,t;K,T)\right.$$

$$\left.-\,(K-se^x)]\nu(s,x,t)\,dx\right\}=0. \tag{11.22}$$

We now assume that the log price process has independent increments, i.e. is additive or equivalently that $\sigma(S,t)$ and $\nu(S,x,t)$ are both independent of the stock price S. Then, for each fixed t and T, the exercise boundary is a linearly homogeneous function of the strike price:

$$\underline{s}(t;T,\lambda K)=\lambda\underline{s}(t;T,K),\ \text{ for all } \lambda\geq 0.$$

Setting $\lambda=\frac{1}{K}$ implies that:

$$\underline{s}(t;T,K)=K\underline{s}(t;T,1). \tag{11.23}$$

For each fixed s, t and T, the condition $s>\underline{s}(t;T,K)$ is thus equivalent to the condition $K<\frac{s}{\underline{s}(t;T,1)}=\frac{sK}{\underline{s}(t;T,K)}\equiv\overline{K}(s,t;T)$. We refer to the output of this function as the *critical strike price*. For each fixed s, t and T, the critical strike price is the lowest strike price K at which the put is exercised early. Note that the critical strike price depends on s but is independent of K. For an American put, the critical strike price is bounded above by s. In addition, note that the geometric mean of the two critical prices is just the geometric mean of the stock price and strike price:

$$\sqrt{\underline{s}(t;T,K)\overline{K}(s,t;T)}=\sqrt{sK}. \tag{11.24}$$

The additivity of the log price process implies that the function $\Pi(s,t;K,T)$ is linearly homogeneous in s and K. It follows from Euler's theorem that:

$$\Pi(s,t,K,T)=s\frac{\partial}{\partial s}\Pi(s,t;K,T)+K\frac{\partial}{\partial K}\Pi(s,t;K,T). \tag{11.25}$$

Differentiation with respect to s and K and some obvious algebra establishes that:

$$s^2\frac{\partial^2}{\partial s^2}\Pi(s,t;K,T)=K^2\frac{\partial^2}{\partial K^2}\Pi(s,t;K,T). \tag{11.26}$$

Dropping the dependence of $\sigma(S,t)$ and $\nu(S,x,t)$ on S and substituting equations (11.25) and (11.26) into equation (11.22) implies:

$$\frac{\partial\Pi(s,t;K,T)}{\partial t}+\frac{\sigma^2(t)K^2}{2}\frac{\partial^2\Pi(s,t;K,T)}{\partial K^2}$$

$$-\,[r(t)-q(t)]K\frac{\partial\Pi(s,t;K,T)}{\partial K}-q(t)\Pi(s,t;K,T)$$

$$+\int_{-\infty}^{\infty}\left[\Pi(s,t;Ke^{-x},T)-\Pi(s,t;K,T)-\frac{\partial}{\partial K}\Pi(s,t;K,T)K(e^{-x}-1)\right]e^{x}\nu(x,t)\,dx$$

$$+1(k>\overline{k}(s,t;T))\left\{r(t)K-q(t)s-\int_{\ln(\overline{k}(s,t;T)/K)}^{\infty}[\Pi(s,t;Ke^{-x},T)\right.$$

$$\left.-(Ke^{-x}-s)]e^{x}\nu(x,t)\,dx\right\}=0. \tag{11.27}$$

We note that one can fix s and T at say s_0 and T_0 and just solve the above problem in the K, t plane if desired. In this case, the terminal condition is:

$$\Pi(s_0,T_0;K,T_0)=\max[0,K-s_0],\quad K>0. \tag{11.28}$$

The Dirichlet boundary conditions are:

$$\lim_{K\uparrow\infty}\Pi(s_0,t;K,T_0)=\sim K-s_0,\quad t\in[0,T_0] \tag{11.29}$$

$$\lim_{K\downarrow 0}\Pi(s_0,t;K,T_0)=0,\quad t\in[0,T_0]. \tag{11.30}$$

Alternatively, these Dirichlet conditions can be replaced by:

$$\lim_{K\uparrow\infty}\Pi_{kk}(s_0,t;KT_0)=0,\quad t\in[0,T_0] \tag{11.31}$$

$$\lim_{K\downarrow 0}\Pi_{kk}(s_0,t;K,T_0)=0,\quad t\in[0,T_0]. \tag{11.32}$$

The smooth fit conditions are:

$$\lim_{K\uparrow\overline{K}(s,t;T_0)}\Pi(s_0,t;K,T_0)=\overline{K}(s_0,t;T_0)-s_0,\quad t\in[0,T_0] \tag{11.33}$$

$$\lim_{K\uparrow\overline{K}(s,t;T_0)}\frac{\partial\Pi(s_0,t;K,T_0)}{\partial K}=1,\quad t\in[0,T_0]. \tag{11.34}$$

Figure 11.4 plots American put values in the DEVG model against strike price and calendar time.

We note that setting jumps to zero reduces the PIDE to a PDE arising in the special case of the time-dependent Black–Scholes model. If one wishes to value American options in this model for multiple strikes and maturities and with fixed time and spot, it is much more efficient to solve the hybrid problem of this section once for each T than it is to solve the usual backward problem once for each K and once for each T, as is usually done.

11.5 THE FORWARD FREE BOUNDARY PROBLEM

We now assume that we have both stationarity and additivity. In other words, the log price is a Lévy process and $r(t)$, $q(t)$, $\sigma(S,t)$ and $\nu(S,x,t)$ are all independent of both time t

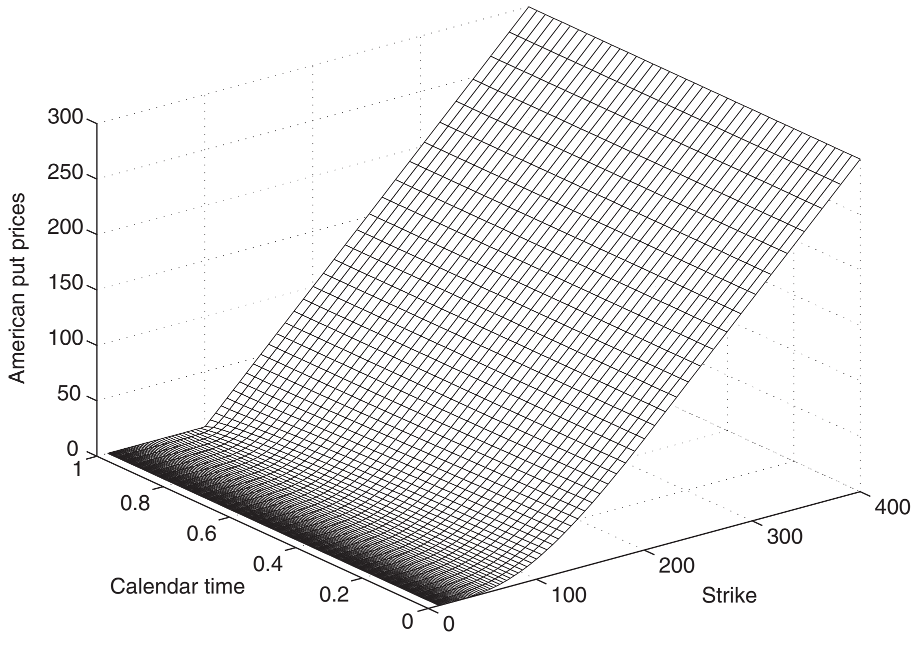

Figure 11.4 American put values in the DEVG model against stock price and calendar time. Critical stock prices are computed from the DEVG model for the following inputs: $r = 0.06$; $q = 0.02$; $\sigma = 0.4$; $s = 0.3$; $v = 0.25$; $\theta = -0.3$; $K_0 = 110$; $T_0 = 1$. The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400. The value of the American put at the initial stock price of $S_0 = 100$ is \$23.9875

and the stock price S. Stationarity implies that the function $\Pi(s, t; K, T)$ depends on t and T only through $T - t$. It thus follows that:

$$\frac{\partial}{\partial t}\Pi(s, t; K, T) = -\frac{\partial}{\partial T}\Pi(s, t; K, T). \tag{11.35}$$

Substituting equation (11.35) into equation (11.27) implies:

$$\begin{aligned} -\frac{\partial \Pi(s, t; K, T)}{\partial T} &+ \frac{\sigma^2 K^2}{2}\frac{\partial^2 \Pi(s, t; K, T)}{\partial K^2} - (r - q)K\frac{\partial \Pi(s, t; K, T,)}{\partial K} - q\Pi(s, t; K, T) \\ &+ \int_{-\infty}^{\infty}\left[\Pi(s, t; Ke^{-x}, T) - \Pi(s, t; K, T) - \frac{\partial}{\partial K}\Pi(s, t; K, T)K(e^{-x} - 1)\right]e^{x}\nu(x)\,dx \\ &+ 1(k > \overline{k}(s, t; T))\left\{rK - qs - \int_{\ln(\overline{k}(s,t;T)/K)}^{\infty}[\Pi(s, t; Ke^{-x}, T)\right. \\ &\qquad \left. - (Ke^{-x} - s)]e^{x}\nu(x)\,dx\right\} = 0. \end{aligned} \tag{11.36}$$

We note that one can fix s and t at say s_0 and t_0 and just solve the above problem in the K, T plane if desired. In this case, the initial condition is:

$$\Pi(s_0, t_0; K, t_0) = \max[0, K - s_0], \quad K > 0. \tag{11.37}$$

The Dirichlet boundary conditions are:

$$\lim_{K\uparrow\infty} \Pi(s_0, t_0; K, T) \sim K - S_0, \quad T \in [t_0, \overline{T}] \tag{11.38}$$

$$\lim_{K\downarrow 0} \Pi(s_0, t_0; K, T) = 0, \quad T \in [t_0, \overline{T}]. \tag{11.39}$$

Alternatively, these Dirichlet conditions can be replaced by:

$$\lim_{K\uparrow\infty} \Pi_{kk}(s_0, t_0; K, T) = 0, \quad T \in [t_0, \overline{T}] \tag{11.40}$$

$$\lim_{K\downarrow 0} \Pi_{kk}(s_0, t_0; K, T) = 0, \quad T \in [t_0, \overline{T}]. \tag{11.41}$$

The smooth fit conditions are:

$$\lim_{K\uparrow\overline{K}(s,t_0;T)} \Pi(s_0, t_0; K, T) = \overline{K}(s_0, t_0; T) - s_0, \quad T \in [t_0, \overline{T}] \tag{11.42}$$

$$\lim_{K\uparrow\overline{K}(s,t_0;T)} \frac{\partial\Pi(s_0, t_0; K, T)}{\partial K} = 1, \quad T \in [t_0, \overline{T}]. \tag{11.43}$$

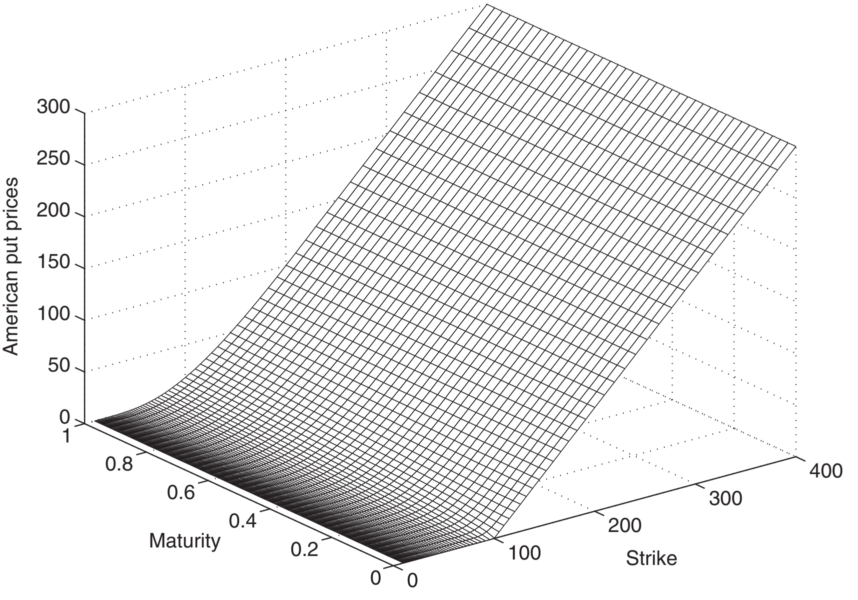

Figure 11.5 American put values in the DEVG model against stock price and maturity. American put values are computed from the DEVG model for the following inputs: $r = 0.06$; $q = 0.02$; $\sigma = 0.4$; $s = 0.3$; $\upsilon = 0.25$; $\theta = -0.3$; $K_0 = 110$; $T_0 = 1$. The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400. The value of the American put at the initial stock price of $S_0 = 100$ is \$23.9785

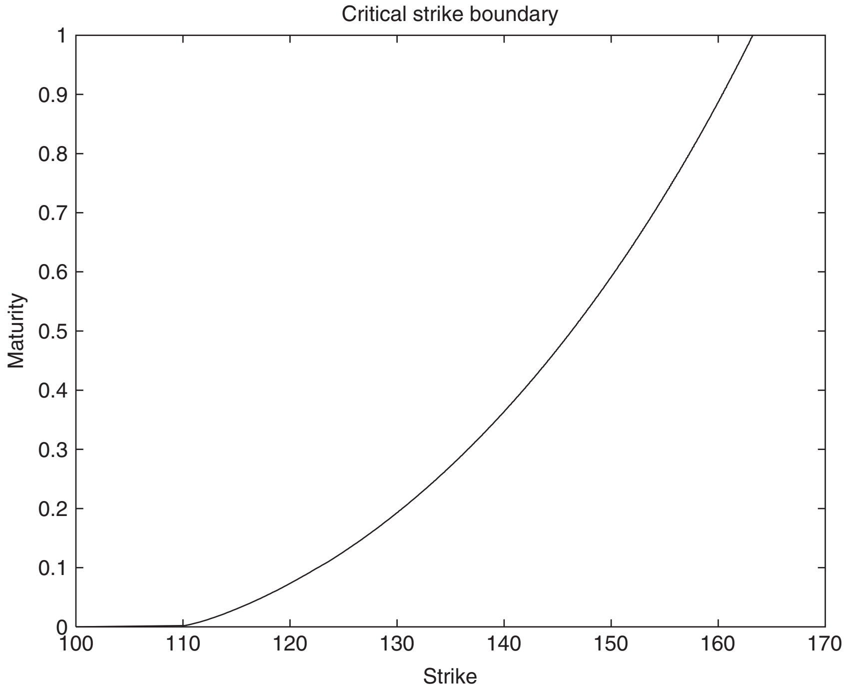

Figure 11.6 Critical strike boundary in the DEVG model. Critical strike prices are computed from the DEVG model for the following inputs: $r = 0.06$; $q = 0.02$; $\sigma = 0.4$; $s = 0.3$; $v = 0.25$; $\theta = -0.3$; $K_0 = 110$; $T_0 = 1$. The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400

Figure 11.5 plots American put values in the DEVG model against strike price and maturity. The value of the American put at the initial stock price of $S_0 = 100$ is \$23.9875 from the backward problem and \$23.9785 from the forward problem. The small difference is due to numerical error since the difference gets even smaller as we increase the number of time and spatial steps. Figure 11.6 plots critical strike prices against maturity using the same inputs.

11.6 SUMMARY AND FUTURE RESEARCH

We first reviewed the backward PIDE governing the arbitrage-free price of an American put option when the underlying spot price process is Markov in itself. By imposing various restrictions on the process, we then derived three new PIDEs for American put values. In particular, by assuming stationarity, we derived a forward PIDE in maturities with spot price still an independent variable. By alternatively assuming that the evolution coefficients for the proportional process are independent of spot, we derived a backward PIDE with the strike price as an independent variable. Finally, by assuming that the log price of the underlying is a Lévy process, we derived the forward PIDE for arbitrage-free American put values. We numerically solved this forward PIDE for the case of the diffusion extended VG model and found very close agreement to the numerical solution of the backward PIDE. A longer version of this paper, downloadable from www.math.nyu.edu/research/carrp/papers/pdf, contains an appendix detailing the finite difference scheme used to numerically solve the forward PIDE for American put options.

It is clear how to apply our analysis to American calls or more generally to payoffs which are both monotone and linearly homogeneous in spot and strike. It should be possible to extend our analysis to barrier options in which the payoff is linearly homogeneous in some subset of spot, strike, barrier, or rebate. An open problem is the forward equation for American options when the evolution parameters depend on stock price and/or time. It would also be interesting to extend our univariate approach to additional state variables besides the stock price. If the extra state variable is another asset price, then bivariate American options could be handled. If the extra state variable is a path statistic, then many path-dependent options could be handled. If the extra state variable is the current level of a randomly evolving volatility process, then our approach would encompass stochastic volatility and GARCH models for which there is considerable empirical support. In the interests of brevity, we defer this research to future work.

APPENDIX: DISCRETIZATION OF FORWARD EQUATION FOR AMERICAN OPTIONS

This appendix shows how finite differences can be used to numerically solve the following forward PIDE governing American put values:

$$\frac{\partial P(s,t;K,T)}{\partial T} - \frac{\sigma^2}{2}K^2\frac{\partial^2 P(s,t;K,T)}{\partial K^2} + (r-q)K\frac{\partial P(s,t;K,T)}{\partial K} + qP(s,t;K,T) \quad (11.44)$$

$$-\int_{-\infty}^{+\infty}\left[P(s,t;Ke^{-y},T) - P(s,t;K,T) - \frac{\partial P(s,t;K,T)}{\partial K}K(e^{-y}-1)\right]e^{y}\nu(y)\,dy \quad (11.45)$$

$$-\mathbf{1}_{K>\overline{K}(s,t;T)}\left\{rK - qs - \int_{\ln(K/\overline{K}(s,t;T))}^{\infty}\left[P(s,t;Ke^{-y},T) - (Ke^{-y}-s)\right]e^{y}\nu(y)\,dy\right\} = 0 \quad (11.46)$$

We illustrate the solution in the diffusion extended VG model for which the Lévy density has the form:

$$\nu(y)\,dy = \frac{\exp(\theta y/\sigma^2)}{\nu\,|y|}\exp\left(-\frac{\sqrt{\frac{2}{\nu}+\frac{\theta^2}{\sigma^2}}}{\sigma}\,|y|\right). \quad (11.47)$$

Notice that this Lévy density explodes as y approaches zero from either direction. As a result, special measures will have to be taken when approximating the integral containing this Lévy density. One can show that:

$$\int_{-\infty}^{+\infty}(e^{-y}-1)e^{y}\nu(y)\,dy = \omega$$

where:

$$\omega \equiv \frac{1}{\nu}\ln(1-\theta\nu-\sigma^2\,\nu/2). \quad (11.48)$$

Dropping the arguments s and t to simplify notation, we can rewrite equation (11.46) as:

$$\frac{\partial P(K,T)}{\partial T} - \frac{\sigma^2}{2}K^2\frac{\partial^2 P(K,T)}{\partial K^2} + (r-q+\omega)K\frac{\partial P(K,T)}{\partial K} + qP(K,T)$$
$$-\int_{-\infty}^{+\infty}\big((P(Ke^{-y},T) - P(K,T)\big)\, e^y \nu(y)\, dy$$
$$-\mathbf{1}_{K>\overline{K}(T)}\left\{rK - qs - \int_{\ln(K/\overline{K}(T))}^{\infty}\big[P(Ke^{-y},T) - (Ke^{-y}-s)\big]\, e^y\nu(y)\, dy\right\} = 0$$

By making the change of variable $x = \ln K$ we have

$$\begin{aligned} p(x,T) &= P(K,T),\\ \frac{\partial p}{\partial x}(x,T) &= K\frac{\partial P}{\partial K}(K,T),\\ \frac{\partial^2 p}{\partial x^2}(x,T) - \frac{\partial p}{\partial x}(x,T) &= K^2\frac{\partial^2 P}{\partial K^2}(K,T),\\ p(x-y,T) &= P(Ke^{-y},T), \end{aligned}$$

and hence we obtain the following PIDE for $p(x,T)$,

$$\frac{\partial p}{\partial T}(x,T) - \frac{\sigma^2}{2}\frac{\partial^2 p(x,T)}{\partial x^2} + (r - q + \frac{\sigma^2}{2} + \omega)\frac{\partial p}{\partial x}(x,T) + qp(x,T)$$
$$-\int_{-\infty}^{+\infty}(p(x-y,T) - p(x,T))\,\tilde{\nu}(y)\, dy$$
$$-\mathbf{1}_{x>\overline{x}(T)}\left\{re^x - qs - \int_{x-\overline{x}(T)}^{\infty}\big(p(x-y,T) - (e^{x-y}-s)\big)\,\tilde{\nu}(y)\, dy\right\} = 0,$$

where:

$$\begin{aligned} \tilde{\nu}(y) &= \frac{e^{-\tilde{\lambda}_p y}}{\nu y}\mathbf{1}_{y>0} + \frac{e^{-\tilde{\lambda}_n |y|}}{\nu|y|}\mathbf{1}_{y<0},\\ \tilde{\lambda}_p &= \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}} - \frac{\theta}{\sigma^2} - 1,\\ \tilde{\lambda}_n &= \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}} + \frac{\theta}{\sigma^2} + 1. \end{aligned}$$

This PIDE is solved subject to the initial condition

$$p(x,0) = (e^x - s)^+, \tag{11.49}$$

and the (zero gamma) boundary conditions

$$\frac{\partial^2 p}{\partial x^2}(-\infty,T) - \frac{\partial p}{\partial x}(-\infty,T) = 0 \quad \forall T, \tag{11.50}$$

$$\frac{\partial^2 p}{\partial x^2}(+\infty,T) - \frac{\partial p}{\partial x}(+\infty,T) = 0 \quad \forall T. \tag{11.51}$$

Discretization of PIDE

In our finite difference discretization, we adopt a mixed approach. For the jump terms, we use an explicit approach so that the matrix to be inverted at each time step is tri-diagonal. To evaluate the integrals, we apply an analytical approach to handle the singularity at zero. On the rest of the PIDE, a fully implicit approach is used. We consider M equally spaced sub-intervals in the T-direction. For the x-direction, we assume N equally spaced sub-intervals on $[x_{\min}, x_{\max}]$. Thus, we have the following mesh on $[x_{\min}, x_{\max}] \times [0, \overline{T}]$

$$D = \Big\{(x_i, T_j) \in \mathbb{R}^+ \times \mathbb{R}^+ \,|x_i = (x_{\min} + i\Delta x, i = 0, 1, \ldots, N, \\ T_j = j\Delta T, j = 0, 1, \ldots, M,\ \Delta x = (x_{\min} - x_{\max})/N, \Delta T = \overline{T}/M\Big\}.$$

Let $p_{i,j}$ be the finite difference approximation of the values of $p(x_i, T_j)$ on D. We obtain the following difference equation at point (x_i, T_{j+1})

$$\frac{1}{\Delta T}\big(p_{i,j+1} - p_{i,j}\big) - \frac{\sigma^2}{2}\frac{p_{i+1,j+1} - 2p_{i,j+1} + p_{i-1,j+1}}{\Delta x^2} + qp_{i,j+1} \\ + \left(r - q + \frac{\sigma^2}{2} + \omega\right)\frac{1}{2\Delta x}(p_{i+1,j+1} - p_{i-1,j+1}) \\ - \int_{-\infty}^{+\infty}\big(p(x_i - y, T_j) - p(x_i, T_j)\big)\,\tilde{\nu}(y)\,dy \\ - \mathbf{1}_{x_i > x(T_j)}\left\{rK - qe^{x_i} - \int_{x_i - x(T_j)}^{\infty}\big[p(x_i - y, T_j) - (K - e^{x_i + y})\big]\,\tilde{\nu}(y)\,dy\right\} = 0.$$

Equivalently, we have:

$$(-B - A)p_{i-1,j+1} + (1 + 2B + q\Delta T)p_{i,j+1} + (-B + A)p_{i+1,j+1} = \\ p_{i,j} + \Delta T\int_{-\infty}^{+\infty}\big(p(x_i - y, T_j) - p_{i,j}\big)\,\tilde{\nu}(y)\,dy \\ + \Delta T \times \mathbf{1}_{x_i > x(T_j)}\left\{re^{x_i} - qs - \int_{x_i - x(T_j)}^{\infty}\big[p(x_i - y, T_j) - (e^{x_i - y} - s)\big]\,\tilde{\nu}(y)\,dy\right\}, \tag{11.52}$$

where:

$$A = \left(r - q + \frac{\sigma^2}{2} + \omega\right)\frac{\Delta T}{2\Delta x},$$

$$B = \frac{\sigma^2}{2}\frac{\Delta T}{\Delta x^2},$$

$$p_{i,0} = (e^{x_i} - s)^+,$$

$$x(T_0) = \ln s,$$

and

$$x(T_j) = \min_{x_i}\{x_i : p_{i,j} - (e^{x_i} - s)^+ < 0\} \quad \text{for } j = 1, \ldots, M.$$

For the first integral on the right-hand side of equation (11.52), we decompose the range of integration into six parts:

$$\int_{-\infty}^{+\infty} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy = \int_{-\infty}^{x_i - x_N} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy$$

$$+ \int_{x_i - x_N}^{-\Delta x} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy$$

$$+ \int_{-\Delta x}^{0} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy$$

$$+ \int_{0}^{+\Delta x} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy$$

$$+ \int_{+\Delta x}^{x_i - x_0} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy$$

$$+ \int_{x_i - x_0}^{+\infty} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy$$

The six integrals are evaluated as:

$$\int_{-\Delta x}^{0} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy \cong \frac{1}{\nu \Delta x \tilde{\lambda}_n}(1 - e^{-\tilde{\lambda}_n \Delta x})(p_{i+1,j} - p_{i,j}),$$

$$\int_{0}^{\Delta x} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy \cong \frac{1}{\nu \Delta x \tilde{\lambda}_p}(1 - e^{-\tilde{\lambda}_p \Delta x})(p_{i-1,j} - p_{i,j}).$$

$$\int_{x_i - x_N}^{-\Delta x} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy$$

$$= \frac{1}{\nu} \sum_{k=1}^{N-i-1} \left(p_{i+k,j} - p_{i,j} - k(p_{i+k+1,j} - p_{i+k,j})\right)$$

$$\times \left\{\text{expint}(k \Delta x \tilde{\lambda}_n) - \text{expint}((k+1) \Delta x \tilde{\lambda}_n)\right\}$$

$$+ \frac{1}{\tilde{\lambda}_n \nu \Delta x} \sum_{k=1}^{N-i-1} \left(p_{i+k+1,j} - p_{i+k,j}\right) \left(e^{-\tilde{\lambda}_n k \Delta x} - e^{-\tilde{\lambda}_n (k+1) \Delta x}\right)$$

where:

$$\text{expint}(x) \equiv \int_{x}^{\infty} \frac{e^{-t}}{t} dt \qquad (11.53)$$

is the exponential integral.

$$\int_{\Delta x}^{x_i - x_0} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy$$

$$= \sum_{k=1}^{i-1} \frac{1}{\nu} \left(p_{i-k,j} - p_{i,j} - k(p_{i-k-1,j} - p_{i-k,j})\right) \left\{\text{expint}(k\Delta x \lambda_p) - \text{expint}((k+1)\Delta x \lambda_p)\right\}$$

$$+ \sum_{k=1}^{i-1} \frac{p_{i-k-1,j} - p_{i-k,j}}{\lambda_p \nu \Delta x} \left(e^{-\lambda_p k \Delta x} - e^{-\lambda_p (k+1)\Delta x}\right).$$

$$\int_{-\infty}^{x_i - x_N} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy = \frac{e^{x_i}}{\nu} \text{expint}((N-i)\Delta x(\tilde{\lambda}_n - 1))$$

$$- \frac{s + p_{i,j}}{\nu} \text{expint}((N-i)\Delta x \tilde{\lambda}_n).$$

$$\int_{x_i - x_0}^{\infty} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y)\, dy = -\frac{1}{\nu} p_{i,j} \text{expint}(i \Delta x \lambda_p).$$

The integral inside the Heaviside term in equation (11.52) is treated in the same manner as the other integral. Therefore, we have:

$$\int_{x_i - x(T_j)}^{\infty} \left[p(x_i - y, T_j) - (e^{x_i - y} - s)\right] \tilde{\nu}(y)\, dy$$

$$= \frac{1}{\nu} \sum_{k=i-l}^{i-1} \left(p_{i-k,j} - k(p_{i-k-1,j} - p_{i-k,j})\right) \left(\text{expint}(k\Delta x \tilde{\lambda}_p) - \text{expint}((k+1)\Delta x \tilde{\lambda}_p)\right)$$

$$+ \sum_{k=i-l}^{i-1} \frac{p_{i-k-1,j} - p_{i-k,j}}{\tilde{\lambda}_p \nu \Delta x} \left(e^{-\tilde{\lambda}_p k \Delta x} - e^{-\tilde{\lambda}_p (k+1)\Delta x}\right)$$

$$- \frac{1}{\nu} \left\{e^{x_i} \text{expint}((i-l)\Delta x(\tilde{\lambda}_p + 1)) - s\ \text{expint}((i-l)\Delta x \tilde{\lambda}_p)\right\}$$

Difference equation

Putting all of the pieces together, we obtain the following difference equation at the point (x_i, T_{j+1})

$$E p_{i-1,j+1} + F p_{i,j+1} + G p_{i+1,j+1} = p_{i,j} + \frac{\Delta T}{\nu} R_{i,j} + \Delta T \mathbf{1}_{x_i > x(T_j)} H_{i,j} \tag{11.54}$$

where

$$
\begin{aligned}
E &= -A - B - B_p,\\
F &= 1 + q\Delta T + 2B + B_n + B_p + \frac{\Delta T}{\nu}(\text{expint}(i\Delta x\tilde{\lambda}_p) + \text{expint}((N-i)\Delta x\tilde{\lambda}_n)),\\
G &= A - B - B_n,
\end{aligned}
$$

$$
\begin{aligned}
R_{i,j} = &\sum_{k=1}^{N-i-1} \big(p_{i+k,j} - p_{i,j} - k(p_{i+k+1,j} - p_{i+k,j})\big)\\
&\times \{\text{expint}(k\Delta x\lambda_n) - \text{expint}((k+1)\Delta x\lambda_n)\}\\
&+ \sum_{k=1}^{N-i-1} \frac{p_{i+k+1,j} - p_{i+k,j}}{\lambda_n \Delta x}(e^{-\tilde{\lambda}_n k\Delta x} - e^{-\tilde{\lambda}_n (k+1)\Delta x})\\
&+ \sum_{k=1}^{i-1} (p_{i-k,j} - p_{i,j} - k(p_{i-k-1,j} - p_{i-k,j}))\\
&\times \big\{\text{expint}(k\Delta x\lambda_p) - \text{expint}((k+1)\Delta x\lambda_p)\big\}\\
&+ \sum_{k=1}^{i-1} \frac{p_{i-k-1,j} - p_{i-k,j}}{\lambda_p \Delta x}(e^{-\tilde{\lambda}_p k\Delta x} - e^{-\tilde{\lambda}_p (k+1)\Delta x})\\
&+ e^{x_i}\text{expint}((N-i)\Delta x(\lambda_n - 1)) - s\ \text{expint}((N-i)\Delta x\lambda_n),
\end{aligned}
$$

$$
\begin{aligned}
H_{i,j} &= re^{x_i} - qs\\
&- \sum_{k=i-l}^{i-1} \frac{1}{\nu}\big(p_{i-k,j} - k(p_{i-k-1,j} - p_{i-k,j})\big)\big(\text{expint}(k\Delta x\lambda_p) - \text{expint}((k+1)\Delta x\lambda_p)\big)\\
&- \sum_{k=i-l}^{i-1} \frac{p_{i-k-1,j} - p_{i-k,j}}{\lambda_p \nu \Delta x}(e^{-\tilde{\lambda}_p k\Delta x} - e^{-\tilde{\lambda}_p (k+1)\Delta x})\\
&+ \frac{1}{\nu}\big\{e^{x_i}\text{expint}((i-l)\Delta x(\lambda_p + 1)) - s\ \text{expint}((i-l)\Delta x\lambda_p)\big\},
\end{aligned}
$$

and

$$
\begin{aligned}
B_n &= \frac{\Delta T}{\nu\Delta x\tilde{\lambda}_n}(1 - e^{-\tilde{\lambda}_n\Delta x}),\\
B_p &= \frac{\Delta T}{\nu\Delta x\tilde{\lambda}_p}(1 - e^{-\tilde{\lambda}_p\Delta x}).
\end{aligned}
$$

The initial condition (equation (11.49)) and boundary conditions (equations (11.50) and (11.51)) are discretized in the usual manner. A standard finite difference solver can then be used to solve the boundary value problem.

REFERENCES

[1] Andersen, L. and Andreasen, J. (1999), "Jumping smiles," *Risk*, **12**(11), 65–68.

[2] Anderson, T.G. Benzoni, L. and Lund, J. (2002), "An empirical investigation of continuous-time equity return models," *Journal of Finance*, **57**, 1239–1284.

[3] Andreasen, J. (1998), "Implied modelling, stable implementation, hedging, and duality," *Working Paper*, University of Aarhus, Aarhus, Denmark.

[4] Andreasen, J. and Carr, P. (2002), "Put call reversal," *Working Paper*, New York University, New York, USA.

[5] Black, F. and Scholes, M. (1973), "The pricing of options and corporate liabilities," *Journal of Political Economy*, **81**, 637–654.

[6] Buraschi, A. and Dumas, B. (2001), "The forward valuation of compound options," *Journal of Derivatives*, **9**(1), 8–17.

[7] Carr, P., Geman, H. Madan, D. and Yor, M. (2002), "The fine structure of asset returns: An empirical investigation, *Journal of Business*, **75**, 305–332.

[8] Carr, P.P. and Wu, L. (2002), "The finite moment logstable process and option pricing," *Working Paper*, New York University, New York, USA.

[9] Chriss, N. (1996), "Transatlantic trees," *Risk*, **9**(7), 45–48.

[10] Dumas, B., Fleming, J. and Whaley, R. (1998), "Implied volatilities: empirical tests," *Journal of Finance*, **53**, 2059–2106.

[11] Dupire, B. (1994), "Pricing with a smile," *Risk*, **7**(1), 18–20.

[12] Eberlein, E., Keller, U. and Prause, K. (1998), "New insights into smile, mispricing, and value at risk: the hyperbolic model," *Journal of Business*, **71**, 371–406.

[13] Esser, A. and Schlag, C. (2002), "A note on forward and backward partial differential equations for derivative contracts with forwards as underlyings," in J. Hakala and U. Wystup (Eds), *Foreign Exchange Risk*, Risk Publications, London, pp. 115–124.

[14] Hirsa, A. and Madan, D.B. (2003), "Pricing American options under Variance Gamma," *The Journal of Computational Finance*, **7**(2).

[15] Kou, S.G. (2002), "A jump-diffusion model for option pricing," *Management Science*, **48**, 1086–1101.

[16] Madan, D.B., Carr, P.P. and Chang, E. (1998), "The variance gamma process and option pricing," *European Financial Review*, **2**, 79–105.

[17] Merton, R.C. (1976), "Option pricing when underlying stock returns are discontinuous," *Journal of Financial Economics*, **3**, 125–144.

12
Numerical Valuation of American Options Under the CGMY Process

Ariel Almendral
Norwegian Computing Center, Oslo, Norway

Abstract

American put options written on an underlying stock following a Carr–Madan–Geman–Yor (CGMY) process are considered. It is known that American option prices satisfy a Partial Integro-Differential Equation (PIDE) on a moving domain. These equations are reformulated as a Linear Complementarity Problem, and solved iteratively by an implicit–explicit type of iteration based on a convenient splitting of the Integro-Differential operator. The solution to the discrete complementarity problems is found by the Brennan–Schwartz algorithm and computations are accelerated by the Fast Fourier Transform. The method is illustrated throughout a series of numerical experiments.

12.1 INTRODUCTION

In this paper, we propose a numerical method to compute American put options, when the underlying asset is modeled by the Carr–Madan–Geman–Yor (CGMY) process considered in Carr *et al*. (2002) [8]. Our contribution is to show experimentally that the implicit–explicit method proposed in Cont and Voltchkova (2003) [12] for European options may be successfully applied to the computation of American options under Lévy models. A similar splitting was already proposed in Hirsa and Madan (2004) [13] for the computation of the American price under the Variance Gamma (VG) process (see also, Anon (2004) [3]).

Matache *et al*. (2003) [17] have previously studied the American pricing problem under the CGMY process. They considered a variational inequality formulation combined with a convenient wavelet basis to compress the stiffness matrix. The approach here is different: we essentially work with a formulation as a Linear Complementarity Problem (LCP), and use standard finite differences. To deal with the singularity of the jump measure at the origin, we first approximate the problem by another problem, where small jumps are substituted by a small Brownian component. Next, we solve the approximated problem iteratively, where for each time step one needs to solve tridiagonal linear complementarity problems. The Fast Fourier Transform (FFT) plays also an important role when computing the convolution integrals fast. The sequence of linear complementarity problems are solved with the help of a simple algorithm proposed by Brennan and Schwartz (1977) [6], that works well for the particular case of a put option. We have also verified numerically the recent results in Alili and Kyprianou (2004) [1] on the smooth-fit principle for general Lévy processes.

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

A statistical study of financial time series in Carr *et al.* (2002) [8] shows that the diffusion component could in most cases be neglected, provided that the remaining part of the process is of infinite activity and finite variation. We concentrate precisely on the finite variation case, but also allow for a diffusion component, that may be safely omitted without affecting the pricing algorithm.

In Section 12.2, we briefly introduce the CGMY process, the European and American put option problem, and the related PIDEs. For further information on Lévy processes in finance, we refer the reader to the books by Cont and Tankov (2004) [11] and Schoutens (2003) [20]. An approximation to the equation with a discretization by finite differences is exposed in Section 12.3 and numerical results are presented in Section 12.4.

12.2 THE CGMY PROCESS AS A LÉVY PROCESS

A Lévy process is a stochastic process with stationary, independent increments. The Lévy–Khintchine theorem (see Sato (2001) [19]) provides a characterization of Lévy processes in terms of the characteristic function of the process, namely, there exists a measure ν such that, for all $z \in \mathbf{R}$ and $t \geq 0$, $E(e^{izL_t}) = \exp(t\phi(z))$, where

$$\phi(z) = i\gamma z - \frac{\sigma^2 z^2}{2} + \int_{\mathbf{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|\leq 1\}})d\nu(x). \tag{12.1}$$

Here $\sigma \geq 0$, $\gamma \in \mathbf{R}$ and ν is a measure on $\mathbf{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbf{R}} \min(1, x^2)d\nu(x) < \infty$.

Consider a Lévy process $\{L_t\}_{t\geq 0}$ of the form

$$L_t = (r - q + \mu)t + \sigma W_t + Z_t, \tag{12.2}$$

where r and q are the risk-free interest rate and the continuous dividend paid by the asset, respectively. This process has a drift term controlled by μ, a Brownian component $\{W_t\}_{t\geq 0}$ and a pure-jump component $\{Z_t\}_{t\geq 0}$. In this paper, we focus on the case where the Lévy measure in equation (12.1) associated to the pure-jump component can be written as $d\nu(x) = k(x)dx$, where the weight $k(x)$ is defined as

$$k(x) = \begin{cases} C\dfrac{\exp(-G|x|)}{|x|^{1+Y}} & \text{if} \quad x < 0, \\[2ex] C\dfrac{\exp(-M|x|)}{|x|^{1+Y}} & \text{if} \quad x > 0, \end{cases} \tag{12.3}$$

for constants $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$. The process $\{Z_t\}_{t\geq 0}$ is known in the literature as the CGMY process (Carr *et al.* (2002)) [8]; it generalizes a jump-diffusion model by Kou (2002) [15] ($Y = -1$) and the VG process (Carr *et al.* (1998)) [10] ($Y = 0$). The CGMY process is, in turn, a particular case of the Kobol process studied by Boyarchenko and Levendorskiĭ (2002) [5] and Carr *et al.* (2003) [7], where the constant C is allowed to take on different values on the positive and negative semiaxes.

The characteristic function of the CGMY process may be computed explicitly (see Boyarchenko and Levendorskiĭ (2002) [5] and Carr *et al.* (2002) [8]). In this paper, we consider *only* those processes having infinite activity and finite variation, excluding the VG process, that is, $0 < Y < 1$. In such a situation one has

$$\phi(z) = (r - q + \mu)iz - \frac{\sigma^2}{2}z^2 + C\Gamma(-Y)\left\{(M - iz)^Y - M^Y + (G + iz)^Y - G^Y\right\}. \quad (12.4)$$

A market model

Let a market consist of one risky asset $\{S_t\}_{t\geq 0}$ and one bank account $\{B_t\}_{t\geq 0}$. Let us assume that the asset process $\{S_t\}_{t\geq 0}$ evolves according to the geometric law

$$S_t = S_0 \exp(L_t), \quad (12.5)$$

where $\{L_t\}_{t\geq 0}$ is the Lévy process defined in equation (12.2), and the bank account follows the law $B_t = \exp(rt)$. Assume next the existence of some Equivalent Martingale Measure Q (a measure with the same null sets as the market probability, for which the discounted processes $\{e^{-(r-q)t}S_t\}_{t\geq 0}$ are martingales). In this paper, one works only with a risk-neutral measure Q, where the drift of the Lévy process has been changed. The EMM-condition $E_Q[S_t] = S_0 e^{t(r-q)}$ implies $\phi(-i) = r - q$, and so we get the following risk-neutral form for μ:

$$\omega := -\frac{\sigma^2}{2} - C\Gamma(-Y)\left\{(M - 1)^Y - M^Y + (G + 1)^Y - G^Y\right\}. \quad (12.6)$$

We keep the same notation for the risk-neutral parameters G and M. The other parameters σ, C and Y are the same in the risk-neutral world (see, e.g. Cont and Tankov (2004) [11] and Raible (2000) [18]). Note that M must be larger than one for ω to be well defined.

12.2.1 Options in a Lévy market

12.2.1.1 European vanilla options

Consider a European put option on the asset $\{S_t\}_{t\geq 0}$, with time to expiration T, and strike price K. Let us define the price of a European put option by the formula:

$$v(\tau, s) = e^{-r\tau} E_Q\left[(K - sH_\tau)^+\right], \quad 0 \leq s < \infty, \quad 0 \leq \tau \leq T, \quad (12.7)$$

where the process $\{H_\tau\}_{\tau\geq 0}$ is the underlying risk-neutral process starting at 1, given by

$$H_\tau := \exp\left[(r - q + \omega)\tau + \sigma W_\tau + Z_\tau\right]. \quad (12.8)$$

Note that τ means time to expiration $T - t$.

We will not work directly with the asset price s, but rather with its logarithm. Thus, let $x = \ln s$, and define the new function

$$u(\tau, x) := v(\tau, e^x). \quad (12.9)$$

From a generalization of Ito's formula it follows that u satisfies the following Cauchy problem:

$$\begin{cases} u_\tau - \mathcal{L}u = 0, & \tau \in (0, T], \quad x \in \mathbf{R}, \\ u(0, x) = (K - e^x)^+, & x \in \mathbf{R}, \end{cases} \tag{12.10}$$

where $\mathcal{L}$ is an integro-differential operator of the form

$$\begin{aligned} \mathcal{L}\varphi := & \frac{\sigma^2}{2}\varphi_{xx} + \left(r - q - \frac{\sigma^2}{2}\right)\varphi_x - r\varphi \\ & + \int_{\mathbf{R}} \left[\varphi(\tau, x + y) - \varphi(\tau, x) - (e^y - 1)\varphi_x(\tau, x)\right] k(y)\, dy. \end{aligned} \tag{12.11}$$

For a derivation of equation (12.10), see Boyarchenko and Levendorskiĭ (2002) [5] or Raible (2000) [18].

12.2.1.2 American vanilla options

Consider an American put option written on the underlying asset $\{S_t\}_{t\geq 0}$. The price may be found by solving an optimal stopping problem of the form:

$$v(\tau, s) = \sup_{\tau' \in \mathcal{S}_{0,\tau}} E_Q\left[e^{-r\tau'}(K - sH_{\tau'})^+\right]. \tag{12.12}$$

Here $\mathcal{S}_{0,\tau}$ denotes the set of stopping times taking values in $[0, \tau]$ and $\{H_\tau\}_{\tau\geq 0}$ is the process in equation (12.8). The corresponding function u (cf. equation (12.9)) satisfies the free-boundary value problem (Boyarchenko and Levendorskiĭ (2002) [5] and Matache *et al.* (2003) [17]):

$$\begin{cases} u_\tau - \mathcal{L}u = 0, & \tau > 0, \quad x > \widetilde{c}(\tau), \\ u(\tau, x) = K - e^x, & \tau > 0, \quad x \leq \widetilde{c}(\tau), \\ u(\tau, x) \geq (K - e^x)^+, & \tau > 0, \quad x \in \mathbf{R}, \\ u_\tau - \mathcal{L}u \geq 0, & \tau > 0, \quad x \in \mathbf{R}, \\ u(0, x) = (K - e^x)^+, & x \in \mathbf{R}, \end{cases} \tag{12.13}$$

where the operator $\mathcal{L}$ is defined in equation (12.11) and the free-boundary is given by

$$\widetilde{c}(\tau) = \inf\left\{x \in \mathbf{R} \mid u(\tau, x) > (K - e^x)^+\right\}, \quad \tau \in (0, T]. \tag{12.14}$$

The set $\{x \in \mathbf{R} \mid x \leq \widetilde{c}(\tau)\}$ is the exercise region for the logarithmic prices. Hence, for asset prices $s \leq \exp(\widetilde{c}(\tau))$, the American put should be exercised.

12.3 NUMERICAL VALUATION OF THE AMERICAN CGMY PRICE

The function $\widetilde{c}(\tau)$ is not known *a priori*, and needs to be found as part of the solution. Thus, rather than solving equation (12.13) directly, it is more convenient to use another formulation as a so-called Linear Complementarity Problem:

$$\begin{cases} u_\tau - \mathcal{L}u \geq 0 & \text{in } (0,T)\times\mathbf{R}, \\ u \geq \psi & \text{in } [0,T]\times\mathbf{R}, \\ (u_\tau - \mathcal{L}u)\,(u-\psi) = 0 & \text{in } (0,T)\times\mathbf{R}, \\ u(0,x) = \psi(x), \end{cases} \tag{12.15}$$

where the initial condition is given by

$$\psi(x) := (K - e^x)^+. \tag{12.16}$$

Note that the dependency on the free-boundary $\widetilde{c}(\tau)$ has disappeared, but instead we are left with a set of inequalities. The discretization and numerical solution of equation (12.15) is from now on our main goal. The free-boundary is obtained after computing the solution, by making use of equation (12.14).

12.3.1 Discretization and solution algorithm

The main idea of the method is to approximate the operator (equation (12.11)) by truncating the integral term close to zero and infinity. The truncation around infinity is harmless, as long as a sufficiently large interval is chosen and the price is substituted by the option's intrinsic value outside the computational domain. However, the truncation around zero gives rise to an artificial diffusion that must be taken into account. More precisely, the operator $\mathcal{L}$ may be split into the sum of two operators: the first one containing the Black and Scholes operator and the second accounting for the jumps, namely, $\mathcal{L} = \mathcal{L}_{BS} + \mathcal{L}_J$. The jump integral part is in turn split into the sum of one operator $\mathcal{P}^\epsilon$ for the integration variable in a neighborhood of the origin, and $\mathcal{Q}^\epsilon$ for the complementary domain. For $\mathcal{P}^\epsilon$, we use Taylor's expansion to write the following approximation:

$$\begin{aligned} (\mathcal{P}^\epsilon\varphi)(\tau,x) &:= \int_{|y|\leq\epsilon} \big[\varphi(\tau,x+y) - \varphi(\tau,x) - (e^y-1)\varphi_x(\tau,x)\big]\,k(y)\,dy \\ &= \int_{|y|\leq\epsilon} \big[\varphi(\tau,x+y) - \varphi(\tau,x) - y\varphi_x(\tau,x) - (e^y-1-y)\varphi_x(\tau,x)\big]\,k(y)\,dy \\ &\approx (\widetilde{\mathcal{P}}^\epsilon\varphi)(\tau,x) := \frac{\sigma^2(\epsilon)}{2}\varphi_{xx}(\tau,x) - \frac{\sigma^2(\epsilon)}{2}\varphi_x(\tau,x), \end{aligned}$$

with the notation:

$$\sigma^2(\epsilon) = \int_{|y|\leq\epsilon} y^2 k(y)dy. \tag{12.17}$$

That is, $\mathcal{P}^\epsilon$ has been approximated by a convection–diffusion operator $\widetilde{\mathcal{P}}^\epsilon$, with a small diffusion coefficient $\sigma^2(\epsilon)$.

The operator $\mathcal{Q}^\epsilon$ is simply split into a sum, given that this operation is now allowed away from the origin:

$$(\mathcal{Q}^\epsilon \varphi)(\tau, x) := \int_{|y|\geq\epsilon} \left[\varphi(\tau, x+y) - \varphi(\tau, x) - (e^y - 1)\varphi_x(\tau, x)\right] k(y)dy$$
$$= (\mathcal{J}^\epsilon \varphi)(\tau, x) - \lambda(\epsilon)\varphi(\tau, x) + \omega(\epsilon)\varphi_x(\tau, x), \tag{12.18}$$

where we have written $\mathcal{J}^\epsilon$ for the convolution term, and

$$\lambda(\epsilon) = \int_{|y|\geq\epsilon} k(y)dy, \tag{12.19}$$

$$\omega(\epsilon) = \int_{|y|\geq\epsilon} (1 - e^y)k(y)dy. \tag{12.20}$$

Remark 12.3.1 *These operations have a probabilistic meaning: the pure-jump process has been approximated by a compound Poisson process plus a small Brownian component. As proved in Asmussen and Rosiński (2001) [4], this approximation is valid, if and only if,* $\sigma(\epsilon)/\epsilon \to \infty$*, as* $\epsilon \to 0$*. Note that this condition implies* $0 < Y < 1$*, excluding therefore the VG process and processes with infinite activity.*

An approximation result in Cont and Voltchkova (2003) [12] states the following. Let $\mathcal{L}^\epsilon := \mathcal{L}_{BS} + \widetilde{\mathcal{P}}^\epsilon + \mathcal{Q}^\epsilon$ and u^ϵ be the solution of the Cauchy problem

$$\begin{cases} u^\epsilon_\tau - \mathcal{L}^\epsilon u^\epsilon = 0, \\ u(0, x) = \psi(x), \end{cases} \tag{12.21}$$

and then there exists a constant $C > 0$ such that $|u(\tau, x) - u^\epsilon(\tau, x)| < C\epsilon$, for all τ and x. We use here – without proof – the same approximation to numerically solve an American put option. An indication that this approximation works also for American options is shown in Figure 12.1, where one observes that the exercise boundary tends to the theoretical perpetual exercise price, when the time to expiration τ is taken large. The proof of this fact is thus an open problem.

Let us focus now on the problem shown in equation (12.15), but with $\mathcal{L}^\epsilon$ instead of $\mathcal{L}$. One possible idea to discretize this new problem is to apply Euler's scheme in time combined with an implicit–explicit iteration in space. Let the time interval $[0, T]$ be divided into L equal parts, i.e. $\tau_j = j\Delta\tau$ $(j = 0, 1, \ldots, L)$ with $\Delta\tau = T/L$ and define the functions $u^j \approx u(\tau_j, x)$. Let operator $\mathcal{L}^\epsilon$ be split as $\mathcal{L}^\epsilon = \mathcal{A} + \mathcal{B}$. We consider the following sequence of problems:

$$\begin{cases} \dfrac{u^{j+1}}{\Delta\tau} - \mathcal{A}u^{j+1} \geq d^j := \dfrac{u^j}{\Delta\tau} + \mathcal{B}u^j, \\ u^{j+1} \geq \psi, \\ \left(\dfrac{u^{j+1}}{\Delta\tau} - \mathcal{A}u^{j+1} - d^j\right)(u^{j+1} - \psi) = 0, \\ u^0 = \psi. \end{cases} \tag{12.22}$$

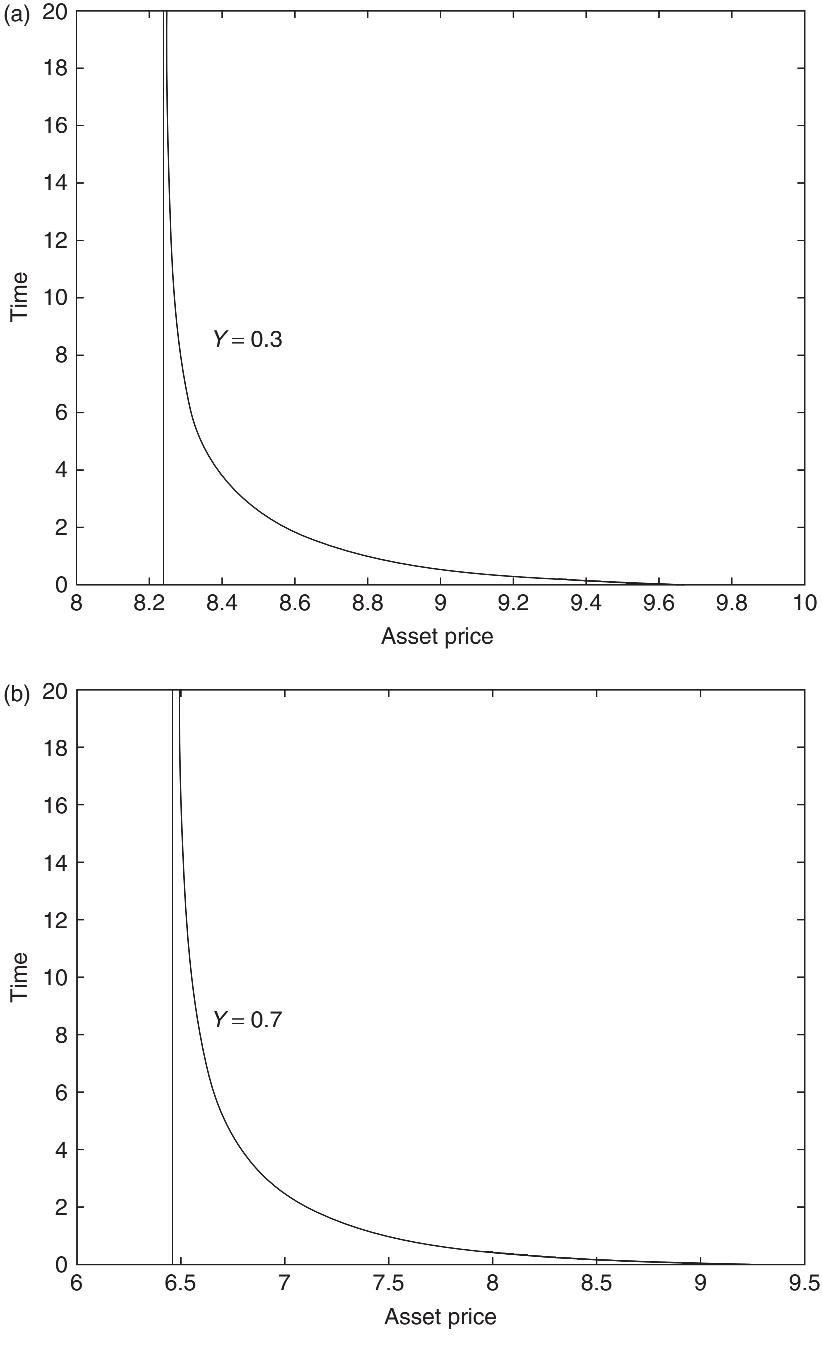

Figure 12.1 Exercise boundary and perpetual boundary for two different values of the parameter Y, i.e. 0.3 (a) and 0.7 (b): $\sigma = 0$; $r = 0.1$; $q = 0$; $K = 10$; $T = 20$; $C = 1$; $G = 7$; $M = 9$

That is, given the function u^j, we compute u^{j+1} by solving these integro-differential inequalities. A natural choice for the splitting of $\mathcal{L}^\epsilon$ is the following:

$$\mathcal{A}\varphi := \frac{\sigma^2+\sigma^2(\epsilon)}{2}\varphi_{xx} + \left[r - q - \frac{\sigma^2+\sigma^2(\epsilon)}{2} + \omega(\epsilon)\right]\varphi_x - r\varphi \tag{12.23}$$

$$\mathcal{B}\varphi := \mathcal{J}^\epsilon\varphi - \lambda(\epsilon)\varphi. \tag{12.24}$$

Observe that the integral term is treated explicitly, whereas the differential part is treated implicitly. This method imposes a stability restriction on the time step; see Cont and Voltchkova (2003) [12] for a discussion of this issue for the European case.

12.3.1.1 Spatial discretization of $\mathcal{A}$

Consider a computational domain of the form $[0, T] \times [x_{min}, x_{max}]$. Let $\ln K \in [x_{min}, x_{max}]$ and define the uniform spatial grid $x_i = x_{min} + ih$ $(i = 0, \ldots, N)$ where $h = (x_{max} - x_{min})/N$. Once we have defined the grid, we can discretize $\mathcal{A}$ by standard second-order schemes. For the first and second derivatives, the central scheme and the standard 3-point scheme are chosen, respectively. Namely, after introducing the notation $\delta_1(\varphi) := [\varphi_{i+1} - \varphi_{i-1}]/2h$ and $\delta_2(\varphi) := [\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}]/h^2$, where $\varphi_i := \varphi(x_i)$ $(i = 0, 1, \ldots, N)$, we may write

$$(\mathcal{A}\varphi)_i = \beta\delta_2(\varphi) + \gamma\delta_1(\varphi) - r\varphi_i, \tag{12.25}$$

with the quantities β and γ defined as

$$\beta := \frac{\sigma^2+\sigma^2(\epsilon)}{2}, \tag{12.26}$$

$$\gamma := r - q - \frac{\sigma^2+\sigma^2(\epsilon)}{2} + \omega(\epsilon). \tag{12.27}$$

We obtain the following coefficients for the implicit part

$$a = -\frac{\beta}{h^2} + \frac{\gamma}{2h}, \tag{12.28}$$

$$b = \frac{1}{\Delta\tau} + r + \frac{2\beta}{h^2}, \tag{12.29}$$

$$c = -\frac{\beta}{h^2} - \frac{\gamma}{2h}. \tag{12.30}$$

The tridiagonal matrix T associated to the implicit part has constant diagonals: b is on the main diagonal, a is on the subdiagonal and c is on the superdiagonal.

From now on, the parameter ϵ is taken as the mesh-size h. The artificial diffusion $\sigma^2(h)$ (cf. Matache *et al.* (2003) [17]) may be approximated by the composite trapezoidal rule on the intervals $[-h, 0]$ and $[0, h]$. This gives

$$\sigma^2(h) \approx \frac{[k(h) + k(-h)]\,h^3}{2}. \tag{12.31}$$

The quantities $\lambda(h)$ and $\omega(h)$ are approximated in the next section.

12.3.1.2 *Spatial discretization of $\mathcal{B}$*

The discretization of $\mathcal{B}$ involves the discretization of $\mathcal{J}^\epsilon$, since $\mathcal{B}\varphi = \mathcal{J}^\epsilon\varphi - \lambda(\epsilon)\varphi$. The discretization of $\mathcal{J}^\epsilon$ is explained in detail in Anon (2004) [3]. Briefly, the idea is to truncate the integral to a finite domain and then apply the composite trapezoidal rule, i.e.,

$$\begin{aligned} J_i := (\mathcal{J}^\epsilon\varphi)_i &= \int_{|y|\geq h} \varphi(x_i + y)k(y)dy \\ &\approx \int_{h\leq|y|\leq Mh} \varphi(x_i + y)k(y)dy \\ &\approx h \sum_{m=-M}^{M} \varphi_{i+m}k_m\rho_m, \quad i = 0, 1, \ldots, N, \end{aligned} \tag{12.32}$$

where $k_m = k(mh)$ for $m \neq 0$ and we let $k_0 = 0$. The coefficients obtained from applying the trapezoidal rule are:

$$\rho_m = \begin{cases} 1/2 & \text{if } m \in \{-M, -1, 1, M\}, \\ 1 & \text{otherwise.} \end{cases}$$

It is important to substitute φ by the payoff function ψ outside the computational domain. The computation of the numbers J_i constitutes the main burden of the method, but thanks to the FFT algorithm, this may be carried out efficiently (see next section). However, N must be an even number, and $M = N/2$, to be able to express this convolution in matrix–vector notation.

Finally, we may use the composite trapezoidal rule to compute an approximation to the numbers $\lambda(h)$ and $\omega(h)$ by simply taking φ in equation (12.32) as 1 and $e^y - 1$, respectively.

12.3.1.3 *Fast convolution by FFT*

The Fast Fourier Transform is an algorithm that evaluates the Discrete Fourier Transform (DFT) of a vector $f = [f_0, f_2 \ldots, f_{R-1}]$ in $O(R \log R)$ operations.

The Discrete Fourier Transform is defined as:

$$F_k = \sum_{n=0}^{R-1} f_n e^{-i2\pi nk/R}, \quad k = 0, 1, \ldots, R. \tag{12.33}$$

One of the multiple applications of the DFT is in computing convolutions. Let us first introduce the concept of circulant convolution. Let $\{x_m\}$ and $\{y_m\}$ be two sequences with period R. The convolution sequence $z := x * y$ is defined component-wise as

$$z_n = \sum_{m=0}^{R-1} x_{m-n} y_m. \tag{12.34}$$

We now use the FFT to compute the vector $[z_0, \ldots, z_{R-1}]$. The periodic structure of x allows the derivation of the following simple relation:

$$Z_k = X_k \cdot Y_k, \tag{12.35}$$

where X, Y and Z denote the Discrete Fourier Transform of the sequences x, y and z, respectively. That is, the DFT applied to the convolution sequence is equal to the product of the transforms of the original two sequences. The vector $[z_0, \ldots, z_{R-1}]$ may be recovered by means of the Inverse Discrete Fourier Transform (IDFT):

$$z_n = \frac{1}{R}\sum_{k=0}^{R-1} Z_k e^{i2\pi kn/R}, \quad n = 0, 1, \ldots, R. \tag{12.36}$$

In the language of matrices, a circulant convolution may be seen as the product of a circulant matrix times a vector. For example, let $R = 3$, and use the periodicity $x_k = x_{k+R}$ to write equation (12.34) as

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_2 & x_0 & x_1 \\ x_1 & x_2 & x_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}. \tag{12.37}$$

A circulant matrix is thus a matrix in which each row is a ‘circular’ shift of the previous row.

We are interested in the convolution shown in equation (12.32), where the vector φ is not periodic. The associated matrix is a so-called Toeplitz matrix, which by definition is a matrix that is constant along diagonals. A circulant matrix is hence a particular type of Toeplitz matrix. The next idea is to embed a Toeplitz matrix into a circulant matrix. As an example, let $M = 1$ and $N = 2$, so that the matrix-vector notation for equation (12.32) reads

$$\begin{bmatrix} \varphi_1 & \varphi_0 & \varphi_{-1} \\ \varphi_2 & \varphi_1 & \varphi_0 \\ \varphi_3 & \varphi_2 & \varphi_1 \end{bmatrix} \begin{bmatrix} k_1/2 \\ k_0 \\ k_{-1}/2 \end{bmatrix}. \tag{12.38}$$

The matrix above may be embedded in a circulant matrix C of size 5 in the following way. (For computational efficiency of the FFT algorithm, it is advisable to use a circulant matrix whose size is a power of 2.):

$$C = \left[\begin{array}{ccc|cc} \varphi_1 & \varphi_0 & \varphi_{-1} & \varphi_3 & \varphi_2 \\ \varphi_2 & \varphi_1 & \varphi_0 & \varphi_{-1} & \varphi_3 \\ \varphi_3 & \varphi_2 & \varphi_1 & \varphi_0 & \varphi_{-1} \\ \hline \varphi_{-1} & \varphi_3 & \varphi_2 & \varphi_1 & \varphi_0 \\ \varphi_0 & \varphi_{-1} & \varphi_3 & \varphi_2 & \varphi_1 \end{array}\right]. \tag{12.39}$$

If we define the vector $\eta := [k_1/2, k_0, k_{-1}/2, 0, 0]^T$, then the product (equation (12.38)) is the vector consisting of the first three elements in the product $C\eta$. As explained before, a product of a circulant matrix and a vector may be efficiently obtained by applying the FFT algorithm.

As a summary, following the ideas explained above, it is possible to compute the convolution (equation (12.32)), with $M = N/2$, by 'embedding' the resulting matrix into a circulant matrix. The product of a circulant matrix and a vector is carried out in three FFT operations, namely, two DFT and one IDFT.

In Almendral and Oosterlee (2003) [2], we applied the FFT algorithm in the computation of European options for Merton's model and Kou's model, and in Anon (2004) [3] to find the American price under the Variance Gamma process. For further details on the computation of convolutions by FFT we refer the reader to Van Loan (1992) [21].

12.3.1.4 Boundary conditions

We used points on the boundary when discretizing the differential operator $\mathcal{A}$. This means that the vector d^j needs to be updated. For a put option, this is done by updating the first and the last entries of d^j as follows:

$$d_1^j \leftarrow d_1^j - a(K - e^{x_{min}}), \quad d_{N-1}^j \leftarrow 0. \tag{12.40}$$

12.3.1.5 Discrete LCP

We are now in position to write the discrete inequalities that correspond to the discretization of equation (12.22):

$$\begin{cases} Tu^{j+1} \geq d^j, \\ u^{j+1} \geq \psi, \\ (Tu^{j+1} - d^j, u^{j+1} - \psi) = 0, \\ u^0 = \psi, \end{cases} \tag{12.41}$$

for $j = 0, 1, \ldots, L-1$. The matrix T has entries given by equations (12.28)–(12.30), $d_i^j = u_i^j/\Delta\tau + (\mathcal{J}^\epsilon u^j)_i - \lambda(\epsilon)u_i^j$ $(i = 1, \ldots, N-1)$ with the update shown in equation (12.40) and ψ is the vector $[\psi_1, \psi_2, \ldots, \psi_{N-1}]^T$, with $\psi_i = \psi(x_i)$ (cf. Lewis (2001) [16]). The same letter ψ is used to simplify the notation.

We proceed now to explain a simple algorithm to solve equation (12.41).

12.3.1.6 Brennan–Schwartz algorithm for a put option

Let a tridiagonal matrix

$$T = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \tag{12.42}$$

and vectors $d = [d_1, \ldots, d_n]^T$ and $\psi = [\psi_1, \ldots, \psi_n]^T$ be given. Consider the following problem: find a vector u satisfying the system

$$\begin{cases} Tu \geq d, \\ u \geq \psi, \\ (Tu - d, u - \psi) = 0. \end{cases} \tag{12.43}$$

The following algorithm to find u in equation (12.43) was proposed by Brennan and Schwartz (1977) [6] (for put options) and discussed in detail by Jaillet *et al.* (1990) [14]:

- Step 1: Compute recursively a vector $\widetilde{b}$ as

$$\begin{aligned} \widetilde{b}_n &= b_n, \\ \widetilde{b}_{j-1} &= b_{j-1} - c_{j-1}a_j/\widetilde{b}_j, \quad j = n, \ldots, 2. \end{aligned}$$

- Step 2: Compute recursively a vector $\widetilde{d}$ as

$$\begin{aligned} \widetilde{d}_n &= d_n, \\ \widetilde{d}_{j-1} &= d_{j-1} - c_{j-1}\tilde{d}_j/\widetilde{b}_j, \quad j = n, \ldots, 2. \end{aligned}$$

- Step 3: Compute u forward as follows:

$$\begin{aligned} u_1 &= \max\left[\widetilde{d}_1/b_1 \ , \ \psi_1\right], \\ u_j &= \max\left[\left(\widetilde{d}_j - a_j u_{j-1}\right)/\widetilde{b}_j \ , \ \psi_j\right], \quad j = 2, \ldots, n. \end{aligned}$$

We apply these three steps with $a_i = a$, $b_i = b$ and $c_i = c$, with a, b and c as in equations (12.28)–(12.30). The splitting proposed in equations (12.23) and (12.24) does not, in general, guarantee the validity of the Brennan–Schwartz algorithm. However, the convection term may be moved to the explicit part of the splitting, so that the conditions of the Brennan–Schwartz algorithm hold Almendral and Oosterlee (2003) [2]. The solutions obtained in both ways are the same, to within the discretization error.

12.4 NUMERICAL EXPERIMENTS

In this section, European and American option prices are computed numerically. In the first experiment, we compute an European option (problem (12.21)) and compare it with the solution obtained by the Carr–Madan formula in Carr and Madan (1999) [9]; see also the appendix in this paper, formula (12.9). Both solutions are compared in the ℓ_∞-norm, and the results are shown in Table 12.1. A linear convergence rate is observed, and note that the algorithm computes the European price with an error of one cent in about one second.

Table 12.1 Linear convergence to exact solution in ℓ_∞-norm and CPU times on a Pentium IV, 1.7 GHz. The parameters are as follows: $r = 0$; $q = 0$; $K = 10$; $T = 1$; $C = 1$; $G = 7$; $M = 9$; $Y = 0.7$

N	L	ℓ_∞-error	CPU-time (s)
50	5	0.2675	0.22
100	10	0.1281	0.31
200	20	0.0459	0.34
400	40	0.0160	1.06

A second experiment concerns the verification of the theoretical perpetual exercise price against the asymptotic behavior of the free boundary for some large time to expiry. The asymptotic value s^* of the American put was verified with the aid of a formula in Boyarchenko and Levendorskiĭ (2002) [5], (Theorem 3.2 and Theorem 5.1):

$$s^* = \exp(x^*) = K \exp\left\{ -\frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{\ln\left[r + q + \phi_0(z)\right]}{z^2 + iz} dz \right\}, \tag{12.44}$$

with ρ a positive number (not arbitrary, see Boyarchenko and Levendorskiĭ [5]) and $\phi_0(z)$ is given by equation (12.47) (see below). Figure 12.1 above shows two examples of exercise boundaries and their corresponding theoretical asymptotic values. In these examples, $\rho = 1$ gives the right value.

In the next two experiments, we examine the behavior of the option price and free boundary for different values of Y and M. We conclude from Figures 12.2 and 12.3 that the American option price is an increasing function of Y and a decreasing function of M. We mention that the results shown in Figure 12.2 are in accordance with the numerical tests in Matache *et al*. (2003) [17] (Figure 6).

The last test is designed to verify the smooth-fit principle. According to Alili and Kyprianou (2004) [1], the smooth-fit principle holds for perpetual American put options in the bounded variation case considered here, if and only if, the drift $r - q + \omega$ is negative, or an additional condition on the jump measure is satisfied for zero drift. In Figure 12.4(a), we show the numerical derivative v_s at time $T = 1$, for a set of parameters giving negative drift. In this case, we have smooth-fit. For a second set of parameters chosen such that the drift is positive, we see a discontinuous derivative in Figure 12.4(b) and so there is no smooth-fit.

APPENDIX: ANALYTIC FORMULA FOR EUROPEAN OPTION PRICES

We include here the analytic expression given in Lewis (2001) [16] for European options, adapted to the case of a CGMY process:

$$u(t, x) = \frac{e^{-rt}}{2\pi} \int_{i\alpha-\infty}^{i\alpha+\infty} \exp\left[-izx + t\phi_0(-z)\right] \hat{\psi}(z) dz, \tag{12.45}$$

where $\hat{\psi}(z)$ is the generalized Fourier transform of the payoff ψ, which for a put option is given by

$$\hat{\psi}(z) = -\frac{K^{iz+1}}{z^2 - iz}, \tag{12.46}$$

and the risk-neutral characteristic function ϕ_0 to be used is obtained by substituting μ by ω from equation (12.6) into equation (12.4), i.e.

$$\begin{aligned} \phi_0(z) = {} & (r - q + \omega)iz - \frac{\sigma^2}{2}z^2 \\ & + C\Gamma(-Y)\left\{(M - iz)^Y - M^Y + (G + iz)^Y - G^Y\right\}. \end{aligned} \tag{12.47}$$

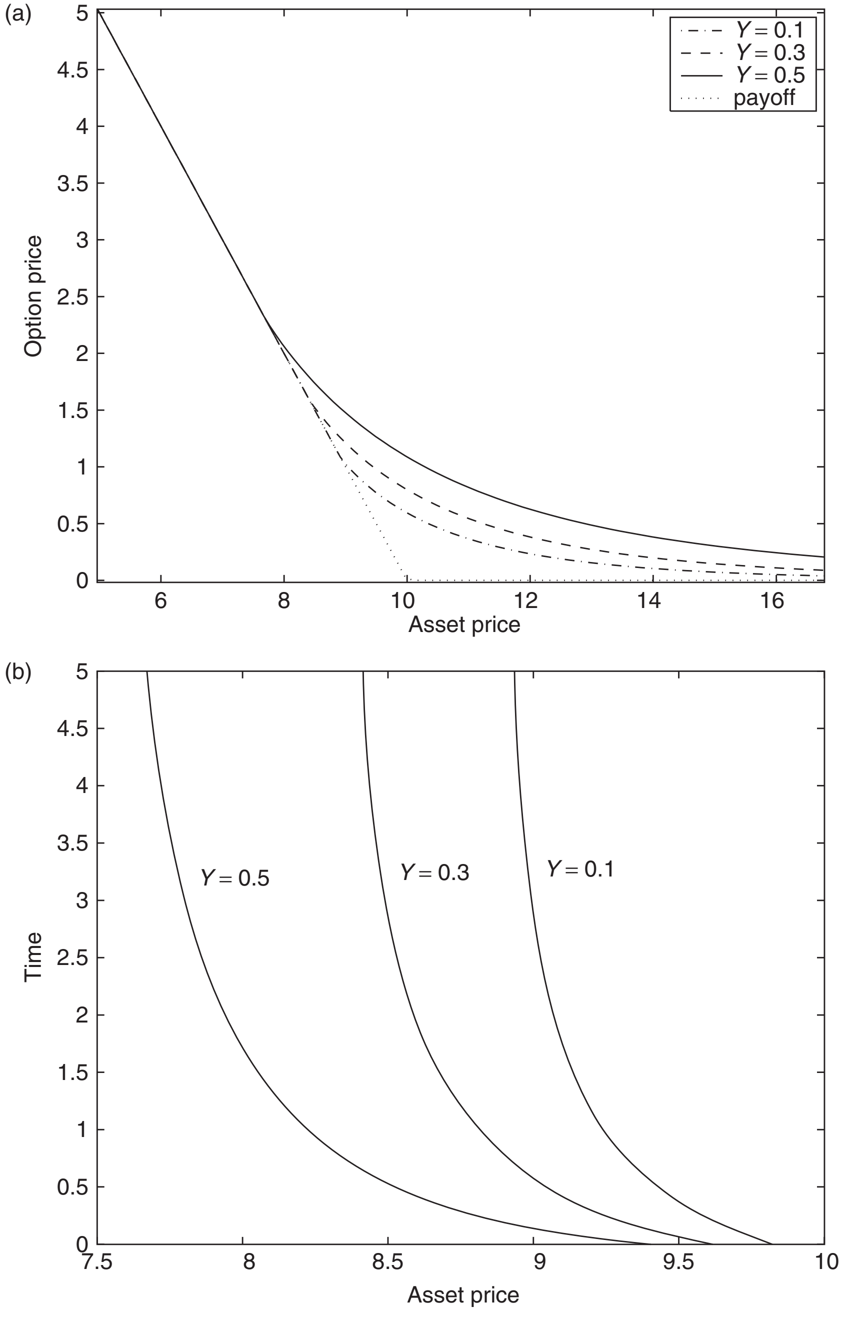

Figure 12.2 (a) Option prices for different values of the parameter Y, i.e. 0.1, 0.3 and 0.7, and (b) the corresponding exercise boundaries: $\sigma = 0$; $r = 0.1$; $q = 0$; $K = 10$; $T = 5$; $C = 1$; $G = 7.8$; $M = 8.2$

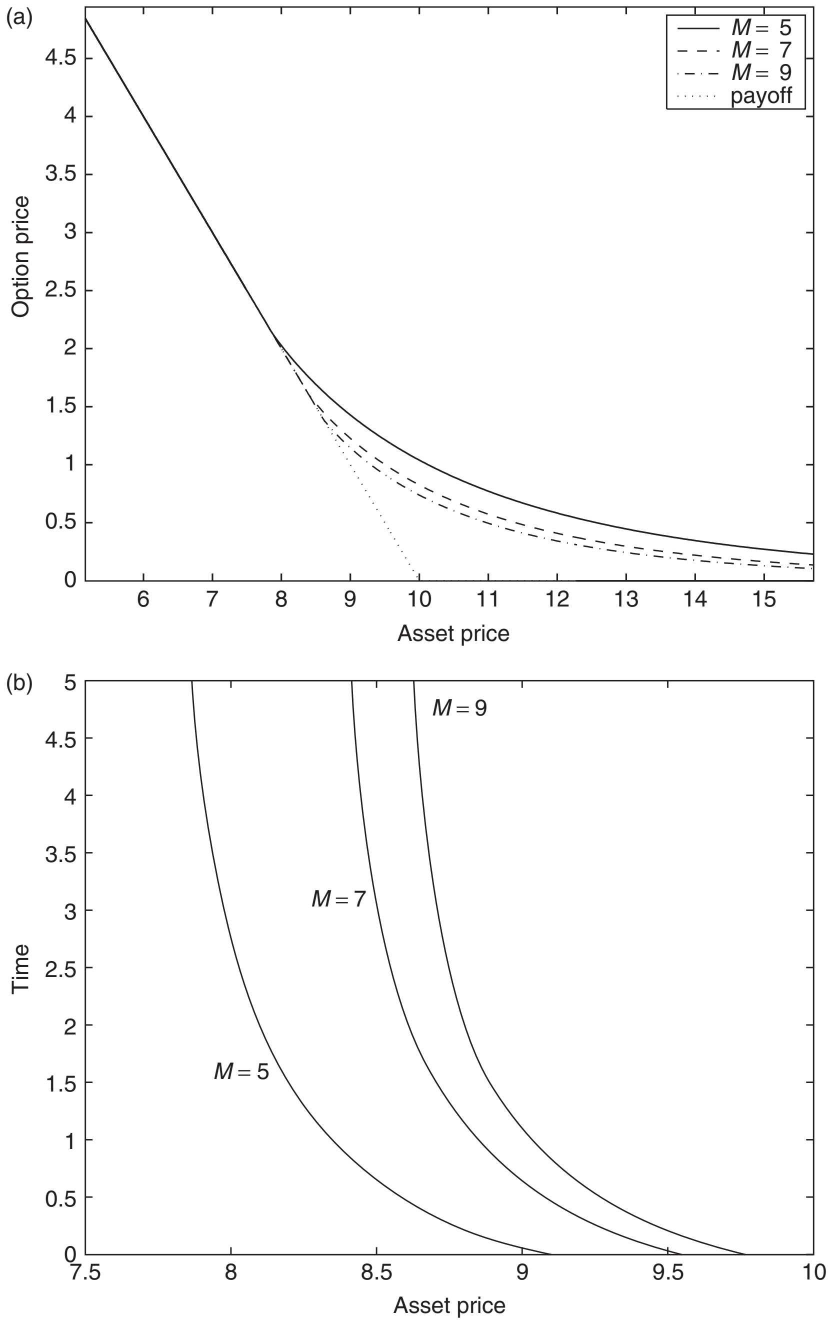

Figure 12.3 (a) Option prices for different values of the parameter M, i.e. 5, 7 and 9, and (b) the corresponding exercise boundaries: $\sigma = 0$; $r = 0.1$; $q = 0$; $K = 10$; $T = 5$; $C = 1$; $G = 7$; $Y = 0.2$

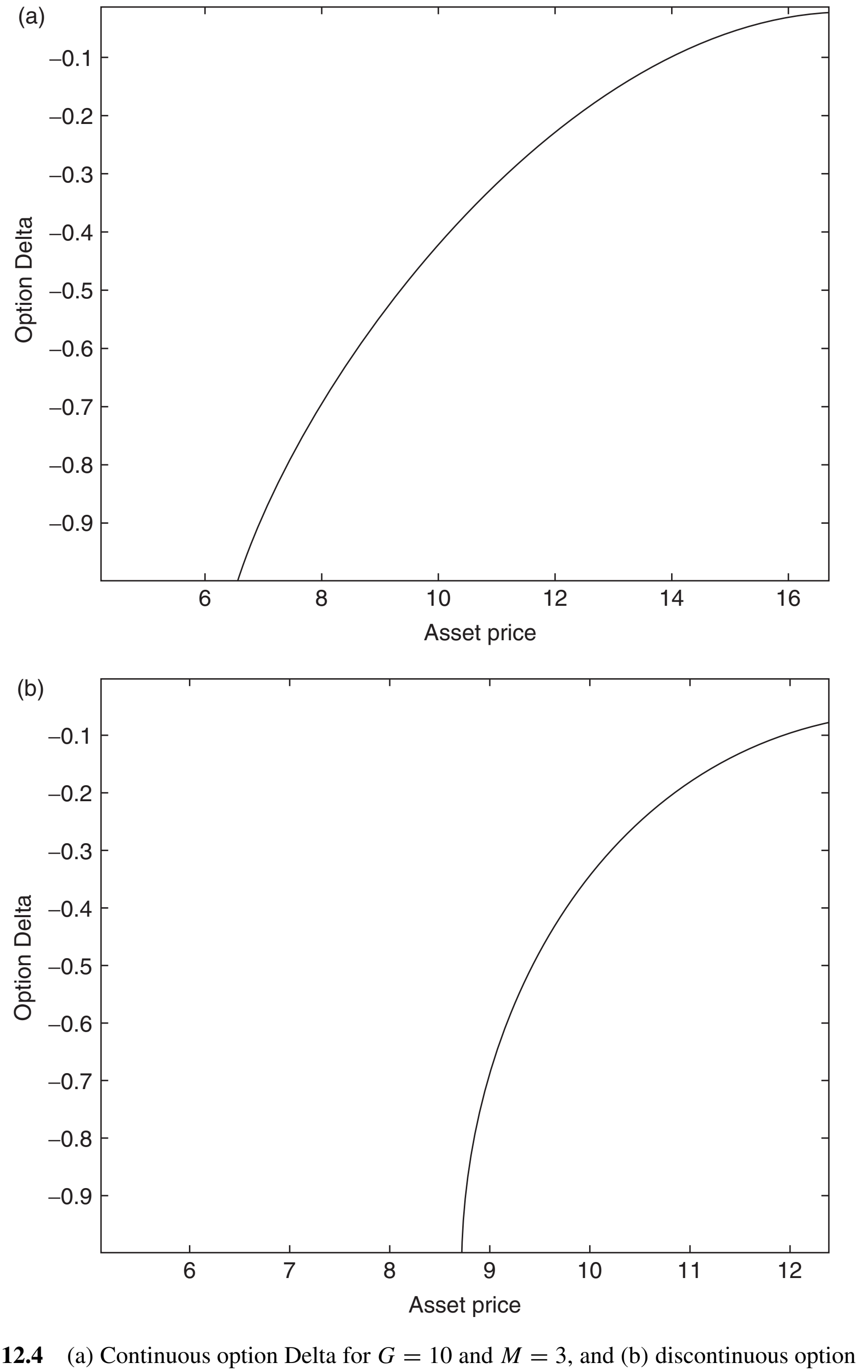

Figure 12.4 (a) Continuous option Delta for $G = 10$ and $M = 3$, and (b) discontinuous option Delta for $G = 7$ and $M = 9$: $\sigma = 0$; $r = 0.1$; $q = 0$; $K = 10$; $T = 1$; $C = 1$

The constant α in equation (12.9) is determined by the region of validity of equation (12.46) together with the strip of regularity of equation (12.47). In this case, we may pick $\alpha \in (-G, 0)$. A method using the FFT algorithm was proposed in Carr and Madan (1999) [9] to evaluate an analogous version of equation (12.45).

REFERENCES

[1] Alili, L. and Kyprianou, A.E. (2004) "Some remarks on first passage of Lévy processes, the American put and pasting principles", to be published.

[2] Almendral, A. and Oosterlee, C.W. (2003), "Numerical valuation of options with jumps in the underlying", *Technical Report*, Delft University of Technology, Delft, The Netherlands; to be published.

[3] Almendral, A. (2004), "On American options under the Variance Gamma process", *Technical Report*, Delft University of Technology, Delft, The Netherlands: to be published.

[4] Asmussen, S. and Rosiński, J. (2001), "Approximations of small jumps of Lévy processes with a view towards simulation", *Journal of Applied Probability*, **38**, 482–493.

[5] Boyarchenko, S.I. and Levendorskiĭ, S.Z. (2002), *Non-Gaussian Merton–Black–Scholes Theory*, Advanced Series on Statistical Science and Applied Probability, vol. 9, World Scientific, River Edge, NJ, USA.

[6] Brennan, M.J. and Schwartz, E.S. (1977), "The valuation of the American put option", *Journal of Finance*, **32**, 449–462.

[7] Carr, P., Geman, H., Madan, D.B. and Yor, M. (2003), "Stochastic volatility for Lévy processes", *Mathematical Finance*, **13**, 345–382.

[8] Carr, P.P., Geman, H., Madan, D.B. and Yor, M. (2002), "The fine structure of asset returns: An empirical investigation", *Journal of Business*, **75**, 305–332.

[9] Carr, P.P. and Madan, D.B. (1999), "Option valuation using the Fast Fourier Transform", *Journal of Computational Finance*, **2**, 61–73.

[10] Carr, P.P., Madan, D.B. and Chang, E.C. (1998), "The Variance Gamma process and option pricing", *European Finance Review*, **2**, 79–105.

[11] Cont, R. and Tankov, P. (2004), *Financial Modelling with Jump Processes*, Chapman & Hall, Boca Raton, FL, USA.

[12] Cont, R. and Voltchkova, E. (2003), "A finite difference scheme for option pricing in jump diffusion and exponential Lévy models", *Technical Report 513*, CMAP, Palaiseau, France.

[13] Hirsa, A. and Madan, D.B. (2004), "Pricing American options under Variance Gamma", *Journal of Computational Finance*, **7**.

[14] Jaillet, P., Lamberton, D. and Lapeyre, B. (1990), "Variational inequalities and the pricing of American options", *Acta Applied Mathematica*, **21**, 263–289.

[15] Kou, S.K. (2002), "A jump diffusion model for option pricing", *Management Science*, **48**, 1086–1101.

[16] Lewis, A.L. (2001), "A simple option formula for general jump-diffusion and other exponential Lévy processes", in *Proceedings of the Annual CAP Workshop on Derivative Securities and Risk Management*.

[17] Matache, A.M., Nitsche, P.A. and Schwab, C. (2003), "Wavelet Galerkin pricing of American options on Lévy driven assets, *Working Paper*, ETH, Zürich, Switzerland.

[18] Raible, S. (2000), "Lévy processes in finance: theory, numerics and empirical Facts", *Ph.D Thesis*, Institute für Mathematische Stochastik, Albert-Ludwigs-Universität Freiburg, Freiburg, Germany.

[19] Sato, K.-I. (2001), "Basic results on Lévy processes", in *Lévy Processes*, Birkhäuser, Boston, MA, USA, pp. 3–37.

[20] Schoutens, W. (2003), *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley, Chichester, UK.

[21] van Loan, C. (1992), *Computational Frameworks for the Fast Fourier Transform*, Frontiers in Applied Mathematics, vol. 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA.

13 Convertible Bonds: Financial Derivatives of Game Type

Jan Kallsen
Munich University of Technology, Garching, Germany

and

Christoph Kühn
Johann Wolfgang Goethe-Universität, Frankfurt am Main, Germany

Abstract

A convertible bond is a security that the holder can convert into a specified number of underlying shares. In addition, very often the issuer can recall the bond, paying some compensation, or force the holder to convert it immediately. Therefore, the pricing problem has also a game-theoretic aspect. When modelling convertible (callable) bonds within the framework of a firm value model, they can be considered as an example of a standard game contingent claim as long as no dividends are distributed to the equity holders.

This article reviews the classical as well as some recent literature in this field. Furthermore, we introduce a mathematically rigorous concept of no-arbitrage price *processes* for these kinds of derivatives, which explicitly incorporates the feature that the contract can be terminated by both counterparties prematurely. We compare this dynamic conception to price derivatives with the static one by Karatzas and Kou (1998) [18].

13.1 INTRODUCTION

A *game contingent claim (GCC)*, as introduced in Kifer (2000) [20], is a contract between a seller A and a buyer B which can be terminated by A and exercised by B at any time $t \in [0, T]$ up to a maturity date T when the contract is terminated anyway. More precisely, the contract may be specified in terms of two stochastic processes $(L_t)_{t\in[0,T]}$, $(U_t)_{t\in[0,T]}$ with

$$L_t \leq U_t \text{ for } t \in [0, T) \text{ and } L_T = U_T. \tag{13.1.1}$$

If A terminates the contract at time t before it is exercised by B, she has to pay B the amount U_t. If B exercises the option before it is terminated by A, he is paid L_t. An example is a put option of game type with constant penalty $\delta > 0$. If S^1 denotes the price process of the underlying and K the strike price, then $L_t = (K - S_t^1)^+$ and $U_t = (K - S_t^1)^+ + \delta 1_{\{t<T\}}$. In the Black–Scholes market, the value function of this finite expiry put option was characterized via mixtures of other exotic options by Kühn and Kyprianou (2003a) [22]. However, sometimes the payoff processes L and U themselves depend on the 'market price process' of the GCC (e.g. for convertible bonds if the dividends paid to the equity holders

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

depend on the stock price – see Section 13.3). To cover this we extend the definition of L and U as follows.

Definition 13.1.1 *L and U are mappings*

$$L, U : [0, T] \times \Omega \times \mathcal{S} \longrightarrow \mathbb{R}_+ \tag{13.1.2}$$

satisfying the conditions (13.1.1), where $\mathcal{S}$ is the set of $\mathbb{R}$-valued semimartingales representing the possible price processes of the GCC. Fixing a price process $\widetilde{S} \geq 0$, we identify the triplet $(L, U, \widetilde{S})$ with an $(\mathbb{R}_+ \cup \{+\infty\})^3$-valued stochastic process. L and U only have to be adapted processes with càdlàg paths. We say that (L, U) is exogenous *if the mappings in (13.1.2) do not depend on their third argument.*

With American options, the right to terminate the contract is restricted to the buyer B. Formally, they can be interpreted as GCCs by setting $U_t := \infty$ for $t \in [0, T)$. We obtain a European claim by setting additionally $L_t := 0$ for $t \in [0, T)$ and $L_T = H$ for some nonnegative $\mathcal{F}_T$-measurable random variable H. This allows us to consider nearly every option as an example of a GCC. Exceptions are passport options, or swing-options (multiple exercisable options) which equip the holder with more extensive rights to influence the payoff. At least heuristically, GCCs incorporate a Dynkin game: if seller A selects stopping time σ as cancellation time and buyer B chooses stopping time τ as exercise time, then A pledges to pay B at time $\sigma \wedge \tau$ the amount

$$R(\tau, \sigma) = L_\tau 1_{\{\tau \leq \sigma\}} + U_\sigma 1_{\{\sigma < \tau\}}. \tag{13.1.3}$$

The paper is organized as follows. In Section 13.2, we introduce two different concepts to define no-arbitrage prices for GCCs. In Section 13.3 we review the classical as well as some recent literature on convertible bonds, the most prominent example for GCCs.

Throughout, we use the notation of Jacod and Shiryaev (1987) [14] (henceforth JS) and Jacod (1979 and 1980) [12] [13] . The components of a vector are denoted by superscripts. Increasing processes are identified with their corresponding Lebesgue–Stieltjes measure. $L(S)$ denotes the set of vector-valued predictable processes φ which are integrable with respect to the vector-valued semimartingale S (cf. JS, Section III.6.17). Stochastic integrals are written in dot notation, i.e. $\varphi \cdot S_t$ means $\int_0^t \varphi_s \, dS_s$.

Our mathematical framework for a frictionless market model is as follows: fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ in the sense of JS (Section I.1.2). $\mathcal{T}_t$ denotes the set of all $[t, T]$-valued stopping times and $\mathcal{T} := \mathcal{T}_0$. We consider traded *securities* $1, \ldots, d$ whose price processes are expressed in terms of multiples of a traded *numeraire* security 0. Put differently, these securities are modelled by their discounted *price process* $S := (S^1, \ldots, S^d)$. We assume that S is a $\mathbb{R}^d$-valued semimartingale. *Trading strategies* are modelled by $\mathbb{R}^d$-valued, predictable stochastic processes $\varphi = (\varphi^1, \ldots, \varphi^d) \in L(S)$, where φ_t^i denotes the number of shares of security i in the investor's portfolio at time t. A strategy $\varphi \in L(S)$ belongs to the set Θ of all *admissible* strategies if its discounted *wealth process* $V(\varphi) := v_0 + \varphi \cdot S$ is bounded from below by some constant (possibly depending on φ). In the whole article, we assume that the 'underlying market' $(S^1, \ldots, S^d)$ satisfies the condition *no free lunch with vanishing risk (NFLVR)*. From Delbaen and Schachermayer (1998) [7], this is equivalent to the condition that the set

$$\mathcal{M}^e := \{Q \sim P \mid S \text{ is a } Q\text{-}\sigma\text{-martingale}\}$$

is nonempty. S is a σ-martingale iff there is a sequence of predictable sets $(D_n)_{n\in\mathbb{N}} \subset [0, T] \times \Omega$ with $D_n \nearrow [0, T] \times \Omega$, for $n \nearrow \infty$, and the processes $1_{D_n} \cdot S$ are martingales for all $n \in \mathbb{N}$. This generalization of a *local* martingale becomes necessary if S is not locally bounded. For background on σ-localization and the related classes of processes, we refer the reader to Delbaen and Schachermayer (1998) [7], Kabanov (1997) [15], and Kallsen (2003) [16].

13.2 NO-ARBITRAGE PRICING FOR GAME CONTINGENT CLAIMS

There are different ways to define no-arbitrage prices for GCCs. Perhaps the most important distinguishing feature is the difference whether to look only at *initial* arbitrage-free prices of the GCC or at whole arbitrage-free price *processes*. Taking the static point of view (which is usually done in lectures on mathematical finance) corresponds to the assumption that only buy-and-hold strategies in the derivative are allowed, whereas the underlyings $(S^1, \dots, S^d)$ can be traded dynamically (to make use of the replication property in complete models as, e.g. in the Black–Scholes model). For American and game options premature exercising is of course modelled. However, for the static approach it is not necessary that there is a liquid market for the option during the whole term $[0, T]$. Therefore, this approach is particularly well suited for over-the-counter trades, and is treated in Section 13.2.1.

On the other hand, one may assume that the option becomes a liquid and negotiable security that can be traded together with the underlyings on the market (during the whole term $[0, T]$). This corresponds to a dynamic point of view where we want to determine a derivative price *process* $S^{d+1} = (S_t^{d+1})_{t\in[0,T]}$. For given $S^1, \dots, S^d$ the process S^{d+1} should be determined such that the joint market $(S^1, \dots, S^d, S^{d+1})$ is arbitrage-free in some sense. We treat this in Section 13.2.2. For derivative pricing based on utility (rather than arbitrage) arguments, the same distinction can be made between dynamic and static trading (see Kühn (2004) [21] and Kallsen and Kühn (2004) [17]). One should note that the results of this section are also relevant for American and even for European contingent claims which can be interpreted as special cases of a GCC.

13.2.1 Static no-arbitrage prices

We want to define static initial no-arbitrage prices for the GCC (L, U). Having in mind that there may not exist a liquid market for the option, we assume that the payoff processes are exogenous, i.e. L and U do not depend in turn on the price process of the GCC which may not even exist in this context.

The following definition is quite similar to Definition 4.2 in Karatzas and Kou (1998) [18] for American contingent claims.

Definition 13.2.1 *Suppose that $u \in \mathbb{R}$ is the price of the GCC at time $t = 0$. We say that u admits an arbitrage opportunity if there exists either*

(i) a pair $(\varphi, \sigma) \in \Theta \times \mathcal{T}$ such that

$$x + (\varphi \cdot S)_{t\wedge\sigma} - R(t, \sigma) \geq 0, \quad \forall t \in [0, T]$$

for some $x < u$ (seller-arbitrage), or *(ii) a pair* $(\varphi, \tau) \in \Theta \times \mathcal{T}$ *such that*

$$-x + (\varphi \cdot S)_{\tau \wedge t} + R(\tau, t) \geq 0, \quad \forall t \in [0, T]$$

for some $x > u$ *(buyer-arbitrage).*

Condition (i) would allow the seller/writer to make a riskless profit regardless of the exercising strategy of the option holder (he need not even know it) and, analogously, (ii) allows the buyer/holder to make a riskless profit in the same sense.

The following theorem characterizes the set of no-arbitrage prices for GCCs. It follows from the results in Karatzas and Zamfirescu (2003) [19], Kifer (2000) [20], Lepeltier and Maingueneau (1984) [24] (henceforth LM) and Föllmer and Kabanov (1998) [8].

Theorem 13.2.2 *Let*

$$\sup_{Q \in \mathcal{M}^e} E_Q(\sup_{t \in [0,T]} L_t) < \infty. \tag{13.2.1}$$

(Note that we allow for $U = +\infty$ *on* $[0, T)$*). Then, the set of no-arbitrage prices in the sense of Definition 13.2.1 is given by the closed interval* $[h_{low}, h_{up}]$*, where*

$$h_{up} = \inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} \sup_{Q \in \mathcal{M}^e} E_Q\left(R(\tau, \sigma)\right) = \sup_{\tau \in \mathcal{T}_0} \sup_{Q \in \mathcal{M}^e} \inf_{\sigma \in \mathcal{T}_0} E_Q\left(R(\tau, \sigma)\right) \tag{13.2.2}$$

and

$$h_{low} = \sup_{\tau \in \mathcal{T}_0} \inf_{\sigma \in \mathcal{T}_0} \inf_{Q \in \mathcal{M}^e} E_Q\left(R(\tau, \sigma)\right) = \inf_{\sigma \in \mathcal{T}_0} \inf_{Q \in \mathcal{M}^e} \sup_{\tau \in \mathcal{T}_0} E_Q\left(R(\tau, \sigma)\right). \tag{13.2.3}$$

If L *has no negative and* U *no positive jumps, then the supremum over all* $\tau \in \mathcal{T}_0$ *and the infimum over all* $\sigma \in \mathcal{T}_0$ *are attained in equations (13.2.2) and (13.2.3), respectively.*

Remark 13.2.3 *The interchangeability of the infima and suprema in equations (13.2.2) and (13.2.3) is non-trivial and has essentially been shown in Karatzas and Zamfirescu (2003) [19] and LM.*

Remark 13.2.4 *The closedness of the interval (connected with the strict inequality in Definition 13.2.1) has technical reasons. For the European case, e.g. the lower end point of a non-degenerate interval would already lead to an arbitrage opportunity for the option buyer in the usual sense.*

Remark 13.2.5 *In complete markets, i.e. when a unique* σ*-martingale measure* Q *exists, Theorem 13.2.2 implies that there is a unique no-arbitrage price given by*

$$h = \inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} E_Q\left(R(\tau, \sigma)\right) = \sup_{\tau \in \mathcal{T}_0} \inf_{\sigma \in \mathcal{T}_0} E_Q\left(R(\tau, \sigma)\right).$$

This is precisely the situation of a zero-sum Dynkin stopping game *(see Kifer (2000) [20]).*

The buyer wants to maximize

$$\max_{\tau\in\mathcal{T}_0} E_Q\left(R(\tau,\sigma)\right),$$

while the seller wants to minimize

$$\min_{\sigma\in\mathcal{T}_0} E_Q\left(R(\tau,\sigma)\right).$$

Proof of Theorem 13.2.2. Since the situation for GCC is symmetric, we may restrict ourselves to the assertion related to h_{up}.

Step 1: First of all, we have to show that

$$\inf_{\sigma\in\mathcal{T}_0}\ \sup_{\tau\in\mathcal{T}_0}\ \sup_{Q\in\mathcal{M}^e} E_Q\left(R(\tau,\sigma)\right) = \sup_{\tau\in\mathcal{T}_0}\ \sup_{Q\in\mathcal{M}^e}\ \inf_{\sigma\in\mathcal{T}_0} E_Q\left(R(\tau,\sigma)\right). \tag{13.2.4}$$

To do this we have to generalize Theorem 3.3 in Karatzas and Zamfirescu (2003) [19] to game contingent claims. In addition, the latter theorem is stated only for quasi-left continuous payoff processes, but it also holds for the general case of càdlàg processes. As this generalization is straightforward we only sketch the main steps.

Similar to that in Karatzas and Zamfirescu (2003) [19], we define the generalized (lower) Dynkin value process by a càdlàg version of

$$\underline{V}_t = ess\ \ sup_{Q\in\mathcal{M}^e}\ ess\ \ sup_{\tau\in\mathcal{T}_t}\ ess\ \ inf_{\sigma\in\mathcal{T}_t} E_Q\left(R(\tau,\sigma)|\mathcal{F}_t\right)$$

and for all $\varepsilon > 0$ we define the recall times

$$\sigma^\varepsilon = \inf\{t \geq 0 \mid \underline{V}_t \geq U_t - \varepsilon\}.$$

Analogously to Proposition 3.1 in Karatzas and Zamfirescu (2003) [19] and Theorem 11 in LM, one can show that for all $\varepsilon > 0$ the stopped process $\underline{V}^{\sigma^\varepsilon}$ is a Q-supermartingale w.r.t. all $Q \in \mathcal{M}^e$. Note that in LM the payoff processes L and U are supposed to be bounded, but the results still hold under the weaker condition (13.2.1) (see Theorem 1.1 in Kühn and Kyprianou (2003b) [23].

By this supermartingale property and $L \leq \underline{V}$ we have for all $Q \in \mathcal{M}^e$, $\tau \in \mathcal{T}_0$

$$E_Q\left(R(\tau,\sigma^\varepsilon)\right) \leq E_Q\left(L_\tau 1_{\{\tau\leq\sigma^\varepsilon\}} + \underline{V}_{\sigma^\varepsilon} 1_{\{\sigma^\varepsilon<\tau\}}\right) + \varepsilon \leq E_Q\left(\underline{V}_{\tau\wedge\sigma^\varepsilon}\right) + \varepsilon \leq \underline{V}_0 + \varepsilon.$$

This immediately implies equation (13.2.4).

If U has no positive jumps, then this step also holds with $\varepsilon = 0$ (again, by Karatzas and Zamfirescu (2003) [19] and LM) and σ^0 is the optimal recall time.

Step 2: It is well-known that for the value $\inf_{\sigma\in\mathcal{T}_0}\ \sup_{\tau\in\mathcal{T}_0} E_Q\left(R(\tau,\sigma)\right)$ of a stochastic game (with respect to a single probability measure Q), the payoff in the marginal case $\tau = \sigma$ (both agents stopping at the same time) is irrelevant, as long as it belongs to the interval $[L, U]$. Again, the same holds for worst case stochastic games. Thus we can w.l.o.g. define R from equation (13.1.3) in each case such that $t \mapsto R(t,\sigma)$ and $t \mapsto R(\tau, t)$, respectively, are càdlàg.

Let $\varepsilon > 0$ and $\sigma^\star \in \mathcal{T}_0$ be a recalling time with $h_{\sigma^\star} := \sup_{\tau\in\mathcal{T}_0} \sup_{Q\in\mathcal{M}^e} E_Q\,(R(\tau,\sigma^\star)) \le h_{up} + \varepsilon$. Define an American contingent claim (ACC) by the càdlàg process $X = (X_t)_{t\in[0,T]}$, where

$$X_t = L_t 1_{\{t<\sigma^\star\}} + U_{\sigma^\star} 1_{\{t\ge\sigma^\star\}}. \tag{13.2.5}$$

By the general theory on ACCs (cf. Föllmer and Kabanov (1998) [8]), we know that there is a $\varphi \in \Theta$ such that

$$h_{\sigma^\star} + \varphi \cdot S \ge X.$$

Thus, a derivative price lying in the interval (h_{up}, ∞) allows for seller-arbitrage. On the other hand, for each $x \in \mathbb{R}$, $\sigma^\star \in \mathcal{T}_0$, and X defined as in equation (13.2.5), $x + \varphi \cdot S \ge X$ implies that $x \ge \sup_{\tau\in\mathcal{T}_0} \sup_{Q\in\mathcal{M}^e} E_Q\,(R(\tau,\sigma^\star)) \ge h_{up}$ (cf. Föllmer and Kabanov (1998) [8]). Thus, the interval $(-\infty, h_{up}]$ excludes a seller-arbitrage in the sense of Definition 13.2.1.

13.2.2 No-arbitrage price processes

Now we want to determine derivative price *processes* $S^{d+1} = (S_t^{d+1})_{t\in[0,T]}$. The extended market $(S^1, \ldots, S^d, S^{d+1})$ should still be arbitrage-free in a sense specified later on. For GCCs, the dynamic approach seems to induce a quite complex situation. A GCC can be both traded on the market in continuous time and the respective holder and writer can insist on their right to exercise resp. recall the option. This means that there are market transactions as well as the claiming of contractually guaranteed exercising rights. However, it turns out that the admissible market transactions are more comprehensive than the contractually provided exercising possibilities. For example, a holder who wants to exercise an American option at time t (which yields the reward L_t) can alternatively resell the option on the market getting the amount $S_t^{d+1} \ge L_t$. Thus, exercising need not be considered explicitly.

Trading in GCCs corresponds to trading under some constraints. Investors may not be able to hold arbitrary amounts of GCCs because these contracts can be cancelled. If the market price approaches the upper cancellation value U, it may happen that all options vanish from the market because they are terminated by the sellers. So, a long position in the option is no longer feasible. Conversely, all derivative contracts may be exercised by the claim holders if the market price coincides with the exercise value L. This terminates short positions in the claim. However, as long as the derivative price stays above the exercise value, nobody will exercise the option because selling it on the market yields a higher reward. Similarly, there is no danger that the seller of a GCC cancels the contract as long as the cancellation value exceeds the market price. Summing up, derivative traders are facing trading constraints $\widetilde{\Theta}$ given by

$$\begin{aligned}\widetilde{\Theta} := \{&\varphi \in L(S):\ \varphi\cdot S \text{ is bounded from below and we have } \forall(t,\omega)\in[0,T]\times\Omega \\ &\varphi_t^{d+1}(\omega)\ge 0 \text{ if } S_{t-}^{d+1}(\omega)=L_{t-}(\omega) \text{ and } \varphi_t^{d+1}(\omega)\le 0 \text{ if } S_{t-}^{d+1}(\omega)=U_{t-}(\omega)\}.\end{aligned} \tag{13.2.6}$$

To keep the exposition simple, we work with only one GCC and assume that L is non-negative.

Definition 13.2.6 *A derivative price process S^{d+1} is arbitrage-free for the GCC $(L, U) = (L(S^{d+1}), U(S^{d+1})) = (L_t, U_t)_{t\in[0,T]}$ iff the following conditions are satisfied:*

1. *S^{d+1} is a semimartingale with $L \le S^{d+1} \le U$.*
2. *The market $S = (S^0, \dots, S^d, S^{d+1})$ with a constrained set of trading strategies $\widetilde{\Theta}$ satisfies the following* no free lunch with vanishing risk (NFLVR) *condition: 0 is the only non-negative element of the $L^\infty(\Omega, \mathcal{F}, P)$-closure of the set $C := \{f \in L^\infty(\Omega, \mathcal{F}, P) : f \le \varphi \cdot S_T$ for some $\varphi \in \widetilde{\Theta}\}$. (Note that this is a straightforward extension of the usual NFLVR condition in Delbaen and Schachermayer (1998) [7], Definition 2.8, to markets containing a game contingent claim.)*
3. *For any $v_0 \in \mathbb{R}$, $\varphi \in \widetilde{\Theta}$, the following implication holds:*

$$L \le v_0 + \varphi \cdot S \Rightarrow S^{d+1} \le v_0 + \varphi \cdot S.$$

Remark 13.2.7 *Even for bounded European claims in the Black–Scholes (BS) model, we cannot derive unique arbitrage-free derivative prices based on conditions (1) and (2) alone. The reason for this is that wealth processes are supposed to be bounded from below. In the BS model, it is possible to construct suicide strategies ('bad' doubling strategies) such that the corresponding wealth process $V(\varphi)$ satisfies $V_0(\varphi) = 1$ and $V_T(\varphi) = 0$ P-a.s., see e.g. Harrison and Pliska (1981) [9]. Such a wealth process $V(\varphi)$ is bounded from below but not from above. $V(\varphi)$ is a strict local martingale with respect to the unique equivalent martingale measure in the BS model. However, it does not generate an arbitrage opportunity because $-\varphi$ is not an admissible strategy. Thus, $V(\varphi)$ is a derivative price process for the European claim $H = 0$ satisfying conditions (1) and (2), but not (3).*

Remark 13.2.8 *Let $\mathcal{K}_0$ be the set of all terminal wealths which are attainable with initial capital* 0, *i.e.*

$$\mathcal{K}_0 := \left\{g \in L^0(\Omega, \mathcal{F}, P) : \exists\varphi \in \widetilde{\Theta} \text{ such that } g = \varphi \cdot S_T\right\}.$$

In Delbaen and Schachermayer (1998) [7] an element $g \in \mathcal{K}_0$ is called maximal *if $h \in \mathcal{K}_0$ and $h \ge g$ imply that $h = g$. For the European case, condition (3) is equivalent to the condition that $S_T^{d+1} - S_0^{d+1}$ is a maximal element in the set $\mathcal{K}_0$ of terminal wealths which are attainable in the enlarged market $S = (S^1, \dots, S^d, S^{d+1})$ with initial capital* 0. *Namely, if $S_T^{d+1} - S_0^{d+1}$ is maximal there exists a measure $Q \in \widetilde{\mathcal{M}}^e$ such that S^{d+1} is a Q-martingale (cf. Theorem 5.2 in Delbaen and Schachermayer (1998) [7]). Thus, $v_0 + \varphi \cdot S_T - S_T^{d+1} \ge 0$ implies that $v_0 + \varphi \cdot S - S^{d+1} \ge 0$, as $\varphi \cdot S - S^{d+1}$ is a Q-supermartingale, and we arrive at condition (3). On the other hand, assuming that for every $v_0 \in \mathbb{R}$, $\varphi \in \widetilde{\Theta}$ the implication*

$$S_T^{d+1} \le v_0 + \varphi \cdot S_T \Rightarrow S^{d+1} \le v_0 + \varphi \cdot S \tag{13.2.7}$$

holds, we can take a maximal element $\varphi \cdot S_T$, $\varphi \in \widetilde{\Theta}$ which superhedges S_T^{d+1} (cf. Lemma 5.13 in Delbaen and Schachermayer (1998) [7]). By the right-hand side of equation (13.2.7) we have that the process S^{d+1} is bounded from above by a Q-martingale for some $Q \in \widetilde{\mathcal{M}}^e$. Thus, $S_T^{d+1} - S_0^{d+1}$ is a maximal element in the enlarged market.

The following theorem characterizes the set of no-arbitrage prices for GCCs. It is more or less a direct consequence of the results in Delbaen and Schachermayer (1998) [7]. In addition, a couple of arguments are borrowed from Kallsen and Kühn (2004) [17].

Theorem 13.2.9 *Let*

$$\sup_{Q\in\mathcal{M}^e} E_Q\left(\sup_{t\in[0,T]} L_t\right) < \infty. \tag{13.2.8}$$

(Note that we allow for $U = +\infty$ on $[0, T)$). Then, S^{d+1} is an arbitrage-free price process for the GCC (L, U) in the sense of Definition 13.2.6, if and only if, it is a semimartingale and satisfies

$$\begin{aligned} S_t^{d+1} &= \textit{ess inf}_{\sigma\in\mathcal{T}_t}\ \textit{ess sup}_{\tau\in\mathcal{T}_t} E_Q\left(R(\tau,\sigma)|\mathcal{F}_t\right) \\ &= \textit{ess sup}_{\tau\in\mathcal{T}_t}\ \textit{ess inf}_{\sigma\in\mathcal{T}_t} E_Q\left(R(\tau,\sigma)|\mathcal{F}_t\right) \end{aligned} \tag{13.2.9}$$

for some $Q \in \mathcal{M}^e$.

Remark 13.2.10 *There are several conditions on the processes L and U ensuring that the right-continuous version of S^{d+1} in equation (13.2.9) is a semimartingale (regardless of the chosen pricing measure Q). One condition is*

$$U > L \quad \text{on } [0,T)\times\Omega, \text{ and } \quad U_- > L_- \quad \text{on } [0,T]\times\Omega, \tag{13.2.10}$$

which holds true for the American case ($U_t = \infty$ for $t < T$) or for the callable put with constant penalty (see Kühn and Kyprianou 2003a [22]). For other sufficient conditions, see Kallsen and Kühn (2004) [17].

Proof. Ad $\Rightarrow$: *Step 1:* Let S^{d+1} be an arbitrage-free price process. By condition (2) and the fundamental theorem of asset pricing there exists a probability measure $Q \in \mathcal{M}^e$ (i.e. $Q \sim P$ and $S^1, \ldots, S^d$ are Q-σ-martingales) such that $1_{\{S_-^{d+1}=L_-\}} \cdot S^{d+1}$ is a Q-σ-supermartingale, $1_{\{L_-<S_-^{d+1}<U_-\}} \cdot S^{d+1}$ is a Q-σ-martingale, and $1_{\{S_-^{d+1}=U_-\}} \cdot S^{d+1}$ is a Q-σ-submartingale. The assertion can be verified by following the lines of the proof of Theorem 1.1 in Delbaen and Schachermayer (1998) [7] by taking the long- and shortselling constraints into account.

Let (B, C, ν) be the characteristics of the semimartingale $S = (S^1, \ldots, S^d, S^{d+1})$ relative to some truncation function $h : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$. By JS, II.2.9, there exists some predictable process $A \in \mathcal{A}_{loc}^+$, some predictable $\mathbb{R}^{d+1}$-valued process b, some predictable $\mathbb{R}^{(d+1)\times(d+1)}$-valued process c whose values are non-negative, symmetric matrices, and some transition kernel F from $(\Omega\times[0,T],\mathcal{P})$ into $(\mathbb{R}^{d+1},\mathcal{B}^{d+1})$ such that

$$B = b\cdot A, \quad C = c\cdot A, \quad \nu = A\otimes F.$$

We call (b, c, F, A) the *differential characteristics* of S. If $\int |x^{d+1} - h^{d+1}(x)|F(dx) < \infty$, one can interpret $b_t^{d+1} + \int (x^{d+1} - h^{d+1}(x))F_t(dx)$ as a drift rate of S^{d+1}. A non-positive, non-negative or vanishing drift corresponds to a σ-supermartingale, σ-submartingale or σ-martingale, respectively. Therefore, we have that $\int |x^{d+1} - h^{d+1}(x)|F(dx) < \infty$ and

$$\begin{aligned} &b^{d+1} + \int (x^{d+1} - h^{d+1}(x))F(dx) \le 0 \quad \text{on the set } \{S_-^{d+1} = L_-\} \\ &b^{d+1} + \int (x^{d+1} - h^{d+1}(x))F(dx) = 0 \quad \text{on the set } \{L_- < S_-^{d+1} < U_-\} \\ &b^{d+1} + \int (x^{d+1} - h^{d+1}(x))F(dx) \ge 0 \quad \text{on the set } \{S_-^{d+1} = U_-\} \end{aligned} \tag{13.2.11}$$

with each inequality holding $(P \otimes A)$-almost everywhere. Define

$$\widetilde{\mathcal{M}}^e := \left\{ \widetilde{Q} \in \mathcal{M}^e : \ S^{d+1} \text{ satisfies the drift conditions (13.2.11) with respect to } \widetilde{Q} \right\}.$$

Since $Q \in \widetilde{\mathcal{M}}^e$, this set is nonempty.

Step 2: Due to equation (13.2.8) we have $v_0 := \sup_{\widetilde{Q} \in \widetilde{\mathcal{M}}^e} E_{\widetilde{Q}} \left(\sup_{t \in [0,T]} L_t \right) < \infty$. By Theorem 5.1 and Lemma 5.13 in Delbaen and Schachermayer (1998) [7], there is a strategy $\varphi \in \widetilde{\Theta}$ within the enlarged market such that $v_0 + \varphi \cdot S_T \geq \sup_{t \in [0,T]} L_t$ and $\varphi \cdot S_T$ is a *maximal element* (for a definition, see Remark 13.2.8) in the enlarged market for initial capital 0. From Theorem 5.2 in Delbaen and Schachermayer (1998) [7] it follows that there exists some $\widetilde{Q} \in \widetilde{\mathcal{M}}^e$ such that $\varphi \cdot S$ is a $\widetilde{Q}$-martingale. We obtain

$$v_0 + \varphi \cdot S_t \geq v_0 + E_{\widetilde{Q}} \left(\sup_{s \in [0,T]} L_s \middle| \mathcal{F}_t \right) \geq v_0 + L_t.$$

Due to condition (3), this implies that $S^{d+1} \leq v_0 + \varphi \cdot S$, i.e. the derivative price is bounded from above by some $\widetilde{Q}$-martingale. This, together with the drift conditions, implies that $S_t^{d+1} = \mathit{ess\ inf}_{\sigma \in \mathcal{T}_t} \ \mathit{ess\ sup}_{\tau \in \mathcal{T}_t} E_{\widetilde{Q}} \left(R(\tau, \sigma) | \mathcal{F}_t \right)$ (cf. Step 5 in the proof of Theorem 3.2 in Kallsen and Kühn (2004) [17]).

Ad $\Leftarrow$: Let S^{d+1} be a càdlàg version of $\mathit{ess\ inf}_{\sigma \in \mathcal{T}_t} \ \mathit{ess\ sup}_{\tau \in \mathcal{T}_t} E_Q \left(R(\tau, \sigma) | \mathcal{F}_t \right)$ for some $Q \in \mathcal{M}^e$ and assume that this process is a semimartingale. We obtain that S^{d+1} satisfies the drift conditions (13.2.11) and from the form of the constraints it follows that

$$\varphi^\top \left(b + \int (x - h(x)) F(dx) \right) \leq 0, \quad (P \otimes A)\text{-almost everywhere,}$$

for any strategy φ within the market $S = (S^1, \ldots, S^d, S^{d+1})$, cf. Steps 2 and 4 in the proof of Theorem 3.2 in Kallsen and Kühn (2004) [17]. As $\varphi \cdot S$ is bounded from below, this implies that $\varphi \cdot S$ is a Q-supermartingale, i.e. $E_Q(V_T(\varphi)) \leq v_0$ (see Proposition 3.5 in Kallsen (2003) [16]. Therefore, the enlarged market satisfies NFLVR (condition (2)). Assume that $v_0 + \varphi \cdot S \geq L$. As the process $t \mapsto \ \mathit{ess\ sup}_{\tau \in \mathcal{T}_t} E_Q \left(L_\tau | \mathcal{F}_t \right)$ is the smallest Q-supermartingale dominating L, we obtain that

$$v_0 + \varphi \cdot S_t \geq \ \mathit{ess\ sup}_{\tau \in \mathcal{T}_t} E_Q \left(L_\tau | \mathcal{F}_t \right) \geq \mathit{ess\ inf}_{\sigma \in \mathcal{T}_t} \ \mathit{ess\ sup}_{\tau \in \mathcal{T}_t} E_Q \left(R(\tau, \sigma) | \mathcal{F}_t \right) = S_t^{d+1}$$

for any $t \in [0, T]$, i.e. condition (3) is satisfied.

As in the previous section, the set of initial prices is typically convex as the following result shows.

Proposition 13.2.11 *Suppose that*

$$\sup_{Q \in \mathcal{M}^e} E_Q \left(\sup_{t \in [0,T]} U_t \right) < \infty \tag{13.2.12}$$

(or, alternatively, $U = \infty$ and condition (13.2.8)). Moreover, assume that condition (13.2.10) holds. Then, the set of initial prices

$$\left\{ \inf_{\sigma \in \mathcal{T}_0} \ \sup_{\tau \in \mathcal{T}_0} E_Q \left(R(\tau, \sigma) \right) : Q \in \mathcal{M}^e \right\}$$

is convex.

Proof. Let $Q_0, Q_1 \in \mathcal{M}^e$. For $\lambda \in [0,1]$ define $Q_\lambda := \lambda Q_1 + (1-\lambda)Q_0$. We show that the mapping $\pi : \lambda \mapsto \inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} E_{Q_\lambda}(R(\tau,\sigma))$ is Lipschitz with a Lipschitz constant $c = \sup_{Q \in \mathcal{M}^e} E_Q\left(\sup_{t\in[0,T]} U_t\right)$, which implies that π assumes any value between $\pi(0)$ and $\pi(1)$. Indeed, for $\lambda, \lambda' \in [0,1]$ we have that

$$\begin{aligned}|E_{Q_{\lambda'}}(R(\tau,\sigma)) - E_{Q_\lambda}(R(\tau,\sigma))| &= |\lambda' - \lambda||E_{Q_1}(R(\tau,\sigma)) - E_{Q_0}(R(\tau,\sigma))| \\ &\leq |\lambda' - \lambda| c\end{aligned}$$

for any $\sigma, \tau \in \mathcal{T}_0$. Consequently, we have that $|\pi(\lambda') - \pi(\lambda)| \leq |\lambda' - \lambda| c$. In the American case (i.e. $U = \infty$), the assertion follows similarly by substituting L_τ for $R(\tau,\sigma)$ and using $c = \sup_{Q\in\mathcal{M}^e} E_Q\left(\sup_{t\in[0,T]} L_t\right)$.

From Theorem 13.2.2 and Proposition 13.2.11, it follows that the sets of arbitrage-free prices in the sense of Definitions 13.2.1 resp. 13.2.6 are essentially the same (i.e. up to the interval limits).

Remark 13.2.12 *It is interesting to note that the set of arbitrage-free* price processes *is generally* not *convex, not even in the European case. As an example, consider a two-period model with one asset besides the constant numeraire, namely a stock with initial price* $S_0^1 = 1$. *At time* $t = 1$, *four states of nature can be distinguished, referring to a stock price* $S_1^1 = 0.9, 0.95, 1.05,$ *and* 1.1, *respectively, all assumed with positive probability under* P. *As regards* $t = 2$, *we suppose that* S_2^1/S_1^1 *has, e.g. a standard normal law. We consider a European call expiring at* $T = 2$ *with strike price* $K = 2$, *i.e.* $S_2^2 = (S_2^1 - 2)^+$ *for any arbitrage-free price process* S^2 *of the call. Due to the choice of the law in the second period, any option price* S_1^2 *in the open interval* $(0, S_1^1)$ *is consistent with the absence of arbitrage.*

One possible choice is

$$S_1^2 = \begin{cases} 0.8 & \text{on } \{S_1^1 = 0.9 \text{ or } 1.1\} \\ 0.1 & \text{on } \{S_1^1 = 0.95 \text{ or } 1.05\}. \end{cases}$$

If we have $Q(S_1^1 = 0.9) = Q(S_1^1 = 1.1) = 0.4$ *and* $Q(S_1^1 = 0.95) = Q(S_1^1 = 1.05) = 0.1$ *under the corresponding pricing measure* Q, *we obtain an initial call price* $S_0^2 = 0.66$.

Alternatively, we consider a call price process $\tilde{S}^2$ *with*

$$\tilde{S}_1^2 = \begin{cases} 0.1 & \text{on } \{S_1^1 = 0.9 \text{ or } 1.1\} \\ 0.8 & \text{on } \{S_1^1 = 0.95 \text{ or } 1.05\}. \end{cases}$$

If we have $\tilde{Q}(S_1^1 = 0.9) = \tilde{Q}(S_1^1 = 1.1) = 0.1$ *and* $\tilde{Q}(S_1^1 = 0.95) = \tilde{Q}(S_1^1 = 1.05) = 0.4$ *under the corresponding pricing measure* $\tilde{Q}$, *we obtain again an initial call price* $\tilde{S}_0^2 = 0.66$.

However, the convex combination $\overline{S}^2 = \frac{1}{2}(S^2 + \tilde{S}^2)$ *of these price processes allows for arbitrage. Indeed, we have* $\overline{S}_0^2 = 0.66$ *and* $\overline{S}_1^2 = 0.45$ *in any state of nature.*

13.3 CONVERTIBLE BONDS

This section reviews some literature on convertible bonds. Within a firm value model the pricing problem is treated in Sîrbu *et al.* (2004) [29]. In contrast to earlier articles by Brennan and Schwartz (1977) [2] and Ingersoll (1977a,b) [10], [11], this paper includes the case that earlier conversion can be optimal which necessitates to address a nontrivial free boundary problem.

Assume that a firm issues a convertible bond. At each subsequent time, the bondholder can decide whether to continue to hold the bond, thereby collecting coupons, or to convert it into a predetermined number of stocks. On the other hand, anytime the firm may redeem the convertible at a call price or force an untimely conversion into stocks.

Let us first analyse convertible bonds by using a firm value model. Assume that $V = (V_t)_{t\in[0,T]}$ is the total value of the firm and one stock and one convertible bond are the only assets issued by the firm. Then, the firm value splits into the stock price S (total equity value) and the value of the convertible bond D (debt capital), i.e.

$$V_t = S_t + D_t, \quad t \in [0, T] \quad (\text{resp. } t \in [0, \infty)). \tag{13.3.1}$$

Under the Miller–Modigliani hypothesis (see Miller and Modigliani (1958, 1961) [27] [28]), changes in corporate capital structure do not affect the firm value. Economically, a convertible bond is something between an ordinary bond and a stock. The holder can convert this bond into $\gamma \in \mathbb{R}_+$ stocks and the issuing firm can recall the bond prematurely by paying the amount $K \in \mathbb{R}_+$ and at the same time allowing the holder still to convert. Otherwise, at T the bondholder receives the amount 1 (if the firm is not overindebted). We assume that $1 < K$. As both S and D are tradeable securities (D possibly under some long- resp. shortselling constraints, cf. conditions (13.2.6)) the firm value V is in principle also tradeable. Thus, in view of Theorem 13.2.9, we model it directly under some martingale measure Q, i.e. assume that V satisfies under Q 'up to bankruptcy of the firm or exercise' the SDE

$$dV_t = V_t(r\,dt + \sigma\, dW_t) \quad - \underbrace{c\,dt}_{\text{bond dividends}} - \underbrace{\delta S_t\, dt}_{\text{equity dividends}},$$

where r is the default-free interest rate. Bankruptcy can occur because coupon payments do not vanish when the firm value gets small. By contrast, stock dividends are proportional to the current stock price (and therefore vanish when the firm value tends to zero). Assume that $\delta < r$ and the market containing V is complete. The aim is to determine an equilibrium model for the total equity value S and the debt value D. We obtain the following discounted payoff processes. If the bondholder stops prematurely (i.e. converts the bond into stocks), he obtains the payoff (including the coupons which are already paid)

$$\begin{aligned} L_t &= \exp(-rt)\gamma S_t + c\int_0^t \exp(-ru)\, du \\ &= \exp(-rt)\frac{\gamma}{1+\gamma}V_t + \frac{c}{r}\big[1 - \exp(-rt)\big], \quad t \in [0, T). \end{aligned} \tag{13.3.2}$$

The firm can terminate the contract by paying the amount (including previous coupons)

$$U_t = \exp(-rt)\max\{K, \gamma S_t\} + \frac{c}{r}\big[1 - \exp(-rt)\big], \quad t \in [0, T), \tag{13.3.3}$$

and $L_T = U_T = \exp(-rT)\max\{1, \gamma S_T\} + \frac{1}{r}\big[1 - \exp(-rT)\big]$. Obviously, we make use of the general definition of a GCC incorporating a fixed point problem, cf. conditions (13.1.2).

Namely, V_t depends on the dividends $\delta S_u\, du$ paid up to t to the equity holders and these dividends depend on the division between the stock price S_u and the price D_u of the convertible bond. However, D_u, $u \le t$, also depend on V_t (as D is the dynamic value of a Dynkin game and V influences the payoffs).

Example 13.3.1 (Perpetual case) *For the perpetual case the solution is given in Sîrbu et al. (2004) [29]. There are three regions with qualitative different solutions, namely,* $K \in [0, \frac{c}{r}), [\frac{c}{r}, \frac{c}{\delta}], (\frac{c}{\delta}, \infty)$. *Recall that* $\delta < r$. *Assume that* $\gamma S_0 < K$. *The very last time to recall or exercise the option is when* γS_t *hits* K *(when* $L_t = U_t$ *and the game must be stopped). Define*

$$T_{ult} = \inf\{t \ge 0 \mid \gamma S_t \ge K\}. \tag{13.3.4}$$

1. *For* $K \ge \frac{c}{r}$, *the writer does not stop before* T_{ult}. *Heuristically, this can be seen by a local comparison of the payment streams: by recalling at* t *instead of recalling at* $t + \Delta t$, *the writer can avoid paying coupons* $c\Delta t$, *but on the other hand he has an interest rate loss* $rK\Delta t$, *which is larger. Since, in addition, the discounted price of the convertible cannot exceed* $\exp(-rt)\max\{K, \gamma S_t\}$ *there is no incentive for the writer to stop before* T_{ult}. *Therefore, the game reduces to the optimal stopping problem of optimal coversion by the holder.*
2. *For* $K \le \frac{c}{\delta}$, *there is no reason for the holder to stop before* T_{ult}. *This can again be seen by a local comparison of the payment streams: by converting at* t *instead of converting at* $t + \Delta t$, *the holder gains the stock dividends* $\gamma S_t \delta \Delta t$ *but does without the larger bond dividends* $c\Delta t$ *(as long as* $t < T_{ult}$*). Since, in addition, the process* $t \mapsto \exp(-rt)S_t + \delta \int_0^t \exp(-ru)S_u\, du$ *is a martingale and exercising at time* t *deletes further possibilities given by the claim, there is no incentive for the holder to stop before* T_{ult}. *Therefore, the game reduces to the optimal stopping problem of optimal recalling by the writer.*

Summing up, in the middle interval $[\frac{c}{r}, \frac{c}{\delta}]$ *both players stop at* T_{ult}. *In the left interval the writer and in the right interval the holder could stop earlier.*

Remark 13.3.2 *In the perpetual model the nature of (risky) equity versus (safer) debt capital is reflected by the fact that the stock's dividend rate is proportional to the stock price, whereas the coupons payment rate is constant subject to the firm value being positive.*

Remark 13.3.3 *Even though, within the limits of the model,* T_{ult} *from definition (13.3.4) is the latest reasonable stopping time, empirical literature says that firms often wait until the conversion price* γS_t *is much higher than the call price* K *before they recall. There are quite different reasons for this (cf. Asquith and Mullins (1991) [1] and Sîrbu et al. (2004) [29]).*

1. *If the writer wants to call the bond he has to announce this and the investor has typically 30 days to decide whether he wants to convert it or to obtain the call price* K. *If calling takes place when* $\gamma S_t = K$, *this time delay can become quite important. The investor can condition this decision on the evolution of the stock price after the announcement. Thus, one should replace the buy-back value* $\max\{K, \gamma S_t\}$ *by* $\gamma S_t + V_E(\gamma S_t, \Delta)$ *where* V_E *is the value of a European(!) put option with strike* K *and maturity* Δ. Δ *is the duration of this notice period.*

2. *Taxes are different for coupons payments (for the bond) and dividend payments for stocks. Thus, it can be unprofitable for the firm to cancel the bond.*
3. *'Sleeping investors' which do not convert their bonds optimally should not be awaked by a calling signal.*

Of course, for practical use many other features have to be taken into account. We refer the reader to McNee (1999) [26] and McConnell and Schwartz (1986) [25]. Other aspects from game theory arise when looking at a firm which has issued *several* convertible bonds. In view of equation (13.3.1), it becomes evident that optimal conversion strategies of the different bondholders are mutually dependent (see Constantinides (1984) [4], Constantinides and Rosenthal (1984) [5] and Bühler and Koziol (2002) [3]).

The structural (firm value) models have the drawback that the firm value is not directly observable. Therefore, from a practical point of view it is more convenient to start with the stochastic process describing the stock price process. However, S and D interact. But if the proportion of the total capital taken by the convertible bond is small, the fixed point effect described above can be neglected. This leads to so-called *reduced form models*, where the payoff processes L and U are derived from S as in equations (13.3.2) and (13.3.3), but S is directly given, e.g. by $S_0 = s_0 > 0$ and

$$dS_t = S_t\left[(r-\delta)\,dt + \widetilde{\sigma}\,d\widetilde{W}_t\right],$$

where $\widetilde{W}$ is a standard Brownian motion (see Davis and Lischka (2002) [6] and the references therein).

13.4 CONCLUSIONS

The static and the dynamic no-arbitrage approach lead to quite similar results. The initial arbitrage-free prices are in both cases essentially given by all Dynkin values

$$\sup_{\tau\in\mathcal{T}_0}\ \inf_{\sigma\in\mathcal{T}_0} E_Q\left(R(\tau,\sigma)\right) = \inf_{\sigma\in\mathcal{T}_0}\ \sup_{\tau\in\mathcal{T}_0} E_Q\left(R(\tau,\sigma)\right),$$

where $Q \in \mathcal{M}^e$ (see Theorem 13.2.2 and Proposition 13.2.11).

However, from a conceptual point of view it seems desirable to have derivative price processes rather than just initial derivative prices. In some cases, this is even essential to define the payoff processes of the GCC. Then an approach allowing for intermediate trades in the derivative as in Section 13.2.2 is needed.

ACKNOWLEDGEMENT

The authors wish to thank an anonymous referee for valuable comments.

REFERENCES

[1] Asquith, P. and Mullins, D. (1991), "Convertible debt: corporate call policy and voluntary conversion", *Journal of Finance*, **46**, 1273–1289.

[2] Brennan, M.J. and Schwartz, E.S. (1977), "Convertible bonds: valuation and optimal strategies for call and conversion", *Journal of Finance*, **32**, 1699–1715.

[3] Bühler, W. and Koziol, C. (2002), "Valuation of convertible bonds under sequential conversion", *Schmalenbach Business Review*, **54**, 302–334.

[4] Constantinides, G. (1984), "Warrant exercise and bond conversion in competitive markets", *Journal of Financial Economics*, **13**, 371–397.

[5] Constantinides, G. and Rosenthal, R. (1984), "Strategic analysis of the competitive exercise of certain financial options", *Journal of Economic Theory*, **32**; 128–138.

[6] Davis, M.H.A. and Lischka, F.R. (2002), "Convertible bonds with market risk and credit risk", in R. Chan, Y.-K. Kwok, D. Yao, and Q. Zhang (Eds), *Applied Probability*, Studies in Advanced Mathematics, American Mathematical Society/International Press, New York, NY, USA, pp. 45–58.

[7] Delbaen, F. and Schachermayer, W. (1998), "The fundamental theorem of asset pricing for unbounded stochastic processes", *Mathematische Annalen*, **312**, 215–250.

[8] Föllmer, H. and Kabanov, Y. (1998), "Optional decomposition and Lagrange multipliers", *Finance and Stochastics*, **2**, 69–81.

[9] Harrison, M. and Pliska, S. (1981), "Martingales and stochastic integrals in the theory of continuous trading", *Stochastic Processes and their Applications*, **11**, 215–260.

[10] Ingersoll, J.E. (1977a), "A contingent-claims valuation of convertible securities", *Journal of Financial Economics*, **4**, 289–322.

[11] Ingersoll, J.E. (1977b), "An examination of corporate call policies on convertible securities", *Journal of Finance*, **32**, 463–478.

[12] Jacod, J. (1979), *Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Mathematics, Vol. 714, Springer-Verlag, Berlin, Germany.

[13] Jacod, J. (1980), "Intégrales stochastiques par rapport à une semi-martingale vectorielle et changements de filtration", in J. Azéma and M. Yov (Eds), *Séminaire de Probabilités XIV, 1978/79*, Lecture Notes in Mathematics, Vol. 784, Springer-Verlag, Berlin, Germany, pp. 161–172.

[14] Jacod, J. and Shiryaev, A. (1987), *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, Germany.

[15] Kabanov, Yu. (1997), "On the FTAP of Kreps–Delbaen–Schachermayer", in Yu. M. Kabanov, B.L. Rozovski and A.N. Shiryaev (Eds), *Statistics and Control of Stochastic Processes (Moscow, 1995/1996)*, World Scientific, River Edge, NJ, USA, pp. 191–203.

[16] Kallsen, J. (2003), "σ-localization and σ-martingales", *Theory of Probability and Its Applications*, **48**, 177–188.

[17] Kallsen, J. and Kühn, C. (2004), "Pricing derivatives of American and game type in incomplete markets", *Finance and Stochastics*, **8**, 261–284.

[18] Karatzas, I. and Kou, S.G. (1998), "Hedging American contingent claims with constrained portfolios", *Finance and Stochastics*, **2**, 215–258.

[19] Karatzas, I. and Zamfirescu, M. (2003), "Optimal stopping with model-uncertainty", Preprint.

[20] Kifer, Y. (2000), "Game options", *Finance and Stochastics*, **4**, 443–463.

[21] Kühn, C. (2004), "Game contingent claims in complete and incomplete markets", *Journal of Mathematical Economics*, **40**, 889–902.

[22] Kühn, C. and Kyprianou, A. (2003a), "Callable puts as composite exotic options", Preprint.

[23] Kühn, C. and Kyprianou, A. (2003b), "Pricing Israeli options: a pathwise approach", Preprint.

[24] Lepeltier, J. and Maingueneau, M. (1984), "Le jeu de dynkin en théorie générale sans l'hypothèse de Mokobodski", *Stochastics*, **13**, 25–44.

[25] McConnell, J.J. and Schwartz, E.S. (1986), "LYON taming", *Journal of Finance*, **16**, 561–576.

[26] McNee, A. (1999), "The challenge of equity derivatives", *Risk*, (Technology Supplement), 29–31.

[27] Miller, M.H. and Modigliani, F. (1958), "The cost of capital, corporation finance, and the theory of investment", *The American Economic Review*, **48**, 261–297.

[28] Miller, M.H. and Modigliani, F. (1961), "Dividend policy, growth and the valuation of shares", *Journal of Business*, **34**, 411–433.

[29] Sîrbu, M., Pikovsky, I. and Shreve, S. (2004), "Perpetual convertible bonds", *SIAM Journal on Control and Optimization*, **45**, 58–85.

14 The Spread Option Optimal Stopping Game

Pavel V. Gapeev
Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

Abstract

We present a solution to an optimal stopping game for geometric Brownian motion with gain functions having the form of payoff functions of spread options. The method of proof is based on reducing the initial problem to a free-boundary problem and solving the latter by means of the smooth-fit condition. The derived result can be interpreted as pricing the (perpetual) spread game option in the Black–Merton–Scholes model.

14.1 INTRODUCTION

Optimal stopping games (usually called *Dynkin's games*) were introduced and studied by Dynkin (1969) [7]. The purely probabilistic theory of such games was developed in Frid (1969) [9], Kifer (1971a, b) [18] [19], Neveu (1975) [27], Elbakidze (1976) [8], Krylov (1971) [21], Bismut (1977) [5], Stettner (1982) [33], Alario-Nazaret *et al*. (1982) [1], Morimoto (1984) [26], Lepeltier and Mainguenau (1984) [25] and others. This approach was based on applying the martingale theory for solving a generalization of the optimal stopping problem introduced by Snell (1952) [32]. The analytical theory of stochastic differential games with stopping times in Markov diffusion models was developed in Bensoussan and Friedman (1974, 1977) [3] [4] and Friedman (1973) [10] (see also Friedman (1976) [11] Chapter XVI). This approach for studying the value functions and saddle points of such games was based on using the theory of variational inequalities and free-boundary problems for partial differential equations. Cvitanić and Karatzas (1996) [6] established a connection between the values of optimal stopping games and the solutions of backward stochastic differential equations with reflection and provided a pathwise approach to these games. Karatzas and Wang (2001) [17] studied such games in a more general non-Markovian setting and brought them into connection with bounded-variation optimal control problems.

Recently, Kifer (2000) [20] introduced the concept of a *game* (or *Israeli*) *option* generalizing the concept of an American option by also allowing the seller to cancel the option prematurely, but at the expense of some penalty. It was shown that the problem of pricing and hedging such options can be reduced to solving an associated optimal stopping game. Kyprianou (2004) [24] obtained explicit expressions for the value functions of two classes of perpetual game option problems. Kühn and Kyprianou (2003a,b) [22] [23] characterized the value functions of the finite expiry versions of these classes of options via mixtures of other exotic options using martingale arguments and then produced the same for a more

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott

general class of finite expiry game options via a pathwise pricing formulae. Kallsen and Kühn (2004, 2005) [15] [16] applied the neutral valuation approach to American and game options in incomplete markets and introduced a mathematically rigorous dynamic concept to define no-arbitrage prices for game contingent claims. Further calculations for game options were recently done by Baurdoux and Kyprianou (2004) [2]. In this present paper, we introduce the perpetual *spread game option* problem and find sufficient conditions for the existence of a (nontrivial) closed form solution to the problem.

The paper is organized as follows. In Section 14.2, we give a formulation of the spread option optimal stopping game in the Black–Merton–Scholes model and discuss its economic interpretation. In Section 14.3, we formulate the corresponding free-boundary problem for the infinitesimal operator of geometric Brownian motion and derive sufficient conditions for the existence of a unique solution to the problem. In Section 14.4, we verify that under certain relations on the parameters of the model the solution of the free-boundary problem turns out to be a solution of the initial optimal stopping game. In Section 14.5, we give some remarks and mention another question arising from the spread game option problem.

14.2 FORMULATION OF THE PROBLEM

For a precise probabilistic formulation of the problem, let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$ started at zero. It is assumed that the price of a risky asset (e.g. a stock) on a financial market is described by a geometric Brownian motion $X = (X_t)_{t \geq 0}$ defined by:

$$X_t = x \, \exp((r - \theta^2/2)\, t + \theta \, B_t) \tag{14.2.1}$$

and hence solving the stochastic differential equation:

$$dX_t = rX_t \, dt + \theta X_t \, dB_t \quad (X_0 = x) \tag{14.2.2}$$

where $r > 0$ is the interest rate, $\theta > 0$ is the volatility coefficient, and $x > 0$ is given and fixed. The main purpose of this present paper is to find a solution to the following optimal stopping game for the time-homogeneous (strong) Markov process X having the value function:

$$\begin{aligned} V_*(x) &= \inf_{\sigma} \sup_{\tau} E_x[e^{-(\lambda+r)(\sigma \wedge \tau)}(G_1(X_\sigma)\, I(\sigma < \tau) + G_2(X_\tau)\, I(\tau \leq \sigma))] \\ &= \sup_{\tau} \inf_{\sigma} E_x[e^{-(\lambda+r)(\sigma \wedge \tau)}(G_1(X_\sigma)\, I(\sigma < \tau) + G_2(X_\tau)\, I(\tau \leq \sigma))] \end{aligned} \tag{14.2.3}$$

where P_x is a probability measure under which the process X defined in equations (14.2.1) and (14.2.2) starts at some $x > 0$, the infimum and supremum are taken over all finite stopping times σ and τ of the process X (i.e. stopping times with respect to $(\mathcal{F}_t^X)_{t \geq 0}$ denoting the natural filtration of X: $\mathcal{F}_t^X = \sigma\{X_u \,|\, 0 \leq u \leq t\}$, $t \geq 0$), $\lambda > 0$ is a discounting rate, and the functions $G_i(x)$ are defined by:

$$G_i(x) = (x - L_i)\, I(L_i \leq x < K_i) + (K_i - L_i)\, I(x \geq K_i) \tag{14.2.4}$$

for all $x > 0$ with some constants L_i and K_i such that $0 < L_i < K_i$, $i = 1, 2$, as well as $L_1 < L_2$, $K_1 < K_2$ and $K_1 - L_1 = K_2 - L_2$. We will derive sufficient conditions for the existence of a nontrivial closed form solution to the problem (equation (14.2.3)). Note that the existence of a unique value (equation (14.2.3)) was proved in Lepeltier and Mainguenau (1984) [25] and Kifer (2000) [20]. This fact will be re-proved in Theorem 4.1 below under certain conditions on the parameters of the model. It also follows from equation (14.2.3) that the inequalities $G_2(x) \leq V_*(x) \leq G_1(x)$ hold for all $x > 0$.

We will search for optimal stopping times in the problem (equation (14.2.3)) of the following form:

$$\sigma_* = \inf\{t \geq 0 \mid X_t \leq A_*\} \tag{14.2.5}$$

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq B_*\} \tag{14.2.6}$$

for some numbers A_* and B_* such that $L_1 \leq A_* \leq D_1$ and $D_2 \leq B_* \leq K_2$ hold with $D_i = L_i(\lambda + r)/\lambda$, $i = 1, 2$ (for an explanation of the latter inequalities see the text following equation (14.4.5) below). In this connection, the points A_* and B_* are called *optimal stopping boundaries*. Note that in this case, A_* is the largest number from $L_1 \leq x \leq D_1$ such that $V_*(x) = G_1(x)$, and B_* is the smallest number from $D_2 \leq x \leq K_2$ such that $V_*(x) = G_2(x)$. The pair of stopping times (σ_*, τ_*) is usually called a *saddle point* of the optimal stopping game.

On a financial market, there are investors speculating for a rise of stock prices (so-called 'bulls' playing on the *increase*) and investors speculating for a fall of stock prices (so-called 'bears' playing on the *decrease*), and their strategies on the market are asymmetric (see, e.g. Shiryaev (1999) [31] Chapter I, Section 1c). In order to restrict their losses and gains simultaneously, the investors playing on the *increase* may turn to a strategy consisting of buying a call option with a strike price L_2 and selling a call option with a higher strike price $K_2 > L_2$, while the investors playing on the *decrease* may turn to a strategy consisting of selling a call option with a strike price L_1 and buying a call option with a higher strike price $K_1 > L_1$. Such combinations are called spread options of 'bull' and 'bear', respectively, and their payoff functions are given by $G_2(x)$ and $-G_1(x)$ from equation (14.2.4), where x denotes the stock price (see Shiryaev (1999) [31] Chapter VI, Section 4e). In this present paper, we consider a contingent claim with arbitrary (random) times of exercise τ and cancellation σ, where according to the conditions of the claim the buyer can choose the exercise time τ and in case $\tau \leq \sigma$ gets the value $G_2(X_\tau)$ from the seller, and the seller can choose the cancellation time σ and in case $\sigma < \tau$ gives the value $G_1(X_\sigma)$ to the buyer. Then, by virtue of the fact that P_x is a martingale measure for the given market model (see, e.g. Shiryaev *et al*. (1994) [29] Section 1, Shiryaev (1999) [31] Chapter VII, Section 3g, and Kifer (2000) [20] Section 3), the value (equation (14.2.3)) may be interpreted as a rational (fair) price of the mentioned contingent claim in the given model. We also observe that from the structure of the problem (equation (14.2.3)) it is intuitively clear that the buyer wants to stop when the process X comes close to L_1 (from above) while the seller wants to stop when the process X comes close to K_2 (from below) without waiting too long because of the punishment of discounting.

Taking into account the arguments stated above, we will call the presented contingent claim a *spread game option*. Note that the structure of the given option differs from the structure of the game options considered in Kifer (2000) [20] and Kyprianou (2004) [24].

14.3 SOLUTION OF THE FREE-BOUNDARY PROBLEM

By means of standard arguments, it is shown that the infinitesimal operator $\mathbb{L}$ of the process X acts on an arbitrary function F from the class C^2 on $(0, \infty)$ according to the rule:

$$(\mathbb{L}F)(x) = rx\, F'(x) + (\theta^2 x^2/2)\, F''(x) \tag{14.3.1}$$

for all $x > 0$. In order to find explicit expressions for the unknown value function $V_*(x)$ from equation (14.2.3) and the boundaries A_* and B_* from equations (14.2.5) and (14.2.6), using the results of general theory of optimal stopping problems for continuous time Markov processes as well as taking into account the results about the connection between optimal stopping games and free-boundary problems (see, e.g. Grigelionis and Shiryaev (1966) [12] and Shiryaev (1978) [30] Chapter III, Section 8; as well as Bensoussan and Friedman (1974, 1977) [3] [4]), we can formulate the following *free-boundary problem*:

$$(\mathbb{L}V)(x) = (\lambda + r)V(x) \quad \text{for} \quad A < x < B \tag{14.3.2}$$

$$V(A+) = A - L_1, \quad V(B-) = B - L_2 \quad (\textit{continuous fit}) \tag{14.3.3}$$

$$V(x) = G_1(x) \quad \text{for} \quad 0 < x < A, \quad V(x) = G_2(x) \quad \text{for} \quad x > B \tag{14.3.4}$$

$$G_2(x) < V(x) < G_1(x) \quad \text{for} \quad A < x < B \tag{14.3.5}$$

where $L_1 \leq A \leq D_1$ and $D_2 \leq B \leq K_2$ with $D_i = L_i(\lambda + r)/\lambda$, $i = 1, 2$. Moreover, we also assume that the following conditions hold:

$$V'(A+) = V'(B-) = 1 \quad (\textit{smooth fit}). \tag{14.3.6}$$

By means of straightforward calculations it is shown (see, e.g. Shiryaev *et al.* (1994) [29] Section 8 or Shiryaev (1999) [31] Chapter VIII, Section 2a) that the general solution of equation (14.3.2) takes the form:

$$V(x) = C_1\, x^{\gamma_1} + C_2\, x^{\gamma_2} \tag{14.3.7}$$

where C_1 and C_2 are some arbitrary constants, and $\gamma_1 < 0 < 1 < \gamma_2$ are defined by:

$$\gamma_i = \left(\frac{1}{2} - \frac{r}{\theta^2}\right) + (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r}{\theta^2}\right)^2 + \frac{2(\lambda + r)}{\theta^2}} \tag{14.3.8}$$

for $i = 1, 2$. In this case, using the conditions (14.3.3), we get:

$$C_1\, A^{\gamma_1} + C_2\, A^{\gamma_2} = A - L_1, \quad C_1\, B^{\gamma_1} + C_2\, B^{\gamma_2} = B - L_2 \tag{14.3.9}$$

from where we find that in equation (14.3.7) we have:

$$C_1 = \frac{(A - L_1)(B/A)^{\gamma_2} - (B - L_2)}{A^{\gamma_1}[(B/A)^{\gamma_2} - (B/A)^{\gamma_1}]} \tag{14.3.10}$$

$$C_2 = \frac{B - L_2 - (A - L_1)(B/A)^{\gamma_1}}{A^{\gamma_2}[(B/A)^{\gamma_2} - (B/A)^{\gamma_1}} \tag{14.3.11}$$

and hence, the solution of the system (14.3.2)–(14.3.4) takes the form:

$$V(x; A, B) = \frac{(A - L_1)(B/A)^{\gamma_2} - (B - L_2)}{(B/A)^{\gamma_2} - (B/A)^{\gamma_1}} \left(\frac{x}{A}\right)^{\gamma_1} + \frac{B - L_2 - (A - L_1)(B/A)^{\gamma_1}}{(B/A)^{\gamma_2} - (B/A)^{\gamma_1}} \left(\frac{x}{A}\right)^{\gamma_2} \tag{14.3.12}$$

for all $A < x < B$. Then, using the assumed smooth-fit conditions (14.3.6) we obtain:

$$\gamma_1 C_1 A^{\gamma_1 - 1} + \gamma_2 C_2 A^{\gamma_2 - 1} = 1, \quad \gamma_1 C_1 B^{\gamma_1 - 1} + \gamma_2 C_2 B^{\gamma_2 - 1} = 1 \tag{14.3.13}$$

from where, by virtue of the equalities (14.3.10) and (14.3.11), after some straightforward transformations we may conclude that the boundaries A and B should satisfy the following system of equations:

$$\left(\frac{B}{A}\right)^{\gamma_1} = \frac{(\gamma_2 - 1)B - \gamma_2 L_2}{(\gamma_2 - 1)A - \gamma_2 L_1} \tag{14.3.14}$$

$$\left(\frac{B}{A}\right)^{\gamma_2} = \frac{(1 - \gamma_1)B + \gamma_1 L_2}{(1 - \gamma_1)A + \gamma_1 L_1} \tag{14.3.15}$$

which is equivalent to the system:

$$\frac{(\gamma_2 - 1)A - \gamma_2 L_1}{A^{\gamma_1}} = \frac{(\gamma_2 - 1)B - \gamma_2 L_2}{B^{\gamma_1}} \tag{14.3.16}$$

$$\frac{(1 - \gamma_1)A + \gamma_1 L_1}{A^{\gamma_2}} = \frac{(1 - \gamma_1)B + \gamma_1 L_2}{B^{\gamma_2}} \tag{14.3.17}$$

where $L_1 \leq A \leq D_1$ and $D_2 \leq B \leq K_2$ with $D_i = L_i(\lambda + r)/\lambda$, $i = 1, 2$ (for an explanation of the latter inequalities, see the text following equation (14.4.5) below).

In order to find sufficient conditions for the existence and uniqueness of a solution of the system of equations (14.3.16) and (14.3.17) for $L_1 \leq A \leq D_1$ and $D_2 \leq B \leq K_2$, let us use the idea of proof of the existence and uniqueness of solution of the system of equations (4.85) from Shiryaev (1978) [30] Chapter IV, Section 2. For this, let us define the functions $I_k(A)$ and $J_k(B)$, $k = 1, 2$, by:

$$I_1(A) = \frac{(\gamma_2 - 1)A - \gamma_2 L_1}{A^{\gamma_1}} \tag{14.3.18}$$

$$J_1(B) = \frac{(\gamma_2 - 1)B - \gamma_2 L_2}{B^{\gamma_1}} \tag{14.3.19}$$

$$I_2(A) = \frac{(1 - \gamma_1)A + \gamma_1 L_1}{A^{\gamma_2}} \tag{14.3.20}$$

$$J_2(B) = \frac{(1 - \gamma_1)B + \gamma_1 L_2}{B^{\gamma_2}} \tag{14.3.21}$$

for all A and B such that $L_1 \leq A \leq D_1$ and $D_2 \leq B \leq K_2$. By virtue of the fact that for the derivatives of the functions (14.3.18)–(14.3.21) the following expressions hold:

$$I_1'(A) = -\frac{(\gamma_1 - 1)(\gamma_2 - 1)(A - D_1)}{A^{\gamma_1+1}} < 0 \tag{14.3.22}$$

$$I_2'(A) = \frac{(\gamma_1 - 1)(\gamma_2 - 1)(A - D_1)}{A^{\gamma_2+1}} > 0 \tag{14.3.23}$$

for all $L_1 < A < D_1$ as well as:

$$J_1'(B) = -\frac{(\gamma_1 - 1)(\gamma_2 - 1)(B - D_2)}{B^{\gamma_1+1}} > 0 \tag{14.3.24}$$

$$J_2'(B) = \frac{(\gamma_1 - 1)(\gamma_2 - 1)(B - D_2)}{B^{\gamma_2+1}} < 0 \tag{14.3.25}$$

for all $D_2 < B < K_2$, we may therefore conclude that $I_1(A)$ decreases and $I_2(A)$ increases on the interval (L_1, D_1), while $J_1(B)$ increases and $J_2(B)$ decreases on the interval (D_2, K_2).

Let us further assume that the following conditions are satisfied:

$$\frac{(\gamma_2 - 1)L_1 - \gamma_2 L_1}{L_1^{\gamma_1}} \leq \frac{(\gamma_2 - 1)K_2 - \gamma_2 L_2}{K_2^{\gamma_1}} \tag{14.3.26}$$

$$\frac{(1 - \gamma_1)L_1 + \gamma_1 L_1}{L_1^{\gamma_2}} \leq \frac{(1 - \gamma_1)K_2 + \gamma_1 L_2}{K_2^{\gamma_2}} \tag{14.3.27}$$

and observe that by means of straightforward calculations it can be verified that the following inequalities hold:

$$\frac{(\gamma_2 - 1)D_1 - \gamma_2 L_1}{D_1^{\gamma_1}} \geq \frac{(\gamma_2 - 1)D_2 - \gamma_2 L_2}{D_2^{\gamma_1}} \tag{14.3.28}$$

$$\frac{(1 - \gamma_1)D_1 + \gamma_1 L_1}{D_1^{\gamma_2}} \geq \frac{(1 - \gamma_1)D_2 + \gamma_1 L_2}{D_2^{\gamma_2}}. \tag{14.3.29}$$

Then, it is easily seen that there exist A_1 and A_2 such that $L_1 \leq A_1 \leq A_2 \leq D_1$ and being uniquely determined from the following equations:

$$\frac{(1 - \gamma_1)A_1 + \gamma_1 L_1}{A_1^{\gamma_2}} = \frac{(1 - \gamma_1)K_2 + \gamma_1 L_2}{K_2^{\gamma_2}} \tag{14.3.30}$$

$$\frac{(1 - \gamma_1)A_2 + \gamma_1 L_1}{A_2^{\gamma_2}} = \frac{(1 - \gamma_1)D_2 + \gamma_1 L_2}{D_2^{\gamma_2}}. \tag{14.3.31}$$

In this case, from the system (14.3.16) and (14.3.17) it follows that for each A such that $A_1 \leq A \leq A_2$ there exist unique values $B_1(A)$ and $B_2(A)$, and according to the implicit function theorem, for the derivatives the following expressions hold:

$$B_1'(A) = \frac{I_1'(A)}{J_1'(B)} = \frac{A - D_1}{B - D_2}\left(\frac{B}{A}\right)^{\gamma_1+1} < 0 \tag{14.3.32}$$

$$B_2'(A) = \frac{I_2'(A)}{J_2'(B)} = \frac{A - D_1}{B - D_2}\left(\frac{B}{A}\right)^{\gamma_2+1} < 0 \tag{14.3.33}$$

from where it directly follows that:

$$\frac{B_2'(A)}{B_1'(A)} = \frac{A_1'(B)}{A_2'(B)} = \left(\frac{B}{A}\right)^{\gamma_2-\gamma_1} > 1 \tag{14.3.34}$$

for all $L_1 \leq A_1 \leq A \leq A_2 \leq D_1$. We also observe that by means of standard arguments it is shown that the inequalities $D_2 = B_2(A_2) \leq B_1(A_2) \leq B_1(A_1) \leq B_2(A_1) = K_2$ hold. Taking into account the properties (14.3.32)–(14.3.34), we may therefore conclude that the system of equations (14.3.16) and (14.3.17) admits a unique solution A_* and B_* such that $L_1 \leq A_* \leq D_1$ and $D_2 \leq B_* \leq K_2$ with $D_i = L_i(\lambda + r)/\lambda$, $i = 1, 2$, so that, under the added conditions (14.3.26) and (14.3.27), the solution of the system (14.3.2)–(14.3.4)+(14.3.6) exists and is unique.

14.4 MAIN RESULT AND PROOF

Taking into account the facts proved above, let us now formulate the main assertion of the paper.

Theorem 4.1. *Let the process X be given by equations (14.2.1) and (14.2.2). Assume that the parameters* r, θ, λ *and* L_i, K_i, $i = 1, 2$, *are such that* $0 < L_i < K_i$, $i = 1, 2$, *as well as* $L_1 < L_2$, $K_1 < K_2$, $K_1 - L_1 = K_2 - L_2$, $L_2(\lambda + r)/\lambda \leq K_2$, *and the conditions (14.3.26) and (14.3.27) are satisfied. Then, the value function of the problem (14.2.3) takes the expression:*

$$V_*(x) = \begin{cases} G_1(x), & \text{if } 0 < x \leq A_* \\ V(x; A_*, B_*), & \text{if } A_* < x < B_* \\ G_2(x), & \text{if } x \geq B_* \end{cases} \tag{14.4.1}$$

and the optimal stopping times σ_* *and* τ_* *have the structure (14.2.5) and (14.2.6), where the function* $V(x; A, B)$ *is explicitly given by equation (14.3.12), and the optimal boundaries* A_* *and* B_* *satisfy the inequalities* $L_1 \leq A_* \leq L_1(\lambda + r)/\lambda$ *and* $L_2(\lambda + r)/\lambda \leq B_* \leq K_2$ *and are uniquely determined from the system of equations (14.3.16) and (14.3.17) (see Figure 14.1).*

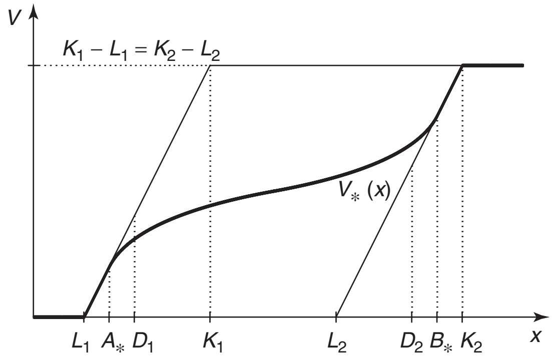

Figure 14.1 A computer drawing of the value function $V_*(x)$ and the optimal stopping boundaries A_* and B_*

Proof. Let us show that the function (14.4.1) coincides with the value function (14.2.3) and the stopping times σ_* and τ_* from equations (14.2.5) and (14.2.6) with the boundaries A_* and B_* specified above are optimal. For this, let us denote by $V(x)$ the right-hand side of the expression (14.4.1). In this case, by means of straightforward calculations and the assumptions above it follows that the function $V(x)$ satisfies the system (14.3.2)–(14.3.4) and the conditions (14.3.6) as well as represents a difference of two convex functions where the latter is easily seen from equation (14.3.12). Then, by applying the Itô–Tanaka–Meyer formula (see, e.g. Jacod (1979) [13] Chapter V, Theorem 5.52, or Protter (1992) [28] Chapter IV, Theorem 51) to $e^{-(\lambda+r)t}V(X_t)$ we obtain:

$$\begin{aligned} e^{-(\lambda+r)t}\, V(X_t) &= V(x) + M_t \\ &\quad + \int_0^t e^{-(\lambda+r)s}\,(\mathbb{L}V - (\lambda+r)V)(X_s) I(X_s \neq L_1, X_s \neq K_2)\, ds \\ &\quad + \frac{1}{2}\int_0^t e^{-(\lambda+r)s}\, I(X_s = L_1)\, d\ell_s^{L_1} - \frac{1}{2}\int_0^t e^{-(\lambda+r)s}\, I(X_s = K_2)\, d\ell_s^{K_2} \end{aligned} \tag{14.4.2}$$

where the processes $(\ell_t^{L_1})_{t\ge 0}$ and $(\ell_t^{K_2})_{t\ge 0}$, the local time of X at the points L_1 and K_2, are defined by:

$$\ell_t^{L_1} = P_x - \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t I(L_1 - \varepsilon < X_s < L_1 + \varepsilon)\,\theta^2 X_s^2\, ds \tag{14.4.3}$$

$$\ell_t^{K_2} = P_x - \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t I(K_2 - \varepsilon < X_s < K_2 + \varepsilon)\,\theta^2 X_s^2\, ds \tag{14.4.4}$$

and the process $(M_t)_{t\ge 0}$ given by:

$$M_t = \int_0^t e^{-(\lambda+r)s}\, V'(X_s) I(X_s \neq L_1, X_s \neq K_2)\,\theta X_s\, dB_s \tag{14.4.5}$$

is a local martingale under the measure P_x with respect to $(\mathcal{F}_t^X)_{t\ge 0}$.

By virtue of the arguments from the previous section, we may conclude that $(\mathbb{L}V - (\lambda+r)V)(x) \le 0$ for all $x > A_*$, $x \neq B_*$, $x \neq K_2$, and $(\mathbb{L}V - (\lambda+r)V)(x) \ge 0$ for all $0 < x < B_*$, $x \neq L_1$, $x \neq A_*$, where the boundaries A_* and B_* satisfy the inequalities $L_1 \le A_* \le L_1(\lambda+r)/\lambda$ and $L_2(\lambda+r)/\lambda \le B_* \le K_2 = K_1 - L_1 + L_2$. Moreover, by means of straightforward calculations, it is shown that we have $V'(x; A_*, B_*)$ on the interval (A_*, B_*), and thus the property (14.3.5) also holds that together with equations (14.3.3) and (14.3.4) yields $V(x) \ge G_2(x)$ and $V(x) \le G_1(x)$ for all $x > 0$. By virtue of the fact that the time spent by the process X at the points L_1, A_*, B_* and K_2 is of Lebesgue measure zero, from the expression (14.4.2) it therefore follows that the inequalities:

$$e^{-(\lambda+r)(\sigma_*\wedge\tau)}\, G_2(X_{\sigma_*\wedge\tau}) \le e^{-(\lambda+r)(\sigma_*\wedge\tau)}\, V(X_{\sigma_*\wedge\tau}) \le V(x) + M_{\sigma_*\wedge\tau} \tag{14.4.6}$$

$$e^{-(\lambda+r)(\sigma\wedge\tau_*)}\, G_1(X_{\sigma\wedge\tau_*}) \ge e^{-(\lambda+r)(\sigma\wedge\tau_*)}\, V(X_{\sigma\wedge\tau_*}) \ge V(x) + M_{\sigma\wedge\tau_*} \tag{14.4.7}$$

are satisfied for any finite stopping times σ and τ of the process X.

Let $(\tau_n)_{n\in\mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process $(M_t)_{t\ge 0}$. Then, by using inequalities (14.4.6) and (14.4.7) and taking the expectations with respect to

P_x, by means of the optional sampling theorem (see, e.g. Jacod and Shiryaev (1987) [14] Chapter I, Theorem 1.39) we get:

$$E_x\Big[e^{-(\lambda+r)(\sigma_*\wedge\tau\wedge\tau_n)}\Big(G_1(X_{\sigma_*})I(\sigma_* < \tau\wedge\tau_n) + G_2(X_{\tau\wedge\tau_n})I(\tau\wedge\tau_n \le \sigma_*)\Big)\Big] \le E_x\Big[e^{-(\lambda+r)(\sigma_*\wedge\tau\wedge\tau_n)}\,V(X_{\sigma_*\wedge\tau\wedge\tau_n})\Big] \le V(x) + E_x\big[M_{\sigma_*\wedge\tau\wedge\tau_n}\big] = V(x) \qquad (14.4.8)$$

$$E_x\Big[e^{-(\lambda+r)(\sigma\wedge\tau_*\wedge\tau_n)}\Big(G_1(X_{\sigma\wedge\tau_n})I(\sigma\wedge\tau_n < \tau_*) + G_2(X_{\tau_*})I(\tau_* \le \sigma\wedge\tau_n)\Big)\Big] \ge E_x\Big[e^{-(\lambda+r)(\sigma\wedge\tau_*\wedge\tau_n)}\,V(X_{\sigma\wedge\tau_*\wedge\tau_n})\Big] \ge V(x) + E_x\big[M_{\sigma\wedge\tau_*\wedge\tau_n}\big] = V(x) \qquad (14.4.9)$$

for all $x > 0$. Hence, letting n go to infinity and using Fatou's lemma, we obtain that for any finite stopping times σ and τ the inequalities:

$$E_x\Big[e^{-(\lambda+r)(\sigma_*\wedge\tau)}\Big(G_1(X_{\sigma_*})I(\sigma_* < \tau) + G_2(X_\tau)I(\tau \le \sigma_*)\Big)\Big] \le V(x) \le E_x\Big[e^{-(\lambda+r)(\sigma\wedge\tau_*)}\Big(G_1(X_\sigma)I(\sigma < \tau_*) + G_2(X_{\tau_*})I(\tau_* \le \sigma)\Big)\Big] \qquad (14.4.10)$$

hold for all $x > 0$.

In order to show that the equalities in expression (14.4.10) are attained at σ_* and τ_* from equations (14.2.5) and (14.2.6), let us use the fact that the function $V(x)$ solves the equation (14.3.2) for all $A_* < x < B_*$. In this case, by the expression (14.4.2) and the structure of the stopping times σ_* and τ_*, it follows that the equality:

$$e^{-(\lambda+r)(\sigma_*\wedge\tau_*\wedge\tau_n)}\,V(X_{\sigma_*\wedge\tau_*\wedge\tau_n}) = V(x) + M_{\sigma_*\wedge\tau_*\wedge\tau_n} \qquad (14.4.11)$$

holds, from where, by using the expressions (14.4.6) and (14.4.7), we may conclude that the inequalities:

$$-(K_1 - L_1) \le M_{\sigma_*\wedge\tau_*\wedge\tau_n} \le K_2 - L_2 \qquad (14.4.12)$$

are satisfied for all $x > 0$, where $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for $(M_t)_{t\ge 0}$. Hence, letting n go to infinity in the expression (14.4.11) and using the conditions (14.3.3), as well as the obviously fulfilled property $P_x[\sigma_*\wedge\tau_* < \infty] = 1$ (see, e.g. Shiryaev *et al.* (1994) [29] Section 8, or Shiryaev (1999), [31] Chapter VIII, Section 2a), by means of the Lebesgue bounded convergence theorem we obtain the equality:

$$E_x\Big[e^{-(\lambda+r)(\sigma_*\wedge\tau_*)}\Big(G_1(X_{\sigma_*})\,I(\sigma_* < \tau_*) + G_2(X_{\tau_*})\,I(\tau_* \le \sigma_*)\Big)\Big] = V(x) \qquad (14.4.13)$$

for all $x > 0$, which together with (14.4.10) directly implies the desired assertion. □

14.5 CONCLUSIONS

Recall that throughout the paper and particularly in the proof of Theorem 4.1 we have used the assumption that $L_2(\lambda + r)/\lambda \le K_2$ among others. When the latter condition fails to hold but $L_1(\lambda + r)/\lambda \le K_1$ holds, let us set $B_* = K_2$ in equation (14.2.6) and consider the problem (14.2.3) as an optimal stopping problem for the seller. In this case, we can also formulate the free-boundary problem (equations (14.3.2)–(14.3.5)), where $L_1 \le A \le D_1$ and $B = K_2$ with $D_1 = L_1(\lambda + r)/\lambda$, and assume that the following condition holds:

$$V'(A+) = 1 \quad (\textit{smooth fit}). \tag{14.5.1}$$

By means of the same arguments as in Section 14.3, by using the assumed smooth-fit condition (14.5.1), it can be shown that the boundary A should satisfy the following equation:

$$\frac{\gamma_1}{A}\,\frac{(A-L_1)(K_2/A)^{\gamma_2}-(K_2-L_2)}{(K_2/A)^{\gamma_2}-(K_2/A)^{\gamma_1}}+\frac{\gamma_2}{A}\,\frac{(K_2-L_2)-(A-L_1)(K_2/A)^{\gamma_1}}{(K_2/A)^{\gamma_2}-(K_2/A)^{\gamma_1}}=1. \tag{14.5.2}$$

In order to find sufficient conditions for the existence and uniqueness of solution of the equation (14.5.2) let us define the function $H(A)$ by:

$$\begin{aligned} H(A) &= [(\gamma_1-1)A-\gamma_1 L_1](K_2/A)^{\gamma_2} \\ &\quad - [(\gamma_2-1)A-\gamma_2 L_1](K_2/A)^{\gamma_1} + (\gamma_2-\gamma_1)(K_2-L_2) \end{aligned} \tag{14.5.3}$$

for all A such that $L_1 \le A \le D_1$. By virtue of the fact that for the derivative of the function (14.5.3) the following expression holds:

$$H'(A) = -\frac{(\gamma_1-1)(\gamma_2-1)(A-D_1)}{A}\left(\left(\frac{K_2}{A}\right)^{\gamma_2}-\left(\frac{K_2}{A}\right)^{\gamma_1}\right) < 0 \tag{14.5.4}$$

for all $L_1 < A < D_1$, we may therefore conclude that $H(A)$ decreases on the interval (L_1, D_1). It thus follows that, if the following conditions are satisfied:

$$[(\gamma_1-1)L_1-\gamma_1 L_1](K_2/L_1)^{\gamma_2}-[(\gamma_2-1)L_1-\gamma_2 L_1](K_2/L_1)^{\gamma_1} \ge (\gamma_1-\gamma_2)(K_2-L_2) \tag{14.5.5}$$

$$[(\gamma_1-1)D_1-\gamma_1 L_1](K_2/D_1)^{\gamma_2}-[(\gamma_2-1)D_1-\gamma_2 L_1](K_2/D_1)^{\gamma_1} \le (\gamma_1-\gamma_2)(K_2-L_2) \tag{14.5.6}$$

then the equation (14.5.2) admits a unique solution A_* such that $L_1 \le A_* \le D_1$, and so that the solution of the system (14.3.2)–(14.3.4)+(14.5.1) with $B = K_2$ exists and is unique. Taking into account the arguments above, let us formulate the following assertion.

Proposition 5.1. *Let the process X be given by equations (14.2.1) and (14.2.2). Assume that the parameters r, θ, λ and L_i, K_i, $i = 1, 2$, are such that $0 < L_i < K_i$, $i = 1, 2$, as well as $L_1 < L_2$, $K_1 < K_2$, $K_1 - L_1 = K_2 - L_2$, $L_1(\lambda + r)/\lambda \le K_1$, $L_2(\lambda + r)/\lambda > K_2$, and the conditions (14.5.5) and (14.5.6) are satisfied. Then, the value function of the problem (14.2.3) takes the expression (14.4.1) and the optimal stopping times σ_* and τ_* have the structure (14.2.5) and (14.2.6) with $B_* = K_2$, where the function $V(x; A, B)$ is explicitly given by*

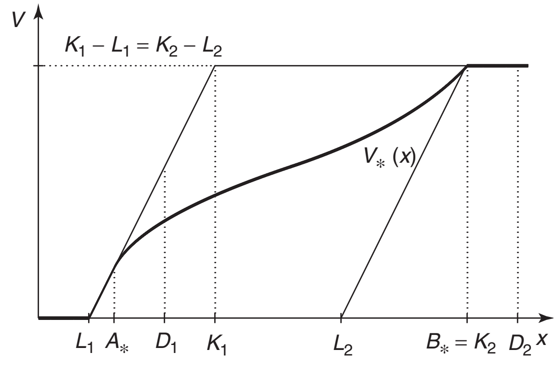

Figure 14.2 A computer drawing of the value function $V_*(x)$ and the optimal stopping boundaries A_* and K_2

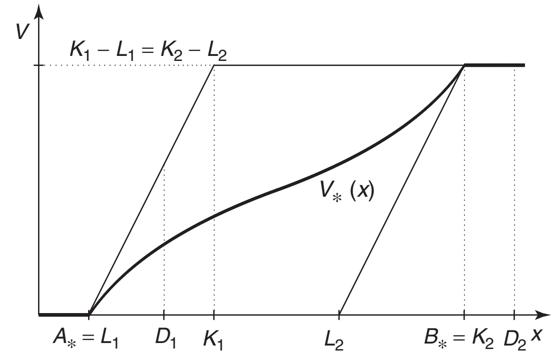

Figure 14.3 A computer drawing of the value function $V_*(x)$ and the optimal stopping boundaries L_1 and K_2

equation (14.3.12), and A_* satisfying the inequalities $L_1 \le A_* \le L_1(\lambda + r)/\lambda$ is determined as a unique solution of the equation (14.5.2) (see Figure 14.2).

The verification of this assertion can be carried out by means of a slight modification of the arguments from the proof of Theorem 4.1, using also the facts that the condition (14.5.6) implies that $V'(K_2; D_1, K_2) < 1$ and the function $V'(K_2; A, K_2)$ is increasing in A on the interval (L_1, D_1). It is seen that the smooth-fit condition at the point B_* breaks down in this case. We also note that when the condition (14.5.5) fails to hold, almost the same arguments show that (even when the condition $L_1(\lambda + r)/\lambda \le K_1$ fails to hold too) the assertion of Proposition 5.1 remains true with $A_* = L_1$, while the smooth-fit condition at A_* also breaks down (see Figure 14.3).

Remark 5.2. We also mention that another interesting but difficult question is to present a complete description of the behavior of the optimal stopping boundaries A_* and B_* from equations (14.2.5) and (14.2.6) under the changing of the parameters of the model.

ACKNOWLEDGMENTS

The author thanks Andreas E. Kyprianou and two anonymous Referees for valuable comments and suggestions.

REFERENCES

[1] Alario-Nazaret, M., Lepeltier, J.P. and Marchal, B. (1982), "Dynkin games", in *Stochastic Differential Games (Bad Honnef)*, Lecture Notes in Control and Information Sciences, Vol. 43, Springer-Verlag, Berlin, Germany, pp. 23–32.

[2] Baurdoux, E. and Kyprianou, A. (2004), "Further calculations for Israeli options", *Stochastics and Stochastic Reports*, to be published.

[3] Bensoussan, A and Friedman, A. (1974), "Non-linear variational inequalities and differential games with stopping times", *Journal of Functional Analysis*, **16**, 305–352.

[4] Bensoussan, A and Friedman, A. (1977), "Nonzero-sum stochastic differential games with stopping times and free-boundary problems", *Transactions of the American Mathematical Society*, **231**, 275–327.

[5] Bismut, J.M. (1977), "Sur un problème de Dynkin", *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **39**, 31–53.

[6] Cvitanić, J. and Karatzas, I. (1996), "Backward stochastic differential equations with reflection and Dynkin games", *Annals of Probability*, **24**, 2024–2056.

[7] Dynkin, E.B. (1969), "Game variant of a problem on optimal stopping", *Soviet Mathematical Doklady*, **10**, 270–274.

[8] Elbakidze, N.V. (1976), "Construction of the cost and optimal policies in a game problem of stopping a Markov process", *Theory of Probability and its Applications*, **16**, 163–168.

[9] Frid, E.B. (1969), "Optimal stopping of a Markov chain of two persons with opponent interests", *Theory of Probability and its Applications*, **14**, 746–749.

[10] Friedman, A. (1973), "Stochastic games and variational inequalities", *Archive for Rational Mechanics and Analysis*, **51**, 321–346.

[11] Friedman, A. (1976), *Stochastic Differential Equations and Applications II*, Academic Press, New York, NY, USA.

[12] Grigelionis, B.I. and Shiryaev, A.N. (1966), "On Stefan's problem and optimal stopping rules for Markov processes", *Theory of Probability and its Applications*, **11**, 541–558.

[13] Jacod, J. (1979), *Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Mathematics, Vol. 714, Springer-Verlag, Berlin, Germany.

[14] Jacod, J. and Shiryaev, A.N. (1987), *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, Germany.

[15] Kallsen, J. and Kühn, C. (2004), "Pricing derivatives of American and game type in incomplete markets", *Finance and Stochastics*, **8**, 261–284.

[16] Kallsen, J. and Kühn, C. (2005), "Convertible bonds: financial derivatives of game type", in A.E. Kyprianou, W. Schoutens and P. Wilmott (Eds), *Exotic Option Pricing and Advanced Lévy Models*, Wiley, Chichester, UK, pp. 277–291.

[17] Karatzas, I. and Wang, H. (2001), "*Connection Between Bounded-Variation Control and Dynkin games*", in J.L. Menaldi *et al.* (Eds), *Optimal Control and Partial Differential Equations* (Volume in Honour of Alain Bensoussan), IOS Press, Amsterdam, The Netherlands, pp. 363–373.

[18] Kifer, Yu.M. (1971a), "Optimal stopped games", *Theory of Probability and its Applications*, **16**, 183–188.

[19] Kifer, Yu.M. (1971b), "Optimal stopping in games with continuous time", *Theory of Probability and its Applications*, **16**, 545–550.

[20] Kifer, Y. (2000), "Game options", *Finance and Stochastics*, **4**, 443–463.

[21] Krylov, N.V. (1971), "Control of Markov processes and W-spaces", *Izvestija*, **5**, 233–266.

[22] Kühn, C. and Kyprianou, A. (2003a), "Callable puts as composite exotic options", to be published.

[23] Kühn, C. and Kyprianou, A. (2003b), "Pricing Israeli options: a pathwise approach", to be published.

[24] Kyprianou, A. (2004), "Some calculations for Israeli options", *Finance and Stochastics*, **8**, 73–86.

[25] Lepeltier, J.P. and Mainguenau, M.A. (1984), "Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski", *Stochastics*, **13**, 25–44.

[26] Morimoto, H. (1984), "Dynkin games and martingale methods", *Stochastics*, **13**, 213–228.

[27] Neveu, J. (1975), *Discrete-Parameter Martingales*, North-Holland, Amsterdam, The Netherlands.

[28] Protter, Ph. (1992), *Stochastic Integration and Differential Equations: A New Approach*, Applications of Mathematics, Vol. 21, Springer-Verlag, Berlin, Germany.

[29] Shiryaev, A.N., Kabanov, Y.M., Kramkov, D.O. and Melnikov, A.V. (1994), "On the pricing of options of European and American types, II. Continuous time", *Theory of Probability and its Applications*, **39**, 61–102.

[30] Shiryaev, A.N. (1978), *Optimal Stopping Rules*, Springer-Verlag, Berlin, Germany.

[31] Shiryaev, A.N. (1999), *Essentials of Stochastic Finance*, World Scientific, Singapore.

[32] Snell, J.L. (1952), "Applications of martingale systems theorems", *TAMS*, **73**, 293–312.

[33] Stettner, L. (1982), "Zero-sum Markov games with stopping and impulsive strategies", *Applied Mathematics and Optimization*, **9**, 1–24.

Index

Exotic Option Pricing and Advanced Lévy Models. Edited by A. E. Kyprianou, W. Schoutens and P. Wilmott