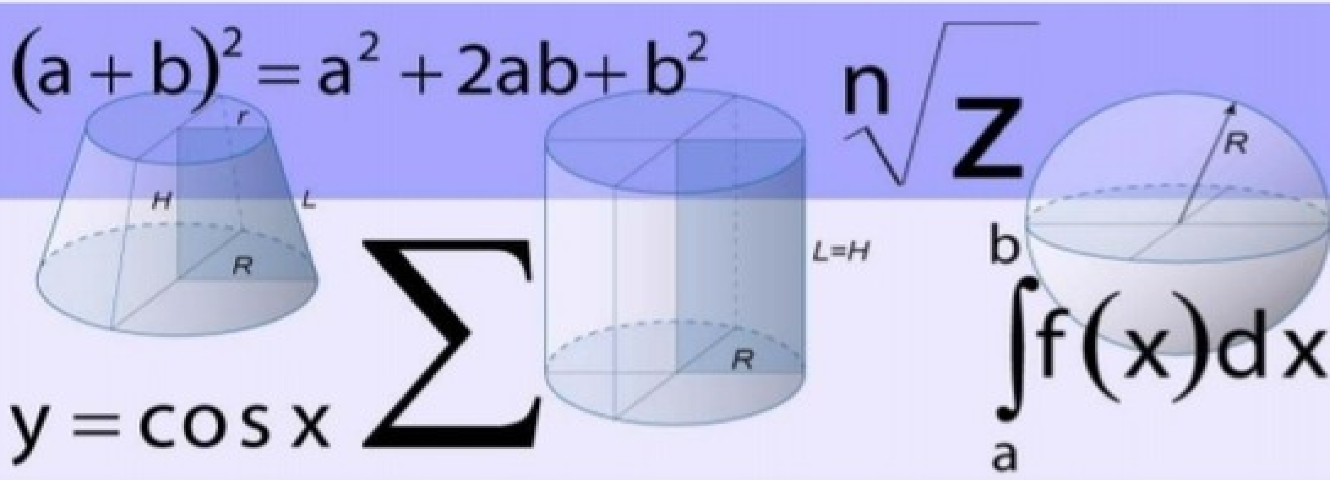

1300 Math Formulas

Alex Svirin, Ph.D.

=

qÜáë=é~ÖÉ=áë=áåíÉåíáçå~ääó=äÉÑí=Ää~åâK=

i Preface

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=
=

qÜáë= Ü~åÇÄççâ= áë= ~= ÅçãéäÉíÉ= ÇÉëâíçé= êÉÑÉêÉåÅÉ= Ñçê= ëíì - ÇÉåíë= ~åÇ= ÉåÖáåÉÉêëK= fí= Ü~ë= ÉîÉêóíÜáåÖ= Ñêçã= ÜáÖÜ= ëÅÜççä= ã~íÜ=íç=ã~íÜ=Ñçê=~Çî~åÅÉÇ=ìåÇÉêÖê~Çì~íÉë=áå=ÉåÖáåÉÉêáåÖI= ÉÅçåçãáÅëI=éÜóëáÅ~ä=ëÅáÉåÅÉëI=~åÇ=ã~íÜÉã~íáÅëK=qÜÉ=ÉÄçç â= Åçåí~áåë=ÜìåÇêÉÇë=çÑ=Ñçêãì~ëI=í~ÄäÉëI=~åÇ=ÑáÖìêÉë=Ñêçã= kìãÄÉê= pÉíëI= ^äÖÉÄê~I= dÉçãÉíêóI= qêáÖçåçãÉíêóI= j~íêáÅÉë= ~åÇ= aÉíÉêãáå~åíëI= sÉÅíçêëI= ^å~äóíáÅ= dÉçãÉíêóI= `~äÅìäìëI= aáÑÑÉêÉåíá~ä=bèì~íáçåëI=pÉêáÉëI=~åÇ=mêçÄ~Äáäáíó=qÜÉçêóK== qÜÉ= ëíêìÅîêÉÇ= í~ÄäÉ= çÑ= ÅçåíÉåíëI= äáåâëI= ~åÇ= ä~óçìí= ã~âÉ= ÑáåÇáåÖ= íÜÉ= êÉäÉî~åí= áåÑçêã~íáçå= èìáÅâ= ~åÇ= é~áåäÉëëI= ëç= áí= Å~å=ÄÉ=ìëÉÇ=~ë=~å=ÉîÉêóÇ~ó=çåäáåÉ=êÉÑÉêÉåÅÉ=ÖìáÇÉK===

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1 krj_bo=pbqp=
NKN= pÉí=fÇÉåíáíáÉë==1=
NKO= pÉíë=çÑ=kìãÄÉêë==5=
NKP= _~ëáÅ=fÇÉåíáíáÉë==7=
NKQ= `çãéäÉñ=kìãÄÉêë==8=

=
2 ^idb_o^=
OKN= c~ÅíçêáåÖ=cçêãìä~ë==12=
OKO= mêçÇìÅí=cçêãìä~ë==13=
OKP= mçïÉêë==14=
OKQ= oççíë==15=
OKR= içÖ~êáíÜãë==16=
OKS= bèì~íáçåë==18=
OKT= fåÉèì~äáíáÉë==19=
OKU= `çãéçìåÇ=fåíÉêÉëí=cçêãìä~ë==22=

=
3 dbljbqov=
PKN= oáÖÜí=qêá~åÖäÉ==24=
PKO= fëçëÅÉäÉë=qêá~åÖäÉ==27=
PKP= bèìáä~íÉê~ä=qêá~åÖäÉ==28=
PKQ= pÅ~äÉåÉ=qêá~åÖäÉ==29=
PKR= pèì~êÉ==33=
PKS= oÉÅí~åÖäÉ==34=
PKT= m~ê~ääÉäçÖê~ã==35=
PKU= oÜçãÄìë==36=
PKV= qê~éÉòçáÇ==37=
PKNM= fëçëÅÉäÉë=qê~éÉòçáÇ==38=

PKNN= fëçëÅÉäÉë=qê~éÉòçáÇ=ïáíÜ=fåëÅêáÄÉÇ=`áêÅäÉ==40=
PKNO= qê~éÉòçáÇ=ïáíÜ=fåëÅêáÄÉÇ=`áêÅäÉ==41=

iii

PKNP= háíÉ==42=
PKNQ= `óÅääáÅ=nì~Çêáä~íÉê~ä==43=
PKNR= q~åÖÉåíá~ä=nì~Çêáä~íÉê~ä==45=
PKNS= dÉåÉê~ä=nì~Çêáä~íÉê~ä==46=
PKNT= oÉÖìä~ê=eÉñ~Öçå==47=
PKNU= oÉÖìä~ê=mçäóÖçå==48=
PKNV= `áêÅäÉ==50=
PKOM= pÉÅíçê=çÑ=~=`áêÅäÉ==53=
PKON= pÉÖãÉåí=çÑ=~=`áêÅäÉ==54=
PKOO= `ìÄÉ==55=
PKOP= oÉÅí~åÖìä~ê=m~ê~ääÉäÉéáéÉÇ==56=
PKOQ= mêáëã==57=
PKOR= oÉÖìä~ê=qÉíê~ÜÉÇêçå==58=
PKOS= oÉÖìä~ê=móê~ãáÇ==59=
PKOT= cêìëíìã=çÑ=~=oÉÖìä~ê=móê~ãáÇ==61=
PKOU= oÉÅí~åÖìä~ê=oáÖÜí=tÉÇÖÉ==62=
PKOV= mä~íçåáÅ=pçäáÇë==63=
PKPM= oáÖÜí=`áêÅìä~ê=`óäáåÇÉê==66=
PKPN=
oáÖÜí=`áêÅìä~ê=`óäáåÇÉê=ïáíÜ=~å=lÄäáèìÉ=mä~åÉ=c~ÅÉ==68=
PKPO= oáÖÜí=`áêÅìä~ê=`çåÉ==69=
PKPP= cêìëíìã=çÑ=~=oáÖÜí=`áêÅìä~ê=`çåÉ==70=
PKPQ= péÜÉêÉ==72=
PKPR= péÜÉêáÅ~ä=`~é==72=
PKPS= péÜÉêáÅ~ä=pÉÅíçê==73=
PKPT= péÜÉêáÅ~ä=pÉÖãÉåí==74=
PKPU= péÜÉêáÅ~ä=tÉÇÖÉ==75=
PKPV= bääáéëçáÇ==76=
PKQM= `áêÅìä~ê=qçêìë==78=
==

4 qofdlkljbqov=
QKN= o~Çá~å=~åÇ=aÉÖêÉÉ=jÉ~ëìêÉë=çÑ=^åÖäÉë==80= QKO=
aÉÑáåáíáçåë=~åÇ=dê~éÜë=çÑ=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==81=

QKP= páÖåë=çÑ=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==86=
QKQ= qêáÖçåçãÉíêáÅ=cìåÅíáçåë=çÑ=`çããçå=^åÖäÉë==87= QKR=
jçëí=fãéçêí~åí=cçêãìä~ë==88=
QKS= oÉÇìÅíáçå=cçêãìä~ë==89=
QKT= mÉêáçÇáÅáíó=çÑ=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==90=
QKU= oÉä~íáçåë=ÄÉíïÉÉå=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==90= QKV=
^ÇÇáíáçå=~åÇ=pìÄíê~Åíáçå=cçêãìä~ë==91=
QKNM= açìÄäÉ=^åÖäÉ=cçêãìä~ë==92=
QKNN= jìäíáéäÉ=^åÖäÉ=cçêãìä~ë==93=
QKNO= e~äÑ=^åÖäÉ=cçêãìä~ë==94=
QKNP= e~äÑ=^åÖäÉ=q~åÖÉåí=fÇÉåíáíáÉë==94=
QKNQ=
qê~åëÑçêãáåÖ=çÑ=qêáÖçåçãÉíêáÅ=bñéêÉëëáçåë=íç=mêçÇìÅí==95=
QKNR=
qê~åëÑçêãáåÖ=çÑ=qêáÖçåçãÉíêáÅ=bñéêÉëëáçåë=íç=pìã==97===
QKNS= mçïÉêë=çÑ=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==98=
QKNT= dê~éÜë=çÑ=fåîÉêëÉ=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==99=
QKNU=
mêáåÅáé~ä=s~äìÉë=çÑ=fåîÉêëÉ=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==102=
QKNV=
oÉä~íáçåë=ÄÉíïÉÉå=fåîÉêëÉ=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==103=
QKOM= qêáÖçåçãÉíêáÅ=bèì~íáçåë==106=
QKON= oÉä~íáçåë=íç=eóéÉêÄçäáÅ=cìåÅíáçåë==106=
==

5 j^qof`bp=^ka=abqbojfk^kqp=
RKN= aÉíÉêãáå~åíë==107=
RKO= mêçéÉêíáÉë=çÑ=aÉíÉêãáå~åíë==109=
RKP= j~íêáÅÉë==110=
RKQ= léÉê~íáçåë=ïáíÜ=j~íêáÅÉë==111=
RKR= póëíÉãë=çÑ=iáåÉ~ê=bèì~íáçåë==114=
==

6 sb`qlop=
SKN= sÉÅíçê=`ççêÇáå~íÉë==118=
SKO= sÉÅíçê=^ÇÇáíáçå==120=
SKP= sÉÅíçê=pìÄíê~Åíáçå==122=
SKQ= pÅ~ä~êåÖ=sÉÅíçêë==122=

SKR= pÅ~ä~ê=mêçÇìÅí==123=
SKS= sÉÅíçê=mêçÇìÅí==125=
SKT= qêáéäÉ=mêçÇìÅí=127=
==

7 ^k^ivqf`=dbljbqov=
TKN= låÉ=-aáãÉåëáçå~ä=`çêÇáå~íÉ=póëíÉã==130=

v

TKO= qïç=-aáãÉåëáçå~ä=`ççêÇáå~íÉ=póëíÉã==131= TKP=
píê~áÖÜí=iáåÉ=áå=mä~åÉ==139=
TKQ= `áêÅäÉ==149=
TKR= bääáéëÉ==152=
TKS= eóéÉêÄçä~==154=
TKT= m~ê~Äçä~==158=
TKU= qÜêÉÉ=-aáãÉåëáçå~ä=`ççêÇáå~íÉ=póëíÉã==161= TKV=
mä~åÉ==165=
TKNM= píê~áÖÜí=iáåÉ=áå=pé~ÅÉ==175=
TKNN= nì~ÇêáÅ=pìêÑ~ÅÉë==180=
TKNO= péÜÉêÉ==189=
==

8 afccbobkqf^i=`^i`rirp=
UKN= cìåÅíáçåë=~åÇ=qÜÉáê=dê~éÜë==191=
UKO= iáãáíë=çÑ=cìåÅíáçåë==208=
UKP= aÉÑáåáíáçå=~åÇ=mêçéÉêíáÉë=çÑ=íÜÉ=aÉêáî~íáîÉ==209=
UKQ= q~ÄäÉ=çÑ=aÉêáî~íáîÉë==211=
UKR= eáÖÜÉê=lêÇÉê=aÉêáî~íáîÉë==215=
UKS= ^ééäáÅ~íáçåë=çÑ=aÉêáî~íáîÉ==217=
UKT= aáÑÑÉêÉåíá~ä==221=
UKU= jìäíáî~êá~ÄäÉ=cìåÅíáçåë==222=
UKV= aáÑÑÉêÉåíá~ä=léÉê~íçêë==225=
==

9 fkqbdo^i=`^i`rirp=
VKN= fåÇÉÑáåáíÉ=fåíÉÖê~ä==227=
VKO= fåíÉÖê~äë=çÑ=o~íáçå~ä=cìåÅíáçåë==228=
VKP= fåíÉÖê~äë=çÑ=fêê~íáçå~ä=cìåÅíáçåë==231=
VKQ= fåíÉÖê~äë=çÑ=qêáÖçåçãÉíêáÅ=cìåÅíáçåë==237=
VKR= fåíÉÖê~äë=çÑ=eóéÉêÄçäáÅ=cìåÅíáçåë==241=
VKS=
fåíÉÖê~äë=çÑ=bñéçåÉåíá~ä=~åÇ=içÖ~êáíÜãáÅ=cìåÅíáçåë==242=
VKT= oÉÇìÅíáçå=cçêãìä~ë==243=

VKU= aÉÑáåáíÉ=fåíÉÖê~ä==247=
VKV= fãéêçéÉê=fåíÉÖê~ä==253=
VKNM= açìÄäÉ=fåíÉÖê~ä==257=
VKNN= qêáéäÉ=fåíÉÖê~ä==269=
VKNO= iáåÉ=fåíÉÖê~ä==275=
VKNP= pìêÑ~ÅÉ=fåíÉÖê~ä==285=
==

10 afccbobkqf^i=bnr^qflkp=
NMKN= cáêëí=lêÇÉê=lêÇáå~êó=aáÑÑÉêÉåíá~ä=bèì~íáçåë==295=
NMKO= pÉÅçåÇ=lêÇÉê=lêÇáå~êó=aáÑÑÉêÉåíá~ä=bèì~íáçåë==298=
NMKP= pçãÉ=m~êíá~ä=aáÑÑÉêÉåíá~ä=bèì~íáçåë==302= ==

11 pbofbp=
NNKN= ^êáíÜãÉíáÅ=pÉêáÉë==304=
NNKO= dÉçãÉíêáÅ=pÉêáÉë==305=
NNKP= pçãÉ=cáåáíÉ=pÉêáÉë==305=
NNKQ= fåÑáåáíÉ=pÉêáÉë==307=
NNKR= mêçéÉêíáÉë=çÑ=`çåîÉêÖÉåí=pÉêáÉë==307=
NNKS= `çåîÉêÖÉåÅÉ=qÉëíë==308=
NNKT= ^äíÉêå~íáåÖ=pÉêáÉë==310=
NNKU= mçïÉê=pÉêáÉë==311=
NNKV=
aáÑÑÉêÉåíá~íáçå=~åÇ=fåíÉÖê~íáçå=çÑ=mçïÉê=pÉêáÉë==312=
NNKNM= q~óäçê=~åÇ=j~Åä~ìêáå=pÉêáÉë==313=
NNKNN= mçïÉê=pÉêáÉë=bñé~åëáçåë=Ñçê=pçãÉ=cìåÅíáçåë==314=
NNKNO= _áåçãá~ä=pÉêáÉë==316=
NNKNP= cçìêáÉê=pÉêáÉë==316=
==

12 mol_^_fifqv=
NOKN= mÉêãìí~íáçåë=~åÇ=`çãÄáå~íáçåë==318=
NOKO= mêçÄ~Äáäáíó=cçêãìä~ë==319=
==
=

= =

=

qÜáë=é~ÖÉ=áë=áåíÉåíáçå~ääó=äÉÑí=Ää~åâK=

=

=
=
=
=

1.1 Set Identities

=
pÉíëW=^I=_I=`=
råáîÉêë~ä=ëÉíW=f=
`çãéäÉãÉåí=W= ^′ =
mêçéÉê=ëìÄëÉíW=^⊂_==
bãéíó=ëÉíW=∅=
råáçå=çÑ=ëÉíëW=^∪_=
fåíÉêëÉÅíáçå=çÑ=ëÉíëW=^∩_= aáÑÑÉêÉåÅÉ=çÑ=ëÉíëW= ^y_= =

=
1. ^⊂f=
=
2. ^⊂^=
=
3. ^=_=áÑ= ^⊂_=~åÇ=_⊂ ^.=
=
4. bãéíó=pÉí=
∅ ⊂ ^=
=
5. råáçå=çÑ=pÉíë==

`=^U_= ñö{}=

Figure 1. =

6. `çãäìí~íáîáíó=

^U_=_U^=

=

7. ^ëëçÅá~íáîáíó=

^U()( )U`=

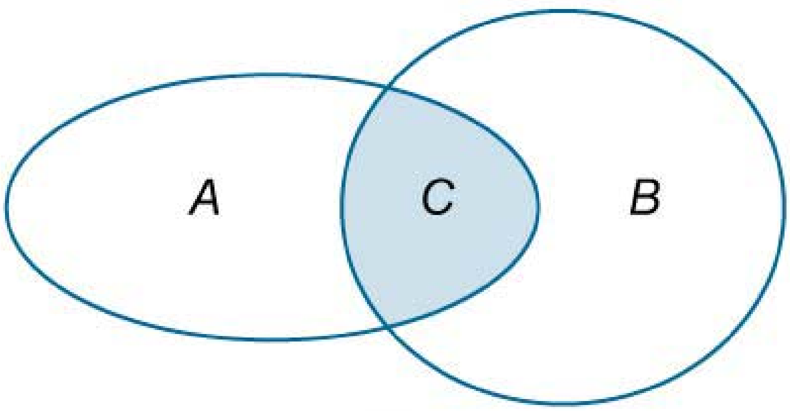
=

8. fåíÉêëÉÅíáçå=çÑ=pÉíë=

` =^U_= ñöñ∈^{}==

=

= =====



= **Figure 2.** =

9. `çãàìí~íáîáíó=

^∩_=_∩^=

=

10. ^ëëçÅá~íáîáíó=

^∩()( )∩`=

11. aáëíêáÄìíáîâáíó=

^∪()( ) (^∪`)I=

^∩()( ) (^∩`)K=

=

12. fÇÉãéçíÉåÅó=

^ = ^**I==**

^ = ^**=**

=

13. açãáå~íáçå=

^∩∅ =∅**I=**

^ =**f f=**

=

14. fÇÉåíáíó=

^ =^**I==**

^ =**f** ^

=

15. `çãéäÉãÉåí=

^=′ ñ∈föñ∉{}

=

16. **`çãéäÉãÉåí=çÑ=fåíÉêëÉÅíáçå=~åÇ=råáçå** ^ =′**fI==**
^∩ ^=′∅=

= **17.** **aÉ=jçêÖ~å∞ë=i~ïë**

^()_′ =^′∩_′I==
()′=^′∪_′=

=

18. **aáÑÑÉêÉåÅÉ=çÑ=pÉíë**

`=_y^= ñöñ∈{}=

Figure 3. =

19. ()

= **20.** _**y**^ =_∩ ^′
= **21.** ^**y** ^=∅
= **22.** ^**y** =^=**áÑ=** ^∩_ =∅ .

_
=
=====

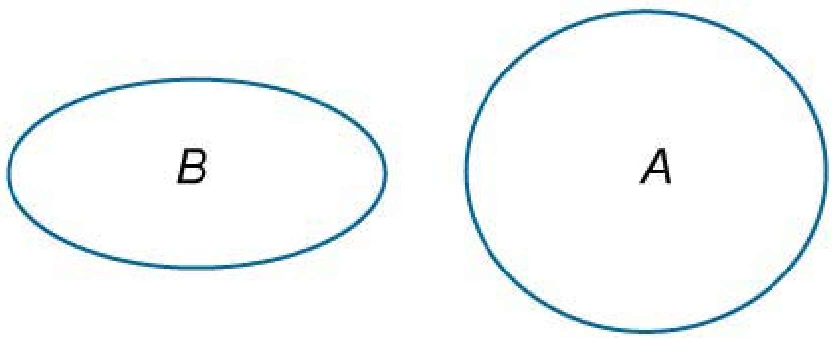


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Figure 4.
=

23. () ( ) ( )

24. ^=′**fy**^ **25.** `~**êíÉëá~å=mêçÇìÅí**

`=^×_=**(){}**

=

1.2 Sets of Numbers

=
k~îiê~ä=åìãÄÉêëW=k=
tÜçäÉ=åìãÄÉêëW=

k =
M
fåíÉÖÉêëW=w=
mçëáíáîÉ=áåíÉÖÉêëW=
+
w=
kÉÖ~íáîÉ=áåíÉÖÉêëW=
−
w=

o~íáçå~ä=åìãÄÉêëW=n=
oÉ~ä=åìãÄÉêëW=o==
`çãéäÉñ=åìãÄÉêëW=`==
=
=

26. k~îiê~ä=kìãÄÉêë

`çìåíáåÖ=åìãÄÉêëW {NIOIPIK}K=

27. tÜçäÉ=kìãÄÉêë

`çìåíáåÖ=åìãÄÉêë=~åÇ=òÉêçW=k {MINIOIPIK}K=M=

=
28. fåíÉÖÉêë
tÜçäÉ=åìãÄÉêë=~åÇ=íÜÉáê=çééçëáíÉë=~åÇ=òÉêçW=

w^+ =k {}IOIPIK I= w^- ={}I−PI−OI N I= K

w

=w^{-} ∪{}∪w^{+} ={KI−PI−OI−NIMINIOIPIK}

M K=
=
29. o~íáçå~ä=kìãÄÉêë
oÉéÉ~íáåÖ=çê=íÉêãáå~íáåÖ=ÇÉÅáã~äëW==
=
□□□
ö
ñ
=
~
ñnK=
Ä
∈w ~åÇ Ä∈w ~åÇ Ä≠M□□□ =

30. fêê~íáçå~ä=kìãÄÉêë
kçåêÉéÉ~íáåÖ=~åÇ=åçåíÉêãáå~íáåÖ=ÇÉÅáã~äëK
31. oÉ~ä=kìãÄÉêë==
råáçå=çÑ=ê~íáçå~ä=~åÇ=áêê~íáçå~ä=åìãÄÉêëW=oK= =
32. `çãéäÉñ=kìãÄÉêë

` = ñ+áóöñ∈o ~åÇ{}I==

ïÜÉêÉ=á=áë=íÜÉ=áã~Öáå~êó=ìåáíK

=

33. k⊂w ⊂n⊂o⊂`=

=

===

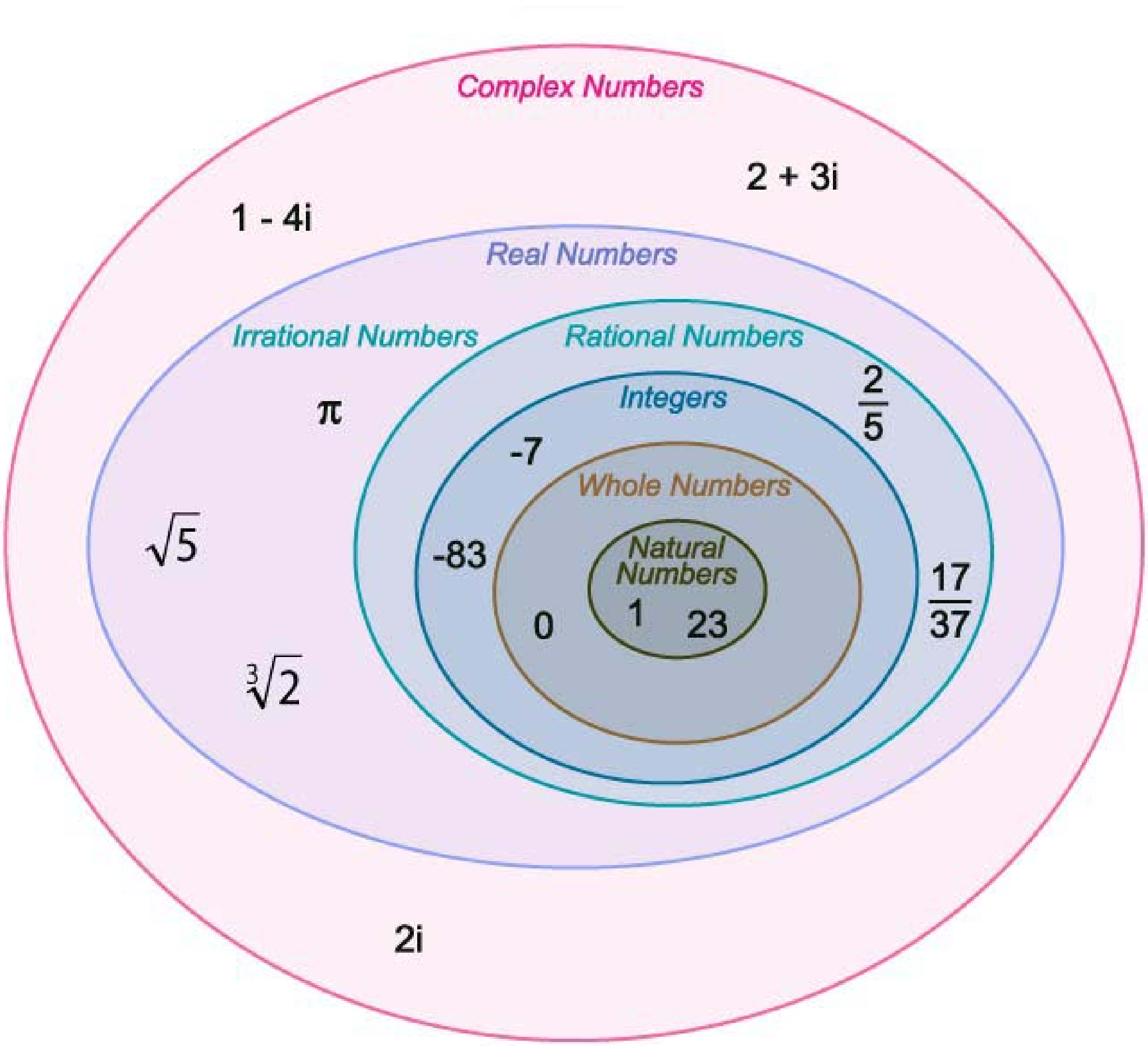

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Figure 5.

=

=

=

=

=

1.3 Basic Identities

=
oÉ~ä=åìãÄÉêëW=~I=ÄI=Å=
=
=
34. ^ÇÇáíáîÉ=fÇÉåíáíó= ~ =~ =

35. ^ÇÇáíáîÉ=fåîÉêëÉ=

~+()=M=

=
= 36. `çããìí~íáîÉ=çÑ=^ÇÇáíáçå= ~+Ä =Ä+~ =
= 37. ^ëëçÅá~íáîÉ=çÑ=^ÇÇáíáçå=

()+Å=~+( )=

=

38. aÉÑáåáíáçå=çÑ=pìÄíê~Åíáçå=

~−Ä=~ +()=

= 39. jìäíáéäáÅ~íáîÉ=fÇÉåíáíó=

~ =~ =

=

40. jìäíáéäáÅ~íáîÉ=fåîÉêëÉ=

N

~ =NI=~≠M~

=

41. jìäíáéäáÅ~íáçå=qáãÉë=M

~

=

M M

= 42. `çããìí~íáîÉ=çÑ=jìäíáéäáÅ~íáçå= ~·Ä=Ä·~

= 43. ^ëëçÅá~íáîÉ=çÑ=jìäíáéäáÅ~íáçå=

() ( )

=
44. aáëíêáÄìíáîÉ=i~ï=

~ Ä()Å =~Ä+~Å=

=
45. aÉÑáåáíáçå=çÑ=aáîáëáçå=
~ ·~N=Ä Ä

=
=
=

1.4 Complex Numbers

=

k~îìê~ä=åìãÄÉêW=å=

fã~Öáå~êó=ìåáíW=á=

`çãéäÉñ=åìãÄÉêW=ò=

oÉ~ä=é~êíW=~I=Å=

fã~Öáå~êó=é~êíW=ÄáI=Çá=

jçÇìäìë=çÑ=~=ÅçãéäÉñ=åìãÄÉêW=êI=$_N$êI=$_O$ê=

^êÖìãÉåí=çÑ=~=ÅçãéäÉñ=åìãÄÉêW=ϕ I=$_N$ I=$_O$ =

=

=

46.

á

^{N}==$á^R$==$á^{Qå+N}$

= =

$á^O$ −=N^S −=$N^{Qå+O}$ − = $á^P$ −=$á^T$ −=$á^{Qå+P}$ = $á^Q$==N^U==$N^{Qå}$= =

=

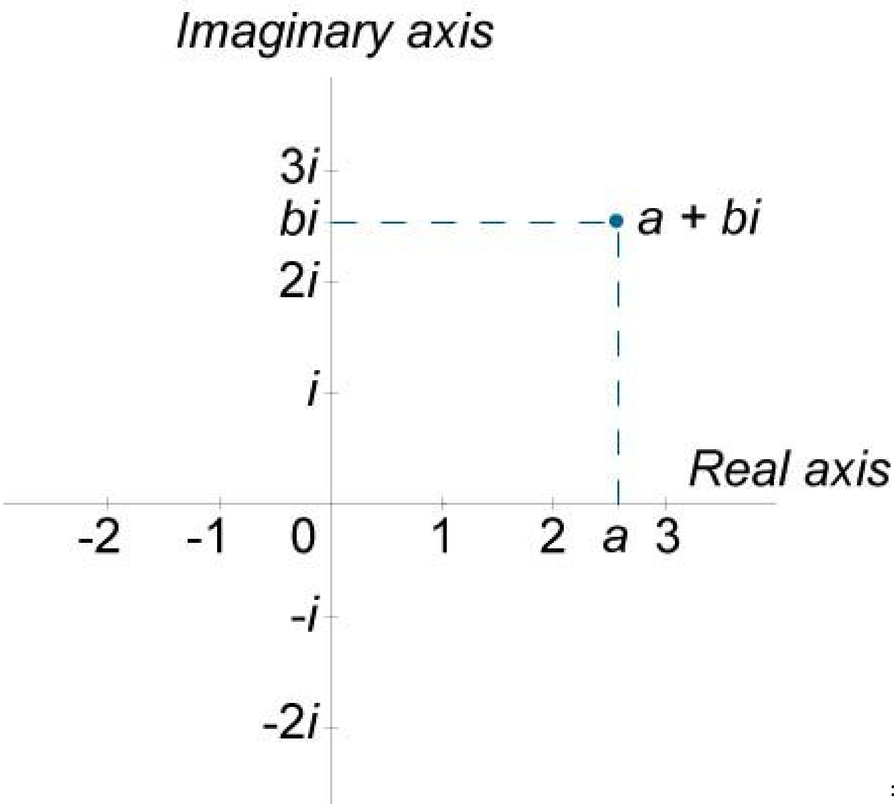
47. **ò** +**Äá=**

~

=

48. **`çãéäÉñ=mä~åÉ=**

= =====



= **Figure 6.** =

49. **()( ) ( ) ( )á=**

=

50. ()( ) ( ) ( )á=

=

51. ()( ) (~Å−ÄÇ) (~Ç+ÄÅ)á=

=

52. ~ +Äá= ~Å+ÄÇ+ÄÅ −~Ç·á=Å +Çá ÅO +ÇO ÅO +ÇO

=

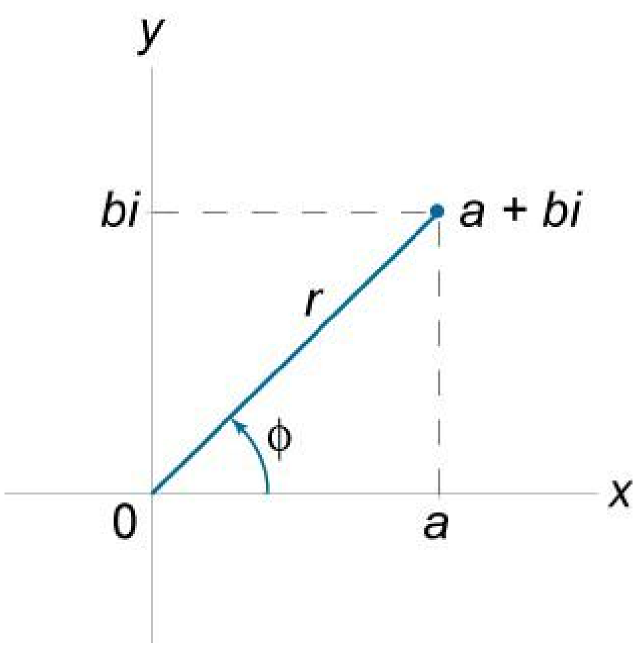
53. `çåàìÖ~íÉ=`çãéäÉñ=kìãÄÉêë=

|||||||

~ +Äá −Äá=

=

54. ~ =êÅçëϕI=Ä=êëáåϕ==



==

Figure 7.

=

55. mçä~ê=mêÉëÉåí~íáçå=çÑ=`çãéäÉñ=kìãÄÉêë=

~+Äá=ê()=

=

56. jçÇìäìë=~åÇ=^êÖìãÉåí=çÑ=~=`çãéäÉñ=kìãÄÉê=
fÑ=~+Äá=áë=~=ÅçãéäÉñ=åìãÄÉêI=íÜÉå=
ê= ~O +ÄO =EãçÇìäìëFI==
ϕ=~êÅí~åÄ =E~êÖìãÉåíFK=~

=

57. mêçÇìÅí=áå=mçä~ê=oÉéêÉëÉåí~íáçå=

ò() ( )=N·òO =êN ÅçëϕN +áëáåϕN êO ÅçëϕO
+áëáåϕO =êNêO Åçë(ϕN +ϕO)+áëáå(ϕN +ϕO)[]=

=
58. `çåàìÖ~íÉ=kìãÄÉêë=áå=mçä~ê=oÉéêÉëÉåí~íáçå=

||||||||| |||||||||| []=ê Åçë( ) () ()áëáå ê Åçë áëáå

=
59. fåîÉêëÉ=çÑ=~=`çãéäÉñ=kìãÄÉê=áå=mçä~ê=oÉéêÉëÉåí~íáçå=

N N []=ê() () ()

60. nìçíáÉåí=áå=mçä~ê=oÉéêÉëÉåí~íáçå=
ò
N
=
ê

()
N ÅçëϕN+áëáåϕN =ê() () ( )[]=òO êO ÅçëϕO +áëáåϕO êO

=
61. mçïÉê=çÑ=~=`çãéäÉñ=kìãÄÉê=
ò

å = ê()áëáå []å =êå[Åçë å( )+áëáå å( )

Åçë]=
=
62. cçêãìä~=±aÉ=jçáîêÉ≤=

()å=Åçë( )+áëáå( )=

=

63. kíÜ=oççí=çÑ=~=`çãéäÉñ=kìãÄÉê=

å ò =å ê()=å ê□Åçëϕ+Oπâ +áëáåϕ+Oπâ □I==□□ å å □□

ïÜÉêÉ==

â =MINIOIKIå−NK==

=

64. bìäÉê∞ë=cçêãìä~=

Éáñ =Åçëñ+áëáåñ=

=

Chapter 2 Algebra

=

=

=

=

2.1 Factoring Formulas

=
oÉ~ä=åìãÄÉêëW=~I=ÄI=Å==
k~îìê~ä=åìãÄÉêW=å=

=
=

65. ()( )Ä^{O}

~ ~ Ä ~ Ä =
=

66. ()Ä ~ Ä (~^{O}+~Ä+Ä^{O}

~)$^{=}$
=

67. ()Ä ~ Ä (~^{O} −~Ä+Ä^{O}

~)$^{=}$
=

68. ~^{Q} −Ä^{Q} =(~^{O} −Ä^{O})(~^{O} +Ä^{O}) ( )( )Ä ~ +Ä (~^{O} +Ä^{O}

~)$^{=}$ =

69. ()Ä ~ Ä (~^{Q} +~^{P}Ä+~^{O}Ä^{O} +~Ä^{P} +Ä^{Q}

~)$^{=}$
=

70. **()Ä = ~ Ä (~Q −~PÄ+~OÄO −~ÄP +ÄQ**

~)=
=

71. **fÑ=å=áë=çÇÇI=íÜÉå=)K==(~å−N −~å−OÄ +~å−PÄO −K−~Äå−O +Äå−N~å +Äå =()**

=

72. **fÑ=å=áë=ÉîÉåI=íÜÉå==)I==(~å−N +~å−OÄ+~å−PÄO +K+~Äå−O +Äå−N~å −Äå =() ~å +Äå =()(~å−N −~å−OÄ +~å−PÄO −K+~Äå−O −Äå−N)K=**

=
=
=

2.2 Product Formulas

oÉ~ä=åìãÄÉêëW=~I=ÄI=Å==
tÜçäÉ=åìãÄÉêëW=åI=â=
=
=

73. ()^{O} =~^{O}−O~Ä+Ä^{O}=

=

74. ()^{O} =~^{O} +O~Ä+Ä^{O}=

=

75. ()^{P} =~^{P} −P~^{O}Ä+P~Ä^{O} −Ä^{P}=

=

76. ()^{P} =~^{P} +P~^{O}Ä+P~Ä^{O} +Ä^{P}=

=

77. $()^{Q} =\sim^{Q} - Q\sim^{P}\ddot{A}+S\sim^{O}\ddot{A}^{O} - Q\sim\ddot{A}^{P} + \ddot{A}^{Q}=$

=

78. ()Q =~Q +Q~PÄ+S~OÄO +Q~ÄP +ÄQ=

=

79. _áåçãá~ä=cçêãìä~=

()å =å`M~å +å`N~å−NÄ+å`O~å−OÄO +K+å`å−N~Äå−N
+å`åÄåI ïÜÉêÉ=å` = å>
=~êÉ=íÜÉ=Äáåçãá~ä=ÅçÉÑÑáÅáÉåíëK= ()â> å−â >

=

80. **()^{O} =~^{O} +Ä^{O} +Å^{O} +O~Ä+O~Å+OÄÅ=**

=

81. ()K^{O} =~^{O} +Ä^{O} +Å^{O} +K+ì^{O} +î^{O} +=

+O$_{K}$()$_{K\ K}$ =

2.3 Powers

=

_~ëÉë=EéçëáíáîÉ=êÉ~ä=åìãÄÉêëFW=~I=Ä==
mçïÉêë=Eê~íáçå~ä=åìãÄÉêëFW=åI=ã=

=

=

82. ~ã~å =~ã+å=

=

83.

~ã

~å =~ ã−å=

=

84. ()ã =~ãÄã=

=

85.

□~ □ã ~ã □□Ä □□ =Äã = =

86.

()

å

= ~**ãå**= =

87. ~**M**=**NI=**~≠**M**= =

88. ~**N**=**N**=

=

89.

~

−ã **N**

= =~ã

=

ã

90. ~ =**å** ~**ã** = =

=

=

=

2.4 Roots

=

_~ëÉëW=~I=Ä==

mçïÉêë=Eê~íáçå~ä=åìãÄÉêëFW=åI=ã= ~ ≥M=Ñçê=ÉîÉå=êççíë=Eå=Oâ

I=â∈kF= =

=

91. å ~Ä=å ~å Ä =

=

92. å ~ã =åã~ãÄå =

=

93. å~=

å ~

å Ä I=Ä≠M=Ä

=

å ~ =åãÄå =åã ~ã

94.

åã ~ã

ã Ä Äå I=Ä≠MK= =

95. = ( )é å~ãé=

= ()å

96. =~=

=

97. å ~ã =åé~ãé = =

ã

98. å ~ã =~å =

=

99. ã å ~= ãå ~ =

= ()ã

100. =å ~ã = 101.
å
~
=
~å−NN å
~ I=~≠MK= =

102. ~ ± Ä =~ + ~O −Ä ±~− ~O −Ä =O O =
103.

N
~ ± Ä= ~ m Ä =~ −Ä

=
=
=

2.5 Logarithms

=
mçëáíáîÉ=êÉ~ä=åìãÄÉêëW=ñI=óI=~I=ÅI=â=
k~îê~ä=åìãÄÉêW=å==
=
=

104. aÉÑáåáíáçå=çÑ=içÖ~êáíÜã=
ó=äçÖ~ ñ =áÑ=~åÇ=çåäó=áÑ=ñ=~ó I=~>MI=~≠NK= =

105. äçÖ~=M=
=
106. äçÖ~ ~ =N=
=

107. äçÖ~ M=□□□−∞ áÑ ~ >N=+∞ áÑ ~ <N
=

108. ()=äçÖ~ ñ+äçÖ~ ó=~

=
109.
äçÖ
ñ
~ ó =äçÖ~ ñ −äçÖ~ ó=

110. ( )~ =åäçÖ~ ñ=

=

111.

äçÖ

å

~ =NäçÖ~ ñ=å

=

112. äçÖ~ ñ =äçÖÅ ñ =äçÖÅ ñ·äçÖ~ ÅI=Å>MI=Å≠NK=äçÖÅ~

=

113. äçÖ~ =N =äçÖÅ~

=

114. ñ=~äçÖ~ñ =

=

115. içÖ~êáíÜã=íç=_~ëÉ=NM=

äçÖNM =äçÖ ñ=

=

116. k~íìê~ä=içÖ~êáíÜã=

äçÖÉ ñ =äåñ I==

□+

N

□

â

ïÜÉêÉ=É=äáãâ□□ =OKTNUOUNUOUK= ∞□□N

=

117. äçÖ ñ =N äåñ =MKQPQOVQ äåñ=äåNM

=

118. äå ñ=N äçÖ ñ=OKPMORURäçÖ ñ=äçÖ É

=

=

=

=

2.6 Equations

=

oÉ~ä=åìãÄÉêëW=~I=ÄI=ÅI=éI=èI=ìI=î= pçäìíáçåëW=

ñI=ñI=óI=óI=ó = N O N O P

=

=

119. iáåÉ~ê=bèì~íáçå=áå=låÉ=s~êá~ÄäÉ= ~ñ =MI=ñ −Ä K==~

=

120. nì~Çê~íáÅ=bèì~íáçå=

~ñO +Äñ +Å =MI=ñNIO =−Ä± ÄO −Q~Å K=O~

=

121. aáëÅêáãáå~åí=

a=ÄO−Q~Å =

=

122. sáÉíÉ∞ë=cçêãìä~ë=

fÑ=ñO +éñ +è =MI=íÜÉå==

ñN +ñO =−éK=ñNñO =è

=

123. ~ñO =MI=ñN=MI=ñO −Ä K=~

=

124. ~ñO =MI=ñNIO =± −Å K=~

=

125. `ìÄáÅ=bèì~íáçåK=`~êÇ~åç∞ë=cçêãìä~K==

óP +éó+è=MI==

óN +îI=óOIP =−N () ()P á I==

O O

ïÜÉêÉ==

ì

=

P

−

è+ è O é O è− è O é O O + P I=î = P − O O + P K== O

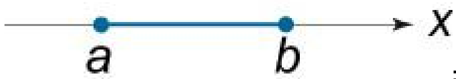
=

=

2.7 Inequalities

s~êá~ÄäÉëW=ñI=óI=ò=
oÉ~ä=åìãÄÉêëW=□□□~IÄIÅIÇ I=ãI=å=~NI~OI~PIKI~å
aÉíÉêãáå~åíëW=aI=
a
ñ
I=
aI=a ==
ó ò
=
=
126. fåÉèì~äáíáÉëI=fåíÉêî~ä=kçí~íáçåë=~åÇ=dê~éÜë== =
fåÉèì~äáíó= fåíÉêî~ä=kçí~íáçå= dê~éÜ=

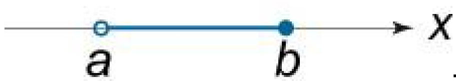
~≤ ñ Ä= []I~=



= ~

≤Ä= ( ]

ñ I~=



= ~

<

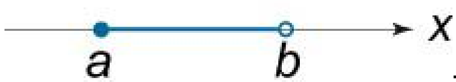
Ä

=

[

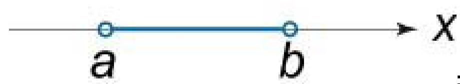
)

I~=



=

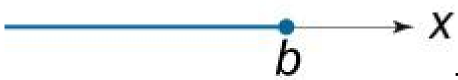
~<ñ Ä= ()=



=

−∞<ñ≤ÄI= ( ]=

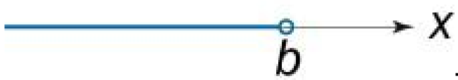
ñ≤Ä =



=

−∞<ñ<ÄI= ()=

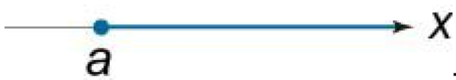
ñ<**Ä** =



=

ñ~I= [ )=

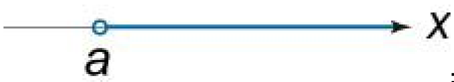
ñ≥~ =



=

~< ñ<∞I= ()=

ñ>~ =



=

127. fÑ=~>ÄI=íÜÉå=Ä<~ K=

=

128. fÑ=~>ÄI=íÜÉå=~ >M=çê=Ä <MK=

=

129. fÑ=~>ÄI=íÜÉå=~+Å >Ä+ÅK=

=

130. fÑ=~>ÄI=íÜÉå=~−Å>Ä−ÅK=

=

131. fÑ=~>Ä=~åÇ=Å>ÇI=íÜÉå=~ +Å>Ä+Ç K=

=

132. fÑ=~>Ä=~åÇ=Å>ÇI=íÜÉå=~ −Ç>Ä−ÅK=

=

133. fÑ=~>Ä=~åÇ=ã>M I=íÜÉå=ã~>ãÄK=

=

134. fÑ=~>Ä=~åÇ=ã>M I=íÜÉå=˜>Ä K=ã ã

=

135. fÑ=~>Ä=~åÇ=ã<MI=íÜÉå=ã~<ãÄK=

=

136. fÑ=~>Ä=~åÇ=ã<MI=íÜÉå=˜<Ä K=ã ã

=

137. fÑ=M<~ Ä=~åÇ=å>MI=íÜÉå=~å<Äå K=

=

138. fÑ=M<~ Ä=~åÇ=å<MI=íÜÉå=~å>ÄåK=

=

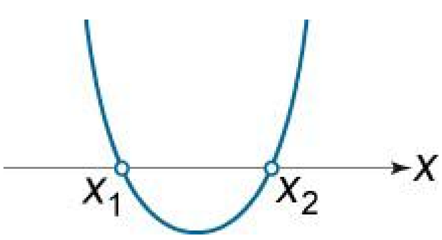
139. fÑ=M<~ ÄI=íÜÉå=å ~<å Ä K=
=

140.
~Ä
+
≤~ Ä I==O
ïÜÉêÉ=~>M=I=Ä>MX=~å=Éèì~äáíó=áë=î~äáÇ=çåäó=áÑ=~=ÄK== =
N
141. ~ ≥OI=ïÜÉêÉ=
~>MX=~å=Éèì~äáíó=í~âÉë=éä~ÅÉ=çåäó=~í=~=NK=
~

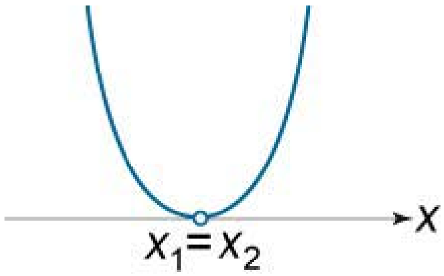
142. å ~N~OK~å ≤~N +~O +K+~å I=ïÜÉêÉ= ~NI~OIKI~å >MK=å
=
143. fÑ=~ñ >M=~åÇ=~>MI=íÜÉå=ñ −Ä K=~
=
144. fÑ=~ñ >M=~åÇ=~<MI=íÜÉå=ñ −Ä K==~

=
145. ~ñO +Äñ +Å>M=
=

=M =M= = =
=



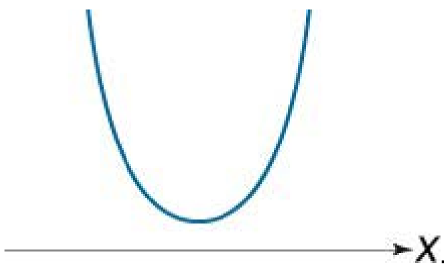
=
ñ<ñNI=ñ>ñO =
= =
=



ñ$_{\mathbf{N}}$<**ñI=ñ**>**ñ**$_{\mathbf{N}}$=

= =

=



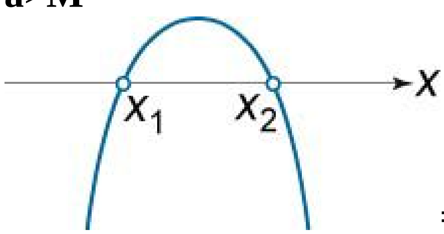
=

−∞<**ñ**<∞=

=

=

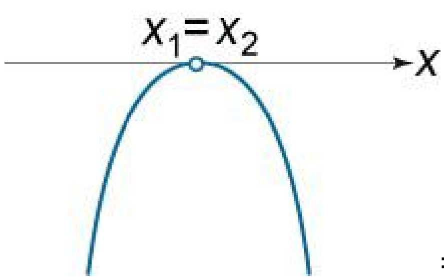
a>**M=**



=$_=$ **ñ**$_{\mathbf{N}}$ <**ñ ñ**$_{\mathbf{O}}$=

=

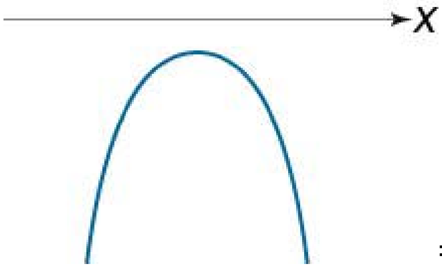
a=**M=**



=$_=$ **ñ**∈∅ =

=

a<M=



== = ñ∈∅ =

146. ~+Ä ≤ ~ + Ä =

=

147. fÑ= ñ<~I=íÜÉå=−~ <ñ ~I=ïÜÉêÉ=~>MK= =

148. fÑ= ñ>~ I=íÜÉå=ñ −~ =~åÇ=ñ>~ I=ïÜÉêÉ=~>MK= =

149. fÑ=ñ^{O}<~ I=íÜÉå= ñ< ~ I=ïÜÉêÉ=~>MK= =

150. fÑ=ñ^{O}>~ I=íÜÉå= ñ> ~ I=ïÜÉêÉ=~>MK= =

() 151. fÑ=Ö >MI=íÜÉå= ( ) ( )>MK= () □□□Ö()≠M

=

152. () M I=íÜÉå= ( ) ( )<MK=Ö < () □□□Ö()≠M

=

=

=

2.8 Compound Interest Formulas

=

cìîêÉ=î~äìÉW=^=

fåáíá~ä=ÇÉéçëáíW=`=

^ååì~ä=ê~íÉ=çÑ=áåíÉêÉëíW=ê=

kìãÄÉê=çÑ=óÉ~êë=áåîÉëíÉÇW=í=

kìãÄÉê=çÑ=íáãÉë=ÅçãéçìåÇÉÇ=éÉê=óÉ~êW=å=

=

=

153. dÉåÉê~ä=`çãéçìåÇ=fåíÉêÉëí=cçêãìä~=

□+

ê

□

åí

^ =`□□å□□ =

154. páãéäáÑáÉÇ=`çãéçìåÇ=fåíÉêÉëí=cçêãìä~=

fÑ= áåíÉêÉëí= áë= ÅçãéçìåÇÉÇ= çåÅÉ= éÉê= óÉ~êI= íÜÉå= íÜÉ= éêÉîáçìë= Ñçêãìä~=ëáãéäáÑáÉë=íçW=

^ +()í K=

=

155. `çåíáåìçìë=`çãéçìåÇ=fåíÉêÉëí=

fÑ=áåíÉêÉëí=áë=ÅçãéçìåÇÉÇ=Åçåíáåì~ääó=Eå ∞FI=íÜÉå== ^=`Éêí K=

=

=

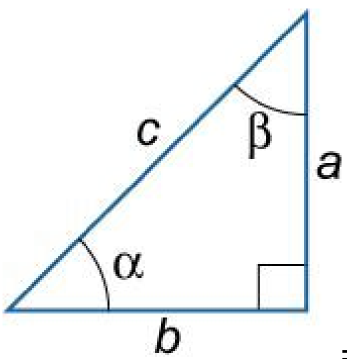
Chapter 3 Geometry

=
=
=
=

3.1 Right Triangle

=

iÉÖë=çÑ=~=êáÖÜí=íêá~åÖäÉW=~I=Ä=
eóéçíÉåìëÉW=Å=
^äíáîìÇÉW=Ü=
jÉÇá~åëW=ã~ I=ãÄI=ã =Å
^åÖäÉëW=αIβ =
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê=
^êÉ~W=p=
=
=



==

Figure 8. =

156. α+β=**VM** =

157. ëáå α=~ =**Åçë**β=**Å**
=

158. Åçëα=**Ä** =**ëáå** β=**Å**
=

159. í~åα =~ =**Åçí**β=**Ä**
=

160. Åçíα =**Ä** =**í~å**β=~
=

161. ëÉÅα=**Å** =**ÅçëÉÅ**β=**Ä**

=

162. ÅçëÉÅα=Å =ëÉÅβ=~

= 163. móíÜ~ÖçêÉ~å=qÜÉçêÉã= ~O +ÄO =ÅO=

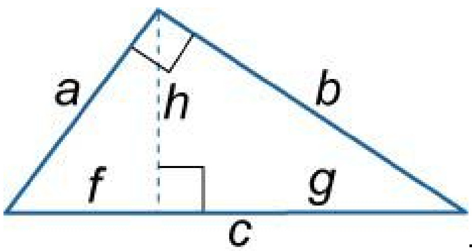
=

164. ~O=ÑÅI=ÄO=ÖÅI==

ïÜÉêÉ= Ñ= ~åÇ= Å= ~êÉ= éêçàÉÅíáçåë= çÑ= íÜÉ= äÉÖë= ~= ~åÇ= ÄI= êÉëéÉÅ- íáîÉäóI=çåíç=íÜÉ=ÜóéçíÉåìëÉ=ÅK=

=

= =====



= Figure 9.

165. ÜO=ÑÖI===

ïÜÉêÉ=Ü=áë=íÜÉ=~äíáíìÇÉ=Ñêçã=íÜÉ=êáÖÜí=~åÖäÉK== =

166.

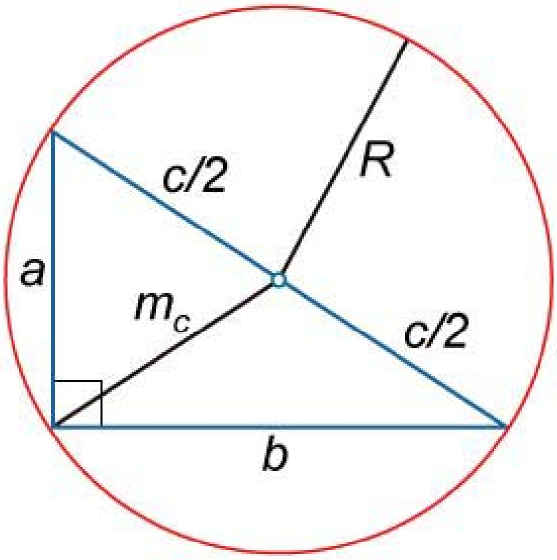
ã

O = O ~O ÄO

~ −Q I=ãO =O I===Ä −Q

ïÜÉêÉ=ã~ =~åÇ=ãÄ=~êÉ=íÜÉ=ãÉÇá~åë=íç=íÜÉ=äÉÖë=~=~åÇ=ÄK==

=



= =

Figure 10.

=

167. ãÅ=Å I==O

ïÜÉêÉ=Aã =áë=íÜÉ=ãÉÇá~å=íç=íÜÉ=ÜóéçíÉåìëÉ=ÅK= =

168. o=Å ãÅ=O

=

169. ê=~Ä~ +Ä −Å =~ +Ä+Å =O

=

170. ~Ä=ÅÜ=

=

171. p=~Ä ÅÜ =O O

=

=

=

3.2 Isosceles Triangle

=

_~ëÉW=~=

iÉÖëW=Ä=

_~ëÉ=~åÖäÉW=β =

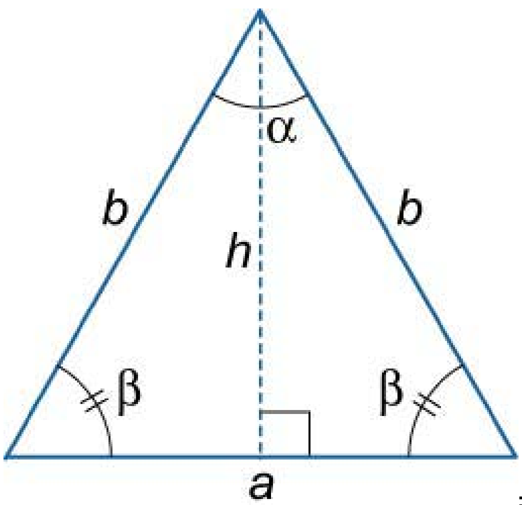
sÉêíÉñ=~åÖäÉW=α=

^äíáíìÇÉ=íç=íÜÉ=Ä~ëÉW=Ü= mÉêáãÉíÉêW=i=

^êÉ~W=p=

=

=



==

Figure 11. =

172.

β

=

VM

α

°− =O

=

173.

Ü

O =O ~O

−Q =

174. **i** +**OÄ= = ~Ü**
=
Ä
O
175. **p**=**O O ëáå**α**=**

=
=
=

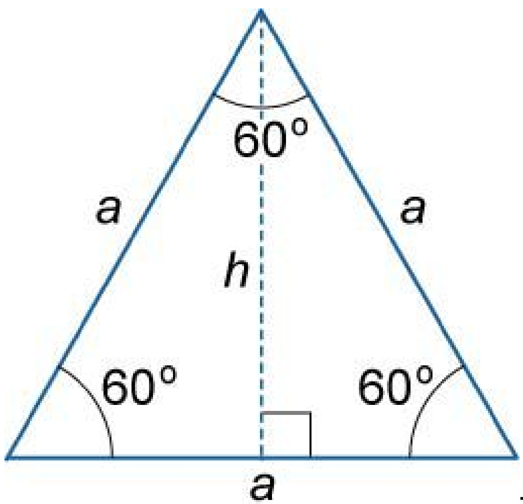
3.3 Equilateral Triangle

=

páÇÉ=çÑ=~=Éèìáä~íÉê~ä=íêá~åÖäÉW=~= ^äíáíìÇÉW=Ü=
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê= mÉêáãÉíÉêW=i=
^êÉ~W=p=

=

=



=$_{=}$

Figure 12. =

176. Ü=~ P =O

177. o=OÜ=~ P =P P

178. ê=

= NÜ = ~ P= o=P S O

= **179.** i=P~ =

=

~Ü

=

~

O

180. p=P =O Q

=
=
=

3.4 Scalene Triangle

E^=íêá~åÖäÉ=ïáíÜ=åç=îïç=ëáÇÉë=Éèì~äF=

=

=

páÇÉë=çÑ=~=íêá~åÖäÉW=~I=ÄI=Å=

pÉãáéÉêáãÉíÉêW= é=~ +Ä+Å ==O

^åÖäÉë=çÑ=~=íêá~åÖäÉW= γ=

^äíáíìÇÉë=íç=íÜÉ=ëáÇÉë=~I=ÄI=ÅW=Ü~IÜÄIÜÅ=

jÉÇá~åë=íç=íÜÉ=ëáÇÉë=~I=ÄI=ÅW=ã~IãÄIãÅ=

_áëÉÅíçêë=çÑ=íÜÉ=~åÖäÉë= γIIW=í~IíÄIíÅ=

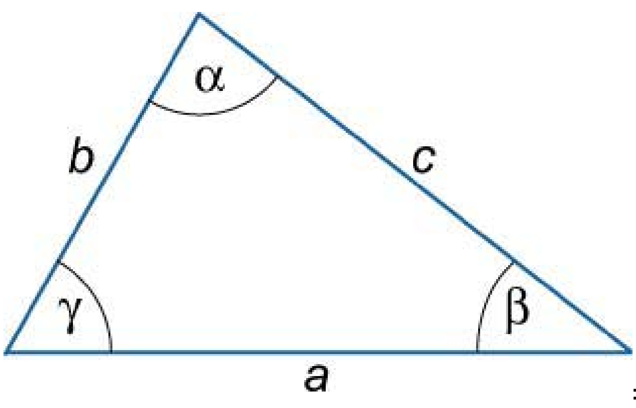
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o=

o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê=

^êÉ~W=p=

=

= **=====**



= **Figure 13.** =

181. α+β+γ =**NUM°=**

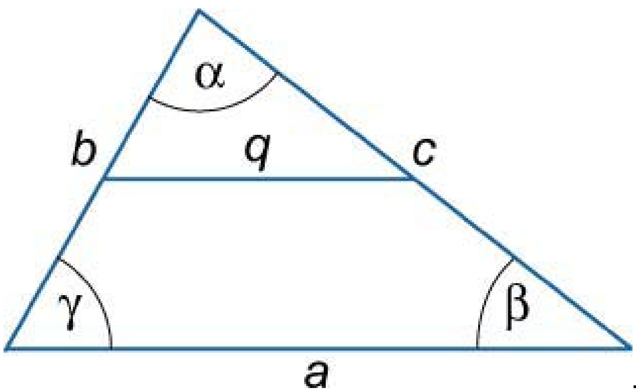
=

182. **~** >**ÅI==**

Ä

Ä >~ I==
~ >ÄK=
=
183. ~ <ÅI==

Ä <~ I==
~ <ÄK=
= **184.** jáÇäáåÉ=
è=$^{\sim}$ I=ööèK=$_{O}$
=
= =====



= Figure 14.

185. i~ï=çÑ=`çëáåÉë=
~^{O} =Ä^{O} +Å^{O} −OÄÅÅçëαI=
Ä^{O} =~^{O} +Å^{O} −O~Å ÅçëβI=
Å^{O} =~^{O} +Ä^{O} −O~ÄÅçë γK=
=

186. i~ï=çÑ=páåÉë=
~ =^{Ä} =^{Å} =OoI==$_{ëáåα\ ëáå\ β\ ëáå\ γ}$
ïÜÉêÉ=o=áë=íÜÉ=ê~Çáìë=çÑ=íÜÉ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉK== =
187.
o
=
~ Ä Å
Oëáå α= Oëáå β=Oëáå γ= ÄÅ = ~Å = ~Ä = ~ÄÅ=OÜ$_{\sim}$ OÜ$_{Ä}$ OÜ$_{Å}$ Qp =
188.

ê
O

()( )( )

= I==é
N= N+ N+ NK=ê Ü~ ÜÄ ÜÅ
189. ëáå =

()( )I=O = ÄÅ
Åçëα= é()I=O ÄÅ
í~åα= ()( )K=O é()

=

190. Ü~ ()( )( )O é é ~ é−Ä é −Å I=~
Ü =O é é()( )( )~ é−Ä é−Å I=Ä Ä
ÜÅ =O é é()( )( )~ é −Ä é−Å K=Å

191. Ü~ =Äëáå γ =ÅëáåβI= ÜÄ =~ëáå γ =ÅëáåαI= ÜÅ =~ëáå β=ÄëáåαK=

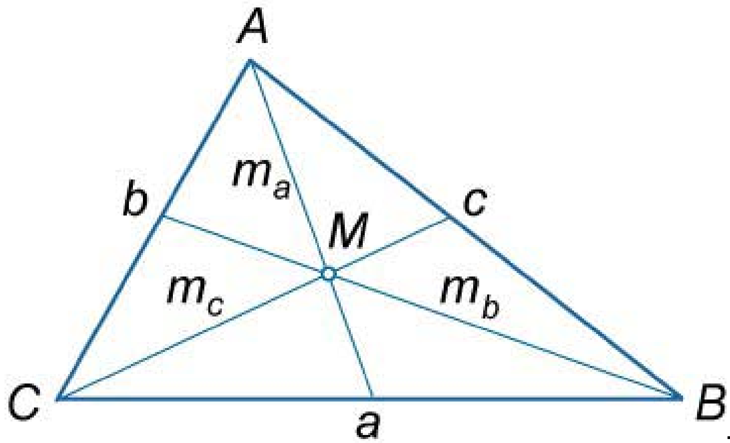
=
O
=
Ä
O +ÅO ~O
192. ã~ O −Q I==

ã
O = ~O +ÅO ÄO Ä O −Q I==

ã
O = ~O +ÄO ÅO Å O −Q K=

=

= =====



= **Figure 15.**

=

193. ^j=Oã~I=_j=OãÄI=`j=OãÅ=EcáÖKNRFK=P P P

=

194. íO ()I==~ = Ä()Å O

íO =Q~Åé()I==Ä ~()Å O

íO =Q~Äé()K=Å ()O

195. p=~Ü~ =ÄÜÄ =ÅÜÅ I==O O O

p= ~Äëáå γ= ~Å ëáåβ= ÄÅëáå αI==O O O

p= é()( )( )=EeÉêçå∞ë=cçêãìä~FI=

p =éêI==

p=~ÄÅI=Qo

p=OoOëáå αëáåβ ëáå γI=

p=éO í~åα í~åβí~åγ K=O O O

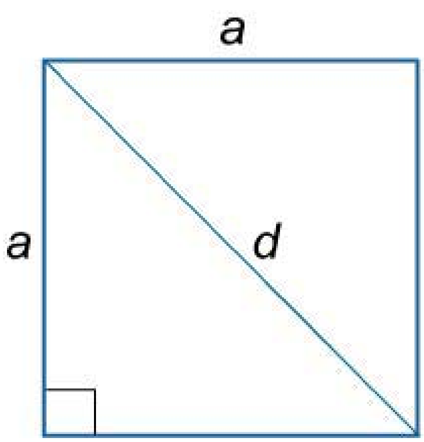
=

=

=

3.5 Square

páÇÉ=çÑ=~=ëèì~êÉW=~=
aá~Öçå~äW=Ç=
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê=
mÉêáãÉíÉêW=i=
^êÉ~W=p=
=



=

Figure 16.

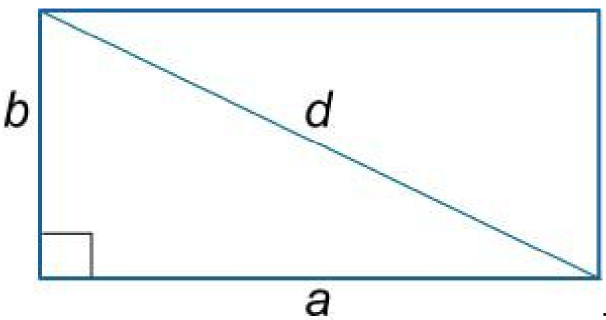
196. Ç=~ O==
197. o=
= Ç ~ O=O O

= 198. ê=~ =O

= 199. i=Q~=
= 200. p=~O=
=
=
=

3.6 Rectangle

=
páÇÉë=çÑ=~=êÉÅí~åÖäÉW=~I=Ä=
aá~Öçå~äW=Ç=
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o= mÉêáãÉíÉêW=i=
^êÉ~W=p=
=
=



==

Figure 17.
=
201. Ç = ~O +ÄO ==
202. o=Ç =O

203. ()=

=

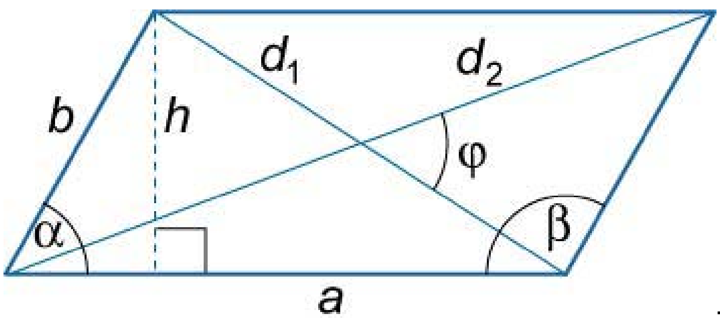
= 204. p=~Ä=
=
=
=

3.7 Parallelogram

=

páÇÉë=çÑ=~=é~ê~ääÉäçÖê~ãW=~I=Ä= aá~Öçå~äëW=Ç$_{N}$ Ç$_{O}$=
`çåëÉÅìíáîÉ=~åÖäÉëW= β**I= ^åÖäÉ=ÄÉíïÉÉå=íÜÉ=Çá~Öçå~äëW=**ϕ **=**
^äíáíìÇÉW=Ü==
mÉêáãÉíÉêW=i=
^êÉ~W=p=
=
=

= =====



= **Figure 18.** =

205. α+β=**NUM**°=

206. **Ç^{O}** +**Ç^{O}** =**O**(**~^{O}** +**Ä^{O}**)=$_{N\ O}$
=
207. **ÄÜ=**

208. ()=

=

= 209. p=~Ü=~Äëáå αI==

p=^{N}Ç$_{N}$Ç$_{O}$ ëáåϕK=$_{O}$

=

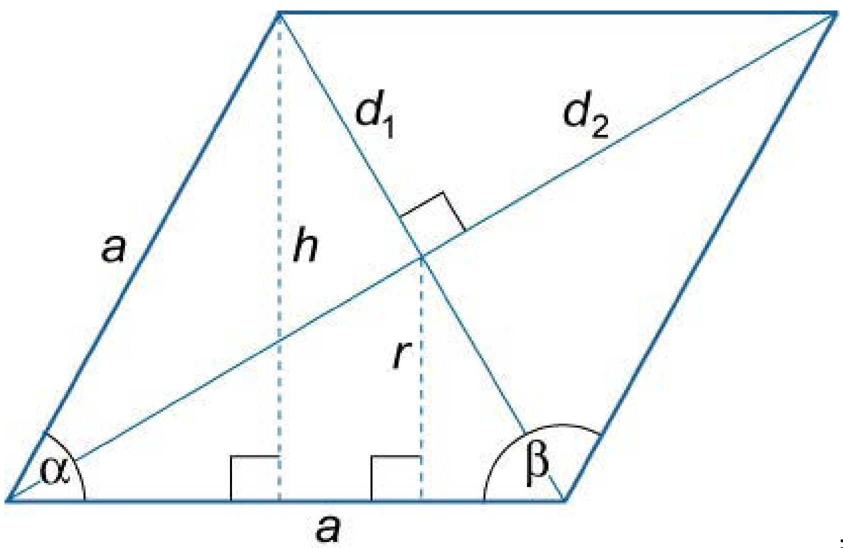
=

=

3.8 Rhombus

=
páÇÉ=çÑ=~=êÜçãÄìëW=~= aá~Öçå~äëW=Ç$_N$ Ç$_O$=
`çåëÉÅìíáîÉ=~åÖäÉëW= βI= ^äíáîÇÉW=e=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê= mÉêáãÉíÉêW=i=
^êÉ~W=p=
=
=

= =====



= Figure 19.

210. α+β=NUM°= =
211. Ç^O +Ç^O =Q~^{O}=$_{N\ O}$
=

212. Ü =~ ëáåα =^{ÇNÇO} =$_{O\sim}$
=
213. ê=^{Ü} =^{ÇNÇO} =$^{\sim}$ ëáå α=$_{O\ Q\sim\ O}$

= 214. i=Q~=
= 215. p=~Ü=~^{O}ëáå αI==

p=^{N}ÇNÇOK=O

=

=

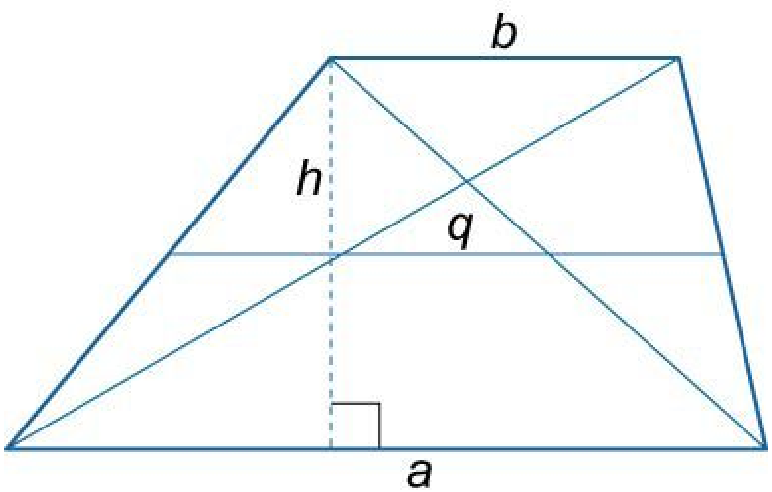
=

3.9 Trapezoid

=

_~ëÉë=çÑ=~=íê~éÉòçáÇW=~I=Ä= jáÇäáåÉW=è=
^äíáîÇÉW=Ü=
^êÉ~W=p=

=



==

Figure 20.

=

216.
è
+**Ä**
= **=O**

=

217. **p**=~+Ä **·Ü** =**èÜ=O**

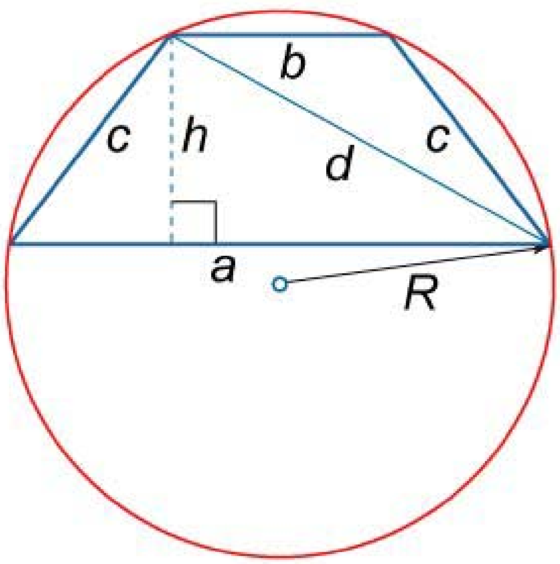
=

=

=

3.10 Isosceles Trapezoid

=
_~ëÉë=çÑ=~=íê~éÉòçáÇW=~I=Ä=
iÉÖW=Å=
jáÇäáåÉW=è=
^äíáîìÇÉW=Ü=
aá~Öçå~äW=Ç=
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o=
^êÉ~W=p=
=



==

Figure 21.
=
218.
è
+Ä
= =o

=

219. Ç = ~Ä+Å^{O} =

=

220. Ü ()Å −^{N} Ä ~^{O} =

Q

=

Å

~Ä

+

Å

O

221. o= =$_{()()}$OÅ−~+Ä OÅ+~−Ä

= **222.** p=$^{~+Ä}$ ·Ü =èÜ=$_{O}$

=

=

=

=

=

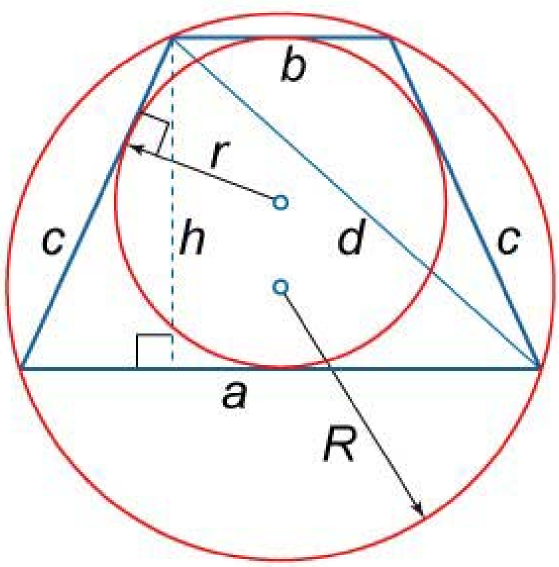
3.11 Isosceles Trapezoid with Inscribed Circle

=

_~ëÉë=çÑ=~=íê~éÉòçáÇW=~I=Ä=
iÉÖW=Å=
jáÇäáåÉW=è=
^äíáîìÇÉW=Ü=
aá~Öçå~äW=Ç=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=o=
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=ê= mÉêáãÉíÉêW=i=
^êÉ~W=p=

=

=



==

Figure 22.

=

223. ~ =OÅ=

=

224. è =~+Ä =Å=O

=

225. Ç^O =Ü^O +Å^O=

226. ê=^Ü ~Ä =_O O

=

ÅÇ

=

ÅÇ

=

Å

N

+

Å

O

227. o=^Å Ü^O +Å^O =^~ +Ä ~ +S+^Ä =_OÜ Qê O ~Ä =_OÜ U Ä ~ =

228. ()=QÅ=

=

229.

p=

() ~Ä=èÜ=ÅÜ=iê==O ·Ü= O O

=

=

=

3.12 Trapezoid with Inscribed Circle

=

_~ëÉë=çÑ=~=íê~éÉòçáÇW=~I=Ä=

i~íÉê~ä=ëáÇÉëW=ÅI=Ç=

jáÇäáåÉW=è=

^äíáíìÇÉW=Ü=

aá~Öçå~äëW=NIÇ=O

^åÖäÉ=ÄÉíïÉÉå=íÜÉ=Çá~Öçå~äëW=φ =

o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê=

o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o=

mÉêáãÉíÉêW=i=

^êÉ~W=p=

=

Figure 23. =

230. ~ +Ä=Å+Ç=

= **231.** è =~+Ä =Å +Ç =O O

232. () ( )=

=

= 233. p=~+Ä ·Ü =Å+Ç·Ü =èÜI==O O

p=NÇNÇO ëáåϕK=O

=

=

=

3.13 Kite

=

páÇÉë=çÑ=~=âáíÉW=~I=Ä=
aá~Öçå~äëW=$_{N}$IÇ=$_{O}$

^åÖäÉëW=
γ

$_{I}$ I =
mÉêáãÉíÉêW=i=
^êÉ~W=p=
=
=

Figure 24. =
234. α+β+**PSMO=**

235. ()=

=

= 236. p=^{ÇNÇO} =$_{O}$

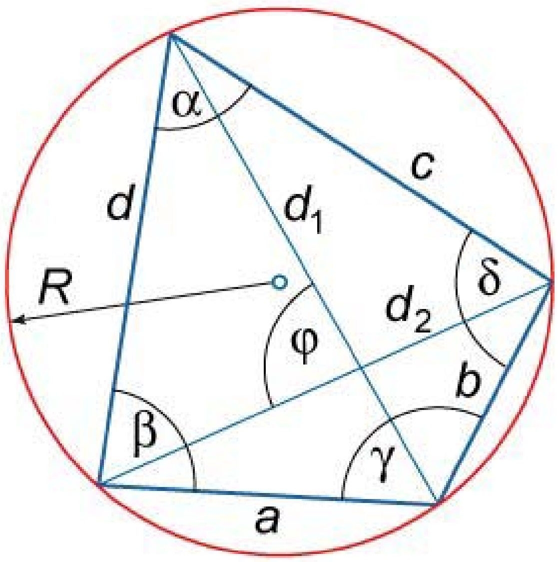
=

=

=

3.14 Cyclic Quadrilateral

páÇÉë=çÑ=~=èì~Çêáä~íÉê~äW=~I=ÄI=ÅI=Ç= aá~Öçå~äëW=Ç$_N$ Ç$_O$=
^åÖäÉ=ÄÉíïÉÉå=íÜÉ=Çá~Öçå~äëW=ϕ = fåíÉêå~ä=~åÖäÉëW=α^β
γ**I**δ**II=**
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o= mÉêáãÉíÉêW=i=
pÉãáéÉêáãÉíÉêW=é==
^êÉ~W=p=



=$_=$

Figure 25.
=

237. α+γ =β+δ=**NUM°=**
=
238. **míçäÉãó∞ë=qÜÉçêÉã=**
~Å =**Ç$_N$Ç$_O$=**
=
239. **i** =**~** +**Ä**+**Å**+**Ç=**
=

240.
o=

()(~Ç+ÄÅ)(~Ä+ÅÇ)I==Q ()( )( )( )

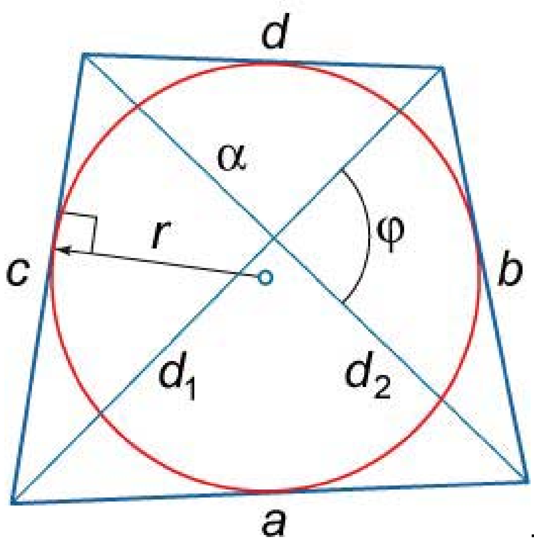
ïÜÉêÉ=é=i K=O
=
241. p=NÇNÇO ëáåϕI==O

p= ()( )( )( )I==

ïÜÉêÉ=é=i K=O
=
=

3.15 Tangential Quadrilateral

=
páÇÉë=çÑ=~=èì~Çêáä~íÉê~äW=~I=ÄI=ÅI=Ç= aá~Öçå~äëW=NIÇ=O
^åÖäÉ=ÄÉíïÉÉå=íÜÉ=Çá~Öçå~äëW=ϕ =
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê=
mÉêáãÉíÉêW=i=
pÉãáéÉêáãÉíÉêW=é==
^êÉ~W=p=
=
=



==

Figure 26. =

242. ~ +Å =Ä+Ç=
=

243. i=~+Ä+Å+Ç=O( ) ( )=

=
244.
ê
=

N O− ()( )O

Oé$^{I==}$ ïÜÉêÉ=é=^{i} K==$_{O}$

245. p=éê=^{N}Ç$_{N}$Ç$_{O}$ ëáå ϕ=$_{O}$

=

=

=

3.16 General Quadrilateral

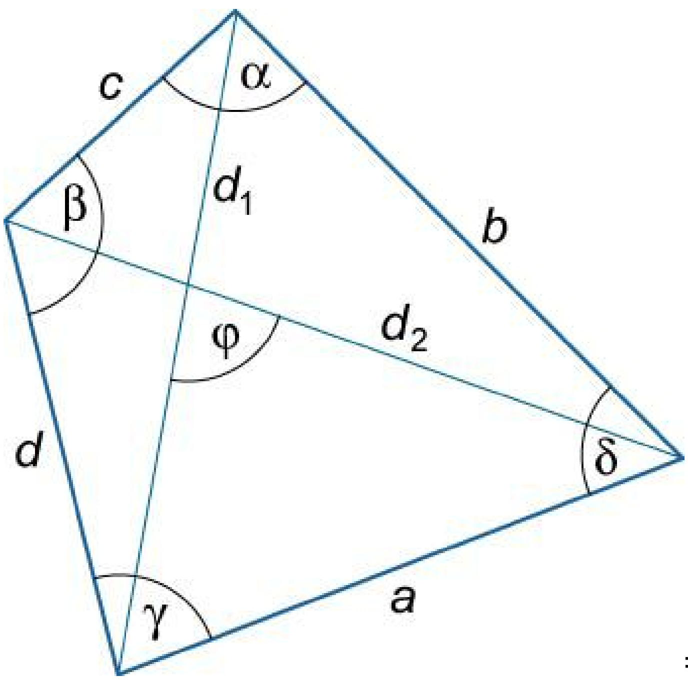
=

páÇÉë=çÑ=~=èì~Çêáä~íÉê~äW=~I=ÄI=ÅI=Ç= aá~Öçå~äëW=ÇN ÇO=
^åÖäÉ=ÄÉíïÉÉå=íÜÉ=Çá~Öçå~äëW=ϕ = fåíÉêå~ä=~åÖäÉëW=αII=
mÉêáãÉíÉêW=i=
^êÉ~W=p=

=

=

= =======



= **Figure 27.** =

246. $\alpha+\beta+\gamma+\delta=$**PSM** =

=

247. i =~ +**Ä**+**Å**+**Ç**=

248. p=**NÇNÇO ëáåϕ=O**

=

=

=

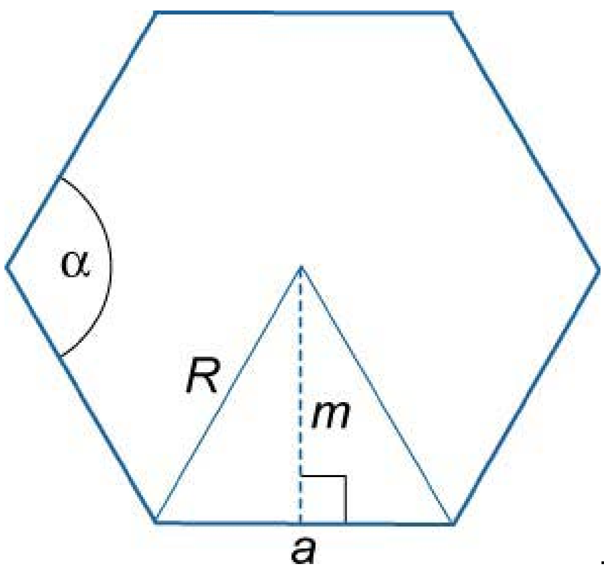
3.17 Regular Hexagon

=

páÇÉW=~=
fåíÉêå~ä=~åÖäÉW=α=
pä~åí=ÜÉáÖÜíW=ã=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê=
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o= mÉêáãÉíÉêW=i=
pÉãáéÉêáãÉíÉêW=é==
^êÉ~W=p=

=

=



==

Figure 28.

=

249. α=**NOM°=**

=

250. **ê=ã~ P =O**

251. **o=~= =**

252. **i=S~=**

= ~

O

253. p=éê=PP I==O

ïÜÉêÉ=é=i K=O

=

=

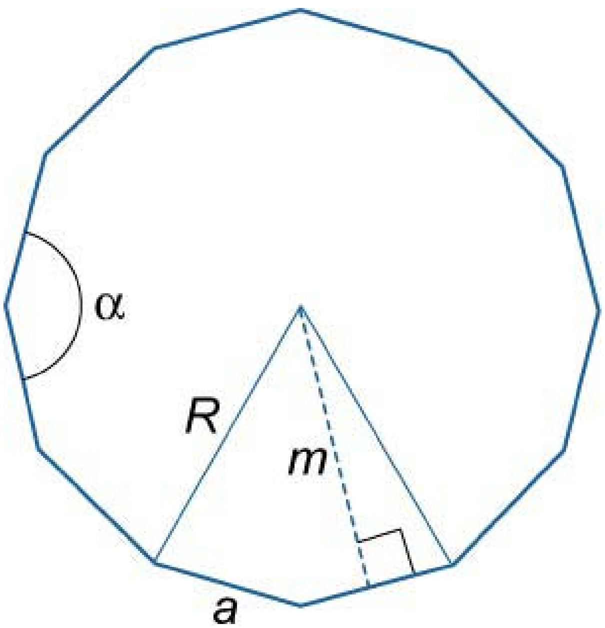
=

3.18 Regular Polygon

=

páÇÉW=~=
kìãÄÉê=çÑ=ëáÇÉëW=å=
fåíÉêå~ä=~åÖäÉW=α=
pä~åí=ÜÉáÖÜíW=ã=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê=
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o= mÉêáãÉíÉêW=i=
pÉãáéÉêáãÉíÉêW=é==
^êÉ~W=p=

=



==

Figure 29.

=

254. α=å−O·NUM°=O

=

255. α=å−O·NUM°=O
=

256. o=~ =
O
ëáå
π
å
= ~
=
o
O
−
~
O
257. ê=ã=π Q = Oí~åå

= 258. i=å~=
= 259. p= åoO πI==O ëáå å

p
=
éê
=
é
o
O_~O

Q I== ïÜÉêÉ=é=i K==O

=
=
=

3.19 Circle

=

o~ÇáìëW=o=

aá~ãÉíÉêW=Ç=

`ÜçêÇW=~=

pÉÅ~åí=ëÉÖãÉåíëW=ÉI=Ñ= q~åÖÉåí=ëÉÖãÉåíW=Ö=

`Éåíê~ä=~åÖäÉW=α= fåëÅêáÄÉÇ=~åÖäÉW=β = mÉêáãÉíÉêW=i=

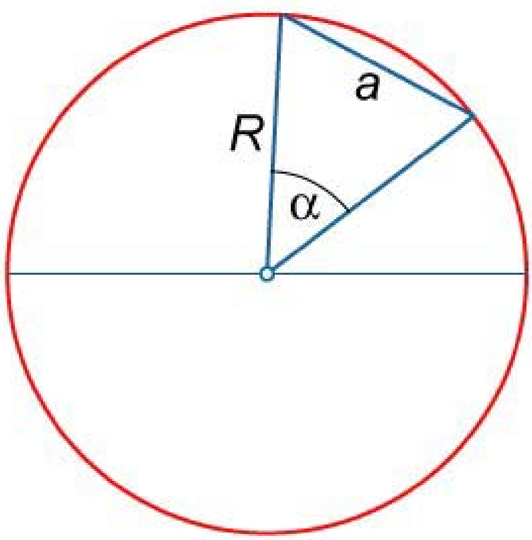
^êÉ~W=p=

=

=

260. ~ =Ooëáå$^{\alpha}$=$_{O}$

=

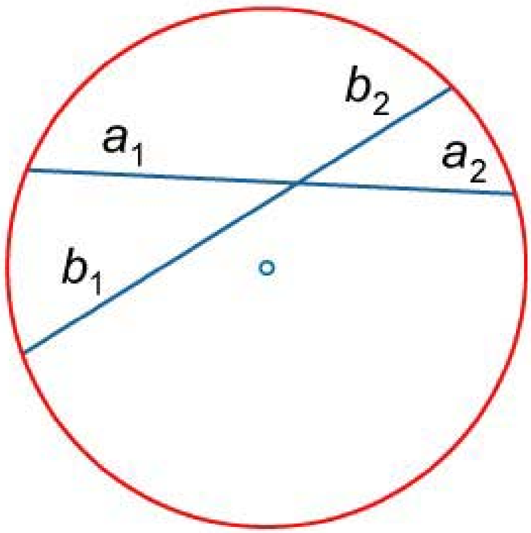


=$_{=}$

Figure 30.

261. ~$_{N}$~ $^{=}$Ä$_{N}$Ä$_{O}$=$_{O}$

=



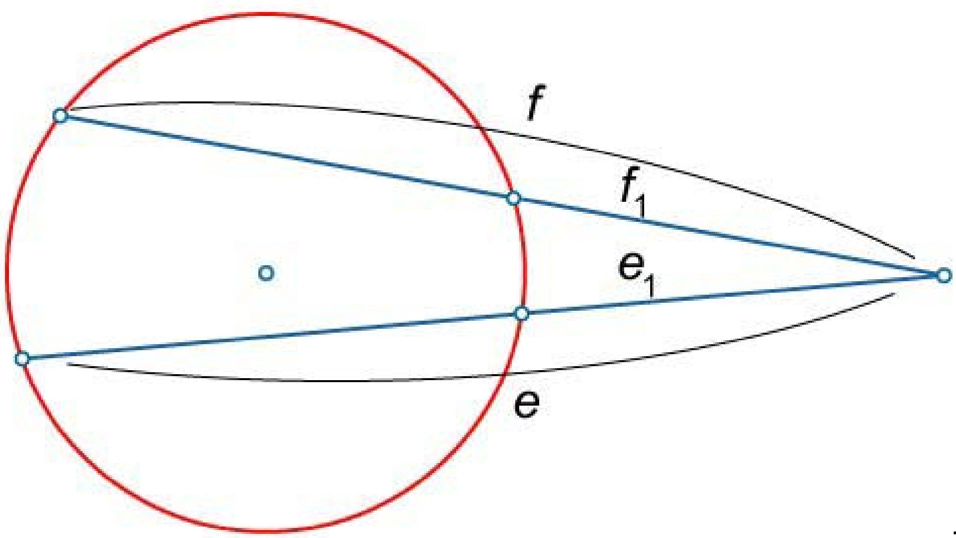
=$_=$

Figure 31. =

262. **ÉÉ$_N$=ÑÑ$_N$=**

=

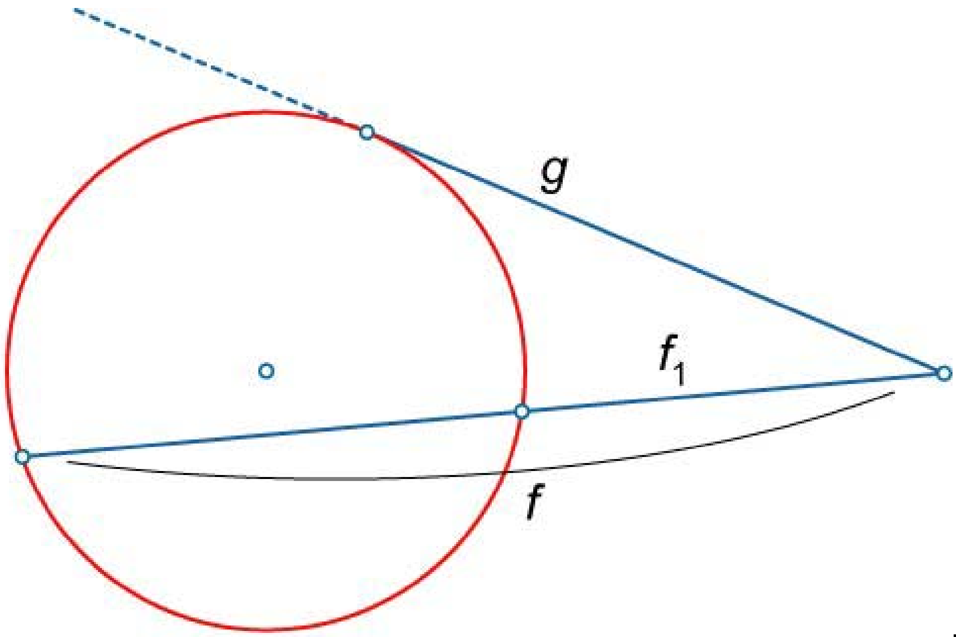
$_=$ =====



= **Figure 32.** =

263. **Ö^O=ÑÑ$_N$=**

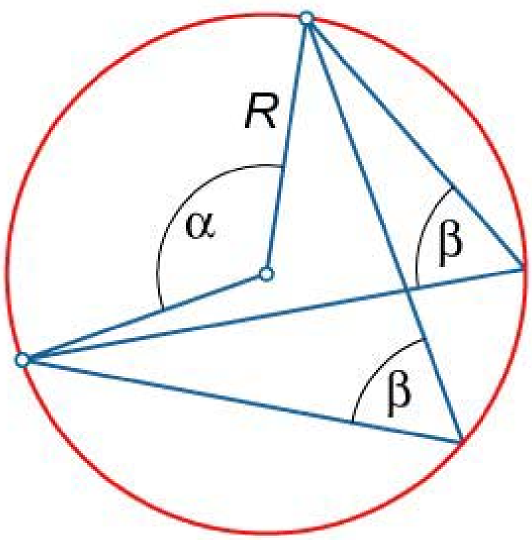
$_=$ =====



= **Figure 33.** =

264. β=α =O

=



==

Figure 34. =

265. i =**O**π**o**=π**Ç=**

=

266. p=π**o**O =πÇO io==Q =O

3.20 Sector of a Circle

=

o~Çáìë=çÑ=~=ÅáêÅäÉW=o=

^êÅ=äÉåÖíÜW=ë=

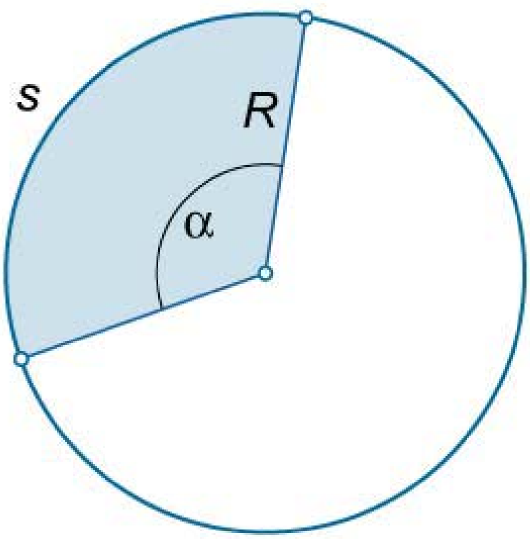
`Éåíê~ä=~åÖäÉ=Eáå=ê~Çá~åëFW=ñ=

`Éåíê~ä=~åÖäÉ=Eáå=ÇÉÖêÉÉëFW=α= mÉêáãÉíÉêW=i=

^êÉ~W=p=

=

=



==

Figure 35. =

267. ë=oñ=

=

268. ë =πoα= NUM°

=

269. i +Oo=

=

270. **p**=**oë** =**oOñ** =πoOα **==O O PSM°**
=

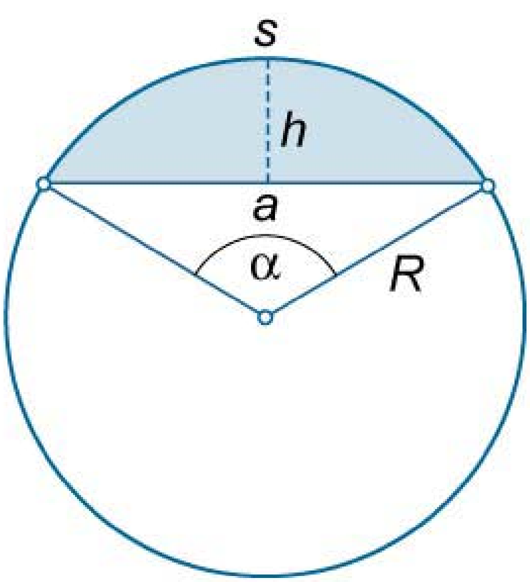
3.21 Segment of a Circle

=

o~Çáìë=çÑ=~=ÅáêÅäÉW=o=
^êÅ=äÉåÖíÜW=ë=
`ÜçêÇW=~=
`Éåíê~ä=~åÖäÉ=Eáå=ê~Çá~åëFW=ñ=
`Éåíê~ä=~åÖäÉ=Eáå=ÇÉÖêÉÉëFW=α=
eÉáÖÜí=çÑ=íÜÉ=ëÉÖãÉåíW=Ü= mÉêáãÉíÉêW=i=
^êÉ~W=p=

=

=



=$_{=}$

Figure 36.

=

271. ~ =**O OÜo** −**Ü**^{O} =

=

272. **Ü** =**o**−^{N} **Qo**^{O} −**~**^{O} **I=Ü**<**o=**$_{O}$

=

273. **i** +

ë **~** =

274.

p
=
N

()

[]

=

o

O □ απ □ **oO I==**

O NUM°O □□ ()□□O

p≈OÜ~K=P

=

=

=

3.22 Cube

=

bÇÖÉW=~==

aá~Öçå~äW=Ç=

o~Çáìë=çÑ=áåëÅêáÄÉÇ=ëéÜÉêÉW=ê=

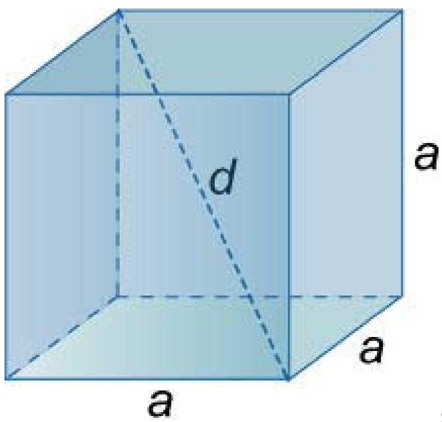
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ëéÜÉêÉW=ê= pìêÑ~ÅÉ=~êÉ~W=p=

sçäìãÉW=s=

=

=

= ===



= **Figure 37.** =

275. Ç=~ P=

=

276. ê=~ =O

277. o=~ P =O

= **278. p=S~O=**

=

279. s=~P==

=

=

=

3.23 Rectangular Parallelepiped

=

bÇÖÉëW=~I=ÄI=Å==

aá~Öçå~äW=Ç=

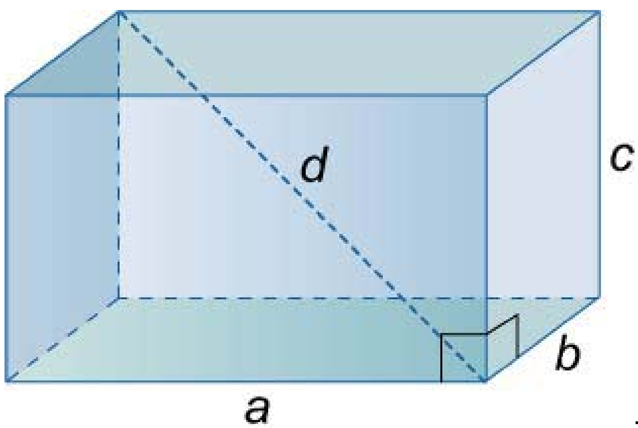
pìêÑ~ÅÉ=~êÉ~W=p=

sçäìãÉW=s=

=

=

= =====



= **Figure 38.** =

280. **Ç** = **~O +ÄO +ÅO =**

281. **()=**

=

= **282.** **s**=**~ÄÅ==**

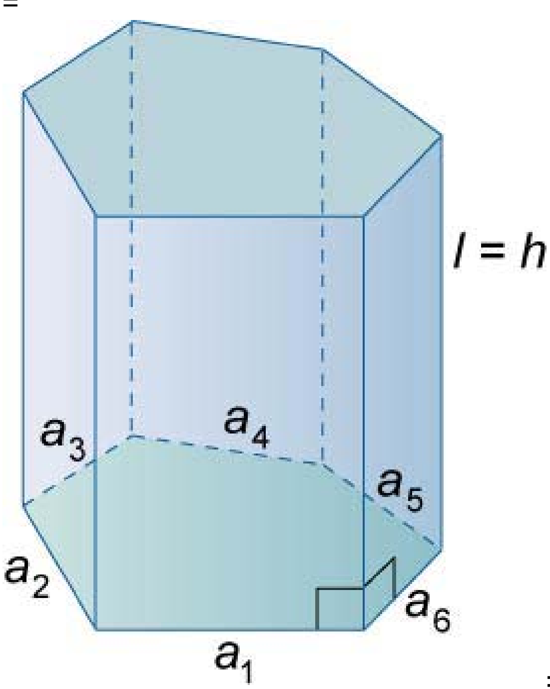
3.24 Prism

=
i~íÉê~ä=ÉÇÖÉW=ä=
eÉáÖÜíW=Ü=
i~íÉê~ä=~êÉ~W=

p= i
^êÉ~=çÑ=Ä~ëÉW=
p= _

qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p= sçäìãÉW=s=
=
=

= =====



= Figure 39. =

283.
p
=
+
p Op_K==i

=

284. **i~íÉê~ä=^êÉ~=çÑ=~=oáÖÜí=mêáëã=**

pi=N()O +P +K å ä=

=

285. i~íÉê~ä=^êÉ~=çÑ=~å=lÄäáèìÉ=mêáëã=

pi=éäI==

ïÜÉêÉ=é=áë=íÜÉ=éÉêáãÉíÉê=çÑ=íÜÉ=Åêçëë=ëÉÅíáçåK=

286. s =p_Ü= =

287. `~î~äáÉêáDë=mêáåÅáéäÉ==

dáîÉå= íïç= ëçäáÇë= áåÅäìÇÉÇ= ÄÉíïÉÉå= é~ê~ääÉä= éä~åÉëK= fÑ= ÉîÉêó=

éä~åÉ=Åêçëë=ëÉÅíáçå=é~ê~ääÉä=íç=íÜÉ=ÖáîÉå=éä~åÉë=Ü~ë=íÜÉ=ë~ãÉ=

~êÉ~=áå=ÄçíÜ=ëçäáÇëI=íÜÉå=íÜÉ=îçäìãÉë=çÑ=íÜÉ=ëçäáÇë=~êÉ=Éèì~äK= =

=

=

3.25 Regular Tetrahedron

=

qêá~åÖäÉ=ëáÇÉ=äÉåÖíÜW=~=

eÉáÖÜíW=Ü=

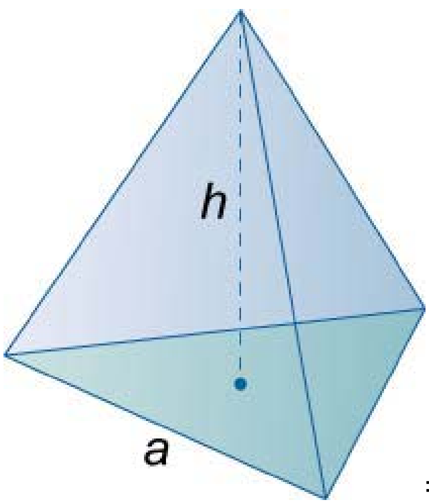
^êÉ~=çÑ=Ä~ëÉW=p_=

pìêÑ~ÅÉ=~êÉ~W=p=

sçäìãÉW=s=

=

=



=$_{=}$

Figure 40. =

288. Ü=^{O} ~=$_{P}$

289.

p

_

=

P~^{O} Q$^{=}$

= **290.** p= P~^{O}=

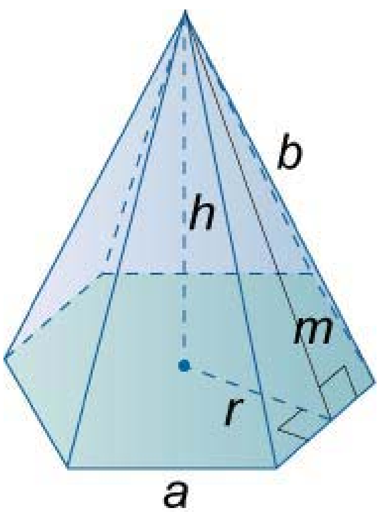
=

291.
S
=
^{N}P_ =$^{\sim P}$

P S O$^{K==}$
=
=
=

3.26 Regular Pyramid

=
páÇÉ=çÑ=Ä~ëÉW=~=
i~íÉê~ä=ÉÇÖÉW=Ä=
eÉáÖÜíW=Ü=
pä~åí=ÜÉáÖÜíW=ã==
kìãÄÉê=çÑ=ëáÇÉëW=å==
pÉãáéÉêáãÉíÉê=çÑ=Ä~ëÉW=é=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ëéÜÉêÉ=çÑ=Ä~ëÉW=ê=
^êÉ~=çÑ=Ä~ëÉW=p_=
i~íÉê~ä=ëìêÑ~ÅÉ=~êÉ~W=p$_i$=
qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=



==

Figure 41. =

292.

ã

=

Ä

O_~O

Q=

=

OëáåO π_~OQÄ

293. Ü =å =

O

ëáå

π

å

=

294. p_i =Nå~ã=Nå~ QÄO _~O =éã=O Q

=

295. p_ =éê=

296. p= +

=

p pi=_

=

297. s =Np_Ü =NéêÜ==P P

=

=

3.27 Frustum of a Regular Pyramid

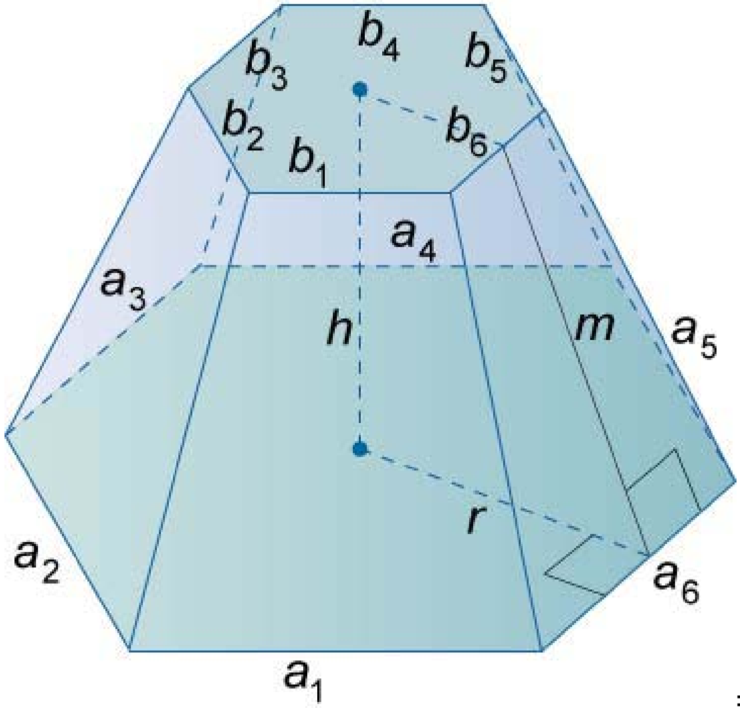
=

_~ëÉ=~åÇ=íçé=ëáÇÉ=äÉåÖíÜëW=□□□~NI~OI~PIKI~å =ÄNIÄOIÄPIKIÄå

eÉáÖÜíW=Ü=
pä~åí=ÜÉáÖÜíW=ã==
^êÉ~=çÑ=Ä~ëÉëW=

p I=p =
N O
i~íÉê~ä=ëìêÑ~ÅÉ=~êÉ~W=
p=
i
mÉêáãÉíÉê=çÑ=Ä~ëÉëW=
mI=m=
N O

pÅ~äÉ=Ñ~ÅíçêW=â=
qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=
=



= =

Figure 42.

=

298. ÄN =ÄO =ÄP =K=Äå =Ä =**â**=~N ~O ~P ~å ~

299. pO=**â**O=pN

300.

()

pN O =i =O

=

= **301.** **p**=**pi** +**pN** +**pO**=

=

302. ()=

P

=

303. **s =ÜpN □ Ä +□Ä□O□=ÜpN []=P □□N+~ □□~**

□□ □□P□ □

=

=

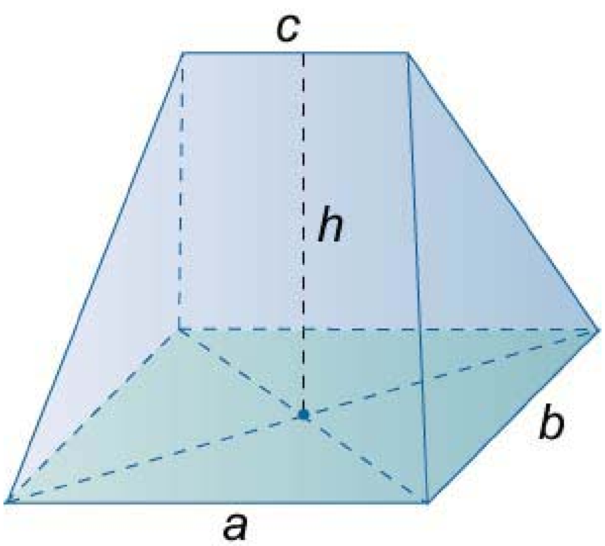
=

3.28 Rectangular Right Wedge

=
páÇÉë=çÑ=Ä~ëÉW=~I=Ä=
qçé=ÉÇÖÉW=Å=
eÉáÖÜíW=Ü=
i~íÉê~ä=ëìêÑ~ÅÉ=~êÉ~W=

p=
i
^êÉ~=çÑ=Ä~ëÉW=
p=
–

qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=



= =

Figure 43.
=

()=N ~ Å QÜO +ÄO +Ä ÜO + ~()ÅO

p =iO
=
305. p_=~Ä=
306. p= +
=
p pi=_
=
307.
s
=

()

ÄÜ=
S

=
=
=

3.29 Platonic Solids

=
bÇÖÉW=~=
o~Çáìë=çÑ=áåëÅêáÄÉÇ=ÅáêÅäÉW=ê=
o~Çáìë=çÑ=ÅáêÅìãëÅêáÄÉÇ=ÅáêÅäÉW=o= pìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=
308. cáîÉ=mä~íçåáÅ=pçäáÇë=

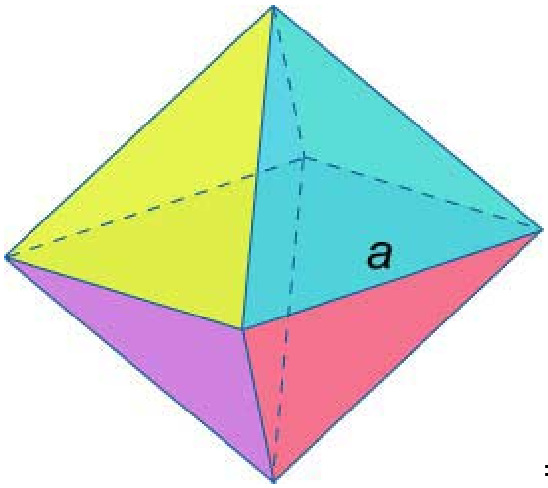
qÜÉ= éä~íçåáÅ= ëçäáÇë= ~êÉ= ÅçåîÉñ= éçäóÜÉÇê~= ïáíÜ=
Éèìáî~äÉåí=
Ñ~ÅÉë=ÅçãéçëÉÇ=çÑ=ÅçåÖêìÉåí=ÅçåîÉñ=êÉÖìä~ê=éçäóÖçåëK==

=
pçäáÇ= kìãÄÉê= kìãÄÉê= kìãÄÉê= pÉÅíáçå= çÑ=sÉêíáÅÉë
çÑ=bÇÖÉë= çÑ=c~ÅÉë=

qÉíê~ÜÉÇêçå== Q= S= Q= PKOR= `ìÄÉ= U= NO= S= PKOO=
lÅí~ÜÉÇêçå= S= NO= U= PKOT=
fÅçë~ÜÉÇêçå= NO= PM= OM= PKOT= açÇÉÅ~ÜÉÇêçå= OM= PM=
NO= PKOT= =
=

Octahedron

=



=$_=$

Figure 44.

=

309. ê=~ S =$_S$

=

310. o=~ O =$_O$

311. p=O~^{O} P= =

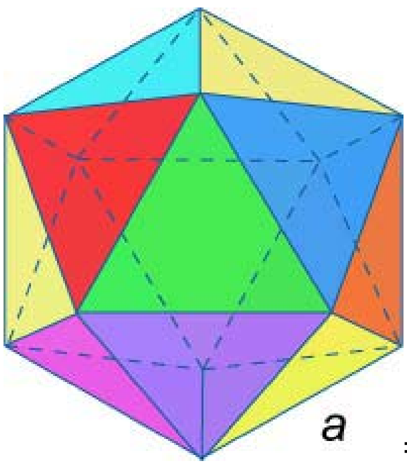
312. s =~P O =$_P$

=

=

Icosahedron

=



=
=

Figure 45. =

313. **ê**=~ **P(P+ R)**=**NO**

=

314. ()=

Q

=

315. p=R~^{O} P=

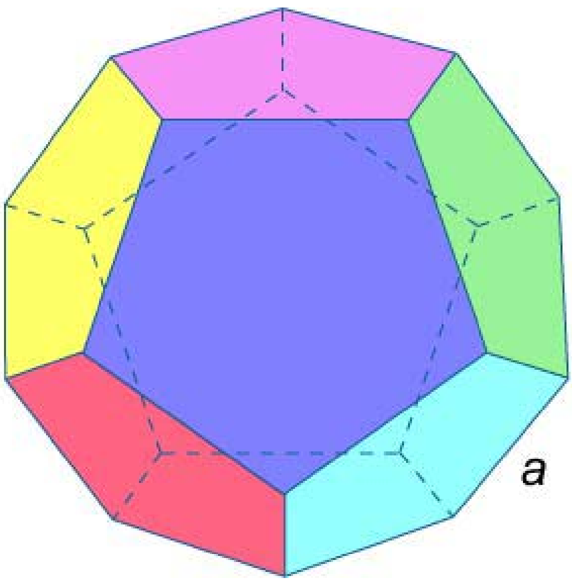
=

316. s =$^{R\sim P(+ R)}$=$_{NO}$

=

Dodecahedron

=



$=_=$

317. ()

Figure 46.

=

ê= =$_{O}$

=

318. o=~ $^{P(N+R)}$=$_{Q}$

=

319. $()=$

$=$

320. $s = \sim^{P}(+T\ R)_{=Q}$

$=$

$=$

$=$

3.30 Right Circular Cylinder

=

o~Çáìë=çÑ=Ä~ëÉW=o=

aá~ãÉíÉê=çÑ=Ä~ëÉW=Ç=

eÉáÖÜíW=e=

i~íÉê~ä=ëìêÑ~ÅÉ=~êÉ~W=

p= i

^êÉ~=çÑ=Ä~ëÉW=

p=

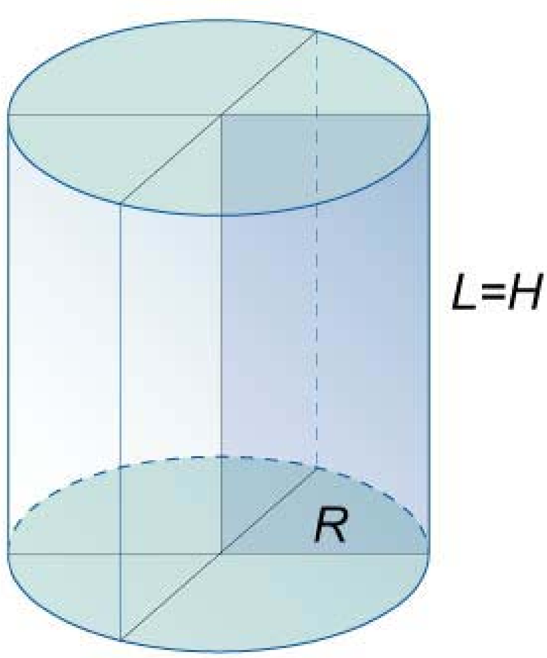
–

qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p= sçäìãÉW=s=

=

=

= =====



=

Figure 47. =

321. p_i π

O oe=

=

322. p=pi+O ()=πÇ +Ç =_ O

=

323. s =p_ πo^{O}e=

=

=

3.31 Right Circular Cylinder with an Oblique Plane Face

=

o~Çáìë=çÑ=Ä~ëÉW=o=

qÜÉ=ÖêÉ~íÉëí=ÜÉáÖÜí=çÑ=~=ëáÇÉW=

Ü =

N

qÜÉ=ëÜçêíÉëí=ÜÉáÖÜí=çÑ=~=ëáÇÉW=

Ü=

O

i~íÉê~ä=ëìêÑ~ÅÉ=~êÉ~W=

p=

i

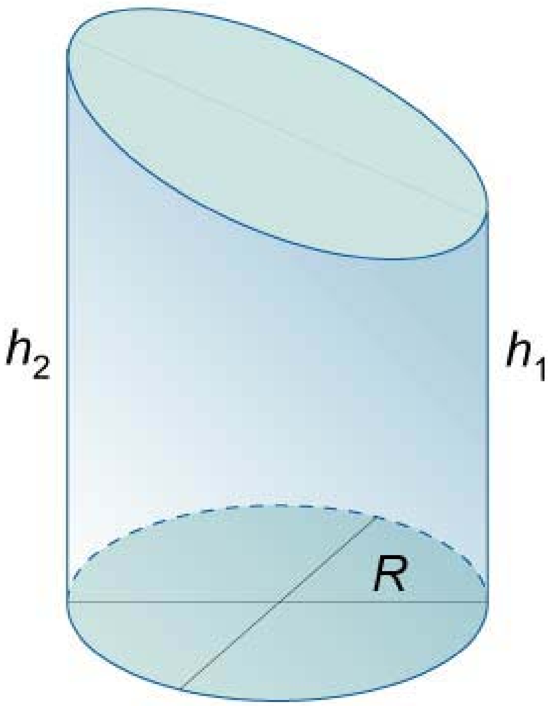
^êÉ~=çÑ=éä~åÉ=ÉåÇ=Ñ~ÅÉëW= p_ =

qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=

sçäìãÉW=s=

=

=



$=_=$

Figure 48. =

324. **pi ()$_N$ +Ü$_O$** =

=

325.

p

_

=

π

o

O +π**o o**^{O} +□□$^{ÜN\ ÜO\ □O□\ −}$

O □□$^=$

□$_O$ □ −**Ü**$_O$ □^{O} □

326. =$_i$ +**p**_ =π**o**□□**Ü**$_N$ **Ü**$_O$**pp=**^{N} □□ □$^{□□\ O\ □□}$ □

=

327. **()**^{πoO}

O$^{\mathbf{N}}$ **O =**

=

=

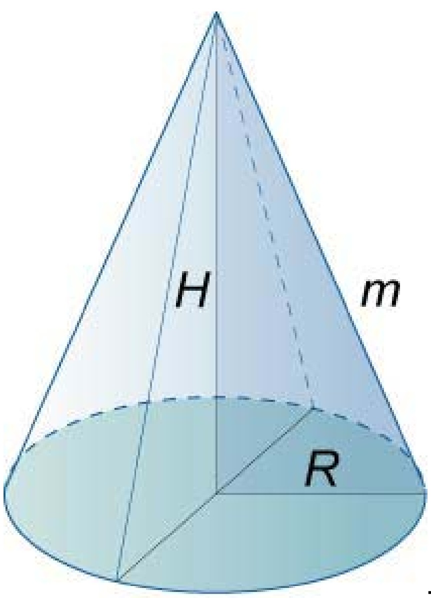
=

3.32 Right Circular Cone

o~Çáìë=çÑ=Ä~ëÉW=o=
aá~ãÉíÉê=çÑ=Ä~ëÉW=Ç=
eÉáÖÜíW=e=
pä~åí=ÜÉáÖÜíW=ã=
i~íÉê~ä=ëìêÑ~ÅÉ=~êÉ~W=p$_i$= ^êÉ~=çÑ=Ä~ëÉW=

p=
_

qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=
=



=

Figure 49.

328. **e**= **ã**^{O} −**o**^{O} =

=

329. **p$_i$** =π**oã** =^{πãÇ} =$_O$

=

330. **p_** π**o**^{O}=

=

331. p=pi + ()=NπÇ□□□ +Ç □=_ O O□□

=

332. s =Np_e=NπoOe=P P

=

=

=

3.33 Frustum of a Right Circular Cone

=
o~Çáìë=çÑ=Ä~ëÉëW=oI=ê=
eÉáÖÜíW=e=
pä~åí=ÜÉáÖÜíW=ã=
pÅ~äÉ=Ñ~ÅíçêW=â=
^êÉ~=çÑ=Ä~ëÉëW=

p I=p =
N O
i~íÉê~ä=ëìêÑ~ÅÉ=~êÉ~W=
p=
i

qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=
=

Figure 50. =

333. ()O =

=
334. º=â=ê
=
335.

pO = oO
pN êO =âO=
=

336. ()=i

=

337. p=p [o^{O} +ê^{O} +ã( )]=$_{N}$ +p$_{O}$ +p$_{i}$ =π

=

338. $()=$

P
=

339. **s =ÜpN □N+o +□o□O□=ÜpN []=P □□ê □□ ê**

□□ □□P□ □

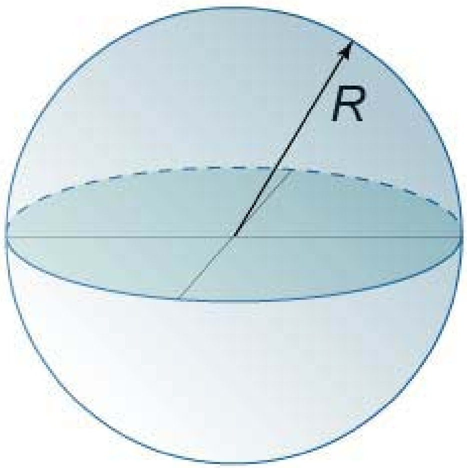
=

=

3.34 Sphere

=

o~ÇáìëW=o=
aá~ãÉíÉêW=Ç= pìêÑ~ÅÉ=~êÉ~W=p= sçäìãÉW=s= =



==

Figure 51. =

340. $p\ \pi o^{O}$=

=

341. $s = ^{Q}\pi o^{P} e = ^{N}\pi Ç^{P} = ^{N}po =_{PSP}$

=

=

=

3.35 Spherical Cap

o~Çáìë=çÑ=ëéÜÉêÉW=o=
o~Çáìë=çÑ=Ä~ëÉW=ê=
eÉáÖÜíW=Ü=
^êÉ~=çÑ=éä~åÉ=Ñ~ÅÉW=

p=
–

^êÉ~=çÑ=ëéÜÉêáÅ~ä=Å~éW= p`=
qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=

Figure 52. =
342.
o
=
êO+ÜO

O Ü=
=
343. **p_** πêO=
=
344. **p** (ÜO +êO)=` =π
=

345. p=p (Ü^{O} +Oê^{O}) (OoÜ+ê^{O})=_ +p` =π

=
346.
s
=
π
π

() (Pê^{O}+Ü^{O})=

S S

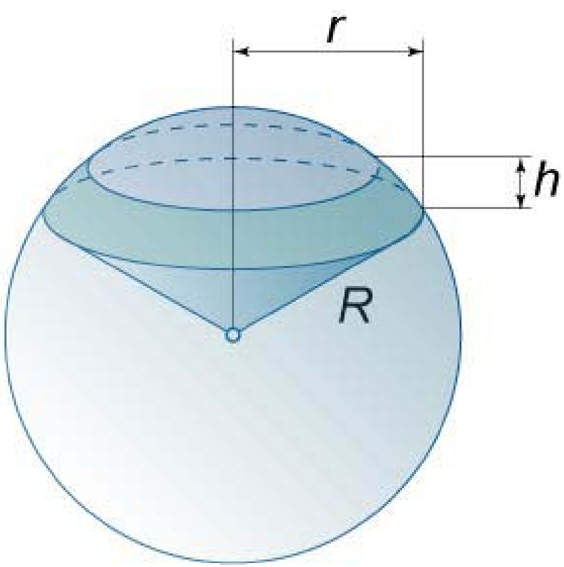
=
=
=

3.36 Spherical Sector

=

o~Çáìë=çÑ=ëéÜÉêÉW=o=
o~Çáìë=çÑ=Ä~ëÉ=çÑ=ëéÜÉêáÅ~ä=Å~éW=ê= eÉáÖÜíW=Ü=
qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=

======



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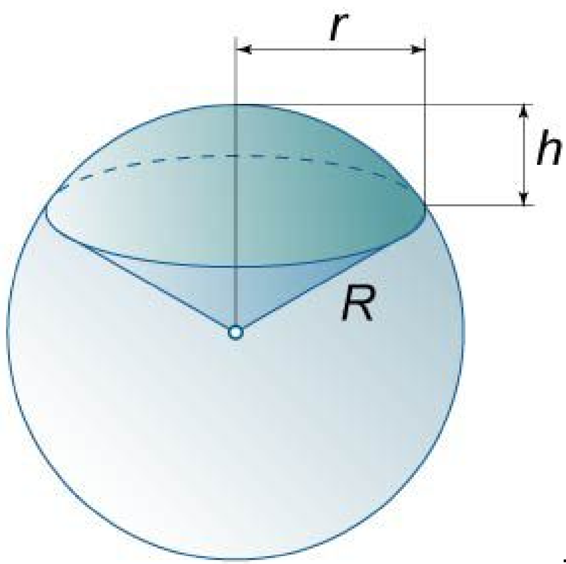


=$_=$

Figure 53. =

347. ()=

=

348. s =^{O}πo^{O}Ü=$_{P}$

=

kçíÉW= qÜÉ= ÖáîÉå= Ñçêãìä~ë= ~êÉ= ÅçêêÉÅí= ÄçíÜ= Ñçê= ±çéÉå≤=~åÇ= ±ÅäçëÉÇ≤=ëéÜÉêáÅ~ä=ëÉÅíçêK=

=

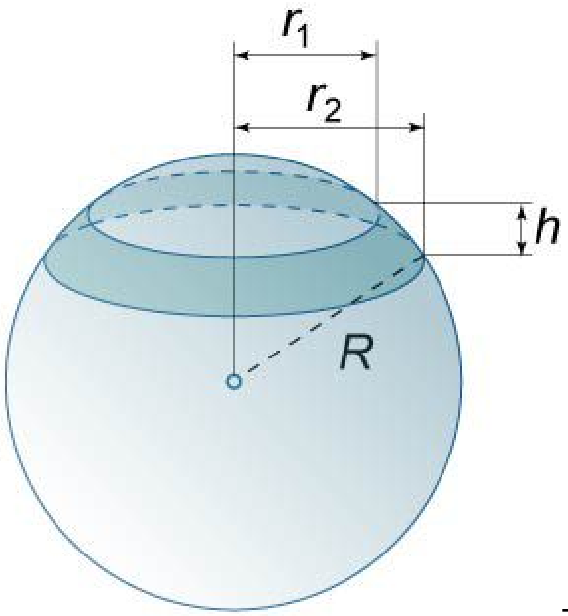
=

=

3.37 Spherical Segment

=
o~Çáìë=çÑ=ëéÜÉêÉW=o=
o~Çáìë=çÑ=Ä~ëÉëW=$_N$êI=$_O$ê=
eÉáÖÜíW=Ü=
^êÉ~=çÑ=ëéÜÉêáÅ~ä=ëìêÑ~ÅÉW=

p =
p
^êÉ~=çÑ=éä~åÉ=ÉåÇ=Ñ~ÅÉëW=
pI=p=
N O
qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
= =====



= **Figure 54.** =

349. **p_p** π**oÜ=**
=
350. **p**=**p** (**OoÜ**+**ê**^{O} +**ê**^{O})=$_p$ +**p**$_N$ +**p**$_O$ =π$_{N\ O}$ =

N

351. **()êN + êO +Ü =**

S

=
=
=

3.38 Spherical Wedge

=

o~ÇáìëW=o=

aáÜÉÇê~ä=~åÖäÉ=áå=ÇÉÖêÉÉëW=ñ=

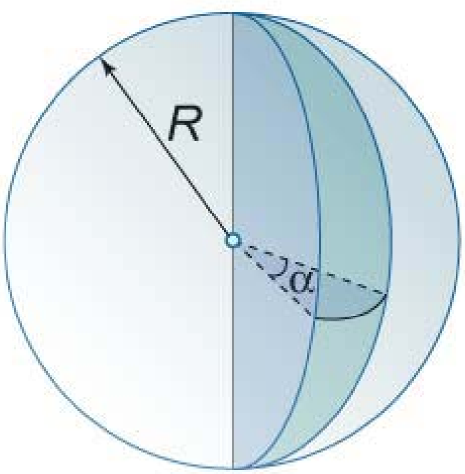
aáÜÉÇê~ä=~åÖäÉ=áå=ê~Çá~åëW=α= ^êÉ~=çÑ=ëéÜÉêáÅ~ä=äìåÉW=

p= i

qçí~ä=ëìêÑ~ÅÉ=~êÉ~W=p=

sçäìãÉW=s=

=



==

Figure 55. =

352.

p
i
=
πo^{O}
VM α=Oo^{O}ñ=

=

353.

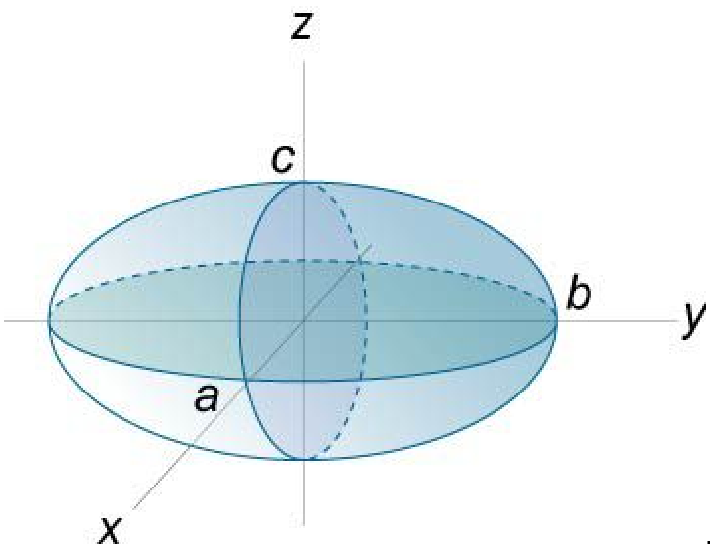
p
=
π
o
O + π**o**^{O}
VM α=π**o**^{O} +**Oo**^{O}**ñ= =**
354. **s** =π**oP** **Oo**^{P}**ñ=**$_{OTM}$α=$_{P}$

=
=
=

3.39 Ellipsoid

=
pÉãá-~ñÉëW=~I=ÄI=Å=
sçäìãÉW=s=

= =======



= Figure 56. =

Q
355. s π~ÄÅ=
P

=
=
=

Prolate Spheroid

=
pÉãá-~ñÉëW=~I=ÄI=Ä=E~>Ä F= pìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=
=

356.
=
π
Ä
 +
~ ~êÅëáåÉ
OpI== É
ïÜÉêÉ=
É
=
~O−ÄO
~ K=
=
357. s =QπÄO~=P

Oblate Spheroid

=
pÉãá-~ñÉëW=~I=ÄI=Ä=E~<Ä F= pìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉW=s=
=
=

 ~~êÅëáåÜÄÉ 358.
p
=
O
π
Ä

Ä
+
 ~
ÄÉL~ I==   Ä
O−~O
ïÜÉêÉ=É=Ä K=
=
359. s =QπÄO~=P

=
=
=

3.40 Circular Torus

=

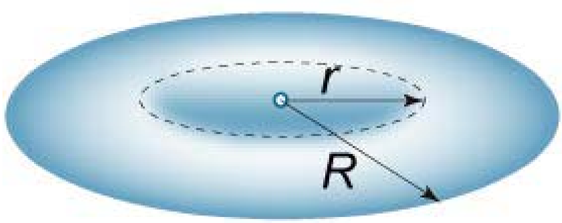
j~àçê=ê~ÇáìëW=o=

jáåçê=ê~ÇáìëW=ê=

pìêÑ~ÅÉ=~êÉ~W=p=

sçäìãÉW=s=

==



=**Picture 57.**

=

360. **p=Q^{O}oê=**

=

361. **s π^{O}oê^{O}=**

=

=

Chapter 4 Trigonometry

=
=
=
=
^åÖäÉëW=α**I=**β **=**
oÉ~ä=åìãÄÉêë=EÅççêÇáå~íÉë=çÑ=~=éçáåíFW=ñI=ó==
tÜçäÉ=åìãÄÉêW=â=
=
=

4.1 Radian and Degree Measures of Angles

=

362. N ê~Ç =NUM° ≈RT°NTDQR?=π

=

363. N°=π ê~Ç≈MKMNTQRP ê~Ç=NUM

=

364. ND=π ê~Ç ≈MKMMMOVN ê~Ç=NUM·SM

=

365.

N

?

=

π

NUM·PSMM ê~Ç≈MKMMMMMR ê~Ç=

=

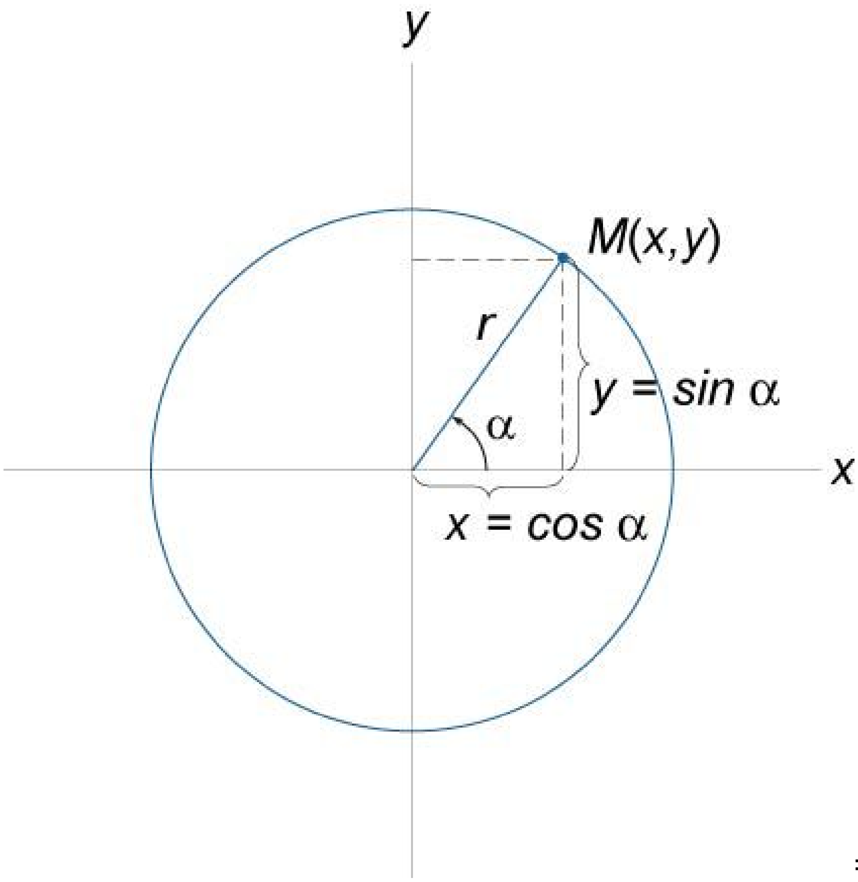
366. = ^åÖäÉ= M= PM= QR= SM= VM= NUM= OTM= PSM== EÇÉÖêÉÉëF=

^åÖäÉ= M=π =π =π =π = π=Pπ = πO= = = Eê~Çá~åëF= S Q P O O

4.2 Definitions and Graphs of Trigonometric Functions

=

==



= **Figure 58.** =

367. **ëáå** =ó =ê

=

368. **Åçë** =ñ =ê

=

369. **í~å** =ó =ñ

=

370. **Åçí** =ñ =ó

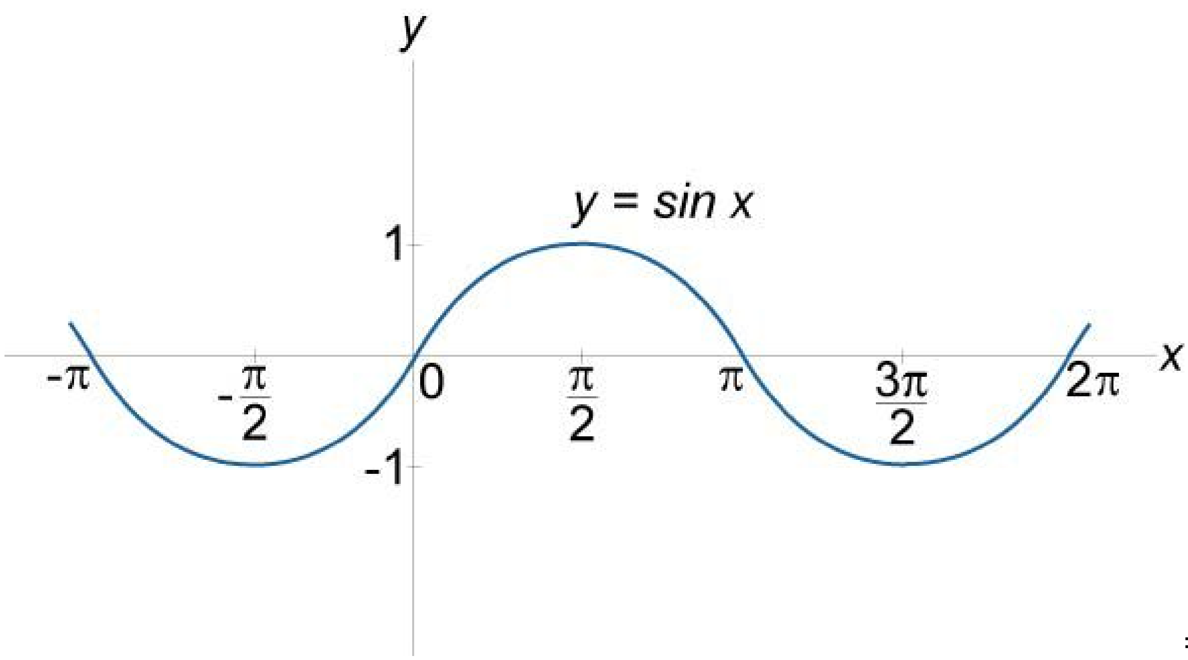
371. **ëÉÅ =ê =ñ =**

372. **ÅçëÉÅ =ê =ó**

=

373. **páåÉ=cìåÅíáçå=**
ó=ëáå ñ I=−N≤ëáåñ≤NK=

=

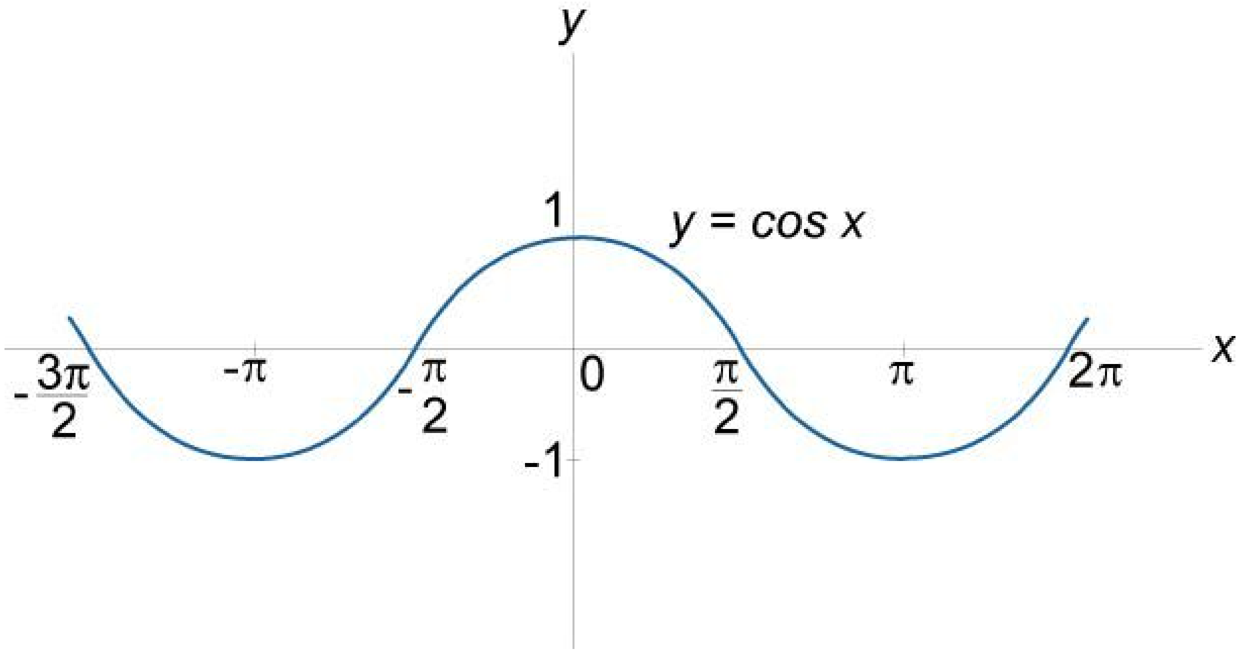


=

Figure 59.

=

374. **`çëáåÉ=cìåÅíáçå==**
ó=Åçë ñI=−N≤Åçëñ≤NK=

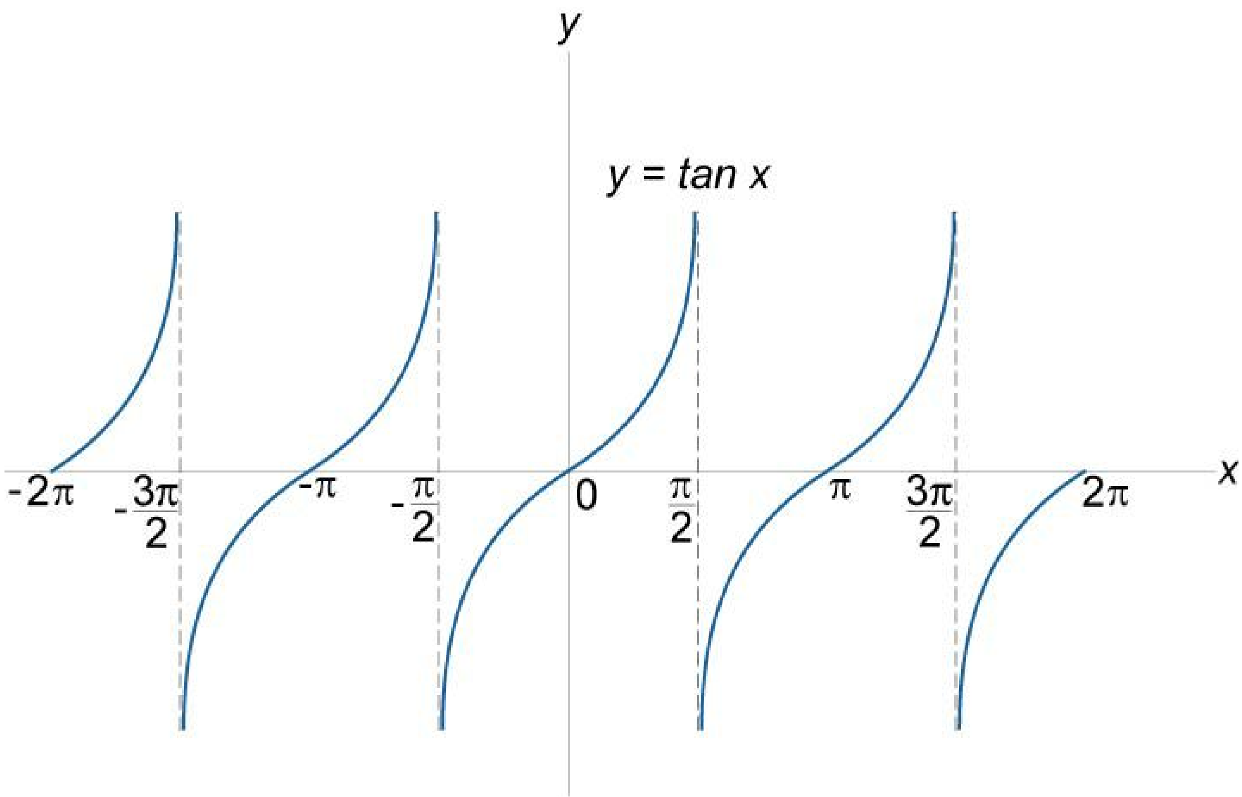


=
Figure 60.
=

375. **q~åÖÉåí=cìåÅíáçå=**

ó=í~å ñI=ñ()Oâ N$^{\pi}$I=−∞≤í~åñ ∞K=O

=



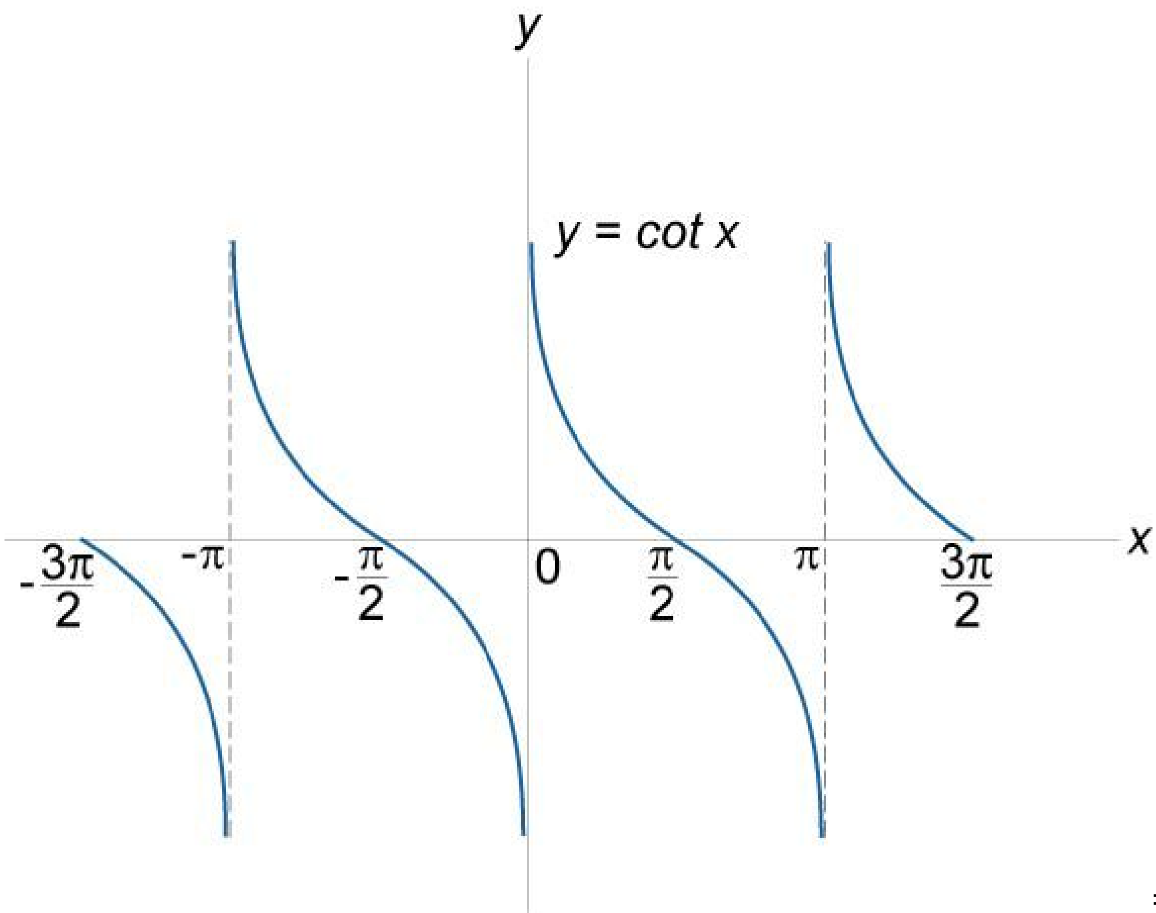
==

Figure 61.

376. **`çí~åÖÉåí=cìåÅíáçå==**

ó=Åçí ñI= πI==−∞≤Åçíñ ≤∞K=

=



==

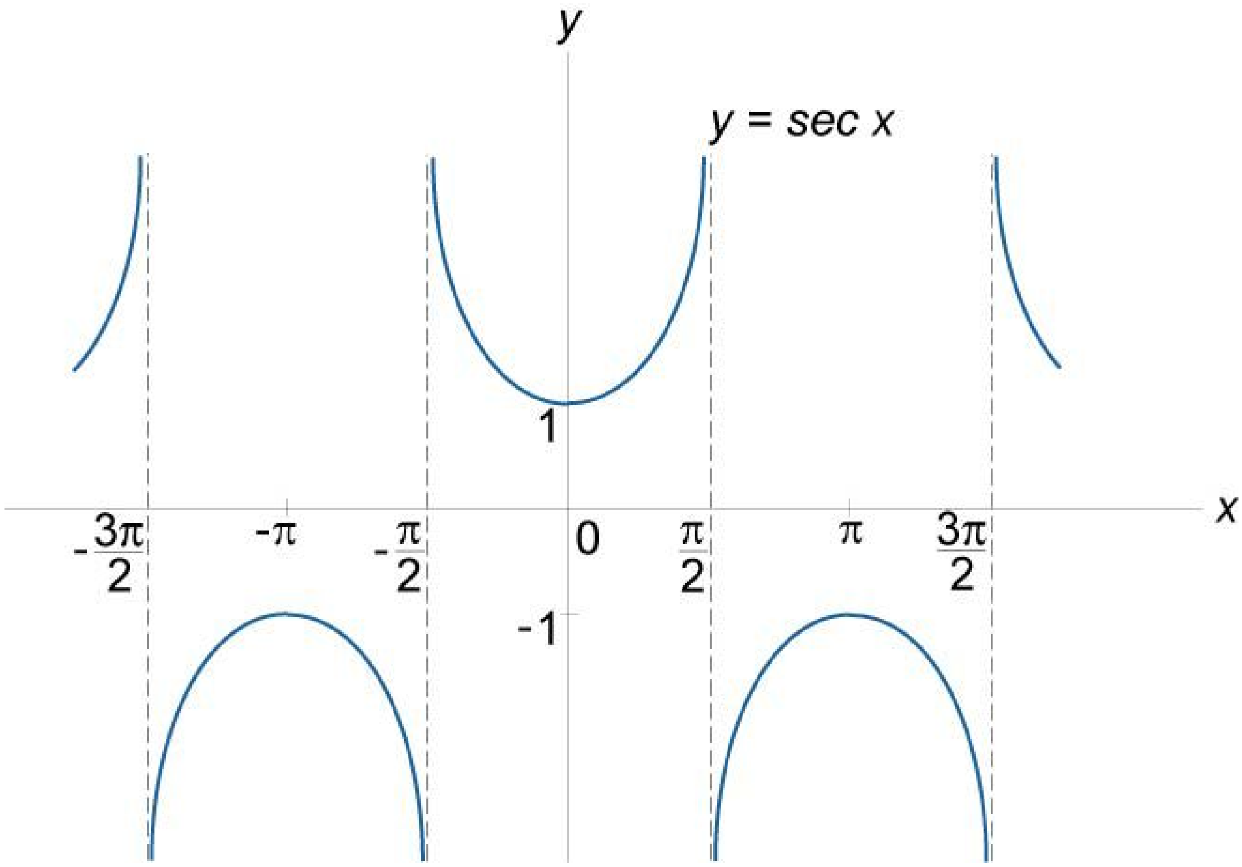
Figure 62.

=

377. pÉÅ~åí=cìåÅíáçå=

ó=ëÉÅñI=()πK=O

==



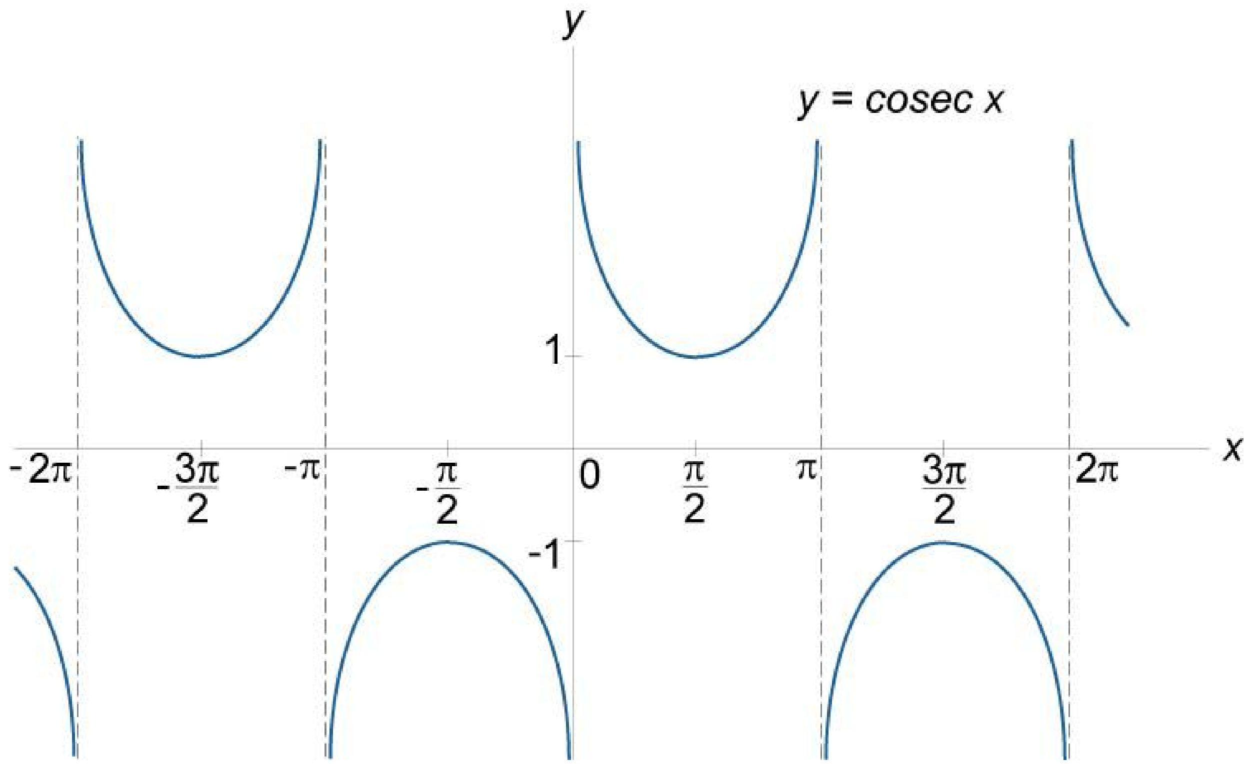
=

Figure 63.

=

378. **`çëÉÅ~åí=cìåÅíáçå==**

ó=**ÅçëÉÅ ñI=ñ** π **â K=**



= **Figure 64.**

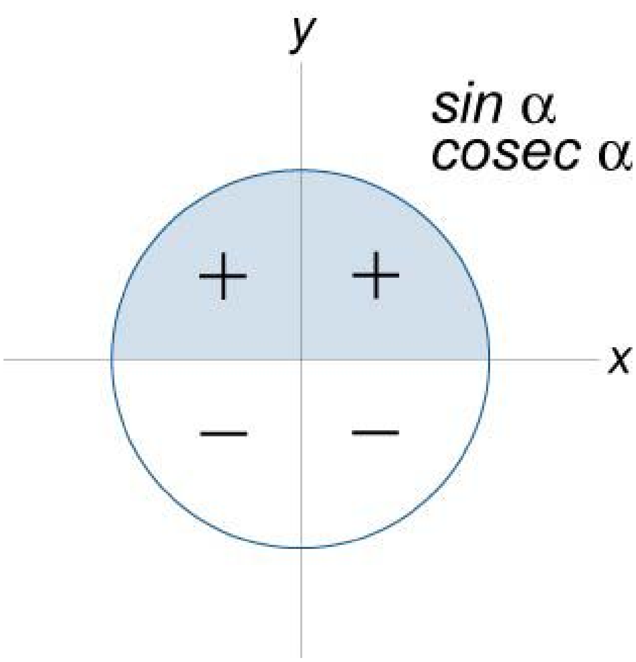
4.3. Signs of Trigonometric Functions

379. =nì~Çê~åí= **páå `çë q~å `çí pÉÅ `çëÉÅ=**

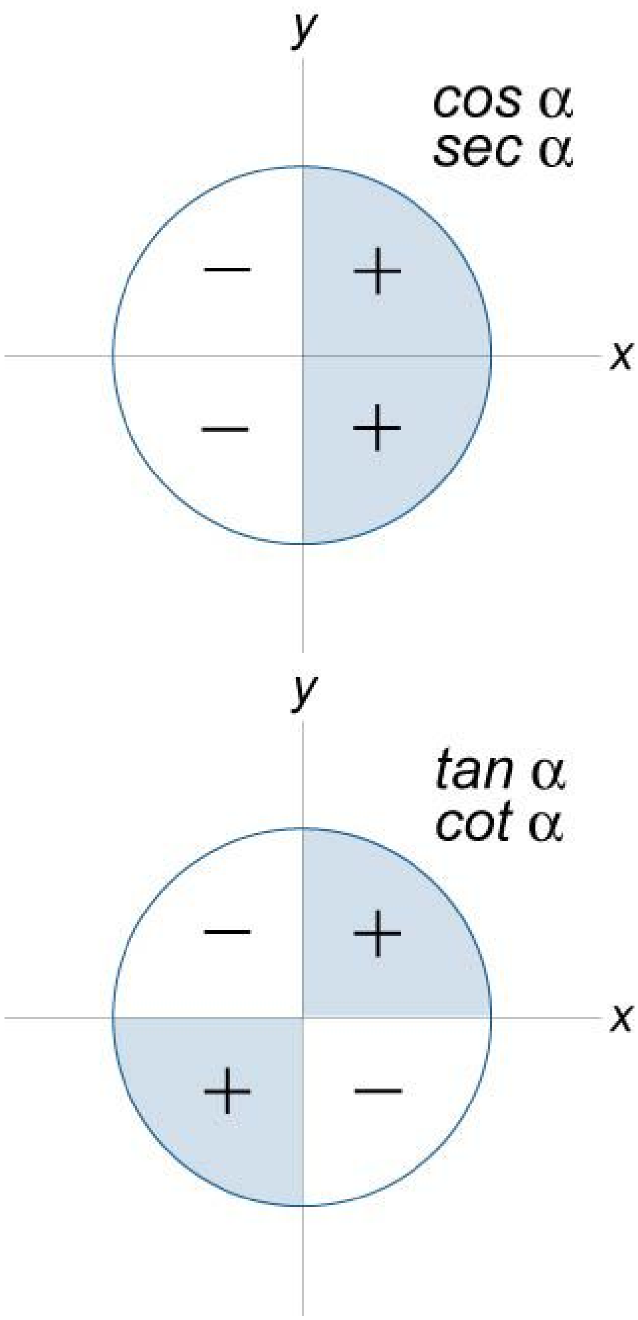
= α= α= α= α= α= α= **f= H= H= H= H= H= H= ff= H= = = = = H= fff= = =**
H= H= = = =

fs= =H= = =H= = =
=

380. =
=



=
=
=
=
=
=
=
=
=



=

Figure 65.

4.4 Trigonometric Functions of Common Angles

381. =
α°= αê~Ç= ëáå α= Åçëα= í~åα = Åçíα ëÉÅα= ÅçëÉÅ α= M= M= M= N=
M= ∞= N= ∞= PM=π = N = P =N = P=O = O=S O O P P
QR= π = O = O =N= N= O = O =Q O O
π = P = N = P=N = O=O =SM=P O O P P
VM=π = N= M= ∞= M= ∞= N=O

O π = P = −N= − P = −N O=O =NOM=P O O P P NUM= π=M= −N= M=
∞=N− = ∞= OTM=Pπ = −N= M= ∞= M= ∞=N− =O

PSM= πO = M= N= M= ∞= N= ∞= =
=
=
=
=
=
=
=
=
=
=
=

382. =
α°= αê~Ç = ëáå α= Åçëα= í~åα= Åçíα=
NR=π = S− O =S+ O = O− P = O+ P =NO Q Q
NU= π = R−N= NM+O R R−O R= R+O R =NM Q Q R

PS=
π = NM−O R R+N= NM−O RR+N NM−O RR Q Q R +N =

NM −O R RQ=

Pπ = R+N= NM−O R R+N

NM

Q

Q

NM

−

O

R

R+N = R−O R TO= Oπ = NM+O R R−N= R+O R = RR Q Q =

TR= Rπ = S+ O= S− O= O+ P= O− P =NO Q Q

=

=

=

4.5 Most Important Formulas

=

383. **ëáå**^{O} α+**Åçë**^{O} α=**N=**

=

384. **ëÉÅ**^{O} α−**í~å**^{O} α=**N=**

=

385. **ÅëÅ**^{O} α−**Åçí**^{O} α=**N=**

=

386. **í~å**α =$^{\text{ëáå α}}$ **=**$_{\text{Åçëα}}$

387. **Åçí**α =$^{\text{Åçëα}}$ **=**$_{\text{ëáå α}}$

=

388. **í~å**α·**Åçí**α=**N=**

=

389. **ëÉÅ**α=$^{\text{N}}$ **=**$_{\text{Åçëα}}$

=

390. **ÅçëÉÅ**α=$^{\text{N}}$ **=**$_{\text{ëáå α}}$

=

=

=

4.6 Reduction Formulas

=

391. = β = **ëáå**β= **Åçë**β= **í~å**β= **Åçí**β= = −α= −**ëáå** α= + **Åçë**α= − **í~å**α= −**Åçí**α=

VM °−α= + **Åçë**α= + **ëáå** α= + **Åçí**α= + **í~å**α = **VM**°+α= + **Åçë**α= −**ëáå** α= −**Åçí**α= − **í~å**α= **NUM**°−α + **ëáå** α= −**Åçë**α= − **í~å**α= −**Åçí**α= **NUM**°+α −**ëáå** α= −**Åçë**α= + **í~å**α = + **Åçí**α= **OTM**°−α −**Åçë**α= −**ëáå** α= + **Åçí**α= + **í~å**α = **OTM**°+α −**Åçë**α= + **ëáå** α= −**Åçí**α= − **í~å**α= **PSM**°−α −**ëáå** α= + **Åçë**α= − **í~å**α= −**Åçí**α=

=

PSM°+α + **ëáå** α= + **Åçë**α= + **í~å**α = + **Åçí**α=

=

=

=

=

4.7 Periodicity of Trigonometric Functions

=

392. () αI=éÉêáçÇ=π=çê= °PSM K=

=

393. () αI=éÉêáçÇ=π=çê= °PSM K=

=

394. í~å()å αI=éÉêáçÇ=π=çê= °

í~å NUM K=

=

395. () αI=éÉêáçÇ=π=çê= °K=

=

=

=

4.8 Relations between Trigonometric Functions

=

396. ëáåα=± N−Åçë^{O} α =±^{N}() =OÅçë^{O}□ π□−N=O □□ O − Q□□

Oí~å^{α}
= =^{O} =
N
+
í~å
O^{α}
O
=

397. Åçëα=± N−ëáå^{O} α =±^{N}() =OÅçë^{O} α −N=O O

N −í~å^{O} α
= =^{O} =

N
+
í~å
O α
O
=
398.
í~å
α
=
ëáåα =± ëÉÅ^{O} α−N=ëáåOα
N+ÅçëOα=N−ÅçëOα =Åçëα ëáåOα Oí~å^{α} = =± N−ÅçëOα = O =N+ÅçëOα N+ í~åO^{α}
O
=

399.
Åçí
α
=
Åçëα=± **ÅëÅO** α−**N**=**N+ÅçëO**α= **ëáåO**α **N−ÅçëO**α=**ëáå**α **ëáåO**α
N−í~åO α
=
=
±
N+ÅçëOα= **O**
α=**N−ÅçëO**α**Oí~å O**
=
N+í~åO α **400.** **ëÉÅ**α=**N** =± **N+í~åO** α =**O** **=Åçë**α **N−í~åO** α **O =**
401. **ÅëÅ**α=**N** =± **N+ÅçíO** α =
N
+
í~å
O α **O** α=**ëáå**α**Oí~å O**

=
=
=

4.9 Addition and Subtraction Formulas

=

402. ()=ëáåαÅçëβ+ëáåβÅçëα=

=

403. ()=ëáåαÅçëβ−ëáåβÅçëα=

=

404. ()=ÅçëαÅçëβ−ëáåαëáåβ=

=

405. ()=ÅçëαÅçëβ+ëáåαëáåβ=

406. ()=í~åα+í~åβ =N−í~åαí~åβ

=

407. ()=í~åα−í~åβ =N+í~åαí~åβ

=

408. ()=N−í~åαí~åβ=í~åα+í~åβ

=

409. ()=N+í~åαí~åβ=í~åα−í~åβ

=

=

=

4.10 Double Angle Formulas

=

410. ëáå Oα=Oëáå α·Åçëα=

=

411. ÅçëOα=ÅçëOα−ëáåOα=N−OëáåO α=OÅçëO α−N=

=

412.

í~å

O

α

=

Oí~åα O

N−í~åO α= Åçíα−í~å α=

=

413. ÅçíOα=ÅçíOα−N=Åçíα−í~å α=OÅçí α O

=

=

=

=

=

4.11 Multiple Angle Formulas

=

414. ëáåPα=Pëáå α−QëáåPα=PÅçëOα·ëáåα−ëáåP α=

=

415. ëáå Qα=Qëáåα·Åçëα−UëáåP α·Åçëα=

=

416. ëáå Rα=Rëáå α−OMëáåPα+NSëáåR α=

=

417. ÅçëPα=QÅçëP α−PÅçëα=ÅçëP α−PÅçëα·ëáåOα=

=

418. ÅçëQα=UÅçëQα−UÅçëO α+N=

=

419. ÅçëRα =NSÅçëR α−OMÅçëPα+RÅçëα=

=

420. í~åPα=Pí~åα−í~åPα=N−Pí~åOα

=

421.

í~å

Q

α

=

Qí~åα−Qí~åP α

N−Sí~åO α+í~åQ α=

=

422. í~åRα= í~åR α−NMí~åP α+Rí~åα=N−NMí~åO α+Rí~åQ α

=

423. ÅçíPα= ÅçíP α−PÅçí α=PÅçíO α−N

=

424. ÅçíQα =N−Sí~åO α+í~åQ α==Qí~å α−Qí~åP α

425. ÅçíRα= N−NMí~åOα+Rí~åQα =í~åR α−NMí~åP α+Rí~åα

=

=

=

4.12 Half Angle Formulas

=

426. ëáå α=± N−Åçëα=O O

=

427. Åçëα=± N+Åçëα=O O

=

428.

í~å

α=± N−Åçëα= ëáå α

N+Åçëα=N−Åçëα =ÅëÅ α−Åçíα=O N+Åçëα ëáå α

=

429.

Åçí

α=± N+Åçëα= ëáå α

N −Åçëα=N+Åçëα =ÅëÅ α+Åçí α=O N−Åçëα ëáå α

=

=

=

4.13 Half Angle Tangent Identities

=
Oí~åα 430. ëáå α=O = N
+
í~å
Oα O

N −í~åO α 431. Åçëα=O =

N
+
í~å
O α O =
Oí~åα 432. í~åα =O = N
−
í~å
Oα O =
N−í~åO α
433. Åçíα =O =
O
í~å
α

O
=
=
=

4.14 Transforming of Trigonometric Expressions to Product

=

434. **ëáå** α+**ëáå**β=**Oëáå**$^{\alpha+\beta}$**Åçë**$^{\alpha-\beta}$**=**$_{\mathbf{O\ O}}$

=

435. **ëáå** α−**ëáå**β=**OÅçë**$^{\alpha+\beta}$**ëáå**$^{\alpha-\beta}$**=**$_{\mathbf{O\ O}}$

=

436. **Åçë**α+**Åçë**β=**OÅçë**$^{\alpha+\beta}$**Åçë**$^{\alpha-\beta}$**=**$_{\mathbf{O\ O}}$

=

437. **Åçë**α−**Åçë**β=−**Oëáå**$^{\alpha+\beta}$**ëáå**$^{\alpha-\beta}$**=**$_{\mathbf{O\ O}}$

438. **í~åα+í~åβ=**$^{\text{ëáå( )}}$ **=**$_{\text{Åçëα·Åçëβ}}$

=

439. **í~åα−í~åβ=**$^{\text{ëáå( )}}$ **=**$_{\text{Åçëα·Åçëβ}}$

=

440. **Åçíα+Åçíβ=**$^{\text{ëáå( )}}$ **=**$_{\text{ëáåα·ëáåβ}}$

=

441. **Åçíα−Åçíβ=**$^{\text{ëáå( )}}$ **=**$_{\text{ëáå α·ëáåβ}}$

=

442. **Åçë**α+**ëáå** α = **OÅçë**□□$^{\pi}$□ α$^{□}$ = **Oëáå**□□$^{\pi}$□ α$^{□}$**=**$_{\mathbf{Q}\ □□\ \mathbf{Q}\ □□}$

=

443. **Åçë**α−**ëáå** α = **Oëáå**□□$^{\pi}$□ α$^{□}$= **OÅçë**□□$^{\pi}$□ α$^{□}$**=**$_{\mathbf{Q}\ □□\ \mathbf{Q}\ □□}$

=

444. í~åα+Åçíβ=Åçë( ) =Åçëα·ëáåβ

=

445. í~åα−Åçíβ=−Åçë( ) =Åçëα·ëáåβ

=

446. N+Åçëα=OÅçë^{Oα}=$_{O}$

=

447. N−Åçëα =Oëáå^{Oα} =$_{O}$

448. N+ëáå α =OÅçë$^{O α}$=Q − O

=

449. N−ëáå α= Oëáå$^{O α}$=Q − O

=

=

=

4.15 Transforming of Trigonometric Expressions to Sum

=

450. ëáåα·ëáåβ=Åçë( ) ( )=O

=

451. Åçëα·Åçëβ=Åçë( ) ( )=O

=

452. ëáåα·Åçëβ=ëáå( ) ( )=O

=

453. í~åα·í~åβ=í~åα+í~åβ=Åçí α+Åçíβ

=

454. Åçíα·Åçíβ=Åçíα+Åçíβ =í~åα+í~åβ

=

455. í~åα·Åçíβ=í~åα+Åçíβ=Åçíα+í~åβ

=

=

4.16 Powers of Trigonometric Functions

=

456. ëáåO α=N−ÅçëOα=O

=

457. ëáåPα= Pëáå α−ëáåPα=Q

=

458. ëáåQ α= ÅçëQα−QÅçëOα+P=U

=

459. ëáåR α=NMëáåα −RëáåPα+ëáå Rα=NS

=

460. ëáåS α=NM−NRÅçëOα+SÅçëQα−ÅçëSα=PO

=

461. ÅçëO α=N+ÅçëOα=O

=

462. ÅçëP α=PÅçëα+ÅçëPα=Q

=

463. ÅçëQ α= ÅçëQα+QÅçëOα+P=U

=

464. ÅçëR α=NMÅçëα+Rëáå Pα+ÅçëRα=NS

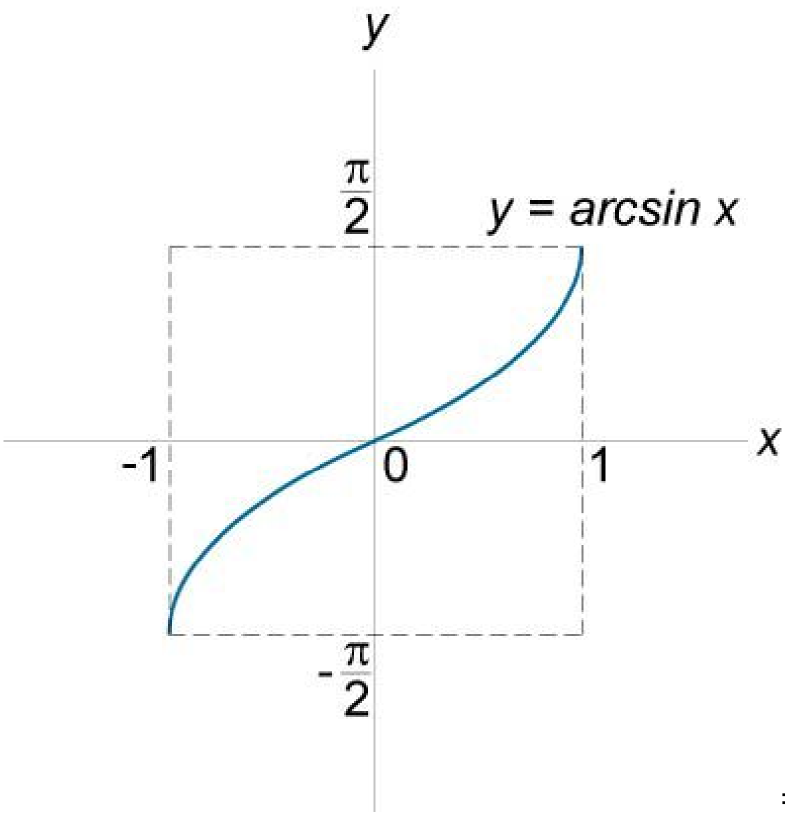
=

465. ÅçëS α=NM+NRÅçëOα+SÅçëQα+ÅçëSα=PO

4.17 Graphs of Inverse Trigonometric Functions

=

466. fåîÉêëÉ=páåÉ=cìåÅíáçå==
ó=~êÅëáåñI=−N≤ñ NI=−π≤~êÅëáåñ≤π K=O O =

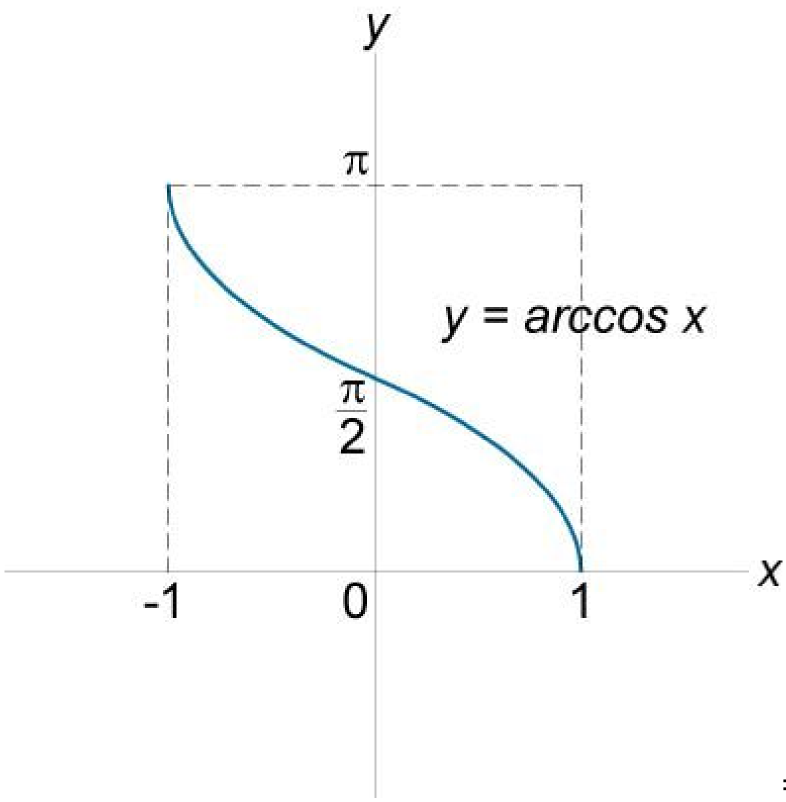


==

Figure 66.

=

467. fåîÉêëÉ=`çëáåÉ=cìåÅíáçå==
ó=~êÅÅçëñ I=−N≤ñ NI=~êÅÅçëMK=



=

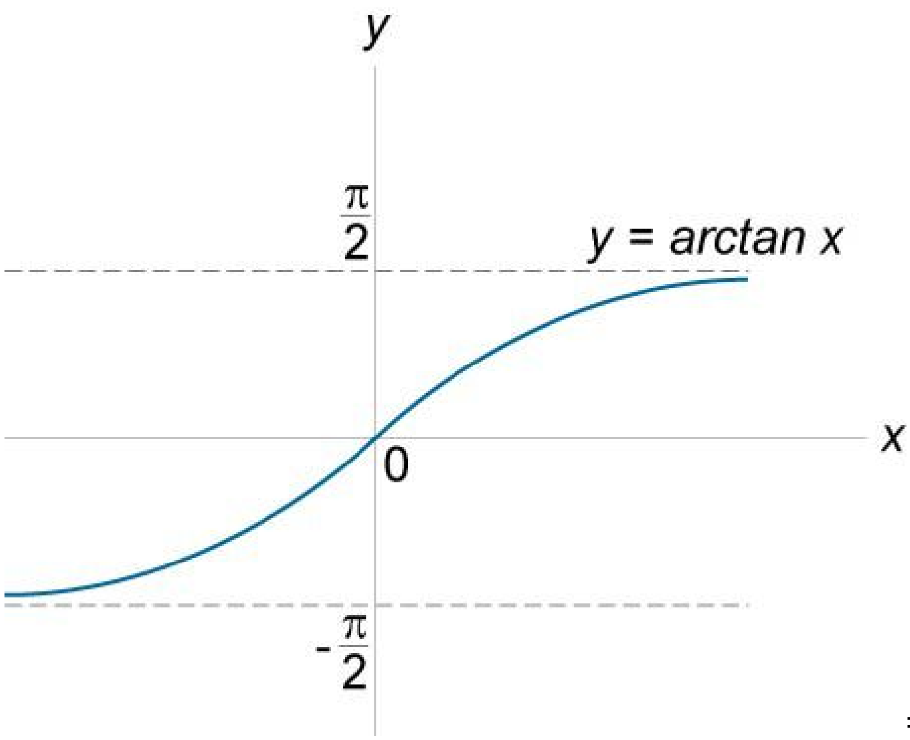
Figure 67.

=

468. fåîÉêëÉ=q~åÖÉåí=cìåÅíáçå==

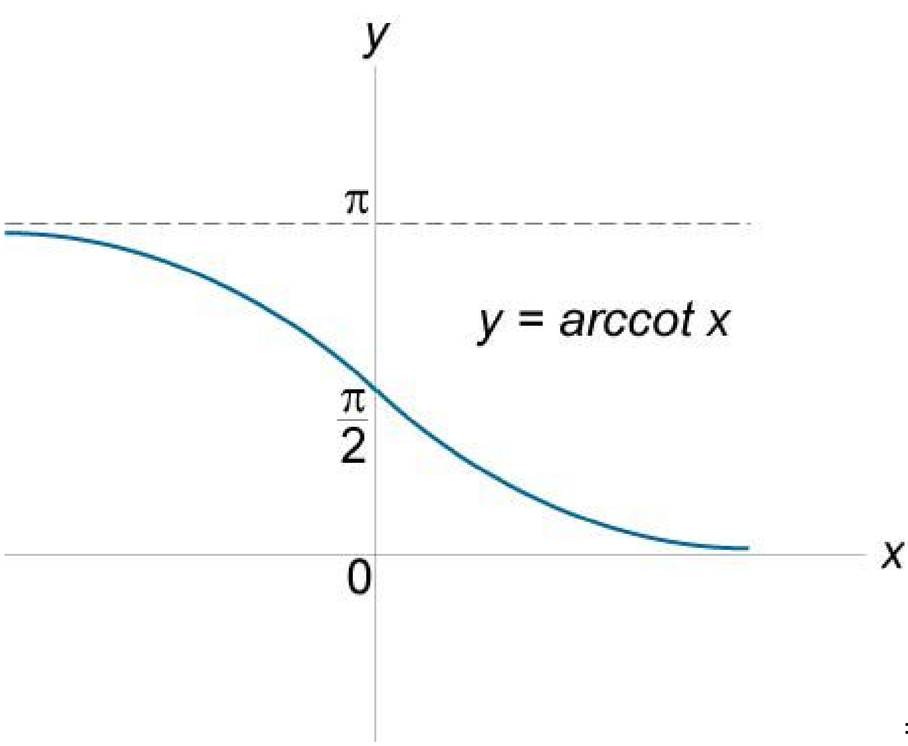
ó=~êÅí~åñI=−∞≤ñ≤∞I=−π<~êÅí~åñ <πK=O O =

= =====



= **469.**

fåîÉêëÉ=`çí~åÖÉåí=cìåÅíáçå==
ó=~êÅÅçíñI=−∞≤ñ≤∞I=~êÅMK=
=====



= **Figure 69.**

=

470. **fåîÉêëÉ=pÉÅ~åí=cìåÅíáçå==**

ó

=

~êÅëÉÅ

=

ñ

I

ñ

∈

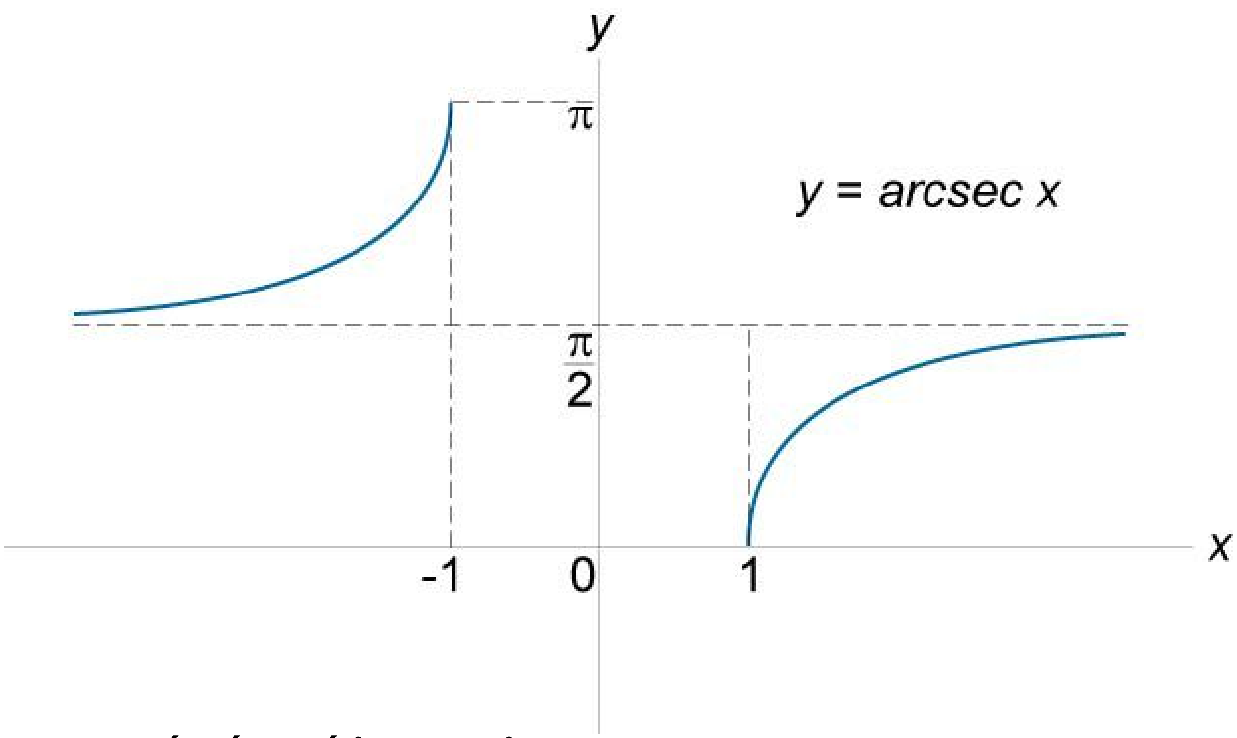
(

][

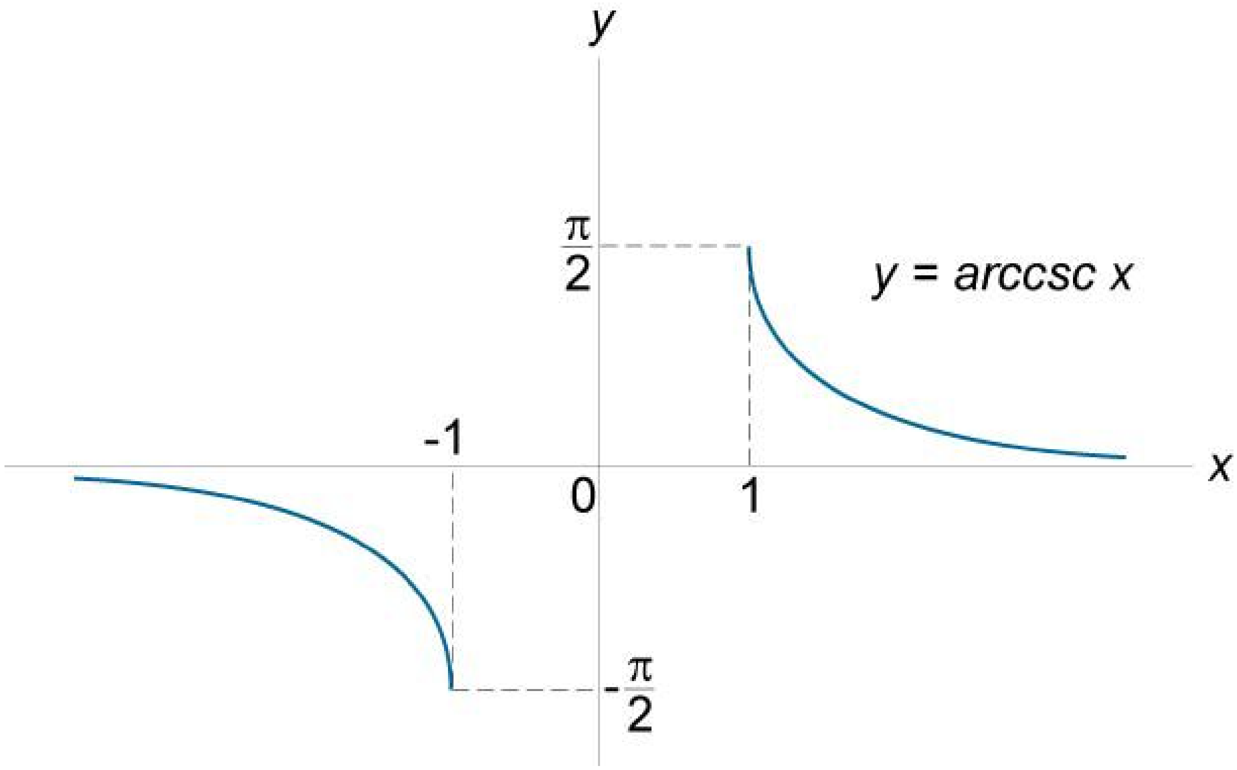
N
I

)

I
~êÅ
ëÉÅ
ñ
∈
□ π
π
□∪□ π**I** □**K□□MIO□□ □□O □□**



ó=~êÅÅëÅ ñI ñ∈( ][ )I ~êÅÅëÅ ñ∈ πIM∪ πK– O MI O



=

Figure 71.

=

=

4.18 Principal Values of Inverse Trigonometric Functions

472. M= N =O = P N=ñ = O O O
~êÅëáåñ = °= PM = QR°= ° VM°
~êÅÅçë
ñ
=
° SM°= QR°= ° °
VM PM M =
ñ = −N −O −P −N= =O O O
~êÅëáåñ =−PM° −QR° −SM°−VM° == =
~êÅÅçëñ= ° NPR°= NRM°= ° == =
473. ñ = M=P N= P= −P −N= − P =P P
~êÅí~åñ= °= ° QR° SM° −PM°−QR° −SM ==
~êÅÅçíñ = ° SM° QR° PM° NOM°= ° NRM°==

=
=
=

4.19 Relations between Inverse Trigonometric Functions

=

474. ()=−~êÅëáåñ=

=

475. ~êÅëáåñ =π−~êÅÅçëñ=O

=

476. ~êÅëáåñ =~êÅÅçë N−ñO I=M≤ ñ NK=

=

477. ~êÅëáåñ =−~êÅÅçë N−ñO I=−N≤ ñ MK=

=

478. ~êÅëáåñ =~êÅí~åñ I=ñO<NK=N−ñO

=

479.

~êÅëáå

ñ

=

~êÅ

Åçí

N−ñO ñ I=M ≤NK= =

480.

~êÅëáå

ñ

=

~êÅ

Åçí

N−ñO

ñ −πI=−N <MK= =

481. ()=π−~êÅÅçëñ=

482. ~êÅÅçëñ =^{π}−~êÅëáåñ=$_{O}$
=
483. ~êÅÅçëñ =~êÅëáå N−ñ^{O} I=M≤ ñ NK= =
484. ~êÅÅçëñ =π−~êÅëáå N−ñ^{O} I=−N≤ ñ MK= =
485.
~êÅÅçë
ñ
=
~êÅí~å
N−ñ^{O}
$_{ñ}$ I=M ≤NK= =
486.
~êÅÅçë
ñ
=
π
+
~êÅí~å
N−ñ^{O}
$_{ñ}$ I=−N <MK= =
487. ~êÅÅçëñ =~êÅÅçí^{ñ} I=−N≤ ñ NK= N−ñ^{O}
=

488. ()=−~êÅí~åñ=

=

489. ~êÅí~åñ =$^{\pi}$ −~êÅÅçíñ=$_{O}$

=

490. ~êÅí~åñ =~êÅëáå^{ñ} =

N+ñ^{O}

=

491. ~êÅí~åñ =~êÅÅçë^{N} I=ñ≥MK= N+ñ^{O}

=

492. ~êÅí~åñ =−~êÅÅçë^{N} I=ñ≤MK= N+ñ^{O}

493. ~êÅí~åñ =$^{\pi}$−~êÅí~å^{N}I=ñ>MK=$_{O\ ñ}$

=

494. ~êÅí~åñ =−$^{\pi}$−~êÅí~å^{N} I=ñ<MK=$_{O\ ñ}$

=

495. ~êÅí~å =~êÅÅçí^{N} I=ñ>MK=$_{ñ}$

=

496. ~êÅí~åñ =~êÅÅçí^{N} −πI=ñ<MK=$_{ñ}$

=

497. ()=π−~êÅÅçíñ=

=

498. ~êÅÅçíñ=$^{\pi}$−~êÅí~åñ=$_{O}$

=

499. ~êÅÅçíñ=~êÅëáå^{N} I=ñ>MK= N+ñ^{O}

=

500. ~êÅÅçíñ=π−~êÅëáå^{N} I=ñ<MK= N+ñ^{O}

=

501. ~êÅÅçíñ=~êÅÅçëñ =
N+ñO

=

502. ~êÅ =~êÅí~åN I=ñ>MK=ñ

=

503. ~êÅÅçíñ=π+~êÅí~åN I=ñ<MK=ñ

=

4.20 Trigonometric Equations

=

tÜçäÉ=åìãÄÉêW=å=

=

=

504. ëáå =~ I= ()å~êÅëáå~ +πå=

=

505. Åçë =~I=ñ =±~êÅÅçë~ +Oπå=

=

506. í~å =~I=ñ =~êÅí~å~ +πå=

=

507. Åçíñ =~I=ñ =~êÅÅçí~ +πå=

=

=

=

4.21 Relations to Hyperbolic Functions

=
fã~Öáå~êó=ìåáíW=á=
=
=

508. () =áëáåÜñ=

=

509. () =áí~åÜñ=

=

510. () −áÅçíÜñ=

=

511. ()áñ =

ëÉÅ ëÉÅÜñ=
=

512. () −áÅëÅÜ ñ=

=

=

Chapter 5 Matrices and Determinants

=
=
=
=
j~íêáÅÉëW=^I=_I=`=
bäÉãÉåíë=çÑ=~=ã~íêáñW=

~I=ÄI=~I=ÄI=Å=
á á áà áà áà
aÉíÉêãáå~åí=çÑ=~=ã~íêáñW=ÇÉí^=
jáåçê=çÑ=~å=ÉäÉãÉåí=
~W=
áà j=áà
`çÑ~Åíçê=çÑ=~å=ÉäÉãÉåí=
~W=
áà `=áà

qê~åëéçëÉ=çÑ=~=ã~íêáñW= ^qI= ^ú =
^Çàçáåí=çÑ=~=ã~íêáñW=~Çà^=
qê~ÅÉ=çÑ=~=ã~íêáñW=íê ^=

fåîÉêëÉ=çÑ=~=ã~íêáñW=
^
−N
=
oÉ~ä=åìãÄÉêW=â=
oÉ~ä=î~êá~ÄäÉëW=
ñ=
á

k~íìê~ä=åìãÄÉêëW=ãI=å===
=
=

5.1 Determinants

=

513. pÉÅçåÇ=lêÇÉê=aÉíÉêãáå~åí=

ÇÉí^=~N ÄN =~NÄO −~OÄN=~O ÄO

= = = = **514. qÜáêÇ=lêÇÉê=aÉíÉêãáå~åí=**

~NN~NO~NP

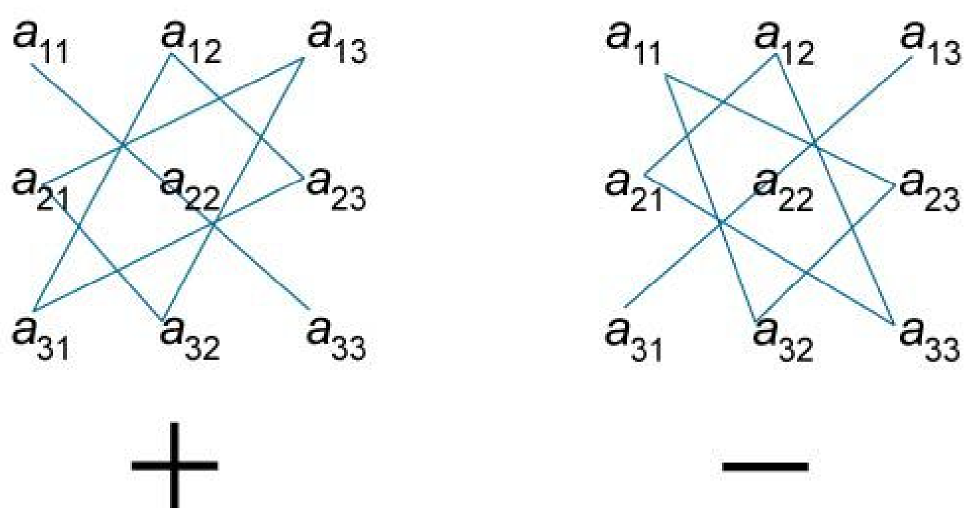
ÇÉí^ = ~ON ~OO ~OP =~NN~OO~PP +~NO~OP~PN +~NP~ON~PO −=

~PN~PO~PP

−~NN~OP~PO −~NO~ON~PP −~NP~OO~PN=

=

515. p~êêìë=oìäÉ=E^êêçï=oìäÉF=



==

Figure 72.

=

516. k-íÜ=lêÇÉê=aÉíÉêãáå~åí=

~NN~NO K**~Nà** K**~Nå**

~ ON~OO K**~Oà** K**~Oå**

ÇÉí^ = K K K K K K**=~áN ~áO** K **~áà** K **~áå**

K K K K K K

~åN~åO K**~åà** K**~åå**

=

517. jáåçê=

qÜÉ=ãáåçê=áàj =~ëëçÅá~íÉÇ=ïáíÜ=íÜÉ=ÉäÉãÉåí=áà~=çÑ=å-íÜ=çêÇÉê=

ã~íêáñ=^=áë=íÜÉ=( )-íÜ= çêÇÉê= ÇÉíÉêãáå~åí= ÇÉêáîÉÇ= Ñêçã=

íÜÉ=ã~íêáñ=^=Äó=ÇÉäÉíáçå=çÑ=áíë=á-íÜ=êçï=~åÇ=à-íÜ=ÅçäìãåK=== 518. `çÑ~Åíçê=

`áà =()^(á+à) jáà=

=

519. i~éä~ÅÉ=bñé~åëáçå=çÑ=å-íÜ=lêÇÉê=aÉíÉêãáå~åí=
i~éä~ÅÉ=Éñé~åëáçå=Äó=ÉäÉãÉåíë=çÑ=íÜÉ=á-íÜ=êçï=

å
ÇÉí^ =∑~áà`áà I=á =NIOIKIåK=
à=N
i~éä~ÅÉ=Éñé~åëáçå=Äó=ÉäÉãÉåíë=çÑ=íÜÉ=à-íÜ=Åçäìãå=
å
ÇÉí^ =∑~áà`áà I=à=NIOIKIåK==
á=N

=

=

=

5.2 Properties of Determinants

=

520.

qÜÉ==î~äìÉ==çÑ=~=ÇÉíÉêãáå~åí=êÉã~áåë==ìåÅÜ~åÖÉÇ=áÑ=êçïë=~êÉ= ÅÜ~åÖÉÇ=íç=Åçäìãåë=~åÇ=Åçäìãåë=íç=êçïëK=

= ~N ~O = ~N ÄN ==ÄN ÄO ~O ÄO

=

521.

fÑ=îïç==êçïë==Eçê=îïç=ÅçäìãåëF=~êÉ==áåíÉêÅÜ~åÖÉÇI=íÜÉ=ëáÖå=çÑ= íÜÉ=ÇÉíÉêãáå~åí=áë=ÅÜ~åÖÉÇK=

~N ÄN =−~O ÄO =~O ÄO ~N ÄN

=

522.

fÑ=îïç=êçïë==Eçê=îïç=ÅçäìãåëF=~êÉ==áÇÉåíáÅ~äI=íÜÉ=î~äìÉ=çÑ=íÜÉ= ÇÉíÉêãáå~åí=áë=òÉêçK=

~N ~N M=~ =

O ~O

523.

fÑ==íÜÉ===ÉäÉãÉåíë==çÑ==~åó=êçï==Eçê=ÅçäìãåF=~êÉ=ãìäíáéäáÉÇ=Äó=====

~==Åçããçå==Ñ~ÅíçêI==íÜÉ==ÇÉíÉêãáå~åí==áë==ãìäíáéäáÉÇ==Äó==íÜ~í= Ñ~ÅíçêK=

â~N âÄN =â ~N ÄN =~O ÄO ~O ÄO

=

524.

fÑ==íÜÉ==ÉäÉãÉåíë==çÑ==~åó==êçï==Eçê==ÅçäìãåF=~êÉ=áåÅêÉ~ëÉÇ=Eçê= ÇÉÅêÉ~ëÉÇFÄó=Éèì~ä=ãìäíáéäÉë=çÑ=íÜÉ=ÅçêêÉëéçåÇáåÖ=ÉäÉãÉåíë= çÑ=~åó=çíÜÉê=êçï==Eçê=ÅçäìãåFI==íÜÉ=î~äìÉ=çÑ=íÜÉ=ÇÉíÉêãáå~å

í= áë=ìåÅÜ~åÖÉÇK=

~$_N$ +âÄ$_N$ Ä$_N$ = ~$_N$ Ä$_N$ =~O +âÄO ÄO ~O ÄO

=

=

=

5.3 Matrices

=

525. aÉÑáåáíáçå=
^å=ã×å=ã~íêáñ=^=áë=~=êÉÅí~åÖìä~ê=~êê~ó=çÑ=ÉäÉãÉåíë=Eåìã-
ÄÉêë=çê=ÑìåÅíáçåëF=ïáíÜ=ã=êçïë=~åÇ=å=ÅçäìãåëK==

~NN~NO K~Nå

^ = ~[]= ~ON ~OO K ~Oå ==áà M M M

~ãN~ ãO K~ ãå

=

526. pèì~êÉ=ã~íêáñ=áë=~=ã~íêáñ=çÑ=çêÇÉê=å×åK==

=

**527. ^=ëèì~êÉ=ã~íêáñ==[
]==áë==ëóããÉíêáÅ==áÑ==~áà=~àáI==áKÉK==áí
==áë=**

**ëóããÉíêáÅ=~Äçìí=íÜÉ=äÉ~ÇáåÖ=Çá~Öçå~äK==
=**

528. ^=ëèì~êÉ=ã~íêáñ=[ ]~=áë=ëâÉï-ëóããÉíêáÅ=áÑ=~áà −~àáK==

529.
aá~Öçå~ä=ã~íêáñ==áë==~=ëèì~êÉ==ã~íêáñ=ïáíÜ=~ää==ÉäÉãÉåíë==òÉêç= ÉñÅÉéí=íÜçëÉ=çå=íÜÉ=äÉ~ÇáåÖ=Çá~Öçå~äK==
=

530.
råáí=ã~íêáñ==áë==~=Çá~Öçå~ä==ã~íêáñ==áå=ïÜáÅÜ=íÜÉ=ÉäÉãÉåíë =çå= íÜÉ=äÉ~ÇáåÖ=Çá~Öçå~ä=~êÉ=~ää=ìåáíóK=qÜÉ=ìåáí=ã~íêáñ=áë=== ======== ÇÉåçíÉÇ=Äó=fK==
=

531. ^=**åìää=ã~íêáñ**=áë=çåÉ=ïÜçëÉ=ÉäÉãÉåíë=~êÉ=~ää=òÉêçK= =
=
=

5.4 Operations with Matrices

=

532.

**qïç=ã~íêáÅÉë=^=~åÇ=_=~êÉ=Éèì~ä=áÑI=~åÇ=çåäó=áÑI=íÜÉó=~êÉ=
ÄçíÜ=
çÑ==íÜÉ==ë~ãÉ==ëÜ~éÉ==ã×å==~åÇ=ÅçêêÉëéçåÇáåÖ=ÉäÉãÉåíë=~ê
É=
Éèì~äK=**

=

533.

**qïç=ã~íêáÅÉë==^=~åÇ=_==Å~å=ÄÉ=~ÇÉÇ=Eçê=ëìÄíê~ÅíÉÇF=çÑI=
~åÇ=
çåäó=áÑI=íÜÉó=Ü~îÉ=íÜÉ=ë~ãÉ=ëÜ~éÉ=ã×åK=fÑ==**

□˜NN˜NO K˜Nå □

^= ~[]= □□□□~ON ~OO K ~Oå □□□□I==áà M M M

□~ãN~ãO K~ãå □
□ÄNNÄNO KÄNå □

[]

_= **Ä** □□□□**ÄON ÄOO** K **ÄOå** □□□□**I==áà** =
M M M
□**ÄãNÄãO** K**Äãå** □

=
=
=
=
□ **~NN**+**ÄNN ~NO**+**ÄNO** K **~Nå** +**ÄNå** □

^+_=□□□□**~ON**+**ÄON ~OO**+**ÄOO** K **~Oå** +**ÄOå** □□□□**K=** M M M
□~**ãN**+ **ÄãN**~ **ãO**+ **ÄãO** K~**ãå**+ **Äãå**□
=

534. fÑ=â=áë=~=ëÅ~ä~êI=~åÇ= [ ]=áë=~=ã~íêáñI=íÜÉå=

â~$_{NN}$ **â~**$_{NO}$ K **â~**$_{Nå}$

â^=[]= □□□□â~ON â~OO K â~Oå □□□□K=

M M M
□**â~ãN â~ãO** K **â~ãå**□

=
535. jìäíáéäáÅ~íáçå=çÑ=qïç=j~íêáÅÉë=
qïç= ã~íêáÅÉë= Å~å= ÄÉ= ãìäíáéäáÉÇ= íçÖÉíÜÉê= çåäó= ïÜÉå= íÜÉ= åìãÄÉê= çÑ= Åçäìãåë= áå= íÜÉ= Ñáêëí= áë= Éèì~ä= íç= íÜÉ= åìãÄÉê= çÑ=
êçïë=áå=íÜÉ=ëÉÅçåÇK==
=
fÑ=
□~NN ~NO K ~Nå □

^ = []= ~ON ~OO K ~Oå I==

M M M

~ãN ~ãO K ~ãå ÄNN ÄNO K ÄNâ

_ = []= □□□□Ä$_{ON}$ Ä$_{OO}$ K Ä$_{Oâ}$ □□□□I=

M M M
□Ä$_{åN}$ Ä$_{åO}$ K Ä$_{åâ}$ □

=
=
=
=

□Å$_{NN}$Å$_{NO}$ KÅ$_{Nâ}$ □
^_ =`= □□□□Å$_{ON}$ Å$_{OO}$ K Å$_{Oâ}$ □□□□I== M M M
□Ä$_{ãN}$Å$_{ãO}$ KÅ$_{ãâ}$ □
ïÜÉêÉ==
å
Å$_{áà}$ =~$_{áN}$Ä$_{Nà}$ +~$_{áO}$Ä$_{Oà}$ +K+~$_{áå}$Ä$_{åà}$ =∑~$_{áλ}$Ä$_{λà}$ =
λ=N

Eã K XâK FK==
=
qÜìë=áÑ=

□Ä$_{N}$□

^ = ~[]=~NN ~NO ~NPI= []á = ÄOI==áà ~ON ~OO ~OP ÄP

íÜÉå==

^_ = ~NN~NO~NP ÄN ~NNÄN~NOÄO~NPÄP K==~ ON ~OO ~OP ·ÄO = ~ONÄN ~OOÄO ~OPÄP ÄP

=

536. qê~åëéçëÉ=çÑ=~=j~íêáñ=

fÑ=íÜÉ=êçïë=~åÇ=Åçäìãåë=çÑ=~=ã~íêáñ=~êÉ=áåíÉêÅÜ~åÖÉÇI=íÜÉå

=

íÜÉ=åÉï=ã~íêáñ=áë=Å~ääÉÇ=íÜÉ=íê~åëéçëÉ=çÑ=íÜÉ=çêáÖáå~ä=ã~íêáñK===

fÑ=^=áë=íÜÉ=çêáÖáå~ä=ã~íêáñI=áíë=íê~åëéçëÉ=áë=ÇÉåçíÉÇ=q^=çê=

^ú K==

=

537. qÜÉ=ã~íêáñ=^=áë=çêíÜçÖçå~ä=áÑ= ^^q=fK==

=

538. fÑ=íÜÉ=ã~íêáñ=éêçÇìÅí=^_=áë=ÇÉÑáåÉÇI=íÜÉå==

()q =_q^qK=

=

539. **^Çàçáåí=çÑ=j~íêáñ=**
fÑ=^=áë=~=ëèì~êÉ=å×åã~íêáñI= áíë= ~ÇàçáåíI= ÇÉåçíÉÇ=
Äó=~Çà^I=
áë=íÜÉ=íê~åëéçëÉ=çÑ=íÜÉ=ã~íêáñ=çÑ=ÅçÑ~Åíçêë=áà`=çÑ=^W=

[]

~Çà = `^{q} K==$_{áà}$

=

540. qê~ÅÉ=çÑ=~=j~íêáñ=

fÑ=^=áë=~=ëèì~êÉ= å×åã~íêáñI= áíë= íê~ÅÉI= ÇÉåçíÉÇ= Äó= íê ^I= áë=
ÇÉÑáåÉÇ=íç=ÄÉ==íÜÉ=ëìã=çÑ==íÜÉ=íÉêãë=çå=íÜÉ=äÉ~ÇáåÖ=Çá~Öçå~äW=

íê ^=~$_{NN}$+~$_{OO}$ +K+~$_{åå}$K=

=

541. fåîÉêëÉ=çÑ=~=j~íêáñ=

fÑ=^=áë=~=ëèì~êÉ=å×åã~íêáñ=ïáíÜ=~=åçåëáåÖìä~ê=ÇÉíÉêãáå~åí=ÇÉí^I=íÜÉå=áíë=áåîÉêëÉ=^$^{−N}$ =áë=ÖáîÉå=Äó=

^−N=~Çà^$_{K=ÇÉí^}$

=

542. fÑ=íÜÉ=ã~íêáñ=éêçÇìÅí=^_=áë=ÇÉÑáåÉÇI=íÜÉå==

()−N=_−N∧−NK=

=

543.

fÑ==^==áë=~=ëèì~êÉ===å×å==ã~íêáñI==íÜÉ==ÉáÖÉåîÉÅíçêë==u===

ë~íáëÑó=

íÜÉ=Éèì~íáçå=

^u λuI==

ïÜáäÉ=íÜÉ=ÉáÖÉåî~äìÉë=λ=ë~íáëÑó=íÜÉ=ÅÜ~ê~ÅíÉêáëíáÅ=Éèì~íáçå

= ^ =MK===

=

=

=

5.5 Systems of Linear Equations

=
=
s~êá~ÄäÉëW=ñI=óI=òI=ñI=ñI =N K
oÉ~ä=åìãÄÉêëW=~NI~OI~PIÄNI~NNI~NOIK=
aÉíÉêãáå~åíëW=aI=

aI=aI=a== ñ ó ò

j~íêáÅÉëW=^I=_I=u=
=
=

544. ~Nñ +ÄNó =ÇN I==~Oñ +ÄOó=ÇO

ñ =añ I=ó =aó =E`ê~ãÉê∞ë=êìäÉFI==a a
ïÜÉêÉ==
a=~N ÄN =~NÄO −~OÄNI==~O ÄO

aÇN ÄN =ÇNÄO −ÇOÄNI==ñ =ÇO ÄO
a~N ÇN =~NÇO −~OÇNK==ó =~O ÇO
=
545. fÑ=a≠MI=íÜÉå=íÜÉ=ëóëíÉã=Ü~ë=~=ëáåÖäÉ=ëçäìíáçåW==
ñ =añ I=ó =aó K=a a
fÑ= a= =~åÇ= añ≠MEçê= aó≠MFI= íÜÉå= íÜÉ= ëóëíÉã= Ü~ë= = åç==
ëçäìíáçåK=
fÑ= a=añ =aó =MI=íÜÉå=íÜÉ=ëóëíÉã=Ü~ë==áåÑáåáíÉäó==ã~åó==
ëçäìíáçåëK=
=

~Nñ +ÄNó+ÅNò =ÇN=
546. ~Oñ +ÄOó+ÅOò =ÇO I==
~Pñ+ÄPó +ÅPò =ÇP

ñ =añ I=ó =aó I=ò=aò =E`ê~ãÉê∞ë=êìäÉFI==a a a

~N ÄN ÅN ÇN ÄN ÅN
a= ~O ÄO ÅO I=añ= ÇO ÄO ÅO I=
~P ÄP ÅP ÇP ÄP ÅP
~N ÇN ÅN ~N ÄN ÇN
aó= ~O ÇO ÅO I=aò= ~O ÄO ÇO K==
~P ÇP ÅP ~P ÄP ÇP

=
547. fÑ=a≠MI=íÜÉå=íÜÉ=ëóëíÉã=Ü~ë=~=ëáåÖäÉ=ëçäìíáçåW== ñ =añ
I=ó =aó I=ò=aò K=a a a
fÑ=a=M=~åÇ=añ≠MEçê=aó≠M=çê=aò≠MFI=íÜÉå=íÜÉ=ëóëíÉã=
Ü~ë=åç=ëçäìíáçåK=
fÑ= a=añ =aó =aò =MI= íÜÉå= íÜÉ= ëóëíÉã= Ü~ë= áåÑáåáíÉäó=
ã~åó=ëçäìíáçåëK=
=

548.
j~íêáñ=cçêã=çÑ=~=póëíÉã=çÑ=å=iáåÉ~ê=bèì~íáçåë=áå============
===== å=råâåçïåë=
qÜÉ=ëÉí=çÑ=äáåÉ~ê=Éèì~íáçåë==

□~NNñN +~NOñO +K+~Nåñå =ÄN□□~ONñN +~OOñO +K+~Oåñå =ÄO
=□KKKKKKKKKKKK□□~åNñN +~åOñO +K+~ååñå =Äå□
Å~å=ÄÉ=ïêáííÉå=áå=ã~íêáñ=Ñçêã=
□~NN ~NO K ~Nå□ □ñN □ □ÄN□
□□□□□
~ON ~OO K ~Oå□□□□□.□□□□□ñO □□□□□=□□□□□ÄO□□□□□I==

M M M M M
□~åN ~åO K ~åå□ □ñå□ □Äå□

áKÉK==
^ =_I==

□~$_{NN}$ ~$_{NO}$ K ~$_{Nå}$ □ □ñ$_{N}$□ □Ä$_{N}$□
^ =□□□□□~^{ON} ~^{OO} K ~^{Oå}□□□□□I=u
=□□□□□ñ^{O}□□□□□I=_=□□□□□Ä^{O}□□□□□K== M M M M M

□ ~$_{åN}$ ~$_{åO}$ K ~$_{åå}$□ □ñ$_{å}$□ □Ä$_{å}$□ =

549. pçäìíáçå=çÑ=~=pÉí=çÑ=iáåÉ~ê=bèì~íáçåë=å×å=

u
=
−

^ N·_I==
ïÜÉêÉ=^$^{−N}$ =áë=íÜÉ=áåîÉêëÉ=çÑ=^K=
=
=

Chapter 6 Vectors

=
=
=
=

→
sÉÅíçêëW=ì I= î I= ï I= ê I=^_ I=£=
sÉÅíçê=äÉåÖíÜW= ì I= î I=£=
råáí=îÉÅíçêëW= á^{r} I= à I=â =

kìää=îÉÅíçêW= M =
`ççêÇáå~íÉë=çÑ=îÉÅíçê=ì W=$_N$Iu=$_N$Iw$_N$
`ççêÇáå~íÉë=çÑ=îÉÅíçê= î W=u$_O$I v$_O$Iw$_O$= pÅ~ä~êëW=λIμ=

aáêÉÅíáçå=ÅçëáåÉëW=
αI= βI= γ
Åçë Åçë Åçë =

^åÖäÉ=ÄÉíïÉÉå=íïç=îÉÅíçêëW= θ =
=
=

6.1 Vector Coordinates

=

550. råáí=sÉÅíçêë=

r

á=()I=r

à=()I=r â=()I=

r= r

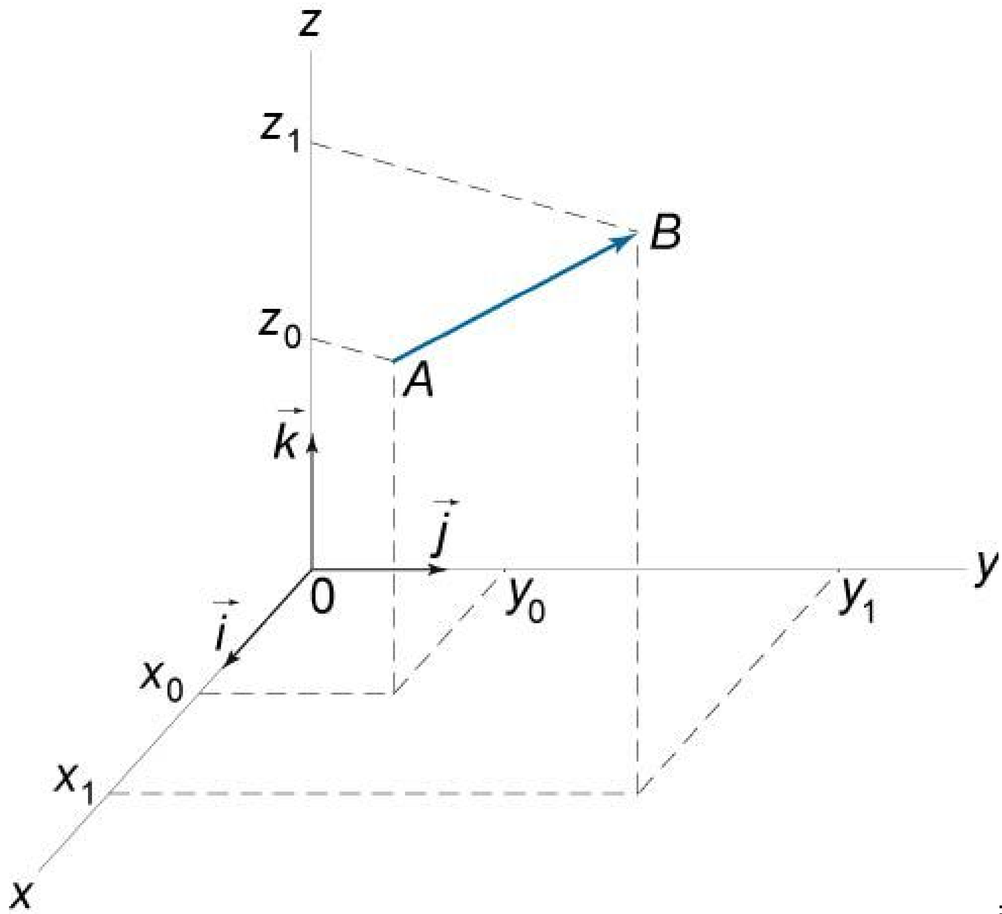
á à = **â** =**NK=**

=

551.

r → r −M +N −M â= ()( )( )r r

N M N
=
=======



= =

Figure 73.
=

552.

r
→

()( )( )

N
M
N
−
M
+ ò
N
− ò
O M=
=
→ →

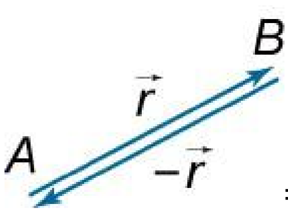
553.

fÑ=

^_

rêI=íÜÉå=_^ r

= =−êK= =

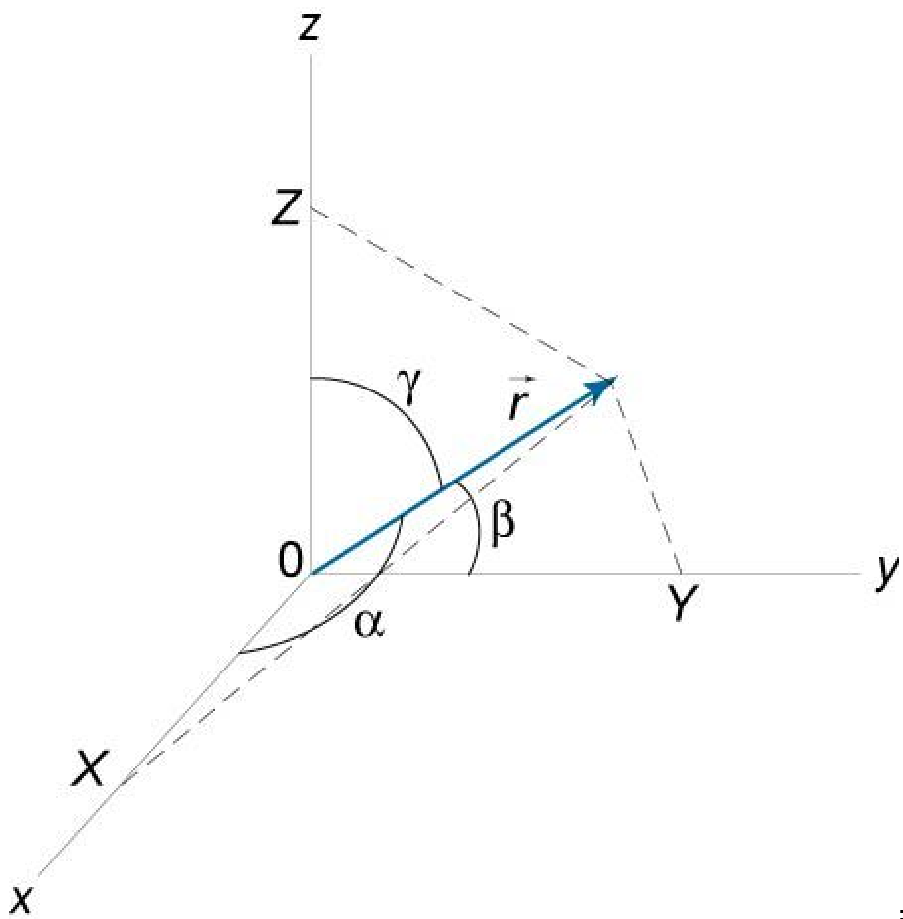


= =

Figure 74. = **554.**

u
r
= **ê ÅçëαI=**
v
r
= **ê ÅçëβI=**
w
r
= **ê ÅçëγK=**

= =====



= **Figure 75.** =

555. fÑ=^{r}() ($^{r}{}_{N}$ u$_{N}$Iv$_{N}$Iw$_{N}$)I=íÜÉå==

u =u$_{N}$I=v=v$_{N}$ I=w=w$_{N}$K==

==

=

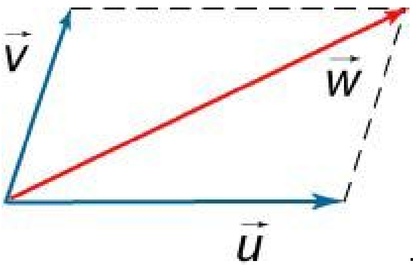
6.2 Vector Addition

=r r r

556. ï + î=

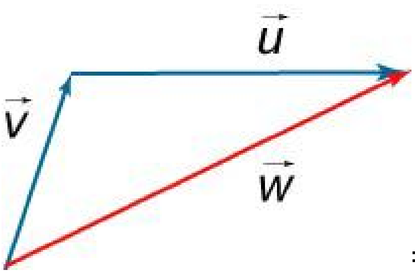
=

= ==



= **Figure 76.**

= ==

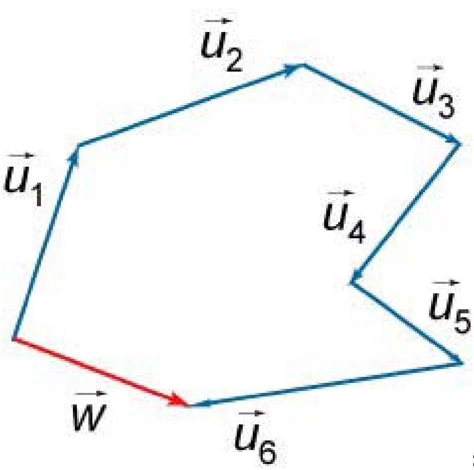


= **Figure 77.**

=r r r r r

557. ï=ì$_{N}$ +ì$_{O}$ +ì$_{P}$ +K+ì$_{å}$= =

= ==



= **Figure 78.** =

558. `çãäìí~íáîÉ=i~ï=

r r r r

ì+î= î+ì=

=

559. **^ëëçÅá~íáîÉ=i~ï=**

r r( )r

ì

()

r+ r= r

î ï ì+ î ï =

=

560.

ì

$$\mathbf{r}_{+}\mathbf{r}_{()}=$$

=

=

=

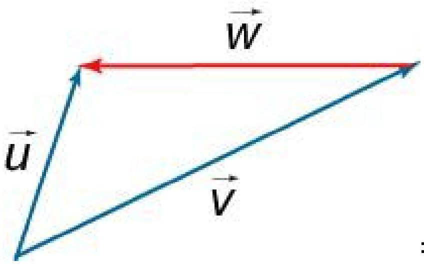
=

=

6.3 Vector Subtraction

=r r r r r r

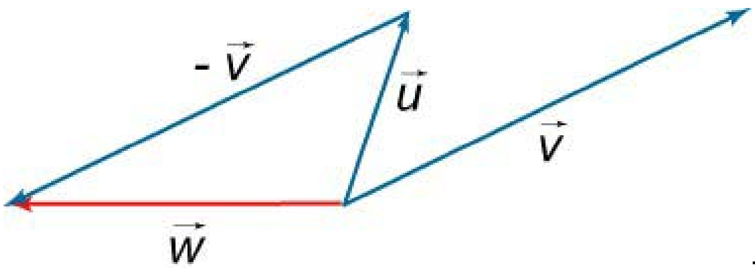
561. ï −î=áÑ= î =ìK= =



==

Figure 79. =

= ==



= **Figure 80.** =

562. r ()r r + −r =

=

563. r r ()=

=r

564. M=M=

=

r r

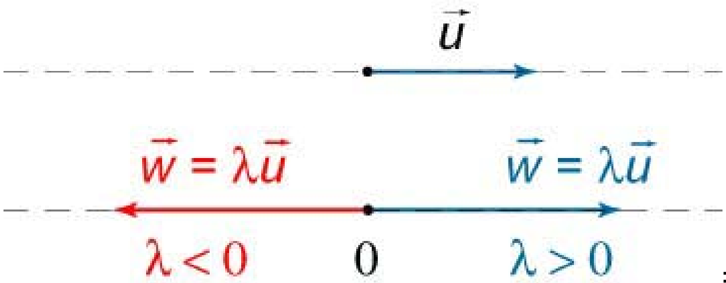
565. ()N−OIvN−vOIN−wO I==

=
=
=

6.4 Scaling Vectors

=r r

566. ï λì=



==

Figure 81.

= r ·r**567.** ï ì =

=

568. $\mathbf{r}$**()=**

= r

569.

λ

ì

=

rìλ=

= r =λ^r^ +μ^r^570. ()ì ì ì=

= r() ( ) ( )^r^ = λμ^r^571. λ μ ì=

572. ()^r^ r= r +^r^ =λì+λî=

=

=

=

6.5 Scalar Product

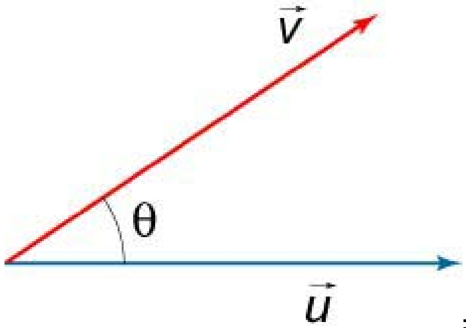
=

573. pÅ~ä~ê=mêçÇìÅí=çÑ=sÉÅíçêë=ì =~åÇ î =

r r r r

ì·î = ì · î ·ÅçëθI==

ïÜÉêÉ=θ =áë=íÜÉ=~åÖäÉ=ÄÉíïÉÉå=îÉÅíçêë=ì =~åÇ î K==== ==



= Figure 82. =

574. pÅ~ä~ê=mêçÇìÅí=áå=`ççêÇáå~íÉ=cçêã=

fÑ=()N vNIwN I= (uOIvOIwO)I=íÜÉå==

r

ìî·r =uNuO +vNvO +wNwOK=

=

575. ^åÖäÉ=_ÉíïÉÉå=qïç=sÉÅíçêë==

fÑ=()

Åçëθ =O O O O O O

)I=íÜÉå==N N N I= (uOIvOIwO

uNuO +vNvO +wNwO K= uN +vN +wN uO +vO +wO =

576. `çããìí~íáîÉ=mêçéÉêíó=

r r r r

ì·î =î·ì=

=

577. **^ëëçÅá~íáîÉ=mêçéÉêíó=**
λ

r()( )r =λμr.r

ì î ì î=
=
578. **aáëíêáÄìíáîÉ=mêçéÉêíó=**

r.()=r.r+r.r

ì ì î ì ï=
=
579.
r r
ì
=M=áÑ=ì Iî =~êÉ=çêíÜçÖçå~ä=Eθ=
î
π FK=O =
580.
r r
ì
>M=áÑ=M<θ<
î
πK=O
r r
581. ì <M=áÑ=π <θ<πK=O
=r r r r
582. ì·î≤ ì· î =
=r r r r
583. ì·î= ì· î =áÑ=ì Iî =~êÉ=é~ê~ääÉä=Eθ=MFK=
= r
584. fÑ=ì=
ì·ì=ìO O O

()uNI vN N I=íÜÉå==

r r rO r
= ìO =uN +vN +wN K=
r r
585. á á à à
=r.r =r.r =â·â =N=

r r

586. **á à à á**

=r.r =r·**â** =**â**·r =**M=**

=

=

=

6.6 Vector Product

=

587. **sÉÅíçê=mêçÇìÅí=çÑ=sÉÅíçêë=ì =~åÇ î =**

r r r

ì = **ïI=ïÜÉêÉ==**

r r r

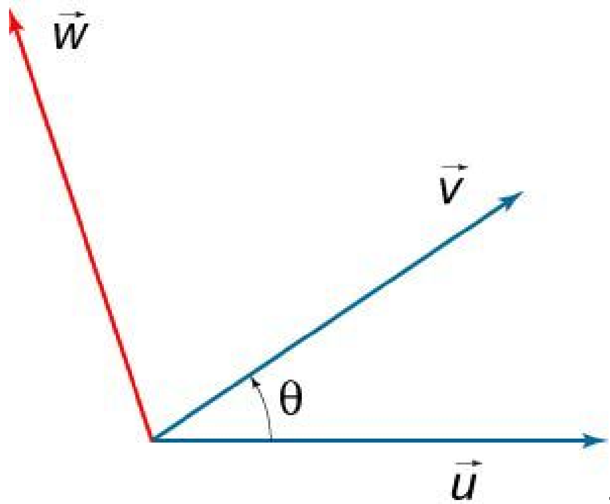
• **ï** = **ì**· **î** ·**ëáåθI=ïÜÉêÉ=M**≤θ≤$_{O}$

• **ï**⊥**=~åÇ=ï**$_{\perp}$

π$_{X=}$

r r r r$_{îX=}$

• **=sÉÅíçêë=ì I=î I= ï =Ñçêã=~=êáÖÜí-Ü~åÇÉÇ=ëÅêÉïK=** $_{=}$ **=======**



= **Figure 83.** $^{=}$ r r^{r}

á à â

588.

r$_{=}$ r$_{\times}$r

ï ì î = **u**$_{N}$ **v**$_{N}$ **w**$_{N}$ = **u**$_{O}$ **v**$_{O}$ **w**$_{O}$

=

589.

r$_{=}$ r

ï ì □□ □□×î=□ **v**$_{N}$ **w**$_{N\ I–}$ **u**$_{N}$ **w**$_{N\ I}$ **u**$_{N}$ **v**$_{N}$ □$_{=}$ □ **v**$_{O}$ **w**$_{O}$ **u**$_{O}$ **w**$_{O}$ **u**$_{O}$ **v**$_{O}$ □ =

590.

p= **ì î ì î**

r×r =r ·r ·**ëáåθ=EcáÖKUPF=**

=

591. **^åÖäÉ=_ÉíïÉÉå=qïç=sÉÅíçêë=EcáÖKUPF=**

r×r

ëáå

θ

=

ì î

r r =**ì · î**

=

592. **kçåÅçããìí~íáîÉ=mêçéÉêíó=**

r r

ì×î=−()==

=

593. **^ëëçÅá~íáîÉ=mêçéÉêíó=**

()( )

=

λμ

r×r

ì î=

=

594. aáëíêáÄìíáîÉ=mêçéÉêíó=

r r r r r

ì×()=ì×î+ì×ï =

=

r r

595. ì =M=áÑ=ì =~åÇ=î =~êÉ=é~ê~ääÉä=Eθ=MFK=

r r r

596. á á à à

=r×r =r×r =â×â =M=

=r r r r r r

597. á =âI=à = á I=â = à =

=

=

=

6.7 Triple Product

=

598. pÅ~ä~ê=qêáéäÉ=mêçÇìÅí=r ×r = r r×r =[]() ( ) ( )

= rrr rrr = îrr r =−rr r =−rrr =−rrr599. [][ ] [ ] [ ] [
] [ ]ïìîïìîïìîï ïîì ìïî =

600. rrr= r r×r =â ìîï =() [ ]î ï

=

601. pÅ~ä~ê=qêáéäÉ=mêçÇìÅí=áå=`çêÇáå~íÉ=cçêã= uN vN wN

r

ì·r()= uO vO wO I==

uP vP wP
ïÜÉêÉ==
r r r

ì= u()IwN I=î (uOI vOIwO)I=ï (upIvpIwp)K==NIvN

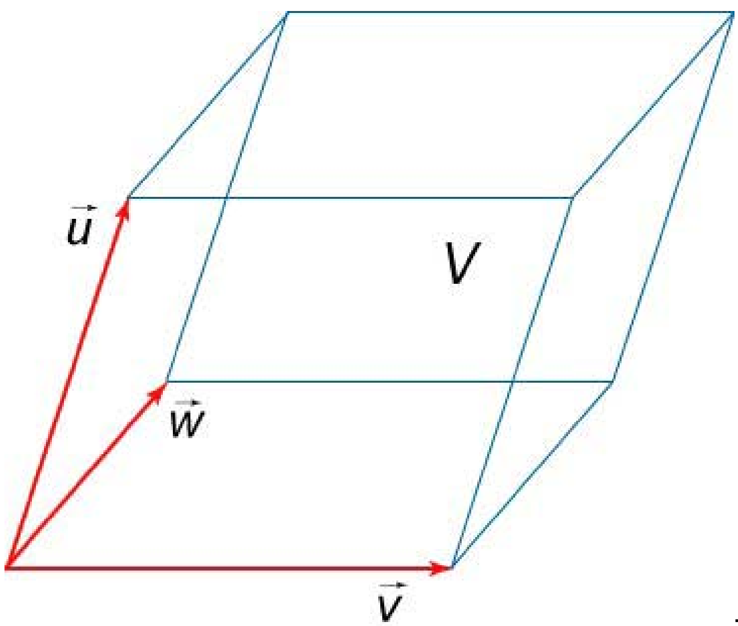
=

602. **sçäìãÉ=çÑ=m~ê~ääÉäÉéáéÉÇ=**

r

s=ì ×()=

= ============



= **Figure 84.** =

603. **sçäìãÉ=çÑ=móê~ãáÇ=**

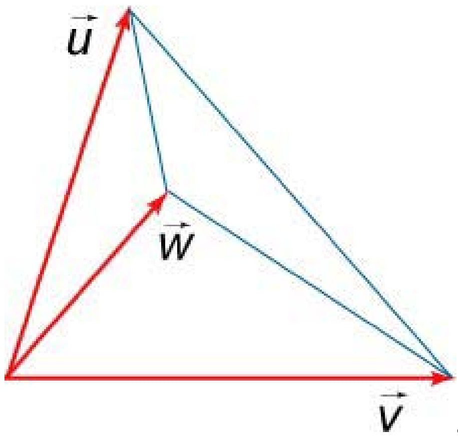
s

=

N r ×r()r

s ì î ï =

=



=
=

Figure 85.

=

604. **fÑ==ì î**

ÇÉéÉåÇÉåí=I=ëç=ï ì î=Ñçê=ëçãÉ=ëÅ~ä~êë=

r ()r×ï =MI=íÜÉå=íÜÉ=îÉÅíçêë==ì I= î I=~åÇ= ï =~êÉ=äáåÉ~êäó=

r =λr +μr λ=~åÇ=μ**K**== = rr

605. **fÑ== ()≠MI=íÜÉå=íÜÉ=îÉÅíçêë==ì I= î I=~åÇ= ï =~êÉ=äáåÉ~êäó=**

áåÇÉéÉåÇÉåíK=

606.

sÉÅíçê=qêáéäÉ=mêçÇìÅí=

r r.rr

ì×()( ) ( ) ï==

= = = = = = = = =

Chapter 7 Analytic Geometry

=

=

=

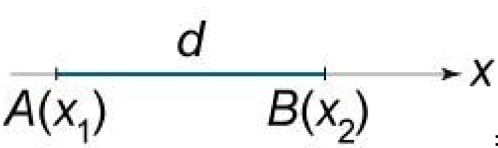
=

7.1 One-Dimensional Coordinate System

=
mçáåí=ÅççêÇáå~íÉëW=
ñ I=ñI=ñI=ó I=óI=ó=
M N O M N O

oÉ~ä=åìãÄÉêW= λ==
áåëí~åÅÉ=ÄÉíïÉÉå=íïç=éçáåíëW=Ç=
=
=

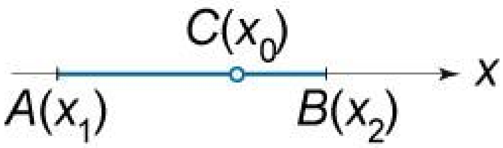
607. **áåëí~åÅÉ=_ÉíïÉÉå=qïç=mçáåíë=**
Ç= ^_= **ñ**$_{\mathbf{O}}$ −**ñ**$_{\mathbf{N}}$ = **ñ**$_{\mathbf{N}}$ −**ñ**$_{\mathbf{O}}$ =
=



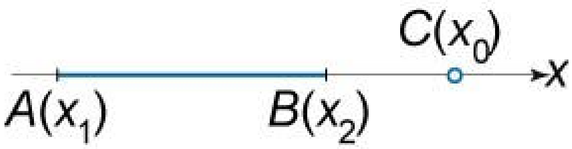
==

Figure 86.
=

608. **áîáÇáåÖ=~=iáåÉ=pÉÖãÉåí=áå=íÜÉ=o~íáç=**λ=
ñ$_{\mathbf{M}}$ =**ñN** +λ**ñO** **I=**λ=^\`**I=**λ≠−**NK=**$_{\mathbf{N}+\lambda}$ \`_
=



$\lambda > 0$ ========



$\lambda < 0$ ==

609. jáÇéçáåí=çÑ=~=iáåÉ=pÉÖãÉåí=

ñM =ñN +ñO I=λ=NK=O

=
=
=

7.2 Two-Dimensional Coordinate System

=

mçáåí=ÅççêÇáå~íÉëW=

ñ I=ñI=ñI=ó I=óI=ó= M N O M N O

mçä~ê=ÅççêÇáå~íÉëW= ϕ=

oÉ~ä=åìãÄÉêW=λ==

mçëáíáîÉ=êÉ~ä=åìãÄÉêëW=~I=ÄI=ÅI==

aáëí~åÅÉ=ÄÉíïÉÉå=íïç=éçáåíëW=Ç=

^êÉ~W=p=

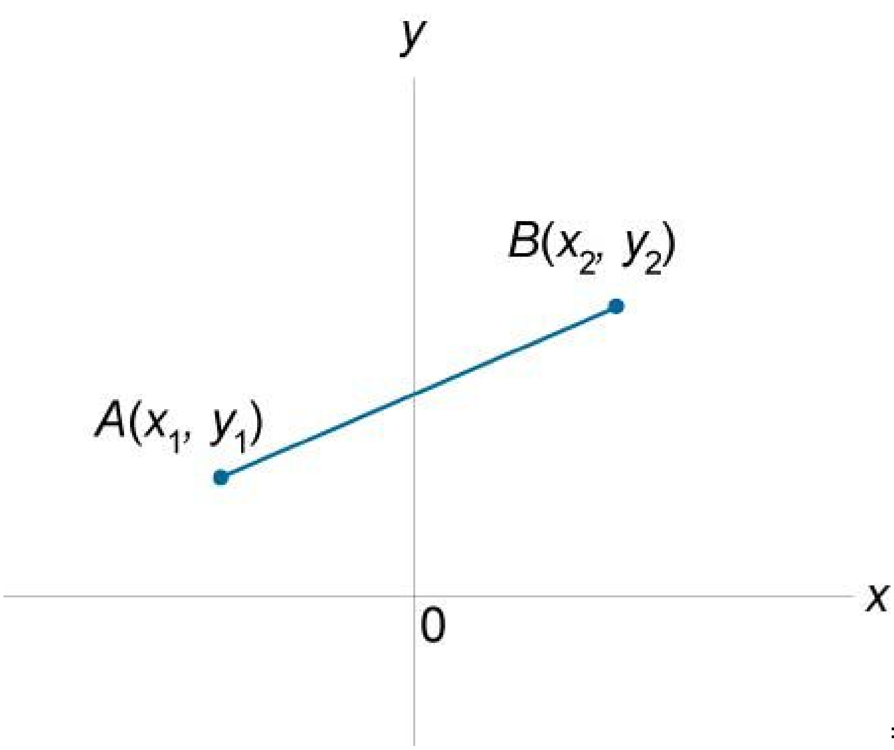
=

=

610. **aáëí~åÅÉ=_ÉíïÉÉå=qïç=mçáåíë=**

Ç= ^_= ñO −ñ()( ) O =N + O N

=



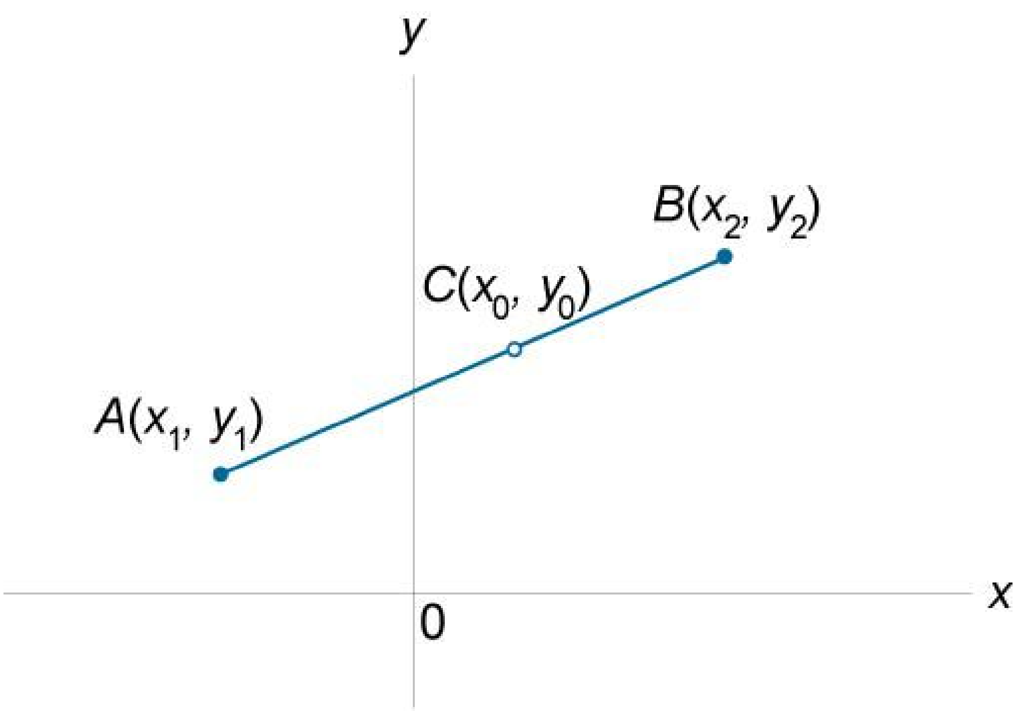
==

611. **aáîáÇáåÖ=~=iáåÉ=pÉÖãÉåí=áå=íÜÉ=o~íáç=λ=** **ñM =ñN +λñO I=óM =óN +λóO I==N+λ N+λ**

λ=^` I=λ≠−NK=`_

=

=======

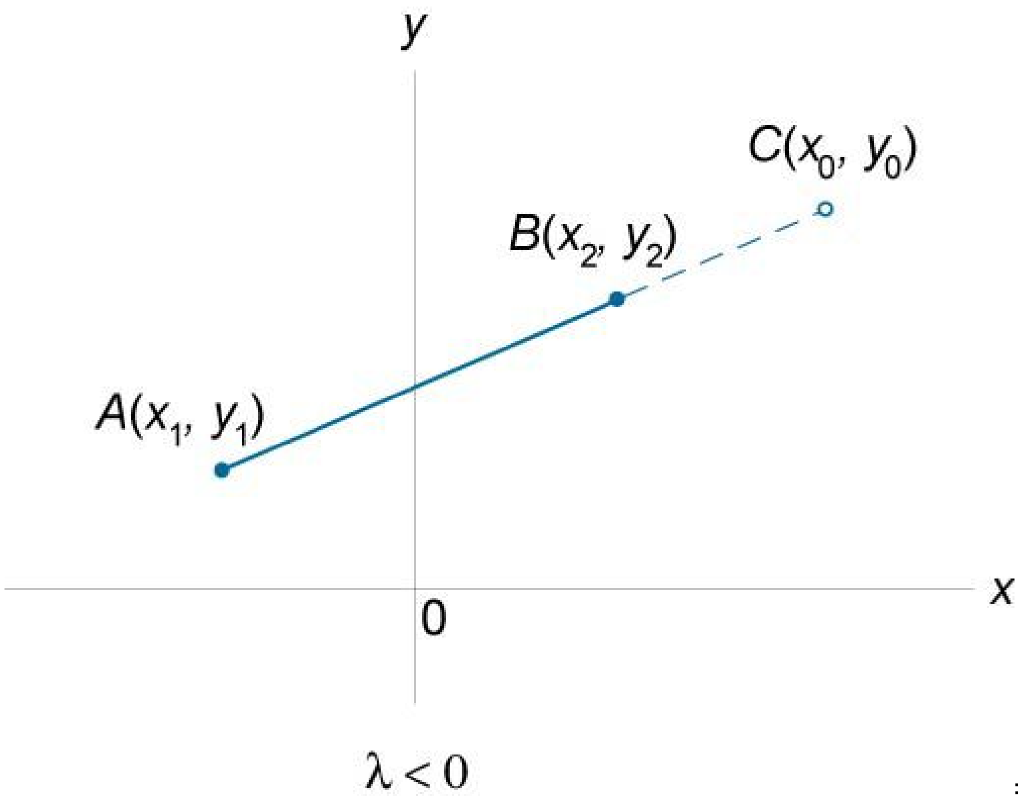


$\lambda > 0$

= =

Figure 89. =

=======



= =

Figure 90.

=

612. **jáÇéçáåí=çÑ=~=iáåÉ=pÉÖãÉåí=**

ñM = ñN +ñO I=óM = óN +óO I=λ=NK=O O

=

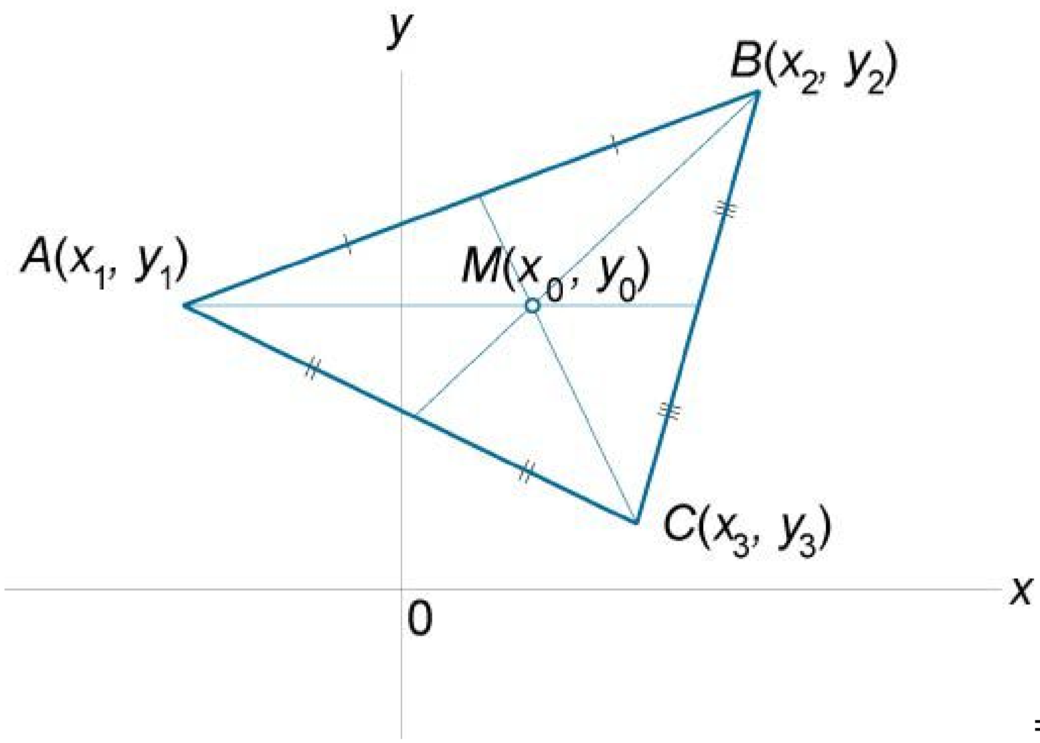
613. **`ÉåíêçáÇ=EfåíÉêëÉÅíáçå=çÑ=jÉÇá~åëF=çÑ=~=qêá~åÖäÉ=**

ñM = ñN +ñO +ñP I=óM = óN +óO +óP I==P

ïÜÉêÉ==() P ( )ñP óP ==~êÉ=îÉêíáÅÉë=çÑ=N óñ^I== ( )IOñ_I==~åÇ==`N O

íÜÉ=íêá~åÖäÉ=^_`K= =

=========



= =

Figure 91.

=

614.

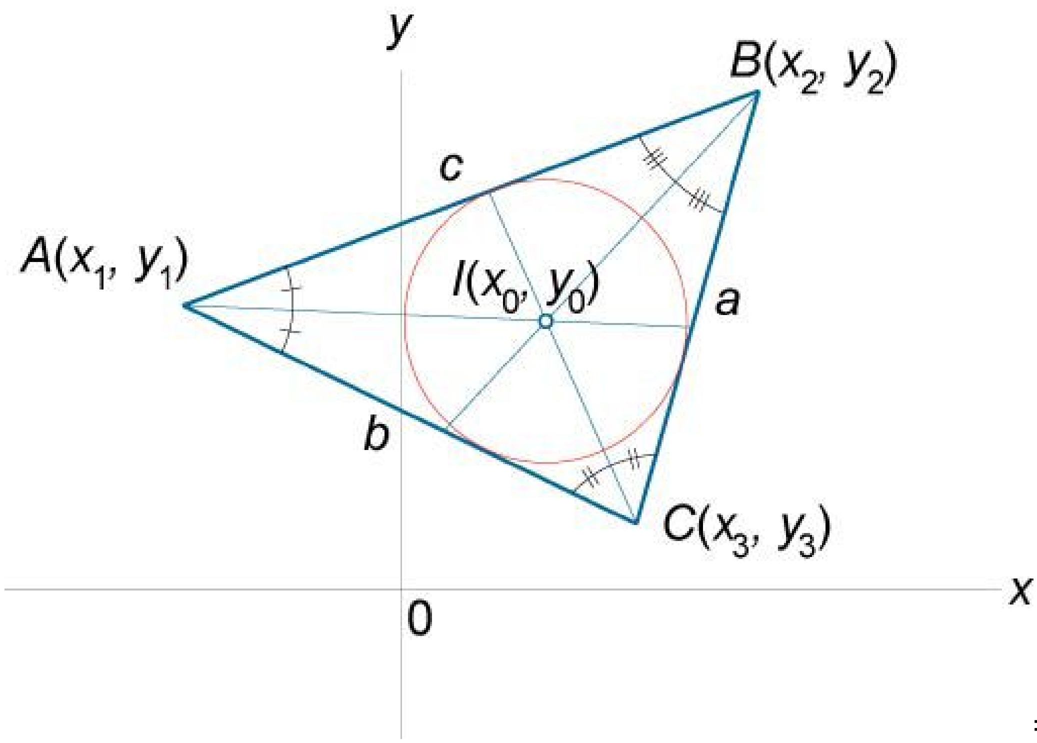
fåÅÉåíÉê=EfåíÉêëÉÅíáçå=çÑ=^åÖäÉ=_áëÉÅíçêëF=çÑ=~=qêá~åÖäÉ=

ñM = ~ñN +ÄñO +ÅñP I=óM = ~óN +ÄóO +ÅóP I==~+Ä+Å ~ +Ä+Å

ïÜÉêÉ=~=_`I=Ä=`^I=Å=^_K==

=

========



= =

Figure 92.

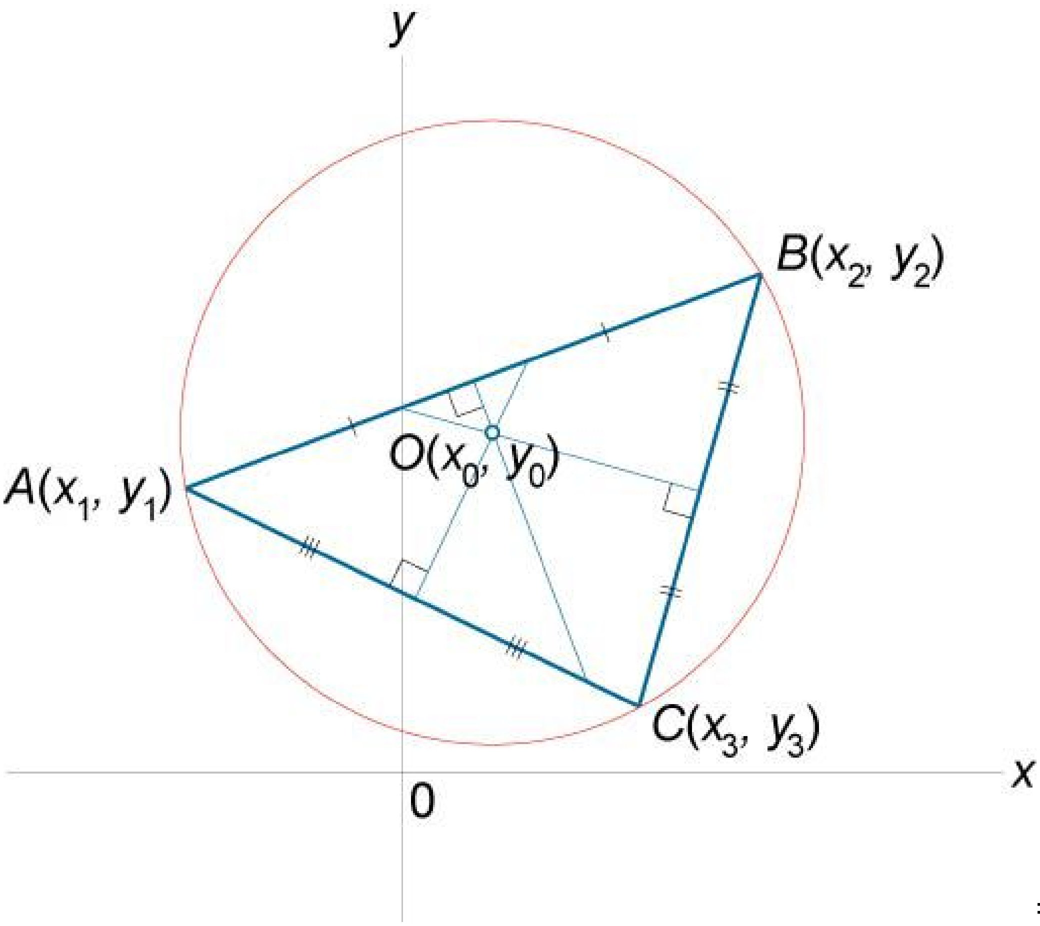
615.

`áêÅìãÅÉåíÉê=EfåíÉêëÉÅíáçå=çÑ=íÜÉ=páÇÉ=mÉêéÉåÇáÅìä~ê=====
=================

_áëÉÅíçêëF=çÑ=~=qêá~åÖäÉ=
ñO +óO óN N ñN ñO +óO NN N N N
ñO +óO óO N ñO ñO +óO NO O O O

ñO +óO óP NI=óM = ñP ñO +óO N=M = P P P Pñ ñN óN N ñN óN N
O ñO óO N O ñO óO N
ñP óP N ñP óP N
=

========



= ==

Figure 93. =

=

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=

=

=

616. lêíÜçÅÉåíÉê=EfåíÉêëÉÅíáçå=çÑ=^äíáîÇÉëF=çÑ=~=qêá~åÖäÉ=

óN ñOñP +óO N ñO +óOóP ñN NN N

óO ñPñN +óO N ñO +óPóN ñO NO O

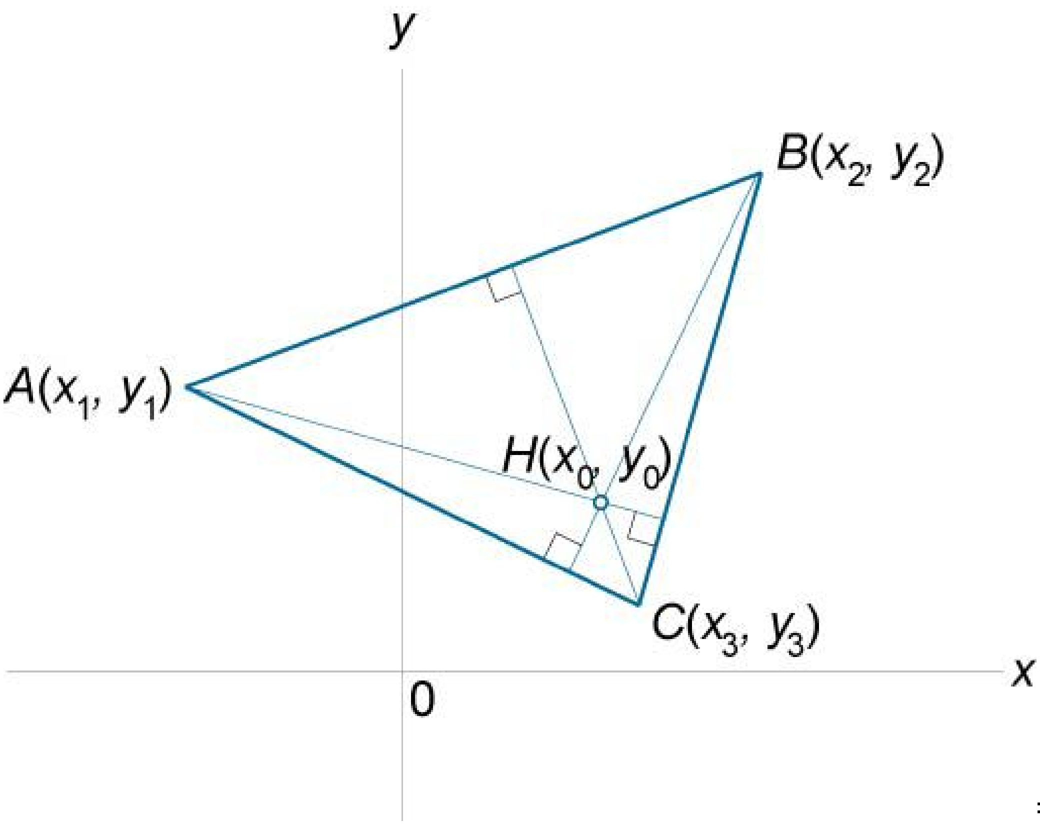
ó

P

ñ

N

ñ
O
+
ó
O N I=ó$_M$ =ñO
ñ P P +óNóO ñP N =M = ñ$_N$ ó$_N$ N ñ$_N$ ó$_N$ N ñ$_O$ ó$_O$ N ñ$_O$ ó$_O$ N ñ$_P$ ó$_P$ N ñ$_P$ ó$_P$ N =
======



= =

Figure 94.
=

617. ^êÉ~=çÑ=~=qêá~åÖäÉ=

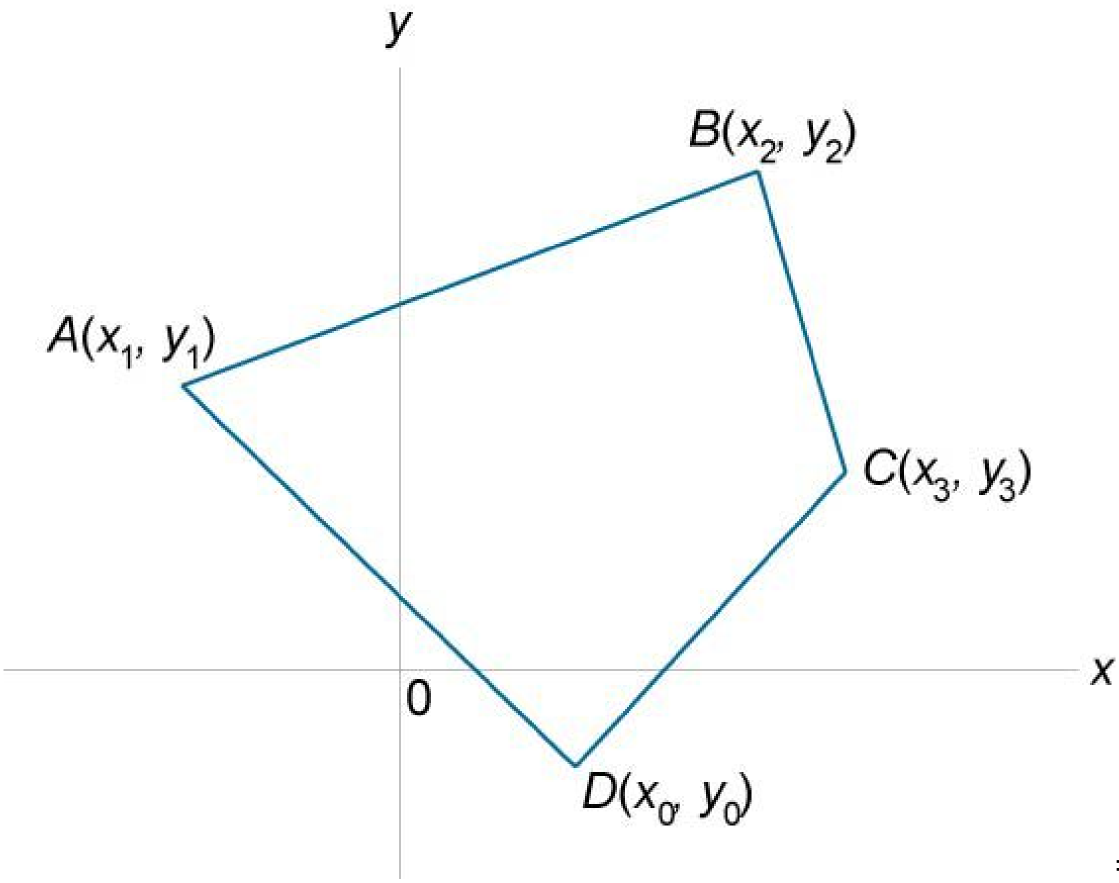
p=()N ñN óN N N ñO−ñN óO−óN =O ñO óO N=()O ñP−ñN óP −óNñP óP N

=
=

618. ^êÉ~=çÑ=~=nì~Çêáä~íÉê~ä=
Np=ñN ñO óN O ñO ñP O óPO

+ ñ()(ñQ óP +óQ) (ñQ −ñN)(óQ +óN)]=P –

=
===

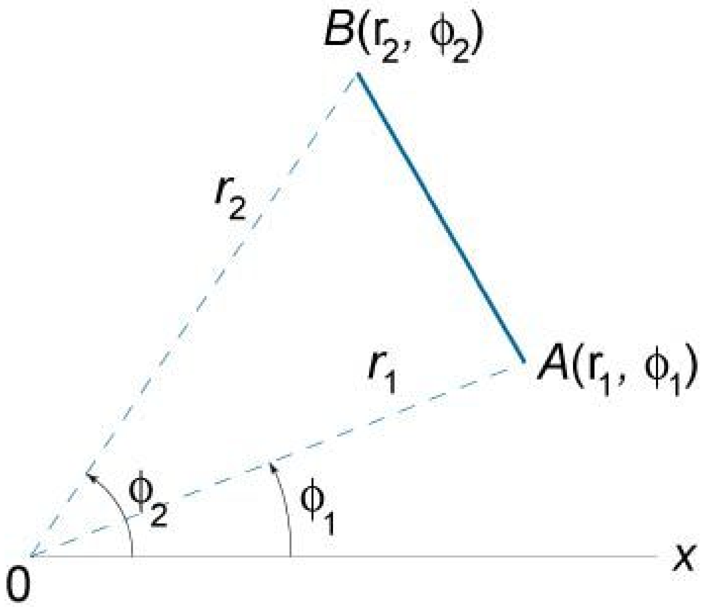


= =

Figure 95.
=
kçíÉW=få=Ñçêãìä~ë=SNTI=SNU=ïÉ=ÅÜççëÉ=íÜÉ=ëáÖå=EHF=çê=E¥F=ëç= íÜ~í=íç=ÖÉí=~=éçëáíáîÉ=~åëïÉê=Ñçê=~êÉ~K==

=
619. **aáëí~åÅÉ=_ÉíïÉÉå=qïç=mçáåíë=áå=mçä~ê=`ççêÇáå~íÉë=**

Ç= ^_= êO +êO −OêNêO Åçë()−ϕN =N O ϕO



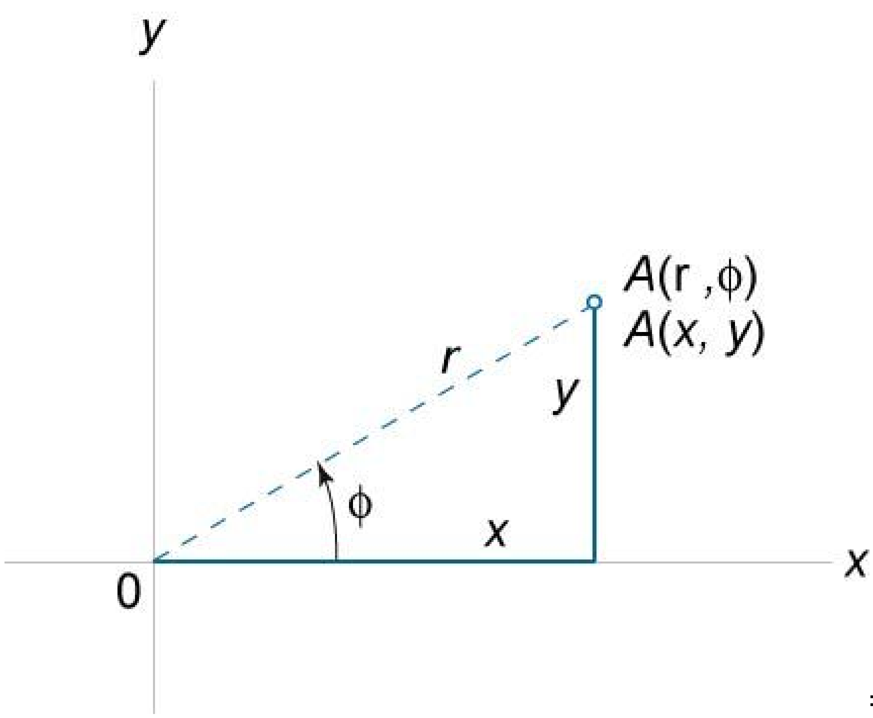
==

Figure 96.

=

620. `çåîÉêíáåÖ=oÉÅí~åÖìä~ê=`ççêÇáå~íÉë=íç=mçä~ê=`ççêÇáå~íÉë= ñ =êÅçëϕI= =êóK=ϕ

=



==

Figure 97.

=

621. `çåîÉêíáåÖ=mçä~ê=`ççêÇáå~íÉë=íç=oÉÅí~åÖìä~ê=`ççêÇáå~íÉë=

ê= ñO +óO I=í~å =ó K=ñ

7.3 Straight Line in Plane

=

mçáåí=ÅççêÇáå~íÉëW=uI=vI=ñI=ñ I=ñ I==ó I=ó I=~ I= ~ I=£==M N M N N O

oÉ~ä=åìãÄÉêëW=âI=~I=ÄI=éI=íI=^I=_I=`I=^I= ^I=£=N O

^åÖäÉëW=αI=β=

^åÖäÉ=ÄÉíïÉÉå=íïç=äáåÉëW=ϕ=

kçêã~ä=îÉÅíçêW= å =

mçëáíáçå=îÉÅíçêëW= êI=~rI=Ä =

=

=

622. **dÉåÉê~ä=bèì~íáçå=çÑ=~=píê~áÖÜí=iáåÉ=**

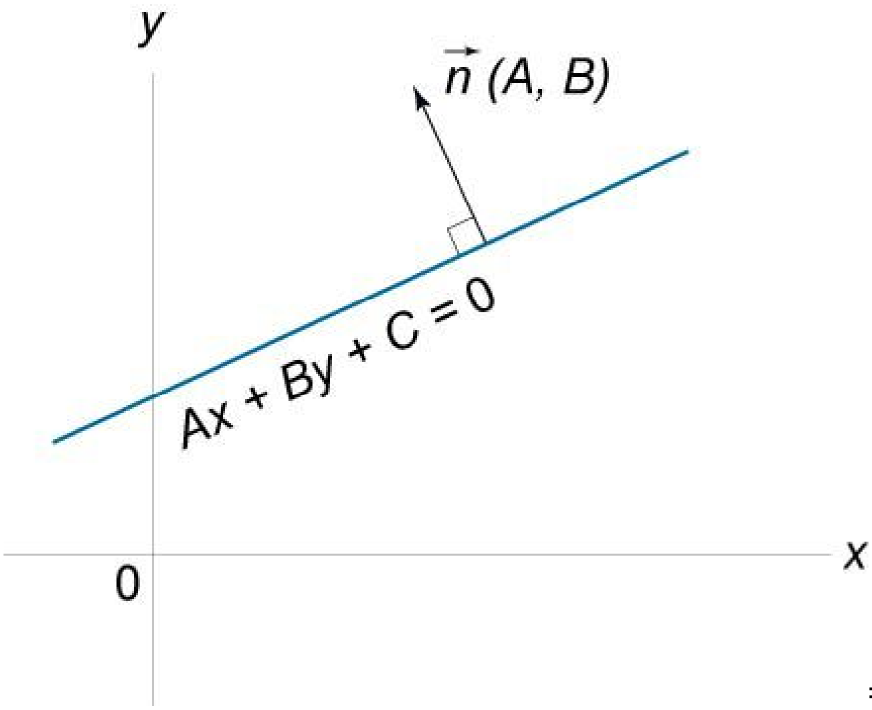
^ñ+_ó+`=M=

=

623. **kçêã~ä=sÉÅíçê=íç=~=píê~áÖÜí=iáåÉ=**

qÜÉ=îÉÅíçê=()=áë=åçêã~ä=íç=íÜÉ=äáåÉ=^ñ+_ó+`=MK=

=

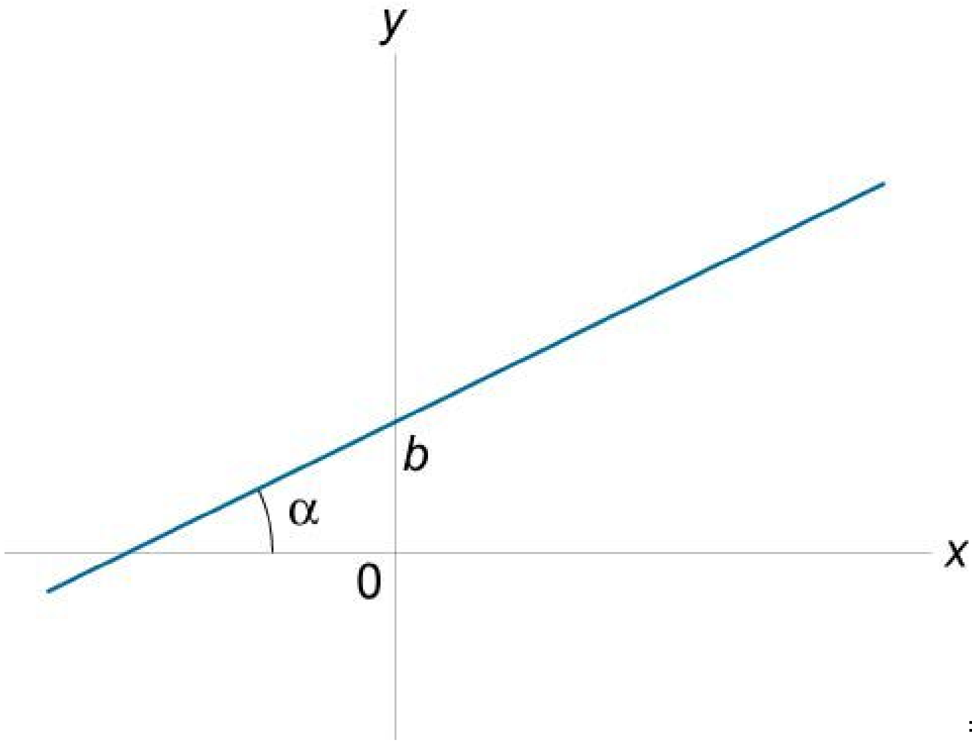


==

Figure 98.

=

624. **bñéäáÅáí=bèì~íáçå=çÑ=~=píê~áÖÜí=iáåÉ=EpäçéÉ-**
fåíÉêÅÉéí=cçêãF= ó +Ä K==
qÜÉ=Öê~ÇáÉåí=çÑ=íÜÉ=äáåÉ=áë= í~åâK= =



==

Figure 99.

=

625. dê~ÇáÉåí=çÑ=~=iáåÉ==

â =í~åα=**óO −óN =ñO −ñN**

=

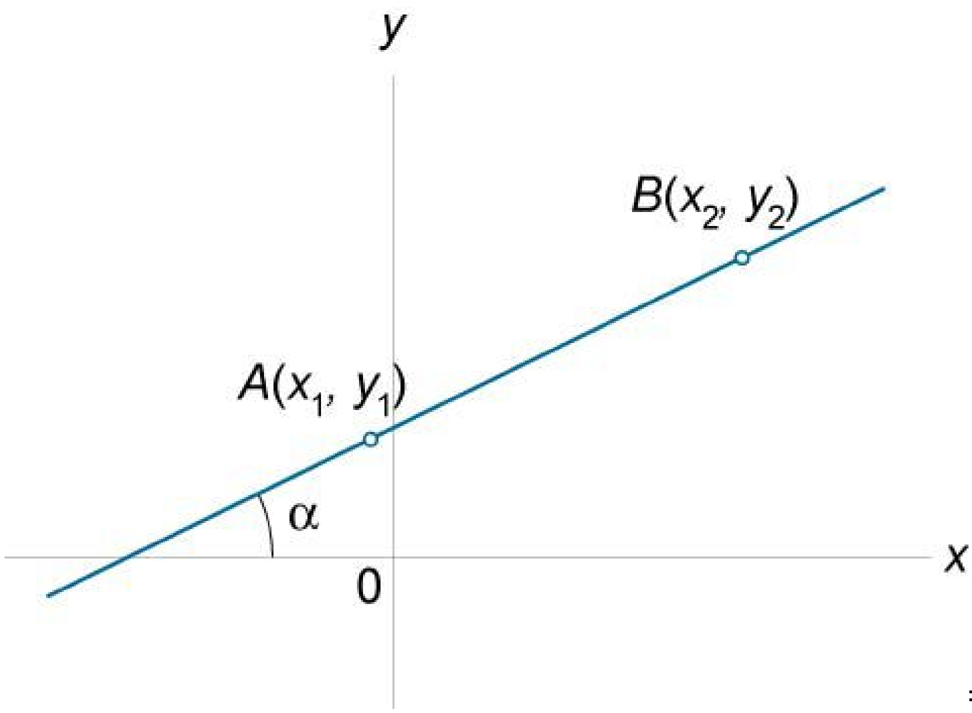
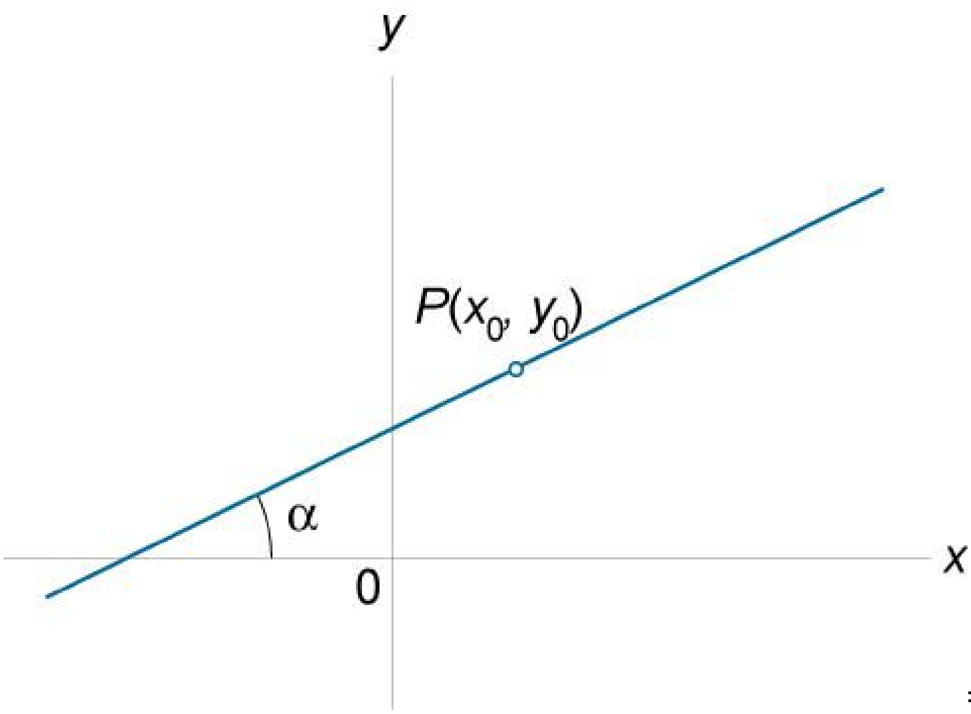


Figure 100. 626.

bèì~íáçå=çÑ=~=iáåÉ=dáîÉå=~=mçáåí=~åÇ=íÜÉ=dê~ÇáÉåí=

ó= óM +â()M I==

ïÜÉêÉ=â=áë=íÜÉ=Öê~ÇáÉåíI=(ñM óM)=áë=~=éçáåí=çå=íÜÉ=äáåÉK=
=



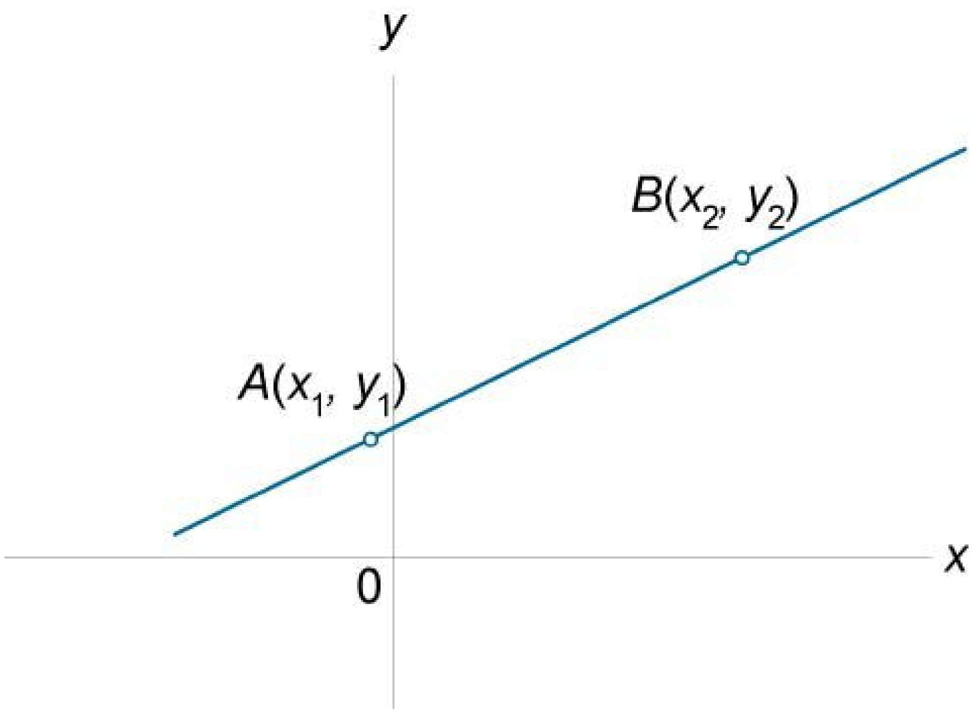
==

Figure 101.
=

627. **bèì~íáçå=çÑ=~=iáåÉ=qÜ~í=m~ëëÉë=qÜêçìÖÜ=qïç=mçáåíë= ó −óN**

= **ñ−ñN ==óO −óN ñO −ñN**

çê=
ñ ó N
ñN óN N =MK=
ñO óO N



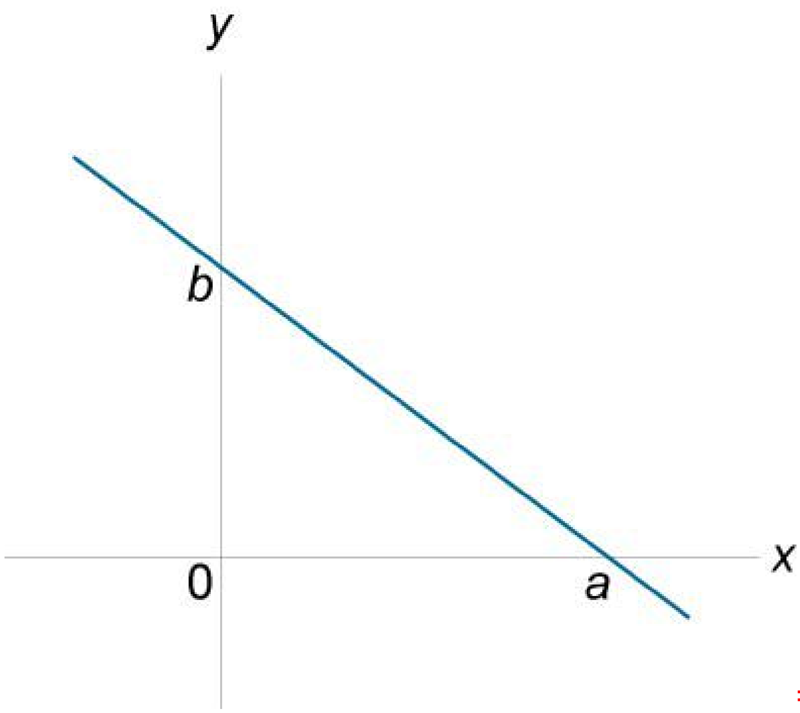
==

Figure 102.
=

628. fåíÉêÅÉéí=cçêã=
ñó
~

=**N=**
Ä

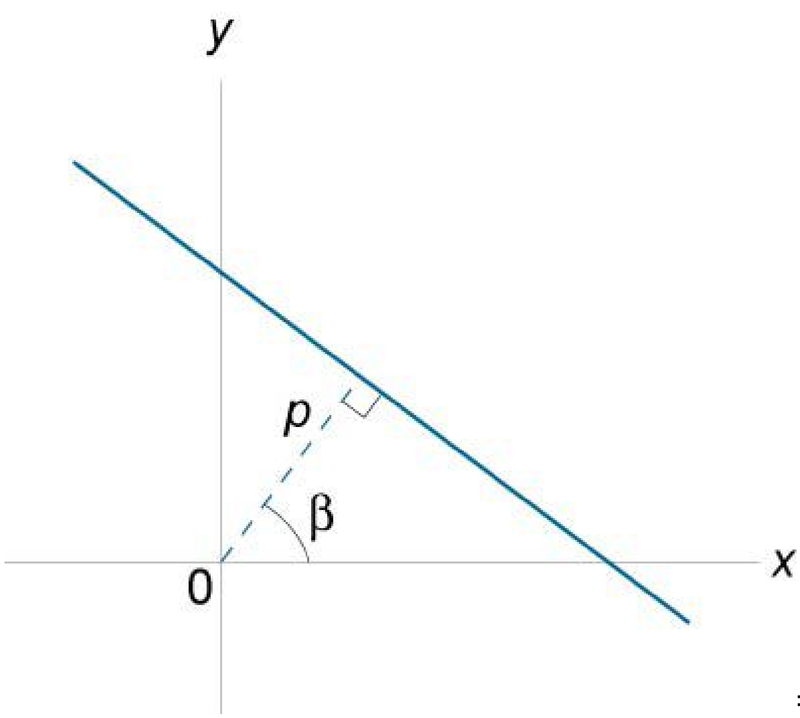
=
629. kçêã~ä=cçêã=
ñÅçëβ+óëáåβ−é=M= =



==

Figure 103.

=



==

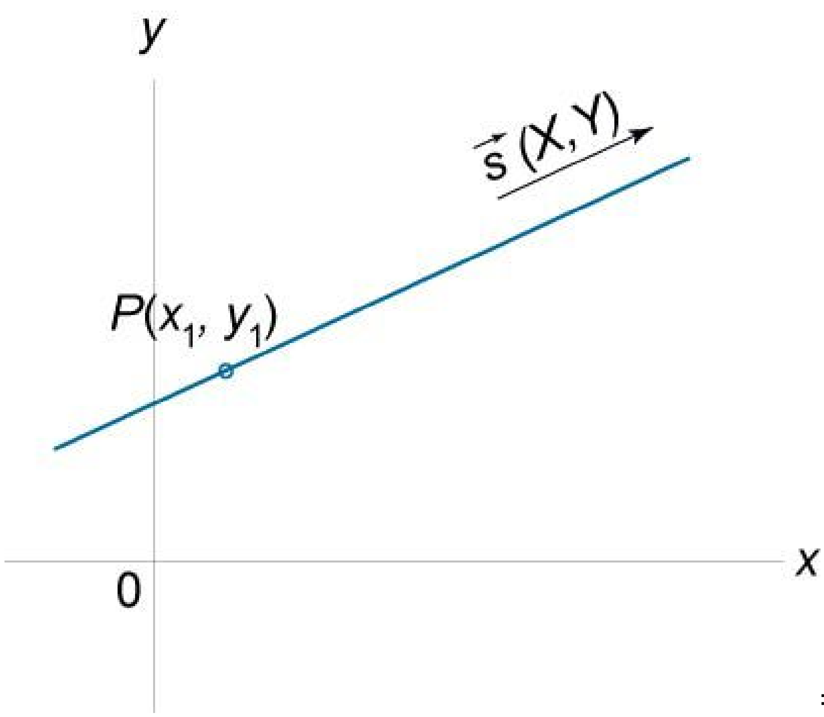
Figure 104.

=

630. **mçáåí=aáêÉÅíáçå=cçêã=**
ñ−ñN −ó óN I==u = v

ïÜÉêÉ=()=áë=íÜÉ=ÇáêÉÅíáçå=çÑ=íÜÉ=äáåÉ=~åÇ= ( )IóNñm=äáÉë=N N

çå=íÜÉ=äáåÉK=



=

Figure 105.
=

631. **sÉêíáÅ~ä=iáåÉ=**
ñ=~ =
=

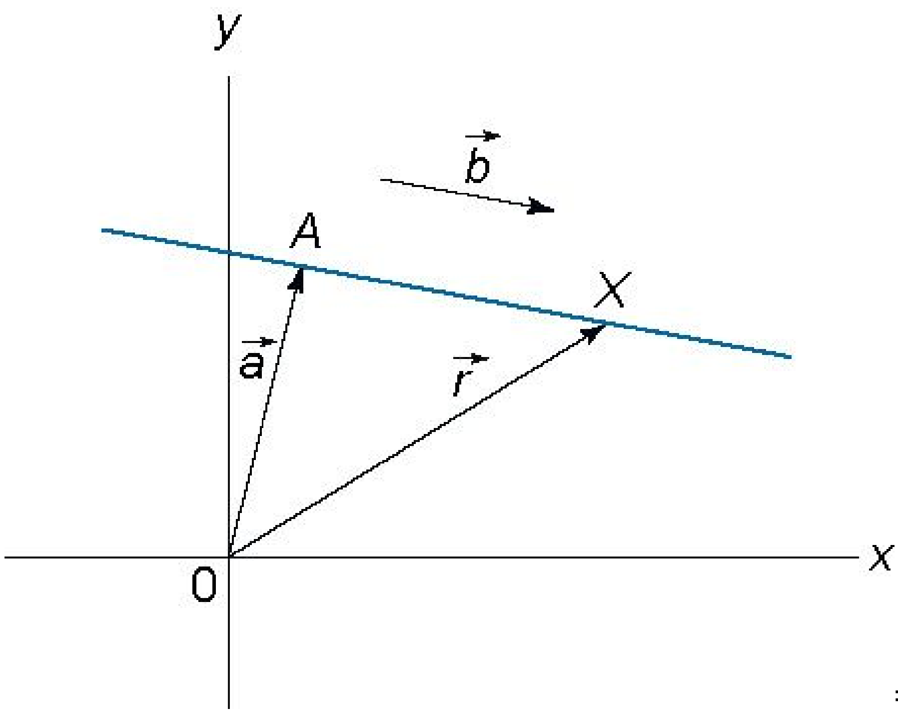
632. **eçêáòçåí~ä=iáåÉ=**
ó=Ä =
=

633. **sÉÅíçê=bèì~íáçå=çÑ=~=píê~áÖÜí=iáåÉ=**
r

= +
r r

í ÄI==ê ~
ïÜÉêÉ==
l=áë=íÜÉ=çêáÖáå=çÑ=íÜÉ=ÅççêÇáå~íÉëI=
u=áë=~åó=î~êá~ÄäÉ=éçáåí=çå=íÜÉ=äáåÉI==
~=áë=íÜÉ=éçëáíáçå=îÉÅíçê=çÑ=~=âåçïå=éçáåí=^=çå=íÜÉ=äáåÉ=I= Ä
=áë=~=âåçïå=îÉÅíçê=çÑ=ÇáêÉÅíáçåI=é~ê~ääÉä=íç=íÜÉ=äáåÉI==
í=áë=~=é~ê~ãÉíÉêI==

→
ê =**lu=áë=íÜÉ=éçëáíáçå=îÉÅíçê=çÑ=~åó=éçáåí=u=çå=íÜÉ=äáåÉK==**



=

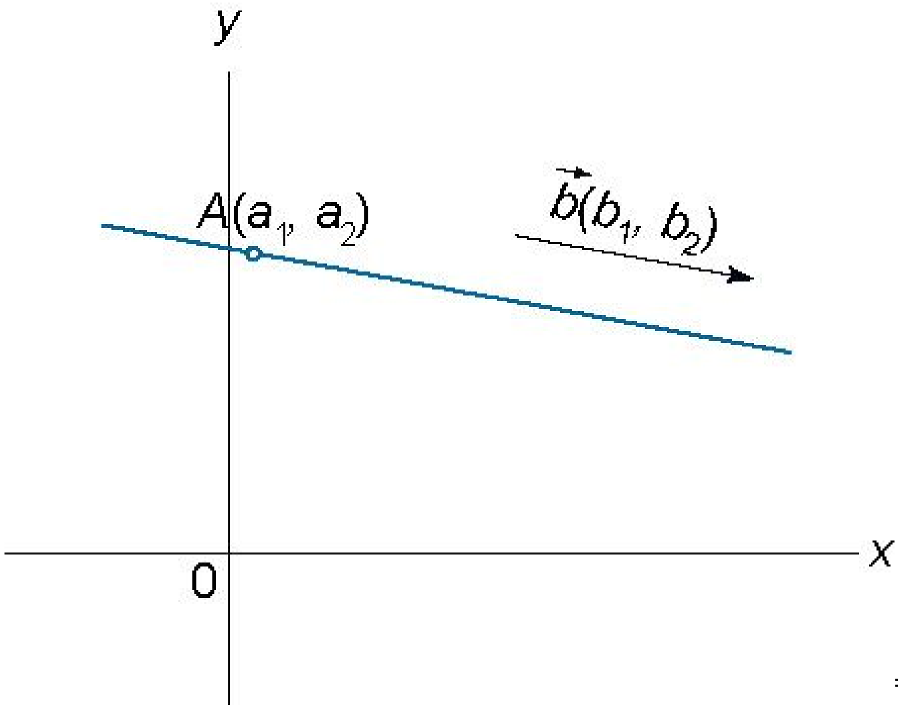
Figure 106.
=
634. píê~áÖÜí=iáåÉ=áå=m~ê~ãÉíêáÅ=cçêã=
ñ =~N +íÄN I==ó=~O +íÄO
ïÜÉêÉ==

()~êÉ=íÜÉ=ÅççêÇáå~íÉë=çÑ=~åó=ìåâåçïå=éçáåí=çå=íÜÉ=äáåÉ
I== ()

~ ~O =~êÉ=íÜÉ=ÅççêÇáå~íÉë=çÑ=~=âåçïå=éçáåí=çå=íÜÉ=äáåÉI==N

()I

Ä =~êÉ=íÜÉ=ÅççêÇáå~íÉë=çÑ=~=îÉÅíçê=é~ê~ääÉä=íç=íÜÉ=äáåÉI==N
ÄO
í=áë=~=é~ê~ãÉíÉêK=



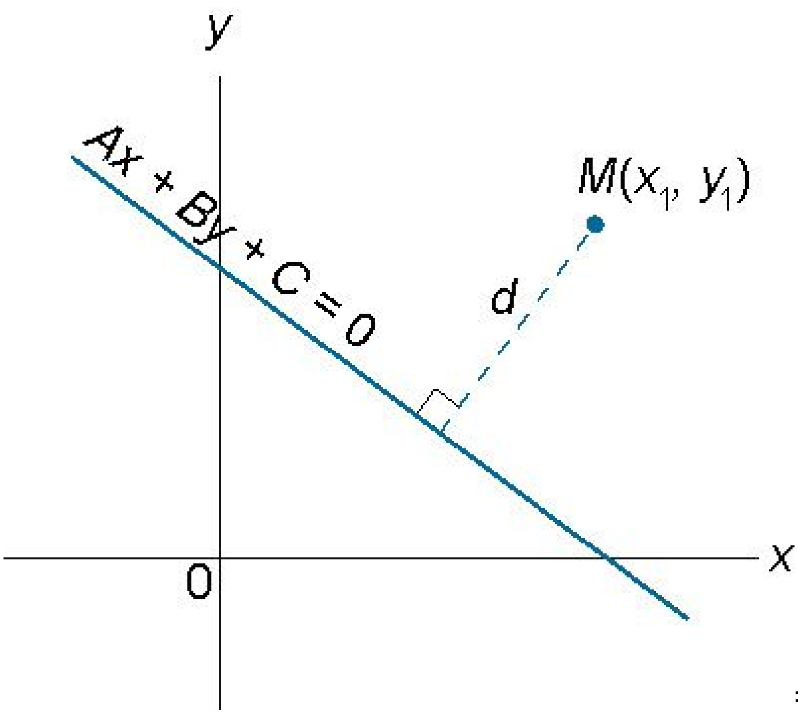
=Figure 107.

=

635. aáëí~åÅÉ=cêçã=~=mçáåí=qç=~=iáåÉ=

qÜÉ=Çáëí~åÅÉ=Ñêçã=íÜÉ=éçáåí=( )=íç=íÜÉ=äáåÉ=

^ñ+_ó+`=M=áë==
Ç= ^~ +_Ä+`K=
^O +_O
=

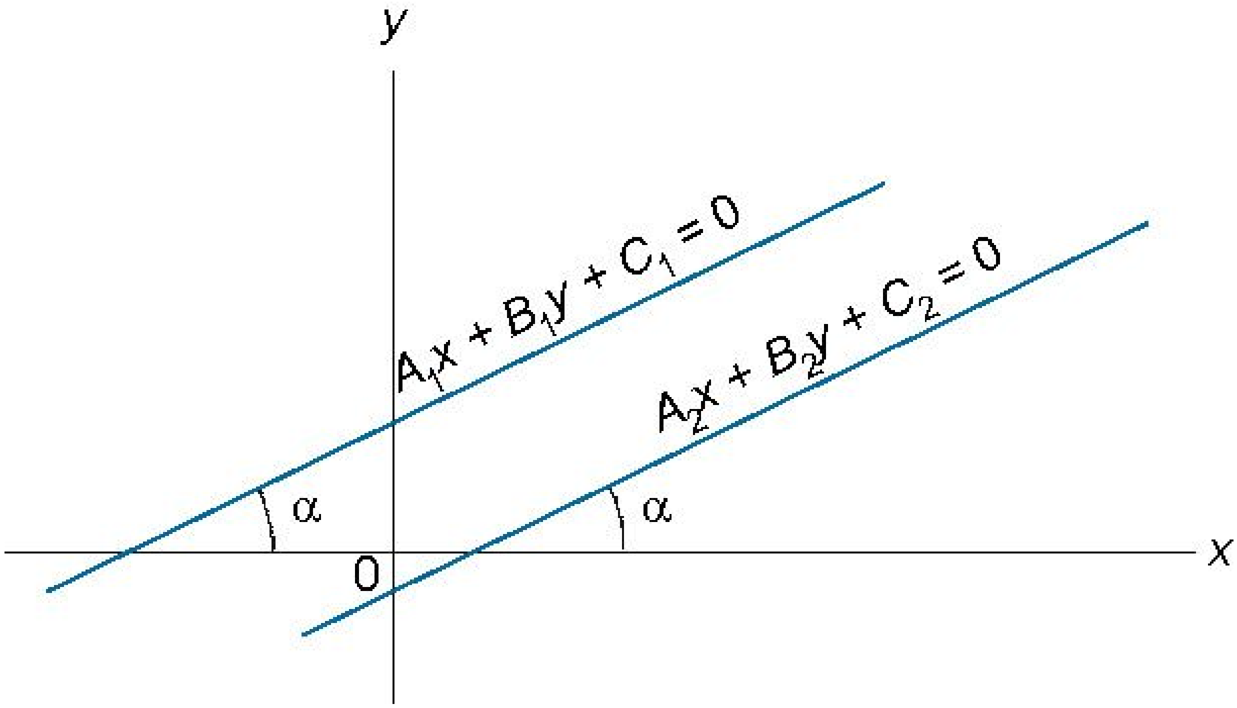


==

Figure 108.

636. m~ê~ääÉä=iáåÉë=
qïç=äáåÉë=ó=âNñ +ÄN=~åÇ=ó=âOñ+ÄO=~êÉ=é~ê~ääÉä=áÑ==
âN=âOK=
qïç= äáåÉë= ^Nñ+_Nó+`N =M=~åÇ= ^Oñ+\_Oó+`O =M=~êÉ=
é~ê~ääÉä=áÑ=

^N=_N K=^O _O
=



==

Figure 109.

=

637. mÉêéÉåÇáÅìä~ê=iáåÉë=

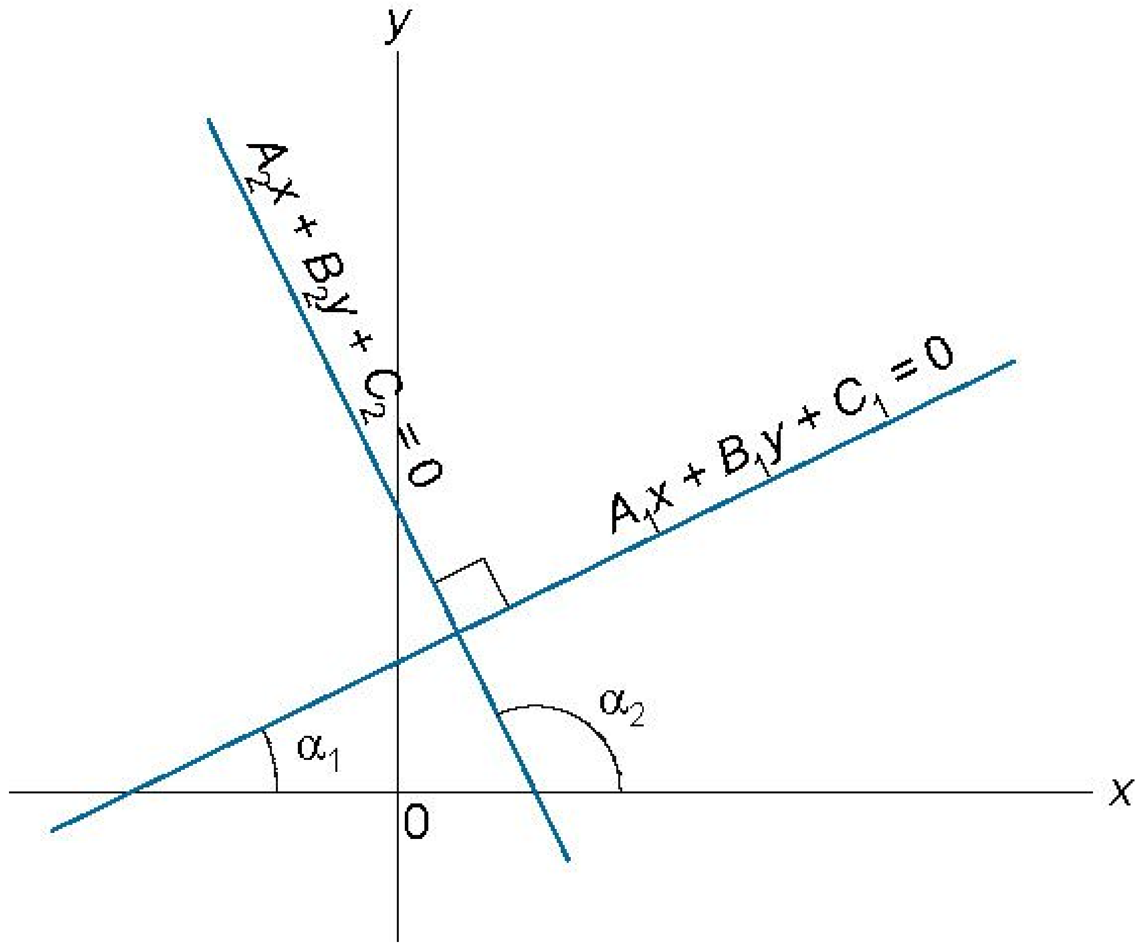
qïç=äáåÉë=ó=âNñ +ÄN=~åÇ=ó=âOñ+ÄO=~êÉ=éÉêéÉåÇáÅìä~ê=áÑ==

âO –N =çêI=Éèìáî~äÉåíäóI=âN O −NK=âN

qïç= äáåÉë= ^Nñ+_Nó+\`N =M=~åÇ= ^Oñ+\_Oó+\`O =M=~êÉ=

éÉêéÉåÇáÅìä~ê=áÑ=

^N^O +_N_O =MK=



==

Figure 110.

=

638. ^åÖäÉ=_ÉíïÉÉå=qïç=iáåÉë=

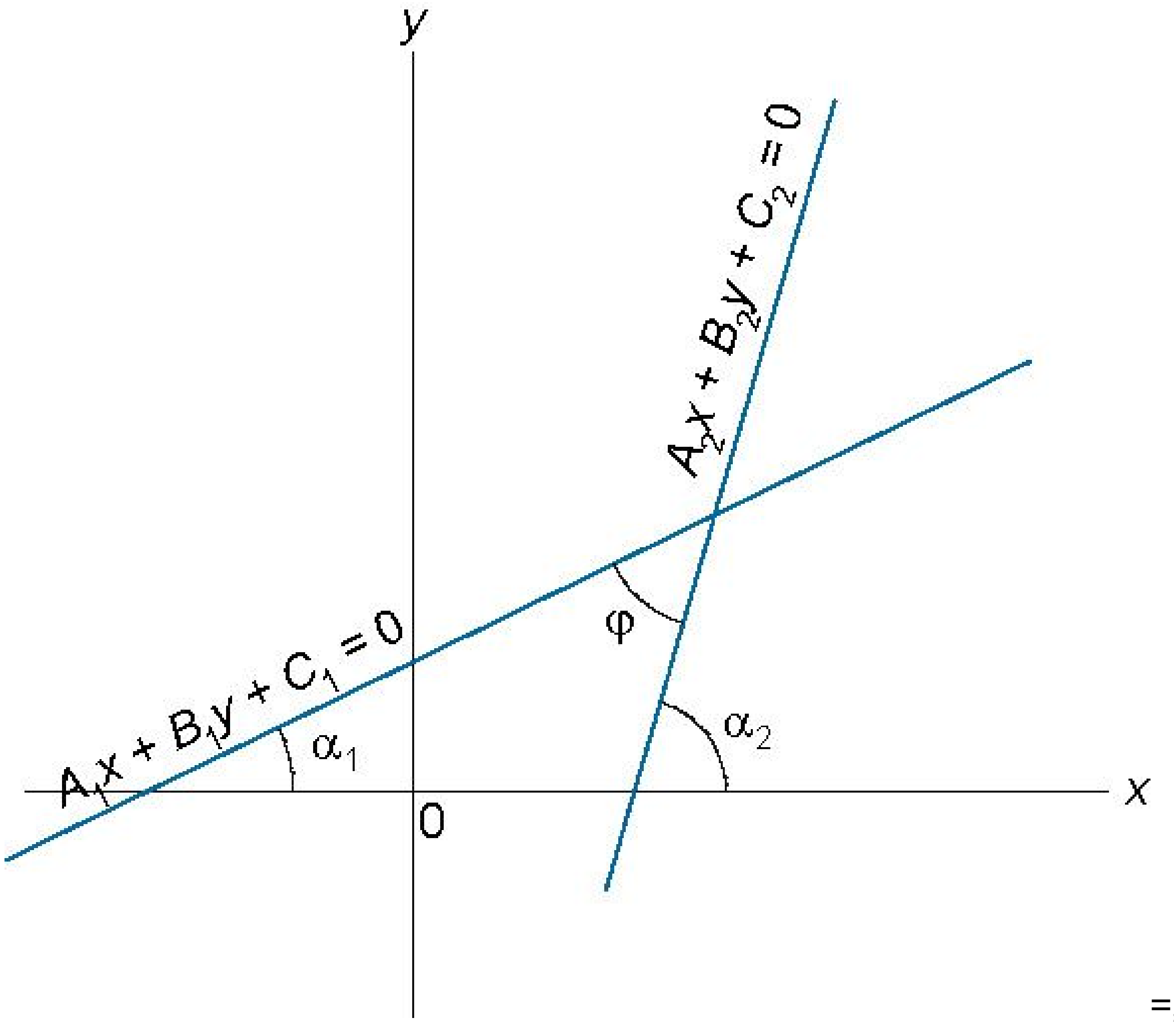
í~åϕ= âO −âN I== +âNâO

Åçëϕ = ^N^O +_N_O K=

^

O +_O . ^O +_O

N N O O



==

Figure 111.

=

639. fåíÉêëÉÅíáçå=çÑ=qïç=iáåÉë=

fÑ=íïç=äáåÉë= ^Nñ+_Nó+`N =M=~åÇ=^Oñ+\_Oó+`O =M=áåíÉê-ëÉÅíI=íÜÉ=áåíÉêëÉÅíáçå=éçáåí=Ü~ë=ÅççêÇáå~íÉë=

ñM = −`N\_O +`O_N **I=óM =** −^N`O +^O`N **K=^N_O −^O_N ^N_O −^O_N**

=

=

=

7.4 Circle

=

o~ÇáìëW=o=

`ÉåíÉê=çÑ=ÅáêÅäÉW=( )I~=

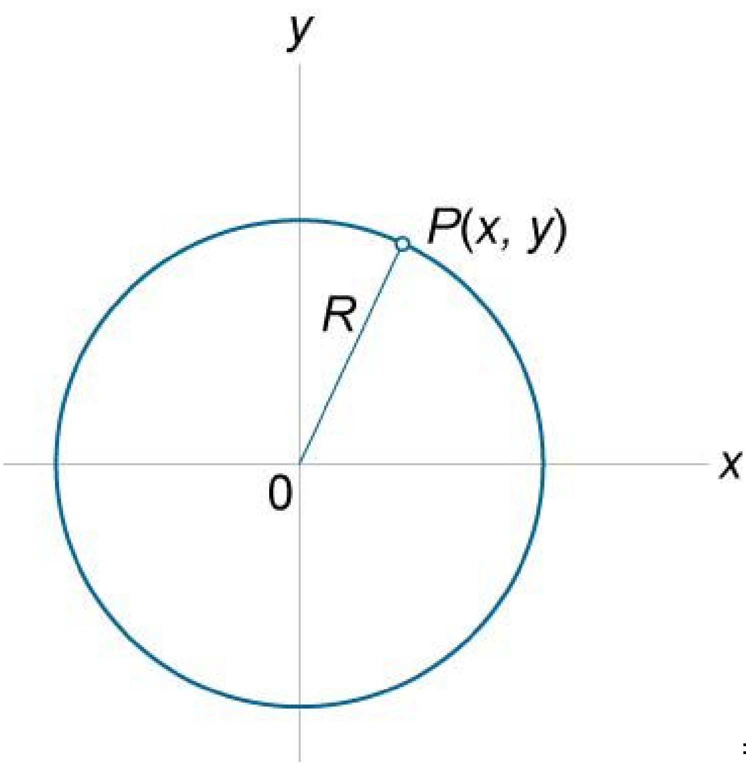
mçáåí=ÅççêÇáå~íÉëW=ñI=óI=ñ I= ó I=£=N N

oÉ~ä=åìãÄÉêëW=^I=_I=`I=aI=bI=cI=í=

640. **bèì~íáçå=çÑ=~=`áêÅäÉ=`ÉåíÉêÉÇ=~í=íÜÉ=lêáÖáå=Epí~åÇ~êÇ= cçêãF=**

ñO +óO =oO=

======



==

Figure 112.

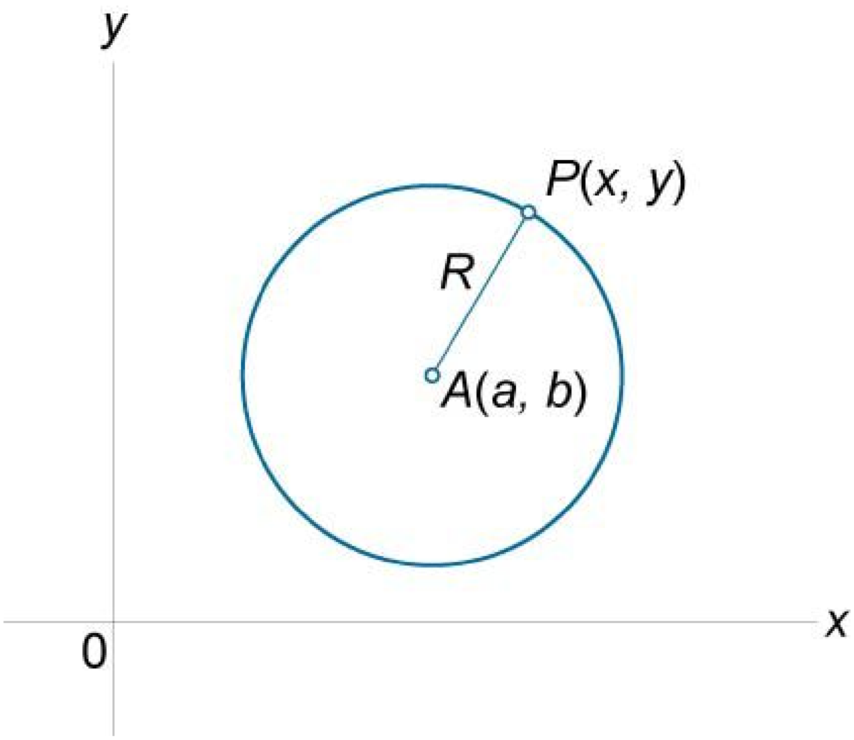
=

641. bèì~íáçå=çÑ=~=`áêÅäÉ=`ÉåíÉêÉÇ=~í=^åó=mçáåí =( )

ñ

()( )ó−Ä^{O} =o^{O}

~



642. qÜêÉÉ=mçáåí=cçêã ñ^{O} +ó^{O} ñ ó N

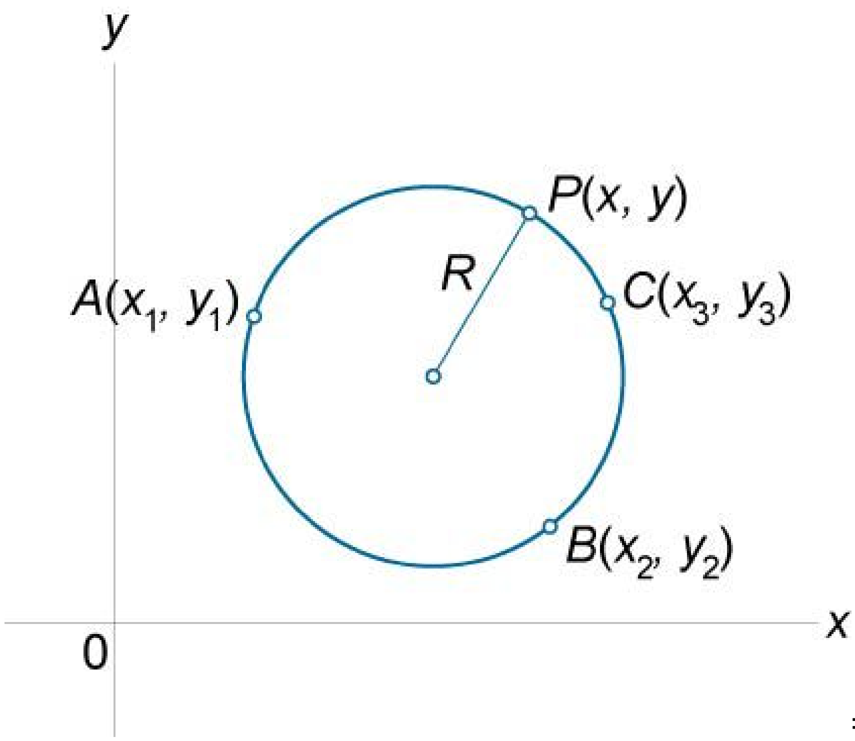
ñ

O +ó^{O}

N N ñN óN N =Mñ^{O} +ó^{O} ñ$_{O}$ ó$_{O}$ N$_{O}$ $_{O}$

ñ^{O} +ó^{O} ñ$_{P}$ ó$_{P}$ N$_{P}$ $_{P}$

=



==

Figure 114.

=

643. m~ê~ãÉíêáÅ=cçêã

□□□ñ =oÅçëíI=íMKó=oëáåí

=

644. dÉåÉê~ä=cçêã

^ñO +^óO +añ+bó +c =M=E^=åçåòÉêçI=aO >O Q^cFK==

qÜÉ=ÅÉåíÉê=çÑ=íÜÉ=ÅáêÅäÉ=Ü~ë=ÅççêÇáå~íÉë=( )I=ïÜÉêÉ==

~ –^{a} I=Ä –^{b} K=$_{O\wedge}$ $_{O\wedge}$
qÜÉ=ê~Çáìë=çÑ=íÜÉ=ÅáêÅäÉ=áë
o= a^{O} –^{O} Q^c$_{KO\wedge}$

=
=
=

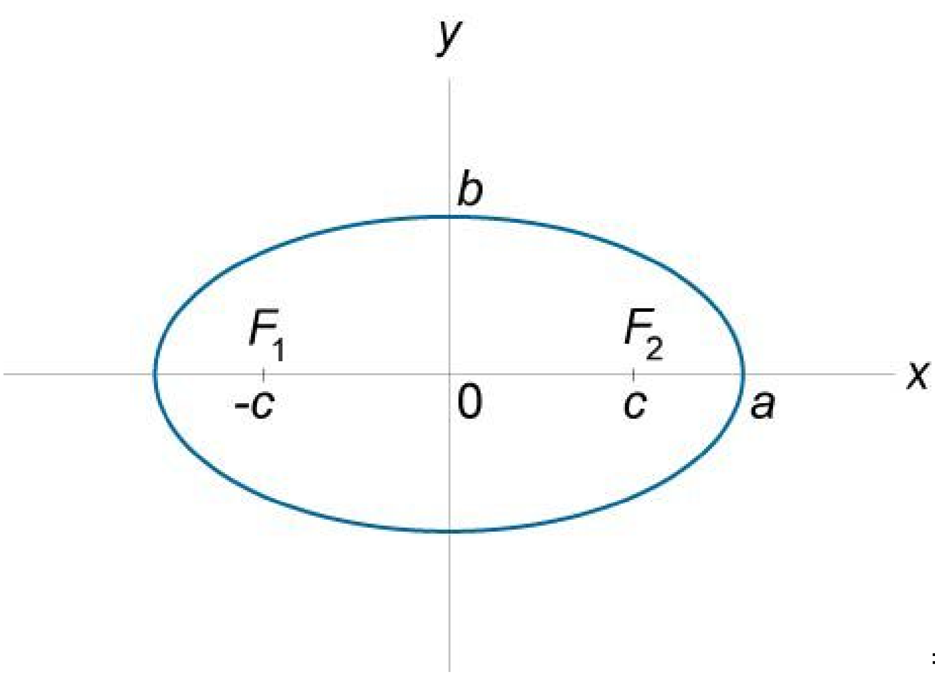
7.5 Ellipse

=
pÉãáã~àçê=~ñáëW=~=
pÉãáãáåçê=~ñáëW=Ä=

cçÅáW=N()I=O ( )=

aáëí~åÅÉ=ÄÉíïÉÉå=íÜÉ=ÑçÅáW=OÅ= =
bÅÅÉåíêáÅáíóW=É==
oÉ~ä=åìãÄÉêëW=^I=_I=`I=aI=bI=cI=í=
mÉêáãÉíÉêW=i=
^êÉ~W=p=
=
=

645. **bèì~íáçå=çÑ=~å=bääáéëÉ=Epí~åÇ~êÇ=cçêãF ñO óO**
~O +ÄO =N

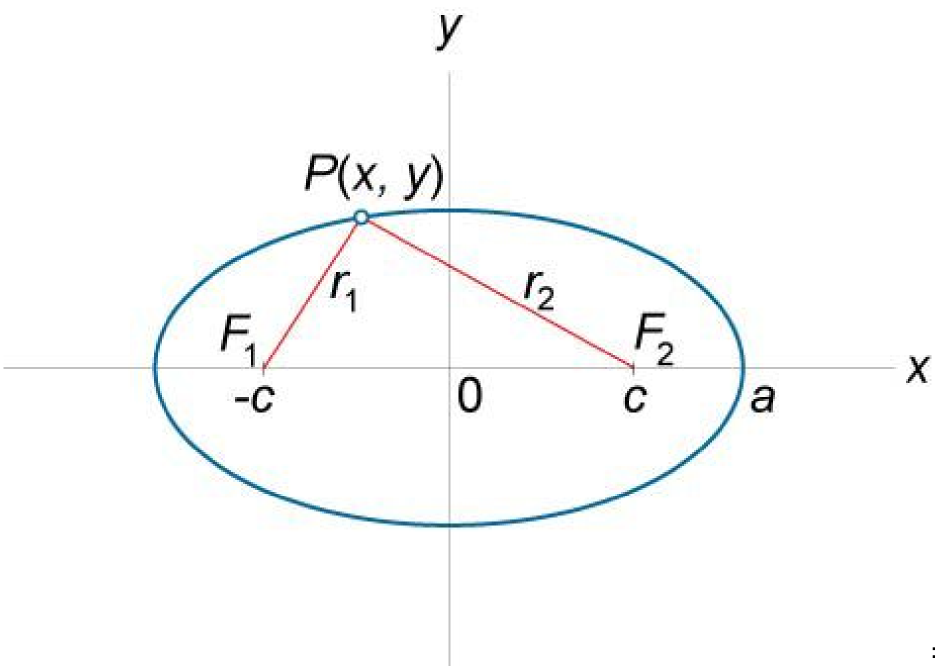


==

646. **êN + =O~I=O**

ïÜÉêÉ== ( )==çå=Nê I==Oê ==~êÉ==Çáëí~åÅÉë==Ñêçã==~åó==éçáåí==

íÜÉ=ÉääáéëÉ=íç=íÜÉ=îïç=ÑçÅáK=
=



==

Figure 116.
=

647. ~O =ÄO +ÅO
=
648. bÅÅÉåíêáÅáíó

Å
É <N=~
=
649. bèì~íáçåë=çÑ=aáêÉÅíêáÅÉë
ñ
=
±
~ =±~O

É Å=

=

650. **m~ê~ãÉíêáÅ=cçêã**

□□□ñ =~Åçëí I=M≤í≤Oπ K ó=Äëåí

=

651. **dÉåÉê~ä=cçêã**

^ñO +_ñó+`óO +añ+bó+c =MI==

ïÜÉêÉ=_O −Q^`<MK=

=

652. **dÉåÉê~ä=cçêã=ïáíÜ=^ñÉë=m~ê~ääÉä=íç=íÜÉ=`ççêÇáå~íÉ=^ñÉë**

^ñO +`óO +añ+bó +c =MI==

ïÜÉêÉ=^`>MK

=

653. **`áêÅìãÑÉêÉåÅÉ**

i=Q~b()I==

ïÜÉêÉ==íÜÉ==ÑìåÅíáçå=b==áë==íÜÉ=ÅçãéäÉíÉ==ÉääáéíáÅ=áåíÉÖê ~ä==çÑ= íÜÉ=ëÉÅçåÇ=âáåÇK==

=

654. **^ééêçñáã~íÉ=cçêãìä~ë=çÑ=íÜÉ=`áêÅìãÑÉêÉåÅÉ**

i =π(NKR()− ~Ä)I==

i=π O()K=

=

655. p π~Ä=

=

=

=

7.6 Hyperbola

=

qê~åëîÉêëÉ=~ñáëW=~=

`çåàìÖ~íÉ=~ñáëW=Ä=

cçÅáW=N()I=cO ( )=

aáëí~åÅÉ=ÄÉíïÉÉå=íÜÉ=ÑçÅáW=OÅ= =
bÅÅÉåíêáÅáíóW=É==
^ëóãéíçíÉëW=ëI=í=
oÉ~ä=åìãÄÉêëW=^I=_I=`I=aI=bI=cI=íI=â=
=
=
656. bèì~íáçå=çÑ=~=eóéÉêÄçä~=Epí~åÇ~êÇ=cçêãF=

ñO óO
~O −ÄO =N=
=

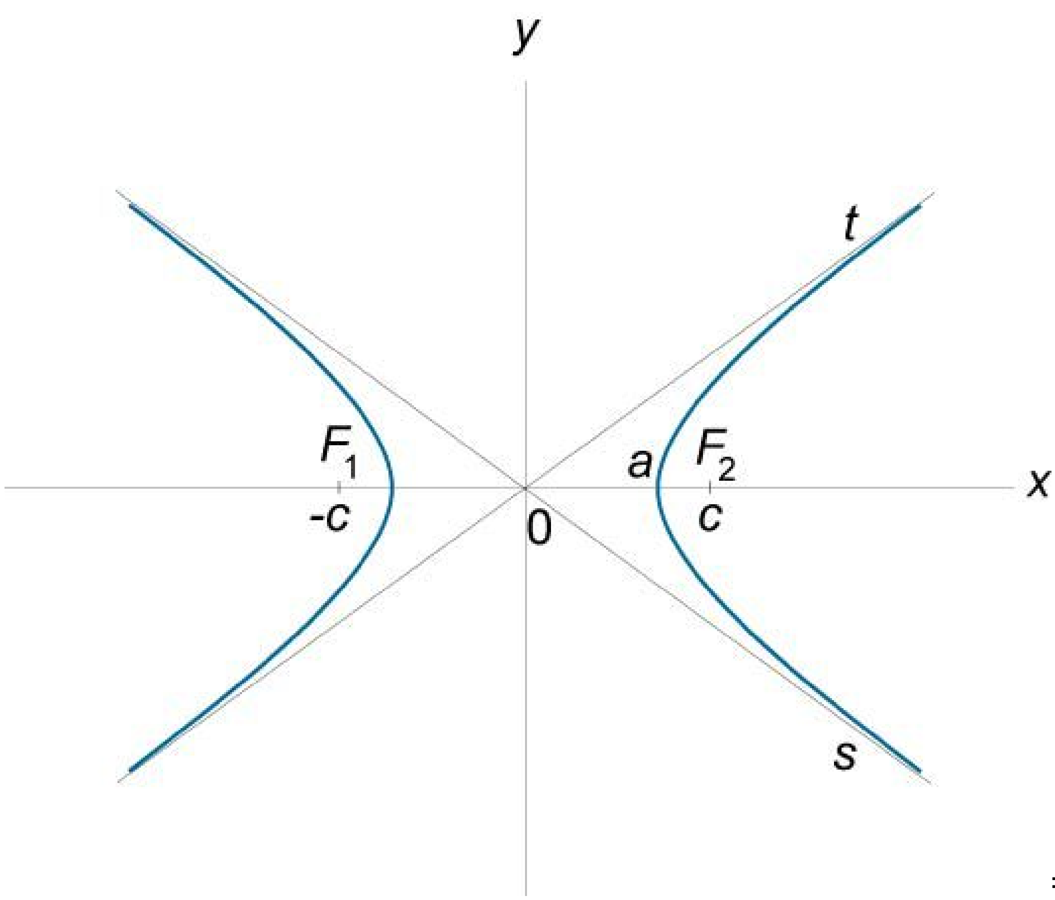


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Figure 117.

=

657. **ê$_{N}$ – $^{=}$O~ I=$_{O}$**

**ïÜÉêÉ==$_{N}$êI==$_{O}$ê ==~êÉ==Çáëí~åÅÉë==Ñêçã==~åó=éçáåí== (
)==çå=**

íÜÉ=ÜóéÉêÄçä~=íç=íÜÉ=ïïç=ÑçÅáK=

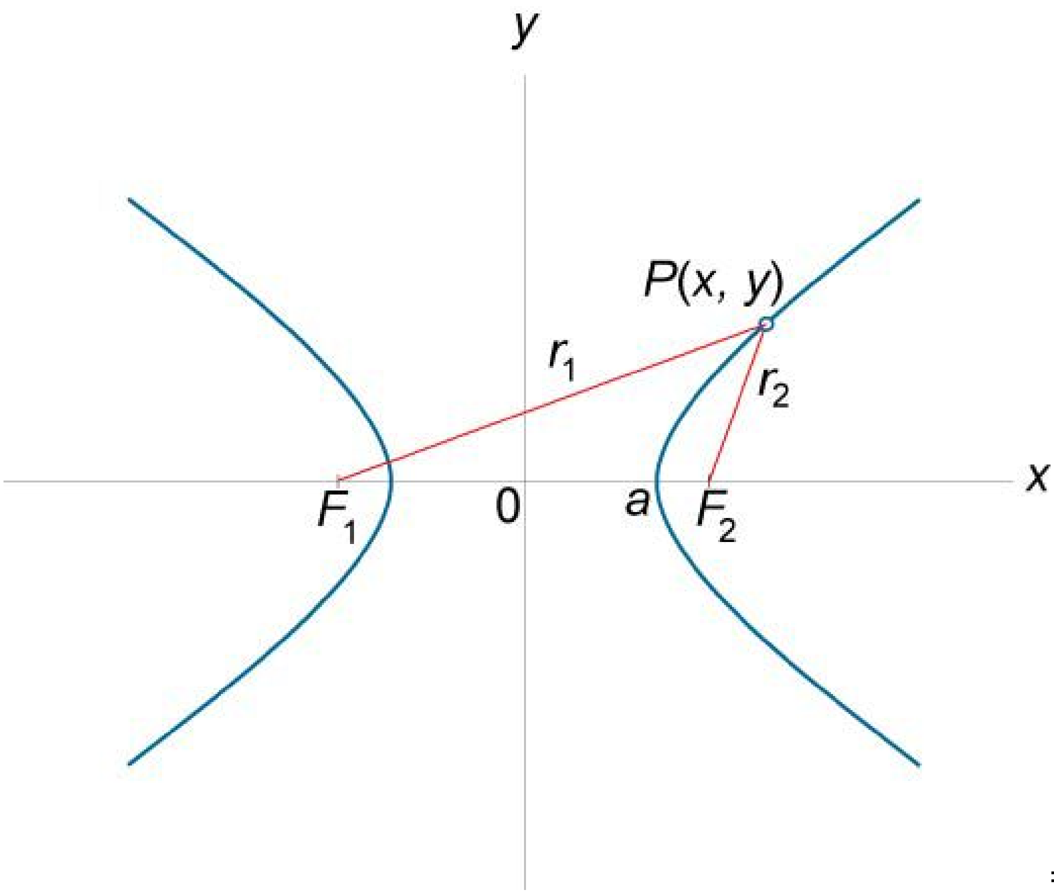

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Figure 118.

=

658. **bèì~íáçåë=çÑ=^ëóãéíçíÉë=**

ó ±^{Ä} ñ=$_{\sim}$

=

659. **Å^{O} +^{O} Ä^{O}=**

=

660. **bÅÅÉåíêáÅáíó**

Å

É >**N=**
~
=
661. bèì~íáçåë=çÑ=aáêÉÅíêáÅÉë

ñ
=
±
~ =±$^{\sim O}$

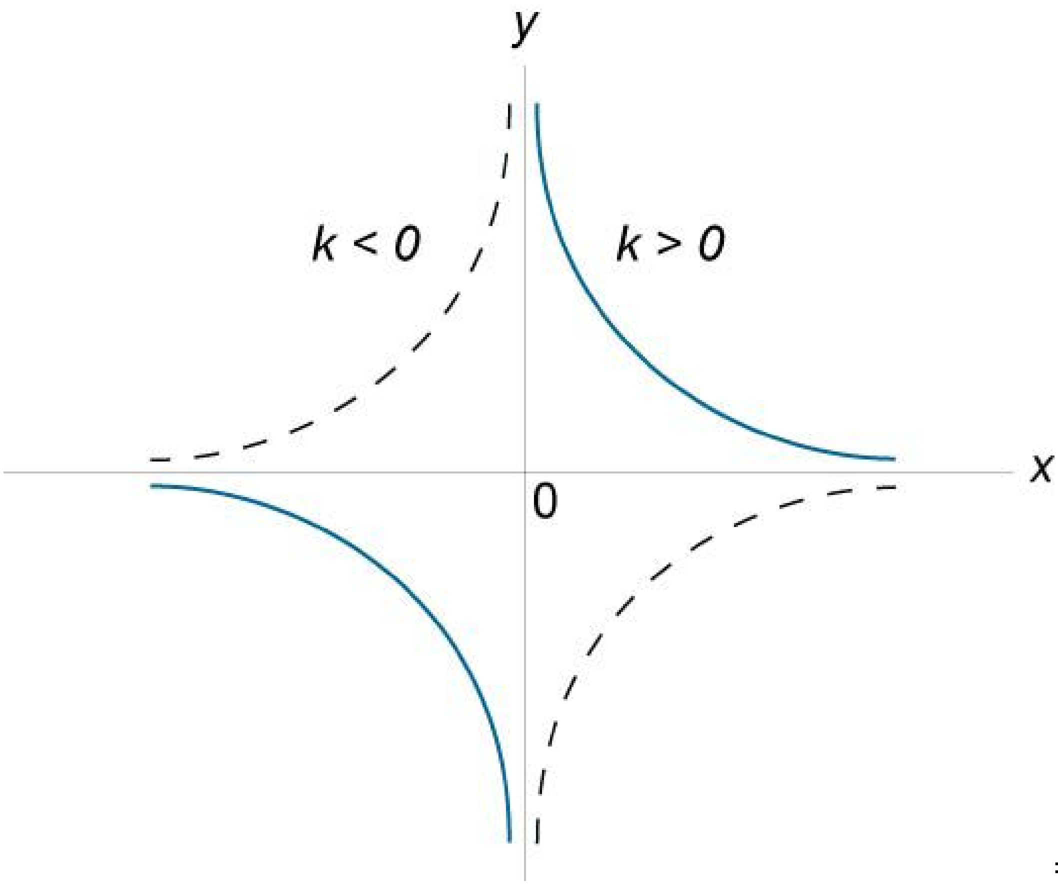
É Å$^{=}$
=
=
662.
m~ê~ãÉíêáÅ=bèì~íáçåë=çÑ=íÜÉ=oáÖÜí=_ê~åÅÜ=çÑ=~=eóéÉêÄçä~=

□□□$^{ñ =\sim}$ **ÅçëÜ í I=M≤í≤OπK**$_{ó=Ä\ ëáåÜí}$

=
663. dÉåÉê~ä=cçêã
^ñ^{O} +**_ñó**+**`ó**^{O} +**añ**+**bó**+**c** =**MI==**
ïÜÉêÉ=_^{O} −**Q^`**>**MK=**
=
664. dÉåÉê~ä=cçêã=ïáíÜ=^ñÉë=m~ê~ääÉä=íç=íÜÉ=`ççêÇáå~íÉ=^ñÉë
^ñ^{O} +**`ó**^{O} +**añ**+**bó** +**c** =**MI==**
ïÜÉêÉ=^`<**MK=**

665. ^ëóãéíçíáÅ=cçêã=
=
É
O
ñó$_{Q}$ **I==**
çê==
=
â
I=ïÜÉêÉ=

â
=
É
O
óñ Q K=
få= íÜáë= Å~ëÉ= I= íÜÉ= ~ëóãéíçíÉë= Ü~îÉ= Éèì~íáçåë= ñ= =~åÇ=
ó=MK==



==

Figure 119.
=
=
=

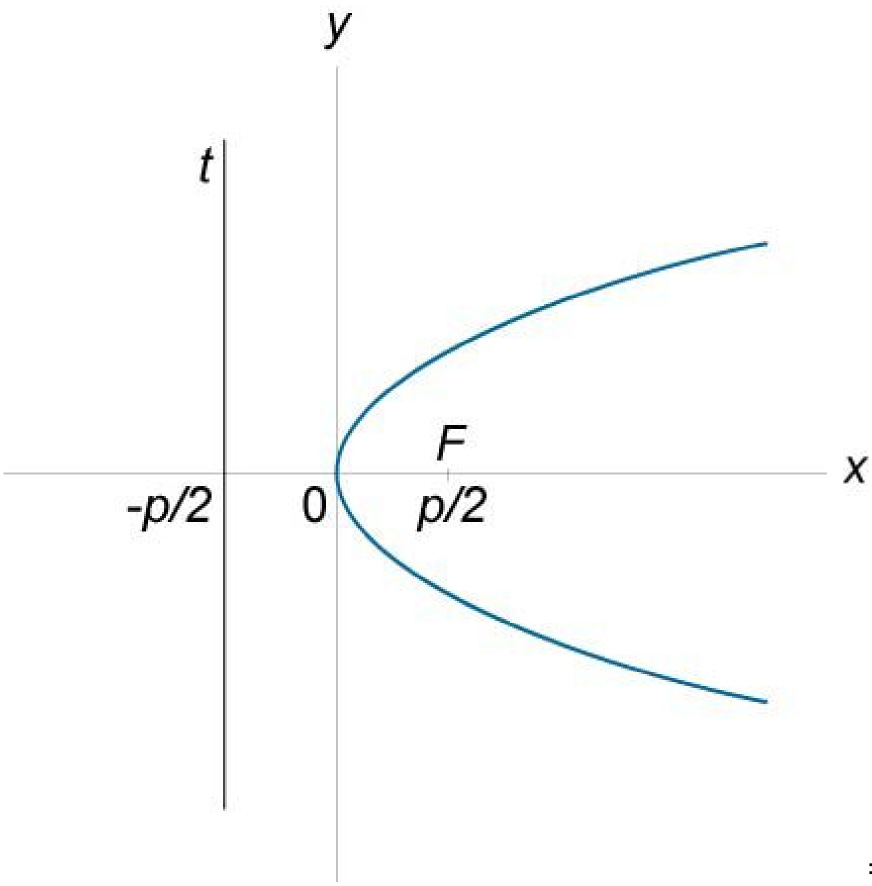
7.7 Parabola

=
cçÅ~ä=é~ê~ãÉíÉêW=é=
cçÅìëW=c=

sÉêíÉñW=() =M M

oÉ~ä=åìãÄÉêëW=^I=_I=`I=aI=bI=cI=éI=~I=ÄI=Å=
=
=

666. bèì~íáçå=çÑ=~=m~ê~Äçä~=Epí~åÇ~êÇ=cçêãF
óO=Oéñ



==

Figure 120.
=
bèì~íáçå=çÑ=íÜÉ=ÇáêÉÅíêáñ

ñ –é I=O
`çêÇáå~íÉë=çÑ=íÜÉ=ÑçÅìë=

□écI=□□O □□
`çç꟪áå~íÉë=çÑ=íÜÉ=îÉêíÉñ=

()

MjK=
=

667. dÉåÉê~ä=cçêã
^ñO +_ñó+`óO +añ+bó+c =MI==
ïÜÉêÉ=_O −Q^`=MK=
=

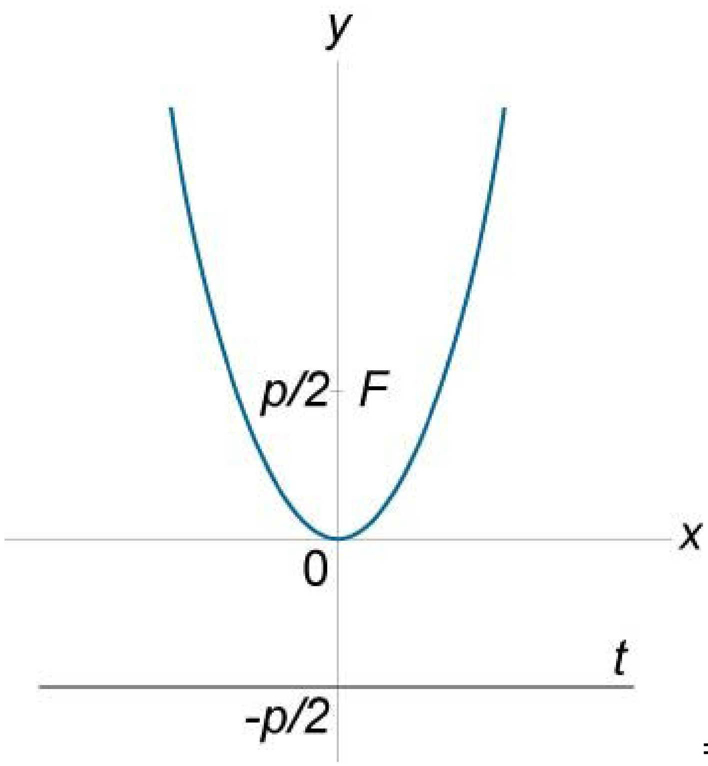
668. ó=~ñOI=é=N K=O~
bèì~íáçå=çÑ=íÜÉ=ÇáêÉÅíêáñ
ó −é I=O
`çç꟪áå~íÉë=çÑ=íÜÉ=ÑçÅìë=

□ é□

McI=
□□ O□□
`çç꟪áå~íÉë=çÑ=íÜÉ=îÉêíÉñ=

()

MjK=
=



==

Figure 121.

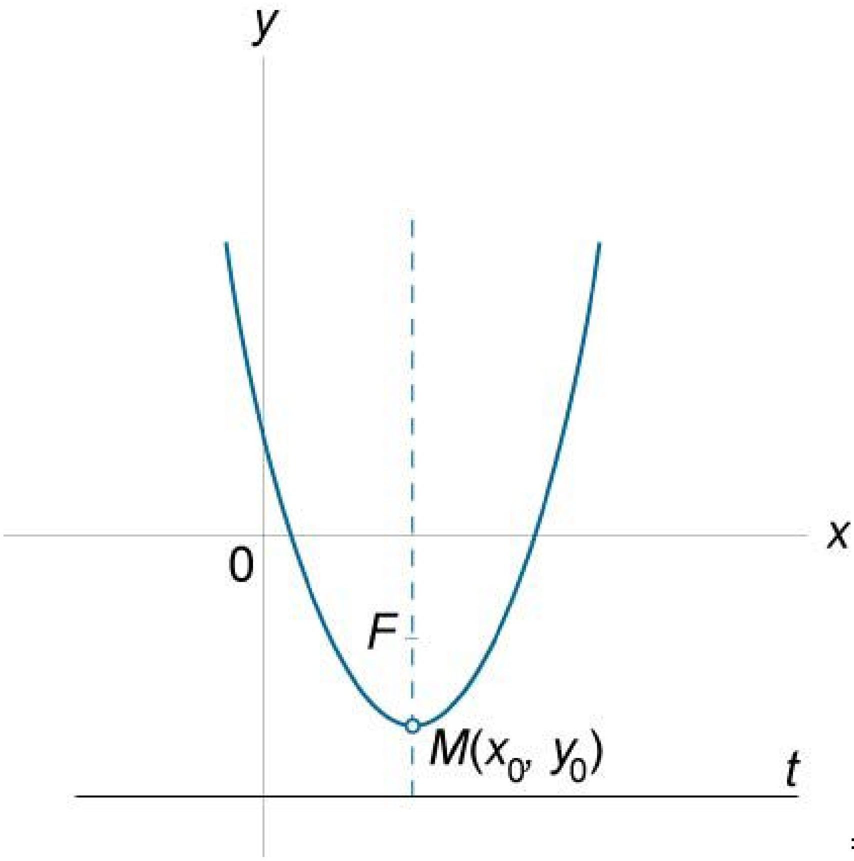
=

669. **dÉåÉê~ä=cçêãI=^ñáë=m~ê~ääÉä=íç=íÜÉ=ó-~ñáë== ^ñO**
+añ+bó+c =M=E^I=b=åçåòÉêçFI==

ó =~ñO +Äñ ÅI=é=N K==O~
bèì~íáçå=çÑ=íÜÉ=ÇáêÉÅíêáñ
ó= óM–é I=O
`ççêÇáå~íÉë=çÑ=íÜÉ=ÑçÅìë=
cñMIóM +éI= O

`ççêÇáå~íÉë=çÑ=íÜÉ=îÉêíÉñ=
Ä
I=
ó
M
=
~ñ
O
+

Äñ
M
+
Å
=
Q
~Å − Ä
O
ñ$_{M}$ −$_{O\sim\ M\ Q\sim}$ K= =



=$_{=}$

Figure 122.
=
=
=

7.8 Three-Dimensional Coordinate System

=

mçáåí=ÅççêÇáå~íÉëW=ñ I=ó I=ò I=ñI=óI= òI=£=M M M N N N

oÉ~ä=åìãÄÉêW=λ==

aáëí~åÅÉ=ÄÉíïÉÉå=íïç=éçáåíëW=Ç=

^êÉ~W=p=

sçäìãÉW=s=

670. aáëí~åÅÉ=_ÉíïÉÉå=qïç=mçáåíë=

Ç

=

^_

=

ñ

O

−

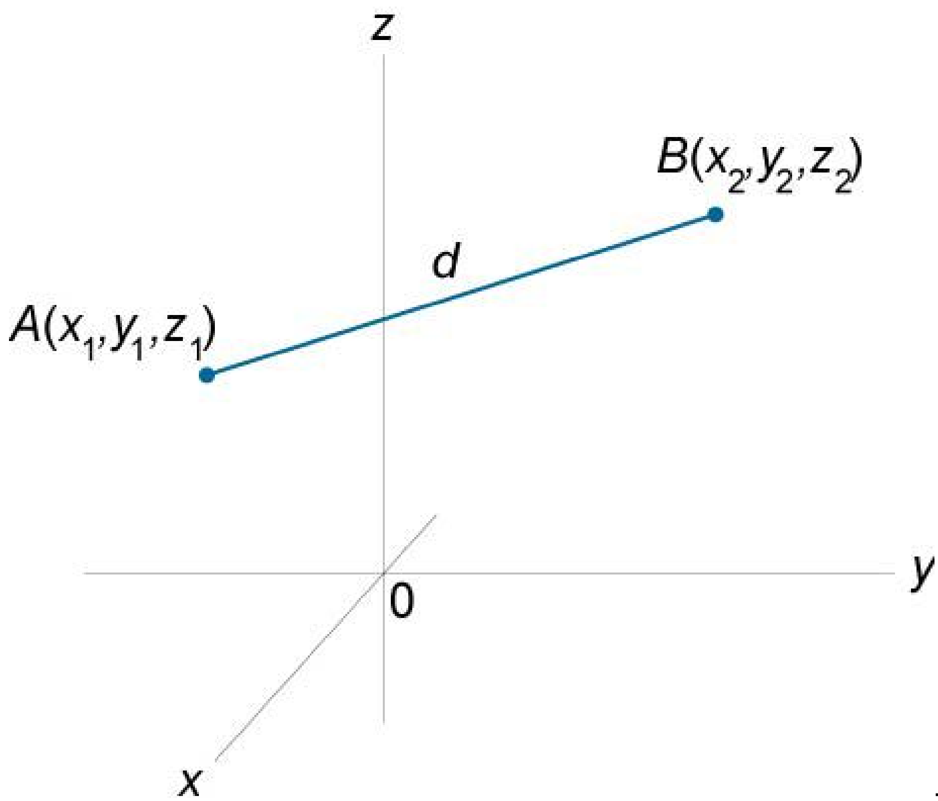
ñ

()( )( )− N + òO − O

N =N + O

=

= ===

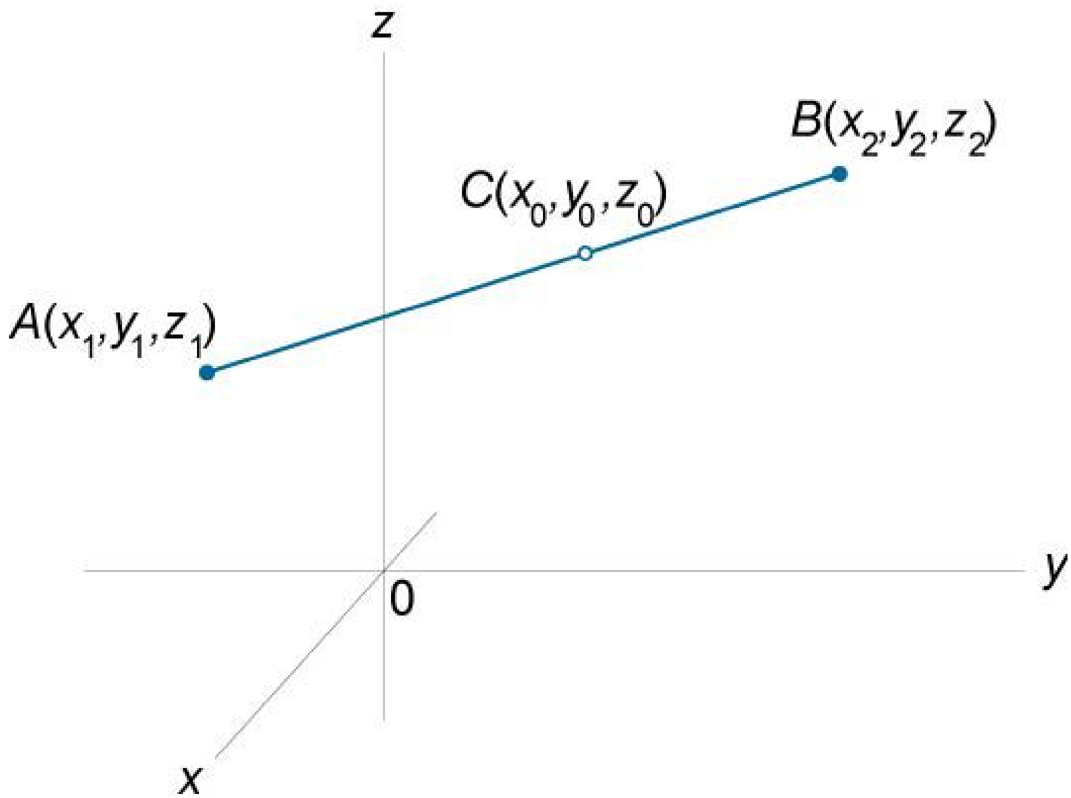


= **Figure 123.**

=

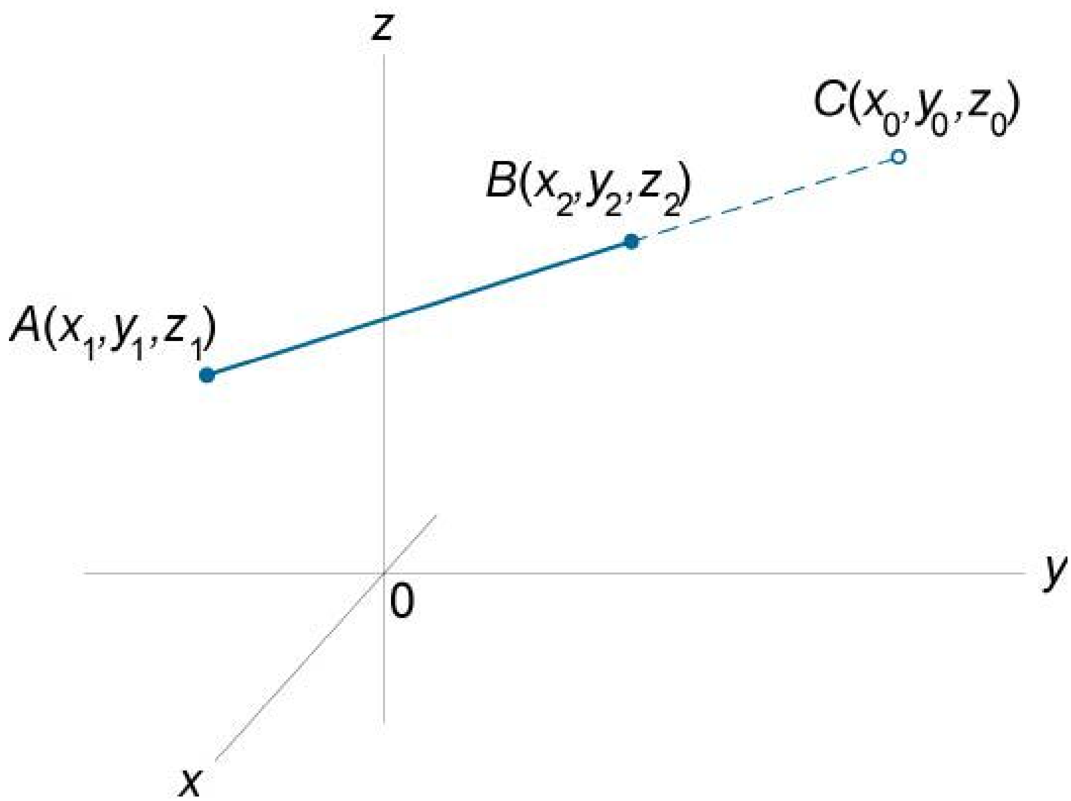
671. **aáîáÇáåÖ=~=iáåÉ=pÉÖãÉåí=áå=íÜÉ=o~íáç=λ= ñ$_M$ =ñN +λñO I=ó$_M$ =óN +λóO I=ò$_M$ =òN +λòO I==$_{N+λ\ N+λ\ N+λ}$ ïÜÉêÉ=**
λ=^` **I=λ≠−NK=**`_

========



$\lambda > 0$ = =

Figure 124. =



$\lambda < 0$

=

Figure 125.

672. **jáÇéçáåí=çÑ=~=iáåÉ=pÉÖãÉåí=**

ñM =ñN + ñO I=óM =óN +óO I=òM =òN +òO I=λ=**NK=O O O**

=

673. ^êÉ~=çÑ=~=qêá~åÖäÉ= (

ñmI=NIòN)qÜÉ=~êÉ~=çÑ=~=íêá~åÖäÉ=ïáíÜ=îÉêíáÅÉë=N N

m

() O ñOIOIòO I=~åÇ= (ñpIópIòp)=áë=ÖáîÉå=Äó==mP

O òN ñN NO ñN óN NO N
óN òN N
p=O óO òO N + òO ñO N + ñO óO N K= óP òP N òP ñP N ñP óP N =
674. sçäìãÉ=çÑ=~=qÉíê~ÜÉÇêçå=

qÜÉ=îçäìãÉ=çÑ=~=íÉíê~ÜÉÇêçå=ïáíÜ=îÉêíáÅÉë= (ñNIóNIòN)I=

() IO (ñPIóPIòP)I=~åÇ=
(ñQIóQIòQ)=áë=ÖáîÉå=Äó==OñmI=OIóO mQ

ñN óN òN N

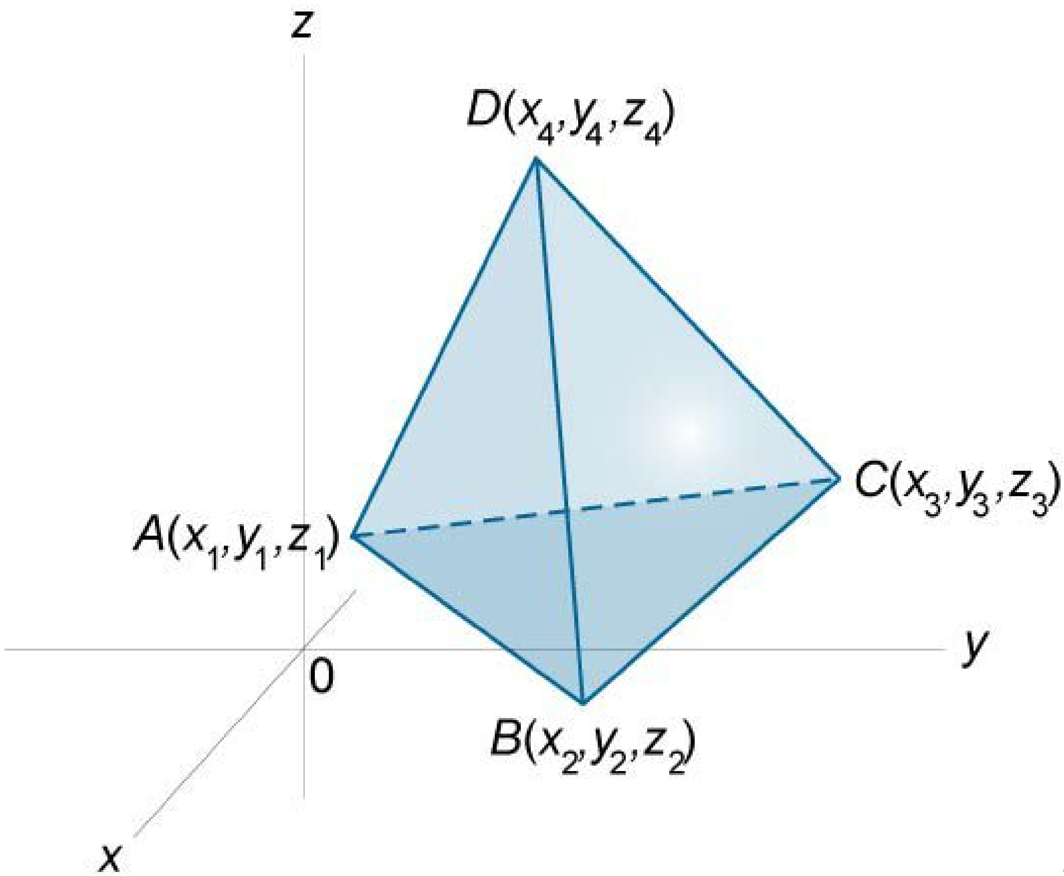
s =±N ñO óO òO NI==S ñP óP òP N
ñQ óQ òQ N
çê=

s
=
±

N
ñN −ñQ óN −óQ òN −òQ
ñO −ñQ óO −óQ òO −òQ K=S ñP −ñQ óP −óQ òP −òQ

kçíÉW=tÉ=ÅÜççëÉ=íÜÉ=ëáÖå=EHF=çê=E¥F=ëç=íÜ~í=íç=ÖÉí=~=éçë áíáîÉ= ~åëïÉê=Ñçê=îçäìãÉK==

====



= =

Figure 126.

=

=

=

7.9 Plane

=

mçáåí=ÅççêÇáå~íÉëW=ñI=óI=òI=ñ I=ó I=ò I=ñ I=ó I= ò I=£=M M M N N N
oÉ~ä=åìãÄÉêëW=^I=_I=`I=aI=^I= ^ I=~I=ÄI=ÅI=~I=~I=λI=éI=íI=£==N
O N O kçêã~ä=îÉÅíçêëW=å I=åN I=åO =

aáêÉÅíáçå=ÅçëáåÉëW=
α
I=
βI= γ
Åçë Åçë =

aáëí~åÅÉ=Ñêçã=éçáåí=íç=éä~åÉW=Ç=
=
=

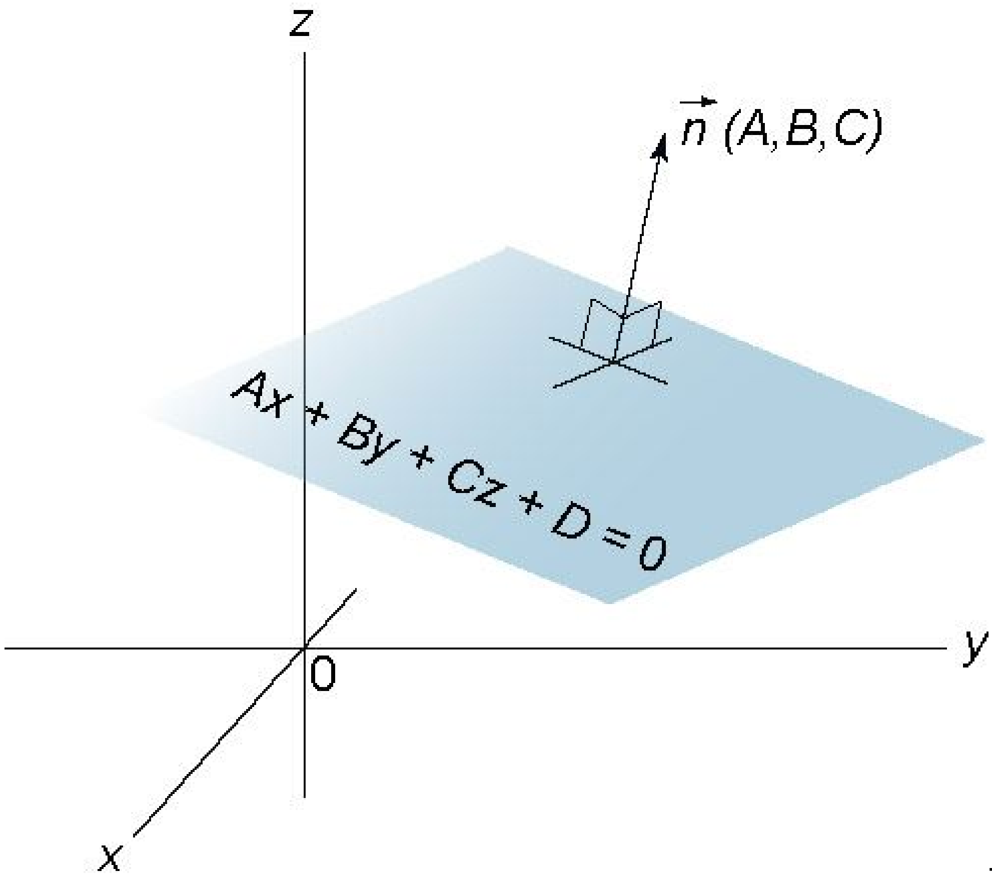
675. dÉåÉê~ä=bèì~íáçå=çÑ=~=mä~åÉ=
^ñ+_ó+`ò+a=M=
=
676. kçêã~ä=sÉÅíçê=íç=~=mä~åÉ=

qÜÉ=îÉÅíçê=()=áë=åçêã~ä=íç=íÜÉ=éä~åÉ=

^ñ+_ó+`ò+a=MK=

=

= ===



= **Figure 127.**

=

677. **m~êíáÅìä~ê=`~ëÉë=çÑ=íÜÉ=bèì~íáçå=çÑ=~=mä~åÉ=**

^ñ+_ó+`ò+a=M=

=

fÑ=^=MI=íÜÉ=éä~åÉ=áë=é~ê~ääÉä=íç=íÜÉ=ñ-~ñáëK=
fÑ=_=MI=íÜÉ=éä~åÉ=áë=é~ê~ääÉä=íç=íÜÉ=ó-~ñáëK=
fÑ=`=MI=íÜÉ=éä~åÉ=áë=é~ê~ääÉä=íç=íÜÉ=ò-~ñáëK=
fÑ=a=MI=íÜÉ=éä~åÉ=äáÉë=çå=íÜÉ=çêáÖáåK==

=

fÑ=^=_ MI=íÜÉ=éä~åÉ=áë=é~ê~ääÉä=íç=íÜÉ=ñó-éä~åÉK= fÑ=_=`

MI=íÜÉ=éä~åÉ=áë=é~ê~ääÉä=íç=íÜÉ=óò-éä~åÉK= fÑ=^=`
MI=íÜÉ=éä~åÉ=áë=é~ê~ääÉä=íç=íÜÉ=ñò-éä~åÉK=

678. mçáåí=aáêÉÅíáçå=cçêã=
^
ñ

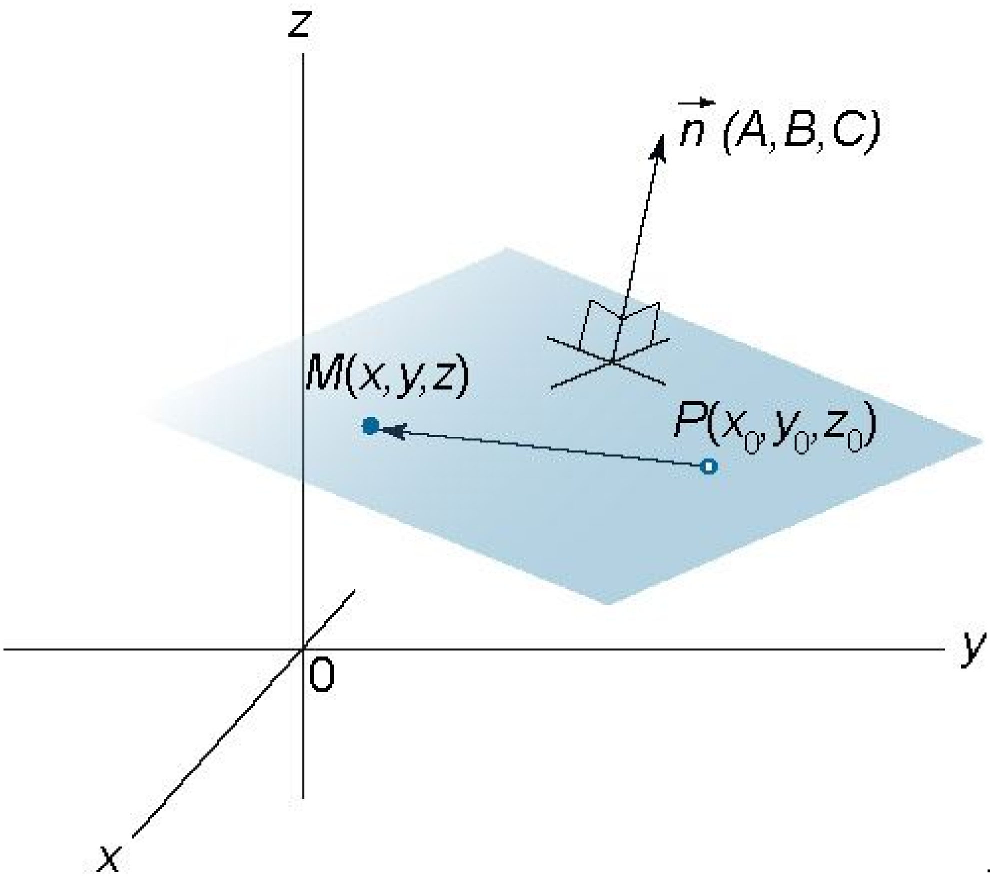
()( ) ( )

_**ñ**$_{M}$ +**_ ó**−$_{M}$ +$_{M}$ =**MI==**

**ïÜÉêÉ=íÜÉ=éçáåí=
(ñMIóMIòM)=äáÉë=áå=íÜÉ=éä~åÉI=~åÇ=íÜÉ=î
ÉÅ- íçê=()=áë=åçêã~ä=íç=íÜÉ=éä~åÉK===**

=

= ====

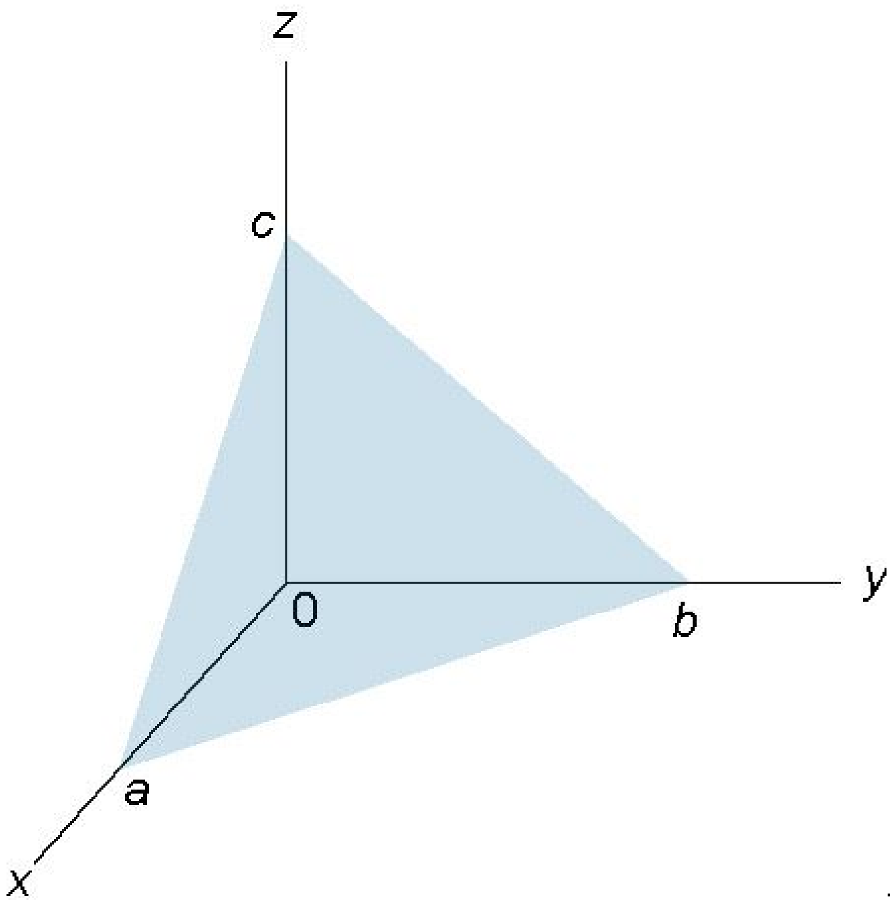


= **Figure 128.** =

679. **fåíÉêÅÉéí=cçêã=**

ñ +ó +ò =**N=**~ Ä Å

= =====



= **Figure 129.** =

680. **qÜêÉÉ=mçáåí=cçêã=**

$ñ−ñ_P$ $ó−ó_P$ $ò −ò_P$

$ñ_N −ñ_P$ $ó_N −ó_P$ $ò_N −ò_P$ =MI== $ñ_O −ñ_P$ $ó_O −ó_P$ $ò_O −ò_P$

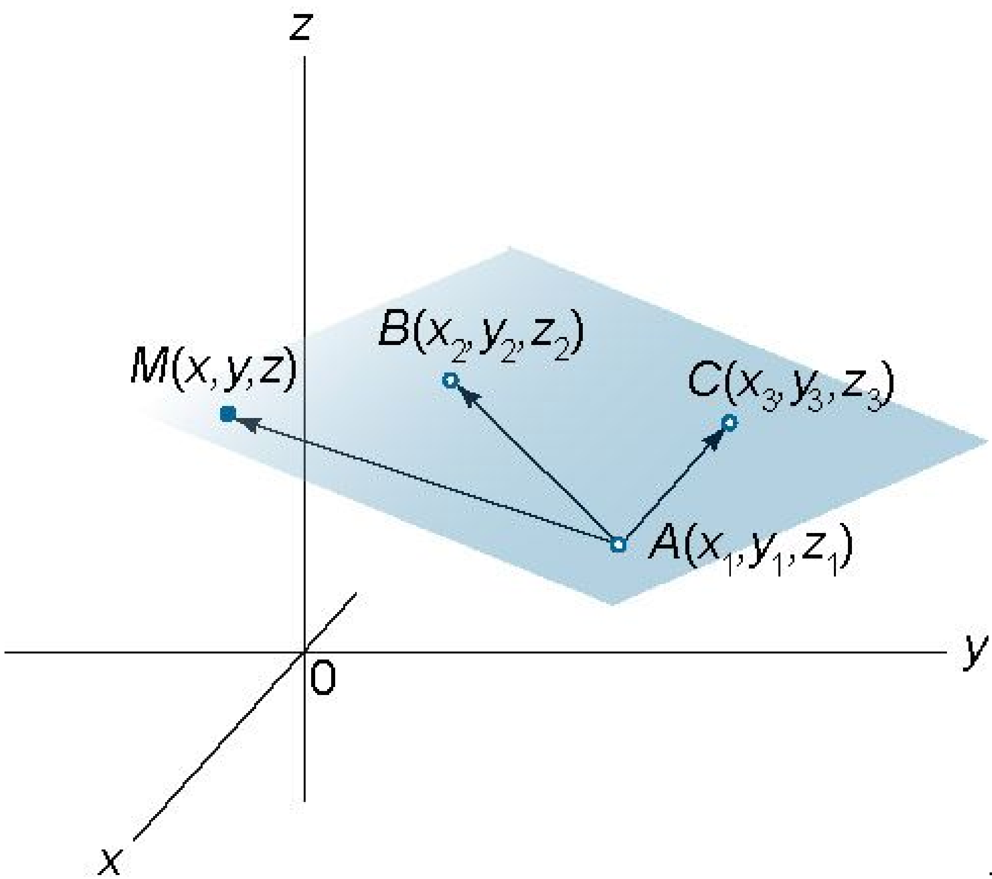
çê==

ñ ó ò N

$ñ_N$ $ó_N$ $ò_N$ N=MK=ñO óO òO N

$ñ_P$ $ó_P$ $ò_P$ N

=====

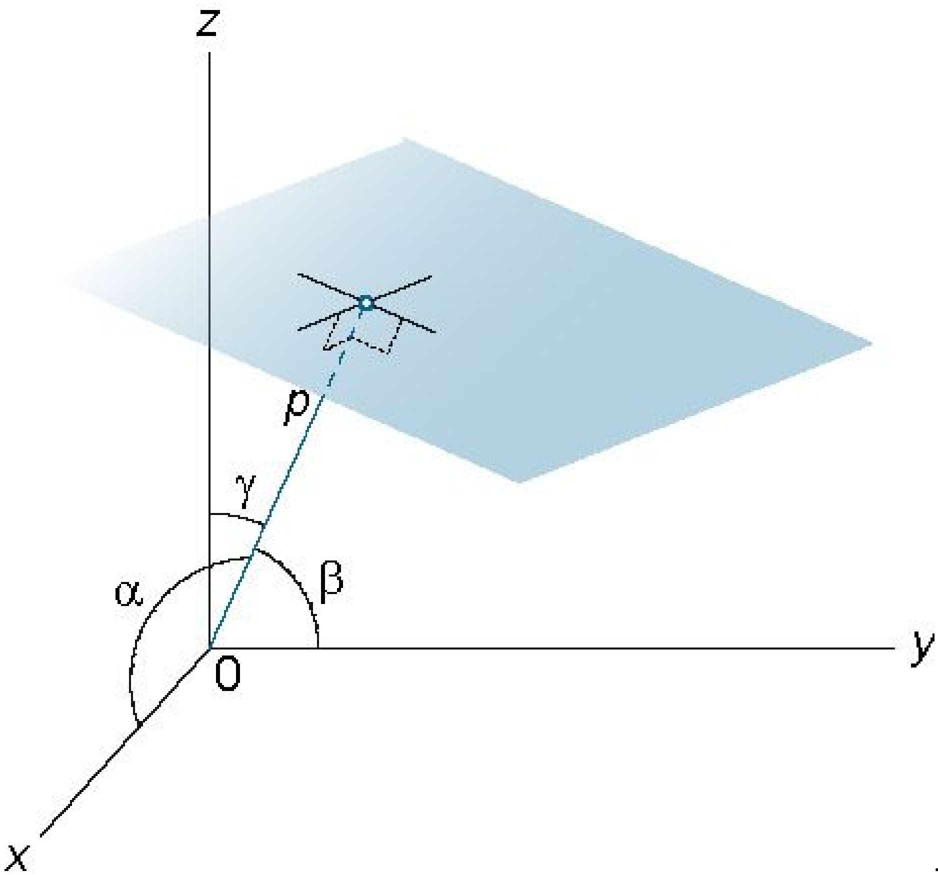


= **Figure 130.**

=

681. **kçêã~ä=cçêã=**
ñÅçëα+óÅçëβ+òÅçëγ −é =MI==
ïÜÉêÉ==é==áë==íÜÉ==éÉêéÉåÇáÅìä~ê==Çáëí~åÅÉ==Ñêçã==íÜÉ=çêá
Öáå=íç= íÜÉ=éä~åÉ=I=~åÇ= αI= βI=
γ=~êÉ=íÜÉ=ÇáêÉÅíáçå=ÅçëáåÉë=
çÑ=~åó=äáåÉ=åçêã~ä=íç=íÜÉ=éä~åÉK==

= ======



= Figure 131.

=

682. **m~ê~ãÉíêáÅ=cçêã=**

□ñ =ñN +~Në+~Oí

□□ó= óN +ÄNë +ÄOíI==

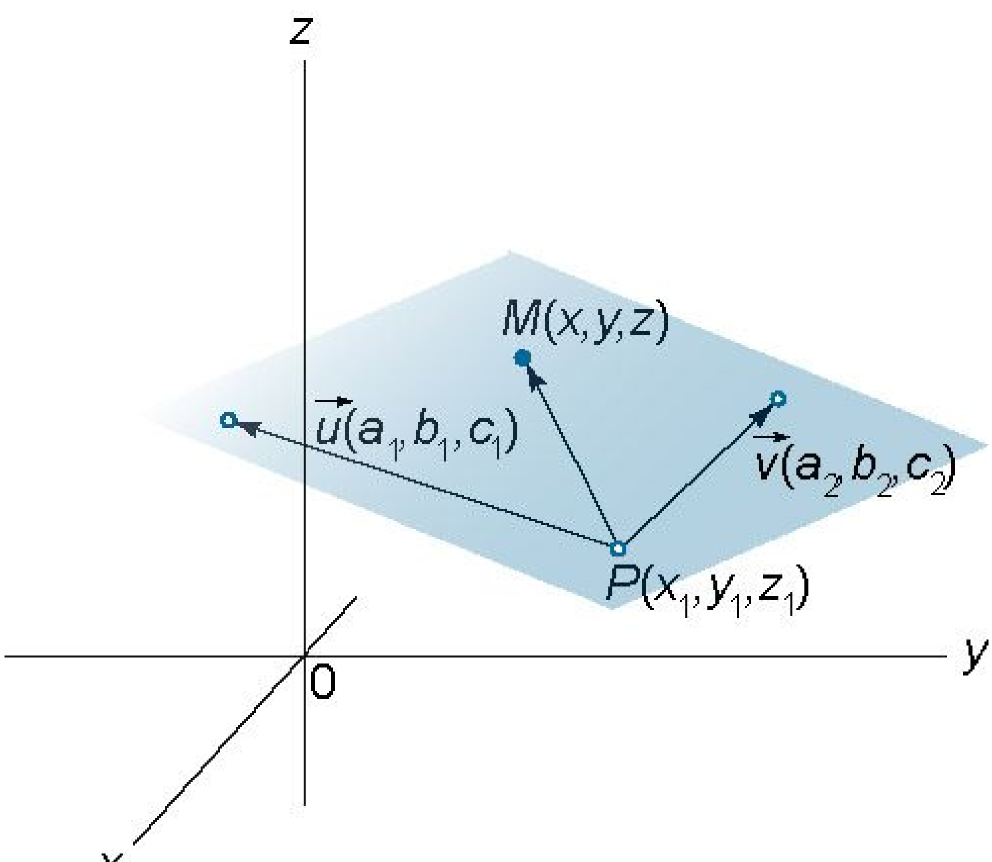
□□ò =òN +ÅNë+ÅOí

ïÜÉêÉ=
()=~êÉ=íÜÉ=ÅççêÇáå~íÉë=çÑ=~åó=ìåâåçïå=éçáåí =çå=

íÜÉ=äáåÉ=I=íÜÉ=éçáåí=
(ñNIóNIòN)=äáÉë=áå=íÜÉ=éä~åÉI=íÜÉ=îÉÅíçêë=

()ÄNINI~=~åÇ=
(~OIÄOIÅO)=~êÉ=é~ê~ääÉä=íç=íÜÉ=éä~åÉK=N

======

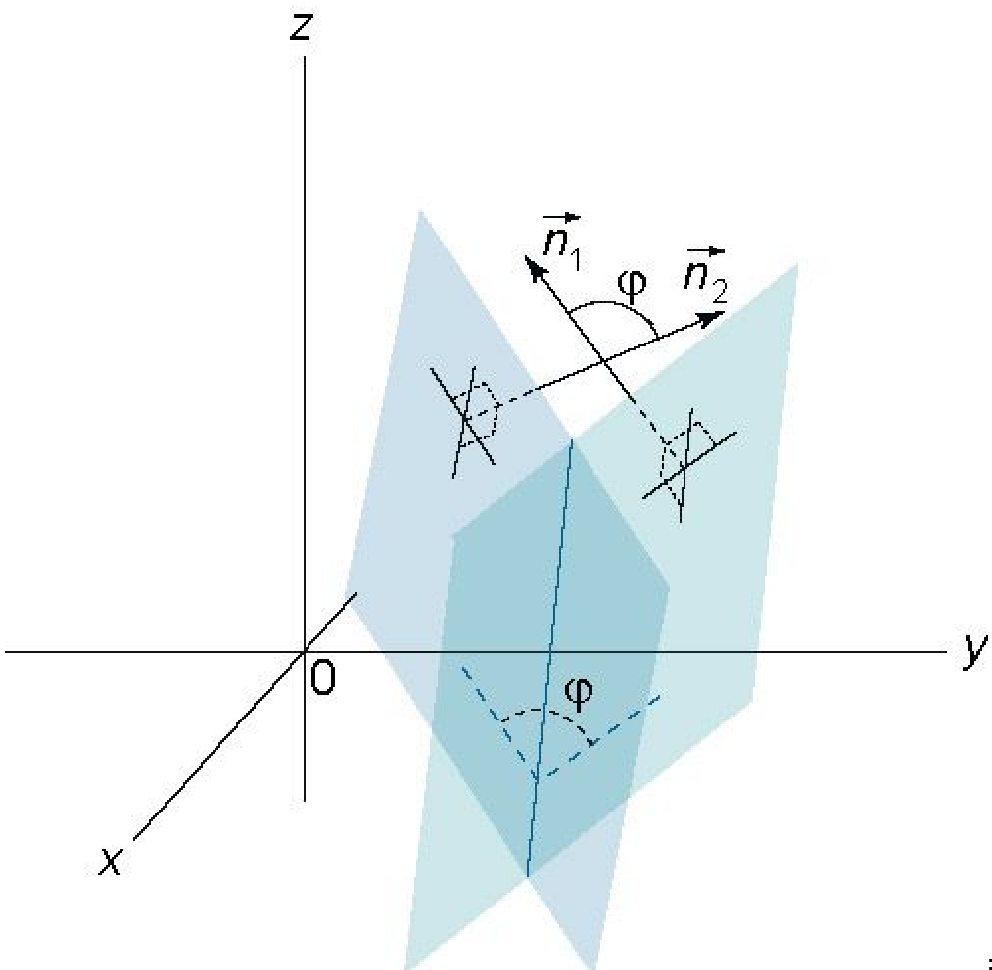


= **Figure 132.**

=

683. **aáÜÉÇê~ä=^åÖäÉ=_ÉíïÉÉå=qïç=mä~åÉë=**
fÑ=íÜÉ=éä~åÉë=~êÉ=ÖáîÉå=Äó==
^Nñ+_Nó+`Nò+aN =MI==
^Oñ +_Oó+`Oò +aO =MI==
íÜÉå=íÜÉ=ÇáÜÉÇê~ä=~åÖäÉ=ÄÉíïÉÉå=íÜÉã=áë==

r.r
Åçë
ϕ
=
å
N
å
^
N
^
O
r
+_N_O +`N`O K=åN .rO = O +_O +`O . ^O +\_O +`OåO ^N N N O O O =======



= **Figure 133.**

=

684. m~ê~ääÉä=mä~åÉë=
qïç=éä~åÉë=^Nñ+_Nó+`Nò +aN =M=~åÇ=
^Oñ +_Oó+`Oò +aO =M=~êÉ=é~ê~ääÉä=áÑ==
^N = _N = `N K=^O \_O `O

=

685. mÉêéÉåÇáÅìä~ê=mä~åÉë=
qïç=éä~åÉë=^Nñ+_Nó+`Nò +aN =M=~åÇ=
^Oñ +_Oó+`Oò +aO =M=~êÉ=éÉêéÉåÇáÅìä~ê=áÑ==
^N^O +_N_O +`N`O =MK=

=

686. bèì~íáçå=çÑ=~=mä~åÉ=qÜêçìÖÜ=
(ñNIóNIòN)=~åÇ=m~ê~ääÉä=qç=

íÜÉ=sÉÅíçêë=()ÄNIÅI~=~åÇ=(~OIÄOIÅO)=EcáÖKNPOF=N
N

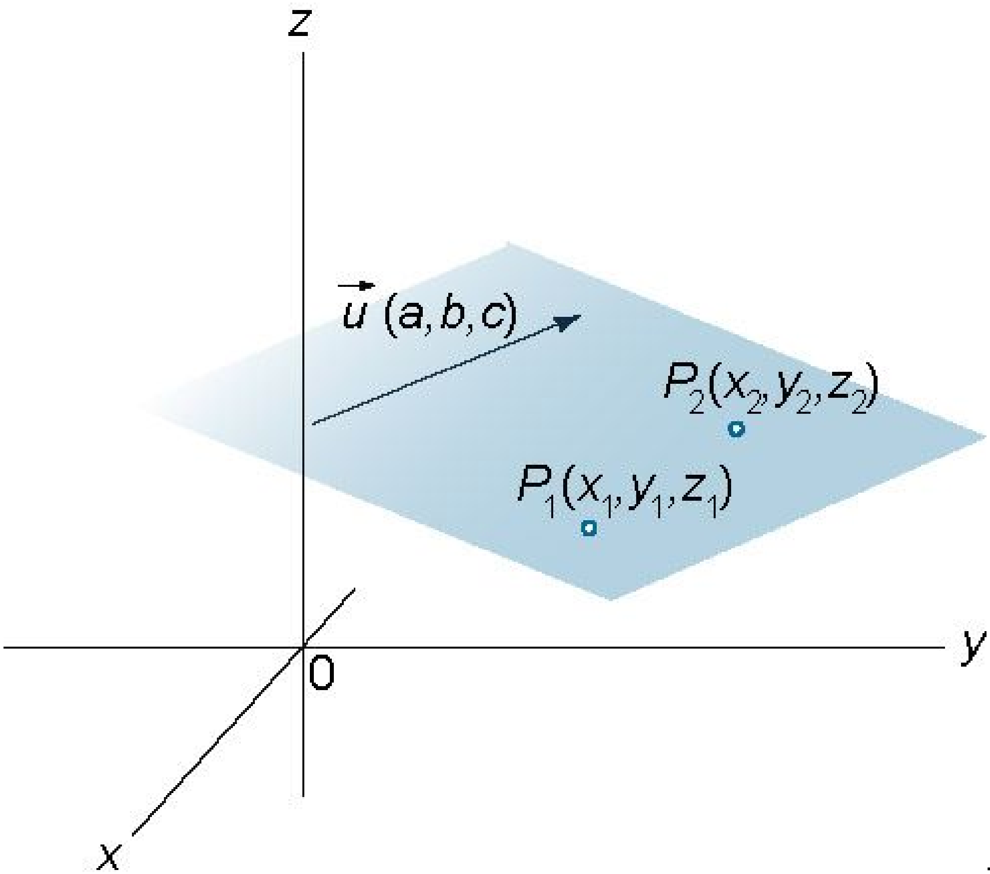
ñ −ñN ó−óN ò−òN
~N ÄN ÅN =M=
~O ÄO ÅO

=
687.
bèì~íáçå=çÑ=~=mä~åÉ=qÜêçìÖÜ=
m

(ñNIóNIòN)=~åÇ=m (ñOIóOIòO)I=N O

~åÇ=m~ê~ääÉä=qç=íÜÉ=sÉÅíçê=( )I~=

ñ−ñN ó−óN ò −òN
ñO −ñN óO −óN òO −òN =M=
~ Ä Å
=



=

Figure 134.

=

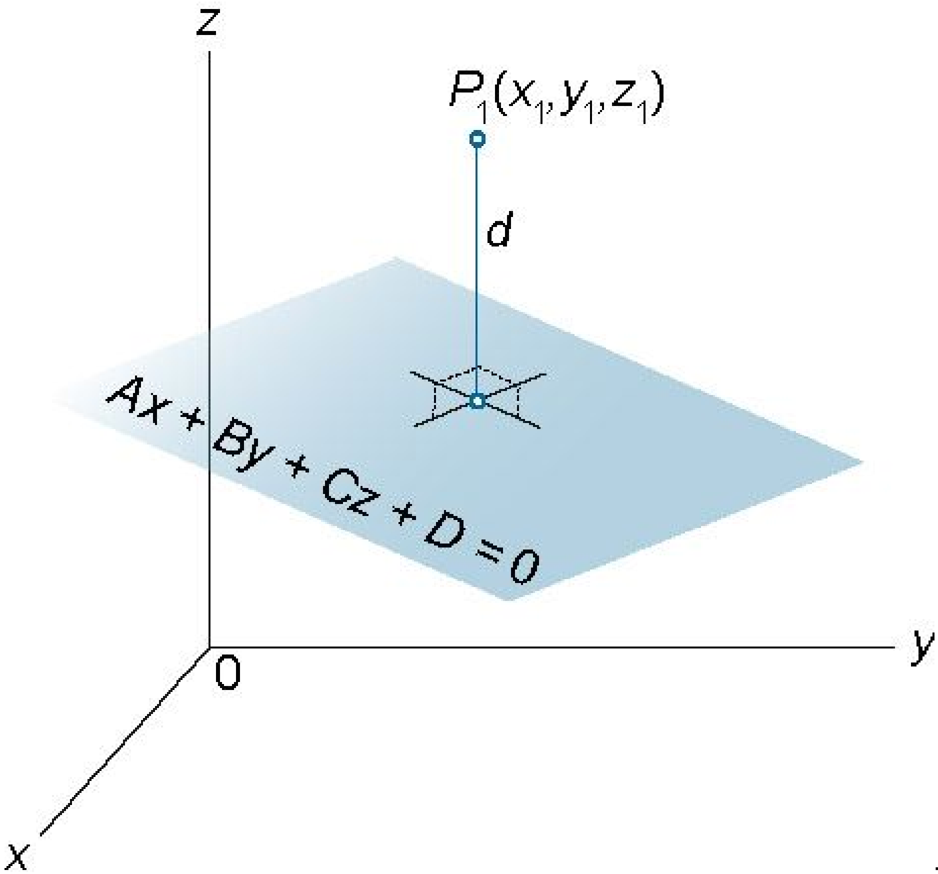
688. aáëí~åÅÉ=cêçã=~=mçáåí=qç=~=mä~åÉ=

qÜÉ=Çáëí~åÅÉ=Ñêçã=íÜÉ=éçáåí= ( )=íç=íÜÉ=éä~åÉ=N
ñNIóNIòN

^ñ+_ó+`ò+a=M=áë==

ç= ^ñN +_óN +`òN +aK= ^O +_O +`O

=



== ======

Figure 135.

=

689. **fåíÉêëÉÅíáçå=çÑ=qïç=mä~åÉë=**

fÑ=íïç=éä~åÉë=^Nñ+_Nó+`Nò +aN =M=~åÇ=

^Oñ +_Oó+`Oò +aO =M= áåíÉêëÉÅíI= íÜÉ= áåíÉêëÉÅíáçå= ëíê~áÖÜí= äáåÉ=áë=ÖáîÉå=Äó=

□ñ =ñN +~í

□□ ó= óN +ÄíI==

□□ò =òN +Åí

çê==

ñ−ñN = ó−óN = ò −òN I==~ Ä Å

ïÜÉêÉ==

~=–N `N I=Ä=`N ^N I=Å=^N –N I==_O `O `O ^O ^O _O

aN `N –ÅaN –NÄ

ñ aO `O aO –O I==N = ~O +ÄO +ÅO

ó Å

aN ^N –~ aN `N aO ^O aO `O I==N = ~O +ÄO +ÅO

~

aN –N –ÄaN ^N aO –O aO ^O K==N = ~O +ÄO +ÅO

ò

=

=

=

7.10 Straight Line in Space

=

mçáåí=ÅççêÇáå~íÉëW=ñI=óI=òI=ñ I=ó I= ò I=£=N N N

aáêÉÅíáçå=ÅçëáåÉëW= αI= βI= γ =

oÉ~ä=åìãÄÉêëW=^I=_I=`I=aI=~I=ÄI=ÅI=~I= ~I=íI=£==

aáêÉÅíáçå=îÉÅíçêë=çÑ=~=äáåÉW=ë

r

I=

ë

r

N O

I= ër =N O

kçêã~ä=îÉÅíçê=íç=~=éä~åÉW= å =

^åÖäÉ=ÄÉíïÉÉå=ïç=äáåÉëW=ϕ=

=

690. **mçáåí=aáêÉÅíáçå=cçêã=çÑ=íÜÉ=bèì~íáçå=çÑ=~=iáåÉ== ñ−ñN =**

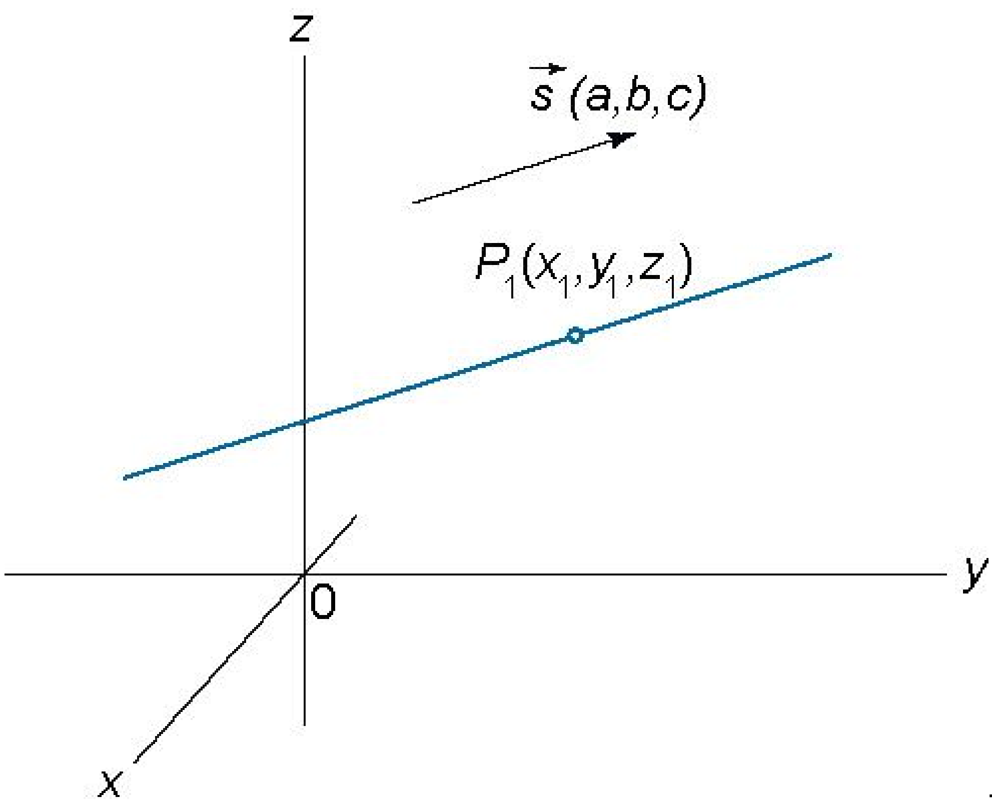
ó−óN = ò −òN I==~ Ä Å

ïÜÉêÉ=íÜÉ=éçáåí=

(m ñNIóNIòN)=äáÉë=çå=íÜÉ=äáåÉI=~åÇ=( )=áë=N

íÜÉ=ÇáêÉÅíáçå=îÉÅíçê=çÑ=íÜÉ=äáåÉK==

= =====



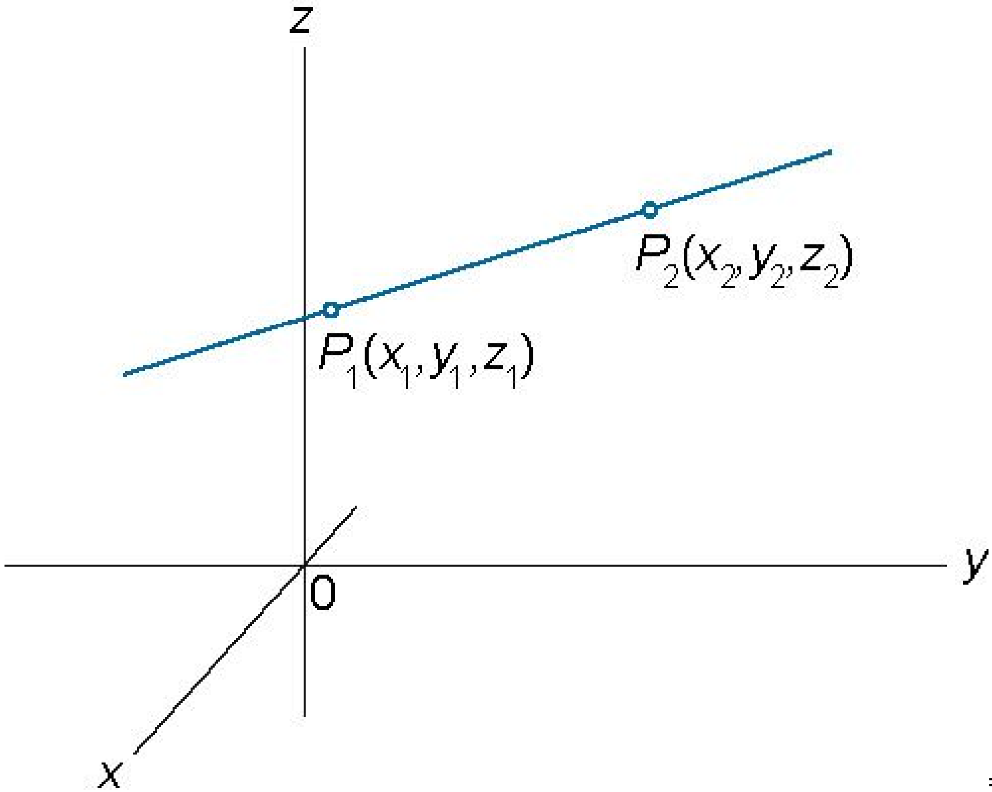
= **Figure 136.** =

691. **qïç=mçáåí=cçêã=**

ñ −ñN = ó−óN = ò −òN ==ñO −ñN óO −óN òO −òN

=

=====



=Figure 137.

692. **m~ê~ãÉíêáÅ=cçêã==**

□**ñ =ñN +íÅçë**α

□□**ó= óN +íÅçë**β **I==**

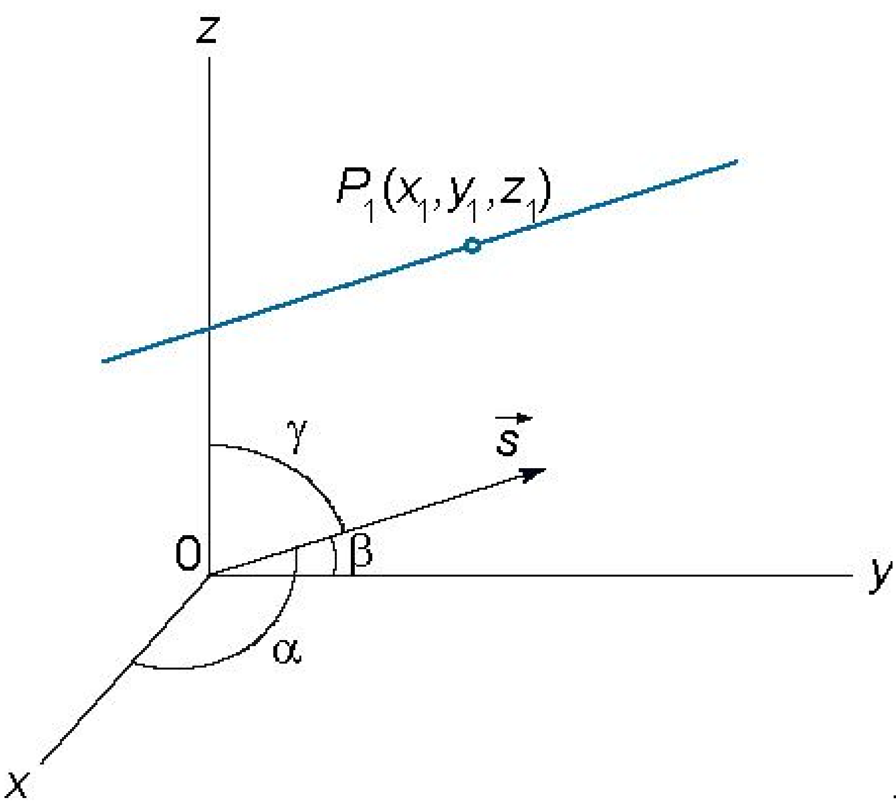
□□**ò =òN +íÅçë**γ

ïÜÉêÉ=íÜÉ=éçáåí=

(

m ñNIóNIòN)=äáÉë=çå=íÜÉ=ëíê~áÖÜí=äáåÉI=N
ÅçëαI= βI= γ=~êÉ=íÜÉ=ÇáêÉÅíáçå=ÅçëáåÉë=çÑ=íÜÉ=ÇáêÉÅíáçå=
îÉÅíçê=çÑ=íÜÉ=äáåÉI=íÜÉ=é~ê~ãÉíÉê=í=áë=~åó=êÉ~ä=åìãÄÉêK== =

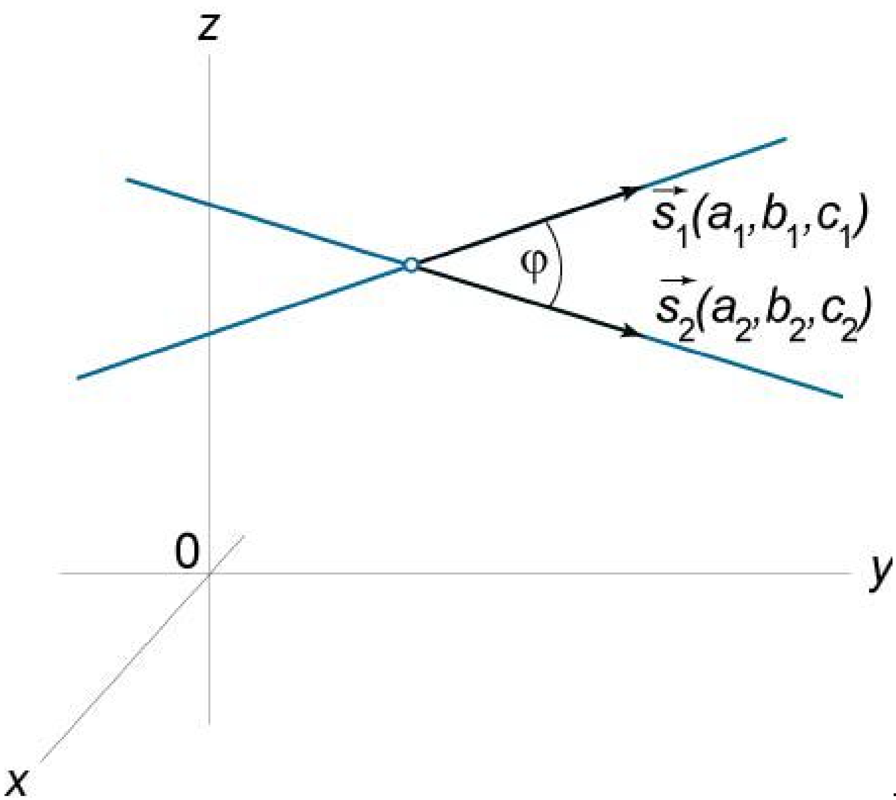
= =====



= Figure 138.

=

693. ^åÖäÉ=_ÉíïÉÉå=qïç=píê~áÖÜí=iáåÉë= Åçë
ϕ
=
rN·r = O +ÄO +ÅO · ~O +ÄO +ÅO =r
ë ëO ~N~O +ÄNÄO +ÅNÅO ·
r
ëN ëO ~N N N O O O
= =====



= **Figure 139.** =

694. m~ê~ääÉä=iáåÉë=

qïç=äáåÉë=~êÉ=é~ê~ääÉä=áÑ==

r r

ë ëO I==N öö

çê==

~N =ÄN =ÅN K=~O ÄO ÅO

=

695. mÉêéÉåÇáÅìä~ê=iáåÉë=

qïç=äáåÉë=~êÉ=é~ê~ääÉä=áÑ==

r ·r =

ë MI==N O

çê==

~N~O +ÄNÄO +ÅNÅO =MK=

=

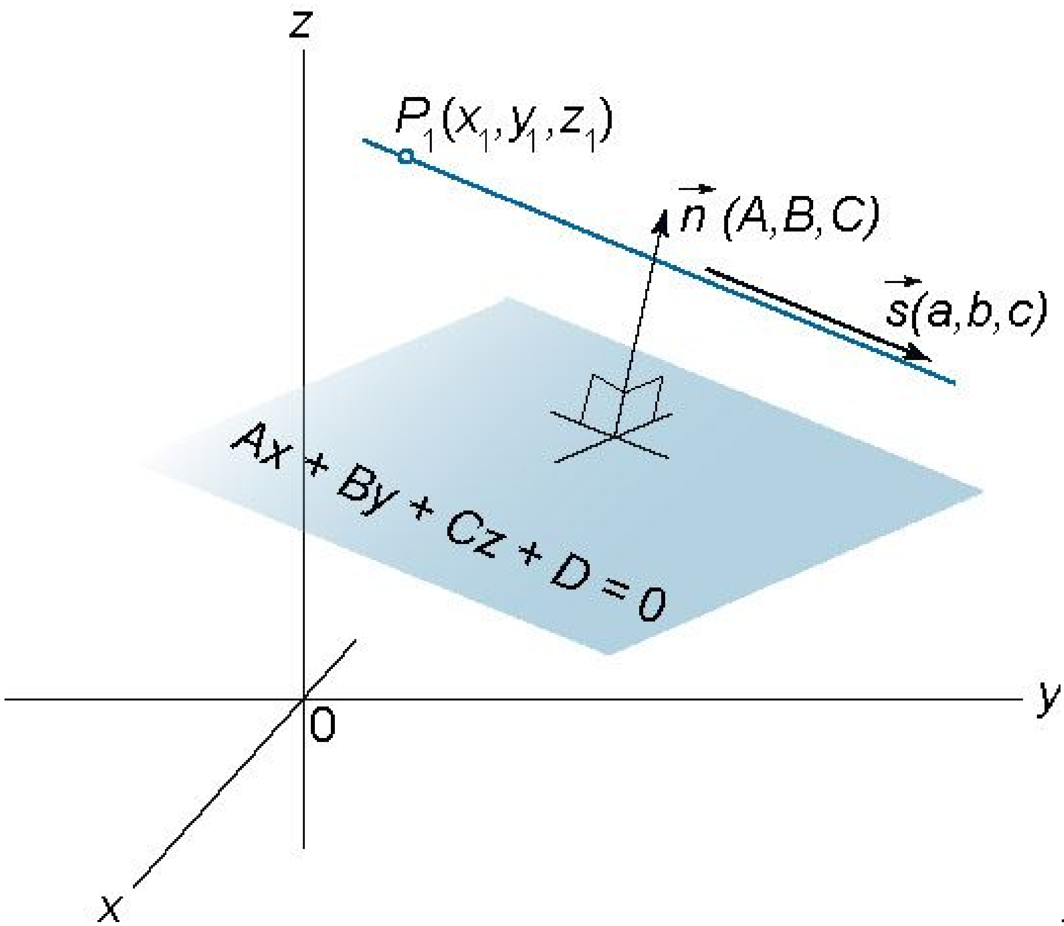
696. fåíÉêëÉÅíáçå=çÑ=qïç=iáåÉë=
qïç=äáåÉë=ñ−ñN =ó−óN =ò−òN =~åÇ=~N ÄN ÅN ñ−ñO =ó−óO =ò−òO
=áåíÉêëÉÅí=áÑ==~O ÄO ÅO

ñO −ñN óO −óN òO −òN
~N ÄN ÅN =MK=
~O ÄO ÅO

=
697. m~ê~ääÉä=iáåÉ=~åÇ=mä~åÉ==
qÜÉ=ëíê~áÖÜí=äáåÉ=ñ−ñN =ó−óN =ò−òN =~åÇ=íÜÉ=éä~åÉ=~ Ä Å
^ñ+_ó+`ò+a=M=~êÉ=é~ê~ääÉä=áÑ=
r

å ⋅r =MI==
çê==
^~ +_Ä +`Å=MK=
=

=====



=

=

698. mÉêéÉåÇáÅìä~ê=iáåÉ=~åÇ=mä~åÉ==

qÜÉ=ëíê~áÖÜí=äáåÉ=ñ−ñN =ó−óN =ò−òN =~åÇ=íÜÉ=éä~åÉ=~ Ä Å

^ñ+_ó+`ò+a=M=~êÉ=éÉêéÉåÇáÅìä~ê=áÑ=

Figure 140. =

r

å

öö

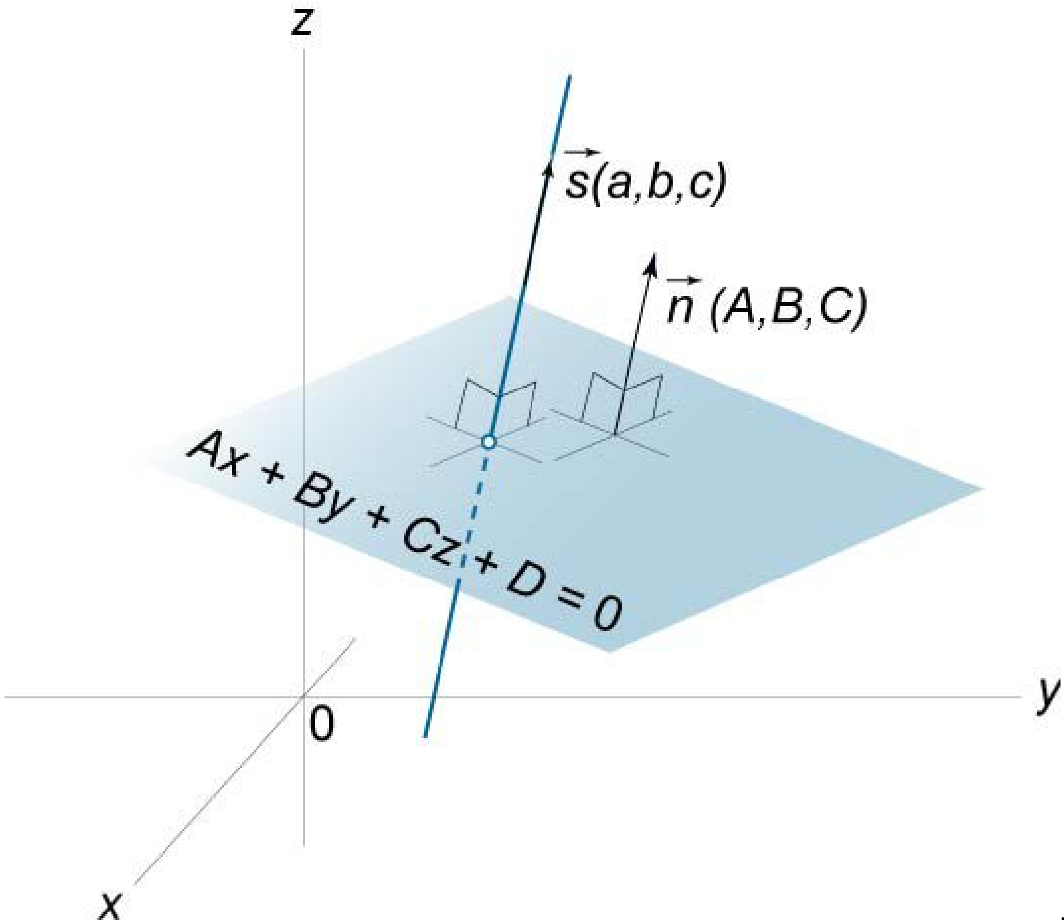
r **ë I==**

çê==

^= – `K=~ Ä Å

=

====



= =

Figure 141.

=

=

7.11 Quadric Surfaces

=

mçáåí=ÅççêÇáå~íÉë=çÑ=íÜÉ=èì~ÇêáÅ=ëìêÑ~ÅÉëW=ñI=óI=ò=
oÉ~ä=åìãÄÉêëW=^I=_I=`I=~I=ÄI=ÅI=â$_N$Iâ$_O$Iâ$_P$I=£= 699.
dÉåÉê~ä=nì~Çê~íáÅ=bèì~íáçå=

^ñ^O +_ó^O +`ò^O +Ocóò +Odòñ+Oeñó+Omñ+Onó+Ooò +a=M =

700. `ä~ëëáÑáÅ~íáçå=çÑ=nì~ÇêáÅ=pìêÑ~ÅÉë=

=

`~ëÉ= o~åâEÉF= o~åâEbF= Δ= â=ëáÖåë= qóéÉ=çÑ=pìêÑ~ÅÉ= N= P= Q= <M= p~ãÉ= oÉ~ä=bääáéëçáÇ= O= P= Q= >M= p~ãÉ= fã~Öáå~êó=bääáéëçáÇ= P= P= Q= >M= aáÑÑÉêÉåí= eóéÉêÄçäçáÇ=çÑ=N=pÜÉÉí= Q= P= Q= <M= aáÑÑÉêÉåí= eóéÉêÄçäçáÇ=çÑ=O=pÜÉÉíë= R= P= P= = aáÑÑÉêÉåí= oÉ~ä=nì~ÇêáÅ=`çåÉ= S= P= P= = p~ãÉ= fã~Öáå~êó=nì~ÇêáÅ=`çåÉ= T= O= Q= <M= p~ãÉ= bääáéíáÅ=m~ê~ÄçäçáÇ= U= O= Q= >M= aáÑÑÉêÉåí= eóéÉêÄçäáÅ=m~ê~ÄçäçáÇ= V= O= P= = p~ãÉ= oÉ~ä=bääáéíáÅ=`óäáåÇÉê= NM= O= P= = p~ãÉ= fã~Öáå~êó=bääáéíáÅ=`óäáåÇÉê= NN= O= P= = aáÑÑÉêÉåí= eóéÉêÄçäáÅ=`óäáåÇÉê= NO= O= O= = aáÑÑÉêÉåí= oÉ~ä=fåíÉêëÉÅíáåÖ=mä~åÉë= NP= O= O= = p~ãÉ= fã~Öáå~êó=fåíÉêëÉÅíáåÖ=mä~åÉë= NQ= N= P= = = m~ê~ÄçäáÅ=`óäáåÇÉê= NR= N= O= = = oÉ~ä=m~ê~ääÉä=mä~åÉë= NS= N= O= = = fã~Öáå~êó=m~ê~ääÉä=mä~åÉë= NT= N= N= = = `çáåÅáÇÉåí=mä~åÉë= =

eÉêÉ==

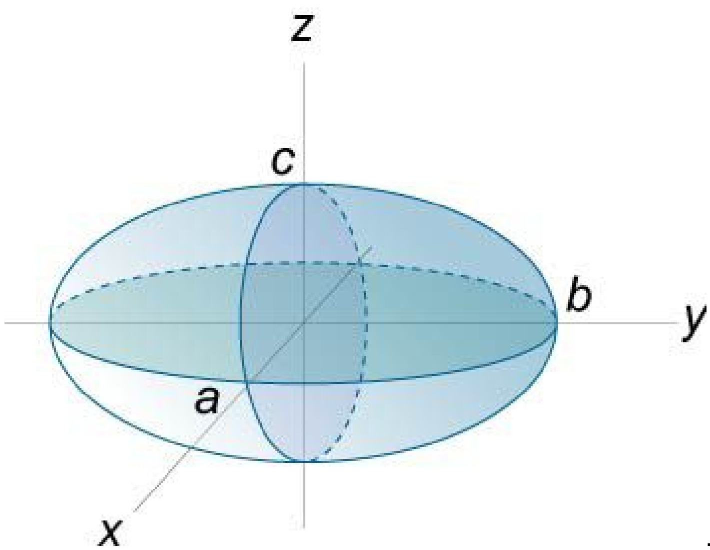
□^ e d□ □^ e n m□

□□□
ÉI=
□□□
=
□□□□□
e _ c n
= **bI=**

d c ` o□□□□□ Δ =ÇÉí( )I==

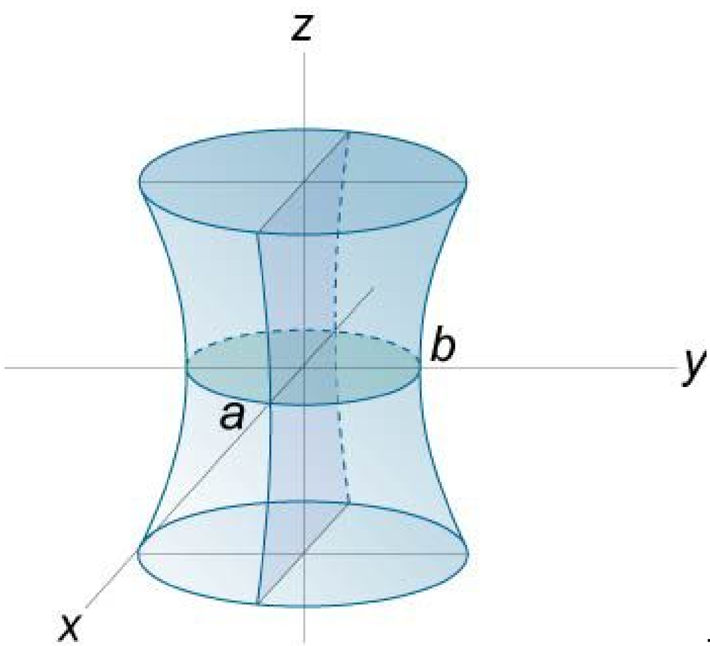
d c ` m n o a
â$_N$Iâ$_O$Iâ$_P$ =~êÉ=íÜÉ=êççíë=çÑ=íÜÉ=Éèì~íáçåI==
^−ñ e d

e _−ñ c =MK=
d c `−ñ
=
701. oÉ~ä=bääáéëçáÇ=E`~ëÉ=NF= ñ^O ó^O ò^O
~O +ÄO +ÅO =N=
=
= =====



= Figure 142. =

702. fã~Öáå~êó=bääáéëçáÇ=E`~ëÉ=OF= ñ^O ó^O ò^O
O +ÄO +ÅO =−N=~
=
703. eóéÉêÄçäçáÇ=çÑ=N=pÜÉÉí=E`~ëÉ=PF= ñ
O ó^O ò^O
+ÄO −ÅO =N=~O
= =====



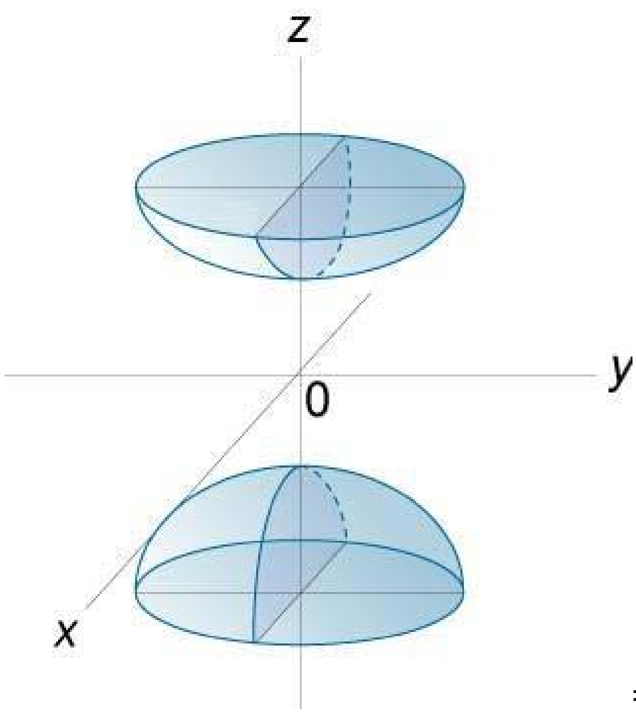
= **Figure 143.** =

704. **eóéÉêÄçäçáÇ=çÑ=O=pÜÉÉíë=E`~ëÉ=QF= ñO óO òO**

~O +ÄO −åO =−N=

=

==



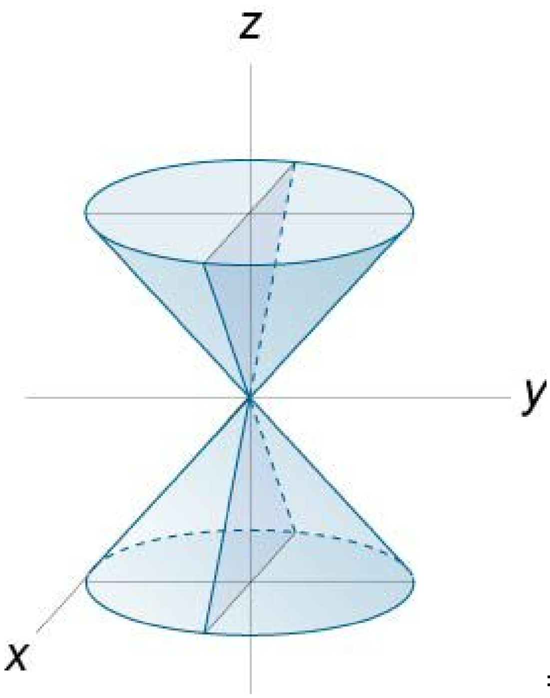
= **Figure 144.**

705. **oÉ~ä=nì~ÇêáÅ=`çåÉ=E`~ëÉ=RF= ñO óO òO**

~O +ÄO −åO =M=

=

= ==



= Figure 145. =

706. fã~Öáå~êó=nì~ÇêáÅ=`çåÉ=E`~ëÉ=SF= ñO óO òO

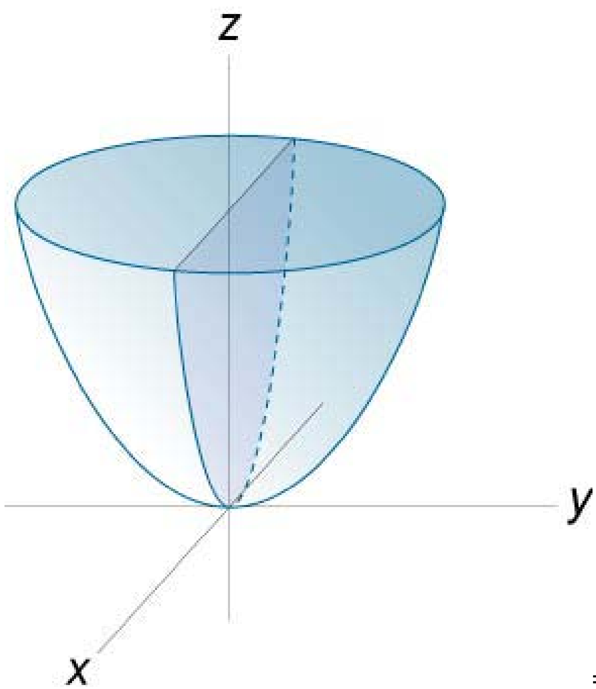
~O +ÄO +ÅO =M=

=

707. bääáéíáÅ=m~ê~ÄçäçáÇ=E`~ëÉ=TF= ñO óO

~O +ÄO −ò =M=

======



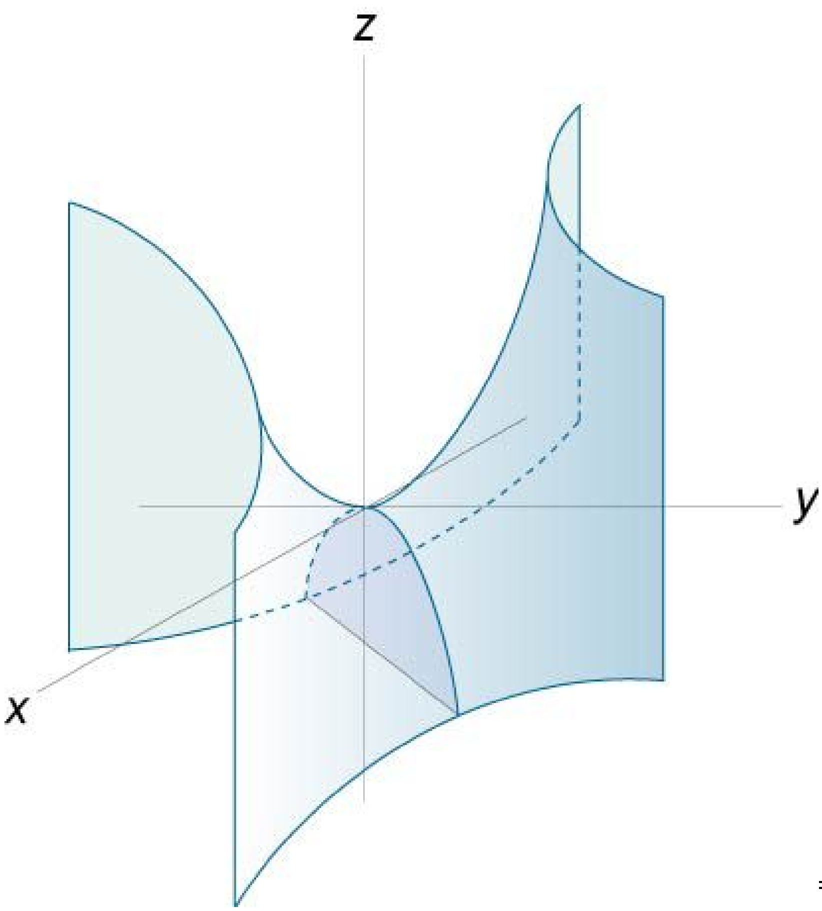
=Figure 146.

708. **eóéÉêÄçäáÅ=m~ê~ÄçäçáÇ=E`~ëÉ=UF=**

ñO óO

~O −ÄO −ò =M=

====



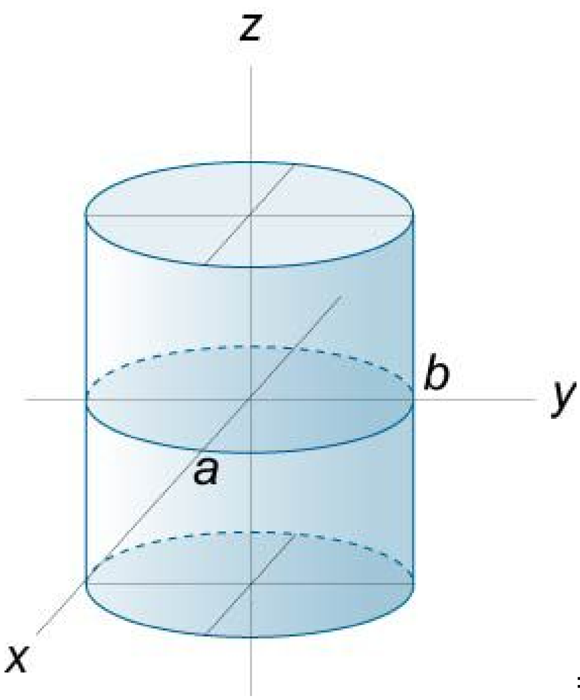
=Figure 147.

709. oÉ~ä=bääáéíáÅ=`óäáåÇÉê=E`~ëÉ=VF= ñO óO

~O +ÄO =N=

=

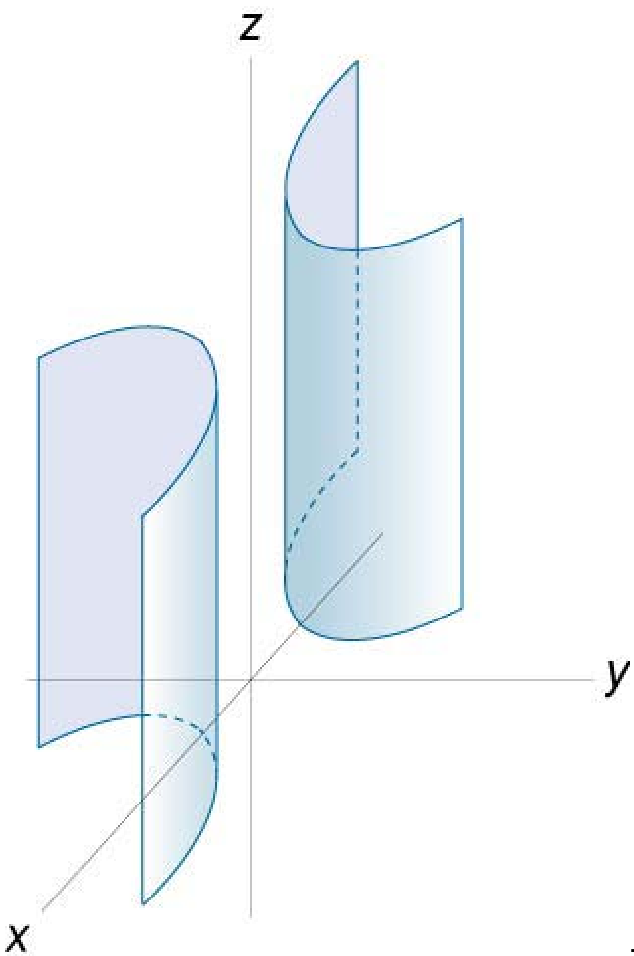
= =====



= **Figure 148.** =

710. **fã~Öáå~êó=bääáéíáÅ=`óäáåÇÉê=E`~ëÉ=NMF= ñO óO**
O +ÄO =−N=~
=

711. **eóéÉêÄçäáÅ=`óäáåÇÉê=E`~ëÉ=NNF= ñ**
O óO
−ÄO =N=~O
= ======



= **Figure 149.** =

712. oÉ~ä=fåíÉêëÉÅíáåÖ=mä~åÉë=E`~ëÉ=NOF= ñO óO

~O −ÄO =M=

=

713. fã~Öáå~êó=fåíÉêëÉÅíáåÖ=mä~åÉë=E`~ëÉ=NPF= ñO óO

~O +ÄO =M=

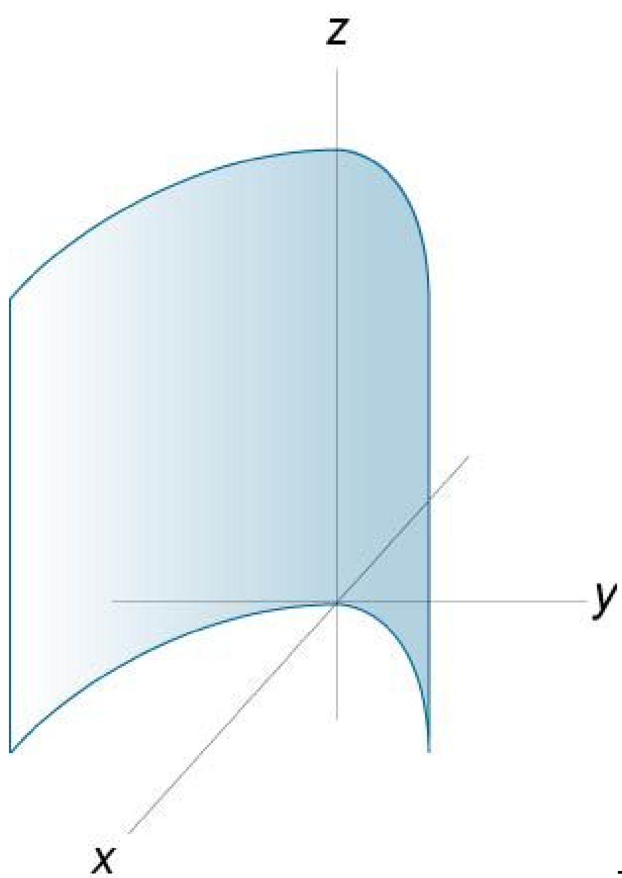
=

714. m~ê~ÄçäáÅ=`óäáåÇÉê=E`~ëÉ=NQF=

ñO

~O −ó=M=

= =====



= Figure 150. =

715. oÉ~ä=m~ê~ääÉä=mä~åÉë=E`~ëÉ=NRF= ñO
O =N=~
=
716. fã~Öáå~êó=m~ê~ääÉä=mä~åÉë=E`~ëÉ=NSF= ñO
~O =−N=

=
717. `çáåÅáÇÉåí=mä~åÉë=E`~ëÉ=NTF=
ñO=M=
=
=
=
=

7.12 Sphere

=

o~Çáìë=çÑ=~=ëéÜÉêÉW=o=

mçáåí=ÅççêÇáå~íÉëW=ñI=óI=òI=ñI=óI= òI=£=

`ÉåíÉê=çÑ=~=ëéÜÉêÉW=

( )

N N N

=

oÉ~ä=åìãÄÉêëW=^I=aI=bI=cI=j=

=

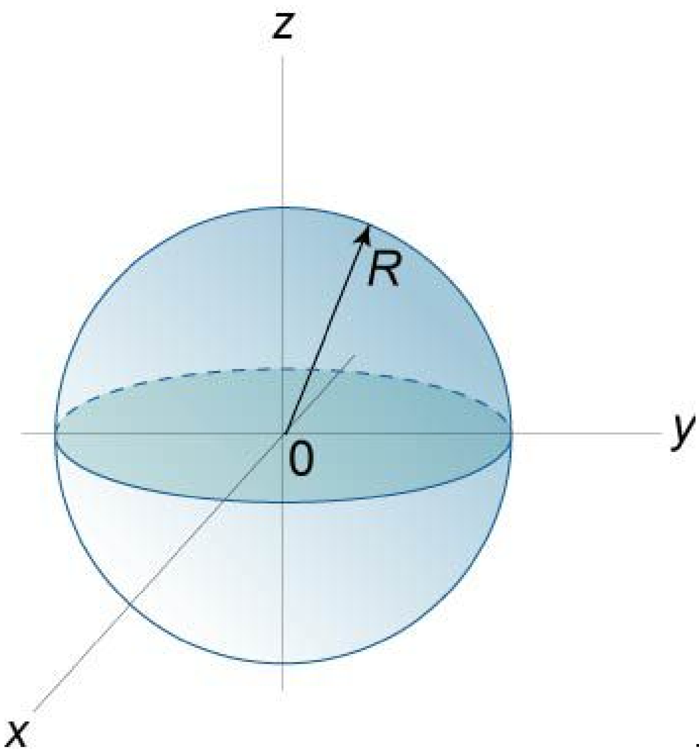
=

718. bèì~íáçå=çÑ=~=péÜÉêÉ=`ÉåíÉêÉÇ=~í=íÜÉ=lêáÖáå=Epí~åÇ~êÇ= cçêãF=

ñ^{O} +ó^{O} +ò^{O} =o^{O}=

=

= ======



= **Figure 151.**

=

719. bèì~íáçå=çÑ=~=`áêÅäÉ=`ÉåíÉêÉÇ=~í=^åó=mçáåí=( )~IÄIÅ

()( ) ( )^{O} =o^{O}

720. aá~ãÉíÉê=cçêã
ñ
−

()( ) ( )(−ñO +N ó−óO + −) ( )( )

ñ =MI==N ñ N O ïÜÉêÉ==

ñmI= () (ñOIóOIòO)=~êÉ=íÜÉ=ÉåÇë=çÑ=~=Çá~ãÉíÉêK==N
N^{Ió}N^{I}òN mO

=

721. cçìê=mçáåí=cçêã=
ñ^{O} +ó^{O} +ò^{O} ñ ó ò N
ñ^{O} +ó^{O} +ñ^{O} ñ$_{N}$ ó$_{N}$ ò$_{N}$ N$_{N N N}$
ñ^{O} +ó^{O} +ñ^{O} ñ$_{O}$ ó$_{O}$ ò$_{O}$ N =M=$_{O O O}$
ñ^{O} +ó^{O} +ñ^{O} ñ$_{P}$ ó$_{P}$ ò$_{P}$ N$_{P P P}$
ñ^{O} +ó^{O} +ñ^{O} ñ$_{Q}$ ó$_{Q}$ ò$_{Q}$ N$_{Q Q Q}$

=
722. dÉåÉê~ä=cçêã
^ñ^{O} +^ó^{O} +^ò^{O} +añ+bó +cò+j=M=E^=áë=åçåòÉêçFK==

qÜÉ=ÅÉåíÉê=çÑ=íÜÉ=ëéÜÉêÉ=Ü~ë=ÅççêÇáå~íÉë=(
)I~I=ïÜÉêÉ==

~ −^{a} I=Ä −^{b} I=Å −^{c} K=$_{O^ O^ O^}$
qÜÉ=ê~Çáìë=çÑ=íÜÉ=ëéÜÉêÉ=áë

$a^{O} + b^{O} + c^{O} - Q \wedge^{O} j_{Ko} = O\wedge$

$=$

Chapter 8 Differential Calculus

=
=
=
=
cìåÅíáçåëW=ÑI=ÖI=óI=ìI=î=
^êÖìãÉåí=EáåÇÉéÉåÇÉåí=î~êá~ÄäÉFW=ñ=
oÉ~ä=åìãÄÉêëW=~I=ÄI=ÅI=Ç=
k~îìê~ä=åìãÄÉêW=å=
^åÖäÉW=α=
fåîÉêëÉ=ÑìåÅíáçåW=$Ñ^{-N}$ =
=
=

8.1 Functions and Their Graphs

=

723. **bîÉå=cìåÅíáçå=**

Ñ =() ()=

=

724. lÇÇ=cìåÅíáçå=

Ñ() ()=

=

725. mÉêáçÇáÅ=cìåÅíáçå=

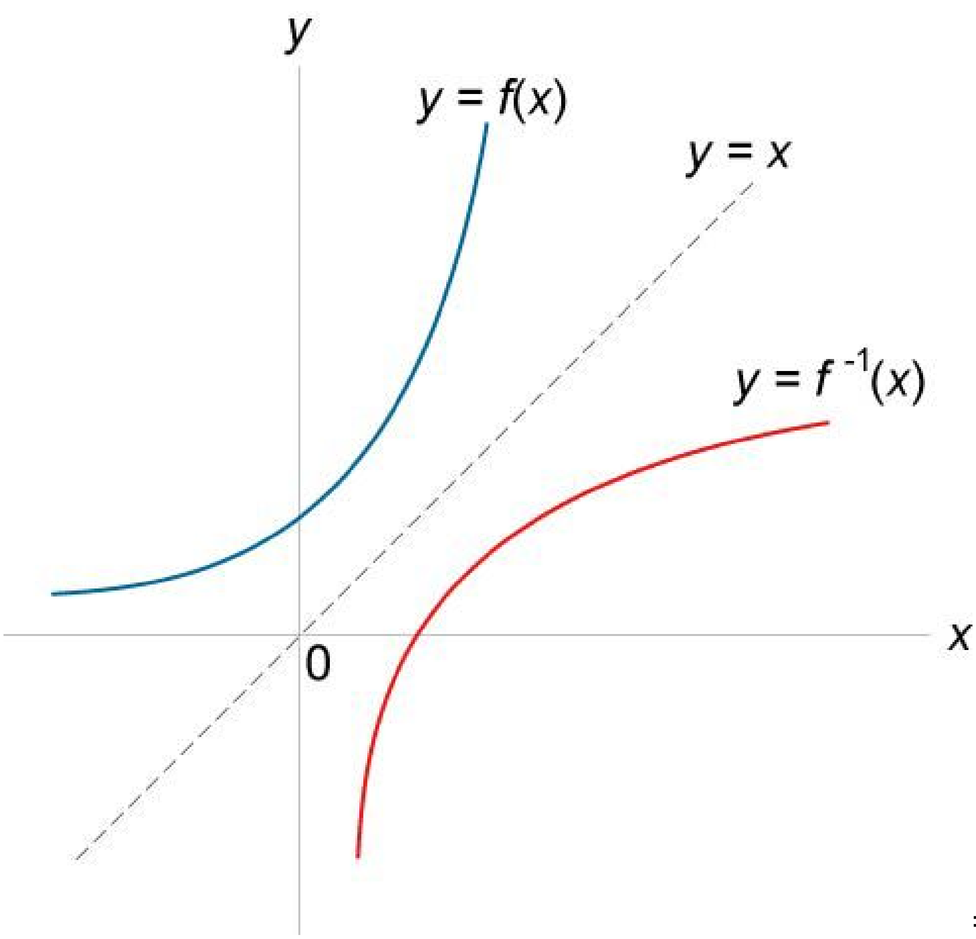
Ñ()()=

=

726. fåîÉêëÉ=cìåÅíáçå=
ó

=Ñ()=áë=~åó=ÑìåÅíáçåI=ñ ( )Ö ó =çê=ó=Ñ$^{-N}$ ñ (
)=áë=áíë=áåîÉêëÉ=

ÑìåÅíáçåK==
= =====



= Figure 152.

=

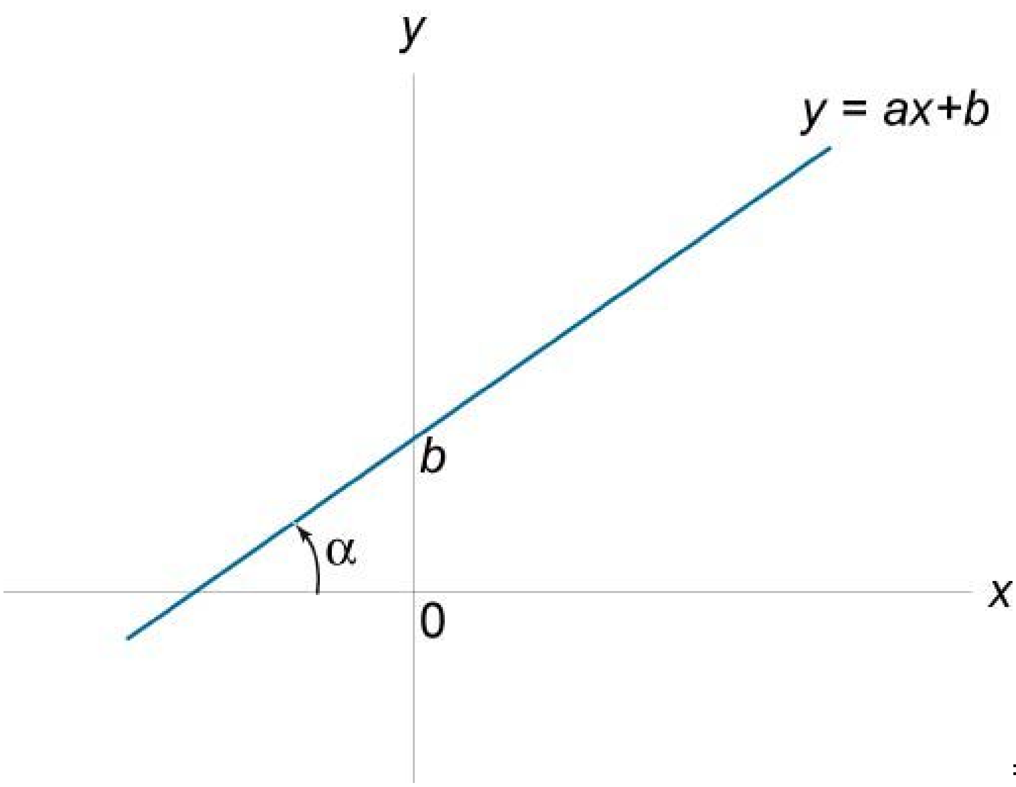
727. `çãéçëáíÉ=cìåÅíáçå=

ó=Ñ()I= ( )I=ó=Ñ Ö( ) =áë=~=ÅçãéçëáíÉ=ÑìåÅíáçåK=

=

728. **iáåÉ~ê=cìåÅíáçå=**

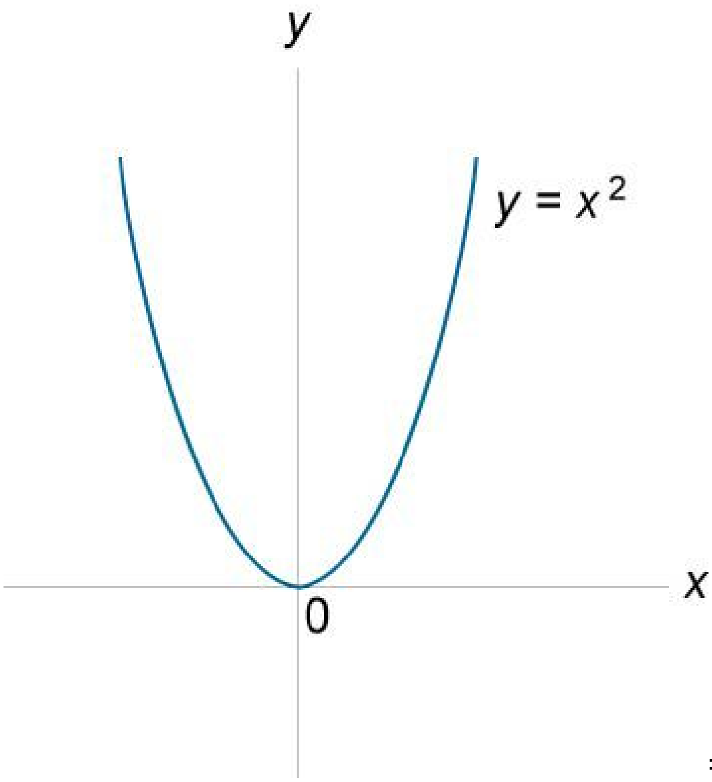
ó
+**ÄI== ñ∈oI==~** = **í~å** α
~ñ **==áë=íÜÉ=ëäçéÉ=çÑ=íÜÉ=äáåÉI==Ä==áë= íÜÉ=ó-áåíÉêÅÉéíK=**
= ======



= **Figure 153.** =

729. **nì~Çê~íáÅ=cìåÅíáçå==**
ó= **ñ**^{O} **I=ñ∈oK=**
=

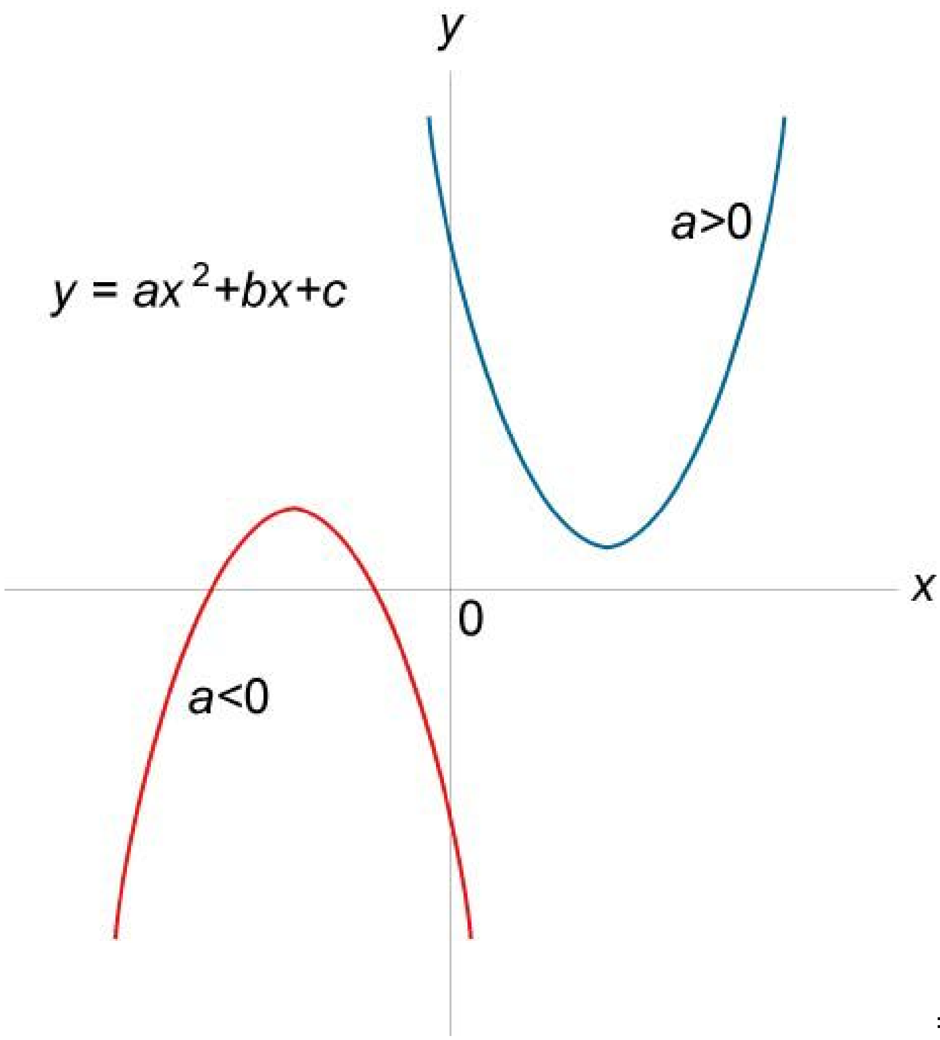
= ======



= **Figure 154.**

730. ó =~ñO +Äñ ÅI=ñ∈oK= =

= ===

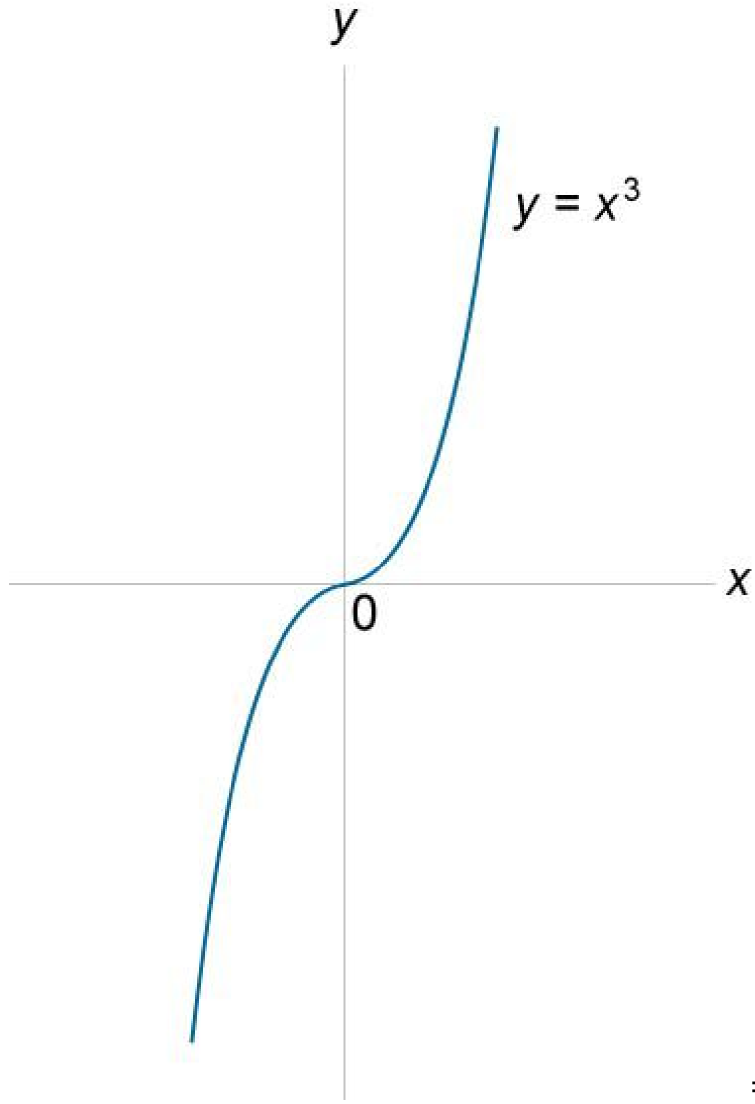


= **Figure 155.** =

731. **\`ìÄáÅ=cìåÅíáçå==**

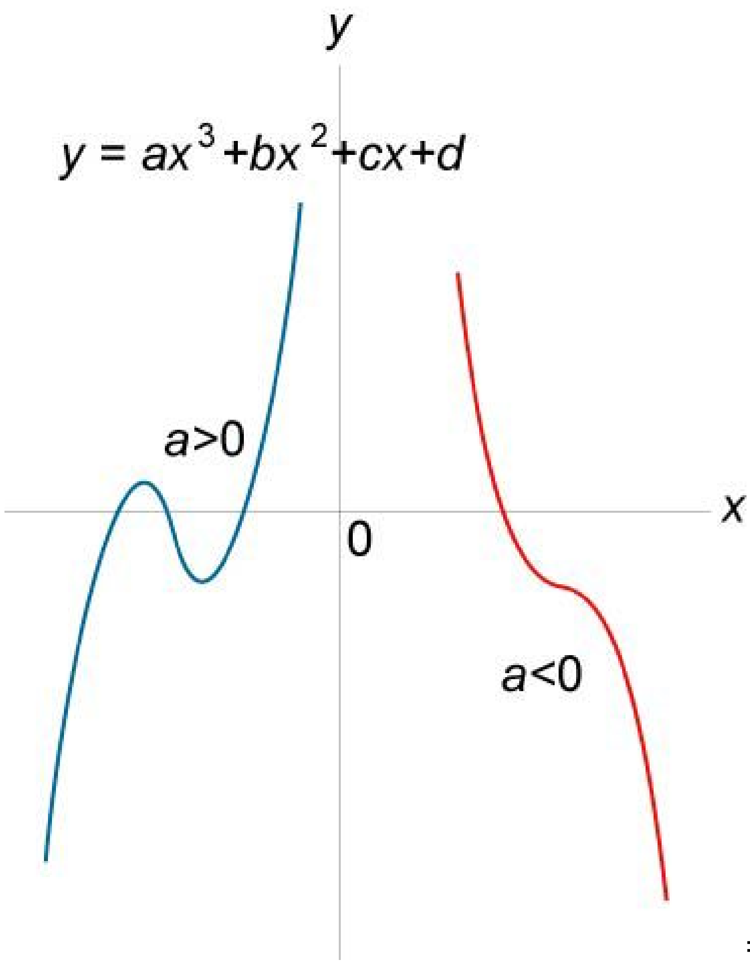
ó= ñP I=ñ∈oK=

==



= **Figure 156.** =

732. **ó =~ñ^{P} +Äñ^{O} +Åñ+ÇI=ñ∈oK=** = ==

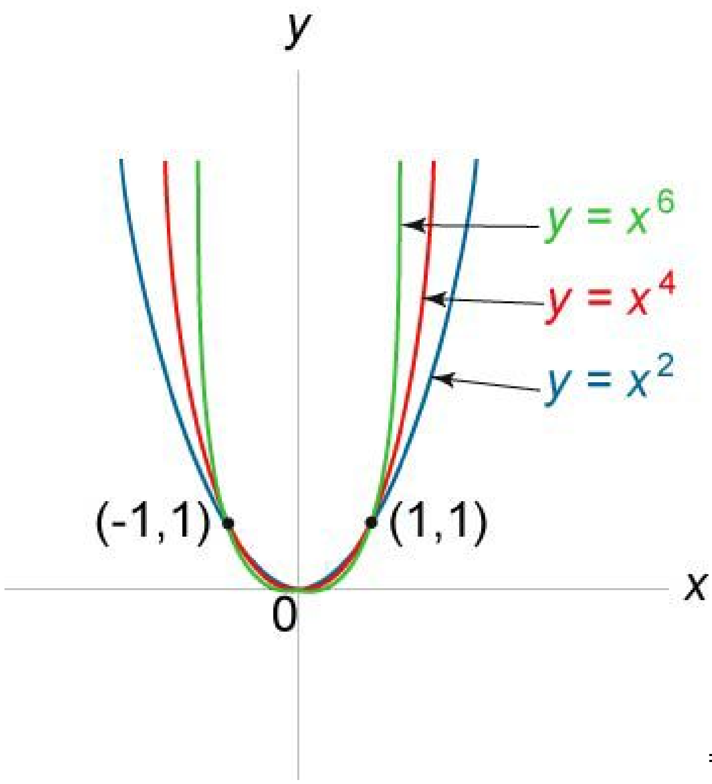


= **Figure 157.** =

733. **mçïÉê=cìåÅíáçå==**

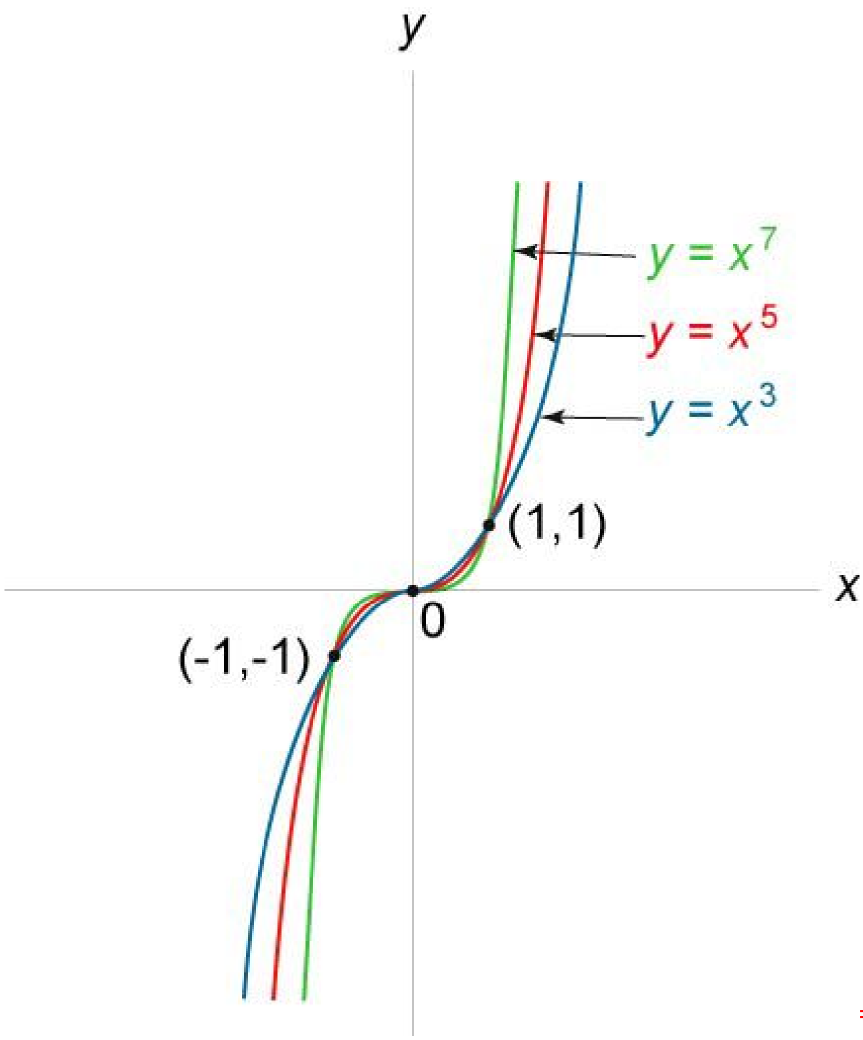
ó= ñåI=å∈kK=

======



=**Figure 158.**

=



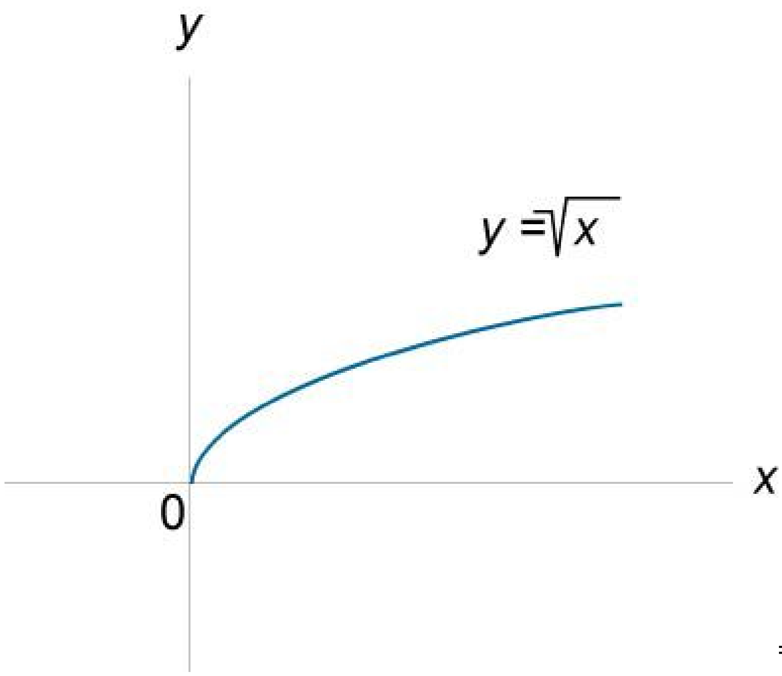
=Figure 159.

734. pèì~êÉ=oççí=cìåÅíáçå==

ó= ñI= [ )K=

=

=======



=**Figure 160.**

=

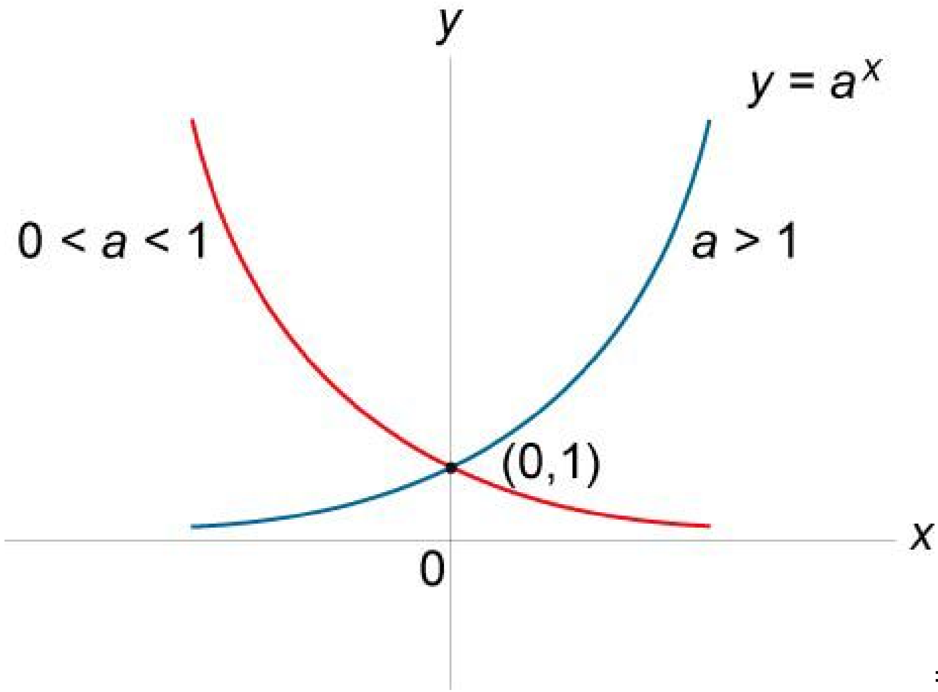
735. bñéçåÉåíá~ä=cìåÅíáçåë=

ó=~ñI= ~>MI=~≠NI=

ó=Éñ =áÑ=~=ÉI=É=OKTNUOUNUOUQSK=

=

==

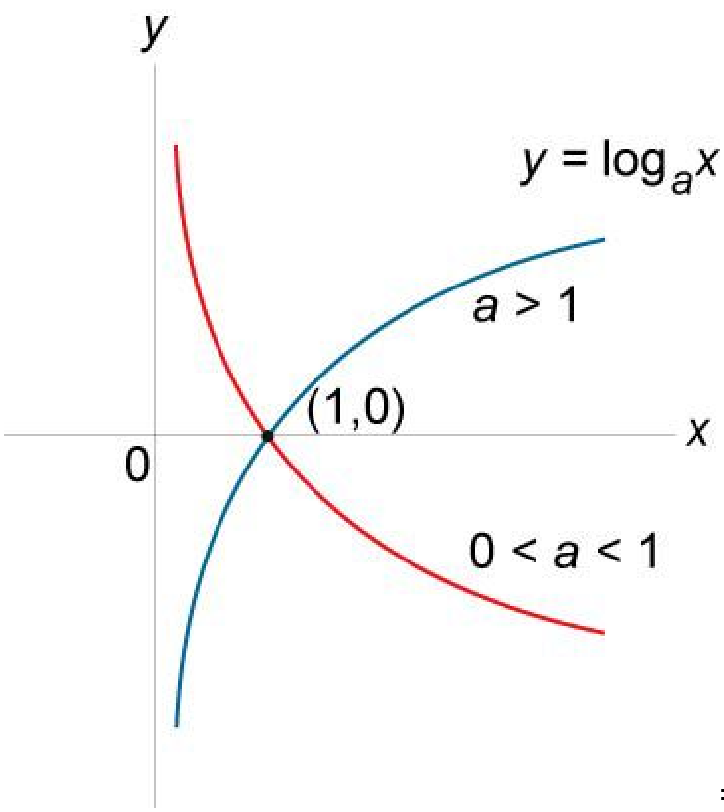


= **736.**

içÖ~êáíÜãáÅ=cìåÅíáçåë=

ó=äçÖ~ ñ I= ( )I= ~>MI=~≠NI=

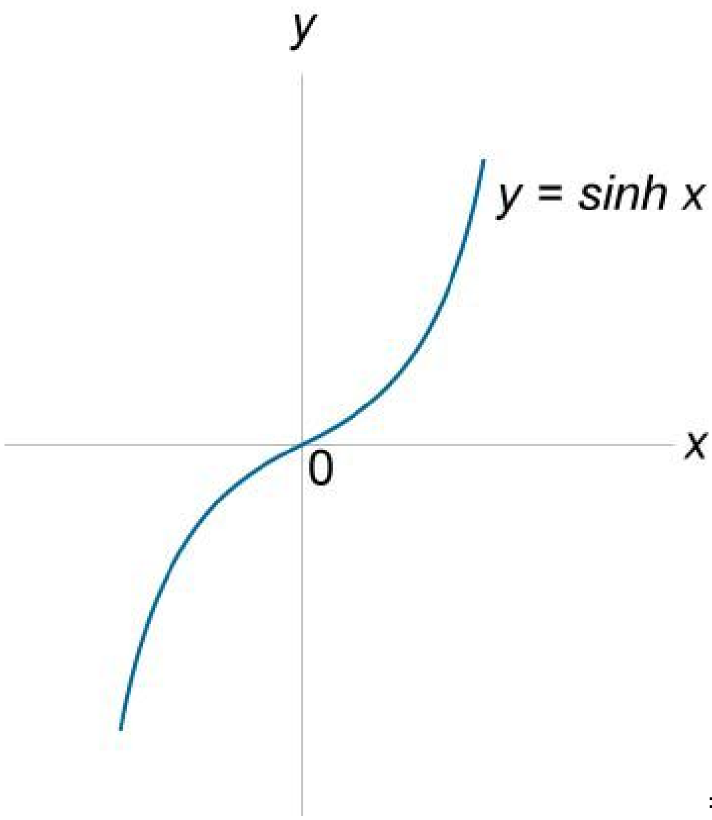
ó=äåñ=áÑ=~=ÉI=ñ>MK=
=



==

Figure 162.
=

737. eóéÉêÄçäáÅ=páåÉ=cìåÅíáçå==
ó
=
ëáåÜ
ñ
I=
ëáåÜ
ñ
=
Éñ −É−ñ
O I=ñ∈oK= = ====



= **Figure 163.**

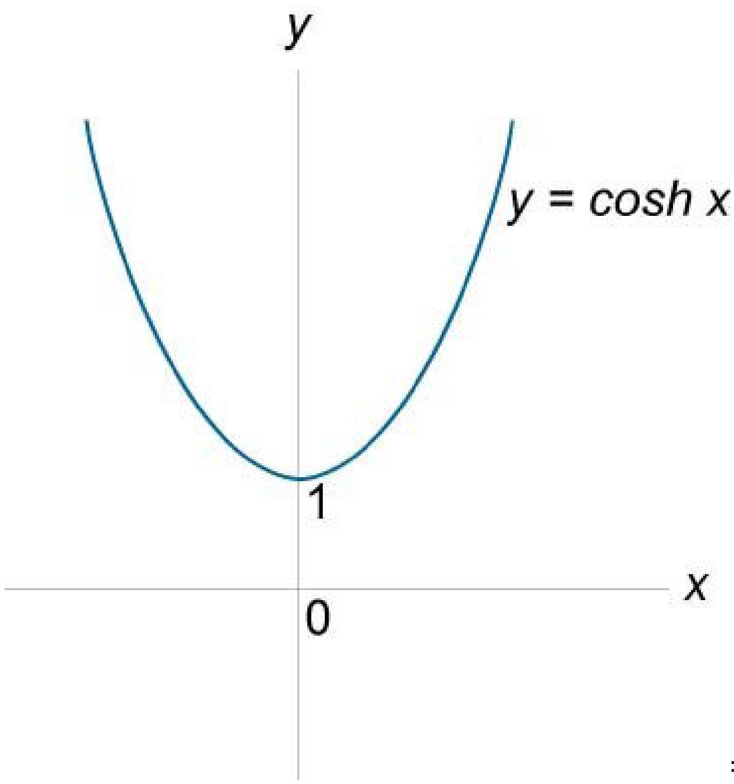
=

738. **eóéÉêÄçäáÅ=`çëáåÉ=cìåÅíáçå==**

É

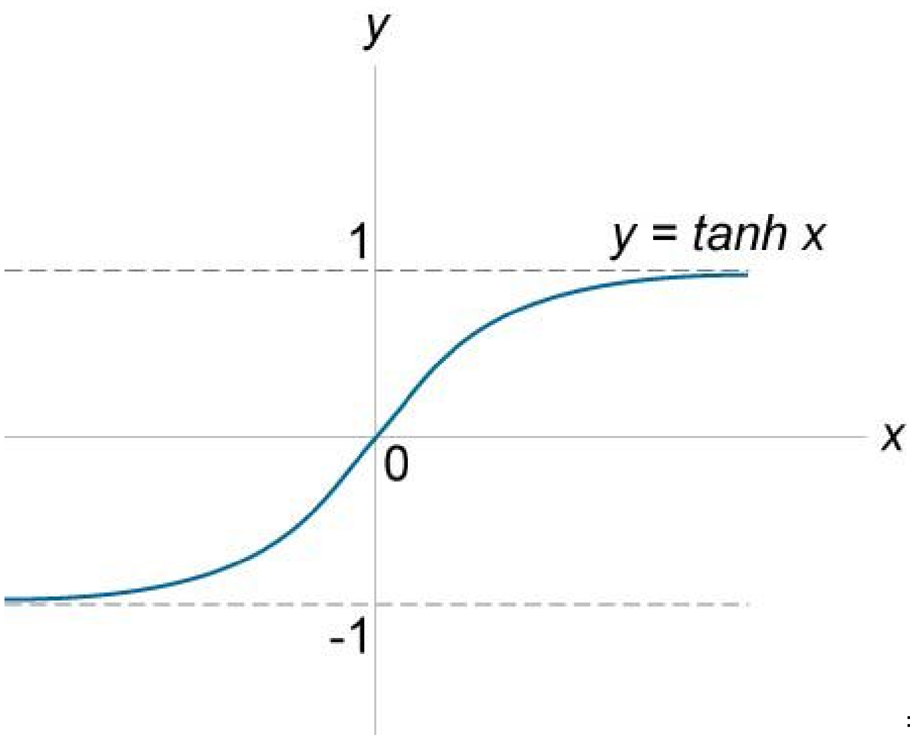
ñ +**É**−**ñ**

ó=**ÅçëÜñ I=ÅçëÜñ** =**O I=ñ∈oK=**



= = ======

739. **eóéÉêÄçäáÅ=q~åÖÉåí=cìåÅíáçå==**
ëáåÜ
ñ
=
É
ñ −É−ñ
ó=í~åÜ ñ I=ó =í~åÜ ñ=ÅçëÜ ñ Éñ +É−ñ I=ñ∈oK=
=
= ======



= **Figure 165.**

=

740. **eóéÉêÄçäáÅ=`çí~åÖÉåí=cìåÅíáçå==**

ÅçëÜ

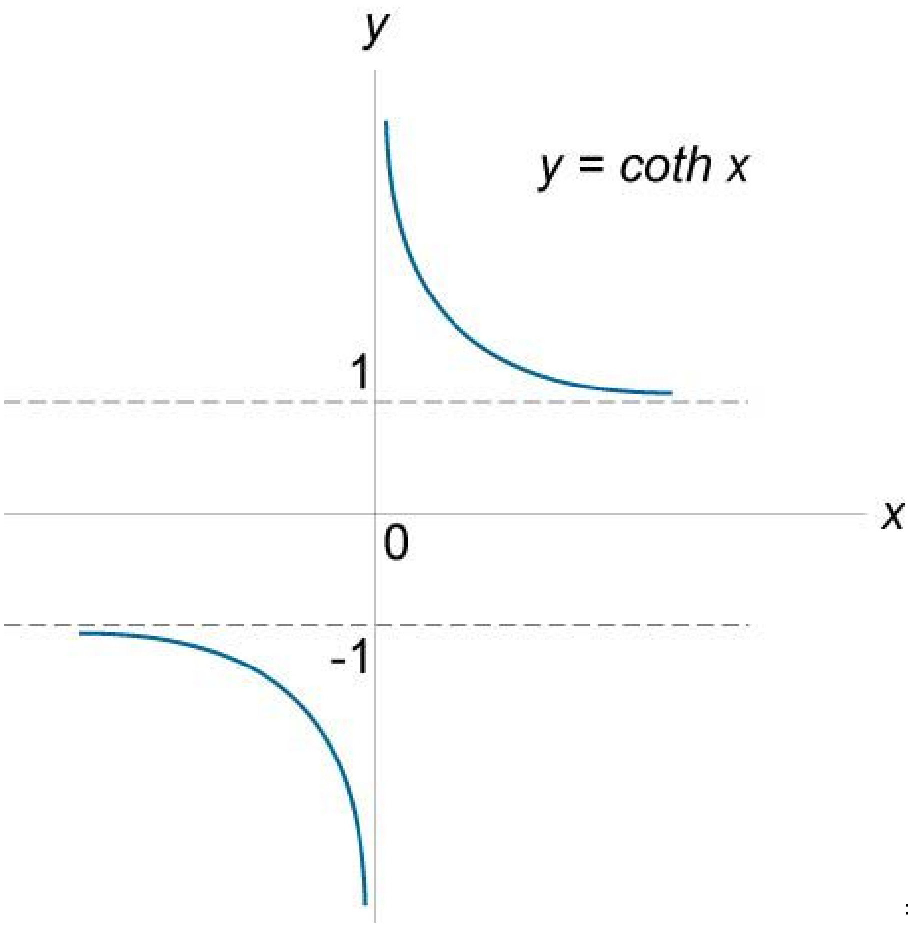
ñ

=

É

ñ +**É**−**ñ**

ó=**ÅçíÜñI=ó** =**ÅçíÜñ** =**ëáåÜ ñ Éñ −É−ñ** **I=ñ∈oI=ñ≠MK=** = **======**



= Figure 166.

=

741. **eóéÉêÄçäáÅ=pÉÅ~åí=cìåÅíáçå==**

ó

=

ëÉÅ

Ü

ñ

I=

ó

=

ëÉÅ

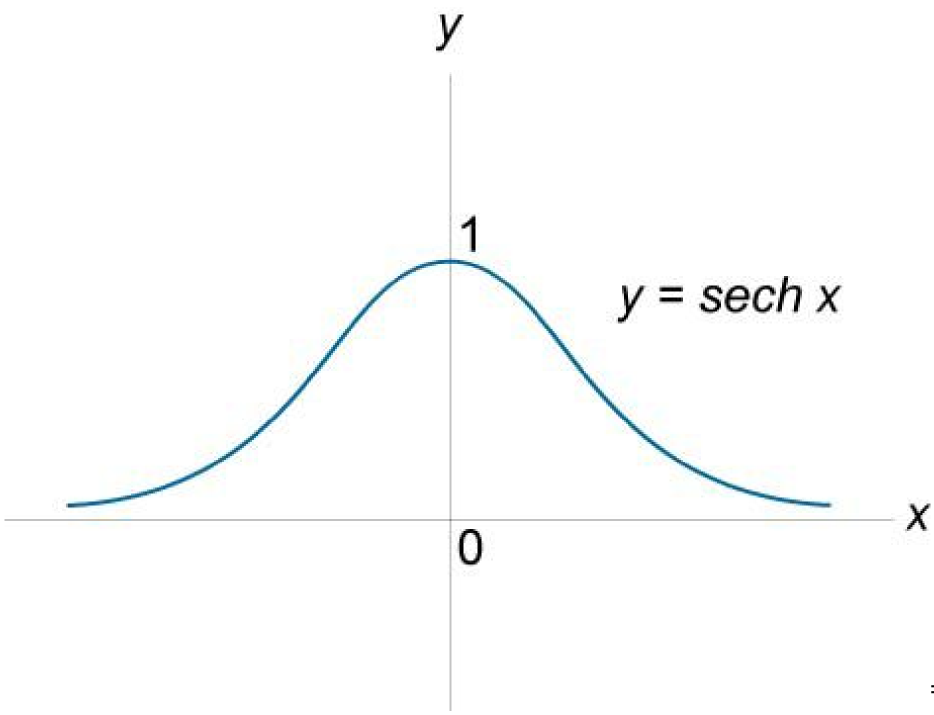
Ü

ñ

=

N= O

Éñ +É−ñ I=ñ∈oK=ÅçëÜ ñ



=Figure 167.

742. eóéÉêÄçäáÅ=`çëÉÅ~åí=cìåÅíáçå==

ó

=

ÅëÅÜ

ñ

I=

ó

=

ÅëÅÜ

ñ

=

N= O

Éñ −É−ñ I=ñ∈oI=ñ≠MK=ëáåÜñ

=

= ======

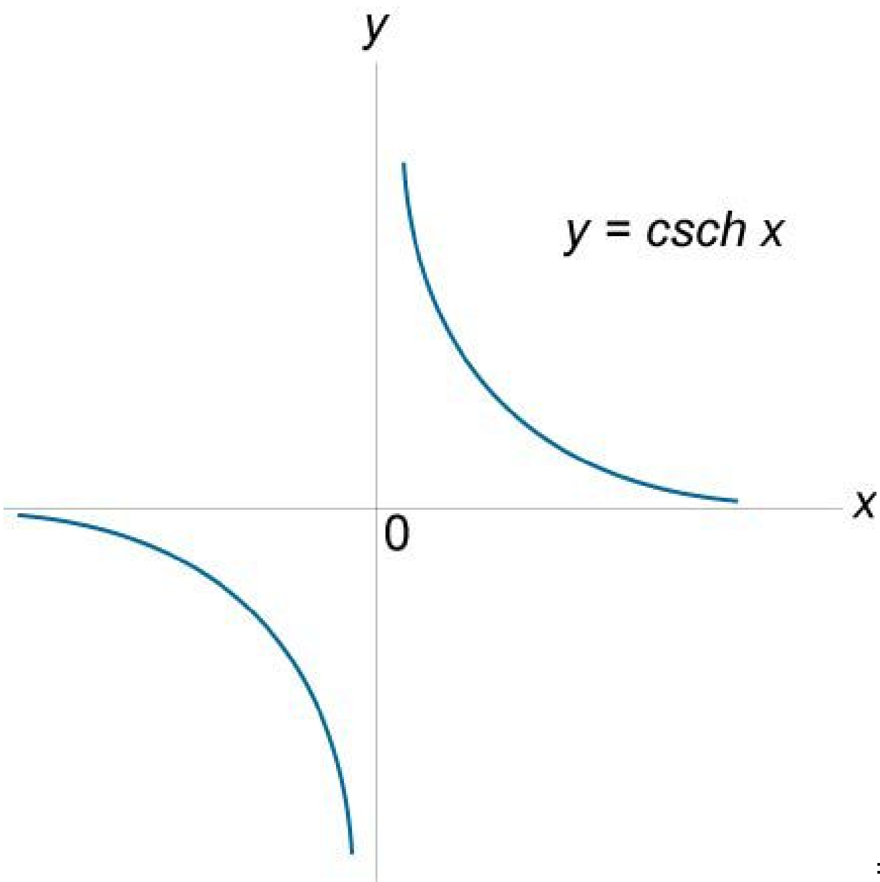
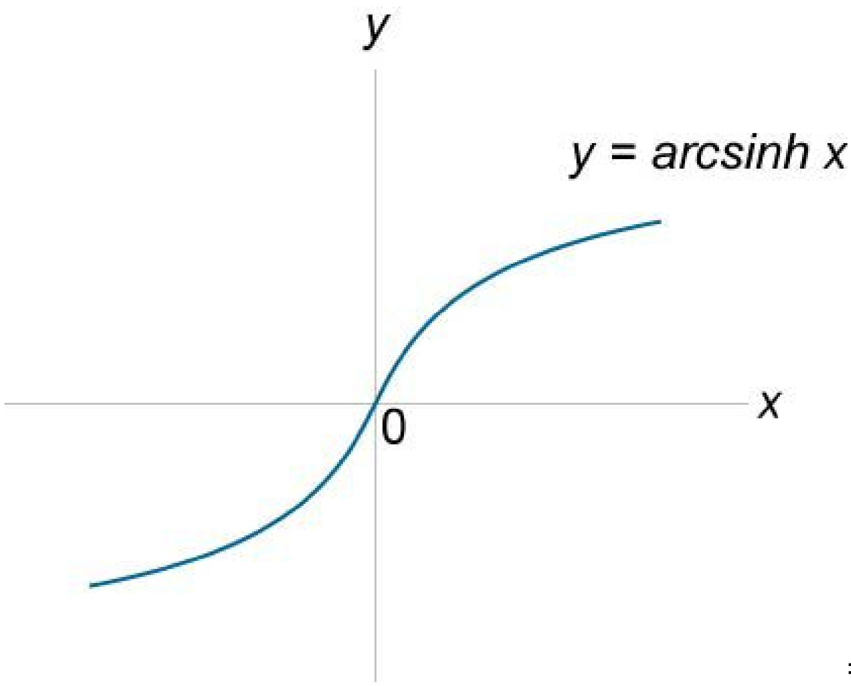


Figure 168.

743. fåîÉêëÉ=eóéÉêÄçäáÅ=páåÉ=cìåÅíáçå== ó=~êÅëáåÜ ñI=ñ∈oK=



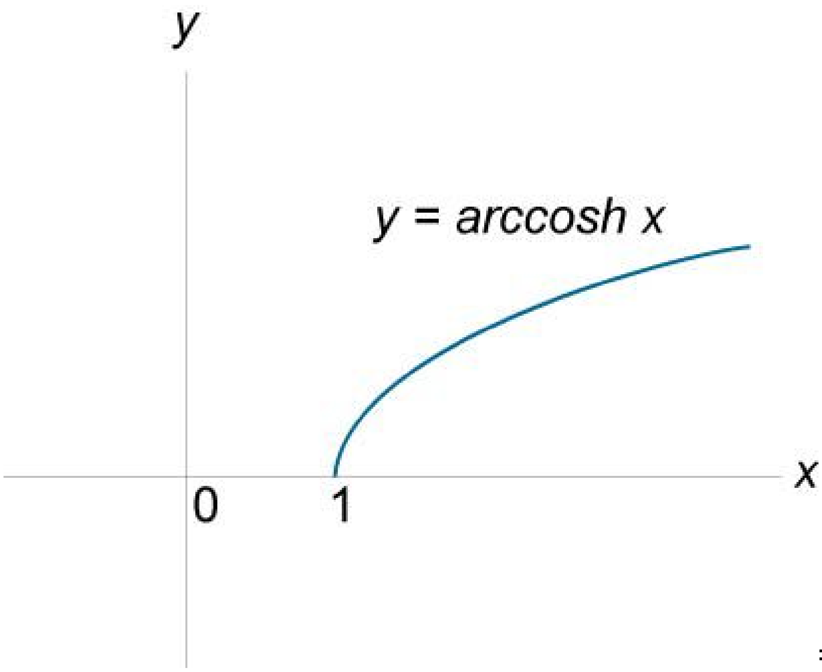
= Figure 169. =

744. fåîÉêëÉ=eóéÉêÄçäáÅ=`ëáåÉ=cìåÅíáçå==

ó=~êÅÅçëÜ ñI= [ )K=

=

= =====

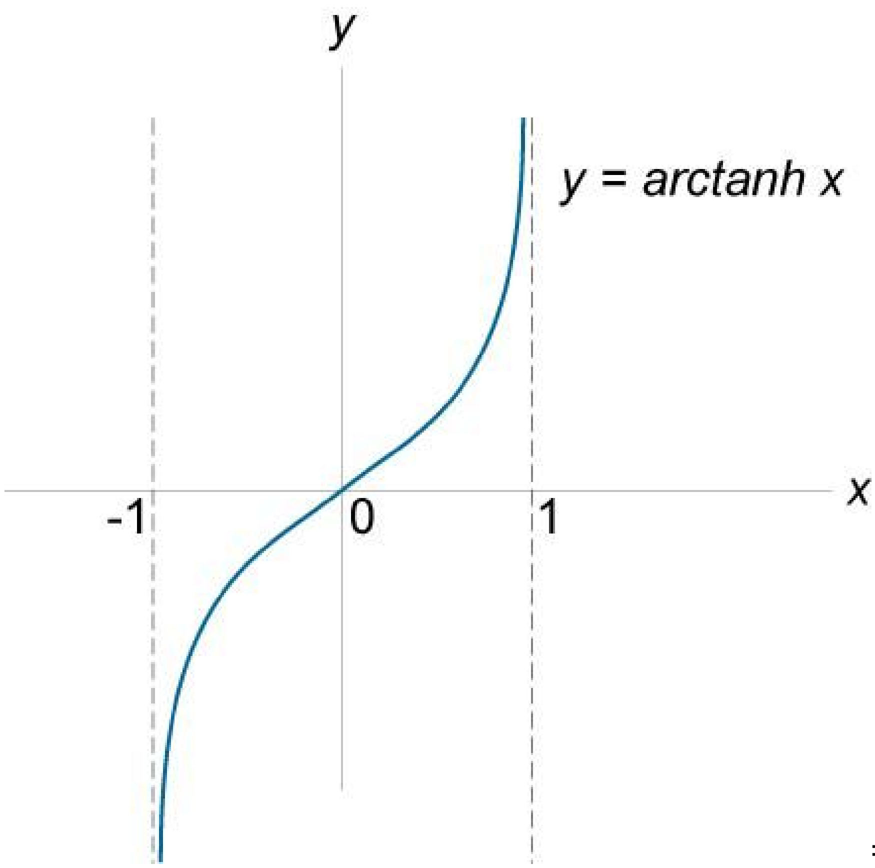


= Figure 170. =

745. **fåîÉêëÉ=eóéÉêÄçäáÅ=q~åÖÉåí=cìåÅíáçå==**

ó=~êÅí~åÜ ñI= ( )K=

= =====

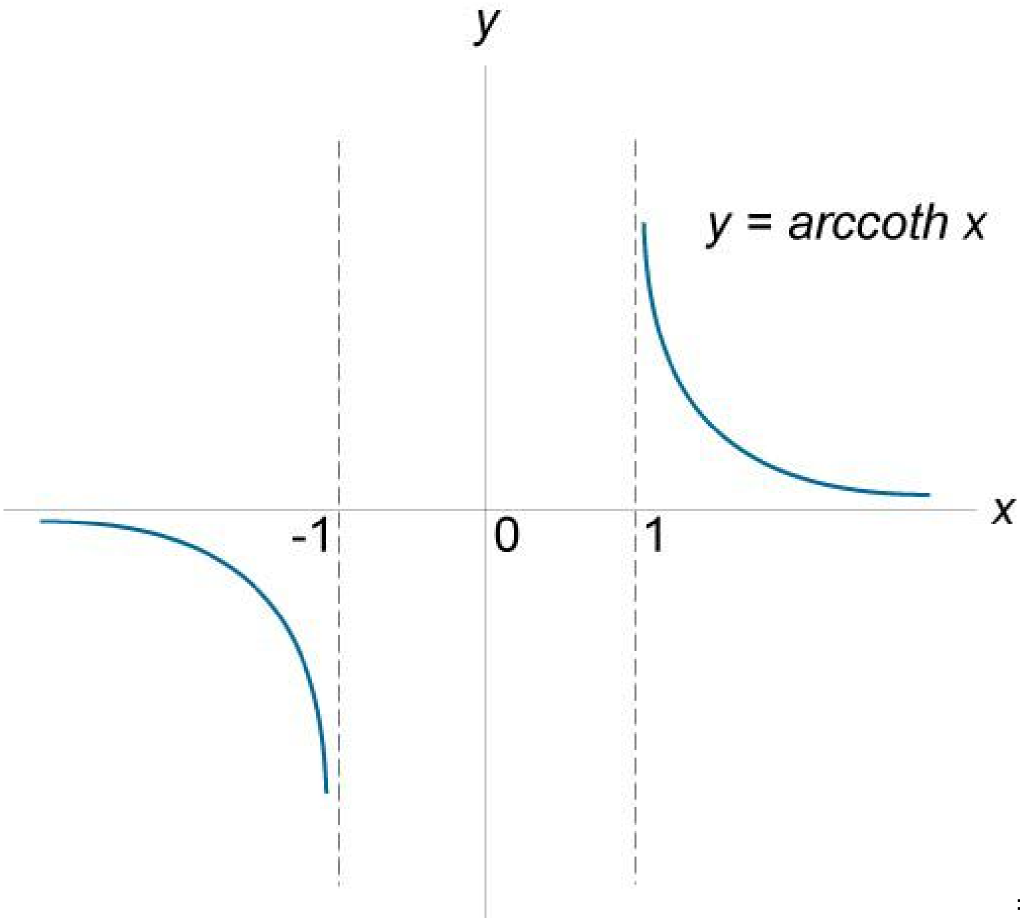


= **Figure 171.** =

746. fåîÉêëÉ=eóéÉêÄçäáÅ=`çí~åÖÉåí=cìåÅíáçå==

ó=~êÅÅçíÜñ I= (−∞I−N) ( )K==

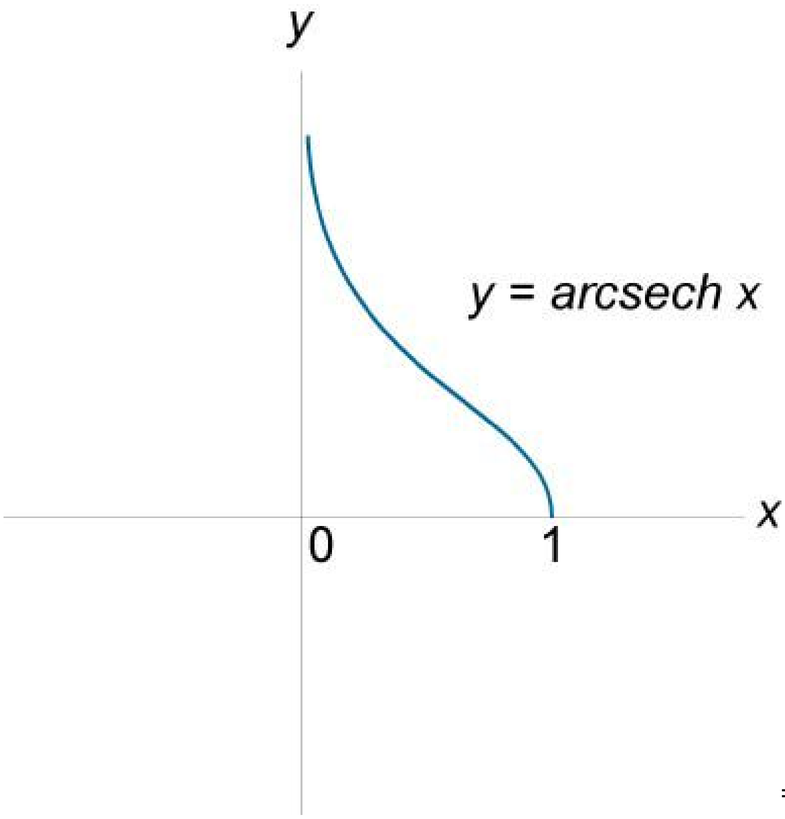
=====



= **Figure 172.** =

747. fåîÉêëÉ=eóéÉêÄçäáÅ=pÉÅ~åí=cìåÅíáçå==

ó=~êÅëÉÅÜ ñI= ( ]K=



=**Figure 173.**

=

748. **fåîÉêëÉ=eóéÉêÄçäáÅ=`çëÉÅ~åí=cìåÅíáçå== ó=~êÅÅëÅÜ**

ñI=ñ∈oI=ñ≠MK==

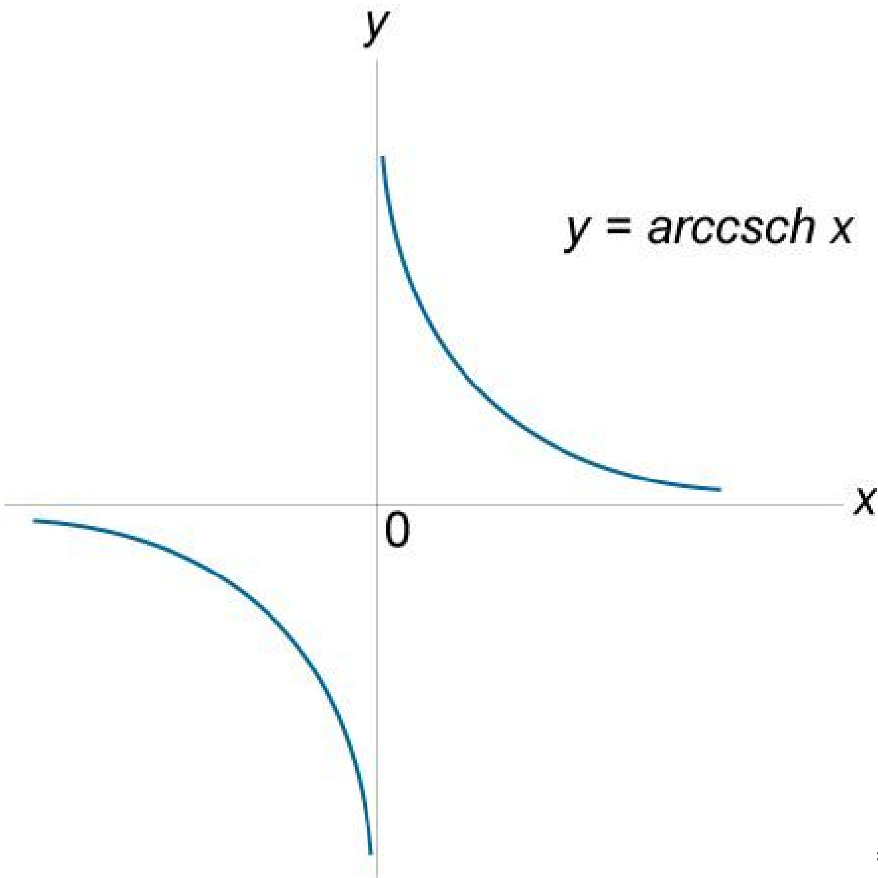


Figure 174.

8.2 Limits of Functions

=

cìåÅíáçåëW=()I=( )=

^êÖìãÉåíW=ñ=
oÉ~ä=Åçåëí~åíëW=~I=â=
=
=

749. () ()[]=äáãÑ( )+äáãÖ( )=

ñ→~ ñ→~ ñ→~
=

750. () ()[]=äáãÑ( )−äáãÖ( )=

ñ→~ ñ→~ ñ→~
=

751. () ()[]=äáãÑ( )⋅äáãÖ( )=

ñ→~ ñ→~ ñ→~

= ( ) ( )≠MK= ñ→~ Ö()=äáãÑ752. ()
äáã ñ→~ I=áÑ=äáãÖ() ñ→~

ñ→~
=

753. ()[]=âäáãÑ( )=

ñ→~ ñ→~
=

754. ()()=Ñ(äáãÖ( ))=

ñ→~ ñ→~
=

755. () ()I=áÑ=íÜÉ=ÑìåÅíáçå=( )=áë=Åçåíáåìçìë=~í=ñ=~ K=

ñ→~
=
756. äáã^{ëáåñ} =N=
ñ→M ñ
=
757. äáã$^{í~åñ}$ =N=
ñ→M ñ
=
758. äáã$^{ëáå−Nñ}$ =N=
ñ→M ñ
759. äáã$^{í~å−N ñ}$ =N=
ñ→M ñ
=
760.
äáã

()=N=

ñ→M ñ
=
761.
äáã
+
N^{ñ}
=É=
ñ→∞ ñ
=
762.
äáã
+
â ^{ñ}

=Éâ=
ñ→∞ ñ
=
763.
äáã
ñ
=N=
ñ→M

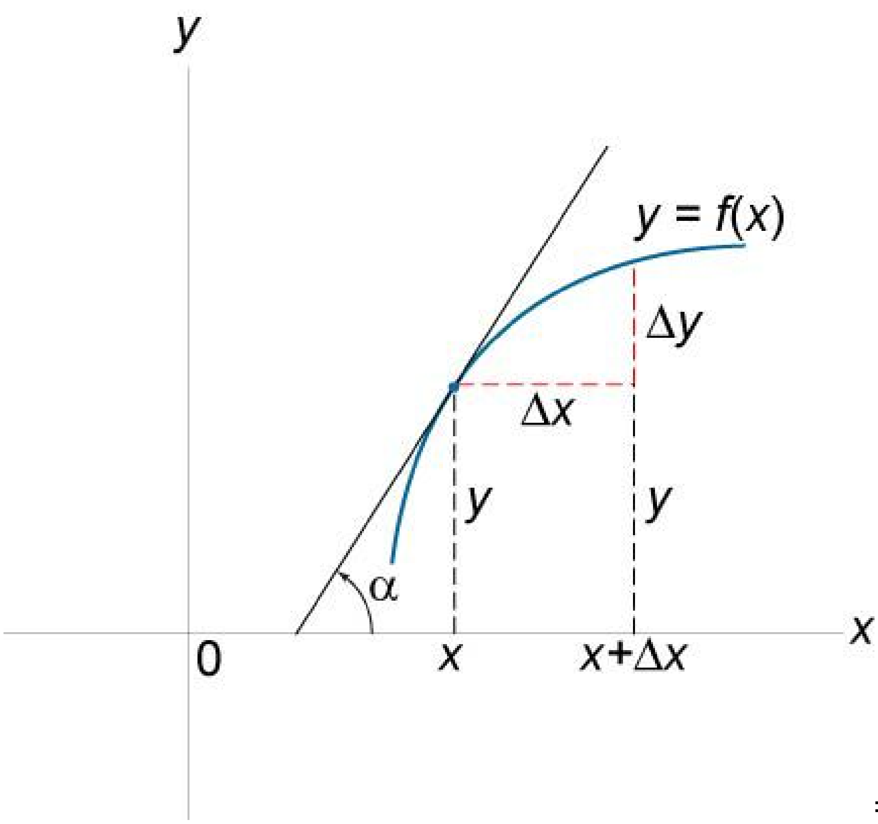
=
=
=

8.3 Definition and Properties of the Derivative

=
cìåÅíáçåëW=ÑI=ÖI=óI=ìI=î=
fåÇÉéÉåÇÉåí=î~êá~ÄäÉW=ñ=
oÉ~ä=Åçåëí~åíW=â=
^åÖäÉW=α=
=
=

764. () () ( )=äáã Δó=Çó==

Δñ→M ΔñΔñ→M Δñ Çñ
= ==



= Figure 175. =

765. Çó = í~åα==Çñ
=

766. ()=Çì+Çî=Çñ Çñ Çñ

=

767. ()=Çì–Çî=Çñ Çñ Çñ

=

768.

()âÇì

= =

Çñ Çñ

=

769. mêçÇìÅí=oìäÉ=

Ç( )= Çì·î+ì· Çî==Çñ Çñ Çñ

=

770. nìçíáÉåí=oìäÉ=

Çì ·î−ì·Çî Ç □ì□= Çñ Çñ=Çñ□□î□□ îO

=

771. `Ü~áå=oìäÉ=

ó=Ñ()()I= ( )I==

Çó = Çó ·Çì K=Çñ Çì Çñ

=

772. aÉêáî~íáîÉ=çÑ=fåîÉêëÉ=cìåÅíáçå=

Çó= NI==Çñ Çñ

Çó

ïÜÉêÉ=()áë=íÜÉ=áåîÉêëÉ=ÑìåÅíáçå=çÑ=( )ñóK==

=

773. oÉÅáéêçÅ~ä=oìäÉ=

Çó

Ç N=− Çñ =Çñ ó óO

=

774. içÖ~êáíÜãáÅ=aáÑÑÉêÉåíá~íáçå=

ó=Ñ()I=äå =äåÑ( )I==

Çó =Ñ ñ ·Ç() ()[]K=Çñ Çñ

=

=

8.4 Table of Derivatives

=
fåÇÉéÉåÇÉåí=î~êá~ÄäÉW=ñ=
oÉ~ä=Åçåëí~åíëW=`I=~I=ÄI=Å=
k~íìê~ä=åìãÄÉêW=å=

775.
Ç

()

=M=
Çñ
=
776.
Ç

()

=N=
Çñ
=

777. Ç()=~=

Çñ
=

778. Ç()=~ñ+Ä=

Çñ
=

779. Ç ()=åñ$^{å−N}$=

Çñ
=
780.

Ç () – å

ñ

å
+
N

=
Çñ

=
781. Ç N =– N = Çñ ñ ñO
=
782.

Ç ()= N

O ñ=Çñ
=
783.

Ç ()= N

åñå−N=Çñå
=

784. Ç ()=N=

Çñ ñ
=
785.

Ç() = N

ñ
äå
~
I=~>MI=~≠NK= Çñ~

786. Ç ()=~ñ äå~I=~>MI=~≠NK=

Çñ
=

787. Ç () =Éñ=

Çñ
=

788. Ç()=Åçëñ=

Çñ
=

789. Ç()=−ëáåñ=

Çñ
=

790. Ç()=N =ëÉÅO ñ=

Çñ ÅçëO ñ
=

791. Ç()=−N −ÅëÅO ñ=

Çñ ëáåO ñ
=

792. Ç()=í~åñ·ëÉÅñ=

Çñ
=

793. Ç()=−Åçíñ·ÅëÅñ=

Çñ
=

794. Ç()=N =

Çñ N−ñO
=

795. Ç()=−N =

Çñ N−ñO
=
796.

Ç()=N

N
+
ñ
O
=
Çñ
797.

Ç()=−N

N
+
ñ
O
=
Çñ
=

798. Ç()=N =

Çñ ñ ñO−N
=

799. Ç()=−N =

Çñ ñ ñO−N =

800. Ç()=ÅçëÜñ=

Çñ
=

801. Ç()=ëáåÜñ=

Çñ
=

802. Ç()=N =ëÉÅÜOñ=

Çñ ÅçëÜO ñ
=

803. Ç()=−N −ÅëÅÜOñ=

Çñ ëáåÜO ñ
=

804. Ç()=−ëÉÅÜñ·í~åÜ ñ=

Çñ
=

805. Ç()=−ÅëÅÜñ·ÅçíÜ ñ=

Çñ
=

806. Ç()=N =

Çñ ñO+N
=

807. Ç()=N =

Çñ ñO−N
808.

Ç()=N

N
−
ñ
O
I= ñ<NK=
Çñ
=
809.

Ç()=−N

ñ
O
−
N
I= ñ>NK=
Çñ
=

810. Ç ()= îì$^{\text{î}-\text{N}}$.Çì +ì$^{\text{î}}$ äåì.Çî =

Çñ Çñ Çñ =

=

=

8.5 Higher Order Derivatives

=
cìåÅíáçåëW=ÑI=óI=ìI=î=
fåÇÉéÉåÇÉåí=î~êá~ÄäÉW=ñ=
k~îìê~ä=åìãÄÉêW=å=
=
=

811. pÉÅçåÇ=ÇÉêáî~íáîÉ=

Ñ=′′()=′Çó =′ Ç Çó = ÇOó = Çñ Çñ Çñ Çñ ÇñO

=
812. eáÖÜÉê-lêÇÉê=ÇÉêáî~íáîÉ=

åó óÑ ()′ÇÑ=

Çñå
=
813.

()

() ()+ **î**( )

=**ì** **=**

=

814.

()

() ()−î( )
=ì =
=
815. iÉáÄåáíò∞ë=cçêãìä~ë=

()′′ =ì′′î+Oì î+′′ìî′′= ()′′′ =ì′′′î+Pì+′′′Pì+′′′ìî′′′=
()
() ()î +åì( )î+′å( )()î+′′K+ìî()

=ì =N·O ì
=

816. ()() =ã> ñã−å=()>

=
817.

()

()
=**å**>=
=

818. **()() ( )>()~ = ñåäå~ =**

=

819. **=()()() ( )>=ñå**

=
820.

()

()
=~ñ **äåå~ =**
=
821.

()

()
=**É**ñ=
=
822.

()

()
=ã^{å}~^{ãñ} ää^{å}~ =
=

823. ()$^{()}$ =ëáå π=ñ+ O

=

824. ()$^{()}$ =Åçë π=ñ + O

=
=

8.6 Applications of Derivative

=
cìåÅíáçåëW=ÑI=ÖI=ó=
mçëáíáçå=çÑ=~å=çÄàÉÅíW=ë==
sÉäçÅáíóW=î=
^ÅÅÉäÉê~íáçåW=ï=
fåÇÉéÉåÇÉåí=î~êá~ÄäÉW=ñ=
qáãÉW=í=
k~íìê~ä=åìãÄÉêW=å=
=
=

825. sÉäçÅáíó=~åÇ=^ÅÅÉäÉê~íáçå=

ë=Ñ()=áë=íÜÉ=éçëáíáçå=çÑ=~å=çÄàÉÅí=êÉä~íáîÉ=íç=~=ÑáñÉÇ=
Åççê

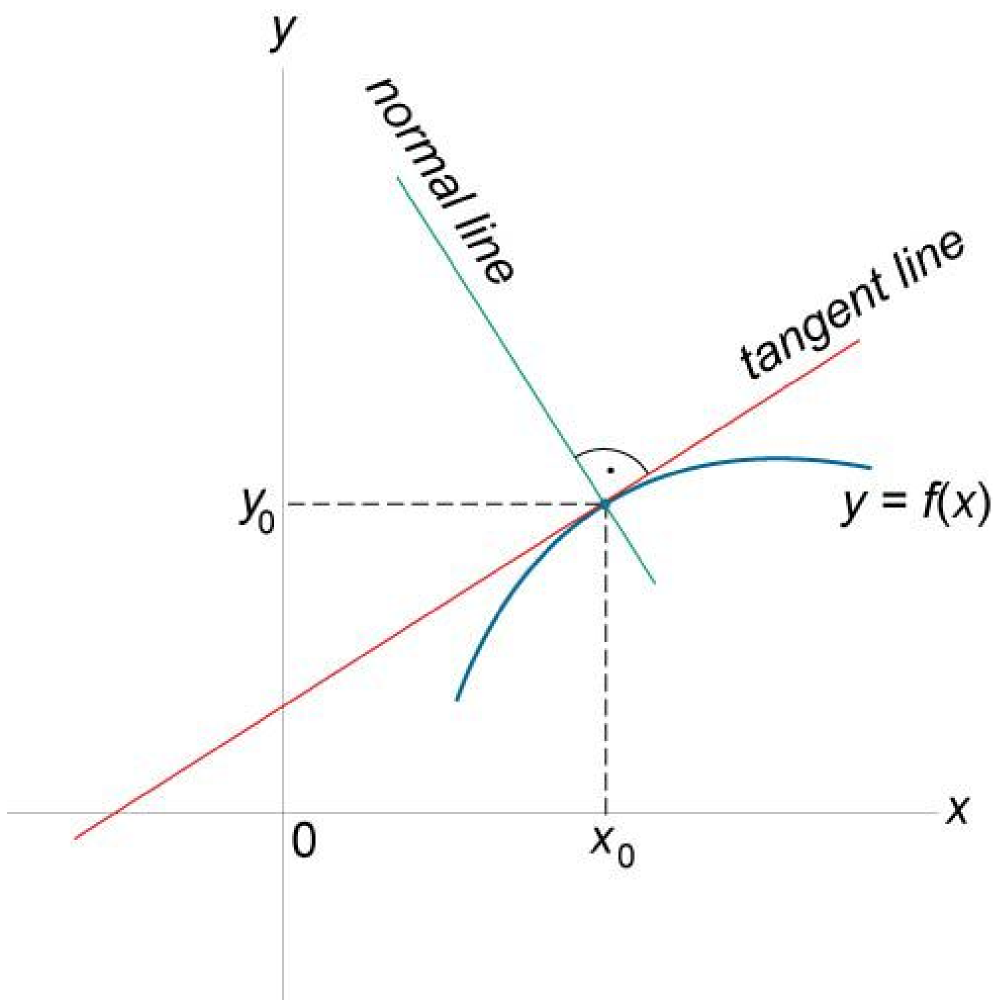
Çáå~íÉ=ëóëíÉã=~í=~=íáãÉ=íI==

î=ë=′Ñ′()=áë=íÜÉ=áåëí~åí~åÉçìë=îÉäçÅáíó=çÑ=íÜÉ=çÄàÉÅíI=

ï =î=′ë=′′Ñ′′(
)=áë=íÜÉ=áåëí~åí~åÉçìë=~ÅÅÉäÉê~íáçå=çÑ=

íÜÉ=çÄàÉÅíK==
=

826. q~åÖÉåí=iáåÉ=()( )M =ó−óM =Ñ′ ñM



= = **Figure 176.**

=

827. kçêã~ä=iáåÉ=

ó

−

ó

M

=

−

N

Ñ′ ñM()() =EcáÖ=NTSF=

=

828. **fåÅêÉ~ëáåÖ=~åÇ=aÉÅêÉ~ëáåÖ=cìåÅíáçåëK==**

fÑ=()M>MI=íÜÉå=ÑEñF=áë=áåÅêÉ~ëáåÖ=~í=

ñO< ñFI=

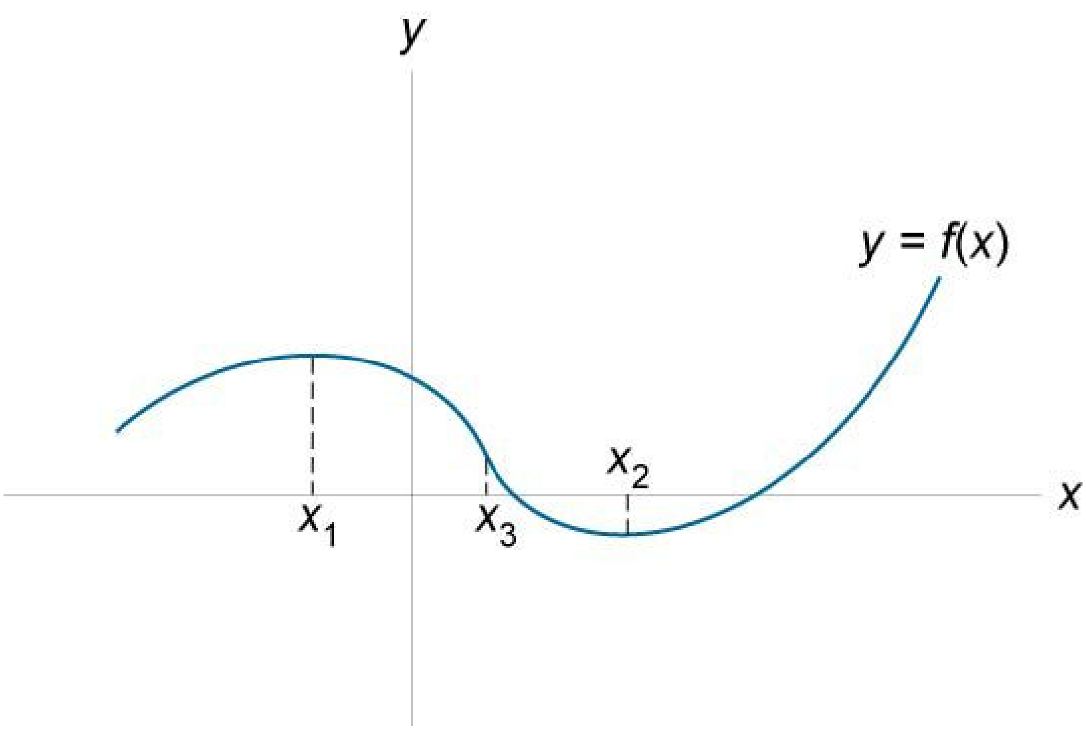
fÑ=$()_{M}$ <MI=íÜÉå=ÑEñF=áë=ÇÉÅêÉ~ëáåÖ=~í=

ñ$_{N}$ < ñ ñ$_{O}$FI=

()M
=ÇçÉë=åçí=Éñáëí=çê=áë=òÉêçI=íÜÉå=íÜÉ=íÉëí=Ñ~áäëK==

ñ K=EcáÖ=NTTI=ñ<ñ$_N$I=$_M$
$_M$ñ K=EcáÖ=NTTI=

fÑ=



=$_=$

Figure 177.
=

829. **içÅ~ä=ÉñíêÉã~=**
^=ÑìåÅíáçå=ÑEñF=Ü~ë=~=äçÅ~ä=ã~ñáãìã=~í=$_N$ñ
=áÑ=~åÇ=çåäó=áÑ= íÜÉêÉ=Éñáëíë=ëçãÉ=áåíÉêî~ä=Åçåí~áåáåÖ=$_N$ñ
=ëìÅÜ=íÜ~í=

Ñ$_N$()
()=Ñçê=~ää=ñ=áå=íÜÉ=áåíÉêî~ä=EcáÖKNTTFK
==

=
^=ÑìåÅíáçå=ÑEñF=Ü~ë=~=äçÅ~ä=ãáåáãìã=~í=$_O$ñ
=áÑ=~åÇ=çåäó=áÑ= íÜÉêÉ=Éñáëíë=ëçãÉ=áåíÉêî~ä=Åçåí~áåáåÖ=$_O$ñ
=ëìÅÜ=íÜ~í=

Ñ$_O$() ()=Ñçê=~ää=ñ=áå=íÜÉ=áåíÉêî~ä=EcáÖKNTTFK=

=
830. `êáíáÅ~ä=mçáåíë=
^=ÅêáíáÅ~ä=éçáåí=çå=ÑEñF=çÅÅìêë=~í=$_M$ñ
=áÑ=~åÇ=çåäó=áÑ=ÉáíÜÉê=

Ñ′

()=áë=òÉêç=çê=íÜÉ=ÇÉêáî~íáîÉ=ÇçÉëå∞í=Éñáëí

K=M

=

831. cáêëí=aÉêáî~íáîÉ=qÉëí=Ñçê=içÅ~ä=bñíêÉã~K=

fÑ=ÑEñF=áë==áåÅêÉ~ëáåÖ==E ( )
>MF=Ñçê==~ää==ñ==áå==ëçãÉ==áåíÉêî~ä=

( ]N ==~åÇ==ÑEñF==áë==ÇÉÅêÉ~ëáåÖ==E ( )
<MF==Ñçê=~ää==ñ=áå=ëçãÉ=

áåíÉêî~ä= =

[
)I==íÜÉå=ÑEñF=Ü~ë=~==äçÅ~ä=ã~ñáãìã==~í==ñ=

N N EcáÖKNTTFK==

832. fÑ=ÑEñF=áë=ÇÉÅêÉ~ëáåÖ=E ( )
<MF=Ñçê=~ää=ñ=áå=ëçãÉ=áåíÉêî~ä=

( ]O =~åÇ=ÑEñF=áë=áåÅêÉ~ëáåÖ=E ( )
>MF=Ñçê=~ää=ñ=áå=ëçãÉ=

áåíÉêî~ä=

[ )I=íÜÉå=ÑEñF=Ü~ë=~=äçÅ~ä=ãáåáãìã=~í=ñ
K==

O O
EcáÖKNTTFK=
=
833. pÉÅçåÇ=aÉêáî~íáîÉ=qÉëí=Ñçê=içÅ~ä=bñíêÉã~K=

fÑ=Ñ()ñN =M=~åÇ=Ñ ( )ñN
<MI=íÜÉå=ÑEñF=Ü~ë=~=äçÅ~ä=ã~ñáãìã=

~í==
ñK=
N

fÑ=()O =M=~åÇ= ( )O
>MI=íÜÉå=ÑEñF=Ü~ë=~=äçÅ~ä=ãáåáãìã=

~í=Oñ K=EcáÖKNTTF=

=

834. **`çåÅ~îáíóK==**

ÑEñF=áë==ÅçåÅ~îÉ=ìéï~êÇ=~í== ()=
áë============Mñ ==áÑ==~åÇ==çåäó==áÑ==

áåÅêÉ~ëáåÖ=~í=Mñ =EcáÖKNTTI=ñP<ñFK===

ÑEñF=áë==ÅçåÅ~îÉ==Ççïåï~êÇ=~í==Mñ
==áÑ=~åÇ=çåäó=áÑ==()==áë===============

ÇÉÅêÉ~ëáåÖ=~í=Mñ K=EcáÖKNTTI=ñ<ñPFK===
=
835. pÉÅçåÇ=aÉêáî~íáîÉ=qÉëí=Ñçê=`çåÅ~îáíóK==
fÑ=

()

M

>

M

I=íÜÉå=ÑEñF=áë=ÅçåÅ~îÉ=ìéï~êÇ=~í=

ñ K==

M

fÑ=

()

M
<
M
I=íÜÉå=ÑEñF=áë=ÅçåÅ~îÉ=Ççïåï~êÇ=~í=
ñ K=
M

fÑ=
()=ÇçÉë=åçí=Éñáëí=çê=áë=òÉêçI=íÜÉå=íÜÉ=íÉëí=Ñ~áäëK=

=
836. **fåÑäÉÅíáçå=mçáåíë=**

fÑ== ( )==ÅÜ~åÖÉë=ëáÖå=~í= ñ=ñpI==íÜÉå=()p
==Éñáëíë==~åÇ==

íÜÉ= éçáåí=

( )()

ñ **IÑ ñP =áë=~å= áåÑäÉÅíáçå= éçáåí= çÑ= íÜÉ= Öê~éÜ= çÑ=P**

Ñ()K=fÑ=()p
=Éñáëíë=~í=íÜÉ=áåÑäÉÅíáçå=éçáåíI=íÜÉå=
()p=M=

EcáÖKNTTFK=

=

837. i∞eçéáí~ä∞ë=oìäÉ=

äáãÑ() =äáãÑ′( ) =áÑ= () ()Ñ ñ äáã Ö ñ □□□M K==

ñ
→
Å
Ö

()

→
Å
Ö
′

()

äáã
ñ→Å ñ →Å ∞

8.7 Differential

=
cìåÅíáçåëW=ÑI=ìI=î=
fåÇÉéÉåÇÉåí=î~êá~ÄäÉW=ñ=

aÉêáî~íáîÉ=çÑ=~=ÑìåÅíáçåW= ( )I= ( )=

oÉ~ä=Åçåëí~åíW=`=

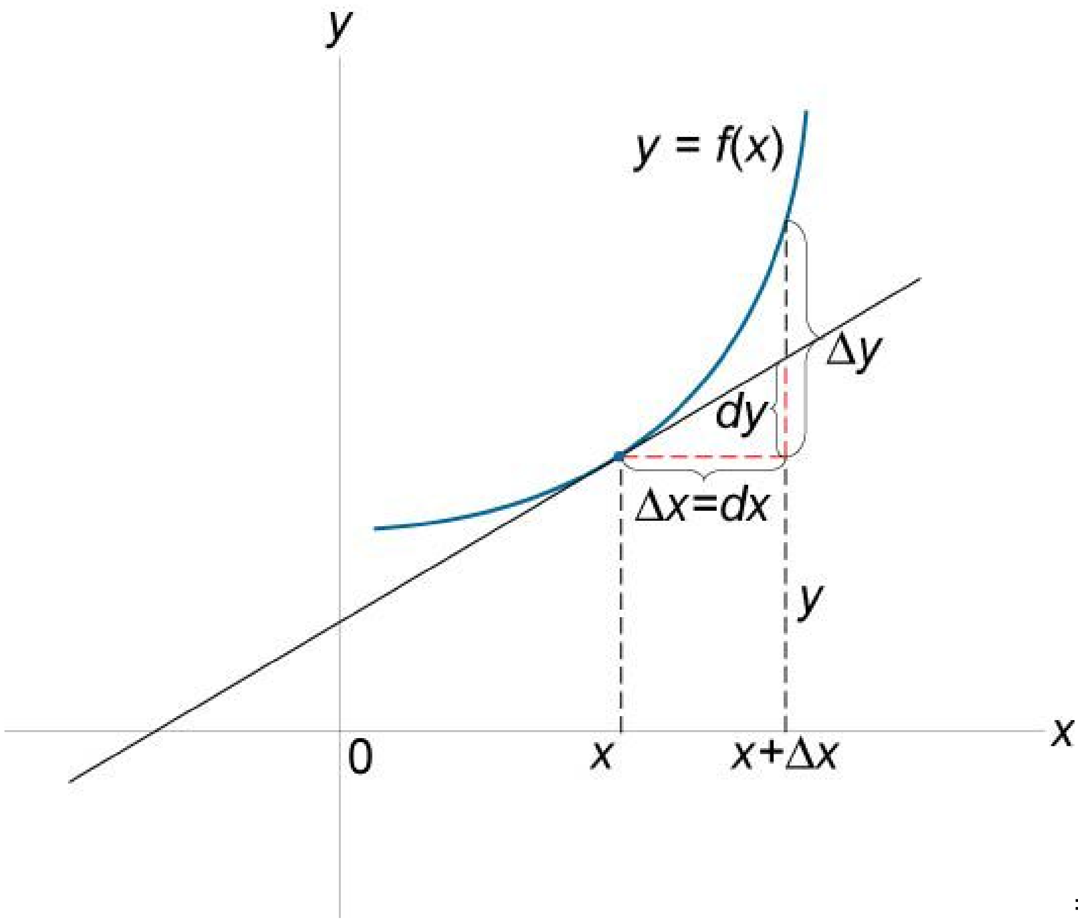
aáÑÑÉêÉåíá~ä=çÑ=ÑìåÅíáçå= ( )W=Çó=

aáÑÑÉêÉåíá~ä=çÑ=ñW=Çñ=
pã~ää=ÅÜ~åÖÉ=áå=ñW=Δñ=
pã~ää=ÅÜ~åÖÉ=áå=óW= Δó=
=
=

838.
Çó
′
=ó Çñ =
=

839. ()() ( ) ñ=

=



==

Figure 178. 840. pã~ää=`Ü~åÖÉ=áå=ó=

Δó=Ñ() ( )=

=

841. **()=Çì+Çî=**

=

842. **()=Çì−Çî=**

=

843. **() =`Çì=**

=

844. **()=îÇì+ìÇî=**

=

845. **Ç□ì□= îÇì−ìÇî=□□ î□□ îO**

=

=

=

8.8 Multivariable Functions

=

cìåÅíáçåë=çÑ=íïç=î~êá~ÄäÉëW=( )ñòI=( )ñÑI=( )ñÖI=()ñÜ==

^êÖìãÉåíëW=ñI=óI=í=
pã~ää=ÅÜ~åÖÉë=áå=ñI=óI=òI=êÉëéÉÅíáîÉäóW=ΔñI=ΔóI=Δò K= =
=

846. **cáêëí=lêÇÉê=m~êíá~ä=aÉêáî~íáîÉë=**
qÜÉ=é~êíá~ä=ÇÉêáî~íáîÉ=ïáíÜ=êÉëéÉÅí=íç=ñ=

∂ Ñ=**Ññ =E~äëç=**∂ò=**òñ FI=**∂ñ ∂ñ
qÜÉ=é~êíá~ä=ÇÉêáî~íáîÉ=ïáíÜ=êÉëéÉÅí=íç=ó=
∂Ñ=**Ñó =E~äëç=**∂ò=**òó FK=**∂ó ∂ó

=

847. **pÉÅçåÇ=lêÇÉê=m~êíá~ä=aÉêáî~íáîÉë=** ∂ □∂Ñ □=∂OÑ =**Ññññ I==**∂ñ
□□∂ñ□□ ∂ñO
∂ □ ∂Ñ □=∂OÑ =**Ñóó I==**∂ó□□∂ó□□ ∂óO□ □
∂ □ ∂Ñ □= ∂OÑ =ÑñóI==∂ó□□∂ñ□□ ∂ó∂ñ
∂ □∂Ñ □=∂OÑ =**Ñóñ K==**∂ñ □□∂ó□□ ∂ñ∂ó□ □
fÑ=íÜÉ=ÇÉêáî~íáîÉë=~êÉ=ÅçåíáåìçìëI=íÜÉå==
∂OÑ = ∂OÑ K==∂ó∂ñ ∂ñ∂ó
=
848. **`Ü~áå=oìäÉë==**

fÑ=() ( )
()=EÖ=áë=~=ÑìåÅíáçå=çÑ=çåÉ=î~êá~ÄäÉ=ÜFI=í
ÜÉå==

∂**Ñ =Ö′()()**∂**Ü I=**∂**Ñ ()()**∂**Ü K==**∂ ∂ ∂

ñ ñ ∂**ó ó**
=

fÑ=() () ( )()I=íÜÉå=()=∂Ñ Çñ+∂Ñ ÇóK==∂ñ Çí ∂ó Çí

=

fÑ=ò () ( )()Ñ ñ ìIî Ió ìIî I=íÜÉå==

∂ò = ∂Ñ ∂ñ + ∂Ñ ∂ó I=∂ò = ∂Ñ ∂ñ + ∂Ñ ∂óK==∂ì ∂ñ ∂ì ∂ó ∂ì ∂î ∂ñ ∂î ∂ó ∂î

=

849. pã~ää=`Ü~åÖÉë=

Δò ≈∂Ñ Δñ +∂Ñ Δó=∂ñ ∂ó

=

850. içÅ~ä=j~ñáã~=~åÇ=jáåáã~=

Ñ( ) óM =áÑ=( ) ( )M óM =

()=Ü~ë=~=äçÅ~ä=ã~ñáãìã=~í=ñM

Ñçê=~ää=

()

=ëìÑÑáÅáÉåíäó=ÅäçëÉ=íç=

( )

ñ ó$_{M}$ K==$_{M}$
=

Ññ()=Ü~ë=~=äçÅ~ä=ãáåáãìã=~í=( ) óM =áÑ=( ) ( )M óM =M

Ñçê=~ää=

()

=ëìÑÑáÅáÉåíäó=ÅäçëÉ=íç=

( )

ñ ó$_M$ K=$_M$

=

851. pí~íáçå~êó=mçáåíë=

∂ Ñ = ∂Ñ =MK=∂ñ ∂ó

içÅ~ä=ã~ñáã~=~åÇ=äçÅ~ä=ãáåáã~=çÅÅìê=~í=ëí~íáçå~êó=éçáåíëK= ==

852. p~ÇÇäÉ=mçáåí=

^=ëí~íáçå~êó==éçáåí==ïÜáÅÜ==áë==åÉáíÜÉê==~==äçÅ~ä==ã~ñáãìã=

åçê=~=äçÅ~ä=ãáåáãìã=

=

853. pÉÅçåÇ=aÉêáî~íáîÉ=qÉëí=Ñçê=pí~íáçå~êó=mçáåíë=

iÉí=

() óM =ÄÉ=~=ëí~íáçå~êó=éçáåí=E∂Ñ =∂Ñ

ñ =MFK==M ∂ñ ∂ó
Ñ
ññ
ñ
M
I

() ( )

a=óM ñó
óM óó

ñMIóM K==Ñóñ ñMI() ( )ñMIóM

=

fÑ=a>MI=() >MI==( ) óM

==áë=~=éçáåí=çÑ=äçÅ~ä=ãáåáã~K=ññ M M ñM

fÑ=a>MI=() <MI==( ) óM

==áë=~=éçáåí=çÑ=äçÅ~ä=ã~ñáã~K=ññ M M ñM

fÑ=
a
<
M
I=

()

ñ ó$_M$ =áë=~=ë~ÇÇäÉ=éçáåíK=$_M$
fÑ=a=MI=íÜÉ=íÉëí=Ñ~áäëK=
=

854. q~åÖÉåí=mä~åÉ=
qÜÉ=Éèì~íáçå=çÑ=íÜÉ=í~åÖÉåí=éä~åÉ=íç=íÜÉ=ëìêÑ~ÅÉ= ñ

()=

~í=()IóMIòM =áë==M

ò

–

ò

M

=

Ñ

()( ) ( )(

ñ ñMIóM ñ −ñM +Ñó ñMIóM ó−óM)K=

855. kçêã~ä=íç=pìêÑ~ÅÉ=

qÜÉ=Éèì~íáçå=çÑ=íÜÉ=åçêã~ä=íç=íÜÉ=ëìêÑ~ÅÉ=

ñ

( )=~í=

()IóMIòM =áë==M

ñ−ñM = ó−óM =ò−òM K=Ñ ()()

ñ M M ó M M−N

=

=

=

8.9 Differential Operators

=

råáí=îÉÅíçêë=~äçåÖ=íÜÉ=ÅççêÇáå~íÉ=~ñÉëW= á^{r}I= à^{r}I= â =

pÅ~ä~ê=ÑìåÅíáçåë=EëÅ~ä~ê=ÑáÉäÇëFW=( )I= (ñNI ñOIKIñå)=

dê~ÇáÉåí=çÑ=~=ëÅ~ä~ê=ÑáÉäÇW=Öê~Ç ìI=∇ì=

aáêÉÅíáçå~ä=ÇÉêáî~íáîÉW=$^{\partial Ñ}$ =$_{\partial ä}$

sÉÅíçê=ÑìåÅíáçå=EîÉÅíçê=ÑáÉäÇFW= (mInIo)=

aáîÉêÖÉåÅÉ=çÑ=~=îÉÅíçê=ÑáÉäÇW=
Çáî
r
c **I=∇ c=**
`ìêä=çÑ=~=îÉÅíçê=ÑáÉäÇW=
Åìêä
r
c**I=∇×c =**

i~éä~Åá~å=çéÉê~íçêW= ∇O =
=
=

856. dê~ÇáÉåí=çÑ=~=pÅ~ä~ê=cìåÅíáçå=
Öê~ÇÑ =∇Ñ =□ ∂Ñ I∂Ñ I∂Ñ □I==□□∂ñ ∂ó ∂ò□□
□ □
Öê~Çì=∇ì=□□
□
□ ∂ì**I** ∂ì**IKI** ∂ì □**K=□□∂ñN ∂ñO ∂ñå □**
=
857. aáêÉÅíáçå~ä=aÉêáî~íáîÉ=
∂Ñ =∂Ñ **Åçë**α+∂Ñ **Åçë**β+∂Ñ **Åçë**γ**I==∂ä ∂ñ ∂ó ∂ò**
ïÜÉêÉ=íÜÉ=ÇáêÉÅíáçå=áë=ÇÉÑáåÉÇ=Äó=íÜÉ=îÉÅíçê=

är**()I=ÅçëO α+ÅçëOβ+ÅçëO γ =NK==**

=
858. aáîÉêÖÉåÅÉ=çÑ=~=sÉÅíçê=cáÉäÇ=
r
Çáî
c
=

∇·c
r= ∂m+ ∂n+∂o=∂ñ ∂ó ∂ò
=
859. `ìêä=çÑ=~=sÉÅíçê=cáÉäÇ=rr r
á à â
r =∇×r = ∂ ∂ ∂ =Åìêäc c ∂ñ ∂ñ ∂ñ
m n o
=
∂o− ∂n r+ ∂m−∂o r+ ∂n−∂m r ∂ó ∂ò á ∂ò ∂ñ à ∂ñ ∂ó â =
=
860. i~éä~Åá~å=léÉê~íçê=
∇OÑ = ∂OÑ + ∂OÑ + ∂OÑ =∂ñO ∂óO ∂òO
=

861. () ( )≡M=

=

862. () ( )≡M=

=

863. () ( )=∇OÑ=

=

864.

()

=

Öê~Ç

( )

–
∇
o

$\mathbf{r}\ (\ ) - \nabla^{\mathbf{Or}}$

$\mathbf{c} = \nabla\ \mathbf{c} = =$

Chapter 9 Integral Calculus

=
=
=
=
cìåÅíáçåëW=ÑI=ÖI=ìI=î=
fåÇÉéÉåÇÉåí=î~êá~ÄäÉëW=ñI=íI=ξ **=**

fåÇÉÑáåáíÉ=áåíÉÖê~ä=çÑ=~=ÑìåÅíáçåW=

∫ ( )

Çñ
I=

∫ ( )

ö ñ Çñ I=£= aÉêáî~íáîÉ=çÑ=~=ÑìåÅíáçåW=

( )ñ I= ( )I=c ( )

ó ñ I=£= oÉ~ä=Åçåëí~åíëW=`I=~I=ÄI=ÅI=ÇI=â=
k~îìê~ä=åìãÄÉêëW=ãI=åI=áI=à=
=
=

9.1 Indefinite Integral

=

865. () ()∫Ñ ñ Çñ =c ñ +`=áÑ=c ( ) ( )

ñ Ñ ñ K=

= ()=′

866. ()∫Ñ ñ ()Ñ ñ =

=

867. ∫ ()ñ Çñ =â∫Ñ( )

âÑ ñ Çñ=

=

868. ∫ () ()Ö ñ []Çñ=∫Ñ( )ñÑ=+∫Ö( )

ñ ñ Çñ

=

869. ∫ () ()Ö ñ []Çñ=∫Ñ( )ñÑ=−∫Ö( )

ñ ñ Çñ

=

870. () ()∫Ñ ~ñ Çñ =Nc ~ñ +`=~

871. ∫() ()N +`=

~

=

872. ∫ ()() ()N O +`=

O

=

873. ()() ()+`=Ñ

=

874. jÉíÜçÇ=çÑ=pìÄëíáîìíáçå=

∫Ñ()Çñ=∫Ñ( )() ( )Çí=áÑ= ( )K=

=

875. fåíÉÖê~íáçå=Äó=m~êíë=

∫

ìÇî =ìî −∫îÇìI==

()

ïÜÉêÉ=

ñìI=

î

( )ñ =~êÉ=ÇáÑÑÉêÉåíá~ÄäÉ=ÑìåÅíáçåëK==

=

=

=

9.2 Integrals of Rational Functions

=
876. ∫~Çñ +`=
=

877.
∫
ñÇñ =
ñO O +`= =
878.
∫
ñ
OÇñ =ñP P +`= =
879.
∫
ñ
éÇñ =ñé+N é +N+`I=é −NK= 880.

∫
()

åÇñ = (~ñ+Ä)å+N

~() +`I=å −NK=

=
881. ∫Çñ =äåñ +`=ñ
=
882. ∫Çñ =Näå~ñ +Ä +`=~ñ+Ä ~
=
883. ∫~ñ+ÄÇñ =~ ñ +ÄÅ −~ÇäåÅñ+Ç +`=Åñ+Ç Å ÅO
=
884.

∫
Çñ= N

~−Ä äåñ+Ä +`I=~≠ÄK= ()( ) ñ+~

=
885.
∫
ñÇñ

~+Äñ =N()+`=ÄO

=

886. ∫ñOÇñ =N N () ( )+~Oäå~+Äñ+`=~+Äñ ÄP O

=

887. ∫Çñ =Näå~ +Äñ +`=() ~ ñ

=

888. ∫Çñ =−N +Ä äå~+Äñ +`=() ~ñ ~O ñ

=

889.

∫

ñÇñ =N äå~ +Äñ +~

~+Äñ+`=()O ÄO

890.

∫

ñOÇñ =N ~+Äñ −O~äå~+Äñ −~O ~ + Äñ

+`=()O ÄP

=

891. ∫Çñ = N + N äå~+Äñ +`=()() ~O ñ

=
892.
∫
Çñ
ñO −N= Näå ñ −N +`=O ñ +N
=
893.
∫
Çñ
N− ñO = NäåN+ñ +`=O N−ñ
=
894.
∫
Çñ
~O −ñO = N äå ~ +ñ +`=O~ ~ −ñ
=
895.
∫
Çñ
ñO −~O = N äå ñ −~ +`=O~ ñ +~
=
896.
∫
Çñ
N+ ñO =í~å−N ñ +`=
=
897.
∫
Çñ
~O +ñO =N í~å−N ñ +`=~ ~
=
898.
∫
ñÇñ
ñO +~O =Näå(ñO +~O)+`=O
=

899.

∫

Çñ

~

+

Äñ

O

=N ~êÅí~å□ñÄ □+`I=~Ä>MK=~Ä □□~ □□

□ □

900.

∫

ñÇñ

~+ÄñO =N äåñO +~ +`=OÄ Ä =

901.

∫

Çñ = N äå~+Äñ O +`= ()ñO

O~

=

902.

∫

Çñ N

~O −ÄOñO = O~Ääå ~ +Äñ +`=~ −Äñ =

903.

∫

~ñO +

Çñ Nää O~ñ+Ä− ÄO −Q~Å+`I= ÄÄñ +Å= O −Q~Å O~ñ+Ä+ ÄO −Q~Å ÄO

−Q~Å >MK=

=

904.

∫

Çñ

~ñ

O

+

Äñ

+

Å

=O ~êÅí~åO~ñ +Ä +`I= Q~Å−ÄO Q~Å −ÄO

ÄO −Q~Å<MK=

=

=

=

9.3 Integrals of Irrational Functions

=
905. ∫Çñ =O ~ñ +Ä +`=~ñ+Ä ~
=

906. ∫ ()O +P+`=

P~
=

907. ()~ñ+Ä+`=∫~ñ+Ä = P~O
908. ∫ñ ~ñ+Ä Çñ =O(P~ñ−OÄ)()PO +`=NR~O

=
909.
∫
Çñ= N

Ä −~Åäå ~ñ+Ä − Ä−~Å +`I==() ~ñ +Ä ~ñ+Ä + Ä−~Å

Ä >MK=
=

910. ∫Çñ =N ~êÅí~å~ñ +Ä +`I==() ~ñ +Ä ~Å −Ä ~Å −Ä

Ä <MK=
=

911. ∫~ñ+Ä ÇñN ()( )−=Åñ+Ç Å
−~Ç−ÄÅäå ~() () +`I=~>MK=Å ~Å

=

912. ∫~ñ+Ä ÇñN −=Åñ+Ç ()( )

Å

−~Ç−ÄÅ ~êÅí~å~(Åñ +Ç)+`I=E~<MI=Å>MFK==Å ~Å Å()
= (U~O −NO~Äñ +NRÄOñO)()P +`=∫ñO ~+Äñ Çñ =O**913.**
NMRÄP

=

914. ()~+Äñ +`=∫~+Äñ = NRÄP

=

915.

∫

Çñ

ñ ~+Äñ= N äå ~+Äñ − ~ +`I=~>MK=~ ~+Äñ + ~

916.

∫

Çñ

ñ ~+Äñ =O ~êÅí~å~ +Äñ +`I=~<MK=_~ _~

=

917. ∫~−ñ ()( )()~êÅëáåñ+Ä +`=Ä+ñ ~+Ä

=

918. ∫~+ñ Çñ=− ()( )()~êÅëáåÄ−ñ +`=Ä−ñ ~+Ä

=

919. ∫N+ñ Çñ =− N−ñO +~êÅëáåñ+`=N−ñ

=

920. ∫Çñ =O~êÅëáå ñ−~ +`=()( ) Ä−~

=

921. ∫ÅñO OÅñ −Ä OÄñ~=

QÅ

Ä

O

+−Q~Å~êÅëáåOÅñ −Ä +`=

U ÅP ÄO +Q~Å

=

922. ∫Çñ =N
~ñO +Äñ +Å ~ äåO~ñ +Ä+O ~(~ñO +Äñ +Å)+`I==
~>MK=
=
923. ∫Çñ =−N
~ñO +Äñ +Å ~ ~êÅëáåO~ñ+Ä ÄO −Q~Å +`I=~<MK=Q~
=
924.
∫
ñ
O +~OÇñ =ñ ñO +~O +~O O O äåñ+ ñO +~O +`= 925. ∫ñ ñO +~OÇñ =N(ñO
+~O)PO +`=P
=
926.
∫
O O +~OÇñ =ñ(OñO +~O)ñO ~O
U ññ=
−
~Q
U äåñ+ ñO +~O +`= =
927.
∫
ñ
O +~O ñO +~O
Çñ =−ñ +äåñ + ñO +~O +`=ñO
=
928. ∫Çñ
ñO +~O =äåñ+ ñO +~O +`=
=
929. ∫ñO +~O ñ +`=ñ Çñ = ñO +~O +~äå+ ñO +~O~
=
930. ∫ñÇñ = ñO +~O +`=
ñO +~O
= 931. ∫ñOÇñ =ñ ñO +~O −~O

ñO +~O O O äåñ+ ñO +~O +`=

= 932. ∫Çñ ñ

ñ ñO +~O =Näå+ ñO +~O +`=~ ~

=

933.

∫

ñ

O −~OÇñ =ñ ñO −~O −~O

O O äåñ + ñO −~O +`= =

934. ∫ñ ñO −~OÇñ =N(ñO −~O)PO +`=P

935. ∫ñO −~O ~ +`=ñ Çñ = ñO −~O +~ ~êÅëáåñ =

936.

∫

ñ

O −~O ñO −~O

Çñ =−ñ +äåñ+ ñO −~O +`=ñO

=

937. ∫Çñ

ñO −~O =äåñ+ ñO −~O +`=

=

938. ∫ñÇñ = ñO −~O +`= ñO −~O

=

939. ∫ñOÇñ =ñ ñO −~O +~O

ñO −~O O O äåñ+ ñO −~O +`=

=

940. ∫Çñ

ñ ñO −~O =−N~êÅëáå~ +`=~ ñ =

941. ∫Çñ =N ñ−~ +`=() ñO −~O ~ ñ+~

=

942. ∫Çñ =−N ñ+~ +`=() ñO−~O ~ ñ−~

=

943. ∫ñO ñO −~O =ñO −~OÇñ +`=~Oñ

=

944.

∫

Çñ=− ñ

O ñO–~O +`=()PO ~

945.

∫ ()

PO Çñ =−ñ(Oñ^{O} −R~^{O})ñ^{O} ~^{O}
U ~ñ=
+
P~^{Q}
U äåñ + ñ^{O} −~^{O} +` =

=
946. ∫~^{O} −ñ^{O}Çñ =ñ ~^{O} −ñ^{O} +~O ñ +`=O O ~êÅëáå~
=
947. ∫ñ ~^{O} −ñ^{O}Çñ =−N(~^{O} −ñ^{O})^{PO} +`=P
=

948. ∫ ()ñ Oñ O ~O −ñO +~Q ñ +`=U U ~êÅëáå~

=

949. ∫~O −ñO ñ +`=ñ Çñ = ~O −ñO +~ äå+ ~O −ñO~

=

950. ∫~O −ñO ~O −ñO ñ +`=ñO Çñ =−ñ −~êÅëáå~ =

951. ∫Çñ =~êÅëáåñ +`=
N−ñO
=

952. ∫Çñ
~O −ñO =ëáåñ +`=~
=

953. ∫ñÇñ =− ~O −ñO +`= ~O −ñO
=

954. ∫ñOÇñ =−ñ ~O −ñO +~O ñ +`= ~O −ñO O O ~êÅëáå~

955. ∫Çñ =−N ~−ñ +`=()ñ ~ ~O −ñO O ~ +ñ

=

956. ∫Çñ =−N ~ +ñ +`=()ñ ~ ~O −ñO O ~ −ñ

=

957. ∫Çñ =O −~O ~êÅëáåÄñ +~ON +`I=Ä>~ K=()ñ+Ä ~O −ñO
Ä ~()

=

958.
N

ñ
+ Ä + ` I =

() ~O−ñO ~ ~

∫Çñ = O −ÄO äå O −ÄO ~O −ñO +~O +Äñ Ä<~ K=
=
959. ∫ñO ~O −ñO =−~O −ñOÇñ +`=~Oñ
=

960. ∫() ()ñ O − ~O −ñO +P~Q ñ +`=

U U ~êÅëáå~ =

961.

∫

Çñ= ñ

O ~O–ñO +`=()PO ~

=
=
=

9.4 Integrals of Trigonometric Functions

=

962. ∫ëáåñÇñ =−Åçëñ+`=

=

963. ∫ÅçëñÇñ=ëáåñ +`=

964. ∫ëáåO ñÇñ =ñ −NëáåOñ +`=O Q

=

965. ∫ÅçëO ñÇñ =ñ +NëáåOñ+`=O Q

=

966. ∫ëáåP ñ Çñ =NÅçëP ñ−Åçëñ+` =N ÅçëPñ−PÅçëñ+`=P NO Q

=

967. ∫ÅçëP ñÇñ =ëáåñ−NëáåP ñ +`=N ëáåPñ+Pëáåñ+`=P NO Q

=

968. ∫Çñ =∫ÅëÅñÇñ =äåí~åñ +`=ëáåñ O

=

969. ∫Çñ =∫ëÉÅñ Çñ =äåí~å π +`=Åçëñ O + Q

=

970. ∫Çñ =∫ÅëÅO ñÇñ =−Åçíñ+`=ëáåO ñ

=

971. ∫Çñ =∫ëÉÅO ñ Çñ =í~åñ+`=ÅçëO ñ

=

972.

∫
Çñ =∫ÅëÅP ñ Çñ =−Åçëñ
OëáåO ñ+ Näåí~å ñ +`=ëáåP ñ O O =

973.

∫
Çñ =∫ëÉÅP ñÇñ =ëáåñ
OÅçëO ñ+ Näåí~å π +`=ÅçëP ñ O O+Q

=

974. ∫ëáåñÅçëñ Çñ =−NÅçëOñ+`=Q

975. ∫ëáåO ñÅçëñ Çñ =NëáåP ñ +`=P

=

976. ∫ëáåñÅçëO ñ Çñ =−NÅçëP ñ +`=P

=

977. ∫ëáåO ñÅçëO ñ Çñ =ñ −N ëáåQñ +`=U PO

=

978. ∫í~åñÇñ =−äåÅçëñ +`=

=

979. ∫ëáåñ Çñ =N +`=ëÉÅñ+`=ÅçëO ñ Åçëñ

=

980. ∫ëáåO ñÇñ =äåí~å π −ëáåñ+`=Åçëñ O + Q

=

981. ∫í~åO ñ Çñ =í~åñ−ñ+`=

=

982. ∫ÅçíñÇñ =äåëáå ñ +`=

=

983. ∫Åçëñ Çñ =−N +` =−ÅëÅñ +`=ëáåO ñ ëáåñ

=

984. ∫ÅçëO ñ Çñ =äåí~åñ +Åçëñ +`=ëáåñ O

=

985. ∫ÅçíO ñ Çñ =−Åçíñ−ñ+`=

=

986.

∫

Çñ

Åçëñëáåñ =äåí~åñ +`=

987.

∫

Çñ
ëáå^O ñÅçëñ=− N +äåí~å π +`=ëáåñ O+ Q
=
988.
∫
Çñ
ëáåñÅçë^O ñ= N +äåí~åñ +`=Åçëñ O
=
989.
∫
Çñ
ëáåO ñÅçëO ñ =í~åñ−Åçíñ+`=

= ( )ñ +ëáå( )ñ +`I=∫ëáåãñëáåñ Çñ =−ëáå990. O() O()

ã^O≠å^OK=

= ( )ñ −Åçë( )ñ +`I=∫ëáåãñÅçëåñ Çñ =−Åçë991. O() O()

ã^O≠å^OK=

= ( )ñ +ëáå( )ñ +`I=∫ÅçëãñÅçëåñ Çñ =ëáå992. O() O()

ã^O≠å^OK=
=

993. ∫ëÉÅñí~åñÇñ =ëÉÅñ +`=
=
994. ∫ÅëÅ ñÅçíñÇñ =−ÅëÅñ +`=
=

995. ∫ëáåñÅçë^å ñÇñ =−Åçë^(å+N) ñ +`=_(å+N) =
996. ∫ëáå^å ñÅçëñÇñ =ëáå^(å+N) ñ +`=_(å+N)

997. ∫~êÅëáåñÇñ =ñ~êÅëáåñ+ N−ñ^{O}+`= =
998. ∫~êÅÅçëñÇñ = ñ~êÅÅçëñ− N−ñ^{O}+`= =

999. ∫~êÅí~åñÇñ =ñ ~êÅí~åñ−^{N}äå( ) `=$_{O}$

=

1000. **∫~êÅÅçíñÇñ =ñ ~êÅÅçíñ +Näå( ) `=O**

=

=

=

9.5 Integrals of Hyperbolic Functions

=

1001. ∫ëáåÜñÇñ=ÅçëÜñ +`=

=

1002. ∫ÅçëÜñÇñ =ëáåÜñ +`=

=

1003. ∫í~åÜñÇñ =äåÅçëÜñ+`=

=

1004. ∫ÅçíÜñÇñ =äåëáåÜñ +`=

=

1005. ∫ëÉÅÜ^{O}ñÇñ =í~åÜñ +`=

=

1006. ∫ÅëÅÜ^{O}ñÇñ =−ÅçíÜñ +`=

=

1007. ∫ëÉÅÜñí~åÜ ñÇñ=−ëÉÅÜñ+`=

1008. ∫ÅëÅÜñÅçíÜñÇñ =−ÅëÅÜñ+`=

=

=

=

9.6 Integrals of Exponential and Logarithmic Functions

=

1009. ∫Éñ Çñ =Éñ +`=

=

1010.

∫

~

ñÇñ =~ñ

äå~ +`=

=

1011.

∫

É

~ñ Çñ =É~ñ

~ +`=

=

1012.

()Çñ=É~ñ

∫ñÉ ~O − +`=

=

1013. ∫äåñ Çñ =ñäåñ−ñ+`=

=

1014.

∫

Çñ

ñäåñ =ääåñ +`=

=

1015.

∫

ñ

åäåñ Çñ = **+N ääñ −N**

ñ +`=å+N ()O

=

1016. **∫É~ñ ëáåÄñ Çñ** = **~ëáåÄñ −ÄÅçëÄñÉ~ñ +`=~O +ÄO**

1017. **∫É~ñ ÅçëÄñ Çñ** = **~ÅçëÄñ +ÄëáåÄñ É~ñ +`=~O +ÄO**

=

=

=

9.7 Reduction Formulas

=
1018. ∫ñåÉãñÇñ =N ñåÉãñ −å ∫ñå−NÉãñÇñ=ã ã
=
1019.

∫
É
ãñ Éãñ ã∫Éãñ

Çñ=− + å−N ÇñI=å≠NK= ()ñå−N −N ñ

=
1020. ∫ëáåÜå ñÇñ =N ëáåÜå−N ñÅçëÜñ −å−N∫ëáåÜå−O ñÇñ=å å
=

1021. ∫Çñ =− ÅçëÜñ −å−O∫Çñ I=å≠NK= ()ëáåÜå−Nñ å−N
ëáåÜå−O ñ

=
1022. ∫ÅçëÜå ñÇñ =N ëáåÜñÅçëÜå−N ñÅçëÜñ +å−N∫ÅçëÜå−O ñÇñ=å å
=

1023. ∫Çñ =− ëáåÜñ +å−O∫Çñ I=å≠NK= ()ÅçëÜå−N ñ å−N
ÅçëÜå−O ñ

=
1024. ∫ëáåÜå ñÅçëÜã ñÇñ =ëáåÜå+N ñÅçëÜã−N ñ =å+ã
+ã−N ∫ëáåÜå ñÅçëÜã−O ñÇñ =å+ã
=
1025. ∫ëáåÜå ñÅçëÜã ñÇñ =ëáåÜå−N ñÅçëÜã+N ñ =å+ã

−å−N ∫ëáåÜå−O ñÅçëÜã ñÇñ =å+ã
=
1026.
∫
í~åÜ
å ñÇñ =−N
å−Ní~åÜå−N +∫í~åÜå−O ñÇñI=å≠NK= =
1027.
∫
ÅçíÜ
å ñÇñ =−N
å−NÅçíÜå−N +∫ÅçíÜå−O ñÇñI=å≠NK=

=
1028. ∫ëÉÅÜåñÇñ =ëÉÅÜå−Oñí~åÜñ +å−O∫ëÉÅÜå−OñÇñI=å≠NK=å−N å−N
=
1029. ∫ëáåå ñÇñ=−Nëáåå−N ñÅçëñ +å−N∫ëáåå−O ñÇñ=å å
=

1030. ∫Çñ =− Åçëñ +å−O∫Çñ I=å≠NK=ëáå ()å N ëáåå−N ñ å−N
ëáåå−O

ñ ñ

=
1031. ∫Åçëå ñÇñ =N ëáåñÅçëå−N ñ +å−N∫Åçëå−O ñÇñ=å å
=

1032. ∫Çñ = ëáåñ +å−O∫Çñ I=å≠NK= ()ñ å N Åçëå−N ñ å−N
Åçëå−O

Åçë ñ
=
1033. ∫ëáåå ñÅçëã ñÇñ =ëáåå+N ñÅçëã−N ñ =å+ã

+ã−N ∫ëáåå ñÅçëã−O ñÇñ =å+ã
=
1034. ∫ëáåå ñÅçëã ñÇñ =−ëáåå−N ñÅçëã+N ñ =å+ã +å−N ∫ëáåå−O ñÅçëã ñÇñ
=å+ã
=
1035.
∫
í~å
å ñÇñ =N
å−Ní~åå−N −∫í~åå−O ñÇñI=å≠NK= =
1036.
∫
Åçí
å ñÇñ=−N
å−NÅçíå−N − ∫Åçíå−O ñÇñI=å≠NK=

=
1037. ∫ëÉÅåñÇñ =ëÉÅå−O ñí~åñ +å−O∫ëÉÅå−O ñÇñI=å≠NK=å−N å−N
=
1038. ∫ÅëÅåñÇñ =−ÅëÅå−O ñÅçíñ +å−O∫ÅëÅå−O ñÇñI=å≠NK=å−N å−N
=
1039. ∫ñå äåã ñÇñ =ñå+N äåã ñ −ã ∫ñå äåã−N ñÇñ=å+N å+N
=

1040. ∫äåã ñÇñ = − äåã ñ +ã ∫äåã−N ñ ÇñI=å≠NK=ñ ()å N ñå−N
−N ñå

=
1041. ∫äåå ñÇñ= ñäåå ñ−å∫äåå−N ñÇñ=
=
1042. ∫ñåëáåÜñÇñ =ñå ÅçëÜñ −å∫ñå−N ÅçëÜñÇñ=
=
1043. ∫ñå ÅçëÜñÇñ =ñå ëáåÜñ−å∫ñå−N ëáåÜñÇñ=
=

1044. ∫ñ^{å}ëáå ñÇñ=−ñ^{å} Åçëñ +å∫ñ$^{å−N}$ ÅçëñÇñ=

=

1045. ∫ñ^{å}ÅçëñÇñ =ñ^{å} ëáåñ −å∫ñ$^{å−N}$ ëáåñÇñ=

1046.

∫

ñ

å

ëáå

−

N

ñÇñ

=

ñ

å+N N∫ñå+N

ëáå$^{−N}$ñ −$_{å+N\ N−ñO}$ Çñ=$_{å+N}$

=

1047.

∫

ñ

å

Åçë

−

N

ñÇñ

=

ñ

å+N N∫ñå+N

Åçë$^{−N}$ ñ+$_{å+N\ N−ñO}$ Çñ=$_{å+N}$

=

1048.

∫

ñ

å

í~å

−

N

ñÇñ

=

ñ
å+N N∫ñå+N
í~å−N ñ−å+N N+ñO Çñ=å+N
=
1049.
∫
ñåÇñ= ñ− Ä∫Çñ
~ñå +Ä=~ñå + Ä ~ ~
=

1050. ∫Çñ = −O~ñ−Ä =()å()()( )å−N

− O()~ ∫Çñ I=å≠NK=()N ÄO −Q~Å ()()å−N

å
=

1051. ∫Çñ = ñ + Oå−P ∫Çñ I()å()() ()Oå−N ()å−N

å≠NK=
=

1052. ∫Çñ =− ñ =()å ()()å−N

–
Oå−P

Oå()N ~O ∫Çñ I=å≠NK=()å−N

= = = =

9.8 Definite Integral

=
Ä Ä

aÉÑáåáíÉ=áåíÉÖê~ä=çÑ=~=ÑìåÅíáçåW=∫()Çñ I=∫ ()Çñ I=£=

~ ~

å ()ñÑ==oáÉã~åå=ëìãW=∑á á

á=N
pã~ää=ÅÜ~åÖÉëW=Δñ$_{á}$ ==

^åíáÇÉêáî~íáîÉëW=( )I= ( )==

iáãáíë=çÑ=áåíÉÖê~íáçåëW=~I=ÄI=ÅI=Ç=

=

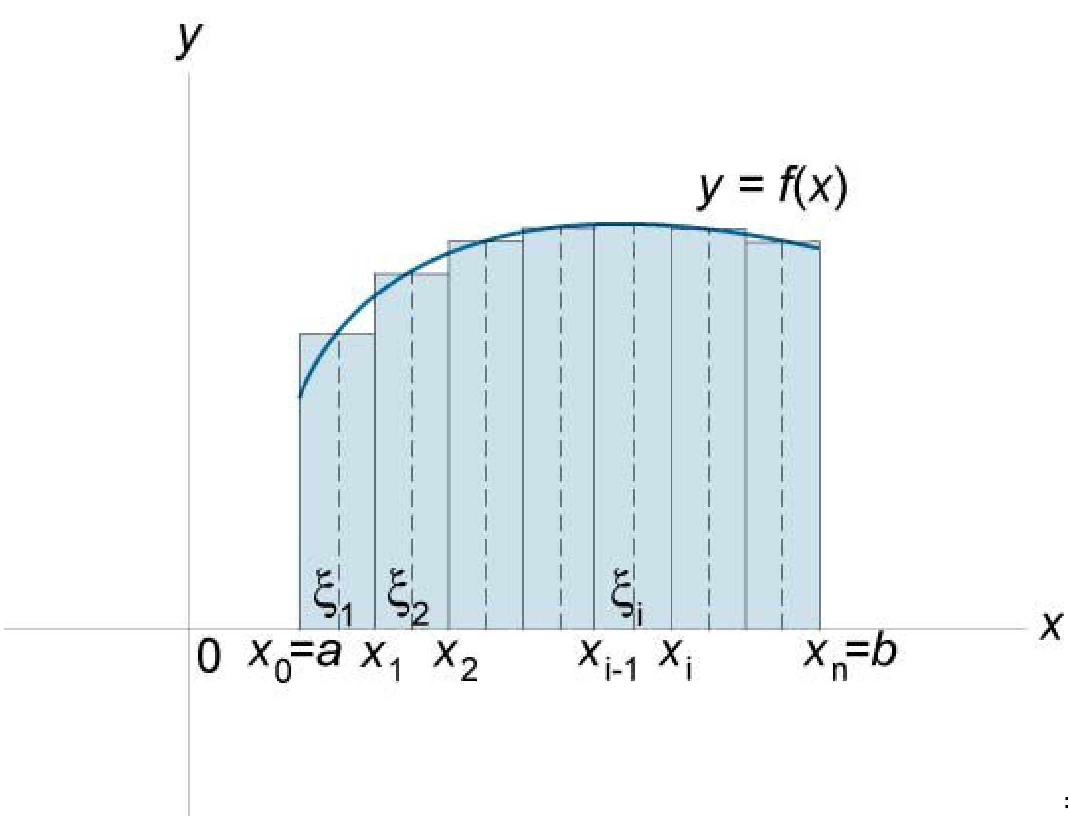
1053. =

Ä å ( )Δñá I==∫()Çñ = äáã ∑Ñá

~ ã~ñå Δ∞ →M á=Nñá

ïÜÉêÉ==Δñá =ñá −ñá−NI==ñá−N ≤ξá ≤ñáK==

=



==

Figure 179.

Ä

1054. **∫NÇñ =Ä−~ =**

~

=

Ä Ä

1055. ∫ **() ()**∫**Ñ ñ Çñ=**

~ ~
=

Ä[] Ä Ä

1056.

∫

() ()

() ()

Çñ = ∫**Ö ñ Çñ**∫**ñÑ=**

~ ~ ~ **=**

Ä[] Ä Ä

1057.

∫
() ()
() ()

Çñ =∫ñÑ=∫Çñ
~ ~ ~
=
~

1058. ∫()Çñ =M=

~
=
Ä ~

1059. ∫() ()∫ñÑ=

~ Ä
=
Ä Å Ä

1060. ∫() () ()∫Ñ ∫ñÑ=Ñçê=~ <Å ÄK=

~ ~ Å
=
1061.

Ä [ ]K=∫()Çñ≥M=áÑ=( ) ≥M=çå=

~
=
1062.

Ä [ ]K=∫()Çñ≤M=áÑ=( ) ≤M=çå=

~

=
=
=
1063. cìåÇ~ãÉåí~ä=qÜÉçêÉã=çÑ=`~äÅìäìë=

Ä
∫

Ñ() () () ()=áÑ= ( ) ( )K=

~
~
=
1064. jÉíÜçÇ=çÑ=pìÄëíáîìíáçå==

fÑ=ñ ()Ö í I=íÜÉå==

Ä Ç
∫
Çñ

=∫() ()()()

Ñ ñÑI==
~ Å
ïÜÉêÉ=
Å

=Ö−N()I=Ç=Ö−N( )

~ Ä K=
=

1065. fåíÉÖê~íáçå=Äó=m~êíë=
ÄÄ Ä

∫ìÇî =() ∫îÇì=

~ ~
=
1066. qê~éÉòçáÇ~ä=oìäÉ=
Ä
Ä
−
~
□

Ñ ñ()() ()□∫()Çñ= Oå □□

å−N ñÑ=
M å ∑á □□~ á=N
= Figure 180.
=
1067. páãéëçå∞ë=oìäÉ==

Ä Ä−~+ () ( ) ()+∫() ()å M QÑ ñNñÑ=O P

~ P
+
O
Ñ

()+K+QÑ( ) ( )

ñ Ñ ñå]I==Q ñå−N
ïÜÉêÉ==
ñá =~ +Ä−~ áI=á =MINIOIKIåK==å
= Figure 181. =
1068. ^êÉ~=råÇÉê=~=`ìêîÉ=

Ä
p

=∫Ñ ñ() () ()c Ä −

Çñ c ~ I==
~
ïÜÉêÉ=
c

() ( )

ñ Ñ ñ K=
= Figure 182.
=
1069. ^êÉ~=_ÉíïÉÉå=qïç=`ìêîÉë=
Ä

p=∫Ñ ñ() ()Ö ñ []Çñ () () () ()c Ä −d Ä −c ~ +d ~ I==

~
ïÜÉêÉ=
c

() ( )Ñ ñ I=d ( ) ( )

ñ ñ Ö ñ K=
=

Figure 183.
=
=
=

9.9 Improper Integral

=
Ä

1070. qÜÉ=ÇÉÑáåáíÉ=áåíÉÖê~ä== ∫()Çñ
=áë=Å~ääÉÇ=~å=áãéêçéÉê=áåíÉÖê~ä=

~
áÑ==
• ~=çê=Ä=áë=áåÑáåáíÉI=

•

()==Ü~ë==çåÉ==çê==ãçêÉ=éçáåíë=çÑ==ÇáëÅçåíá åìáíó= =====áå=íÜÉ=áåíÉêî~ä=[ ]I~K=

=

1071. fÑ=()=áë=~=Åçåíáåìçìë=ÑìåÅíáçå=çå=[ )I=íÜÉå==

∞ å

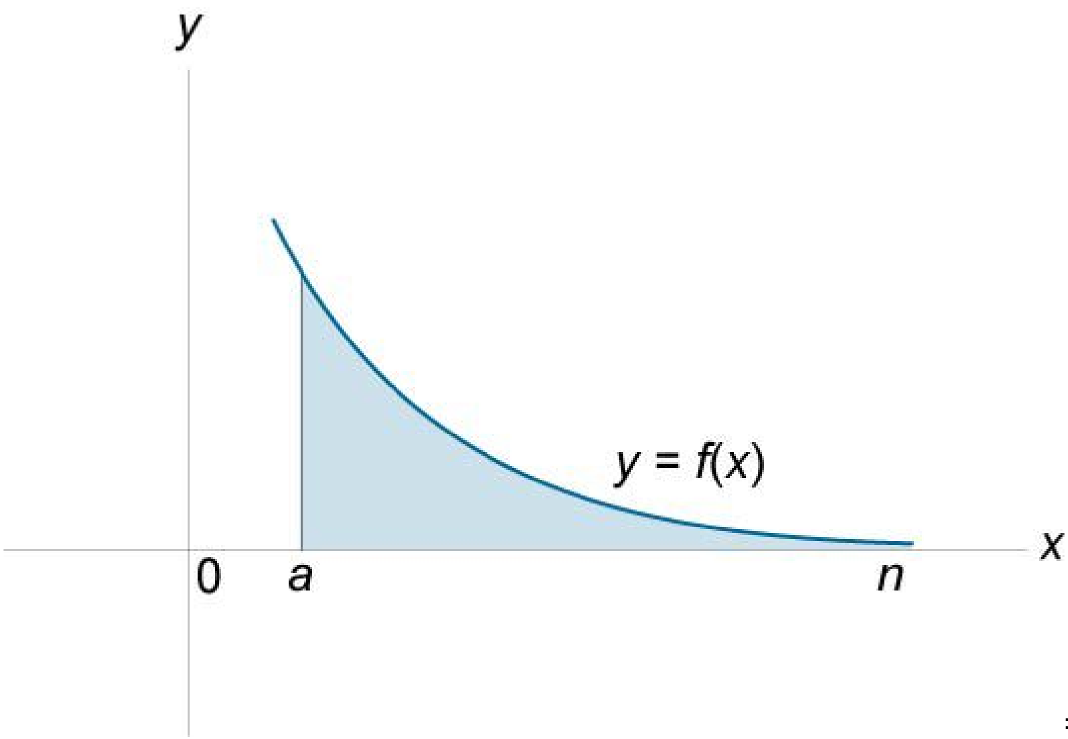
∫

()ñÑK=∫Ñ()Çñ

å→∞

~ ~

1072. fÑ=



=

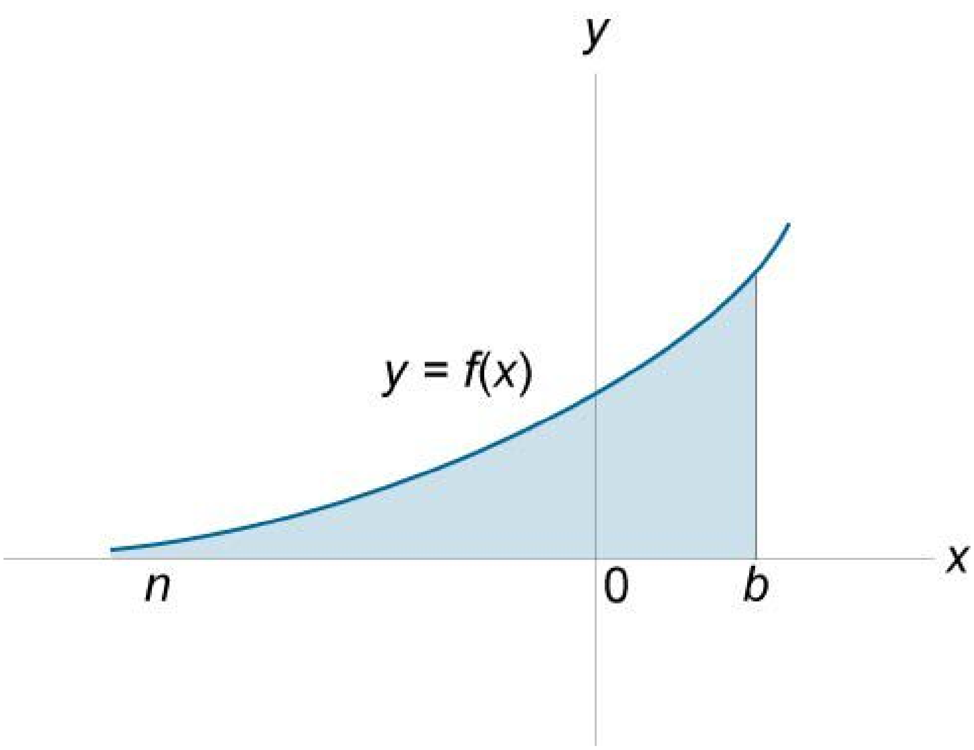
Figure 184.

= ()=áë=~=Åçåíáåìçìë=ÑìåÅíáçå=çå=( ]I=íÜÉå==

Ä Ä

∫()Çñ =ñÑK=∫Ñ()Çñ

–
∞
å→ −∞
å
=

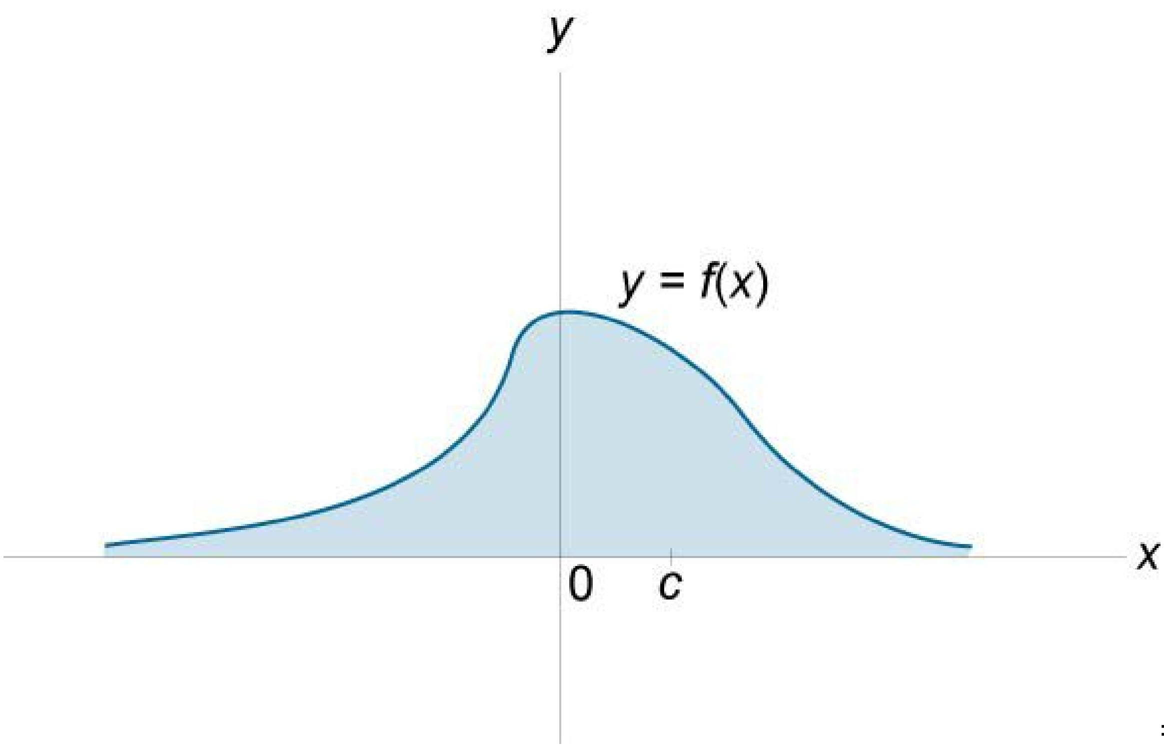


==
kçíÉ=W=qÜÉ=áãéêçéÉê=áåíÉÖê~äë=áå=NMTNI=NMTO=~êÉ=ÅçåîÉêÖÉåí=
áÑ=íÜÉ=äáãáíë=Éñáëí=~åÇ=~êÉ=ÑáåáíÉX=çíÜÉêïáëÉ=íÜÉ=áåíÉÖê~äë=~êÉ= ÇáîÉêÖÉåíK=
=

∞ Å ∞

1073. ∫ () () ()∫ñÑ=∫Ñ Çñ

−∞ −∞ Å
=



==

Figure 186.
=
fÑ=Ñçê=ëçãÉ=êÉ~ä=åìãÄÉê=ÅI=ÄçíÜ=çÑ=íÜÉ=áåíÉÖê~äë=áå=íÜÉ=ê áÖÜí=

∞

ëáÇÉ=~êÉ=ÅçåîÉêÖÉåíI=íÜÉå= íÜÉ= áåíÉÖê~ä= ∫ ()Çñ= áë= ~äëç=====

−∞
ÅçåîÉêÖÉåíX=çíÜÉêïáëÉ=áí=áë=ÇáîÉêÖÉåíK=
=
1074. **`çãé~êáëçå=qÜÉçêÉãë=**

iÉí==()=~åÇ==( )==ÄÉ==Åçåíáåìçìë==ÑìåÅíáçåë==çå=íÜÉ=ÅäçëÉ Ç=

áåíÉêî~ä=[ )K=pìééçëÉ=íÜ~í= ( ) ( )=Ñçê=~ää=ñ=áå=
[
)

I~K=
∞ ∞

• **fÑ=∫()Çñ=áë=ÅçåîÉêÖÉåíI=íÜÉå=∫ ()ñÖ=áë=~äëç=**

~ ~
=====ÅçåîÉêÖÉåíI=
∞ ∞

• **fÑ=∫ ()Çñ =áë=ÇáîÉêÖÉåíI=íÜÉå=∫()Çñ =áë=~äëç=ÇáîÉêÖÉåíK=**

~ ~

=
1075. **^ÄëçäìíÉ=`çåîÉêÖÉåÅÉ=**
=

∞ ∞

fÑ=∫ ()Çñ=áë=ÅçåîÉêÖÉåíI=íÜÉå=íÜÉ=áåíÉÖê~ä∫()Çñ =áë=~Äëç-

~ ~
äìíÉäó=ÅçåîÉêÖÉåíK===
=
1076. **aáëÅçåíáåìçìë=fåíÉÖê~åÇ=**

iÉí=
()=ÄÉ=~=ÑìåÅíáçå=ïÜáÅÜ=áë=Åçåíáåìçìë=çå=íÜ É=áåíÉêî~ä==

[
~
I

)=Äìí=áë=ÇáëÅçåíáåìçìë=~í=ñ=

Ä **ÄK=qÜÉå==**

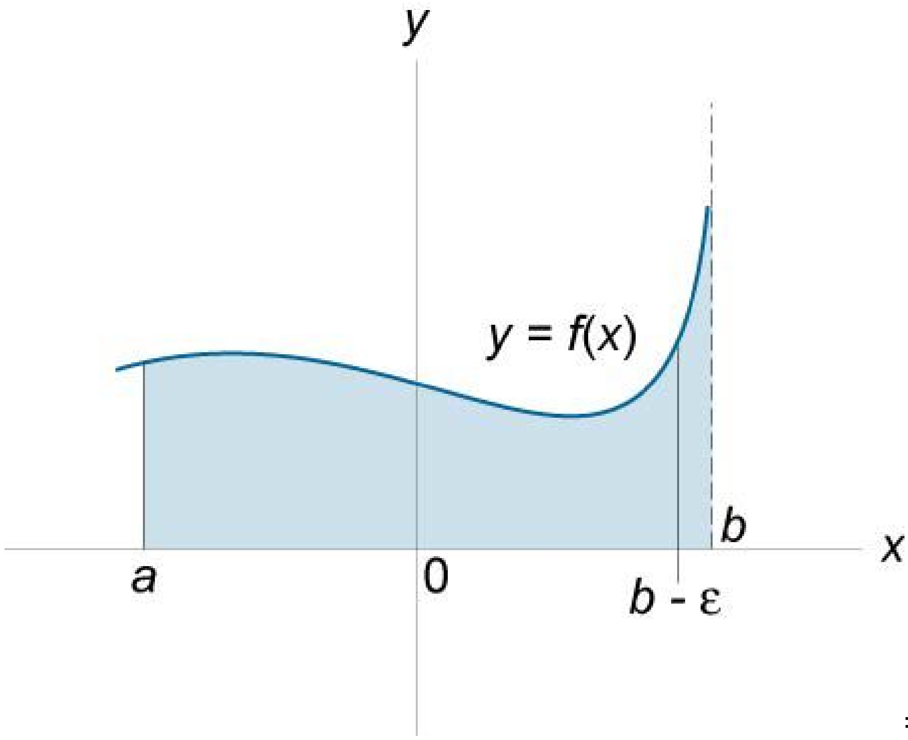
Ä ()Ä−ε

∫

ñÑ=∫Ñ()Çñ

ε→**M**+
~ ~

=



==

1077. **iÉí=**
()=ÄÉ=~=ÅçåíáåìçÌë=ÑìåÅíáçå=Ñçê=~ää=êÉ~ä=åìãÄÉêë==ñ==
áå=

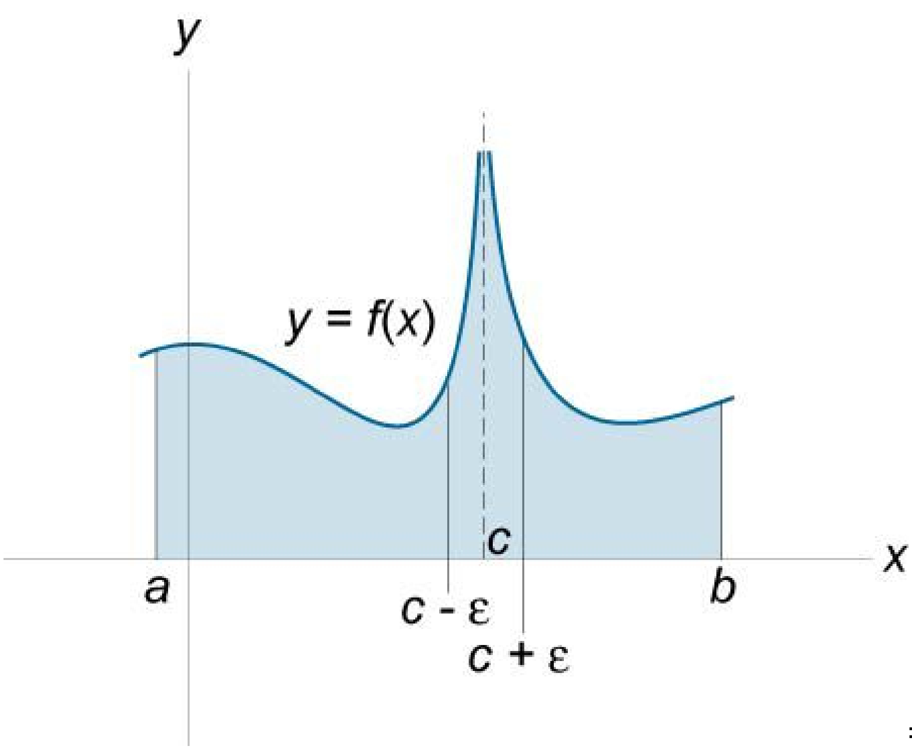
íÜÉ=áåíÉêî~ä==
[]==ÉñÅÉéí==Ñçê==ëçãÉ=éçáåí==Å==áå=(
)K=qÜÉå=

Ä Å−ε Ä

∫()Çñ = ääáã ∫Ñ()ñÑK=∫Ñ()Çñ

~ ε→**M**+~ δ→**M**+**Å**+δ

=



==

Figure 188.

=

=

=

9.10 Double Integral

=

cìåÅíáçåë=çÑ=íïç=î~êá~ÄäÉëW=( )I=( )I=£=

açìÄäÉ=áåíÉÖê~äëW= ∫∫ ( )ñÑI=∫∫ ( )ñÖI=£=

o o

ã å ()Δñáìх=àΣΣ^{Δ}

oáÉã~åå=ëìãW=
á==N
pã~ää=ÅÜ~åÖÉëW=Δñ$_{á}$ I=Δó$_{à}$=

oÉÖáçåë=çÑ=áåíÉÖê~íáçåW=oI=p==
mçä~ê=ÅççêÇáå~íÉëW=êI=θ=
^êÉ~W=^=
pìêÑ~ÅÉ=~êÉ~W=p=
sçäìãÉ=çÑ=~=ëçäáÇW=s=
j~ëë=çÑ=~=ä~ãáå~W=ã=

aÉåëáíóW=()=

cáêëí=ãçãÉåíëW=
jI=
ñ j=$_{ó}$
jçãÉåíë=çÑ=áåÉêíá~W=
ñ
fI=
ó
fI=
f =
M
`Ü~êÖÉ=çÑ=~=éä~íÉW=n=

`Ü~êÖÉ=ÇÉåëáíóW= ( )=

`ççêÇáå~íÉë=çÑ=ÅÉåíÉê=çÑ=ã~ëëW=ñ I= ó =
^îÉê~ÖÉ=çÑ=~=ÑìåÅíáçåW=μ=
=

1078. **aÉÑáåáíáçå=çÑ=açìÄäÉ=fåíÉÖê~ä=**

qÜÉ=ÇçìÄäÉ=áåíÉÖê~ä=çîÉê=~=êÉÅí~åÖäÉ=[ ] [ ]=áë=ÇÉÑáåÉÇ=

íç=ÄÉ==

∬

Ñ

()

ã å

Ç^ = äáãM∑∑Ñ ì() ΔñáΔóà I==

[][ ]

ã~ñ
Δ
ñ
á
→

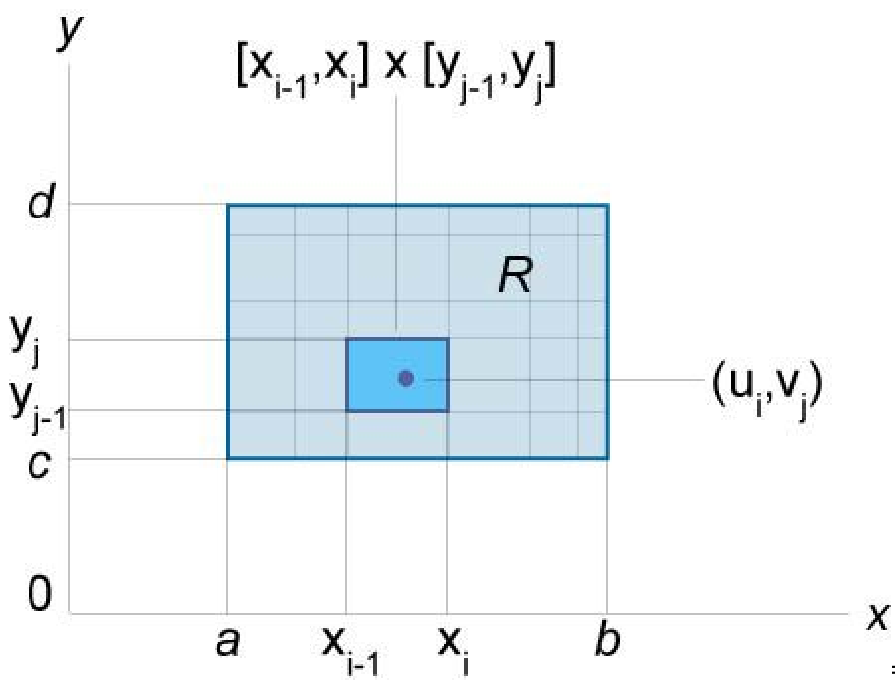
á à
á N==à N
ïÜÉêÉ=

( ) ã~ñΔóà→M

á à =áë=ëçãÉ=éçáåí=áå=íÜÉ=êÉÅí~åÖäÉ=

ñ()Iñá ×(óà−NIóà)I=~åÇ=Δñá =ñá −ñá−NI=Δóà =óà
−óà−NK=á−N

=
===



=Figure 189.

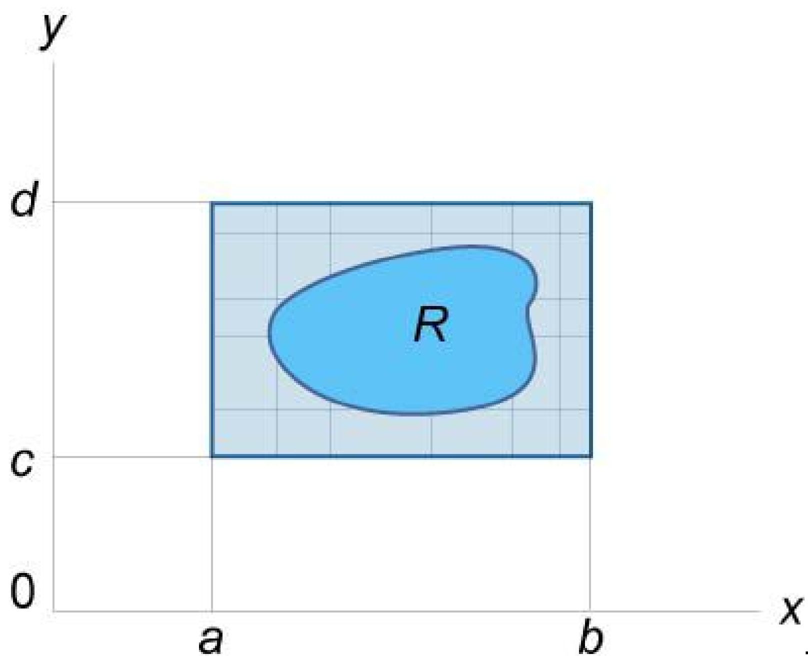
qÜÉ=ÇçìÄäÉ=áåíÉÖê~ä=çîÉê=~=ÖÉåÉê~ä=êÉÖáçå=o=áë==

∬ ()ñÑI==∬Ö( )Ç^

o [][ ]

ïÜÉêÉ=êÉÅí~åÖäÉ=[ ] [ ]=Åçåí~áåë=oI== Ö( )( )=áÑ=( )=áë=áå=o=~åÇ=( ) =M =çíÜÉêïáëÉK=

=



==

Figure 190.

=

1079.

$\iint$

$() ()$

[]

Ç^
=

∬

Ñ

( )

Ç^
+

∬

( )

ñÑ=
o o o
=
1080.

$\iint$

() ()

[]

Ç^
=

∫∫

Ñ

( )

Ç^

–

∫∫

( )

ñÑ=
o o o
=

1081. ∫∫ **()Ç^=â∫∫Ñ( )Ç^I==**

o o
ïÜÉêÉ=â=áë=~=Åçåëí~åíK=
=
1082.
fÑ=

() ()

=çå=oI=íÜÉå=

∬

( )

Ç^
≤

∬

( )

ñÑK=
o o =

1083. fÑ=() ≥M=çå=o=~åÇ=p⊂oI=íÜÉå=

=

∬

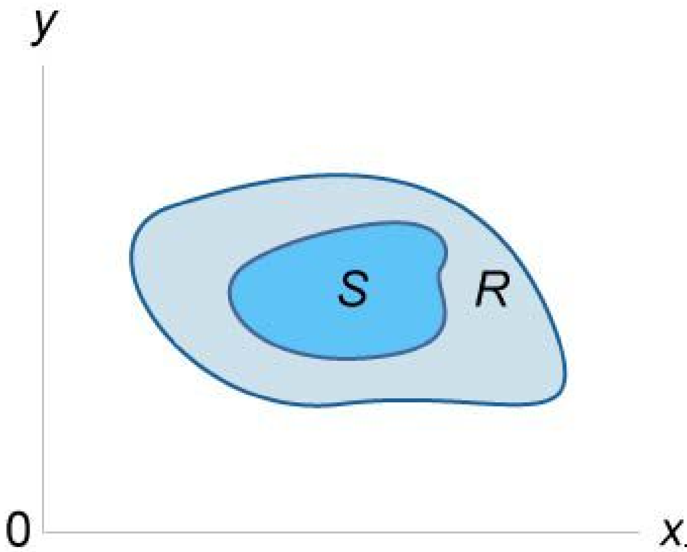
()

Ç^
≤

∬

( )

ñÑK=
p o
=
1084. fÑ=



==

Figure 191.

= () ≥M=çå=o=~åÇ=o=~åÇ=p=~êÉ=åçå-çîÉêä~ééáåÖ=

êÉÖáçåëI=íÜÉå=

∬

( )

Ç^
=

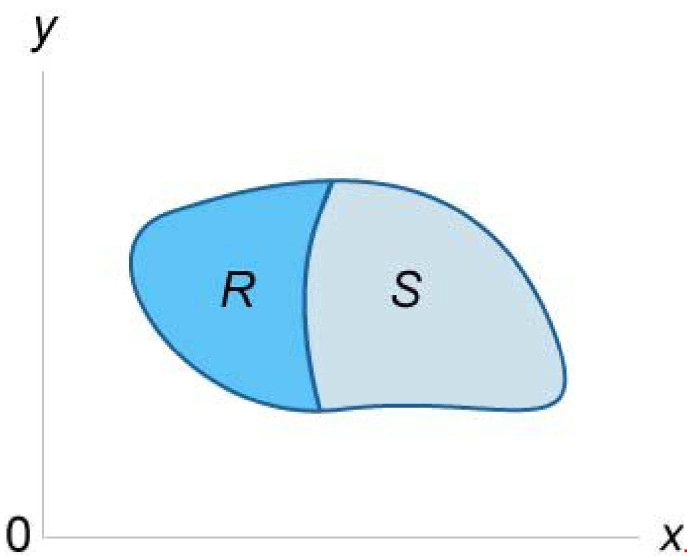
$\iint$

Ñ

( )
( )

Ç^ +∫∫ñÑK==

o∪p o p
eÉêÉ=o∪p=áë=íÜÉ=ìåáçå=çÑ=íÜÉ=êÉÖáçåë=o=~åÇ=pK= =



=
=

Figure 192.
=

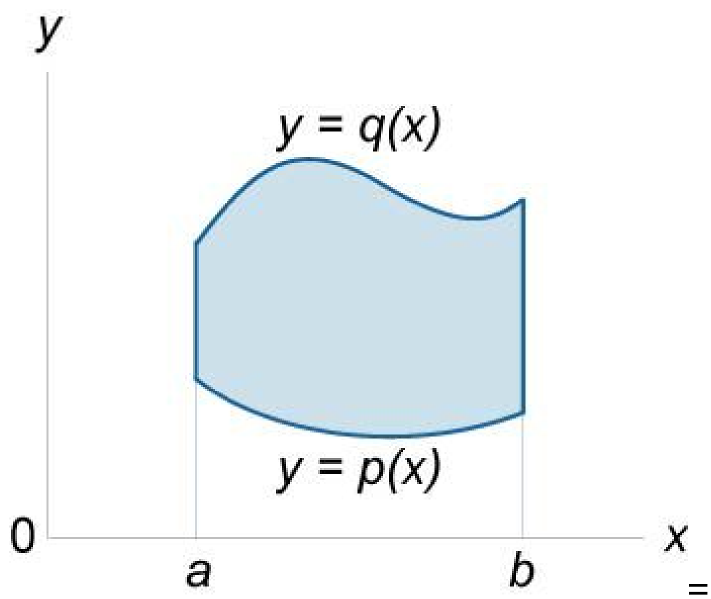
1085. fíÉê~íÉÇ=fåíÉÖê~äë=~åÇ=cìÄáåá∞ë=qÜÉçêÉã= Äè( )

∫∫ () ()∫∫ñÑ==

o
~
é
ñ
()
Ñçê=~=êÉÖáçå=çÑ=íóéÉ=fI==

o=( ){}K=

=



==

Figure 193.

=

ç î( )

∬ () ()∫∫ñÑ==

o
Å
ì
ó
()

Ñçê=~=êÉÖáçå=çÑ=íóéÉ=ffI=

o=( ){}K=

= **Figure 194.**

=

1086. açìÄäÉ=fåíÉÖê~äë=çîÉê=oÉÅí~åÖìä~ê=oÉÖáçåë=

=

fÑ=o=áë=íÜÉ=êÉÅí~åÖìä~ê=êÉÖáçå=[ ] [ ]I=íÜÉå==

ç Ä

() () ()

Ä□Ç □Çñ ∫∫ Çñ□Çó∫∫ñÑK==□□ □□∫∫

o ~□Å □ Å□~ □ =

få=íÜÉ=ëéÉÅá~ä=Å~ëÉ=ïÜÉêÉ=íÜÉ=áåíÉÖê~åÇ=(
)=Å~å=ÄÉ=ïêáí- íÉå=~ë=()()ñÖ=ïÉ=Ü~îÉ==

∬

( )

ÇñÇó

=

∬

Ö

()()

ÇñÇó
=
Ä Ç

∫
() ()

ñÑK==
o
o

~

∫

å

=

1087. `Ü~åÖÉ=çÑ=s~êá~ÄäÉë=

∬Ñ ñIó ÇñÇó ∬() ()()[]( )ÇìÇî I==

o p ∂()

∂ñ ∂ñ

ïÜÉêÉ=() =∂∂ ∂î ≠Máë= íÜÉ= à~ÅçÄá~å= çÑ= íÜÉ= íê~åë-∂() ó

∂ì ∂î

Ñçêã~íáçåë=() ( )I=~åÇ=p=áë=íÜÉ=éìääÄ~Åâ=çÑ=o=ïÜáÅÜ= Å~å=ÄÉ=ÅçãéìíÉÇ=Äó= ( )I= ( )=áåíç=íÜÉ=ÇÉÑáåá-

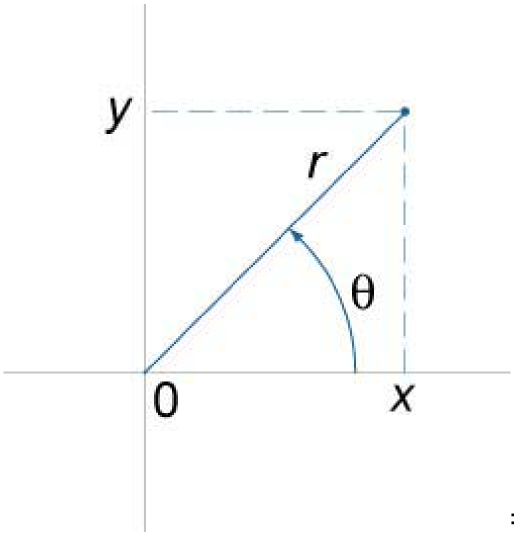
íáçå=çÑ=oK==

=

1088. mçä~ê=`ççêÇáå~íÉë=

ñ =êÅçëθI=êóK==

=



=
=

Figure 195.

=

1089. **açìÄäÉ=fåíÉÖê~äë=áå=mçä~ê=`ççêÇáå~íÉë= =**
qÜÉ=aáÑÑÉêÉåíá~ä=ÇñÇó=Ñçê=mçä~ê=`ççêÇáå~íÉë=áë==

ÇñÇó =∂()ÇêÇθ=êÇêÇθK==∂()

=

iÉí=íÜÉ=êÉÖáçå=o=áë=ÇÉíÉêãáåÉÇ=~ë=Ñçääçïë=W=

≤ ()ÖMI=( ) α≤θ ≤βI=ïÜÉêÉ=β−α≤OK==

qÜÉå==
β Ü( )

() ( )

∫∫

Ió ÇñÇó =∫∫ñÑK=

θ
o α **Ö**()
=

Figure 196.
=

fÑ=íÜÉ=êÉÖáçå=o=áë=íÜÉ=éçä~ê=êÉÅí~åÖäÉ=ÖáîÉå=Äó== M≤**~**
≤**ê**≤**ÄI=**α≤θ ≤β**I=ïÜÉêÉ=**β−α≤**O**π**I===**

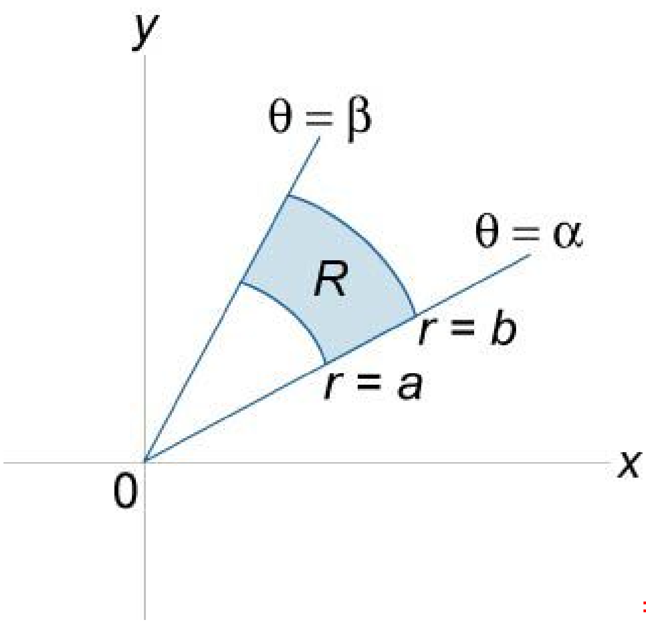
íÜÉå==
β **Ä**

∫∫

Ñ
ñ
I
ó

() ( )∫∫Ñ êÅçë Iêëáå êÇêÇθK==

ÇñÇó
o α ~
=



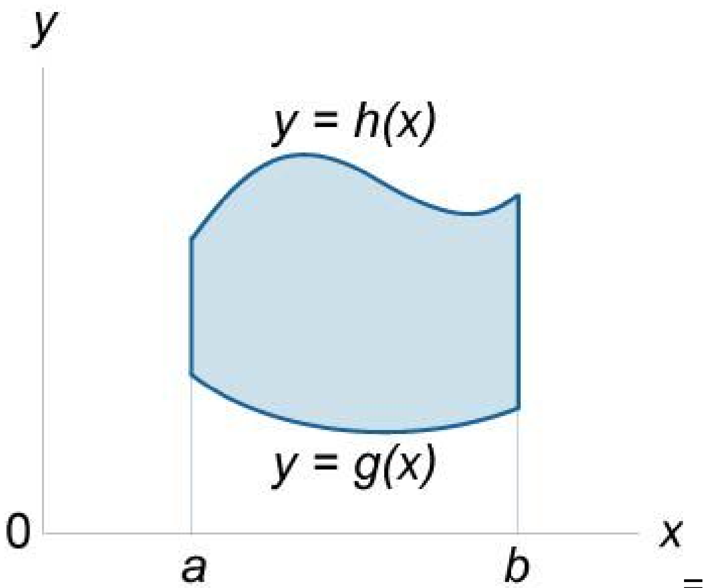
=
=

Figure 197.
=

1090. ^êÉ~=çÑ=~=oÉÖáçå=
Ä Ñ()

^=∫∫ÇóÇñ =EÑçê=~=íóéÉ=f=êÉÖáçåFK=

~
Ö
ñ
()
=



=

Figure 198.
=

∧ =∬

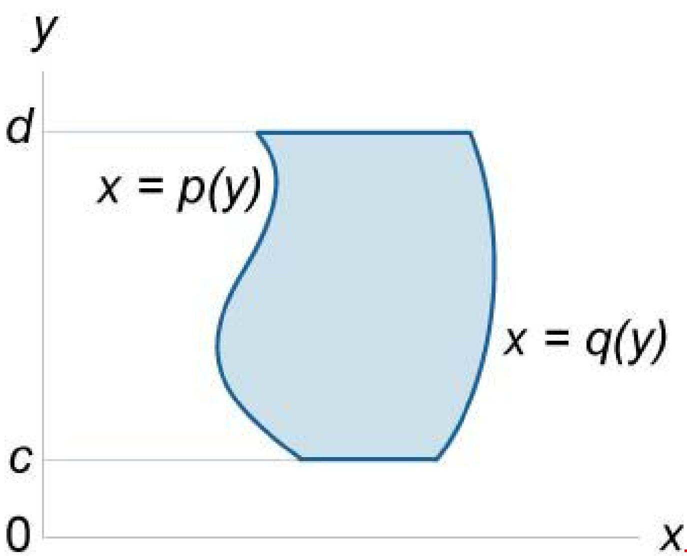
ç è()

ÇñÇó =EÑçê=~=íóéÉ=ff=êÉÖáçåFK=

ó

Å é()

=



= = **Figure 199.** =

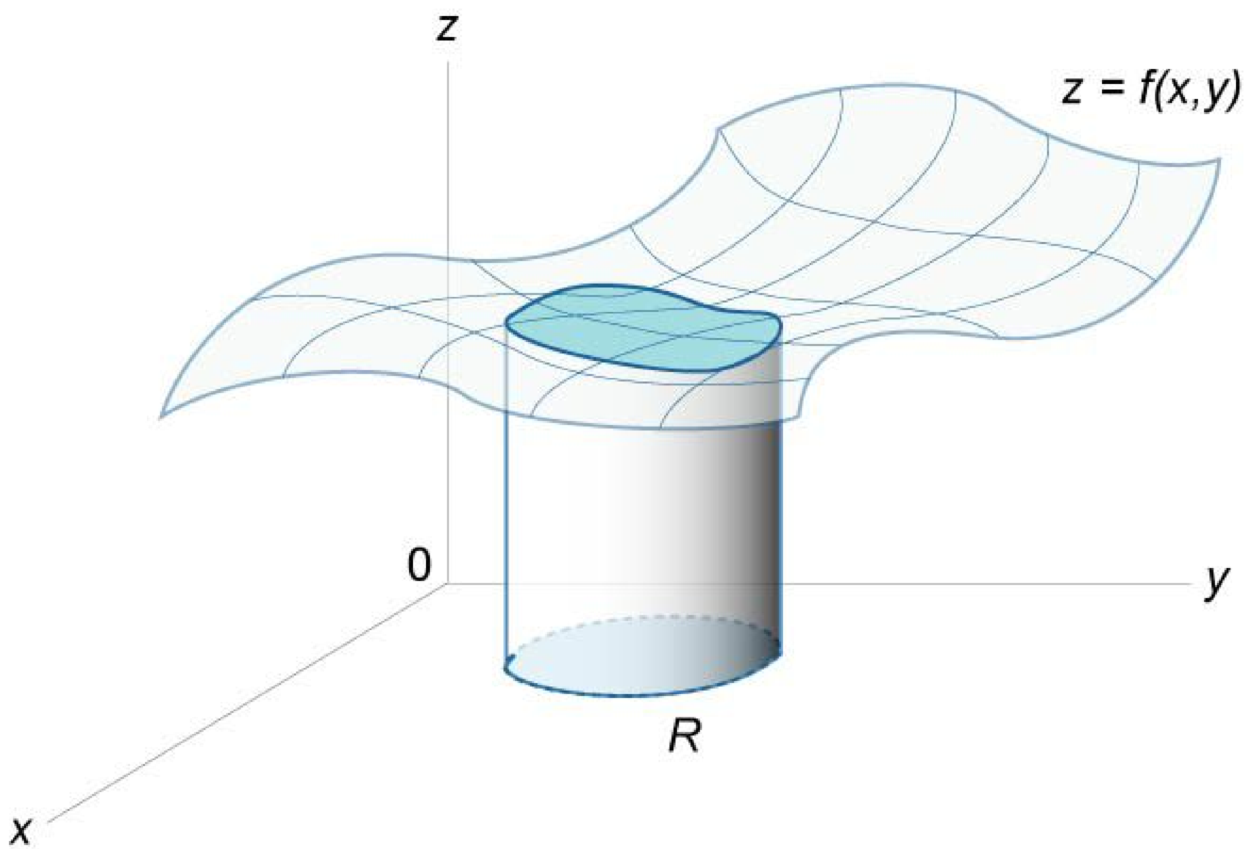
1091. **sçäìãÉ=çÑ=~=pçäáÇ=**

=

∬

()

ÑsK==
o
=



==
Figure 200.
=

fÑ=o=áë=~=íóéÉ=f=êÉÖáçå=ÄçìåÇÉÇ=Äó=ñ=~ I=ñ=Ä I= ()I=
ó=Ö()I=íÜÉå==

Ä Ö( )

() ()

=∫∫ÑsK==∫∫Ñ ñI ÇóÇñ

o
~
Ü
ñ
()
=

fÑ=o=áë=~=íóéÉ=ff=êÉÖáçå=ÄçìåÇÉÇ=Äó=ó=ÅI=ó=Ç I= ()I=

ñ=é()I=íÜÉå=

Ç è( )

() ()

=∫∫ÑsK==∫∫Ñ ñI ÇñÇó

o
Å
é
ó
()

fÑ==()
()==çîÉê==~==êÉÖáçå==oI==íÜÉå==íÜÉ==îçäìãÉ==çÑ=

íÜÉ= ëçäáÇ= ÄÉíïÉÉå= òN=Ñ( )=~åÇ= òO=Ö( )=çîÉê=o=áë=

ÖáîÉå=Äó=

=

$\iint$

() ( )

ÑsK==[]Ç^

o

=

1092. ^êÉ~=~åÇ=sçäìãÉ=áå=mçä~ê=`ççêÇáå~íÉë=
fÑ=p=áë=~=êÉÖáçå=áå=íÜÉ=ñó-éä~åÉ=ÄçìåÇÉÇ=Äó= αI= βI=

= () ()

ÜêI= ÖêI==
íÜÉå==
β Ö()
=Ç^^I== êÇêÇθ

∬

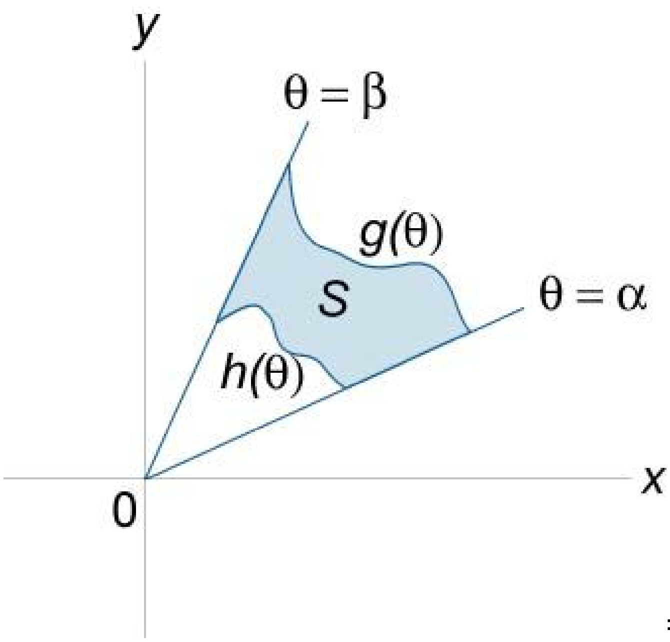
=∬θ

p α Ü()

=∬() θÑsK==

p

=



==

Figure 201.

=

1093. pìêÑ~ÅÉ=^êÉ~=

p

=

∬

N

+

□∂**ò** □**O** □∂**ò**□**O**

□□

∂

ñ

□□

+□□∂**ó**□□ **ÇñÇó =**

o □ □

1094. j~ëë=çÑ=~=i~ãáå~=

=

∫∫

()

ñãI==
o
ïÜÉêÉ==íÜÉ==ä~ãáå~==çÅÅìéáÉë==~==êÉÖáçå==o=~åÇ==áíë==ÇÉå
ëáíó==~í=

~=éçáåí=EñIóF=áë=( )K===

=

1095. **jçãÉåíë=**

qÜÉ=ãçãÉåí=çÑ=íÜÉ=ä~ãáå~=~Äçìí=íÜÉ==ñ-~ñáë==áë=ÖáîÉå=Äó=Ñçê-ãìä~=

ójK==ñ =∫∫ρ()Ç^

o

=

qÜÉ=ãçãÉåí=çÑ=íÜÉ=ä~ãáå~=~Äçìí=íÜÉ=ó-~ñáë=áë=

ñjK=∫∫ρ()Ç^ó

o

=

qÜÉ=ãçãÉåí=çÑ=áåÉêíá~=~Äçìí=íÜÉ=ñ-~ñáë=áë=

ñ

=

∫∫

O

()

ófK==
o
=
qÜÉ=ãçãÉåí=çÑ=áåÉêíá~=~Äçìí=íÜÉ=ó-~ñáë=áë= ó

=ñfK==O ()

ñIó Ç^

∫∫

o

= qÜÉ=éçä~ê=ãçãÉåí=çÑ=áåÉêíá~=áë=

M

=

∫∫
()

ñfK==( )ñIó Ç^

o
=
1096. `ÉåíÉê=çÑ=j~ëë=

∫∫ñρ( )Ç^

= ó = N ∫∫jñI==ã ã o ∫∫ρ()ñIó Ç^

o

∫∫óρ( )Ç^

= ñ = N ∫∫jóK==ã ã o ∫∫ρ()ñIó Ç^

o
=
1097. `Ü~êÖÉ=çÑ=~=mä~íÉ=

()

=∫∫ñnI==

o
ïÜÉêÉ=ÉäÉÅíêáÅ~ä=ÅÜ~êÖÉ=áë=ÇáëíêáÄìíÉÇ=çîÉê=~=êÉÖáçå=o=~
åÇ=áíë=

**ÅÜ~êÖÉ=ÇÉåëáíó=~í=~=éçáåí=EñIóF=áë= (
)K===**

=

1098. **^îÉê~ÖÉ=çÑ=~=cìåÅíáçå=**

µ =N∫∫Ñ()Ió Ç^I==p ñ

o

ïÜÉêÉ= ∫∫pK==

o

=
=
=

9.11 Triple Integral

=

cìåÅíáçåë=çÑ=íÜêÉÉ=î~êá~ÄäÉëW=( )ñÑI=( )I=£=
qêáéäÉ=áåíÉÖê~äëW=∫∫∫ ( )ñÑI=∫∫∫ ( )ÇsI=£=

d d
ã å é
oáÉã~åå=ëìãW=

()

á
I
î
ìÑ=

à â ñá óà òâ∑∑∑

á== =N
pã~ää=ÅÜ~åÖÉëW=Δñá I=ΔóàI=Δòâ =

iáãáíë=çÑ=áåíÉÖê~íáçåW=~I=ÄI=ÅI=ÇI=êI=ë=
oÉÖáçåë=çÑ=áåíÉÖê~íáçåW=dI=qI=p==
`óäáåÇêáÅ~ä=ÅççêÇáå~íÉëW=êI=θI=ò=
péÜÉêáÅ~ä=ÅççêÇáå~íÉëW= ê I=θI=ϕ=
sçäìãÉ=çÑ=~=ëçäáÇW=s=
j~ëë=çÑ=~=ëçäáÇW=ã==

aÉåëáíóW=()=

`ççêÇáå~íÉë=çÑ=ÅÉåíÉê=çÑ=ã~ëëW=ñ I= ó I= ò =
cáêëí=ãçãÉåíëW=jI=jI=jñò =ñó óò
jçãÉåíë=çÑ=áåÉêíá~W=

ñó
fI=
óò
fI=
f
ñò
I=
f I= fI=f I=f =
ñ ó ò M
=
=
1099. aÉÑáåáíáçå=çÑ=qêáéäÉ=fåíÉÖê~ä=

qÜÉ=íêáéäÉ=áåíÉÖê~ä=çîÉê=~=é~ê~ääÉäÉéáéÉÇ =[ ] [ ][ ]=

áë=ÇÉÑáåÉÇ=íç=ÄÉ==

∭

Ñ

()

ã å é

Çs= äáã→MΣΣΣÑ() ΔñáΔóàΔòâ I

[][ ][ ]
ã~ñ
Δ
ñ
á à â á→Má N à== =â Nã~ñ Δóà N
(
ì
á
I
î
à
I
ï
â
)
ã~ñ Δòâ→M
ïÜÉêÉ= =áë=ëçãÉ=éçáåí=áå=íÜÉ=é~ê~ääÉäÉéáéÉÇ=

ñ() (óà−NIóà)(òâ−NIòâ)I=~åÇ=Δñá =ñá −ñá−NI=á−N á

Δóà =óà −óà−NI=Δòâ =òâ −òâ−NK=
=

1100. ∫∫∫() ( )[]Çs=∫∫∫Ñ( )Çs+∫∫∫ Ö( )Çs

d d d =

1101. ∫∫∫() ( )[]Çs=∫∫∫Ñ( )Çs−∫∫∫Ö( )Çs

d d d
=

1102. ∫∫∫ ()Çs=â∫∫∫ Ñ( )ÇsI==

d d
ïÜÉêÉ=â=áë=~=Åçåëí~åíK=
=
1103.
fÑ==
Ñ

()ñIóIò ≥M =~åÇ=d=~åÇ=q=~êÉ=åçåçîÉêä~ééáåÖ=Ä~ëáÅ=

êÉÖáçåëI=íÜÉå==

∭

()IóIò Çs=∭ Ñ( )ñÑK==+∭ Ñ( )

ñ ñIóIò Çs
d∪q d q
eÉêÉ=d∪q=áë=íÜÉ=ìåáçå=çÑ=íÜÉ=êÉÖáçåë=d=~åÇ=qK= =
1104. bî~äì~íáçå=çÑ=qêáéäÉ=fåíÉÖê~äë=Äó=oÉéÉ~íÉÇ=fåíÉÖê~äë=

fÑ=íÜÉ=ëçäáÇ=d=áë=íÜÉ=ëÉí=çÑ=éçáåíë=(
)=ëìÅÜ=íÜ~í=

= ñIó ∈oI χ() ()≤ò ≤χO( )I=íÜÉå==N

=∫∫∫() ( )χO() ÇñÇóI==∫∫ ∫ Çò

d o χN ()
ïÜÉêÉ=o=áë=éêçàÉÅíáçå=çÑ=d=çåíç=íÜÉ=ñó-éä~åÉK=
=

fÑ=íÜÉ=ëçäáÇ=d=áë=íÜÉ=ëÉí=çÑ=éçáåíë=( )=ëìÅÜ=íÜ~í=

~

≤ ñ≤ÄI ϕ$_N$()≤ó ≤ϕ$_O$( ) ( )I$_N$ ñIó ≤ò ≤χ$_O$( )

ñ ñ ñIó I=íÜÉå==

Ä□ϕO()□χO() □ □∭ Ñ()IóIò ÇñÇóÇò= □□∬ ∫□□ Ñ()

ñ ñIóIò Çò□□Çó□□Çñ ==

d ~ □ϕN()□χN() □ □

=

1105. qêáéäÉ=fåíÉÖê~äë=çîÉê=m~ê~ääÉäÉéáéÉÇ=

fÑ=d=áë=~=é~ê~ääÉäÉéáéÉÇ=[ ] [ ] [ ]I=íÜÉå=

ÄÇë ÇóÇñK==∫∫∫Ñ() ()

∫∫∫ Çò

d ~ Å ê
=

få==íÜÉ=ëéÉÅá~ä=Å~ëÉ==ïÜÉêÉ=íÜÉ=áåíÉÖê~åÇ==( )==Å~å=ÄÉ=

ïêáííÉå=~ë=() ( ) ( )ñÖ=ïÉ=Ü~îÉ==

□Ä □□Ç □□ë □

∫∫∫

() () () ()

ñÑK==

d

∫Ö ñ ∫

~ Å ê

=

1106. `Ü~åÖÉ=çÑ=s~êá~ÄäÉë=

∭ ()ñÑ=

d

= ∭ Ñ ñ()()( )[] ( ) ÇñÇóÇòI= p ∂()

∂ñ ∂ñ ∂ñ
∂ì ∂î ∂ï

ïÜÉêÉ==() =∂ó ∂ó ∂ó ≠M==áë==íÜÉ==à~ÅçÄá~å==çÑ=∂() ∂ì ∂î
∂ï

∂ò ∂ò ∂ò
∂ì ∂î ∂ï

íÜÉ= íê~åëÑçêã~íáçåë=( ) (ìIîIï)I=~åÇ=p=áë=íÜÉ=éìää-

Ä~Åâ= çÑ= d= ïÜáÅÜ= Å~å= ÄÉ= ÅçãéìíÉÇ= Äó= ñ=ñ(ìIîIï)I=

ó=ó()=

ò=ò()=áåíç=íÜÉ=ÇÉÑáåáíáçå=çÑ=dK=

= =

1107. qêáéäÉ=fåíÉÖê~äë=áå=`óäáåÇêáÅ~ä=`ççêÇáå~íÉë=

qÜÉ=ÇáÑÑÉêÉåíá~ä=ÇñÇóÇò=Ñçê=ÅóäáåÇêáÅ~ä=ÅççêÇáå~íÉë=áë=

=

ÇñÇóÇò=∂()ÇêÇθÇò=êÇêÇθÇòK==∂()

=

iÉí=íÜÉ=ëçäáÇ=d=áë=ÇÉíÉêãáåÉÇ=~ë=ÑçääçïëW=

() ()N ≤ò≤χO( )I=

ïÜÉêÉ=o=áë=éêçàÉÅíáçå=çÑ=d=çåíç=íÜÉ=ñó-éä~åÉK=qÜÉå==

∭

()

ÇñÇóÇò
=

∭

Ñ

(

θ

ñÑ=θ**Iò**)**êÇêÇ**θ**Çò**

d p

□χ**O**()

=□êÇêÇθK=∫∫ ∫Ñ()Çò□□

o()θ □χN() □

eÉêÉ=p=áë=íÜÉ=éìääÄ~Åâ=çÑ=d=áå=ÅóäáåÇêáÅ~ä=ÅççêÇáå~íÉëK=

=

1108. qêáéäÉ=fåíÉÖê~äë=áå=péÜÉêáÅ~ä=`ççêÇáå~íÉë=

qÜÉ=aáÑÑÉêÉåíá~ä=ÇñÇóÇò=Ñçê=péÜÉêáÅ~ä=`ççêÇáå~íÉë=áë==

ÇñÇóÇò=∂()ÇêÇθÇϕ =êO ëáåθÇêÇθÇϕ==∂()

=

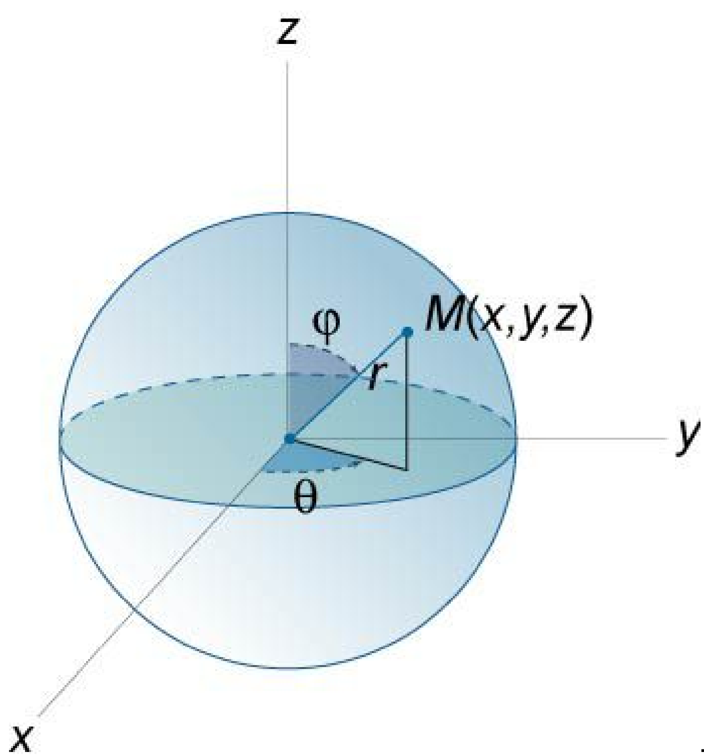
∭ ()ñÑ=

==========∭()ê^{O}êÑI==

p

ïÜÉêÉ=íÜÉ=ëçäáÇ=p=áë=íÜÉ=éìääÄ~Åâ=çÑ=d=áå=ëéÜÉêáÅ~ä=ÅççêÇá -
å~íÉëK=qÜÉ=~åÖäÉ==θ==ê~åÖÉë==Ñêçã==M=íç==πI==íÜÉ==~åÖäÉ==ϕ= ê~åÖÉë=Ñêçã=M=íç=πK==
=

= =======



= Figure 202. =

1109. sçäìãÉ=çÑ=~=pçäáÇ=

=∫∫∫s=

d

=

1110. sçäìãÉ=áå=`óäáåÇêáÅ~ä=`ççêÇáå~íÉë=

s = ∭ êÇêÇθÇò=

p()
=

1111. sçäìãÉ=áå=péÜÉêáÅ~ä=`ççêÇáå~íÉë=

s =∭ ê^{O} ëáåθÇêÇθÇϕ=

p()

=

1112. j~ëë=çÑ=~=pçäáÇ=

=

∫∫∫

()

ñãI==
d
ïÜÉêÉ= íÜÉ= ëçäáÇ= çÅÅìéáÉë= ~= êÉÖáçå= d= ~åÇ= áíë= ÇÉåëáíó=
~í============

~=éçáåí=()=áë=( )K===

=

1113. `ÉåíÉê=çÑ=j~ëë=çÑ=~=pçäáÇ=

ñ =jóò I=ó =jñò I=ò =jñó I==ã ã ã

ïÜÉêÉ==

óò

ñjI==()Çs

$\iiint$

d

ójI==()Çsñò ∭

d

òj=∫∫∫µ()Çsñó

d
~êÉ==íÜÉ==Ñáêëí==ãçãÉåíë==~Äçìí==íÜÉ==ÅççêÇáå~íÉ=éä~åÉë=ñ=
MI=

ó=MI=ò=MI=êÉëéÉÅíáîÉäóI==( )ñIóIò
=áë=íÜÉ=ÇÉåëáíó=ÑìåÅíáçåK==

=
1114. jçãÉåíë=çÑ=fåÉêíá~=~Äçìí==íÜÉ==ñó-éä~åÉ=Eçê=ò=MFI=óò-éä~åÉ==
EMFI=~åÇ=ñò-éä~åÉ=Eó=MF=

=

∭

O ()

ñó òfI==
d

ñfI==^{O} ()Çs$_{óò}$ ∭

d
=

∫∫∫

O ()

ñò ófK==
d
=
1115. jçãÉåíë=çÑ=fåÉêíá~=~Äçìí=íÜÉ=ñ-~ñáëI=ó-~ñáëI=~åÇ=ò-~ñáë=
=
ñó
+
f
ñò
=

∭

(
ò

O +óO)( )

ñ ffI==
d
=
ñó
+
f
óò
=

∫∫∫

(
ò

O+ñO)( )

ó ffI==
ò
=
ñò
+
f
óò
=

∫∫∫

(
ó
O
+
ñ
O

)
( )

ffK==
d
=

1116. mçä~ê=jçãÉåí=çÑ=fåÉêíá~=)( )

M =ñó +fóò +fñò =∭ (ñ^{O} +ó^{O} +ò^{O} ff=

d

=

=

9.12 Line Integral

=

pÅ~ä~ê=ÑìåÅíáçåëW=( )ñcI=( )ñcI=( )ñÑ=

pÅ~ä~ê=éçíÉåíá~äW=( )=

`ìêîÉëW=`I=
`I=`=
N O

iáãáíë=çÑ=áåíÉÖê~íáçåëW=~I=ÄI= αI=β=
m~ê~ãÉíÉêëW=íI=ë=
mçä~ê=ÅççêÇáå~íÉëW= ê I=θ=

sÉÅíçê=ÑáÉäÇW= (mInIo)= mçëáíáçå=îÉÅíçêW=( )=

råáí=îÉÅíçêëW= á^{r}I= àI=â I=τ=
^êÉ~=çÑ=êÉÖáçåW=p=
iÉåÖíÜ=çÑ=~=ÅìêîÉW=i=
j~ëë=çÑ=~=ïáêÉW=ã=

aÉåëáíóW=()I=( )=

`çêÇáå~íÉë=çÑ=ÅÉåíÉê=çÑ=ã~ëëW=ñ I= ó I= ò =
cáêëí=ãçãÉåíëW=$^{jI=jI=}$j$_{ñò}$ =$_{ñó óò}$
jçãÉåíë=çÑ=áåÉêíá~W=

f I= fI=f =
ñ ó ò

sçäìãÉ=çÑ=~=ëçäáÇW=s=
tçêâW=t=
j~ÖåÉíáÅ=ÑáÉäÇW= _ =
`ìêêÉåíW=f=
bäÉÅíêçãçíáîÉ=ÑçêÅÉW=ε=
j~ÖåÉíáÅ=ÑäìñW=ψ=
1117. iáåÉ=fåíÉÖê~ä=çÑ=~=pÅ~ä~ê=cìåÅíáçå=$_{r\ r}$

iÉí=~=ÅìêîÉ=`=ÄÉ=ÖáîÉå=Äó=íÜÉ=îÉÅíçê=ÑìåÅ íáçå= ( )I=

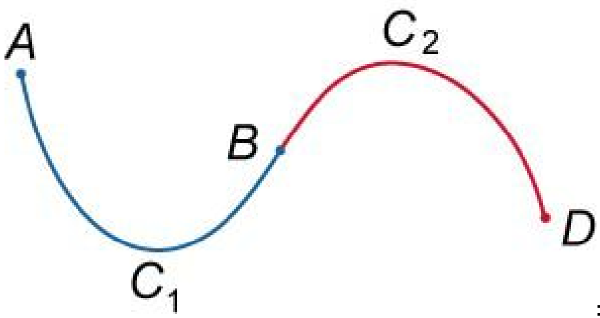
M≤ë
pI=~åÇ=~=ëÅ~ä~ê=ÑìåÅíáçå=c=áë=ÇÉÑáåÉÇ=çîÉê=íÜÉ=ÅìêîÉ=`K==
qÜÉå==

p r()() ( )∫c ñIó ∫cÇëI==∫c

M``
ïÜÉêÉ=Çë=áë=íÜÉ=~êÅ=äÉåÖíÜ=ÇáÑÑÉêÉåíá~äK==
=
1118. ∫Çëc=∫cÇë

$\int$

`N∪`O `N `O
=



==

Figure 203.
=r r

1119.

fÑ=íÜÉ=ëãççíÜ=ÅìêîÉ=`=áë=é~ê~ãÉíêáòÉÇ=Äó= ( )I=

αíI=íÜÉå==
β

∫c( )Çë=∫c() () ()()()() ()() ()()OÇíK=

` α
=
1120. fÑ=`=áë=~=ëãççíÜ=ÅìêîÉ=áå=íÜÉ=ñó-
éä~åÉ=ÖáîÉå=Äó=íÜÉ=Éèì~íáçå= ó

=Ñ()I=~ ≤

ñ ñ ÄI=íÜÉå==
Ä
∫
c

() ()() ()()Ió Çë ∫c ñIÑ ñ N Ñ ñOÇñ K==

ñ
` ~
=
1121. iáåÉ=fåíÉÖê~ä=çÑ=pÅ~ä~ê=cìåÅíáçå=áå=mçä~ê=`ççêÇáå~íÉë= β
Çê O

∫c ñIó Çë=∫() ( ) êO+ÇθÇθI==

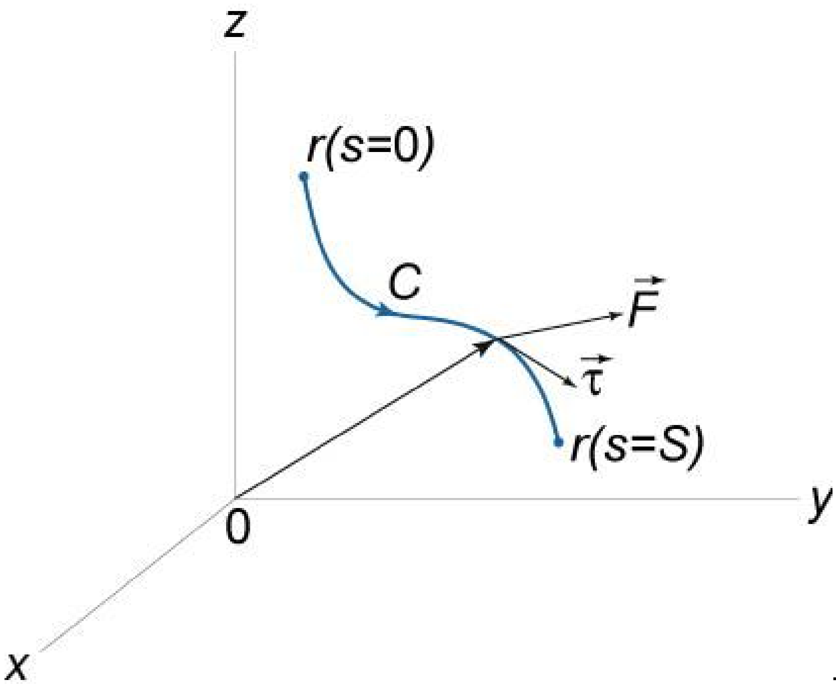
` α
ïÜÉêÉ=íÜÉ=ÅìêîÉ=`=áë=ÇÉÑáåÉÇ=Äó=íÜÉ=éçä~ê=ÑìåÅíáçå=êθ K= =
1122. iáåÉ=fåíÉÖê~ä=çÑ=sÉÅíçê=cáÉäÇ=r r

iÉí=~=ÅìêîÉ=`=ÄÉ=ÇÉÑáåÉÇ=Äó=íÜÉ=îÉÅíçê= ÑìåÅíáçå= ()I=

M≤ë pK=qÜÉå==
Ç
r

r=τ=()==Çë

áë=íÜÉ=ìåáí=îÉÅíçê=çÑ=íÜÉ=í~åÖÉåí=äáåÉ=íç=íÜáë=ÅìêîÉK== =



==

Figure 204.
=

iÉí= ~= îÉÅíçê= ÑáÉäÇ= (mInIo)=áë=ÇÉÑáåÉÇ=çîÉê=íÜÉ=ÅìêîÉ=`K=

qÜÉå=íÜÉ=äáåÉ=áåíÉÖê~ä=çÑ=íÜÉ=îÉÅíçê=ÑáÉäÇ=
c=~äçåÖ=íÜÉ=ÅìêîÉ= `=áë==
p
∫
mÇñ

+nÇó +oÇò =∫m()

Åçë nÅçë oÅçë ÇëK=
` M
= 1123. mêçéÉêíáÉë=çÑ=iáåÉ=fåíÉÖê~äë=çÑ=sÉÅíçê=cáÉäÇë=

∫() ()∫c·Çê I==

_` `

ïÜÉêÉ=-`=ÇÉåçíÉ==íÜÉ=ÅìêîÉ=ïáíÜ=íÜÉ=çééçëáíÉ=çêáÉåí~íáçåK= r

( ) ( ) ( ) ( )

r r= r
∫c ∫c·Ç ∫c·Çr I==∫ê
` `N∪ `O `N `O
ïÜÉêÉ=`=áë=íÜÉ=ìåáçå=çÑ=íÜÉ=ÅìêîÉë=
N
`=~åÇ=
`K== O
=

1124. fÑ=íÜÉ=ÅìêîÉ=`=áë=é~ê~ãÉíÉêáòÉÇ=Äó=r( ) ( ) ( ) ( ) I=

αíI=íÜÉå==
∫mÇñ +nÇó +oÇò ==
`
β

=m() () ()()Çñ+n() () ()()Çó+o() () ()()ÇòÇí∫ Çí Çí Çíα

=

1125. fÑ=`=äáÉë=áå=íÜÉ=ñó-
éä~åÉ=~åÇ=ÖáîÉå=Äó=íÜÉ=Éèì~íáçå= ()I=

íÜÉå==

Äm ñIÑ()() ()()n ñIÑ ñÇÑ ÇñK==∫mÇñ +nÇó =∫ ñ Çñ `
~

=
1126. dêÉÉå∞ë=qÜÉçêÉã=

∂n _∂m ÇñÇó= ∫mÇñ +nÇóI==∬ ∂ñ ∂ó

o r `

=r +r= áë= ~= Åçåíáåìçìë= îÉÅíçê= ÑìåÅ-ïÜÉêÉ= c () ( ) à

íáçå=ïáíÜ==Åçåíáåìçìë==Ñáêëí=é~êíá~ä==ÇÉêáî~íáîÉë=∂mI=∂n=áå=~=∂ó ∂ñ

ëçãÉ= Ççã~áå= oI= ïÜáÅÜ= áë= ÄçìåÇÉÇ= Äó= ~= ÅäçëÉÇI= éáÉÅÉïáëÉ= ëãççíÜ=Åìêîé=`K==

=

1127. ^êÉ~=çÑ=~=oÉÖáçå=o=_çìåÇÉÇ=Äó=íÜÉ=`ìêîÉ=`=

p= ∫∫ÇñÇó =N∫ñÇó−óÇñ=

oO `
=
1128. r qÜÉ=äáåÉ=áåíÉÖê~ä=çÑ=~=îÉÅíçê=ÑìåÅíáçå= c =má à
m~íÜ=fåÇÉéÉåÇÉåÅÉ=çÑ=iáåÉ=fåíÉÖê~äë=r r +nr +oâ=áë=
ë~áÇ=íç=ÄÉ=é~íÜ=áåÇÉéÉåÇÉåíI=áÑ=~åÇ=çåäó=áÑ=mI=nI=~åÇ=o=
~êÉ= Åçåíáåìçìë= áå= ~= Ççã~áå= aI= ~åÇ= áÑ= íÜÉêÉ= Éñáëíë=
ëçãÉ= ëÅ~ä~ê=

ÑìåÅíáçå==ì ( )ì ñIóIò
==E~==ëÅ~ä~ê==éçíÉåíá~äF==áå==a==ëìÅÜ=íÜ~í=

r
c=Öê~ÇìI=çê=∂ì=mI=∂ì=nI=∂ì=oK==∂ñ ∂ó ∂ò
qÜÉå==
∫
rr

c()⋅r ∫mÇñ +nÇó +oÇò =ì( ) ( )K==

``
=
1129. qÉëí=Ñçê=~=`çåëÉêî~íáîÉ=cáÉäÇ=
r
^=îÉÅíçê=ÑáÉäÇ=çÑ=íÜÉ=Ñçêã=c
ÑáÉäÇK=qÜÉ=äáåÉ=áåíÉÖê~ä=çÑ=~=îÉÅíçê=ÑìåÅíáçå=cá à
=Öê~Çì=áë=Å~ääÉÇ=~=ÅçåëÉêî~íáîÉ= r =mr +nr+r oâ=
áë=é~íÜ=áåÇÉéÉåÇÉåí=áÑ=~åÇ=çåäó=áÑ==
r rr
á à â
r = ∂ ∂ ∂=r
ÅìêäcÑ∂ñ ∂ó ∂ò MK==

m n o
=
fÑ=íÜÉ=äáåÉ=áåíÉÖê~ä=áë=í~âÉå=áå=ñó-éä~åÉ=ëç=íÜ~í==

∫mÇñ +nÇó =ì( ) ( )I==

`

íÜÉå=íÜÉ=íÉëí=Ñçê=ÇÉíÉêãáåáåÖ=áÑ=~=îÉÅíçê=ÑáÉäÇ=áë=ÅçåëÉê
î~íáîÉ= Å~å=ÄÉ=ïêáííÉå=áå=íÜÉ=Ñçêã==
∂m=∂nK==∂ó ∂ñ
1130. iÉåÖíÜ=çÑ=~=`ìêîÉ=

β Çr () ∫ Çñ O Çó O Çò Oβ

i =∫Çë=∫ê + Çí + Çí ÇíI= ` α Çíα Çí
ïÜÉêÉ=`=á~=~=éáÉÅÉïáëÉ=ëãççíÜ=ÅìêîÉ=ÇÉëÅêáÄÉÇ=Äó=íÜÉ=éçëá
-

íáçå=îÉÅíçê=()I=α≤í ≤βK=

=

fÑ=íÜÉ=ÅìêîÉ=`=áë=ïïç-ÇáãÉåëáçå~äI=íÜÉå=

β Çr () ∫ Çñ O Çó Oβ

i =∫Çë=∫ê + Çí ÇíK==
` α Çíα Çí

= ( )=áå=íÜÉ=ñó-
fÑ=íÜÉ=ÅìêîÉ=`=áë=íÜÉ=Öê~éÜ=çÑ=~=ÑìåÅíáçå=

éä~åÉ=()I=íÜÉå==

Ä □Çó□O
i =∫N+□□Çñ□□ ÇñK==
~
=
1131. iÉåÖíÜ=çÑ=~=`ìêîÉ=áå=mçä~ê=`ççêÇáå~íÉë=
β **□Çê □O**
i =∫□□Çθ□□ +êO ÇθI==
α

ïÜÉêÉ=íÜÉ=ÅìêîÉ=`=áë=ÖáîÉå=Äó=íÜÉ=Éèì~íáçå= ( )êêI=

α≤θ ≤β**=áå=éçä~ê=ÅççêÇáå~íÉëK===**
=
1132. j~ëë=çÑ=~=táêÉ=

()

=∫ñãI==
`

ïÜÉêÉ=()=áë=íÜÉ=ã~ëë=éÉê=ìåáí=äÉåÖíÜ=çÑ=íÜÉ=ïáêÉK=

=
fÑ=`=áë=~=ÅìêîÉ=é~ê~ãÉíêáòÉÇ=Äó=íÜÉ=îÉÅíçê=ÑìåÅíáçå r

ê() () () ( ) I==íÜÉå==íÜÉ==ã~ëë==Å~å==ÄÉ==ÅçãéìíÉÇ=Äó=

íÜÉ=Ñçêãìä~=
β **□Çñ □O □Çó□O □Çò□O**

ã=∫ρ() () ()()□□Çí□□ +□□Çí□□ +□□Çí□□ ÇíK==

α

=
fÑ=`=áë=~=ÅìêîÉ=áå=ñó-
éä~åÉI=íÜÉå=íÜÉ=ã~ëë=çÑ=íÜÉ=ïáêÉ=áë=ÖáîÉå= Äó==

=

∫
()

ñãI=
`
çê=
β ÇñÇó

ã=∫ρ() ()() Çí Çí Çí=Eáå=é~ê~ãÉíêáÅ=ÑçêãFK=

α
=
1133. `ÉåíÉê=çÑ=j~ëë=çÑ=~=táêÉ=
ñ =jóò I=ó =jñò I=ò =jñó I=ã ã ã
ïÜÉêÉ==

()

=∫ñjI==óò
`

()

=∫ójI==ñò
`

()

=∫òjK=ñó
`

=

1134. jçãÉåíë=çÑ=fåÉêíá~=
qÜÉ=ãçãÉåíë=çÑ=áåÉêíá~=~Äçìí=íÜÉ=ñ-~ñáëI=ó-~ñáëI=~åÇ=ò-~ñáë=
~êÉ=ÖáîÉå=Äó=íÜÉ=Ñçêãìä~ë=

= ∫()( )Çëñ ófI==

`

=∫()( )Çëó ñfI==

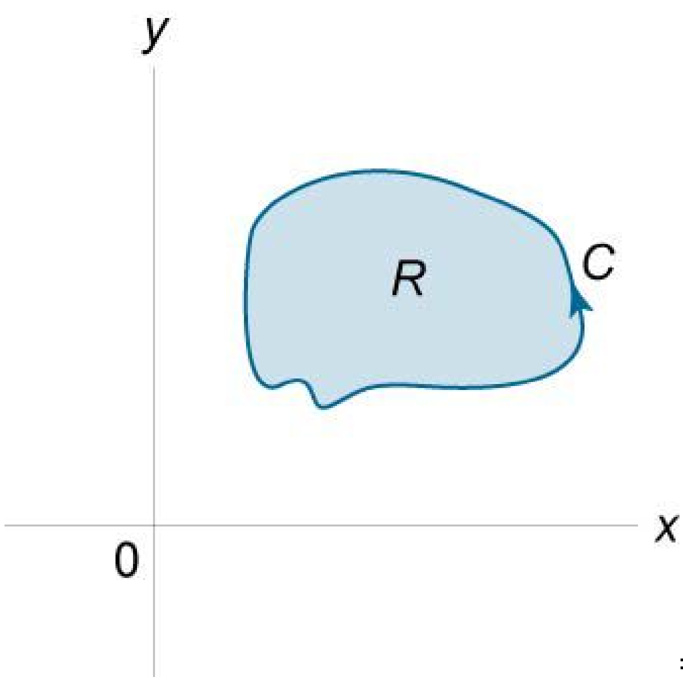
`

=∫()( )Çëò ñfK==

`

1135. ^êÉ~=çÑ=~=oÉÖáçå=_çìåÇÉÇ=Äó=~=`äçëÉÇ=`ìêîÉ= p= ∫ñÇó
=−∫óÇñ=N∫ñÇó−óÇñK==
``O`

=



==

Figure 205.

=

fÑ=íÜÉ=ÅäçëÉÇ=ÅìêîÉ=`=áë=ÖáîÉå=áå=é~ê~ãÉíêáÅ=Ñçêã= r

ê() () ()

I=íÜÉå=íÜÉ=~êÉ~=Å~å=ÄÉ=Å~äÅìä~íÉÇ=Äó=íÜÉ=Ñçê-

ãìä~=

β

p =

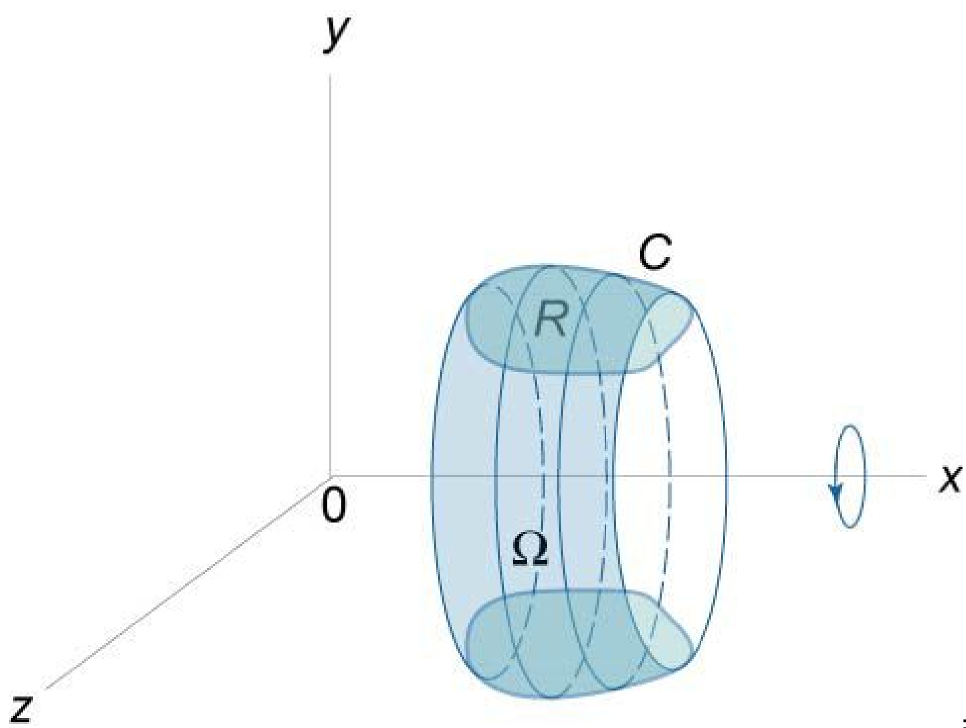
β Çó Çí = −∫ó()Çñ Çí =Nβñ íÇó() ()Çñ ÇíK==∫ñ()Çí α Çí

O∫ Çí Çí α α

=

1136. sçäìãÉ=çÑ=~=pçäáÇ=cçêãÉÇ=Äó=oçí~íáåÖ=~=`äçëÉÇ=`ìêîÉ=~Äçìí=íÜÉ=ñ-~ñáë=

s =−π∫**ó**O**Çñ** =−**O**π∫**ñóÇó** =−π∫**OñóÇó** +**ó**O**Çñ ==**



=

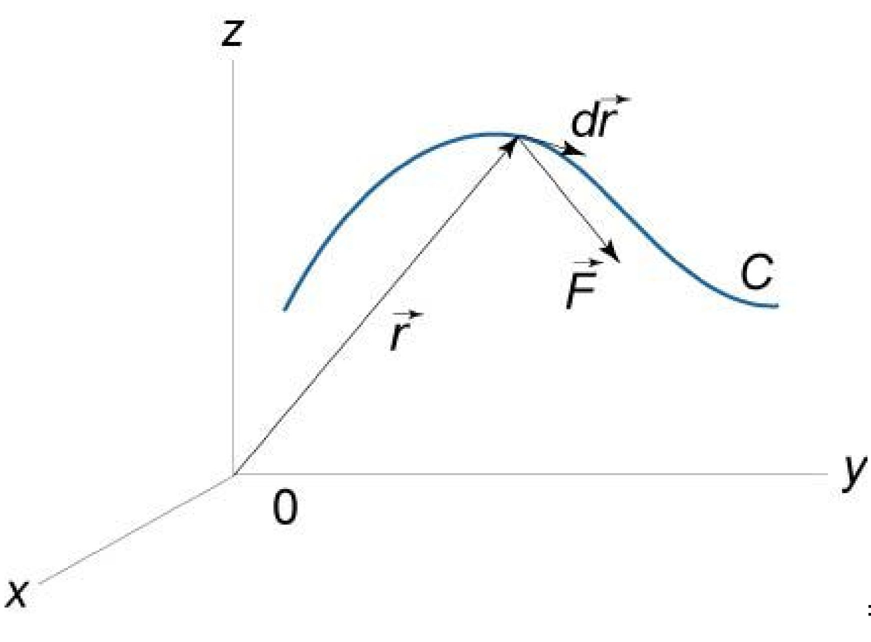
Figure 206.

=

1137. **tçêâ=**

tçêâ=ÇçåÉ=Äó=~=ÑçêÅÉ= c
=çå=~å=çÄàÉÅí=ãçîáåÖ=~äçåÖ=~=ÅìêîÉ=
`=áë=ÖáîÉå=Äó=íÜÉ=äáåÉ=áåíÉÖê~ä=

t
=
∫
r
·
Ç
ê
r**I=**
`
ïÜÉêÉ= c
=áë=íÜÉ=îÉÅíçê=ÑçêÅÉ=ÑáÉäÇ=~ÅíáåÖ=çå=íÜÉ=çÄàÉÅíI= Ç
r **ê** **=áë= íÜÉ=ìåáí=í~åÖÉåí=îÉÅíçêK==**



==

Figure 207.
fÑ=íÜÉ=çÄàÉÅí=áë=ãçîÉÇ=~äçåÖ=~=ÅìêîÉ=`=áå=íÜÉ=ñó-
éä~åÉI=íÜÉå= t
=

∫
r
c
·
Ç
ê
r =∫mÇñ +nÇó I==
``

=
fÑ=~=é~íÜ=`=áë=ëéÉÅáÑáÉÇ=Äó=~=é~ê~ãÉíÉê=í=Eí=çÑíÉå=ãÉ~åë=
íáãÉFI=íÜÉ=Ñçêãìä~=Ñçê=Å~äÅìä~íáåÖ=ïçêâ=ÄÉÅçãÉë=

β

t□m() () ()()Çñ +n() () ()()Çó +o() () ()()Çò□ÇíI∫

α □□ Çí Çí Çí □□ ïÜÉêÉ=í=ÖçÉë=Ñêçã=α=íç=βK==
=

fÑ=~=îÉÅíçê=ÑáÉäÇ= c = áë= ÅçåëÉêî~íáîÉ= ~åÇ= (
)=áë=~=ëÅ~ä~ê=

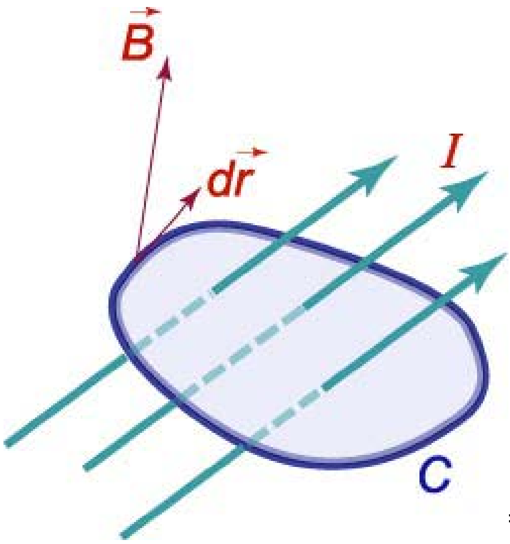
éçíÉåíá~ä=çÑ=íÜÉ=ÑáÉäÇI=íÜÉå=íÜÉ=ïçêâ=çå=~å=çÄàÉÅí=ãçîáåÖ=
Ñêçã=^=íç=_=Å~å=ÄÉ=ÑçìåÇ=Äó=íÜÉ=Ñçêãìä~=

t=ì() ( )K==

=
1138. ^ãéÉêÉ∞ë=i~ï=
r
_·Çê =μMfK==
`

qÜÉ=äáåÉ=áåíÉÖê~ä=çÑ=~=ã~ÖåÉíáÅ=ÑáÉäÇ=_
=~êçìåÇ=~=ÅäçëÉÇ=é~íÜ=
`=áë=Éèì~ä=íç=íÜÉ=íçí~ä=ÅìêêÉåí=f=ÑäçïáåÖ=íÜêçìÖÜ=íÜÉ=~êÉ~=
ÄçìåÇÉÇ=Äó=íÜÉ=é~íÜK==

=



=Figure 208.

1139. c~ê~Ç~ó∞ë=i~ï=

r

εÇψ =∫b·Çê =−Çí`

=

qÜÉ= ÉäÉÅíêçãçíáîÉ= ÑçêÅÉ=
EÉãÑF=ε=áåÇìÅÉÇ=~êçìåÇ=~=ÅäçëÉÇ=
äççé=`=áë=Éèì~ä=íç=íÜÉ=ê~íÉ=çÑ=íÜÉ=ÅÜ~åÖÉ=çÑ=ã~ÖåÉíáÅ=Ñäì
ñ=ψ= é~ëëáåÖ=íÜêçìÖÜ=íÜÉ=äççéK===

=

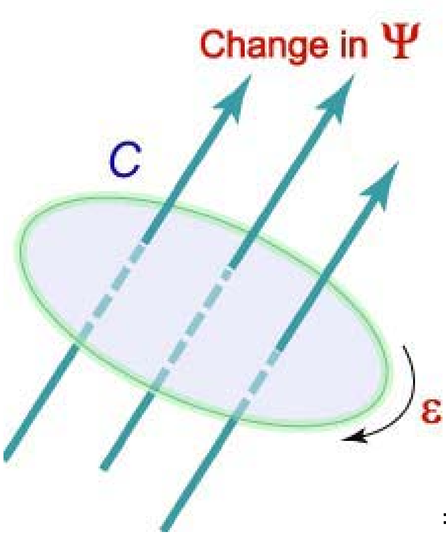


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Figure 209. =

=

=

9.13 Surface Integral

=

pÅ~ä~ê=ÑìåÅíáçåëW=( )I=( )=

mçëáíáçå=îÉÅíçêëW=( )I=( )=

råáí=îÉÅíçêëW= á^{r}I= àI= â =
pìêÑ~ÅÉW=p=

sÉÅíçê=ÑáÉäÇW= (mInIo)=

r r
aáîÉêÖÉåÅÉ=çÑ=~=îÉÅíçê=ÑáÉäÇW=Çáî c ·c =
r r
`ìêä=çÑ=~=îÉÅíçê=ÑáÉäÇW=Åìêäc ×c ==
sÉÅíçê=ÉäÉãÉåí=çÑ=~=ëìêÑ~ÅÉW=
Ç
r
p =

kçêã~ä=íç=ëìêÑ~ÅÉW= å =
pìêÑ~ÅÉ=~êÉ~W=^=
j~ëë=çÑ=~=ëìêÑ~ÅÉW=ã=

aÉåëáíóW=()=

`ççêÇáå~íÉë=çÑ=ÅÉåíÉê=çÑ=ã~ëëW=ñ I= ó I= ò =
cáêëí=ãçãÉåíëW=
jI=jI=j=
ñó óò ñò

jçãÉåíë=çÑ=áåÉêíá~W=$_{ñó}$fI=$_{óò}$fI=$_{ñò}$fI=$_{ñ}$fI=$_{ó}$fI=$_{ò}$f=
sçäìãÉ=çÑ=~=ëçäáÇW=s=

cçêÅÉW=c =
dê~îáí~íáçå~ä=Åçåëí~åíW=d=

cäìáÇ=îÉäçÅáíóW=( )=

cäìáÇ=ÇÉåëáíóW=ρ=
r

mêÉëëìêÉW=() =

j~ëë=ÑäìñI=ÉäÉÅíêáÅ=ÑäìñW=Φ=
pìêÑ~ÅÉ=ÅÜ~êÖÉW=n=

`Ü~êÖÉ=ÇÉåëáíóW= ( )=

j~ÖåáîÇÉ=çÑ=íÜÉ=ÉäÉÅíêáÅ=ÑáÉäÇW=b =
=
=

1140. **pìêÑ~ÅÉ=fåíÉÖê~ä=çÑ=~=pÅ~ä~ê=cìåÅíáçå=**
iÉí=~=ëìêÑ~ÅÉ=p=ÄÉ=ÖáîÉå=Äó=íÜÉ=éçëáíáçå=îÉÅíçê=

r() () ( ) ( )^{r}**+**$^{r\ r}$

$_{ê}$ + **âI==**
ïÜÉêÉ=

()ìIî = ê~åÖÉë= çîÉê= ëçãÉ= Ççã~áå= ( )ìIî =çÑ=íÜÉ=ìî-

a
éä~åÉK=

qÜÉ==ëìêÑ~ÅÉ==áåíÉÖê~ä==çÑ==~==ëÅ~ä~ê=ÑìåÅíáçå==( )ñÑ=çîÉê=

íÜÉ=ëìêÑ~ÅÉ=p=áë=ÇÉÑáåÉÇ=~ë== r ∂r

∬

Ñ
ñ
I
ó
I
ò

( ) ( )()()()

Ñ
ñ
ì
I
î
I
ó
ì
I
î
I
ò
ì
I
î
∂ê × ê
∂ î ÇìÇî I==
p
Çp ∂ìa()r ∂r

ïÜÉêÉ=íÜÉ=é~êíá~ä=ÇÉêáî~íáîÉë=
∂ê =~åÇ=ê
∂î =~êÉ=ÖáîÉå=Äó==∂ì

∂
r

ê ∂**ñ() () ()r+** ∂**ó r** ∂**ò rI==**∂**ì =** ∂**ì** ∂ **â**

ì ∂**ì**
∂
r

ê ∂**ñ() () ()r** ∂**ó r** ∂**ò r=**∂**î =** ∂**î â**

∂**î** ∂**î**
∂
r
~åÇ=ê ∂**ê =áë=íÜÉ=Åêçëë=éêçÇìÅíK==**∂**ì×**∂**î**
=

1141.

fÑ==íÜÉ==ëìêÑ~ÅÉ==p==áë==ÖáîÉå=Äó==íÜÉ= Éèì~íáçå= ( )=ïÜÉêÉ=

ò()==áë==~==ÇáÑÑÉêÉåíá~ÄäÉ==ÑìåÅíáçå==áå=íÜÉ=Ççã~áå =()ñaI=

íÜÉå==

∫∫

Ñ

() ()()

N
+
□∂ò□O □∂ò□O
□□
∂
ñ
□□
+□□∂ó□□ ÇñÇóK==
p a() □ □

=
1142. pìêÑ~ÅÉ=fåíÉÖê~ä=çÑ=íÜÉ=sÉÅíçê=cáÉäÇ= c
=çîÉê=íÜÉ=pìêÑ~ÅÉ=p=
• fÑ=p=áë=çêáÉåíÉÇ=çìíï~êÇI=íÜÉå==
=====

∫∫

r r
c c

r ()·Çp=∫∫r( )·åÇp=

p p

r ()()()()·□∂r ∂r

====== = cê ê□ÇìÇî K==
a() □□∂ì× ∂î□□
=
· fÑ=p=áë=çêáÉåíÉÇ=áåï~êÇI=íÜÉå==

=====∫∫r()·Çp

r r

c= ∫∫r( )·åÇp =

p p

r ()()()()·□∂r ∂r

====== = cê ê□ÇìÇî K==
a() □□∂î× ∂ì□□
= r
Çp=åÇp
==áë==Å~ääÉÇ==íÜÉ==îÉÅíçê=ÉäÉãÉåí=çÑ=íÜÉ=ëìêÑ~ÅÉK==açí=
ãÉ~åë==íÜÉ==ëÅ~ä~ê==éêçÇìÅí==çÑ==íÜÉ==~ééêçéêá~íÉ==îÉÅíçêëK
= qÜÉ=é~êíá~ä=ÇÉêáî~íáîÉë=

∂
r ∂r
ê =~åÇ==ê
∂î =~êÉ=ÖáîÉå=Äó==∂ì

∂r ∂ñ() () ()r+ ∂ó r+ ∂òr

∂ì âI==ê = ∂ì ∂ì ∂ì

∂r ∂ñ() () ()r+ ∂ó r+ ∂òr

∂î âK==ê = ∂î ∂î ∂î
=

1143.
fÑ==íÜÉ==ëìêÑ~ÅÉ==p==áë==ÖáîÉå=Äó=íÜÉ=Éèì~íáçå= ( )I=ïÜÉêÉ=

ò()==áë==~==ÇáÑÑÉêÉåíá~ÄäÉ==ÑìåÅíáçå==áå==íÜÉ=Ççã~áå=()I=

íÜÉå==

• fÑ=p=áë=çêáÉåíÉÇ= ìéï~êÇI=áKÉK=íÜÉ=â-íÜ= ÅçãéçåÉåí= çÑ= íÜÉ= åçêã~ä=îÉÅíçê=áë=éçëáíáîÉI=íÜÉå===
=====

∬

r

()

r r r

·Çp= ∬c( )·åÇp =

p p

====== ()rr .–∂ò r –∂ò r +âÇñÇóI== ∂ñ á ∂ó à

a()
=

• fÑ=p=áë=çêáÉåíÉÇ=Ççïåï~êÇI=áKÉK=íÜÉ=â-
íÜ=ÅçãéçåÉåí=çÑ=íÜÉ= åçêã~ä=îÉÅíçê=áë=åÉÖ~íáîÉI=íÜÉå===
=====

∬

r

()

.

r

Ç

p

=

∫∫

r( ).r

c åÇp
p p
======

()

r
.
∂**ò** r+ ∂**ò** r− r
ÇñÇóK==

a
()
∂
ñ
á ∂**ó**à â
=

1144. ∫∫() ∫∫mÇóÇò+nÇòÇñ+oÇñÇó=

p p

=∫∫()ÅçëmI==

p

ïÜÉêÉ=()ñmI= ( )ñnI= ( )=~êÉ=íÜÉ=ÅçãéçåÉåíë=çÑ=

íÜÉ=îÉÅíçê= ÑáÉäÇ= c K==
Åçë
αI= βI= γ

Åçë Åçë ==~êÉ=íÜÉ= ~åÖäÉë= ÄÉíïÉÉå=íÜÉ=çìíÉê=ìååáí=
åçêã~ä=îÉÅíçê= å=~åÇ=íÜÉ=ñ-~ñáëI=ó-~ñáëI=~åÇ=ò-~ñáëI=êÉëéÉÅí-
áîÉäóK=

1145.
fÑ=íÜÉ=ëìêÑ~ÅÉ=p=áë=ÖáîÉå=áå=é~ê~ãÉíêáÅ=Ñçêã=Äó=íÜÉ=îÉÅíçê
=

ê()( ) ( )
()I==íÜÉå==íÜÉ==ä~ííÉê=Ñçêãìä~=Å~å=ÄÉ=

ïêáííÉå=~ë==
m n o

∬r()Çp= ∬mÇóÇò+nÇòÇñ +oÇñÇó = ∬∂ñ ∂ó ∂ò ÇìÇîI

p p a()∂ì∂ì∂ì∂ñ ∂ó ∂ò ∂î ∂î ∂î

ïÜÉêÉ=()=ê~åÖÉë=çîÉê=ëçãÉ=Ççã~áå= ( )=çÑ=íÜÉ=ìî-

éä~åÉK=

=

1146. aáîÉêÖÉåÅÉ=qÜÉçêÉã=

$$\iint$$

r

c

r·Çp= ∭()ÇsI==

p d
ïÜÉêÉ==
r

c() () ( ) ( ) ===

áë==~==îÉÅíçê==ÑáÉäÇ==ïÜçëÉ==ÅçãéçåÉåíë==mI==nI==~åÇ==o==
Ü~îÉ== Åçåíáåìçìë=é~êíá~ä=ÇÉêáî~íáîÉëI==

∇
r= ∂**m**+ ∂**n**+ ∂**o**
·c ==∂**ñ** ∂**ó** ∂**ò**

áë==íÜÉ==
ÇáîÉêÖÉåÅÉ
==çÑ== c
I==~äëç==ÇÉåçíÉÇ==
Çáî
r
c K==qÜÉ==ëóãÄçä=

∫∫
=áåÇáÅ~íÉë==íÜ~í=íÜÉ=ëìêÑ~ÅÉ=áåíÉÖê~ä=áë= í~âÉå=çîÉê=~=ÅäçëÉÇ=

ëìêÑ~ÅÉK==

=

1147. aáîÉêÖÉåÅÉ=qÜÉçêÉã=áå=`ççêÇáå~íÉ=cçêã=

∂m +∂n +∂o ÇñÇóÇòK==∬mÇóÇò
+nÇñÇò+oÇñÇó =∭ ∂ñ ∂ó ∂ò

p d
=

1148. **píçâÉ∞ë=qÜÉçêÉã=()·ÇpI==∫r r**

c·Ç$_{ê}$

r r =∫∫ ∇×c

` p
ïÜÉêÉ==
r

c() () ( ) ( ) ===

**áë==~=îÉÅíçê==ÑáÉäÇ==ïÜçëÉ==ÅçãéçåÉåíë==mI==nI==~åÇ=o==Ü~
îÉ= Åçåíáåìçìë=é~êíá~ä=ÇÉêáî~íáîÉëI==**
r r r
á à â
r ∂ ∂ ∂= ∂**o**− ∂**n** r+ ∂**m**− ∂**o** r+ ∂**n**− ∂**m** r ∇×c= ∂**ñ** ∂**ñ** ∂**ñ** ∂**ó** ∂**ò** **á** ∂**ò**
∂**ñ** **à** ∂**ñ** ∂**ó** **â** **m n o** **áë=íÜÉ=**
Åìêä
=çÑ=c
I=~äëç=ÇÉåçíÉÇ=
Åìêä
r
c**K==**
**qÜÉ=ëóãÄçä==∫=áåÇáÅ~íÉë=íÜ~í=íÜÉ=äáåÉ=áåíÉÖê~ä=áë=í~âÉå=çî
Éê= ~=ÅäçëÉÇ=ÅìêîÉK==**
=
1149. **píçâÉ∞ë=qÜÉçêÉã=áå=`ççêÇáå~íÉ=cçêã=**
∫mÇñ +nÇó +oÇò =
`
=

∬

∂o −∂n ÇóÇò+ ∂m −∂o ÇòÇñ+ ∂n −∂m ÇñÇó

p

∂

ó

∂

ò

∂ò ∂ñ ∂ñ ∂ó

==

1150. pìêÑ~ÅÉ=^êÉ~=

^=

∬

p
=
1151. fÑ=íÜÉ=ëìêÑ~ÅÉ=p=áë=é~ê~ãÉíÉêáòÉÇ=Äó=íÜÉ=îÉÅíçê==

r() () ( ) ( )r+r+r

ê âI==
íÜÉå=íÜÉ=ëìêÑ~ÅÉ=~êÉ~=áë==
^
=
∂
r∂r
ê × ê
∂
î
ÇìÇîI==
a()∂ì

ïÜÉêÉ==
()==áë==íÜÉ==Çç\~áå==ïÜÉêÉ==íÜÉ=ëìêÑ~ÅÉ=
()=áë=

ÇÉÑáåÉÇK==

1152. fÑ=p=áë=ÖáîÉå=ÉñéäáÅáíäó=Äó=íÜÉ=ÑìåÅíáçå=(
)I==íÜÉå==íÜÉ==ëìê-

Ñ\~ÅÉ=\~êÉ\~=áë==
^
=
N
\+
□ ∂**ò** □**O** □∂**ò** □**O**
□□
∂
ñ
□□
\+□□∂**ó**□□ **ÇñÇó I==**
a() □ □

ïÜÉêÉ=
()=áë=íÜÉ=éêçàÉÅíáçå=çÑ=íÜÉ=ëìêÑ~ÅÉ=p=çåíç=íÜÉ=ñó-

éä~åÉK==
=
1153. j~ëë=çÑ=~=pìêÑ~ÅÉ=
=

∫∫

µ

()

ñãI==
p

ïÜÉêÉ=()=áë=íÜÉ=ã~ëë=éÉê=ìåáí=~êÉ~==EÇÉåëáíó=ÑìåÅ-

íáçåFK=
=

1154. **`ÉåíÉê=çÑ=j~ëë=çÑ=~=pÜÉää=**
ñ =**jóò** **I=ó** =**jñò** **I=ò** =**jñó** **I==ã ã ã**

ïÜÉêÉ==
=

∬

()

óò ñjI==
p
=

∫∫

()

ñò ójI==
p

òj=∫∫µ()Çpñó

p
~êÉ==íÜÉ==Ñáêëí==ãçãÉåíë==~Äçìí===íÜÉ=ÅççêÇáå~íÉ=éä~åÉë=ñ=
MI=

ó=MI=ò=MI==êÉëéÉÅíáîÉäóK==(
)=áë=íÜÉ=ÇÉåëáíó=ÑìåÅíáçåK=

=
1155. jçãÉåíë=çÑ=fåÉêíá~=~Äçìí=íÜÉ=ñó-éä~åÉ=Eçê=ò=MFI==óò-éä~åÉ==
E ñ=MFI=~åÇ=ñò-éä~åÉ=Eó=MF=

=

∬

O ()

ñó µ**òfI==**
p
=

∬

O ()

óò ñfI==
ñò
=

∬

O

()

ófK=
p
=
1156. jçãÉåíë=çÑ=fåÉêíá~=~Äçìí=íÜÉ=ñ-~ñáëI=ó-~ñáëI=~åÇ=ò-~ñáë=

=∫∫()( )Çpñ ófI==

p

=∫∫()( )Çpó ñfI==

p

=∫∫()( )Çpò ñfK==

p
=
1157. sçäìãÉ=çÑ=~=pçäáÇ=_çìåÇÉÇ=Äó=~=`äçëÉÇ=pìêÑ~ÅÉ=

s =N ∫∫ñÇóÇò+óÇñÇò +òÇñÇó ==p p

=

1158. dê~îáí~íáçå~ä=cçêÅÉ=

r=dã∫∫µ()r

cê Çp I=

p êP

ïÜÉêÉ=ã=áë=~=ã~ëë=~í=~=éçáåí= ñ$_{M}$Ió$_{M}$Iò$_{M}$

=çìíëáÇÉ=íÜÉ=ëìêÑ~ÅÉI== r

ê = ñ−ñ$_{M}$Ió−ó$_{M}$Iò −ò$_{M}$ I==

µ( )=áë=íÜÉ=ÇÉåëáíó=ÑìåÅíáçåI==

~åÇ=d=áë=Öê~îáí~íáçå~ä=Åçåëí~åíK=

=

1159. mêÉëëìêÉ=cçêÅÉ=

r ∫∫é()ÇpI==

p r

ïÜÉêÉ=íÜÉ=éêÉëëìêÉ==( ) ==~Åíë==çå==íÜÉ==ëìêÑ~ÅÉ==p==ÖáîÉå==Äó=

íÜÉ=éçëáíáçå=îÉÅíçê=êK=

=

1160. cäìáÇ=cäìñ=E~Åêçëë=íÜÉ=ëìêÑ~ÅÉ=pF=

Φ

=

∬

r

()

·ÇpI==
r r
c=
ïÜÉêÉ=
r
r

()=áë=íÜÉ=ÑäìáÇ=îÉäçÅáíóK=

==
1161. j~ëë=cäìñ=E~Åêçëë=íÜÉ=ëìêÑ~ÅÉ=pF=

Φ
=

∬

r

()

·ÇpI==
p r r
ïÜÉêÉ=c ρ**î=áë=íÜÉ=îÉÅíçê=ÑáÉäÇI=**ρ**=áë=íÜÉ=ÑäìáÇ=ÇÉåëáíóK= =**
1162. pìêÑ~ÅÉ=`Ü~êÖÉ=

()

=∫∫ñnI==

p

ïÜÉêÉ=()ñIó =áë=íÜÉ=ëìêÑ~ÅÉ=ÅÜ~êÖÉ=ÇÉåëáíóK=

=

1163. **d~ìëë∞=i~ï=**

qÜÉ=ÉäÉÅíêáÅ=Ñäìñ=íÜêçìÖÜ=~åó=ÅäçëÉÇ=ëìêÑ~ÅÉ=áë=éêçéçêíáçå
~ä=
íç=íÜÉ=ÅÜ~êÖÉ=n=ÉåÅäçëÉÇ=Äó=íÜÉ=ëìêÑ~ÅÉ=

Φ

$\iint$

r

b·Çp=^{n} I==

p ε_{M}

ïÜÉêÉ==

Φ=**áë=íÜÉ=ÉäÉÅíêáÅ=ÑäìñI==**

b =áë=íÜÉ=ã~ÖåáîÇÉ=çÑ=íÜÉ=ÉäÉÅíêáÅ=ÑáÉäÇ=ëíêÉåÖíÜI=

ε_{M} =**UIUR**×**NM**$^{-NO}$ **c =áë=éÉêãáííáîáíó=çÑ=ÑêÉÉ=ëé~ÅÉK==ã**

=

Chapter 10 Differential Equations

=
=
=
=
cìåÅíáçåë=çÑ=çåÉ=î~êá~ÄäÉW=óI=éI=èI=ìI=ÖI=ÜI=dI=eI=êI=ò==
^êÖìãÉåíë=EáåÇÉéÉåÇÉåí=î~êá~ÄäÉëFW=ñI=ó=

cìåÅíáçåë=çÑ=íïç=î~êá~ÄäÉëW=( )ñÑI= ( )ñjI= ( )ñk=

cáêëí=çêÇÉê=ÇÉêáî~íáîÉW= ó′I=ì′I= ó I= ÇíÇó I=£=
Ç
O
pÉÅçåÇ=çêÇÉê=ÇÉêáî~íáîÉëW= ó′′ I= óf I=£=&& I=ÇíO
∂
ì
I=
∂
O
m~êíá~ä=ÇÉêáî~íáîÉëW=ì I=£=∂í ∂ñO
k~îê~ä=åìãÄÉêW=å=
m~êíáÅìä~ê=ëçäìíáçåëW=óI= ó=N é

oÉ~ä=åìãÄÉêëW=âI=íI=`I=`I= `I=éI=èI=αI=β=N O
oççíë=çÑ=íÜÉ=ÅÜ~ê~ÅíÉêáëíáÅ=Éèì~íáçåëW=N I=O = qáãÉW=í=
qÉãéÉê~îêÉW=qI=p=

mçéìä~íáçå=ÑìåÅíáçåW=( )==

j~ëë=çÑ=~å=çÄàÉÅíW=ã=
píáÑÑåÉëë=çÑ=~=ëéêáåÖW=â=
aáëéä~ÅÉãÉåí=çÑ=íÜÉ=ã~ëë=Ñêçã=ÉèìáäáÄêáìãW=ó=
^ãéäáíìÇÉ=çÑ=íÜÉ=ÇáëéäÅ~ÅÉãÉåíW=^=
cêÉèìÉåÅóW=ω=
a~ãéáåÖ=ÅçÉÑÑáÅáÉåíW=γ=
mÜ~ëÉ=~åÖäÉ=çÑ=íÜÉ=Çáëéä~ÅÉãÉåíW=δ =
^åÖìä~ê=Çáëéä~ÅÉãÉåíW=θ=
mÉåÇìäìã=äÉåÖíÜW=i=
^ÅÅÉäÉê~íáçå=çÑ=Öê~îáíóW=Ö=
`ìêêÉåíW=f=
oÉëáëí~åÅÉW=o=
fåÇìÅí~åÅÉW=i=
`~é~Åáí~åÅÉW=`=
=
=

10.1 First Order Ordinary Differential Equations

=
1164. iáåÉ~ê=bèì~íáçåë=

Çó +é ñ() ()è ñ K==Çñ ó

=
qÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë==

ó=∫ì()()Çñ+`I==ì()

ïÜÉêÉ==

()()Éñé(∫ñìK=)

=
1165. pÉé~ê~ÄäÉ=bèì~íáçåë=

Çó =Ñ() ()()=Çñ

=
qÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë=ÖáîÉå=Äó=

∫Çó() ()Çñ+`I==Ü

çê=

e() ()+`K=

=
=
1166. eçãçÖÉåÉçìë=bèì~íáçåë=

qÜÉ= ÇáÑÑÉêÉåíá~ä= Éèì~íáçå=Çó ()= áë= ÜçãçÖÉåÉçìëI= áÑ=

íÜÉ=ÑìåÅíáçå=

()

Çñ
=áë=ÜçãçÖÉåÉçìëI=íÜ~í=áë==

Ñ()()K==

=
qÜÉ=ëìÄëíáíìíáçå=ò=ó =EíÜÉå=ó=òñF=äÉ~Çë=íç=íÜÉ=ëÉé~ê~ÄäÉ=ñ
Éèì~íáçå=

ñÇò +ò =Ñ()K==Çñ

=
1167. _Éêåçìääá=bèì~íáçå=

Çó +é ñ() ()è ñ óåK==Çñ ó

=
qÜÉ=ëìÄëíáíìíáçå=ò =óN−å=äÉ~Çë=íç=íÜÉ=äáåÉ~ê=Éèì~íáçå==

Çò + N()() ()()å é ñ ò = N−å è ñ K==Çñ

=
1168. oáÅÅ~íá=bèì~íáçå=

Çó =é() () ()óO=Çñ

=
fÑ=~=é~êíáÅìä~ê=ëçäìíáçå=Nó =áë=âåçïåI=íÜÉå=íÜÉ=ÖÉåÉê~ä=ëçäì-
íáçå=Å~å=ÄÉ=çÄí~áåÉÇ=ïáíÜ=íÜÉ=ÜÉäé=çÑ=ëìÄëíáíìíáçå=

ò =N I=ïÜáÅÜ=äÉ~Çë=íç=íÜÉ=Ñáêëí=çêÇÉê=äáåÉ~ê=Éèì~íáçå== −óN

Çò =−è ñ +() ()[] ()K==Çñ O N

=
=
1169. bñ~Åí=~åÇ=kçåÉñ~Åí=bèì~íáçåë=
qÜÉ=Éèì~íáçå==

j() ( )Çó=M==

áë=Å~ääÉÇ=Éñ~Åí=áÑ==
∂j= ∂kI==∂ó ∂ñ

~åÇ= åçåÉñ~Åí=çíÜÉêïáëÉK=
=
qÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë==

∫j()Çñ +∫k( )Çó =`K==

=
1170. o~Çáç~ÅíáîÉ=aÉÅ~ó=
Çó −âóI==Çí

ïÜÉêÉ= ()=áë=íÜÉ=~ãçìåí=çÑ=ê~Çáç~ÅíáîÉ=ÉäÉãÉåí=~í= íáãÉ=íI=â=

áë=íÜÉ=ê~íÉ=çÑ=ÇÉÅ~óK==
=
qÜÉ=ëçäìíáçå=áë==
ó

()=óMÉ−âíI=ïÜÉêÉ=óM= í ó( )=áë=íÜÉ=áåáíá~ä=~ãçìåíK=

=
1171. kÉïíçå∞ë=i~ï=çÑ=`ççäáåÖ=

Çq=−â()I==Çí
ïÜÉêÉ=
()=áë=íÉãéÉê~îìêÉ=çÑ=~å=çÄàÉÅí=~í=íáãÉ=íI=p=áë=íÜÉ
=

íÉãéÉê~îìêÉ=çÑ=íÜÉ=ëìêêçìåÇáåÖ=ÉåîáêçåãÉåíI=â=áë=~=éçëá - íáîÉ=Åçåëí~åíK==
=
qÜÉ=ëçäìíáçå=áë=

q( )()M É−âíI==

ïÜÉêÉ=M ()= áë= íÜÉ= áåáíá~ä= íÉãéÉê~îìêÉ= çÑ= íÜÉ= çÄàÉÅí= ~í=

íáãÉ=í=MK==
=
1172. mçéìä~íáçå=aóå~ãáÅë=EiçÖáëíáÅ=jçÇÉäF=
Çm =âm□□□−m □I==Çí j□□

ïÜÉêÉ=
()=áë=éçéìä~íáçå=~í=íáãÉ=íI=â=áë=~=éçëáíáîÉ=Åçåëí~åíI=

j=áë=~=äáãáíáåÖ=ëáòÉ=Ñçê=íÜÉ=éçéìä~íáçåK==
=
qÜÉ=ëçäìíáçå=çÑ=íÜÉ=ÇáÑÑÉêÉåíá~ä=Éèì~íáçå=áë==

m

()

=
jmM

mM()mM É−âí I=ïÜÉêÉ=mM=m( )=áë=íÜÉ=áåáíá~ä=éçéì-

ä~íáçå=~í=íáãÉ= í=MK===
=
=
=

10.2 Second Order Ordinary Differential Equations

=

1173. **eçãçÖÉåÉçìë=iáåÉ~ê=bèì~íáçåë=ïáíÜ=`çåëí~åí=`çÉÑÑáÅáÉåíë==**

ó+′′**éó**+′**èó**=**MK==**

qÜÉ=ÅÜ~ê~ÅíÉêáëíáÅ=Éèì~íáçå=áë==

λ**O** +**é**λ+**è**=**MK==**

=

fÑ=$_N$ **=~åÇ=**$_O$ **=~êÉ=ÇáëíáåÅí=êÉ~ä=êççíë=çÑ=íÜÉ=ÅÜ~ê~ÅíÉêáëíáÅ=Éèì~íáçåI=íÜÉå=íÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë==**

ó=**`**$_N$**É**$^{\lambda N ñ}$ +**`**$_O$**É**$^{\lambda O ñ}$**I=ïÜÉêÉ==**

$_N$**`** **=~åÇ=**$_O$**`** **=~êÉ=áåíÉÖê~íáçå=Åçåëí~åíëK==**

==

fÑ=λ$_N$ =λ$_O$ =−**é** **I=íÜÉå=íÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë==**$_O$

()

`
O
ñ
É
−
é

ó = **`Oñ** **K==**$_N$ +

=

fÑ=$_N$ **=~åÇ=**$_O$ **=~êÉ=ÅçãéäÉñ=åìãÄÉêëW=**

λ$_N$ =α+β**áI=**λ$_O$ =α−β**á** **I=ïÜÉêÉ==**

é

I=

β

=

Q
è
−
é
O
α=−O O I==
íÜÉå=íÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë==
ó
=
αñ

É `()β +O β K==NÅçë

=
1174.
fåÜçãçÖÉåÉçìë=iáåÉ~ê=bèì~íáçåë=ïáíÜ=`çåëí~åí=================
======
`çÉÑÑáÅáÉåíë==

ó+′′éó+′èó= ( )K==

=
qÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë=ÖáîÉå=Äó==

ó
=
+

ó óÜI=ïÜÉêÉ==é
ó =áë==~=é~êíáÅìä~ê==ëçäìíáçå=çÑ==íÜÉ=áåÜçãçÖÉåÉçìë=Éèì~íáçå=é
~åÇ=Üó
=áë=íÜÉ=ÖÉåÉê~ä=ëçäìíáçå=çÑ=íÜÉ=~ëëçÅá~íÉÇ=ÜçãçÖÉåÉ-
çìë=Éèì~íáçå=EëÉÉ=íÜÉ=éêÉîáçìë=íçéáÅ=NNTPFK=
=
fÑ=íÜÉ=êáÖÜí=ëáÇÉ=Ü~ë=íÜÉ=Ñçêã==
Ñ

() ()
( )()

m ñ Åçëβñ+N β I==N
íÜÉå=íÜÉ==é~êíáÅìä~ê=ëçäìíáçå=éó =áë=ÖáîÉå=Äó==
ó
é

=ñâÉαñ( )()ñ Åçëβñ+oO ñ ëáåβñ I==

oN

ïÜÉêÉ= íÜÉ= éçäóåçãá~äë= ( )=~åÇ= ( )=Ü~îÉ=íç=ÄÉ=ÑçìåÇ=N O

Äó=ìëáåÖ=íÜÉ=**ãÉíÜçÇ=çÑ=ìåÇÉíÉêãáåÉÇ=ÅçÉÑÑáÅáÉåíë**K==

- fÑ=α+ á=áë=åçí=~=êççí=çÑ=íÜÉ=ÅÜ~ê~ÅíÉêáëíáÅ=Éèì~íáçåI=íÜÉå= íÜÉ=éçïÉê=â=MI=
- fÑ=α+ á=áë=~=ëáãéäÉ=êççíI=íÜÉå=â=NI=
- fÑ=α+ á=áë=~=ÇçìÄäÉ=êççíI=íÜÉå=â=OK==

=

1175. aáÑÑÉêÉåíá~ä=bèì~íáçåë=ïáíÜ=ó=jáëëáåÖ=

ó=′′Ñ()K==

pÉí=ì ′ó K=qÜÉå=íÜÉ=åÉï=Éèì~íáçå=ë~íáëÑáÉÇ=Äó=î=áë==

ì=′Ñ()I==

ïÜáÅÜ=áë=~=Ñáêëí=çêÇÉê=ÇáÑÑÉêÉåíá~ä=Éèì~íáçåK=

1176. aáÑÑÉêÉåíá~ä=bèì~íáçåë=ïáíÜ=ñ=jáëëáåÖ=

ó=′′Ñ()K==

pÉí=
ì
′
=ó K=páåÅÉ==
ó=′′Çì= Çì Çó=ìÇìI==Çñ Çó Çñ Çó
ïÉ=Ü~îÉ=
Çì

ì =Ñ()I==

Çó
ïÜáÅÜ=áë=~=Ñáêëí=çêÇÉê=ÇáÑÑÉêÉåíá~ä=Éèì~íáçåK==
=

1177. cêÉÉ=råÇ~ãéÉÇ=sáÄê~íáçåë=
qÜÉ=ãçíáçå=çÑ=~=j~ëë=çå=~=péêáåÖ=áë=ÇÉëÅêáÄÉÇ=Äó=íÜÉ=Éèì~- íáçå==
ã+âó =MI==&&
ïÜÉêÉ==
ã=áë=íÜÉ=ã~ëë=çÑ=íÜÉ=çÄàÉÅíI=
â=áë=íÜÉ=ëíáÑÑåÉëë=çÑ=íÜÉ=ëéêáåÖI=
ó=áë=Çáëéä~ÅÉãÉåí=çÑ=íÜÉ=ã~ëë=Ñêçã=ÉèìáäáÄêáìãK= =
qÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë==

ó=^Åçë()I==

ïÜÉêÉ==
^=áë=íÜÉ=~ãéäáíìÇÉ=çÑ=íÜÉ=Çáëéä~ÅÉãÉåíI=
=áë=íÜÉ=ÑìåÇ~ãÉåí~ä=ÑêÉèìÉåÅóI=íÜÉ=éÉêáçÇ=áë=q =Oπ I=M ωM

δ =áë=éÜ~ëÉ=~åÖäÉ=çÑ=íÜÉ=Çáëéä~ÅÉãÉåíK=
qÜáë=áë=~å=Éñ~ãéäÉ=çÑ=ëáãéäÉ=Ü~êãçåáÅ=ãçíáçåK== =

1178. cêÉÉ=a~ãéÉÇ=sáÄê~íáçåë=
ã ó+γ ó +âó=MI=ïÜÉêÉ==&& &
γ=áë=íÜÉ=Ç~ãéáåÖ=ÅçÉÑÑáÅáÉåíK==
qÜÉêÉ=~êÉ=P=Å~ëÉë=Ñçê=íÜÉ=ÖÉåÉê~ä=ëçäìíáçåW=
=
`~ëÉ=NK γ O >Qâã =EçîÉêÇ~ãéÉÇF=

ó()=^É λNí +_É λOí I==

ïÜÉêÉ==
−
γ
−
γ

O −Qâã I=λO = −γ+ γ O
λ −Qâã K==N = Oã Oã
=
`~ëÉ=OK=γ O =QâãEÅêáíáÅ~ääó=Ç~ãéÉÇF=

ó() ( )É λí I==

ïÜÉêÉ==
λ
=

–
γ

O ãK=
=
`~ëÉ=PK=γO<Qâã=EìåÇÉêÇ~ãéÉÇF==

γ
ó
í
=
É

() ()

–Oãí∧Åçë I=ïÜÉêÉ==
ω= Q γO K==
=

1179. páãéäÉ=mÉåÇìäìã=
ÇOθ +Ö θ=MI=ÇíO i

ïÜÉêÉ= θ= áë= íÜÉ= ~åÖìä~ê= Çáëéä~ÅÉãÉåíI= i= áë= íÜÉ= éÉåÇìäìã= äÉåÖíÜI=Ö=áë=íÜÉ=~ÅÅÉäÉê~íáçå=çÑ=Öê~îáíóK=
=
qÜÉ=ÖÉåÉê~ä=ëçäìíáçå=Ñçê=ëã~ää=~åÖäÉë=θ=áë==

θ()=θã~ñ ëáåÖíI=íÜÉ=éÉêáçÇ=áë=q πi K==i Ö

=
1180. oi`=`áêÅìáí=
Ç

O

if +oÇf+Nf =s′()=ωbM Åçë( )í I=ÇíO Çí ` í ïÜÉêÉ=f=áë=íÜÉ=ÅìêêÉåí=áå=~å=oi`=ÅáêÅìáí=ïáíÜ=~å=~Å=îçäí~ÖÉ=

ëçìêÅÉ=()=bM ω( )K===

=
qÜÉ=ÖÉåÉê~ä=ëçäìíáçå=áë==

f()=`NÉêNí +`OÉêOí +^ëáå(ωí−)I=

ïÜÉêÉ==
−o± oO −Qi
ê

N
I
O

=
`
OiI==
^=ωbM I==
□□
i
□ −
N□O
□□
+
o
OωO
`
ϕ=~êÅí~å□iω −N □I==□□ o o`ω□□

`I= ` =~êÉ=Åçåëí~åíë=ÇÉéÉåÇáåÖ=çå=áåáíá~ä=ÅçåÇáíáçåëK=N O
=
=
=

10.3. Some Partial Differential Equations

=

1181. qÜÉ=i~éä~ÅÉ=bèì~íáçå=

∂Oì +∂Oì =M=∂ñO ∂óO

~ééäáÉë=íç=éçíÉåíá~ä=ÉåÉêÖó=ÑìåÅíáçå= (
)==Ñçê==~==ÅçåëÉê-

î~íáîÉ=ÑçêÅÉ=ÑáÉäÇ=áå=íÜÉ=ñó -éä~åÉK= m~êíá~ä=
ÇáÑÑÉêÉåíá~ä= Éèì~-
íáçåë=çÑ=íÜáë=íóéÉ=~êÉ=Å~ääÉÇ=ÉääáéíáÅK==
=

1182. qÜÉ=eÉ~í=bèì~íáçå=

∂Oì+ ∂Oì= ∂ì=∂ñO ∂óO ∂í

~ééäáÉë= íç= íÜÉ= íÉãéÉê~îìêÉ= ÇáëíêáÄìíáçå= (
)=áå=íÜÉ=ñó-

éä~åÉ=ïÜÉå=ÜÉ~í=áë=~ääçïÉÇ=íç=Ñäçï=Ñêçã=ï~êã=~êÉ~ë=íç=Åççä=
çåÉëK=qÜÉ=Éèì~íáçåë=çÑ=íÜáë=íóéÉ=~êÉ=Å~ääÉÇ=é~ê~ÄçäáÅK==
=

1183. qÜÉ=t~îÉ=bèì~íáçå=

∂Oì+∂Oì=∂Oì=∂ñO ∂óO ∂íO

~ééäáÉë=íç=íÜÉ=Çáëéä~ÅÉãÉåí=

( )

ì ñIó =çÑ=îáÄê~íáåÖ=ãÉãÄê~åÉë= ~åÇ= çíÜÉê= ï~îÉ= ÑìåÅíáçåëK=
qÜÉ= Éèì~íáçåë= çÑ= íÜáë= íóéÉ= ~êÉ= Å~ääÉÇ=ÜóéÉêÄçäáÅK==
=
=

Chapter 11 Series

=

=

=

=

11.1 Arithmetic Series

=
fåáíá~ä=íÉêãW=
~=
N

kíÜ=íÉêãW=å~=
aáÑÑÉêÉåÅÉ=ÄÉíïÉÉå=ëìÅÅÉëëáîÉ=íÉêãëW=Ç=
kìãÄÉê=çÑ=íÉêãë=áå=íÜÉ=ëÉêáÉëW=å=
pìã=çÑ=íÜÉ=Ñáêëí=å=íÉêãëW=

p=
å
=
=

1184. ( )Ç=å =~å−N +Ç =~å−O +OÇ =K=~N +~

=
1185. ~N +~å =~O +~å−N =K=~á +~å+N−á=
=

1186. ~á =~á−N+~á+N =O

= ( )N Ç ·å=å =~N +~å ·å=O~N + å **1187.** pO O

=
=
=
=

11.2 Geometric Series

=
fåáíá~ä=íÉêãW=
~ =
N

kíÜ=íÉêãW=$_{å}$~=
`çãäçå=ê~íáçW=è=
kìãÄÉê=çÑ=íÉêãë=áå=íÜÉ=ëÉêáÉëW=å=
pìã=çÑ=íÜÉ=Ñáêëí=å=íÉêãëW=

p =
å

pìã=íç=áåÑáåáíóW=p=
=
=

1188. ~$_{å}$ =è~$_{å−N}$=~$_{N}$è$^{å−N}$=
=
1189. ~$_{N}$·~$_{å}$ =~$_{O}$ ·~$_{å−N}$ =K=~$_{á}$ ·~$_{å+N−á}$=
=
1190. ~$_{á}$ = ~$_{á−N}$·~$_{á+N}$ =

= ( )=å =~åè−~N =~N1191. pè −N è−N

=

1192.

p

=

äáã

p

å

=

~N

N−è= ∞

cçê= è<NI=íÜÉ=ëìã=p=ÅçåîÉêÖÉë=~ë= ∞K= =

=

=

11.3 Some Finite Series

=
kìãÄÉê=çÑ=íÉêãë=áå=íÜÉ=ëÉêáÉëW=å=
=

1193. N+O+P+K+å=å( )=O

=

1194. O+Q+S+ ( )=K+Oå=å

=

1195. ()=åO=

K
=
1196.
â + â+N + â+

()( ) ( ) (Oâ+å−N)=K O

=

1197. NO +OO +PO +K+åO =å( )( )=S

=
1198.
N
P
+
O
P

+
P
P
+
K
+
å
P
=

+( )O

O =

1199. **() (QåO−N)=K = P**

=

1200. **NP +PP +RP + (OåO −N)=K+( )**

=
1201. **N**+**N**+**N** +**N** +K+**N** =**O=O Q U Oå** K
=
1202.

N N N

N·O+ N + P·Q+K+ å()+K=N=O·P

=

1203. **N+N +N +N +K+N +K=É=N> O> P> ()>**

=
=

11.4 Infinite Series

=

pÉèìÉåÅÉW= ~{}=å

cáêëí=íÉêãW=
~=
N

kíÜ=íÉêãW=å~=
=
=

1204. **fåÑáåáíÉ=pÉêáÉë=**
∞
∑
~~=
å N ~O K+~**å** K
å=N
=

1205. **kíÜ=m~êíá~ä=pìã=**
å
på =∑~**å** =~**N** +~**O** +K+~**å=**
å=N
=

1206. **`çåîÉêÖÉåÅÉ=çÑ=fåÑáåáíÉ=pÉêáÉë=**
∞
∑
~
å
=**i I=áÑ=äáã** =
p i =
å=N å→∞**å**
=

1207. **kíÜ=qÉêã=qÉëí=**
∞
• **fÑ=íÜÉ=ëÉêáÉë=**∑~**å áë=ÅçåîÉêÖÉåíI=íÜÉå=äáã** =**MK==**
å=N å→∞**å**

• **fÑ=äáã ≠MI=íÜÉå=íÜÉ=ëÉêáÉë=áë=ÇáîÉêÖÉåíK=**

å→∞**å**

=

=

=

11.5 Properties of Convergent Series

=

∞ ∞

`çåîÉêÖÉåí=pÉêáÉëW=∑~å=^I=∑Äå=_=

å=N å=N

oÉ~ä=åìãÄÉêW=Å=

∞ ∞ ∞

1208. **()+å =∑~å +∑Äå =^+_=∑å**

å=N å=N å=N

=

∞ ∞

1209. **∑Å~å =∑~å =Å^K==**

å=N å=N

=

=

=

11.6 Convergence Tests

=

1210. qÜÉ=`çãé~êáëçå=qÉëí=

∞ ∞

iÉí=∑å~=~åÇ=∑Äå =ÄÉ=ëÉêáÉë=ëìÅÜ=íÜ~í=M<~

≤Äå=Ñçê=~ää=åK==å

å=N å=N

∞ ∞

• fÑ=∑Äå áë=ÅçåîÉêÖÉåí=íÜÉå=∑~å áë=~äëç=ÅçåîÉêÖÉåíK==

å=N å=N

∞ ∞

• fÑ=∑~å áë=ÇáîÉêÖÉåí=íÜÉå=∑Äå áë=~äëç=ÇáîÉêÖÉåíK=

å=N å=N

=

1211. qÜÉ=iáãáí=`çãé~êáëçå=qÉëí=

∞ ∞

iÉí=∑~å =~åÇ=∑Äå =ÄÉ=ëÉêáÉë=ëìÅÜ=íÜ~í=å~ =~åÇ=åÄ =~êÉ=éçëá-

å=N å=N

íáîÉ=Ñçê=~ää=åK==

~ ∞ ∞

• fÑ= äáãM=íÜÉå=∑å~=~åÇ=∑Äå ~êÉ= ÉáíÜÉê= ÄçíÜ=

å ∞ Äå å=N å=N

ÅçåîÉêÖÉåí=çê=ÄçíÜ=ÇáîÉêÖÉåíK=

∞ ∞

• fÑ=äáã~å =M=íÜÉå=∑Äå ÅçåîÉêÖÉåí=áãéäáÉë=íÜ~í=∑å~=áë=

å→∞ Äå å=N å=N

~äëç=ÅçåîÉêÖÉåíK=

•

fÑ=

äáã

~

∞ ∞

å =∞=íÜÉå=∑Äå ÇáîÉêÖÉåí=áãéäáÉë=íÜ~í=∑å~=áë=

å→∞ Äå å=N å=N

~äëç=ÇáîÉêÖÉåíK=

=

1212. **é-ëÉêáÉë=**

∞

é-ëÉêáÉë=∑åéN =ÅçåîÉêÖÉë=Ñçê=é>N=~åÇ=ÇáîÉêÖÉë=Ñçê=

å=N

M ≤NK=

=

1213. **qÜÉ=fåíÉÖê~ä=qÉëí=**

iÉí=()= ÄÉ= ~= ÑìåÅíáçå= ïÜáÅÜ= áë= ÅçåíáåìçìëI= éçëáíáîÉI= ~åÇ=

ÇÉÅêÉ~ëáåÖ=Ñçê=~ää=ñ≥NK=qÜÉ=ëÉêáÉë==

∞ () () () () ()K+Ñ å +K=∑Ñ

å=N
∞
ÅçåîÉêÖÉë=áÑ=
∫
Ñ

()

ñ Çñ ÅçåîÉêÖÉëI=~åÇ=ÇáîÉêÖÉë=áÑ= N

å () ∞ ∞K=∫ñÑ=~ë=å

N
=
1214. qÜÉ=o~íáç=qÉëí=
∞
iÉí=∑~å =ÄÉ=~=ëÉêáÉë=ïáíÜ=éçëáíáîÉ=íÉêãëK=
å=N
•
fÑ=
äáã
~
∞
å+N <N=íÜÉå=∑~å áë=ÅçåîÉêÖÉåíK=
å→∞ ~å å=N
∞
• fÑ=äáã~å+N >N=íÜÉå=∑~å =áë=ÇáîÉêÖÉåíK=
å→∞ ~å å=N
∞

• fÑ=äáã~å+N =N=íÜÉå=∑å~ =ã~ó=ÅçåîÉêÖÉ=çê=ÇáîÉêÖÉ=~åÇ=
å→∞ ~å å=N
íÜÉ= ê~íáç= íÉëí= áë= áåÅçåÅäìëáîÉX= ëçãÉ= çíÜÉê= íÉëíë= ãìëí= ÄÉ= ìëÉÇK==
=
1215. qÜÉ=oççí=qÉëí=
∞
iÉí=∑~å =ÄÉ=~=ëÉêáÉë=ïáíÜ=éçëáíáîÉ=íÉêãëK=
å=N
∞
• fÑ=äáã å ~å <N=íÜÉå=∑~å =áë=ÅçåîÉêÖÉåíK=
å→∞å=N
∞
• fÑ=äáã å ~å >N=íÜÉå=∑~å =áë=ÇáîÉêÖÉåíK=
å→∞å=N
∞
• fÑ=äáã å ~å =N=íÜÉå=∑å~ =ã~ó=ÅçåîÉêÖÉ=çê=ÇáîÉêÖÉI=Äìí=
å→∞å=N

åç=ÅçåÅäìëáçå=Å~å=ÄÉ=Çê~ïå=Ñêçã=íÜáë=íÉëíK=
=
=
=

11.7 Alternating Series

=

1216. qÜÉ=^äíÉêå~íáåÖ=pÉêáÉë=qÉëí=EiÉáÄåáò∞ë=qÜÉçêÉãF=

=

iÉí={}=ÄÉ=~=ëÉèìÉåÅÉ=çÑ=éçëáíáîÉ=åìãÄÉêë=ëìÅÜ=íÜ~í=å

~ <~å=Ñçê=~ää=åK=å+N

äáã =MK==

å→∞å

∞ ()∞

å ~N=~åÇ= ∑ ()å−N~N=qÜÉå=íÜÉ=~äíÉêå~íáåÖ=ëÉêáÉë= ∑å

å

å=N å=N

ÄçíÜ=ÅçåîÉêÖÉK====

=

1217. ^ÄëçäìíÉ=`çåîÉêÖÉåÅÉ=

∞

• ^= ëÉêáÉë=∑å~=áë= ~ÄëçäìíÉäó= ÅçåîÉêÖÉåí= áÑ= íÜÉ= ëÉêáÉë=

å=N

∞

∑~å =áë=ÅçåîÉêÖÉåíK==

å=N

∞

• fÑ=íÜÉ=ëÉêáÉë=∑~å áë=~ÄëçäìíÉäó=ÅçåîÉêÖÉåí=íÜÉå=áí=áë=Ååç-

å=N

îÉêÖÉåíK=

=

1218. `çåÇáíáçå~ä=`çåîÉêÖÉåÅÉ=

∞

^= ëÉêáÉë=∑å~áë= ÅçåÇáíáçå~ääó= ÅçåîÉêÖÉåí= áÑ= íÜÉ= ëÉêáÉë=

áë=
å=N

ÅçåîÉêÖÉåí=Äìí=áë=åçí=~ÄëçäìíÉäó=ÅçåîÉêÖÉåíK=

11.8 Power Series

=
oÉ~ä=åìãÄÉêëW=ñI=
ñ =
M

∞ ∞ () åmçïÉê=ëÉêáÉëW=∑ñ~I=å

å ∑åñ~=M
å=M å=M

tÜçäÉ=åìãÄÉêW=å=
o~Çáìë=çÑ=`çåîÉêÖÉåÅÉW=o=
=

1219. mçïÉê=pÉêáÉë=áå=ñ=
∞
∑
å ~
åñ~=M ~Nñ ~Oñ O K+~åñ å +K
å=M
=
1220. mçïÉê=pÉêáÉë=áå=
∑~å ñ −ñM ~M N
(ñ−ñM)=

∞ () ()() ()ñ ñM O ñ ñM K å ñ ñ å +

M K
å=M

=

1221. fåíÉêî~ä=çÑ=`çåîÉêÖÉåÅÉ===
qÜÉ=ëÉí=çÑ=íÜçëÉ=î~äìÉë=çÑ=ñ=Ñçê=ïÜáÅÜ=íÜÉ=ÑìåÅíáçå=

∞
Ñ

()

=

()~å ñ å M
=áë=ÅçåîÉêÖÉåí=áë=Å~ääÉÇ==íÜÉ==áåíÉêî~ä=ç
Ñ=∑

å=M
ÅçåîÉêÖÉåÅÉK=

1222. o~Çáìë=çÑ=`çåîÉêÖÉåÅÉ=
fÑ=íÜÉ=áåíÉêî~ä=çÑ=ÅçåîÉêÖÉåÅÉ=áë==(ñ_M −oIñ_M
+o)==Ñçê==ëçãÉ=
o≥MI=íÜÉ=o=áë=Å~ääÉÇ==íÜÉ=ê~Çáìë=çÑ=ÅçåîÉêÖÉåÅÉK==fí=áë
=ÖáîÉå= ~ë=

N =çê=o=äáã~å K==
å→∞
o=äáãå ~å å→∞ ~å+N

=
=
=

11.9 Differentiation and Integration of Power Series

= ( )

`çåíáåìçìë=ÑìåÅíáçåW=ñÑ=
∞
mçïÉê=ëÉêáÉëW=
∑
ñ~=å
å
å=M

tÜçäÉ=åìãÄÉêW=å=
o~Çáìë=çÑ=`çåîÉêÖÉåÅÉW=o=
=
=

1223. **aáÑÑÉêÉåíá~íáçå=çÑ=mçïÉê=pÉêáÉë=**
iÉí=

()

∞
=∑~ñÑ=Ñçê=ñå =~M +~Nñ+~OñO +K ñ<oK==å
å=M

qÜÉåI==Ñçê= ñ<oI=( )= áë= ÅçåíáåìçìëI= íÜÉ= ÇÉêáî~íáîÉ=()=

Éñáëíë=~åÇ=
′

()

=
Ç
+ **Ç**
~ñÑ=~Nñ +Ç **~Oñ^O** +KÇñ **M Çñ Çñ**
∞
=~N +**O~Oñ** +**P~Pñ^O** +K=∑**å~åñ^å−N K=**
å=N
=
=
1224. **fåíÉÖê~íáçå=çÑ=mçïÉê=pÉêáÉë=**
iÉí=

()

∞
=∑~åñ^å =~M +~Nñ+~Oñ^O +K=**Ñçê= ñ<oK==** å=M

qÜÉåI==Ñçê= ñ<oI=íÜÉ=áåÇÉÑáåáíÉ=áåíÉÖê~ä=∫ ( )Çñ

Éñáëíë=~åÇ== ∫Ñ()Çñ =∫~MÇñ+∫~NñÇñ+∫~Oñ^OÇñ +K=

ñ
Oñ P ∞**ñå+N**
=~Mñ+~N O +~O P +K=∑~å å+N+`K=
å=M

=
=

=

11.10 Taylor and Maclaurin Series

=

tÜçäÉ=åìãÄÉêW=å=

aáÑÑÉêÉåíá~ÄäÉ=ÑìåÅíáçåW=( )=

oÉã~áåÇÉê=íÉêãW=

o=

å

=

=

1225. **q~óäçê=pÉêáÉë=**

()

∞

=

∑

Ñ

()

()( ) () ()( ) ( )( )

O

Ñ +

å

>

O >

K

å=**M**

==

\+

Ñ()()( )**å**

å> +o$_å$K==

=

1226. qÜÉ=oÉã~áåÇÉê=^ÑíÉê=åHN=qÉêãë=áë=ÖáîÉå=Äó== =

Ñ

()()( )$^{å+N}$

o$_å$ ()> I===~ <ñ K=

=

1227. j~Åä~ìêáå=pÉêáÉë= Ñ

()

∞

=

∑

Ñ

()

() () () ( )

ñ

OÑ ( )()ñ^å

+K+M +o$_å$

å=M å> O> å>

=

=

=

=

11.11 Power Series Expansions for Some Functions

=
tÜçäÉ=åìãÄÉêW=å=
oÉ~ä=åìãÄÉêW=ñ=
=
=

1228.
É
ñ =N+ñ +ñO ñP ñå
O>+ P>+K+ å>K=
=
1229.
~

ñ =N+ñäå~+( ) ( )P ( )å

N
>
O
>
+
P
>
+
K
+
+
å> K= =
1230.

()

=
ñ
−

ñO ñP ñQ ( )åñå+N

O +P −Q +K+å+N ±KI=−N ≤NK= =
1231. ää N+ñ =O□ñ+ñP ñR ñT □I= ñ<NK=N−ñ □□ P +R +T +K□□
□ □
=
1232. ääñ =O□ñ−N+N□ñ−N□P N□ñ−N□R □I=ñ>MK=□□ñ+N P□□ñ+N□□ +R□□ñ+N□□
K□□ □ □ =
1233.
Åçë
ñ
=
N
−
ñ

O ñQ ñS ( )åñOå

+
Q
>
−
S
>
+
K
+
±

O() K=O> å >

1234.
ëáå
ñ
=
ñ
−

ñP ñR ñT ( )NåñOå+N

P
>
+
R
>
−
T
>
+
K
+
±

()> K=

= 1235. í~åñ=ñ +ñP OñR NTñT SOñV πK=P +NR +PNR +OUPR +KI= ñ< O
=

1236. Åçíñ=N −ñ +ñP OñR OñT I= πñK=ñ P QR+VQR +QTOR +K =
1237.
~êÅëáå
ñ
=
ñ
+

ñP N·PñR N·P·RK( )ñOå+N O·P+ O·Q·R+K+ O·Q·SK()( )+
KI=

ñ<NK=
=

1238. ~êÅÅçëñ=π –□ñ +ñP N·PñR N·P·RK( )ñOå+N □IO □□
O·P +O·Q·R +K+O·Q·SK()( ) +K□□

□ □ ñ<NK=
=
1239.
~êÅí~å
ñ
=
ñ
−

ñP ñR ñT ( )åñOå+N

P +R −T +K+Oå+N ±KI= ñ≤NK= =
1240.
ÅçëÜ
ñ
=
N
+
ñO ñQ ñS ñOå

O> +Q> +S> +K+()> + K=

=
1241.
ëáåÜ
ñ

=

ñ

+

ñ^{P} ñ^{R} ñ^{T} ñ$^{Oå+N}$

P>$^{+}$ R>$^{+}$ T>$^{+}$K+()>$^{+}$ K=

=

=

=

=

11.12 Binomial Series

=
tÜçäÉ=åìãÄÉêëW=åI=ã=
oÉ~ä=åìãÄÉêW=ñ=
`çãÄáå~íáçåëW=

å`=
ã
=
=

1242. ()å =N+å`Nñ+å`Oñ O +K+ã`åñã +K+ñå=

=
1243.
å

()K[å−( )

`]I= ñ<NK=ã = ã>
=
1244.
N
N+ñ =N−ñ +ñO −ñP +KI= ñ<NK=
=
1245.
N
N−ñ =N+ñ +ñO +ñP +KI= ñ<NK=
=
1246.
N
+
ñ
=

N
+
ñ
−
ñ
O N·Pñ[P] **N·P·Rñ**[Q]
+ **O·Q·S**−**O·Q·S·U**+K**I= ñ≤NK=O O·Q**
=
1247.
P
N
+
ñ
=
N
+
ñ
−
N
·
O
ñ

O N·O·Rñ[P] **N·O·R·Uñ**[Q]
+ **P·S·V**− **P·S·V·NO**+K**I= ñ≤NK=P P·S**
=
=
=

11.13 Fourier Series

=

fåíÉÖê~äëW=( )=

cçìêáÉê=ÅçÉÑÑáÅáÉåíëW=
~ I=~ I=Ä =
M å å
tÜçäÉ=åìãÄÉêW=å==

1248.

()()~ ∞

= +∑~å åñÑ=O å=N

=

1249.

~

å

=

N

π

∫Ñ()Åçëåñ Çñ==π−π

=

1250.

Ä

å

=

N

π

∫Ñ()ëáååñ Çñ ==π−π

=

=

Chapter 12 Probability

=
=
=
=

12.1 Permutations and Combinations

=

mÉêãìí~íáçåëW=

åm=

ã

`çãÄáå~íáçåëW=

å`=

ã

tÜçäÉ=åìãÄÉêëW=åI=ã=

=

=

1251. **c~Åíçêá~ä=**

å>=N ()( )å= K·O·P

M>=N=

=

1252. **åmå =å>=**

=

1253. **åm = å> = ()>**

=

1254. **_áåçãá~ä=`çÉÑÑáÅáÉåí=**

å`ã =□å□=ã>()>=

□

ã

□□

□

=

1255. å`ã =å `å−ã=
=
1256. å`ã +å `ã+N=å+N `ã+N=
å>

□□

1257. å`M +å`N +å`O +K+å `å =Oå=
=
1258. m~ëÅ~ä∞ë=qêá~åÖäÉ=
=
oçï=M= ====== N= ===== = oçï=N= = = = = =N==N= = = = = = oçï=O=
==== N= = O= = N= === = oçï=P= = = =N==P==P= = N= = = = oçï=Q= =
= N= = Q== S==Q==N= = = oçï=R= = N= = R= = NM= = NM= = R= =
N= = oçï=S= N= = S= = NR= = OM= = NR= = S= = N= =
=
=

12.2 Probability Formulas

=
bîÉåíëW=^I=_=
mêçÄ~ÄáäáíóW=m=
o~åÇçã=î~êá~ÄäÉëW=uI=vI=w=
s~äìÉë=çÑ=ê~åÇçã=î~êá~ÄäÉëW=ñI=óI=ò=
bñéÉÅíÉÇ=î~äìÉ=çÑ=uW=μ=
^åó=éçëáíáîÉ=êÉ~ä=åìãÄÉêW=ε==
pí~åÇ~êÇ=ÇÉîá~íáçåW=σ=
s~êá~åÅÉW=^{O} **=**

aÉåëáíó=ÑìåÅíáçåëW=( )ñÑI=( )íÑ=

=
=
1259. **mêçÄ~Äáäáíó=çÑ=~å=bîÉåí=**
m

() =^{ã} I==

^ **å**

ïÜÉêÉ==
ã=áë=íÜÉ=åìãÄÉê=çÑ=éçëëáÄäÉ=éçëáíáîÉ=çìíÅçãÉëI==
å=áë=íÜÉ=íçí~ä=åìãÄÉê=çÑ=éçëëáÄäÉ=çìíÅçãÉëK=
1260. **o~åÖÉ=çÑ=mêçÄ~Äáäáíó=s~äìÉë=**

M≤m()≤N=

=

1261. **`Éêí~áå=bîÉåí=**

m

( ) =

^ **N=**

=

1262. **fãéçëëáÄäÉ=bîÉåí=**

m( ) =M=

=

1263. **`çãéäÉãÉåí=**

m( ) ()=

=

1264. **fåÇÉéÉåÇÉåí=bîÉåíë=**

m( )()I==

m( )()=

=

1265. ^ÇÇáíáçå=oìäÉ=Ñçê=fåÇÉéÉåÇÉåí=bîÉåíë=

m( )() ( )=

=

1266. jìäíáéäáÅ~íáçå=oìäÉ=Ñçê=fåÇÉéÉåÇÉåí=bîÉåíë=

m( )() ( )=

=

1267. dÉåÉê~ä=^ÇÇáíáçå=oìäÉ=

m( )() ( ) (^∩_)I==

ïÜÉêÉ==
^∪_ =áë=íÜÉ=ìåáçå=çÑ=ÉîÉåíë=^=~åÇ=_I== ^∩_
=áë=íÜÉ=áåíÉêëÉÅíáçå=çÑ=ÉîÉåíë=^=~åÇ=_K= =

1268. `çåÇáíáçå~ä=mêçÄ~Äáäáíó=

m()()=
m()

=

1269. **()() ( ) ( ) ( )=**

1270. **i~ï=çÑ=qçí~ä=mêçÄ~Äáäáíó=**
ã
m
^

() ( )( )

= m _á má I==∑

á=N
ïÜÉêÉ=á_ =áë=~=ëÉèìÉåÅÉ=çÑ=ãìíì~ääó=ÉñÅäìëáîÉ=ÉîÉåíëK== =
1271. _~óÉë∞=qÜÉçêÉã=

m_L^()() ( )=m()

=
1272. _~óÉë∞=cçêãìä~=

() () ( )

á
L
^
_mI==

ã () ( )á∑m _á

â=N

ïÜÉêÉ==
á_ =áë=~=ëÉí=çÑ=ãìíì~ääó=ÉñÅäìëáîÉ=ÉîÉåíë=EÜóéçíÜÉëÉëFI=
^ =áë=íÜÉ=Ñáå~ä=ÉîÉåíI==
m

( )

_ =~êÉ=íÜÉ=éêáçê=éêçÄ~ÄáäáíáÉëI=á

m( )=~êÉ=íÜÉ=éçëíÉêáçê=éêçÄ~ÄáäáíáÉëK=á

=

1273. i~ï=çÑ=i~êÖÉ=kìãÄÉêë=

m□ på −μ ≥ε□→M=~ë= ∞I==□□ å □□

m□ på −μ <ε□→N=~ë= ∞I==□□ å □□

ïÜÉêÉ==

åp =áë=íÜÉ=ëìã=çÑ=ê~åÇçã=î~êá~ÄäÉëI=

å=áë=íÜÉ=åìãÄÉê=çÑ=éçëëáÄäÉ=çìíÅçãÉëK=

=

1274. `ÜÉÄóëÜÉî=fåÉèì~äáíó=

m

()

( )

≤ I==
ïÜÉêÉ=

()

εO
=áë=íÜÉ=î~êá~åÅÉ=çÑ=uK=
1275. kçêã~ä=aÉåëáíó=cìåÅíáçå= N
É
−
()O
ϕ

()

=
OσO

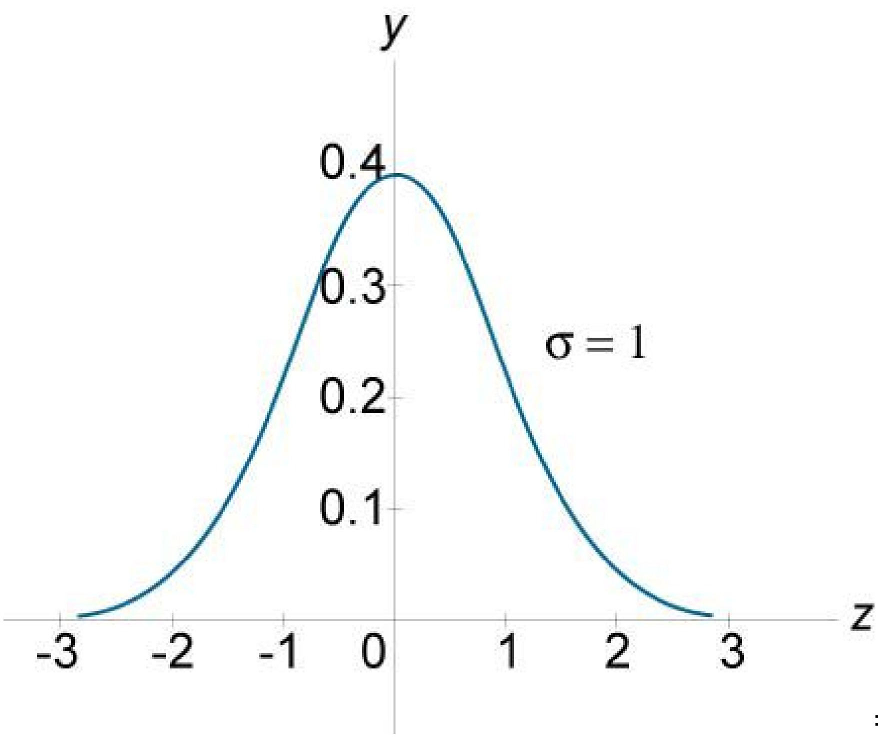
σ OπI== ïÜÉêÉ=ñ=áë=~=é~êíáÅìä~ê=çìíÅçãÉK= =

1276. pí~åÇ~êÇ=kçêã~ä=aÉåëáíó=cìåÅíáçå=

()

=
N
É
−
ò O
ϕO
ò Oπ=
^îÉê~ÖÉ=î~äìÉ=μ =MI=ÇÉîá~íáçå=σ =NK= =

= =====



= Figure 210.

=

1277. **pí~åÇ~êÇ=w=s~äìÉ=**

uw=−µ

σ

=

1278. **`ìãìä~íáîÉ=kçêã~ä=aáëíêáÄìíáçå=cìåÅíáçå=**

cñ ÉOσO ÇíI==

−

()

=

N ñ −()**O**

σ **O**π∫∞

ïÜÉêÉ==

ñ=áë=~=é~êíáÅìä~ê=çìíÅçãÉI==

í =áë=~=î~êá~ÄäÉ=çÑ=áåíÉÖê~íáçåK=

=

1279.

()

□α−μ□ □β−μ□

umI=
□□ σ □□ □□ σ □□
ïÜÉêÉ=
u=áë=åçêã~ääó=ÇáëíêáÄìíÉÇ=ê~åÇçã=î~êá~ÄäÉI=
c=áë=Åìãìä~íáîÉ=åçêã~ä=ÇáëíêáÄìíáçå=ÑìåÅíáçåI== m(α<**u**<
)**=áë=áåíÉêî~ä=éêçÄ~ÄáäáíóK=**

=
1280.

()

□ε □

umI==
□□σ □□
ïÜÉêÉ==
u=áë=åçêã~ääó=ÇáëíêáÄìíÉÇ=ê~åÇçã=î~êá~ÄäÉI=
c=áë=Åìãìä~íáîÉ=åçêã~ä=ÇáëíêáÄìíáçå=ÑìåÅíáçåK= =

1281. `ìãìä~íáîÉ=aáëíêáÄìíáçå=cìåÅíáçå=
ñ

() ( ) ()

ñcI== Ñ í Çí
−
∫∞
ïÜÉêÉ=í=áë=~=î~êá~ÄäÉ=çÑ=áåíÉÖê~íáçåK=
=

1282. _Éêåçìääá=qêá~äë=mêçÅÉëë=
μ =åé= I=σ^O=åéè I==
ïÜÉêÉ==
å =áë=~=ëÉèìÉåÅÉ=çÑ=ÉñéÉêáãÉåíëI==
é =áë=íÜÉ=éêçÄ~Äáäáíó=çÑ=ëìÅÅÉëë=çÑ=É~ÅÜ=ÉñéÉêáãÉåíI=

è =áë=íÜÉ=éêçÄ~Äáäáíó=çÑ=Ñ~áäìêÉI=
è
−
N éK= =
1283. _áåçãá~ä=aáëíêáÄìíáçå=cìåÅíáçå=======

Ä□□()=□å□éâèå−âI==

â

μ =åéI=σ^{O}=åéè I=

ñ()()^{å}I==

ïÜÉêÉ==
å=áë=íÜÉ=åìãÄÉê=çÑ=íêá~äë=çÑ=ëÉäÉÅíáçåëI=
é=áë=íÜÉ=éêçÄ~Äáäáíó=çÑ=ëìÅÅÉëëI=
è=áë=íÜÉ=éêçÄ~Äáäáíó=çÑ=Ñ~áäìêÉI=è −é K=
=

1284. dÉçãÉíêáÅ=aáëíêáÄìíáçå=

m( )=è$^{à−N}$éI==

μ =^{N} I=σ^{O} =^{è} I==$_{é\ éO}$

ïÜÉêÉ==
q=áë=íÜÉ=Ñáêëí=ëìÅÅÉëëÑìä=ÉîÉåí=áë=íÜÉ=ëÉêáÉëI=
à=áë=íÜÉ=ÉîÉåí=åìãÄÉêI=
é=áë=íÜÉ=éêçÄ~Äáäáíó=íÜ~í=~åó=çåÉ=ÉîÉåí=áë=ëìÅÅÉëëÑìäI==
è=áë=íÜÉ=éêçÄ~Äáäáíó=çÑ=Ñ~áäìêÉI=è −é K=
=

1285. mçáëëçå=aáëíêáÄìíáçå=
m
u

()

â
≈

λ^â^

â> É^−λ^I=λ=åéI==

µ =λI=^O^ λ I==

ïÜÉêÉ==

λ=áë=íÜÉ=ê~íÉ=çÑ=çÅÅìêêÉåÅÉI=

â=áë=íÜÉ=åìãÄÉê=çÑ=éçëáíáîÉ=çìíÅçãÉëK=

=

1286. aÉåëáíó=cìåÅíáçå==

Ä

()()

~m==∫Ñ ñ Çñ

~

=

1287. `çåíáåìçìë=råáÑçêã=aÉåëáíó=

Ñ = N I=µ= ~+ÄI== −~ O

ïÜÉêÉ=Ñ=áë=íÜÉ=ÇÉåëáíó=ÑìåÅíáçåK=

=

1288. bñéçåÉåíá~ä=aÉåëáíó=cìåÅíáçå=

Ñ()=λÉ^−λí^I= λI=σ^O^ λ^O^=

ïÜÉêÉ=í=áë=íáãÉI=λ=áë=íÜÉ=Ñ~áäìêÉ=ê~íÉK=

=

1289. bñéçåÉåíá~ä=aáëíêáÄìíáçå=cìåÅíáçå=

c

()=N−É^−λí^

í I==

ïÜÉêÉ=í=áë=íáãÉI=λ=áë=íÜÉ=Ñ~áäìêÉ=ê~íÉK=

=

1290. **bñéÉÅíÉÇ=s~äìÉ=çÑ=aáëÅêÉíÉ=o~åÇçã=s~êá~ÄäÉë=** µ
=

()

å
∑
ñ
ubI==
áéá
á=**N**
ïÜÉêÉ=ññ =áë=~=é~êíáÅìä~ê=çìíÅçãÉI=é =áë=áíë=éêçÄ~ÄáäáíóK= =
1291. **bñéÉÅíÉÇ=s~äìÉ=çÑ=`çåíáåìçìë=o~åÇçã=s~êá~ÄäÉë=** ∞

() ()

µ =**ub==**

∫

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1292. **mêçéÉêíáÉë=çÑ=bñéÉÅí~íáçåë=**

b( )() ( )I== b( )() ( )I=

b()()I==

b( )() ( )I==

ïÜÉêÉ=Å=áë=~=Åçåëí~åíK=
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1293. ( ) ()+µ^{O}I==

ïÜÉêÉ==

µ =b u()=áë=íÜÉ=ÉñéÉÅíÉÇ=î~äìÉI===

s()=áë=íÜÉ=î~êá~åÅÉK=

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=
1294. j~êâçî=fåÉèì~äáíó=

m() ()I==

â
ïÜÉêÉ=â=áë=ëçãÉ=Åçåëí~åíK=
=

1295. s~êá~åÅÉ=çÑ=aáëÅêÉíÉ=o~åÇçã=s~êá~ÄäÉë=
σ
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ïÜÉêÉ==
$_{á}$ñ =áë=~=é~êíáÅìä~ê=çìíÅçãÉI=
$_{á}$é =áë=áíë=éêçÄ~ÄáäáíóK=

=
1296. s~êá~åÅÉ=çÑ=`çåíáåìçìë=o~åÇçã=s~êá~ÄäÉë= σ
O
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s

() ( )

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∞

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1297. mêçéÉêíáÉë=çÑ=s~êá~åÅÉ=

s()() ( )I==

s()() ( )I= s()()I=

s() ()I=

ïÜÉêÉ=Å=áë=~=ÅçåëíK=

=

1298. pí~åÇ~êÇ=aÉîá~íáçå=

a() ( ) ()[u−μ^{O}]=

=

1299. `çî~êá~åÅÉ=

Åçî()=b)[] ( ) ( ) ( )I==

ïÜÉêÉ==
u =áë=ê~åÇçã=î~êá~ÄäÉI==

s()=áë=íÜÉ=î~êá~åÅÉ=çÑ=uI==

µ**=áë=íÜÉ=ÉñéÉÅíÉÇ=î~äìÉ=çÑ=u=çê=vK=**
=
1300. `çêêÉä~íáçå=

ρ() = Åçî( ) I==s()()

ïÜÉêÉ==

s()=áë=íÜÉ=î~êá~åÅÉ=çÑ=uI==

s()=áë=íÜÉ=î~êá~åÅÉ=çÑ=vK=

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iççâ=Ñçê=çíÜÉê=Ü~åÇÄççâë=~åÇ=ëçäîÉÇ=éêçÄäÉã=ÖìáÇÉë=~í=
ïïïKã~íÜ-ÉÄççâëKÅçãK=

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