CONTENTS

Using *McGraw-Hill Ryerson Calculus & Advanced Functions, Solutions*

McGraw-Hill Ryerson Calculus & Advanced Functions, *Solutions* provides complete model solutions to the following:
for each numbered section of *McGraw-Hill Ryerson Calculus & Advanced Functions*,
- every odd numbered question in the *Practise*
- all questions in the *Apply, Solve, Communicate*

Solutions are also included for all questions in these sections:
- Review
- Chapter Check
- Problem Solving: Using the Strategies

Note that solutions to the Achievement Check questions are provided in *McGraw-Hill Ryerson Calculus & Advanced Functions*, Teacher's Resource.

Teachers will find the completeness of the *McGraw-Hill Ryerson Calculus & Advanced Functions*, *Solutions* helpful in planning students' assignments. Depending on their level of ability, the time available, and local curriculum constraints, students will probably only be able to work on a selection of the questions presented in the *McGraw-Hill Ryerson Calculus & Advanced Functions* student text. A review of the solutions provides a valuable tool in deciding which problems might be better choices in any particular situation. The solutions will also be helpful in deciding which questions might be suitable for extra practice of a particular skill.

In mathematics, for all but the most routine of practice questions, multiple solutions exist. The methods used in *McGraw-Hill Ryerson Calculus & Advanced Functions*, *Solutions* are generally modelled after the examples presented in the student text. Although only one solution is presented per question, teachers and students are encouraged to develop as many different solutions as possible. An example of such a question is Page 30, Question 7, parts b) and c). The approximate values can be found by substitution as shown or by using the Value operation on the graphing calculator. Discussion and comparison of different methods can be highly rewarding. It leads to a deeper understanding of the concepts involved in solving the problem and to a greater appreciation of the many connections among topics.

Occasionally different approaches are used. This is done deliberately to enrich and extend the reader's insight or to emphasize a particular concept. In such cases, the foundations of the approach are supplied. Also, in a few situations, a symbol that might be new to the students is introduced. For example in Chapter 3 the dot symbol is used for multiplication. When a graphing calculator is used, there are often multiple ways of obtaining the required solution. The solutions provided here sometimes use different operations than the one shown in the book. This will help to broaden students' skills with the calculator.

There are numerous complex numerical expressions that are evaluated in a single step. The solutions are developed with the understanding that the reader has access to a scientific calculator, and one has been used to achieve the result. Despite access to calculators, numerous problems offer irresistible challenges to develop their solution in a manner that avoids the need for one, through the order in which algebraic simplifications are performed. Such challenges should be encouraged.

There are a number of situations, particularly in the solutions to Practise questions, where the reader may sense a repetition in the style of presentation. The solutions were developed with an understanding that a solution may, from time to time, be viewed in isolation and as such might require the full treatment.

The entire body of *McGraw-Hill Ryerson Calculus & Advanced Functions*, *Solutions* was created on a home computer in Te$_x$tures. Graphics for the solutions were created with the help of a variety of graphing software, spreadsheets, and graphing calculator output captured to the computer. Some of the traditional elements of the accompanying graphic support are missing in favour of the rapid capabilities provided by the electronic tools. Since many students will be working with such tools in their future careers, discussion of the features and interpretation of these various graphs and tables is encouraged and will provide a very worthwhile learning experience. Some solutions include a reference to a web site from which data was obtained. Due to the dynamic nature of the Internet, it cannot be guaranteed that these sites are still operational.

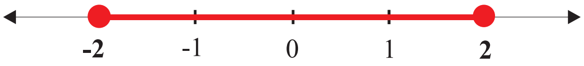
CHAPTER 1 Functions and Models

1.1 Functions and Their Use in Modelling

Practise

Section 1.1 Page 18 Question 1

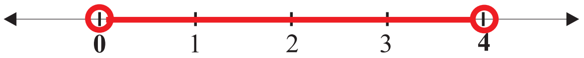
a) $x \in [-2, 2]$



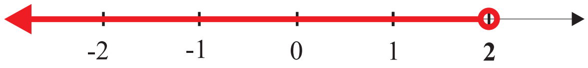
b) $x \in [4, 13]$

c) $x \in (-4, -1)$

d) $x \in (0, 4)$

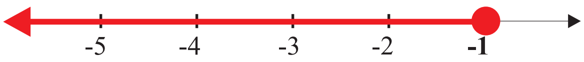


e) $x \in (-\infty, 2)$

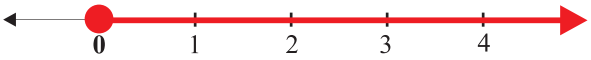


f) $x \in (-1, \infty)$

g) $x \in (-\infty, -1]$



h) $x \in [0, \infty)$

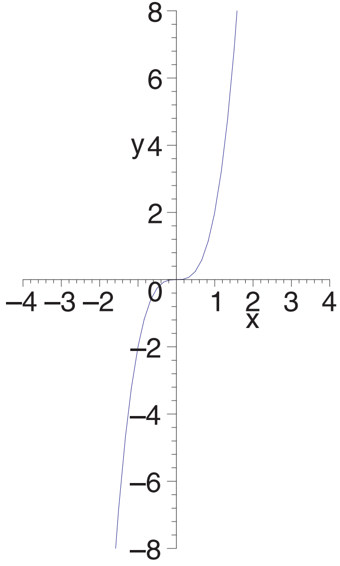


Section 1.1 Page 18 Question 3

a)

$$\begin{aligned} f(-x) &= 2(-x)^3 \\ &= -2x^3 \\ &= -f(x) \end{aligned}$$

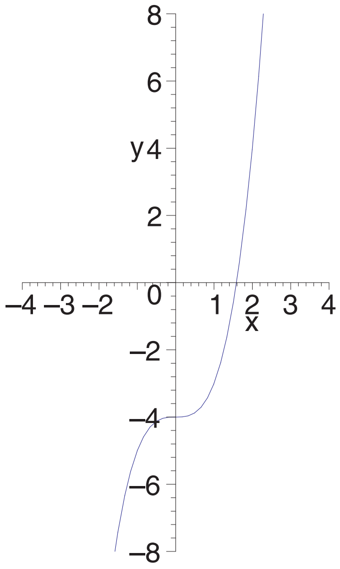
$f(x) = 2x^3$ is odd.



b)

$$\begin{aligned} g(-x) &= (-x)^3 - 4 \\ &= -x^3 - 4 \\ &\neq g(x) \text{ or } -g(x) \end{aligned}$$

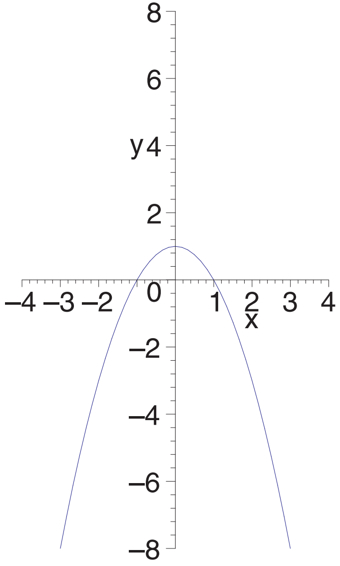
$g(x) = -x^3 - 4$ is neither even nor odd.



c)

$$\begin{aligned} h(-x) &= 1 - (-x)^2 \\ &= 1 - x^2 \\ &= h(x) \end{aligned}$$

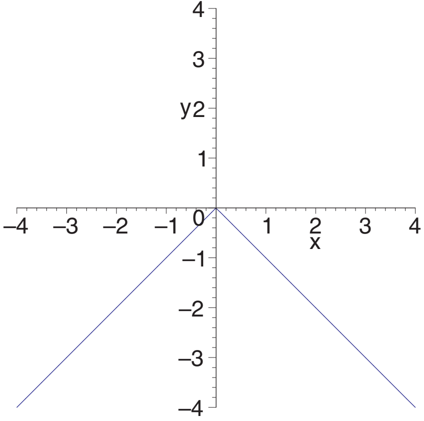
$h(x) = 1 - x^2$ is even.



Section 1.1 Page 19 Question 5

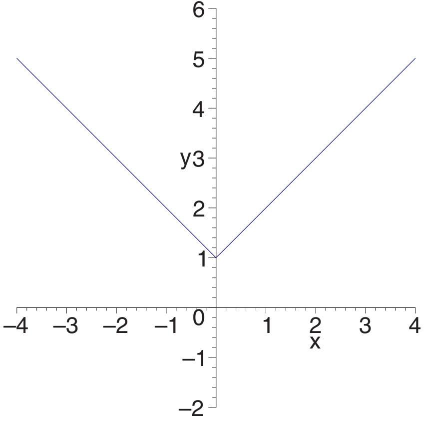
a)
$$\begin{aligned} f(-x) &= -\lvert -x\rvert \\ &= -\lvert x\rvert \\ &= f(x) \end{aligned}$$

The function $f(x) = -\lvert x\rvert$ is even.



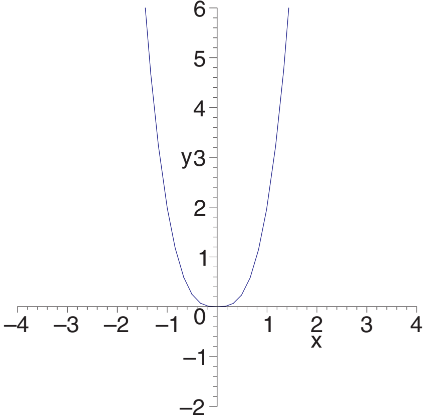
b)
$$\begin{aligned} g(-x) &= \lvert -x\rvert + 1 \\ &= \lvert x\rvert + 1 \\ &= g(x) \end{aligned}$$

The function $g(x) = \lvert x\rvert + 1$ is even.



c)
$$\begin{aligned} h(-x) &= \left\lvert 2(-x)^3 \right\rvert \\ &= \left\lvert 2x^3 \right\rvert \\ &= h(x) \end{aligned}$$

The function $h(x) = \left\lvert 2x^3 \right\rvert$ is even.



Section 1.1 Page 19 Question 7

a) Use $f(x) = \dfrac{1}{x^2}$.

i)
$$\begin{aligned} f(3) &= \frac{1}{3^2} \\ &= \frac{1}{9} \end{aligned}$$

ii)
$$\begin{aligned} f(-3) &= \frac{1}{(-3)^2} \\ &= \frac{1}{9} \end{aligned}$$

iii)
$$\begin{aligned} f\left(\frac{1}{3}\right) &= \frac{1}{\left(\frac{1}{3}\right)^2} \\ &= \frac{1}{\frac{1}{9}} \\ &= 9 \end{aligned}$$

iv)
$$\begin{aligned} f\left(\frac{1}{4}\right) &= \frac{1}{\left(\frac{1}{4}\right)^2} \\ &= \frac{1}{\frac{1}{16}} \\ &= 16 \end{aligned}$$

v)
$$\begin{aligned} f\left(\frac{1}{k}\right) &= \frac{1}{\left(\frac{1}{k}\right)^2} \\ &= \frac{1}{\frac{1}{k^2}} \\ &= k^2 \end{aligned}$$

vi)
$$\begin{aligned} f\left(\frac{k}{1+k}\right) &= \frac{1}{\left(\frac{k}{1+k}\right)^2} \\ &= \frac{1}{\frac{k^2}{(1+k)^2}} \\ &= \frac{(1+k)^2}{k^2} \end{aligned}$$

b) Use $f(x) = \dfrac{x}{1-x}$.

i)
$$\begin{aligned} f(3) &= \frac{3}{1-3} \\ &= -\frac{3}{2} \end{aligned}$$

ii)
$$\begin{aligned} f(-3) &= \frac{-3}{1-(-3)} \\ &= -\frac{3}{4} \end{aligned}$$

iii)
$$\begin{aligned} f\left(\frac{1}{3}\right) &= \frac{\frac{1}{3}}{1-\frac{1}{3}} \\ &= \frac{\frac{1}{3}}{\frac{2}{3}} \\ &= \frac{1}{2} \end{aligned}$$

iv)
$$\begin{aligned} f\left(\frac{1}{4}\right) &= \frac{\frac{1}{4}}{1-\frac{1}{4}} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{1}{3} \end{aligned}$$

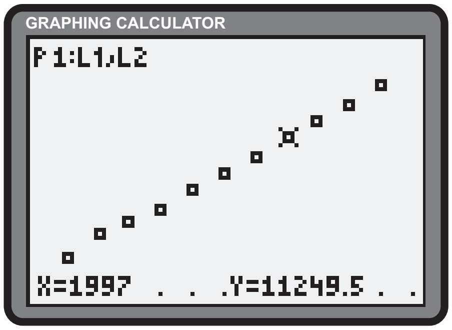
v)
$$\begin{aligned} f\left(\frac{1}{k}\right) &= \frac{\frac{1}{k}}{1-\frac{1}{k}} \\ &= \frac{\frac{1}{k}}{\frac{k-1}{k}} \\ &= \frac{1}{k-1} \end{aligned}$$

vi)
$$\begin{aligned} f\left(\frac{k}{1+k}\right) &= \frac{\frac{k}{1+k}}{1-\frac{k}{1+k}} \\ &= \frac{\frac{k}{1+k}}{\frac{(1+k)-k}{1+k}} \\ &= k \end{aligned}$$

Section 1.1 Page 19 Question 9

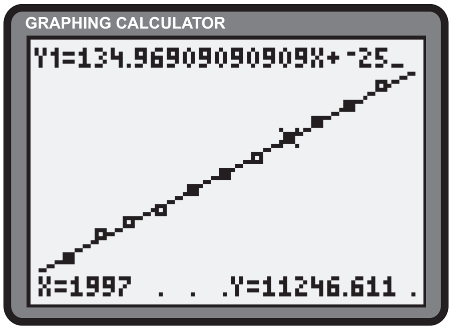
Verbally: Answers will vary.

Visual representation with a scatter plot:



```
P1:L1,L2
X=1997  .  .  .Y=11249.5  .  .
```

Algebraic representation using linear regression:

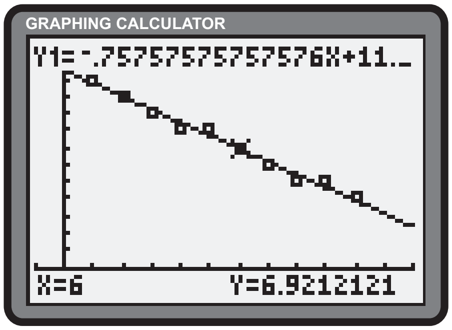


```
Y1=134.96909090909X+⁻25_
X=1997  .  .  .Y=11246.611  .
```

The graphing calculator suggests the function $f(x) \doteq 134.9690x - 258\ 290$.

Section 1.1 Page 19 Question 11

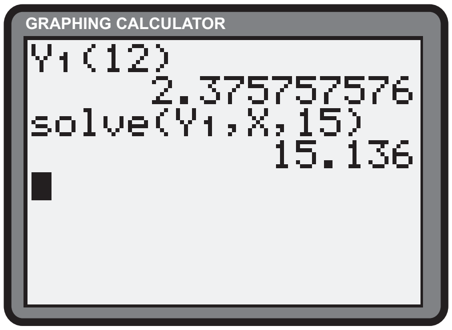
a) The calculator suggests $y = -0.\overline{75}x + 11.4\overline{6}$ as the line of best fit.



```
Y1=⁻.75757575757576X+11._
X=6          Y=6.9212121
```

b) $y(12) \doteq 2.3\overline{75}$

c) When $y = 0$, $x \doteq 15.136$.



```
Y1(12)
         2.375757576
solve(Y1,X,15)
              15.136
```

Apply, Solve, Communicate

Section 1.1 Page 20 Question 13

Consider the function $h(x)$, where

$$h(x) = f(x)g(x)$$

If $f(x)$ and $g(x)$ are odd functions, then

$$\begin{aligned} h(-x) &= f(-x)g(-x) \\ &= -f(x)[-g(x)] \\ &= f(x)g(x) \\ &= h(x) \end{aligned}$$

The product of two odd functions is an even function.

Section 1.1 Page 20 Question 14

Consider the function $h(x)$, where

$$h(x) = \frac{f(x)}{g(x)}$$

If $f(x)$ and $g(x)$ are even functions, then

$$\begin{aligned} h(-x) &= \frac{f(-x)}{g(-x)} \\ &= \frac{f(x)}{g(x)} \\ &= h(x) \end{aligned}$$

The quotient of two even functions is an even function.

Section 1.1 Page 20 Question 15

The product of an odd function and an even function is an odd function. Consider $h(x)$, where

$$h(x) = f(x)g(x)$$

If $f(x)$ is odd and $g(x)$ is even, then

$$\begin{aligned} h(-x) &= f(-x)g(-x) \\ &= -f(x)g(x) \\ &= -h(x) \end{aligned}$$

Section 1.1 Page 20 Question 16

a) Since $y = f(x)$ is odd, $f(-x) = -f(x)$ for all x in the domain of f. Given that 0 is in the domain of f, we have $f(-0) = -f(0) \Rightarrow f(0) = -f(0) \Rightarrow 2f(0) = 0 \Rightarrow f(0) = 0$.

b) An odd function for which $f(0) \neq 0$ is $f(x) = \dfrac{|x|}{x}$

Section 1.1 Page 20 Question 17

a) Let V be the value, in dollars, of the computer equipment after t years. From the given information, the points $(t, V) = (0, 18\,000)$ and $(t, V) = (4, 9000)$ are on the linear function. The slope of the line is $m = \dfrac{9000 - 18\,000}{4 - 0}$ or -2250. The linear model can be defined by the function $V(t) = 18\,000 - 2250t$.

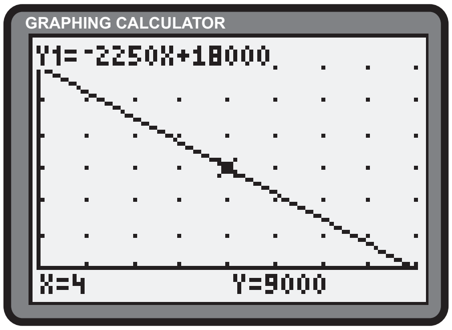
b) $V(6) = 18\,000 - 2250(6)$ or \$4500

c) The domain of the function in the model is $t \in [0, 8]$. (After 8 years the equipment is worthless.)

d) The slope represents the annual depreciation of the computer equipment.

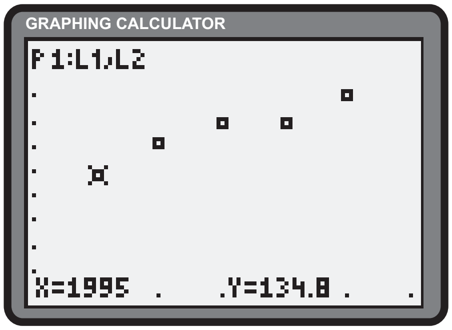
f) Since the function is given to be linear, its slope does not change.

e)



Section 1.1 Page 20 Question 18

a)



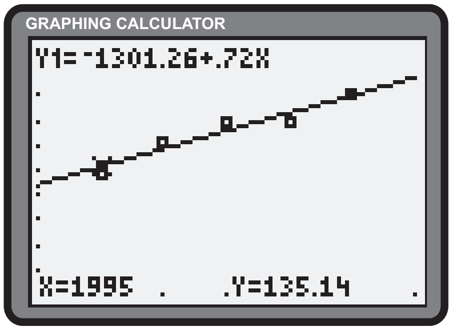
c) The model suggests the population reaches 145 940 in the year 2010.

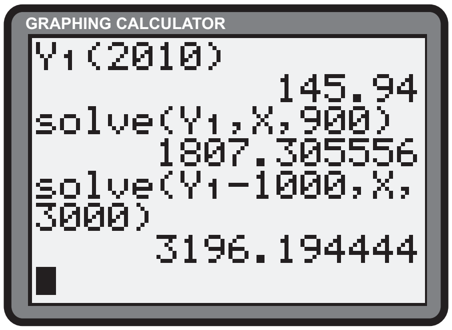
d) The model suggests the population was 0 in 1807. No.

e) The model suggests the population will reach 1 000 000 in the year 3196. Answers will vary.

f) No. Explanations will vary.

b) The line of best fit is $y = 0.72x - 1301.26$.

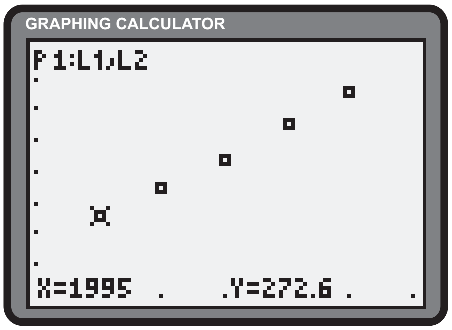




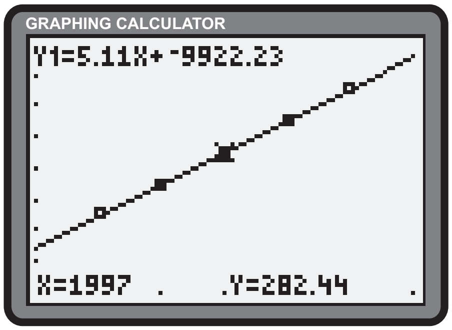
```
Y₁(2010)
          145.94
solve(Y₁,X,900)
     1807.305556
solve(Y₁-1000,X,
3000)
     3196.194444
```

Section 1.1 Page 20 Question 19

a)



b) The line of best fit is $y = 5.11x - 9922.23$.

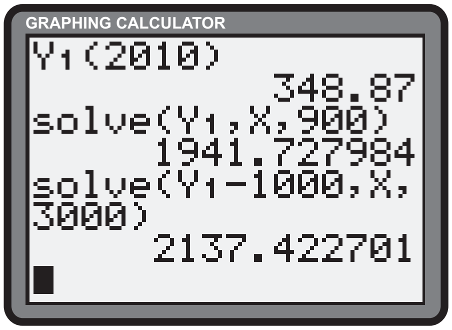


c) The model suggests the population will reach 348 870 in the year 2010.

d) The model suggests the population was 0 in 1941.

e) The model suggests the population will reach 1 000 000 in the year 2137. No.

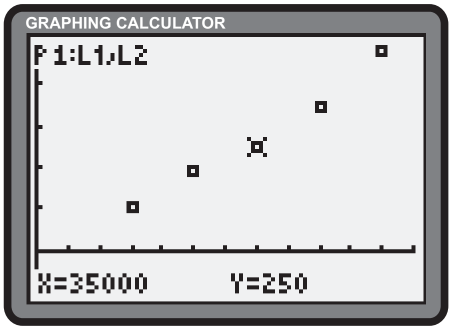
f) Explanations will vary.



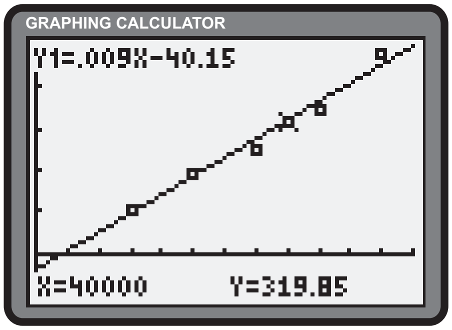
```
Y₁(2010)
              348.87
solve(Y₁,X,900)
         1941.727984
solve(Y₁-1000,X,
3000)
         2137.422701
```

Section 1.1 Page 20 Question 20

a)



b) The line of best fit is $y = 0.009x - 40.15$.



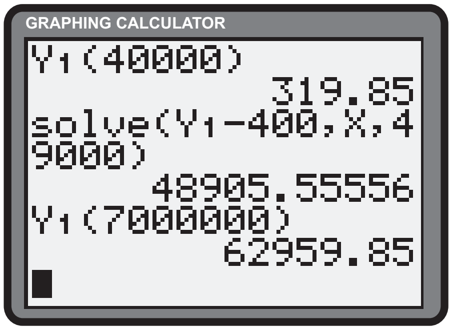
c) Estimates will vary. The model suggests annual pet expenses of approximately \$319.85.

d) Estimates will vary. The model suggests an annual income of approximately \$48 905.56 results in annual pet expenses of \$400.

e) The model predicts pet expenses of approximately \$62 959.85. Explanations will vary.

f) The model suggests an annual income of \$ – 40.15. Explanations will vary.

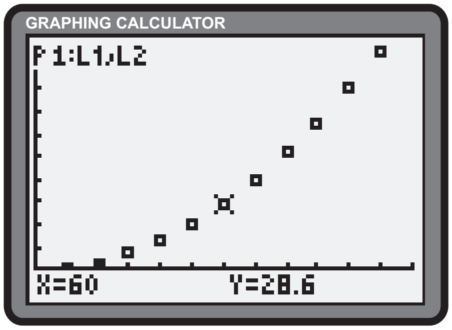
g) No. Explanations will vary.



```
Y₁(40000)
              319.85
solve(Y₁-400,X,4
9000)
         48905.55556
Y₁(7000000)
            62959.85
```

Section 1.1 Page 20 Question 21

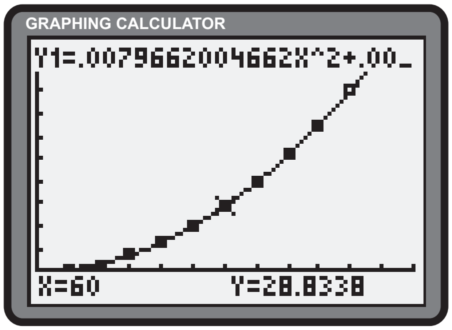
a)



c) As speed increases the slope of the curve increases.

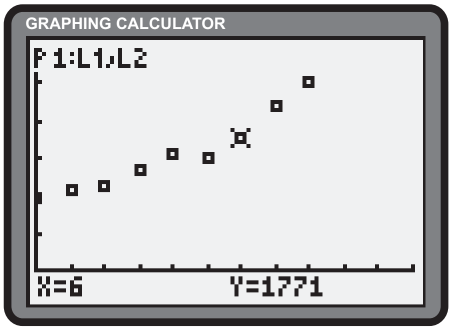
d) Answers will vary. This curve is steeper, and its slope increases more quickly.

b) Answers will vary. Using the quadratic regression feature of the calculator yields an approximation of $d = 0.008s^2 + 0.002s + 0.059$.



Section 1.1 Page 20 Question 22

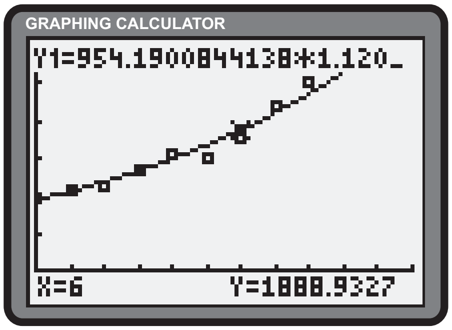
a) A scatter plot of the data appears below.

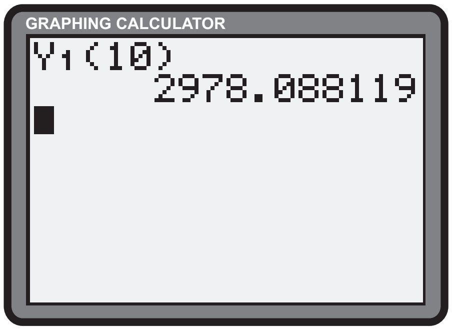


c) The slope increases with time.

d) The model predicts the value of the investment after 10 years to be approximately \$2978.09.

b) The **ExpReg feature** of the calculator approximates the data with $V = 954.19(1.12)^t$.





```
Y₁(10)
           2978.088119
```

Section 1.1 Page 20 Question 23

All constant functions are even functions. The constant function $f(x) = 0$ is both even and odd.

Section 1.1 Page 20 Question 24

The sum of an odd function and an even function can be neither odd nor even, unless one of the functions is $y = 0$. Let $f(x)$ be even and $g(x)$ be odd, and let $h(x) = f(x) + g(x)$.

If $h(x)$ is even, then

$$\begin{aligned} h(-x) &= h(x) \\ f(-x) + g(-x) &= f(x) + g(x) \\ f(x) - g(x) &= f(x) + g(x) \\ 2g(x) &= 0 \\ g(x) &= 0 \end{aligned}$$

If $h(x)$ is odd, then

$$\begin{aligned} h(-x) &= -h(x) \\ f(-x) + g(-x) &= -f(x) - g(x) \\ f(x) - g(x) &= -f(x) - g(x) \\ 2f(x) &= 0 \\ f(x) &= 0 \end{aligned}$$

Section 1.1 Page 20 Question 25

Yes; only the function $f(x) = 0$. If a function $f(x)$ is both even and odd, then

$$\begin{aligned} f(-x) &= f(x) \\ f(-x) &= -f(x) \end{aligned}$$

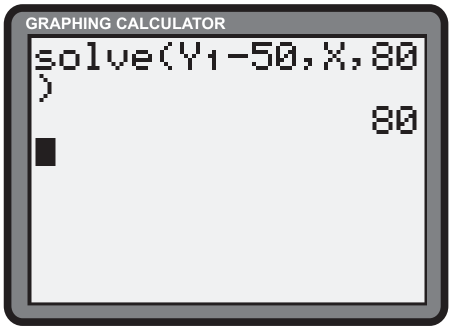
Thus,

$$\begin{aligned} f(x) &= -f(x) \\ 2f(x) &= 0 \\ f(x) &= 0 \end{aligned}$$

Section 1.1 Page 20 Question 26

a) A possible domain is $x \in [0, 100)$. Explanations may vary.

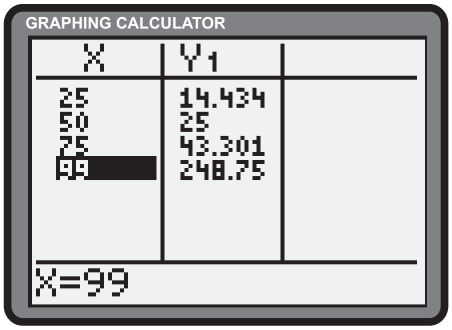
c) The calculator confirms that an estimate of 80% of pollutant can be removed for \$50 000.



GRAPHING CALCULATOR

```
solve(Y₁-50,X,80
)
                80
```

b) Define $y_1 = \dfrac{25\sqrt{x}}{\sqrt{100 - x}}$. The costs of removal for the percent given appear in the table below as \$14 434, \$25 000, \$43 301 and \$248 750.

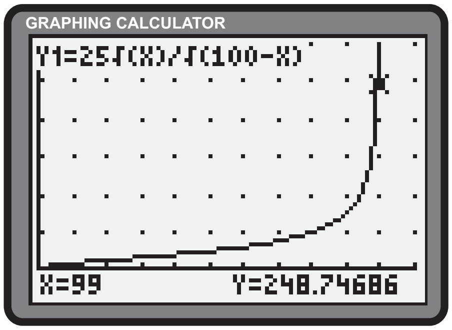


GRAPHING CALCULATOR

X	Y₁	
25	14.434	
50	25	
75	43.301	
99	248.75	

X=99

d) Since an attempt to evaluate $C(100)$ results in division by zero, the model suggests that no amount of money will remove all the pollutant.

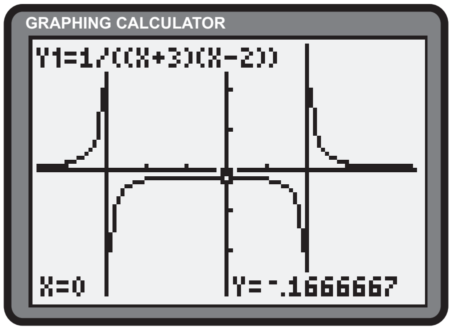


1.2 Lies My Graphing Calculator Tells Me

Apply, Solve, Communicate

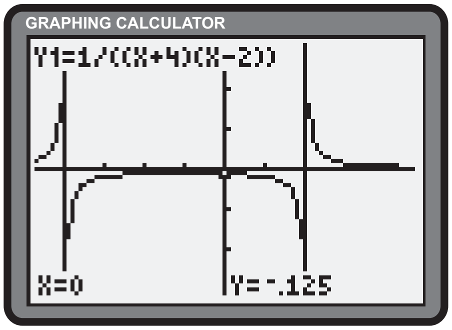
Section 1.2 Page 28 Question 1

a) The equations of the two vertical asymptotes are $x = -3$ and $x = 2$. There are no x-intercepts. The y-intercept is $-\frac{1}{6}$.



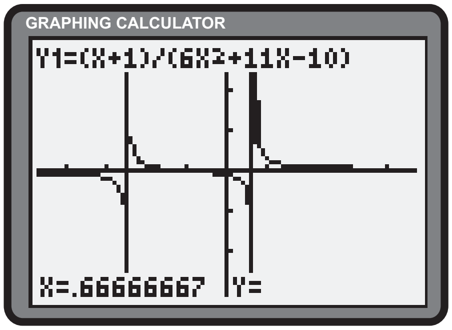
```
Y1=1/((X+3)(X-2))
X=0          Y=⁻.1666667
```

b) Factoring gives $y = \frac{1}{(x+4)(x-2)}$. The equations of the two vertical asymptotes are $x = -4$ and $x = 2$. There are no x-intercepts. The y-intercept is -0.125 or $-\frac{1}{8}$.



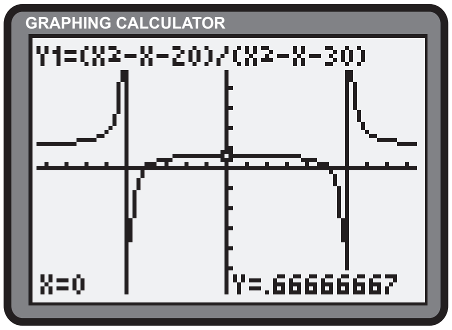
```
Y1=1/((X+4)(X-2))
X=0          Y=⁻.125
```

c) Factoring gives $y = \frac{x+1}{(2x+5)(3x-2)}$. The equations of the two vertical asymptotes are $x = -\frac{5}{2}$ and $x = \frac{2}{3}$. Setting $y = 0$ yields an x-intercept of -1. Setting $x = 0$ yields a y-intercept of $-\frac{1}{10}$.



```
Y1=(X+1)/(6X²+11X-10)
X=.66666667  Y=
```

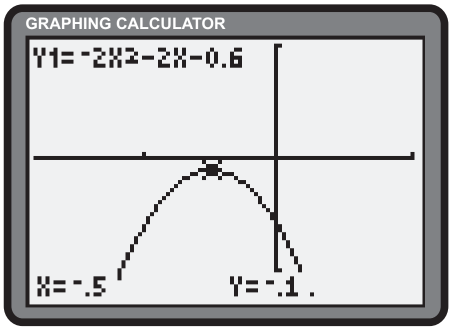
d) Factoring the numerator and denominator gives $y = \frac{(x-5)(x+4)}{(x-6)(x+5)}$. The equations of the two vertical asymptotes are $x = -5$ and $x = 6$. The x-intercepts are -4 and 5. The y-intercept is $\frac{2}{3}$.



```
Y1=(X²-X-20)/(X²-X-30)
X=0          Y=.66666667
```

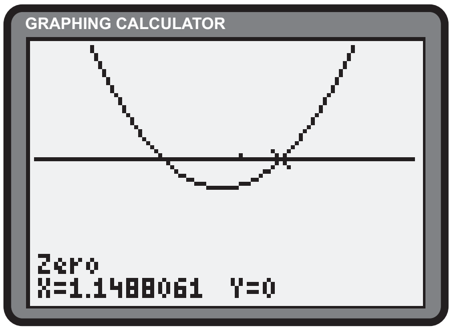
Section 1.2 Page 28 Question 2

a) $y = -2x^2 - 2x - 0.6$ has no x-intercepts.



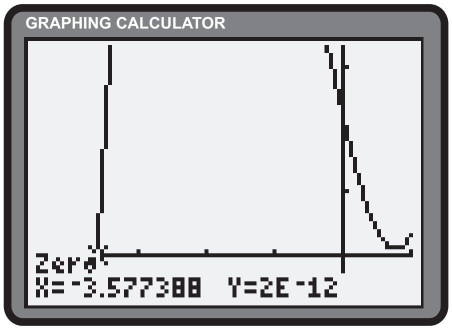
```
Y1=⁻2X²-2X-0.6
X=⁻.5        Y=⁻.1
```

b) The **Zero operation** of the graphing calculator reveals x-intercepts of approximately 0.731 and 1.149.



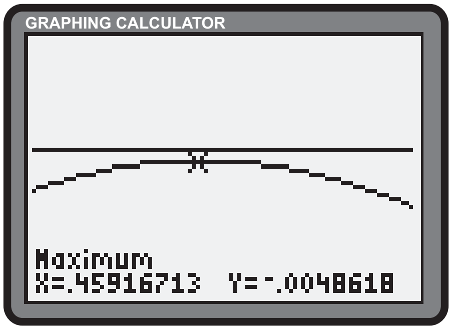
```
Zero
X=1.1488061  Y=0
```

c) $y = x^3 + 2x^2 - 5x + 2.3$ has an x-intercept of approximately -3.578.



Zero
X=-3.577388 Y=2E-12

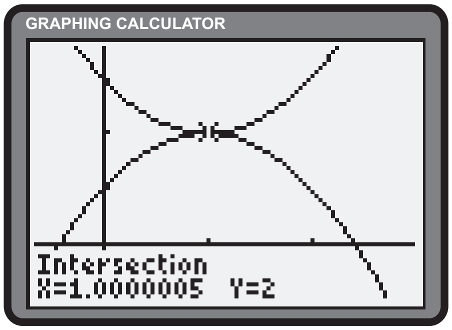
d) The **Zero operation** of the graphing calculator reveals an x-intercept of approximately 3.582. Zooming in on the interval around $x = 0.459$ several times reveals no point of intersection of the function with the x-axis.



Maximum
X=.45916713 Y=-.0048618

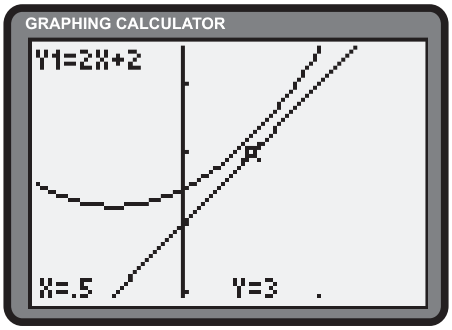
Section 1.2 Page 28 Question 3

a) The **Intersect operation** suggests a point of intersection at $(1, 2)$. Substitution confirms this result.



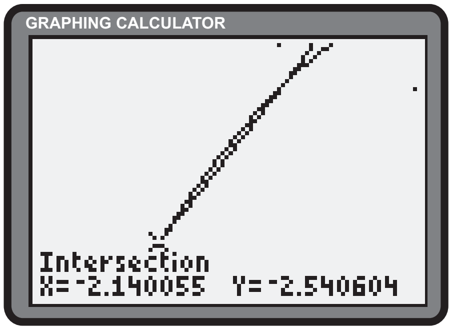
Intersection
X=1.0000005 Y=2

b) Zooming in on the interval around $x = 0.5$ reveals that the curves do not intersect.

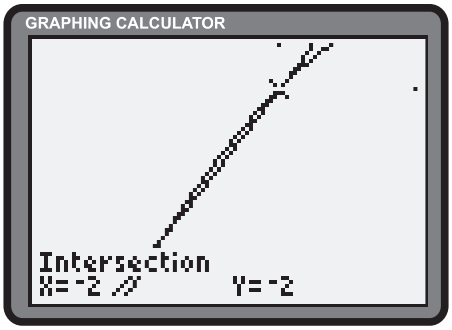


Y1=2X+2
X=.5 Y=3

c) Zooming in on the interval around $x = -2$ reveals that the curves intersect twice in this neighbourhood. The coordinates are $(-2, -2)$ and approximately $(-2.1401, -2.5406)$. The calculator identifies the third point of intersection with the approximate coordinates $(5.1401, 77.541)$.

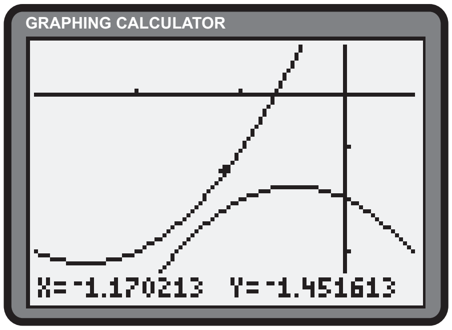


Intersection
X=-2.140055 Y=-2.540604



Intersection
X=-2 Y=-2

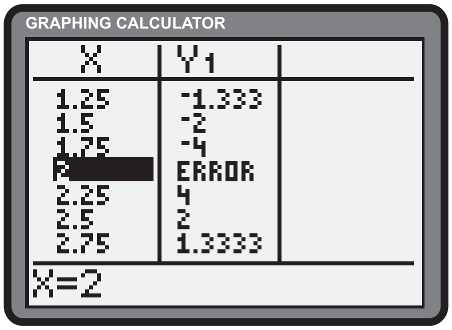
d) Zooming in on the interval around $x = -1$ reveals that the curves do not intersect.



X=-1.170213 Y=-1.451613

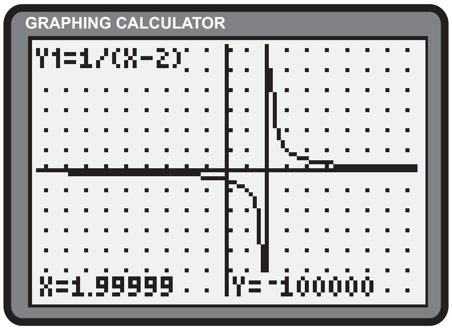
Section 1.2 Page 28 Question 4

a) To avoid division by zero, $x - 2 \neq 0$, so the domain is $x \in \Re,\ x \neq 2$. The range is $y \in \Re,\ y \neq 0$.



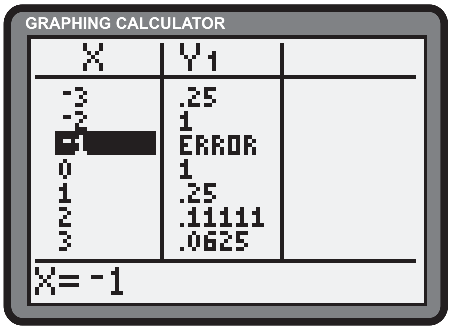
X	Y1
1.25	-1.333
1.5	-2
1.75	-4
2	ERROR
2.25	4
2.5	2
2.75	1.3333

```
X=2
```



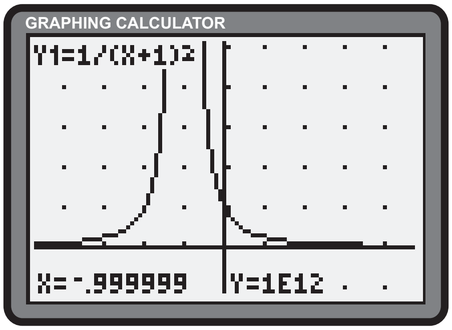
```
Y1=1/(X-2)
X=1.99999    Y=-100000
```

b) To avoid division by zero, $x + 1 \neq 0$, so the domain is $x \in \Re,\ x \neq -1$. The range is $y \in (0, \infty)$.



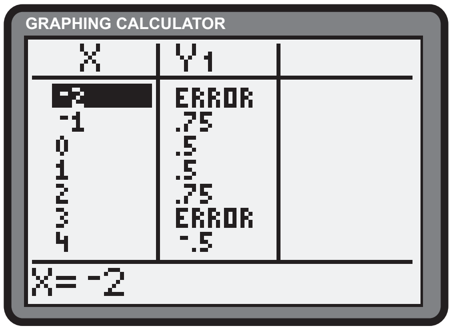
X	Y1
-3	.25
-2	1
-1	ERROR
0	1
1	.25
2	.11111
3	.0625

```
X= -1
```



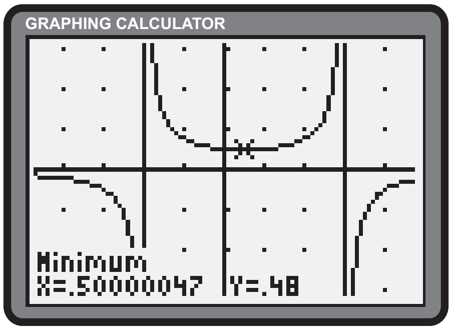
```
Y1=1/(X+1)²
X=-.999999    Y=1E12
```

c) The function can be rewritten as $y = \dfrac{-3}{(x+2)(x-3)}$. To avoid division by zero, $(x+2)(x-3) \neq 0$, so the domain is $x \in \Re,\ x \neq -2, 3$. The range is $y \in (-\infty, 0)$ or $y \in [0.48, +\infty)$.



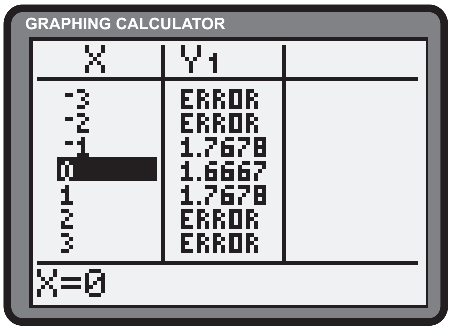
X	Y1
-2	ERROR
-1	.75
0	.5
1	.5
2	.75
3	ERROR
4	-.5

```
X= -2
```



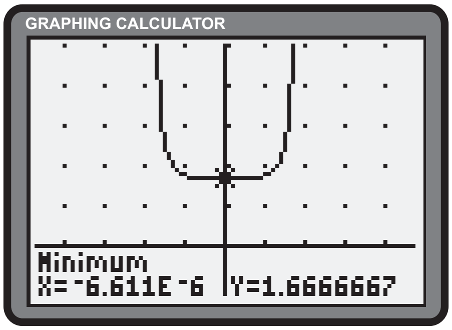
```
Minimum
X=.50000047    Y=.48
```

d) To avoid a negative radicand, $9 - x^4 \geq 0$, so $x \in \left[-\sqrt{3}, \sqrt{3}\right]$. To avoid division by zero, a further restriction limits the domain to $x \in \left(-\sqrt{3}, \sqrt{3}\right)$. The range is $y \in \left[\dfrac{5}{3}, +\infty\right)$.



X	Y1
-3	ERROR
-2	ERROR
-1	1.7678
0	1.6667
1	1.7678
2	ERROR
3	ERROR

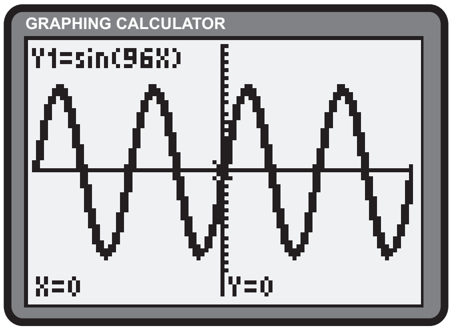
```
X=0
```



```
Minimum
X=-6.611E-6    Y=1.6666667
```

Section 1.2 Page 28 Question 5

a)



Y1=sin(96X)

X=0 Y=0

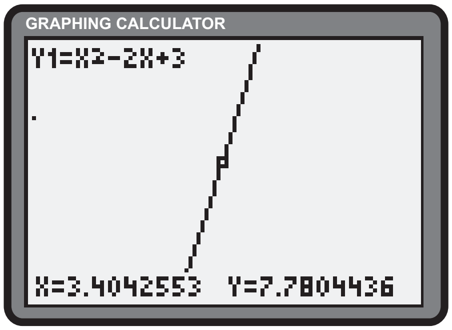
b) Answers may vary.

c) It will not work in any window. In the window given in part a), it will work for $y = \cos(2x)$ and $y = \cos(96x)$.

Section 1.2 Page 28 Question 6

For each of these solutions, use the **Zoom menu** until the segment of the graph appears linear.

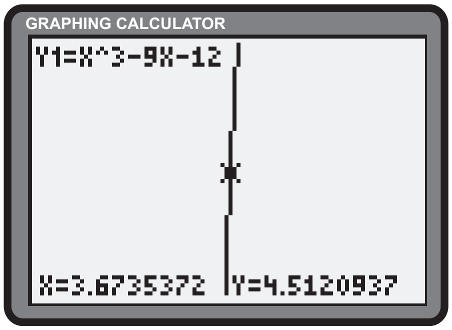
a) Answers may vary.



Y1=X²-2X+3

X=3.4042553 Y=7.7804436

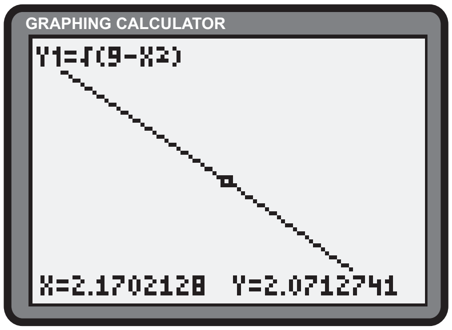
b) Answers may vary.



Y1=X^3-9X-12

X=3.6735372 Y=4.5120937

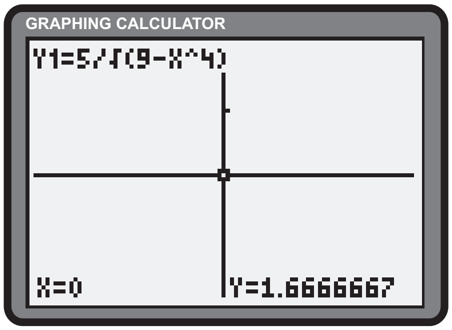
c) Answers may vary.



Y1=√(9-X²)

X=2.1702128 Y=2.0712741

d) Answers may vary.

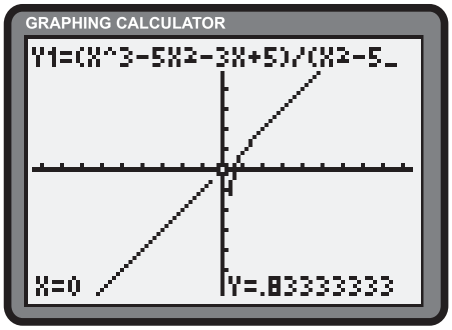


Y1=5/√(9-X^4)

X=0 Y=1.6666667

Section 1.2 Page 28 Question 7

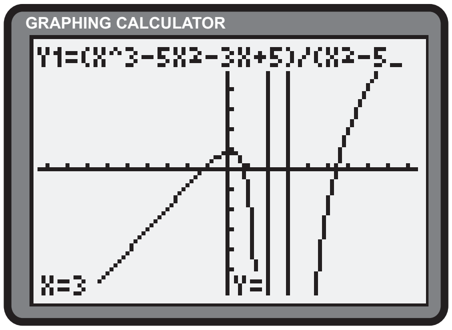
In the window $x \in [-94, 94]$, $y \in [-62, 62]$, the graph appears linear as $|x| \to \infty$.



Y1=(X^3-5X²-3X+5)/(X²-5_

X=0 Y=.83333333

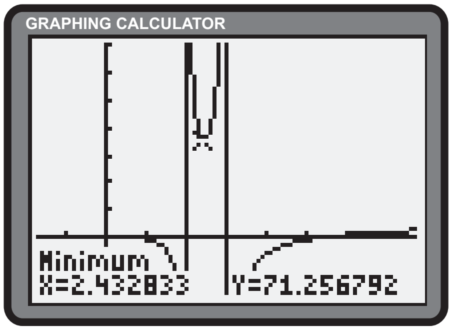
In the window $x \in [-9.4, 9.4]$, $y \in [-6.2, 6.2]$, the vertical asymptotes of $x = 2$ and $x = 3$ are revealed.



Y1=(X^3-5X²-3X+5)/(X²-5_

X=3 Y=

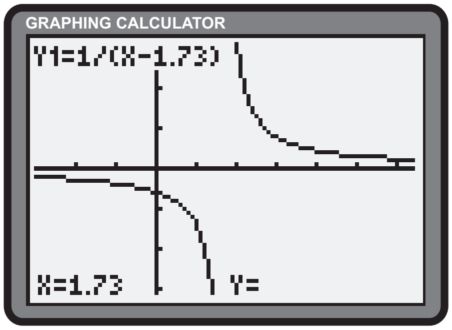
In the window $x \in [-1.7, 7.7]$, $y \in [-40, 140]$, the part of the function between the vertical asymptotes is highlighted.

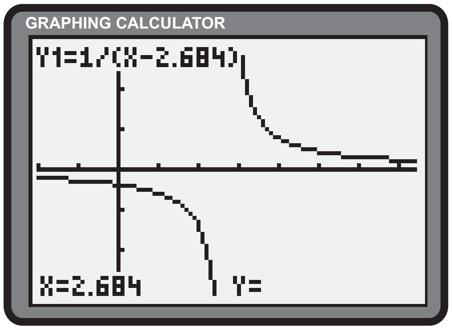


Section 1.2 Page 28 Question 8

a) The graphing of a function is due in part to the calculator sampling elements within the domain from Xmin to Xmax. The function $y = \dfrac{1}{x - 1.73}$ is undefined at $x = 1.73$. To graph the function properly, one sample must fall exactly at 1.73 so the discontinuity in the graph can be detected. Translating the **ZDecimal window** to the right 1.73 units results in a correct graph. Thus, use the window $x \in [-2.97, 6.43]$, $y \in [-3.1, 3.1]$

b) The graphing of a function is due in part to the calculator sampling elements within the domain from Xmin to Xmax. The function $y = \dfrac{1}{x - 2.684}$ is undefined at $x = 2.684$. To graph the function properly, one sample must fall exactly at 2.684 so the discontinuity in the graph can be detected. Translating the **ZDecimal window** to the right 2.684 units results in a correct graph. Thus, use the window $x \in [-2.016, 7.384]$, $y \in [-3.1, 3.1]$

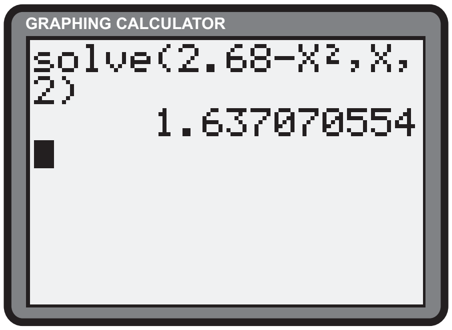




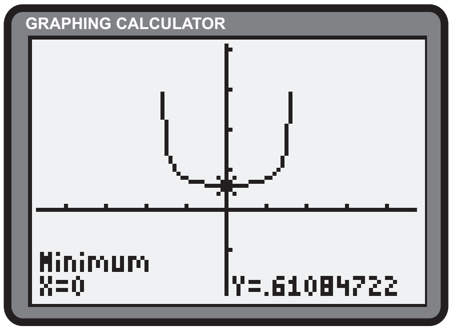
Section 1.2 Page 28 Question 9

The domain is restricted to $2.68 - x^2 > 0$ or $x^2 < 2.68$. Thus, $|x| < \sqrt{2.68}$, or $|x| < 1.6371$.

The **Minimum operation** of the calculator helps approximate the range: $y \in [0.6108, \infty)$.



```
solve(2.68-X²,X,
2)
      1.637070554
█
```



Review of Key Concepts

1.1 Functions and Their Use in Modelling

Section Review Page 30 Question 1

a)

x	y
−2	0
−1	0.5
1	3
3	1.5

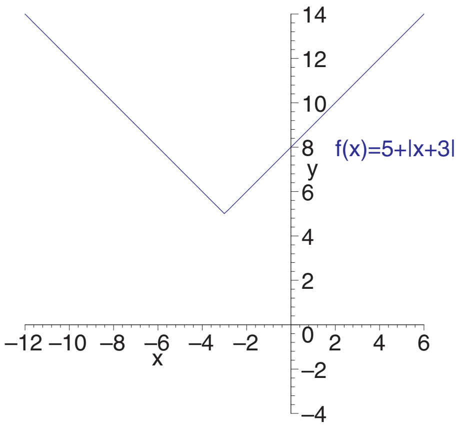
b) Domain: $x \in [-5, \infty)$

Section Review Page 30 Question 2

a)

x	y
−6	$5 + \lvert -6 + 3 \rvert = 8$
−5	$5 + \lvert -5 + 3 \rvert = 7$
−4	$5 + \lvert -4 + 3 \rvert = 6$
−3	$5 + \lvert -3 + 3 \rvert = 5$
−2	$5 + \lvert -2 + 3 \rvert = 6$
−1	$5 + \lvert -1 + 3 \rvert = 7$
0	$5 + \lvert 0 + 3 \rvert = 8$

b)



c) The function can be described as the sum of 5 and the distance from x to -3 on a number line.

Section Review Page 30 Question 3

a) $x \le 0$ is written as $x \in (-\infty, 0]$.
b) $-4 < x$ is written as $x \in (-4, +\infty)$.
c) $-5 \le x \le 5$ is written as $x \in [-5, 5]$.

Section Review Page 30 Question 4

a) For all but $x = \pm 2$, $f(x) = f(-x)$. $f(x)$ is neither even nor odd.
b) For each x-value, $g(x) = -g(-x)$. $g(x)$ is an odd function.
c) For each x-value, $h(x) = h(-x)$. $h(x)$ is an even function.

Section Review Page 30 Question 5

a)

$$\begin{aligned} f(-x) &= (-x)^2 + (-x) \\ &= x^2 - x \\ &\neq f(x) \text{ or } -f(x) \end{aligned}$$

$f(x)$ is neither even nor odd.

b)

$$\begin{aligned} g(-x) &= \left| (-x)^2 - 3 \right| \\ &= \left| x^2 - 3 \right| \\ &= g(x) \end{aligned}$$

$g(x)$ is an even function.

c)

$$\begin{aligned} h(-x) &= 5(-x) \\ &= -5x \\ &= -h(x) \end{aligned}$$

$h(x)$ is an odd function.

d)

$$\begin{aligned} r(-x) &= (-x)^2 - |-x| \\ &= x^2 - |x| \\ &= r(x) \end{aligned}$$

$r(x)$ is an even function.

e)

$$\begin{aligned} s(-x) &= \frac{1}{(-x)^3} \\ &= -\frac{1}{x^3} \\ &= -s(x) \end{aligned}$$

$s(x)$ is an odd function.

f)

$$\begin{aligned} t(-x) &= \left((-x)^3\right)^3 \\ &= -\left(x^3\right)^3 \\ &= -t(x) \end{aligned}$$

$t(x)$ is an odd function.

Section Review Page 30 Question 6

Consider the function $h(x)$, where

$$h(x) = f(x)g(x)$$

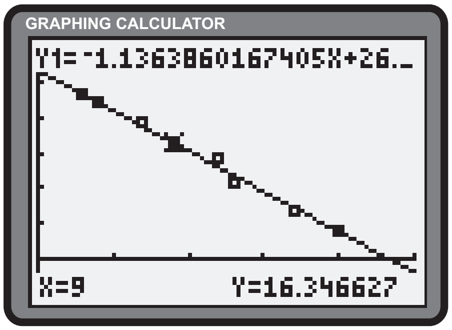
If $f(x)$ and $g(x)$ are even functions, then (1) becomes,

$$\begin{aligned} h(-x) &= f(-x)g(-x) \\ &= f(x)g(x) \\ &= h(x) \end{aligned}$$

The product of two even functions is an even function.

Section Review Page 30 Question 7

a) The calculator suggests $d \doteq -1.14t + 26.57$ as the line of best fit.



Y1=⁻1.1363860167405X+26._

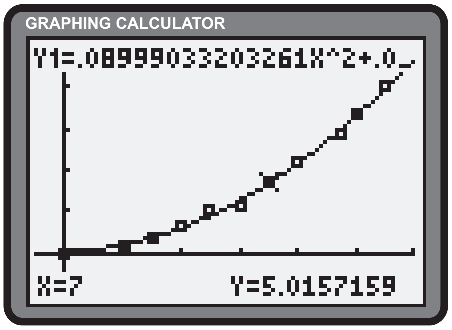
X=9 Y=16.346627

b) $d(14) = -1.14(14) + 26.57$
$= 10.61$

c) $d(23) = -1.14(23) + 26.57$
$= 0.35$

Section Review Page 30 Question 8

The **QuadReg feature** suggests the model $y \doteq 0.09x^2 + 0.08x + 0.03$.

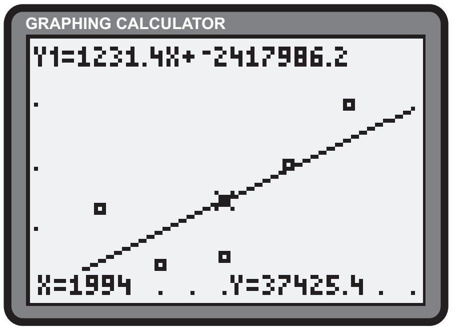


Y1=.08999033203261X^2+.0_

X=7 Y=5.0157159

Section Review Page 31 Question 9

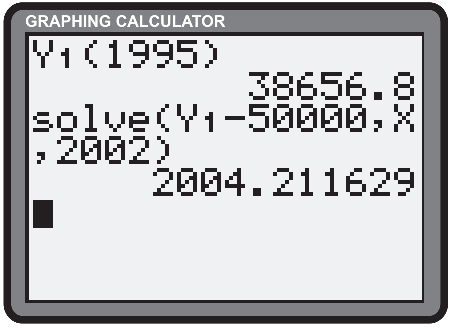
a) The calculator suggests $P \doteq 1231.4t - 2\,417\,986.2$ as the line of best fit, where P is the number of passengers, in thousands, and t is the year.



```
Y1=1231.4X+⁻2417986.2
X=1994          Y=37425.4
```

b) $P(1995) \doteq 38\,656\,800$. This result is higher than the actual value.

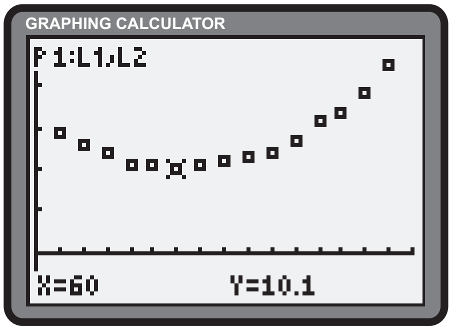
c) The model suggests that the number of passengers will reach 50 000 000 in 2004. No. Answers will vary.



```
Y1(1995)
          38656.8
solve(Y1-50000,X
,2002)
      2004.211629
```

Section Review Page 31 Question 10

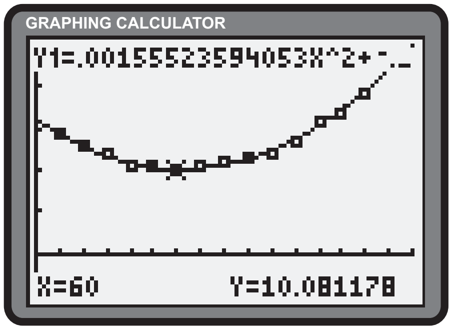
a)



```
P1:L1,L2
X=60          Y=10.1
```

b) The data suggest the best fuel economy of 10.1 L/100 km is achieved at 60 km/h.

c) The **QuadReg feature** suggests the model $F \doteq 0.001\,56s^2 - 0.192\,68s + 16.043\,16$.



```
Y1=.00155523594053X^2+⁻.
X=60          Y=10.081178
```

d) The curve would be higher, and the slope would be steeper. The lowest point might change.

e) Translation of the curve downward would yield reduced fuel consumption for the same speeds.

Section Review Page 31 Question 11

a) To avoid division by zero, $x + 1 \neq 0$. The domain is $x \in \Re,\ x \neq -1$.

b) To avoid division by zero, $3 - x \neq 0$. The domain is $x \in \Re,\ x \neq 3$.

c) To avoid a negative radicand, $x + 1 \geq 0$. The domain is $x \in [-1, +\infty)$.

d) To avoid a negative radicand and division by zero, $3 - x > 0$. The domain is $x \in (-\infty, 3)$.

e) To avoid a negative radicand, $|x| - 1 \geq 0$. The domain is $x \in (-\infty, -1]$ or $x \in [1, +\infty)$.

f) $x^2 + 3 > 0$ for all real numbers. The domain is $x \in \Re$.

g) The denominator can be written as $(x - 1)^2$. To avoid division by zero, $(x - 1)^2 \neq 0$. The domain is $x \in \Re,\ x \neq 1$.

h) The denominator can be written as $(x + 3)(x - 2)$. To avoid division by zero, $(x + 3)(x - 2) \neq 0$. The domain is $x \in \Re,\ x \neq -3, 2$.

1.2 Lies My Graphing Calculator Tells Me

Section Review Page 31 Question 12

Answers will vary.

Section Review Page 31 Question 13

Answers will vary.

Chapter Test

Section Chapter Test   Page 32   Question 1

a) For $f(x) = x^2$,

i) $f(1) = 1^2$ or 1

ii) $f(-1) = (-1)^2$ or 1

iii) $f(2) = (2)^2$ or 4

iv) $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2$ or $\frac{1}{4}$

b) For $f(x) = 1 - x^3$,

i) $f(1) = 1 - 1^3$ or 0

ii) $f(-1) = 1 - (-1)^3$ or 2

iii) $f(2) = 1 - (2)^3$ or -7

iv) $f\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^3$ or $\frac{7}{8}$

Section Chapter Test   Page 32   Question 2

a) $-4 < x < 10$ is written $x \in (-4, 10)$.   **b)** $x \leq 5$ is written $x \in (-\infty, 5]$.   **c)** $0 \leq x$ is written $x \in [0, +\infty)$.

Section Chapter Test   Page 32   Question 3

a) Since the graph of the function is rotationally symmetric with respect to the origin, the function is odd.

b) Since the graph of the function is symmetric with respect to neither the origin nor the y-axis, the function is neither odd nor even.

c) Since the graph of the function is symmetric with respect to the y-axis, the function is even.

Section Chapter Test   Page 32   Question 4

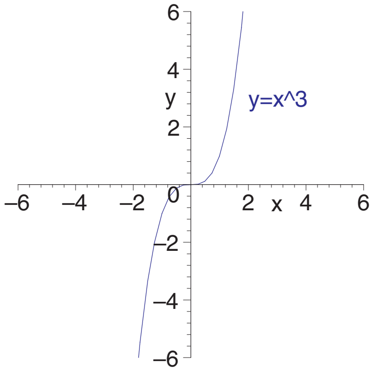
For each x-value, since $f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$, $f(x)$ is neither even nor odd.

For each x-value, $g(x) = g(-x)$. $g(x)$ is an even function.

For each x-value, $h(x) = -h(-x)$. $h(x)$ is an odd function.

Section Chapter Test   Page 32   Question 5

a)



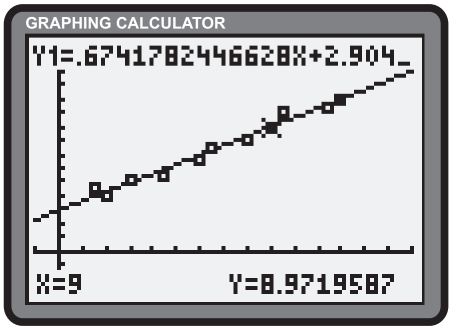
b) The slope has large positive values, decreases to 0 at $x = 0$, and then increases to large positive values.

c) The function is rotationally symmetric with respect to the origin (odd).

d) $y = x^5$ has a sharper turn on $(-1, 1)$, and is steeper outside this interval.

Section Chapter Test   Page 32   Question 6

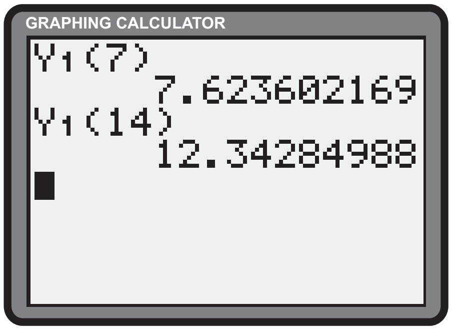
a) The calculator suggests the model $V \doteq 0.674r + 2.904$.



```
Y1=.6741782446628X+2.904_
X=9                Y=8.9719587
```

b) $V(7) \doteq 7.624$

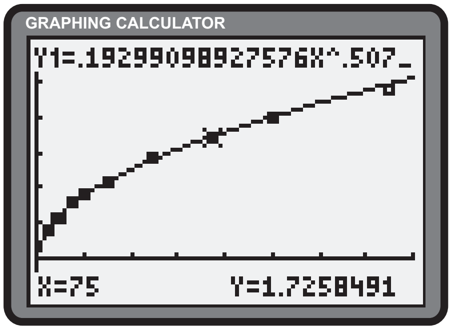
c) $V(14) \doteq 12.343$

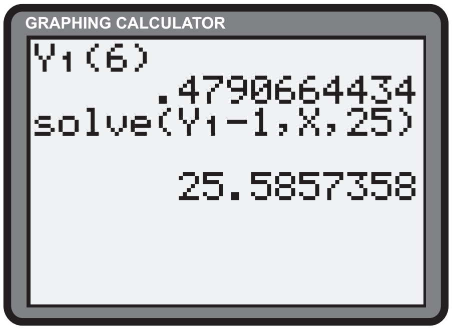


```
Y1(7)
      7.623602169
Y1(14)
      12.34284988
```

Section Chapter Test Page 32 Question 7

a)





```
Y₁(6)
          .4790664434
solve(Y₁-1,X,25)
           25.5857358
```

b) Answers may vary. The **PwrReg feature** of the calculator suggests the model $P \doteq 0.193l^{0.507}$, where P is the period in seconds and l is the length of the pendulum in centimetres.

c) $P(6) \doteq 0.479$

d) The **Solve feature** of the calculator suggests a $l \doteq 25.586$ cm would yield a period of 1 s.

e) Shortening the pendulum reduces the period; the clock takes less time for each “tick”.

Section Chapter Test Page 33 Question 8

a) To avoid division by 0, $x^2 - 1 \neq 0$. The domain of $f(x)$ is $x \in \Re,\ x \neq \pm 1$.

b) To avoid a negative radicand and division by 0, $x + 3 > 0$. The domain of $f(x)$ is $x \in (-3, +\infty)$.

c) The denominator can be written as $(x-3)(x+1)$. To avoid division by 0, $(x-3)(x+1) \neq 0$. The domain of $f(x)$ is $x \in \Re,\ x \neq -1, 3$.

d) Since $x^2 + x + 1 > 0$ for all real numbers, the domain of $f(x)$ is $\Re$.

Section Chapter Test Page 33 Question 9

a)

$$\begin{aligned} f(-x) &= (-x)^2 + (-x)^4 \\ &= x^2 + x^4 \\ &= f(x) \end{aligned}$$

$f(x)$ is an even function.

b)

$$\begin{aligned} g(-x) &= |(-x) - 1| \\ &= |-x - 1| \\ &= |x + 1| \\ &\neq g(x) \text{ or } -g(x) \end{aligned}$$

$g(x)$ is neither even nor odd.

c)

$$\begin{aligned} h(-x) &= -7(-x) \\ &= 7x \\ &= -h(x) \end{aligned}$$

$h(x)$ is an odd function.

d)

$$\begin{aligned} r(-x) &= (-x)^3 + |-x| \\ &= -x^3 + |x| \\ &\neq r(x) \text{ or } -r(x) \end{aligned}$$

$r(x)$ is neither even nor odd.

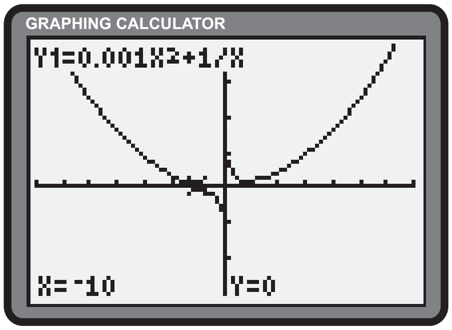
e)

$$\begin{aligned} s(-x) &= \frac{1 + (-x)^2}{(-x)^2} \\ &= \frac{1 + x^2}{x^2} \\ &= s(x) \end{aligned}$$

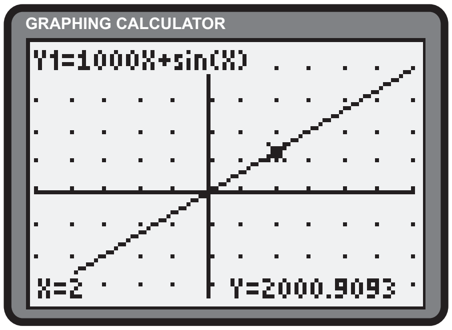
$s(x)$ is an even function.

Section Chapter Test Page 33 Question 10

a) Answers will vary. $x \in [-70, 70]$ and $y \in [-3, 4]$



b) $x \in [-5, 6]$ and $y \in [-5000, 7000]$



Challenge Problems

Section Challenge Problems Page 34 Question 1

Let x be the number of hits the player has made to this point in the season.

$$\frac{x}{322} = 0.289$$
$$x = 0.289(322) \tag{1}$$

Let y be the number of hits remaining to achieve a batting average of .300.

$$\frac{x+y}{322+53} = 0.300 \tag{2}$$

Substitute (1) into (2).

$$\begin{aligned}\frac{0.289(322)+y}{375} &= 0.3\\ y &= 0.3(375) - 0.289(322)\\ &\doteq 19.4\end{aligned}$$

The batter must have 20 hits to achieve a batting average of .300. This equates to a batting average of $\frac{20}{53}$ or 0.377 for the rest of the season.

Section Challenge Problems Page 34 Question 2

Let x be the date in the top left corner of the 3 by 3 square. Expressions for the remaining dates appear in the respective cells of the diagram. Let T be the total of the dates.

x	$x+1$	$x+2$
$x+7$	$x+8$	$x+9$
$x+14$	$x+15$	$x+16$

$$\begin{aligned}T &= x + (x+1) + (x+2) + (x+7) + (x+8) + (x+9) + (x+14) + (x+15) + (x+16)\\ &= 9x + 72\\ &= 9(x+8)\end{aligned}$$

Section Challenge Problems Page 34 Question 3

Let T be the equivalent temperature on both scales.

$$\begin{aligned}\frac{T-32}{212-32} &= \frac{T-0}{100-0}\\ 100T - 3200 &= 180T\\ -80T &= 3200\\ T &= -40\end{aligned}$$

−40° is an equivalent temperature on both scales.

Section Challenge Problems Page 34 Question 4

The amount of tape remaining on the reel is proportional to the area, A, of tape showing. Let r be the distance from the centre to the outer edge of the reel of tape.

a)
$$\begin{aligned}\frac{A_{\text{remaining}}}{A_{\text{original}}} &= \frac{3}{4}\\ \frac{\pi r^2 - \pi(2)^2}{\pi(6)^2 - \pi(2)^2} &= \frac{3}{4}\\ \frac{\pi r^2 - 4\pi}{36\pi - 4\pi} &= \frac{3}{4}\\ \frac{\pi r^2 - 4\pi}{32\pi} &= \frac{3}{4}\\ \pi r^2 - 4\pi &= 24\pi\\ \pi r^2 &= 28\pi\\ r &= \pm 2\sqrt{7}\end{aligned}$$

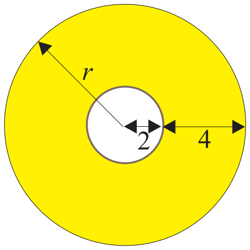
Since $r \geq 0$, $r = 2\sqrt{7}$.

b)
$$\begin{aligned}\frac{A_{\text{remaining}}}{A_{\text{original}}} &= \frac{1}{2}\\ \frac{\pi r^2 - \pi(2)^2}{\pi(6)^2 - \pi(2)^2} &= \frac{1}{2}\\ \frac{\pi r^2 - 4\pi}{36\pi - 4\pi} &= \frac{1}{2}\\ \frac{\pi r^2 - 4\pi}{32\pi} &= \frac{1}{2}\\ \pi r^2 - 4\pi &= 16\pi\\ \pi r^2 &= 20\pi\\ r &= \pm 2\sqrt{5}\end{aligned}$$

Since $r \geq 0$, $r = 2\sqrt{5}$.

c)
$$\begin{aligned}\frac{A_{\text{remaining}}}{A_{\text{original}}} &= \frac{1}{4}\\ \frac{\pi r^2 - \pi(2)^2}{\pi(6)^2 - \pi(2)^2} &= \frac{1}{4}\\ \frac{\pi r^2 - 4\pi}{36\pi - 4\pi} &= \frac{1}{4}\\ \frac{\pi r^2 - 4\pi}{32\pi} &= \frac{1}{4}\\ \pi r^2 - 4\pi &= 8\pi\\ \pi r^2 &= 12\pi\\ r &= \pm 2\sqrt{3}\end{aligned}$$

Since $r \geq 0$, $r = 2\sqrt{3}$.



Section Challenge Problems Page 34 Question 5

Double the first number, triple the second number, and add the two resulting numbers together.

Section Challenge Problems Page 34 Question 6

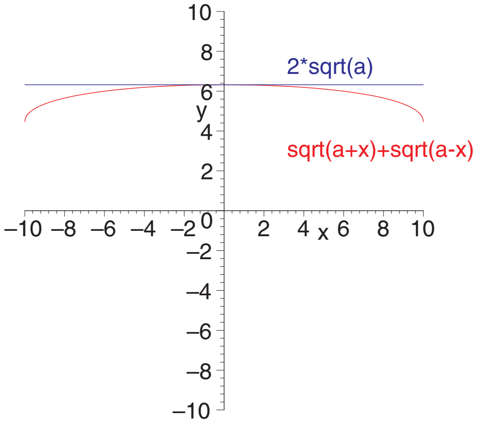
$$\begin{aligned}(\sqrt{a+x}+\sqrt{a-x})^2 &= (a+x)+(a-x)+2\sqrt{a-x}\sqrt{a+x}\\ &= 2a+2\sqrt{a-x}\sqrt{a+x}\\ &= 2(a+\sqrt{a^2-x^2})\end{aligned}$$

Since $x \in (0, a]$, $\sqrt{a^2-x^2} < \sqrt{a^2}$, thus,

$$\begin{aligned}(\sqrt{a+x}+\sqrt{a-x})^2 &< 2\left(a+\sqrt{a^2}\right)\\ &< 2(a+a)\\ &< 4a\\ &< (2\sqrt{a})^2 \qquad (1)\end{aligned}$$

Take the square root of both sides of (1).

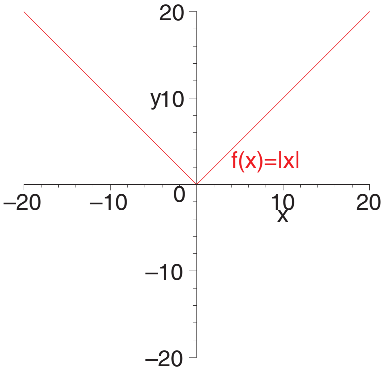
$$\sqrt{a+x}+\sqrt{a-x} < 2\sqrt{a}$$



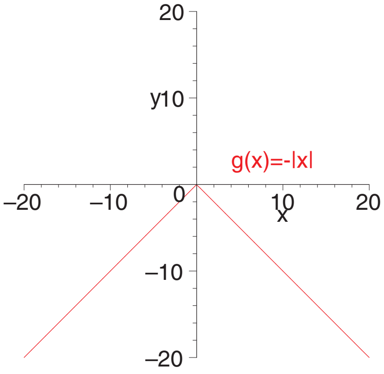
Section Challenge Problems Page 34 Question 7

The graph can be developed by applying a sequence of transformations.

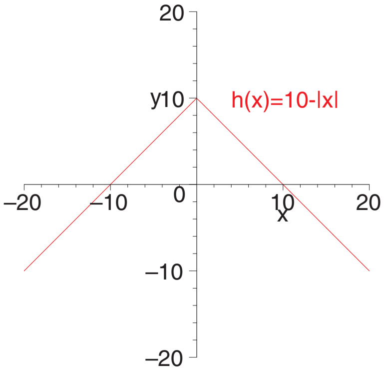
i) Graph the function $f(x) = |x|$.



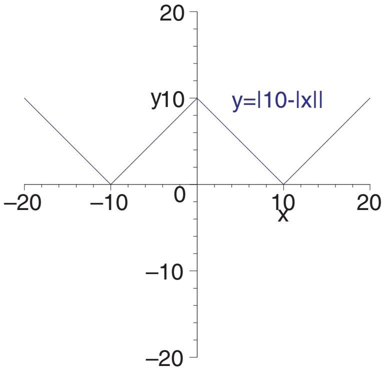
ii) Reflect the graph in the x-axis: $g(x) = -|x|$.



iii) Translate upward 10 units: $h(x) = 10 - |x|$.

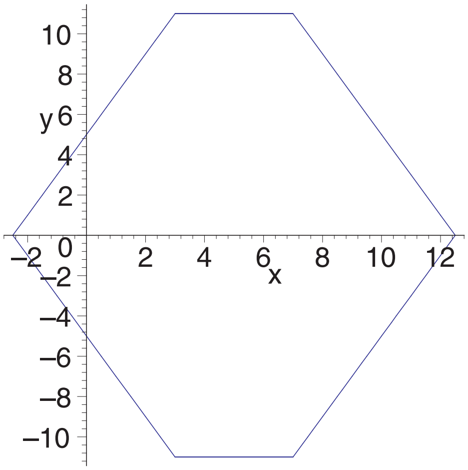


iv) Apply the absolute value transformation: $y = |10 - |x||$.



Section Challenge Problems Page 34 Question 8

Rewrite the relation as $|y| = 15 - |x-3| - |x-7|$. Use the definition of absolute value to reconstruct the relation as a piecewise set.



$$|y| = \begin{cases} 15 - |x-3| - |x-7| & ; y \geq 0 \\ -(15 - |x-3| - |x-7|) & ; y < 0 \end{cases} \quad (1)$$

$$|x-3| = \begin{cases} x-3 & ; x \geq 3 \\ 3-x & ; x < 3 \end{cases} \quad (2)$$

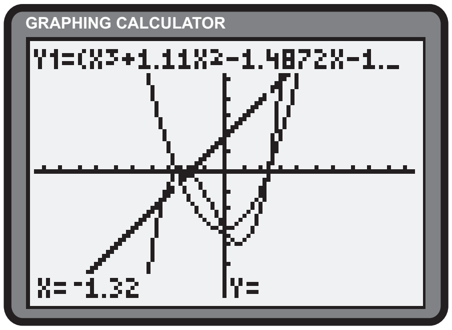
$$|x-7| = \begin{cases} x-7 & ; x \geq 7 \\ 7-x & ; x < 7 \end{cases} \quad (3)$$

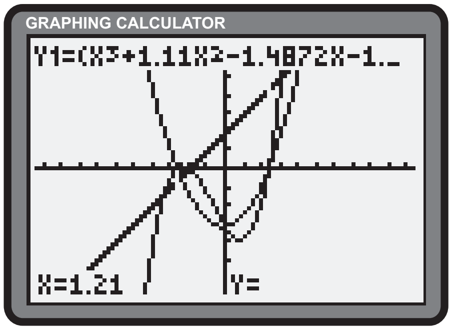
Assemble the pieces within the respective regions defined by the intersections of the intervals. The results are summarized in the table below.

Interval	***y* < 0**	***y* ≥ 0**
$(-\infty, 3)$	$y = (3-x) + (7-x) - 15$ $= -2x - 5$	$y = 15 - (3-x) - (7-x)$ $= 2x + 5$
$[3, 7)$	$y = (x-3) + (7-x) - 15$ $= -11$	$y = 15 - (x-3) - (7-x)$ $= 11$
$[7, \infty)$	$y = (x-3) + (x-7) - 15$ $= 2x - 25$	$y = 15 - (x-3) - (x-7)$ $= -2x + 25$

Section Challenge Problems Page 34 Question 9

The calculator assists in determining the roots of the numerator and denominator. The roots of $x^3 + 1.11x^2 - 1.4872$ are -1.32, -1 and 1.21. The roots of $x^2 + 0.11x - 1.5972$ are -1.32 and 1.21. These results define holes in $y = \dfrac{x^3 + 1.11x^2 - 1.4872}{x^2 + 0.11x - 1.5972}$ at $x = -1.32$ and $x - 1.21$. For the holes to appear visually, window settings must ensure that these domain values are sampled. Since both values are multiples of 0.11, suitable window settings for the domain would be $[-47(0.11), 47(0.11)]$ or $[-5.17, 5.17]$. To reflect proportionality, the range should be set to $[-31(0.11), 31(0.11)]$ or $[-3.41, 3.41]$. The holes are depicted in the thick graphs below.





Using the Strategies

Section Problem Solving Page 37 Question 1

Once the female captain is identified, there are 3 remaining females from which 2 must be chosen. There are 3 ways this can satisfied. From the four males, 3 must be chosen. There are 4 possible combinations for the male members.

As a result there are 4 × 3 or 12 possible combinations of members that could represent the school.

Section Problem Solving Page 37 Question 2

Fill the 5-L container and empty it into the 9-L container; then fill the 5-L container again and pour water into the 9-L container to fill it. There is now 1 L of water in the 5-L container. Empty the 9-L container, pour the 1 L of water from the 5-L container into the 9-L container, refill the 5-L container and pour it into the 9-L container. There are now 6 L of water in the 9-L container.

Section Problem Solving Page 37 Question 3

Juan and Sue cross in 2 min; Sue returns in 2 min; Alicia and Larry cross in 8 min; Juan returns in 1 min; Juan and Sue cross in 2 min. The total time to cross is 15 min. Hint: Alicia and Larry must cross together, and someone must be on the opposite side to return the flashlight.

Section Problem Solving Page 37 Question 4

From each vertex of convex n-gon, $n-3$ diagonals can be drawn to non-adjacent vertices. Since each diagonal must be counted only once, a convex n-gon has $\frac{n(n-3)}{2}$ diagonals. Let the number of sides in the polygons be x and y respectively. Solve the following system of equations.

$$\begin{aligned} x+y &= 11 \\ y &= 11-x \qquad (1) \\ \frac{x(x-3)}{2}+\frac{y(y-3)}{2} &= 14 \\ x^2-3x+y^2-3y &= 28 \qquad (2) \end{aligned}$$

Substitute (1) into (2).

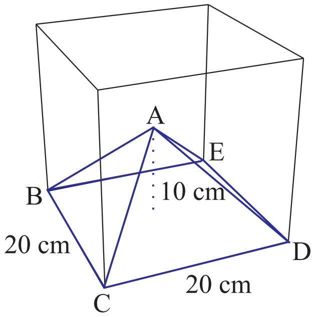
$$\begin{aligned} x^2-3x+(11-x)^2-3(11-x) &= 28 \\ x^2-3x+121-22x+x^2-33+3x &= 28 \\ 2x^2-22x+60 &= 0 \\ x^2-11x+30 &= 0 \\ (x-5)(x-6) &= 0 \\ x &= 5 \text{ or } 6 \end{aligned}$$

the convex polygons that satisfy the requirements are a pentagon and a hexagon.

Section Problem Solving Page 37 Question 5

a) The top vertex of each pyramid meets at the centre of the cube with each face of the cube being a base of the pyramid.

b) The dimensions are 20 cm by 20 cm by 10 cm.



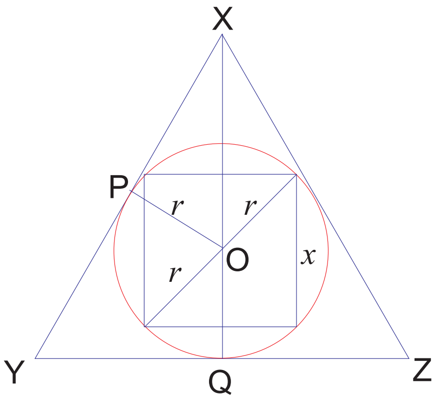
Section Problem Solving Page 37 Question 6

Yes. The following table gives the first month in the year offering a Friday the 13th.

January 1st	First month (not a leap year)	First month (leap year)
Sunday	Friday January 13	Friday January 13
Monday	Friday April 13	Friday September 13
Tuesday	Friday September 13	Friday June 13
Wednesday	Friday June 13	Friday March 13
Thursday	Friday February 13	Friday February 13
Friday	Friday August 13	Friday May 13
Saturday	Friday May 13	Friday October 13

Section Problem Solving Page 37 Question 7

Let r be the length of the diagonal of the inner square. The radius of the circle is then r. Determine the side length of the square.



$$\begin{aligned} x^2 + x^2 &= (2r)^2 \\ 2x^2 &= 4r^2 \\ x^2 &= 2r^2 \end{aligned}$$

Construct OP such that OP ⊥ XY. In $\triangle$XOP,

$$\begin{aligned} \sin \angle \mathrm{OXP} &= \frac{\mathrm{OP}}{\mathrm{OX}} \\ \mathrm{OX} &= \frac{\mathrm{OP}}{\sin \angle \mathrm{OXP}} \\ &= \frac{r}{\sin 30°} \\ &= 2r \\ \tan \angle \mathrm{OXP} &= \frac{\mathrm{OP}}{\mathrm{PX}} \\ \mathrm{PX} &= \frac{\mathrm{OP}}{\tan \angle \mathrm{OXP}} \\ &= \frac{r}{\tan 30°} \\ &= \sqrt{3}r \end{aligned}$$

Since XQ = XO + OQ, the height of $\triangle$XYZ is $2r + r$ or $3r$. Since QY = PX = $\sqrt{3}r$, the ratio of the areas can be determined.

$$\begin{aligned} \frac{A_{\text{triangle}}}{A_{\text{square}}} &= \frac{(\sqrt{3}r)(3r)}{2r^2} \\ &= \frac{3\sqrt{3}}{2} \end{aligned}$$

The ratio of the area of the triangle to the area of the square is $\frac{3\sqrt{3}}{2}$.

CHAPTER 2 Polynomials

2.2 Dividing a Polynomial by a Polynomial

Practise

Section 2.2 Page 51 Question 1

a)

$$\begin{array}{r} x+\ 5 \\ x+3\overline{)x^2+8x+15} \\ \underline{x^2+3x} \\ 5x+15 \\ \underline{5x+15} \\ 0 \end{array}$$

Restriction: $x \neq -3$

b)

$$\begin{array}{r} a-\ 2 \\ a-5\overline{)a^2-7a+10} \\ \underline{a^2-5a} \\ -2a+10 \\ \underline{-2a+10} \\ 0 \end{array}$$

Restriction: $a \neq 5$

c)

$$\begin{array}{r} y+\ 3 \\ y-4\overline{)y^2-\ \ y-12} \\ \underline{y^2-4y} \\ 3y-12 \\ \underline{3y-12} \\ 0 \end{array}$$

Restriction: $y \neq 4$

d)

$$\begin{array}{r} t-\ 2 \\ t+2\overline{)t^2+0t-\ 4} \\ \underline{t^2+2t} \\ -2t-\ 4 \\ \underline{-2t-\ 4} \\ 0 \end{array}$$

Restriction: $t \neq -2$

Section 2.2 Page 51 Question 3

a)

$$\begin{array}{r} x+\ 3 \\ 2x+5\overline{)2x^2+11x+15} \\ \underline{2x^2+\ 5x} \\ 6x+15 \\ \underline{6x+15} \\ 0 \end{array}$$

Restriction: $x \neq -\frac{5}{2}$

b)

$$\begin{array}{r} y+3 \\ 3y-1\overline{)3y^2+8y-3} \\ \underline{3y^2-y} \\ 9y-3 \\ \underline{9y-3} \\ 0 \end{array}$$

Restriction: $y \neq \frac{1}{3}$

c)

$$\begin{array}{r} r-6 \\ 5r-1\overline{)5r^2-31r+6} \\ \underline{5r^2-\ \ \ r} \\ -30r+6 \\ \underline{-30r+6} \\ 0 \end{array}$$

Restriction: $r \neq \frac{1}{5}$

d)

$$\begin{array}{r} 2t-3 \\ 2t+1\overline{)4t^2-4t-3} \\ \underline{4t^2+2t} \\ -6t-3 \\ \underline{-6t-3} \\ 0 \end{array}$$

Restriction: $t \neq -\frac{1}{2}$

e)

$$\begin{array}{r} 2r-\ 7 \\ 3r-2\overline{)6r^2-25r+14} \\ \underline{6r^2-\ 4r} \\ -21r+14 \\ \underline{-21r+14} \\ 0 \end{array}$$

Restriction: $r \neq \frac{2}{3}$

f)

$$\begin{array}{r} 2x+3 \\ 5x+3\overline{)10x^2+21x+9} \\ \underline{10x^2+\ 6x} \\ 15x+9 \\ \underline{15x+9} \\ 0 \end{array}$$

Restriction: $x \neq -\frac{3}{5}$

g)

$$\begin{array}{r}
2x+\ \ 5 \\
4x-3\overline{)8x^2+14x+15} \\
\underline{8x^2-\ \ 6x} \\
20x+15 \\
\underline{20x-15} \\
30
\end{array}$$

Restriction: $x \neq \frac{3}{4}$

Section 2.2 Page 51 Question 5

a)

$$\begin{array}{r}
2x^2+\ \ 3 \\
x-1\overline{)2x^3-2x^2+3x-3} \\
\underline{2x^3-2x^2} \\
3x-3 \\
\underline{3x-3} \\
0
\end{array}$$

Restriction: $x \neq 1$

b)

$$\begin{array}{r}
3z^2+\ \ 5 \\
z+2\overline{)3z^3+6z^2+5z+10} \\
\underline{3z^3+6z^2} \\
5z+10 \\
\underline{5z+10} \\
0
\end{array}$$

Restriction: $z \neq -2$

c)

$$\begin{array}{r}
2m^2-\ \ 3 \\
2m+1\overline{)4m^3+2m^2-6m-3} \\
\underline{4m^3+2m^2} \\
-6m-3 \\
\underline{-6m-3} \\
0
\end{array}$$

Restriction: $m \neq -\frac{1}{2}$

d)

$$\begin{array}{r}
3n^2-\ \ 4 \\
2n-3\overline{)6n^3-9n^2-8n+12} \\
\underline{6n^3-9n^2} \\
-8n+12 \\
\underline{-8n+12} \\
0
\end{array}$$

Restriction: $n \neq \frac{3}{2}$

e)

$$\begin{array}{r}
2d+3 \\
5d^2+0d+2\overline{)10d^3+15d^2+4d+6} \\
\underline{10d^3+\ \ 0d^2+4d} \\
15d^2+6 \\
\underline{15d^2+6} \\
0
\end{array}$$

Restriction: none

f)

$$\begin{array}{r}
4x-3 \\
2x^2+0x+1\overline{)8x^3-6x^2+4x-3} \\
\underline{8x^3+0x^2+4x} \\
-6x^2-3 \\
\underline{-6x^2-3} \\
0
\end{array}$$

Restriction: none

g)

$$\begin{array}{r}
4s+1 \\
3s^2+0s-5\overline{)12s^3+3s^2-20s-5} \\
\underline{12s^3+0s^2-20s} \\
3s^2-5 \\
\underline{3s^2-5} \\
0
\end{array}$$

Restriction: $s \neq \pm\frac{\sqrt{15}}{3}$

h)

$$\begin{array}{r}
3t-4 \\
7t^2+0t-2\overline{)21t^3-28t^2-6t+8} \\
\underline{21t^3+\ \ 0t^2-6t} \\
-28t^2+8 \\
\underline{-28t^2+8} \\
0
\end{array}$$

Restriction: $t \neq \pm\frac{\sqrt{14}}{7}$

Section 2.2 Page 52 Question 7

Since multiplication is commutative, the divisor and quotient are interchangeable.

a) dividend: 255; divisor: 11; quotient: 23; remainder: 2

b) dividend: $8y^3 + 6y^2 - 4y - 5$; divisor: $4y + 3$; quotient: $2y^2 - 1$; remainder: -2

c) dividend: $x^2 + x + 3$; divisor: x; quotient: $x + 1$; remainder: 3

Apply, Solve, Communicate

Section 2.2 Page 52 Question 9

a) i)

$$\begin{array}{r}
x^2 + x + 1 \\
x - 1\,\overline{)\,x^3 + 0x^2 + 0x - 1} \\
\underline{x^3 - x^2} \phantom{{}+0x-1} \\
x^2 + 0x \phantom{{}-1} \\
\underline{x^2 - x} \phantom{{}-1} \\
x - 1 \\
\underline{x - 1} \\
0
\end{array}$$

Restriction: $x \neq 1$

ii)

$$\begin{array}{r}
x^2 + 2x + 4 \\
x - 2\,\overline{)\,x^3 + 0x^2 + 0x - 8} \\
\underline{x^3 - 2x^2} \phantom{{}+0x-8} \\
2x^2 + 0x \phantom{{}-8} \\
\underline{2x^2 - 4x} \phantom{{}-8} \\
4x - 8 \\
\underline{4x - 8} \\
0
\end{array}$$

Restriction: $x \neq 2$

iii)

$$\begin{array}{r}
x^2 + 3x + 9 \\
x - 3\,\overline{)\,x^3 + 0x^2 + 0x - 27} \\
\underline{x^3 - 3x^2} \phantom{{}+0x-27} \\
3x^2 + 0x \phantom{{}-27} \\
\underline{3x^2 - 9x} \phantom{{}-27} \\
9x - 27 \\
\underline{9x - 27} \\
0
\end{array}$$

Restriction: $x \neq 3$

b)

$$x^2 + x + 1 = 1^0(x^2) + 1^1(x) + 1^2(1)$$
$$x^2 + 2x + 4 = 2^0(x^2) + 2^1(x) + 2^2(1);$$ The coefficients are a geometric sequence.
$$x^2 + 3x + 9 = 3^0(x^2) + 3^1(x) + 3^2(1)$$

c)

$$\begin{array}{r}
x^2 + 4x + 16 \\
x - 4\,\overline{)\,x^3 + 0x^2 + 0x - 64} \\
\underline{x^3 - 4x^2} \phantom{{}+0x-64} \\
4x^2 + 0x \phantom{{}-64} \\
\underline{4x^2 - 16x} \phantom{{}-64} \\
16x - 64 \\
\underline{16x - 64} \\
0
\end{array}$$

Restriction: $x \neq 4$

$$\begin{array}{r}
x^2 + 5x + 25 \\
x - 5\,\overline{)\,x^3 + 0x^2 + 0x - 125} \\
\underline{x^3 - 5x^2} \phantom{{}+0x-125} \\
5x^2 + 0x \phantom{{}-125} \\
\underline{5x^2 - 25x} \phantom{{}-125} \\
25x - 125 \\
\underline{25x - 125} \\
0
\end{array}$$

Restriction: $x \neq 5$

Section 2.2 Page 52 Question 10

a)

$$\begin{array}{r}
x^4 + x^3 + x^2 + x + 1 \\
x - 1\,\overline{)\,x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 1} \\
\underline{x^5 - x^4} \phantom{{}+0x^3+0x^2+0x-1} \\
x^4 + 0x^3 \phantom{{}+0x^2+0x-1} \\
\underline{x^4 - x^3} \phantom{{}+0x^2+0x-1} \\
x^3 + 0x^2 \phantom{{}+0x-1} \\
\underline{x^3 - x^2} \phantom{{}+0x-1} \\
x^2 + 0x \phantom{{}-1} \\
\underline{x^2 - x} \phantom{{}-1} \\
x - 1 \\
\underline{x - 1} \\
0
\end{array}$$

Thus,

$(x^5 - 1) \div (x - 1) = x^4 + x^3 + x^2 + x + 1$

for $x \neq 1$.

b)

$$\begin{array}{r}
x^4 + 2x^3 + 4x^2 + 8x + 16 \\
x - 2\,\overline{)\,x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 32} \\
\underline{x^5 - 2x^4} \phantom{{}+0x^3+0x^2+0x-32} \\
2x^4 + 0x^3 \phantom{{}+0x^2+0x-32} \\
\underline{2x^4 - 4x^3} \phantom{{}+0x^2+0x-32} \\
4x^3 + 0x^2 \phantom{{}+0x-32} \\
\underline{4x^3 - 8x^2} \phantom{{}+0x-32} \\
8x^2 + 0x \phantom{{}-32} \\
\underline{8x^2 - 16x} \phantom{{}-32} \\
16x - 32 \\
\underline{16x - 32} \\
0
\end{array}$$

Thus,

$(x^5 - 32) \div (x - 2) = x^4 + 2x^3 + 4x^2 + 8x + 16$

for $x \neq 2$.

Section 2.2 Page 52 Question 11

a)

$$\begin{array}{rr}
 & 3x+2 \\ \cline{2-2}
2x^2+0x+1 & \big)\,6x^3+4x^2+4x+3 \\
 & \underline{6x^3+0x^2+3x\phantom{{}+3}} \\
 & 4x^2+x+3 \\
 & \underline{4x^2+0x+2} \\
 & x+1
\end{array}$$

Thus, $(6x^3+4x^2+4x+3) \div (2x^2+1) = 3x+2+\dfrac{x+1}{2x^2+1}$ for $x \in \Re$.

b)

$$\begin{array}{rr}
 & y-3 \\ \cline{2-2}
3y^2+0y+1 & \big)\,3y^3-9y^2+0y-3 \\
 & \underline{3y^3+0y^2+y\phantom{{}-3}} \\
 & -9y^2-y-3 \\
 & \underline{-9y^2+0y-3} \\
 & -y
\end{array}$$

Thus, $(3y^3-9y^2-3) \div (3y^2+1) = y-3-\dfrac{y}{3y^2+1}$ for $y \in \Re$.

c)

$$\begin{array}{rr}
 & t+2 \\ \cline{2-2}
2t^2+0t-4 & \big)\,2t^3+4t^2-4t-19 \\
 & \underline{2t^3+0t^2-4t\phantom{{}-19}} \\
 & 4t^2+0t-19 \\
 & \underline{4t^2+0t-8} \\
 & -11
\end{array}$$

Thus, $(2t^3+4t^2-4t-19) \div (2t^2-4) = t+2-\dfrac{11}{2t^2-4}$ for $t \neq \pm\sqrt{2}$.

d)

$$\begin{array}{rr}
 & 3d^2\qquad\quad-2 \\ \cline{2-2}
2d^2+0d-3 & \big)\,6d^4+0d^3-13d^2+d+4 \\
 & \underline{6d^4+0d^3-9d^2}\phantom{{}+d+4} \\
 & -4d^2+d+4 \\
 & \underline{-4d^2+0d+6} \\
 & d-2
\end{array}$$

Thus, $(6d^4-13d^2+d+4) \div (2d^2-3) = 3d^2-2+\dfrac{d-2}{2d^2-3}$ for $d \neq \pm\dfrac{\sqrt{6}}{2}$.

Section 2.2 Page 52 Question 12

a) The value of the remainder depends on the value of the variable.
b) The value of the remainder depends on the value of the variable.
c) The value of the remainder does not depend on the value of the variable, because the remainder is constant.
d) The value of the remainder depends of the value of the variable.

Section 2.2 Page 52 Question 13

a)

$$\begin{array}{r} x-\ \ 4 \\ x+3\,\overline{)\,x^2-\ \ x-\ \ k} \\ \underline{x^2+3x} \\ -4x-\ \ k \\ \underline{-4x-12} \\ -k+12 \end{array}$$

Set the remainder equal to 0 and solve for k.

$$\begin{aligned} -k+12 &= 0 \\ -k &= -12 \\ k &= 12 \end{aligned}$$

$x+3$ divides x^2-x-k evenly if $k=12$.

b)

$$\begin{array}{r} 3y+2 \\ 2y-1\,\overline{)\,6y^2+\ \ y-k} \\ \underline{6y^2-3y} \\ 4y-k \\ \underline{4y-2} \\ -k+2 \end{array}$$

Set the remainder equal to 0 and solve for k.

$$\begin{aligned} -k+2 &= 0 \\ -k &= -2 \\ k &= 2 \end{aligned}$$

$2y-1$ is a factor of $6y^2+y-k$ if $k=2$.

c)

$$\begin{array}{r} 4x+13 \\ x-1\,\overline{)\,4x^2+9x+\ \ k} \\ \underline{4x^2-4x} \\ 13x+\ \ k \\ \underline{13x-13} \\ k+13 \end{array}$$

Set the remainder equal to 3 and solve for k.

$$\begin{aligned} k+13 &= 3 \\ k &= -10 \end{aligned}$$

$x-1$ divides $4x^2+9x+k$ with a remainder of 3 if $k=-10$.

d)

$$\begin{array}{r} x^2+3x+1 \\ 2x+1\,\overline{)\,2x^3+7x^2+5x-k} \\ \underline{2x^3+\ \ x^2} \\ 6x^2+5x \\ \underline{6x^2+3x} \\ 2x-k \\ \underline{2x+1} \\ -k-1 \end{array}$$

Set the remainder equal to -5 and solve for k.

$$\begin{aligned} -k-1 &= -5 \\ -k &= -4 \\ k &= 4 \end{aligned}$$

$2x+1$ divides $2x^3+7x^2+5x-k$ with a remainder of -5 if $k=4$.

Section 2.2 Page 52 Question 14

The area of the triangle is given by $A=\frac{1}{2}bh$, where h is the height and b is the base. Thus, $b=\frac{2A}{h}$.

$$\begin{aligned} A &= 6x^2-5x-4 \text{ and} \\ h &= 3x-4 \end{aligned}$$

Thus,

$$\begin{aligned} b &= \frac{2(6x^2-5x-4)}{3x-4} \\ &= \frac{2(3x-4)(2x+1)}{3x-4} \\ &= 2(2x+1) \\ &= 4x+2 \end{aligned}$$

The base of the triangle is represented by $4x+2$.

Section 2.2 Page 52 Question 15

The area of the trapezoid is given by $A = \frac{1}{2}h(a + b)$, where h is the height and a and b are the bases. Thus, $h = \frac{2A}{a + b}$.

$$\begin{aligned} A &= 12y^2 - 11y + 2 \\ a &= 5y - 3 \\ b &= 3y + 1 \end{aligned}$$

Thus,

$$\begin{aligned} h &= \frac{2(12y^2 - 11y + 2)}{(5y - 3) + (3y + 1)} \\ &= \frac{2(3y - 2)(4y - 1)}{8y - 2} \\ &= \frac{2(3y - 2)(4y - 1)}{2(4y - 1)} \\ &= 3y - 2 \end{aligned}$$

The height of the trapezoid is represented by $3y - 2$.

Section 2.2 Page 52 Question 16

The expression $\frac{5x^2 + 14x - 3}{x + 3}$ is defined for all real numbers except $x = -3$. The expression $5x - 1$ is defined for *all* real numbers.

Section 2.2 Page 53 Question 17

The amount, A, of the investment, P, after n months can be expressed as $A = P(1 + i)^n$, where i is the interest rate per period, expressed as a decimal.

a) Determine the amount for $P = 500$, $i = 0.01$, and $n = 10$.

$$\begin{aligned} A(10) &= 500(1 + 0.01)^{10} \\ &\doteq 552.31 \end{aligned}$$

After 10 months, the investment is worth approximately \$552.21.

b) Determine the amount for $P = 500$, $i = 0.01$, and $n = 25$.

$$\begin{aligned} A(10) &= 500(1 + 0.01)^{25} \\ &\doteq 641.22 \end{aligned}$$

After 25 months, the investment is worth approximately \$641.22.

c) Determine the amount for $P = 500$, $i = 0.01$, and $n = 12(10)$ or 120.

$$\begin{aligned} A(120) &= 500(1 + 0.01)^{120} \\ &\doteq 1650.19 \end{aligned}$$

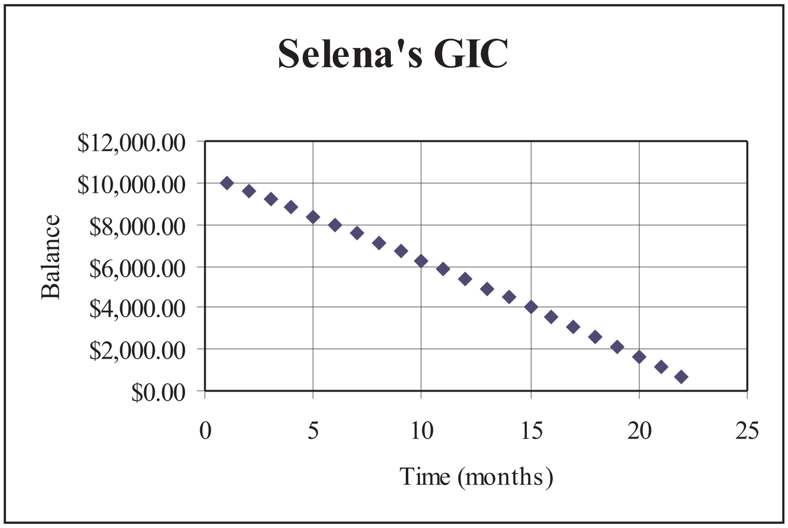
After 10 years, the investment is worth approximately \$1650.19.

Section 2.2 Page 53 Question 18

a) At the end of the 22nd month, Selena withdraws $211.37 and the account is exhausted.

Month	Principal	Interest	Savings	Withdrawal	Balance
1	$10,000.00	$100.00	$10,100.00	$500.00	$9,600.00
2	$9,600.00	$96.00	$9,696.00	$500.00	$9,196.00
3	$9,196.00	$91.96	$9,287.96	$500.00	$8,787.96
4	$8,787.96	$87.88	$8,875.84	$500.00	$8,375.84
5	$8,375.84	$83.76	$8,459.60	$500.00	$7,959.60
6	$7,959.60	$79.60	$8,039.19	$500.00	$7,539.19
7	$7,539.19	$75.39	$7,614.59	$500.00	$7,114.59
8	$7,114.59	$71.15	$7,185.73	$500.00	$6,685.73
9	$6,685.73	$66.86	$6,752.59	$500.00	$6,252.59
10	$6,252.59	$62.53	$6,315.11	$500.00	$5,815.11
11	$5,815.11	$58.15	$5,873.27	$500.00	$5,373.27
12	$5,373.27	$53.73	$5,427.00	$500.00	$4,927.00
13	$4,927.00	$49.27	$4,976.27	$500.00	$4,476.27
14	$4,476.27	$44.76	$4,521.03	$500.00	$4,021.03
15	$4,021.03	$40.21	$4,061.24	$500.00	$3,561.24
16	$3,561.24	$35.61	$3,596.85	$500.00	$3,096.85
17	$3,096.85	$30.97	$3,127.82	$500.00	$2,627.82
18	$2,627.82	$26.28	$2,654.10	$500.00	$2,154.10
19	$2,154.10	$21.54	$2,175.64	$500.00	$1,675.64
20	$1,675.64	$16.76	$1,692.40	$500.00	$1,192.40
21	$1,192.40	$11.92	$1,204.32	$500.00	$704.32
22	$704.32	$7.04	$711.37	$500.00	$211.37

b)



c) The x-intercept (Time in months) lies somewhere between 22.4 and 22.5. This defines the time when the balance in the account is $0.

Section 2.2 Page 53 Question 19

a) i)
$$\begin{aligned}\frac{x^2-1}{x-1} &= \frac{(x-1)(x+1)}{x-1}\\ &= x+1,\ x \neq 1\end{aligned}$$

ii)
$$\begin{aligned}\frac{x^3-1}{x-1} &= \frac{(x-1)(x^2+x+1)}{x-1}\\ &= x^2+x+1,\ x \neq 1\end{aligned}$$

iii)
$$\begin{aligned}\frac{x^4-1}{x-1} &= \frac{(x-1)(x^3+x^2+x+1)}{x-1}\\ &= x^3+x^2+x+1,\ x \neq 1\end{aligned}$$

iv)
$$\begin{aligned}\frac{x^5-1}{x-1} &= \frac{(x-1)(x^4+x^3+x^2+x+1)}{x-1}\\ &= x^4+x^3+x^2+x+1,\ x \neq 1\end{aligned}$$

b)
$$\begin{aligned}\frac{x^n-1}{x-1} &= \frac{(x-1)(x^{n-1}+x^{n-2}+\ldots+x^2+x+1)}{x-1}\\ &= x^{n-1}+x^{n-2}+\ldots+x^2+x+1,\ x \neq 1\end{aligned}$$

c) Answers may vary.

Section 2.2 Page 53 Question 20

a) This is a geometric series with a common ratio of x.

b) Since the series $1+x+x^2+x^3+\ldots+x^k$ results when simplifying the quotient $\frac{x^{k+1}-1}{x-1}$, the formula for the sum of the geometric series with 1 as the first term and x as the common ratio is $\frac{x^{k+1}-1}{x-1}$, $x \neq 1$.

c) i) At the end of 1 month, the total value of the investment is $S_1 = 1$.

ii) At the end of 2 months, the total value of the investment is $S_2 = 1 + (1 + r)$.

iii) At the end of 3 months, the total value of the investment is $S_3 = 1 + (1 + r) + (1 + r)^2$ or $\frac{(1+r)^3-1}{(1+r)-1}$.

iv) At the end of k months, the total value is $S_k = 1 + (1 + r) + (1 + r)^2 + \ldots + (1 + r)^{k-1}$ or $\frac{(1+r)^k-1}{r}$.

d) Use $k = 120$ and $r = 0.01$.

$$\begin{aligned}S_k &= \frac{(1+r)^k-1}{r}\\ S_{120} &= \frac{(1+0.01)^{120}-1}{0.01}\\ &= \frac{(1.01)^{120}-1}{0.01}\\ &\doteq 230.038\ 69\end{aligned}$$

The total value of the investment is approximately \$230.04.

e) Each of the 1000 dollars accumulates in the same manner as the \$1 in part d). The total value of this investment is \$1000 × 230.038 69 or \$230 038.69.

Section 2.2 Page 53 Question 21

a) No. When dividing a polynomial by a linear polynomial, the remainder will be a constant (lesser degree).

b) No. Yes. When dividing a polynomial by a quadratic polynomial, the remainder can be either a linear polynomial or a constant (lesser degree).

c) The division of polynomials is considered complete when the degree of the remainder is less than the degree of the divisor.

Section 2.2 Page 53 Question 22

a) Every three months, Mario transfers $\$150 \times \frac{(1.002)^3-1}{0.002}$ or \$450.90.

b) At the end of seven years, the total value of Mario's GIC is $\$450.90 \times \frac{(1.013)^{28}-1}{0.013}$ or \$15 112.18.

2.3 The Remainder Theorem

Practise

Section 2.3 Page 59 Question 1

a)
$$\begin{aligned} P(x) &= 2x^2 - 3x - 2 \\ P(1) &= 2(1)^2 - 3(1) - 2 \\ &= 2 - 3 - 2 \\ &= -3 \end{aligned}$$

b)
$$\begin{aligned} P(x) &= 2x^2 - 3x - 2 \\ P(0) &= 2(0)^2 - 3(0) - 2 \\ &= 0 - 0 - 2 \\ &= -2 \end{aligned}$$

c)
$$\begin{aligned} P(x) &= 2x^2 - 3x - 2 \\ P(2) &= 2(2)^2 - 3(2) - 2 \\ &= 8 - 6 - 2 \\ &= 0 \end{aligned}$$

d)
$$\begin{aligned} P(x) &= 2x^2 - 3x - 2 \\ P(-2) &= 2(-2)^2 - 3(-2) - 2 \\ &= 8 + 6 - 2 \\ &= 12 \end{aligned}$$

Section 2.3 Page 59 Question 3

a)
$$\begin{aligned} P(x) &= x^2 - 5x - 3 \\ P(2) &= 2^2 - 5(2) - 3 \\ &= 4 - 10 - 3 \\ &= -9 \end{aligned}$$

The remainder is −9.

b)
$$\begin{aligned} P(x) &= 2x^2 + x - 10 \\ P(2) &= 2(2)^2 + 2 - 10 \\ &= 8 + 2 - 10 \\ &= 0 \end{aligned}$$

The remainder is 0.

c)
$$\begin{aligned} P(x) &= x^3 + 2x^2 - 8x + 1 \\ P(2) &= 2^3 + 2(2)^2 - 8(2) + 1 \\ &= 8 + 8 - 16 + 1 \\ &= 1 \end{aligned}$$

The remainder is 1.

d)
$$\begin{aligned} P(x) &= x^3 - 3x^2 + 5x - 2 \\ P(2) &= 2^3 - 3(2)^2 + 5(2) - 2 \\ &= 8 - 12 + 10 - 2 \\ &= 4 \end{aligned}$$

The remainder is 4.

e)
$$\begin{aligned} P(x) &= 2x^3 + 3x^2 - 9x - 10 \\ P(2) &= 2(2)^3 + 3(2)^2 - 9(2) - 10 \\ &= 16 + 12 - 18 - 10 \\ &= 0 \end{aligned}$$

The remainder is 0.

f)
$$\begin{aligned} P(x) &= 3x^3 - 12x + 2 \\ P(2) &= 3(2)^3 - 12(2) + 2 \\ &= 24 - 24 + 2 \\ &= 2 \end{aligned}$$

The remainder is 2.

Section 2.3 Page 59 Question 5

a)
$$\begin{aligned} P(x) &= x^2 + 2x + 4 \\ P(2) &= (2)^2 + 2(2) + 4 \\ &= 4 + 4 + 4 \\ &= 12 \end{aligned}$$

The remainder is 12.

b)
$$\begin{aligned} P(n) &= 4n^2 + 7n - 5 \\ P(-3) &= 4(-3)^2 + 7(-3) - 5 \\ &= 36 - 21 - 5 \\ &= 10 \end{aligned}$$

The remainder is 10.

c)
$$\begin{aligned} P(x) &= x^3 + 2x^2 - 3x + 1 \\ P(1) &= 1^3 + 2(1)^2 - 3(1) + 1 \\ &= 1 + 2 - 3 + 1 \\ &= 1 \end{aligned}$$

The remainder is 1.

d)
$$\begin{aligned} P(x) &= x^3 + 6x^2 - 3x^2 - x + 8 \\ P(-1) &= (-1)^3 + 6(-1)^2 - 3(-1)^2 - (-1) + 8 \\ &= -1 + 6 - 3 + 1 + 8 \\ &= 11 \end{aligned}$$

The remainder is 11.

e)
$$\begin{aligned} P(w) &= 2w^3 + 3w^2 - 5w + 2 \\ P(-3) &= 2(-3)^3 + 3(-3)^2 - 5(-3) + 2 \\ &= -54 + 27 + 15 + 2 \\ &= -10 \end{aligned}$$

The remainder is −10.

f)
$$\begin{aligned} P(y) &= y^3 - 8 \\ P(-2) &= (-2)^3 - 8 \\ &= -8 - 8 \\ &= -16 \end{aligned}$$

The remainder is −16.

g)
$$\begin{aligned} P(x) &= x^3 + x^2 + 3 \\ P(-4) &= (-4)^3 + (-4)^2 + 3 \\ &= -64 + 16 + 3 \\ &= -45 \end{aligned}$$

The remainder is −45.

h)
$$\begin{aligned} P(x) &= 1 - x^3 \\ P(1) &= 1 - 1^3 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The remainder is 0.

i)
$$\begin{aligned} P(m) &= m^4 - 2m^3 + m^2 + 12m - 6 \\ P(2) &= 2^4 - 2(2)^3 + 2^2 + 12(2) - 6 \\ &= 16 - 16 + 4 + 24 - 6 \\ &= 22 \end{aligned}$$

The remainder is 22.

j)
$$\begin{aligned} P(t) &= 2 - 3t + t^2 + t^3 \\ P(4) &= 2 - 3(4) + 4^2 + 4^3 \\ &= 2 - 12 + 16 + 64 \\ &= 70 \end{aligned}$$

The remainder is 70.

k)
$$\begin{aligned} P(y) &= 2y^4 - 3y^2 + 1 \\ P(3) &= 2(3)^4 - 3(3)^2 + 1 \\ &= 162 - 27 + 1 \\ &= 136 \end{aligned}$$

The remainder is 136.

l)
$$\begin{aligned} P(x) &= 2 - x + x^2 - x^3 - x^4 \\ P(-2) &= 2 - (-2) + (-2)^2 - (-2)^3 - (-2)^4 \\ &= 2 + 2 + 4 + 8 - 16 \\ &= 0 \end{aligned}$$

The remainder is 0.

m)
$$\begin{aligned} P(x) &= 3x^2 - \sqrt{2}x + 3 \\ P(-\sqrt{2}) &= 3(-\sqrt{2})^2 - \sqrt{2}(-\sqrt{2}) + 3 \\ &= 6 + 2 + 3 \\ &= 11 \end{aligned}$$

The remainder is 11.

n)
$$\begin{aligned} P(r) &= 2r^4 - 4r^2 - 9 \\ P(\sqrt{3}) &= 2(\sqrt{3})^4 - 4(\sqrt{3})^2 - 9 \\ &= 18 - 12 - 9 \\ &= -3 \end{aligned}$$

The remainder is −3.

Section 2.3 Page 59 Question 7

a)
$$\begin{aligned} P(x) &= kx^2 + 3x + 1 \\ P(-2) &= 3 \\ k(-2)^2 + 3(-2) + 1 &= 3 \\ 4k - 5 &= 3 \\ 4k &= 8 \\ k &= 2 \end{aligned}$$

b)
$$\begin{aligned} P(x) &= x^3 + x^2 + kx - 17 \\ P(2) &= 3 \\ 2^3 + 2^2 + k(2) - 17 &= 3 \\ 2k - 5 &= 3 \\ 2k &= 8 \\ k &= 4 \end{aligned}$$

c)
$$\begin{aligned} P(x) &= x^3 + 4x^2 - x + k \\ P(1) &= 3 \\ 1^3 + 4(1)^2 - 1 + k &= 3 \\ k &= -1 \end{aligned}$$

d)
$$\begin{aligned} P(x) &= x^3 + kx^2 + x + 2 \\ P(-1) &= 3 \\ (-1)^3 + k(-1)^2 + (-1) + 2 &= 3 \\ k &= 3 \end{aligned}$$

Section 2.3 Page 59 Question 9

$$\begin{aligned} P(x) &= px^3 - x^2 + qx - 2 \\ P(1) &= 0 \\ p(1)^3 - 1^2 + q(1) - 2 &= 0 \\ p - 1 + q - 2 &= 0 \\ p + q &= 3 && (1) \\ P(-2) &= -18 \\ p(-2)^3 - (-2)^2 + q(-2) - 2 &= -18 \\ -8p - 4 - 2q - 2 &= -18 \\ 8p + 2q &= 12 \\ 4p + q &= 6 && (2) \end{aligned}$$

Subtract (1) from (2).

$$\begin{aligned} 3p &= 3 \\ p &= 1 && (3) \end{aligned}$$

Substitute (3) into (1).

$$\begin{aligned} 1 + q &= 3 \\ q &= 2 \end{aligned}$$

A check by substitution confirms $p = 1$ and $q = 2$.

Apply, Solve, Communicate

Section 2.3 Page 59 Question 11

Let $P(x) = 2x^3 + 4x^2 - kx + 5$.

$$\begin{aligned} P(-3) &= 2(-3)^3 + 4(-3)^2 - k(-3) + 5 \\ &= -54 + 36 + 3k + 5 \\ &= 3k - 13 && (1) \end{aligned}$$

Let $Q(y) = 6y^3 - 3y^2 + 2y + 7$.

$$\begin{aligned} Q\left(\frac{1}{2}\right) &= 6\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) + 7 \\ &= \frac{3}{4} - \frac{3}{4} + 1 + 7 \\ &= 8 && (2) \end{aligned}$$

Equate (1) and (2).

$$\begin{aligned} 3k - 13 &= 8 \\ 3k &= 21 \\ k &= 7 \end{aligned}$$

Section 2.3 Page 60 Question 12

$$\begin{aligned} P(x) &= kx^3 - 3x^2 + 5x - 8 && (1) \\ P(2) &= 22 \\ k(2)^3 - 3(2)^2 + 5(2) - 8 &= 22 \\ 8k - 12 + 10 - 8 &= 22 \\ 8k &= 32 \\ k &= 4 && (2) \end{aligned}$$

Substitute (2) into (1).

$$P(x) = 4x^3 - 3x^2 + 5x - 8$$

Determine the remainder of $P(x)$ when divided by $x + 1$.

$$\begin{aligned} P(-1) &= 4(-1)^3 - 3(-1)^2 + 5(-1) - 8 \\ &= -4 - 3 - 5 - 8 \\ &= -20 \end{aligned}$$

The remainder is -20.

Section 2.3 Page 60 Question 13

a)
$$\begin{aligned} A(h) &= h^2 + 0.5h \\ A\left(\frac{7}{2}\right) &= \left(\frac{7}{2}\right)^2 + \frac{1}{2}\left(\frac{7}{2}\right) \\ &= \frac{49}{4} + \frac{7}{4} \\ &= \frac{56}{4} \\ &= 14 \end{aligned}$$

The remainder is 14.

b) When the height is $\frac{7}{2}$, the area is 14.

Section 2.3 Page 60 Question 14

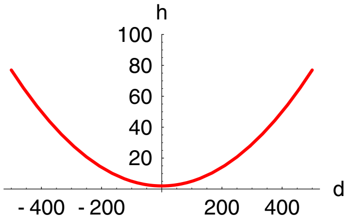
a)
$$\begin{aligned} P(n) &= 6n^2 - 5n + 8 \\ P\left(-\frac{1}{2}\right) &= 6\left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) + 8 \\ &= \frac{6}{4} + \frac{5}{2} + \frac{16}{2} \\ &= \frac{3}{2} + \frac{5}{2} + \frac{16}{2} \\ &= 12 \end{aligned}$$

The remainder is 12.

Section 2.3 Page 60 Question 15

a)
$$\begin{aligned} h(d) &= 0.0003d^2 + 2 \\ h(500) &= 0.0003(500)^2 + 2 \\ &= 77 \end{aligned}$$

The remainder is 77.



b)
$$\begin{aligned} h(d) &= 0.0003d^2 + 2 \\ h(-500) &= 0.0003(-500)^2 + 2 \\ &= 77 \end{aligned}$$

The remainder is 77.

c) The results are equal.

d) At 500 m either side of the centre point, the cable is 77 m above the roadway.

Section 2.3 Page 60 Question 16

a)
$$\begin{array}{r} -0.017d + 0.45 \\ d - 50\overline{\smash{)}-0.017d^2 + 1.3d + 2.5} \\ \underline{-0.017d^2 + 0.85d} \\ 0.45d + 2.5 \\ \underline{0.45d - 22.5} \\ 25 \end{array}$$

b) When the horizontal distance is 50 m, the height is 25 m.

c)
$$\begin{array}{r} -0.017d - 0.06 \\ d - 80\overline{\smash{)}-0.017d^2 + 1.3d + 2.5} \\ \underline{-0.017d^2 + 1.36d} \\ -0.06d + 2.5 \\ \underline{-0.06d - 4.8} \\ -2.3 \end{array}$$

d) Since the hammer cannot have a negative height, it must have landed closer than 80 m from the thrower.

Section 2.3 Page 60 Question 17

a)
$$\begin{array}{r} 0.5t + 3 \\ t - 2\overline{\smash{)}0.5t^2 + 2t + 0} \\ \underline{0.5t^2 - t} \\ 3t + 0 \\ \underline{3t - 6} \\ 6 \end{array}$$

$Q(t) = 0.5t + 3$, $R = 6$. $Q(t)$ is the average number of tickets issued between 2 h and time t. R is the number of tickets issued in the first 2 h.

b) Answers will vary.

c) The formula is impossible. It would require issuing a negative number of tickets in each of the last 3 h.

Section 2.3 Page 60 Question 18

a) Answers will vary.

b)

$$\begin{array}{r}
0.1t + 10.5 \\
t-5\overline{)\,0.1t^2 + \;10t + \;\;25} \\
\underline{0.1t^2 - 0.5t} \\
10.5t + \;\;25 \\
\underline{10.5t - 52.5} \\
77.5
\end{array}$$

$Q(t) = 0.1t + 10.5$, $R = 77.5$. $Q(t)$ represents the average number of trees sold from December 5 to day t. R is the number of trees sold by December 5.

c) No. The demand for trees falls sharply after December 24.

Section 2.3 Page 60 Question 19

$$\begin{aligned}
f(1) &= f(-2) \\
1^3 + 6(1)^2 + k(1) - 4 &= (-2)^3 + 6(-2)^2 + k(-2) - 4 \\
1 + 6 + k &= -8 + 24 - 2k \\
3k &= 9 \\
k &= 3
\end{aligned}$$

Section 2.3 Page 60 Question 20

$$\begin{aligned}
P(x) &= x^2 + 5x + 7 \\
P(-k) &= 3 \\
(-k)^2 + 5(-k) + 7 &= 3 \\
k^2 - 5k + 7 &= 3 \\
k^2 - 5k + 4 &= 0 \\
(k-1)(k-4) &= 0 \\
k &= 1 \text{ or } 4
\end{aligned}$$

Section 2.3 Page 60 Question 21

a) $x-a$ is a factor of bx^2+cx+d and $ba^2+ca+d = 0$.

b)

$$\begin{aligned}
P(x) &= bx^2 + cx + d \\
P(a) &= 0 \\
b(a)^2 + c(a) + d &= 0 \\
ba^2 + ca + d &= 0 \\
a &= \frac{-c \pm \sqrt{c^2 - 4bd}}{2b}
\end{aligned}$$

Section 2.3 Page 60 Question 22

a) Quotient: $-Q(x)$; Remainder: R.

b) Answers will vary.

$$\begin{aligned}
\frac{x^2 + 2x + 3}{x - 1} &= x + 3 + \frac{6}{x - 1} \\
\frac{x^2 + 2x + 3}{-x + 1} &= -x - 3 + \frac{6}{-x + 1}
\end{aligned}$$

c)

$$\begin{aligned}
P(x) &= (x - b)Q(x) + R \\
\frac{P(x)}{x - b} &= Q(x) + \frac{R}{x - b} \\
-\frac{P(x)}{x - b} &= -\left(Q(x) + \frac{R}{x - b}\right) \\
\frac{P(x)}{b - x} &= -Q(x) + \frac{R}{b - x}
\end{aligned}$$

Section 2.3 Page 61 Question 23

Answers will vary.

a)

$$\begin{aligned}
P(x) &= (x + 1)(x - 3) - 4 \\
&= x^2 - 2x - 7
\end{aligned}$$

Check.

$$\begin{aligned}
P(3) &= (3 + 1)(3 - 3) - 4 \\
&= -4
\end{aligned}$$

b)

$$\begin{aligned}
Q(x) &= (x^2 + x + 1)(x + 2) + 3 \\
&= x^3 + 2x^2 + x^2 + 2x + x + 2 + 3 \\
&= x^3 + 3x^2 + 3x + 5
\end{aligned}$$

Check.

$$\begin{aligned}
Q(-2) &= (-2)^3 + 3(-2)^2 + 3(-2) + 5 \\
&= -8 + 12 - 6 + 5 \\
&= 3
\end{aligned}$$

c)

$$\begin{aligned}
R(x) &= x^3(2x - 1) + 1 \\
&= 2x^4 - x^3 + 1
\end{aligned}$$

Check.

$$\begin{aligned}
R\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^3 + 1 \\
&= \frac{1}{8} - \frac{1}{8} + 1 \\
&= 1
\end{aligned}$$

Section 2.3 Page 61 Question 24

a)

$$
\begin{array}{r}
x^2 \qquad -1 \\
x^2+0x+1\,\big)\overline{x^4+0x^3+0x^2+0x-1} \\
\underline{x^4+0x^3+x^2} \\
-x^2+0x-1 \\
\underline{-x^2+0x-1} \\
0
\end{array}
$$

Quotient: x^2-1; Remainder: 0

b)

$$
\begin{array}{r}
x+2 \\
x^2+0x+1\,\big)\overline{x^3+2x^2+5x+1} \\
\underline{x^3+0x^2+x} \\
2x^2+4x+1 \\
\underline{2x^2+0x+2} \\
4x-1
\end{array}
$$

Quotient: $x+2$; Remainder: $4x-1$

c)

$$
\begin{array}{r}
x^6 \qquad +x^4 \qquad +x^2 \qquad +1 \\
x^2+0x-1\,\big)\overline{x^8+0x^7+0x^6+0x^5+0x^4+0x^3+0x^2+0x-1} \\
\underline{x^8+0x^7-x^6} \\
x^6+0x^5+0x^4 \\
\underline{x^6+0x^5-x^4} \\
x^4+0x^3+0x^2 \\
\underline{x^4+0x^2-x^2} \\
x^2+0x-1 \\
\underline{x^2+0x-1} \\
0
\end{array}
$$

Quotient: $x^6+x^4+x^2+1$; Remainder: 0; Restriction: $x \neq \pm 1$

d)

$$
\begin{array}{r}
2x-3 \\
x^2-x+0\,\big)\overline{2x^3-5x^2+6x-7} \\
\underline{2x^3-2x^2+0x} \\
-3x^2+6x-7 \\
\underline{-3x^2+3x+0} \\
3x-7
\end{array}
$$

Quotient: $2x-3$; Remainder: $3x-7$; Restriction: $x \neq 0, 1$

Section 2.3 Page 61 Question 25

The remainder upon division of $P(x)$ by x^2+a can be found by replacing x^2 with $-a$ wherever it occurs in $P(x)$. The remainder upon division of $P(x)$ by x^2+bx can be found by replacing x^2 with $-bx$ wherever it occurs in $P(x)$. The conjectures can be justified using the division statements $P(x) = Q(x)(x^2+a)+R(x)$ and $P(x) = Q(x)(x^2+bx)+R(x)$.

2.4 The Factor Theorem

Practise

Section 2.4 Page 68 Question 1

a)
$$\begin{aligned} P(x) &= x^3 - 3x^2 + 4x - 2 \\ P(1) &= 1^3 - 3(1)^2 + 4(1) - 2 \\ &= 1 - 3 + 4 - 2 \\ &= 0 \end{aligned}$$

Since $P(1) = 0$, $x - 1$ is a factor of $P(x)$.

b)
$$\begin{aligned} P(x) &= 2x^3 - x^2 - 3x - 2 \\ P(1) &= 2(1)^3 - 1^2 - 3(1) - 2 \\ &= 2 - 1 - 3 - 2 \\ &= -4 \end{aligned}$$

Since $P(1) \neq 0$, $x - 1$ is not a factor of $P(x)$.

c)
$$\begin{aligned} P(x) &= 3x^3 - x - 3 \\ P(1) &= 3(1)^3 - (1) - 3 \\ &= 3 - 1 - 3 \\ &= -1 \end{aligned}$$

Since $P(1) \neq 0$, $x - 1$ is not a factor of $P(x)$.

d)
$$\begin{aligned} P(x) &= 2x^3 + 4x^2 - 5x - 1 \\ P(1) &= 2(1)^3 + 4(1)^2 - 5(1) - 1 \\ &= 2 + 4 - 5 - 1 \\ &= 0 \end{aligned}$$

Since $P(1) = 0$, $x - 1$ is a factor of $P(x)$.

Section 2.4 Page 68 Question 3

a)
$$\begin{aligned} P(x) &= x^3 + 2x^2 + 2x + 1 \\ P(-1) &= (-1)^3 + 2(-1)^2 + 2(-1) + 1 \\ &= -1 + 2 - 2 + 1 \\ &= 0 \end{aligned}$$

Since $P(-1) = 0$, $x + 1$ is a factor of $P(x)$.

b)
$$\begin{aligned} P(x) &= x^3 - 3x^2 + 4x - 4 \\ P(2) &= 2^3 - 3(2)^2 + 4(2) - 4 \\ &= 8 - 12 + 8 - 4 \\ &= 0 \end{aligned}$$

Since $P(2) = 0$, $x - 2$ is a factor of $P(x)$.

c)
$$\begin{aligned} P(m) &= m^3 - 3m^2 + m - 3 \\ P(3) &= 3^3 - 3(3)^2 + 3 - 3 \\ &= 27 - 27 + 3 - 3 \\ &= 0 \end{aligned}$$

Since $P(3) = 0$, $m - 3$ is a factor of $P(m)$.

d)
$$\begin{aligned} P(x) &= x^3 + 7x^2 + 17x + 15 \\ P(-3) &= (-3)^3 + 7(-3)^2 + 17(-3) + 15 \\ &= -27 + 63 - 51 + 15 \\ &= 0 \end{aligned}$$

Since $P(-3) = 0$, $x + 3$ is a factor of $P(x)$.

e)
$$\begin{aligned} P(x) &= 4x^2 - 9x - 9 \\ P(3) &= 4(3)^2 - 9(3) - 9 \\ &= 36 - 27 - 9 \\ &= 0 \end{aligned}$$

Since $P(3) = 0$, $x - 3$ is a factor of $P(x)$.

f)
$$\begin{aligned} P(x) &= 6x^2 - 11x - 17 \\ P(-1) &= 6(-1)^2 - 11(-1) - 17 \\ &= 6 + 11 - 17 \\ &= 0 \end{aligned}$$

Since $P(-1) = 0$, $x + 1$ is a factor of $P(x)$.

g)
$$\begin{aligned} P(y) &= 2y^3 - 5y^2 + 2y + 1 \\ P(1) &= 2(1)^3 - 5(1)^2 + 2(1) + 1 \\ &= 2 - 5 + 2 + 1 \\ &= 0 \end{aligned}$$

Since $P(1) = 0$, $y - 1$ is a factor of $P(y)$.

h)
$$\begin{aligned} P(x) &= x^3 - 6x - 4 \\ P(-2) &= (-2)^3 - 6(-2) - 4 \\ &= -8 + 12 - 4 \\ &= 0 \end{aligned}$$

Since $P(-2) = 0$, $x + 2$ is a factor of $P(x)$.

Section 2.4 Page 68 Question 5

a)

$$\begin{aligned} P(x) &= 2x^3 + x^2 + 2x + 1 \\ P\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) + 1 \\ &= -\frac{1}{4} + \frac{1}{4} - 1 + 1 \\ &= 0 \end{aligned}$$

Since $P\left(-\frac{1}{2}\right) = 0,\ 2x + 1$ is a factor of $P(x)$.

b)

$$\begin{aligned} P(x) &= 2x^3 - 3x^2 - 2x + 3 \\ P\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 - 3\left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right) + 3 \\ &= \frac{27}{4} - \frac{27}{4} - 3 + 3 \\ &= 0 \end{aligned}$$

Since $P\left(\frac{3}{2}\right) = 0,\ 2x - 3$ is a factor of $P(x)$.

c)

$$\begin{aligned} P(y) &= 3y^3 + 8y^2 + 3y - 2 \\ P\left(\frac{1}{3}\right) &= 3\left(\frac{1}{3}\right)^3 + 8\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) - 2 \\ &= \frac{1}{9} + \frac{8}{9} + 1 - 2 \\ &= 0 \end{aligned}$$

Since $P\left(\frac{1}{3}\right) = 0,\ 3y - 1$ is a factor of $P(y)$.

d)

$$\begin{aligned} P(n) &= 6n^3 - 7n^2 + 1 \\ P\left(-\frac{1}{3}\right) &= 6\left(-\frac{1}{3}\right)^3 - 7\left(-\frac{1}{3}\right)^2 + 1 \\ &= -\frac{2}{9} - \frac{7}{9} + 1 \\ &= 0 \end{aligned}$$

Since $P\left(-\frac{1}{3}\right) = 0,\ 3n + 1$ is a factor of $P(n)$.

e)

$$\begin{aligned} P(x) &= 8x^2 - 2x - 1 \\ P\left(\frac{1}{2}\right) &= 8\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) - 1 \\ &= 2 - 1 - 1 \\ &= 0 \end{aligned}$$

Since $P\left(\frac{1}{2}\right) = 0,\ 2x - 1$ is a factor of $P(x)$.

f)

$$\begin{aligned} P(x) &= 3x^3 - x^2 - 3x + 1 \\ P\left(\frac{1}{3}\right) &= 3\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right) + 1 \\ &= \frac{1}{9} - \frac{1}{9} - 1 + 1 \\ &= 0 \end{aligned}$$

Since $P\left(\frac{1}{3}\right) = 0,\ 3x - 1$ is a factor of $P(x)$.

Section 2.4 Page 68 Question 7

a) The possible roots are $\pm 1, \pm\frac{1}{2}, \pm 3, \pm\frac{3}{2}$.

$$\begin{aligned} P(x) &= 2x^3 - 9x^2 + 10x - 3 \\ P(1) &= 2(1)^3 - 9(1)^2 + 10(1) - 3 \\ &= 2 - 9 + 10 - 3 \\ &= 0 \end{aligned}$$

$$\begin{array}{r|rrrr} 1 & 2 & -9 & 10 & -3 \\ & & 2 & -7 & 3 \\ \hline & 2 & -7 & 3 & 0 \end{array}$$

Since $P(1) = 0,\ x - 1$ is one factor of $P(x)$. Synthetic division yields another factor of $2x^2 - 7x + 3$.

$$\begin{aligned} P(x) &= (x - 1)(2x^2 - 7x + 3) \\ &= (x - 1)(2x - 1)(x - 3) \end{aligned}$$

So, $2x^3 - 9x^2 + 10x - 3 = (x - 1)(2x - 1)(x - 3)$.

b) The possible roots are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}$.

$$\begin{aligned} P(y) &= 4y^3 - 7y - 3 \\ P(-1) &= 4(-1)^3 - 7(-1) - 3 \\ &= -4 + 7 - 3 \\ &= 0 \end{aligned}$$

$$\begin{array}{r|rrrr} -1 & 4 & 0 & -7 & -3 \\ & & -4 & 4 & 3 \\ \hline & 4 & -4 & -3 & 0 \end{array}$$

Since $P(-1) = 0,\ y + 1$ is one factor of $P(y)$. Synthetic division yields another factor of $4y^2 - 4y - 3$.

$$\begin{aligned} P(y) &= (y + 1)(4y^2 - 4y - 3) \\ &= (y + 1)(2y + 1)(2y - 3) \end{aligned}$$

So, $4y^3 - 7y - 3 = (y + 1)(2y + 1)(2y - 3)$.

c) The possible roots are $\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 3, \pm 6$.

$$\begin{aligned} P(x) &= 3x^3 - 4x^2 - 17x + 6 \\ P(-2) &= 3(-2)^3 - 4(-2)^2 - 17(-2) + 6 \\ &= -24 - 16 + 34 + 6 \\ &= 0 \end{aligned}$$

$$\begin{array}{r|rrrr} -2 & 3 & -4 & -17 & 6 \\ & & -6 & 20 & -6 \\ \hline & 3 & -10 & 3 & 0 \end{array}$$

Since $P(-2) = 0,\ x + 2$ is one factor of $P(x)$. Synthetic division yields another factor of $3x^2 - 10x + 3$.

$$\begin{aligned} P(x) &= (x + 2)(3x^2 - 10x + 3) \\ &= (x + 2)(3x - 1)(x - 3) \end{aligned}$$

So, $3x^3 - 4x^2 - 17x + 6 = (x + 2)(3x - 1)(x - 3)$.

d) The possible roots are $\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}, \pm 8, \pm\frac{8}{3}$.

$$\begin{aligned} P(x) &= 3x^3 - 2x^2 - 12x + 8 \\ P(2) &= 3(2)^3 - 2(2)^2 - 12(2) + 8 \\ &= 24 - 8 - 24 + 8 \\ &= 0 \end{aligned}$$

$$\begin{array}{r|rrrr} 2 & 3 & -2 & -12 & 8 \\ & & 6 & 8 & -8 \\ \hline & 3 & 4 & -4 & 0 \end{array}$$

Since $P(2) = 0,\ x - 2$ is one factor of $P(x)$. Synthetic division yields another factor of $3x^2 + 4x - 4$.

$$\begin{aligned} P(x) &= (x - 2)(3x^2 + 4x - 4) \\ &= (x - 2)(3x - 2)(x + 2) \end{aligned}$$

So, $3x^3 - 2x^2 - 12x + 8 = (x - 2)(3x - 2)(x + 2)$.

e) The possible roots are $\pm 1, \pm\frac{1}{2}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm 4, \pm 6, \pm 12$.

$$\begin{aligned} P(x) &= 2x^3 + 13x^2 + 23x + 12 \\ P(-1) &= 2(-1)^3 + 13(-1)^2 + 23(-1) + 12 \\ &= -2 + 13 - 23 + 12 \\ &= 0 \end{aligned}$$

$$\begin{array}{r|rrrr} -1 & 2 & 13 & 23 & 12 \\ & & -2 & -11 & -12 \\ \hline & 2 & 11 & 12 & 0 \end{array}$$

Since $P(-1) = 0,\ x + 1$ is one factor of $P(x)$. Synthetic division yields another factor of $2x^2 + 11x + 12$.

$$\begin{aligned} P(x) &= (x + 1)(2x^2 + 11x + 12) \\ &= (x + 1)(2x + 3)(x + 4) \end{aligned}$$

So, $2x^3 + 13x^2 + 23x + 12 = (x + 1)(2x + 3)(x + 4)$.

f) The possible roots are $\pm 1, \pm\frac{1}{2}, \pm 2, \pm 5, \pm\frac{5}{2}, \pm 10$.

$$\begin{aligned} P(x) &= 2x^3 - 3x^2 + 3x - 10 \\ P(2) &= 2(2)^3 - 3(2)^2 + 3(2) - 10 \\ &= 16 - 12 + 6 - 10 \\ &= 0 \end{aligned}$$

2	2	-3	3	-10
		4	2	10
	2	1	5	0

Since $P(2) = 0,\ x - 2$ is one factor of $P(x)$. Synthetic division yields another factor of $2x^2 + x + 5$.

$$P(x) = (x - 2)(2x^2 + x + 5)$$

$2x^2 + x + 5$ cannot be factored. So, $2x^3 + 13x^2 + 23x + 12 = (x - 2)(2x^2 + x + 5)$.

g) The possible roots are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 3, \pm\frac{3}{2}, \pm 5, \pm\frac{5}{2}, \pm\frac{5}{3}, \pm\frac{5}{6}, \pm 15, \pm\frac{15}{2}$.

$$\begin{aligned} P(x) &= 6x^3 - 11x^2 - 26x + 15 \\ P(3) &= 6(3)^3 - 11(3)^2 - 26(3) + 15 \\ &= 162 - 99 - 78 + 15 \\ &= 0 \end{aligned}$$

3	6	-11	-26	15
		18	21	-15
	6	7	-5	0

Since $P(3) = 0,\ x - 3$ is one factor of $P(x)$. Synthetic division yields another factor of $6x^2 + 7x - 5$.

$$\begin{aligned} P(x) &= (x - 3)(6x^2 + 7x - 5) \\ &= (x - 3)(2x - 1)(3x + 5) \end{aligned}$$

So, $6x^3 - 11x^2 - 26x + 15 = (x - 3)(2x - 1)(3x + 5)$.

h) The possible roots are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2$.

$$\begin{aligned} P(y) &= 4y^3 + 8y^2 - y - 2 \\ P(-2) &= 4(-2)^3 + 8(-2)^2 - (-2) - 2 \\ &= -32 + +32 + 2 - 2 \\ &= 0 \end{aligned}$$

-2	4	8	-1	-2
		-8	0	2
	4	0	-1	0

Since $P(-2) = 0,\ y + 2$ is one factor of $P(y)$. Synthetic division yields another factor of $4y^2 - 1$.

$$\begin{aligned} P(y) &= (y + 2)(4y^2 - 1) \\ &= (y + 2)(2y - 1)(2y + 1) \end{aligned}$$

So, $4y^3 + 8y^2 - y - 2 = (y + 2)(2y - 1)(2y + 1)$.

i) The coefficients suggest a factor by grouping strategy.

$$\begin{aligned} P(x) &= 4x^3 + 3x^2 - 4x - 3 \\ &= x^2(4x + 3) - 1(4x + 3) \\ &= (4x + 3)(x^2 - 1) \\ &= (4x + 3)(x - 1)(x + 1) \end{aligned}$$

j) The possible roots are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 5, \pm\frac{5}{2}, \pm\frac{5}{3}, \pm\frac{5}{6}$.

$$\begin{aligned} P(w) &= 6w^3 + 16w^2 - 21w + 5 \\ P\left(\frac{1}{3}\right) &= 6\left(\frac{1}{3}\right)^3 + 16\left(\frac{1}{3}\right)^2 - 21\left(\frac{1}{3}\right) + 5 \\ &= \frac{2}{9} + \frac{16}{9} - 7 + 5 \\ &= 0 \end{aligned}$$

Since $P\left(\frac{1}{3}\right) = 0$, $3w - 1$ is one factor of $P(w)$. Long division yields another factor of $2w^2 + 6w - 5$. Since $b^2 - 4ac = 76$, which is not a perfect square, $2w^2 + 6w - 5$ cannot be factored further.

So, $6w^3 + 16w^2 - 21w + 5 = (3w - 1)(2w^2 + 6w - 5)$.

Apply, Solve, Communicate

Section 2.4 Page 68 Question 8

For this solution, the model $A(t) = 3t^2 + 2t$ defines the total amount of forest cut down after t years.

a) $A(2) = 3(2)^2 + 2(2)$ or 16. $A(2)$ is the number of millions of hectares of forest cut down in 2002.

b)

$$\begin{aligned} A(5) - A(2) &= 3(5)^2 + 2(5) - \left[3(2)^2 + 2(2\right] \\ &= 85 - 16 \\ &= 69 \end{aligned}$$

$A(5) - A(2)$ is the number of millions of hectares of forest cut down between 2002 and 2005.

c)

$$\begin{aligned} \frac{A(5) - A(2)}{5 - 2} &= \frac{3(5)^2 + 2(5) - \left[3(2)^2 + 2(2\right]}{3} \\ &= \frac{85 - 16}{3} \\ &= \frac{69}{3} \\ &= 23 \end{aligned}$$

$\frac{A(5) - A(2)}{5 - 2}$ is the *average* area of forest cut down between 2002 and 2005.

d) $\frac{A(t) - A(2)}{t - 2}$ is the *average* area of forest cut down between 2002 and $2000 + t$.

e) Answers may vary.

Section 2.4 Page 68 Question 9

a) i)

$$\begin{aligned} f(x) - f(b) &= f(x) - f(3) \\ &= x^2 - 3^2 \\ &= x^2 - 9 \\ &= (x - 3)(x + 3) \end{aligned}$$

ii)

$$\begin{aligned} \frac{f(x) - f(b)}{x - b} &= \frac{f(x) - f(3)}{x - 3} \\ &= \frac{(x - 3)(x + 3)}{x - 3} \\ &= x + 3,\ x \neq 3 \end{aligned}$$

b) i)

$$\begin{aligned} f(x) - f(b) &= f(x) - f(2) \\ &= x^2 + 2x - 1 - (2^2 + 2(2) - 1) \\ &= x^2 + 2x - 1 - 7 \\ &= x^2 + 2x - 8 \\ &= (x + 4)(x - 2) \end{aligned}$$

ii)

$$\begin{aligned} \frac{f(x) - f(b)}{x - b} &= \frac{f(x) - f(3)}{x - 2} \\ &= \frac{(x + 4)(x - 2)}{x - 2} \\ &= x + 4,\ x \neq 2 \end{aligned}$$

c) i)

$$\begin{aligned} f(x) - f(b) &= f(x) - f(-1) \\ &= 2x^3 + 4x^2 - 5x - 7 - (2(-1)^3 + 4(-1)^2 - 5(-1) - 7) \\ &= 2x^3 + 4x^2 - 5x - 7 \\ &= (x + 1)(2x^2 + 2x - 7) \end{aligned}$$

ii)

$$\begin{aligned} \frac{f(x) - f(b)}{x - b} &= \frac{f(x) - f(-1)}{x - (-1)} \\ &= \frac{(x + 1)(2x^2 + 2x - 7)}{x + 1} \\ &= 2x^2 + 2x - 7,\ x \neq 1 \end{aligned}$$

Section 2.4 Page 68 Question 10

a) $\dfrac{f(x)}{x-b} = Q(x) + \dfrac{R(x)}{x-b}$

b)
$$\begin{aligned}\frac{f(x)}{x-b} &= Q(x) + \frac{R(x)}{x-b}\\ f(x) &= Q(x)(x-b) + R(x)\\ R(x) &= f(x) - Q(x)(x-b)\\ R(b) &= f(b) - Q(b)(b-b)\\ &= f(b)\end{aligned}$$

The division statement can be rewritten as $\dfrac{f(x)}{x-b} = Q(x) + \dfrac{f(b)}{x-b}$.

c) $Q(x) = \dfrac{f(x)-f(b)}{x-b}$

d) Yes; reasons will vary. The numerator is always divisible by the denominator, $x - b$.

e) $\dfrac{f(x)-f(b)}{x-b}$ defines the slope of the line from the point $(b, f(b))$ to the point $(x, f(x))$.

Section 2.4 Page 69 Question 11

For this solution, the model $M(t) = 5t^3 + 2t^2 + 10t$ defines the total emissions after t years.

a)
$$\begin{aligned}M(t) &= 5t^3 + 2t^2 + 10t\\ M(4) &= 5(4)^3 + 2(4)^2 + 10(4)\\ &= 320 + 32 + 40\\ &= 392\end{aligned}$$

The mass of emissions in the year 2004 is 392 million tonnes.

b)
$$\begin{aligned}M(7) - M(4) &= 5(7)^3 + 2(7)^2 + 10(7) - \left[5(4)^3 + 2(4)^2 + 10(4)\right]\\ &= 1715 + 98 + 70 - 392\\ &= 1491\end{aligned}$$

The total mass of emissions from 2004 to 2007 is 1491 million tonnes.

c)
$$\begin{aligned}\frac{M(7) - M(4)}{7-4} &= \frac{1491}{3}\\ &= 497\end{aligned}$$

The average mass of emissions from 2004 to 2007 is 497 million tonnes per year.

d) $\dfrac{M(t) - M(1)}{t-1}$ represents the average mass emitted over the period from 2001 to $2000 + t$, where $t > 1$.

Section 2.4 Page 69 Question 12

a)
$$\begin{aligned}P(x) &= x^4 + 4x^3 - 7x^2 - 34x - 24\\ P(-1) &= (-1)^4 + 4(-1)^3 - 7(-1)^2 - 34(-1) - 24\\ &= 1 - 4 - 7 + 34 - 24\\ &= 0\end{aligned}$$

$x + 1$ is one factor of $P(x)$. Synthetic division yields another factor of $x^3 + 3x^2 - 10x - 24$.

$$\begin{aligned}Q(x) &= x^3 + 3x^2 - 10x - 24\\ Q(-2) &= (-2)^3 + 3(-2)^2 - 10(-2) - 24\\ &= -8 + 12 + 20 - 24\\ &= 0\end{aligned}$$

$x + 2$ is also a factor of $P(x)$. Synthetic division yields another factor of $x^2 + x - 12$. Summarizing,

$$\begin{aligned}P(x) &= (x+1)Q(x)\\ &= (x+1)(x+2)(x^2 + x - 12)\\ &= (x+1)(x+2)(x+4)(x-3)\end{aligned}$$

So, $x^4 + 4x^3 - 7x^2 - 34x - 24 = (x+1)(x+2)(x+4)(x-3)$.

b) The expression can be factored using a strategy similar to a). Factoring by grouping can also be employed.

$$\begin{aligned}P(x) &= x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12\\ &= x^4(x+3) - 5x^2(x+3) + 4(x+3)\\ &= (x+3)(x^4 - 5x^2 + 4)\\ &= (x+3)(x^2-1)(x^2-4)\\ &= (x+3)(x+1)(x-1)(x+2)(x-2)\end{aligned}$$

So, $x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12 = (x+3)(x+1)(x-1)(x+2)(x-2)$.

c) Factor by grouping.

$$\begin{aligned}P(x) &= 8x^3 + 4x^2 - 2x - 1\\ &= 4x^2(2x+1) - 1(2x+1)\\ &= (2x+1)(4x^2-1)\\ &= (2x+1)(2x+1)(2x-1)\\ &= (2x+1)^2(2x-1)\end{aligned}$$

So, $8x^3 + 4x^2 - 2x - 1 = (2x+1)^2(2x-1)$.

d) Factor by grouping.

$$\begin{aligned}P(x) &= 8x^3 - 12x^2 - 2x + 3\\ &= 4x^2(2x-3) - 1(2x-3)\\ &= (2x-3)(4x^2-1)\\ &= (2x-3)(2x+1)(2x-1)\end{aligned}$$

So, $8x^3 - 12x^2 - 2x + 3 = (2x-3)(2x+1)(2x-1)$.

Section 2.4 Page 69 Question 13

$$\begin{aligned}P(x) &= x^4 + 6x^3 + 11x^2 + 6x\\ &= x(x^3 + 6x^2 + 11x + 6)\\ &= xQ(x)\\ Q(-1) &= (-1)^3 + 6(-1)^2 + 11(-1) + 6\\ &= -1 + 6 - 11 + 6\\ &= 0\end{aligned}$$

To this point, x and $x+1$ are factors of $P(x)$. Synthetic division of $Q(x)$ by $x+1$ reveals

$$\begin{aligned}P(x) &= xQ(x)\\ &= x(x+1)(x^2 + 5x + 6)\\ &= x(x+1)(x+2)(x+3)\end{aligned}$$

So, $x^4 + 6x^3 + 11x^2 + 6x = x(x+1)(x+2)(x+3)$. The expression is the product of four consecutive integers.

Section 2.4 Page 69 Question 14

a)

$$\begin{aligned}P(x) &= 4x^4 - 3x^3 - 2x^2 + kx - 9\\ P(3) &= 0\\ 4(3)^4 - 3(3)^3 - 2(3)^2 + k(3) - 9 &= 0\\ 324 - 81 - 18 + 3k - 9 &= 0\\ 3k &= -216\\ k &= -72\end{aligned}$$

b)

$$\begin{aligned}P(x) &= kx^3 - 10x^2 + 2x + 3\\ P(3) &= 0\\ k(3)^3 - 10(3)^2 + 2(3) + 3 &= 0\\ 27k - 90 + 9 &= 0\\ 27k &= 81\\ k &= 3\end{aligned}$$

Section 2.4 Page 69 Question 15

a)

$$\begin{aligned}P(2) = 0 &\iff x - 2 \text{ is a factor.}\\ P(-4) = 0 &\iff x + 4 \text{ is a factor.}\\ P\left(-\frac{1}{3}\right) = 0 &\iff 3x + 1 \text{ is a factor.}\end{aligned}$$

b) No. A third-degree polynomial can have at most three factors.

Section 2.4 Page 69 Question 16

$$
\begin{aligned}
P(x) &= 6x^3 + mx^2 + nx - 5 \\
P(-1) &= 0 \\
6(-1)^3 + m(-1)^2 + n(-1) - 5 &= 0 \\
-6 + m - n - 5 &= 0 \\
m - n &= 11 && (1) \\
P(1) &= -4 \\
6(1)^3 + m(1)^2 + n(1) - 5 &= -4 \\
6 + m + n - 5 &= -4 \\
m + n &= -5 && (2)
\end{aligned}
$$

Add (1) and (2).

$$
\begin{aligned}
2m &= 6 \\
m &= 3 && (3)
\end{aligned}
$$

Substitute (3) into (2).

$$
\begin{aligned}
3 + n &= -5 \\
n &= -8
\end{aligned}
$$

A check by substitution confirms $m = 3$ and $n = -8$.

Section 2.4 Page 69 Question 17

Factor by grouping.

$$
\begin{aligned}
P(x) &= x^3 - ax^2 + bx^2 - abx + cx - ac \\
&= x^2(x - a) + bx(x - a) + c(x - a) \\
&= (x - a)(x^2 + bx + c)
\end{aligned}
$$

Thus, $x - a$ is a factor of $x^3 - ax^2 + bx^2 - abx + cx - ac$.

Section 2.4 Page 69 Question 18

a) i)

$$
\begin{aligned}
P(x) &= x^3 - 1 \\
P(1) &= 1^3 - 1 \\
&= 0
\end{aligned}
$$

$x - 1$ is a factor of $x^3 - 1$. Synthetic division reveals another factor of $x^2 + x + 1$.

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

ii)

$$
\begin{aligned}
P(x) &= x^3 + 1 \\
P(-1) &= (-1)^3 + 1 \\
&= 0
\end{aligned}
$$

$x + 1$ is a factor of $x^3 + 1$. Synthetic division reveals another factor of $x^2 - x + 1$.

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

iii)

$$
\begin{aligned}
P(x) &= x^3 - 27 \\
P(3) &= 3^3 - 27 \\
&= 0
\end{aligned}
$$

$x - 3$ is a factor of $x^3 - 27$. Synthetic division reveals another factor of $x^2 + 3x + 9$.

$$x^3 - 27 = (x - 3)(x^2 + 3x + 9)$$

iv)

$$
\begin{aligned}
P(x) &= x^3 + 64 \\
P(-4) &= (-4)^3 + 64 \\
&= 0
\end{aligned}
$$

$x + 4$ is a factor of $x^3 + 64$. Synthetic division reveals another factor of $x^2 - 4x + 16$.

$$x^3 + 64 = (x + 4)(x^2 - 4x + 16)$$

v)

$$\begin{aligned} P(x) &= 8x^3 - 1 \\ P\left(\frac{1}{2}\right) &= 8\left(\frac{1}{2}\right)^3 - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$2x - 1$ is a factor of $8x^3 - 1$. Synthetic division reveals another factor of $4x^2 + 2x + 1$.

$$8x^3 - 1 = (2x - 1)(4x^2 + 2x + 1)$$

vi)

$$\begin{aligned} P(x) &= 64x^3 + 1 \\ P\left(-\frac{1}{4}\right) &= 64\left(-\frac{1}{4}\right)^3 + 1 \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

$4x+1$ is a factor of $64x^3+1$. Synthetic division reveals another factor of $16x^2 - 4x + 1$.

$$64x^3 + 1 = (4x + 1)(16x^2 - 4x + 1)$$

b) The expressions have the same signs. $x + y$ is a factor of $x^3 + y^3$. The other factor is $x^2 - xy + y^2$.
c) The expressions have the same signs. $x - y$ is a factor of $x^3 - y^3$. The other factor is $x^2 + xy + y^2$.

d)

$$\begin{aligned} P(x) &= 8x^3 + 125 \\ &= (2x)^3 + (5)^3 \\ &= (2x + 5)((2x)^2 - (2x)(5) + (5)^2) \\ &= (2x + 5)(4x^2 - 10x + 25) \end{aligned}$$

$$\begin{aligned} Q(x) &= 27x^3 - 64 \\ &= (3x)^3 - (4)^3 \\ &= (3x - 4)((3x)^2 + (3x)(4) + (4)^2) \\ &= (3x - 4)(9x^2 + 12x + 16) \end{aligned}$$

e)

$$\begin{aligned} P(x) &= x^6 + y^6 \\ &= (x^2)^3 + (y^2)^3 \\ &= (x^2 + y^2)(x^4 - x^2y^2 + y^4) \end{aligned}$$

f) Answers may vary. $x - y$ is not a factor of $x^{3n} + y^{3n}$; $x + y$ is a factor of $x^{3n} + y^{3n}$ if and only if n is odd.
g) Answers may vary. $x - y$ is a factor of $x^{3n} - y^{3n}$; $x + y$ is a factor of $x^{3n} - y^{3n}$ if and only if n is even.

Section 2.4 Page 69 Question 19

a) Both $x + 1$ and $x - 1$ are factors of $x^{100} - 1$.

$$\begin{aligned} P(x) &= x^{100} - 1 \\ P(\pm 1) &= (\pm 1)^{100} - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

c) Let $P(x) = x^n - 1$. Since $P(1) = 0$, $x - 1$ is a factor of every polynomial of the form $x^n - 1$.
d) Let $P(x) = x^n + 1$. Since $P(-1) = 0$ for all odd n, $x + 1$ is a factor of every polynomial of the form $x^n + 1$, where n is odd.

b)

$$\begin{aligned} P(x) &= x^{99} + 1 \\ P(-1) &= (-1)^{99} + 1 \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

$x + 1$ is a factor of $x^{99} + 1$.

$$\begin{aligned} P(1) &= (1)^{99} + 1 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Since $P(1) \neq 0,$ $x - 1$ is not a factor of $x^{99} + 1$.

Section 2.4 Page 69 Question 20

$$\begin{aligned}
V(r) &= \frac{4}{3}\pi(r^3 + 9r^2 + 27r + 27)\\
V(-3) &= \frac{4}{3}\pi((-3)^3 + 9(-3)^2 + 27(-3) + 27)\\
&= \frac{4}{3}\pi(-27 + 81 - 81 + 27)\\
&= 0
\end{aligned}$$

Since $V(-3) = 0$, $r + 3$ is a factor of $V(r)$. Synthetic division of $V(r)$ by $r + 3$ reveals another factor of $r^2 + 6r + 9$.

$$\begin{aligned}
V(r) &= \frac{4}{3}\pi(r + 3)(r^2 + 6r + 9)\\
&= \frac{4}{3}\pi(r + 3)(r + 3)(r + 3)\\
&= \frac{4}{3}\pi(r + 3)^3
\end{aligned}$$

The radius of the larger sphere is $r + 3$.

Section 2.4 Page 70 Question 21

a) i)

$$\begin{aligned}
P(x) &= x^4 + y^4\\
P(-y) &= (-y)^4 + y^4\\
&= y^4 + y^4\\
&= 2y^4
\end{aligned}$$

$x + y$ is not a factor of $x^4 + y^4$.

$$\begin{aligned}
P(y) &= (y)^4 + y^4\\
&= 2y^4
\end{aligned}$$

$x - y$ is not a factor of $x^4 + y^4$.

ii)

$$\begin{aligned}
P(x) &= x^4 - y^4\\
P(-y) &= (-y)^4 - y^4\\
&= y^4 - y^4\\
&= 0
\end{aligned}$$

$x + y$ is a factor of $x^4 - y^4$.

$$\begin{aligned}
P(y) &= (y)^4 - y^4\\
&= 0
\end{aligned}$$

$x - y$ is a factor of $x^4 - y^4$.

iii)

$$\begin{aligned}
P(x) &= x^5 + y^5\\
P(-y) &= (-y)^5 + y^5\\
&= -y^5 + y^5\\
&= 0
\end{aligned}$$

$x + y$ is a factor of $x^5 + y^5$.

$$\begin{aligned}
P(y) &= y^5 + y^5\\
&= 2y^5
\end{aligned}$$

$x - y$ is a not factor of $x^5 + y^5$.

iv)

$$\begin{aligned}
P(x) &= x^5 - y^5\\
P(-y) &= (-y)^5 - y^5\\
&= -y^5 - y^5\\
&= -2y^5
\end{aligned}$$

$x + y$ is not a factor of $x^5 - y^5$.

$$\begin{aligned}
P(y) &= y^5 - y^5\\
&= 0
\end{aligned}$$

$x - y$ is a factor of $x^5 - y^5$.

v)

$$\begin{aligned}
P(x) &= x^6 + y^6\\
P(-y) &= (-y)^6 + y^6\\
&= y^6 + y^6\\
&= 2y^6
\end{aligned}$$

$x + y$ is not a factor of $x^6 + y^6$.

$$\begin{aligned}
P(y) &= y^6 + y^6\\
&= 2y^6
\end{aligned}$$

$x - y$ is not a factor of $x^6 + y^6$.

vi)

$$\begin{aligned}
P(x) &= x^6 - y^6\\
P(-y) &= (-y)^6 - y^6\\
&= y^6 - y^6\\
&= 0
\end{aligned}$$

$x + y$ is a factor of $x^6 - y^6$.

$$\begin{aligned}
P(y) &= y^6 - y^6\\
&= 0
\end{aligned}$$

$x - y$ is a factor of $x^6 - y^6$.

vii)

$$\begin{aligned} P(x) &= x^7 + y^7 \\ P(-y) &= (-y)^7 + y^7 \\ &= -y^7 + y^7 \\ &= 0 \end{aligned}$$

$x + y$ is a factor of $x^7 + y^7$.

$$\begin{aligned} P(y) &= y^7 + y^7 \\ &= 2y^7 \end{aligned}$$

$x - y$ is a not factor of $x^7 + y^7$.

viii)

$$\begin{aligned} P(x) &= x^7 - y^7 \\ P(-y) &= (-y)^7 - y^7 \\ &= -y^7 - y^7 \\ &= -2y^7 \end{aligned}$$

$x + y$ is not a factor of $x^7 - y^7$.

$$\begin{aligned} P(y) &= y^7 - y^7 \\ &= 0 \end{aligned}$$

$x - y$ is a factor of $x^7 - y^7$.

b) $x + y$ is a factor of $x^n + y^n$ if n is odd.

$x - y$ is a factor of $x^n - y^n$ if n is odd.

$x + y$ and $x - y$ are factors of $x^n - y^n$ if n is even.

Neither $x + y$ nor $x - y$ is a factor of $x^n + y^n$ if n is even.

c) Since $x^8 - y^8$ is of the form $x^n - y^n$, where n is even, both $x + y$ and $x - y$ are factors.
d) Since $x^{11} + y^{11}$ is of the form $x^n + y^n$, where n is odd, only $x + y$ is a factor.

Section 2.4 Page 70 Question 22

If there is more than one zero, there must be three zeros. Since they are all integers, the factors must be of the form $x - m$ or $m - x$, where m is an integer. When these factors are multiplied, a must be ± 1.

Section 2.4 Page 70 Question 23

a)

$$\begin{aligned} P(x) &= ax^3 + bx^2 + cx + d \\ P(1) &= 0 \\ a(1)^3 + b(1)^2 + c(1) + d &= 0 \\ a + b + c + d &= 0 \end{aligned}$$

If $x - 1$ is a factor of $P(x)$, $a + b + c + d = 0$.

b) i) Since $3 + 5 - 6 - 2 = 0$, $x - 1$ is a factor of $3x^3 + 5x^2 - 6x - 2$.
ii) Since $2 - 9 - 1 - 8 \neq 0$, $x - 1$ is not a factor of $2x^3 - 9x^2 - x - 8$.
iii) Since $-5 + 4 + 1 = 0$, $x - 1$ is a factor of $-5x^3 + 4x + 1$.

Section 2.4 Page 70 Question 24

$x + y$ is a factor of $P(x)$ if and only if $P(-y) = 0$.

$$\begin{aligned} P(x) &= x^2(y^2 - 1) - y^2(1 + x^2) + x^2 + y^2 \\ P(-y) &= (-y)^2(y^2 - 1) - y^2(1 + (-y)^2) + (-y)^2 + y^2 \\ &= y^2(y^2 - 1) - y^2(1 + y^2) + y^2 + y^2 \\ &= y^4 - y^2 - y^2 - y^4 + y^2 + y^2 \\ &= 0 \end{aligned}$$

Section 2.4 Page 70 Question 25

a)

$$\begin{aligned} 3x^2 - 1 - \frac{2}{x^2} &= \frac{1}{x^2}(3x^4 - x^2 - 2) \\ &= \frac{1}{x^2}(3x^2 + 2)(x^2 - 1) \\ &= \frac{1}{x^2}(3x^2 + 2)(x - 1)(x + 1) \end{aligned}$$

b) $2x^2 - xy - 3y^2 = (2x - 3y)(x + y)$

c)

$$\begin{aligned} P(x) &= x^3 - 6x + \frac{11}{x} - \frac{6}{x^3} \\ &= \frac{1}{x^3}(x^6 - 6x^4 + 11x^2 - 6) \\ &= \frac{1}{x^3}((x^2)^3 - 6(x^2)^2 + 11(x^2) - 6) \end{aligned}$$

Since $P(1) = 0$, $x^2 - 1$ is a factor of $P(x)$. Use synthetic division.

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\begin{aligned} P(x) &= \frac{1}{x^3}(x^2 - 1)((x^2)^2 - 5(x^2) + 6) \\ &= \frac{1}{x^3}(x - 1)(x + 1)(x^2 - 3)(x^2 - 2) \end{aligned}$$

d)

$$\begin{aligned} P(y) &= 15 - 23\frac{x}{y} + 9\frac{x^2}{y^2} - \frac{x^3}{y^3} \\ &= \frac{1}{y^3}(15y^3 - 23xy^2 + 9x^2y - x^3) \end{aligned}$$

Since $P(x) = 0$, $y - x$ is a factor of $P(y)$. Use synthetic division.

$$\begin{array}{r|rrrr} x & 15 & -23x & 9x^2 & -x^3 \\ & & 15x & -8x^2 & x^3 \\ \hline & 15 & -8x & x^2 & 0 \end{array}$$

$$\begin{aligned} P(y) &= \frac{1}{y^3}(y - x)(15y^2 - 8xy + x^2) \\ &= \frac{1}{y^3}(y - x)(3y - x)(5y - x) \end{aligned}$$

2.5 Roots of Polynomial Equations

Practise

Section 2.5 Page 79 Question 1

a)

$$\begin{aligned}(x+1)(x-4)(x+5) &= 0\\ x &= -1,\ 4,\ \text{or } -5\end{aligned}$$

b)

$$\begin{aligned}(x-2)(x-7)(x+6) &= 0\\ x &= 2,\ 7,\ \text{or } -6\end{aligned}$$

c)

$$\begin{aligned}x(x+3)(x-8) &= 0\\ x &= 0,\ -3,\ \text{or } 8\end{aligned}$$

d)

$$\begin{aligned}(x+6)(x-3)^2 &= 0\\ x &= -6 \text{ or } 3 \text{ (double)}\end{aligned}$$

Section 2.5 Page 79 Question 3

a)

$$\begin{aligned}P(x) &= 0\\ x^3+3x^2-x-3 &= 0\\ x^2(x+3)-1(x+3) &= 0\\ (x+3)(x^2-1) &= 0\\ x &= -3,\ -1,\ \text{or } 1\end{aligned}$$

Check.

$$\begin{aligned}P(-3) &= (-3)^3+3(-3)^2-(-3)-3\\ &= -27+27+3-3\\ &= 0\\ P(-1) &= (-1)^3+3(-1)^2-(-1)-3\\ &= -1+3+1-3\\ &= 0\\ P(1) &= 1^3+3(1)^2-1-3\\ &= 1+3-1-3\\ &= 0\end{aligned}$$

The solutions are $-3,\ -1$, and 1.

b)

$$\begin{aligned}P(x) &= 0\\ x^3-3x^2-4x+12 &= 0\\ x^2(x-3)-4(x-3) &= 0\\ (x-3)(x^2-4) &= 0\\ x &= 3, \pm 2\end{aligned}$$

Check.

$$\begin{aligned}P(3) &= 3^3-3(3)^2-4(3)+12\\ &= 27-27-12+12\\ &= 0\\ P(-2) &= (-2)^3-3(-2)^2-4(-2)+12\\ &= -8-12+8+12\\ &= 0\\ P(2) &= 2^3-3(2)^2-4(2)+12\\ &= 8-12-8+12\\ &= 0\end{aligned}$$

The solutions are 3 and ±2.

c)

$$\begin{aligned}P(t) &= t^3+2t^2-7t+4\\ P(1) &= 1^3+2(1)^2-7(1)+4\\ &= 1+2-7+4\\ &= 0\end{aligned}$$

$t-1$ is a factor of $P(t)$. Synthetic division reveals the factor t^2+3t-4.

$$\begin{aligned}P(t) &= 0\\ (t-1)(t^2+3t-4) &= 0\\ (t-1)(t+4)(t-1) &= 0\\ t &= 1 \text{ or } -4\end{aligned}$$

Check.

$$\begin{aligned}P(-4) &= (-4)^3+2(-4)^2-7(-4)+4\\ &= -64+32+28+4\\ &= 0\end{aligned}$$

The solutions are -4 and 1.

d)

$$\begin{aligned}P(y) &= 0\\ y^3-3y^2-16y+48 &= 0\\ y^2(y-3)-16(y-3) &= 0\\ (y-3)(y^2-16) &= 0\\ y &= 3, \pm 4\end{aligned}$$

Check.

$$\begin{aligned}P(3) &= 3^3-3(3)^2-16(3)+48\\ &= 27-27-48+48\\ &= 0\\ P(-4) &= (-4)^3-3(-4)^2-16(-4)+48\\ &= -64-48+64+48\\ &= 0\\ P(4) &= 4^3-3(4)^2-16(4)+48\\ &= 64-48-64+48\\ &= 0\end{aligned}$$

The solutions are 3 and ±4.

e)

$$\begin{aligned} P(a) &= a^3 - 4a^2 + a + 6 \\ P(2) &= 2^3 - 4(2)^2 + 2 + 6 \\ &= 8 - 16 + 8 \\ &= 0 \end{aligned}$$

$a - 2$ is a factor of $P(a)$. Synthetic division reveals the factor $a^2 - 2a - 3$.

$$\begin{aligned} P(a) &= 0 \\ (a-2)(a^2 - 2a - 3) &= 0 \\ (a-2)(a-3)(a+1) &= 0 \\ a &= 2,\ 3,\ \text{or } -1 \end{aligned}$$

Check.

$$\begin{aligned} P(3) &= 3^3 - 4(3)^2 + 3 + 6 \\ &= 27 - 36 + 9 \\ &= 0 \\ P(-1) &= (-1)^3 - 4(-1)^2 - 1 + 6 \\ &= -1 - 4 - 1 + 6 \\ &= 0 \end{aligned}$$

The solutions are $2,\ 3$, and -1.

f)

$$\begin{aligned} P(x) &= x^3 - 4x^2 - 3x + 18 \\ P(-2) &= (-2)^3 - 4(-2)^2 - 3(-2) + 18 \\ &= -8 - 16 + 6 + 18 \\ &= 0 \end{aligned}$$

$x + 2$ is a factor of $P(x)$. Synthetic division reveals the factor $x^2 - 6x + 9$.

$$\begin{aligned} P(x) &= 0 \\ (x+2)(x^2 - 6x + 9) &= 0 \\ (x+2)(x-3)^3 &= 0 \\ x &= -2, 3 \text{ (double)} \end{aligned}$$

Check.

$$\begin{aligned} P(3) &= 3^3 - 4(3)^2 - 3(3) + 18 \\ &= 27 - 36 - 9 + 18 \\ &= 0 \end{aligned}$$

The solutions are -2 and 3 (double).

Section 2.5 Page 79 Question 5

a)

$$\begin{aligned} P(x) &= 0 \\ 2x^3 - 3x^2 - 2x &= 0 \\ x(2x^2 - 3x - 2) &= 0 \\ x(2x+1)(x-2) &= 0 \\ x &= 0,\ -\frac{1}{2},\ \text{or } 2 \end{aligned}$$

Check.

$$\begin{aligned} P(0) &= 2(0)^3 - 3(0)^2 - 2(0) \\ &= 0 \\ P\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) \\ &= -\frac{1}{4} - \frac{3}{4} + 1 \\ &= 0 \\ P(2) &= 2(2)^3 - 3(2)^2 - 2(2) \\ &= 16 - 12 - 4 \\ &= 0 \end{aligned}$$

The solutions are $0,\ -\frac{1}{2},$ and 2.

b)

$$\begin{aligned} P(x) &= 0 \\ 3x^3 - 10x^2 + 3x &= 0 \\ x(3x^2 - 10x + 3) &= 0 \\ x(3x-1)(x-3) &= 0 \\ x &= 0,\ \frac{1}{3},\ \text{or } 3 \end{aligned}$$

Check.

$$\begin{aligned} P(0) &= 3(0)^3 - 10(0)^2 + 3(0) \\ &= 0 \\ P\left(\frac{1}{3}\right) &= 3\left(\frac{1}{3}\right)^3 - 10\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) \\ &= \frac{1}{9} - \frac{10}{9} + 1 \\ &= 0 \\ P(3) &= 3(3)^3 - 10(3)^2 + 3(3) \\ &= 81 - 90 + 9 \\ &= 0 \end{aligned}$$

The solutions are $0,\ \frac{1}{3},$ and 3.

c)

$$\begin{aligned} P(z) &= 0 \\ 9z^3 - 4z &= 0 \\ z(9z^2 - 4) &= 0 \\ z(3z + 2)(3z - 2) &= 0 \\ z &= 0,\ -\frac{2}{3},\ \text{or}\ \frac{2}{3} \\ P(0) &= 9(0)^3 - 4(0) \\ &= 0 \end{aligned}$$

Check.

$$\begin{aligned} P\left(-\frac{2}{3}\right) &= 9\left(-\frac{2}{3}\right)^3 - 4\left(-\frac{2}{3}\right) \\ &= -\frac{72}{27} + \frac{8}{3} \\ &= -\frac{8}{3} + \frac{8}{3} \\ &= 0 \\ P\left(\frac{2}{3}\right) &= 9\left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right) \\ &= \frac{72}{27} - \frac{8}{3} \\ &= 0 \end{aligned}$$

The solutions are 0, $-\frac{2}{3}$, and $\frac{2}{3}$.

d)

$$\begin{aligned} P(x) &= 0 \\ 16x^3 + 8x^2 + x &= 0 \\ x(16x^2 + 8x + 1) &= 0 \\ x(4x + 1)^2 &= 0 \\ x &= 0,\ -\frac{1}{4}\ \text{(double)} \end{aligned}$$

Check.

$$\begin{aligned} P(0) &= 16(0)^3 + 8(0)^2 + (0) \\ &= 0 \\ P\left(-\frac{1}{4}\right) &= 16\left(-\frac{1}{4}\right)^3 + 8\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right) \\ &= -\frac{16}{64} + \frac{8}{16} - \frac{1}{4} \\ &= 0 \end{aligned}$$

The solutions are 0 and $-\frac{1}{4}$ (double).

Section 2.5 Page 79 Question 7

a) i)

$$\begin{aligned} P(x) &= 0 \\ x^3 - 8x &= 0 \\ x(x^2 - 8) &= 0 \\ x &= 0 \\ x^2 - 8 &= 0 \\ x^2 &= 8 \\ x &= \pm 2\sqrt{2} \end{aligned}$$

The roots are 0, $\pm 2\sqrt{2}$.

ii) The roots are 0 and $\pm 2\sqrt{2} \doteq \pm 2.83$.

b) i)

$$\begin{aligned} P(x) &= x^3 - 10x + 3 \\ P(3) &= 3^3 - 10(3) + 3 \\ &= 27 - 30 + 3 \\ &= 0 \end{aligned}$$

$x - 3$ is a factor of $P(x)$. Synthetic division reveals another factor of $x^2 + 3x - 1$. Use the quadratic formula to determine the remaining roots.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{13}}{2} \end{aligned}$$

The roots are 3, $\frac{-3 \pm \sqrt{13}}{2}$.

ii) The roots are 3 and approximately 0.30 and −3.30.

c) i)

$$\begin{aligned} P(x) &= x^3 - 6x^2 + 6x + 8 \\ P(4) &= 4^3 - 6(4)^2 + 6(4) + 8 \\ &= 64 - 96 + 24 + 8 \\ &= 0 \end{aligned}$$

$x - 4$ is a factor of $P(x)$. Synthetic division reveals another factor of x^2-2x-2. Use the quadratic formula to determine the remaining roots.

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{2 \pm \sqrt{12}}{2} \\ &= \frac{2 \pm 2\sqrt{3}}{2} \\ &= 1 \pm \sqrt{3} \end{aligned}$$

The roots are 4, $1 \pm \sqrt{3}$.

ii) The roots are 4 and approximately 2.73 and −0.73.

d) i)

$$\begin{aligned} P(x) &= x^3 + 3x^2 - 15x - 25 \\ P(-5) &= (-5)^3 + 3(-5)^2 - 15(-5) - 25 \\ &= -125 + 75 + 75 - 25 \\ &= 0 \end{aligned}$$

$x + 5$ is a factor of $P(x)$. Synthetic division reveals another factor of $x^2 - 2x - 5$. Use the quadratic formula to determine the remaining roots.

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)} \\ &= \frac{2 \pm \sqrt{24}}{2} \\ &= 1 \pm \sqrt{6} \end{aligned}$$

The roots are −5, $1 \pm \sqrt{6}$.

ii) The roots are −5 and approximately 3.45 and −1.45.

e) i)

$$\begin{aligned} P(v) &= v^3 + 5v^2 - 18 \\ P(-3) &= (-3)^3 + 5(-3)^2 - 18 \\ &= -27 + 45 - 18 \\ &= 0 \end{aligned}$$

$v + 3$ is a factor $P(v)$. Synthetic division reveals another factor of $v^2 + 2v - 6$. Use the quadratic formula to determine the remaining roots.

$$\begin{aligned} v &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-6)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{28}}{2} \\ &= \frac{-2 \pm 2\sqrt{7}}{2} \\ &= -1 \pm \sqrt{7} \end{aligned}$$

The roots are −3, $-1 \pm \sqrt{7}$.

ii) The roots are −3 and approximately 1.65 and −3.65.

f) i)

$$\begin{aligned} P(x) &= x(x + 4)(x + 1) - 4 \\ &= x^3 + 5x^2 + 4x - 4 \\ P(-2) &= (-2)^3 + 5(-2)^2 + 4(-2) - 4 \\ &= -8 + 20 - 8 - 4 \\ &= 0 \end{aligned}$$

$x + 2$ is a factor of $P(x)$. Synthetic division reveals another factor of $x^2 + 3x - 2$. Use the quadratic formula to determine the remaining roots.

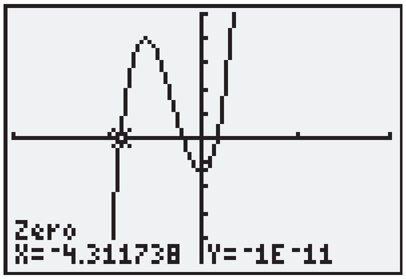
$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{17}}{2} \end{aligned}$$

The roots are −2, $\frac{-3 \pm \sqrt{17}}{2}$.

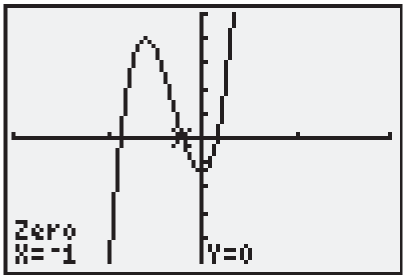
ii) The roots are −2 and approximately 0.56 and −3.56.

Section 2.5 Page 79 Question 9

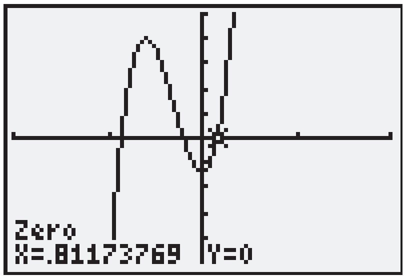
a) The roots are −4.31, −1, and 0.81.



```
Zero
X=-4.311738  Y=-1E-11
```

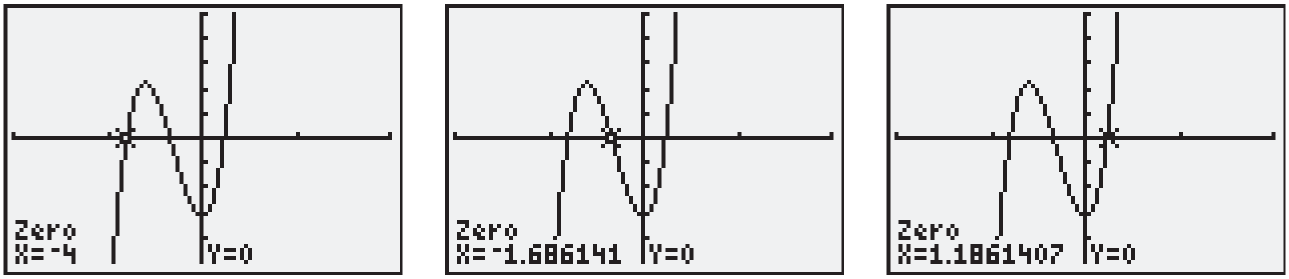


```
Zero
X=-1         Y=0
```

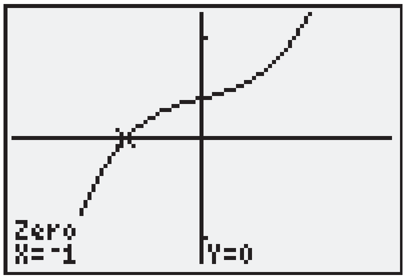


```
Zero
X=.81173769  Y=0
```

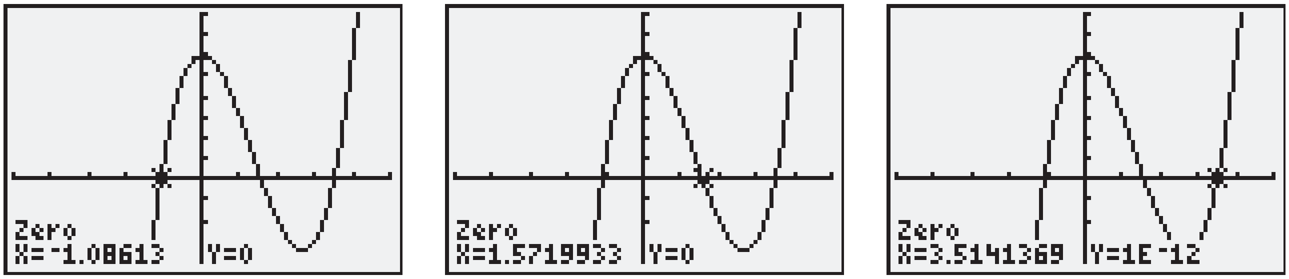
b) The roots are −4, −1.69, and 1.19.



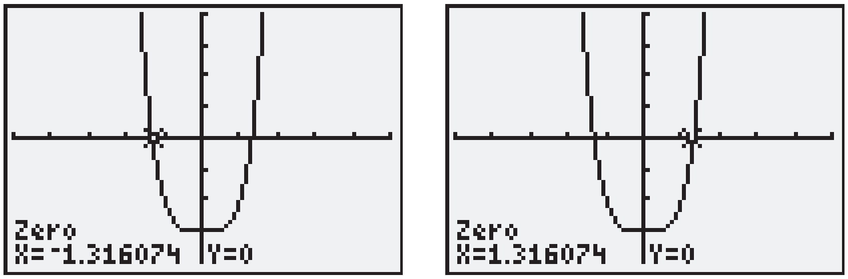
c) The root is −1.



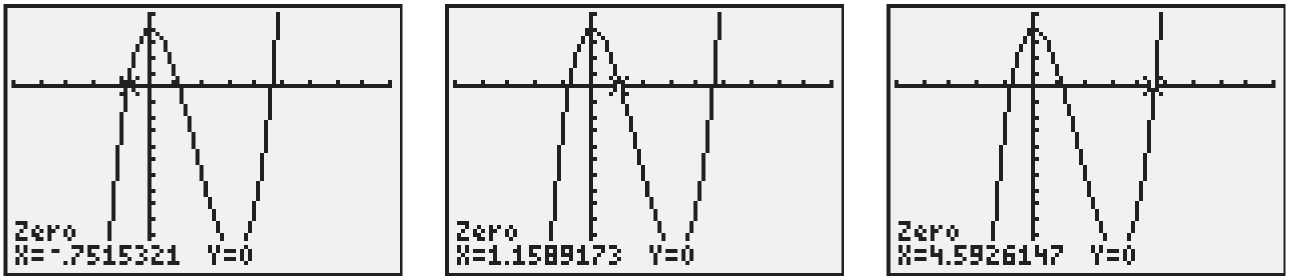
d) The roots are −1.09, 1.57, and 3.51.



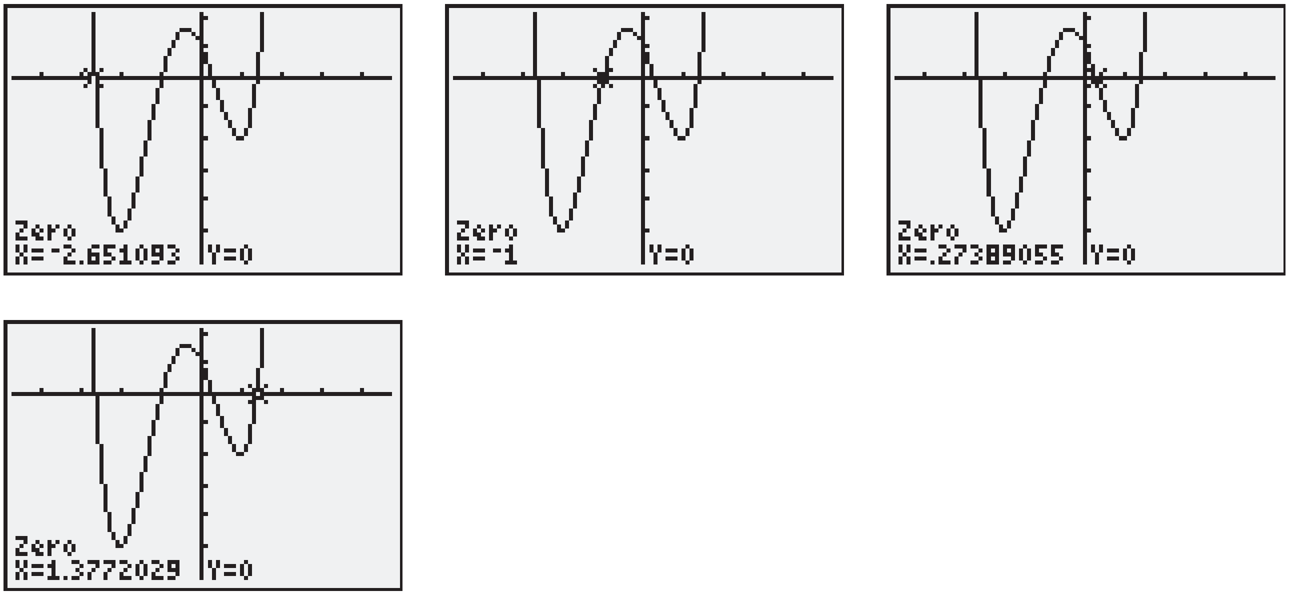
e) The roots are ±1.32.



f) The roots are −0.75, 1.16, and 4.59.



g) The roots are −2.65, −1, 0.27, and 1.38.



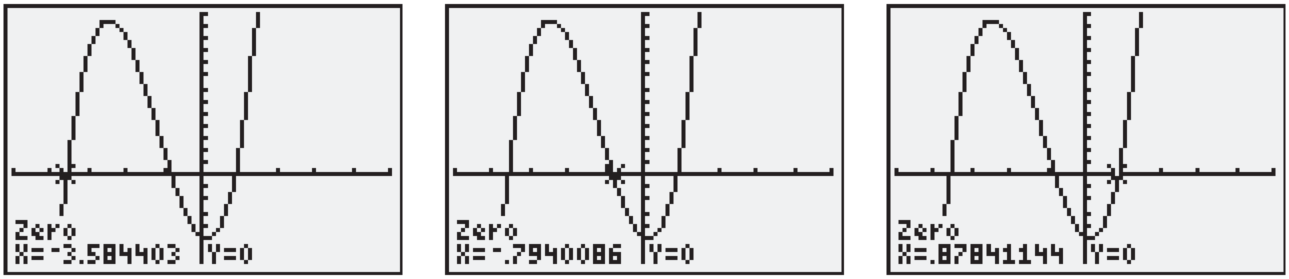
```
Zero
X=⁻2.651093  Y=0
```

```
Zero
X=⁻1         Y=0
```

```
Zero
X=.27389055  Y=0
```

```
Zero
X=1.3772029  Y=0
```

h) The roots are −3.58, −0.79, and 0.88.



```
Zero
X=⁻3.584403  Y=0
```

```
Zero
X=⁻.7940086  Y=0
```

```
Zero
X=.87841144  Y=0
```

Apply, Solve, Communicate

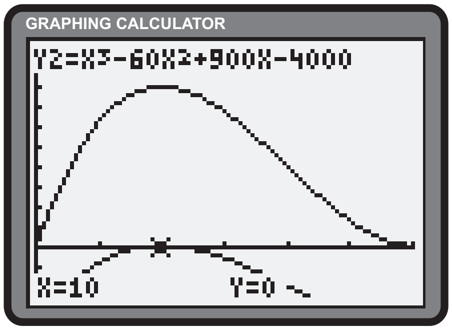
Section 2.5 Page 79 Question 11

From the diagram, the dimensions of the box are determined to be x by $60 - 2x$ by $\frac{60-2x}{2}$ cm. Let $V(x)$ be the volume of the box, in cubic centimetres, in terms of the height, x, $x \in (0, 30)$. Determine the volume of the box.

$$\begin{aligned} V(x) &= x(60-2x)\left(\frac{60-2x}{2}\right) \\ &= x(30-x)(60-2x) \\ &= 2x^2 - 120x^2 + 1800x \qquad (1) \end{aligned}$$

Determine x for a volume of 8000 cm^3.

$$\begin{aligned} V(x) &= 8000 \\ 2x^2 - 120x^2 + 1800x &= 8000 \\ x^3 - 60x^2 + 900x - 4000 &= 0 \qquad (2) \end{aligned}$$



GRAPHING CALCULATOR

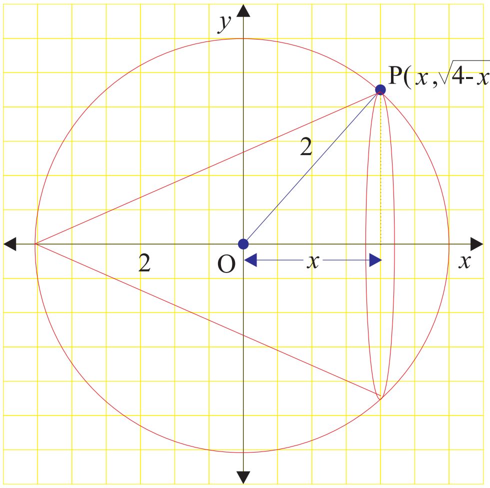
```
Y2=X³-60X²+900X-4000
X=10          Y=0
```

Polynomials (1) and (2) appear in the adjacent graphic. The **Zero operation** of the graphing calculator reveals that 10 is the only root of $x^3 - 60x^2 + 900x - 4000$ in the interval $(0, 30)$. The value of x is 10 cm.

Section 2.5 Page 80 Question 12

a) Let the point P be the general point on the function $f(x) = \sqrt{4 - x^2}$. The base radius of the cone can be expressed as $r = \sqrt{4 - x^2}$. The height of the cone can be expressed as $h = x + 2$.

$$V(r, h) = \frac{1}{3}\pi r^2 h$$
$$V\left(\sqrt{4 - x^2}, x + 2\right) = \frac{1}{3}\pi\left(\sqrt{4 - x^2}\right)^2 (x + 2)$$
$$V(x) = \frac{1}{3}\pi(4 - x^2)(x + 2)$$



b) The cone remains inscribed in the sphere and both the radius and height of the sphere remain non-negative over the domain $x \in [-2, 2]$.

c)
$$V(x) = 0$$
$$\frac{1}{3}\pi(4 - x^2)(x + 2) = 0$$
$$\frac{1}{3}\pi(2 - x)(x + 2)^2 = 0$$
$$x = \pm 2$$

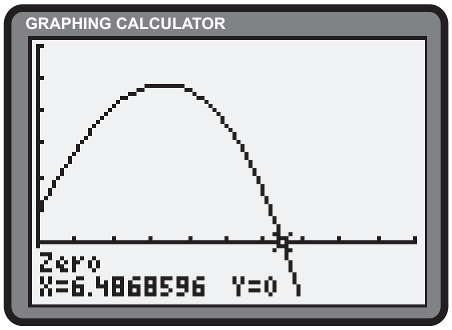
For the roots $x = \pm 2$, the volume of the cone is zero.

Section 2.5 Page 80 Question 13

a) The base radius of the cone can be expressed as $r = \sqrt{100 - x^2}$. The height of the cone can be expressed as $h = x + 10$.

$$V(r, h) = \frac{1}{3}\pi r^2 h$$
$$V\left(\sqrt{100 - x^2}, x + 10\right) = \frac{1}{3}\pi\left(\sqrt{100 - x^2}\right)^2 (x + 10)$$
$$V(x) = \frac{1}{3}\pi(100 - x^2)(x + 10)$$

b)
$$V(x) = 1000$$
$$\frac{1}{3}\pi(100 - x^2)(x + 10) = 1000$$
$$\frac{1}{3}\pi(100x + 1000 - x^3 - 10x^2) - 1000 = 0$$
$$x \doteq 6.49$$

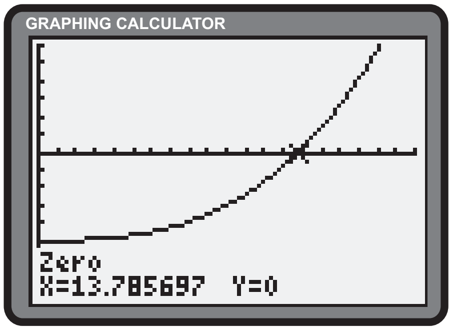


```
Zero
X=6.4868596  Y=0
```

For the volume to be 1000 m^3, $x \doteq 6.49$ m. The radius of the cone is $r = \sqrt{100 - 6.49^2}$ or 7.61 m. The height is $h = 10 + 6.49$ or 16.49 m.

c) The dimensions of the pyramid are x by x by $x+2$. Let V be the volume.

$$V = 1000$$
$$\frac{1}{3}x^2(x + 2) = 1000$$
$$\frac{1}{3}(x^3 + 2x^2) - 1000 = 0$$
$$x \doteq 13.79$$



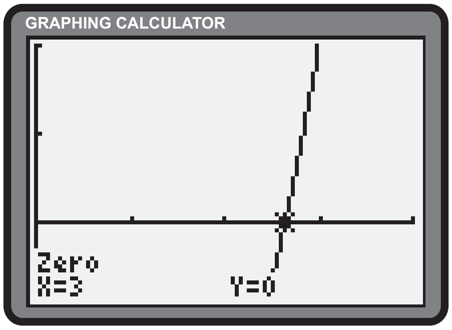
```
Zero
X=13.785697  Y=0
```

For the volume to be 1000 m^3, the side lengths of the base should be 13.79 m. The height should be 15.79 m.

Section 2.5 Page 80 Question 14

$$\begin{aligned} P(x) &= 3P(0) \\ x^3 + x^2 - 2x + 10 &= 3(10) \\ x^3 + x^2 - 2x - 20 &= 0 \qquad (1) \end{aligned}$$

Setting the **Float operation** of the graphing calculator to 0 and using the **Zero operation** reveals the solution to (1).



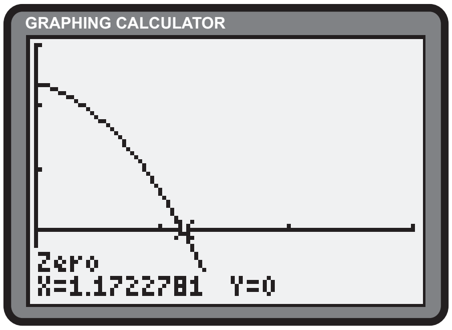
Zero
X=3 Y=0

City council will pass the bylaw 3 years from now.

Section 2.5 Page 80 Question 15

$$\begin{aligned} P(x) &= 0.001 \\ -x^3 - 5x^2 - 3x + 12 &= 0.001 \\ -x^3 - 5x^2 - 3x + 11.999 &= 0 \qquad (1) \end{aligned}$$

The **Zero operation** reveals the solution to (1).



Zero
X=1.1722781 Y=0

The population of fish declines to 1000 after 1.1723 decades, or about 12 years.

Section 2.5 Page 80 Question 16

a) The possible zeros are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$.

$$\begin{aligned} P(x) &= 8x^3 - 12x^2 + 6x - 1 \\ P\left(\frac{1}{2}\right) &= 8\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) - 1 \\ &= 1 - 3 + 3 - 1 \\ &= 0 \end{aligned}$$

$2x - 1$ is a factor $P(x)$. Division reveals another factor of $4x^2 - 4x + 1$. This is a perfect square trinomial.

$$\begin{aligned} P(x) &= 0 \\ 8x^3 - 12x^2 + 6x - 1 &= 0 \\ (2x - 1)(4x^2 - 4x + 1) &= 0 \\ (2x - 1)(2x - 1)^2 &= 0 \end{aligned}$$

The roots are $\frac{1}{2}$, $\frac{1}{2}$, and , $\frac{1}{2}$.

b) The possible zeros are

$$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{5}, \pm\frac{1}{6} \pm \frac{1}{10}, \pm\frac{1}{15}, \pm\frac{1}{30}$$

$$\begin{aligned} P(x) &= 30x^3 + 19x^2 - 1 \\ P\left(\frac{1}{5}\right) &= 30\left(\frac{1}{5}\right)^3 + 19\left(\frac{1}{5}\right)^2 - 1 \\ &= \frac{6}{25} + \frac{19}{25} - 1 \\ &= 0 \end{aligned}$$

$5x - 1$ is a factor of $P(x)$. Division reveals another factor of $6x^2 + 5x + 1$.

$$\begin{aligned} P(x) &= 0 \\ 30x^3 + 19x^2 - 1 &= 0 \\ (5x - 1)(6x^2 + 5x + 1) &= 0 \\ (5x - 1)(2x + 1)(3x + 1) &= 0 \end{aligned}$$

The roots are $\frac{1}{5}$, $-\frac{1}{2}$, and $-\frac{1}{3}$.

c) Factor by grouping.

$$\begin{aligned} P(y) &= 0 \\ 12y^3 - 4y^2 - 27y + 9 &= 0 \\ 4y^2(3y-1) - 9(3y-1) &= 0 \\ (3y-1)(4y^2-9) &= 0 \\ (3y-1)(2y-3)(2y+3) &= 0 \\ y &= \frac{1}{3} \text{ or } \pm\frac{3}{2} \end{aligned}$$

The roots are $\frac{1}{3}$, $\pm\frac{3}{2}$.

d) The possible zeros are $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{3}{8}, \pm9, \pm\frac{9}{2}, \pm\frac{9}{4}, \pm\frac{9}{8}, \pm27, \pm\frac{27}{2}, \pm\frac{27}{4}, \pm\frac{27}{8}$.

$$\begin{aligned} P(a) &= 8a^3 + 27 \\ P\left(-\frac{3}{2}\right) &= 8\left(-\frac{3}{2}\right)^3 + 27 \\ &= -27 + 27 \\ &= 0 \end{aligned}$$

$2a + 3$ is a factor of $P(a)$. Division reveals another factor of $4a^2 - 6a + 9$. Use the quadratic formula to determine the remaining roots.

$$\begin{aligned} a &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(4)(9)}}{2(4)} \\ &= \frac{6 \pm \sqrt{-108}}{8} \\ &= \frac{6 \pm 6i\sqrt{3}}{8} \\ &= \frac{3 \pm 3i\sqrt{3}}{4} \end{aligned}$$

The roots are $-\frac{3}{2}$, $\frac{3 \pm 3i\sqrt{3}}{4}$.

e)

$$\begin{aligned} \frac{x^3}{2} - \frac{x}{3} &= 0 \\ 3x^3 - 2x &= 0 \\ x(3x^2 - 2) &= 0 \\ x &= 0 \\ 3x^2 - 2 &= 0 \\ 3x^2 &= 2 \\ x^2 &= \frac{2}{3} \\ x &= \pm\frac{\sqrt{2}}{\sqrt{3}} \\ &= \pm\frac{\sqrt{6}}{3} \end{aligned}$$

The roots are 0, $\frac{\pm\sqrt{6}}{3}$.

f)

$$\begin{aligned} x^3 - \frac{13x^2}{4} + x &= -3 \\ 4x^3 - 13x^2 + 4x + 12 &= 0 \\ P(x) &= 4x^3 - 13x^2 + 4x + 12 \end{aligned}$$

The possible zeros are $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm2, \pm3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm4, \pm6, \pm12$.

$$\begin{aligned} P(2) &= 4(2)^3 - 13(2)^2 + 4(2) + 12 \\ &= 32 - 52 + 8 + 12 \\ &= 0 \end{aligned}$$

$x - 2$ is a factor of $P(x)$. Synthetic division reveals another factor of $4x^2 - 5x - 6$.

$$\begin{aligned} P(x) &= 0 \\ 4x^3 - 13x^2 + 4x + 12 &= 0 \\ (x-2)(4x^2 - 5x - 6) &= 0 \\ (x-2)(x-2)(4x+3) &= 0 \end{aligned}$$

The roots are 2 (double) and $-\frac{3}{4}$.

g)

$$\begin{aligned}1-\frac{1}{x}&=\frac{1}{x^3}-\frac{1}{x^2}\\x^3-x^2&=1-x\\x^3-x^2+x-1&=0\\x^2(x-1)+1(x-1)&=0\\(x-1)(x^2+1)&=0\\x&=1\\x^2&=-1\\x&=\pm i\end{aligned}$$

The roots are $1, i,$ and $-i$.

h)

$$\begin{aligned}\frac{2}{x-2}-\frac{1}{x-1}&=x\\2(x-1)-1(x-2)&=x(x-2)(x-1)\\2x-2-x+2&=x^3-3x^2+2x\\x^3-3x^2+x&=0\\x(x^2-3x+1)&=0\\x&=0\\x&=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)}\\&=\frac{3\pm\sqrt{5}}{2}\end{aligned}$$

The roots are 0 and $\frac{3\pm\sqrt{5}}{2}$.

Section 2.5 Page 80 Question 17

a)

$$\begin{aligned}P(x)&=x^3+kx^2-10x-24\\P(-2)&=0\\(-2)^3+k(-2)^2-10(-2)-24&=0\\-8+4k+20-24&=0\\4k&=12\\k&=3\end{aligned}$$

$x+2$ is a factor of $P(x)=x^3+3x^2-10x-24$. Synthetic division reveals another factor of x^2+x-12.

$$\begin{aligned}(x+2)(x^2+x-12)&=0\\(x+2)(x+4)(x-3)&=0\\x&=-2,-4,\text{ or }3\end{aligned}$$

The roots are $-2, -4,$ and 3.

b)

$$\begin{aligned}P(x)&=3x^3+4x^2+kx-2\\P(-2)&=0\\3(-2)^3+4(-2)^2-2k-2&=0\\-24+16-2k-2&=0\\-2k&=10\\k&=-5\end{aligned}$$

$x+2$ is a factor of $P(x)=3x^3+4x^2-5x-2$. Synthetic division reveals another factor of $3x^2-2x-1$. Factoring reveals the remaining factors.

$$\begin{aligned}(x+2)(3x^2-2x-1)&=0\\(x+2)(3x+1)(x-1)&=0\\x&=-2,-\frac{1}{3},\text{ or }1\end{aligned}$$

The roots are $-2, -\frac{1}{3},$ and 1.

Section 2.5 Page 81 Question 18

a)

$$\begin{aligned}x^3-4x^2+kx&=0\\x(x^2-4x+k)&=0\\x&=0\\x^2-4x+k&=0\end{aligned}$$

For two equal roots, determine the zeros of the discriminant.

$$\begin{aligned}(-4)^2-4(1)(k)&=0\\16-4k&=0\\k&=4\end{aligned}$$

$x^3-4x^2+4x=0$ has two equal roots.

b)

$$\begin{aligned}x^3-4x^2+4x&=0\\x(x^2-4x+4)&=0\\x(x-2)^2&=0\\x&=0\text{ and }2\end{aligned}$$

Section 2.5 Page 81 Question 19

Let the three consecutive integers be $x - 1$, x, and $x + 1$.

$$\begin{aligned}(x-1)(x)(x+1) &= -504\\ x(x^2-1)+504 &= 0\\ x^3-x+504 &= 0\end{aligned}$$

Let $P(x) = x^3 - x + 504$.

$$\begin{aligned}P(-8) &= (-8)^3-(-8)+504\\ &= -512+8+504\\ &= 0\end{aligned}$$

One root is -8. The polynomial can be expressed as $P(x) = (x + 8)(x^2 - 8x + 63)$. Use of the quadratic formula on $x^2 - 8x + 63$, yields no further integer roots. The consecutive integers are $-9, -8,$ and -7.

Section 2.5 Page 81 Question 20

a)

$$\begin{aligned}P(x) &= 0\\ x^4-4x^3+x^2+6x &= 0\\ x(x^3-4x^2+x+6) &= 0\\ x &= 0\\ x^3-4x^2+x+6 &= 0\end{aligned}$$

Let $Q(x) = x^3 - 4x^2 + x + 6$.

$$\begin{aligned}Q(2) &= 2^3-4(2)^2+2+6\\ &= 8-16+2+6\\ &= 0\end{aligned}$$

$x - 2$ is a factor of $Q(x)$. Synthetic division reveals another factor of $x^2 - 2x - 3$.

$$\begin{aligned}P(x) &= 0\\ xQ(x) &= 0\\ x(x-2)(x^2-2x-3) &= 0\\ x(x-2)(x-3)(x+1) &= 0\\ x &= -1, 0, 2, \text{ or } 3\end{aligned}$$

The roots are $-1, 0, 2,$ and 3.

b)

$$\begin{aligned}P(x) &= x^4+2x^3-7x^2-8x+12\\ P(1) &= 1^4+2(1)^3-7(1)-8(1)+12\\ &= 1+2-7-8+12\\ &= 0\end{aligned}$$

$x - 1$ is a factor of $P(x)$. Synthetic division reveals another factor of $Q(x) = x^3 + 3x^2 - 4x - 12$. Factor $Q(x)$ by grouping.

$$\begin{aligned}P(x) &= 0\\ (x-1)Q(x) &= 0\\ (x-1)[x^2(x+3)-4(x+3)] &= 0\\ (x-1)(x+3)(x^2-4) &= 0\\ (x-1)(x+3)(x-2)(x+2) &= 0\\ x &= 1, -3, 2, \text{ or } -2\end{aligned}$$

The roots are $-3, -2, 1,$ and 2.

c)
$$\begin{aligned} P(t) &= t^4 - 2t^3 - 11t^2 + 12t + 36 \\ P(3) &= 3^4 - 2(3)^3 - 11(3)^2 + 12(3) + 36 \\ &= 81 - 54 - 99 + 36 + 36 \\ &= 0 \end{aligned}$$

$t - 3$ is a factor of $P(t)$. Synthetic division reveals another factor of $Q(t) = t^3 + t^2 - 8t - 12$.

$$\begin{aligned} Q(3) &= 3^3 + 3^2 - 8(3) - 12 \\ &= 27 + 9 - 24 - 12 \\ &= 0 \end{aligned}$$

$t - 3$ is a factor of $Q(t)$. Synthetic division reveals another factor of $t^2 + 4t + 4$.

$$\begin{aligned} P(t) &= 0 \\ (t-3)(t-3)(t^2+4t+4) &= 0 \\ (t-3)^2(t+2)^2 &= 0 \\ t &= 3 \text{ or } -2 \end{aligned}$$

The roots are -2 and 3.

d)
$$\begin{aligned} P(x) &= 0 \\ x^4 - 4x^2 + 3 &= 0 \\ (x^2-3)(x^2-1) &= 0 \\ x^2 - 3 &= 0 \\ x &= \pm\sqrt{3} \\ x^2 - 1 &= 0 \\ x &= \pm 1 \end{aligned}$$

The roots are ± 1 and $\pm\sqrt{3}$.

e)
$$\begin{aligned} P(x) &= x^4 - x^3 - 10x^2 + 10x + 12 \\ P(2) &= 2^4 - 2^3 - 10(2)^2 + 10(2) + 12 \\ &= 16 - 8 - 40 + 20 + 12 \\ &= 0 \end{aligned}$$

$x - 2$ is a factor of $P(x)$. Synthetic division reveals another factor of $Q(x) = x^3 + x^2 - 8x - 6$.

$$\begin{aligned} Q(x) &= x^3 + x^2 - 8x - 6 \\ Q(-3) &= (-3)^3 + (-3)^2 - 8(-3) - 6 \\ &= -27 + 9 + 24 - 6 \\ &= 0 \end{aligned}$$

$x + 3$ is a factor of $Q(x)$. Synthetic division reveals another factor of $R(x) = x^2-2x-2$. Use the quadratic formula to determine the roots of $R(x)$.

$$\begin{aligned} R(x) &= 0 \\ x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{2 \pm \sqrt{12}}{2} \\ &= \frac{2 \pm 2\sqrt{3}}{2} \\ &= 1 \pm \sqrt{3} \end{aligned}$$

The roots are $2, -3, 1 \pm \sqrt{3}$.

f)
$$\begin{aligned} P(x) &= 0 \\ x^4 - 1 &= 0 \\ (x^2-1)(x^2+1) &= 0 \\ x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \\ x^2 + 1 &= 0 \\ x^2 &= -1 \\ x &= \pm i \end{aligned}$$

The roots are $\pm 1, \pm i$.

g) $P(x) = 2x^4 + 5x^3 + 3x^2 - x - 1$

$$\begin{aligned} P(-1) &= 2(-1)^4 + 5(-1)^3 + 3(-1)^2 - (-1) - 1 \\ &= 2 - 5 + 3 + 1 - 1 \\ &= 0 \end{aligned}$$

$x + 1$ is a factor of $P(x)$. Synthetic division reveals another factor of $Q(x) = 2x^3 + 3x^2 - 1$.

$$\begin{aligned} Q(x) &= 2x^3 + 3x^2 - 1 \\ Q(-1) &= 2(-1)^3 + 3(-1)^2 - 1 \\ &= -2 + 3 - 1 \\ &= 0 \end{aligned}$$

$x + 1$ is a factor of $Q(x)$. Synthetic division reveals another factor of $R(x) = 2x^2 + x - 1$. Factor $R(x)$ to obtain the remaining roots.

$$\begin{aligned} R(x) &= 0 \\ 2x^2 + x - 1 &= 0 \\ (2x - 1)(x + 1) &= 0 \\ x &= \frac{1}{2} \text{ or } -1 \end{aligned}$$

The roots are -1 and $\frac{1}{2}$.

h) $P(y) = 4y^4 - 8y^3 - 3y^2 + 5y + 2$

$$\begin{aligned} P(1) &= 4(1)^4 - 8(1)^3 - 3(1)^2 + 5(1) + 2 \\ &= 4 - 8 - 3 + 5 + 2 \\ &= 0 \end{aligned}$$

$y - 1$ is a factor of $P(y)$. Synthetic division reveals another factor of $Q(y) = 4y^3 - 4y^2 - 7y - 2$.

$$\begin{aligned} Q(y) &= 4y^3 - 4y^2 - 7y - 2 \\ Q(2) &= 4(2)^3 - 4(2)^2 - 7(2) - 2 \\ &= 32 - 16 - 14 - 2 \\ &= 0 \end{aligned}$$

$y - 2$ is a factor of $P(y)$. Synthetic division reveals another factor of $R(y) = 4y^2 + 4y + 1$. Factor $R(y)$ to obtain the remaining roots.

$$\begin{aligned} R(y) &= 0 \\ 4y^2 + 4y + 1 &= 0 \\ (2y + 1)^2 &= 0 \\ y &= -\frac{1}{2} \end{aligned}$$

The roots are $1, 2,$ and $-\frac{1}{2}$.

Section 2.5 Page 81 Question 21

Let $V(x)$ be the volume of the solid, in cubic centimetres.

$$\begin{aligned} V(x) &= (x - 1)(x - 2)(x + 3) \\ &= (x - 1)(x^2 + x - 6) \\ &= x^3 - 7x + 6 \end{aligned}$$

Given a volume of 42 cm^3, we can solve for x.

$$\begin{aligned} V(x) &= 42 \\ x^3 - 7x + 6 &= 42 \\ x^3 - 7x - 36 &= 0 \end{aligned}$$

Let $P(x) = x^3 - 7x - 36$.

$$\begin{aligned} P(4) &= 4^3 - 7(4) - 36 \\ &= 64 - 28 - 36 \\ &= 0 \\ P(x) &= (x - 4)(x^2 + 4x + 9) \end{aligned}$$

Thus, $x = 4$ is the only real root of $P(x) = 0$. From $x = 4$, the dimensions of the solid are 2 cm by 3 cm by 7 cm.

Section 2.5 Page 81 Question 22

Let $V(x)$ be the volume of the box, in cubic centimetres.

$$\begin{aligned} V(x) &= (x)(x)(x + 12) \\ &= x^2(x + 12) \\ &= x^3 + 12x^2 \end{aligned}$$

Given a volume of 135 cm^3, we can solve for x.

$$\begin{aligned} V(x) &= 135 \\ x^3 + 12x^2 &= 135 \\ x^3 + 12x^2 - 135 &= 0 \end{aligned}$$

Let $P(x) = x^3 + 12x^2 - 135$.

$$\begin{aligned} P(3) &= 3^3 + 12(3)^2 - 135 \\ &= 27 + 108 - 135 \\ &= 0 \end{aligned}$$

Thus, $x = 3$ is the only positive root of $P(x) = 0$. From $x = 3$, the dimensions of the toothpaste box are 3 cm by 3 cm by 15 cm.

Section 2.5 Page 81 Question 23

Let h be the height of the box, in centimetres. Let $V(h)$ be the volume of the box, in cubic centimetres.

$$\begin{aligned} V(h) &= (h)(h + 15)(h + 20) \\ &= h(h^2 + 35h + 300) \\ &= h^3 + 35h^2 + 300h \end{aligned}$$

Given a volume of 2500 cm^3, we can solve for h.

$$\begin{aligned} V(h) &= 2500 \\ h^3 + 35h^2 + 300h &= 2500 \\ h^3 + 35h^2 + 300h - 2500 &= 0 \end{aligned}$$

Let $P(h) = h^3 + 35h^2 + 300h - 2500$.

$$\begin{aligned} P(5) &= 5^3 + 35(5)^2 + 300(5) - 2500 \\ &= 0 \end{aligned}$$

Thus, $h = 5$ is the only real root of $P(h) = 0$. The dimensions of the CD-ROM box are 5 cm by 20 cm by 25 cm.

Section 2.5 Page 81 Question 24

Let x be the uniform amount by which each dimension is increased. Let $V(x)$ be the volume of the box, in cubic centimetres.

$$\begin{aligned} V(x) &= (x + 5)(x + 10)(x + 10) \\ &= (x + 5)(x^2 + 20x + 100) \\ &= x^3 + 25x^2 + 200x + 500 \end{aligned}$$

Given a volume of 1008 cm^3, the value of x can be determined.

$$\begin{aligned} V(x) &= 1008 \\ x^3 + 25x^2 + 200x + 500 &= 1008 \\ x^3 + 25x^2 + 200x - 508 &= 0 \end{aligned}$$

Let $P(x) = x^3 + 25x^2 + 200x - 508$.

$$\begin{aligned} P(2) &= 2^3 + 25(2)^2 + 200(2) - 508 \\ &= 8 + 100 + 400 - 508 \\ &= 0 \end{aligned}$$

Thus, $x = 2$ is a root of $P(x) = 0$. From $x = 2$, the dimensions of the new prism are 7 cm by 12 cm by 12 cm.

Section 2.5 Page 81 Question 25

Let h be the height of the building, in metres. The length can be expressed as $3h$. The width is also $3h$. Let $V(h)$ be the volume of the building, in cubic metres.

$$\begin{aligned} V(h) &= (h)(3h)(3h) \\ &= 9h^3 \end{aligned}$$

Given a volume of 0.009×10^9 or 9 000 000 m^3, we can solve for h.

$$\begin{aligned} V(h) &= 9\,000\,000 \\ 9h^3 &= 9\,000\,000 \\ h^3 &= 1\,000\,000 \\ h &= 100 \end{aligned}$$

From $h = 100$, the dimensions of the building are 100 m by 300 m by 300 m.

Section 2.5 Page 81 Question 26

Let d be the depth of the sandbox, in metres. The width can be expressed as $12d$. The length can be expressed as $12d + 1$. Let $V(d)$ be the volume of the sandbox, in cubic metres.

$$\begin{aligned} V(d) &= 3 \\ d(12d)(12d+1) &= 3 \\ 144d^3 + 12d^2 - 3 &= 0 \\ 48d^3 + 4d^2 - 1 &= 0 \end{aligned}$$

Let $P(d) = 48d^3 + 4d^2 - 1$.

$$\begin{aligned} P\left(\frac{1}{4}\right) &= 48\left(\frac{1}{4}\right)^3 + 4\left(\frac{1}{4}\right)^2 - 1 \\ &= \frac{48}{64} + \frac{4}{16} - 1 \\ &= 0 \\ P(d) &= (4d-1)(12d^2 + 4d + 1) \end{aligned}$$

Thus, $d = \frac{1}{4}$ is the only real root of P. From $d = \frac{1}{4}$, the dimensions are 0.25 m by 3 m by 4 m if the bottom is included. Let C be the cost of the sandbox.

$$\begin{aligned} C &= 2\left[2\left(\frac{1}{4}\right)(3) + 2\left(\frac{1}{4}\right)(4) + 3(4)\right] \\ &= 2\left(\frac{3}{2} + 2 + 12\right) \\ &= 31 \end{aligned}$$

If the bottom of the sandbox is required, the total cost is \$31. For no bottom, $C = 2\left(\frac{3}{2} + 2\right)$ or \$7.

Section 2.5 Page 81 Question 27

Factor by grouping.

$$\begin{aligned} x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12 &= 0 \\ x^4(x+3) - 5x^2(x+3) + 4(x+3) &= 0 \\ (x+3)(x^4 - 5x^2 + 4) &= 0 \\ (x+3)(x^2-4)(x^2-1) &= 0 \\ (x+3)(x-2)(x+2)(x-1)(x+1) &= 0 \\ x &= -3, 2, -2, 1, -1 \end{aligned}$$

The roots of $P(x)$ are $-3, -2, -1, 1,$ and 2.

Section 2.5 Page 81 Question 28

Answers may vary. For a real root of 7, the equation must have a linear factor of $x - 7$. An additional quadratic factor with a discriminant less than zero will provide the two imaginary roots. For example,

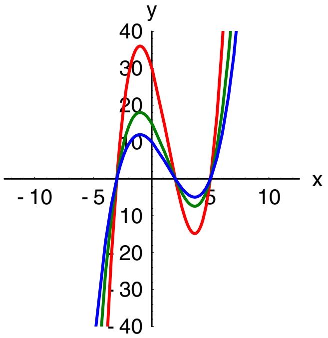
$$\begin{aligned} P(x) &= 0 \\ (x - 7)(x^2 + x + 1) &= 0 \\ x^3 - 6x^2 - 6x - 7 &= 0 \end{aligned}$$

The cubic equation $x^3 - 6x^2 - 6x - 7 = 0$ has a real root of 7 and two imaginary roots.

Section 2.5 Page 81 Question 29

a) Zeros of 5, 2, and -3 require factors of $x-5$, $x-2$, and $x + 3$. Stretching the function in a vertical direction generates a family of cubic equations with the same roots.

b) Answers will vary. For example, the coefficient of x^3 may be specified; the y-intercept may be specified.



Section 2.5 Page 81 Question 30

Yes. It has three real roots; some may have multiplicity greater than one.

Section 2.5 Page 81 Question 31

a) Zeros of $\sqrt{5}$, $-\sqrt{5}$, and -1 require factors of $x - \sqrt{5}$, $x + \sqrt{5}$, and $x + 1$ respectively.

$$\begin{aligned} P(x) &= (x - \sqrt{5})(x + \sqrt{5})(x + 1) \\ &= (x^2 - 5)(x + 1) \\ &= x^3 + x^2 - 5x - 5 \end{aligned}$$

Since $P(0) = -5$, the cubic function $P(x) = x^3 + x^2 - 5x - 5$ fulfills the requirements.

b) Stretching the cubic function vertically by a factor of 2 will, among other effects, double the y-intercept.

$$\begin{aligned} Q(x) &= 2P(x) \\ &= 2(x^3 + x^2 - 5x - 5) \\ &= 2x^3 + 2x^2 - 10x - 10 \end{aligned}$$

The cubic function $Q(x) = 2x^3 + 2x^2 - 10x - 10$ has x-intercepts of $\sqrt{5}$, $-\sqrt{5}$, and -1. It has a y-intercept of -10.

Section 2.5 Page 81 Question 32

a) Never true. Imaginary roots always come in conjugate pairs.

b) Always true. n-th degree polynomials always have n roots when real and imaginary roots are considered.

c) Sometimes true if the graph of the quartic does not intersect the x-axis as in the case of $x^4 + 1 = 0$.

d) Sometimes true. For example, $(x - 1)(x - \sqrt{5})(x + \sqrt{5}) = 0$.

e) Sometimes true. For example, $(x - \sqrt{2})^4 = 0$.

f) Never true. Imaginary roots occur in conjugate pairs and an imaginary number never equals its conjugate.

Section 2.5 Page 82 Question 33

a) No. If the polynomial has real coefficients, complex roots always occur in conjugate pairs.

b) No. If the polynomial has real coefficients, complex roots always occur in conjugate pairs.

c) If the polynomial has real coefficients, there will always be an even number of complex roots.

Section 2.5 Page 82 Question 34

a) If a polynomial has exactly one real root, then n must be odd. This allows for the possibility of an even number of complex roots.

b) If a polynomial has exactly two real roots, then n must be even. This allows for the possibility of an even number of complex roots.

Section 2.5 Page 82 Question 35

a) No. Complex roots always occur in conjugate pairs as in $a \pm bi$, $a, b \in \Re$.

b) Yes. The other complex root is $1 - i$.

c) Yes. The other complex roots are the complex conjugates of the given roots, $1 - i$ and $3 - 2i$.

Section 2.5 Page 82 Question 36

Complex roots always occur in conjugate pairs as in $a \pm bi, a, b \in \Re$.

2.6 Polynomial Functions and Inequalities

Practise

Section 2.6 Page 92 Question 1

Estimates to the nearest tenth may vary for local minima and maxima.

a) Domain: $\Re$. Range: $\Re$. Local maximum: $(1.4, 0.4)$. Local minimum: $(2.6, -0.4)$. y-int.: -6.

b) Domain: $\Re$. Range: $\Re$. Local maximum: $(2, 6)$. Local minimum: $(0, 2)$. y-int.: 2.

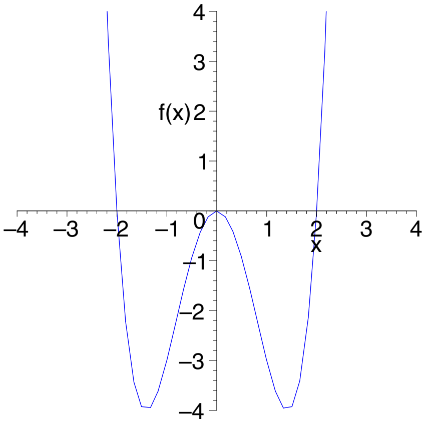
c) Domain: $\Re$. Range: $y \geq -9$. Local maximum: $(-0.5, 1.8)$. Local minima: $(-2.3, -9)$ and $(1.2, -9)$. y-int.: 0.

d) Domain: $\Re$. Range: $y \leq 8.9$. Local maxima: $(-0.3, 6)$ and $(3.9, 8.9)$. Local minimum: $(1.8, -9.9)$. y-int.: 5.

Section 2.6 Page 93 Question 3

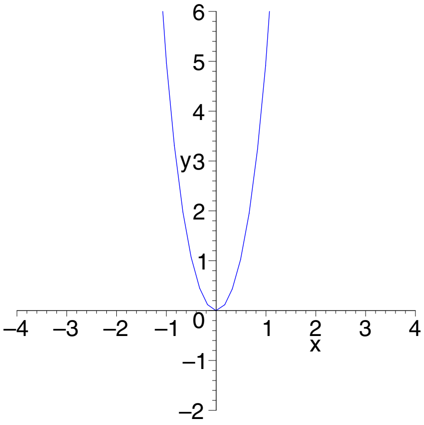
a) i) Zeros: $-2, 0$ (double) , 2
ii) Domain: $\Re$; Range: $y \geq -4$
iii) y-intercept: 0
iv) Local maximum: $(0, 0)$; Local minima: $(\pm 1.4, -4)$
v) Symmetry: even
vi) The left-most y-values are positive. The right-most y-values are positive.

$f(x) = x^2(x^2 - 4)$



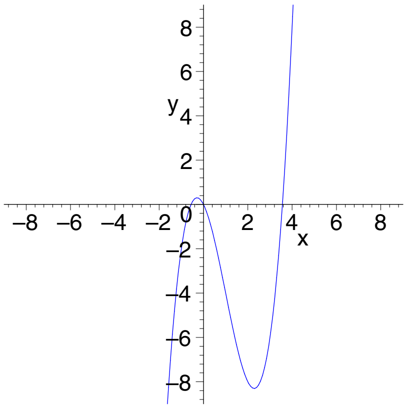
b) i) Zeros: 0 (double)
ii) Domain: $\Re$; Range: $y \geq 0$
iii) y-intercept: 0
iv) Local maximum: none; Local minimum $(0, 0)$
v) Symmetry: even
vi) The left-most y-values are positive. The right-most y-values are positive.

$y = x^2(x^2 + 4)$



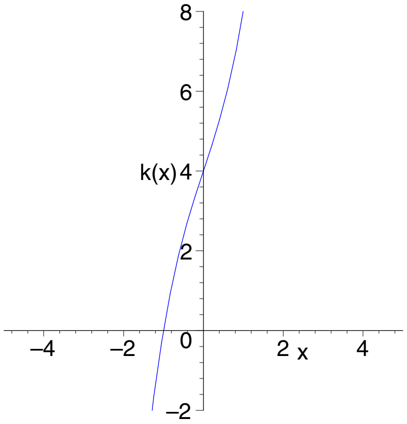
c) i) Zeros: $-0.6, 0, 3.6$
ii) Domain: $\Re$; Range: $\Re$
iii) y-intercept: 0
iv) Local maximum: $(-0.3, 0.3)$
Local minimum: $(2.3, -8.3)$
v) Symmetry: none
vi) The left-most y-values are negative. The right-most y-values are positive.

$y = x^3 - 3x^2 - 2x$



d) i) Zeros: -1
ii) Domain: $\Re$; Range: $\Re$
iii) y-intercept: 4
iv) Local maximum: none; Local minimum: none
v) Symmetry: none
vi) The left-most y-values are negative. The right-most y-values are positive.

$k(x) = x^3 + 3x + 4$



e) i) Zeros: $-1.6, 1.6$

ii) Domain: $\Re$; Range: $y \leq 6$

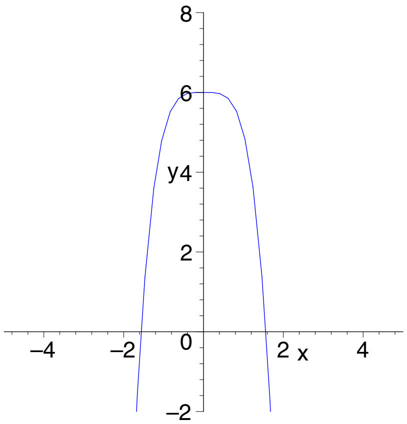
iii) y-intercept: 6

iv) Local maximum: $(0, 6)$; Local minimum: none

v) Symmetry: even

vi) The left-most y-values are negative. The right-most y-values are negative.

$y = -x^4 + 6$



f) i) Zeros: $0, 1.6$

ii) Domain: $\Re$; Range: $\Re$

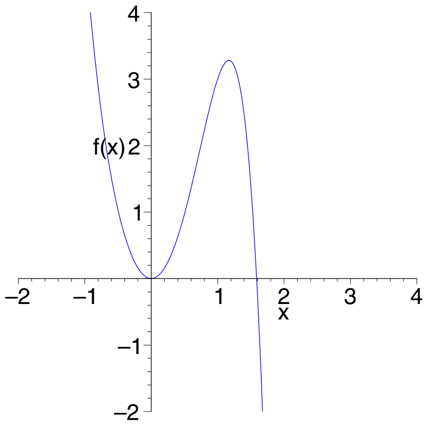
iii) y-intercept: 0

iv) Local maximum: $(1.2, 3.3)$;
Local minimum: $(0, 0)$

v) Symmetry: none

vi) The left-most y-values are positive. The right-most y-values are negative.

$f(x) = -x^5 + 4x^2$



g) i) Zeros: -1 (double), 4

ii) Domain: $\Re$; Range: $\Re$

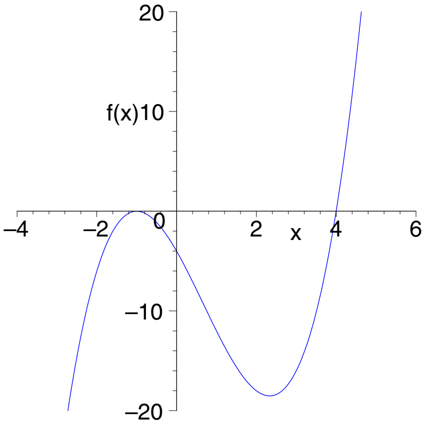
iii) y-intercept: -4

iv) Local maximum: $(-1, 0)$;
Local minimum: $(2.3, -18.5)$

v) Symmetry: none

vi) The left-most y-values are negative. The right-most y-values are positive.

$f(x) = (x-4)(x+1)(x+1)$



h) i) Zeros: $2, 4, 6$

ii) Domain: $\Re$; Range: $\Re$

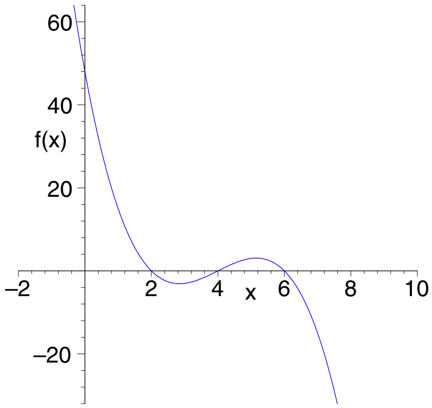
iii) y-intercept: 48

iv) Local maximum: $(5.2, 3.1)$;
Local minimum: $(2.8, -3.1)$

v) Symmetry: none

vi) The left-most y-values are positive. The right-most y-values are negative.

$f(x) = -(x-2)(x-4)(x-6)$



i) i) Zeros: $-2, 0, 1, 3$

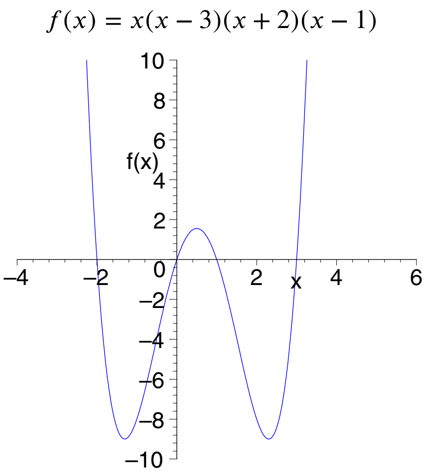
ii) Domain: $\Re$; Range: $y \ge -9$

iii) y-intercept: 0

iv) Local maximum: $(0.5, 1.6)$;
Local minima: $(-1.3, -9)$ and $(2.3, -9)$

v) Symmetry: none

vi) The left-most y-values are positive. The right-most y-values are positive.


$f(x) = x(x-3)(x+2)(x-1)$

j) i) Zeros: -3 (double), $0, 1$

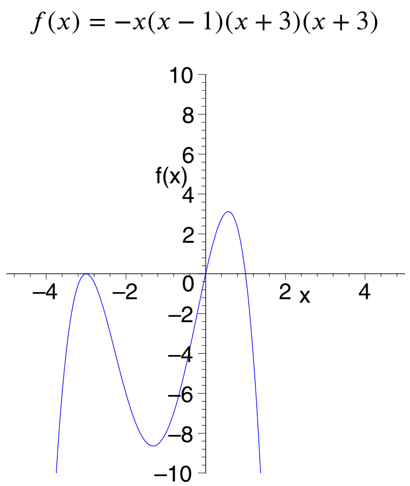
ii) Domain: $\Re$; Range: $y \le 3.1$

iii) y-intercept: 0

iv) Local maxima: $(-3, 0)$, $(0.6, 3.1)$;
Local minimum: $(-1.3, -8.6)$

v) Symmetry: none

vi) The left-most y-values are negative. The right-most y-values are negative.

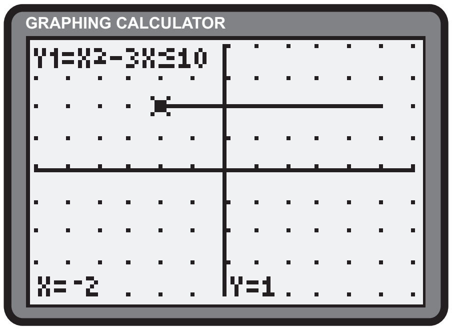

$f(x) = -x(x-1)(x+3)(x+3)$

Section 2.6 Page 93 Question 5

a) The roots of the equation $(x-3)(x+2) = 0$ are $x = 3$ and $x = -2$. An interval chart yields the solution $-2 < x < 3$.

	−2		3	
Intervals	(- ,-2)	(-2,3)		(3,)
Test values	-3	0		4
Sign	+	0 –	0	+

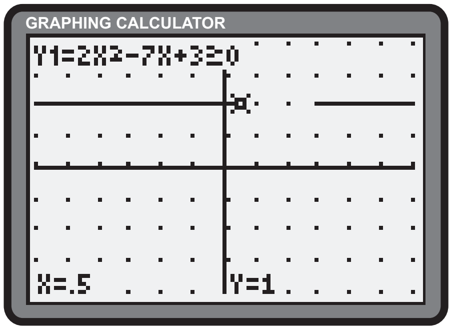
c) The graphing calculator yields $-2 \le x \le 5$ as the solution to the inequality $x^2 - 3x \le 10$.

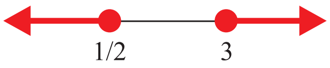


b) The roots of the equation, $(2x-1)(2x+3) = 0$ are $x = \frac{1}{2}$ and $x = -\frac{3}{2}$. An interval chart yields the solution $-\frac{3}{2} \le x \le \frac{1}{2}$.

	−3/2		1/2	
Intervals	(- ,-3/2)	(-3/2,1/2)		(1/2,)
Test values	-2	0		1
Sign	+	0 –	0	+

d) The graphing calculator yields $x \le \frac{1}{2}$ or $x \ge 3$ as the solution to the inequality $2x^2 - 7x + 3 \ge 0$.

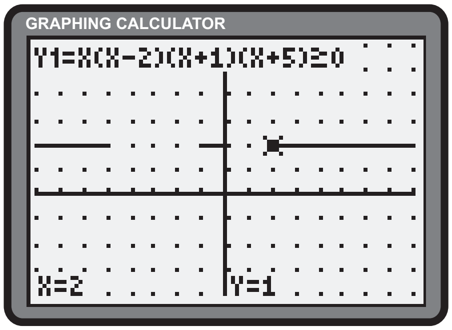




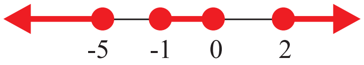
e) The roots of the equation, $x(x-2)(x-2) = 0$ are $x = 0$ and $x = 2$. An interval chart yields the solution $x \geq 0$.

		0		2	
Intervals	(- ,0)		(0,2)		(2,)
Test values	-2		1		3
Sign	−	0	+	0	+

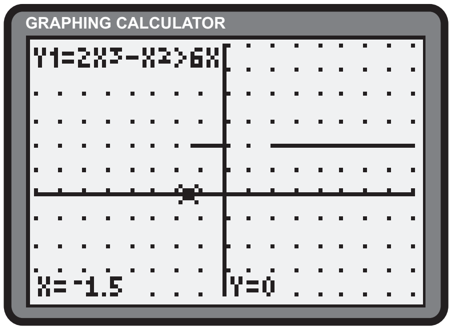
g) The calculator yields $x \leq -5$ or $-1 \leq x \leq 0$ or $x \geq 2$ as the solution to $x(x-2)(x+1)(x+5) \geq 0$.



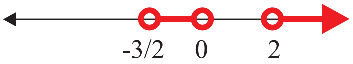
```
Y1=X(X-2)(X+1)(X+5)≥0
X=2          Y=1
```



i) The calculator yields $-\frac{3}{2} < x < 0$ or $x > 2$ as the solution to the inequality $2x^3 - x^2 > 6x$.

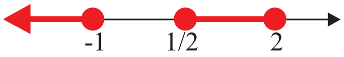


```
Y1=2X³-X²>6X
X=⁻1.5       Y=0
```

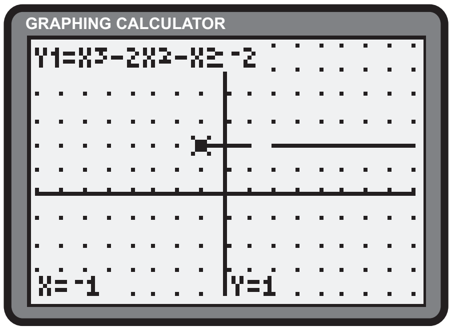


f) The roots of the equation, $(2x-1)(x+1)(x-2) = 0$ are $x = \frac{1}{2}$, $x = -1$, and $x = 2$. An interval chart yields the solution $x \leq -1$ or $\frac{1}{2} \leq x \leq 2$.

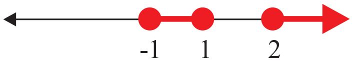
		-1		1/2		2	
Intervals	(- ,-1)		(-1,1/2)		(1/2,2)		(2,)
Test values	-2		0		1		3
Sign	−	0	+	0	−	0	+



h) The calculator yields $-1 \leq x \leq 1$ or $x \geq 2$ as the solution to the inequality $x^3 - 2x^2 - x \geq -2$.

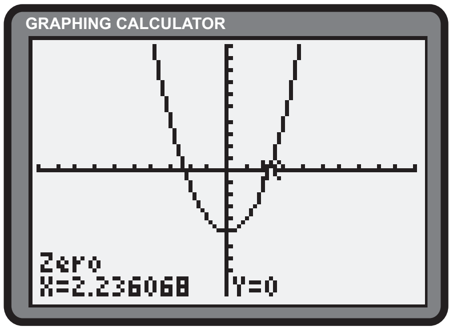


```
Y1=X³-2X²-X≥⁻2
X=⁻1         Y=1
```



Section 2.6 Page 93 Question 7

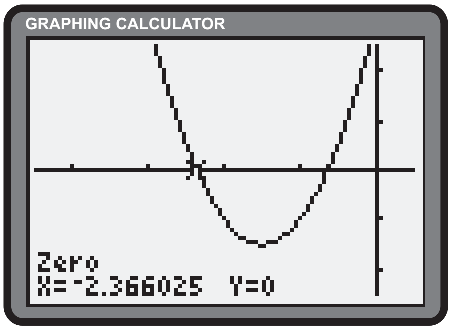
a) The zeros of $y = x^2 - 5$ are $x \pm 2.24$.



Zero
X=2.236068 Y=0

The solution to $x^2 - 5 > 0$ is $x < -2.24$ or $x > 2.24$.

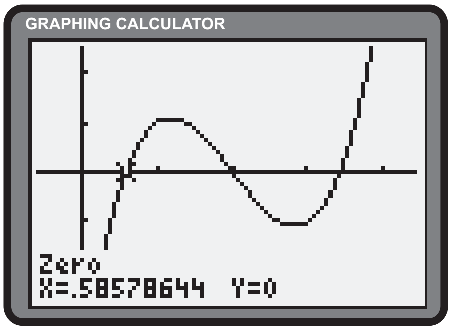
c) The zeros of $y = 2x^2 + 6x + 3$ are $x = -2.37$ and $x = -0.63$.



Zero
X=-2.366025 Y=0

The solution to $2x^2 + 6x + 3 > 0$ is $x < -2.37$ or $x > -0.63$.

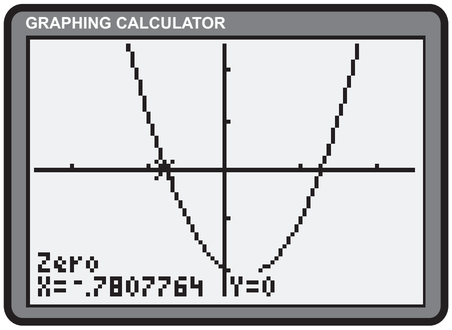
e) The zeros of $y = x^3 - 6x^2 + 10x - 4$ are $x = 0.59$, $x = 2$ and $x = 3.41$.



Zero
X=.58578644 Y=0

The solutions to $x^3 - 6x^2 + 10x - 4 \leq 0$ are $x \leq 0.59$ or $2 \leq x \leq 3.41$.

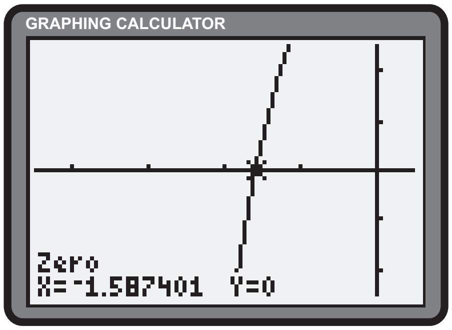
b) The zeros of $y = 2x^2 - x - 2$ are $x = -0.78$ and $x = 1.28$.



Zero
X=-.7807764 Y=0

The solution to $2x^2 - x - 2 \geq 0$ is $x \leq -0.78$ or $x \geq 1.28$.

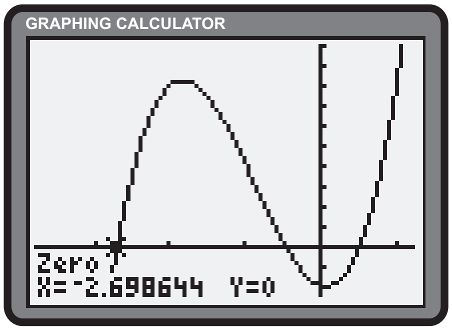
d) The zeros of $y = x^3 + 4$ is $x = -1.59$.



Zero
X=-1.587401 Y=0

The solution to $x^3 + 4 > 0$ is $x > -1.59$.

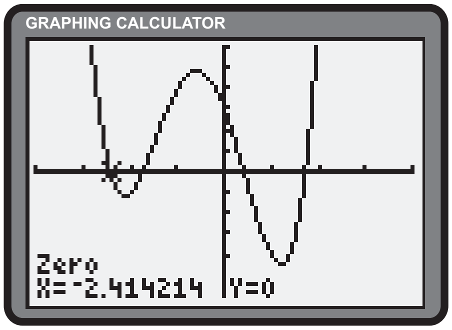
f) The zeros of $y = 3x^3 + 8x^2 - x - 2$ are $x = -2.70$, $x = -0.48$, and $x = 0.51$.



Zero
X=-2.698644 Y=0

The solutions to $3x^3 + 8x^2 - x < 2$ are $x < -2.70$ and $-0.48 < x < 0.51$.

g) The zeros of $y = x^4 + 2x^3 - 4x^2 - 6x + 3$ are $x = -2.41$, $x = -1.73$, $x = 0.41$, and $x = 1.73$.



Zero
X=-2.414214 Y=0

The solutions to $x^4 + 2x^3 - 4x^2 - 6x \leq -3$ are $-2.41 \leq x \leq -1.73$ and $0.41 \leq x \leq 1.73$.

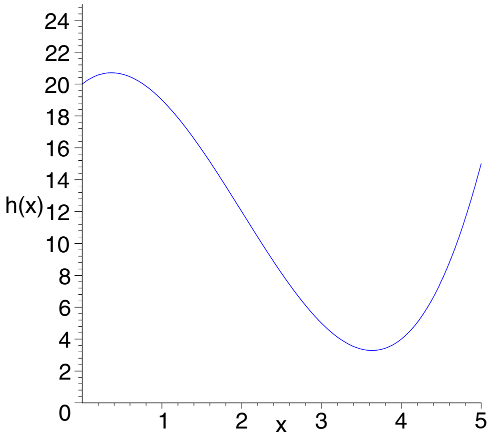
Apply, Solve, Communicate

Section 2.6 Page 93 Question 8

a) $f(x) \leq g(x) \iff x \leq -2.3$ or $x \geq 1.3$
b) $f(x) > g(x) \iff x < -1.1$ or $1.1 < x < 3.9$
c) $f(x) \geq g(x) \iff -1 \leq x \leq 0.5$
d) $f(x) < g(x) \iff -2 < x < -0.6$ or $x > 0.5$

Section 2.6 Page 94 Question 9

a) The graph of $h(x) = x^3 - 6x^2 + 4x + 20$ appears below.



b) From the graph it appears that over the interval $0 \leq x \leq 5$, $h(x)$ has a local maximum in the neighbourhood of $x = 0.4$ and a local minimum in the neighbourhood of $x = 3.6$. The vertical distance, d, can be given as

$$\begin{aligned} d &= h(0.4) - h(3.6) \\ &\doteq 17.408 \end{aligned}$$

The vertical distance is approximately 17.4 m.

Section 2.6 Page 94 Question 10

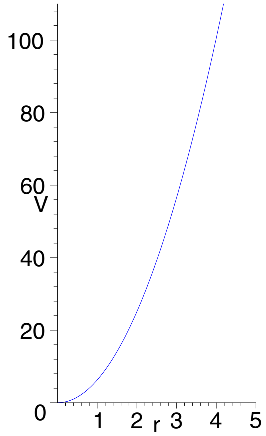
a) From $V(r, h) = \pi r^2 h$ and $h = 2r$ we obtain

$$\begin{aligned} V(r, h) &= \pi r^2 h \\ V(r) &= \pi r^2 (2r) \\ &= 2\pi r^3 \end{aligned}$$

The volume of the cylinder can be expressed as $V(r) = 2\pi r^3$.

c) The domain is $r \geq 0$ and the range is $V \geq 0$.

b) $V = 2\pi r^2$



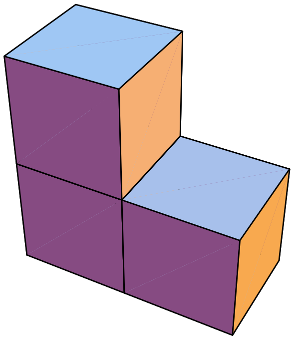
Section 2.6 Page 94 Question 11

a) Each cube has six faces. There are four hidden faces. Therefore there are $3 \times 6 - 4$ or 14 exposed faces. Each exposed face has an area of x^2 square units. The surface area of the object can be expressed as $A(x) = 14x^2$. The solid is composed of three cubes, each with a volume of x^3 cubic units. The total volume of the solid can be expressed as $V(x) = 3x^3$ cubic units.

b)

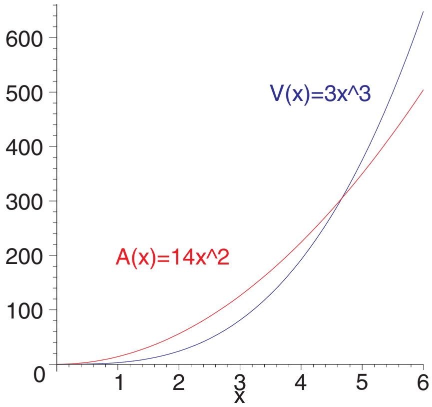
$$\begin{aligned} A(x) &= V(x) \\ 14x^2 &= 3x^3 \\ 3x^3 - 14x^2 &= 0 \\ x^2(3x - 14) &= 0 \\ x &= 0, 0, \text{ or } \frac{14}{3} \end{aligned}$$

Since $x > 0$, the surface area and volume will have the same numerical value if $x = \frac{14}{3}$.



c) The three roots from part b) yield the two test intervals, $0 < x < \frac{14}{3}$ and $x > \frac{14}{3}$. Using $x = 1$ in the first test interval we find $A(1) = 14$ and $V(1) = 3$. Thus, $A(x) > V(x)$ in the interval $0 < x < \frac{14}{3}$. Using $x = 5$ in the second test interval gives $A(5) = 350$ and $V(5) = 375$, therefore $A(x) < V(x)$ in the interval $x > \frac{14}{3}$.

d) The x-intercepts of the graph of $A(x) - V(x)$ are the values for which the surface area and volume have the same numerical value. The domain over which $A(x) - V(x) > 0$ is the set of values for which the numerical value for surface area is greater than that for the volume. The domain over which $A(x) - V(x) < 0$ is the set of values for which the numerical value for surface area is less than that for the volume.



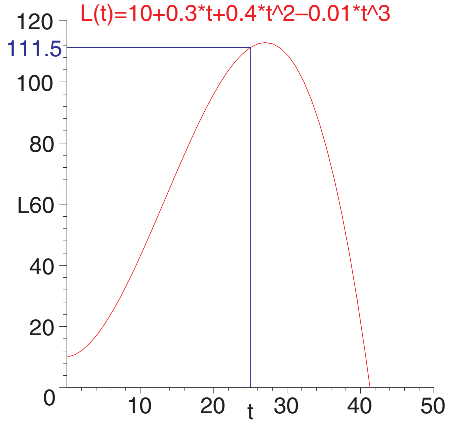
Section 2.6 Page 95 Question 12

a) Answers may vary: $0 \le t \le 40$.

c) The rate is approximately 110 lm at 25°C.

d) Fireflies produce light energy at the greatest rate at approximately 27°C.

b)



Section 2.6 Page 95 Question 13

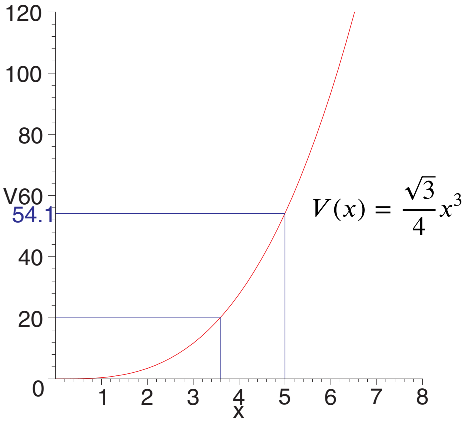
a) The volume, $V(x)$, is the area of the equilateral triangle times the length.

$$\begin{aligned} V(x) &= \frac{x}{2}\left(\frac{\sqrt{3}x}{2}\right) \times x \\ &= \frac{\sqrt{3}x^3}{4} \end{aligned}$$

The volume of the triangular prism can be given by $V(x) = \frac{\sqrt{3}}{4}x^3$.

b) The domain of $V(x)$ is restricted to $x > 0$.

c)



d)

$$\begin{aligned} V(5) &= \frac{\sqrt{3}}{4}(5)^3 \\ &\doteq 54.13 \end{aligned}$$

The volume is approximately 54.1 cm^3.

e)

$$\begin{aligned} V(x) &= 20 \\ \frac{\sqrt{3}}{4}x^3 &= 20 \\ x^3 &= \frac{80\sqrt{3}}{3} \\ x &= \sqrt[3]{\frac{80\sqrt{3}}{3}} \\ &\doteq 3.59 \end{aligned}$$

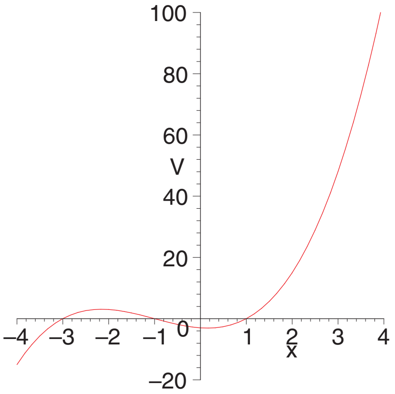
The edge length is approximately 3.6 cm.

Section 2.6 Page 95 Question 14

a) A graph of the function $V(x) = (x+3)(x+1)(x-1)$ can be used to solve the inequality $V(x) > 0$. The solution is $-3 < x < -1$ or $x > 1$.

b) Since all edge lengths must be positive, the edge length $x-1$ requires $x > 1$.

c) The condition $-3 < x < -1$ is discarded because it would result in negative values for two of the dimensions.



Section 2.6 Page 95 Question 15

Answers may vary.

a) A parabola opening upward with roots at -1 and 2 can be defined by the equation $y = (x+1)(x-2)$. The interval of the domain over which the parabola lies on or below the x-axis can be defined by the inequality $(x+1)(x-2) \leq 0$.

b) A parabola opening upward with roots at $-\frac{1}{2}$ and 4 can be defined by the equation $y = (2x+1)(x-4)$. The interval of the domain over which the parabola lies above the x-axis can be defined by the inequality $(2x+1)(x-4) > 0$.

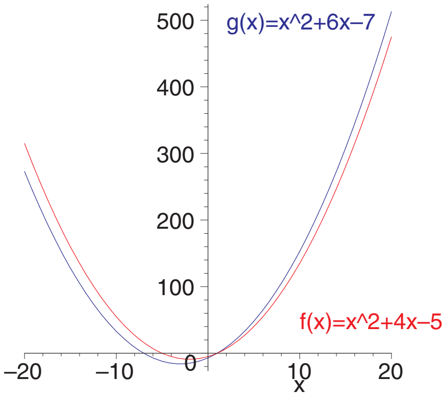
Section 2.6 Page 95 Question 16

Answers will vary.

Section 2.6 Page 95 Question 17

$$\begin{aligned} x^2 + 4x - 5 &> 0 \\ (x+5)(x-1) &> 0 \\ x < -5 \text{ or } x &> 1 \qquad (1) \\ x^2 + 6x - 7 &\le 0 \\ (x+7)(x-1) &\le 0 \\ -7 \le x &\le 1 \qquad (2) \end{aligned}$$

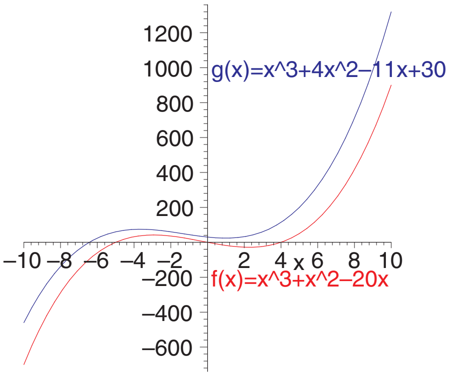
Comparison of (1) and (2) leads to the solution $-7 \le x < -5$.

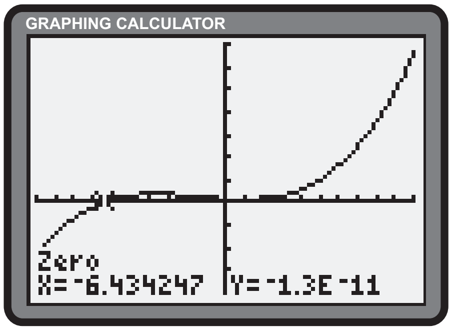


Section 2.6 Page 95 Question 18

$$\begin{aligned} x^3 + x^2 - 20x &\le 0 \\ x(x^2 + x - 20) &\le 0 \\ x(x+5)(x-4) &\le 0 \\ x < -5 \text{ or } 0 \le x &\le 4 \end{aligned}$$

The graphing calculator reveals a zero of $x = -6.43$ for the function $g(x) = x^3 + 4x^2 - 11x + 30$.





Comparison of the intervals leads to the solution $x < -6.43$.

Section 2.6 Page 95 Question 19

a) Determine r in terms of h.

$$\begin{aligned} 3(2\pi r) + 4h &= 6 \\ 3\pi r + 2h &= 3 \\ r &= \frac{3-2h}{3\pi} \qquad (1) \end{aligned}$$

The radius is $r = \dfrac{3-2h}{3\pi}$. Determine S.

$$\begin{aligned} S &= 2(\pi r^2) + 2\pi r h \\ &= 2\pi r(r+h) \qquad (2) \end{aligned}$$

Substitute (1) into (2).

$$\begin{aligned} S(h) &= 2\pi\left(\frac{3-2h}{3\pi}\right)\left(\frac{3-2h}{3\pi}+h\right) \\ &= \frac{2(3-2h)^2}{9\pi} + \frac{2h(3-2h)}{3} \\ &= \frac{18-24h+8h^2}{9\pi} + 2h - \frac{4}{3}h^2 \end{aligned}$$

The surface area is $S = \dfrac{18-24h+8h^2}{9\pi} + 2h - \dfrac{4}{3}h^2$.

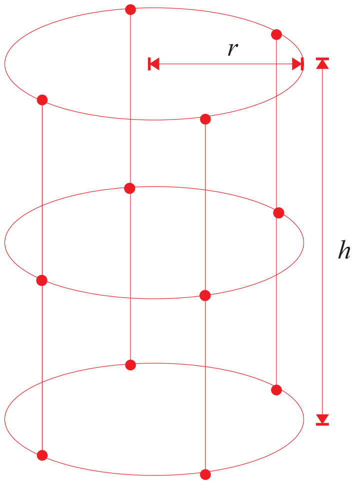
Determine V.

$$V(r,h) = \pi r^2 h \qquad (3)$$

Substitute (1) in (3).

$$\begin{aligned} V(h) &= \pi\left(\frac{3-2h}{3\pi}\right)^2 h \qquad (3) \\ &= \frac{h(3-2h)^2}{9\pi} \\ &= \frac{9h-12h^2+4h^3}{9\pi} \end{aligned}$$

The volume is $V = \dfrac{9h-12h^2+4h^3}{9\pi}$.



b)

$$\begin{aligned} r &\ge 0.1 \\ \frac{3-2h}{3\pi} &\ge 0.1 \\ 3-2h &\ge 0.3\pi \\ -2h &\ge -3+0.3\pi \\ h &\le 1.5-0.15\pi \end{aligned}$$

The height restriction is $h \le 1.5 - 0.15\pi$.

Review of Key Concepts

2.2 Dividing a Polynomial by a Polynomial

Section Review Page 107 Question 3

a)

$$\begin{array}{r} x+\ \ 8 \\ x+2\overline{)x^2+10x+16} \\ \underline{x^2+2x}\qquad\ \ \\ 8x+16 \\ \underline{8x+16} \\ 0 \end{array}$$

Restriction: $x \neq -2$

b)

$$\begin{array}{r} y^2+2y+2 \\ y-1\overline{)y^3+y^2+0y-2} \\ \underline{y^3-y^2}\qquad\qquad\ \ \\ 2y^2+0y\qquad \\ \underline{2y^2-2y}\qquad \\ 2y-2 \\ \underline{2y-2} \\ 0 \end{array}$$

Restriction: $y \neq 1$

c)

$$\begin{array}{r} 2m-1 \\ 2m+3\overline{)4m^2+4m-1} \\ \underline{4m^2+6m}\qquad \\ -2m-1 \\ \underline{-2m-3} \\ 2 \end{array}$$

Restriction: $m \neq -\frac{3}{2}$

d)

$$\begin{array}{r} x^2-2x+3 \\ x+3\overline{)x^3+\ \ x^2-3x-9} \\ \underline{x^3+3x^2}\qquad\qquad\ \ \\ -2x^2-3x\qquad \\ \underline{-2x^2-6x}\qquad \\ 3x-9 \\ \underline{3x+9} \\ -18 \end{array}$$

Restriction: $x \neq -3$

e)

$$\begin{array}{r} 3y-2 \\ y^2+0y+2\overline{)3y^3-2y^2+12y-9} \\ \underline{3y^3+0y^2+\ \ 6y}\qquad \\ -2y^2+\ \ 6y-9 \\ \underline{-2y^2+\ \ 0y-4} \\ 6y-5 \end{array}$$

Restriction: none

Section Review Page 107 Question 4

a) Let w be the width of the business card.

$$\begin{aligned} w &= \frac{\text{area}}{\text{length}} \\ w(x) &= \frac{28x^2-15x+2}{7x-2} \\ &= \frac{(7x-2)(4x-1)}{7x-2} \\ &= 4x-1 \end{aligned}$$

Restriction: $x \neq \frac{2}{7}$. The width is $4x - 1$.

b)

$$\begin{aligned} w(x) &= 4x-1 \\ w(13) &= 4(13)-1 \\ &= 51 \end{aligned}$$

The width is 51 mm. The length is 7(13) − 2 or 89 mm.

Section Review Page 107 Question 5

a)

$$\begin{array}{r} x^2-\ \ x-1 \\ x+3\overline{)x^3+2x^2-4x-3} \\ \underline{x^3+3x^2}\qquad\qquad\ \ \\ -\ x^2-4x\qquad \\ \underline{-\ x^2-3x}\qquad \\ -\ x-3 \\ \underline{-\ x-3} \\ 0 \end{array}$$

Restriction: $x \neq -3$

b)

$$\begin{array}{r} x^2-\ \ 3x-5 \\ 2x-1\overline{)2x^3-7x^2-\ \ 7x+5} \\ \underline{2x^3-\ \ x^2}\qquad\qquad\quad \\ -6x^2-\ \ 7x\qquad \\ \underline{-6x^2+\ \ 3x}\qquad \\ -10x+5 \\ \underline{-10x+5} \\ 0 \end{array}$$

Restriction: $x \neq \frac{1}{2}$

c)

$$\begin{array}{r} x^3\qquad\quad-\ \ 3x+1 \\ 3x-4\overline{)3x^4-4x^3-9x^2+15x-4} \\ \underline{3x^4-4x^3}\qquad\qquad\qquad\quad \\ -9x^2+15x\qquad \\ \underline{-9x^2+12x}\qquad \\ 3x-4 \\ \underline{3x-4} \\ 0 \end{array}$$

Restriction: $x \neq \frac{4}{3}$

2.3 The Remainder Theorem

Section Review Page 107 Question 6

a)
$$\begin{aligned} P(x) &= x^2 + 5x - 8 \\ P(2) &= 2^2 + 5(2) - 8 \\ &= 4 + 10 - 8 \\ &= 6 \end{aligned}$$

The remainder is 6.

b)
$$\begin{aligned} P(m) &= 3m^2 + 7m + 1 \\ P(-3) &= 3(-3)^2 + 7(-3) + 1 \\ &= 27 - 21 + 1 \\ &= 7 \end{aligned}$$

The remainder is 7.

c)
$$\begin{aligned} P(y) &= y^3 - 5y^2 - 3y + 1 \\ P(-1) &= (-1)^3 - 5(-1)^2 - 3(-1) + 1 \\ &= -1 - 5 + 3 + 1 \\ &= -2 \end{aligned}$$

The remainder is −2.

d)
$$\begin{aligned} P(x) &= 2x^2 + 5x + 11 \\ P\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) + 11 \\ &= \frac{1}{2} + \frac{5}{2} + 11 \\ &= 3 + 11 \\ &= 14 \end{aligned}$$

The remainder is 14.

e)
$$\begin{aligned} P(y) &= 8y^3 + 12y^2 - 4y + 5 \\ P\left(-\frac{3}{2}\right) &= 8\left(-\frac{3}{2}\right)^3 + 12\left(-\frac{3}{2}\right)^2 - 4\left(-\frac{3}{2}\right) + 5 \\ &= -27 + 27 + 6 + 5 \\ &= 11 \end{aligned}$$

The remainder is 11.

Section Review Page 107 Question 7

a)
$$\begin{aligned} P(x) &= x^3 - 4x^2 + kx - 2 \\ P(1) &= -1 \\ 1^3 - 4(1)^2 + k(1) - 2 &= -1 \\ 1 - 4 + k - 2 &= -1 \\ k &= 4 \end{aligned}$$

The remainder is −1 when $k = 4$.

b)
$$\begin{aligned} P(x) &= x^3 - 3x^2 - 6x + k \\ P(-2) &= -1 \\ (-2)^3 - 3(-2)^2 - 6(-2) + k &= -1 \\ -8 - 12 + 12 + k &= -1 \\ k &= 7 \end{aligned}$$

The remainder is −1 when $k = 7$.

Section Review Page 107 Question 8

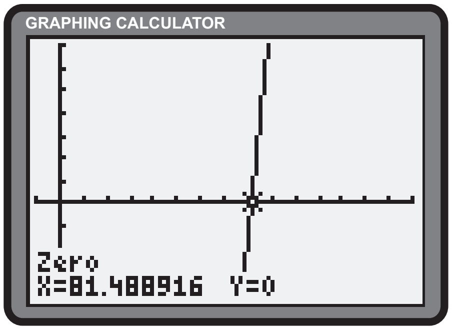
a) Answers will vary.

b)
$$\begin{array}{r} 0.5t + 25 \\ t - 10\overline{)0.5t^2 + 20t + 50} \\ \underline{0.5t^2 - 5t} \phantom{{}+ 50} \\ 25t + 50 \\ \underline{25t - 250} \\ 300 \end{array}$$

Restriction: $t \neq 10$. $Q(t) = 0.5t + 25$. $R = 300$. $Q(t)$ is the average sales between the 10th day after release and the tth day after release. R is the total sales 10 days after release.

c) Answers will vary.

d)
$$\begin{aligned} C(t) &= 5000 \\ 0.5t^2 + 20t + 50 &= 5000 \\ 0.5t^2 + 20t - 4950 &= 0 \\ t^2 + 40t - 9900 &= 0 \end{aligned}$$



Zero
X=81.488916 Y=0

No. The book becomes a bestseller on the 82nd day.

Section Review Page 108 Question 9

a)

$$\begin{aligned} P(x) &= 3x^3 - 2x^2 + x + k \\ P(2) &= 0 \\ 3(2)^3 - 2(2)^2 + 2 + k &= 0 \\ 24 - 8 + 2 + k &= 0 \\ 18 + k &= 0 \\ k &= -18 \end{aligned}$$

b)

$$\begin{aligned} P(x) &= 8x^3 + 6x^2 + 2x + k \\ P\left(-\frac{3}{2}\right) &= -15 \\ 8\left(-\frac{3}{2}\right)^3 + 6\left(-\frac{3}{2}\right)^2 + 2\left(-\frac{3}{2}\right) + k &= -15 \\ -27 + \frac{27}{2} - 3 + k &= -15 \\ -\frac{33}{2} + k &= -15 \\ k &= \frac{3}{2} \end{aligned}$$

c)

$$\begin{aligned} P(x) &= 4x^2 + kx - 15 \\ P(-5) &= 10 \\ 4(-5)^2 + k(-5) - 15 &= 10 \\ 100 - 5k - 15 &= 10 \\ -5k + 85 &= 10 \\ 5k &= 75 \\ k &= 15 \end{aligned}$$

2.4 The Factor Theorem

Section Review Page 108 Question 10

a)

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 5x + 6 \\ P(1) &= 1^3 - 2(1)^2 - 5(1) + 6 \\ &= 1 - 2 - 5 + 6 \\ &= 0 \end{aligned}$$

Since $P(1) = 0$, $x - 1$ is a factor of $x^3 - 2x^2 - 5x + 6$.

b)

$$\begin{aligned} P(y) &= y^4 + 4y^3 - 9y^2 - 16y + 20 \\ P(-2) &= (-2)^4 + 4(-2)^3 - 9(-2)^2 - 16(-2) + 20 \\ &= 16 - 32 - 36 + 32 + 20 \\ &= 0 \end{aligned}$$

Since $P(-2) = 0$, $y + 2$ is a factor of $y^4 + 4y^3 - 9y^2 - 16y + 20$.

c)

$$\begin{aligned} P(n) &= 2n^3 - n^2 - 4n + 3 \\ P\left(-\frac{3}{2}\right) &= 2\left(-\frac{3}{2}\right)^3 - \left(-\frac{3}{2}\right)^2 - 4\left(-\frac{3}{2}\right) + 3 \\ &= -\frac{27}{4} - \frac{9}{4} + 6 + 3 \\ &= -9 + 6 + 3 \\ &= 0 \end{aligned}$$

Since $P\left(-\frac{3}{2}\right) = 0$, $2n + 3$ is a factor of $2n^3 - n^2 - 4n + 3$.

d)

$$\begin{aligned} P(z) &= 3z^3 + 17z^2 + 18z - 8 \\ P\left(\frac{1}{3}\right) &= 3\left(\frac{1}{3}\right)^3 + 17\left(\frac{1}{3}\right)^2 + 18\left(\frac{1}{3}\right) - 8 \\ &= \frac{1}{9} + \frac{17}{9} + 6 - 8 \\ &= 0 \end{aligned}$$

Since $P\left(\frac{1}{3}\right) = 0$, $3z - 1$ is a factor of $3z^3 + 17z^2 + 18z - 8$.

Section Review Page 108 Question 11

a)
$$\begin{aligned} P(x) &= x^3 - x^2 - 5x - 3 \\ P(3) &= 3^3 - 3^2 - 5(3) - 3 \\ &= 27 - 9 - 15 - 3 \\ &= 0 \end{aligned}$$

$x - 3$ is a factor of $P(x)$. Synthetic division of $P(x)$ by $x - 3$ reveals another factor of $x^2 + 2x + 1$.

$$\begin{aligned} P(x) &= (x - 3)(x^2 + 2x + 1) \\ x^3 - x^2 - 5x - 3 &= (x - 3)(x + 1)^2 \end{aligned}$$

b)
$$\begin{aligned} P(x) &= x^3 + 5x^2 + 3x - 4 \\ P(-4) &= (-4)^3 + 5(-4)^2 + 3(-4) - 4 \\ &= -64 + 80 - 12 - 4 \\ &= 0 \end{aligned}$$

$x + 4$ is a factor of $P(x)$. Synthetic division of $P(x)$ by $x + 4$ reveals another factor of $x^2 + x - 1$.

$$x^3 + 5x^2 + 3x - 4 = (x + 4)(x^2 + x - 1)$$

c) Factor by grouping.

$$\begin{aligned} P(x) &= 2x^3 - x^2 - 2x + 1 \\ &= x^2(2x - 1) - 1(2x - 1) \\ &= (2x - 1)(x^2 - 1) \\ &= (2x - 1)(x - 1)(x + 1) \end{aligned}$$

$$2x^3 - x^2 - 2x + 1 = (2x - 1)(x - 1)(x + 1)$$

d)
$$\begin{aligned} P(x) &= 3x^3 + 13x^2 - 16 \\ P(1) &= 3(1)^3 + 13(1)^2 - 16 \\ &= 3 + 13 - 16 \\ &= 0 \end{aligned}$$

$x - 1$ is a factor of $P(x)$. Synthetic division of $P(x)$ by $x - 1$ reveals another factor of $3x^2 + 16x + 16$.

$$\begin{aligned} P(x) &= (x - 1)(3x^2 + 16x + 16) \\ 3x^3 + 13x^2 - 16 &= (x - 1)(x + 4)(3x + 4) \end{aligned}$$

Section Review Page 108 Question 12

Answers may vary. Since $V(4) = 0$, one possible dimension is $x + 4$. Division reveals the other two dimensions.

$$\begin{aligned} V &= 2x^3 - x^2 - 22x - 24 \\ &= (x - 4)(2x^2 + 7x + 6) \\ &= (x - 4)(x + 2)(2x + 3) \end{aligned}$$

4	2	-1	-22	-24
		8	28	24
	2	7	6	0

Possible dimensions are $x - 4$, $x + 2$, and $2x + 3$.

2.5 Roots of Polynomial Equations

Section Review Page 108 Question 13

a)
$$\begin{aligned} y^3 &= 9y \\ y^3 - 9y &= 0 \\ y(y^2 - 9) &= 0 \\ y(y - 3)(y + 3) &= 0 \\ y &= 0 \text{ or } \pm 3 \end{aligned}$$

Check $y = 0$.

$$\begin{aligned} \textbf{L.S.} &= 0^3 \\ &= 0 \\ \textbf{R.S.} &= 9(0) \\ &= 0 \\ &= \textbf{L.S.} \end{aligned}$$

Check $y = -3$.

$$\begin{aligned} \textbf{L.S.} &= (-3)^3 \\ &= -27 \\ \textbf{R.S.} &= 9(-3) \\ &= -27 \\ &= \textbf{L.S.} \end{aligned}$$

Check $y = 3$.

$$\begin{aligned} \textbf{L.S.} &= (3)^3 \\ &= 27 \\ \textbf{R.S.} &= 9(3) \\ &= 27 \\ &= \textbf{L.S.} \end{aligned}$$

The solutions are 0 and ±3.

b)

$$\begin{aligned} P(n) &= n^3 - 3n - 2 \\ P(-1) &= (-1)^3 - 3(-1) - 2 \\ &= -1 + 3 - 2 \\ &= 0 \end{aligned}$$

$n + 1$ is a factor of $P(n)$. Synthetic division by $n + 1$ reveals another factor of $n^2 - n - 2$.

$$\begin{aligned} P(n) &= n^3 - 3n - 2 \\ &= (n + 1)(n^2 - n - 2) \\ &= (n + 1)(n + 1)(n - 2) \\ &= (n + 1)^2(n - 2) \end{aligned}$$

Check $n = 2$.

$$\begin{aligned} \textbf{L.S.} &= 2^3 - 3(2) - 2 \\ &= 8 - 6 - 2 \\ &= 0 \\ &= \textbf{R.S.} \end{aligned}$$

The solutions are -1 and 2.

c)

$$\begin{aligned} 3w^2 + 11w &= 2w^3 + 6 \\ 2w^3 - 3w^2 - 11w + 6 &= 0 \end{aligned}$$

Let $P(w) = 2w^3 - 3w^2 - 11w + 6$.

$$\begin{aligned} P(3) &= 2(3)^3 - 3(3)^2 - 11(3) + 6 \\ &= 54 - 27 - 33 + 6 \\ &= 0 \end{aligned}$$

$w - 3$ is a factor of $P(w)$. Synthetic division reveals another factor of $2w^2 + 3w - 2$.

$$\begin{aligned} P(w) &= 2w^3 - 3w^2 - 11w + 6 \\ &= (w - 3)(2w^2 + 3w - 2) \\ &= (w - 3)(2w - 1)(w + 2) \end{aligned}$$

Check $w = \frac{1}{2}$.

$$\begin{aligned} \textbf{L.S.} &= 3\left(\frac{1}{2}\right)^2 + 11\left(\frac{1}{2}\right) \\ &= \frac{3}{4} + \frac{11}{2} \\ &= \frac{25}{4} \\ \textbf{R.S.} &= 2\left(\frac{1}{2}\right)^3 + 6 \\ &= \frac{1}{4} + 6 \\ &= \frac{25}{4} \\ &= \textbf{L.S.} \end{aligned}$$

Check $w = -2$.

$$\begin{aligned} \textbf{L.S.} &= 3(-2)^2 + 11(-2) \\ &= 12 - 22 \\ &= -10 \\ \textbf{R.S.} &= 2(-2)^3 + 6 \\ &= -16 + 6 \\ &= -10 \\ &= \textbf{L.S.} \end{aligned}$$

The solutions are 3, $\frac{1}{2}$, and -2.

d)

$$\begin{aligned} 3x^3 - 2 &= 8x^3 - 7x \\ 5x^3 - 7x + 2 &= 0 \end{aligned}$$

Let $P(x) = 5x^3 - 7x + 2$.

$$\begin{aligned} P(1) &= 5(1)^3 - 7(1) + 2 \\ &= 5 - 7 + 2 \\ &= 0 \end{aligned}$$

$x - 1$ is a factor of $P(x)$. Synthetic division reveals another factor of $5x^2 + 5x - 2$.

$$\begin{aligned} (x-1)(5x^2 + 5x - 2) &= 0 \\ x - 1 &= 0 \\ x &= 1 \\ 5x^2 + 5x - 2 &= 0 \\ x &= \frac{-(5) \pm \sqrt{(5)^2 - 4(5)(-2)}}{2(5)} \\ &= \frac{-5 \pm \sqrt{65}}{10} \end{aligned}$$

Check $x = \frac{-5 + \sqrt{65}}{10}$.

$$\begin{aligned} \textbf{L.S.} &= 3\left(\frac{-5 + \sqrt{65}}{10}\right)^3 - 2 \\ &= \frac{21\sqrt{65} - 265}{50} \\ \textbf{R.S.} &= 8\left(\frac{-5 + \sqrt{65}}{10}\right)^3 - 7\left(\frac{-5 + \sqrt{65}}{10}\right) \\ &= \frac{21\sqrt{65} - 265}{50} \\ &= \textbf{L.S.} \end{aligned}$$

Check $x = \frac{-5 - \sqrt{65}}{10}$.

$$\begin{aligned} \textbf{L.S.} &= 3\left(\frac{-5 - \sqrt{65}}{10}\right)^3 - 2 \\ &= \frac{-21\sqrt{65} - 265}{50} \\ \textbf{R.S.} &= \frac{-21\sqrt{65} - 265}{50} \\ &= \textbf{L.S.} \end{aligned}$$

Section Review Page 108 Question 14

a)

$$\begin{aligned} P(x) &= x^3 - 3x^2 + x + 1 \\ P(1) &= 1^3 - 3(1)^2 + 1 + 1 \\ &= 1 - 3 + 1 + 1 \\ &= 0 \end{aligned}$$

$x - 1$ is a factor of $P(x)$. Synthetic division reveals another factor of $x^2 - 2x - 1$. Use the quadratic formula to determine the other roots.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{2 \pm \sqrt{8}}{2} \\ &= \frac{2 \pm 2\sqrt{2}}{2} \\ &= 1 \pm \sqrt{2} \end{aligned}$$

The exact roots are $1,\ 1 \pm \sqrt{2}$.

b)

$$\begin{aligned} 4y^3 - 19y &= 6 \\ P(y) &= 4y^3 - 19y - 6 \\ P(-2) &= 4(-2)^3 - 19(-2) - 6 \\ &= -32 + 38 - 6 \\ &= 0 \end{aligned}$$

$y + 2$ is a factor of $P(y)$. Synthetic division reveals another factor of $4y^2 - 8y - 3$. Use the quadratic formula to determine the other roots.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(-3)}}{2(4)} \\ &= \frac{8 \pm \sqrt{112}}{8} \\ &= \frac{8 \pm 4\sqrt{7}}{8} \\ &= \frac{2 \pm \sqrt{7}}{2} \end{aligned}$$

The exact roots are $-2, \frac{2 \pm \sqrt{7}}{2}$.

c) Factor by grouping.

$$\begin{aligned} x^3 + x^2 + 16x + 16 &= 0 \\ x^2(x + 1) + 16(x + 1) &= 0 \\ (x + 1)(x^2 + 16) &= 0 \\ x + 1 &= 0 \\ x &= -1 \\ x^2 + 16 &= 0 \\ x^2 &= -16 \\ x &= \pm\sqrt{-16} \\ &= \pm 4i \end{aligned}$$

The exact roots are $-1, \pm 4i$.

d)

$$\begin{aligned} m^3 - m^2 &= 2m + 12 \\ m^3 - m^2 - 2m - 12 &= 0 \\ P(m) &= m^3 - m^2 - 2m - 12 \\ P(3) &= 3^3 - 3^2 - 2(3) - 12 \\ &= 27 - 9 - 6 - 12 \\ &= 0 \end{aligned}$$

$m - 3$ is a factor of $P(m)$. Synthetic division reveals another factor of $m^2 + 2m + 4$. Use the quadratic formula to determine the other roots.

$$\begin{aligned} m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{-12}}{2} \\ &= \frac{-2 \pm 2i\sqrt{3}}{2} \\ &= -1 \pm i\sqrt{3} \end{aligned}$$

The exact roots are $3, -1 \pm i\sqrt{3}$.

Section Review   Page 108   Question 15

Let d be the depth of the cereal box in centimetres. The width can be expressed as $4d$. The height can be expressed as $4d + 5$.

$$\begin{aligned} V(d) &= 2500 \\ d(4d)(4d+5) &= 2500 \\ 16d^3 + 20d^2 &= 2500 \\ 16d^3 + 20d^2 - 2500 &= 0 \\ 4d^3 + 5d^2 - 625 &= 0 \end{aligned} \tag{1}$$

Let $P(d) = 4d^3 + 5d^2 - 625$. Try $d = 5$.

$$\begin{aligned} P(5) &= 4(5)^3 + 5(5)^2 - 625 \\ &= 0 \end{aligned}$$

$d - 5$ is a factor of $P(d)$. Synthetic division reveals another factor of $4d^2 + 25d + 125$. Use the quadratic formula to determine the other roots.

$$\begin{aligned} d &= \frac{-25 \pm \sqrt{25^2 - 4(4)(125)}}{2(4)} \\ &= \frac{-25 \pm \sqrt{-1375}}{8} \end{aligned}$$

$d = 5$ is the only real root, so the dimensions are 5 cm by 20 cm by 25 cm.

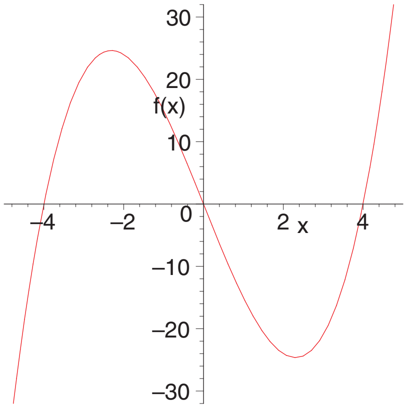
Section Review   Page 108   Question 16

a) Answers will vary.
b) No; an exponential function has no zeros.
c) i) No; an exponential function has no zeros.

2.6 Polynomial Functions and Inequations

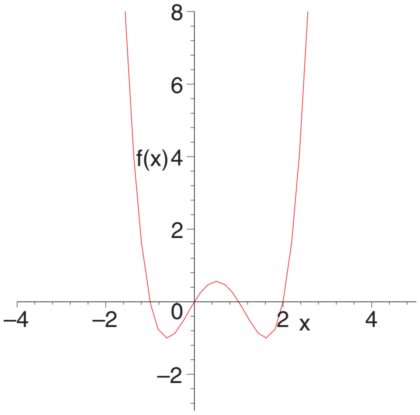
Section Review   Page 108   Question 17

a) $f(x) = x(x+4)(x-4)$



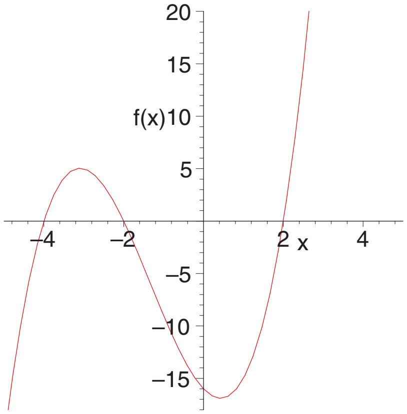
i) Domain: $\Re$; Range: $\Re$
ii) y-intercept: 0
iii) Zeros: $-4, 0, 4$
iv) $f(x) > 0$: $(-4, 0)$ and $(4, \infty)$;
$f(x) \leq 0$: $(-\infty, -4]$ and $[0, 4]$
v) Symmetry: odd
vi) The left-most y-values are negative. The right-most y-values are positive.

b) $f(x) = x(x-1)(x+1)(x-2)$



i) Domain: $\Re$; Range: $y \geq -1$
ii) y-intercept: 0
iii) Zeros: $-1, 0, 1, 2$
iv) $f(x) > 0$: $(-\infty, -1)$, $(0, 1)$, and $(2, \infty)$;
$f(x) \leq 0$: $[-1, 0]$ and $[1, 2]$
v) Symmetry: none
vi) The left-most y-values are positive. The right-most y-values are positive.

c) $f(x) = (x - 2)(x + 2)(x + 4)$



i) Domain: $\Re$; Range: $\Re$

ii) y-intercept: -16

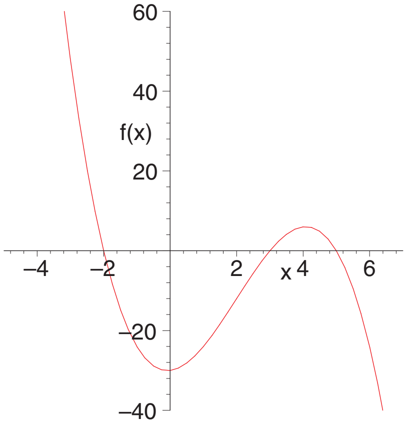
iii) Zeros: $-4, -2, 2$

iv) $f(x) > 0$: $(-4, -2)$ and $(2, \infty)$;
$f(x) \leq 0$: $(-\infty, -4]$ and $[-2, 2]$

v) Symmetry: none

vi) The left-most y-values are negative. The right-most y-values are positive.

d) $f(x) = -(x + 2)(x - 3)(x - 5)$



i) Domain: $\Re$; Range: $\Re$

ii) y-intercept: -30

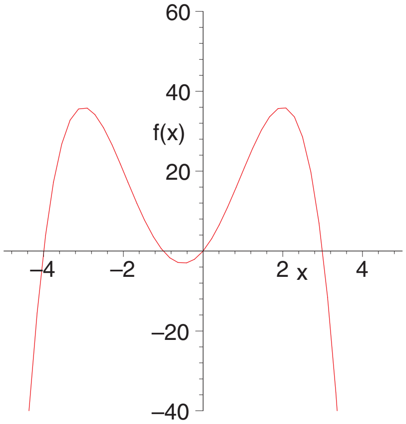
iii) Zeros: $-2, 3, 5$

iv) $f(x) > 0$: $(-\infty, -2)$ and $(3, 5)$;
$f(x) \leq 0$: $[-2, 3]$ and $[5, \infty)$

v) Symmetry: none

vi) The left-most y-values are positive. The right-most y-values are negative.

e) $f(x) = -x(x - 3)(x + 1)(x + 4)$



i) Domain: $\Re$; Range: $y \leq 36$

ii) y-intercept: 0

iii) Zeros: $-4, -1, 0, 3$

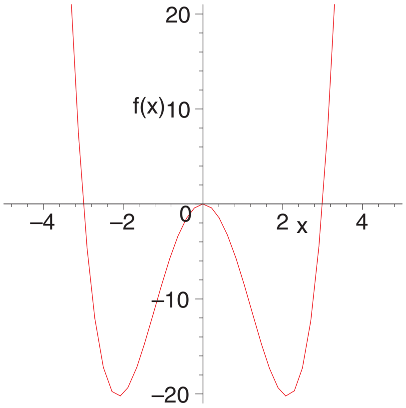
iv) $f(x) > 0$: $(-4, -1)$ and $(0, 3)$;
$f(x) \leq 0$: $(-\infty, -4]$, $[-1, 0]$, and $[3, \infty)$

v) Symmetry: none

vi) The left-most y-values are negative. The right-most y-values are negative.

Section Review Page 108 Question 18

a) $f(x) = x^2(x^2 - 9)$



i) Domain: $\Re$; Range: $y \geq -20.3$

ii) y-intercept: 0

iii) Zeros: $-3, 0$ (double), 3

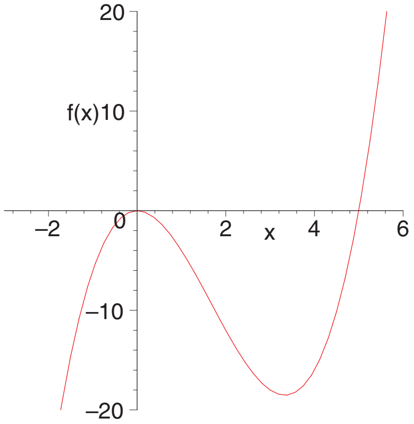
iv) Local maximum: $(0, 0)$;

Local minima: $(-2.1, -20.3)$ and $(2.1, -20.3)$

v) Symmetry: even

vi) The left-most y-values are positive. The right-most y-values are positive.

b) $f(x) = x^2(x - 5)$



i) Domain: $\Re$; Range: $\Re$

ii) y-intercept: 0

iii) Zeros: 0 (double) , 5

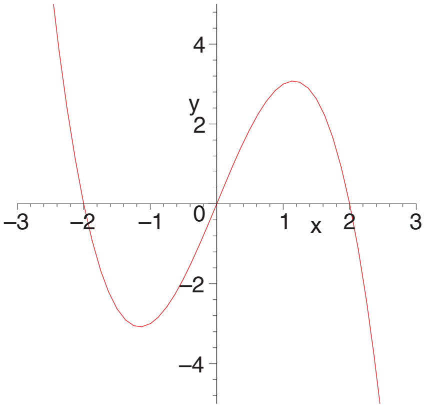
iv) Local maximum: $(0, 0)$;

Local minimum: $(3.3, -18.5)$

v) Symmetry: none

vi) The left-most y-values are negative. The right-most y-values are positive.

c) $y = -x(x^2 - 4)$



i) Domain: $\Re$; Range: $\Re$

ii) y-intercept: 0

iii) Zeros: $-2, 0, 2$

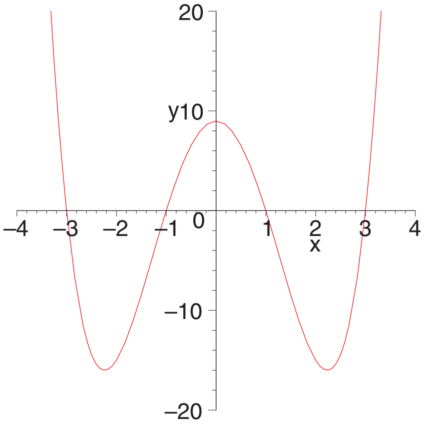
iv) Local maximum: $(1.2, 3.1)$;

Local minimum: $(-1.2, -3.1)$

v) Symmetry: odd

vi) The left-most y-values are positive. The right-most y-values are negative.

d) $y = (x^2 - 9)(x^2 - 1)$



i) Domain: $\Re$; Range: $y \geq -16$

ii) y-intercept: 9

iii) Zeros: $-3, -1, 1, 3$

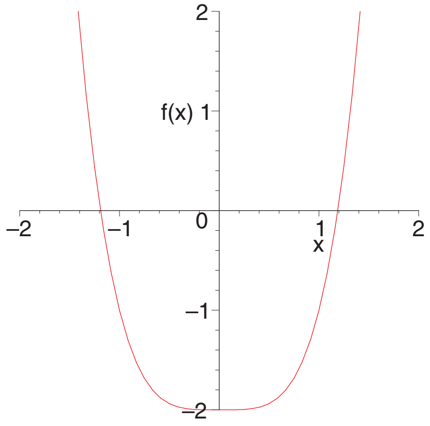
iv) Local maximum: $(0, 9)$;

Local minima: $(-2.2, -16)$, $(2.2, -16)$

v) Symmetry: even

vi) The left-most y-values are positive. The right-most y-values are positive.

e) $f(x) = x^4 - 2$



i) Domain: $\Re$; Range: $y \geq -2$

ii) y-intercept: -2

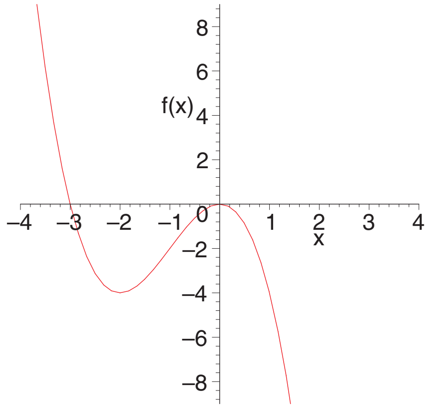
iii) Zeros: $-1.2, 1.2$

iv) Local maximum: none;
Local minimum: $(0, -2)$

v) Symmetry: even

vi) The left-most y-values are positive. The right-most y-values are positive.

f) $f(x) = -x^3 - 3x^2$



i) Domain: $\Re$; Range: $\Re$

ii) y-intercept: 0

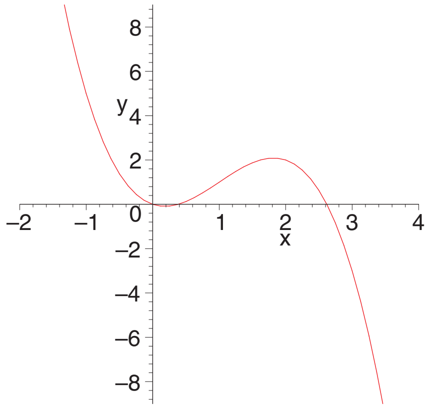
iii) Zeros: $-3, 0$ (double)

iv) Local maximum: $(0, 0)$;
Local minimum: $(-2, -4)$

v) Symmetry: none

vi) The left-most y-values are positive. The right-most y-values are negative.

g) $y = -x^3 + 3x^2 - x$



i) Domain: $\Re$; Range: $\Re$

ii) y-intercept: 0

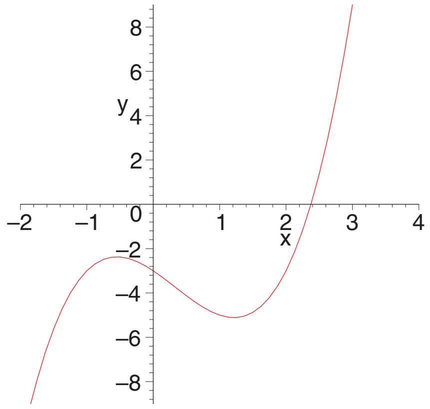
iii) Zeros: $0, 0.4, 2.6$

iv) Local maximum: $(1.8, 2.1)$;
Local minimum: $(0.2, -0.1)$

v) Symmetry: none

vi) The left-most y-values are positive. The right-most y-values are negative.

h) $y = x^3 - x^2 - 2x - 3$



i) Domain: $\Re$; Range: $\Re$

ii) y-intercept: -3

iii) Zeros: 2.4

iv) Local maximum: $(-0.5, -2.4)$;
Local minimum: $(1.2, -5.1)$

v) Symmetry: none

vi) The left-most y-values are negative. The right-most y-values are positive.

Section Review Page 109 Question 19

a) The roots of the equation $x(x + 3) = 0$ are $x = 0$ and $x = -3$. An interval chart yields the solution $x < -3$ or $x > 0$.

		−3		0	
Intervals	(- ,-3)		(-3,0)		(0,)
Test values	-4		-1		1
Sign	+	0	−	0	+

b) The roots of the equation $(x + 4)(x - 1) = 0$ are $x = -4$ and $x = 1$. An interval chart yields the solution $-4 \leq x \leq 1$.

		−4		1	
Intervals	(- ,-4)		(-4,1)		(1,)
Test values	-5		0		2
Sign	+	0	−	0	+

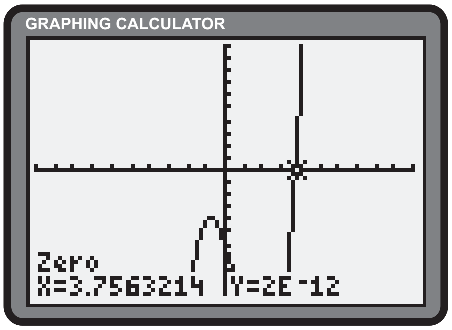
c) The roots of the equation $x^2 + 6x + 9 = 0$ are $x = -3$ (double). An interval chart yields the solution $x = -3$.

		−3	
Intervals	(- ,-3)		(-3,)
Test values	-5		0
Sign	+	0	+

d) The roots of the equation $(x+2)(x-2)(x+4) = 0$ are $x = -4$, $x = -2$, and $x = 2$. An interval chart yields the solution $-4 \leq x \leq -2$ or $x > 2$.

		−4		−2		2	
Intervals	(- ,-4)		(-4,-2)		(-2,2)		(2,)
Test values	-5		-3		0		3
Sign	−	0	+	0	−	0	+

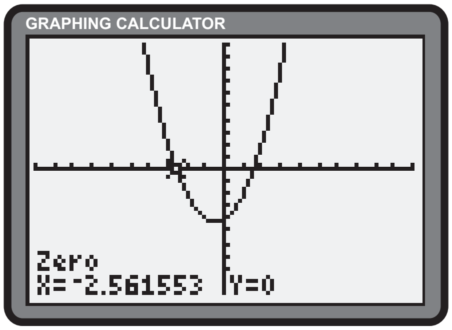
e) The root of the equation $x^3 - 2x^2 - 5x - 6 = 0$ is approximately 3.76. The solution to $x^3 - 2x^2 - 5x > 6$ is $x > 3.76$.



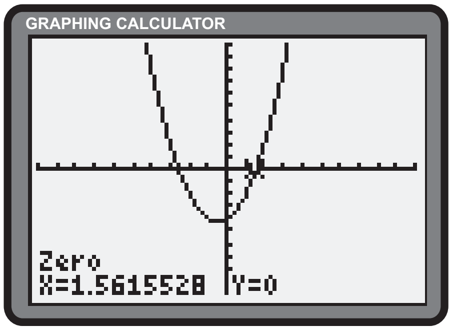
Zero
X=3.7563214 Y=2E-12

Section Review Page 109 Question 20

a) The solution to $x^2 + x - 4 > 0$ is $x < -2.56$ or $x > 1.56$.

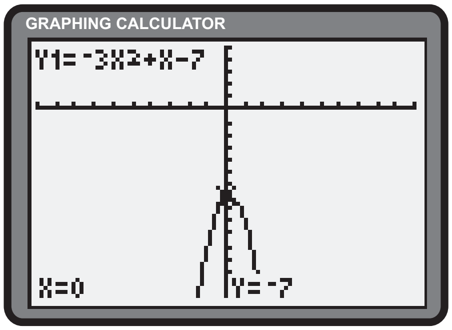


Zero
X=-2.561553 Y=0



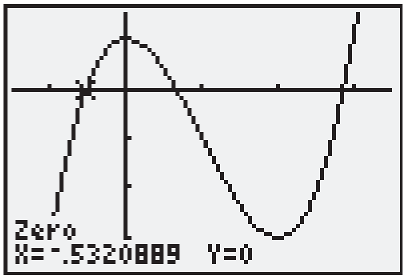
Zero
X=1.5615528 Y=0

b) The inequality can be written as $-3x^2 + x - 7 \leq 0$. The solution is any real number.

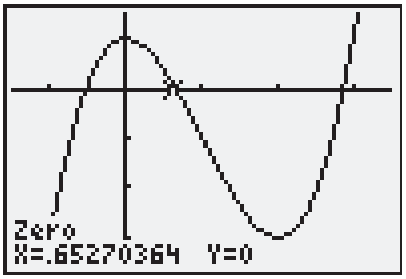


```
Y1=⁻3X²+X-7
X=0          Y=⁻7
```

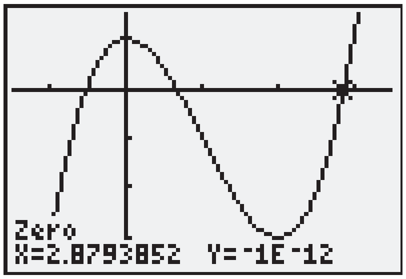
c) The inequality can be written as $x^3 - 3x^2 + 1 \geq 0$. The solution is $-0.53 \leq x \leq 0.65$ or $x \geq 2.88$.



```
Zero
X=⁻.5320889  Y=0
```

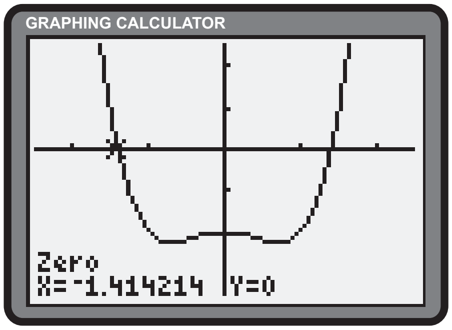


```
Zero
X=.65270364  Y=0
```

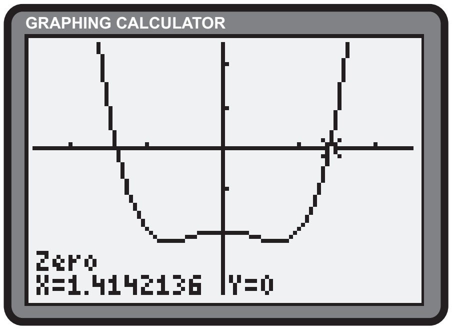


```
Zero
X=2.8793852  Y=⁻1E⁻12
```

d) The inequality can be written as $x^4 - x^2 - 2 < 0$. The solution is $-1.41 < x < 1.41$.



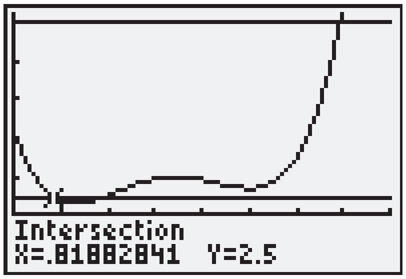
```
Zero
X=⁻1.414214  Y=0
```



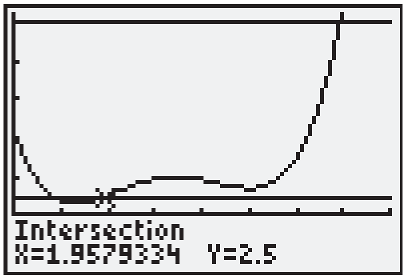
```
Zero
X=1.4142136  Y=0
```

Section Review Page 109 Question 21

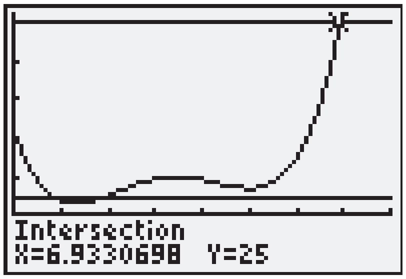
The **Intersect operation** of the graphing calculator reveals that the blood sugar levels lie outside the normal range over the intervals $t \in (0.82, 1.96)$ and $t \in (6.93, 8]$.



```
Intersection
X=.81882841  Y=2.5
```



```
Intersection
X=1.9579334  Y=2.5
```

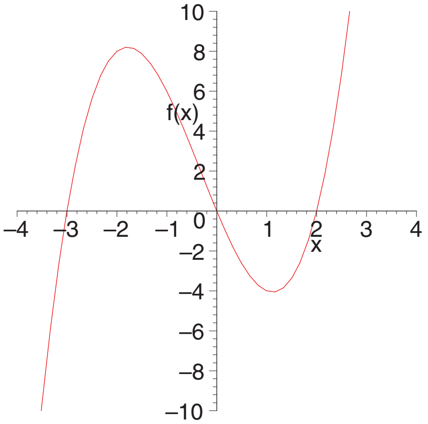


```
Intersection
X=6.9330698  Y=25
```

Chapter Test

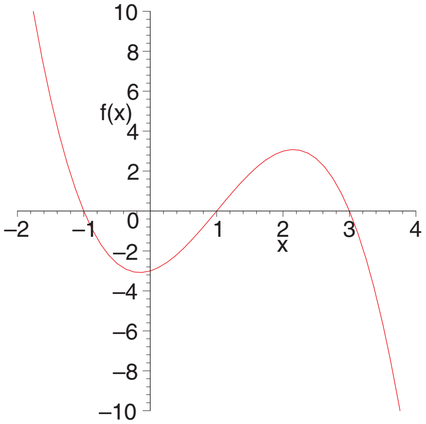
Section Chapter Test Page 111 Question 1

a) $f(x) = x(x-2)(x+3)$



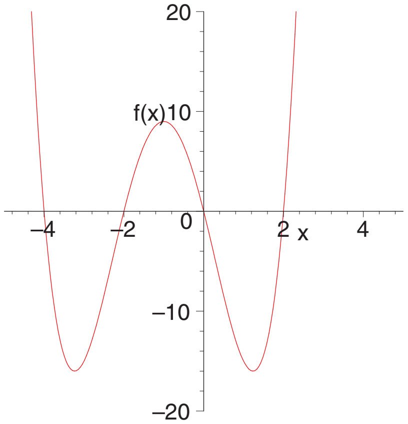
- **i)** Domain: $\Re$; Range: $\Re$
- **ii)** 3 distinct zeros: $-3, 0, 2$
- **iii)** At all the x-intercepts, the curve crosses the x-axis.
- **iv)** Turning points: 1 maximum, 1 minimum
- **v)** The left-most y-values are negative. The right-most y-values are positive.
- **vi)** Symmetry: none

b) $f(x) = -(x+1)(x-1)(x-3)$



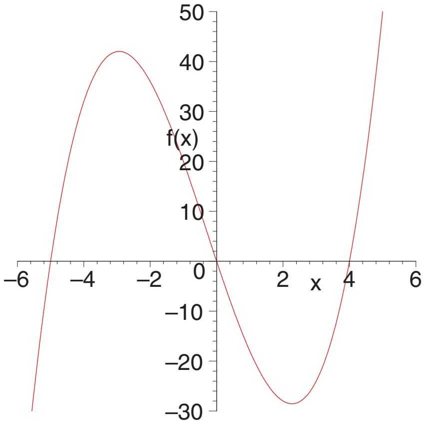
- **i)** Domain: $\Re$; Range: $\Re$
- **ii)** 3 distinct zeros: $-1, 1, 3$
- **iii)** At all the x-intercepts, the curve crosses the x-axis.
- **iv)** Turning points: 1 minimum, 1 maximum
- **v)** The left-most y-values are positive. The right-most y-values are negative.
- **vi)** Symmetry: none

c) $f(x) = x(x+2)(x-2)(x+4)$



- **i)** Domain: $\Re$; Range: $y \geq -16$
- **ii)** 4 distinct zeros: $-4, -2, 0, 2$
- **iii)** At all the x-intercepts, the curve crosses the x-axis.
- **iv)** Turning points: minimum, maximum, minimum
- **v)** The left-most y-values are positive. The right-most y-values are positive.
- **vi)** Symmetry: none

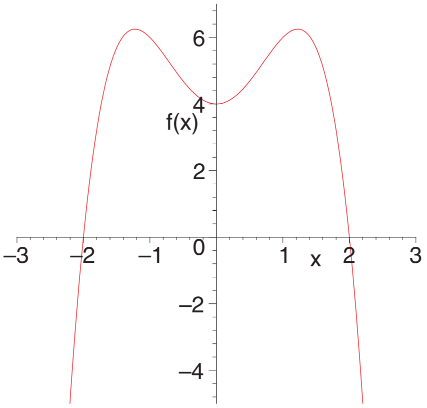
d) $f(x) = x^3 + x^2 - 20x = x(x+5)(x-4)$



- **i)** Domain: $\Re$; Range: $\Re$
- **ii)** 3 distinct zeros: $-5, 0, 4$
- **iii)** At all the x-intercepts, the curve crosses the x-axis.
- **iv)** Turning points: 1 maximum, 1 minimum
- **vi)** The left-most y-values are negative. The right-most y-values are positive.
- **v)** Symmetry: none

e)

$$\begin{aligned} f(x) &= -x^4+3x^2+4\\ &= -(x^4-3x^2-4)\\ &= -(x^2+1)(x^2-4)\\ &= -(x^2+1)(x-2)(x+2) \end{aligned}$$



i) Domain: $\Re$; Range: $y \le \frac{25}{4}$
ii) 2 distinct zeros: $-2, 2$
iii) At all the x-intercepts, the curve crosses the x-axis.
iv) Turning points: maximum, minimum, maximum
vi) The left-most y-values are negative. The right-most y-values are negative.
v) Symmetry: even

Section Chapter Test Page 111 Question 2

a)

$$\begin{array}{r}
x^2+3x+1\\
x-2\overline{)x^3+\ x^2-5x+2}\\
\underline{x^3-2x^2}\\
3x^2-5x\\
\underline{3x^2-6x}\\
x+2\\
\underline{x-2}\\
4
\end{array}$$

Restriction: $x \neq 2$

b)

$$\begin{array}{r}
y-1\\
2y^2+0y+1\overline{)2y^3-2y^2+\ y+0}\\
\underline{2y^3+0y^2+\ y}\\
-2y^2+0y+0\\
\underline{-2y^2+0y-1}\\
1
\end{array}$$

Restriction: none

c)

$$\begin{array}{r}
3x^2-\ 5x+\ 2\\
2x+3\overline{)6x^3-\ x^2-11x+10}\\
\underline{6x^3+9x^2}\\
-10x^2-11x\\
\underline{-10x^2-15x}\\
4x+10\\
\underline{4x+6}\\
4
\end{array}$$

Restriction: $x \neq -\frac{3}{2}$

Section Chapter Test Page 111 Question 3

$$\begin{array}{r}
2x-\ 3\\
3x^2-4x-5\overline{)6x^3-17x^2+\ 2x+15}\\
\underline{6x^3-\ 8x^2-10x}\\
-9x^2+12x+15\\
\underline{-9x^2+12x+15}\\
0
\end{array}$$

The height of the box is $2x-3$.

Section Chapter Test Page 111 Question 4

a)

$$\begin{aligned} Q(p) &= p^3+4p^2-2p+5\\ Q(-5) &= (-5)^3+4(-5)^2-2(-5)+5\\ &= -125+100+10+5\\ &= -10 \end{aligned}$$

When p^3+4p^2-2p+5 is divided by $p+5$, the remainder is -10.

b)

$$\begin{aligned} P(y) &= 4y^3+y^2-12y-5\\ P\left(-\frac{1}{4}\right) &= 4\left(-\frac{1}{4}\right)^3+\left(-\frac{1}{4}\right)^2-12\left(-\frac{1}{4}\right)-5\\ &= -\frac{1}{16}+\frac{1}{16}+3-5\\ &= -2 \end{aligned}$$

When $4y^3+y^2-12y-5$ is divided by $4y+1$, the remainder is -2.

Section Chapter Test Page 111 Question 5

$$\begin{aligned}
P(x) &= 3x^3 + mx^2 + nx + 2 \\
P(2) &= -8 \\
3(2)^3 + m(2)^2 + n(2) + 2 &= -8 \\
24 + 4m + 2n + 2 &= -8 \\
4m + 2n &= -34 \\
2m + n &= -17 \qquad (1) \\
P(-3) &= -88 \\
3(-3)^3 + m(-3)^2 + n(-3) + 2 &= -88 \\
-81 + 9m - 3n + 2 &= -88 \\
9m - 3n &= -9 \\
3m - n &= -3 \qquad (2)
\end{aligned}$$

Add (1) and (2).

$$\begin{aligned}
5m &= -20 \\
m &= -4 \qquad (3)
\end{aligned}$$

Substitute (3) into (1).

$$\begin{aligned}
2(-4) + n &= -17 \\
-8 + n &= -17 \\
n &= -9
\end{aligned}$$

The values are $m = -4$ and $n = -9$.

Section Chapter Test Page 111 Question 6

a)

$$\begin{aligned}
P(x) &= x^3 - 5x^2 - x + 5 \\
P(5) &= 5^3 - 5(5)^2 - 5 + 5 \\
&= 125 - 125 \\
&= 0
\end{aligned}$$

Since $P(5) = 0$, $x - 5$ is a factor of $P(x)$.

b)

$$\begin{aligned}
P(n) &= 3n^3 + n^2 - 38n + 24 \\
P\left(\frac{2}{3}\right) &= 3\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 - 38\left(\frac{2}{3}\right) + 24 \\
&= \frac{8}{9} + \frac{4}{9} - \frac{76}{3} + 24 \\
&= \frac{12}{9} - \frac{76}{3} + 24 \\
&= \frac{4}{3} - \frac{76}{3} + 24 \\
&= -24 + 24 \\
&= 0
\end{aligned}$$

Since $P\left(\frac{2}{3}\right) = 0$, $3n - 2$ is a factor of $P(n)$.

Section Chapter Test Page 111 Question 7

a)

$$\begin{aligned}
P(x) &= x^3 + 2x^2 - 21x + 18 \\
P(1) &= 1^3 + 2(1)^2 - 21(1) + 18 \\
&= 1 + 2 - 21 + 18 \\
&= 0
\end{aligned}$$

$x - 1$ is a factor of $P(x)$. Synthetic division reveals another factor of $x^2 + 3x - 18$.

$$\begin{aligned}
P(x) &= (x - 1)(x^2 + 3x - 18) \\
&= (x - 1)(x + 6)(x - 3)
\end{aligned}$$

$x^3 + 2x^2 - 21x + 18 = (x - 1)(x + 6)(x - 3)$

b)

$$\begin{aligned}
P(x) &= 3x^3 - 10x^2 - 9x + 4 \\
P(-1) &= 3(-1)^3 - 10(-1)^2 - 9(-1) + 4 \\
&= -3 - 10 + 9 + 4 \\
&= 0
\end{aligned}$$

$x + 1$ is a factor of $P(x)$. Synthetic division reveals another factor of $3x^2 - 13x + 4$.

$$\begin{aligned}
P(x) &= (x + 1)(3x^2 - 13x + 4) \\
&= (x + 1)(3x - 1)(x - 4)
\end{aligned}$$

$3x^3 - 10x^2 - 9x + 4 = (x + 1)(3x - 1)(x - 4)$

Section Chapter Test Page 111 Question 8

The possible factors are $x \pm 1, x \pm 5, 2x \pm 1, 2x \pm 5, 3x \pm 1, 3x \pm 5, 6x \pm 1, 6x \pm 5$

Section Chapter Test Page 111 Question 9

a) Factor by grouping.

$$\begin{aligned} y^3 - 3y^2 &= 4y - 12 \\ y^3 - 3y^2 - 4y + 12 &= 0 \\ y^2(y-3) - 4(y-3) &= 0 \\ (y-3)(y^2-4) &= 0 \\ y - 3 &= 0 \\ y &= 3 && (1) \\ y^2 - 4 &= 0 \\ y^2 &= 4 \\ y &= \pm 2 && (2) \end{aligned}$$

The roots are 3, 2, and −2.

b) $m^3 - 5m = 5m^2 - 1$

$$\begin{aligned} m^3 - 5m^2 - 5m + 1 &= 0 \\ P(m) &= m^3 - 5m^2 - 5m + 1 \\ P(-1) &= (-1)^3 - 5(-1)^2 - 5(-1) + 1 \\ &= -1 - 5 + 5 + 1 \\ &= 0 \end{aligned}$$

$m+1$ is a factor of $P(m)$. Synthetic division reveals another factor of $m^2 - 6m + 1$. Use the quadratic formula to determine the remaining roots.

$$\begin{aligned} m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{6 \pm \sqrt{32}}{2} \\ &= \frac{6 \pm 4\sqrt{2}}{2} \\ &= 3 \pm 2\sqrt{2} \end{aligned}$$

The roots are −1 and $3 \pm 2\sqrt{2}$.

c) $x^3 + 4x^2 + 9x + 10 = 0$

$$\begin{aligned} P(x) &= x^3 + 4x^2 + 9x + 10 \\ P(-2) &= (-2)^3 + 4(-2)^2 + 9(-2) + 10 \\ &= -8 + 16 - 18 + 10 \\ &= 0 \end{aligned}$$

$x + 2$ is a factor of $P(x)$. Synthetic division reveals another factor of x^2+2x+5. Use the quadratic formula to determine the remaining roots.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

The roots are −2 and $-1 \pm 2i$.

d) $3y^3 - 28y^2 = 8 - 33y$

$$\begin{aligned} P(y) &= 3y^3 - 28y^2 + 33y - 8 \\ P(1) &= 3(1)^3 - 28(1)^2 + 33(1) - 8 \\ &= 3 - 28 + 33 - 8 \\ &= 0 \end{aligned}$$

$y - 1$ is a factor of $P(y)$. Synthetic division reveals another factor of $3y^2 - 25y + 8$.

$$\begin{aligned} P(y) &= 0 \\ (y-1)(3y^2 - 25y + 8) &= 0 \\ (y-1)(3y-1)(y-8) &= 0 \\ y &= 1, \frac{1}{3} \text{ or } 8 \end{aligned}$$

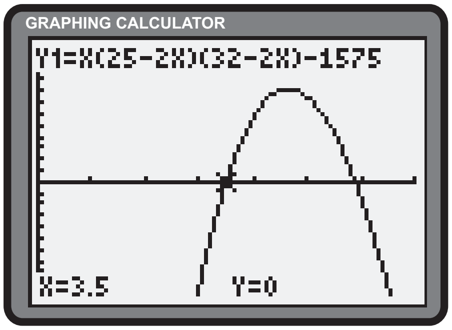
The roots are 1 and $\frac{1}{3}$, and 8.

Section Chapter Test Page 111 Question 10

Let h be the height of the box, in centimetres. The width and length are $25 - 2h$ cubic centimetres and $32 - 2h$ cubic centimetres respectively. Let V be the volume of the box, in cubic centimetres.

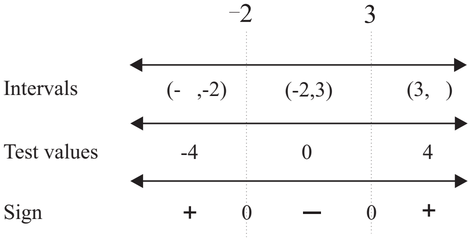
$$\begin{aligned} V &= 1575 \\ h(25-2h)(32-2h) &= 1575 \\ h(25-2h)(32-2h) - 1575 &= 0 \qquad (1) \end{aligned}$$

The graphing calculator reveals the root of (1). The side lengths of the squares should be 3.5 cm, since $h \le 5$.



Section Chapter Test Page 111 Question 11

a) The roots of the equation, $x^2-x-6 = 0$ are $x = -2$ and $x = 3$. An interval chart yields the solution $x \le -2$ or $x \ge 3$.

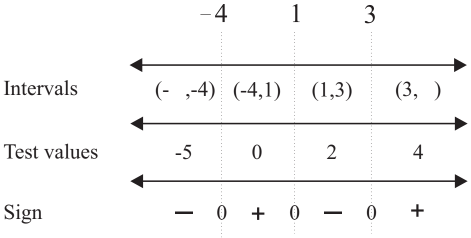


Intervals	(- ,-2)	(-2,3)	(3,)
Test values	-4	0	4
Sign	+ 0	– 0	+

b)

$$\begin{aligned} -x^3 + 13x &< 12 \\ -x^3 + 13x - 12 &< 0 \\ x^3 - 13x + 12 &> 0 \\ (x-1)(x^2+x-12) &> 0 \\ (x-1)(x+4)(x-3) &> 0 \end{aligned}$$

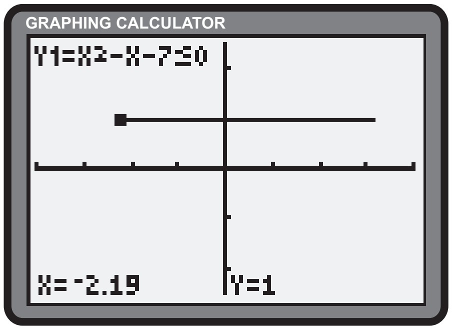
The roots of the equation, $(x - 1)(x + 4)(x - 3) = 0$ are $x = 1$, $x = -4$, and $x = 3$. An interval chart yields the solution, $-4 \le x \le 1$ or $x > 3$.

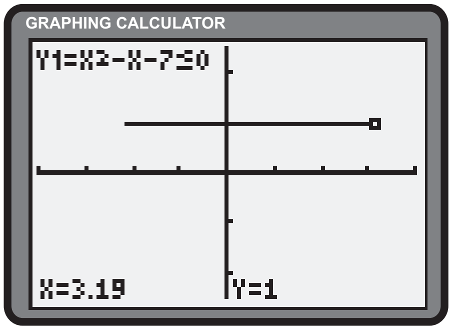


Intervals	(- ,-4)	(-4,1)	(1,3)	(3,)
Test values	-5	0	2	4
Sign	– 0	+ 0	– 0	+

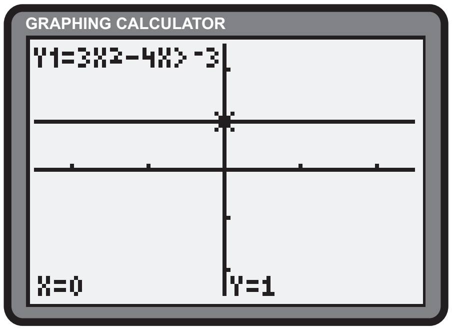
Section Chapter Test Page 111 Question 12

a) The graphing calculator reveals the solution $-2.19 \le x \le 3.19$.





b) The graphing calculator suggests all real numbers as the solution.

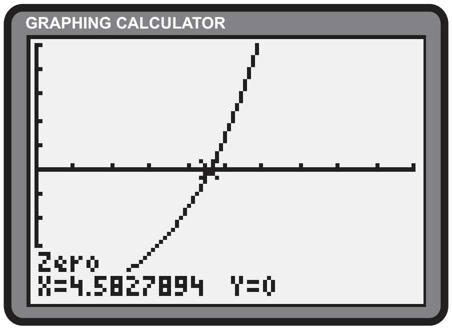


Section Chapter Test   Page 111   Question 16

a)

$$\begin{aligned} A(t) &= 500 \\ 4t^3 + 5t^2 + 10 &= 500 \\ 4t^3 + 5t^2 - 490 &= 0 \end{aligned}$$

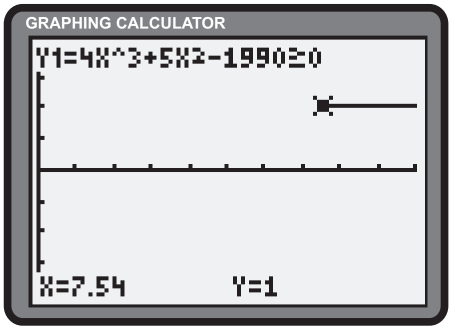
The graphing calculator suggests that 500 000 ha of farmland will be paved over after 4.6 years.



b)

$$\begin{aligned} A(t) &\geq 2000 \\ 4t^3 + 5t^2 + 10 &\geq 2000 \\ 4t^3 + 5t^2 - 1990 &\geq 0 \end{aligned}$$

The graphing calculator suggests that 2 000 000 ha of farmland will be paved over after 7.5 years.



Challenge Problems

Section Challenge Problems Page 113 Question 1

$$
\begin{aligned}
f(0) &= 5 \\
f(1) &= \frac{3}{2}f(0) - \frac{1}{2} \\
&= \frac{3}{2}(5) - \frac{1}{2} && (1) \\
f(2) &= \frac{3}{2}f(1) - \frac{1}{2} \\
&= \frac{3}{2}\left(\frac{3}{2}(5) - \frac{1}{2}\right) - \frac{1}{2} \\
&= \left(\frac{3}{2}\right)^2 (5) - \frac{5}{4} && (2) \\
f(3) &= \frac{3}{2}f(2) - \frac{1}{2} \\
&= \frac{3}{2}\left(\left(\frac{3}{2}\right)^2 (5) - \frac{5}{4}\right) - \frac{1}{2} \\
&= \left(\frac{3}{2}\right)^3 (5) - \frac{19}{8} && (3)
\end{aligned}
$$

The sequence suggested by (1), (2), and (3) leads to the following generalization.

$$
\begin{aligned}
f(n) &= \left(\frac{3}{2}\right)^n (5) - \frac{3^n - 2^n}{2^n} \\
&= \left(\frac{3}{2}\right)^n (5) - \frac{3^n}{2^n} + 1 \\
&= \left(\frac{3}{2}\right)^n (5 - 1) + 1 \\
&= 4\left(\frac{3}{2}\right)^n + 1
\end{aligned}
$$

Section Challenge Problems Page 113 Question 2

$$
\begin{aligned}
f(x) &= 3x^2 - 2ax + b \\
f(a) &= b \\
3a^2 - 2a^2 + b &= b \\
a^2 &= 0 \\
a &= 0 && (1)
\end{aligned}
$$

$$
\begin{aligned}
f(b) &= a \\
3b^2 - 2ab + b &= a && (2)
\end{aligned}
$$

Substitute (1) into (2).

$$
\begin{aligned}
3b^2 + b &= 0 \\
b(3b + 1) &= 0 \\
b &= -\frac{1}{3},\ a \neq b \\
a + b &= 0 - \frac{1}{3} \\
&= -\frac{1}{3}
\end{aligned}
$$

Section Challenge Problems Page 113 Question 3

Let s and l be the lengths of the shorter and longer sides of the rectangle, respectively. From the given information, the length of the diagonal can be expressed as $s + l - \frac{l}{5}$ or $s + \frac{4l}{5}$. Use the Pythagorean theorem.

$$
\begin{aligned}
s^2 + l^2 &= \left(s + \frac{4l}{5}\right)^2 \\
s^2 + l^2 &= s^2 + \frac{8sl}{5} + \frac{16l^2}{25} \\
25l^2 &= 40sl + 16l^2 \\
9l^2 - 40sl &= 0 \\
l(9l - 40s) &= 0 \\
9l - 40s &= 0,\ l \neq 0 \\
40s &= 9l \\
\frac{s}{l} &= \frac{9}{40}
\end{aligned}
$$

The ratio of the shorter side to the longer side is 9 : 40.

Section Challenge Problems Page 113 Question 4

Let h be the height of the point.

$$
\begin{aligned}
h &= x^2 + y^2 + 4x - 6y \\
&= (x^2 + 4x + 4) - 4 + (y^2 - 6y + 9) - 9 \\
&= (x + 2)^2 + (y - 3)^2 - 13 && (1)
\end{aligned}
$$

The lowest possible value of (1) is -13 at $(-2, 3)$.

Section Challenge Problems   Page 113   Question 5

If c is the number of clicks counted in s seconds, the speed, v, of the train will be $\frac{10c}{s}$ metres per second. Convert v to kilometres per hour.

$$\begin{aligned} v &= \frac{10c}{s} \times \frac{\frac{1}{1000}}{\frac{1}{3600}} \\ &= \frac{10c}{s} \times \frac{3600}{1000} \\ &= \frac{10c}{s} \times \frac{36}{10} \qquad (1) \end{aligned}$$

From (1), it is evident that $v = c$ if the passenger counts clicks for 36 s.

Section Challenge Problems   Page 113   Question 6

a)

$$\begin{aligned} (4 + i)(-4 - i) &= -16 - 4i - 4i - i^2 \\ &= -16 - 8i + 1 \\ &= -15 - 8i \end{aligned}$$

The square roots of $-15 + 8i$ are $4 + i$ and $-4 - i$.

b) Let $z = x + yi$ be the square root of the complex number $a + bi$.

$$\begin{aligned} (x + yi)^2 &= a + bi \\ (x^2 - y^2) + (2xy)i &= a + bi \end{aligned}$$

Comparison yields a system of equations.

$$\begin{aligned} x^2 - y^2 &= a \\ 2xy &= b \end{aligned}$$

Given a and b, the solution to this system can be determined.

Determine the square roots of $5 + 3i$.

$$\begin{aligned} x^2 - y^2 &= 5 \qquad (1) \\ 2xy &= 3 \\ y &= \frac{3}{2x} \qquad (2) \end{aligned}$$

Substitute (2) in (1).

$$\begin{aligned} x^2 - \left(\frac{3}{2x}\right)^2 &= 5 \\ x^2 - \frac{9}{4x^2} &= 5 \\ 4x^4 - 9 &= 20x^2 \\ 4x^4 - 20x^2 - 9 &= 0 \\ x^2 &= \frac{20 \pm \sqrt{400 + 144}}{8} \\ &= \frac{20 \pm \sqrt{544}}{8} \\ &= \frac{5 + \sqrt{34}}{2},\ x^2 \geq 0 \\ x &\doteq \pm 2.327 \qquad (3) \end{aligned}$$

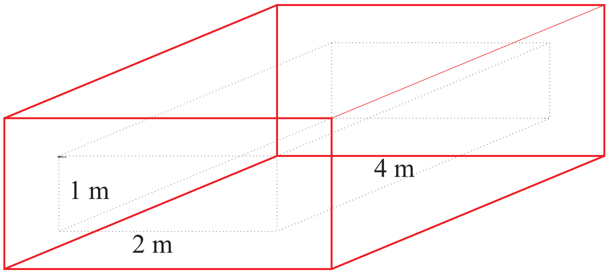
Substitute (3) in (2).

$$\begin{aligned} y &= \frac{3}{2\,(\pm 2.327)} \\ &= \pm 0.645 \end{aligned}$$

The square roots of $5 + 3i$ are $\pm(2.327 + 0.645i)$.

Section Challenge Problems Page 113 Question 7

Let x be the uniform thickness of the shielding, in metres. Let V be the volume of the shielding.

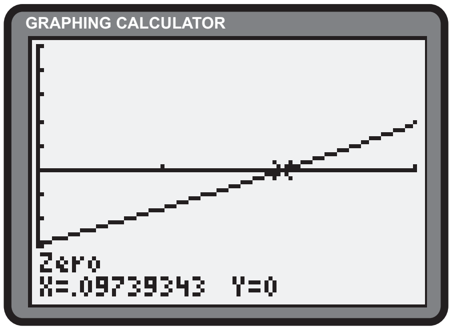


$$\begin{aligned} V(x) &= (2x+1)(2x+2)(2x+4) - (1)(2)(4) \\ &= 4(2x+1)(x+1)(x+2) - 8 \\ &= 4(2x+1)(x^2+3x+2) - 8 \\ &= 4(2x^3+7x^2+7x+2) - 8 \\ &= 8x^3+28x^2+28x \end{aligned}$$

The volume of the shielding is $8x^3+28x^2+28x$ m^3. Determine x to the nearest centimetre.

$$\begin{aligned} V(x) &= 3 \\ 8x^3+28x^2+28x &= 3 \\ 8x^3+28x^2+28x-3 &= 0 \end{aligned}$$

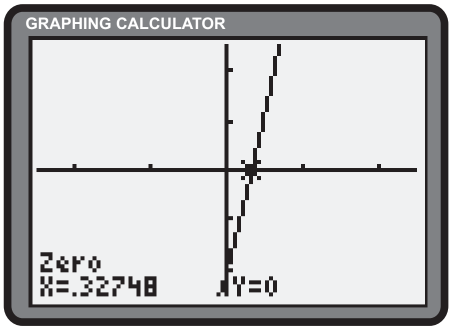
The graphing calculator suggests the thickness of the shielding is approximately 10 cm.



```
Zero
X=.09739343  Y=0
```

Section Challenge Problems Page 113 Question 8

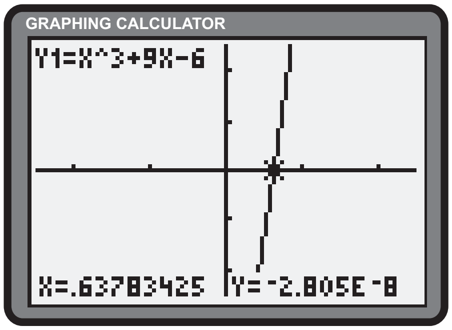
$$\begin{aligned} P(x) &= x^3+6x-2 \\ P\left(\sqrt[3]{4}-\sqrt[3]{2}\right) &= \left(\sqrt[3]{4}-\sqrt[3]{2}\right)^3 + 6\left(\sqrt[3]{4}-\sqrt[3]{2}\right) - 2 \\ &= 4 - 3\sqrt[3]{32} + 3\sqrt[3]{16} - 2 + 6\sqrt[3]{4} - 6\sqrt[3]{2} - 2 \\ &= 6\sqrt[3]{4} + 6\sqrt[3]{2} + 6\sqrt[3]{4} - 6\sqrt[3]{2} \\ &= 0 \end{aligned}$$



```
Zero
X=.32748      Y=0
```

Cardan suggests comparing the cubic form $x^3 + mx = n$ with $x^3 + 3stx = s^3 - t^3$, solving for s and t, and expressing one root as $s - t$. Rewrite $x^3 + 9x - 6 = 0$ as $x^3 + 9x = 6$. Find s and t such that

$$\begin{aligned} 3st &= 9 \\ s^3 - t^3 &= 6 \\ s &= \frac{3}{t} \\ \frac{27}{t^3} - t^3 &= 6 \\ t^6 + 6t^3 - 27 &= 0 \\ (t^3+9)(t^3-3) &= 0 \end{aligned}$$



```
Y1=X^3+9X-6
X=.63783425  Y=-2.805E-8
```

$$\begin{aligned} t^3 + 9 &= 0 \\ t &= -\sqrt[3]{9} \\ s &= -\frac{3}{9^{\frac{1}{3}}} \\ &= -\frac{3}{3^{\frac{2}{3}}} \\ &= -\sqrt[3]{3} \end{aligned}$$

$$\begin{aligned} t^3 - 3 &= 0 \\ t &= \sqrt[3]{3} \\ s &= \frac{3}{3^{\frac{1}{3}}} \\ &= 3^{\frac{2}{3}} \\ &= \sqrt[3]{9} \end{aligned}$$

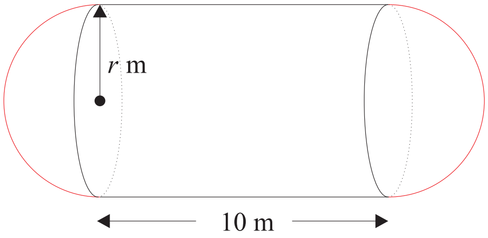
One root is given by $s - t = -\sqrt[3]{3} - (-\sqrt[3]{9})$ or $\sqrt[3]{9} - \sqrt[3]{3}$. It can be shown that for $st > 0$, only one real root exists.

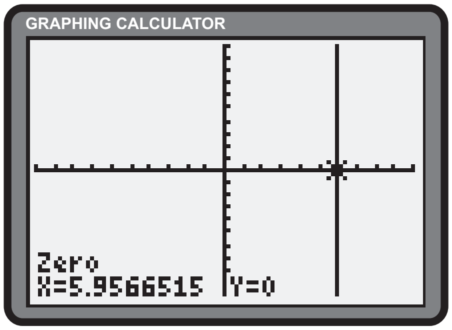
Section Challenge Problems Page 113 Question 9

Let r be the radius of the hemispheres and the cylinder, in metres. Let V be the volume of the tank.

$$\begin{aligned} V(r) &= 2000 \\ \frac{4}{3}\pi r^3 + \pi r^2(10) &= 2000 \\ 4\pi r^3 + 30\pi r^2 &= 6000 \\ \pi\left(4r^3 + 30r^2\right) - 6000 &= 0 \end{aligned}$$

The graphing calculator offers 5.96 m as the solution to the radius of the tank.

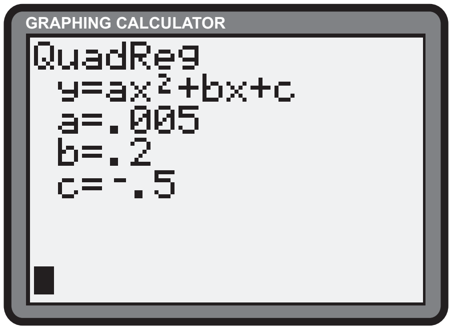




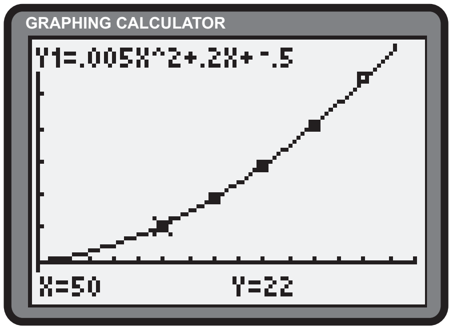
```
Zero
X=5.9566515  Y=0
```

Section Challenge Problems Page 113 Question 10

The quadratic regression feature of the graphing calculator models the data as $y = \frac{1}{200}x^2 + \frac{1}{5}x - \frac{1}{2}$.



```
QuadReg
 y=ax²+bx+c
 a=.005
 b=.2
 c=⁻.5
```



```
Y1=.005X^2+.2X+⁻.5
X=50          Y=22
```

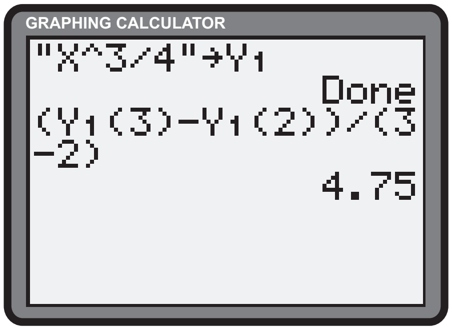
CHAPTER 3 Limits

3.1 From Secants to Tangents

Practise

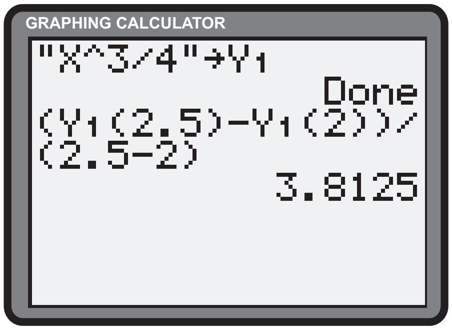
Section 3.1 Page 127 Question 1

a) **i)** $m_{PQ} = 4.75$



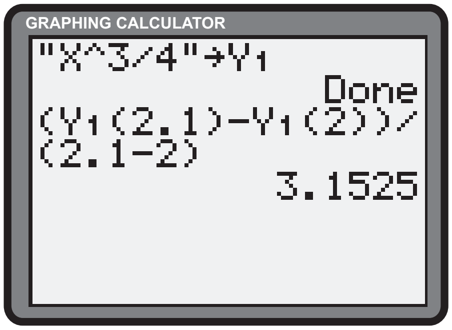
```
"X^3/4"→Y₁
              Done
(Y₁(3)-Y₁(2))/(3
-2)
              4.75
```

ii) $m_{PQ} = 3.8125$



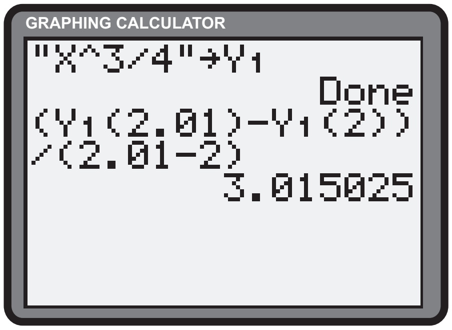
```
"X^3/4"→Y₁
              Done
(Y₁(2.5)-Y₁(2))/
(2.5-2)
            3.8125
```

iii) $m_{PQ} = 3.1525$



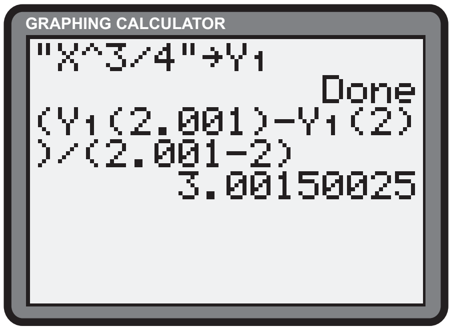
```
"X^3/4"→Y₁
              Done
(Y₁(2.1)-Y₁(2))/
(2.1-2)
            3.1525
```

iv) $m_{PQ} = 3.015\,025$



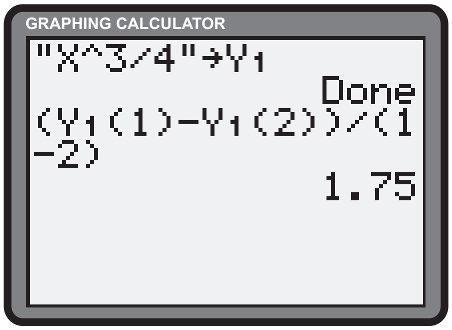
```
"X^3/4"→Y₁
              Done
(Y₁(2.01)-Y₁(2))
/(2.01-2)
          3.015025
```

v) $m_{PQ} = 3.001\,500\,25$



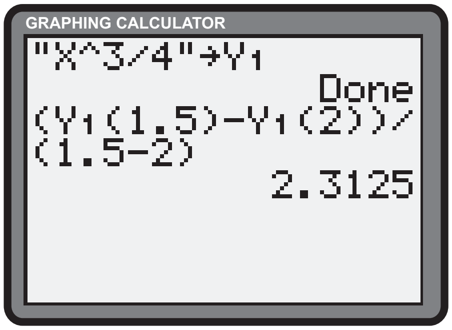
```
"X^3/4"→Y₁
              Done
(Y₁(2.001)-Y₁(2)
)/(2.001-2)
        3.00150025
```

vi) $m_{PQ} = 1.75$



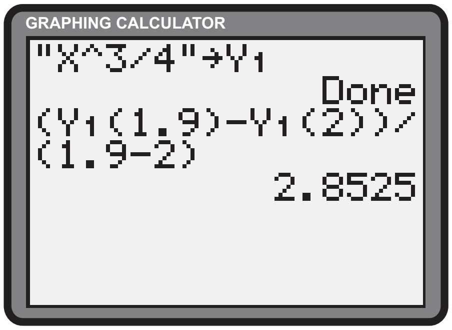
```
"X^3/4"→Y₁
              Done
(Y₁(1)-Y₁(2))/(1
-2)
              1.75
```

vii) $m_{PQ} = 2.3125$



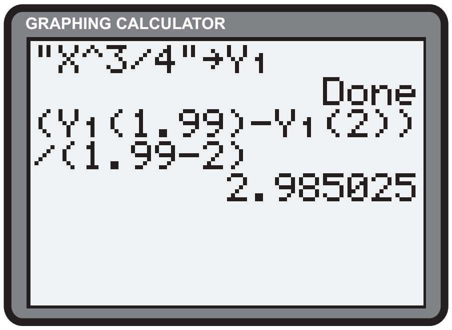
```
"X^3/4"→Y₁
              Done
(Y₁(1.5)-Y₁(2))/
(1.5-2)
            2.3125
```

viii) $m_{PQ} = 2.8525$



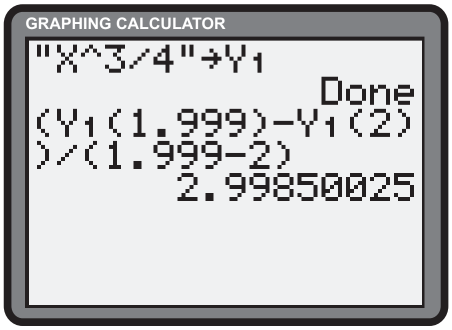
```
"X^3/4"→Y₁
              Done
(Y₁(1.9)-Y₁(2))/
(1.9-2)
            2.8525
```

ix) $m_{PQ} = 2.985\ 025$



```
"X^3/4"→Y1
                Done
(Y1(1.99)-Y1(2))
/(1.99-2)
            2.985025
```

x) $m_{PQ} = 2.998\ 500\ 25$



```
"X^3/4"→Y1
                Done
(Y1(1.999)-Y1(2)
)/(1.999-2)
          2.99850025
```

b) The first five x-values approach 2 from above. The last five x-values approach 2 from below.

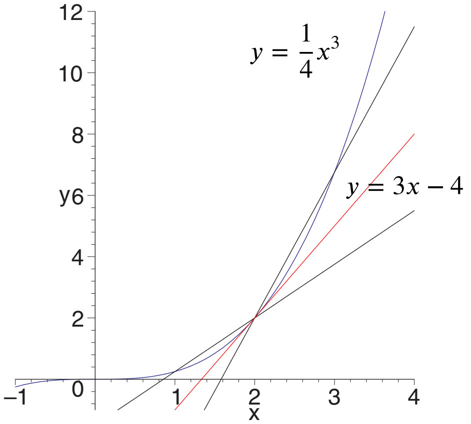
c) The slope of the tangent to the curve at P(2, 2) appears to be 3.

d)

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 3(x - 2) \\ y &= 3x - 4 \end{aligned}$$

The equation of the tangent is $y = 3x - 4$.

e)

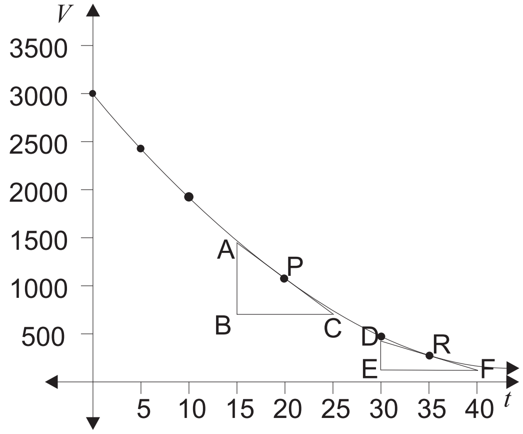


Apply, Solve, Communicate

Section 3.1 Page 128 Question 3

a) The slope of the graph of V versus t represents the rate, in L/min, at which the water is draining from the tank.

b)



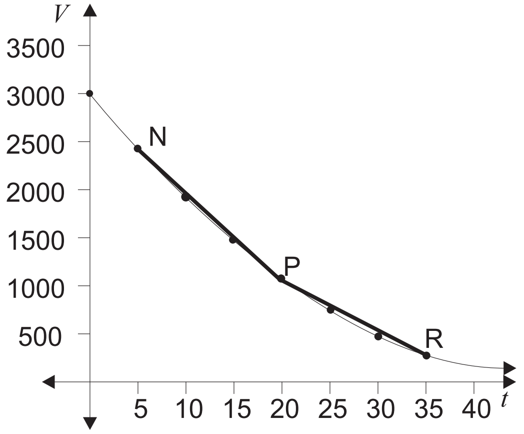
Estimates may vary. Draw an approximation to the tangent at P(20, 1080) and calculate its slope.

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{1450 - 725}{15 - 25} \\ &= -72.5 \end{aligned}$$

Draw an approximation to the tangent at R(35, 270) and calculate its slope.

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{450 - 120}{30 - 40} \\ &= -33 \end{aligned}$$

c)



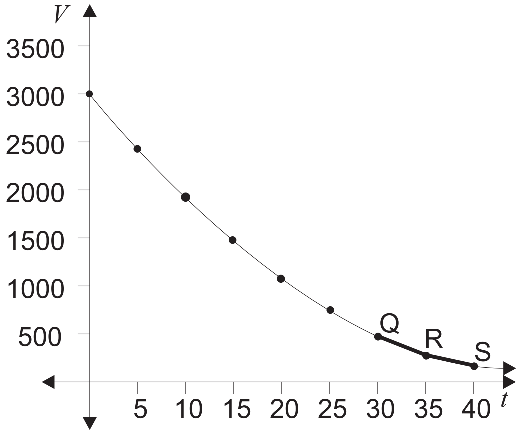
Determine the slopes of secants NP and PR.

$$m_{\mathrm{NP}} = \frac{2430 - 1080}{5 - 20}$$
$$= -90$$
$$m_{\mathrm{PR}} = \frac{1080 - 270}{20 - 35}$$
$$= -54$$

The slope of the tangent at P is approximated by averaging the slopes of the secants NP and PR.

$$m_{\mathrm{P}} = \frac{-90 + (-54)}{2}$$
$$= -72$$

d)



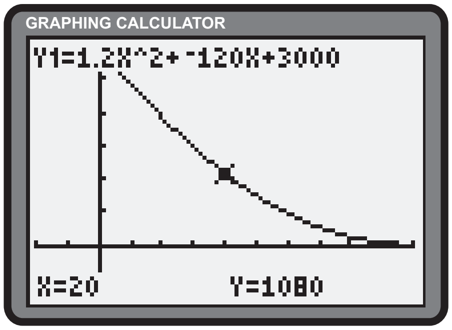
Determine the slopes of secants QR and RS.

$$m_{\mathrm{QR}} = \frac{480 - 270}{30 - 35}$$
$$= -42$$
$$m_{\mathrm{RS}} = \frac{270 - 120}{35 - 40}$$
$$= -30$$

The slope of the tangent at R is approximated by averaging the slopes of the secants QR and RS.

$$m_{\mathrm{R}} = \frac{-42 + (-30)}{2}$$
$$= -36$$

e)

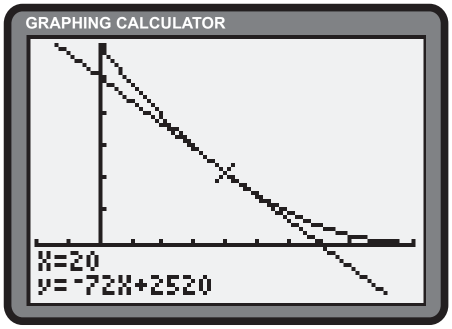


```
Y1=1.2X^2+⁻120X+3000
X=20          Y=1080
```

Quadratic regression yields the following equation.

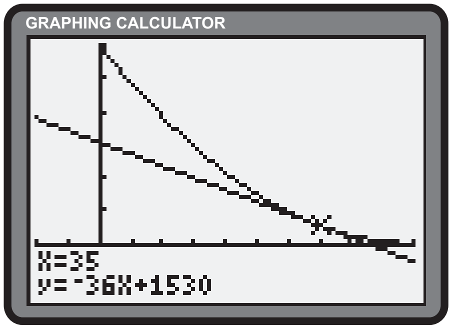
$$V(t) = 1.2t^2 - 120t + 3000$$

The **Tangent operation** yields a slope of −72 at P as compared to values of −72.5 and −72 from b) and c).



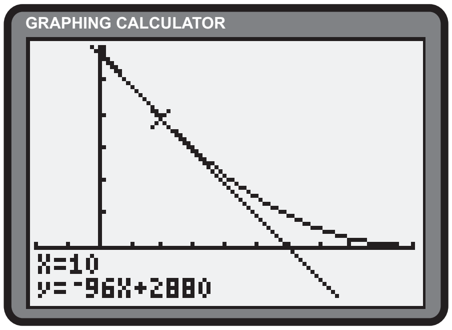
```
X=20
y=⁻72X+2520
```

The **Tangent operation** yields a slope of −36 at R as compared to values of −33 and −36 from b) and d).



```
X=35
y=⁻36X+1530
```

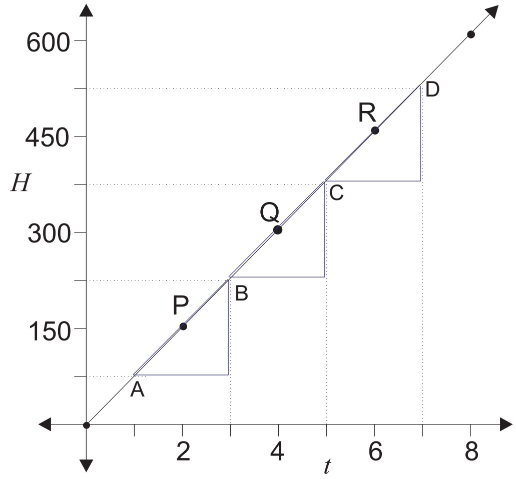
f) The **Tangent operation** gives a slope of −96 at $t = 10$. Thus, water is flowing from the tank at a rate of 96 L/min.



```
X=10
y=⁻96X+2880
```

Section 3.1 Page 128 Question 4

a)



i) At $t = 2$,
$$\begin{aligned} m_{\text{P}} &= \frac{H_{\text{B}} - H_{\text{A}}}{t_{\text{B}} - t_{\text{A}}} \\ &= \frac{225 - 75}{3 - 1} \\ &= 75 \end{aligned}$$

ii) At $t = 4$,
$$\begin{aligned} m_{\text{Q}} &= \frac{H_{\text{C}} - H_{\text{B}}}{t_{\text{C}} - t_{\text{B}}} \\ &= \frac{375 - 225}{5 - 3} \\ &= 75 \end{aligned}$$

iii) At $t = 6$,
$$\begin{aligned} m_{\text{R}} &= \frac{H_{\text{D}} - H_{\text{C}}}{t_{\text{D}} - t_{\text{C}}} \\ &= \frac{525 - 375}{7 - 5} \\ &= 75 \end{aligned}$$

b) i)
$$\begin{aligned} m_{\text{P}} &= \frac{m_{\text{AP}} + m_{\text{PB}}}{2} \\ &= \frac{\frac{150 - 75}{2 - 1} + \frac{225 - 150}{3 - 2}}{2} \\ &= \frac{75 + 75}{2} \\ &= 75 \end{aligned}$$

ii)
$$\begin{aligned} m_{\text{Q}} &= \frac{m_{\text{BQ}} + m_{\text{QC}}}{2} \\ &= \frac{\frac{300 - 225}{4 - 3} + \frac{375 - 300}{5 - 4}}{2} \\ &= \frac{75 + 75}{2} \\ &= 75 \end{aligned}$$

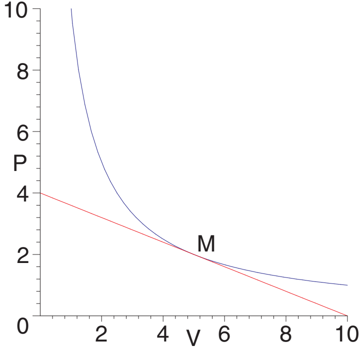
iii)
$$\begin{aligned} m_{\text{R}} &= \frac{m_{\text{QR}} + m_{\text{RD}}}{2} \\ &= \frac{\frac{525 - 450}{7 - 6} + \frac{450 - 375}{6 - 5}}{2} \\ &= \frac{75 + 75}{2} \\ &= 75 \end{aligned}$$

c) No. No.

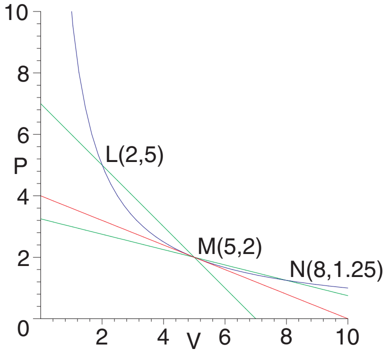
d) Answers will vary.

Section 3.1 Page 128 Question 5

a) From the intercepts, the slope of the tangent at M can be estimated as $-\frac{4}{10}$ or $-\frac{2}{5}$.



b) Two points on the function are L(2, 5) and N(8, 1.25). The slope of the secant LM is $\frac{2-5}{5-2}$ or -1. The slope of the secant MN is $\frac{2-1.25}{5-8}$ or -0.25. The average of the slopes of the two secants is $\frac{-1+(-0.25)}{2}$ or -0.625.



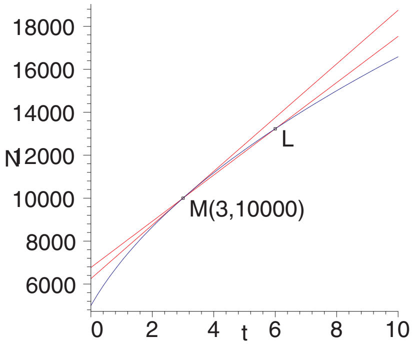
c) The results from the table below suggest that the slope of the tangent at M is -0.4.

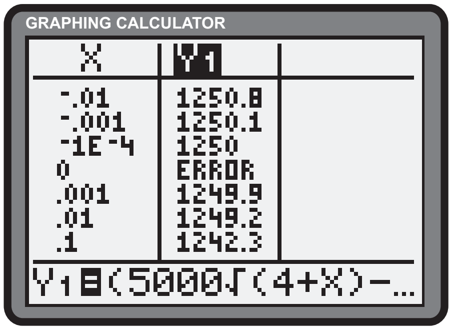
Point	L_1	L_2	L_3	M	N_3	N_2	N_1
(P,V)	$\left(4.9, \frac{10}{4.9}\right)$	$\left(4.99, \frac{10}{4.99}\right)$	$\left(4.999, \frac{10}{4.999}\right)$	$(5, 2)$	$\left(5.001, \frac{10}{5.001}\right)$	$\left(5.01, \frac{10}{5.01}\right)$	$\left(5.1, \frac{10}{5.1}\right)$
Slope of Secant	$\doteq -0.4082$	$\doteq -0.4008$	$\doteq -0.4001$		$\doteq -0.3999$	$\doteq -0.3992$	$\doteq -0.3922$

d) Answers will vary.

Section 3.1 Page 128 Question 6

Let the point M represent the population of the small town at 3 years. The coordinates of M are (3, 10 000). Let L be a point on $N = 5000\sqrt{1+t}$ near M with coordinates $(3 + x, 5000\sqrt{1+3+x}$ or $(3 + x, 5000\sqrt{4+x})$. The slope of the secant LM can be defined as $y = \frac{5000\sqrt{4+x} - 10\,000}{x}$.





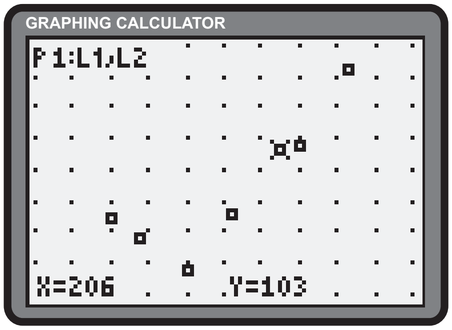
```
X      Y1
-.01   1250.8
-.001  1250.1
-1E-4  1250
0      ERROR
.001   1249.9
.01    1249.2
.1     1242.3
Y1=(5000√(4+X)-...
```

L approaches M as x approaches zero. From the table above, the slopes of the secants approach 1250. The value 1250 can be interpreted as the rate at which the population is growing with respect to time after 3 years.

Section 3.1 Page 128 Question 7

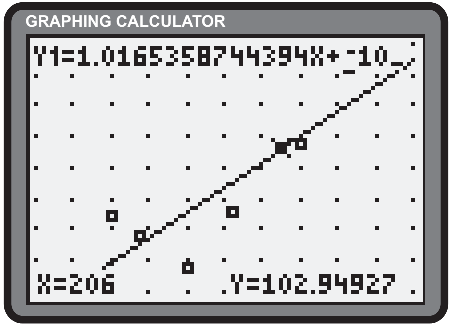
Answers will vary.

a)



```
P1:L1,L2
X=206     Y=103
```

b) Linear regression yields $M(h) \doteq 1.017h - 106.457$.



```
Y1=1.0165358744394X+⁻10
X=206     Y=102.94927
```

c) The slope of the tangent at $x = 200$ is approximately 1.017, the slope of the line.

d) The **Tangent operation** returns a result of approximately 1.017, as expected.

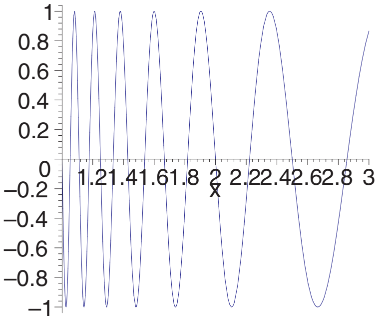
Section 3.1 Page 129 Question 8

a) The slope of the secant, $m_{\text{PQ}}(x)$, joining P(2, 0) and Q$(x, f(x))$, can be expressed as

$$\begin{aligned} m_{\text{PQ}}(x) &= \frac{y_{\text{Q}} - y_{\text{P}}}{x_{\text{Q}} - x_{\text{P}}} \\ &= \frac{f(x) - 0}{x - 2} \\ &= \frac{f(x)}{x - 2} \qquad (1) \end{aligned}$$

Use (1) and the calculator to determine the slopes of the secant PQ.

c)



i) $$\begin{aligned} m_{\text{PQ}}(1.5) &= \frac{f(1.5)}{1.5 - 2} \\ &\doteq 1.732 \end{aligned}$$

ii) $$\begin{aligned} m_{\text{PQ}}(1.6) &= \frac{f(1.6)}{1.6 - 2} \\ &= -2.5 \end{aligned}$$

iii) $$\begin{aligned} m_{\text{PQ}}(1.7) &= \frac{f(1.7)}{1.7 - 2} \\ &\doteq 2.246 \end{aligned}$$

iv) $$\begin{aligned} m_{\text{PQ}}(1.8) &= \frac{f(1.8)}{1.8 - 2} \\ &\doteq 1.710 \end{aligned}$$

v) $$\begin{aligned} m_{\text{PQ}}(1.9) &= \frac{f(1.9)}{1.9 - 2} \\ &\doteq -9.966 \end{aligned}$$

vi) $$\begin{aligned} m_{\text{PQ}}(2.5) &= \frac{f(2.5)}{2.5 - 2} \\ &= 0 \end{aligned}$$

vii) $$\begin{aligned} m_{\text{PQ}}(2.4) &= \frac{f(2.4)}{2.4 - 2} \\ &\doteq 2.165 \end{aligned}$$

viii) $$\begin{aligned} m_{\text{PQ}}(2.3) &= \frac{f(2.3)}{2.3 - 2} \\ &\doteq 2.723 \end{aligned}$$

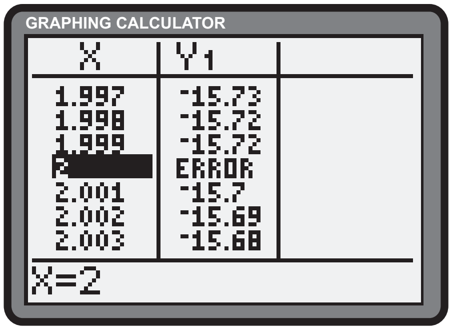
ix) $$\begin{aligned} m_{\text{PQ}}(2.2) &= \frac{f(2.2)}{2.2 - 2} \\ &\doteq -1.409 \end{aligned}$$

x) $$\begin{aligned} m_{\text{PQ}}(2.1) &= \frac{f(2.1)}{2.1 - 2} \\ &\doteq -9.972 \end{aligned}$$

b) The slopes do not appear to be getting closer to a unique value, from either side of (2, 0).

d) Answers may vary.

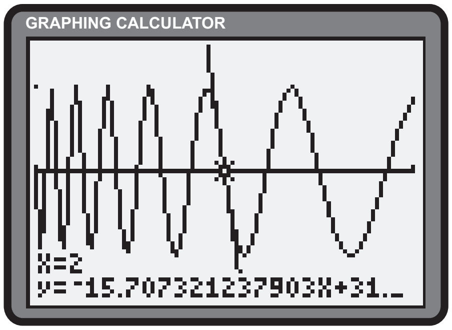
e) To achieve an accurate estimate, values of the x-coordinate of Q must be taken very close to 2. Using (1) and a calculator, define $\mathrm{Y}_1 = \dfrac{\sin\left(\dfrac{20\pi}{x}\right)}{x-2}$. Use the Table feature to display the values on both sides of 2.



GRAPHING CALCULATOR

X	Y1
1.997	⁻15.73
1.998	⁻15.72
1.999	⁻15.72
	ERROR
2.001	⁻15.7
2.002	⁻15.69
2.003	⁻15.68

X=2



GRAPHING CALCULATOR

X=2
y=⁻15.7073212379O3X+31._

The slope of the tangent at P(2, 0) is estimated to be approximately -15.71 or -5π. The **Tangent operation** confirms this result.

3.2 Using Limits to Find Tangents

Practise

Section 3.2 Page 137 Question 1

a) $m_{\mathrm{PQ}} = \dfrac{f(x)-f(2)}{x-2}$

b) $m_{\mathrm{P}} = \lim\limits_{x\to 2}\dfrac{f(x)-f(2)}{x-2}$

Section 3.2 Page 137 Question 3

a) i)

$$\begin{aligned}
m &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\\
&= \lim_{x\to 2}\frac{x^2-4x+5-1}{x-2}\\
&= \lim_{x\to 2}\frac{x^2-4x+4}{x-2}\\
&= \lim_{x\to 2}\frac{(x-2)^2}{x-2}\\
&= \lim_{x\to 2}(x-2),\ x\neq 2\\
&= 2-2\\
&= 0
\end{aligned}$$

ii)

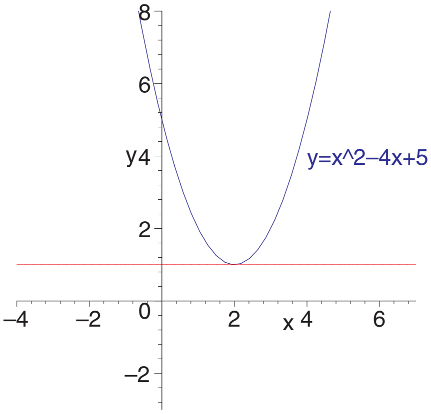
$$\begin{aligned}
m &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to 0}\frac{(2+h)^2-4(2+h)+5-1}{h}\\
&= \lim_{h\to 0}\frac{4+4h+h^2-8-4h+4}{h}\\
&= \lim_{h\to 0}\frac{h^2}{h}\\
&= \lim_{h\to 0}h,\ h\neq 0\\
&= 0
\end{aligned}$$

The slope of the tangent to $f(x) = x^2-4x+5$ at (2, 1) is 0.

b) The equation of the tangent at (2, 1) is given by

$$\begin{aligned}
y-y_1 &= m(x-x_1)\\
y-1 &= 0(x-2)\\
y-1 &= 0\\
y &= 1
\end{aligned}$$

c)



d) Answers will vary.

Section 3.2 Page 137 Question 5

For each of the following, let *m* be the slope of the tangent.

a)

$$\begin{aligned} m &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2}\frac{x^2+2-6}{x-2} \\ &= \lim_{x\to 2}\frac{x^2-4}{x-2} \\ &= \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} \\ &= \lim_{x\to 2}(x+2),\ x\neq 2 \\ &= 4 \end{aligned}$$

The slope of the tangent at $(2, 6)$ is 4.

b)

$$\begin{aligned} m &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to 1}\frac{x^2-6x+9-4}{x-1} \\ &= \lim_{x\to 1}\frac{x^2-6x+5}{x-1} \\ &= \lim_{x\to 1}\frac{(x-5)(x-1)}{x-1} \\ &= \lim_{x\to 1}(x-5),\ x\neq 1 \\ &= -4 \end{aligned}$$

The slope of the tangent at $(1, 4)$ is -4.

c)

$$\begin{aligned} m &= \lim_{x\to 7}\frac{f(x)-f(7)}{x-7} \\ &= \lim_{x\to 7}\frac{\sqrt{x-3}-2}{x-7}\cdot\frac{\sqrt{x-3}+2}{\sqrt{x-3}+2} \\ &= \lim_{x\to 7}\frac{x-3-4}{(x-7)\left(\sqrt{x-3}+2\right)} \\ &= \lim_{x\to 7}\frac{x-7}{(x-7)\left(\sqrt{x-3}+2\right)} \\ &= \lim_{x\to 7}\frac{1}{\sqrt{x-3}+2},\ x\neq 7 \\ &= \frac{1}{\sqrt{7-3}+2} \\ &= \frac{1}{4} \end{aligned}$$

The slope of the tangent at $(7, 2)$ is $\frac{1}{4}$.

d)

$$\begin{aligned} m &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}\cdot\frac{2x}{2x} \\ &= \lim_{x\to 2}\frac{2-x}{2x(x-2)} \\ &= -\lim_{x\to 2}\frac{1}{2x},\ x\neq 2 \\ &= -\frac{1}{4} \end{aligned}$$

The slope of the tangent at $\left(2, \frac{1}{2}\right)$ is $-\frac{1}{4}$.

e)

$$\begin{aligned} m &= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} \\ &= \lim_{x\to 3}\frac{\frac{1}{x-2}-1}{x-3}\cdot\frac{x-2}{x-2} \\ &= \lim_{x\to 3}\frac{1-(x-2)}{(x-3)(x-2)} \\ &= \lim_{x\to 3}\frac{3-x}{(x-3)(x-2)} \\ &= \lim_{x\to 3}\frac{-1}{x-2},\ x\neq 3 \\ &= \frac{-1}{1} \\ &= -1 \end{aligned}$$

The slope of the tangent at $(3, 1)$ is -1.

f)

$$\begin{aligned} m &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to 1}\frac{\frac{1}{x^2}-1}{x-1}\cdot\frac{x^2}{x^2} \\ &= \lim_{x\to 1}\frac{1-x^2}{x^2(x-1)} \\ &= \lim_{x\to 1}\frac{(1+x)(1-x)}{x^2(x-1)} \\ &= \lim_{x\to 1}\frac{-(1+x)}{x^2},\ x\neq 1 \\ &= \frac{-2}{1} \\ &= -2 \end{aligned}$$

The slope of the tangent at $(1, 1)$ is -2.

g)

$$\begin{aligned} m &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2}\frac{\frac{x}{1-x}+2}{x-2}\cdot\frac{1-x}{1-x} \\ &= \lim_{x\to 2}\frac{x+2(1-x)}{(1-x)(x-2)} \\ &= \lim_{x\to 2}\frac{2-x}{(1-x)(x-2)} \\ &= \lim_{x\to 2}\frac{-1}{1-x},\ x\neq 2 \\ &= \frac{-1}{1-2} \\ &= 1 \end{aligned}$$

The slope of the tangent at $(2, -2)$ is 1.

Section 3.2 Page 137 Question 7

a) Determine the slope of the tangent, m, at the general point $(a, a^2 - 8a + 12)$.

$$\begin{aligned} m &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a}\frac{x^2-8x+12-(a^2-8a+12)}{x-a} \\ &= \lim_{x\to a}\frac{x^2-a^2-8x+8a}{x-a} \\ &= \lim_{x\to a}\frac{(x+a)(x-a)-8(x-a)}{x-a} \\ &= \lim_{x\to a}(x+a-8),\ x\neq a \\ &= 2a-8 \end{aligned}$$

The slope of the tangent at $x = a$ is $2a - 8$.

h)

$$\begin{aligned} m &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to 1}\frac{\frac{1}{\sqrt{x}}-1}{x-1}\cdot\frac{\sqrt{x}}{\sqrt{x}} \\ &= \lim_{x\to 1}\frac{1-\sqrt{x}}{\sqrt{x}(x-1)} \\ &= \lim_{x\to 1}\frac{1-\sqrt{x}}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}-1)} \\ &= \lim_{x\to 1}\frac{-1}{\sqrt{x}(\sqrt{x}+1)},\ x\neq 1 \\ &= \frac{-1}{\sqrt{1}(1+1)} \\ &= -\frac{1}{2} \end{aligned}$$

The slope of the tangent at $(1, 1)$ is $-\frac{1}{2}$.

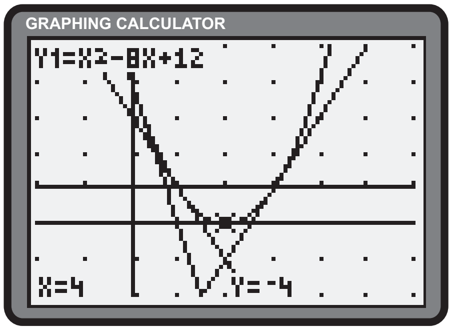
i) The slope of the tangent at $x = 0$ is $2(0) - 8$ or -8.

ii) The slope of the tangent at $x = 2$ is $2(2) - 8$ or -4.

iii) The slope of the tangent at $x = 4$ is $2(4) - 8$ or 0.

iv) The slope of the tangent at $x = 6$ is $2(6) - 8$ or 4.

b)



Apply, Solve, Communicate

Section 3.2 Page 137 Question 8

a) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2}\frac{x-2}{x-2} \\ &= \lim_{x\to 2} 1,\ x \neq 2 \\ &= 1 \end{aligned}$$

Determine the equation of the tangent at $x = 2$.

$$\begin{aligned} y - f(2) &= 1(x-2) \\ y - 2 &= x - 2 \\ y &= x \end{aligned}$$

The equation of the tangent is $y = x$.

ii) Since the equation of the function and the tangent are the same, the requirement that a tangent touch at only one point is not met. The definition is incorrect.

b) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} \\ &= \lim_{x\to 0}\frac{x^3-0}{x} \\ &= \lim_{x\to 0} x^2,\ x \neq 0 \\ &= 0 \end{aligned}$$

Determine the equation of the tangent at $x = 0$.

$$\begin{aligned} y - f(0) &= 0(x-0) \\ y - 0 &= 0 \\ y &= 0 \end{aligned}$$

The equation of the tangent is $y = 0$.

ii) Since the tangent defined by $y = 0$ crosses the cubic function, $f(x) = x^3$, at $(0, 0)$, the definition is incorrect.

Section 3.2 Page 137 Question 9

a) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} \\ &= \lim_{x\to 3}\frac{x^2-9-0}{x-3} \\ &= \lim_{x\to 3}\frac{x^2-9}{x-3} \\ &= \lim_{x\to 3}\frac{(x+3)(x-3)}{x-3} \\ &= \lim_{x\to 3}(x+3),\ x \neq 3 \\ &= 6 \end{aligned}$$

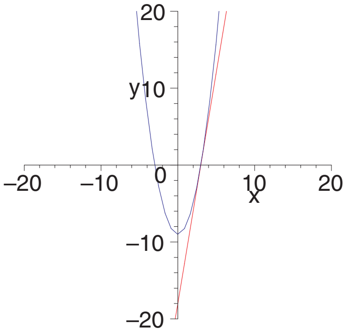
The slope of the tangent at $(3, 0)$ is 6.

ii) Determine the equation of the tangent at $(3, 0)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= 6(x-3) \\ y &= 6x - 18 \end{aligned}$$

The equation of the tangent is $y = 6x - 18$.

iii)



b) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to -2}\frac{g(x)-g(-2)}{x-(-2)} \\ &= \lim_{x\to -2}\frac{x^2+4x-1-(-5)}{x+2} \\ &= \lim_{x\to -2}\frac{x^2+4x+4}{x+2} \\ &= \lim_{x\to -2}\frac{(x+2)^2}{x+2} \\ &= \lim_{x\to -2}(x+2),\ x \neq -2 \\ &= 0 \end{aligned}$$

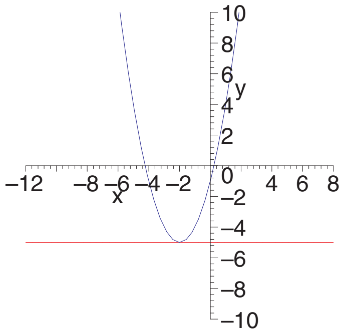
The slope of the tangent at $(-2, -5)$ is 0.

ii) Determine the equation of the tangent at $(-2, -5)$.

$$\begin{aligned} y - (-5) &= 0(x - (-2)) \\ y + 5 &= 0 \\ y &= -5 \end{aligned}$$

The equation of the tangent is $y = -5$.

iii)



c) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x \to 2} \frac{y(x) - y(2)}{x - 2} \\ &= \lim_{x \to 2} \frac{4 - x^2 - 0}{x - 2} \\ &= \lim_{x \to 2} \frac{-(x^2 - 4)}{x - 2} \\ &= \lim_{x \to 2} \frac{-(x + 2)(x - 2)}{x - 2} \\ &= \lim_{x \to 2} (-x - 2),\ x \neq 2 \\ &= -2 - 2 \\ &= -4 \end{aligned}$$

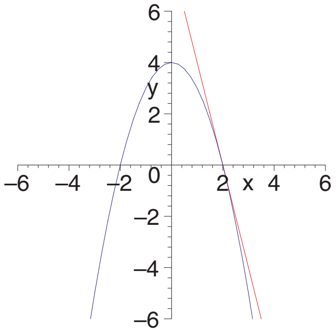
The slope of the tangent at $(2, 0)$ is -4.

ii) Determine the equation of the tangent at $(2, 0)$.

$$\begin{aligned} y - 0 &= -4(x - 2) \\ y &= -4x + 8 \end{aligned}$$

The equation of the tangent is $y = -4x + 8$.

iii)



d) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x \to 2} \frac{x^3 - 2 - 6}{x - 2} \\ &= \lim_{x \to 2} \frac{x^3 - 8}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} \\ &= \lim_{x \to 2} (x^2 + 2x + 4),\ x \neq 2 \\ &= 12 \end{aligned}$$

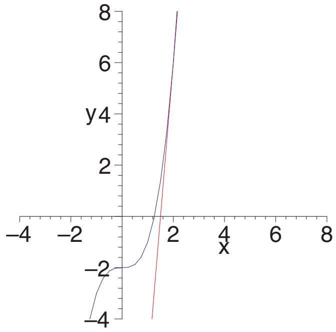
The slope of the tangent at $(2, 6)$ is 12.

ii) Determine the equation of the tangent at $(2, 6)$.

$$\begin{aligned} y - 6 &= 12(x - 2) \\ y - 6 &= 12x - 24 \\ y &= 12x - 18 \end{aligned}$$

The equation of the tangent is $y = 12x - 18$.

iii)



e) i) Let m be the slope of the tangent.

$$\begin{aligned}
m &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\\
&= \lim_{x\to 2}\frac{8-x^3-0}{x-2}\\
&= \lim_{x\to 2}\frac{-(x^3-8)}{x-2}\\
&= \lim_{x\to 2}\frac{-(x-2)(x^2+2x+4)}{x-2}\\
&= \lim_{x\to 2}(-x^2-2x-4),\ x\neq 2\\
&= -4-4-4\\
&= -12
\end{aligned}$$

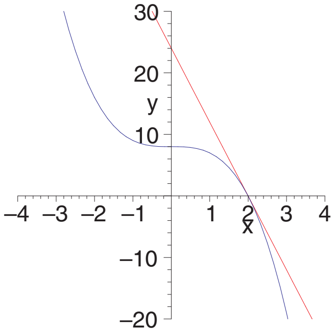
The slope of the tangent at $(2, 0)$ is -12.

ii) Determine the equation of the tangent at $(2, 0)$.

$$\begin{aligned}
y-0 &= -12(x-2)\\
y &= -12x+24
\end{aligned}$$

The equation of the tangent is $y = -12x + 24$.

iii)



f) i) Let m be the slope of the tangent.

$$\begin{aligned}
m &= \lim_{x\to 2}\frac{y(x)-y(2)}{x-2}\\
&= \lim_{x\to 2}\frac{\frac{1}{1-x}-1}{x-2}\\
&= \lim_{x\to 2}\frac{\frac{1}{1-x}+1}{x-2}\cdot\frac{1-x}{1-x}\\
&= \lim_{x\to 2}\frac{1+1-x}{(x-2)(1-x)}\\
&= \lim_{x\to 2}\frac{2-x}{(x-2)(1-x)}\\
&= \lim_{x\to 2}\frac{-1}{x-1},\ x\neq 2\\
&= 1
\end{aligned}$$

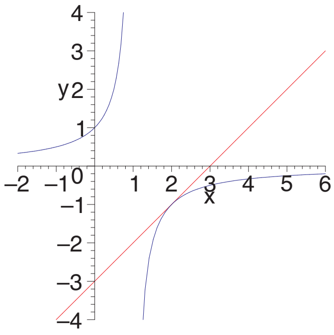
The slope of the tangent at $(2, -1)$ is 1.

ii) Determine the equation of the tangent at $(2, -1)$.

$$\begin{aligned}
y-(-1) &= 1(x-2)\\
y+1 &= x-2\\
y &= x-3
\end{aligned}$$

The equation of the tangent is $y = x - 3$.

iii)



g) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to 2}\frac{y(x)-y(2)}{x-2} \\ &= \lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}\cdot\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} \\ &= \lim_{x\to 2}\frac{x+2-4}{(x-2)(\sqrt{x+2}+2)} \\ &= \lim_{x\to 2}\frac{x-2}{(x-2)\sqrt{x+2}+2} \\ &= \lim_{x\to 2}\frac{1}{\sqrt{x+2}+2},\ x\neq 2 \\ &= \frac{1}{\sqrt{4}+2} \\ &= \frac{1}{4} \end{aligned}$$

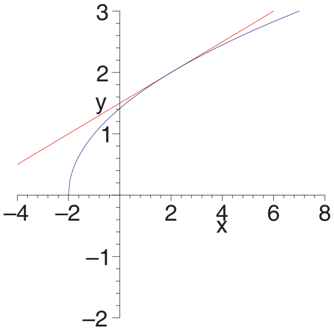
The slope of the tangent at $(2, 2)$ is $\frac{1}{4}$.

ii) Determine the equation of the tangent at $(2, 2)$.

$$\begin{aligned} y-2 &= \frac{1}{4}(x-2) \\ y-2 &= \frac{1}{4}x-\frac{1}{2} \\ y &= \frac{1}{4}x+\frac{3}{2} \end{aligned}$$

The equation of the tangent is $y = \frac{1}{4}x + \frac{3}{2}$.

iii)



h) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to 1}\frac{h(x)-h(1)}{x-1} \\ &= \lim_{x\to 1}\frac{x^4-1}{x-1} \\ &= \lim_{x\to 1}\frac{(x-1)(x^3+x^2+x+1)}{x-1} \\ &= \lim_{x\to 1}x^3+x^2+x+1,\ x\neq 1 \\ &= 4 \end{aligned}$$

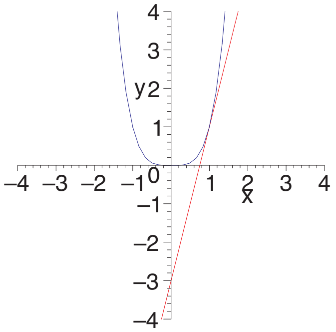
The slope of the tangent at $(1, 1)$ is 4.

ii) Determine the equation of the tangent at $(1, 1)$.

$$\begin{aligned} y-1 &= 4(x-1) \\ y-1 &= 4x-4 \\ y &= 4x-3 \end{aligned}$$

The equation of the tangent is $y = 4x - 3$.

iii)



Section 3.2 Page 138 Question 10

a) Substitution confirms that $(1, 1)$ is on the graph of the function. Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to1}\frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to1}\frac{2x^2-3x+2-1}{x-1} \\ &= \lim_{x\to1}\frac{2x^2-3x+1}{x-1} \\ &= \lim_{x\to1}\frac{(2x-1)(x-1)}{x-1} \\ &= \lim_{x\to1}2x-1,\ x\neq1 \\ &= 1 \end{aligned}$$

The slope of the tangent at $(1, 1)$ is 1. Determine the equation of the tangent at $(1, 1)$.

$$\begin{aligned} y-1 &= 1(x-1) \\ y &= x-1+1 \\ y &= x \end{aligned}$$

The equation of the tangent is $y = x$.

c) Substitution confirms that $(2, 3)$ is on the graph of the function. Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to2}\frac{g(x)-g(2)}{x-2} \\ &= \lim_{x\to2}\frac{\frac{x+1}{x-1}-3}{x-2}\cdot\frac{x-1}{x-1} \\ &= \lim_{x\to2}\frac{x+1-3(x-1)}{(x-2)(x-1)} \\ &= \lim_{x\to2}\frac{-2x+4}{(x-2)(x-1)} \\ &= \lim_{x\to2}\frac{-2(x-2)}{(x-2)(x-1)} \\ &= \lim_{x\to2}\frac{-2}{x-1},\ x\neq2 \\ &= -2 \end{aligned}$$

The slope of the tangent at $(2, 3)$ is -2. Determine the equation of the tangent at $(2, 3)$.

$$\begin{aligned} y-3 &= -2(x-2) \\ y &= -2x+4+3 \\ y &= -2x+7 \end{aligned}$$

The equation of the tangent is $y = -2x + 7$.

b) Substitution confirms that $(0, 0)$ is on the graph of the function. Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to0}\frac{g(x)-g(0)}{x-0} \\ &= \lim_{x\to0}\frac{x^2-x^3-0}{x} \\ &= \lim_{x\to0}\frac{x^2(1-x)}{x} \\ &= \lim_{x\to0}[x(1-x)],\ x\neq0 \\ &= 0 \end{aligned}$$

The slope of the tangent at $(0, 0)$ is 0. Determine the equation of the tangent at $(0, 0)$.

$$\begin{aligned} y-0 &= 0(x-0) \\ y &= 0 \end{aligned}$$

The equation of the tangent is $y = 0$.

d) Substitution confirms that $(0, -1)$ is on the graph of the function. Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to0}\frac{f(x)-f(0)}{x-0} \\ &= \lim_{x\to0}\frac{\frac{x^2-4}{x+4}-(-1)}{x} \\ &= \lim_{x\to0}\frac{\frac{x^2-4}{x+4}+1}{x}\cdot\frac{x+4}{x+4} \\ &= \lim_{x\to0}\frac{x^2-4+x+4}{x(x+4)} \\ &= \lim_{x\to0}\frac{x^2+x}{x(x+4)} \\ &= \lim_{x\to0}\frac{x(x+1)}{x(x+4)} \\ &= \lim_{x\to0}\frac{x+1}{x+4},\ x\neq0 \\ &= \frac{1}{4} \end{aligned}$$

The slope of the tangent at $(0, -1)$ is $\frac{1}{4}$. Determine the equation of the tangent at $(0, -1)$.

$$\begin{aligned} y-(-1) &= \frac{1}{4}(x-0) \\ y+1 &= \frac{1}{4}x \\ y &= \frac{1}{4}x-1 \end{aligned}$$

The equation of the tangent is $y = \frac{1}{4}x - 1$.

e) Substitution confirms that $(4, 5)$ is on the graph of the function. Let m be the slope of the tangent.

$$\begin{aligned}
m &= \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} \\
&= \lim_{x\to 4} \frac{\sqrt{x^2+9} - 5}{x - 4} \cdot \frac{\sqrt{x^2+9} + 5}{\sqrt{x^2+9} + 5} \\
&= \lim_{x\to 4} \frac{x^2 + 9 - 25}{(x-4)(\sqrt{x^2+9} + 5)} \\
&= \lim_{x\to 4} \frac{x^2 - 16}{(x-4)(\sqrt{x^2+9} + 5)} \\
&= \lim_{x\to 4} \frac{(x-4)(x+4)}{(x-4)(\sqrt{x^2+9} + 5)} \\
&= \lim_{x\to 4} \frac{x+4}{\sqrt{x^2+9} + 5}, \ x \neq 4 \\
&= \frac{8}{\sqrt{25} + 5} \\
&= \frac{8}{10} \\
&= \frac{4}{5}
\end{aligned}$$

The slope of the tangent at $(4, 5)$ is $\frac{4}{5}$. Determine the equation of the tangent at $(4, 5)$.

$$\begin{aligned}
y - 5 &= \frac{4}{5}(x - 4) \\
y &= \frac{4}{5}x - \frac{16}{5} + \frac{25}{5} \\
y &= \frac{4}{5}x + \frac{9}{5}
\end{aligned}$$

The equation of the tangent is $y = \frac{4}{5}x + \frac{9}{5}$.

f) Substitution confirms that $\left(4, \frac{1}{2}\right)$ is on the graph of the function. Let m be the slope of the tangent.

$$\begin{aligned}
m &= \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} \\
&= \lim_{x\to 4} \frac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x - 4} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} \\
&= \lim_{x\to 4} \frac{2 - \sqrt{x}}{2\sqrt{x}(x-4)} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}} \\
&= \lim_{x\to 4} \frac{4 - x}{2\sqrt{x}(x-4)(2 + \sqrt{x})} \\
&= \lim_{x\to 4} \frac{-1}{2\sqrt{x}(2 + \sqrt{x})}, \ x \neq 4 \\
&= \frac{-1}{2(2)(2+2)} \\
&= -\frac{1}{16}
\end{aligned}$$

The slope of the tangent at $\left(4, \frac{1}{2}\right)$ is $-\frac{1}{16}$. Determine the equation of the tangent at $\left(4, \frac{1}{2}\right)$.

$$\begin{aligned}
y - \frac{1}{2} &= -\frac{1}{16}(x - 4) \\
y &= -\frac{1}{16}x + \frac{1}{4} + \frac{1}{2} \\
y &= -\frac{1}{16}x + \frac{3}{4}
\end{aligned}$$

The equation of the tangent is $y = -\frac{1}{16}x + \frac{3}{4}$.

Section 3.2 Page 138 Question 11

a) i) The y-coordinate is $y(a)$ or a^2. Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to a} \frac{y(x) - y(a)}{x - a} \\ &= \lim_{x\to a} \frac{x^2 - a^2}{x - a} \\ &= \lim_{x\to a} \frac{(x-a)(x+a)}{x-a} \\ &= \lim_{x\to a}(x+a),\ x \neq a \\ &= a + a \\ &= 2a \end{aligned}$$

The slope of the tangent at (a, a^2) is $2a$.

ii) Determine the slope of the tangent at each x-coordinate.

$$\begin{aligned} m_{-2} &= 2(-2) \\ &= -4 \\ m_{-1} &= 2(-1) \\ &= -2 \\ m_0 &= 2(0) \\ &= 0 \\ m_1 &= 2(1) \\ &= 2 \\ m_2 &= 2(2) \\ &= 4 \end{aligned}$$

iii) As x increases, the slope of the function increases linearly.

b) i) The y-coordinate is $f(a)$ or a^3. Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x\to a} \frac{x^3 - a^3}{x - a} \\ &= \lim_{x\to a} \frac{(x-a)(x^2 + ax + a^2)}{x-a} \\ &= \lim_{x\to a}(x^2 + ax + a^2),\ x \neq a \\ &= a^2 + a^2 + a^2 \\ &= 3a^2 \end{aligned}$$

The slope of the tangent at (a, a^3) is $3a^2$.

ii) Determine the slope of the tangent at each x-coordinate.

$$\begin{aligned} m_{-2} &= 3(-2)^2 \\ &= 12 \\ m_{-1} &= 3(-1)^2 \\ &= 3 \\ m_0 &= 3(0)^2 \\ &= 0 \\ m_1 &= 3(1)^2 \\ &= 3 \\ m_2 &= 3(2)^2 \\ &= 12 \end{aligned}$$

iii) As x increases, the slope of the function decreases to 0 at $x = 0$ and then increases.

c) i) The y-coordinate is $y(a)$ or $a^2 + 2a - 3$. Let m be the slope of the tangent.

$$\begin{aligned}
m &= \lim_{x \to a} \frac{y(x) - y(a)}{x - a} \\
&= \lim_{x \to a} \frac{x^2 + 2x - 3 - (a^2 + 2a - 3)}{x - a} \\
&= \lim_{x \to a} \frac{x^2 + 2x - 3 - a^2 - 2a + 3}{x - a} \\
&= \lim_{x \to a} \frac{x^2 - a^2 + 2x - 2a}{x - a} \\
&= \lim_{x \to a} \frac{(x - a)(x + a) + 2(x - a)}{x - a} \\
&= \lim_{x \to a}(x + a + 2),\ x \neq a \\
&= a + a + 2 \\
&= 2a + 2
\end{aligned}$$

The slope of the tangent at $(a, a^2 + 2a - 3)$ is $2a + 2$.

ii) Determine the slope of the tangent at each x-coordinate.

$$\begin{aligned}
m_{-2} &= 2(-2) + 2 \\
&= -4 + 2 \\
&= -2 \\
m_{-1} &= 2(-1) + 2 \\
&= -2 + 2 \\
&= 0 \\
m_0 &= 2(0) + 2 \\
&= 0 + 2 \\
&= 2 \\
m_1 &= 2(1) + 2 \\
&= 2 + 2 \\
&= 4 \\
m_2 &= 2(2) + 2 \\
&= 4 + 2 \\
&= 6
\end{aligned}$$

iii) As x increases, the slope of the tangent increases linearly.

d) i) The y-coordinate is $g(a)$ or $\frac{6}{a - 3}$. Let m be the slope of the tangent.

$$\begin{aligned}
m &= \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x \to a} \frac{\frac{6}{x-3} - \frac{6}{a-3}}{x - a} \cdot \frac{(x - 3)(a - 3)}{(x - 3)(a - 3)} \\
&= \lim_{x \to a} \frac{6(a - 3) - 6(x - 3)}{(x - a)(x - 3)(a - 3)} \\
&= \lim_{x \to a} \frac{6a - 18 - 6x + 18}{(x - a)(x - 3)(a - 3)} \\
&= \lim_{x \to a} \frac{6(a - x)}{(x - a)(x - 3)(a - 3)} \\
&= \lim_{x \to a} \frac{-6}{(x - 3)(a - 3)},\ x \neq a \\
&= -\frac{6}{(a - 3)^2}
\end{aligned}$$

The slope of the tangent at $\left(a, \frac{6}{a - 3}\right)$ is $-\frac{6}{(a - 3)^2}$.

ii) Determine the slope of the tangent at each x-coordinate.

$$\begin{aligned}
m_{-2} &= -\frac{6}{(-2 - 3)^2} \\
&= -\frac{6}{25} \\
m_{-1} &= -\frac{6}{(-1 - 3)^2} \\
&= -\frac{6}{16} \\
&= -\frac{3}{8} \\
m_0 &= -\frac{6}{(0 - 3)^2} \\
&= -\frac{6}{9} \\
&= -\frac{2}{3} \\
m_1 &= -\frac{6}{(1 - 3)^2} \\
&= -\frac{6}{4} \\
&= -\frac{3}{2} \\
m_2 &= -\frac{6}{(2 - 3)^2} \\
&= -\frac{6}{1} \\
&= -6
\end{aligned}$$

iii) As x increases, the slope of the tangent decreases as x approaches 3, and then increases for $x > 3$.

e) i) The y-coordinate is $y(a)$ or $\sqrt{a^2+1}$. Let m be the slope of the tangent.

$$
\begin{aligned}
m &= \lim_{x\to a}\frac{y(x)-y(a)}{x-a}\\
&= \lim_{x\to a}\frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}\cdot\frac{\sqrt{x^2+1}+\sqrt{a^2+1}}{\sqrt{x^2+1}+\sqrt{a^2+1}}\\
&= \lim_{x\to a}\frac{x^2+1-(a^2+1)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}\\
&= \lim_{x\to a}\frac{x^2-a^2}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}\\
&= \lim_{x\to a}\frac{(x-a)(x+a)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}\\
&= \lim_{x\to a}\frac{x+a}{\sqrt{x^2+1}+\sqrt{a^2+1})},\ x\neq a\\
&= \frac{a+a}{\sqrt{a^2+1}+\sqrt{a^2+1}}\\
&= \frac{2a}{2\sqrt{a^2+1}}\\
&= \frac{a}{\sqrt{a^2+1}}
\end{aligned}
$$

The slope of the tangent at $\left(a, \sqrt{a^2+1}\right)$ is $\dfrac{a}{\sqrt{a^2+1}}$.

ii) Determine the slope of the tangent at each x-coordinate.

$$
\begin{aligned}
m_{-2} &= \frac{-2}{\sqrt{(-2)^2+1}}\\
&= -\frac{2}{\sqrt{5}}\\
m_{-1} &= \frac{-1}{\sqrt{(-1)^2+1}}\\
&= \frac{-1}{\sqrt{2}}\\
m_0 &= \frac{0}{\sqrt{0^2+1}}\\
&= 0\\
m_1 &= \frac{1}{\sqrt{1^2+1}}\\
&= \frac{1}{\sqrt{2}}\\
m_2 &= \frac{2}{\sqrt{2^2+1}}\\
&= \frac{2}{\sqrt{5}}
\end{aligned}
$$

iii) As x increases, the slope of the tangent increases.

f) i) The y-coordinate is $h(a)$ or $\dfrac{1}{a^2+1}$. Let m be the slope of the tangent.

$$
\begin{aligned}
m &= \lim_{x\to a}\frac{h(x)-h(a)}{x-a}\\
&= \lim_{x\to a}\frac{\frac{1}{x^2+1}-\frac{1}{a^2+1}}{x-a}\cdot\frac{(x^2+1)(a^2+1)}{(x^2+1)(a^2+1)}\\
&= \lim_{x\to a}\frac{a^2+1-(x^2+1)}{(x-a)(x^2+1)(a^2+1)}\\
&= \lim_{x\to a}\frac{a^2-x^2}{(x-a)(x^2+1)(a^2+1)}\\
&= \lim_{x\to a}\frac{(a-x)(a+x)}{(x-a)(x^2+1)(a^2+1)}\\
&= \lim_{x\to a}\frac{-(a+x)}{(x^2+1)(a^2+1)}\\
&= \frac{-(a+a)}{(a^2+1)(a^2+1)}\\
&= -\frac{2a}{(a^2+1)^2}
\end{aligned}
$$

The slope of the tangent at $\left(a, \dfrac{1}{a^2+1}\right)$ is $-\dfrac{2a}{(a^2+1)^2}$.

ii) Determine the slope of the tangent at each x-coordinate.

$$
\begin{aligned}
m_{-2} &= -\frac{2(-2)}{((-2)^2+1)^2}\\
&= \frac{4}{25}\\
m_{-1} &= -\frac{2(-1)}{((-1)^2+1)^2}\\
&= \frac{1}{2}\\
m_0 &= -\frac{2(0)}{(0^2+1)^2}\\
&= 0\\
m_1 &= -\frac{2(1)}{(1^2+1)^2}\\
&= -\frac{1}{2}\\
m_2 &= -\frac{2(2)}{(2^2+1)^2}\\
&= -\frac{4}{25}
\end{aligned}
$$

iii) As x increases, the slope of the tangent increases to a maximum of 0.5 at $x = -1$, decreases to a minimum of -0.5 at $x = 1$, and then approaches 0 as x increases without bound. The slope also approaches 0 as x decreases without bound.

Section 3.2 Page 138 Question 12

Determine the slope of the tangent m with respect to x at a distance $x = a$ metres.

$$\begin{aligned}
m &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{-\dfrac{5.38 \times 10^{19}}{x} - \left(-\dfrac{5.38 \times 10^{19}}{a}\right)}{x - a} \\
&= \left(-5.38 \times 10^{19}\right) \lim_{x \to a} \frac{\dfrac{1}{x} - \dfrac{1}{a}}{x - a} \cdot \frac{ax}{ax} \\
&= \left(-5.38 \times 10^{19}\right) \lim_{x \to a} \frac{a - x}{ax(x - a)} \\
&= \left(-5.38 \times 10^{19}\right) \lim_{x \to a} \frac{-1}{ax},\ x \neq a \\
&= \left(-5.38 \times 10^{19}\right) \left(-\frac{1}{a^2}\right) \\
&= \frac{5.38 \times 10^{19}}{a^2} \qquad (1)
\end{aligned}$$

Use (1) in each of the following.

a)

$$\begin{aligned}
m &= \frac{5.38 \times 10^{19}}{(6.77 \times 10^3)^2} \\
&\doteq 1.17 \times 10^6
\end{aligned}$$

The slope of the tangent is approximately 1.17×10^6.

b)

$$\begin{aligned}
m &= \frac{5.38 \times 10^{19}}{(6.6 \times 10^6)^2} \\
&\doteq 1.24 \times 10^6
\end{aligned}$$

The slope of the tangent is approximately 1.24×10^6.

c)

$$\begin{aligned}
m &= \frac{5.38 \times 10^{19}}{(6.5 \times 10^3)^2} \\
&\doteq 1.27 \times 10^6
\end{aligned}$$

The slope of the tangent is approximately 1.27×10^6.

d)

$$\begin{aligned}
m &= \frac{5.38 \times 10^{19}}{(6.38 \times 10^6)^2} \\
&\doteq 1.32 \times 10^6
\end{aligned}$$

The slope of the tangent is approximately 1.32×10^6.

Section 3.2 Page 138 Question 13

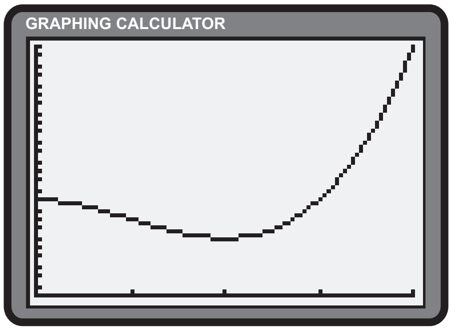
Let g be the acceleration due to gravity.

$$\begin{aligned}
g &= \frac{1.32 \times 10^6}{135\,000} \\
&\doteq 9.8
\end{aligned}$$

This result is similar to the average gravitational force of acceleration at the surface of the Earth of 9.8 m/s^2.

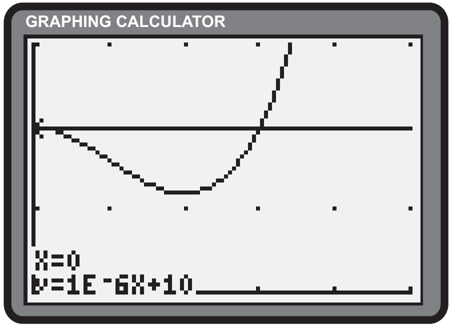
Section 3.2 Page 138 Question 14

a) $V(t) = t^3 - 3t^2 + 10$



b) Yes.

c) The slope of the tangent at $t = 0$ is 0.



d) The investor should reinvest at $t = 2$, where the slope of the tangent is 0.

e)

$$\begin{aligned} m &= \lim_{t\to 2}\frac{V(t)-V(2)}{t-2} \\ &= \lim_{t\to 2}\frac{t^3-3t^2+10-(2^3-3(2)^2+10)}{t-2} \\ &= \lim_{t\to 2}\frac{t^3-3t^2+10-6}{t-2} \\ &= \lim_{t\to 2}\frac{t^3-3t^2+4}{t-2} \\ &= \lim_{t\to 2}\frac{(t-2)(t^2-t-2)}{t-2} \\ &= \lim_{t\to 2}(t^2-t-2);\ t \neq 2 \\ &= 4-2-2 \\ &= 0 \end{aligned}$$

The slope of the tangent at $t = 2$ is 0.

f) Answers may vary.

Section 3.2 Page 138 Question 15

a) The y-coordinate where $x = a$ is $y(a) = 2a^2 + 3a$. Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a}\frac{2x^2+3x-(2a^2+3a)}{x-a} \\ &= \lim_{x\to a}\frac{2(x^2-a^2)+3(x-a)}{x-a} \\ &= \lim_{x\to a}(2(x+a)+3),\ x \neq a \\ &= 2(a+a)+3 \\ &= 4a+3 \end{aligned}$$

The slope of the tangent at $x = a$ is $4a + 3$.

b) The slope of the line $y = 14x - 6$ is 14.

$$\begin{aligned} m &= 14 \\ 4a+3 &= 14 \\ 4a &= 11 \\ a &= \frac{11}{4} \\ y\left(\frac{11}{4}\right) &= 2\left(\frac{11}{4}\right)^2 + 3\left(\frac{11}{4}\right) \\ &= \frac{187}{8} \end{aligned}$$

The tangent to $y = 2x^2 + 3x$ is parallel to the line $y = 14x - 6$ at the point $\left(\frac{11}{4}, \frac{187}{8}\right)$.

Section 3.2 Page 138 Question 16

a)

x	**f(x)**
10	$1.001\,000\,45 \times 10^{10}$
1	$1.010\,045\,12$
0.1	$1.104\,622\,13 \times 10^{-10}$
0.01	$2.597\,742\,46 \times 10^{-20}$
0.001	1.024×10^{-27}

b) $\lim_{x\to 0}(x + 0.001)^{10} = 0$

c)

x	**f(x)**
0.0001	$2.593\,742\,46 \times 10^{-30}$
0.000 01	$1.104\,622\,13 \times 10^{-30}$
0.000 001	$1.010\,045\,12 \times 10^{-30}$
0.000 000 1	$1.001\,000\,45 \times 10^{-30}$

d) No.

e) Answers may vary.

f) Answers may vary.

g) Answers may vary.

Section 3.2 Page 139 Question 17

a) Let $(a, a^2 + 6a + 9)$ be the point where the tangent has a slope of 4. Let m be the slope of the tangent.

$$\begin{aligned}
m &= 4 \\
\lim_{x\to a} \frac{f(x) - f(a)}{x - a} &= 4 \\
\lim_{x\to a} \frac{x^2 + 6x + 9 - (a^2 + 6a + 9)}{x - a} &= 4 \\
\lim_{x\to a} \frac{x^2 - a^2 + 6(x - a)}{x - a} &= 4 \\
\lim_{x\to a} \frac{(x - a)(x + a) + 6(x - a)}{x - a} &= 4 \\
\lim_{x\to a}(x + a + 6) &= 4,\ x \neq a \\
a + a + 6 &= 4 \\
2a + 6 &= 4 \\
2a &= -2 \\
a &= -1
\end{aligned}$$

The point where the slope of the tangent is 4 is $(-1, (-1)^2 + 6(-1) + 9)$ or $(-1, 4)$. Determine the equation of the tangent using the point-slope formula.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 4 &= 4(x - (-1)) \\
y - 4 &= 4x + 4 \\
y &= 4x + 8
\end{aligned}$$

The equation of the tangent is $y = 4x + 8$.

b) Let (a, a^3) be the points where the tangent has a slope of 12. Let m be the slope of the tangent.

$$\begin{aligned}
m &= 12 \\
\lim_{x\to a} \frac{f(x) - f(a)}{x - a} &= 12 \\
\lim_{x\to a} \frac{x^3 - a^3}{x - a} &= 12 \\
\lim_{x\to a} \frac{(x - a)(x^2 + ax + a^2)}{x - a} &= 12 \\
\lim_{x\to a}(x^2 + ax + a^2) &= 12,\ x \neq a \\
a^2 + a^2 + a^2 &= 12 \\
3a^2 &= 12 \\
a^2 &= 4 \\
a &= \pm 2
\end{aligned}$$

The points where the slope of the tangent is 12 are $(2, 2^3)$ or $(2, 8)$ and $(-2, (-2)^3)$ or $(-2, -8)$. Determine the equation of the tangent at $(2, 8)$ using the point-slope formula.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 8 &= 12(x - 2) \\
y - 8 &= 12x - 24 \\
y &= 12x - 16
\end{aligned}$$

$y = 12x - 16$ is the equation of the tangent at $(2, 8)$. Determine the equation of the tangent at $(-2, -8)$ using the point-slope formula.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - (-8) &= 12(x - (-2)) \\
y + 8 &= 12x + 24 \\
y &= 12x + 16
\end{aligned}$$

The equation of the tangent at $(-2, -8)$ is $y = 12x + 16$.

Section 3.2 Page 139 Question 18

Determine the zeros of the function.

$$\begin{aligned} y &= 0 \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x &= 0, \pm 1 \end{aligned} \tag{1}$$

Determine the slope of the tangent, m, at the general point $(a, a^3 - a)$.

$$\begin{aligned} m &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{x^3 - x - (a^3 - a)}{x - a} \\ &= \lim_{x \to a} \frac{x^3 - a^3 - (x - a)}{x - a} \\ &= \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2) - (x - a)}{x - a} \\ &= \lim_{x \to a} (x^2 + ax + a^2 - 1),\ x \neq a \\ &= a^2 + a^2 + a^2 - 1 \\ &= 3a^2 - 1 \end{aligned} \tag{2}$$

The slope of the tangent at $x = -1$ is $3(-1)^2 - 1$ or 2. The equation of the tangent at $(-1, 0)$ is

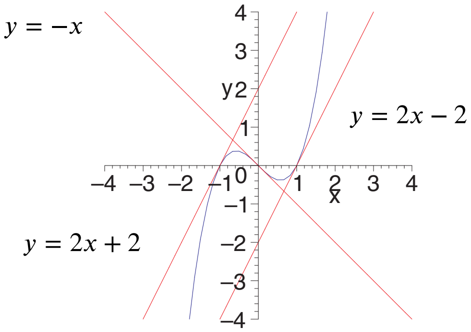
$$\begin{aligned} y - 0 &= 2(x - (-1)) \\ y &= 2x + 2 \end{aligned} \tag{3}$$

The slope of the tangent at $x = 0$ is $3(0)^2 - 1$ or -1. The equation of the tangent at $(0, 0)$ is

$$\begin{aligned} y - 0 &= -1(x - 0) \\ y &= -x \end{aligned} \tag{4}$$

The slope of the tangent at $x = 1$ is $3(1)^2 - 1$ or 2. The equation of the tangent at $(1, 0)$ is

$$\begin{aligned} y - 0 &= 2(x - 1) \\ y &= 2x - 2 \end{aligned} \tag{4}$$



Section 3.2 Page 139 Question 19

Let $y_1 = \frac{1}{2}x^2$ and $y_2 = 1 - \frac{1}{2}x^2$. Determine the x-coordinates of the points of intersection.

$$\begin{aligned} y_1 &= y_2 \\ \frac{1}{2}x^2 &= 1 - \frac{1}{2}x^2 \\ x^2 &= 1 \\ x &= \pm 1 \\ y(1) &= \frac{1}{2} \\ y(-1) &= -\frac{1}{2} \end{aligned}$$

The points of intersection are $\left(-1, \frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$. Let m_1 be the slope of the tangent to y_1 at $x = a$.

$$\begin{aligned} m_1 &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{\frac{1}{2}x^2 - \frac{1}{2}a^2}{x - a} \\ &= \lim_{x \to a} \frac{(x + a)(x - a)}{2(x - a)} \\ &= \lim_{x \to a} \frac{x + a}{2},\ x \neq a \\ &= \frac{a + a}{2} \\ &= a \end{aligned}$$

The slopes of the tangents to y_1 at $x = \pm 1$ are $m_1 = \pm 1$, respectively. Let m_2 be the slope of the tangent to y_2 at $x = a$.

$$\begin{aligned} m_2 &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{1 - \frac{1}{2}x^2 - \left(1 - \frac{1}{2}a^2\right)}{x - a} \\ &= \lim_{x \to a} \frac{-(x^2 - a^2)}{2(x - a)} \\ &= \lim_{x \to a} \frac{-(x + a)(x - a)}{2(x - a)} \\ &= \lim_{x \to a} -\frac{x + a}{2},\ x \neq a \\ &= -\frac{a + a}{2} \\ &= -a \end{aligned}$$

The slopes of the tangents to y_2 at $x = \pm 1$ are $m_2 = \mp 1$, respectively. At the point of intersection where $x = -1$, since $m_1 \times m_2 = (-1)(1)$ or -1, the tangents are perpendicular. At the point of intersection where $x = 1$, since $m_1 \times m_2 = (1)(-1)$ or -1, the tangents are perpendicular.

Section 3.2 Page 139 Question 20

a) Determine the rate of change of E with respect to x at a distance $x = a$ metres.

$$\begin{aligned} F &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x\to a} \frac{-\frac{4\times10^{17}}{x} - \left(-\frac{4\times10^{17}}{a}\right)}{x - a} \\ &= \left(-4\times10^{17}\right) \lim_{x\to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} \cdot \frac{ax}{ax} \\ &= \left(-4\times10^{17}\right) \lim_{x\to a} \frac{a - x}{ax(x - a)} \\ &= \left(-4\times10^{17}\right) \lim_{x\to a} \frac{-1}{ax},\ x \neq a \\ &= \left(-4\times10^{17}\right)\left(-\frac{1}{a^2}\right) \\ &= \frac{4\times10^{17}}{a^2} \qquad (1) \end{aligned}$$

Determine the radius of the orbit of the satellite.

$$\begin{aligned} a &= 2(6.38\times10^6) + 6.38\times10^6 \\ &= 1.914\times10^7 \qquad (2) \end{aligned}$$

Substitute (2) in (1).

$$\begin{aligned} F &= \frac{4\times10^{17}}{(1.914\times10^7)^2} \\ &\doteq 1092 \end{aligned}$$

The force of gravity on the satellite is approximately 1092 N.

b) Result (1) suggests that a increases, the force of gravity decreases. At infinite distances from Earth the gravity becomes infinitesimally small, but it would never be zero.

3.3 The Limit of a Function

Practise

Section 3.3 Page 153 Question 1

a) $\lim_{x\to-1} f(x) = 3$ **b)** $\lim_{x\to0} f(x) = 4^-$ **c)** $\lim_{x\to2} f(x) = 0$

d) The $\lim_{x\to4} f(x)$ does not exist because neither the left-hand limit nor the right-hand limit exists at $x = 4$.

e) $\lim_{x\to6} f(x) \doteq 0.5$

Section 3.3 Page 153 Question 3

a) $\lim_{x\to-2} g(x) = 12$ **b)** $\lim_{x\to0} g(x) = 1$ **c)** $\lim_{x\to2^-} g(x) = -3^+$

d) $g(2)$ does not exist because there is a hole at $x = 2$.

e) $\lim_{x\to2^+} g(x) = -3^+$ **f)** $\lim_{x\to2} g(x) = -3$ **g)** $\lim_{x\to5^-} f(x) = 6^-$

h) $g(5) = 6$

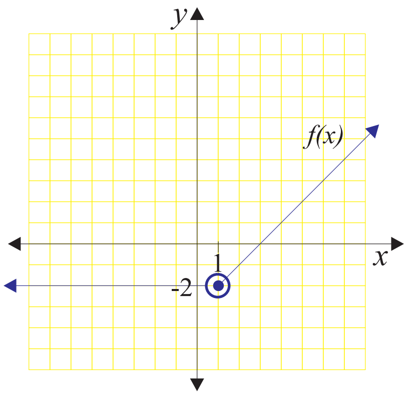
i) $\lim_{x\to5^+} g(x) = 12^-$ **j)** $\lim_{x\to5} g(x)$ does not exist because $\lim_{x\to5^-} g(x) \neq \lim_{x\to5^+} g(x)$.

Section 3.3 Page 153 Question 5

a)
$$\begin{aligned}\lim_{x\to1^-} f(x) &= \lim_{x\to1^-} -2\\ &= -2\end{aligned}$$

b)
$$\begin{aligned}\lim_{x\to1^+} f(x) &= \lim_{x\to1^+} (x-3)\\ &= 1-3\\ &= -2\end{aligned}$$

c) Since $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = -2$, $\lim_{x\to1} f(x) = -2$.

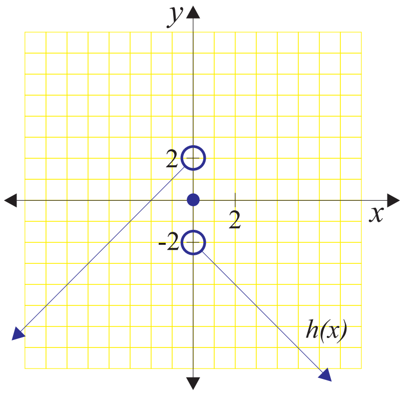


Section 3.3 Page 154 Question 7

a)
$$\begin{aligned}\lim_{x\to0^-} h(x) &= \lim_{x\to0^-} (x+2)\\ &= 0+2\\ &= 2\end{aligned}$$

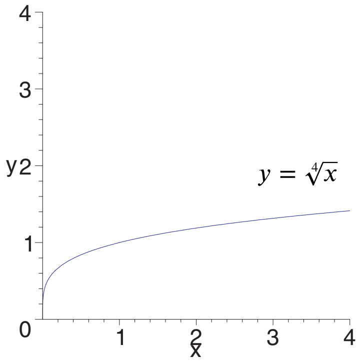
b)
$$\begin{aligned}\lim_{x\to0^+} h(x) &= \lim_{x\to0^+} (-x-2)\\ &= 0-2\\ &= -2\end{aligned}$$

c) Since $\lim_{x\to0^-} h(x) \neq \lim_{x\to0^+} h(x)$, $\lim_{x\to0} h(x)$ does not exist.

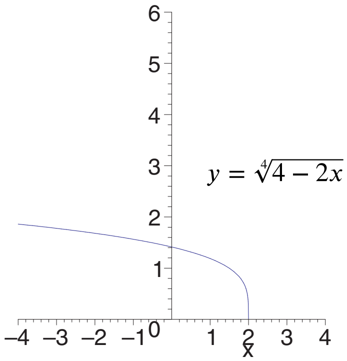


Section 3.3 Page 153 Question 9

a)
$$\lim_{x\to0^+} \sqrt[4]{x} = 0^+$$



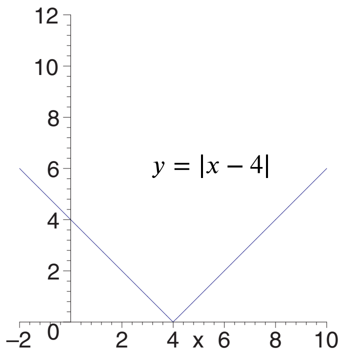
b) Since $\lim_{x\to2^+} \sqrt[4]{4-2x}$ does not exist, $\lim_{x\to2} \sqrt[4]{4-2x}$ does not exist.



c)
$$\lim_{x\to4^-} |x-4| = 0^+$$

d)
$$\lim_{x\to4^+} |x-4| = 0^+$$

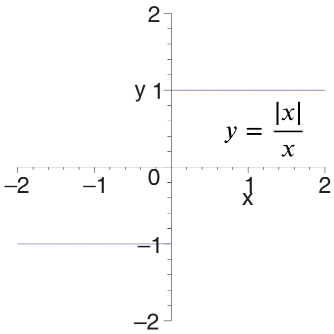
e) Since $\lim_{x\to4^-} |x-4| = \lim_{x\to4^+} |x-4|$, $\lim_{x\to4} |x-4| = 0$.



f)
$$\begin{aligned}\lim_{x\to0^-}\frac{|x|}{x} &= \lim_{x\to0^-}\frac{-x}{x}\\ &= \lim_{x\to0^-}-1\\ &= -1\end{aligned}$$

g)
$$\begin{aligned}\lim_{x\to0^+}\frac{|x|}{x} &= \lim_{x\to0^+}\frac{x}{x}\\ &= \lim_{x\to0^+}1\\ &= 1\end{aligned}$$

h) Since $\lim\limits_{x\to0^-}\frac{|x|}{x} \neq \lim\limits_{x\to0^+}\frac{|x|}{x}$, $\lim\limits_{x\to0}\frac{|x|}{x}$ does not exist.



Section 3.3 Page 154 Question 11

a)
$$\begin{aligned}\lim_{x\to3}\frac{x^2-9}{x-3} &= \lim_{x\to3}\frac{(x+3)(x-3)}{x-3}\\ &= \lim_{x\to3}(x+3),\ x\neq3\\ &= 6\end{aligned}$$

b)
$$\begin{aligned}\lim_{x\to2}\frac{x-2}{x^2-4} &= \lim_{x\to2}\frac{x-2}{(x+2)(x-2)}\\ &= \lim_{x\to2}\frac{1}{x+2},\ x\neq2\\ &= \frac{1}{4}\end{aligned}$$

c)
$$\begin{aligned}\lim_{x\to4}\frac{x^2-6x+8}{x-4} &= \lim_{x\to4}\frac{(x-2)(x-4)}{x-4}\\ &= \lim_{x\to4}(x-2),\ x\neq4\\ &= 2\end{aligned}$$

d)
$$\begin{aligned}\lim_{x\to-2}\frac{x+2}{x^2-3x-10} &= \lim_{x\to-2}\frac{x+2}{(x-5)(x+2)}\\ &= \lim_{x\to-2}\frac{1}{x-5},\ x\neq4,\ x\neq-2\\ &= -\frac{1}{7}\end{aligned}$$

e)
$$\begin{aligned}\lim_{x\to-3}\frac{2x^2+7x+3}{x^2-9} &= \lim_{x\to-3}\frac{(2x+1)(x+3)}{(x-3)(x+3)}\\ &= \lim_{x\to-3}\frac{2x+1}{x-3},\ x\neq-3\\ &= \frac{-6+1}{-6}\\ &= \frac{5}{6}\end{aligned}$$

f)
$$\begin{aligned}\lim_{x\to4}\frac{x^2+2x-24}{x^2-6x+8} &= \lim_{x\to4}\frac{(x+6)(x-4)}{(x-2)(x-4)}\\ &= \lim_{x\to4}\frac{x+6}{x-2},\ x\neq4\\ &= \frac{10}{2}\\ &= 5\end{aligned}$$

g)
$$\begin{aligned}\lim_{x\to1}\frac{5x^2-3x-2}{3x^2-7x+4} &= \lim_{x\to1}\frac{(x-1)(5x+2)}{(x-1)(3x-4)}\\ &= \lim_{x\to1}\frac{5x+2}{3x-4},\ x\neq1\\ &= -7\end{aligned}$$

h)
$$\begin{aligned}\lim_{x\to2}\frac{x^3-8}{x^2-4} &= \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}\\ &= \lim_{x\to2}\frac{x^2+2x+4}{x+2},\ x\neq2\\ &= \frac{12}{4}\\ &= 3\end{aligned}$$

i)

$$\begin{aligned}\lim_{x\to3}\frac{x-3}{x^3-27}&=\lim_{x\to3}\frac{x-3}{(x-3)(x^2+3x+9)}\\&=\lim_{x\to3}\frac{1}{x^2+3x+9},\ x\neq3\\&=\frac{1}{9+9+9}\\&=\frac{1}{27}\end{aligned}$$

j)

$$\begin{aligned}\lim_{x\to-2}\frac{x^3+8}{x+2}&=\lim_{x\to-2}\frac{(x+2)(x^2-2x+4)}{x+2}\\&=\lim_{x\to-2}(x^2-2x+4),\ x\neq-2\\&=4+4+4\\&=12\end{aligned}$$

k)

$$\begin{aligned}\lim_{x\to4}\frac{x-4}{\sqrt{x}-2}&=\lim_{x\to4}\frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2}\\&=\lim_{x\to4}\sqrt{x}+2,\ x\neq4\\&=\sqrt{4}+2\\&=4\end{aligned}$$

l)

$$\begin{aligned}\lim_{x\to3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}&=\lim_{x\to3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\cdot\frac{3x}{3x}\\&=\lim_{x\to3}\frac{3-x}{3x(x-3)}\\&=\lim_{x\to3}\frac{-1}{3x},\ x\neq3\\&=-\frac{1}{9}\end{aligned}$$

Apply, Solve, Communicate

Section 3.3 Page 154 Question 12

a)

$$\begin{aligned}\lim_{x\to-2}[f(x)+g(x)]&=f(-2)+g(-2)\\&=3+(-1)\\&=2\end{aligned}$$

b)

$$\begin{aligned}\lim_{x\to-2}[f(x)g(x)]&=f(-2)g(-2)\\&=3(-1)\\&=-3\end{aligned}$$

c) The limit does not exist as it leads to the square root of a negative number.

d)

$$\begin{aligned}\lim_{x\to0}[f(x)-g(x)]&=f(0)-g(0)\\&=1-2\\&=-1\end{aligned}$$

e)

$$\begin{aligned}\lim_{x\to0}\frac{f(x)}{g(x)}&=\frac{f(0)}{g(0)}\\&=\frac{1}{2}\end{aligned}$$

f)

$$\begin{aligned}\lim_{x\to0}\frac{g(x)}{f(x)}&=\frac{g(0)}{f(0)}\\&=\frac{2}{1}\\&=2\end{aligned}$$

g) The limit does not exist since $\lim_{x\to4^-}f(x)\neq\lim_{x\to4^+}f(x)$ (jump discontinuity).

h) The limit does not exist since $\lim_{x\to4}f(x)$ does not exist.

i)

$$\begin{aligned}\lim_{x\to4^-}\frac{f(x)}{g(x)}&=\frac{\lim_{x\to4^-}f(x)}{\lim_{x\to4^-}g(x)}\\&=\frac{3^-}{1^-}\\&=3\end{aligned}$$

j)

$$\begin{aligned}\lim_{x\to7^-}\frac{g(x)}{f(x)}&=\frac{\lim_{x\to7^-}g(x)}{\lim_{x\to7^-}f(x)}\\&=\frac{0}{6}\\&=0\end{aligned}$$

Section 3.3 Page 155 Question 13

a) The polynomial $3x+1$ is continuous over its domain. Since $\lim_{x\to2}(3x+1)=7\neq8$, f is discontinuous at $x=2$.

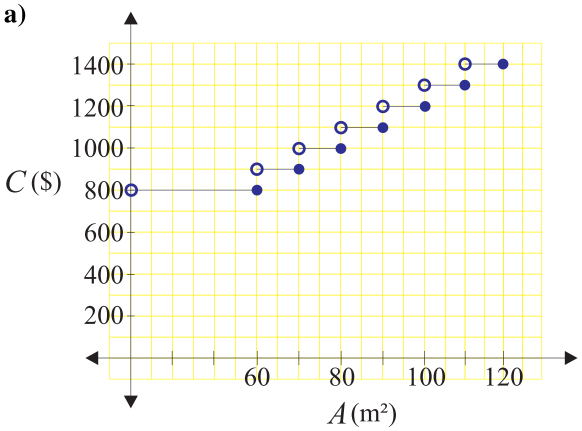
b) Since $x+2$, x^2+2, and $2x+4$ are polynomials, they are continuous over their domains. f is continuous at $x=0$ since $\lim_{x\to0^-}(x+2)=\lim_{x\to0^+}(x^2+2)=2$. f is discontinuous at $x=3$ since $\lim_{x\to3^-}(x^2+2)=11$ and $\lim_{x\to3^+}(2x+4)=10$.

c) x, $-x^3$, and $-x$ are continuous over all real numbers. Since $\lim_{x\to-1^-}x=-1$ and $\lim_{x\to-1^+}-x^3=1$, f is discontinuous at $x=-1$. Since $\lim_{x\to1^-}-x^3=-1$ and $\lim_{x\to1^+}-x=-1$, f is continuous at $x=1$.

d) Since x^2-1, x^2+2x+1, and x^2-2x+1 are polynomials, they are continuous over their domains. Since $\lim_{x\to-1^-}(x^2-1)=0^+$ and $\lim_{x\to-1^+}(x^2+2x+1)=0^+$, f is continuous at $x=-1$. Since $\lim_{x\to1^-}(x^2+2x+1)=4^-$ and $\lim_{x\to1^+}(x^2-2x+1)=0^+$, f is discontinuous at $x=1$.

Section 3.3 Page 155 Question 14

a)



b) i) $\lim_{A \to 60^-} C(A) = 800$

ii) $\lim_{A \to 60^+} C(A) = 900$

iii) $\lim_{A \to 60} C(A)$ does not exist.

iv) $\lim_{A \to 70^-} C(A) = 900$

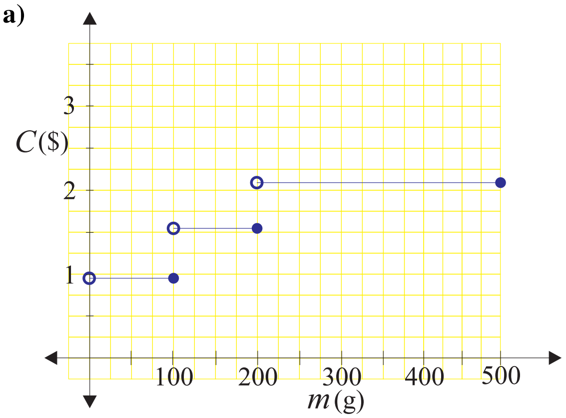
v) $\lim_{A \to 70^+} C(A) = 1000$

vi) $\lim_{A \to 70} C(A)$ does not exist.

c) Answers may vary.

Section 3.3 Page 155 Question 15

a)



b) i) $\lim_{m \to 100^-} C(m) = 0.94$

ii) $\lim_{m \to 100^+} C(m) = 1.55$

iii) $\lim_{m \to 100} C(m)$ does not exist.

iv) $\lim_{m \to 200^-} C(m) = 1.55$

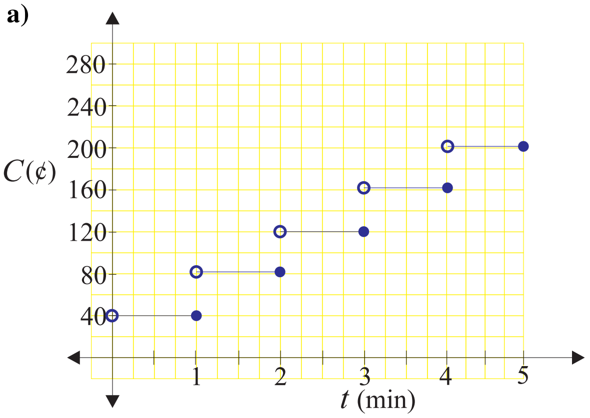
v) $\lim_{m \to 200^+} C(m) = 2.05$

vi) $\lim_{m \to 200} C(m)$ does not exist.

c) Answers may vary.

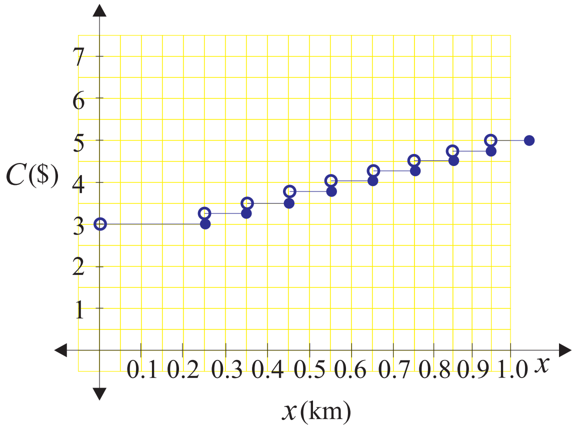
Section 3.3 Page 155 Question 16

a)



b) $\lim_{x \to a} C(t)$ is not defined for $a \in N$ because, at each value, the function has a jump discontinuity.

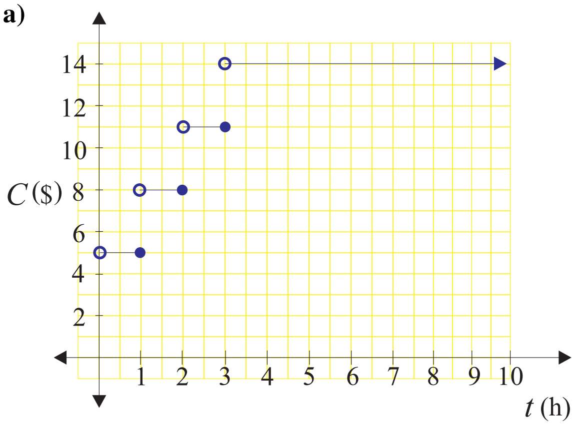
Section 3.3 Page 155 Question 17



The discontinuities are at $x = 0.25 + n(0.1)$ where n, is a whole number.

Section 3.3 Page 155 Question 18

a)

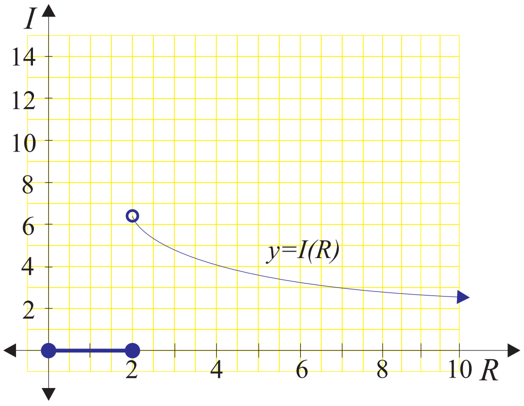


b) The discontinuities of $C(t)$ are at $t = 1, 2$, and 3.

Section 3.3 Page 155 Question 19

a) $I(R) = \frac{2(5+R)}{R}(1 - H(2-R));\ R \geq 0$

b)



c) **i)** $\lim_{R\to 2^-} I(R) = 0$

ii) $\lim_{R\to 2^+} I(R) = 7^-$

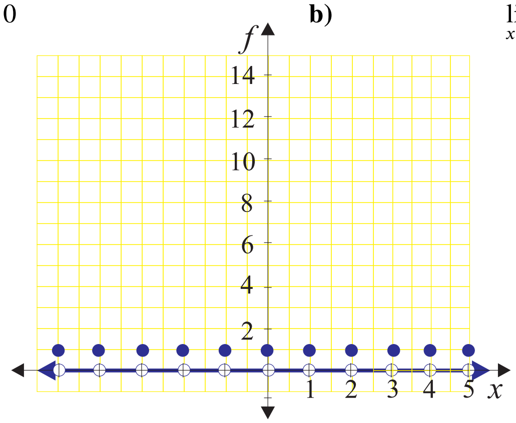
iii) $I(2) = 0$

iv) Since $\lim_{R\to 2^-} I(R) \neq \lim_{R\to 2^+} I(R)$, $\lim_{R\to 2} I(R)$ does not exist.

Section 3.3 Page 155 Question 20

a)

$$\begin{aligned}\lim_{x\to\frac{1}{2}} f(x) &= \lim_{x\to\frac{1}{2}} 0\\ &= 0\end{aligned}$$



b)

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{\frac{x\to 1}{2}} 0\\ &= 0\end{aligned}$$

Section 3.3 Page 155 Question 21

a)

$$\begin{aligned}\lim_{x\to\sqrt{3}} \frac{x^2-3}{3-\sqrt{6+x^2}} &= \lim_{x\to\sqrt{3}} \left[\frac{x^2-3}{3-\sqrt{6+x^2}} \cdot \frac{3+\sqrt{6+x^2}}{3+\sqrt{6+x^2}}\right]\\ &= \lim_{x\to\sqrt{3}} \frac{(x^2-3)\left(3+\sqrt{6+x^2}\right)}{9-(6+x^2)}\\ &= \lim_{x\to\sqrt{3}} \frac{(x^2-3)\left(3+\sqrt{6+x^2}\right)}{3-x^2}\\ &= \lim_{x\to\sqrt{3}} -\left(3+\sqrt{6+x^2}\right),\ x \neq 3\\ &= -\left(3+\sqrt{6+\left(\sqrt{3}\right)^2}\right)\\ &= -\left(3+\sqrt{9}\right)\\ &= -(3+3)\\ &= -6\end{aligned}$$

b)

$$\begin{aligned}\lim_{x\to 8}\frac{x-8}{\sqrt[3]{x}-2} &= \lim_{x\to 8}\frac{\left(\sqrt[3]{x}\right)^3-2^3}{\sqrt[3]{x}-2}\\ &= \lim_{x\to 8}\frac{\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x^2}+2\sqrt[3]{x}+4\right)}{\sqrt[3]{x}-2}\\ &= \lim_{x\to 8}(\sqrt[3]{x^2}+2\sqrt[3]{x}+4);\ x\neq 8\\ &= \sqrt[3]{64}+2\sqrt[3]{8}+4\\ &= 4+4+4\\ &= 12\end{aligned}$$

c)

$$\begin{aligned}\lim_{x\to 3}\frac{\sqrt{7-x}-2}{1-\sqrt{4-x}} &= \lim_{x\to 3}\left[\frac{\sqrt{7-x}-2}{1-\sqrt{4-x}}\cdot\frac{1+\sqrt{4-x}}{1+\sqrt{4-x}}\cdot\frac{\sqrt{7-x}+2}{\sqrt{7-x}+2}\right]\\ &= \lim_{x\to 3}\frac{(7-x-4)\left(1+\sqrt{4-x}\right)}{(1-(4-x))\left(\sqrt{7-x}+2\right)}\\ &= \lim_{x\to 3}\frac{(3-x)\left(1+\sqrt{4-x}\right)}{(x-3)\left(\sqrt{7-x}+2\right)}\\ &= \lim_{x\to 3}\frac{-\left(1+\sqrt{4-x}\right)}{\sqrt{7-x}+2};\ x\neq 3\\ &= \frac{-(1+1)}{2+2}\\ &= -\frac{1}{2}\end{aligned}$$

Section 3.3 Page 155 Question 22

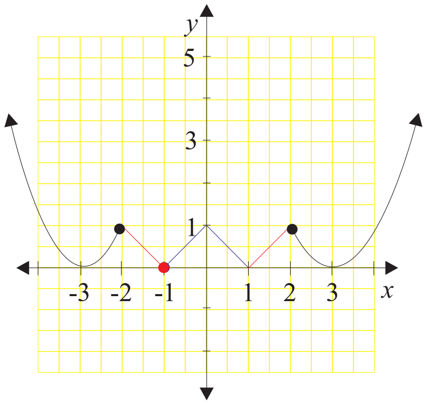
Answers may vary. For example, $f(x)=\dfrac{1+x}{x}$ and $g(x)=-\dfrac{1}{x}$.

Section 3.3 Page 155 Question 23

For simplicity, let $x=z^6$. As $x\to 1$, so, too, does $z\to 1$.

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1} &= \lim_{z\to 1}\frac{\sqrt{z^6}-1}{\sqrt[3]{z^6}-1}\\ &= \lim_{z\to 1}\frac{z^3-1}{z^2-1}\\ &= \lim_{z\to 1}\frac{(z-1)(z^2+z+1)}{(z-1)(z+1)}\\ &= \lim_{z\to 1}\frac{z^2+z+1}{z+1},\ z\neq 1\\ &= \frac{1+1+1}{1+1}\\ &= \frac{3}{2}\end{aligned}$$

Section 3.3 Page 155 Question 24



The function $f(x)$, is continuous over all real numbers.

Section 3.3 Page 155 Question 25

Both $(cx - 1)^3$ and $c^2x^2 - 1$ are continuous over all real numbers. Check at $x = 2$.

$$\begin{aligned} f(2) &= c^2(2)^2 - 1 \\ &= 4c^2 - 1 \\ \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-}(cx - 1)^3 \\ &= (2c - 1)^3 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+}(c^2x^2 - 1) \\ &= 4c^2 - 1 \end{aligned}$$

For $\lim\limits_{x\to 2} f(x)$ to exist,

$$\begin{aligned} (2c - 1)^3 &= 4c^2 - 1 \\ 8c^3 - 4c^2 + 2c - 1 &= 4c^2 - 1 \\ 8c^3 - 8c^2 + 2c &= 0 \\ c(4c^2 - 4c + 1) &= 0 \\ c(2c - 1)^2 &= 0 \\ c &= 0, \frac{1}{2} \end{aligned}$$

The function $f(x)$ is continuous over all real numbers if $c = 0$ or $c = \frac{1}{2}$.

3.4 Rates of Change

Apply, Solve, Communicate

Section 3.4 Page 168 Question 1

a) average velocity

$$\begin{aligned}\text{average velocity} &= \frac{\Delta y}{\Delta t}\\ &= \frac{3(3+h)^2-8-[3(3)^2-8]}{3+h-3}\\ &= \frac{3(9+6h+h^2)-27}{h}\\ &= \frac{27+18h+3h^2-27}{h}\\ &= 18+3h,\ h\neq 0\end{aligned}$$

The average velocity over the interval $t \in [3, 3+h]$ is $18+3h$.

b) Since

$$v(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$v(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \qquad (1)$$

Substitute the result from a) into (1).

$$\begin{aligned}&= \lim_{h\to 0}(18+3h),\ h\neq 0\\ &= 18\end{aligned}$$

The instantaneous velocity at $t = 3$ is 18.

Section 3.4 Page 168 Question 2

a) The average velocity over the interval $t \in [a, a+h]$ can be expressed as $\dfrac{f(a+h)-f(a)}{h}$.

b) The instantaneous velocity at $t = a$ can be expressed as,

$$\begin{aligned}\text{instantaneous velocity} &= \lim_{h\to 0}\text{ average velocity}\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\end{aligned}$$

Section 3.4 Page 168 Question 3

a) i)

$$\begin{aligned}s(2) &= -2(2)^2+3(2)+1\\ &= -8+6+1\\ &= -1 &(1)\\ s(1) &= -2(1)^2+3(1)+1\\ &= -2+3+1\\ &= 2 &(2)\end{aligned}$$

On the interval $t \in [1, 2]$,

$$\text{average velocity} = \frac{s(2)-s(1)}{2-1} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned}&= \frac{-1-2}{1}\\ &= -3\end{aligned}$$

Over $t \in [1, 2]$, the average velocity is -3 m/s.

ii)

$$\begin{aligned}s(1.5) &= -2(1.5)^2+3(1.5)+1\\ &= -4.5+4.5+1\\ &= 1 &(1)\\ s(1) &= -2(1)^2+3(1)+1\\ &= -2+3+1\\ &= 2 &(2)\end{aligned}$$

On the interval $t \in [1, 1.5]$,

$$\text{average velocity} = \frac{s(1.5)-s(1)}{1.5-1} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned}&= \frac{1-2}{0.5}\\ &= -2\end{aligned}$$

Over $t \in [1, 1.5]$, the average velocity is -2 m/s.

iii)

$$\begin{aligned} s(1.1) &= -2(1.1)^2 + 3(1.1) + 1 \\ &= -2.42 + 3.3 + 1 \\ &= 1.88 && (1) \\ s(1) &= -2(1)^2 + 3(1) + 1 \\ &= -2 + 3 + 1 \\ &= 2 && (2) \end{aligned}$$

On the interval $t \in [1, 1.1]$,

$$\text{average velocity} = \frac{s(1.1) - s(1)}{1.1 - 1} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{1.88 - 2}{0.1} \\ &= -1.2 \end{aligned}$$

Over $t \in [1, 1.1]$, the average velocity is -1.2 m/s.

iv)

$$\begin{aligned} s(1.05) &= -2(1.05)^2 + 3(1.05) + 1 \\ &= -2.205 + 3.15 + 1 \\ &= 1.945 && (1) \\ s(1) &= -2(1)^2 + 3(1) + 1 \\ &= -2 + 3 + 1 \\ &= 2 && (2) \end{aligned}$$

On the interval $t \in [1, 1.05]$,

$$\text{average velocity} = \frac{s(1.05) - s(1)}{1.05 - 1} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{1.945 - 2}{0.05} \\ &= -1.1 \end{aligned}$$

Over $t \in [1, 1.05]$, the average velocity is -1.1 m/s.

v)

$$\begin{aligned} s(1.01) &= -2(1.01)^2 + 3(1.01) + 1 \\ &= -2.0402 + 3.03 + 1 \\ &= 1.9898 && (1) \\ s(1) &= -2(1)^2 + 3(1) + 1 \\ &= -2 + 3 + 1 \\ &= 2 && (2) \end{aligned}$$

On the interval $t \in [1, 1.01]$,

$$\text{average velocity} = \frac{s(1.01) - s(1)}{1.01 - 1} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{1.9898 - 2}{0.01} \\ &= -1.02 \end{aligned}$$

Over $t \in [1, 1.01]$, the average velocity is -1.02 m/s.

b)

$$\begin{aligned} v(1) &= \lim_{h\to 0} \frac{s(1+h) - s(1)}{h} \\ &= \lim_{h\to 0} \frac{-2 - 4h - 2h^2 + 3 + 3h + 1 - 2}{h} \\ &= \lim_{h\to 0} \frac{-h - 2h^2}{h} \\ &= \lim_{h\to 0} -1 - 2h \\ &= -1 \end{aligned}$$

The instantaneous velocity at $t = 1$ is -1 m/s.

Section 3.4 Page 168 Question 4

a) i)

$$\begin{aligned} y(3) &= 40(3) - 4.9(3)^2 \\ &= 120 - 44.1 \\ &= 75.9 && (1) \\ y(2) &= 40(2) - 4.9(2)^2 \\ &= 80 - 19.6 \\ &= 60.4 && (2) \end{aligned}$$

On the interval $t \in [2, 3]$,

$$\text{average velocity} = \frac{y(3) - y(2)}{3 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{75.9 - 60.4}{1} \\ &= 15.5 \end{aligned}$$

Over $t \in [2, 3]$, the average velocity is 15.5 m/s.

ii)

$$\begin{aligned} y(2.5) &= 40(2.5) - 4.9(2.5)^2 \\ &= 100 - 30.625 \\ &= 69.375 && (1) \\ y(2) &= 40(2) - 4.9(2)^2 \\ &= 80 - 19.6 \\ &= 60.4 && (2) \end{aligned}$$

On the interval $t \in [2, 2.5]$,

$$\text{average velocity} = \frac{y(2.5) - y(2)}{2.5 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{69.375 - 60.4}{0.5} \\ &= 17.95 \end{aligned}$$

Over $t \in [2, 2.5]$, the average velocity is 17.95 m/s.

iii)

$$\begin{aligned} y(2.1) &= 40(2.1) - 4.9(2.1)^2 \\ &= 84 - 21.609 \\ &= 62.391 && (1) \\ y(2) &= 40(2) - 4.9(2)^2 \\ &= 80 - 19.6 \\ &= 60.4 && (2) \end{aligned}$$

On the interval $t \in [2, 2.1]$,

$$\text{average velocity} = \frac{y(2.1) - y(2)}{2.1 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{62.391 - 60.4}{0.1} \\ &= 19.91 \end{aligned}$$

Over $t \in [2, 2.1]$, the average velocity is 19.91 m/s.

iv)

$$\begin{aligned} y(2.05) &= 40(2.05) - 4.9(2.05)^2 \\ &= 82 - 20.592\,25 \\ &= 61.407\,75 && (1) \\ y(2) &= 40(2) - 4.9(2)^2 \\ &= 80 - 19.6 \\ &= 60.4 && (2) \end{aligned}$$

On the interval $t \in [2, 2.05]$,

$$\text{average velocity} = \frac{y(2.05) - y(2)}{2.05 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{61.407\,75 - 60.4}{0.05} \\ &= 20.155 \end{aligned}$$

Over $t \in [2, 2.05]$, the average velocity is 20.155 m/s.

v)

$$\begin{aligned} y(2.01) &= 40(2.01) - 4.9(2.01)^2 \\ &= 80.4 - 19.796\,49 \\ &= 60.603\,51 && (1) \\ y(2) &= 40(2) - 4.9(2)^2 \\ &= 80 - 19.6 \\ &= 60.4 && (2) \end{aligned}$$

On the interval, $t \in [2, 2.01]$,

$$\text{average velocity} = \frac{y(2.01) - y(2)}{2.01 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{60.603\,51 - 60.4}{0.01} \\ &= 20.351 \end{aligned}$$

Over $t \in [2, 2.01]$, the average velocity is 20.351 m/s.

b)

$$\begin{aligned} v(1) &= \lim_{h \to 0} \frac{y(2+h) - y(2)}{h} \\ &= \lim_{h \to 0} \frac{40(2+h) - 4.9(2+h)^2 - 60.4}{h} \\ &= \lim_{h \to 0} \frac{80 + 40h - 19.6 - 19.6h - 4.9h^2 - 60.4}{h} \\ &= \lim_{h \to 0} \frac{20.4h - 4.9h^2}{h} \\ &= \lim_{h \to 0} (20.4 - 4.9h),\ h \neq 0 \\ &= 20.4 \end{aligned}$$

The instantaneous velocity at $t = 2$ is 20.4 m/s.

Section 3.4 Page 168 Question 5

a) i)

$$\begin{aligned} s(4) &= -4^2 + 6(4) + 5 \\ &= -16 + 24 + 5 \\ &= 13 && (1) \\ s(2) &= -2^2 + 6(2) + 5 \\ &= -4 + 12 + 5 \\ &= 13 && (2) \end{aligned}$$

On the interval $t \in [2, 4]$,

$$\text{average velocity} = \frac{s(4) - s(2)}{4 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{13 - 13}{2} \\ &= 0 \end{aligned}$$

Over $t \in [2, 4]$, the average velocity is 0 m/s.

ii)

$$\begin{aligned} s(3) &= -3^2 + 6(3) + 5 \\ &= -9 + 18 + 5 \\ &= 14 && (1) \\ s(2) &= -2^2 + 6(2) + 5 \\ &= -4 + 12 + 5 \\ &= 13 && (2) \end{aligned}$$

On the interval $t \in [2, 3]$,

$$\text{average velocity} = \frac{s(3) - s(2)}{3 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{14 - 13}{1} \\ &= 1 \end{aligned}$$

Over $t \in [2, 3]$, the average velocity is 1 m/s.

a) iii)

$$\begin{aligned} s(2.5) &= -2.5^2 + 6(2.5) + 5 \\ &= -6.25 + 15 + 5 \\ &= 13.75 && (1) \\ s(2) &= -2^2 + 6(2) + 5 \\ &= -4 + 12 + 5 \\ &= 13 && (2) \end{aligned}$$

On the interval $t \in [2, 2.5]$,

$$\text{average velocity} = \frac{s(2.5) - s(2)}{2.5 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{13.75 - 13}{0.5} \\ &= 1.5 \end{aligned}$$

Over $t \in [2, 2.5]$, the average velocity is 1.5 m/s.

iv)

$$\begin{aligned} s(2.1) &= -2.1^2 + 6(2.1) + 5 \\ &= -4.41 + 12.6 + 5 \\ &= 13.19 && (1) \\ s(2) &= -2^2 + 6(2) + 5 \\ &= -4 + 12 + 5 \\ &= 13 && (2) \end{aligned}$$

On the interval, $t \in [2, 2.1]$,

$$\text{average velocity} = \frac{s(2.1) - s(2)}{2.1 - 2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{13.19 - 13}{0.1} \\ &= 1.9 \end{aligned}$$

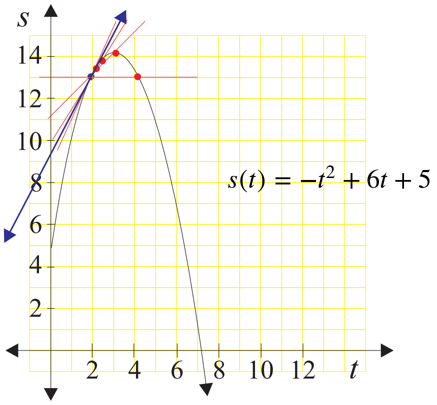
Over $t \in [2, 2.1]$, the average velocity is 1.9 m/s.

b)

$$\begin{aligned} v(2) &= \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} \\ &= \lim_{h \to 0} \frac{-(2+h)^2 + 6(2+h) + 5 - (-2^2 + 6(2) + 5)}{h} \\ &= \lim_{h \to 0} \frac{-4 - 4h - h^2 + 12 + 6h + 4 - 12}{h} \\ &= \lim_{h \to 0} \frac{-h^2 + 2h}{h} \\ &= \lim_{h \to 0} (-h + 2),\ h \neq 0 \\ &= 2 \end{aligned}$$

The instantaneous velocity at $t = 2$ is 2 m/s.

c)



d) The tangent appears in the graph above as the line with arrows at either end.

Section 3.4 Page 169 Question 6

a) i)

$$\begin{aligned} \text{average velocity} &= \frac{s(5) - s(3)}{5 - 3} \\ &= \frac{5^2 - 3^2}{2} \\ &= \frac{25 - 9}{2} \\ &= 8 \end{aligned}$$

Over $t \in [3, 5]$, the average velocity is 8 m/s.

ii)

$$\begin{aligned} \text{average velocity} &= \frac{s(4) - s(3)}{4 - 3} \\ &= \frac{4^2 - 3^2}{1} \\ &= 16 - 9 \\ &= 7 \end{aligned}$$

Over $t \in [3, 4]$, the average velocity is 7 m/s.

iii)

$$\begin{aligned} \text{average velocity} &= \frac{s(3.5) - s(3)}{3.5 - 3} \\ &= \frac{3.5^2 - 3^2}{0.5} \\ &= \frac{12.25 - 9}{0.5} \\ &= 6.5 \end{aligned}$$

Over $t \in [3, 3.5]$, the average velocity is 6.5 m/s.

iv)

$$\begin{aligned} \text{average velocity} &= \frac{s(3.1) - s(3)}{3.1 - 3} \\ &= \frac{3.1^2 - 3^2}{0.1} \\ &= \frac{9.61 - 9}{0.1} \\ &= 6.1 \end{aligned}$$

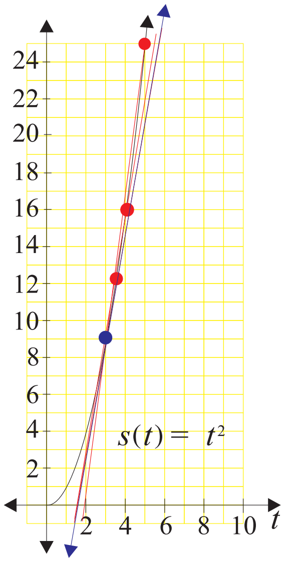
Over $t \in [3, 3.1]$, the average velocity is 6.1 m/s.

b)

$$\begin{aligned} v(3) &= \lim_{h\to 0}\frac{s(3+h)-s(3)}{h} \\ &= \lim_{h\to 0}\frac{(3+h)^2-3^2}{h} \\ &= \lim_{h\to 0}\frac{9+6h+h^2-9}{h} \\ &= \lim_{h\to 0}\frac{6h+h^2}{h} \\ &= \lim_{h\to 0}(6+h),\ h\neq 0 \\ &= 6 \end{aligned}$$

The instantaneous velocity at $t = 3$ is 6 m/s.

c)



d) The tangent appears in the diagram as the line with arrows at either end.

e) Answers may vary.

f) Answers may vary.

Section 3.4 Page 169 Question 7

$$\begin{aligned} v(a) &= \lim_{h\to 0}\frac{s(a+h)-s(a)}{h} \\ &= \lim_{h\to 0}\frac{0.4(a+h)^2+5(a+h)-(0.4a^2+5a)}{h} \\ &= \lim_{h\to 0}\frac{0.4\left(a^2+2ah+(h)^2\right)+5a+5h-0.4a^2-5a}{h} \\ &= \lim_{h\to 0}\frac{0.4a^2+0.8ah+0.4h^2+5h-0.4a^2}{h} \\ &= \lim_{h\to 0}\frac{h(0.8a+5+0.4h)}{h} \\ &= \lim_{h\to 0}(0.8a+5+0.4h),\ h\neq 0 \\ &= 0.8a+5 \end{aligned}$$

The velocity of the car at time $t = a$ is $0.8a+5$ metres per second.

The velocity at time $t = 1$ is $0.8(1)+5$ or 5.8 m/s.

The velocity at time $t = 2$ is $0.8(2)+5$ or 6.6 m/s.

The velocity at time $t = 3$ is $0.8(3)+5$ or 7.4 m/s.

Section 3.4 Page 169 Question 8

For the following solutions, let $\overline{T}_{[a,b]}$ represent the average rate of change of temperature over the closed interval from $t = a$ to $t = b$.

a) i)

$$\begin{aligned} \overline{T}_{[30,50]} &= \frac{T(50)-T(30)}{50-30} \\ &= \frac{5.3-6.5}{20} \\ &= \frac{-1.2}{20} \\ &= -0.06 \end{aligned}$$

The average rate of change of temperature with respect to time is −0.06 °C/min.

ii)

$$\begin{aligned} \overline{T}_{[30,40]} &= \frac{T(40)-T(30)}{40-30} \\ &= \frac{5.7-6.5}{10} \\ &= \frac{-0.8}{10} \\ &= -0.08 \end{aligned}$$

The average rate of change of temperature with respect to time is −0.08 °C/min.

iii)
$$\begin{aligned}\overline{T}_{[10,30]} &= \frac{T(30) - T(10)}{30 - 10}\\ &= \frac{6.5 - 12}{20}\\ &= \frac{-5.5}{20}\\ &= -0.275\end{aligned}$$

The average rate of change of temperature with respect to time is −0.275 °C/min.

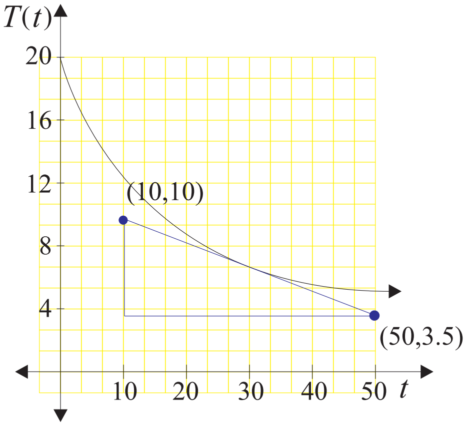
iv)
$$\begin{aligned}\overline{T}_{[20,30]} &= \frac{T(30) - T(20)}{30 - 20}\\ &= \frac{6.5 - 8.3}{10}\\ &= \frac{-1.8}{10}\\ &= -0.18\end{aligned}$$

The average rate of change of temperature with respect to time is −0.18 °C/min.

b) Answers may vary.

$$\begin{aligned}\text{instantaneous rate of change} &\doteq \frac{3.5 - 10}{50 - 10}\\ &= -0.1625\end{aligned}$$

Using the approximate endpoints on the tangent (10, 10) and (50, 3.5), the instantaneous rate of change of T at $t = 30$ is approximately −0.16°C/min.



Section 3.4 Page 169 Question 9

For the following solutions, let $\overline{T}_{[a,b]}$ represent the average rate of change of temperature over the closed interval from $t = a$ to $t = b$.

a) i)
$$\begin{aligned}\overline{T}_{[2,6]} &= \frac{T(6) - T(2)}{6 - 2}\\ &= \frac{9.0 - 5.0}{4}\\ &= \frac{4.0}{4}\\ &= 1.0\end{aligned}$$

The average rate of change of temperature with respect to time is 1.0 °C/h.

ii)
$$\begin{aligned}\overline{T}_{[0,2]} &= \frac{T(2) - T(0)}{2 - 0}\\ &= \frac{5.0 - 3.5}{2}\\ &= \frac{1.5}{2}\\ &= 0.75\end{aligned}$$

The average rate of change of temperature with respect to time is 0.75 °C/h.

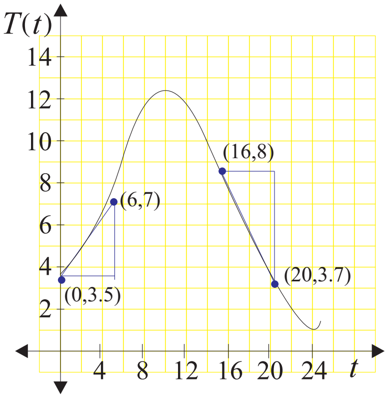
iii)
$$\begin{aligned}\overline{T}_{[2,4]} &= \frac{T(4) - T(2)}{4 - 2}\\ &= \frac{7.3 - 5.0}{2}\\ &= \frac{2.3}{2}\\ &= 1.15\end{aligned}$$

The average rate of change of temperature with respect to time is 1.15 °C/h.

b) Answers may vary.

$$\begin{aligned}\text{Instantaneous rate of change} &\doteq \frac{7-3.5}{6-0}\\ &\doteq 0.58\end{aligned}$$

Using the approximate endpoints of the tangent (0, 3.5) and (6, 7), the instantaneous rate of change of T at $t = 2$ is approximately 0.58°C/h.



c) i)

$$\begin{aligned}\overline{T}_{[18,22]} &= \frac{T(22)-T(18)}{22-18}\\ &= \frac{1.4-6.4}{4}\\ &= \frac{-5.0}{4}\\ &= -1.25\end{aligned}$$

The average rate of change of temperature with respect to time is −1.25°C/h.

ii)

$$\begin{aligned}\overline{T}_{[18,20]} &= \frac{T(20)-T(18)}{20-18}\\ &= \frac{4.0-6.4}{2}\\ &= \frac{-2.4}{2}\\ &= -1.2\end{aligned}$$

The average rate of change of temperature with respect to time is −1.2°C/h.

iii)

$$\begin{aligned}\overline{T}_{[16,18]} &= \frac{T(18)-T(16)}{18-16}\\ &= \frac{6.4-8.7}{2}\\ &= \frac{-2.3}{2}\\ &= -1.15\end{aligned}$$

The average rate of change of temperature with respect to time is −1.15°C/h.

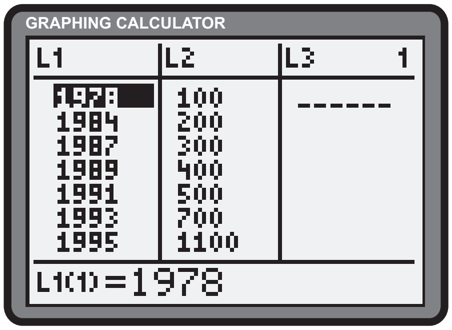
d) Answers may vary.

$$\begin{aligned}\text{Instantaneous rate of change} &\doteq \frac{3.7-8}{20-16}\\ &\doteq -1.075\end{aligned}$$

Using the approximate endpoints of the tangent (16, 8) and (20, 3.7), the instantaneous rate of change of T at $t = 18$ is approximately −1.08°C/h.

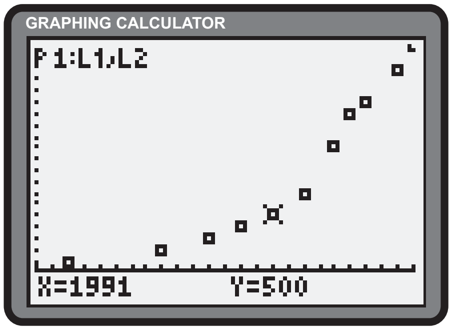
Section 3.4 Page 169 Question 10

a)



L1	L2	L3 1
1978	100	------
1984	200	
1987	300	
1989	400	
1991	500	
1993	700	
1995	1100	

L1(1)=1978



X=1991 Y=500

b) For the following solutions, let $\overline{N}_{[a,b]}$ represent the average rate of change in the number of outlets over the closed interval from $t = a$ to $t = b$.

i)

$$\begin{aligned}\overline{N}_{[1984,1991]} &= \frac{N(1991)-N(1984)}{1991-1984}\\ &= \frac{500-200}{7}\\ &\doteq 43\end{aligned}$$

The average rate of change in the number of outlets is approximately 43 outlets per year.

ii)

$$\begin{aligned}\overline{N}_{[1991,1997]} &= \frac{N(1997)-N(1991)}{1997-1991}\\ &= \frac{1500-500}{6}\\ &\doteq 167\end{aligned}$$

The average rate of change in the number of outlets is approximately 167 outlets per year.

iii)
$$\begin{aligned}\overline{N}_{[1989,1991]} &= \frac{N(1991)-N(1989)}{1991-1989}\\ &= \frac{500-400}{2}\\ &= \frac{100}{2}\\ &= 50\end{aligned}$$

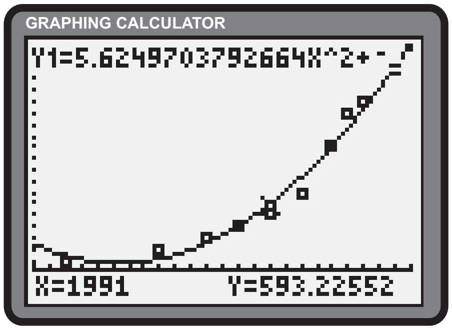
The average rate of change in the number of outlets is 50 outlets per year.

iv)
$$\begin{aligned}\overline{N}_{[1991,1993]} &= \frac{N(1993)-O(1991)}{1993-1991}\\ &= \frac{700-500}{2}\\ &= \frac{200}{2}\\ &\doteq 100\end{aligned}$$

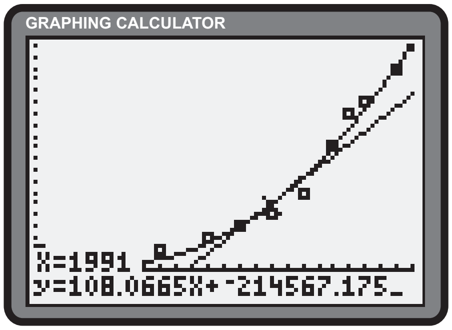
The average rate of change in the number of outlets is 100 outlets per year.

c) The quadratic regression feature yields an approximate equation of,

$$N(t) = 5.625t^2 - 22\,290t + 22\,080\,000$$



d) Using the regression function determined in part c), the **Tangent operation** yields an instantaneous rate of change of 108 outlets per year as compared the average of $\frac{43+50+100+167}{4}$ or 90 outlets per year from part b).



Section 3.4 Page 170 Question 11

a) For the following solutions, let $\overline{P}_{[a,b]}$ represent the average rate of change in the population over the closed interval from $t = a$ to $t = b$.

i)
$$\begin{aligned}\overline{P}_{[1994,1996]} &= \frac{P(1996)-P(1994)}{1996-1994}\\ &= \frac{423\,800-416\,500}{2}\\ &= \frac{7300}{2}\\ &= 3650\end{aligned}$$

The average rate of growth of the population is 3650 people per year.

ii)
$$\begin{aligned}\overline{P}_{[1995,1996]} &= \frac{P(1996)-P(1995)}{1996-1995}\\ &= \frac{423\,800-420\,100}{1}\\ &\doteq 3700\end{aligned}$$

The average rate of growth of the population is 3700 people per year.

iii)
$$\begin{aligned}\overline{P}_{[1996,1998]} &= \frac{P(1998)-P(1996)}{1998-1996}\\ &= \frac{436\,200-423\,800}{2}\\ &= \frac{12\,400}{2}\\ &= 6200\end{aligned}$$

The average rate of growth of the population is 6200 people per year.

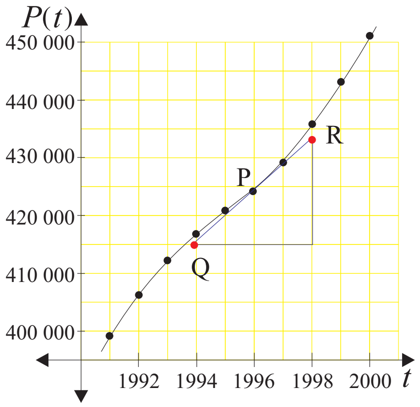
iv)
$$\begin{aligned}\overline{P}_{[1996,1997]} &= \frac{P(1997)-P(1996)}{1997-1996}\\ &= \frac{429\,800-423\,800}{1}\\ &\doteq 6000\end{aligned}$$

The average rate of growth of the population is 6000 people per year.

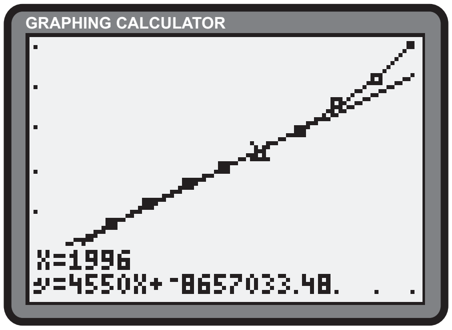
b) The points on the tangent drawn at P(1996, 423 800) are Q(1994, 415 000) and R(1998, 433 000). The slope, m, of the tangent at P can be estimated as follows.

$$\begin{aligned} m &\doteq \frac{433\,000 - 415\,000}{1998 - 1994} \\ &= \frac{1800}{4} \\ &= 4500 \end{aligned}$$

The instantaneous growth rate in 1996 was 4500 people per year.



c) The **Tangent operation** yields an instantaneous growth rate of approximately 4550 people per year in 1996.



```
X=1996
y=4550X+⁻8657033.48
```

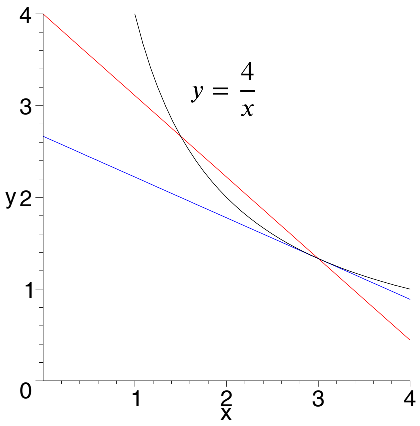
d) Averaging the results from part a) yields a growth rate of 4887.5 people per year.

Section 3.4 Page 170 Question 12

a)

$$\begin{aligned} \text{Average rate of change} &= \frac{y(3) - y(2)}{3 - 2} \\ &= \frac{\frac{4}{3} - \frac{4}{2}}{1} \\ &= \frac{8 - 12}{6} \\ &= \frac{-4}{6} \\ &= -\frac{2}{3} \end{aligned}$$

The average rate of change over $x \in [2, 3]$ is $-\frac{2}{3}$.



b)

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{x \to 3} \frac{\frac{4}{x} - \frac{4}{3}}{x - 3} \\ &= \lim_{x \to 3} \frac{12 - 4x}{3x(x - 3)} \\ &= \lim_{x \to 3} \frac{-4(x - 3)}{3x(x - 3)} \\ &= \lim_{x \to 3} \frac{-4}{3x} \\ &= -\frac{4}{9} \end{aligned}$$

The instantaneous rate of change at $x = 3$ is $-\frac{4}{9}$. The tangent is shown in the graph above.

Section 3.4 Page 170 Question 13

The volume, V, of the crystal with edge length x can be defined by the function $V(x) = x^3$. Let $\overline{V}_{[a,b]}$ represent the average rate of change of volume with respect to edge length over the closed interval from $x = a$ to $x = b$.

a) i)

$$\begin{aligned}\overline{V}_{[4,5]} &= \frac{5^3 - 4^3}{5 - 4} \\ &= \frac{125 - 64}{1} \\ &= 61\end{aligned}$$

The average rate of change of volume with respect to edge length over the interval $x \in [4, 5]$ is 61 mm^3/mm.

iii)

$$\begin{aligned}\overline{V}_{[4,4.01]} &= \frac{4.01^3 - 4^3}{4.01 - 4} \\ &= \frac{64.481\,201 - 64}{0.01} \\ &= 48.1201\end{aligned}$$

The average rate of change of volume with respect to edge length over the interval $x \in [4, 4.01]$ is 48.1201 mm^3/mm.

ii)

$$\begin{aligned}\overline{V}_{[4,4.1]} &= \frac{4.1^3 - 4^3}{4.1 - 4} \\ &= \frac{68.921 - 64}{0.1} \\ &= 49.21\end{aligned}$$

The average rate of change of volume with respect to edge length over the interval $x \in [4, 4.1]$ is 49.21 mm^3/mm.

b) The instantaneous rate of change of volume with respect to edge length when $x = 4$ is determined as follows.

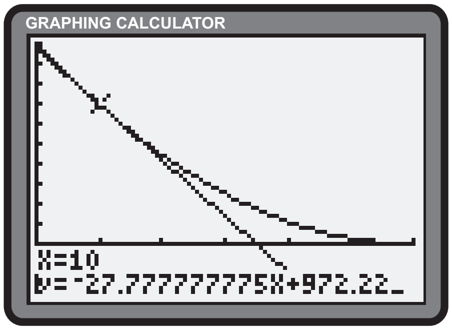
$$\begin{aligned}\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} &= \lim_{x \to 4} \frac{V(x) - V(4)}{x - 4} \\ &= \lim_{x \to 4} \frac{x^3 - 4^3}{x - 4} \\ &= \lim_{x \to 4} \frac{(x - 4)(x^2 + 4x + 16)}{x - 4} \\ &= \lim_{x \to 4}(x^2 + 4x + 16),\ x \neq 4 \\ &= 16 + 16 + 16 \\ &= 48\end{aligned}$$

The instantaneous rate of change of volume with respect to edge length when $x = 4$ is 48 mm^3/mm.

Section 3.4 Page 170 Question 14

The function can be simplified as follows.

$$\begin{aligned}V(t) &= 1000\left(1 - \frac{t}{60}\right)^2 \\ &= 1000\left(\frac{60 - t}{60}\right)^2 \\ &= \frac{1000}{3600}(60 - t)^2 \\ &= \frac{5}{18}(60 - t)^2\end{aligned}$$



GRAPHING CALCULATOR

```
X=10
Y=⁻27.777777775X+972.22_
```

The instantaneous rate of change at $t = 10$ can be determined.

$$\begin{aligned}\lim_{h \to 0} \frac{\Delta V}{h} &= \lim_{t \to 10} \frac{V(t) - V(10)}{t - 10} \\ &= \lim_{t \to 10} \frac{\frac{5}{18}(60 - t)^2 - \frac{5}{18}(60 - 10)^2}{t - 10} \\ &= \frac{5}{18} \cdot \lim_{t \to 10} \frac{(60 - t)^2 - 50^2}{t - 10} \\ &= \frac{5}{18} \cdot \lim_{t \to 10} \frac{(60 - t - 50)(60 - t + 50)}{t - 10} \\ &= \frac{5}{18} \cdot \lim_{t \to 10} \frac{(10 - t)(110 - t)}{t - 10} \\ &= \frac{5}{18} \cdot \lim_{t \to 10} -(110 - t),\ t \neq 10 \\ &= \frac{5}{18}(-100) \\ &\doteq -27.78\end{aligned}$$

The instantaneous rate of change of volume with respect to time at $t = 10$ is approximately −27.78 L/min.

Section 3.4 Page 170 Question 15

a) i)
$$\begin{aligned} P(110) &= 8000 + 20(110) + 0.10(110)^2 \\ &= 8000 + 2200 + 1210 \\ &= 11\,410 && (1) \\ P(100) &= 8000 + 20(100) + 0.10(100)^2 \\ &= 8000 + 2000 + 1000 \\ &= 11\,000 && (2) \end{aligned}$$

Let $\overline{P}_{[a,b]}$ represent the average rate of change of profit with respect to the number of units produced in the interval from $x = a$ to $x = b$.

$$\overline{P}_{[100,110]} = \frac{P(110) - P(100)}{110 - 100} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{11\,410 - 11\,000}{10} \\ &= \frac{410}{10} \\ &= 41 \end{aligned}$$

Over $x \in [100, 110]$, the average profit is \$41/unit.

iii)
$$\begin{aligned} P(101) &= 8000 + 20(101) + 0.10(101)^2 \\ &= 8000 + 2020 + 1020.1 \\ &= 11\,040.1 && (1) \\ P(100) &= 8000 + 20(100) + 0.10(100)^2 \\ &= 8000 + 2000 + 1000 \\ &= 11\,000 && (2) \end{aligned}$$

Let $\overline{P}_{[a,b]}$ represent the average rate of change of profit with respect to the number of units produced from interval from $x = a$ to $x = b$.

$$\overline{P}_{[100,101]} = \frac{P(101) - P(100)}{101 - 100} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{11\,040.1 - 11\,000}{1} \\ &= 40.1 \end{aligned}$$

Over $x \in [100, 101]$, the average profit is \$40.1/unit.

ii)
$$\begin{aligned} P(105) &= 8000 + 20(105) + 0.10(105)^2 \\ &= 8000 + 2100 + 1102.50 \\ &= 11\,202.5 && (1) \\ P(100) &= 8000 + 20(100) + 0.10(100)^2 \\ &= 8000 + 2000 + 1000 \\ &= 11\,000 && (2) \end{aligned}$$

Let $\overline{P}_{[a,b]}$ represent the average rate of change of profit with respect to the number of units produced from interval from $x = a$ to $x = b$.

$$\overline{P}_{[100,105]} = \frac{P(105) - P(100)}{105 - 100} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned} &= \frac{11\,202.5 - 11\,000}{5} \\ &= \frac{202.5}{5} \\ &= 40.5 \end{aligned}$$

Over $x \in [100, 105]$, the average profit is \$40.50/unit.

b) The instantaneous rate of change of profit with respect to the number of units produced at $x = 100$ is determined as follows.

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\Delta P}{\Delta x} &= \lim_{x \to 100} \frac{P(x) - P(100)}{x - 100} \\ &= \lim_{x \to 100} \frac{8000 + 20x + 0.10x^2 - 11\,000}{x - 100} \\ &= \lim_{x \to 100} \frac{0.10x^2 + 20x - 3000}{x - 100} \\ &= \lim_{x \to 100} \frac{(x - 100)(0.1x + 30)}{x - 100} \\ &= \lim_{x \to 100} 0.1x + 30 \\ &= 40 \end{aligned}$$

The instantaneous rate of change of profit with respect to the number of units produced at $x = 100$ is \$40/unit.

Section 3.4 Page 170 Question 16

a) Let $\overline{v}_{[a,b]}$ be the average velocity in the interval from $t = a$ to $t = b$.

i)
$$\begin{aligned} \overline{v}_{[2,3]} &= \frac{s(3) - s(2)}{3 - 2} \\ &= \frac{40(3) - 0.83(3)^2 - [40(2) - 0.83(2)^2]}{1} \\ &= 120 - 9(0.83) - 80 + 4(0.83) \\ &= 40 - 5(0.83) \\ &= 35.85 \end{aligned}$$

The average velocity over $t \in [2, 3]$ is 35.85 m/s.

ii)
$$\begin{aligned} \overline{v}_{[2,2.5]} &= \frac{s(2.5) - s(2)}{2.5 - 2} \\ &= \frac{40(2.5) - 0.83(2.5)^2 - [40(2) - 0.83(2)^2]}{0.5} \\ &= 2[100 - 6.25(0.83) - 80 + 4(0.83)] \\ &= 2[20 - 2.25(0.83)] \\ &= 36.265 \end{aligned}$$

The average velocity over $t \in [2, 2.5]$ is 36.265 m/s.

iii)

$$\begin{aligned}\bar{v}_{[2,2.1]} &= \frac{s(2.1) - s(2)}{2.1 - 2}\\ &= \frac{40(2.1) - 0.83(2.1)^2 - [40(2) - 0.83(2)^2]}{0.1}\\ &= 10[84 - 4.41(0.83) - 80 + 4(0.83)]\\ &= 10[4 - 0.41(0.83)]\\ &= 36.597\end{aligned}$$

The average velocity over $t \in [2, 2.1]$ is 36.597 m/s.

iv)

$$\begin{aligned}\bar{v}_{[2,2.05]} &= \frac{s(2.05) - s(2)}{2.05 - 2}\\ &= \frac{40(2.05) - 0.83(2.05)^2 - [40(2) - 0.83(2)^2]}{0.05}\\ &= 20[82 - 4.2025(0.83) - 80 + 4(0.83)]\\ &= 20[2 - 0.2025(0.83)]\\ &= 36.6385\end{aligned}$$

The average velocity over $t \in [2, 2.05]$ is 36.6385 m/s.

v)

$$\begin{aligned}\bar{v}_{[2,2.01]} &= \frac{s(2.01) - s(2)}{2.01 - 2}\\ &= \frac{40(2.01) - 0.83(2.01)^2 - [40(2) - 0.83(2)^2]}{0.01}\\ &= 100[80.4 - 4.0401(0.83) - 80 + 4(0.83)]\\ &= 20[0.4 - 0.0401(0.83)]\\ &= 36.6717\end{aligned}$$

The average velocity over $t \in [2, 2.01]$ is 36.6717 m/s.

b)

$$\begin{aligned}v(t) &= \lim_{h\to 0} \frac{\Delta s}{h}\\ v(2) &= \lim_{h\to 0} \frac{s(2+h) - s(2)}{h}\\ &= \lim_{h\to 0} \frac{40(2+h) - 0.83(2+h)^2 - [40(2) - 0.83(2)^2]}{h}\\ &= \lim_{h\to 0} \frac{40(2+h-2) - 0.83[(2+h)^2 - 2^2]}{h}\\ &= \lim_{h\to 0} \frac{40h - 0.83[(2+h-2)(2+h+2)]}{h}\\ &= \lim_{h\to 0} \frac{40h - 0.83h(4+h)}{h}\\ &= \lim_{h\to 0} 40 - 0.83(4+h)\\ &= 40 - 0.83(4)\\ &= 36.68\end{aligned}$$

The instantaneous velocity at $t = 2$ s is 36.68 m/s.

c)

$$\begin{aligned}
v(a) &= \lim_{h\to 0}\frac{s(a+h)-s(a)}{h}\\
&= \lim_{h\to 0}\frac{40(a+h)-0.83(a+h)^2-[40a-0.83a^2]}{h}\\
&= \lim_{h\to 0}\frac{40(a+h-a)-0.83[(a+h)^2-a^2]}{h}\\
&= \lim_{h\to 0}\frac{40h-0.83[(a+h-a)(a+h+a)]}{h}\\
&= \lim_{h\to 0}\frac{40h-0.83h(2a+h)}{h}\\
&= \lim_{h\to 0}[40-0.83(2a+h)],\ h\neq 0\\
&= 40-1.66a
\end{aligned}$$

The instantaneous velocity at $t = a$ seconds is $40 - 1.66a$ metres per second.

d) The projectile will hit the moon when $s = 0$.

$$\begin{aligned}
s(t) &= 0\\
40t-0.83t^2 &= 0\\
t(40-0.83t) &= 0
\end{aligned}$$

The projectile was launched at $t = 0$. Determine the other root.

$$\begin{aligned}
40-0.83t &= 0\\
t &= \frac{40}{0.83}\\
&\doteq 48.193
\end{aligned}$$

e)

$$\begin{aligned}
v(a) &= 40-1.66a\\
v(48.193) &= 40-1.66(48.193)\\
&= -40.000\,38
\end{aligned}$$

The projectile will hit the moon with a velocity of approximately −40 m/s.

The projectile will hit the surface of the moon at $t = 48.193$ s.

f) Determine the time taken to achieve the maximum height.

$$\begin{aligned}
v(t) &= 0\\
40-1.66t &= 0\\
1.66t &= 40\\
t &\doteq 24.096
\end{aligned}$$

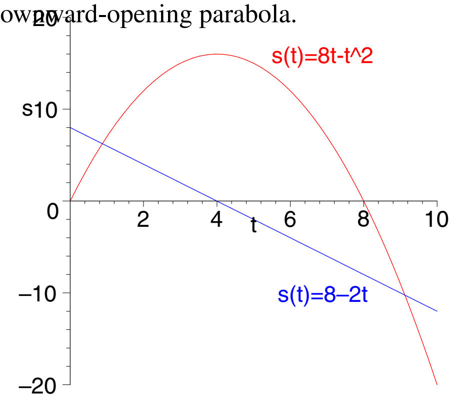
The projectile will reach its maximum height at approximately $t = 24.096$. Determine the height at this instant.

$$\begin{aligned}
s(t) &= 40t-0.83t^3\\
s(24.096) &= 40(24.096)-0.83(24.096)^2\\
&\doteq 481.93
\end{aligned}$$

The projectile reaches a maximum height of approximately 481.93 m.

Section 3.4 Page 171 Question 17

a) The position-time graph is depicted below as the downward-opening parabola.



b)
$$\begin{aligned}
v(a) &= \lim_{h\to 0}\frac{s(a+h)-s(a)}{h}\\
&= \lim_{h\to 0}\frac{8(a+h)-(a+h)^2-(8a-a^2)}{h}\\
&= \lim_{h\to 0}\frac{8(a+h-a)-[(a+h)^2-a^2]}{h}\\
&= \lim_{h\to 0}\frac{8h-(a+h-a)(a+h+a)}{h}\\
&= \lim_{h\to 0}(8-(2a+h)),\ h\neq 0\\
&= 8-2a
\end{aligned}$$

The velocity-time graph is depicted in the graph by the linear function $v(t) = 8 - 2t$.

Section 3.4 Page 171 Question 18

a)
$$\begin{aligned}
F &= \lim_{h\to 0}\frac{E(x_0+h)-E(x_0)}{h}\\
&= \lim_{h\to 0}\frac{\frac{90}{x_0+h}-\frac{90}{x}}{h}\\
&= 90\cdot\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x_0+h}-\frac{1}{x_0}\right)\\
&= 90\cdot\lim_{h\to 0}\frac{1}{h}\left(\frac{x_0-(x_0+h)}{x_0(x_0+h)}\right)\\
&= 90\cdot\lim_{h\to 0}\frac{1}{h}\cdot\frac{-h}{x_0(x_0+h)}\\
&= 90\cdot\lim_{h\to 0}\frac{-1}{x_0(x_0+h)},\ h\neq 0\\
&= -\frac{90}{{x_0}^2}
\end{aligned}$$

The electric force of repulsion between two spheres at a distance of $x = x_0$ metres is $-\dfrac{90}{{x_0}^2}$ newtons.

b) Substitute each value of x into the result obtained in part a).

i) The electric force of repulsion at a distance of 2.0 m is $-\dfrac{90}{2.0^2}$ or −22.5 N.

ii) The electric force of repulsion at a distance of 10.0 m is $-\dfrac{90}{10.0^2}$ or −0.9 N.

iii) The electric force of repulsion at a distance of 100.0 m is $-\dfrac{90}{100.0^2}$ or −0.009 N.

c) A distance of 2 cm is equivalent to 0.02 m. The electric force of repulsion at a distance of 0.02 m is $-\dfrac{90}{0.02^2}$ or 225 000 N.

d) Since the denominator gets larger and larger as the spheres move farther and farther apart, both the energy and the force get very small.

Section 3.4 Page 171 Question 19

No. A measure of an object's velocity includes it's direction. For example, a ball travelling upward has a positive value for velocity. The same ball falling to Earth has a negative value for velocity. The speed of the ball is simply the magnitude of the velocity, independent of the direction it is moving.

Review of Key Concepts

3.1 From Secants to Tangents

Section Review Page 173 Question 1

Answers may vary.

a) Let Q be $\left(1+\Delta x, (1+\Delta x)^2+(1+\Delta x)\right)$.

i) Develop a simplified expression for m_{PQ}.

$$\begin{aligned} m_{\text{PQ}} &= \frac{y_{\text{Q}}-y_{\text{P}}}{x_{\text{Q}}-x_{\text{P}}} \\ &= \frac{(1+\Delta x)^2+(1+\Delta x)-2}{1+\Delta x-1} \\ &= \frac{1+2\Delta x+(\Delta x)^2+1+\Delta x-2}{\Delta x} \\ &= \frac{3\Delta x+(\Delta x)^2}{\Delta x} \\ &= 3+\Delta x, \Delta x \neq 0 \end{aligned}$$

Δx	Q	$m_{\text{PQ}} = 3+\Delta x$
−1.0	$(0, 1)$	$3-1=2$
−0.1	$(0.9, 1.71)$	$3-0.1=2.9$
−0.01	$(0.99, 1.9701)$	$3-0.01=2.99$
−0.001	$(0.999, 1.997\,001)$	$3-0.001=2.999$
−0.0001	$(0.9999, 1.999\,700)$	$3-0.0001=2.9999$
0.0001	$(1.0001, 2.000\,300)$	$3+0.0001=3.0001$
0.001	$(1.001, 2.003\,001)$	$3+0.001=3.001$
0.01	$(1.01, 2.0301)$	$3+0.01=3.01$
0.1	$(1.1, 2.31)$	$3+0.1=3.1$
1.0	$(2, 6)$	$3+1=4$

ii) From the result of part i), the slope of the tangent at $P(1, 2)$ is estimated to be 3.

iii)

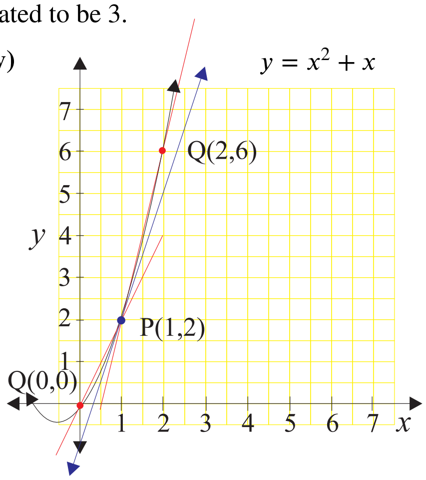
$$y-y_1 = m(x-x_1) \qquad (1)$$

Substitute $P(x_1, y_1) = (1, 2)$ and $m = 3$ into (1).

$$\begin{aligned} y-2 &= 3(x-1) \\ y-2 &= 3x-3 \\ y &= 3x-1 \end{aligned}$$

The equation of the tangent to the curve at P is $y = 3x-1$.

iv)



b) Let Q be $\left(2+\Delta x, (2+\Delta x)^3-2(2+\Delta x)\right)$.

i) Develop a simplified expression for m_{PQ}.

$$\begin{aligned} m_{\text{PQ}} &= \frac{y_{\text{Q}}-y_{\text{P}}}{x_{\text{Q}}-x_{\text{P}}} \\ &= \frac{(2+\Delta x)^3-2(2+\Delta x)-4}{2+\Delta x-2} \\ &= \frac{8+12\Delta x+6(\Delta x)^2+(\Delta x)^3-4-2\Delta x-4}{\Delta x} \\ &= \frac{10\Delta x+6(\Delta x)^2+(\Delta x)^3}{\Delta x} \\ &= 10+6\Delta x+(\Delta x)^2, \Delta x \neq 0 \end{aligned}$$

Δx	Q	$m_{\text{PQ}} = 10+6\Delta x+(\Delta x)^2$
−1.0	$(1, -1)$	5
−0.1	$(1.9, 3.059)$	9.41
−0.01	$(1.99, 3.900\,599)$	9.9401
−0.001	$(1.999, 3.990\,00)$	9.994 001
−0.0001	$(1.9999, 3.999\,000)$	9.999 400
0.0001	$(2.0001, 4.001\,000)$	10.000 600
0.001	$(2.001, 4.010\,006)$	10.006 001
0.01	$(2.01, 4.100\,601)$	10.0601
0.1	$(2.1, 5.061)$	10.61
1.0	$(3, 21)$	17

ii) From the results of part i), the slope of the tangent at $P(1, 2)$ is estimated to be 10.

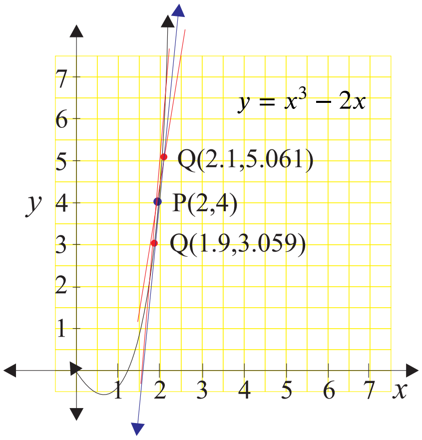
iii) $$y - y_1 = m(x - x_1) \qquad (1)$$

Substitute $P(x_1, y_1) = (2, 4)$ and $m = 10$ into (1).

$$y - 4 = 10(x - 2)$$
$$y - 4 = 10x - 20$$
$$y = 10x - 16$$

The equation of the tangent to the curve at P is $y = 10x - 16$.

iv)



c) Let Q be $\left(3 + \Delta x, \sqrt{3 + \Delta x + 1}\right)$.

i) Develop a simplified expression for m_{PQ}.

$$\begin{aligned} m_{PQ} &= \frac{y_Q - y_P}{x_Q - x_P} \\ &= \frac{\sqrt{4 + \Delta x} - 2}{3 + \Delta x - 3} \\ &= \frac{\sqrt{4 + \Delta x} - 2}{\Delta x} \cdot \frac{\sqrt{4 + \Delta x} + 2}{\sqrt{4 + \Delta x} + 2} \\ &= \frac{4 + \Delta x - 4}{\Delta x\left(\sqrt{4 + \Delta x} + 2\right)} \\ &= \frac{1}{\sqrt{4 + \Delta x} + 2};\ \Delta x \neq 0 \end{aligned}$$

ii) From the result of part i), the slope of the tangent at P(3, 2) is estimated to be 0.25.

Δx	Q	$m_{PQ} = \dfrac{1}{\sqrt{4 + \Delta x} + 2}$
−1.0	(2, 1.732 051)	0.267 949
−0.1	(2.9, 1.974 842)	0.251 582
−0.01	(2.99, 1.997 498)	0.250 156
−0.001	(2.999, 1.999 750)	0.250 016
−0.0001	(2.9999, 1.999 975)	0.250 002
0.0001	(3.0001, 2.000 025)	0.249 998
0.001	(3.001, 2.000 250)	0.249 984
0.01	(3.01, 2.002 498)	0.249 844
0.1	(3.1, 2.024 846)	0.248 457
1.0	(4, 2.236 068)	0.236 068

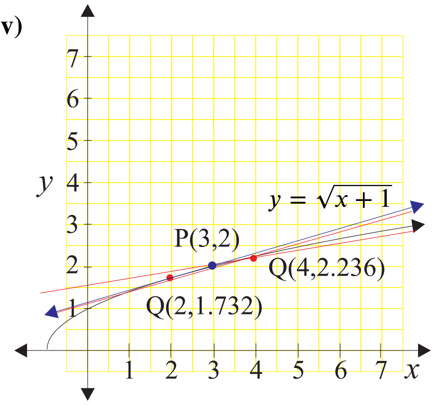
iii) $$y - y_1 = m(x - x_1) \qquad (1)$$

Substitute $P(x_1, y_1) = (3, 2)$ and $m = 0.25$ into (1).

$$y - 2 = 0.25(x - 3)$$
$$4y - 8 = x - 3$$
$$y = \frac{1}{4}x + \frac{5}{4}$$

The equation of the tangent to the curve at P is $y = \frac{1}{4}x + \frac{5}{4}$.

iv)



Section Review Page 173 Question 2

a) **i)** Use P(0.7, 2.896) and Q(0.6, 2.542).

$$\begin{aligned} m_{\mathrm{PQ}} &= \frac{2.542 - 2.896}{0.6 - 0.7} \\ &= \frac{-0.354}{-0.1} \\ &= 3.54 \end{aligned}$$

The slope of the secant PQ is 3.54.

ii) Use P(0.7, 2.896) and Q(0.8, 3.233).

$$\begin{aligned} m_{\mathrm{PQ}} &= \frac{3.233 - 2.896}{0.8 - 0.7} \\ &= \frac{0.337}{0.1} \\ &= 3.37 \end{aligned}$$

The slope of the secant PQ is 3.37.

b) Let m_{P} represent the slope of the tangent at P.

$$\begin{aligned} m_{\mathrm{P}} &= \frac{3.54 + 3.37}{2} \\ &= \frac{6.91}{2} \\ &= 3.455 \end{aligned}$$

The slope of the tangent at P(0.7, 2.896) is estimated to be 3.455.

c) **i)** Use P(1.5, 5.174) and Q(1.4, 4.937).

$$\begin{aligned} m_{\mathrm{PQ}} &= \frac{4.937 - 5.174}{1.4 - 1.5} \\ &= \frac{-0.237}{-0.1} \\ &= 2.37 \end{aligned}$$

The slope of the secant PQ is 2.37.

ii) Use P(1.5, 5.174) and Q(1.6, 5.400).

$$\begin{aligned} m_{\mathrm{PQ}} &= \frac{5.400 - 5.174}{1.6 - 1.5} \\ &= \frac{0.226}{0.1} \\ &= 2.26 \end{aligned}$$

The slope of the secant PQ is 2.26.

Let m_{P} represent the slope of the tangent at P.

$$\begin{aligned} m_{\mathrm{P}} &= \frac{2.37 + 2.26}{2} \\ &= \frac{4.63}{2} \\ &= 2.315 \end{aligned}$$

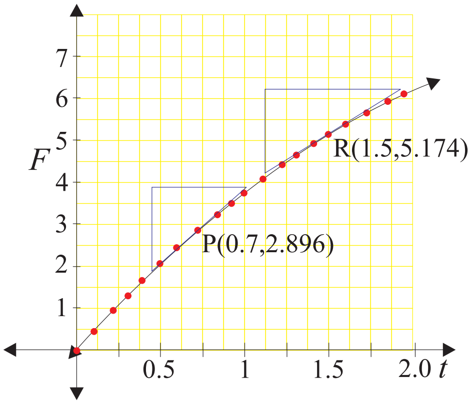
The slope of the tangent at P(1.5, 5.174) is estimated to be 2.315.

d) Answers may vary. Let two points on the tangent sketched at P(0.7, 2.896) be (0.45, 1.94) and (1.05, 3.95).

$$\begin{aligned} m_{\mathrm{P}} &= \frac{3.95 - 1.94}{1.05 - 0.45} \\ &= 3.35 \end{aligned}$$

The slope of the tangent at P is estimated to be 3.35.
Let the two points on the tangent at R(1.5, 5.174) be (1.1, 4.2) and (1.9, 6.4).

$$\begin{aligned} m_{\mathrm{P}} &= \frac{6.4 - 4.2}{1.9 - 1.1} \\ &= 2.75 \end{aligned}$$



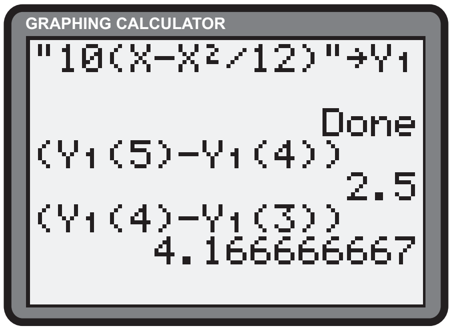
The slope of the tangent at R is estimated to be 2.75.

e) Answers may vary depending on the method used. The slope of the secant drawn from (0.5, 2.169) to (1.0, 3.859) is $\frac{3.859 - 2.169}{1.0 - 0.5}$ or 3.38. The slope of the secant drawn from (1.0, 3.859) to (1.5, 5.174) is $\frac{5.174 - 3.859}{1.5 - 1.0}$ or 2.63. Averaging the slopes of these secants yields an estimate of $\frac{3.38 + 2.63}{2}$ or 3.01 for the slope of the tangent at (1.0, 3.859).

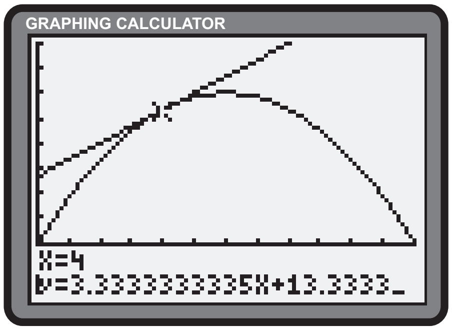
Section Review Page 173 Question 3

a) The slope of the function represents the rate at which the body is eliminating the medicine.

b) The graph below left suggests the slope of the secant cutting $A(t)$ at $t = 4$ and $t = 5$ is 2.5 and the slope of the secant from $t = 3$ to $t = 4$ is approximately 4.167. Averaging these results yields an estimate of $\frac{2.5 + 4.167}{2}$ or 3.335. This compares to a result of 3.333 using the **Tangent operation** of the graphing calculator (below right).

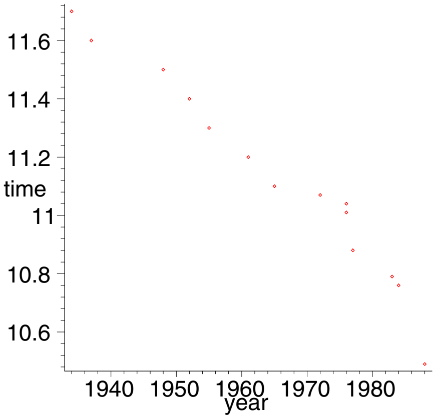


```
"10(X-X²/12)"→Y₁
                Done
(Y₁(5)-Y₁(4))
                 2.5
(Y₁(4)-Y₁(3))
         4.166666667
```



```
X=4
y=3.3333333335X+13.3333_
```

Section Review Page 173 Question 4



a) Answers will vary.

b) The average rate of change from (1937, 11.6) to (1948, 11.5) is $\frac{11.5 - 11.6}{1948 - 1937}$, or approximately −0.009 s/year.

The average rate of change from (1948, 11.5) to (1952, 11.4) is $\frac{11.4 - 11.5}{1952 - 1948}$ or approximately −0.025 s/year.

Averaging these two slopes yields an estimate for the rate of change in 1948 of $\frac{-0.009 - 0.25}{2}$ or −0.017 s/year.

The average rate of change from (1983, 10.79) to (1984, 10.76) is $\frac{10.76 - 10.79}{1984 - 1983}$ or −0.03 s/year.

The average rate of change from (1984, 10.76) to (1988, 10.49) is $\frac{10.49 - 10.76}{1988 - 1984}$ or −0.0675 s/year.

Averaging these two slopes yields an estimate for the rate of change in 1984 of $\frac{-0.03 - 0.0675}{2}$ or −0.048 75 s/year.

c) Answers may vary.

3.2 Using Limits to Find Tangents

Section Review Page 174 Question 5

a)

$$\begin{aligned} m &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{x^2 + 1 - 5}{x - 2} \\ &= \lim_{x\to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x - 2} \\ &= \lim_{x\to 2}(x + 2),\ x \neq 2 \\ &= 4 \end{aligned}$$

The slope of the tangent to $y = x^2 + 1$ at (2, 5) is 4.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= 4(x - 2) \\ y - 5 &= 4x - 8 \\ y &= 4x - 3 \end{aligned}$$

The equation of the tangent is $y = 4x - 3$.

b)

$$\begin{aligned} m &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \to 1} \frac{x^2 + 4x + 4 - 9}{x - 1} \\ &= \lim_{x \to 1} \frac{x^2 + 4x - 5}{x - 1} \\ &= \lim_{x \to 1} \frac{(x - 1)(x + 5)}{x - 1} \\ &= \lim_{x \to 1}(x + 5),\ x \neq 1 \\ &= 6 \end{aligned}$$

The slope of the tangent to $y = x^2 + 4x + 4$ at $(1, 9)$ is 6.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 9 &= 6(x - 1) \\ y - 9 &= 6x - 6 \\ y &= 6x + 3 \end{aligned}$$

The equation of the tangent is $y = 6x + 3$.

c)

$$\begin{aligned} m &= \lim_{x \to 6} \frac{f(x) - f(6)}{x - 6} \\ &= \lim_{x \to 6} \frac{\sqrt{x + 3} - 3}{x - 6} \cdot \frac{\sqrt{x + 3} + 3}{\sqrt{x + 3} + 3} \\ &= \lim_{x \to 6} \frac{x + 3 - 9}{(x - 6)(\sqrt{x + 3} + 3)} \\ &= \lim_{x \to 6} \frac{1}{\sqrt{x + 3} + 3},\ x \neq 6 \\ &= \frac{1}{3 + 3} \\ &= \frac{1}{6} \end{aligned}$$

The slope of the tangent to $y = \sqrt{x + 3}$ at $(6, 3)$ is $\frac{1}{6}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= \frac{1}{6}(x - 6) \\ y &= \frac{1}{6}x + 2 \end{aligned}$$

The equation of the tangent is $y = \frac{1}{6}x + 2$.

d)

$$\begin{aligned} m &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{1 - x^3 - 1}{x} \\ &= \lim_{x \to 0} \frac{x^3}{x} \\ &= \lim_{x \to 0} x^2,\ x \neq 0 \\ &= 0 \end{aligned}$$

The slope of the tangent to $y = 1 - x^3$ at $(0, 1)$ is 0.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 0(x - 1) \\ y &= 1 \end{aligned}$$

The equation of the tangent is $y = 1$.

e)

$$\begin{aligned} m &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x \to 3} \frac{\frac{1}{x-2} - 1}{x - 3} \\ &= \lim_{x \to 3} \frac{1 - (x - 2)}{(x - 2)(x - 3)} \\ &= \lim_{x \to 3} \frac{3 - x}{(x - 2)(x - 3)} \\ &= \lim_{x \to 3} \frac{-1}{x - 2}, x \neq 3 \\ &= -1 \end{aligned}$$

The slope of the tangent to $y = \frac{1}{x - 2}$ at $(3, 1)$ is -1.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -1(x - 3) \\ y - 1 &= -x + 3 \\ y &= -x + 4 \end{aligned}$$

The equation of the tangent is $y = -x + 4$.

f)

$$\begin{aligned} m &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to 1}\frac{\frac{1}{\sqrt{x}}-1}{x-1} \\ &= \lim_{x\to 1}\frac{1-\sqrt{x}}{\sqrt{x}(x-1)} \\ &= \lim_{x\to 1}\frac{1-\sqrt{x}}{\sqrt{x}(\sqrt{x}-1)(\sqrt{x}+1)} \\ &= \lim_{x\to 1}\frac{-1}{\sqrt{x}(\sqrt{x}+1)},\ n \neq 1 \\ &= \frac{-1}{1(1+1)} \\ &= -\frac{1}{2} \end{aligned}$$

The slope of the tangent to $y = \frac{1}{\sqrt{x}}$ at $(1, 1)$ is $-\frac{1}{2}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -\frac{1}{2}(x-1) \\ y &= -\frac{1}{2}x + \frac{3}{2} \end{aligned}$$

The equation of the tangent is $y = -\frac{1}{2}x + \frac{3}{2}$.

Section Review Page 174 Question 6

a)

$$\begin{aligned} m &= \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} \\ &= \lim_{h\to 0}\frac{3-(-1+h)^2-2}{h} \\ &= \lim_{h\to 0}\frac{3-(1-2h+h^2)-2}{h} \\ &= \lim_{h\to 0}\frac{h(2-h)}{h} \\ &= \lim_{h\to 0} 2-h,\ h \neq 0 \\ &= 2 \end{aligned}$$

The slope of the tangent to $y = 3 - x^2$ at $(-1, 2)$ is 2.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 2(x-(-1)) \\ y - 2 &= 2x + 2 \\ y &= 2x + 4 \end{aligned}$$

The equation of the tangent is $y = 2x + 4$.

b)

$$\begin{aligned} m &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0}\frac{(2+h)^2-2(2+h)+1-1}{h} \\ &= \lim_{h\to 0}\frac{4+4h+h^2-4-2h}{h} \\ &= \lim_{h\to 0}\frac{2h+h^2}{h} \\ &= \lim_{h\to 0}\frac{h(2+h)}{h} \\ &= \lim_{h\to 0}(2+h),\ h \neq 0 \\ &= 2 \end{aligned}$$

The slope of the tangent to $y = x^2 - 2x + 1$ at $(2, 1)$ is 2.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x-2) \\ y - 1 &= 2x - 4 \\ y &= 2x - 3 \end{aligned}$$

The equation of the tangent is $y = 2x - 3$.

c)

$$\begin{aligned}
m &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(1+h)^2+3}-2}{h} \\
&= \lim_{h\to 0} \left[\frac{\sqrt{4+2h+h^2}-2}{h} \cdot \frac{\sqrt{4+2h+h^2}+2}{\sqrt{4+2h+h^2}+2}\right] \\
&= \lim_{h\to 0} \frac{4+2h+h^2-4}{h\left(\sqrt{4+2h+h^2}+2\right)} \\
&= \lim_{h\to 0} \frac{2+h}{\sqrt{4+2h+h^2}+2},\ h \neq 0 \\
&= \frac{2}{2+2} \\
&= \frac{1}{2}
\end{aligned}$$

The slope of the tangent to $y = \sqrt{x^2+3}$ at $(1, 2)$ is $\frac{1}{2}$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 2 &= \frac{1}{2}(x-1) \\
y &= \frac{1}{2}x + \frac{3}{2}
\end{aligned}$$

The equation of the tangent is $y = \frac{1}{2}x + \frac{3}{2}$.

d)

$$\begin{aligned}
m &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{4+h}-\frac{1}{4}}{h} \\
&= \lim_{h\to 0} \frac{4-(4+h)}{4h(4+h)} \\
&= \lim_{h\to 0} \frac{-h}{4h(4+h)} \\
&= \lim_{h\to 0} \frac{-1}{4(4+h)},\ h \neq 0 \\
&= -\frac{1}{16}
\end{aligned}$$

The slope of the tangent to $y = \frac{1}{x}$ at $\left(4, \frac{1}{4}\right)$ is $-\frac{1}{16}$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \frac{1}{4} &= -\frac{1}{16}(x-4) \\
y &= -\frac{1}{16}x + \frac{1}{2}
\end{aligned}$$

The equation of the tangent is $y = -\frac{1}{16}x + \frac{1}{2}$.

e)

$$\begin{aligned}
m &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1+h+1}{1+h+3}-\frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{h+2}{h+4}-\frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{2(h+2)-(h+4)}{2h(h+4)} \\
&= \lim_{h\to 0} \frac{2h+4-h-4}{2h(h+4)} \\
&= \lim_{h\to 0} \frac{1}{2(h+4)},\ h \neq 0 \\
&= \frac{1}{8}
\end{aligned}$$

The slope of the tangent to $y = \frac{x+1}{x+3}$ at $\left(1, \frac{1}{2}\right)$ is $\frac{1}{8}$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \frac{1}{2} &= \frac{1}{8}(x-1) \\
y &= \frac{1}{8}x + \frac{3}{8}
\end{aligned}$$

The equation of the tangent is $y = \frac{1}{8}x + \frac{3}{8}$.

f)

$$\begin{aligned}
m &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{\frac{3}{(2+h)^2-1}-1}{h} \\
&= \lim_{h\to 0} \frac{3-[(2+h)^2-1]}{h[(2+h)^2-1]} \\
&= \lim_{h\to 0} \frac{3-(4+4h+h^2-1)}{h[(2+h)^2-1]} \\
&= \lim_{h\to 0} \frac{-h(4+h)}{h[(2+h)^2-1]} \\
&= \lim_{h\to 0} -\frac{4+h}{(2+h)^2-1},\ h \neq 0 \\
&= -\frac{4}{4-1} \\
&= -\frac{4}{3}
\end{aligned}$$

The slope of the tangent to $y = \dfrac{3}{x^2-1}$ at (2, 1) is $-\dfrac{4}{3}$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 1 &= -\frac{4}{3}(x-2) \\
y &= -\frac{4}{3}x + \frac{11}{3}
\end{aligned}$$

The equation of the tangent is $y = -\frac{4}{3}x + \frac{11}{3}$.

Section Review Page 174 Question 7

a) i) Let m be the slope of the tangent.

$$\begin{aligned}
m &= \lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)} \\
&= \lim_{x\to -1} \frac{(x-3)^2-16}{x+1} \\
&= \lim_{x\to -1} \frac{x^2-6x+9-16}{x+1} \\
&= \lim_{x\to -1} \frac{x^2-6x-7}{x+1} \\
&= \lim_{x\to -1} \frac{(x+1)(x-7)}{x+1} \\
&= \lim_{x\to -1} (x-7),\ x \neq 1 \\
&= -8
\end{aligned}$$

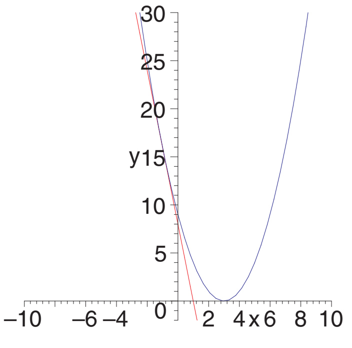
The slope of the tangent at (−1, 16) is −8.

ii) Determine the equation of the tangent at (−1, 16).

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 16 &= -8(x+1) \\
y - 16 &= -8x - 8 \\
y &= -8x + 8
\end{aligned}$$

The equation of the tangent is $y = -8x + 8$.

iii)



b) i) Let m be the slope of the tangent.

$$\begin{aligned}
m &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\
&= \lim_{x\to 2} \frac{x^3-8}{x-2} \\
&= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} \\
&= \lim_{x\to 2} (x^2+2x+4),\ x \neq 2 \\
&= 12
\end{aligned}$$

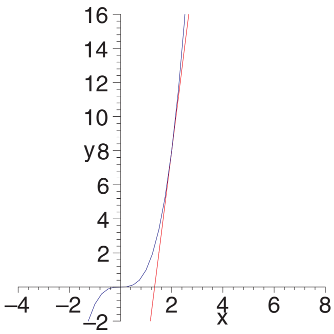
The slope of the tangent at (2, 8) is 12.

ii) Determine the equation of the tangent at (2, 8).

$$\begin{aligned}
y - 8 &= 12(x-2) \\
y - y_1 &= m(x - x_1) \\
y - 8 &= 12x - 24 \\
y &= 12x - 16
\end{aligned}$$

The equation of the tangent is $y = 12x - 16$.

iii)



c) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to 11} \frac{f(x)-f(11)}{x-11} \\ &= \lim_{x\to 11} \frac{\sqrt{x-2}-3}{x-11}\cdot\frac{\sqrt{x-2}+3}{\sqrt{x-2}+3} \\ &= \lim_{x\to 11} \frac{x-2-9}{(x-11)(\sqrt{x-2}+3)} \\ &= \lim_{x\to 11} \frac{1}{\sqrt{x-2}+3};\ x \neq 11 \\ &= \frac{1}{\sqrt{9}+3} \\ &= \frac{1}{6} \end{aligned}$$

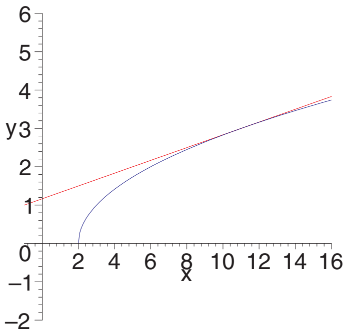
The slope of the tangent at $(11, 3)$ is $\frac{1}{6}$.

ii) Determine the equation of the tangent at $(11, 3)$.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-3 &= \frac{1}{6}(x-11) \\ y &= \frac{1}{6}x+\frac{7}{6} \end{aligned}$$

The equation of the tangent is $y = \frac{1}{6}x + \frac{7}{6}$.

iii)



d) i) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to -\frac{1}{2}} \frac{f(x)-f\left(-\frac{1}{2}\right)}{x-\left(-\frac{1}{2}\right)} \\ &= \lim_{x\to -\frac{1}{2}} \frac{\frac{1}{x+1}-2}{x+\frac{1}{2}} \\ &= \lim_{x\to -\frac{1}{2}} \frac{1-2(x+1)}{\left(x+\frac{1}{2}\right)(x+1)} \\ &= \lim_{x\to -\frac{1}{2}} \frac{-2x-1}{\left(x+\frac{1}{2}\right)(x+1)} \\ &= \lim_{x\to -\frac{1}{2}} \frac{-2\left(x+\frac{1}{2}\right)}{(x+1)\left(x+\frac{1}{2}\right)} \\ &= \lim_{x\to -\frac{1}{2}} \frac{-2}{x+1},\ x \neq \frac{1}{2} \\ &= \frac{-2}{\frac{1}{2}} \\ &= -4 \end{aligned}$$

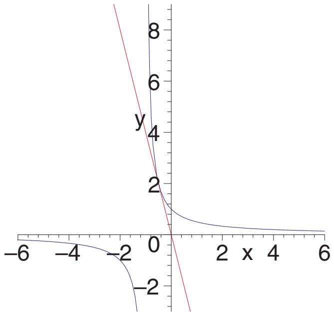
The slope of the tangent at $\left(-\frac{1}{2}, 2\right)$ is -4.

ii) Determine the equation of the tangent at $\left(-\frac{1}{2}, 2\right)$.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-2 &= -4\left(x+\frac{1}{2}\right) \\ y-2 &= -4x-2 \\ y &= -4x \end{aligned}$$

The equation of the tangent is $y = -4x$.

iii)



Section Review Page 174 Question 8

a) Let m be the slope of the tangent.

$$\begin{aligned} m &= \lim_{x\to a} \frac{y(x) - y(a)}{x - a} \\ &= \lim_{x\to a} \frac{x^2 - 3x - 10 - (a^2 - 3a - 10)}{x - a} \\ &= \lim_{x\to a} \frac{x^2 - a^2 - 3x + 3a}{x - a} \\ &= \lim_{x\to a} \frac{(x - a)(x + a) - 3(x - a)}{x - a} \\ &= \lim_{x\to a}(x + a - 3),\ x \neq a \\ &= 2a - 3 \end{aligned}$$

The slope of the tangent at $x = a$ is $2a - 3$.

b)

$$\begin{aligned} m_{-1} &= 2(-1) - 3 \\ &= -5 \\ m_0 &= 2(0) - 3 \\ &= -3 \\ m_1 &= 2(1) - 3 \\ &= -1 \\ m_2 &= 2(2) - 3 \\ &= 1 \\ m_3 &= 2(3) - 3 \\ &= 3 \end{aligned}$$

c) The slope of the tangent is zero when $2a - 3 = 0$ or when $x = \frac{3}{2}$. The y-coordinate of the point is $-\frac{49}{4}$.

Section Review Page 174 Question 9

a)
$$\begin{aligned} m &= \lim_{x\to 2.5} \frac{y(x) - y(2.5)}{x - 2.5} \\ &= \lim_{x\to 2.5} \frac{2x^2 - 8x + 12 - (2(2.5)^2 - 8(2.5) + 12)}{x - 2.5} \\ &= \lim_{x\to 2.5} \frac{2x^2 - 8x + 12 - 2(2.5)^2 + 8(2.5) - 12}{x - 2.5} \\ &= 2 \cdot \lim_{x\to 2.5} \frac{x^2 - 4x - (2.5)^2 + 4(2.5)}{x - 2.5} \\ &= 2 \cdot \lim_{x\to 2.5} \frac{x^2 - (2.5)^2 - 4(x - 2.5)}{x - 2.5} \\ &= 2 \cdot \lim_{x\to 2.5} \frac{(x - 2.5)(x + 2.5) - 4(x - 2.5)}{x - 2.5} \\ &= 2 \cdot \lim_{x\to 2.5} x + 2.5 - 4,\ x \neq 2.5 \\ &= 2 \cdot \lim_{x\to 2.5} x - 1.5 \\ &= 2(1) \\ &= 2 \end{aligned}$$

The slope of the tangent to $y = 2x^2 - 8x + 12$ at $x = 2.5$ is 2. At $x = 2.5$, $y = 2(2.5)^2 - 8(2.5) + 12$ or 4.5.

$$\begin{aligned} y - 4.5 &= 2(x - 2.5) \\ y - 4.5 &= 2x - 5 \\ y &= 2x - 0.5 \end{aligned}$$

The equation of the tangent is $y = 2x - 0.5$.

b) The x-intercept of the tangent in part a) is $\frac{0.5}{2}$ or $\frac{1}{4}$. This result would place the skater on a path to the net, so the goaltender should take evasive action.

3.3 The Limit of a Function

Section Review Page 174 Question 10

a) $\lim_{x\to -4^-} f(x) = 2^+$

b) $\lim_{x\to -4^+} f(x) = -3^+$

c) $f(-4) = -3$

d) $\lim_{x\to -4} f(x)$ does not exist because $\lim_{x\to -4^-} f(x) \neq \lim_{x\to -4^+} f(x)$.

e) $\lim_{x\to 2^-} f(x) = 3^-$

f) $\lim_{x\to 2^+} f(x) = 3^-$

g) $f(2) = 0$

h) $\lim_{x\to 2} f(x) = 3$

i) $\lim_{x\to 5^-} f(x) = -3^+$

j) $\lim_{x\to 5^+} f(x) = -3^+$

k) $f(5) = -3$

l) $\lim_{x\to 5} f(x) = -3^+$

Section Review Page 174 Question 11

a) **i)** $f(x)$ is discontinuous at $x = -4$ (jump discontinuity).

ii) $f(x)$ is discontinuous at $x = 2$ (removable discontinuity).

iii) $f(x)$ is continuous at $x = 5$.

b) No, unless the type of discontinuity is known.

Section Review Page 175 Question 12

a) $\lim_{x \to -3^-} g(x) = 3^-$

b) $\lim_{x \to -3^+} g(x) = 2^+$

c) $g(-3) = 3$

d) $\lim_{x \to -3} g(x)$ does not exist because $\lim_{x \to -3^-} g(x) \neq \lim_{x \to -3^+} g(x)$.

e) $\lim_{x \to 2^-} g(x) = 4^-$

f) $\lim_{x \to 2^+} g(x) = 4^+$

g) $g(2) = 4$.

h) $\lim_{x \to 2} g(x) = 4$

i) $\lim_{x \to 4^-} g(x) = 3^+$

j) $\lim_{x \to 4^+} g(x) = 3^-$

k) $g(4) = 4$

l) $\lim_{x \to 4} g(x) = 3$

m) $g(5)$ does not exist (hole).

n) $\lim_{x \to 5^+} g(x)$ does not exist since $g(x)$ is not defined for $x \geq 5$.

Section Review Page 175 Question 13

a) **i)** $g(x)$ is discontinuous at $x = -3$ (jump discontinuity).

ii) $g(x)$ is continuous at $x = 2$.

iii) $g(x)$ is discontinuous at $x = 4$ (removable discontinuity).

iv) $g(x)$ is discontinuous at $x = 5$ because the function is not defined for $x \geq 5$.

b) **i)** The value of $g(a)$ does not affect the value of $\lim_{x \to a} g(x)$.

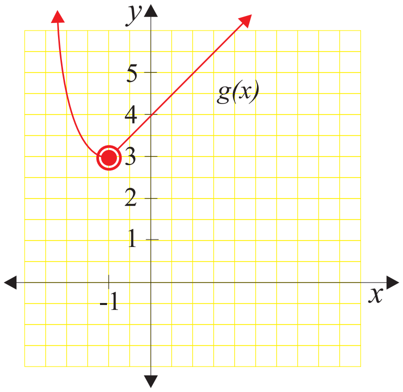
ii) $g(x)$ is continuous at $x = a$ if and only if $g(a) = \lim_{x \to a} g(x)$.

Section Review Page 175 Question 14

a)
$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} (x^2 + 2)$$
$$= 3^+$$

b)
$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} (x + 4)$$
$$= 3^+$$

c)
$$\lim_{x \to -1} g(x) = 3^+$$

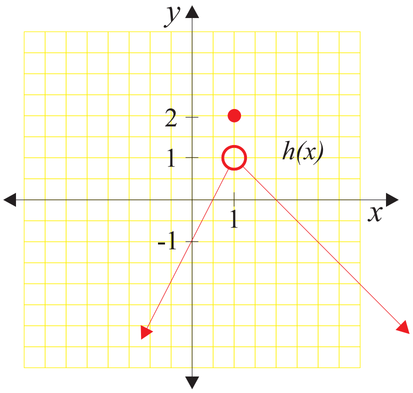


Section Review Page 175 Question 15

a)
$$\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} (2x - 1)$$
$$= 1^-$$

b)
$$\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} (-x + 2)$$
$$= 1^-$$

c)
$$\lim_{x \to -1} h(x) = 1^-$$

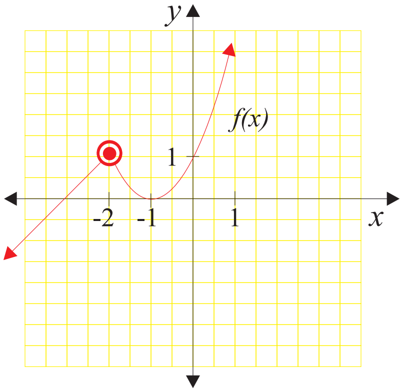


Section Review Page 175 Question 16

a)
$$\begin{aligned}\lim_{x\to-2^-} f(x) &= \lim_{x\to-2^-}(x+3)\\ &= 1^-\end{aligned}$$

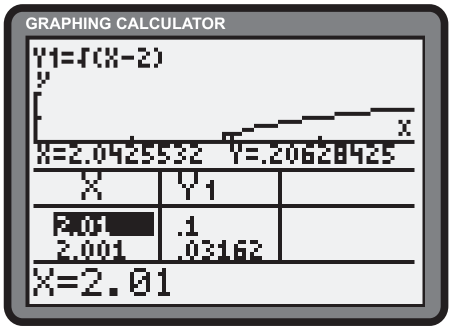
b)
$$\begin{aligned}\lim_{x\to-2^+} f(x) &= \lim_{x\to-2^+}(x+1)^2\\ &= 1^-\end{aligned}$$

c)
$$\lim_{x\to-2} h(x) = 1^-$$



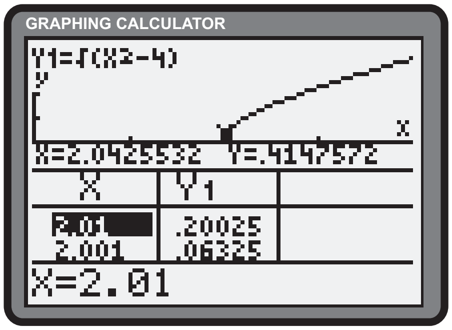
Section Review Page 175 Question 17

a) Defining $Y_1 = \sqrt{x-2}$ and using the Table feature of the calculator suggests $\lim_{x\to2^+}\sqrt{x-2} = 0^+$.



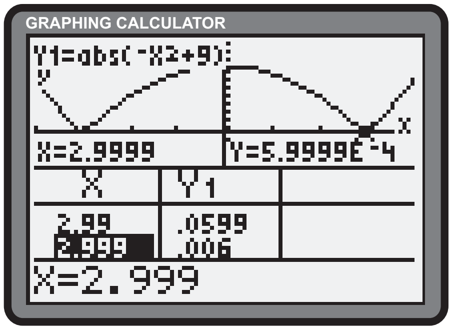
```
Y1=√(X-2)
X=2.0425532  Y=.20628425
  X    | Y1
 2.01  | .1
 2.001 | .03162
X=2.01
```

b) Defining $Y_1 = \sqrt{x^2-4}$ and using the Table feature of the calculator suggests $\lim_{x\to2^+}\sqrt{x^2-4} = 0^+$.



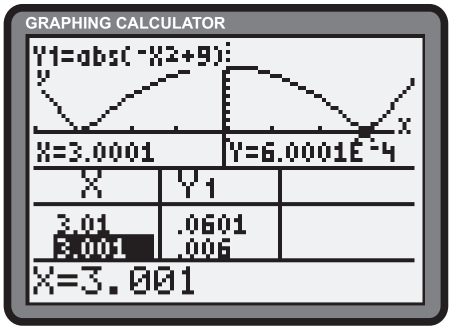
```
Y1=√(X²-4)
X=2.0425532  Y=.4147572
  X    | Y1
 2.01  | .20025
 2.001 | .06325
X=2.01
```

c) Defining $Y_1 = \left|-x^2+9\right|$ and using the Table feature of the calculator suggests $\lim_{x\to3^-}\left|-x^2+9\right| = 0^+$.



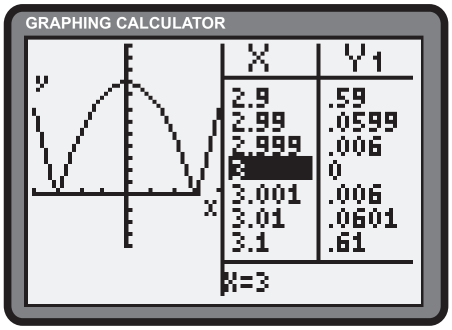
```
Y1=abs(-X²+9)
X=2.9999    Y=5.9999E-4
  X    | Y1
 2.99  | .0599
 2.999 | .006
X=2.999
```

d) Defining $Y_1 = \left|-x^2+9\right|$ and using the Table feature of the calculator suggests $\lim_{x\to3^+}\left|-x^2+9\right| = 0^+$.



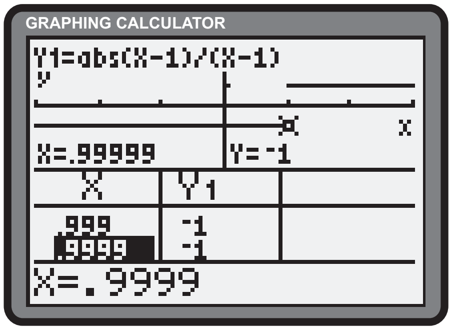
```
Y1=abs(-X²+9)
X=3.0001    Y=6.0001E-4
  X    | Y1
 3.01  | .0601
 3.001 | .006
X=3.001
```

e) Since $\lim\limits_{x\to3^-}\left|-x^2+9\right| = \lim\limits_{x\to3^+}\left|-x^2+9\right| = 0^+$, $\lim\limits_{x\to3}\left|-x^2+9\right| = 0.$



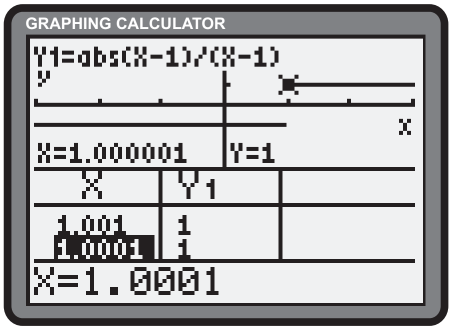
```
X       Y1
2.9     .59
2.99    .0599
2.999   .006
3       0
3.001   .006
3.01    .0601
3.1     .61
X=3
```

f) Defining $Y_1 = \dfrac{|x-1|}{x-1}$ and using the Table feature of the calculator suggests $\lim\limits_{x\to1^-}\dfrac{|x-1|}{x-1} = -1.$



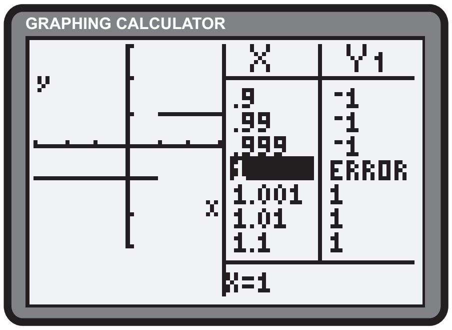
```
Y1=abs(X-1)/(X-1)
X=.99999        Y=-1
X       Y1
.999    -1
.9999   -1
X=.9999
```

g) Defining $Y_1 = \dfrac{|x-1|}{x-1}$ and using the Table feature in Split Screen mode of the graphing calculator suggests, $\lim\limits_{x\to1^+}\dfrac{|x-1|}{x-1} = 1.$



```
Y1=abs(X-1)/(X-1)
X=1.000001      Y=1
X       Y1
1.001   1
1.0001  1
X=1.0001
```

h) Since $\lim\limits_{x\to1^-}\dfrac{|x-1|}{x-1} \neq \lim\limits_{x\to1^+}\left|-x^2+9\right|$, $\lim\limits_{x\to1}\dfrac{|x-1|}{x-1}$ does not exist.



```
X       Y1
.9      -1
.99     -1
.999    -1
1       ERROR
1.001   1
1.01    1
1.1     1
X=1
```

Section Review Page 175 Question 18

a)

$$\begin{aligned}\lim_{x\to2}(4x-1) &= 4(2)-1\\ &= 7\end{aligned}$$

b)

$$\begin{aligned}\lim_{x\to3}(2x^2-4x+6) &= 2(3)^2-4(3)+6\\ &= 18-12+6\\ &= 12\end{aligned}$$

c)

$$\begin{aligned}\lim_{x\to-1}\frac{x^2+3x-4}{\sqrt{x+2}} &= \frac{(-1)^2+3(-1)-4}{\sqrt{-1+2}}\\ &= \frac{1-3-4}{\sqrt{1}}\\ &= -6\end{aligned}$$

d)

$$\begin{aligned}\lim_{x\to4}\frac{x^2+9}{\sqrt{x}+3} &= \frac{4^2+9}{\sqrt{4}+3}\\ &= \frac{25}{5}\\ &= 5\end{aligned}$$

e)

$$\begin{aligned}\lim_{x\to3}\frac{x^2-5x+6}{x^2+4x-21} &= \lim_{x\to3}\frac{(x-3)(x-2)}{(x-3)(x+7)}\\ &= \lim_{x\to3}\frac{x-2}{x+7},\ x\neq3\\ &= \frac{3-2}{3+7}\\ &= \frac{1}{10}\end{aligned}$$

f)

$$\begin{aligned}\lim_{x\to4}\frac{x^2-2x-8}{x^2-7x+12} &= \lim_{x\to4}\frac{(x-4)(x+2)}{(x-4)(x-3)}\\ &= \lim_{x\to4}\frac{x+2}{x-3},\ x\neq4\\ &= \frac{4+2}{4-3}\\ &= 6\end{aligned}$$

g)
$$\begin{aligned}\lim_{x\to4}\frac{x-4}{\sqrt{x^2-16}}&=\lim_{x\to4}\left[\frac{x-4}{\sqrt{x^2-16}}\cdot\frac{\sqrt{x^2-16}}{\sqrt{x^2-16}}\right]\\&=\lim_{x\to4}\frac{(x-4)\sqrt{x^2-16}}{x^2-16}\\&=\lim_{x\to4}\frac{(x-4)\sqrt{x^2-16}}{(x-4)(x+4)}\\&=\lim_{x\to4}\frac{\sqrt{x^2-16}}{x+4},\ x\neq4\\&=\frac{\sqrt{16-16}}{4+4}\\&=0\end{aligned}$$

h)
$$\begin{aligned}\lim_{h\to0}\frac{\sqrt{3+h}-\sqrt{3}}{h}&=\lim_{h\to0}\left[\frac{\sqrt{3+h}-\sqrt{3}}{h}\cdot\frac{\sqrt{3+h}+\sqrt{3}}{\sqrt{3+h}+\sqrt{3}}\right]\\&=\lim_{h\to0}\frac{3+h-3}{h\left(\sqrt{3+h}+\sqrt{3}\right)}\\&=\lim_{h\to0}\frac{1}{\sqrt{3+h}+\sqrt{3}},\ h\neq0\\&=\frac{1}{\sqrt{3}+\sqrt{3}}\\&=\frac{1}{2\sqrt{3}}\end{aligned}$$

i)
$$\begin{aligned}\lim_{x\to0}\frac{(2+x)^3-8}{x}&=\lim_{x\to0}\frac{8+12x+6x^2+x^3-8}{x}\\&=\lim_{x\to0}(12+6x+x^2),\ x\neq0\\&=12\end{aligned}$$

Section Review Page 175 Question 19

a) Since substitution yields a constant other than zero, divided by zero, the limit does not exist.

b)
$$\begin{aligned}\lim_{x\to0}\frac{\frac{4}{2+x}-2}{x}&=\lim_{x\to0}\frac{4-2(2+x)}{h(2+x)}\\&=\lim_{x\to0}\frac{4-4-2x}{x(2+x)}\\&=\lim_{x\to0}\frac{-2}{2+x},\ x\neq0\\&=-1\end{aligned}$$

c)
$$\begin{aligned}\lim_{x\to1}\frac{x^3-1}{x^3-2x^2-5x+6}&=\lim_{x\to1}\frac{(x-1)(x^2+x+1)}{(x-1)(x^2-x-6)}\\&=\lim_{x\to1}\frac{x^2+x+1}{x^2-x-6},\ x\neq1\\&=\frac{3}{-6}\\&=-\frac{1}{2}\end{aligned}$$

d)
$$\begin{aligned}\lim_{x\to-3^+}\sqrt{x^3+27}&=\sqrt{-27^++27}\\&=0^+\end{aligned}$$

e) $\lim_{x\to-3^-}\sqrt{x^3+27}$ does not exist because $\sqrt{x^3+27}$ is only defined for $x\geq3$.

f) $\lim_{x\to-3}\sqrt{x^3+27}$ does not exist because the left-hand limit in part e) does not exist.

g)
$$\begin{aligned}\lim_{x\to2}\frac{3x^2-5x-2}{2x^2-9x+10}&=\lim_{x\to2}\frac{(x-2)(3x+1)}{(x-2)(2x-5)}\\&=\lim_{x\to2}\frac{3x+-1}{2x-5},\ x\neq2\\&=\frac{7}{-1}\\&=-7\end{aligned}$$

h) Examine the left- and right-hand limits separately.

For $x\leq3$, $|3-x|=3-x$, hence,

$$\begin{aligned}\lim_{x\to3^-}\frac{|3-x|}{3-x}&=\lim_{x\to3^-}\frac{3-x}{3-x}\\&=\lim_{x\to3^-}1,\ x\neq3\\&=1\end{aligned}$$

For $x>3$, $|3-x|=-(3-x)$, hence,

$$\begin{aligned}\lim_{x\to3^+}\frac{|3-x|}{3-x}&=\lim_{x\to3^+}\frac{-(3-x)}{3-x}\\&=\lim_{x\to3^+}-1,\ x\neq3\\&=-1\end{aligned}$$

Since $\lim_{x\to3^-}\frac{|3-x|}{3-x}\neq\lim_{x\to3^+}\frac{|3-x|}{3-x}$, $\lim_{x\to3}\frac{|3-x|}{3-x}$ does not exist.

i)
$$\begin{aligned}\lim_{x\to -1}\frac{x^3+1}{x^3+3x^2+3x+1} &= \lim_{x\to -1}\frac{(x+1)(x^2-x+1)}{(x+1)^3}\\ &= \lim_{x\to -1}\frac{x^2-x+1}{(x+1)^2}\end{aligned}$$

Since substitution yields a constant other than 0, divided by zero, the limit does not exist.

3.4 Rates of Change

Section Review Page 176 Question 20

a) i)
$$\begin{aligned}h(3) &= 150-4.9(3)^2\\ &= 150-44.1\\ &= 105.9 && (1)\\ h(2) &= 150-4.9(2)^2\\ &= 150-19.6\\ &= 130.4 && (2)\end{aligned}$$

On the interval $t \in [2, 3]$,

$$\text{average velocity} = \frac{s(3)-s(2)}{3-2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned}&= \frac{105.9-130.4}{1}\\ &= -24.5\end{aligned}$$

Over $t \in [2, 3]$, the average velocity is -24.5 m/s.

ii)
$$\begin{aligned}h(2.1) &= 150-4.9(2.1)^2\\ &= 150-21.609\\ &= 128.391 && (1)\\ h(2) &= 150-4.9(2)^2\\ &= 150-19.6\\ &= 130.4 && (2)\end{aligned}$$

On the interval $t \in [2, 2.1]$,

$$\text{average velocity} = \frac{s(2.1)-s(2)}{2.1-2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned}&= \frac{128.391-130.4}{0.1}\\ &= -20.09\end{aligned}$$

Over $t \in [2, 2.1]$, the average velocity is -20.09 m/s.

iii)
$$\begin{aligned}h(2.01) &= 150-4.9(2.01)^2\\ &= 150-19.796\,49\\ &= 130.203\,51 && (1)\\ h(2) &= 150-4.9(2)^2\\ &= 150-19.6\\ &= 130.4 && (2)\end{aligned}$$

On the interval $t \in [2, 2.01]$,

$$\text{average velocity} = \frac{s(2.01)-s(2)}{2.01-2} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned}&= \frac{130.203\,51-130.4}{0.01}\\ &= -19.649\end{aligned}$$

Over $[2, 2.01]$, the average velocity is -19.649 m/s.

b)
$$\begin{aligned}v(2) &= \lim_{\Delta t\to 0}\frac{\Delta h}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{h(2+\Delta t)-h(2)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{150-4.9(2+\Delta t)^2-(150-4.9(2)^2)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{-4.9[(2+\Delta t)^2-2^2]}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{-4.9(2+\Delta t-2)(2+\Delta t+2)}{\Delta t}\\ &= \lim_{\Delta t\to 0}[-4.9(4+\Delta t)],\ \Delta t \neq 0\\ &= -19.6\end{aligned}$$

The instantaneous velocity at $t = 2$ is -19.6 m/s.

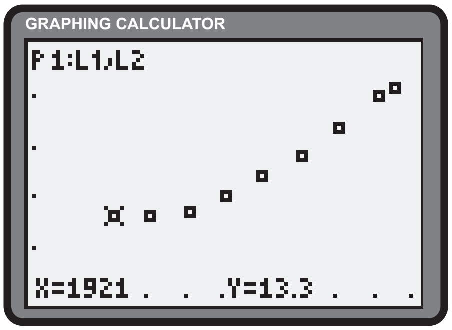
Section Review Page 176 Question 21

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\Delta P}{\Delta t} &= \lim_{\Delta t\to 0}\frac{P(5+\Delta t)-P(5)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{(5+\Delta t)^2-0.7(5+\Delta t)+10-(5^2-0.7(5)+10)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{25+10\Delta t+(\Delta t)^2-3.5-0.7\Delta t-25+3.5}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\Delta t(\Delta t+9.3)}{\Delta t}\\ &= \lim_{\Delta t\to 0}(\Delta t+9.3),\ \Delta t\neq 0\\ &= 9.3\end{aligned}$$

The rate of change of the population after 5 s is 9.3 bacteria/s.

Section Review Page 176 Question 22

a)



P1:L1,L2

X=1921 . . .Y=13.3 . . .

b) For the following solutions, let $\overline{E}_{[a,b]}$ represent the average rate of change in the life expectancy over the closed interval from $t = a$ to $t = b$.

i)
$$\begin{aligned}\overline{E}_{[1941,1961]} &= \frac{E(1961)-E(1941)}{1961-1941}\\ &= \frac{14.8-13.4}{20}\\ &= \frac{1.4}{20}\\ &= 0.07\end{aligned}$$

The average rate of change in the life expectancy is 0.07 years of life per year.

ii)
$$\begin{aligned}\overline{E}_{[1921,1941]} &= \frac{E(1941)-E(1921)}{1941-1921}\\ &= \frac{13.4-13.3}{20}\\ &= \frac{0.1}{20}\\ &= 0.005\end{aligned}$$

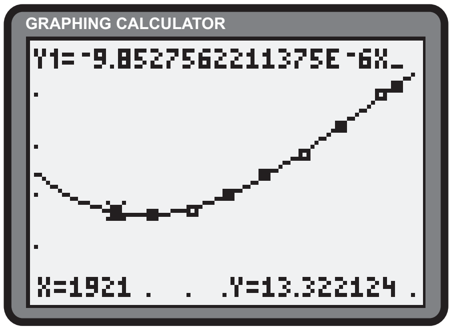
The average rate of change in the life expectancy is 0.005 years of life per year.

iii)
$$\begin{aligned}\overline{E}_{[1931,1941]} &= \frac{E(1941)-E(1931)}{1941-1931}\\ &= \frac{13.4-13.3}{10}\\ &= \frac{0.1}{10}\\ &= 0.01\end{aligned}$$

The average rate of change in the life expectancy is 0.01 years of life per year.

c) The cubic regression feature yields a cubic function with equation

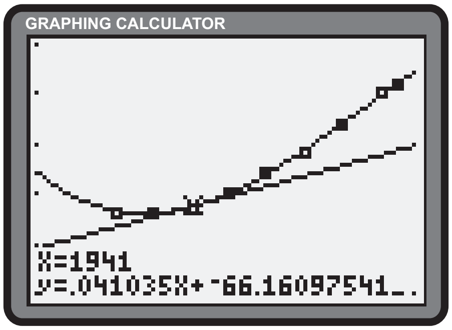
$$y = -9.85 \times 10^6 x^3 + 0.059x^2 - 117x + 77\,400$$



Y1=⁻9.8527562211375E⁻6X_

X=1921 . . .Y=13.322124 .

d) The **Tangent operation** returns an instantaneous rate of change of approximately 0.041 years of life per year. Averaging the results from part b) yields $\frac{0.07+0.005+0.01}{3}$ or approximately 0.0283 years of life per year in 1941.



X=1941

y=.041035X+⁻66.16097541_.

Chapter Test

Section Chapter Test Page 177 Question 1

a)

$$\begin{aligned} m_{PQ} &= \frac{y_Q - y_P}{x_Q - x_P} \\ &= \frac{12 - 5}{3 - 2} \\ &= 7 \end{aligned}$$

b)

$$\begin{aligned} m_P &= \lim_{h \to 0} \frac{(2 + h)^2 + 2(2 + h) - 3 - 5}{h} \\ &= \lim_{h \to 0} \frac{4 + 4h + (h)^2 + 4 + 2h - 8}{h} \\ &= \lim_{h \to 0} \frac{h(4 + h + 2)}{h} \\ &= \lim_{h \to 0} (6 + h),\ h \neq 0 \\ &= 6 \end{aligned}$$

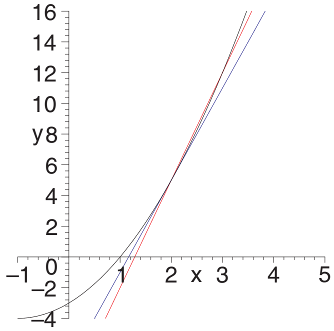
The slope of the tangent at P is 6.

c)

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= 6(x - 2) \\ y - 5 &= 6x - 12 \\ y &= 6x - 7 \end{aligned}$$

The equation of the tangent at P is $y = 6x - 7$.

d)



Section Chapter Test Page 177 Question 2

a) i)

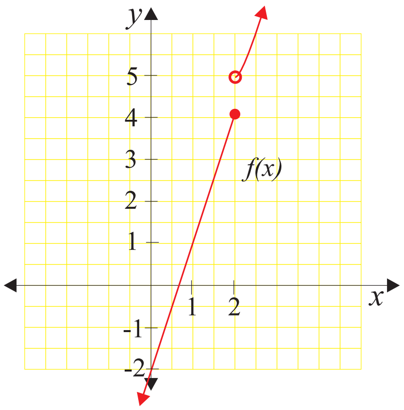
$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (3x - 2) \\ &= 3(2) - 2 \\ &= 4^- \end{aligned}$$

ii)

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^-} (x^2 + 1) \\ &= 2^2 + 1 \\ &= 5^+ \end{aligned}$$

iii) Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, $\lim_{x \to 2} f(x)$ does not exist.

b)



c) The polynomial pieces, $3x - 2$ and $x^2 + 1$, are continuous over all real numbers. Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, $f(x)$ has a jump discontinuity at $x = 2$.

Section Chapter Test Page 177 Question 3

a)

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 1}{x^2 + 3x - 4} &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{(x - 1)(x + 4)} \\ &= \lim_{x \to 1} \frac{x + 1}{x + 4},\ x \neq 1 \\ &= \frac{1 + 1}{1 + 4} \\ &= \frac{2}{5} \end{aligned}$$

b)

$$\begin{aligned} \lim_{x \to 2} \frac{x^3 - 8}{3x^2 - 4x - 4} &= \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4)}{(x - 2)(3x + 2)} \\ &= \lim_{x \to 2} \frac{x^2 + 2x + 4}{3x + 2},\ x \neq 2 \\ &= \frac{12}{8} \\ &= \frac{3}{2} \end{aligned}$$

c)
$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{x+5}}{x-1} &= \frac{\sqrt{9}}{3}\\ &= \frac{3}{3}\\ &= 1\end{aligned}$$

d)
$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{4-x}-2}{x} &= \lim_{x\to 0}\left[\frac{\sqrt{4-x}-2}{x}\cdot\frac{\sqrt{4-x}+2}{\sqrt{4-x}+2}\right]\\ &= \lim_{x\to 0}\frac{4-x-4}{x\left(\sqrt{4-x}+2\right)}\\ &= \lim_{x\to 0}\frac{-1}{\sqrt{4-x}+2},\ x\neq 0\\ &= \frac{-1}{2+2}\\ &= -\frac{1}{4}\end{aligned}$$

e)
$$\begin{aligned}\lim_{x\to -3}\frac{x^3+27}{x^3-7x+6} &= \lim_{x\to -3}\frac{(x+3)(x^2-3x+9)}{(x+3)(x^2-3x+2)}\\ &= \lim_{x\to -3}\frac{x^2-3x+9}{x^2-3x+2},\ x\neq -3\\ &= \frac{9+9+9}{9+9+2}\\ &= \frac{27}{20}\end{aligned}$$

f)
$$\begin{aligned}\lim_{x\to 1}\frac{\frac{1}{\sqrt{x}}-1}{x-1} &= \lim_{x\to 1}\frac{1-\sqrt{x}}{\sqrt{x}(x-1)}\\ &= \lim_{x\to 1}\frac{1-\sqrt{x}}{\sqrt{x}(\sqrt{x}-1)(\sqrt{x}+1)}\\ &= \lim_{x\to 1}\frac{-1}{\sqrt{x}(\sqrt{x}+1)},\ x\neq 1\\ &= -\frac{1}{2}\end{aligned}$$

Section Chapter Test Page 177 Question 4

a)
$$\begin{aligned}m_{\mathrm{P}} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{3(1+\Delta x)^2+5(1+\Delta x)+2-10}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{3+6\Delta x+3\,(\Delta x)^2+5+5\Delta x-8}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\Delta x(11+3\Delta x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}(11+3\Delta x),\ \Delta x\neq 0\\ &= 11\end{aligned}$$

The slope of the tangent at $(1, 10)$ is 11.

b)
$$\begin{aligned}m_{\mathrm{P}} &= \lim_{x\to -2}\frac{f(x)-f(-2)}{x-(-2)}\\ &= \lim_{x\to 0}\frac{x^3+8}{x+2}\\ &= \lim_{x\to 0}\frac{(x+2)(x^2-2x+4)}{x+2}\\ &= \lim_{x\to 0}(x^2-2x+4),\ x\neq -2\\ &= 12\end{aligned}$$

The slope of the tangent at $(-2, 0)$ is 12.

c)
$$\begin{aligned}m_{\mathrm{P}} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\sqrt{3(1+\Delta x)+1}-2}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\sqrt{4+3\Delta x}-2}{\Delta x}\cdot\frac{\sqrt{4+3\Delta x}+2}{\sqrt{4+3\Delta x}+2}\\ &= \lim_{\Delta x\to 0}\frac{4+3\Delta x-4}{\Delta x\left(\sqrt{4+3\Delta x}+2\right)}\\ &= \lim_{\Delta x\to 0}\frac{3}{\sqrt{4+3\Delta x}+2},\ \Delta x\neq 0\\ &= \frac{3}{2+2}\\ &= \frac{3}{4}\end{aligned}$$

The slope of the tangent at $(1, 2)$ is $\frac{3}{4}$.

d)
$$\begin{aligned}m_{\mathrm{P}} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\frac{2(0+\Delta x)^2-5}{0+\Delta x+2}-\left(-\frac{5}{2}\right)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\frac{2(\Delta x)^2-5}{\Delta x+2}+\frac{5}{2}}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{4\,(\Delta x)^2-10+5\Delta x+10}{2\Delta x\,(\Delta x+2)}\\ &= \lim_{\Delta x\to 0}\frac{\Delta x\,(4\Delta x+5)}{2\Delta x\,(\Delta x+2)}\\ &= \lim_{\Delta x\to 0}\frac{4\Delta x+5}{2\,(\Delta x+2)},\ \Delta x\neq 0\\ &= \frac{5}{4}\end{aligned}$$

The slope of the tangent at $\left(0, -\frac{5}{2}\right)$ is $\frac{5}{4}$.

Section Chapter Test Page 177 Question 5

a)

$$\begin{aligned}\text{average velocity} &= \frac{\Delta s}{\Delta t}\\ &= \frac{s(2)-s(1)}{2-1}\\ &= \frac{2(2)^2-11(2)+15-(2(1)^2-11(1)+15)}{1}\\ &= 8-22+15-(2-11+15)\\ &= -5\end{aligned}$$

The average velocity over $t \in [1, 2]$ is -5 m/s.

b)

$$\begin{aligned}v(t) &= \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}\\ v(1) &= \lim_{\Delta t\to 0}\frac{s(1+\Delta t)-s(1)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{2(1+\Delta t)^2-11(1+\Delta t)+15-(2(1)^2-11(1)+15)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{2+4\Delta t+2\,(\Delta t)^2-11-11\Delta t+15-6}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\Delta t(-7+2\Delta t)}{\Delta t}\\ &= \lim_{\Delta t\to 0} -7+2\Delta t,\ \Delta t\neq 0\\ &= -7\end{aligned}$$

The instantaneous velocity at $t = 1$ is -7 m/s.

Section Chapter Test Page 177 Question 6

The formula for the area, A, of a circle of radius r is $A(r) = \pi r^2$.

$$\begin{aligned}\lim_{\Delta r\to 0}\frac{\Delta A}{\Delta r}\,|r=20 &= \lim_{\Delta r\to 0}\frac{A(20+\Delta r)-A(20)}{\Delta r}\\ &= \lim_{\Delta r\to 0}\frac{\pi(20+\Delta r)^2-\pi(20)^2}{\Delta r}\\ &= \pi\cdot\lim_{\Delta r\to 0}\frac{(20+\Delta r)^2-20^2}{\Delta r}\\ &= \pi\cdot\lim_{\Delta r\to 0}\frac{(20+\Delta r-20)(20+\Delta r+20)}{\Delta r}\\ &= \pi\cdot\lim_{\Delta r\to 0}\frac{\Delta r(40+\Delta r)}{\Delta r}\\ &= \pi\cdot\lim_{\Delta r\to 0} 40+\Delta r\\ &= 40\pi\end{aligned}$$

The instantaneous rate of change of area with respect to radius when $r = 20$ cm is 40π cm^2/cm.

Challenge Problems

Section Challenge Problems Page 178 Question 1

The relation $xy = 4$ can be written as a $y = \frac{4}{x}$. The general coordinates of a point P on the function can be written as $\left(c, \frac{4}{c}\right)$. Determine the slope of the tangent at P.

$$\begin{aligned}\lim_{x\to c}\frac{\frac{4}{x}-\frac{4}{c}}{x-c} &= \lim_{x\to c}\frac{4(c-x)}{x-c}\\ &= \lim_{x\to c}\frac{-4}{cx},\ x \neq c\\ &= -\frac{4}{c^2}\end{aligned}$$

The slope of the tangent at P is $-\frac{4}{c^2}$.

Determine the equation of the tangent at P.

$$\begin{aligned}y-\frac{4}{c} &= -\frac{4}{c^2}(x-c)\\ y &= -\frac{4}{c^2}x+\frac{4}{c}+\frac{4}{c}\\ y &= -\frac{4}{c^2}x+\frac{8}{c}\end{aligned}$$

The x-intercept of the tangent is $\frac{-\frac{8}{c}}{-\frac{4}{c^2}}$ or $2c$. Therefore, the coordinates of A are $(2c, 0)$. The y-intercept of the tangent is $\frac{8}{c}$. Therefore, the coordinates of B are $\left(0, \frac{8}{c}\right)$. Since the area of $\triangle$AOB can be expressed as $\frac{ab}{2}$, the area is $\frac{1}{2} \times 2c \times \frac{8}{c}$ or the constant 8, and, therefore, independent of the position of point P.

Section Challenge Problems Page 178 Question 2

Answers will vary.

Section Challenge Problems Page 178 Question 3

a) Use Heron's formula to determine A_1, the area of the first island.

$$\begin{aligned}s &= \frac{a+b+c}{2}\\ &= \frac{1+1+1}{2}\\ &= \frac{3}{2}\\ A_1 &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}\\ &= \sqrt{\frac{3}{16}}\\ &= \frac{\sqrt{3}}{4}\end{aligned}$$

The area of the first island is $\frac{\sqrt{3}}{4}$ square units.

b) Since the side length of each smaller triangle is one third the side length of the first island, the area of each smaller triangle is one ninth the area of the first island. Let A_2 represent the area of the second island.

$$\begin{aligned}A_2 &= A_1 + 3\left(\frac{1}{9}A_1\right)\\ &= A_1 + \frac{1}{3}A_1\\ &= \frac{4}{3}A_1 \qquad (1)\end{aligned}$$

Substitute $A_1 = \frac{\sqrt{3}}{4}$ into (1).

$$\begin{aligned}A_2 &= \frac{4}{3}\left(\frac{\sqrt{3}}{4}\right)\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

The area of the second island is $\frac{\sqrt{3}}{3}$ square units.

c) Since the side length of each smaller triangle is one ninth the side length of the first island, the ratio of the area of each smaller triangle to the area of the first island is 1 : 81. To each of the 4(3) or 12 sides, a smaller triangle is added. Let A_3 represent the area of the third island.

$$\begin{aligned} A_3 &= A_2 + 4(3)\left(\frac{1}{81}A_1\right) \\ &= A_2 + \frac{4}{27}A_1 \qquad (2) \end{aligned}$$

Substitute $A_1 = \dfrac{\sqrt{3}}{4}$ and $A_2 = \dfrac{\sqrt{3}}{3}$ into (2).

$$\begin{aligned} A_3 &= \frac{\sqrt{3}}{3} + \frac{4}{27}\left(\frac{\sqrt{3}}{4}\right) \\ &= \frac{9\sqrt{3}}{27} + \frac{\sqrt{3}}{27} \\ &= \frac{10\sqrt{3}}{27} \end{aligned}$$

The area of the third island is $\dfrac{10\sqrt{3}}{27}$ square units.

d) The area of the nth island, A_n, can be expressed as a series.

$$\begin{aligned} A_n &= A_1 + 3\cdot\frac{1}{9}A_1 + 4(3)\left(\frac{1}{9}\right)^2 A_1 + 4^2(3)\left(\frac{1}{9}\right)^3 A + \ldots + 4^{n-1}(3)\left(\frac{1}{9}\right)^n A_1 \\ &= A_1\left(1 + \frac{1}{3}\sum_{r=0}^{n}\left(\frac{4}{9}\right)^r\right) \qquad (3) \end{aligned}$$

Substitute $A_1 = \dfrac{\sqrt{3}}{4}$ into (3).

$$= \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3}\sum_{r=0}^{n}\left(\frac{4}{9}\right)^r\right)$$

e) Result (1) from part d) contains a geometric series in which the common ratio $r = \frac{4}{9}$. Since $|r| < 1$, the series will have a limit, and, so too, will the area, as n gets large. Let A be the limiting value of the area.

$$\begin{aligned} A &= \lim_{n\to\infty} A_n \\ &= \lim_{n\to\infty} \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3}\sum_{r=0}^{n}\left(\frac{4}{9}\right)^r\right) \\ &= \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3}\cdot\frac{1}{1-\frac{4}{9}}\right) \\ &= \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3}\cdot\frac{9}{5}\right) \\ &= \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\right) \\ &= \frac{\sqrt{3}}{4}\left(\frac{8}{5}\right) \\ &= \frac{2\sqrt{3}}{5} \end{aligned}$$

The limiting value for the area of this Koch snowflake is $\dfrac{2\sqrt{3}}{5}$ square units.

Section Challenge Problems   Page 178   Question 4

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt[3]{ax+1}-1}{x} &= \lim_{x\to 0}\left[\frac{\sqrt[3]{ax+1}-1}{x}\cdot\frac{\left(\sqrt[3]{ax+1}\right)^2+\sqrt[3]{ax+1}+1}{\left(\sqrt[3]{ax+1}\right)^2+\sqrt[3]{ax+1}+1}\right]\\ &= \lim_{x\to 0}\frac{ax+1-1}{x\left(\sqrt[3]{ax+1}\right)^2+\sqrt[3]{ax+1}+1}\\ &= \lim_{x\to 0}\frac{a}{\left(\sqrt[3]{ax+1}\right)^2+\sqrt[3]{ax+1}+1},\ x\neq 0\\ &= \frac{a}{1+1+1}\\ &= \frac{a}{3}\end{aligned}$$

Section Challenge Problems   Page 178   Question 5

The general equation for an upward-opening parabola with its vertex at the origin is $y = ax^2,\ a > 0$. From the position of the boat at $(-100, 100)$, the value of a can be identified.

$$y = ax^2 \tag{1}$$

Substitute $(x, y) = (-100, 100)$ into (1).

$$\begin{aligned}100 &= a(-100)^2\\ 100 &= 10\,000a\\ a &= \frac{1}{100}\end{aligned}$$

The equation of the path of the ski boat is $y = \frac{x^2}{100}$. Determine the slope of the tangent to the parabola at $x = a$.

$$\begin{aligned}\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{\frac{x^2}{100}-\frac{a^2}{100}}{x-a}\\ &= \frac{1}{100}\cdot\lim_{x\to a}\frac{x^2-a^2}{x-a}\\ &= \frac{1}{100}\cdot\lim_{x\to a}\frac{(x-a)(x+a)}{x-a}\\ &= \frac{1}{100}\cdot\lim_{x\to a}(x+a),\ x\neq a\\ &= \frac{2a}{100}\\ &= \frac{a}{50}\end{aligned}$$

The skier should let go of the tow rope at the point $P\left(a, \frac{a^2}{100}\right)$, where the slope of the tangent to the parabola is equal to the slope of the segment joining P to the dock $D(100, 50)$, $a < 100$.

$$\begin{aligned}\text{slope of tangent at P} &= \text{slope of segment PD}\\ \frac{a}{50} &= \frac{\frac{a^2}{100}-50}{a-100}\\ a^2-100a &= \frac{a^2}{2}-2500\\ 2a^2-200a &= a^2-5000\\ a^2-200a+5000 &= 0\\ a &= \frac{-(-200)\pm\sqrt{(-200)^2-4(1)(5000)}}{2}\\ &= \frac{200\pm\sqrt{20\,000}}{2}\\ &= 100\pm 50\sqrt{2}\end{aligned}$$

Since $0 < a < 100$, the skier should let go of the tow rope at $(x, y) = (100-50\sqrt{2}, 150-100\sqrt{2})$.

Section Challenge Problems Page 178 Question 6

First we argue that the jetpack must be on at the instant of landing. If it were not on, then since gravity acts to accelerate the jumper downward, there is no way that the jumper could reach the ground with 0 speed. So the jumper is falling with some speed, and the jetpack goes on at just the right time to slow the jumper down to 0 speed just as she reaches the ground.

When the jumper leaves the helicopter, her speed is 0 and her acceleration is 9.8 m/s^2 downward. When she hits the ground, her speed is 0 and her acceleration is 4.4 m/s^2 upward. Let the height of the helicopter be H. Let the time from when she leaves the helicopter until she turns on the jetpack be t_1. Imagine measuring Ayida's time with the jetpack from the ground up. Let the time from when she hits the ground until when she turned on the jetpack be t_2. From Galileo's law, Ayida will travel $h_1 = 4.9{t_1}^2$ without the jetpack and $h_2 = 2.2{t_2}^2$ with the jetpack. Thus,

(1) $$H = 4.9{t_1}^2 + 2.2{t_2}^2$$

Ayida's velocity is given by $v = at$, or $v = 9.8t$ without the jetpack, and $v = 4.4t$ with the jetpack. At the moment she turns on the jetpack, these two velocities must be equal. Thus,

$$9.8t_1 = 4.4t_2$$

(2) $$t_2 = \frac{9.8}{4.4}t_1$$

Substitute (2) into (1).

$$H = 4.9{t_1}^2 + 2.2\left(\frac{9.8}{4.4}t_1\right)^2$$

$$t_1 = \sqrt{\frac{H}{4.9 + 2.2\left(\frac{9.8}{4.4}t_1\right)^2}}$$

a) For a helicopter of height 100 m,

$$\begin{aligned} t_1 &= \sqrt{\frac{100}{4.9 + 2.2\left(\frac{9.8}{4.4}t_1\right)^2}} \\ &\doteq 2.51 \end{aligned}$$

Thus, Ayida should turn on her jetpack at $t = 2.51$ s.

$$\begin{aligned} t_2 &= \frac{9.8}{4.4}\sqrt{\frac{100}{4.9 + 2.2\left(\frac{9.8}{4.4}t_1\right)^2}} \\ &\doteq 5.60 \\ t_1 + t_2 &= 2.51 + 5.60 \\ &= 8.11 \end{aligned}$$

Ayida will land after 8.11 s.

b) For a helicopter height of 200 m,

$$\begin{aligned} t_1 &= \sqrt{\frac{200}{4.9 + 2.2\left(\frac{9.8}{4.4}t_1\right)^2}} \\ &\doteq 3.56 \end{aligned}$$

Thus, Ayida should turn on her jetpack at $t = 3.56$ s.

$$\begin{aligned} t_2 &= \frac{9.8}{4.4}\sqrt{\frac{200}{4.9 + 2.2\left(\frac{9.8}{4.4}t_1\right)^2}} \\ &\doteq 7.92 \\ t_1 + t_2 &= 3.56 + 7.92 \\ &= 11.48 \end{aligned}$$

Ayida will land after 11.48 s.

c) To find the maximum height, determine H when $t_2 = 10$, the maximum time allowed for the jetpack to be on.

$$\begin{aligned} t_2 &= \frac{9.8}{4.4}\sqrt{\frac{H}{4.9+2.2\left(\frac{9.8}{4.4}t_1\right)^2}} \\ H &= \left(\frac{4.4t_2}{9.8}\right)^2\left(4.9+2.2\left(\frac{9.8}{4.4}\right)^2\right) \\ &= \left(\frac{4.4(10)}{9.8}\right)^2\left(4.9+2.2\left(\frac{9.8}{4.4}\right)^2\right) \\ &\doteq 378.78 \end{aligned}$$

The maximum height is approximately 379 m.

Using the Strategies

Section Problem Solving Page 181 Question 1

For this purpose, the equator can be assumed to be a circle with circumference, C, where,

$$C = 2\pi r \qquad (1)$$

Adding 1 m to the rope, increases the circumference by 1. Let the new circumference be C^*.

$$\begin{aligned} C^* &= C + 1 \\ &= 2\pi r + 1 \qquad (2) \end{aligned}$$

Divide (2) by 2π to reveal the new radius, r^*.

$$\begin{aligned} \frac{C^*}{2\pi} &= \frac{2\pi r + 1}{2\pi} \\ r^* &= r + \frac{1}{2\pi} \end{aligned}$$

The effect of adding 1 m of rope is to increase the radius by $\frac{1}{2\pi}$ m, or approximately 15.9 cm.

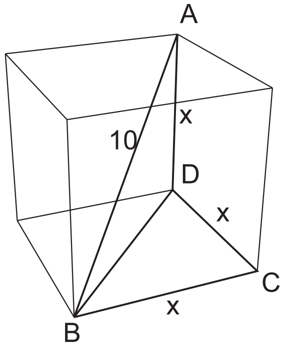
Section Problem Solving Page 181 Question 2

16 moves. Answers may vary. 7 – 2, 1 – 6, 9 – 4, 6 – 7, 2 – 9, 3 – 8, 4 – 3, 8 – 1, 1 – 6, 3 – 8, 7 – 2, 8 – 1, 6 – 7, 9 – 4, 2 – 9, 4 – 3.

Section Problem Solving Page 181 Question 3

Since all 8 vertices of the cube touch the sphere, the body diagonal of the cube is also a diameter of the sphere. In right △ABD, the length of BD can be expressed as

$$\begin{aligned}\text{BD} &= \sqrt{10^2 - x^2}\\ &= \sqrt{100 - x^2} \qquad (1)\end{aligned}$$



Turning to right △BDC, an equation in x can be determined with the help of (1) and the Pythagorean Theorem.

$$\begin{aligned}x^2 + x^2 &= (\sqrt{100 - x^2})^2\\ 2x^2 &= 100 - x^2\\ 3x^2 &= 100\\ x &= \sqrt{\frac{100}{3}} \qquad (2)\end{aligned}$$

The volume of the cube can be determined.

$$\begin{aligned}V(x) &= x^3\\ V\left(\sqrt{\frac{100}{3}}\right) &= \left(\sqrt{\frac{100}{3}}\right)^3\\ &\doteq 192.5\end{aligned}$$

The volume of the cube is about 192.5 cm^3.

Section Problem Solving Page 181 Question 4

△OPQ is a right triangle. Let r be the radius of the inner circle in centimetres. Let A be the area of the shaded region in square centimetres.

$$\begin{aligned}A &= A_{\text{outer circle}} - A_{\text{inner circle}}\\ &= \pi\left(\sqrt{r^2 + 4^2}\right)^2 - \pi r^2\\ &= \pi(r^2 + 16 - r^2)\\ &= 16\pi\end{aligned}$$

The area of the shaded region is 16π cm^2.

Section Problem Solving Page 181 Question 5

Since 3 and 5 are factors of 15, the concern is the remainders obtained by dividing 7 by 3 and 5 respectively. When 7 is divided by the 3, the remainder is 1. When 7 is divided by 5, the remainder is 2. The sum of the remainders is $1 + 2$ or 3.

Section Problem Solving Page 181 Question 7

a) Drawing three lines yields a maximum of 16 parts suggesting the model $5n + 1$ for n lines.
b) Using the result in part a) suggests that 150 lines will yield $5(150) + 1$ or 751 parts.

c)

$$\begin{aligned}5n + 1 &= 116\\ 5n &= 115\\ n &= 23\end{aligned}$$

A maximum of 116 parts can be achieved with 23 lines.

Section Problem Solving Page 181 Question 8

From the fourth point, Dianne and Carys do the same thing. From the third point, it can be concluded then, that Barb behaves differently from both Dianne and Carys. From the fifth point, Erik can't be swimming because Andrew and Dianne would be swimming, but Barb behaves the same as Andrew but differently than Dianne. It is now apparent that Erik is not swimming. Neither, then, are Andrew and Barb. This leaves Dianne and Carys as the only swimmers.

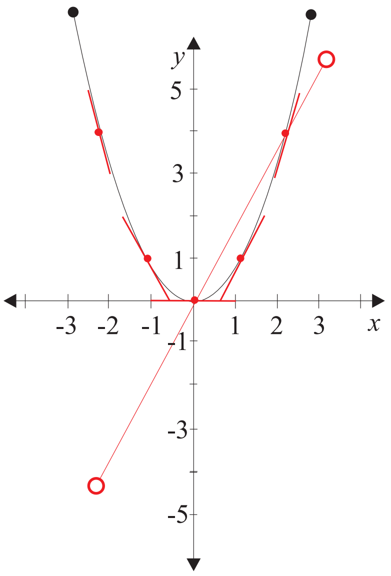
CHAPTER 4 Derivatives

4.1 The Derivative

Practise

Section 4.1 Page 194 Question 1

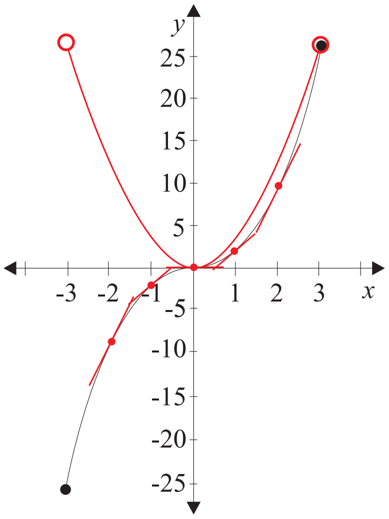
a)



Drawing tangents and measuring the slopes yields the following approximations:

i) $f'(-2) \doteq -4$
ii) $f'(-1) \doteq -2$
iii) $f'(0) \doteq 0$
iv) $f'(1) \doteq 2$
v) $f'(2) \doteq 4$

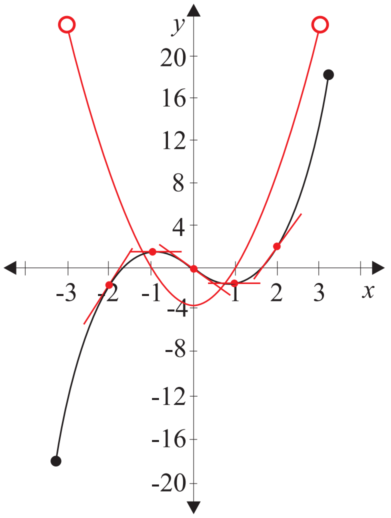
b)



Drawing tangents and measuring the slopes yields the following approximations:

i) $f'(-2) \doteq 12$
ii) $f'(-1) \doteq 3$
iii) $f'(0) \doteq 0$
iv) $f'(1) \doteq 3$
v) $f'(2) \doteq 12$

c)

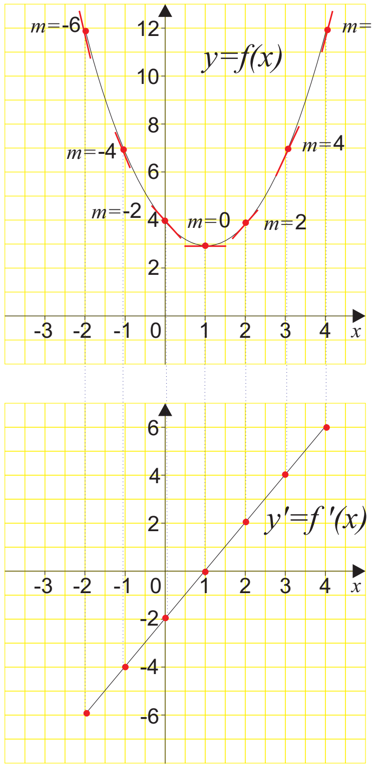


Drawing tangents and measuring the slopes yields the following approximations:

i) $f'(-2) \doteq 9$
ii) $f'(-1) \doteq 0$
iii) $f'(0) \doteq -3$
iv) $f'(1) \doteq 0$
v) $f'(2) \doteq 9$

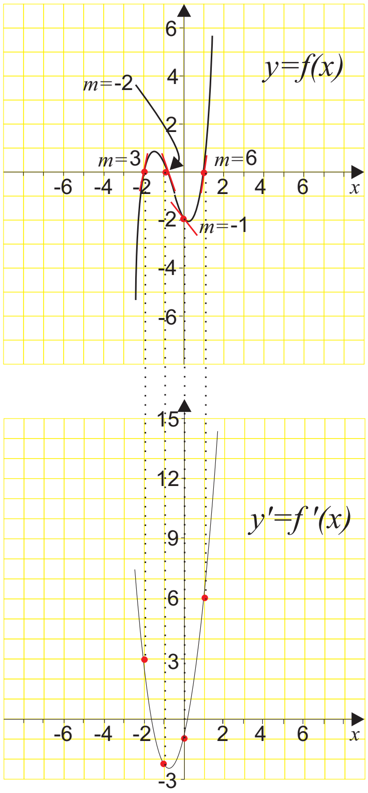
Section 4.1 Page 195 Question 3

a)



b) The degree of the derivative function is 1 (linear).

b)



b) The degree of the derivative function is 2 (quadratic).

Section 4.1 Page 195 Question 5

a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{2(x+h)+3-(2x+3)}{h} \\ &= \lim_{h\to 0}\frac{2x+2h+3-2x-3}{h} \\ &= \lim_{h\to 0}\frac{2h}{h} \\ &= \lim_{h\to 0} 2 \\ &= 2 \end{aligned}$$

b)

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-4-(x^2-4)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{2xh+h^2}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x \end{aligned}$$

c)

$$\begin{aligned} h'(t) &= \lim_{h\to 0}\frac{h(t+h)-h(t)}{h} \\ &= \lim_{h\to 0}\frac{(t+h)^2+3(t+h)-1-(t^2+3t-1)}{h} \\ &= \lim_{h\to 0}\frac{t^2+2th+h^2+3t+3h-1-t^2-3t+1}{h} \\ &= \lim_{h\to 0}\frac{2th+h^2+3h}{h} \\ &= \lim_{h\to 0} 2t+h+3 \\ &= 2t+3 \end{aligned}$$

d)

$$\begin{aligned} N'(b) &= \lim_{h\to 0}\frac{N(b+h)-N(b)}{h} \\ &= \lim_{h\to 0}\frac{(b+h)^3-(b+h)^2-(b^3-b^2)}{h} \\ &= \lim_{h\to 0}\frac{\left[(b+h)^3-b^3\right]-\left[(b+h)^2-b^2\right]}{h} \\ &= \lim_{h\to 0}\frac{\left[3b^2h+3bh^2+h^3\right]-\left[2bh+h^2\right]}{h} \\ &= \lim_{h\to 0}\frac{3b^2h+3bh^2+h^3-2bh-h^2}{h} \\ &= \lim_{h\to 0}(3b^2+3bh+h^2-2b-h) \\ &= 3b^2-2b \end{aligned}$$

e) $$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{y(x+h)-y(x)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^4-x^4}{h}\\ &= \lim_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h}\\ &= \lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h}\\ &= \lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3)\\ &= 4x^3\end{aligned}$$

f) $$\begin{aligned}G'(x) &= \lim_{h\to 0}\frac{G(x+h)-G(x)}{h}\\ &= \lim_{h\to 0}\frac{3(x+h)^2-2(x+h)+1-(3x^2-2x+1)}{h}\\ &= \lim_{h\to 0}\frac{3x^2+6xh+3h^2-2x-2h-3x^2+2x}{h}\\ &= \lim_{h\to 0}\frac{6xh+3h^2-2h}{h}\\ &= \lim_{h\to 0}(6x+3h-2)\\ &= 6x-2\end{aligned}$$

g) $$\begin{aligned}T'(n) &= \lim_{h\to 0}\frac{T(n+h)-T(n)}{h}\\ &= \lim_{h\to 0}\frac{(n+h)-(n+h)^3-(n-n^3)}{h}\\ &= \lim_{h\to 0}\frac{n+h-n^3-3n^2h-3nh^2-h^3-n+n^3}{h}\\ &= \lim_{h\to 0}\frac{h-3n^2h-3nh^2-h^3}{h}\\ &= \lim_{h\to 0}(1-3n^2-3nh-h^2)\\ &= 1-3n^2\end{aligned}$$

h) $$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{4(x+h)-4x}{h}\\ &= \lim_{h\to 0}\frac{4x+4h-4x}{h}\\ &= \lim_{h\to 0}\frac{4h}{h}\\ &= \lim_{h\to 0}4\\ &= 4\end{aligned}$$

i) $$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{y(x+h)-y(x)}{h}\\ &= \lim_{h\to 0}\frac{5-2(x+h)-(5-2x)}{h}\\ &= \lim_{h\to 0}\frac{5-2x-2h-5+2x}{h}\\ &= \lim_{h\to 0}\frac{-2h}{h}\\ &= \lim_{h\to 0}-2\\ &= -2\end{aligned}$$

j) $$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{y(x+h)-y(x)}{h}\\ &= \lim_{h\to 0}\frac{2(x+h)-(x+h)^2-(2x-x^2)}{h}\\ &= \lim_{h\to 0}\frac{2x+2h-x^2-2xh-h^2-2x+x^2}{h}\\ &= \lim_{h\to 0}\frac{2h-2xh-h^2}{h}\\ &= \lim_{h\to 0}(2-2x-h)\\ &= 2-2x\end{aligned}$$

Section 4.1 Page 195 Question 7

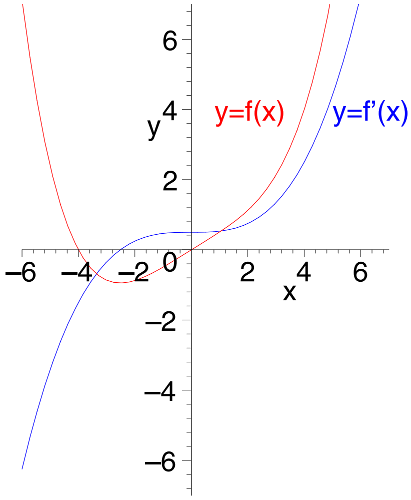
$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\cdot\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}\\ &= \lim_{h\to 0}\frac{x+h+1-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}\\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}\\ &= \frac{1}{2\sqrt{x+1}}\end{aligned}$$

The domain of f is $x \in [-1, \infty)$. The domain of f' is $x \in (-1, \infty)$.

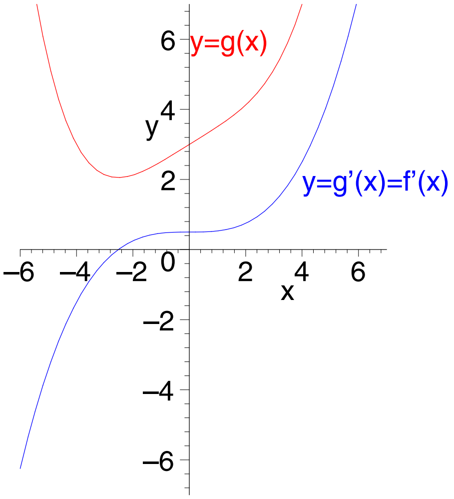
Apply, Solve, Communicate

Section 4.1 Page 196 Question 9

a)



b)



Section 4.1 Page 196 Question 10

a)
$$\begin{aligned} H'(t) &= \lim_{h\to 0}\frac{H(t+h)-H(x)}{h} \\ &= \lim_{h\to 0}\frac{-4.9(t+h)^2+5(t+h)+2-(-4.9t^2+5t+2)}{h} \\ &= \lim_{h\to 0}\frac{-4.9t^2-9.8th-4.9h^2+5t+5h+2+4.9t^2-5t-2)}{h} \\ &= \lim_{h\to 0}\frac{-9.8th-4.9h^2+5h}{h} \\ &= \lim_{h\to 0}(-9.8t-4.9h+5) \\ &= -9.8t+5 \end{aligned}$$

b)
$$\begin{aligned} H'(t) &= -9 \\ -9.8t+5 &= -9 \\ -9.8t &= -14 \\ t &\doteq 1.43 \end{aligned}$$

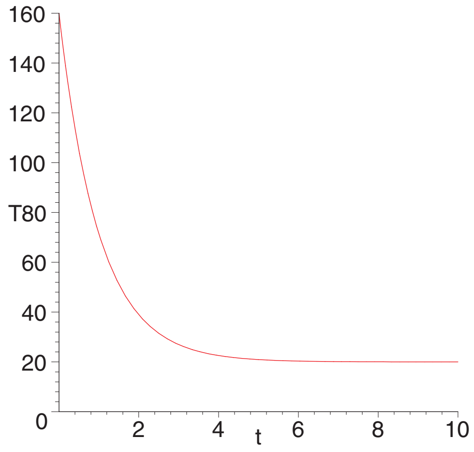
The rate of change of the height is −9 m/s at approximately 1.43 s.

c)
$$\begin{aligned} H'(t) &= 0 \\ -9.8t+5 &= 0 \\ -9.8t &= -5 \\ t &\doteq 0.51 \end{aligned}$$

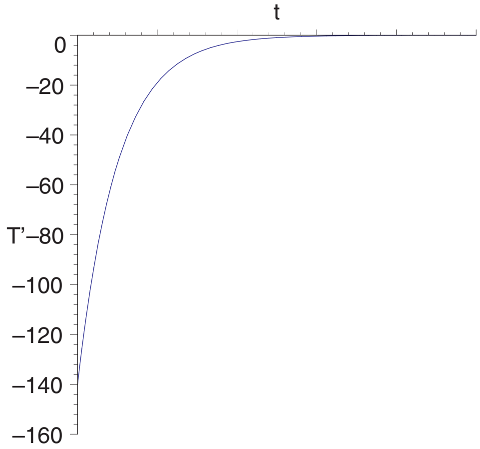
The ball is at rest at approximately 0.51 s.

Section 4.1 Page 196 Question 11

a)



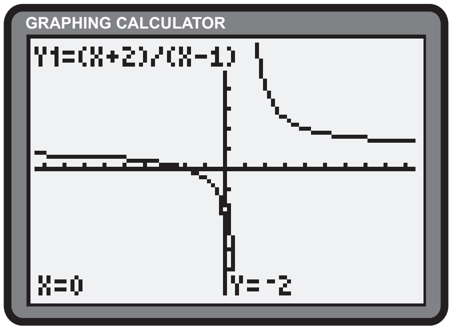
c)

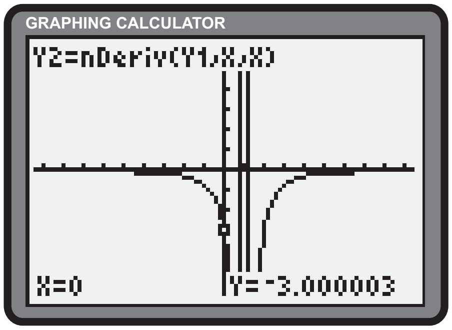


b) The rate of change of the temperature initially offers large negative results, gradually increasing toward zero.

Section 4.1 Page 196 Question 12

$$f(x) = \frac{x+2}{x-1} \qquad\qquad f'(x) = -\frac{3}{(x-1)^2}$$





b) The slopes of the tangents to f are negative over the entire domain. This is reflected in the derivative, since $f'(x) < 0$ for all real numbers except $x = 1$.

Section 4.1 Page 196 Question 13

a)

$$\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} \\
k'(3) &= \lim_{h\to 0} \frac{|3+h-3| - |3-3|}{h} \\
&= \lim_{h\to 0} \frac{|h| - |0|}{h} \\
&= \lim_{h\to 0} \frac{|h|}{h} \qquad (1)
\end{aligned}$$

For $h \in (-\infty, 0)$, $|h| = -h$, so (1) becomes

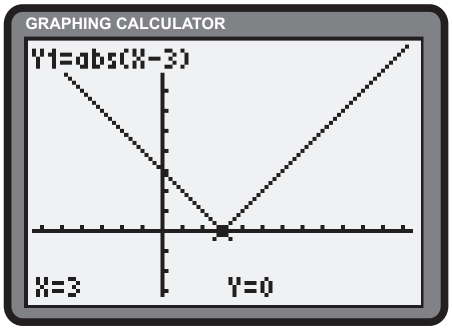
$$\begin{aligned}
&= \lim_{h\to 0^-} \frac{-h}{h} \\
&= \lim_{h\to 0^-} -1 \\
&= -1 \qquad (2)
\end{aligned}$$

For $h \in (0, \infty)$, $|h| = h$, so (1) becomes

$$\begin{aligned}
&= \lim_{h\to 0^+} \frac{h}{h} \\
&= \lim_{h\to 0^+} 1 \\
&= 1 \qquad (3)
\end{aligned}$$

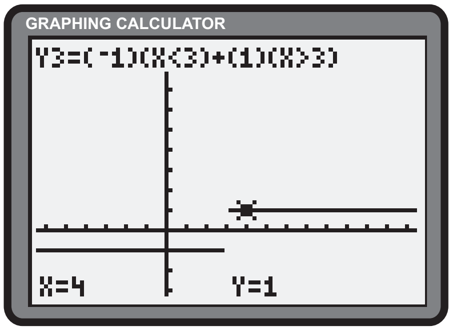
Since the one-sided limits are different, $k'(3)$ does not exist. Therefore, k is not differentiable at $x = 3$.

b)



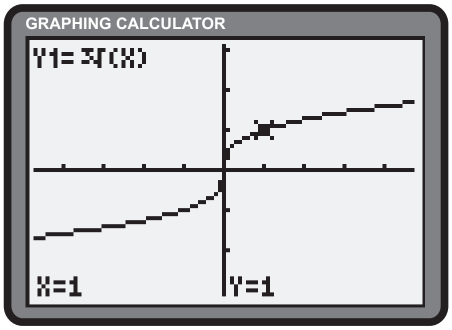
c)

$$k'(x) = \begin{cases} 1 & , x \in (\infty, 3) \\ -1 & , x \in (3, \infty) \\ \text{undefined} & , x = 3 \end{cases}$$



Section 4.1 Page 196 Question 14

a)



b)

$$\begin{aligned}
M'(n) &= \lim_{h\to 0} \frac{M(n+h) - M(n)}{h} \\
M'(0) &= \lim_{h\to 0} \frac{M(0+h) - M(0)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} \\
&= \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}}
\end{aligned}$$

As $h \to 0$, this result increases without bound. Therefore, M is not differentiable at $n = 0$.

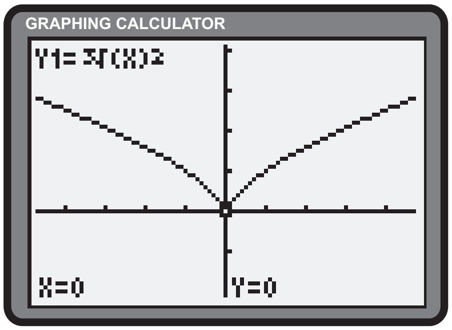
Section 4.1 Page 196 Question 15

a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ f'(0) &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{h^{\frac{2}{3}}}{h} \\ &= \lim_{h\to 0}\frac{1}{\sqrt[3]{h}} \end{aligned}$$

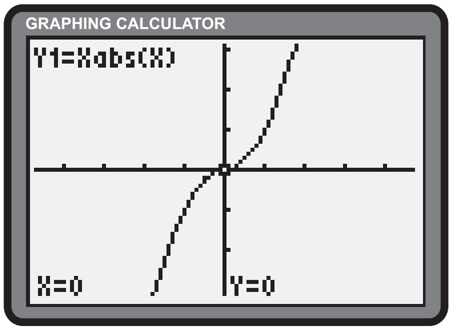
As $h \to 0$, this result increases without bound. Therefore, f is not differentiable at $x = 0$.

b)



Section 4.1 Page 196 Question 16

a)



b) The domain of f' includes all real numbers.

c) Consider two cases. If $x < 0$, $f(x) = x(-x)$ or $-x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{-(x+h)^2-(-x^2)}{h} \\ &= \lim_{h\to 0}\frac{-x^2-2xh-h^2+x^2}{h} \\ &= \lim_{h\to 0}(-2x-h) \\ &= -2x \qquad (1) \end{aligned}$$

If $x \geq 0$, $f(x) = x(x)$ or x^2.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x \qquad (2) \end{aligned}$$

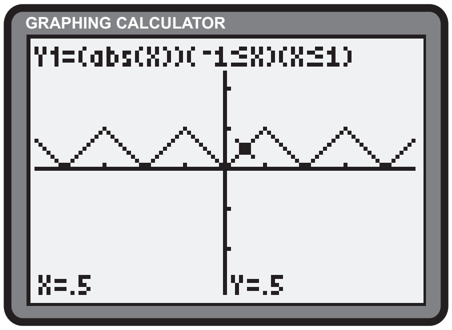
Combining (1) and (2) yields the result, $f'(x) = 2|x|$.

d)

$$\begin{aligned} f'(x) &= 2|x| \\ f'(0) &= 2|0| \\ &= 0 \end{aligned}$$

Section 4.1 Page 196 Question 17

a)



b) f is not differentiable at all integer values within the domain.

4.2 Basic Differentiation Rules

Practise

Section 4.2 Page 204 Question 1

a) $f'(x) = 0$

b) $g'(x) = 5x^4$

c) $h'(x) = 9x^8$

d) $\frac{dy}{dx} = 14x^{13}$

e) $\frac{dy}{dx} = 0$

f) $M'(x) = 1$

g) $h'(t) = 54t^{53}$

h) $A'(h) = 0$

i) $g'(x) = -2x^{-3}$

j) $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

k) $g'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

l) $f'(x) = nx^{n-1}$

Section 4.2 Page 204 Question 3

a)

$$\begin{aligned} f'(x) &= 6x \\ f'(-1) &= 6(-1) \\ &= -6 \end{aligned}$$

b)

$$\begin{aligned} f'(x) &= 3x^2 \\ f'(2) &= 3(2)^2 \\ &= 12 \end{aligned}$$

c)

$$\begin{aligned} g'(x) &= -3x^{-4} \\ g'(1) &= -3(1)^{-4} \\ &= -3 \end{aligned}$$

d)

$$\begin{aligned} g(x) &= 8\sqrt[4]{x} \\ &= 8x^{\frac{1}{4}} \\ g'(x) &= 8\left(\frac{1}{4}\right)x^{\frac{1}{4}-1} \\ &= 2x^{-\frac{3}{4}} \\ g'(1) &= 2(1)^{-\frac{3}{4}} \\ &= 2 \end{aligned}$$

e)

$$\begin{aligned} y &= \frac{18}{x} \\ &= 18x^{-1} \\ \frac{dy}{dx} &= 18(-1)x^{-1-1} \\ &= -18x^{-2} \\ &= -\frac{18}{x^2} \end{aligned}$$

At $x = 3$,

$$\begin{aligned} \frac{dy}{dx} &= -\frac{18}{3^2} \\ &= -2 \end{aligned}$$

f)

$$\begin{aligned} y &= \sqrt{x^3} \\ &= x^{\frac{3}{2}} \\ \frac{dy}{dx} &= \frac{3}{2}x^{\frac{3}{2}-1} \\ &= \frac{3}{2}x^{\frac{1}{2}} \end{aligned}$$

At $x = 4$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{3}{2}(4)^{\frac{1}{2}} \\ &= \frac{3}{2}(2) \\ &= 3 \end{aligned}$$

Section 4.2 Page 204 Question 5

The power rule suggests the derivative of $f(x) = x^{-1}$ is $f'(x) = (-1)x^{-2}$ or $f'(x) = -\frac{1}{x^2}$. This result can be confirmed from first principles.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0}\frac{x - (x+h)}{hx(x+h)} \\ &= \lim_{h\to 0}\frac{-h}{hx(x+h)} \\ &= \lim_{h\to 0}\frac{-1}{x(x+h)} \\ &= -\frac{1}{x^2} \end{aligned}$$

Section 4.2 Page 204 Question 7

a)
$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2+3x) \\ &= 2x+3 \end{aligned}$$

b)
$$\begin{aligned} f'(x) &= \frac{d}{dx}(4x^2-3x) \\ &= 8x-3 \end{aligned}$$

c)
$$\begin{aligned} f'(x) &= \frac{d}{dx}(5x^3-6x^2+2x) \\ &= 15x^2-12x+2 \end{aligned}$$

d)
$$\begin{aligned} g'(x) &= \frac{d}{dx}(14x^5+23x^3-65x) \\ &= 70x^4+69x^2-65 \end{aligned}$$

e)
$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(x^{-1}-3x^3\right) \\ &= (-1)x^{-2}-9x^2 \\ &= -\frac{1}{x^2}-9x^2 \end{aligned}$$

f)
$$\begin{aligned} h(x) &= (x+2)(x-3) \\ &= x^2-x-6 \\ h'(x) &= \frac{d}{dx}(x^2-x-6) \\ &= 2x-1 \end{aligned}$$

g)
$$\begin{aligned} h(x) &= (x-1)^2 \\ &= x^2-2x+1 \\ h'(x) &= \frac{d}{dx}(x^2-2x+1) \\ &= 2x-2 \end{aligned}$$

h)
$$\begin{aligned} f(x) &= (2+x)^3 \\ &= 8+12x+6x^2+x^3 \\ f'(x) &= \frac{d}{dx}(8+12x+6x^2+x^3) \\ &= 12+12x+3x^2 \end{aligned}$$

i)
$$\begin{aligned} f(x) &= a+\frac{b}{x}+\frac{c}{x^2} \\ &= a+bx^{-1}+cx^{-2} \\ f'(x) &= \frac{d}{dx}(a+bx^{-1}+cx^{-2}) \\ &= b(-1)x^{-2}+c(-2)x^{-3} \\ &= -\frac{b}{x^2}-\frac{2c}{x^3} \end{aligned}$$

Section 4.2 Page 205 Question 9

a)
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^3-3x^2+x+3) \\ &= 3x^2-6x+1 \end{aligned}$$

At $x = -1$,
$$\begin{aligned} \frac{dy}{dx} &= 3(-1)^2-6(-1)+1 \\ &= 3+6+1 \\ &= 10 \end{aligned}$$

The slope of the tangent at $x = -1$ is 10.
$$\begin{aligned} y-(-2) &= 10(x-(-1)) \\ y &= 10x+10-2 \\ y &= 10x+8 \end{aligned}$$

The equation of the tangent is $y = 10x + 8$.

b)
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(x^2-4x^{\frac{1}{2}}\right) \\ &= 2x-2x^{-\frac{1}{2}} \end{aligned}$$

At $x = 4$,
$$\begin{aligned} \frac{dy}{dx} &= 2(4)-2(4)^{-\frac{1}{2}} \\ &= 8-1 \\ &= 7 \end{aligned}$$

The slope of the tangent at $x = -4$ is 7.
$$\begin{aligned} y-8 &= 7(x-4) \\ y-8 &= 7x-28 \\ y &= 7x-20 \end{aligned}$$

The equation of the tangent is $y = 7x - 20$.

c)
$$\begin{aligned} y &= \frac{x^4-6x^2}{3x} \\ &= \frac{1}{3}x^3-2x \\ \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{3}x^3-2x\right) \\ &= x^2-2 \end{aligned}$$

At $x = 3$,
$$\begin{aligned} \frac{dy}{dx} &= 3^2-2 \\ &= 7 \end{aligned}$$

The slope of the tangent at $x = 3$ is 7.
$$\begin{aligned} y-3 &= 7(x-3) \\ y-3 &= 7x-21 \\ y &= 7x-18 \end{aligned}$$

The equation of the tangent is $y = 7x - 18$.

d)
$$\begin{aligned} y &= -4+\frac{4}{x}-\frac{8}{x^2} \\ &= -4+4x^{-1}-8x^{-2} \\ \frac{dy}{dx} &= \frac{d}{dx}\left(-4+4x^{-1}-8x^{-2}\right) \\ &= -4x^{-2}+16x^{-3} \end{aligned}$$

At $x = 2$,
$$\begin{aligned} \frac{dy}{dx} &= -4(2)^{-2}+16(2)^{-3} \\ &= -\frac{4}{4}+\frac{16}{8} \\ &= -1+2 \\ &= 1 \end{aligned}$$

The slope of the tangent at $x = 2$ is 1.
$$\begin{aligned} y-(-4) &= 1(x-2) \\ y+4 &= x-2 \\ y &= x-6 \end{aligned}$$

The equation of the tangent is $y = x - 6$.

e)
$$\begin{aligned} y &= (x^2-3)^2 \\ &= x^4 - 6x^2 + 9 \\ \frac{dy}{dx} &= \frac{d}{dx}\left(x^4 - 6x^2 + 9\right) \\ &= 4x^3 - 12x \end{aligned}$$

At $x = -2$,
$$\begin{aligned} \frac{dy}{dx} &= 4(-2)^3 - 12(-2) \\ &= -32 + 24 \\ &= -8 \end{aligned}$$

The slope of the tangent at $x = -2$ is -8.
$$\begin{aligned} y - 1 &= -8(x - (-2)) \\ y - 1 &= -8x - 16 \\ y &= -8x - 15 \end{aligned}$$

The equation of the tangent is $y = -8x - 15$.

f)
$$\begin{aligned} y &= (x+1)^2 \\ &= x^2 + 2x + 1 \\ \frac{dy}{dx} &= \frac{d}{dx}\left(x^2 + 2x + 1\right) \\ &= 2x + 2 \end{aligned}$$

At $x = 1$,
$$\begin{aligned} \frac{dy}{dx} &= 2(1) + 2 \\ &= 4 \end{aligned}$$

The slope of the tangent at $x = 1$ is 4.
$$\begin{aligned} y - 4 &= 4(x - 1) \\ y - 4 &= 4x - 4 \\ y &= 4x \end{aligned}$$

The equation of the tangent is $y = 4x$.

Apply, Solve, Communicate

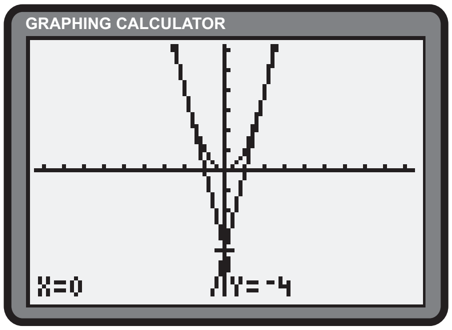
Section 4.2 Page 205 Question 10

$$\begin{aligned} \frac{dy}{dx} &= 24 \\ \frac{d}{dx}(4x^2) &= 24 \\ 8x &= 24 \\ x &= 3 \end{aligned}$$

The slope of the tangent is 24 at $(3, 4(3)^2)$ or $(3, 36)$.

Section 4.2 Page 205 Question 12

a) The tangents intersect at $(0, -4)$.



b) Let the coordinates of one of the points be $P(a, a^2)$. Let Q be the point at $(0, -5)$. At $x = a$,

$$\begin{aligned} \frac{dy}{dx} &= m_{PQ} \\ 2a &= \frac{a^2 - (-5)}{a - 0} \\ 2a^2 &= a^2 + 5 \\ a^2 &= 5 && (1) \\ a &= \pm\sqrt{5} && (2) \end{aligned}$$

From (1) and (2), the coordinates of the points are $\left(\pm\sqrt{5}, 5\right)$.

Section 4.2 Page 205 Question 11

The slope of the line $y = 1 - 4x$ is -4. For the tangent to be perpendicular to the given line, the slope must be $\frac{1}{4}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{4} \\ \frac{d}{dx}(2 - x^{-1}) &= \frac{1}{4} \\ x^{-2} &= \frac{1}{4} \\ \frac{1}{x^2} &= \frac{1}{4} \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

At $x = -2$,
$$\begin{aligned} y &= 2 - \frac{1}{-2} \\ &= \frac{5}{2} \end{aligned}$$

At $x = 2$,
$$\begin{aligned} y &= 2 - \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

The tangent is perpendicular to $y = 1 - 4x$ at $\left(-2, \frac{5}{2}\right)$ and $\left(2, \frac{3}{2}\right)$.

Section 4.2 Page 205 Question 13

a)

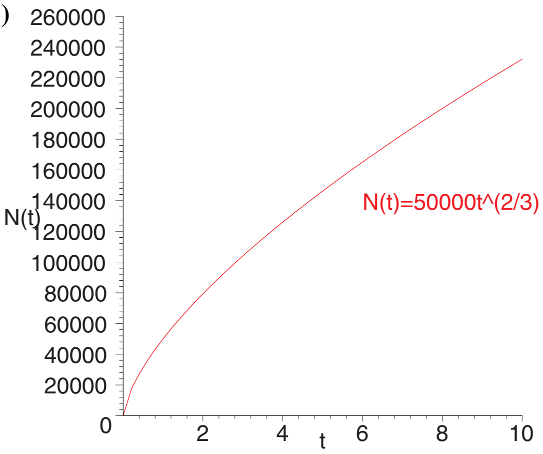
$$\begin{aligned} N(t) &= 50\,000t^{\frac{2}{3}} \\ N'(t) &= \frac{d}{dt}(50\,000t^{\frac{2}{3}}) \\ &= \frac{100\,000}{3}t^{-\frac{1}{3}} \end{aligned}$$

At $t = 7$,

$$\begin{aligned} N'(7) &= \frac{100\,000}{3}(7)^{-\frac{1}{3}} \\ &\doteq 17\,425 \end{aligned}$$

After 7 years, the net worth of the company is increasing at approximately \$17 425 per year.

b)



It would be best to invest at the very beginning ($t = 0$), since the company is growing at the fastest rate.

Section 4.2 Page 205 Question 14

$$\begin{aligned} h &= 1 + 30t - 4.9t^2 \\ v(t) &= \frac{dh}{dt} \\ &= \frac{d}{dt}(1 + 30t - 4.9t^2) \\ &= 30 - 9.8t \end{aligned}$$

The velocity of the ball after t seconds is $v(t) = 30 - 9.8t$.

$$\begin{aligned} v(1) &= 30 - 9.8(1) \\ &= 20.2 \end{aligned}$$

The velocity of the ball after 1 s is 20.2 m/s

$$\begin{aligned} v(3) &= 30 - 9.8(3) \\ &= 30 - 29.4 \\ &= 0.6 \end{aligned}$$

The velocity of the ball after 3 s is 0.6 m/s

$$\begin{aligned} v(5) &= 30 - 9.8(5) \\ &= 30 - 49 \\ &= -19 \end{aligned}$$

The velocity of the ball after 5 s is -19 m/s

Section 4.2 Page 205 Question 15

a)

$$\begin{aligned} G &= \frac{dx}{dt} \\ &= \frac{d}{dt}\left(kt^{\frac{1}{2}}\right) \\ &= \frac{k}{2}t^{-\frac{1}{2}} \end{aligned}$$

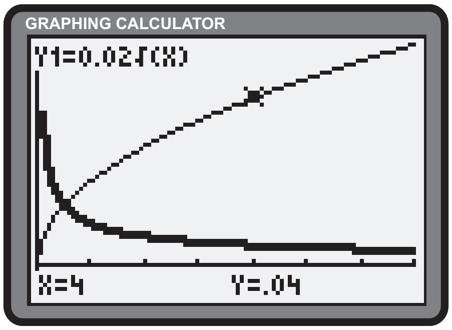
The growth rate is $G = \frac{k}{2}t^{-\frac{1}{2}}$.

b) Given $k = 0.02$, at $t = 4$,

$$\begin{aligned} G &= \frac{0.02}{2}(4)^{-\frac{1}{2}} \\ &\doteq 0.005 \end{aligned}$$

The growth rate is 0.005 cm/year.

c)



As t increases, the thickness continues to increase and the growth rate decreases.

d) Answers will vary.

Section 4.2 Page 205 Question 16

$$\begin{aligned} s &= 20 + 5t + 0.5t^2 \\ v(t) &= \frac{ds}{dt} \\ &= \frac{d}{dt}(20 + 5t + 0.5t^2) \\ &= 5 + t \end{aligned}$$

The velocity of the car after t seconds is $v(t) = 5 + t$.

$$\begin{aligned} v(4) &= 5 + 4 \\ &= 9 \end{aligned}$$

The velocity of the car after 4 s is 9 m/s.

$$\begin{aligned} v(6) &= 5 + 6 \\ &= 11 \end{aligned}$$

The velocity of the car after 6 s is 11 m/s.

$$\begin{aligned} v(10) &= 5 + 10 \\ &= 15 \end{aligned}$$

The velocity of the car after 10 s is 15 m/s.

Section 4.2 Page 205 Question 17

a)

$$\begin{aligned} v(t) &= 31 \\ \frac{d}{dt}(25t + t^2) &= 31 \\ 25 + 2t &= 31 \\ 2t &= 6 \\ t &= 3 \end{aligned}$$

The speeding car is moving at 31 m/s after 3 s.

b) The police car is moving at $20 + 1.5(3)$ or 24.5 m/s at that time.

c) The police car's acceleration and initial velocity are 1.5 m/s^2 and 20 m/s, respectively. Thus, its position function is given by

$$\begin{aligned} d &= \frac{1}{2}(1.5)t^2 + 20t \\ &= 0.75t^2 + 20t \end{aligned}$$

Let x be how far apart the cars are after 3 s.

$$\begin{aligned} x &= 25(3) + 3^2 - (0.75(3)^2 + 20(3)) \\ &= 75 + 9 - 6.75 - 60 \\ &= 17.25 \end{aligned}$$

After 3 s, the vehicles are 17.25 m apart.

d) No, because the two position functions do not intersect after $t = 0$.

Section 4.2 Page 205 Question 18

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{d}{dx}(2x^3 + 3x^2 - 36x + 40) &= 0 \\ 6x^2 + 6x - 36 &= 0 \\ x^2 + x - 6 &= 0 \\ (x + 3)(x - 2) &= 0 \\ x &= -3 \text{ or } 2 \end{aligned}$$

The curve has horizontal tangent slopes at $(-3, y(-3))$ or $(-3, 121)$ and $(2, y(2))$ or $(2, -4)$.

Section 4.2 Page 205 Question 19

$$\begin{aligned} \frac{dy}{dx} &= 2 \\ \frac{d}{dx}(2x^3 + 3x - 4) &= 0 \\ 6x^2 + 3 &= 0 \\ 6x^2 &= -3 \\ x^2 &= -\frac{1}{2} \qquad (1) \end{aligned}$$

There is no real solution to (1).

Section 4.2 Page 205 Question 20

Let the x-coordinate of the point P, where the tangent touches the parabola, be a. Then, the coordinates of the point P are $(a, a^2 + 4)$. If O is the origin, then, at $x = a$,

$$\begin{aligned} \frac{dy}{dx} &= m_{\text{OP}} \\ \frac{d}{dx}(x^2 + 4) &= \frac{a^2 + 4 - 0}{a - 0} \\ 2a &= \frac{a^2 + 4}{a} \\ 2a^2 &= a^2 + 4 \\ a^2 &= 4 \\ a &= \pm 2 \end{aligned}$$

The points whose tangents pass through the origin are $(\pm 2, 8)$. Lines that pass through the origin have the form $y = mx$. At $(-2, 8)$, the slope of the tangent is $2(-2)$ or -4. The equation of the tangent at $(-2, 8)$ is $y = -4x$. At $(2, 8)$, the slope of the tangent is $2(2)$ or 4. The equation of the tangent at $(2, 8)$ is $y = 4x$.

Section 4.2 Page 205 Question 21

Rewrite $xy = 1$ as $y = \frac{1}{x}$. The coordinates of the general point P on the curve $y = \frac{1}{x}$ can be written as $\left(x, \frac{1}{x}\right)$. Define the remote point as $\text{Q}(1, -1)$.

$$\begin{aligned} \frac{dy}{dx} &= m_{\text{PQ}} \\ -\frac{1}{x^2} &= \frac{\frac{1}{x} - (-1)}{x - 1} \\ -\frac{1}{x^2} &= \frac{1 + x}{x(x - 1)} \\ -(x^2 - x) &= x^2 + x^3 \\ -x^2 + x &= x^2 + x^3 \\ x^3 + 2x^2 - x &= 0 \\ x(x^2 + 2x - 1) &= 0 \\ x^2 + 2x - 1 &= 0, \ x \neq 0 \\ x &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-2 \pm 2\sqrt{2}}{2} \\ &= -1 \pm \sqrt{2} \end{aligned}$$

The x-coordinates are $-1 \pm \sqrt{2}$.

Section 4.2 Page 206 Question 22

a)

$$\begin{aligned} C(x) &= 100\,000 + 0.1x + 0.01x^2 \\ C'(x) &= 0.1 + 0.02x \\ C'(101) &= 0.1 + 0.02(101) \\ &= 2.12 \end{aligned}$$

b)

$$\begin{aligned} C(101) &= 100\,000 + 0.1(101) + 0.01(101)^2 \\ &= 100\,112.11 \\ C(100) &= 100\,000 + 0.1(100) + 0.01(100)^2 \\ &= 100\,110 \\ C(101) - C(100) &= 100\,112.11 - 100\,110 \\ &= 2.11 \end{aligned}$$

The cost of producing the 101st copy is \$2.11.

c) $C'(x)$ represents the rate at which the cost is increasing as the xth copy is made. $C'(101)$ is very close to the cost of producing the 101st videotape.

Section 4.2 Page 206 Question 23

a) Since $r = \frac{d}{2}$, the area can be expressed as $A = \pi\left(\frac{d}{2}\right)^2$ or $A(d) = \frac{1}{4}\pi d^2$.

b)

$$\begin{aligned} A'(d) &= 400 \\ \frac{1}{2}\pi d &= 400 \\ d &= \frac{800}{\pi} \\ A\left(\frac{800}{\pi}\right) &= \frac{1}{4}\pi\left(\frac{800}{\pi}\right)^2 \\ &= \frac{1}{4}\pi \cdot \frac{640\ 000}{\pi^2} \\ &= \frac{160\ 000}{\pi} \end{aligned}$$

The area is $\frac{160\ 000}{\pi}$ m^2.

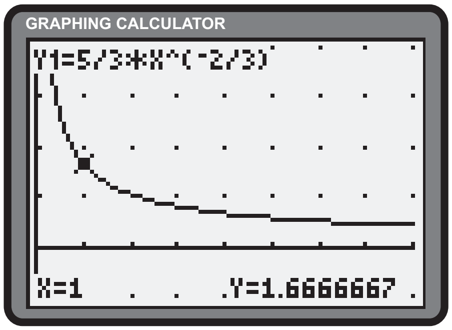
Section 4.2 Page 206 Question 24

a)

$$\begin{aligned} C(x) &= 5x^{\frac{1}{3}} \\ \frac{dC}{dx} &= \frac{5}{3}x^{-\frac{2}{3}} \end{aligned}$$

$\frac{dC}{dx}$ is the rate at which the cost is changing after x units have been produced.

b)



```
Y1=5/3*X^(-2/3)
X=1          Y=1.6666667
```

As x increases, the cost of producing additional units decreases.

Section 4.2 Page 206 Question 25

a)

$$\begin{aligned} Q(x) &= \frac{f(x) - f(a)}{x - a} \\ \frac{f(x)}{x - a} &= Q(x) + \frac{f(a)}{x - a} \end{aligned}$$

From Chapter 2, $Q(x)$ is the polynomial quotient when $f(x)$ is divided by $x - a$, and $f(a)$ is the remainder. Thus, $Q(x)$ is a polynomial.

b)

$$\begin{aligned} Q(x) &= \frac{x^n - a^n}{x - a} \\ &= \frac{(x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})}{x - a} \\ &= x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1} \end{aligned}$$

c)

$$\begin{aligned} \frac{dx^n}{dx} &= \lim_{a \to x} \frac{x^n - a^n}{x - a} \\ &= \lim_{a \to x}(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1}) \\ &= x^{n-1} + x \cdot x^{n-2} + x^2 \cdot x^{n-3} + \ldots + x^{n-1} \\ &= x^{n-1} + x^{n-1} + x^{n-1} + \ldots + x^{n-1},\ (n \text{ times}) \\ &= nx^{n-1} \end{aligned}$$

Section 4.2 Page 206 Question 26

a)

$$\begin{aligned}\frac{dV}{dr} &= \frac{d\pi r^2 h}{dr}\\ &= \pi h \cdot \frac{dr^2}{dr}\\ &= \pi h \cdot 2r\\ &= 2\pi r h\end{aligned}$$

This is the rate at which the volume is changing with respect to the radius of the blood vessel.

b)

$$\begin{aligned}\frac{dV}{dh} &= \frac{d\pi r^2 h}{dh}\\ &= \pi r^2 \cdot \frac{dh}{dh}\\ &= \pi r^2 \cdot 1\\ &= \pi r^2\end{aligned}$$

This is the rate at which the volume is changing with respect to the length of the blood vessel.

Section 4.2 Page 206 Question 27

a) Consider the continuous function $F(x) = f(x)+g(x)$, where $f(x)$ and $g(x)$ are differentiable functions. Determine $F'(x)$.

$$\begin{aligned}F'(x) &= \lim_{h\to0}\frac{F(x+h)-F(x)}{h}\\ &= \lim_{h\to0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h}\\ &= \lim_{h\to0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\ &= \lim_{h\to0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\ &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x)\end{aligned}$$

b) Consider the continuous function $F(x) = f(x)-g(x)$, where $f(x)$ and $g(x)$ are differentiable functions. Determine $F'(x)$.

$$\begin{aligned}F'(x) &= \lim_{h\to0}\frac{F(x+h)-F(x)}{h}\\ &= \lim_{h\to0}\frac{f(x+h)-g(x+h)-[f(x)-g(x)]}{h}\\ &= \lim_{h\to0}\frac{f(x+h)-f(x)-(g(x+h)-g(x))}{h}\\ &= \lim_{h\to0}\left[\frac{f(x+h)-f(x)}{h}-\frac{g(x+h)-g(x)}{h}\right]\\ &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}-\lim_{h\to0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)-g'(x)\end{aligned}$$

c) Determine the derivative of $g(x)$, where $g(x) = cf(x),\ c \in \Re$.

$$\begin{aligned}g'(x) &= \lim_{h\to0}\frac{cf(x+h)-cf(x)}{h}\\ &= c\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\ &= cf'(x)\end{aligned}$$

d) Examples will vary.

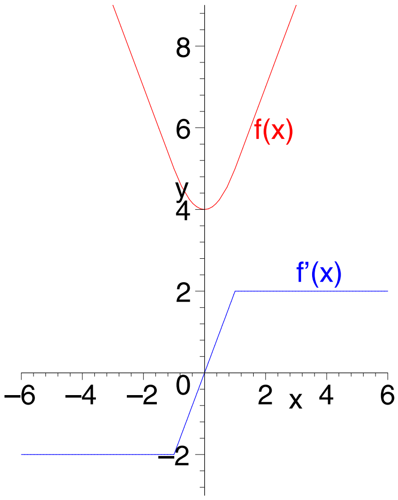
Section 4.2 Page 206 Question 28

a) If $c > 1$ and $x > 0$, the graph of the derivative of $y = cx^n$ is a vertical stretch upward by a factor of c of the graph of the derivative of $y = x^n$, hence greater.

b) If $0 < c < 1$ and $x > 0$, the graph of the derivative of $y = cx^n$ is a vertical compression by a factor of c of the graph of the derivative of $y = x^n$, hence lesser.

Section 4.2 Page 206 Question 29

a)



b) The functions that make up the separate pieces are each differentiable over all real numbers. The only concern is at the transition points. Since the slopes of the tangents for each piece at the respective transition points are the same (−2 and 2 respectively), $f(x)$ is differentiable over all real numbers.

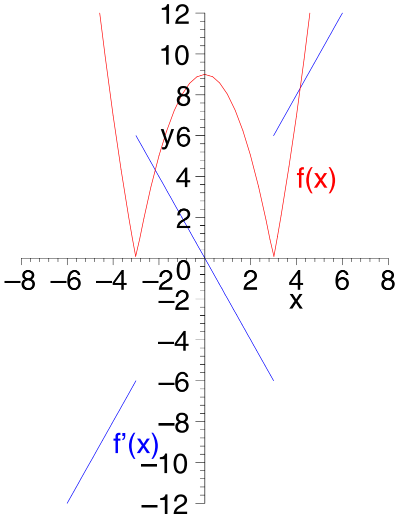
c)

$$f'(x) = \begin{cases} -2 & \text{if } x \in (-\infty, -1) \\ 2x & \text{if } x \in [-1, 1] \\ 2 & \text{if } x \in (1, \infty) \end{cases}$$

For the graph of $f'(x)$, see part a).

Section 4.2 Page 206 Question 30

a)



b) The functions that make up the separate pieces are each differentiable over all real numbers. The only concern is at the transition points. Since the slopes of the tangents for each piece at the respective transition points are different, $f(x)$ is differentiable for all real numbers except −3 and 3.

c)

$$f'(x) = \begin{cases} 2x & \text{if } x \in (-\infty, -3) \\ -2x & \text{if } x \in (-3, 3) \\ 2x & \text{if } x \in (3, \infty) \end{cases}$$

For the graph of $f'(x)$, see part a).

Section 4.2 Page 206 Question 31

Let $x(t)$ be the net position function of the man walking on the bus.

a) Since the directions of bus and man are opposite to one another, the net position function is a difference. Hence, $x(t) = 0.02t^2 - 5t$. The velocity function is $x'(t) = 0.04t - 5$ metres per second.

b) Since the directions of bus and man are the same, the position functions are additive. Hence, $s(t) = 0.02t^2 + 5t$. The velocity function is $x'(t) = 0.04t + 5$ metres per second.

4.3 The Product Rule

Practise

Section 4.3 Page 212 Question 1

a)
$$\begin{aligned} f'(x) &= (x^3 - 2x^2)\frac{d}{dx}(x) + x\frac{d}{dx}(x^3 - 2x^2) \\ &= (x^3 - 2x^2)(1) + x(3x^2 - 4x) \\ &= x^3 - 2x^2 + 3x^3 - 4x^2 \\ &= 4x^3 - 6x^2 \end{aligned}$$

b)
$$\begin{aligned} h'(x) &= (x^2)\frac{d}{dx}(3x + 2) + (3x + 2)\frac{d}{dx}(x^2) \\ &= (x^2)(3) + (3x + 2)(2x) \\ &= 3x^2 + 6x^2 + 4x \\ &= 9x^2 + 4x \end{aligned}$$

c)
$$\begin{aligned} N'(x) &= (x^2 - x)\frac{d}{dx}(x^3 - 2x + 1) + (x^3 - 2x + 1)\frac{d}{dx}(x^2 - x) \\ &= (x^2 - x)(3x^2 - 2) + (x^3 - 2x + 1)(2x - 1) \\ &= 3x^4 - 2x^2 - 3x^3 + 2x + 2x^4 - x^3 - 4x^2 + 2x + 2x - 1 \\ &= 5x^4 - 4x^3 - 6x^2 + 6x - 1 \end{aligned}$$

d)
$$\begin{aligned} g'(x) &= (4x + 1)\frac{d}{dx}(x^2 - 3) + (x^2 - 3)\frac{d}{dx}(4x + 1) \\ &= (4x + 1)(2x) + (x^2 - 3)(4) \\ &= 8x^2 + 2x + 4x^2 - 12 \\ &= 12x^2 + 2x - 12 \end{aligned}$$

e)
$$\begin{aligned} \frac{dy}{dx} &= (x^4 - x^3)\frac{d}{dx}\left(\sqrt{x}\right) + \left(\sqrt{x}\right)\frac{d}{dx}(x^4 - x^3) \\ &= (x^4 - x^3)\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x}(4x^3 - 3x^2) \\ &= \frac{1}{2}x^{\frac{7}{2}} - \frac{1}{2}x^{\frac{5}{2}} + 4x^{\frac{7}{2}} - 3x^{\frac{5}{2}} \\ &= \frac{9}{2}x^{\frac{7}{2}} - \frac{7}{2}x^{\frac{5}{2}} \end{aligned}$$

f)
$$\begin{aligned} f'(x) &= (3x^2 - 2x + 1)\frac{d}{dx}(x - 1) + (x - 1)\frac{d}{dx}(3x^2 - 2x + 1) \\ &= (3x^2 - 2x + 1)(1) + (x - 1)(6x - 2) \\ &= 3x^2 - 2x + 1 + 6x^2 - 2x - 6x + 2 \\ &= 9x^2 - 10x + 3 \end{aligned}$$

g)
$$\begin{aligned} h'(t) &= (5t + t^{-1})\frac{d}{dt}(6t^{-3} - t^4) + (6t^{-3} - t^4)\frac{d}{dt}(5t + t^{-1}) \\ &= (5t + t^{-1})(-18t^{-4} - 4t^3) + (6t^{-3} - t^4)(5 - t^{-2}) \\ &= -90t^{-3} - 20t^4 - 18t^{-5} - 4t^2 + 30t^{-3} - 6t^{-5} - 5t^4 + t^2 \\ &= -60t^{-3} - 25t^4 - 24t^{-5} - 3t^2 \end{aligned}$$

h)
$$\begin{aligned} \frac{dp}{dm} &= \left(m^3 - m^{-1}\right)\frac{d}{dm}\left(m^{-2} + m\right) + \left(m^{-2} + m\right)\frac{d}{dm}\left(m^3 - m^{-1}\right) \\ &= \left(m^3 - m^{-1}\right)\left(-2m^{-3} + 1\right) + \left(m^{-2} + m\right)\left(3m^2 + m^{-2}\right) \\ &= -2 + m^3 + 2m^{-4} - m^{-1} + 3 + m^{-4} + 3m^3 + m^{-1} \\ &= 4m^3 + 1 + 3m^{-4} \end{aligned}$$

i)

$$\begin{aligned} F'(y) &= \left(3y - 4\sqrt{y} + \frac{2}{y}\right)\frac{d}{dy}\left(\sqrt{y}\right) + \left(\sqrt{y}\right)\frac{d}{dy}\left(3y - 4\sqrt{y} + \frac{2}{y}\right) \\ &= \left(3y - 4\sqrt{y} + \frac{2}{y}\right)\left(\frac{1}{2\sqrt{y}}\right) + \left(\sqrt{y}\right)\left(3 - \frac{2}{\sqrt{y}} - \frac{2}{y^2}\right) \\ &= \frac{3}{2}y^{\frac{1}{2}} - 2 + y^{-\frac{3}{2}} + 3y^{\frac{1}{2}} - 2 - 2y^{-\frac{3}{2}} \\ &= \frac{9}{2}y^{\frac{1}{2}} - 4 - y^{-\frac{3}{2}} \end{aligned}$$

j)

$$\begin{aligned} G'(y) &= \left(y + \frac{1}{\sqrt{y}}\right)\frac{d}{dy}\left(y - \sqrt{y}\right) + \left(y - \sqrt{y}\right)\frac{d}{dy}\left(y + \frac{1}{\sqrt{y}}\right) \\ &= \left(y + \frac{1}{\sqrt{y}}\right)\left(1 - \frac{1}{2\sqrt{y}}\right) + \left(y - \sqrt{y}\right)\left(1 - \frac{1}{2}y^{-\frac{3}{2}}\right) \\ &= y - \frac{1}{2}y^{\frac{1}{2}} + y^{-\frac{1}{2}} - \frac{1}{2}y^{-1} + y - \frac{1}{2}y^{-\frac{1}{2}} - y^{\frac{1}{2}} + \frac{1}{2}y^{-1} \\ &= 2y - \frac{3}{2}y^{\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \end{aligned}$$

k)

$$\begin{aligned} \frac{dy}{dx} &= (3x - 5x^4)\frac{d}{dx}\left(x^{\frac{1}{3}}\right) + \left(\sqrt[3]{x}\right)\frac{d}{dx}(3x - 5x^4) \\ &= (3x - 5x^4)\left(\frac{1}{3}x^{-\frac{2}{3}}\right) + \left(x^{\frac{1}{3}}\right)(3 - 20x^3) \\ &= x^{\frac{1}{3}} - \frac{5}{3}x^{\frac{10}{3}} + 3x^{\frac{1}{3}} - 20x^{\frac{10}{3}} \\ &= 4x^{\frac{1}{3}} - \frac{65}{3}x^{\frac{10}{3}} \end{aligned}$$

l)

$$\begin{aligned} c'(k) &= \left(k^2 + 3k^3 - k^4\right)\frac{d}{dk}\left(k^{-2}\right) + \left(k^{-2}\right)\frac{d}{dk}\left(k^2 + 3k^3 - k^4\right) \\ &= \left(k^2 + 3k^3 - k^4\right)\left(-2k^{-3}\right) + \left(k^{-2}\right)\left(2k + 9k^2 - 4k^3\right) \\ &= -2k^{-1} - 6 + 2k + 2k^{-1} + 9 - 4k \\ &= 3 - 2k \end{aligned}$$

m)

$$\begin{aligned} y &= (x^2 - 1)\frac{d}{dx}(x^2 + 1) + (x^2 + 1)\frac{d}{dx}(x^2 - 1) \\ &= (x^2 - 1)(2x) + (x^2 + 1)(2x) \\ &= 2x(x^2 - 1 + x^2 + 1) \\ &= 2x(2x^2) \\ &= 4x^3 \end{aligned}$$

n)

$$\begin{aligned} \frac{dy}{dx} &= (x - 5x^3)\frac{d}{dx}\left(x^{\frac{1}{2}}\right) + \left(\sqrt{x}\right)\frac{d}{dx}(x - 5x^3) \\ &= (x - 5x^3)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + \left(x^{\frac{1}{2}}\right)(1 - 15x^2) \\ &= \frac{1}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{\frac{5}{2}} + x^{\frac{1}{2}} - 15x^{\frac{5}{2}} \\ &= \frac{3}{2}x^{\frac{1}{2}} - \frac{35}{2}x^{\frac{5}{2}} \end{aligned}$$

o)

$$\begin{aligned} \frac{dy}{dx} &= (x^4 - 7x)\frac{d}{dx}\left(2x^{\frac{1}{2}} - 5\right) + \left(2\sqrt{x} - 5\right)\frac{d}{dx}(x^4 - 7x) \\ &= (x^4 - 7x)\left(x^{-\frac{1}{2}}\right) + \left(2x^{\frac{1}{2}} - 5\right)(4x^3 - 7) \\ &= x^{\frac{7}{2}} - 7x^{\frac{1}{2}} + 8x^{\frac{7}{2}} - 14x^{\frac{1}{2}} - 20x^3 + 35 \\ &= 9x^{\frac{7}{2}} - 20x^3 - 21x^{\frac{1}{2}} + 35 \end{aligned}$$

p)

$$\begin{aligned} a'(t) &= \left(t^2 - t^{-1}\right) \frac{d}{dt} \left(6t^{\frac{1}{2}} - 2t^{-2}\right) + \left(6\sqrt{t} - 2t^{-2}\right) \frac{d}{dt} \left(t^2 - t^{-1}\right) \\ &= \left(t^2 - t^{-1}\right) \left(3t^{-\frac{1}{2}} + 4t^{-3}\right) + \left(6t^{\frac{1}{2}} - 2t^{-2}\right) \left(2t + t^{-2}\right) \\ &= 3t^{\frac{3}{2}} + 4t^{-1} - 3t^{-\frac{3}{2}} - 4t^{-4} + 12t^{\frac{3}{2}} + 6t^{-\frac{3}{2}} - 4t^{-1} - 2t^{-4} \\ &= 15t^{\frac{3}{2}} + 3t^{-\frac{3}{2}} - 6t^{-4} \end{aligned}$$

q)

$$\begin{aligned} g'(u) &= \left(bu + au^2\right) \frac{d}{du} \left(au - bu^3\right) + \left(au - bu^3\right) \frac{d}{du} \left(bu + au^2\right) \\ &= \left(bu + au^2\right) \left(a - 3bu^2\right) + \left(au - bu^3\right) (b + 2au) \\ &= abu - 3b^2u^3 + a^2u^2 - 3abu^4 + abu + 2a^2u^2 - b^2u^3 - 2abu^4 \\ &= 2abu + 3a^2u^2 - 4b^2u^3 - 5abu^4 \end{aligned}$$

r)

$$\begin{aligned} A'(v) &= \left(\sqrt{v} + v\right) \frac{d}{dv} \left(v - v^{-1}\right) + \left(v - v^{-1}\right) \frac{d}{dv} \left(v^{\frac{1}{2}} + v\right) \\ &= \left(v^{\frac{1}{2}} + v\right) \left(1 + v^{-2}\right) + \left(v - v^{-1}\right) \left(\frac{1}{2}v^{-\frac{1}{2}} + 1\right) \\ &= v^{\frac{1}{2}} + v^{-\frac{3}{2}} + v + v^{-1} + \frac{1}{2}v^{\frac{1}{2}} + v - \frac{1}{2}v^{-\frac{3}{2}} - v^{-1} \\ &= 2v + \frac{3}{2}v^{\frac{1}{2}} + \frac{1}{2}v^{-\frac{3}{2}} \end{aligned}$$

Section 4.3 Page 212 Question 3

a) i)

$$\begin{aligned} f'(x) &= (2x - 3) \frac{d}{dx} \left(x^2\right) + \left(x^2\right) \frac{d}{dx} (2x - 3) \\ &= (2x - 3)(2x) + \left(x^2\right)(2) \\ f'(2) &= (2(2) - 3)(2(2)) + \left(2^2\right)(2) \\ &= (1)(4) + (4)(2) \\ &= 12 \end{aligned}$$

ii)

$$\begin{aligned} f(x) &= x^2(2x - 3) \\ &= 2x^3 - 3x^2 \\ f'(x) &= \frac{d}{dx} \left(2x^3 - 3x^2\right) \\ &= 6x^2 - 6x \\ f'(2) &= 6(2)^2 - 6(2) \\ &= 24 - 12 \\ &= 12 \end{aligned}$$

iii) Answers will vary.

b) i)

$$\begin{aligned} f'(x) &= \left(x^3 + 4x\right) \frac{d}{dx} (x - 4) + (x - 4) \frac{d}{dx} \left(x^3 + 4x\right) \\ &= \left(x^3 + 4x\right)(1) + (x - 4)\left(3x^2 + 4\right) \\ f'(2) &= \left(2^3 + 4(2)\right) + (2 - 4)\left(3(2)^2 + 4\right) \\ &= (16) + (-2)(16) \\ &= -16 \end{aligned}$$

ii)

$$\begin{aligned} f(x) &= (x - 4)(x^3 + 4x) \\ &= x^4 + 4x^2 - 4x^3 - 16x \\ f'(x) &= \frac{d}{dx} \left(x^4 + 4x^2 - 4x^3 - 16x\right) \\ &= 4x^3 + 8x - 12x^2 - 16 \\ f'(2) &= 4(2)^3 + 8(2) - 12(2)^2 - 16 \\ &= 32 + 16 - 48 - 16 \\ &= -16 \end{aligned}$$

iii) Answers will vary.

c) i)

$$\begin{aligned}
f'(x) &= \left(2-3x+5x^2\right)\frac{d}{dx}\left(x-3x^2\right)+\left(x-3x^2\right)\frac{d}{dx}\left(2-3x+5x^2\right)\\
&= \left(2-3x+5x^2\right)(1-6x)+\left(x-3x^2\right)(-3+10x)\\
f'(2) &= \left(2-3(2)+5(2)^2\right)(1-6(2))+\left(2-3(2)^2\right)(-3+10(2))\\
&= (2-6+20)(-11)+(2-12)(-3+20)\\
&= (16)(-11)+(-10)(17)\\
&= (-176)+(-170)\\
&= -346
\end{aligned}$$

ii)

$$\begin{aligned}
f(x) &= (x-3x^2)(2-3x+5x^2)\\
&= 2x-3x^2+5x^3-6x^2+9x^3-15x^4\\
&= 2x-9x^2+14x^3-15x^4\\
f'(x) &= \frac{d}{dx}\left(2x-9x^2+14x^3-15x^4\right)\\
&= 2-18x+42x^2-60x^3\\
f'(2) &= 2-18(2)+42(2)^2-60(2)^3\\
&= 2-36+168-480\\
&= -346
\end{aligned}$$

iii) Answers will vary.

d) i)

$$\begin{aligned}
f'(x) &= \left(\sqrt{x}+\frac{1}{x}\right)\frac{d}{dx}\left(\sqrt{x}-x\right)+\left(\sqrt{x}-x\right)\frac{d}{dx}\left(\sqrt{x}+\frac{1}{x}\right)\\
&= \left(\sqrt{x}+\frac{1}{x}\right)\left(\frac{1}{2\sqrt{x}}-1\right)+\left(\sqrt{x}-x\right)\left(\frac{1}{2\sqrt{x}}-\frac{1}{x^2}\right)\\
f'(2) &= \left(\sqrt{2}+\frac{1}{2}\right)\left(\frac{1}{2\sqrt{2}}-1\right)+\left(\sqrt{2}-2\right)\left(\frac{1}{2\sqrt{2}}-\frac{1}{2^2}\right)\\
&= \frac{1}{2}-\sqrt{2}+\frac{1}{4\sqrt{2}}-\frac{1}{2}+\frac{1}{2}-\frac{\sqrt{2}}{4}-\frac{1}{\sqrt{2}}+\frac{1}{2}\\
&= 1-\sqrt{2}+\frac{1}{4\sqrt{2}}-\frac{\sqrt{2}}{4}-\frac{1}{\sqrt{2}}\\
&= 1-\frac{8}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}-\frac{2}{4\sqrt{2}}-\frac{4}{4\sqrt{2}}\\
&= 1-\frac{13}{4\sqrt{2}}
\end{aligned}$$

ii)

$$\begin{aligned}
f(x) &= \left(\sqrt{x}-x\right)\left(\sqrt{x}+\frac{1}{x}\right)\\
&= x+x^{-\frac{1}{2}}-x^{\frac{3}{2}}-1\\
f'(x) &= \frac{d}{dx}\left(x+x^{-\frac{1}{2}}-x^{\frac{3}{2}}-1\right)\\
&= 1-\frac{1}{2}x^{-\frac{3}{2}}-\frac{3}{2}x^{\frac{1}{2}}\\
&= 1-\frac{1}{2\sqrt{x^3}}-\frac{3\sqrt{x}}{2}\\
f'(2) &= 1-\frac{1}{2\sqrt{2^3}}-\frac{3\sqrt{2}}{2}\\
&= 1-\frac{1}{4\sqrt{2}}-\frac{12}{4\sqrt{2}}\\
&= 1-\frac{13}{4\sqrt{2}}
\end{aligned}$$

iii) Answers will vary.

Apply, Solve, Communicate

Section 4.3 Page 212 Question 5

a)

$$\begin{aligned}(fg)'(2) &= g(2)f'(2) + f(2)g'(2)\\ &= 3(-1) + 4(2)\\ &= -3 + 8\\ &= 5\end{aligned}$$

b)

$$\begin{aligned}(fg)'(3) &= g(3)f'(3) + f(3)g'(3)\\ f(3)g'(3) &= (fg)'(3) - g(3)f'(3)\\ g'(3) &= \frac{(fg)'(3) - g(3)f'(3)}{f(3)}\\ &= \frac{30 - 4(5)}{-2}\\ &= \frac{10}{-2}\\ &= -5\end{aligned}$$

c)

$$\begin{aligned}(fg)'(-2) &= g(-2)f'(-2) + f(-2)g'(-2)\\ g(-2)f'(-2) &= (fg)'(-2) - f(-2)g'(-2)\\ g(-2) &= \frac{(fg)'(-2) - f(-2)g'(-2)}{f'(-2)}\\ &= \frac{15 - 5(6)}{-3}\\ &= \frac{-15}{-3}\\ &= 5\end{aligned}$$

Section 4.3 Page 212 Question 6

a)

$$\begin{aligned}g'(x) &= f(x)\frac{d}{dx}(x) + (x)\frac{d}{dx}(f(x))\\ &= f(x)(1) + (x)f'(x)\\ &= f(x) + xf'(x)\end{aligned}$$

b)

$$\begin{aligned}g'(x) &= f(x)\frac{d}{dx}\left(\sqrt{x}\right) + \left(\sqrt{x}\right)\frac{d}{dx}(f(x))\\ &= f(x)\left(\frac{1}{2\sqrt{x}}\right) + \left(\sqrt{x}\right)f'(x)\\ &= \frac{f(x)}{2\sqrt{x}} + \sqrt{x}f'(x)\end{aligned}$$

c)

$$\begin{aligned}s'(t) &= f(t)\frac{d}{dt}\left(t^2\right) + \left(t^2\right)\frac{d}{dt}(f(t))\\ &= f(t)(2t) + (t^2)f'(t)\\ &= 2tf(t) + t^2f'(t)\end{aligned}$$

d)

$$\begin{aligned}A'(m) &= f(m)\frac{d}{dm}(m^c) + (m^c)\frac{d}{dm}(f(m))\\ &= f(m)\left(cm^{c-1}\right) + (m^c)f'(m)\\ &= cm^{c-1}f(m) + m^cf'(m)\end{aligned}$$

e)

$$\begin{aligned}g'(x) &= f(x)\frac{d}{dx}\left(\frac{1}{x}\right) + \left(\frac{1}{x}\right)\frac{d}{dx}(f(x))\\ &= f(x)\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x}\right)f'(x)\\ &= -\frac{f(x)}{x^2} + \frac{f'(x)}{x}\end{aligned}$$

f)

$$\begin{aligned}N'(x) &= f(x)\frac{d}{dx}\left((x+2)^2\right) + (x+2)^2\frac{d}{dx}(f(x))\\ &= f(x)\,(2(x+2)) + (x+2)^2f'(x)\\ &= 2(x+2)f(x) + (x+2)^2f'(x)\end{aligned}$$

Section 4.3 Page 212 Question 7

a)

$$\begin{aligned} P(0) &= 12(2(0)^2 + 100)(0 + 20) \\ &= 12(100)(20) \\ &= 24\,000 \end{aligned}$$

The current population of the town is 24 000 people.

$$\begin{aligned} P(6) &= 12(2(6)^2 + 100)(6 + 20) \\ &= 12(172)(26) \\ &= 53\,664 \end{aligned}$$

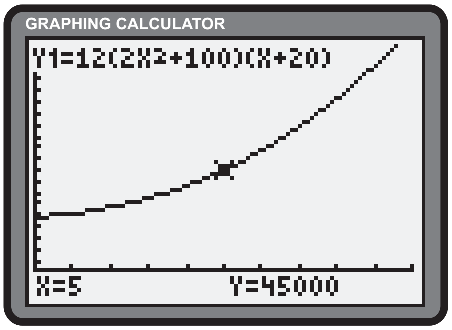
The population in 6 years will be 53 664 people.

b)

$$\begin{aligned} P(t) &= 12(2t^2 + 100)(t + 20) \\ &= 24(t^2 + 50)(t + 20) \\ &= 24(t^3 + 20t^2 + 50t + 1000) \\ \frac{dP}{dt} &= 24(3t^2 + 40t + 50) \\ &= 72t^2 + 960t + 1200 \end{aligned}$$

$\frac{dP}{dt}$ represents the rate of increase of population, in people per year.

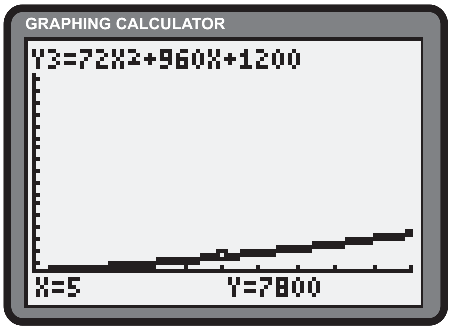
c) $P(t) = 12(2t^2 + 100)(t + 20)$



Y1=12(2X²+100)(X+20)

X=5 Y=45000

$$\frac{dP}{dt} = 72t^2 + 960t + 1200$$



Y3=72X²+960X+1200

X=5 Y=7800

d) Answers will vary.

e) Answers will vary.

Section 4.3 Page 213 Question 8

a) $V = IR$

$$\begin{aligned} V(t) &= \left(4.85 - 0.01t^2\right)(15.00 + 0.11t) \\ &= 72.75 + 0.5335t - 0.15t^2 - 0.0011t^3 \\ V'(t) &= \frac{d}{dt}\left(72.75 + 0.5335t - 0.15t^2 - 0.0011t^3\right) \\ &= 0.5335 - 0.3t - 0.0033t^2 \end{aligned}$$

The rate of change of voltage after t seconds is $0.5335 - 0.3t - 0.0033t^2$ volts per second.

b)

$$\begin{aligned} V'(0) &= 0.5335 - 0.3(0) - 0.0033(0)^2 \\ &= 0.5335 \end{aligned}$$

The rate of change of voltage after 0 seconds is 0.5335 volts per second.

c)

$$\begin{aligned} V'(3) &= 0.5335 - 0.3(3) - 0.0033(3)^2 \\ &= 0.5335 - 0.9 - 0.0297 \\ &= -0.3962 \end{aligned}$$

The rate of change of voltage after 3 seconds is −0.3962 volts per second.

Section 4.3 Page 213 Question 9

$$\begin{aligned} f(x) &= (x^2 - 10)(x + 3) \\ &= x^3 + 3x^2 - 10x - 30 \end{aligned}$$

Determine values of x for which the tangent is horizontal.

$$\begin{aligned} f'(x) &= 0 \\ \frac{d}{dx}\left(x^3 + 3x^2 - 10x - 30\right) &= 0 \\ 3x^2 + 6x - 10 &= 0 \\ x &= \frac{-(6) \pm \sqrt{(6)^2 - 4(3)(-10)}}{2(3)} \\ &= \frac{-6 \pm \sqrt{156}}{6} \\ x &\doteq -3.08 \text{ or } 1.08 \end{aligned}$$

The tangents to the graph are horizontal at approximately −3.08 and 1.08.

Section 4.3 Page 213 Question 10

a)

$$\begin{aligned} R(x) &= xp(x) \\ &= x\left(\frac{750}{\sqrt{x}} - 5\right) \\ &= 750\sqrt{x} - 5x \end{aligned}$$

i) Determine the rate of change of revenue after x calculators are sold.

$$\begin{aligned} R'(x) &= \frac{d}{dx}\left(750\sqrt{x} - 5x\right) \\ &= \frac{750}{2\sqrt{x}} - 5 \\ &= \frac{375}{\sqrt{x}} - 5 \end{aligned}$$

ii) After 1000 calculators are sold,

$$\begin{aligned} R'(1000) &= \frac{375}{\sqrt{1000}} - 5 \\ &= \frac{75}{2\sqrt{10}} - 5 \\ &\doteq 6.86 \end{aligned}$$

iii) After 10 000 calculators are sold,

$$\begin{aligned} R'(10\,000) &= \frac{375}{\sqrt{10\,000}} - 5 \\ &= 3.75 - 5 \\ &= -1.25 \end{aligned}$$

b) The manufacturer takes in less money by selling more calculators, because the price is lower.

Section 4.3 Page 213 Question 11

a)

$$\begin{aligned} (fgh)' &= gh\frac{d}{dx}(f) + f\frac{d}{dx}(gh) \\ &= gh\frac{d}{dx}(f) + f\left[h\frac{d}{dx}(g) + g\frac{d}{dx}(h)\right] \\ &= ghf' + f\left[hg' + gh'\right] \\ &= ghf' + fhg' + fgh' \end{aligned}$$

b)

$$
\begin{aligned}
y &= \sqrt{x}(2x-1)(x^2+3x-2)\\
\frac{dy}{dx} &= (2x-1)(x^2+3x-2)\frac{d}{dx}(\sqrt{x}) + \sqrt{x}(x^2+3x-2)\frac{d}{dx}(2x-1) + \sqrt{x}(2x-1)\frac{d}{dx}\left(x^2+3x-2\right)\\
&= (2x-1)(x^2+3x-2)\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x}(x^2+3x-2)(2) + \sqrt{x}(2x-1)(2x+3)\\
&= \frac{1}{2\sqrt{x}}[(2x-1)(x^2+3x-2) + 4x(x^2+3x-2) + 2x(2x-1)(2x+3)]\\
&= \frac{1}{2\sqrt{x}}(2x^3+6x^2-4x-x^2-3x+2+4x^3+12x^2-8x+8x^3+8x^2-6x)\\
&= \frac{1}{2\sqrt{x}}(14x^3+25x^2-21x+2)\\
&= 7x^{\frac{5}{2}} + \frac{25}{2}x^{\frac{3}{2}} - \frac{21}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}}
\end{aligned}
$$

Section 4.3 Page 213 Question 12

a)

$$
\begin{aligned}
\frac{d}{dx}[f(x)]^2 &= \frac{d}{dx}[f(x)f(x)]\\
&= f(x)\frac{d}{dx}(f(x)) + f(x)\frac{d}{dx}(f(x))\\
&= f(x)f'(x) + f(x)f'(x)\\
&= 2f(x)f'(x)
\end{aligned}
$$

b) i)

$$
\begin{aligned}
y &= (x^2+3x+1)^2\\
\frac{dy}{dx} &= 2(x^2+3x+1)\frac{d}{dx}\left(x^2+3x+1\right)\\
&= 2(x^2+3x+1)(2x+3)
\end{aligned}
$$

ii)

$$
\begin{aligned}
y &= (2-x^3)^2\\
\frac{dy}{dx} &= 2(2-x^3)\frac{d}{dx}\left(2-x^3\right)\\
&= 2(2-x^3)(-3x^2)\\
&= -6x^2(2-x^3)
\end{aligned}
$$

iii)

$$
\begin{aligned}
y &= (1-x)^2(x^3+x)^2\\
\frac{dy}{dx} &= (x^3+x)^2\frac{d}{dx}(1-x)^2 + (1-x)^2\frac{d}{dx}\left(x^3+x\right)^2\\
&= (x^3+x)^2(2)(1-x)(-1) + (1-x)^2(2)(x^3+x)(3x^2+1)\\
&= 2(1-x)(x^3+x)[-(x^3+x) + (1-x)(3x^2+1)]\\
&= 2(1-x)(x^3+x)(-x^3-x+3x^2+1-3x^3-x)\\
&= 2(1-x)(x^3+x)(-4x^3+3x^2-2x+1)
\end{aligned}
$$

Section 4.3 Page 213 Question 13

a)

$$
\begin{aligned}
\frac{d}{dx}[f(x)]^3 &= \frac{d}{dx}[f(x)f(x)f(x)]\\
&= f(x)f(x)\frac{d}{dx}(f(x)) + f(x)f(x)\frac{d}{dx}(f(x)) + f(x)f(x)\frac{d}{dx}(f(x))\\
&= [f(x)]^2 f'(x) + [f(x)]^2 f'(x) + [f(x)]^2 f'(x)\\
&= 3[f(x)]^2 f'(x)
\end{aligned}
$$

b) i)

$$
\begin{aligned}
y &= (x-1)^3\\
\frac{dy}{dx} &= 3(x-1)^2\frac{d}{dx}(x-1)\\
&= 3(x-1)^2(1)\\
&= 3(x-1)^2
\end{aligned}
$$

ii)

$$\begin{aligned} y &= (x^2+3x-1)^3 \\ \frac{dy}{dx} &= 3(x^2+3x-1)^2\frac{d}{dx}(x^2+3x-1) \\ &= 3(x^2+3x-1)^2(2x+3) \end{aligned}$$

iii)

$$\begin{aligned} y &= (\sqrt{x}-x^2)^3 \\ \frac{dy}{dx} &= 3(\sqrt{x}-x^2)^2\frac{d}{dx}(\sqrt{x}-x^2) \\ &= 3(\sqrt{x}-x^2)^2\left(\frac{1}{2\sqrt{x}}-2x\right) \end{aligned}$$

Section 4.3 Page 213 Question 14

$$\begin{aligned} \frac{d\,cf(x)}{dx} &= f(x)\frac{dc}{dx}+c\frac{d\,f(x)}{dx} \\ &= f(x)(0)+cf'(x) \\ &= cf'(x) \end{aligned}$$

4.4 The Quotient Rule

Practise

Section 4.4 Page 218 Question 1

a)

$$\begin{aligned} f'(x) &= \frac{x\frac{d}{dx}(x-1)-(x-1)\frac{d}{dx}x}{x^2} \\ &= \frac{x(1)-(x-1)(1)}{x^2} \\ &= \frac{1}{x^2} \\ f'(2) &= \frac{1}{2^2} \\ &= \frac{1}{4} \end{aligned}$$

Domains of f and f' are the same: $x \neq 0$.

b)

$$\begin{aligned} N'(x) &= \frac{(2x+1)\frac{d}{dx}x^2-x^2\frac{d}{dx}(2x+1)}{(2x+1)^2} \\ &= \frac{(2x+1)(2x)-x^2(2)}{(2x+1)^2} \\ &= \frac{4x^2+2x-2x^2}{(2x+1)^2} \\ &= \frac{2x(x+1)}{(2x+1)^2} \\ N'(2) &= \frac{2(2)(2+1)}{(2(2)+1)^2} \\ &= \frac{12}{25} \end{aligned}$$

Domains of N and N' are the same: $x \neq -\frac{1}{2}$.

c)

$$\begin{aligned} H'(x) &= \frac{(x^2-1)\frac{d}{dx}(2-3x)-(2-3x)\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \\ &= \frac{(x^2-1)(-3)-(2-3x)(2x)}{(x^2-1)^2} \\ &= \frac{-3x^2+3-4x+6x^2}{(x^2-1)^2} \\ &= \frac{3x^2-4x+3}{(x^2-1)^2} \\ H'(2) &= \frac{3(2)^2-4(2)+3}{(2^2-1)^2} \\ &= \frac{7}{9} \end{aligned}$$

Domains of H and H' are the same: $x \neq \pm 1$.

d)

$$\begin{aligned}
g'(x) &= \frac{(x^2+3x-2)\frac{d}{dx}(2x) - (2x)\frac{d}{dx}(x^2+3x-2)}{(x^2+3x-2)^2} \\
&= \frac{(x^2+3x-2)(2) - (2x)(2x+3)}{(x^2+3x-2)^2} \\
&= \frac{2x^2+6x-4-4x^2-6x}{(x^2+3x-2)^2} \\
&= \frac{-2x^2-4}{(x^2+3x-2)^2}
\end{aligned}$$

$$\begin{aligned}
g'(2) &= \frac{-2(2)^2-4}{(2^2+3(2)-2)^2} \\
&= \frac{-12}{64} \\
&= -\frac{3}{16}
\end{aligned}$$

Domains of g and g' are the same: $x \neq \frac{-3 \pm \sqrt{17}}{2}$.

e)

$$\begin{aligned}
M'(x) &= \frac{(3-2x)\frac{d}{dx}(x^2+2) - (x^2+2)\frac{d}{dx}(3-2x)}{(3-2x)^2} \\
&= \frac{(3-2x)(2x) - (x^2+2)(-2)}{(3-2x)^2} \\
&= \frac{6x-4x^2+2x^2+4}{(3-2x)^2} \\
&= \frac{4+6x-2x^2}{(3-2x)^2}
\end{aligned}$$

$$\begin{aligned}
M'(2) &= \frac{4+6(2)-2(2)^2}{(3-2(2))^2} \\
&= \frac{4+12-8}{(3-4)^2} \\
&= 0
\end{aligned}$$

Domains of M and M' are the same: $x \neq \frac{3}{2}$.

f)

$$\begin{aligned}
A'(x) &= \frac{(x^2+2x+1)\frac{d}{dx}(x^3-1) - (x^3-1)\frac{d}{dx}(x^2+2x+1)}{(x^2+2x+1)^2} \\
&= \frac{(x^2+2x+1)(3x^2) - (x^3-1)(2x+2)}{(x^2+2x+1)^2} \\
&= \frac{3x^4+6x^3+3x^2-2x^4-2x^3+2x+2}{(x^2+2x+1)^2} \\
&= \frac{x^4+4x^3+3x^2+2x+2}{(x^2+2x+1)^2} \\
A'(2) &= \frac{2^4+4(2)^3+3(2)^2+2(2)+2}{(2^2+2(2)+1)^2} \\
&= \frac{16+32+12+4+2}{9^2} \\
&= \frac{22}{27}
\end{aligned}$$

Domains of A and A' are the same: $x \neq -1$.

g)

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{(x-1)\frac{d}{dx}\sqrt{x} - \sqrt{x}\frac{d}{dx}(x-1)}{(x-1)^2} \\
&= \frac{(x-1)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(1)}{(x-1)^2} \\
&= \frac{\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}} - \sqrt{x}}{(x-1)^2} \\
&= \frac{-\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}}{(x-1)^2} \\
&= \frac{-x-1}{2\sqrt{x}(x-1)^2} \\
\frac{dy}{dx}_{|x=2} &= \frac{-2-1}{2\sqrt{2}(2-1)^2} \\
&= -\frac{3}{2\sqrt{2}}
\end{aligned}
$$

Domains of y and y' are the same: $x \neq 1$.

h)

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\left(\sqrt{x}-x^2\right)\frac{d}{dx}\left(\sqrt{x}+1\right) - \left(\sqrt{x}+1\right)\frac{d}{dx}\left(\sqrt{x}-x^2\right)}{\left(\sqrt{x}-x^2\right)^2} \\
&= \frac{\left(\sqrt{x}-x^2\right)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}+1\right)\left(\frac{1}{2\sqrt{x}} - 2x\right)}{\left(\sqrt{x}-x^2\right)^2} \\
&= \frac{\frac{\sqrt{x}}{2\sqrt{x}} - \frac{x^2}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}} + 2x\sqrt{x} - \frac{1}{2\sqrt{x}} + 2x}{\left(\sqrt{x}-x^2\right)^2} \\
&= \frac{-\frac{x^2}{2\sqrt{x}} + 2x\sqrt{x} - \frac{1}{2\sqrt{x}} + 2x}{\left(\sqrt{x}-x^2\right)^2} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} \\
&= \frac{-x^2 + 4x^2 - 1 + 4x\sqrt{x}}{2\sqrt{x}(\sqrt{x}-x^2)^2} \\
&= \frac{3x^2 + 4x\sqrt{x} - 1}{2\sqrt{x}(\sqrt{x}-x^2)^2} \\
\frac{dy}{dx}_{|x=2} &= \frac{3(2)^2 + 4(2)\sqrt{2} - 1}{2\sqrt{2}(\sqrt{2}-2^2)^2} \\
&= \frac{12 + 8\sqrt{2} - 1}{2\sqrt{2}(2 + 16 - 8\sqrt{2})} \\
&= \frac{11 + 8\sqrt{2}}{2\sqrt{2}(18 - 8\sqrt{2})} \\
&= \frac{11 + 8\sqrt{2}}{4(9\sqrt{2} - 8)} \cdot \frac{9\sqrt{2} + 8}{9\sqrt{2} + 8} \\
&= \frac{163\sqrt{2} + 232}{392}
\end{aligned}
$$

Domains of y and y' are the same: $x \neq 0, 1$.

i)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x^2+4x+1)\frac{d}{dx}(x^2-3x+2)-(x^2-3x+2)\frac{d}{dx}(x^2+4x+1)}{(x^2+4x+1)^2}\\
&= \frac{(x^2+4x+1)(2x-3)-(x^2-3x+2)(2x+4)}{(x^2+4x+1)^2}\\
&= \frac{2x^3-3x^2+8x^2-12x+2x-3-2x^3-4x^2+6x^2+12x-4x-8}{(x^2+4x+1)^2}\\
&= \frac{7x^2-2x-11}{(x^2+4x+1)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx}_{|x=2} &= \frac{7(2)^2-2(2)-11}{(2^2+4(2)+1)^2}\\
&= \frac{13}{13^2}\\
&= \frac{1}{13}
\end{aligned}$$

Domains of y and y' are the same: $x \neq -2 \pm \sqrt{3}$.

j)

$$\begin{aligned}
f'(x) &= \frac{(x^2+4x+4)\frac{d}{dx}(1)-(1)\frac{d}{dx}(x^2+4x+4)}{(x^2+4x+4)^2}\\
&= \frac{(x^2+4x+4)(0)-(1)(2x+4)}{[(x+2)^2]^2}\\
&= -\frac{2x+4}{(x+2)^4}\\
&= -\frac{2}{(x+2)^3}
\end{aligned}$$

$$\begin{aligned}
f'(2) &= -\frac{2}{(2+2)^3}\\
&= -\frac{2}{64}\\
&= -\frac{1}{32}
\end{aligned}$$

Domains of f and f' are the same: $x \neq -2$.

k)

$$\begin{aligned}
g'(x) &= \frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}\\
&= \frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}\\
&= \frac{acx+ad-acx-bc}{(cx+d)^2}\\
&= \frac{ad-bc}{(cx+d)^2}
\end{aligned}$$

$$\begin{aligned}
g'(2) &= \frac{ad-bc}{(c(2)+d)^2}\\
&= \frac{ad-bc}{(2c+d)^2}
\end{aligned}$$

Domains of g and g' are the same: $x \neq -\frac{d}{c}$.

l)

$$h(x) = \frac{1 + \frac{1}{x}}{x + 1}$$

$$= \frac{\frac{x+1}{x}}{x + 1}$$

$$= \frac{1}{x}$$

$$h'(x) = -\frac{1}{x^2}$$

$$h'(2) = -\frac{1}{2^2}$$

$$= -\frac{1}{4}$$

Domains of h and h' are the same: $x \neq 0, -1$.

m)

$$w'(x) = \frac{(1 - 2x)\frac{d}{dx}(2 + x) - (2 + x)\frac{d}{dx}(1 - 2x)}{(1 - 2x)^2}$$

$$= \frac{(1 - 2x)(1) - (2 + x)(-2)}{(1 - 2x)^2}$$

$$= \frac{1 - 2x + 4 + 2x}{(1 - 2x)^2}$$

$$= \frac{5}{(1 - 2x)^2}$$

$$w'(2) = \frac{5}{(1 - 2(2))^2}$$

$$= \frac{5}{(-3)^2}$$

$$= \frac{5}{9}$$

Domains of w and w' are the same: $x \neq \frac{1}{2}$.

n)

$$f'(x) = \frac{(x^2 - 1)\frac{d}{dx}(x) - (x)\frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)(1) - (x)(2x)}{(x^2 - 1)^2}$$

$$= \frac{x^2 - 1 - 2x^2}{(x^2 - 1)^2}$$

$$= -\frac{x^2 + 1}{(x^2 - 1)^2}$$

$$f'(2) = -\frac{(2)^2 + 1}{(2^2 - 1)^2}$$

$$= -\frac{5}{3^2}$$

$$= -\frac{5}{9}$$

Domains of f and f' are the same: $x \neq \pm 1$.

o)

$$p'(x) = \frac{(x^2+1)\frac{d}{dx}\left(\sqrt{x}\right) - \left(\sqrt{x}\right)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}\right)(2x)}{(x^2+1)^2}$$

$$= \frac{\frac{x^2+1-4x^2}{2\sqrt{x}}}{(x^2+1)^2}$$

$$= \frac{1-3x^2}{2\sqrt{x}(x^2+1)^2}$$

$$p'(2) = \frac{1-3(2)^2}{2\sqrt{2}(2^2+1)^2}$$

$$= \frac{1-12}{2\sqrt{2}(25)}$$

$$= -\frac{11\sqrt{2}}{100}$$

Domains of p and p' include all real numbers.

p)

$$v(x) = \frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$

$$= \frac{\frac{x-1}{x}}{\frac{x+1}{x}}$$

$$= \frac{x-1}{x+1}$$

$$v'(x) = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$

$$= \frac{x+1-x+1}{(x+1)^2}$$

$$= \frac{2}{(x+1)^2}$$

$$v'(2) = \frac{2}{(2+1)^2}$$

$$= \frac{2}{9}$$

Domains of v and v' are the same: $x \neq 0, -1$.

q) $$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2-9)\frac{d}{dx}(x^4)-(x^4)\frac{d}{dx}(x^2-9)}{\left(x^2-9\right)^2}\\ &= \frac{(x^2-9)(4x^3)-(x^4)(2x)}{(x^2-9)^2}\\ &= \frac{2x^3(2x^2-18-x^2)}{(x^2-9)^2}\\ &= \frac{2x^3(x^2-18)}{(x^2-9)^2}\\ \frac{dy}{dx}_{|x=2} &= \frac{2(2)^3(2^2-18)}{(2^2-9)^2}\\ &= \frac{16(4-18)}{(4-9)^2}\\ &= \frac{16(-14)}{(-5)^2}\\ &= -\frac{224}{25}\end{aligned}$$

Domains of y and y' are the same: $x \neq \pm 3$.

r) $$\begin{aligned}f'(x) &= \frac{(x^4-16)\frac{d}{dx}(x^2-3x)-(x^2-3x)\frac{d}{dx}(x^4-16)}{\left(x^4-16\right)^2}\\ &= \frac{(x^4-16)(2x-3)-(x^2-3x)(4x^3)}{(x^4-16)^2}\\ &= \frac{2x^5-3x^4-32x+48-4x^5+12x^4}{(x^4-16)^2}\\ &= \frac{-2x^5+9x^4-32x+48}{(x^4-16)^2}\end{aligned}$$

Since $f(2)$ is undefined, $f'(2)$ is undefined. Domains of f and f' are the same: $x \neq \pm 2$.

Apply, Solve, Communicate

Section 4.4 Page 219 Question 3

a) $$\begin{aligned}\frac{dy}{dx} &= \frac{(4x+3)\frac{d}{dx}(2x+1)-(2x+1)\frac{d}{dx}(4x+3)}{(4x+3)^2}\\ &= \frac{(4x+3)(2)-(2x+1)(4)}{(4x+3)^2}\\ &= \frac{8x+6-8x-4}{(4x+3)^2}\\ &= \frac{2}{(4x+3)^2}\end{aligned}$$

Since both the numerator and denominator of $\frac{2}{(4x+3)^2}$ yield positive results, there are no tangents to $y = \frac{2x+1}{4x+3}$ with negative slopes. The graph is never decreasing. A graph with only tangents of negative slopes is always decreasing.

b) $$\begin{aligned}y &= \frac{x^2}{x^2-x}\\ &= \frac{x}{x-1}\\ \frac{dy}{dx} &= \frac{(x-1)\frac{d}{dx}(x)-x\frac{d}{dx}(x-1)}{(x-1)^2}\\ &= \frac{(x-1)-x}{(x-1)^2}\\ &= \frac{-1}{(x-1)^2}\end{aligned}$$

Since the numerator and denominator of $\frac{-1}{(x-1)^2}$ always yield negative and positive results respectively, there are no tangents to $y = \frac{x^2}{x^2-x}$ with positive slopes. The graph is never increasing. A graph with only tangents with positive slopes is always increasing.

Section 4.4 Page 219 Question 4

$$\begin{aligned}\frac{dy}{dx} &= 0\\ \frac{(2x+1)\frac{d}{dx}(x^2)-(x^2)\frac{d}{dx}(2x+1)}{(2x+1)^2} &= 0\\ \frac{(2x+1)(2x)-(x^2)(2)}{(2x+1)^2} &= 0\\ \frac{4x^2+2x-2x^2}{(2x+1)^2} &= 0\\ \frac{2x^2+2x}{(2x+1)^2} &= 0\\ 2x^2+2x &= 0\\ x^2+x &= 0\\ x(x+1) &= 0\\ x &= 0 \text{ or } -1\end{aligned}$$

At $x = 0$, $y = 0$. At $x = -1$, $y = \frac{(-1)^2}{2(-1)+1}$ or -1. The tangents are horizontal at $(0, 0)$ and $(-1, -1)$. The graph has a local extremum at each point. A graph with only horizontal tangents is a horizontal line.

Section 4.4 Page 219 Question 5

The line $x - y = 2$ has a slope of 1.

$$\begin{aligned}\frac{dy}{dx} &= 1\\ \frac{(x+1)\frac{d}{dx}(x)-(x)\frac{d}{dx}(x+1)}{(x+1)^2} &= 1\\ \frac{(x+1)(1)-(x)(1)}{(x+1)^2} &= 1\\ \frac{x+1-x}{(x+1)^2} &= 1\\ \frac{1}{(x+1)^2} &= 1\\ (x+1)^2 &= 1\\ x+1 &= \pm 1\\ x &= 0 \text{ or } -2\end{aligned}$$

At $x = 0$, $y = 0$. At $x = -2$, $y = \frac{-2}{-2+1}$ or 2. The tangents are parallel to $x - y = 2$ at $(0, 0)$ and $(-2, 2)$.

Section 4.4 Page 219 Question 6

a) $$\begin{aligned}\frac{dy}{dx} &= \frac{f(x)\frac{d}{dx}(1)-(1)\frac{d}{dx}f(x)}{[f(x)]^2}\\ &= \frac{f(x)(0)-(1)(f'(x))}{[f(x)]^2}\\ &= -\frac{f'(x)}{[f(x)]^2}\end{aligned}$$

b) $$\begin{aligned}\frac{dy}{dx} &= \frac{(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}(x)}{(x)^2}\\ &= \frac{xf'(x)-f(x)(1)}{x^2}\\ &= \frac{xf'(x)-f(x)}{x^2}\end{aligned}$$

c) $$\begin{aligned}\frac{dy}{dx} &= \frac{f(x)\frac{d}{dx}(x)-(x)\frac{d}{dx}f(x)}{[f(x)]^2}\\ &= \frac{f(x)(1)-(x)(f'(x))}{[f(x)]^2}\\ &= \frac{f(x)-xf'(x)}{[f(x)]^2}\end{aligned}$$

Section 4.4 Page 219 Question 7

a)

$$\begin{aligned}A'(n) &= \frac{(n)\frac{d}{dn}p(n)-p(n)\frac{d}{dn}(n)}{(n)^2}\\ &= \frac{np'(n)-p(n)(1)}{n^2}\\ A'(n) &= \frac{np'(n)-p(n)}{n^2} \qquad (1)\end{aligned}$$

Productivity declines if the number of workers increases while total value produced remains constant. If there are more workers but nothing more is being produced, extra salaries are being paid for no reason, hence reducing productivity.

b) If average productivity is increasing, more workers will translate into even greater average productivity, thereby increasing total productivity.

c) If $A'(n) < 0$, the company may reduce its workforce in the hope that average productivity will stop declining.

d)

$$p'(n) > A(n) \qquad (2)$$

Substitute $A(n) = \frac{p(n)}{n}$ into (2).

$$\begin{aligned} p'(n) &> \frac{p(n)}{n} \\ np'(n) &> p(n) \\ np'(n) - p(n) &> 0 \qquad (3) \end{aligned}$$

Divide both sides of (3) by n^2.

$$\frac{np'(n) - p(n)}{n^2} > 0 \qquad (4)$$

Substitute (1) from part a) into (4).

$$A'(n) > 0$$

e) Answers will vary.

f) Answers will vary.

Section 4.4 Page 219 Question 8

a) The boat hits the dock when $x(t) = 0$.

$$\begin{aligned} x(t) &= 0 \\ \frac{6(8-t)}{4+t} &= 0 \\ 6(8-t) &= 0 \\ 8 - t &= 0 \\ t &= 8 \end{aligned}$$

The boat hits the dock 8 s after the engine is turned off.

c)

$$\begin{aligned} \frac{dx}{dt}_{|t=8} &= -\frac{72}{(4+8)^2} \\ &= -\frac{72}{12^2} \\ &= -\frac{1}{2} \end{aligned}$$

The boat strikes the dock at $-\frac{1}{2}$ m/s.

e) Answers will vary.

b)

$$\begin{aligned} \frac{dx}{dt} &= \frac{(4+t)\frac{d}{dt}(6(8-t)) - 6(8-t)\frac{d}{dt}(4+t)}{(4+t)^2} \\ &= \frac{(4+t)(-6) - 6(8-t)(1)}{(4+t)^2} \\ &= \frac{-24 - 6t - 48 + 6t}{(4+t)^2} \\ &= -\frac{72}{(4+t)^2} \end{aligned}$$

$\frac{dx}{dt}$ represents the velocity of the boat, in metres per second, as it coasts into the dock.

d)

$$\begin{aligned} \frac{dx}{dt}_{|t=0} &= -\frac{72}{(4+0)^2} \\ &= -\frac{72}{4^2} \\ &= -\frac{9}{2} \end{aligned}$$

The velocity of the boat was $-\frac{9}{2}$ m/s when the engine was shut off.

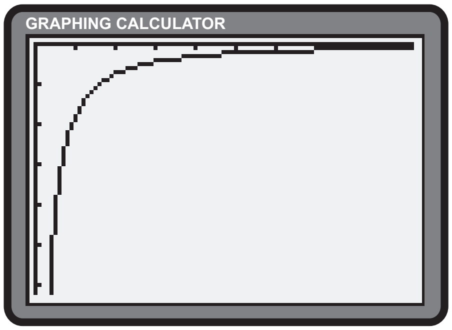
Section 4.4 Page 220 Question 9

a)

$$\begin{aligned} S &= A'(b) \\ &= \frac{\left(1+3.8b^{0.4}\right)\frac{d}{db}\left(32+19.6b^{0.4}\right)-\left(32+19.6b^{0.4}\right)\frac{d}{db}\left(1+3.8b^{0.4}\right)}{\left(1+3.8b^{0.4}\right)^2} \\ &= \frac{\left(1+3.8b^{0.4}\right)\left(7.84b^{-0.6}\right)-\left(32+19.6b^{0.4}\right)\left(1.52b^{-0.6}\right)}{\left(1+3.8b^{0.4}\right)^2} \\ &= \frac{7.84b^{-0.6}+29.792b^{-0.2}-48.64b^{-0.6}-29.792b^{-0.2}}{\left(1+3.8b^{0.4}\right)^2} \\ &= \frac{-40.8b^{-0.6}}{\left(1+3.8b^{0.4}\right)^2} \\ &= \frac{-40.8}{b^{0.6}\left(1+3.8b^{0.4}\right)^2} \end{aligned}$$

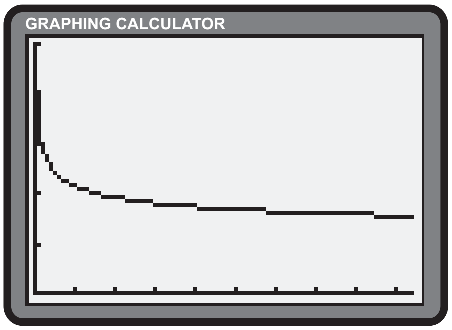
b)

$$S = \frac{-40.8}{b^{0.6}\left(1+3.8b^{0.4}\right)^2}$$



Window variables: $x \in [0, 9.4]$, $y \in [-6.2, 0]$

$$A = \frac{32+19.6b^{0.4}}{1+3.8b^{0.4}}$$



Window variables: $x \in [0, 9.4]$, $y \in [0, 25]$

c) Answers will vary.

Section 4.4 Page 220 Question 10

$$\begin{aligned} M'(t) &= \frac{(t+1.9)\frac{d}{dt}(5.8t)-(5.8t)\frac{d}{dt}(t+1.9)}{(t+1.9)^2} \\ &= \frac{(t+1.9)(5.8)-(5.8t)(1)}{(t+1.9)^2} \\ &= \frac{5.8t+11.02-5.8t}{(t+1.9)^2} \\ &= \frac{11.02}{(t+1.9)^2} \end{aligned}$$

$$\begin{aligned} M'(5) &= \frac{11.02}{(5+1.9)^2} \\ &= \frac{11.02}{(6.9)^2} \\ &\doteq 0.2315 \end{aligned}$$

The rate of change is approximately 0.2315 g/s after 5 s.

Section 4.4 Page 220 Question 11

Methods may vary. One method involves a full simplification of the function before differentiation.

$$\begin{aligned}
y &= \frac{(3x^2+1)(x+1)}{(x+1)^2} \\
&= \frac{3x^2+1}{x+1},\ x \neq -1 \\
\frac{dy}{dx} &= \frac{(x+1)\frac{d}{dx}(3x^2+1) - (3x^2+1)\frac{d}{dx}(x+1)}{(x+1)^2} \\
&= \frac{(x+1)(6x) - (3x^2+1)(1)}{(x+1)^2} \\
&= \frac{6x^2+6x-3x^2-1}{(x+1)^2} \\
&= \frac{3x^2+6x-1}{(x+1)^2}
\end{aligned}$$

A second method involves a partial simplification of the numerator before differentiation.

$$\begin{aligned}
y &= \frac{3x^3+3x^2+x+1}{x^2+2x+1} \\
\frac{dy}{dx} &= \frac{(x^2+2x+1)\frac{d}{dx}(3x^3+3x^2+x+1) - (3x^3+3x^2+x+1)\frac{d}{dx}(x^2+2x+1)}{(x^2+2x+1)^2} \\
&= \frac{(x^2+2x+1)(9x^2+6x+1) - (3x^3+3x^2+x+1)(2x+2)}{(x^2+2x+1)^2} \\
&= \frac{9x^4+6x^3+x^2+18x^3+12x^2+2x+9x^2+6x+1-6x^4-6x^3-6x^3-6x^2-2x^2-2x-2x-2}{(x^2+2x+1)^2} \\
&= \frac{3x^4+12x^3+14x^2+4x-1}{\left((x+1)^2\right)^2} \\
&= \frac{(x+1)(3x^3+9x^2+5x-1)}{(x+1)^4} \\
&= \frac{(x+1)(3x^2+6x-1)}{(x+1)^3} \\
&= \frac{3x^2+6x-1}{(x+1)^2}
\end{aligned}$$

A third method proceeds without simplification.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x^2+2x+1)\frac{d}{dx}\left[(3x^2+1)(x+1)\right] - (3x^2+1)(x+1)\frac{d}{dx}(x^2+2x+1)}{(x^2+2x+1)^2} \\
&= \frac{(x+1)^2\left[(x+1)\frac{d}{dx}(3x^2+1) + (3x^2+1)\frac{d}{dx}(x+1)\right] - (3x^2+1)(x+1)\frac{d}{dx}(x^2+2x+1)}{\left((x+1)^2\right)^2} \\
&= \frac{(x+1)^2\left[(x+1)(6x) + (3x^2+1)(1)\right] - (3x^2+1)(x+1)(2x+2)}{(x+1)^4} \\
&= \frac{(x+1)^2\left(6x^2+6x+3x^2+1\right) - 2(3x^2+1)(x+1)^2}{(x+1)^4} \\
&= \frac{9x^2+6x+1-6x^2-2}{(x+1)^2} \\
&= \frac{3x^2+6x-1}{(x+1)^2}
\end{aligned}$$

Section 4.4 Page 220 Question 12

$$\begin{aligned}\frac{d}{dx}\left(x^{-n}\right) &= \frac{d}{dx}\left(\frac{1}{x^n}\right)\\ &= \frac{(x^n)\frac{d}{dx}(1) - (1)\frac{d}{dx}(x^n)}{(x^n)^2}\\ &= \frac{x^n(0) - (1)\left(nx^{n-1}\right)}{x^{2n}}\\ &= \frac{-nx^{n-1}}{x^{2n}}\\ &= -nx^{n-1-2n}\\ &= -nx^{-n-1}\end{aligned}$$

Section 4.4 Page 220 Question 13

a)

$$\begin{aligned}P'(R) &= 0\\ \frac{(R+0.6)^2\frac{d}{dR}(144R) - (144R)\frac{d}{dR}\left((R+0.6)^2\right)}{\left((R+0.6)^2\right)^2} &= 0\\ \frac{(R+0.6)^2(144) - (144R)(2)(R+0.6)}{(R+0.6)^4} &= 0\\ 144(R+0.6)\left[R+0.6-2R\right] &= 0\\ (R+0.6)(0.6-R) &= 0\\ R &= 0.6,\ R > 0\end{aligned}$$

The rate of change of the power will be 0 when $R = 0.6\ \Omega$.

b) The maximum power will be produced when $R = 0.6\ \Omega$.

4.5 Derivatives of Derivatives

Practise

Section 4.5 Page 225 Question 1

a)
$$\begin{aligned} f'(x) &= 3x^2 - 2 \\ f''(x) &= 6x \end{aligned}$$

b)
$$\begin{aligned} f'(x) &= x^3 + 6x^2 - 10x + 1 \\ f''(x) &= 3x^2 + 12x - 10 \end{aligned}$$

c)
$$\begin{aligned} g(x) &= x^{-1} \\ g'(x) &= (-1)x^{-2} \\ &= -\frac{1}{x^2} \\ g''(x) &= -(-2)x^{-3} \\ &= \frac{2}{x^3} \end{aligned}$$

d)
$$\begin{aligned} m'(t) &= \frac{(t+2)\frac{d}{dt}(1) - (1)\frac{d}{dt}(t+2)}{(t+2)^2} - \frac{d}{dt}(5t) \\ &= \frac{t+2-1}{(t+2)^2} - 5 \\ &= \frac{-1}{(t+2)^2} - 5 \\ m''(t) &= \frac{(t+2)^2\frac{d}{dt}(-1) - (-1)\frac{d}{dt}(t+2)^2}{(t+2)^4} - \frac{d}{dt}(5) \\ &= \frac{2t+4}{(t+2)^4} \\ &= \frac{2}{(t+2)^3} \end{aligned}$$

e)
$$\begin{aligned} y &= x^3 - 2x^2 + x - 2 \\ \frac{dy}{dx} &= 3x^2 - 4x + 1 \\ \frac{d^2y}{dx^2} &= 6x - 4 \end{aligned}$$

f)
$$\begin{aligned} \frac{dy}{dx} &= \frac{(x-2)\frac{d}{dx}(x) - (x)\frac{d}{dx}(x-2)}{(x-2)^2} \\ &= \frac{(x-2)(1) - (x)(1)}{(x-2)^2} \\ &= \frac{-2}{(x-2)^2} \\ \frac{d^2y}{dx^2} &= \frac{(x-2)^2\frac{d(-2)}{dx} - (-2)\frac{d(x-2)^2}{dx}}{(x-2)^4} \\ &= \frac{2(2x-4)}{(x-2)^4} \\ &= \frac{4}{(x-2)^3} \end{aligned}$$

g)
$$\begin{aligned} w(u) &= 6u + 4u^2 - 3u^2 - 2u^3 \\ &= 6u + u^2 - 2u^3 \\ w'(u) &= 6 + 2u - 6u^2 \\ w''(u) &= 2 - 12u \end{aligned}$$

h)
$$\begin{aligned} h'(x) &= \frac{(2x-x^2)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(2x-x^2)}{(2x-x^2)^2} \\ &= \frac{(2x-x^2)(2x) - (x^2-1)(2-2x)}{(2x-x^2)^2} \\ &= \frac{4x^2 - 2x^3 - 2x^2 + 2x^3 + 2 - 2x}{(2x-x^2)^2} \\ &= \frac{2x^2 - 2x + 2}{(2x-x^2)^2} \\ h''(x) &= \frac{(2x-x^2)^2\frac{d}{dx}(2x^2-2x+2) - (2x^2-2x+2)\frac{d}{dx}(4x^2-4x^3+x^4)}{\left((2x-x^2)^2\right)^2} \\ &= \frac{(2x-x^2)^2(4x-2) - (2x^2-2x+2)(8x-12x^2+4x^3)}{(2x-x^2)^4} \\ &= \frac{2(2x-x^2)\left[(2x-x^2)(2x-1) - (2x^2-2x+2)(2-2x)\right]}{(2x-x^2)^4} \\ &= \frac{2\left(4x^2 - 2x - 2x^3 + x^2 - 4x^2 + 4x^3 + 4x - 4x^2 - 4 + 4x\right)}{(2x-x^2)^3} \\ &= \frac{4x^3 - 6x^2 + 12x - 8}{(2x-x^2)^3} \end{aligned}$$

i)

$$\begin{aligned} v(u) &= 6u^{-\frac{1}{2}} \\ v'(u) &= 6\left(-\frac{1}{2}\right)u^{-\frac{3}{2}} \\ &= -3u^{-\frac{3}{2}} \\ &= -\frac{3}{\sqrt{u^3}} \\ v''(u) &= -3\left(-\frac{3}{2}\right)u^{-\frac{5}{2}} \\ &= \frac{9}{2}u^{-\frac{5}{2}} \end{aligned}$$

j)

$$\begin{aligned} M(x) &= x^3 + x^{-2} \\ M'(x) &= 3x^2 + (-2)x^{-3} \\ &= 3x^2 - \frac{2}{x^3} \\ M''(x) &= 6x + 6x^{-4} \\ &= 6x + \frac{6}{x^4} \end{aligned}$$

Section 4.5 Page 225 Question 3

a)

$$\begin{aligned} f'(x) &= 4x^3 \\ f'(3) &= 4(3)^3 \\ &= 108 \end{aligned}$$

The slope of the tangent at $x = 3$ is 108.

$$\begin{aligned} f''(x) &= 12x^2 \\ f''(3) &= 12(3)^2 \\ &= 108 \end{aligned}$$

The rate of change of the slope of the tangent at $x = 3$ is 108.

b)

$$\begin{aligned} g'(x) &= -4 + 2x - 3x^2 \\ g'(3) &= -4 + 2(3) - 3(3)^2 \\ &= -4 + 6 - 27 \\ &= -25 \end{aligned}$$

The slope of the tangent at $x = 3$ is -25.

$$\begin{aligned} g''(x) &= 2 - 6x \\ g''(3) &= 2 - 6(3) \\ &= 2 - 18 \\ &= -16 \end{aligned}$$

The rate of change of the slope of the tangent at $x = 3$ is -16.

c)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x+1)^{-1}}{dx} \\ &= \frac{d(x+1)^{-1}}{d(x+1)} \\ &= -1(x+1)^{-2} \\ &= -\frac{1}{(x+1)^2} \\ \left.\frac{dy}{dx}\right|_{x=3} &= -\frac{1}{(3+1)^2} \\ &= -\frac{1}{16} \end{aligned}$$

The slope of the tangent at $x = 3$ is $-\frac{1}{16}$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{d(x+1)^{-2}}{d(x+1)} \\ &= -1(-2)(x+1)^{-3} \\ &= \frac{2}{(x+1)^3} \\ \left.\frac{d^2y}{dx^2}\right|_{x=3} &= \frac{2}{(3+1)^3} \\ &= \frac{1}{32} \end{aligned}$$

The rate of change of the slope of the tangent at $x = 3$ is $\frac{1}{32}$.

d)

$$\begin{aligned} y &= -2x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= -2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} \\ &= x^{-\frac{3}{2}} \\ \left.\frac{dy}{dx}\right|_{x=3} &= \frac{1}{\sqrt{3^3}} \\ &= \frac{1}{3\sqrt{3}} \end{aligned}$$

The slope of the tangent at $x = 3$ is $\frac{1}{3\sqrt{3}}$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= \left(-\frac{3}{2}\right)x^{-\frac{5}{2}} \\ \left.\frac{d^2y}{dx^2}\right|_{x=3} &= -\frac{3}{2\sqrt{3^5}} \\ &= -\frac{1}{6\sqrt{3}} \end{aligned}$$

The rate of change of the slope of the tangent at $x = 3$ is $-\frac{1}{6\sqrt{3}}$.

e)

$$\begin{aligned} g(x) &= 2x^2 + x - 3 \\ g'(x) &= 4x + 1 \\ g'(3) &= 4(3) + 1 \\ &= 13 \end{aligned}$$

The slope of the tangent at $x = 3$ is 13.

$$\begin{aligned} g''(x) &= 4 \\ g''(3) &= 4 \end{aligned}$$

The rate of change of the slope of the tangent at $x = 3$ is 4.

f)

$$\begin{aligned} f(x) &= x^{-3} \\ f'(x) &= -3x^{-4} \\ f'(3) &= -3(3)^{-4} \\ &= -\frac{1}{27} \end{aligned}$$

The slope of the tangent at $x = 3$ is $-\frac{1}{27}$.

$$\begin{aligned} f''(x) &= 12x^{-5} \\ f''(3) &= 12(3)^{-5} \\ &= \frac{4}{81} \end{aligned}$$

The rate of change of the slope of the tangent at $x = 3$ is $\frac{4}{81}$.

Apply, Solve, Communicate

Section 4.5 Page 225 Question 5

Obtain the first two derivatives of the general quadratic function.

$$\begin{aligned} f(x) &= ax^2 + bx + c && (1) \\ f'(x) &= 2ax + b && (2) \\ f''(x) &= 2a && (3) \end{aligned}$$

Perform back substitution.

$$\begin{aligned} f''(3) &= 8 \\ 2a &= 8 \\ a &= 4 && (4) \end{aligned}$$

Since $f'(3) = 22$, substitute (4) into (2) and solve for b.

$$\begin{aligned} 2(4)(3) + b &= 22 \\ 24 + b &= 22 \\ b &= -2 && (5) \end{aligned}$$

Since $f(3) = 33$, substitute (4) and (5) into (1) and solve for c.

$$\begin{aligned} 4(3)^2 - 2(3) + c &= 33 \\ 36 - 6 + c &= 33 \\ c &= 3 && (6) \end{aligned}$$

The quadratic function that satisfies the given conditions is $f(x) = 4x^2 - 2x + 3$.

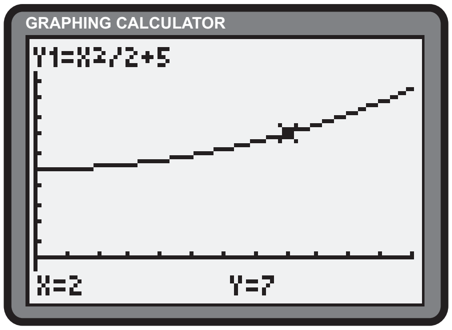
Section 4.5 Page 225 Question 6

a)

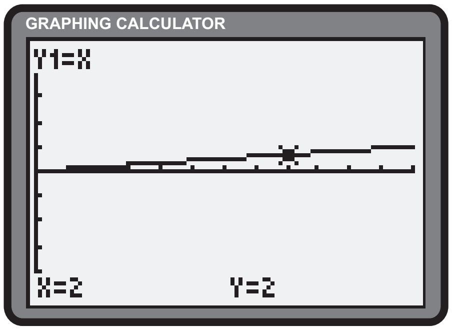
$$\begin{aligned} A(t) &= \frac{1}{2}t^2 + 5 \\ A'(t) &= t \\ A''(t) &= 1 \end{aligned}$$

b) $A''(t)$ is the rate of increase in growth rate, which is therefore steadily increasing.

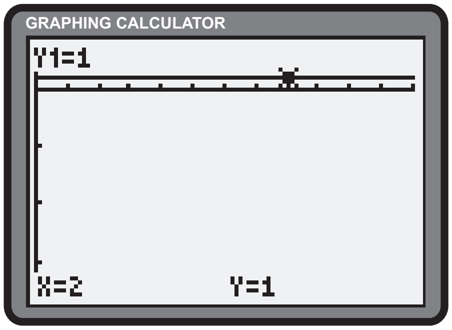
c)



```
Y1=X²/2+5
X=2          Y=7
```

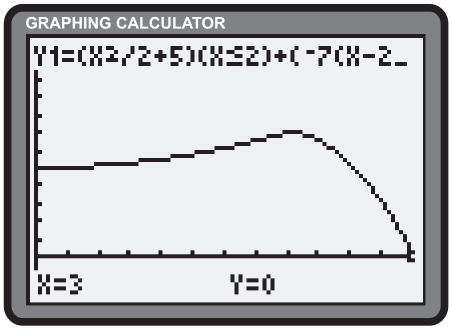


```
Y1=X
X=2          Y=2
```

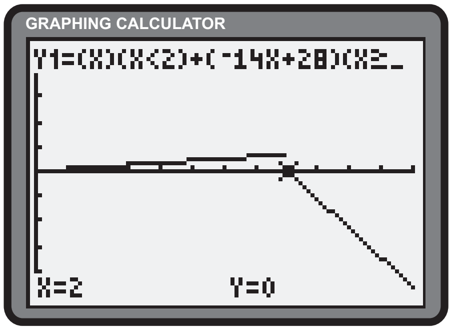


```
Y1=1
X=2          Y=1
```

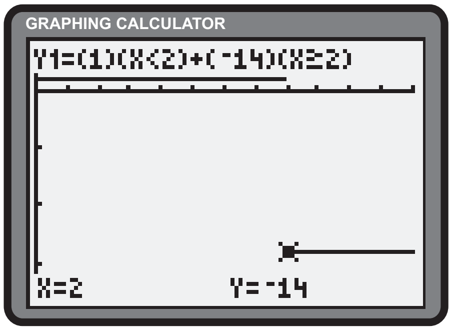
d) Answers will vary



```
Y1=(X²/2+5)(X≤2)+(⁻7(X-2_
X=3          Y=0
```



```
Y1=(X)(X<2)+(⁻14X+28)(X≥_
X=2          Y=0
```



```
Y1=(1)(X<2)+(⁻14)(X≥2)
X=2          Y=⁻14
```

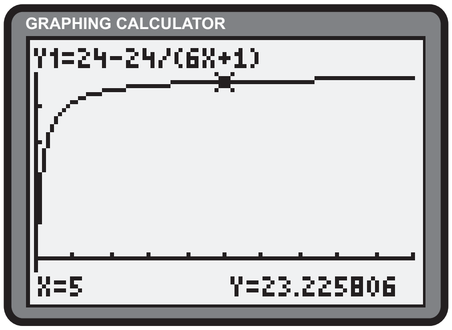
Section 4.5 Page 226 Question 7

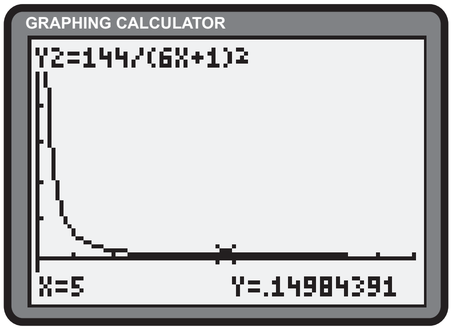
a)

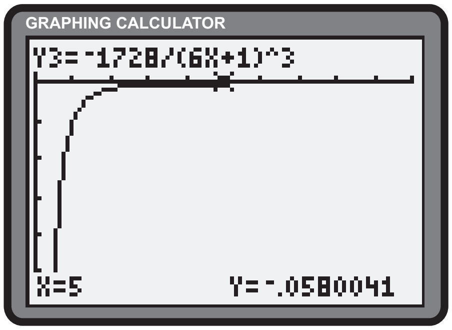
$$\begin{aligned} C'(t) &= \frac{d}{dt}(24) - \frac{(6t+1)\frac{d}{dt}(24) - (24)\frac{d}{dt}(6t+1)}{(6t+1)^2} \\ &= 0 - \frac{(6t+1)(0) - 24(6)}{(6t+1)^2} \\ &= \frac{144}{(6t+1)^2} \\ C''(t) &= \frac{(6t+1)^2\frac{d}{dt}(144) - (144)\left[(6t+1)\frac{d}{dt}(6t+1) + (6t+1)\frac{d}{dt}(6t+1)\right]}{\left((6t+1)^2\right)^2} \\ &= \frac{(6t+1)^2(0) - 144\left[(6t+1)(6) + (6t+1)(6)\right]}{(6t+1)^4} \\ &= -\frac{1728}{(6t+1)^3} \end{aligned}$$

b) It is slowing.

c)







d) Answers will vary.

e) Answers will vary.

Section 4.5 Page 226 Question 8

a)

$$\begin{aligned} h'(t) &= \frac{(8+t^3)\frac{d}{dt}(t^3) - (t^3)\frac{d}{dt}(8+t^3)}{(8+t^3)^2} \\ &= \frac{(8+t^3)(3t^2) - (t^3)(3t^2)}{(8+t^3)^2} \\ &= \frac{3t^2(8+t^3-t^3)}{(8+t^3)^2} \\ &= \frac{24t^2}{(8+t^3)^2} \end{aligned}$$

b)

$$\begin{aligned} h''(t) &= \frac{(8+t^3)^2\frac{d}{dt}(24t^2) - (24t^2)\left[(8+t^3)\frac{d}{dt}(8+t^3) + (8+t^3)\frac{d}{dt}(8+t^3)\right]}{((8+t^3)^2)^2} \\ &= \frac{(8+t^3)^2(48t) - (24t^2)\left[(8+t^3)(3t^2) + (8+t^3)(3t^2)\right]}{(8+t^3)^4} \\ &= \frac{(8+t^3)^2(48t) - 144t^4(8+t^3)}{(8+t^3)^4} \\ &= \frac{48t\left(8+t^3-3t^3\right)}{(8+t^3)^3} \\ &= \frac{48t\left(8-2t^3\right)}{(8+t^3)^3} \end{aligned}$$

$h''(t)$ represents the rate of change of the the growth rate $h'(t)$.

c) The roots of $h''(x)$ include 0 and $\sqrt[3]{4}$ (triple). Over the interval, $(0, \sqrt[3]{4})$, $h''(t) > 0$, so the rate of growth is increasing. Over the interval, $(\sqrt[3]{4}, \infty)$, $h''(t) < 0$, so the rate of growth is decreasing. Therefore, the maximum growth rate occurs at $t = \sqrt[3]{4}$ months.

$$\begin{aligned} h'\left(\sqrt[3]{4}\right) &= \frac{24\left(\sqrt[3]{4}\right)^2}{\left(8 + \left(\sqrt[3]{4}\right)^3\right)^2} \\ &= \frac{24\left(\sqrt[3]{16}\right)}{(8+4)^2} \\ &= \frac{48\sqrt[3]{2}}{144} \\ &= \frac{\sqrt[3]{2}}{3} \end{aligned}$$

The maximum growth rate is $\frac{\sqrt[3]{2}}{3}$ m/month.

d) Answers will vary.

Section 4.5 Page 226 Question 9

a)

$$\begin{aligned} \frac{dy}{dx} &= 4x^3 - 3x^2 + 10x + 2 \\ \frac{d^2y}{dx^2} &= 12x^2 - 6x + 10 \\ \frac{d^3y}{dx^3} &= 24x - 6 \\ \frac{d^4y}{dx^4} &= 24 \\ \frac{d^5y}{dx^5} &= 0 \\ \frac{d^6y}{dx^6} &= 0 \end{aligned}$$

b) The value of $y^{(n)} = 0$ for $n \geq 5$, since the derivative of a constant is 0.

Section 4.5 Page 226 Question 10

$$\begin{aligned} f'(x) &= 5x^4 + 4x^3 + 3x^2 + 2x + 1 \\ f''(x) &= 20x^3 + 12x^2 + 6x + 2 \\ f'''(x) &= 60x^2 + 24x + 6 \\ D_x^4 f(x) &= 120x + 24 \\ D_x^5 f(x) &= 120 \\ D_x^6 f(x) &= 0 \end{aligned}$$

Section 4.5 Page 226 Question 11

$$\begin{aligned}
g'(x) &= \frac{(2x-4)\dfrac{d}{dx}(1) - (1)\dfrac{d}{dx}(2x-4)}{(2x-4)^2} \\
&= \frac{(2x-4)(0) - (1)(2)}{(2x-4)^2} \\
&= -\frac{2}{(2x-4)^2} \\
&= -\frac{1}{2(x-2)^2}
\end{aligned}$$

Expanding $(x-2)^2$ yields $x^2 - 4x + 4$.

$$\begin{aligned}
g''(x) &= -\frac{1}{2} \cdot \frac{(x-2)^2\dfrac{d}{dx}(1) - (1)\dfrac{d}{dx}(x^2 - 4x + 4)}{\left((x-2)^2\right)^2} \\
&= -\frac{1}{2} \cdot \frac{(x-2)^2(0) - (1)(2x-4)}{(x-2)^4} \\
&= \frac{1}{(x-2)^3}
\end{aligned}$$

Expanding $(x-2)^3$ yields $x^3 - 6x^2 + 12x - 8$.

$$\begin{aligned}
g'''(x) &= \frac{(x-2)^3\dfrac{d}{dx}(1) - (1)\dfrac{d}{dx}(x^3 - 6x^2 + 12x - 8)}{\left((x-2)^3\right)^2} \\
&= \frac{(x-2)^3(0) - (1)(3)(3x^2 - 12x + 12)}{(x-2)^6} \\
&= \frac{-3(x-2)^2}{(x-2)^6} \\
&= \frac{-3}{(x-2)^4} \\
g'''(3) &= \frac{-3}{(3-2)^4} \\
&= -3
\end{aligned}$$

Section 4.5 Page 226 Question 12

a)

$$\begin{aligned}
f'(x) &= h(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}h(x) \\
&= h(x)g'(x) + g(x)h'(x) \\
f''(x) &= g'(x)\frac{d}{dx}h(x) + h(x)\frac{d}{dx}g'(x) + h'(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}h'(x) \\
&= g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x) + g(x)h''(x) \\
&= h(x)g''(x) + 2h'(x)g'(x) + g(x)h''(x)
\end{aligned}$$

$$\begin{aligned}
f'''(x) &= g''(x)\frac{d}{dx}h(x) + h(x)\frac{d}{dx}g''(x) + 2\left(g'(x)\frac{d}{dx}h'(x) + h'(x)\frac{d}{dx}g'(x)\right) + h''(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}h''(x) \\
&= g''(x)h'(x) + h(x)g'''(x) + 2\left(g'(x)h''(x) + h'(x)g''(x)\right) + h''(x)g'(x) + g(x)h'''(x) \\
&= g'''(x)h(x) + 3g''(x)h'(x) + 3g'(x)h''(x) + g(x)h'''(x)
\end{aligned}$$

Section 4.5 Page 226 Question 13

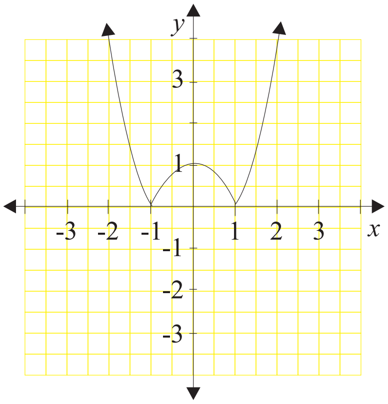
a)

$$f(x) = \begin{cases} x^2 - 1 & , |x| > 1 \\ 1 - x^2 & , |x| \le 1 \end{cases}$$

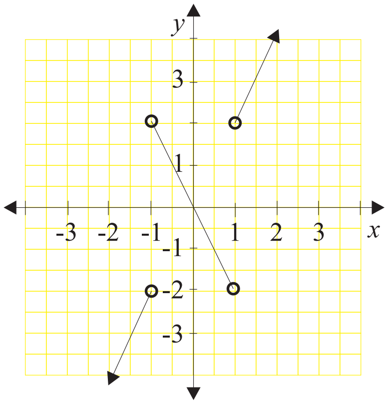
$$f'(x) = \begin{cases} 2x & , |x| > 1 \\ -2x & , |x| < 1 \end{cases} \qquad \text{(domain: } x \neq \pm 1\text{)}$$

$$f''(x) = \begin{cases} 2 & , |x| > 1 \\ -2 & , |x| < 1 \end{cases} \qquad \text{(domain: } x \neq \pm 1\text{)}$$

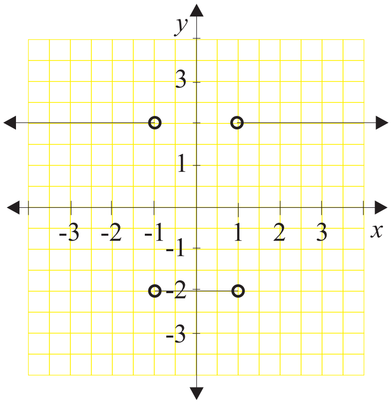
b) $f(x) = \left|x^2 - 1\right|$



$f'(x)$



$f''(x)$



c) The domain of $f(x)$ is $x \in \Re$, while the domains of $f'(x)$ and $f''(x)$ are $x \in \Re$ such that $x \neq \pm 1$.

Section 4.5 Page 226 Question 14

a)

$$\begin{aligned} f(x) &= x^n \\ f'(x) &= nx^{n-1} \\ f''(x) &= n(n-1)x^{n-2} \\ f'''(x) &= n(n-1)(n-2)x^{n-3} \end{aligned}$$

b)

$$\begin{aligned} f^{(n)}(x) &= n(n-1)(n-2)\dots(n-(n-1)) \\ &= n(n-1)(n-2)\dots(1) \\ &= n! \end{aligned}$$

Section 4.5 Page 226 Question 15

a) The number of non-repetitive derivatives of a polynomial of degree n is $n + 1$.

b) The value of the repetitive derivative is 0.

4.6 Velocity and Acceleration

Apply, Solve, Communicate

Section 4.6 Page 237 Question 1

a) The initial velocity of the car is zero.
b) Since the slope of the tangent is positive at A and zero at B, the car is going faster at A.
c) The car is neither speeding up nor slowing down at A. Since the velocity of the car at B is zero, the car is neither speeding up nor slowing down. Since the velocity of the car at C is undefined, the car is neither speeding up nor slowing down.
d) The car is at rest.
e) The car is at rest at its initial starting point.

Section 4.6 Page 237 Question 2

a) Since the slopes of the tangents are positive on the interval from 0 to A, the acceleration is positive.
b) Since the slopes of the tangents are negative on the interval from A to B, the acceleration is negative.
c) Since the slopes of the tangents are positive on the interval from B to C, the acceleration is positive.
d) Since the slopes of the tangents are zero on the interval from C to D, the acceleration is zero.
e) Since the slopes of the tangents are negative on the interval from D to E, the acceleration is negative.

Section 4.6 Page 237 Question 3

a) Since the slopes of the tangents are decreasing on the interval from 0 to A, the velocity is decreasing. Since the velocity is positive, the acceleration must be negative.
b) **i)** Since the slopes of the tangents are positive and decreasing on the interval from A to B, the acceleration is negative.
  ii) Since the slopes of the tangents are negative and increasing on the interval from B to C, the acceleration is positive.
  iii) Since the slopes of the tangents are constant on the interval from C to D, the acceleration is zero.
  iv) Since the slopes of the tangents are positive and decreasing on the interval from D to E, the acceleration is negative.

Section 4.6 Page 237 Question 4

The function c accurately reports the slopes of the tangents to the function a. The function a accurately reports the slopes of the tangents to the function b. The position, velocity, and acceleration functions are b, a, and c respectively.

Section 4.6 Page 238 Question 5

Since $V > 0$ and $a > 0$ for $t \in [0, 3]$, the car is speeding up over this interval. The car is never slowing down.

Section 4.6 Page 238 Question 6

a)
$$s = 9 + 4t$$
$$v = 4$$
$$a = 0$$

b)
$$s = 3t^2 + 2t - 5$$
$$v = 6t + 2$$
$$a = 6$$

c)
$$s = t^3 - 4t^2 + 5t - 7$$
$$v = 3t^2 - 8t + 5$$
$$a = 6t - 8$$

d)

$$\begin{aligned}
s &= \frac{t^3}{t+1}\\
v &= \frac{(t+1)\frac{d}{dt}(t^3) - (t^3)\frac{d}{dt}(t+1)}{(t+1)^2}\\
&= \frac{(t+1)(3t^2) - (t^3)(1)}{(t+1)^2}\\
&= \frac{3t^3 + 3t^2 - t^3}{(t+1)^2}\\
&= \frac{2t^3 + 3t^2}{(t+1)^2}\\
a &= \frac{(t+1)^2\frac{d}{dt}(2t^3 + 3t^2) - (2t^3 + 3t^2)\frac{d}{dt}(t^2 + 2t + 1)}{\left((t+1)^2\right)^2}\\
&= \frac{(t+1)^2(6t^2 + 6t) - (2t^3 + 3t^2)(2t + 2)}{(t+1)^4}\\
&= \frac{2t(t+1)\left[(t+1)(3t+3) - (2t^2 + 3t)\right]}{(t+1)^4}\\
&= \frac{2t(3t^2 + 3t + 3t + 3 - 2t^2 - 3t)}{(t+1)^3}\\
&= \frac{2t(t^2 + 3t + 3)}{(t+1)^3}
\end{aligned}$$

e)

$$\begin{aligned}
s &= (t - 2\sqrt{t})(2t^2 - t + \sqrt{t})\\
&= 2t^3 - t^2 + t\sqrt{t} - 4t^2\sqrt{t} + 2t\sqrt{t} - 2t\\
&= 2t^3 - 4t^2\sqrt{t} - t^2 + 3t\sqrt{t} - 2t\\
&= 2t^3 - 4t^{\frac{5}{2}} - t^2 + 3t^{\frac{3}{2}} - 2t\\
v &= 6t^2 - 10t^{\frac{3}{2}} - 2t + \frac{9}{2}t^{\frac{1}{2}} - 2\\
a &= 12t - 15t^{\frac{1}{2}} - 2 + \frac{9}{4}t^{-\frac{1}{2}}
\end{aligned}$$

f)

$$\begin{aligned}
s &= \frac{6t}{1 + \sqrt{t}}\\
v &= \frac{(1 + \sqrt{t})\frac{d}{dt}(6t) - (6t)\frac{d}{dt}(1 + \sqrt{t})}{(1 + \sqrt{t})^2}\\
&= \frac{(1 + \sqrt{t})(6) - (6t)\left(\frac{1}{2\sqrt{t}}\right)}{(1 + \sqrt{t})^2}\\
&= \frac{3(2 + 2\sqrt{t} - \sqrt{t})}{(1 + \sqrt{t})^2}\\
&= \frac{3(2 + \sqrt{t})}{(1 + \sqrt{t})^2}
\end{aligned}$$

$$\begin{aligned}
a &= 3 \cdot \frac{(1 + \sqrt{t})^2\frac{d}{dt}(2 + \sqrt{t}) - (2 + \sqrt{t})\frac{d}{dt}(1 + 2\sqrt{t} + t)}{\left((1 + \sqrt{t})^2\right)^2}\\
&= 3 \cdot \frac{(1 + \sqrt{t})^2\left(\frac{1}{2\sqrt{t}}\right) - (2 + \sqrt{t})\left(\frac{1}{\sqrt{t}} + 1\right)}{\left(1 + \sqrt{t}\right)^4}\\
&= 3 \cdot \frac{(1 + \sqrt{t})^2\left(\frac{1}{2\sqrt{t}}\right) - 2(2 + \sqrt{t})\left(\frac{1 + \sqrt{t}}{2\sqrt{t}}\right)}{\left(1 + \sqrt{t}\right)^4}\\
&= \frac{3(1 + \sqrt{t})\left[1 + \sqrt{t} - 2(2 + \sqrt{t})\right]}{2\sqrt{t}\left(1 + \sqrt{t}\right)^4}\\
&= \frac{3\left(1 + \sqrt{t} - 4 - 2\sqrt{t}\right)}{2\sqrt{t}\left(1 + \sqrt{t}\right)^3}\\
&= \frac{3\left(-3 - \sqrt{t}\right)}{2\sqrt{t}\left(1 + \sqrt{t}\right)^3}
\end{aligned}$$

Section 4.6 Page 238 Question 7

a)

$$\begin{aligned} s(t) &= 6 + 3t \\ v(t) &= 3 \\ v(4) &= 3 \\ a(t) &= 0 \\ a(4) &= 0 \end{aligned}$$

At $t = 4$, $v = 3$ m/s and $a = 0$ m/s^2.

c)

$$\begin{aligned} s(t) &= t^3 - 6t^2 + 12t \\ v(t) &= 3t^2 - 12t + 12 \\ v(4) &= 3(4)^2 - 12(4) + 12 \\ &= 48 - 48 + 12 \\ &= 12 \\ a(t) &= 6t - 12 \\ a(4) &= 6(4) - 12 \\ &= 12 \end{aligned}$$

At $t = 4$, $v = 12$ m/s and $a = 12$ m/s^2.

e)

$$\begin{aligned} s(t) &= \sqrt{t}(t^2 - 2t) \\ &= t^{\frac{5}{2}} - 2t^{\frac{3}{2}} \\ v(t) &= \frac{5}{2}t^{\frac{3}{2}} - 3t^{\frac{1}{2}} \\ v(4) &= \frac{5}{2}(4)^{\frac{3}{2}} - 3(4)^{\frac{1}{2}} \\ &= \frac{5}{2}(8) - 3(2) \\ &= 20 - 6 \\ &= 14 \\ a(t) &= \frac{15}{4}t^{\frac{1}{2}} - \frac{3}{2}t^{-\frac{1}{2}} \\ a(4) &= \frac{15}{4}(4)^{\frac{1}{2}} - \frac{3}{2}(4)^{-\frac{1}{2}} \\ &= \frac{15}{2} - \frac{3}{4} \\ &= \frac{27}{4} \end{aligned}$$

At $t = 4$, $v = 14$ m/s and $a = \frac{27}{4}$ m/s^2.

b)

$$\begin{aligned} s(t) &= 4t^2 - 7t + 14 \\ v(t) &= 8t - 7 \\ v(4) &= 8(4) - 7 \\ &= 32 - 7 \\ &= 25 \\ a(t) &= 8 \\ a(4) &= 8 \end{aligned}$$

At $t = 4$, $v = 25$ m/s and $a = 8$ m/s^2.

d)

$$\begin{aligned} s(t) &= \frac{3t}{1 + t} \\ v(t) &= \frac{(1 + t)\frac{d}{dt}(3t) - (3t)\frac{d}{dt}(1 + t)}{(1 + t)^2} \\ &= \frac{(1 + t)(3) - (3t)(1)}{(1 + t)^2} \\ &= \frac{3 + 3t - 3t}{(1 + t)^2} \\ &= \frac{3}{(1 + t)^2} \\ v(4) &= \frac{3}{(1 + 4)^2} \\ &= \frac{3}{25} \\ a(t) &= \frac{(1 + t)^2\frac{d}{dt}(3) - (3)\frac{d}{dt}(1 + 2t + t^2)}{\left((1 + t)^2\right)^2} \\ &= \frac{(1 + t)^2(0) - (3)(2 + 2t)}{(1 + t)^4} \\ &= \frac{-6}{(1 + t)^3} \\ a(4) &= \frac{-6}{(1 + 4)^3} \\ &= -\frac{6}{125} \end{aligned}$$

At $t = 4$, $v = \frac{3}{25}$ m/s and $a = -\frac{6}{125}$ m/s^2.

f) $s(t) = \frac{2+3t^2}{t+4}$

$$\begin{aligned}
v(t) &= \frac{(t+4)\frac{d}{dt}(2+3t^2) - (2+3t^2)\frac{d}{dt}(t+4)}{(t+4)^2} \\
&= \frac{(t+4)(6t) - (2+3t^2)(1)}{(t+4)^2} \\
&= \frac{6t^2+24t-2-3t^2}{(t+4)^2} \\
&= \frac{3t^2+24t-2}{(t+4)^2} \\
v(4) &= \frac{3(4)^2+24(4)-2}{(4+4)^2} \\
&= \frac{48+96-2}{64} \\
&= \frac{142}{64} \\
&= \frac{71}{32}
\end{aligned}$$

At $t = 4$, $v = \frac{71}{32}$ m/s.

$$\begin{aligned}
a(t) &= \frac{(t+4)^2\frac{d\,3t^2+24t-2}{dt} - (3t^2+24t-2)\frac{d\,t^2+8t+16}{dt}}{\left((t+4)^2\right)^2} \\
&= \frac{(t+4)^2(6t+24) - (3t^2+24t-2)(2t+8)}{(t+4)^4} \\
&= \frac{2(t+4)\left[3(t+4)(t+4) - (3t^2+24t-2)\right]}{(t+4)^4} \\
&= \frac{2\left(3t^2+24t+48-3t^2-24t+2\right)}{(t+4)^3} \\
&= \frac{2(50)}{(t+4)^3} \\
&= \frac{100}{(t+4)^3} \\
a(4) &= \frac{100}{(4+4)^3} \\
&= \frac{100}{512} \\
&= \frac{25}{128}
\end{aligned}$$

At $t = 4$, $a = \frac{25}{128}$ m/s^2.

Section 4.6 Page 238 Question 8

$$\begin{aligned}
h(t) &= 90 - 10t - 4.9t^2 \\
v(t) &= -10 - 9.8t \\
v(1) &= -10 - 9.8(1) \\
&= -10 - 9.8 \\
&= -19.8
\end{aligned}$$

The velocity after 1 s is −19.8 m/s.

$$\begin{aligned}
v(2) &= -10 - 9.8(2) \\
&= -10 - 19.6 \\
&= -29.6
\end{aligned}$$

The velocity after 2 s is −29.6 m/s.

Section 4.6 Page 238 Question 9

a)

$$\begin{aligned} h(t) &= 29.4t - 4.9t^2 \\ v(t) &= 29.4 - 9.8t \\ v(1) &= 29.4 - 9.8(1) \\ &= 29.4 - 9.8 \\ &= 19.6 \end{aligned}$$

The velocity after 1 s is 19.6 m/s.

$$\begin{aligned} v(2) &= 29.4 - 9.8(2) \\ &= 29.4 - 19.6 \\ &= 9.8 \end{aligned}$$

The velocity after 2 s is 9.8 m/s.

$$\begin{aligned} v(4) &= 29.4 - 9.8(4) \\ &= 29.4 - 39.2 \\ &= -9.8 \end{aligned}$$

The velocity after 4 s is −9.8 m/s.

$$\begin{aligned} v(5) &= 29.4 - 9.8(5) \\ &= 29.4 - 49 \\ &= -19.6 \end{aligned}$$

The velocity after 5 s is −19.6 m/s.

b) At the maximum height the velocity is 0.

$$\begin{aligned} v(t) &= 0 \\ 29.4 - 9.8t &= 0 \\ 9.8t &= 29.4 \\ t &= 3 \end{aligned}$$

The ball reaches its maximum height at 3 s.

c)

$$\begin{aligned} h(3) &= 29.4(3) - 4.9(3)^2 \\ &= 88.2 - 44.1 \\ &= 44.1 \end{aligned}$$

The maximum height of the ball is 44.1 m.

d) The ball is on the ground when its height is 0.

$$\begin{aligned} h(t) &= 0 \\ 29.4t - 4.9t^2 &= 0 \\ t(29.4 - 4.9t) &= 0 \\ t &= 0 \text{ or } 6 \end{aligned}$$

The ball hits the ground after 6 s.

e)

$$\begin{aligned} v(6) &= 29.4 - 9.8(6) \\ &= 29.4 - 58.8 \\ &= -29.4 \end{aligned}$$

The ball hits the ground with a velocity of −29.4 m/s.

Section 4.6 Page 238 Question 10

a)

$$\begin{aligned} s(t) &= \frac{1}{2}t^2 + 15t \\ v(t) &= t + 15 \end{aligned}$$

i)

$$\begin{aligned} v(t) &= 15 \\ t + 15 &= 15 \\ t &= 0 \end{aligned}$$

The initial velocity of the motorcycle is 15 m/s.

ii)

$$\begin{aligned} v(t) &= 25 \\ t + 15 &= 25 \\ t &= 10 \end{aligned}$$

The motorcycle reaches a velocity of 25 m/s after 10 s.

b) The acceleration is $v'(t)$ or $a(t) = 1$ m/s^2.

Section 4.6 Page 238 Question 11

a)
$$
\begin{aligned}
s(t) &= t^2 - 8t + 12\\
v(t) &= 2t - 8\\
v(2) &= 2(2) - 8\\
&= 4 - 8\\
&= -4
\end{aligned}
$$

The velocity after 2 s is −4 m/s.

$$
\begin{aligned}
v(6) &= 2(6) - 8\\
&= 12 - 8\\
&= 4
\end{aligned}
$$

The velocity after 6 s is 4 m/s.

b) The person is at rest when the velocity is 0.

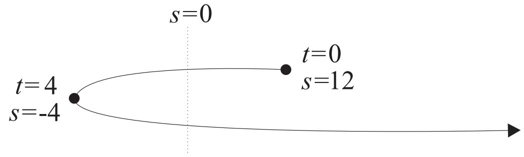
$$
\begin{aligned}
v(t) &= 0\\
2t - 8 &= 0\\
t &= 4
\end{aligned}
$$

The person is at rest at $t = 4$ s.

c)
$$
\begin{aligned}
v(t) &> 4\\
2t - 8 &> 0\\
t &> 4
\end{aligned}
$$

The person is moving in a positive direction when $t > 4$ s.

d)



Section 4.6 Page 238 Question 12

a) i)
$$
\begin{aligned}
s(t) &= 2t^3 - 21t^2 + 60t\\
v(t) &= 6t^2 - 42t + 60\\
v(0) &= 6(0)^2 - 42(0) + 60\\
&= 60
\end{aligned}
$$

The initial velocity of the car is 60 m/s.

ii) The car is at rest when the velocity is 0.

$$
\begin{aligned}
v(t) &= 0\\
6t^2 - 42t + 60 &= 0\\
t^2 - 7t + 10 &= 0\\
(t - 2)(t - 5) &= 0\\
t &= 2 \text{ or } 5
\end{aligned}
$$

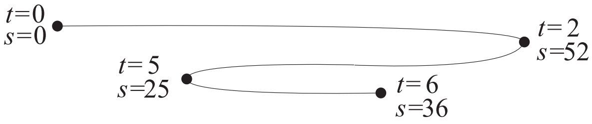
The car is at rest at $t = 2$ s and $t = 5$ s.

iii) The car is moving forward when velocity is positive, or when $(t - 2)(t - 5) > 0$.

		2		5	
$(t-2)$	−	0	+		+
$(t-5)$	−		−	0	+
$(t-2)\,(t-5)$	+	0	−	0	+

The car is moving in a forward direction when $t \in [0, 2)$ and $t \in (5, 6]$.

b)



c) In the first 2 s, the car travels 52 m. Over the interval $t \in [2, 5]$, the car travels 52 − 25 or 27 m. In the sixth second the car travels 36 − 25 or 11 m. The total distance travelled by the car is 52 + 27 + 11 or 90 m.

d) The acceleration of the car is $a(t) = 12t - 42$. This increasing linear function will reach its maximum at $t = 6$ s. The maximum acceleration is 12(6) − 42 or 30 m/s^2.

e) Answers will vary. For example, the maximum acceleration is much greater than the given maximum acceleration for the car.

Section 4.6 Page 238 Question 13

a) The velocity of the rope is $v = 2 - 9.8t$. The velocity is 0 when $t = \frac{10}{49}$ s.

b)

$$\begin{aligned} s &= \frac{-(2) \pm \sqrt{(2)^2 - 4(-4.9)(23)}}{2(-4.9)} \\ &= \frac{-2 \pm \sqrt{4 + 450.8}}{-9.8} \\ &= \frac{2 \pm \sqrt{454.8}}{9.8} \end{aligned}$$

Since $t > 0$, the rope will reach the ground after $\frac{2 + \sqrt{454.8}}{9.8}$ or approximately 2.38 s.

c) The rope strikes the ground with a velocity of approximately $2 - 9.8(2.38)$ or -21.32 m/s.

Section 4.6 Page 239 Question 14

a) The initial position is $s_0 + v_0(0) + \frac{1}{2}g(0)^2$ or s_0.

b)

$$\begin{aligned} s(t) &= s_0 + v_0t + \frac{1}{2}gt^2 \\ v(t) &= v_0 + gt \\ v(0) &= v_0 + g(0) \\ &= v_0 \end{aligned}$$

The initial velocity is v_0 m/s.

c) $a(t) = g$. The acceleration is g m/s^2.

Section 4.6 Page 239 Question 15

Determine the velocity and acceleration functions.

$$\begin{aligned} s(t) &= \frac{1}{3}t^3 - 25t \\ v(t) &= t^2 - 25 \\ a(t) &= 2t \end{aligned}$$

The velocity is 0 at $\sqrt{25}$ or 5 s. The acceleration at this instant is 2(5) or 10 m/s^2.

Section 4.6 Page 239 Question 16

Determine the system of functions describing the motion of the rocket.

$$\begin{aligned} h(t) &= 5t^2 - \frac{1}{6}t^3 \\ v(t) &= 10t - \frac{1}{2}t^2 \\ a(t) &= 10 - t \end{aligned}$$

a) The acceleration is positive when $10 - t > 0$ or when $t \in [0, 10)$ s. The acceleration is negative when $t > 10$ s.

b) The velocity is 0 when $10t - \frac{1}{2}t^2 = 0$ or when $t = 0$ s or $t = 20$ s.

c) The maximum height occurs when the velocity is 0.

$$\begin{aligned} h(20) &= 5(20)^2 - \frac{1}{6}(20)^3 \\ &= 2000 - \frac{8000}{6} \\ &= \frac{12\,000 - 8000}{6} \\ &= \frac{4000}{6} \\ &= \frac{2000}{3} \end{aligned}$$

The maximum height of the rocket is $\frac{2000}{3}$ m.

d) Determine when the rocket strikes the ground.

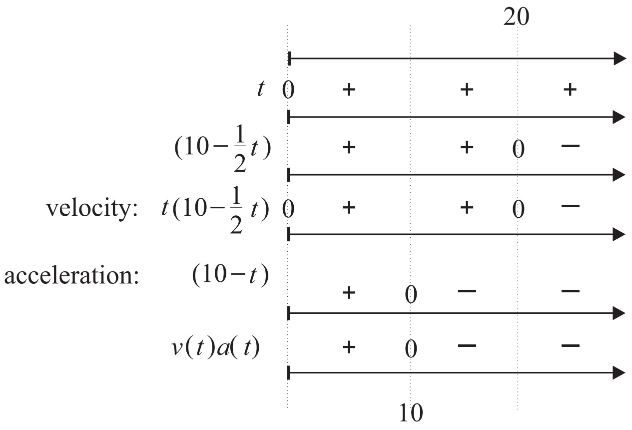
$$\begin{aligned} h(t) &= 0 \\ 5t^2 - \frac{1}{6}t^3 &= 0 \\ t^2\left(5 - \frac{1}{6}t\right) &= 0 \\ 5 - \frac{1}{6}t &= 0 \\ t &= 30 \end{aligned}$$

Determine the velocity at $t = 30$ s.

$$\begin{aligned} v(30) &= 10(30) - \frac{1}{2}(30)^2 \\ &= 300 - 450 \\ &= -150 \end{aligned}$$

The rocket strikes the ground with a velocity of -150 m/s.

e) The rocket is speeding up when $v(t)a(t) > 0$ and slowing down when $v(t)a(t) < 0$. From the line diagrams the rocket is speeding up on the intervals $t \in (0, 10)$ and $t > 20$. The rocket is slowing down on the interval $t \in (10, 20)$.



Section 4.6 Page 239 Question 17

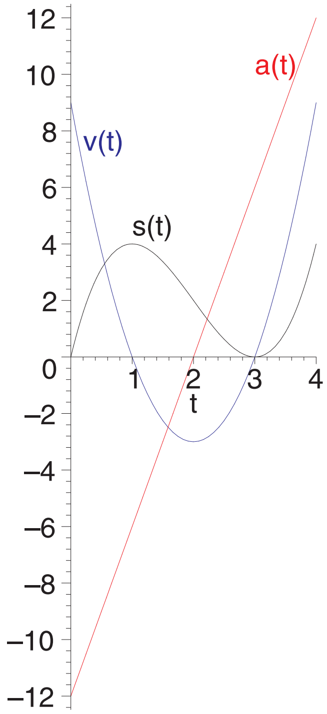
a)

$$\begin{aligned} s(t) &= t^3 - 6t^2 + 9t \\ &= t(t^2 - 6t + 9) \\ &= t(t-3)^2 \\ v(t) &= 3t^2 - 12t + 9 \\ &= 3(t^2 - 4t + 3) \\ &= 3(t-1)(t-3) \\ a(t) &= 6t - 12 \\ &= 6(t-2) \end{aligned}$$

b) From a), the roots of the position function are 0, 3, and 3. The roots of the velocity function are 1 and 3. Since the position is always positive, the cougar is moving toward the origin when the velocity is negative. This occurs over the interval $t \in (1, 3)$. The cougar is moving away from the origin when the velocity is positive. This occurs over the intervals $t \in (0, 1)$ and $t \in (3, \infty)$.

c) The cougar is speeding up when velocity and acceleration have the same sign. This occurs over the intervals $t \in (1, 2)$ and $t \in (3, \infty)$. The cougar is slowing down when the velocity and acceleration have opposite signs. This occurs over the intervals $t \in (0, 1)$ and $t \in (2, 3)$.

d)



Section 4.6 Page 239 Question 18

The only force acting on the stone is acceleration downward due to gravity. Hence $a(t) = -9.8$. The velocity function must, therefore, be $v(t) = -9.8t$. Letting h be the height of the stone above the base of the shaft of the well, in metres, the height function must be $h(t) = -4.9t^2 + 15$. After 1.6 s, $h(1.6) = -4.9(1.6)^2 + 15$ or 2.456 m. Since the height of the shaft is 15 m, the surface of the water is 15 − 2.456 or 12.544 m down the well.

Section 4.6 Page 239 Question 19

a) The horizontal displacement can be given by $s_h = 5.556t$ m, the horizontal velocity by $v_h = 5.556$ m/s, and the horizontal acceleration by $a_h = 0$ m/s^2.

b) Given a vertical acceleration function of $a_v(t) = -9.8$, the vertical velocity is defined by $v_v(t) = -9.8t$. Since the prey is dropped from a height of 30 m, the vertical displacement can be given by $s_v = -4.9t^2 + 30$.

c)

$$\begin{aligned} |v_v| &> h_h \\ |-9.8t| &> 5.556 \\ 9.8t &> 5.556 \\ t &> 0.567 \end{aligned}$$

The vertical speed is greater than the horizontal speed after approximately 0.567 s.

d) Let v be the overall velocity of the prey.

$$\begin{aligned} v^2 &= v_h^2 + v_v^2 \\ &= 5.556^2 + (-9.8t)^2 \\ &= 30.9 + (9.8t)^2 \\ v &= \sqrt{30.9 + (9.8t)^2} \end{aligned}$$

The overall velocity of the prey is given by $v(t) = \sqrt{30.9 + (9.8t)^2}$ m/s.

e)

$$\begin{aligned} a(t) &= \sqrt{(-9.8)^2 + 0^2} \\ &= 9.8 \end{aligned}$$

f) 9.8 m/s^2

4.7 Rates of Change in the Social Sciences

Apply, Solve, Communicate

Section 4.7 Page 249 Question 1

a) The marginal cost function is represented by $C'(x)$.

$$\begin{aligned} C(x) &= 2500x - 1.05x^2 \\ C'(x) &= 2500 - 2.1x \end{aligned}$$

The marginal cost function is $C'(x) = 2500 - 2.1x$.

b) The marginal cost function defines the rate at which costs are increasing (per item) as the xth unit is produced.

c)

$$\begin{aligned} C'(300) &= 2500 - 2.1(300) \\ &= 2500 - 630 \\ &= 1870 \end{aligned}$$

The marginal cost, at the 300-unit level, is \$1870/unit. This result is the cost of producing each unit after 300 units have been produced.

d)

$$\begin{aligned} C(301) - C(300) &= \left[2500(301) - 1.05(301)^2\right] - \left[2500(300) - 1.05(300)^2\right] \\ &= 657\ 368.95 - 655\ 500 \\ &= 1868.95 \end{aligned}$$

The cost of producing the 301st unit is \$1868.95.

e) The cost of producing the 301st unit is very close to the marginal cost after 300 units have been produced.

Section 4.7 Page 249 Question 2

a) The marginal cost function is represented by $C'(x)$.

$$\begin{aligned} C(x) &= 13\ 800 + 950x - 0.003x^2 \\ C'(x) &= 950 - 0.006x \end{aligned}$$

The marginal cost function is $C'(x) = 950 - 0.006x$.

b)

$$\begin{aligned} C'(5500) &= 950 - 0.006(5500) \\ &= 950 - 33 \\ &= 917 \end{aligned}$$

The marginal cost, at the 5500-unit level, is \$917/unit.

c)

$$\begin{aligned} C(5501) - C(5500) &= \left[13\ 800 + 950(5501) - 0.003(5501)^2\right] - \left[13\ 800 + 950(5500) - 0.003(5500)^2\right] \\ &= 916.997 \end{aligned}$$

The cost of producing the 5501st unit is \$916.997.

d) The cost of producing the 5501st unit is very close to the marginal cost after 5500 units have been produced.

Section 4.7 Page 249 Question 3

a) The marginal revenue function is represented by $R'(n)$.

$$\begin{aligned} R(n) &= 5000n - 0.08n^2 \\ R'(n) &= 5000 - 0.16n \end{aligned}$$

The marginal revenue function is $R'(n) = 5000 - 0.16n$.

b) The marginal revenue function defines the rate of change of the revenue with respect to the number of cars sold.

c)

$$\begin{aligned} R'(850) &= 5000 - 0.16(850) \\ &= 5000 - 136 \\ &= 4864 \end{aligned}$$

The marginal revenue, at the 850-unit level, is \$4864/car. This result is the increase in revenue per car realized after 850 cars have been sold.

d)

$$\begin{aligned} R(851) - R(850) &= \left[5000(851) - 0.08(851)^2\right] - \left[5000(850) - 0.08(850)^2\right] \\ &= 4\,197\,063.92 - 4\,192\,200.00 \\ &= 4863.92 \end{aligned}$$

The revenue from the 851st car is \$4863.92, which is very close to the marginal revenue after 850 cars have been sold.

Section 4.7 Page 249 Question 4

a) Let the profit function be $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 19x + 4x^2 - (11x + 2x^2) \\ &= 8x + 2x^2 \end{aligned}$$

The profit function is $P(x) = 8x + 2x^2$.

b) The marginal profit function is $P'(x)$.

$$\begin{aligned} P(x) &= 8x + 2x^2 \\ P'(x) &= 8 + 4x \end{aligned}$$

The marginal profit function is $P'(x) = 8 + 4x$.

c)

$$\begin{aligned} P'(700) &= 8 + 4(700) \\ &= 8 + 2800 \\ &= 2808 \end{aligned}$$

The marginal profit, at the 700-unit level, is \$2808/cup.

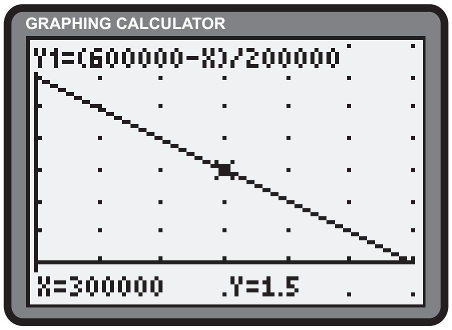
d)

$$\begin{aligned} P(701) - P(700) &= \left[8(701) + 2(701)^2\right] - \left[8(700) + 2(700)^2\right] \\ &= 988\,410 - 985\,600 \\ &= 2810 \end{aligned}$$

The marginal profit from the 701st cup is \$2810, which is very close to $P'(700)$.

Section 4.7 Page 250 Question 5

a)



GRAPHING CALCULATOR

Y1=(600000-X)/200000

X=300000 Y=1.5

b)

$p(x)$	x
0	600 000
\$0.50	500 000
\$1.00	400 000
\$1.50	300 000
\$2.00	200 000
\$2.50	100 000
\$3.00	0

c) Let $R(x)$ represent the revenue function.

$$\begin{aligned} R(x) &= xp(x) \\ &= x\left(\frac{600\,000 - x}{200\,000}\right) \\ &= \frac{1}{200\,000}(600\,000x - x^2) \end{aligned}$$

The revenue function is $R(x) = \frac{1}{200\,000}(600\,000x - x^2)$.

d) The marginal revenue function is $R'(x)$.

$$\begin{aligned} R'(x) &= \frac{1}{200\,000}(600\,000 - 2x) \\ &= \frac{1}{100\,000}(300\,000 - x) \end{aligned}$$

The marginal revenue function is $R'(x) = \frac{1}{100\,000}(300\,000 - x)$.

e)

$$\begin{aligned} R'(200\,000) &= \frac{1}{100\,000}(300\,000 - 200\,000) \\ &= 1 \end{aligned}$$

The marginal revenue when $x = 200\,000$ is \$1.

f) Let the profit function be $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \frac{1}{200\,000}(600\,000x - x^2) - (110\,000 + 0.95x) \\ &= 3x - \frac{x^2}{200\,000} - 110\,000 - 0.95x \\ &= 2.05x - \frac{x^2}{200\,000} - 110\,000 \end{aligned}$$

The profit function is $P(x) = 2.05x - \frac{x^2}{200\,000} - 110\,000$.

g) The marginal profit function is $P'(x)$.

$$\begin{aligned} P(x) &= 2.05x - \frac{x^2}{200\,000} - 110\,000 \\ P'(x) &= 2.05 - \frac{x}{100\,000} \end{aligned}$$

The marginal profit function is $P'(x) = 2.05 - \frac{x}{100\,000}$.

h)

$$\begin{aligned} P'(300\,000) &= 2.05 - \frac{300\,000}{100\,000} \\ &= 2.05 - 3 \\ &= -0.95 \end{aligned}$$

The marginal profit, at the 300 000-unit level, is −\$0.95/hamburger.

Section 4.7 Page 250 Question 6

a)

$$\begin{aligned} N(y) &= 0.2y^4 - 11.5y^3 + 195.85y^2 - 964.325y + 16\,434.625 \\ N'(y) &= 0.8y^3 - 34.5y^2 + 391.7y - 964.325 \end{aligned}$$

The rate of change of the number of births per year is $N'(y) = 0.8y^3 - 34.5y^2 + 391.7y - 964.325$.

b) For the year 1995, $y = 1995 - 1972$ or 23.

$$\begin{aligned} N'(23) &= 0.8(23)^3 - 34.5(23)^2 + 391.7(23) - 964.325 \\ &= -472.125 \end{aligned}$$

The rate of change of the number of births per year in 1995 was approximately −472 births/year.

c) For the year 1995, $y = 1995 - 1972$ or 23.

$$\begin{aligned} N(23) &= \left[0.2(23)^4 - 11.5(23)^3 + 195.85(23)^2 - 964.325(23) + 16\,434.625\right] \\ &= 13\,907.5 \end{aligned}$$

For the year 1994, $y = 1994 - 1972$ or 22.

$$\begin{aligned} N(22) &= \left[0.2(22)^4 - 11.5(22)^3 + 195.85(22)^2 - 964.325(22) + 16\,434.625\right] \\ &= 14\,410.075 \end{aligned}$$

Determine the actual change from 1994 to 1995.

$$\begin{aligned} N(23) - N(22) &= 13\,907.5 - 14\,410.075 \\ &= -502.575 \end{aligned}$$

The actual change from 1994 to 1995 was −502.575.

Section 4.7 Page 250 Question 7

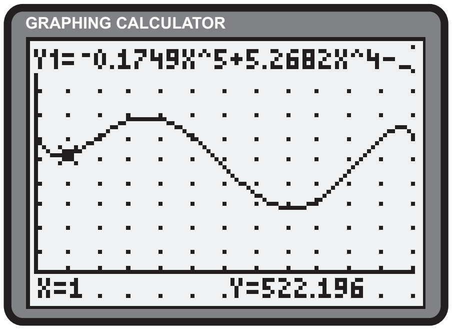
a)

$$\begin{aligned} Q(x) &= -0.1749x^5 + 5.2682x^4 - 53.3763x^3 + 203.679x^2 - 233.2x + 600 \\ Q'(x) &= -0.8745x^4 + 21.0728x^3 - 160.1289x^2 + 407.358x - 233.2 \end{aligned}$$

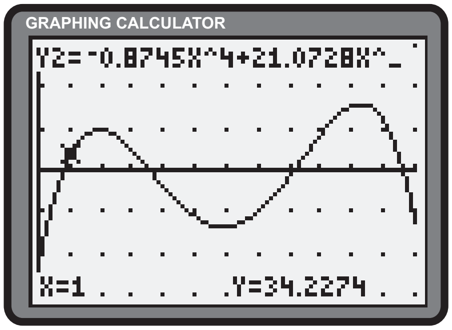
The rate of change is $Q'(x) = -0.8745x^4 + 21.0728x^3 - 160.1289x^2 + 407.358x - 233.2$.

b)

$Q(x)$



$Q'(x)$



c) The quantity of coffee consumed in January is $Q(1) \doteq 522.2$ kg. The quantity of coffee consumed in October is $Q(10) = 451.6$ kg.

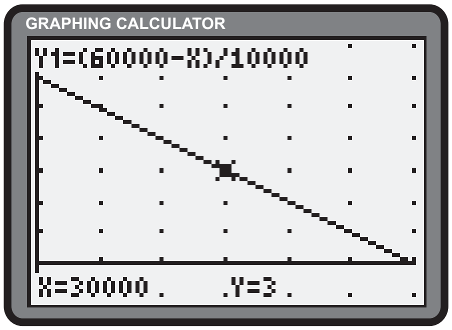
d) The rate of change of the quantity of coffee consumed in January is $Q'(1) \doteq 34.2$ kg/month. The rate of change of the quantity of coffee consumed in October is $Q'(10) \doteq 155.3$ kg/month.

e) Answers will vary.

Section 4.7 Page 250 Question 8

a)

$$p(x) = \frac{60\,000 - x}{10\,000}$$



b)

$p(x)$	x
0	60 000
$1.00	50 000
$2.00	40 000
$3.50	25 000
$4.00	20 000
$5.50	5000
$6.00	0

c) Let $R(x)$ represent the revenue function.

$$\begin{aligned} R(x) &= xp(x) \\ &= x\left(\frac{60\,000 - x}{10\,000}\right) \\ &= \frac{1}{10\,000}(60\,000x - x^2) \end{aligned}$$

The revenue function is $R(x) = \frac{1}{10\,000}(60\,000x - x^2)$.

d) The marginal revenue function is $R'(x)$.

$$\begin{aligned} R'(x) &= \frac{1}{10\,000}(60\,000 - 2x) \\ &= \frac{1}{5000}(30\,000 - x) \end{aligned}$$

The marginal revenue function is $R'(x) = \frac{1}{5000}(30\,000 - x)$.

e)

$$\begin{aligned} R'(5000) &= \frac{1}{5000}(30\,000 - 5000) \\ &= 5 \end{aligned}$$

The marginal revenue when $x = 5000$ is $5/unit.

f) Let the profit function be $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \frac{1}{10\,000}(60\,000x - x^2) - (8000 + 1.2x) \\ &= 6x - \frac{x^2}{10\,000} - 8000 - 1.2x \\ &= 4.8x - \frac{x^2}{10\,000} - 8000 \end{aligned}$$

The profit function is $P(x) = 4.8x - \frac{x^2}{10\,000} - 8000$.

g) The marginal profit function is $P'(x)$.

$$\begin{aligned} P(x) &= 4.8x - \frac{x^2}{10\,000} - 8000 \\ P'(x) &= 4.8 - \frac{x}{5000} \end{aligned}$$

The marginal profit function is $P'(x) = 4.8 - \frac{x}{5000}$.

h)

$$\begin{aligned} P'(20\,000) &= 4.8 - \frac{20\,000}{5000} \\ &= 4.8 - 4 \\ &= 0.8 \end{aligned}$$

The marginal profit, at the 20 000-unit level, is \$0.80/flower pot.

Section 4.7 Page 250 Question 9

a) Let $C(x)$ represent the cost function.

$$\begin{aligned} C(x) &= 5000x - 2.8x^2 \\ C'(x) &= 5000 - 5.6x \end{aligned}$$

The marginal cost of producing x games is $C'(x) = 5000 - 5.6x$.

b) Let $R(x)$ represent the revenue function.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(80 - 0.018x) \\ &= 80x - 0.018x^2 \\ R'(x) &= 80 - 0.036x \end{aligned}$$

The marginal revenue from selling x games is $R'(x) = 80 - 0.036x$.

c) Let $P(x)$ represent the profit function.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 80x - 0.018x^2 - (5000x - 2.8x^2) \\ &= -4920x + 2.782x^2 \\ P'(x) &= -4920 + 5.564x \end{aligned}$$

The marginal profit in selling x games is $P'(x) = -4920 + 5.564x$.

d)

$$\begin{aligned} P'(x) &= -4920 + 5.564x \\ P'(75) &= -4920 + 5.564(75) \\ &= -4502.7 \end{aligned}$$

The marginal profit in selling 75 games is −\$4502.70/game.

Section 4.7 Page 251 Question 10

a) Let $C(x)$ represent the cost function.

$$\begin{aligned} C(x) &= 5000 + 2200x + 0.06x^3 \\ C'(x) &= 2200 + 0.18x^2 \end{aligned}$$

The marginal cost of producing x rolls of carpet is $C'(x) = 2200 + 0.18x^2$.

b) Let $R(x)$ represent the revenue function.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(10\,000 - 0.05x^2) \\ &= 10\,000x - 0.05x^3 \\ R'(x) &= 10\,000 - 0.15x^2 \end{aligned}$$

The marginal revenue from selling x rolls of carpet is $R'(x) = 10\,000 - 0.15x^2$.

c) Let $P(x)$ represent the profit function.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 10\,000x - 0.05x^3 - (5000 + 2200x + 0.06x^3) \\ &= 7800x - 5000 - 0.11x^3 \\ P'(x) &= 7800 - 0.33x^2 \end{aligned}$$

The marginal profit in selling x rolls of carpet is $P'(x) = 7800 - 0.33x^2$.

d)

$$\begin{aligned} P'(x) &= 7800 - 0.33x^2 \\ P'(130) &= 7800 - 0.33(130)^2 \\ &= 2223 \end{aligned}$$

The marginal profit in selling 130 rolls of carpet is $2233/roll.

e)

$$\begin{aligned} C'(x) &= R'(x) \\ 2200 + 0.18x^2 &= 10\,000 - 0.15x^2 \\ 0.33x^2 &= 7800 \\ x^2 &\doteq 23\,636.364 \\ x &\doteq 153.74,\ x > 0 \end{aligned}$$

Marginal cost and marginal revenue are the same at a production level of approximately 154.74 rolls of carpet.

Section 4.7 Page 251 Question 11

Let $p'(x)$ represent the rate of change in price.

$$\begin{aligned} p(x) &= 1000 + 2\sqrt{x} \\ p'(x) &= \frac{1}{\sqrt{x}} \end{aligned}$$

At $x = 500$,

$$\begin{aligned} p'(500) &= \frac{1}{\sqrt{500}} \\ &\doteq 0.044\,72 \end{aligned}$$

The rate of change of price is approximately $0.044 72/unit at a production level of 500 units.

Section 4.7 Page 251 Question 12

a) Let $N'(t)$ represent the rate of change in the number of bacteria per hour.

$$\begin{aligned} N(t) &= 120\,000 + 1000t^3 \\ N'(t) &= 3000t^2 \end{aligned}$$

i) At 1 h, $N(1) = 3000(1)^2$ or 3000 bacteria/h.
ii) At 8 h, $N(8) = 3000(8)^2$ or 192 000 bacteria/h.
iii) At 24 h, $N(24) = 3000(24)^2$ or 1 728 000 bacteria/h.

b) The growth rate is increasing.

Section 4.7 Page 251 Question 13

$$\begin{aligned} Q(I) &= -I^2 - 100I + 2505 \\ Q'(I) &= -2I - 100 \end{aligned}$$

At $I = 60$,

$$\begin{aligned} Q'(60) &= -2(60) - 100 \\ &= -120 - 100 \\ &= -220 \end{aligned}$$

At the $60 000 income level, the rate of change of cartridges demanded is −220/$1000 of income.

Section 4.7 Page 251 Question 14

a) Let $C'(t)$ represent the rate of change of the CPI.

$$C(t) = 0.000\,052t^5 - 0.0005t^4 - 0.0153t^3 + 0.05t^2 + 4.404t + 69.5$$
$$C'(t) = 0.000\,26t^4 - 0.002t^3 - 0.0459t^2 + 0.1t + 4.404$$

The rate of change of the CPI is $C'(t) = 0.000\,26t^4 - 0.002t^3 - 0.0459t^2 + 0.1t + 4.404$.

b) In the year 1984, $t = 0$. $C'(0) = 4.404$. In the year 1999, $t = 1999 - 1984$ or 15.

$$\begin{aligned} C'(15) &= 0.000\,26(15)^4 - 0.002(15)^3 - 0.0459(15)^2 + 0.1(15) + 4.404 \\ &= 1.989 \end{aligned}$$

The rate of change of the CPI in 1999 was \$1.989/year.

c) Determine the actual change from 1984 ($t = 0$) to 1985 ($t = 1$).

$$\begin{aligned} C(1) &= 0.000\,052(1)^5 - 0.0005(1)^4 - 0.0153(1)^3 + 0.05(1)^2 + 4.404(1) + 69.5 \\ &\doteq 73.938 \\ C(0) &= 0.000\,052(0)^5 - 0.0005(0)^4 - 0.0153(0)^3 + 0.05(0)^2 + 4.404(0) + 69.5 \\ &= 69.5 \\ C(1) - C(0) &= 73.938 - 69.5 \\ &= 4.438 \end{aligned}$$

The actual change in the CPI from 1984 to 1985 was 4.438 as compared to 4.404 from the rate of change model.

Determine the actual change from 1998 ($t = 14$) to 1999 ($t = 15$).

$$\begin{aligned} C(15) &= 0.000\,052(15)^5 - 0.0005(15)^4 - 0.0153(15)^3 + 0.05(15)^2 + 4.404(15) + 69.5 \\ &\doteq 109.348 \\ C(14) &= 0.000\,052(14)^5 - 0.0005(14)^4 - 0.0153(14)^3 + 0.05(14)^2 + 4.404(14) + 69.5 \\ &\doteq 107.732 \\ C(15) - C(14) &= 109.348 - 107.732 \\ &= 1.616 \end{aligned}$$

The actual change in the CPI from 1998 to 1999 was 1.616 as compared to 1.989 from the rate of change model.

Section 4.7 Page 251 Question 15

a) Let $N'(d)$ represent the rate of change of transit ridership.

$$N(d) = 21\,000 + 420d - 5d^2$$
$$N'(d) = 420 - 10d$$

The rate of change of transit ridership is $N'(d) = 420 - 10d$ riders per kilometre.

b)

$$\begin{aligned} N'(2) &= 420 - 10(2) \\ &= 420 - 20 \\ &= 400 \\ N'(20) &= 420 - 10(20) \\ &= 220 \end{aligned}$$

At 2 km from downtown the rate of change in ridership is 400 riders/km. At 20 km from downtown the rate of change in ridership is 220 riders/km.

Section 4.7 Page 251 Question 16

Equate the two marginal rates and solve for t.

$$\begin{aligned}\frac{d}{dt}(97.42\sqrt{t}+295.01) &= \frac{d}{dt}(102.58t+403.75)\\ \frac{97.42}{2\sqrt{t}} &= 102.58\\ \sqrt{t} &= \frac{97.42}{205.16}\\ t &= \left(\frac{97.42}{205.16}\right)^2\\ &\doteq 0.2255\end{aligned}$$

The marginal outputs of the cities will be the same in approximately 0.2255 years. The marginal output is \$102.58 million/year.

Section 4.7 Page 251 Question 17

Determine the marginal yield function, $Y'(x)$.

$$\begin{aligned}Y(x) &= -0.08x^2+35x+3575\\ Y'(x) &= -0.16x+35\end{aligned}$$

The marginal yield function is $Y'(x) = -0.16x + 35$ kilograms of wheat per kilogram of seed.

$$\begin{aligned}Y'(75) &= -0.16(75)+35\\ &= 23\end{aligned}$$

When 75 kg of seed are used, the marginal yield is 23 kg wheat/kg seed.

Section 4.7 Page 251 Question 18

a) Let $P'(t)$ represent the growth rate of the bee population.

$$\begin{aligned}P(t) &= 100\sqrt{t}+\frac{200}{t+1}\\ P'(t) &= 100\cdot\frac{1}{2\sqrt{t}}+200\cdot\frac{-1}{(t+1)^2}\\ &= \frac{50}{\sqrt{t}}-\frac{200}{(t+1)^2}\end{aligned}$$

i) After 1 day,

$$\begin{aligned}P'(1) &= \frac{50}{\sqrt{1}}-\frac{200}{(1+1)^2}\\ &= 50-50\\ &= 0\end{aligned}$$

After 1 day, the growth rate is 0 bees/day.

ii) After 10 days,

$$\begin{aligned}P'(10) &= \frac{50}{\sqrt{10}}-\frac{200}{(10+1)^2}\\ &\doteq 14.16\end{aligned}$$

After 10 days, the growth rate is approximately 14.16 bees/day.

iii) After 60 days,

$$\begin{aligned}P'(60) &= \frac{50}{\sqrt{60}}-\frac{200}{(60+1)^2}\\ &\doteq 6.40\end{aligned}$$

After 60 days, the growth rate is approximately 6.40 bees/day

b) The growth rate initially increases and then decreases.

Section 4.7 Page 252 Question 19

Let $P(x)$ represent the profit function.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= xp(x) - C(x) \\ &= x(ax^2 + bx + c) - (dx^3 + ex^2 + fx + g) \\ &= ax^3 + bx^2 + cx - dx^3 - ex^2 - fx - g \\ &= (a - d)x^3 + (b - e)x^2 + (c - f)x - g \end{aligned}$$

The profit function is $P(x) = (a - d)x^3 + (b - e)x^2 + (c - f)x - g$. Let $P'(x)$ represent the marginal profit function.

$$P'(x) = 3(a - d)x^2 + 2(b - e)x + c - f$$

The marginal profit function is $P'(x) = 3(a - d)x^2 + 2(b - e)x + c - f$.

Section 4.7 Page 252 Question 20

Let $C(x)$ be the family of functions representing the cost per pizza. The family of functions can be determined by reversing the derivative rules.

$$\begin{aligned} C'(x) &= 0.000\,25 - \frac{10}{x^2} \\ C(x) &= C_0 + 0.000\,25x + \frac{10}{x} \end{aligned}$$

The family of functions representing the cost per pizza is $C(x) = C_0 + 0.000\,25x + \frac{10}{x}$.

Review of Key Concepts

4.1 The Derivative

Section Review Page 254 Question 1

a) The expression defines the derivative of $f(x) = x^2$ at $x = 3$.
b) The expression defines the derivative of $f(x) = x^3 + 4$ at $x = 1$.
c) The expression defines the derivative of $f(x) = \sqrt{x}$ at $x = 4$.
d) The expression defines the derivative of $f(x) = x^2$ at $x = 1$.
e) The expression defines the derivative of $f(x) = \sqrt{x}$ at $x = 4$.
f) The expression defines the derivative of $f(x) = x^3 + 20$ at $x = -3$.

Section Review Page 254 Question 2

a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ f'(4) &= \lim_{h\to 0} \frac{(4+h)^2 + 5 - (4^2 + 5)}{h} \\ &= \lim_{h\to 0} \frac{16 + 8h + h^2 + 5 - (16 + 5)}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 8h + 21 - 21}{h} \\ &= \lim_{h\to 0}(h + 8),\ h \neq 0 \\ &= 8 \end{aligned}$$

b)

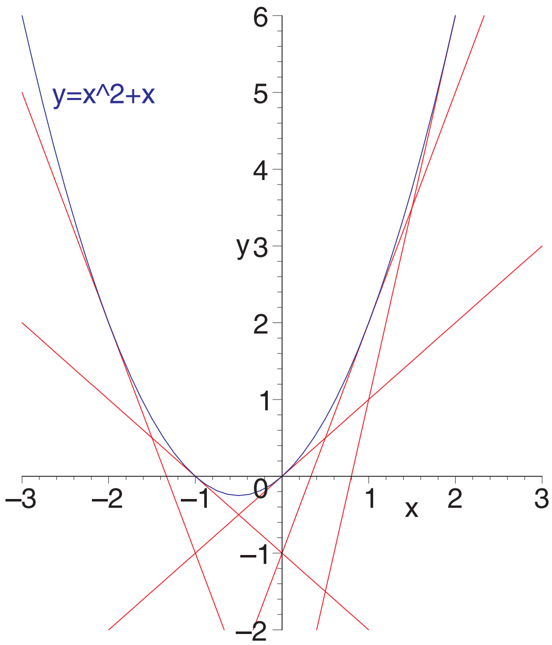
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ f'(4) &= \lim_{h\to 0} \frac{2(4+h)^2 + 3(4+h) - (2(4)^2 + 3(4))}{h} \\ &= \lim_{h\to 0} \frac{2(16 + 8h + h^2) + 12 + 3h - (32 + 12)}{h} \\ &= \lim_{h\to 0} \frac{32 + 16h + 2h^2 + 12 + 3h - 44}{h} \\ &= \lim_{h\to 0} \frac{2h^2 + 19h}{h} \\ &= \lim_{h\to 0}(2h + 19),\ h \neq 0 \\ &= 19 \end{aligned}$$

c)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ f'(4) &= \lim_{h\to 0} \frac{\sqrt{4+h} + (4+h) - \left(\sqrt{4} + 4\right)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} + 4 + h - 6}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2 + h}{h} \\ &= 1 + \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \\ &= 1 + \lim_{h\to 0} \frac{4 + h - 4}{h\left(\sqrt{4+h} + 2\right)} \\ &= 1 + \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} \\ &= 1 + \frac{1}{2 + 2} \\ &= \frac{5}{4} \end{aligned}$$

Section Review   Page 254   Question 3

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(a) &= \lim_{h\to 0} \frac{(a+h)^2+a+h-(a^2+a)}{h} \\
&= \lim_{h\to 0} \frac{a^2+2ah+h^2+a+h-a^2-a}{h} \\
&= \lim_{h\to 0} \frac{h(2a+1+h)}{h} \\
&= \lim_{h\to 0} 2a+1+h \\
&= 2a+1 \\
f'(-2) &= 2(-2)+1 \\
&= -3 \\
f'(-1) &= 2(-1)+1 \\
&= -1 \\
f'(0) &= 2(0)+1 \\
&= 1 \\
f'(1) &= 2(1)+1 \\
&= 3 \\
f'(2) &= 2(2)+1 \\
&= 5
\end{aligned}
$$



Section Review   Page 254   Question 4

a)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(-2) &= \lim_{h\to 0} \frac{4(-2+h)^2-3-(4(-2)^2-3)}{h} \\
&= \lim_{h\to 0} \frac{4(4-4h+h^2)-3-(16-3)}{h} \\
&= \lim_{h\to 0} \frac{16-16h+4h^2-3-13}{h} \\
&= \lim_{h\to 0} \frac{-16h+4h^2}{h} \\
&= \lim_{h\to 0}(-16+4h) \\
&= -16
\end{aligned}
$$

b)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(-2) &= \lim_{h\to 0} \frac{(-2+h)^2-8(-2+h)+16-((-2)^2-8(-2)+16)}{h} \\
&= \lim_{h\to 0} \frac{4-4h+h^2+16-8h+16-(4+16+16)}{h} \\
&= \lim_{h\to 0} \frac{h^2-12h}{h} \\
&= \lim_{h\to 0}(h-12) \\
&= -12
\end{aligned}
$$

c)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(-2) &= \lim_{h\to 0} \frac{\sqrt{(-2+h)+3}-\sqrt{-2+3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{h+1}-1}{h}\cdot\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1} \\
&= \lim_{h\to 0} \frac{h+1-1}{h\left(\sqrt{h+1}+1\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{h+1}+1} \\
&= \frac{1}{2}
\end{aligned}$$

d)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(-2) &= \lim_{h\to 0} \frac{(-2+h)^3+2-((-2)^3+2)}{h} \\
&= \lim_{h\to 0} \frac{-8+12h-6h^2+h^3+2-(-6)}{h} \\
&= \lim_{h\to 0} \frac{12h-6h^2+h^3}{h} \\
&= \lim_{h\to 0}(12-6h+h^2) \\
&= 12
\end{aligned}$$

e)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(-2) &= \lim_{h\to 0} \frac{\dfrac{1}{(-2+h)^3}-\dfrac{1}{(-2)^3}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(h-2)^3}+\dfrac{1}{8}}{h} \\
&= \lim_{h\to 0} \frac{8+(h-2)^3}{8h(h-2)^3} \\
&= \lim_{h\to 0} \frac{8+h^3-6h^2+12h-8}{8h(h-2)^3} \\
&= \lim_{h\to 0} \frac{h^2-6h+12}{8(h-2)^3} \\
&= \frac{12}{-64} \\
&= -\frac{3}{16}
\end{aligned}$$

f)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(-2) &= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{(-2+h)+4}}-\dfrac{1}{\sqrt{-2+4}}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{h+2}}-\dfrac{1}{\sqrt{2}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2}-\sqrt{h+2}}{h\sqrt{h+2}\sqrt{2}}\cdot\frac{\sqrt{2}+\sqrt{h+2}}{\sqrt{2}+\sqrt{h+2}} \\
&= \lim_{h\to 0} \frac{2-(h+2)}{h\sqrt{h+2}\sqrt{2}\left(\sqrt{2}+\sqrt{h+2}\right)} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{h+2}\sqrt{2}\left(\sqrt{2}+\sqrt{h+2}\right)} \\
&= \lim_{h\to 0} \frac{-1}{2\left(2\sqrt{2}\right)} \\
&= -\frac{1}{4\sqrt{2}}
\end{aligned}$$

Section Review Page 254 Question 5

$$\begin{aligned} s'(t) &= \lim_{h\to 0}\frac{s(t+h)-s(t)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{4}(t+h)^2+10(t+h)-\left(\frac{1}{4}t^2+10t\right)}{h} \\ &= \frac{1}{4}\cdot\lim_{h\to 0}\frac{(t+h)^2-t^2}{h}+10\cdot\lim_{h\to 0}\frac{(t+h)-t}{h} \\ &= \frac{1}{4}\cdot\lim_{h\to 0}\frac{t^2+2th+h^2-t^2}{h}+10\cdot\lim_{h\to 0}\frac{h}{h} \\ &= \frac{1}{4}\cdot\lim_{h\to 0}(2t+h)+10\cdot\lim_{h\to 0}1 \\ &= \frac{1}{4}(2t)+10(1) \\ &= \frac{1}{2}t+10 \end{aligned}$$

The velocity of the train after t s is $s'(t) = \frac{1}{2}t + 10$ m/s.

$$\begin{aligned} s'(2) &= \frac{1}{2}(2)+10 \\ &= 11 \end{aligned}$$

After 2 s, the velocity of the train is 11 m/s.

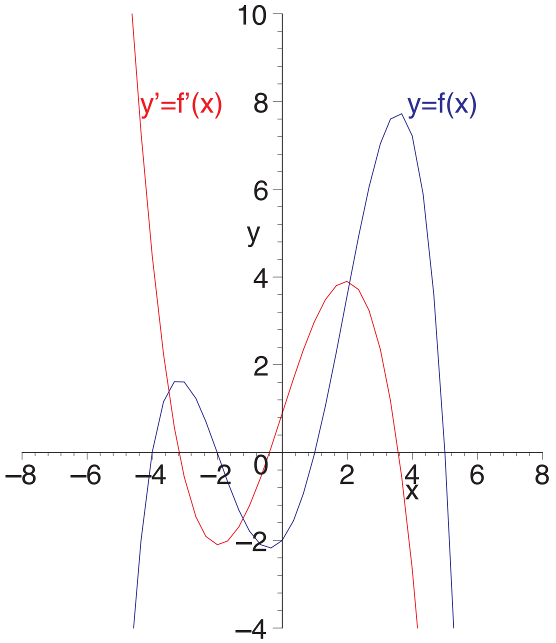
$$\begin{aligned} s'(4) &= \frac{1}{2}(4)+10 \\ &= 2+10 \\ &= 12 \end{aligned}$$

After 4 s, the velocity of the train is 12 m/s.

$$\begin{aligned} s'(6) &= \frac{1}{2}(6)+10 \\ &= 3+10 \\ &= 13 \end{aligned}$$

After 6 s, the velocity of the train is 13 m/s.

Section Review Page 254 Question 6



4.2 Basic Differentiation Rules

Section Review Page 254 Question 7

a)

$$\begin{aligned} y &= x^9 \\ \frac{dy}{dx} &= 9x^8 \end{aligned}$$

b)

$$\begin{aligned} y &= 12x^{\frac{4}{3}} \\ \frac{dy}{dx} &= 16x^{\frac{1}{3}} \end{aligned}$$

c)

$$\begin{aligned} y &= -\frac{3}{x^4} \\ &= -3x^{-4} \\ \frac{dy}{dx} &= -3(-4)x^{-5} \\ &= \frac{12}{x^5} \end{aligned}$$

d)

$$\begin{aligned} f(x) &= \frac{8}{3}\sqrt[4]{x^3} \\ &= \frac{8}{3}x^{\frac{3}{4}} \\ f'(x) &= \frac{8}{3}\left(\frac{3}{4}\right)x^{-\frac{1}{4}} \\ &= \frac{2}{\sqrt[4]{x}} \end{aligned}$$

e)

$$\begin{aligned} f(x) &= -\frac{1}{10}x^{-5} \\ f'(x) &= -\frac{1}{10}(-5)x^{-6} \\ &= \frac{1}{2}x^{-6} \end{aligned}$$

f)

$$\begin{aligned} f(x) &= 3x^{n+1} \\ f'(x) &= 3(n+1)x^{n} \end{aligned}$$

Section Review Page 254 Question 8

a)

$$\begin{aligned} f(x) &= x^2 + 4x \\ f'(x) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(4x) \\ &= 2x + 4 \end{aligned}$$

b)

$$\begin{aligned} f(x) &= 5x^3 - 2x^4 \\ f'(x) &= \frac{d}{dx}(5x^3) - \frac{d}{dx}(2x^4) \\ &= 15x^2 - 8x^3 \end{aligned}$$

c)

$$\begin{aligned} f(x) &= x^5 - 7x^4 + 3x^3 \\ f'(x) &= \frac{d}{dx}(x^5) - \frac{d}{dx}(7x^4) + \frac{d}{dx}(3x^3) \\ &= 5x^4 - 28x^3 + 9x^2 \end{aligned}$$

d)

$$\begin{aligned} g(x) &= \frac{1}{2}x^4 - \frac{1}{x} \\ &= \frac{1}{2}x^4 - x^{-1} \\ g'(x) &= \frac{d}{dx}\left(\frac{1}{2}x^4\right) - \frac{d}{dx}(x^{-1}) \\ &= \frac{1}{2}(4)x^3 - (-1)x^{-2} \\ &= 2x^3 + \frac{1}{x^2} \end{aligned}$$

e)

$$\begin{aligned} g(x) &= (2x-3)^2 \\ &= 4x^2 - 12x + 9 \\ g'(x) &= \frac{d}{dx}(4x^2) - \frac{d}{dx}(12x) + \frac{d}{dx}(9) \\ &= 8x - 12 \end{aligned}$$

f)

$$\begin{aligned} g(x) &= \frac{1}{9}x^{-3} - \frac{1}{6}x^{-2} \\ g'(x) &= \frac{d}{dx}\left(\frac{1}{9}x^{-3}\right) - \frac{d}{dx}\left(\frac{1}{6}x^{-2}\right) \\ &= \frac{1}{9}(-3)x^{-4} - \left(\frac{1}{6}\right)(-2)x^{-3} \\ &= -\frac{1}{3}x^{-4} + \frac{1}{3}x^{-3} \end{aligned}$$

g)

$$\begin{aligned} y &= \frac{1}{3}\sqrt{x} - \frac{2}{\sqrt{x}} \\ &= \frac{1}{3}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{3}x^{\frac{1}{2}}\right) - \frac{d}{dx}\left(2x^{-\frac{1}{2}}\right) \\ &= \frac{1}{3}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} - 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} \\ &= \frac{1}{6\sqrt{x}} + \frac{1}{x\sqrt{x}} \end{aligned}$$

h)

$$\begin{aligned} y &= 6\sqrt[4]{x} - 3\sqrt[3]{x} + \frac{10}{\sqrt[5]{x}} \\ &= 6x^{\frac{1}{4}} - 3x^{\frac{1}{3}} + 10x^{-\frac{1}{5}} \\ \frac{dy}{dx} &= \frac{d}{dx}\left(6x^{\frac{1}{4}}\right) - \frac{d}{dx}\left(3x^{\frac{1}{3}}\right) + \frac{d}{dx}\left(10x^{-\frac{1}{5}}\right) \\ &= 6\left(\frac{1}{4}\right)x^{-\frac{3}{4}} - 3\left(\frac{1}{3}\right)x^{-\frac{2}{3}} + 10\left(-\frac{1}{5}\right)x^{-\frac{6}{5}} \\ &= \frac{3}{2}x^{-\frac{3}{4}} - x^{-\frac{2}{3}} - 2x^{-\frac{6}{5}} \end{aligned}$$

i)

$$\begin{aligned} y &= (x^2 - 2x)^3 \\ &= x^6 - 6x^5 + 12x^4 - 8x^3 \\ \frac{dy}{dx} &= \frac{d}{dx}(x^6) - \frac{d}{dx}(6x^5) + \frac{d}{dx}(12x^4) - \frac{d}{dx}(8x^3) \\ &= 6x^5 - 30x^4 + 48x^3 - 24x^2 \end{aligned}$$

Section Review Page 254 Question 9

a)

$$\begin{aligned} y &= \frac{x^3}{3} - 2x^2 \\ \frac{dy}{dx} &= x^2 - 4x \\ \frac{dy}{dx}_{|x=3} &= 3^2 - 4(3) \\ &= 9 - 12 \\ &= -3 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-9) &= -3(x - 3) \\ y + 9 &= -3x + 9 \\ y &= -3x \end{aligned}$$

The equation of the tangent is $y = -3x$.

b)

$$\begin{aligned} y &= x^2 - 2x + 5 \\ \frac{dy}{dx} &= 2x - 2 \\ \frac{dy}{dx}_{|x=-1} &= 2(-1) - 2 \\ &= -2 - 2 \\ &= -4 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 8 &= -4(x - (-1)) \\ y - 8 &= -4x - 4 \\ y &= -4x + 4 \end{aligned}$$

The equation of the tangent is $y = -4x + 4$.

c)

$$\begin{aligned} y &= \sqrt{x} + \frac{4}{\sqrt{x}} \\ &= x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}} \\ \frac{dy}{dx}_{|x=4} &= \frac{1}{2}(4)^{-\frac{1}{2}} - 2(4)^{-\frac{3}{2}} \\ &= \frac{1}{4} - \frac{1}{4} \\ &= 0 \end{aligned}$$

The tangent is a horizontal line passing through $(4, 4)$.
The equation of the tangent is $y = 4$.

d)

$$\begin{aligned} y &= 2ax^2 + ax \\ \frac{dy}{dx} &= 4ax + a \\ \frac{dy}{dx}_{|x=2} &= 4a(2) + a \\ &= 8a + a \\ &= 9a \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 10a &= 9a(x - 2) \\ y - 10a &= 9ax - 18a \\ y &= 9ax - 8a \end{aligned}$$

The equation of the tangent is $y = 9ax - 8a$.

Section Review Page 255 Question 10

a)

$$\begin{aligned} V(t) &= 2\sqrt{t} + 1 \\ &= 2t^{\frac{1}{2}} + 1 \\ V'(t) &= t^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{t}} \end{aligned}$$

The derivative represents the rate of change in the value ($/month) of the stock.

b) Answers will vary.

c) Answers will vary.

Section Review Page 255 Question 11

$$\begin{aligned} h(t) &= 200 - 4.9t^2 \\ h'(t) &= -9.8t \end{aligned}$$

The velocity of the ball after 1 s is $h'(1) = -9.8(1)$ or -9.8 m/s. The velocity of the ball after 2 s is $h'(2) = -9.8(2)$ or -19.6 m/s. The velocity of the ball after 5 s is $h'(5) = -9.8(5)$ or -49 m/s.

Section Review Page 255 Question 12

The slope of the line $5x + y = 3$ is -5. Parallel lines have the same slopes.

$$\begin{aligned} \frac{d}{dx}(3x^2 - 7x + 4) &= -5 \\ 6x - 7 &= -5 \\ 6x &= 2 \\ x &= \frac{1}{3} \end{aligned}$$

At $x = \frac{1}{3}$,

$$\begin{aligned} y &= 3\left(\frac{1}{3}\right)^2 - 7\left(\frac{1}{3}\right) + 4 \\ &= \frac{1}{3} - \frac{7}{3} + \frac{12}{3} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

The point on the parabola $y = 3x^2 - 7x + 4$ at which the slope of the tangent is parallel to $5x + y = 3$ is $\left(\frac{1}{3}, 2\right)$.

Section Review Page 255 Question 13

The slope of the tangent to $y = x^2 + x$ is $\frac{dy}{dx} = 2x + 1$. The general coordinates of the point P on the parabola are $(x, x^2 + x)$. Let the remote point be Q(2, −3).

$$\begin{aligned} \frac{dy}{dx} &= \text{Slope of segment PQ} \\ 2x + 1 &= \frac{x^2 + x - (-3)}{x - 2} \\ 2x + 1 &= \frac{x^2 + x + 3}{x - 2} \\ 2x^2 - 3x - 2 &= x^2 + x + 3 \\ x^2 - 4x - 5 &= 0 \\ (x + 1)(x - 5) &= 0 \\ x &= -1 \text{ or } 5 \end{aligned}$$

At $x = -1$, the slope of the tangent is $2(-1) + 1$ or -1. The equation of the tangent through $(2, -3)$ with a slope of -1 is

$$\begin{aligned} y - (-3) &= -1(x - 2) \\ y + 3 &= -x + 2 \\ y &= -x - 1 \end{aligned}$$

At $x = 5$, the slope of the tangent is $2(5)+1$ or 11. The equation of the tangent through $(2, -3)$ with a slope of 11 is

$$\begin{aligned} y - (-3) &= 11(x - 2) \\ y + 3 &= 11x - 22 \\ y &= 11x - 25 \end{aligned}$$

4.3 The Product Rule

Section Review Page 255 Question 14

a)
$$\begin{aligned} f'(x) &= (x^2+2x)\frac{d}{dx}(x+1)+(x+1)\frac{d}{dx}(x^2+2x) \\ &= (x^2+2x)(1)+(x+1)(2x+2) \\ &= x^2+2x+2x^2+4x+2 \\ &= 3x^2+6x+2 \end{aligned}$$

b)
$$\begin{aligned} g'(x) &= (3x^2-8x^3)\frac{d}{dx}\left(x^{\frac{1}{2}}\right)+\left(x^{\frac{1}{2}}\right)\frac{d}{dx}(3x^2-8x^3) \\ &= (3x^2-8x^3)\left(\frac{1}{2}\right)x^{-\frac{1}{2}}+\left(x^{\frac{1}{2}}\right)(6x-24x^2) \\ &= \frac{3}{2}x^{\frac{3}{2}}-4x^{\frac{5}{2}}+6x^{\frac{3}{2}}-24x^{\frac{5}{2}} \\ &= \frac{15}{2}x^{\frac{3}{2}}-28x^{\frac{5}{2}} \end{aligned}$$

c)
$$\begin{aligned} s'(t) &= (3t+2t^{-2})\frac{d}{dt}(t^{-2})+(t^{-2})\frac{d}{dt}(3t+2t^{-2}) \\ &= (3t+2t^{-2})(-2)t^{-3}+(t^{-2})(3-4t^{-3}) \\ &= -6t^{-2}-4t^{-5}+3t^{-2}-4t^{-5} \\ &= -3t^{-2}-8t^{-5} \end{aligned}$$

d)
$$\begin{aligned} m'(k) &= (3k-4k^2+k^3)\frac{d}{dk}(k^2-4k)+(k^2-4k)\frac{d}{dk}(3k-4k^2+k^3) \\ &= (3k-4k^2+k^3)(2k-4)+(k^2-4k)(3-8k+3k^2) \\ &= 6k^2-12k-8k^3+16k^2+2k^4-4k^3+3k^2-8k^3+3k^4-12k+32k^2-12k^3 \\ &= 5k^4-32k^3+57k^2-24k \end{aligned}$$

e)
$$\begin{aligned} \frac{dy}{dx} &= (3+2\sqrt{x})\frac{d}{dx}(6x^{\frac{1}{2}}-4)+(6\sqrt{x}-4)\frac{d}{dx}(3+2x^{\frac{1}{2}}) \\ &= (3+2\sqrt{x})(3x^{-\frac{1}{2}})+(6\sqrt{x}-4)(x^{-\frac{1}{2}}) \\ &= (3+2\sqrt{x})\left(\frac{3}{\sqrt{x}}\right)+(6\sqrt{x}-4)\left(\frac{1}{\sqrt{x}}\right) \\ &= \frac{9}{\sqrt{x}}+6+6-\frac{4}{\sqrt{x}} \\ &= 12+\frac{5}{\sqrt{x}} \end{aligned}$$

f)
$$\begin{aligned} \frac{dy}{dx} &= (ax^2-bx)\frac{d}{dx}(ax+b)+(ax+b)\frac{d}{dx}(ax^2-bx) \\ &= (ax^2-bx)(a)+(ax+b)(2ax-b) \\ &= a^2x^2-abx+2a^2x^2-abx+2abx-b^2 \\ &= 3a^2x^2-b^2 \end{aligned}$$

Section Review Page 255 Question 15

a)
$$\begin{aligned} \frac{dy}{dx} &= (x^2-1)\frac{d}{dx}(x+3)+(x+3)\frac{d}{dx}(x^2-1) \\ &= (x^2-1)(1)+(x+3)(2x) \\ &= x^2-1+2x^2+6x \\ &= 3x^2+6x-1 \\ \frac{dy}{dx}_{|x=1} &= 3(1)^2+6(1)-1 \\ &= 3+6-1 \\ &= 8 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-0 &= 8(x-1) \\ y &= 8x-8 \end{aligned}$$

The equation of the tangent is $y = 8x - 8$.

b)

$$\begin{aligned}\frac{dy}{dx} &= (x^3+5)\frac{d}{dx}(x^2+x)+(x^2+x)\frac{d}{dx}(x^3+5)\\ &= (x^3+5)(2x+1)+(x^2+x)(3x^2)\\ &= 2x^4+x^3+10x+5+3x^4+3x^3\\ &= 5x^4+4x^3+10x+5\\ \frac{dy}{dx}_{|x=-2} &= 5(-2)^4+4(-2)^3+10(-2)+5\\ &= 80-32-20+5\\ &= 33\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}y-y_1 &= m(x-x_1)\\ y-(-6) &= 33(x-(-2))\\ y+6 &= 33x+66\\ y &= 33x+60\end{aligned}$$

The equation of the tangent is $y = 33x + 60$.

c)

$$\begin{aligned}\frac{dy}{dx} &= (x+9x^{-1})\frac{d}{dx}(x^{\frac{1}{2}})+(x^{\frac{1}{2}})\frac{d}{dx}(x+9x^{-1})\\ &= (x+9x^{-1})\left(\frac{1}{2}\right)x^{-\frac{1}{2}}+(x^{\frac{1}{2}})(1-9x^{-2})\\ &= \frac{1}{2}x^{\frac{1}{2}}+\frac{9}{2}x^{-\frac{3}{2}}+x^{\frac{1}{2}}-9x^{-\frac{3}{2}}\\ &= \frac{3\sqrt{x}}{2}-\frac{9}{2\left(\sqrt{x}\right)^3}\\ \frac{dy}{dx}_{|x=9} &= \frac{3\sqrt{9}}{2}-\frac{9}{2\left(\sqrt{9}\right)^3}\\ &= \frac{9}{2}-\frac{1}{6}\\ &= \frac{26}{6}\\ &= \frac{13}{3}\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}y-y_1 &= m(x-x_1)\\ y-30 &= \frac{13}{3}(x-9)\\ y &= \frac{13}{3}x-39+30\\ y &= \frac{13}{3}x-9\end{aligned}$$

The equation of the tangent is $y = \frac{13}{3}x - 9$.

d)

$$\begin{aligned}\frac{dy}{dx} &= (4+3x)\frac{d}{dx}(x^{-2})+(x^{-2})\frac{d}{dx}(4+3x)\\ &= (4+3x)(-2x^{-3})+(x^{-2})(3)\\ &= -8x^{-3}-6x^{-2}+3x^{-2}\\ &= -8x^{-3}-3x^{-2}\\ \frac{dy}{dx}_{|x=4} &= -8(4)^{-3}-3(4)^{-2}\\ &= \frac{-8}{64}-\frac{3}{16}\\ &= \frac{-8-12}{64}\\ &= -\frac{5}{16}\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}y-y_1 &= m(x-x_1)\\ y-1 &= -\frac{5}{16}(x-4)\\ y &= -\frac{5}{16}x+\frac{20}{16}+1\\ y &= -\frac{5}{16}x+\frac{36}{16}\\ y &= -\frac{5}{16}x+\frac{9}{4}\end{aligned}$$

The equation of the tangent is $y = -\frac{5}{16}x + \frac{9}{4}$.

Section Review Page 255 Question 16

a)

$$\begin{aligned}(fg)'(x) &= g(x)f'(x)+f(x)g'(x)\\ (fg)'(5) &= g(5)f'(5)+f(5)g'(5)\\ &= (-3)(4)+(2)(3)\\ &= -12+6\\ &= -6\end{aligned}$$

b)

$$\begin{aligned}\frac{d}{dx}(x^2g(x)) &= g(x)\frac{d}{dx}(x^2)+(x^2)\frac{d}{dx}(g(x))\\ &= g(x)(2x)+(x^2)(g'(x))\\ \frac{d}{dx}(x^2g(x))_{|x=5} &= g(5)(2(5))+(5^2)g'(5)\\ &= (-3)(10)+(25)(3)\\ &= -30+75\\ &= 45\end{aligned}$$

c)

$$\begin{aligned}\frac{d}{dx}(x^2f(x)g(x)) &= g(x)\frac{d}{dx}(x^2f(x)) + (x^2f(x))\frac{d}{dx}(g(x))\\ &= g(x)\left[f(x)\frac{d}{dx}(x^2) + (x^2)\frac{d}{dx}(f(x))\right] + (x^2f(x))\frac{d}{dx}(g(x))\\ &= g(x)\left[f(x)(2x) + (x^2)f'(x)\right] + x^2f(x)g'(x)\\ \frac{d}{dx}(x^2f(x)g(x))_{|x=5} &= g(5)\left[f(5)(2(5)) + (5^2)f'(5)\right] + 5^2f(5)g'(5)\\ &= -3\left[(2)(10) + (25)(4)\right] + 25(2)(3)\\ &= -3(20+100) + 150\\ &= -360 + 150\\ &= -210\end{aligned}$$

Section Review Page 255 Question 17

a)

$$\begin{aligned}f(x) &= (x^2+2)g(x)\\ f'(x) &= g(x)\frac{d}{dx}(x^2+2) + (x^2+2)\frac{d}{dx}(g(x))\\ &= g(x)(2x) + (x^2+2)g'(x)\\ &= 2xg(x) + (x^2+2)g'(x)\end{aligned}$$

b)

$$\begin{aligned}f(x) &= x^{-3}g(x)\\ f'(x) &= g(x)\frac{d}{dx}(x^{-3}) + (x^{-3})\frac{d}{dx}(g(x))\\ &= g(x)(-3x^{-4}) + (x^{-3})g'(x)\\ &= -3x^{-4}g(x) + x^{-3}g'(x)\end{aligned}$$

c)

$$\begin{aligned}f(x) &= ax^2g(x)\\ f'(x) &= g(x)\frac{d}{dx}(ax^2) + (ax^2)\frac{d}{dx}(g(x))\\ &= g(x)(2ax) + (ax^2)g'(x)\\ &= 2axg(x) + ax^2g'(x)\end{aligned}$$

4.4 The Quotient Rule

Section Review Page 255 Question 18

a)

$$\begin{aligned}f(x) &= \frac{2x}{x^2+1}\\ f'(x) &= \frac{(x^2+1)\frac{d}{dx}(2x) - (2x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\\ &= \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2}\\ &= \frac{2x^2+2-4x^2}{(x^2+1)^2}\\ &= \frac{2-2x^2}{(x^2+1)^2}\end{aligned}$$

b)

$$\begin{aligned}g(x) &= \frac{x^3}{x-2}\\ g'(x) &= \frac{(x-2)\frac{d}{dx}(x^3) - (x^3)\frac{d}{dx}(x-2)}{(x-2)^2}\\ &= \frac{(x-2)(3x^2) - (x^3)(1)}{(x-2)^2}\\ &= \frac{3x^3-6x^2-x^3}{(x-2)^2}\\ &= \frac{2x^3-6x^2}{(x-2)^2}\end{aligned}$$

c) $h(x) = \dfrac{2x+1}{\sqrt{x}}$

$$\begin{aligned}
h'(x) &= \frac{(\sqrt{x})\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2} \\
&= \frac{(\sqrt{x})(2) - (2x+1)\left(\frac{1}{2\sqrt{x}}\right)}{x} \\
&= \frac{2\sqrt{x} - \sqrt{x} - \frac{1}{2\sqrt{x}}}{x} \\
&= \frac{\sqrt{x} - \frac{1}{2\sqrt{x}}}{x} \\
&= \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}}
\end{aligned}$$

d) $C(x) = \dfrac{x-2}{1-x}$

$$\begin{aligned}
C'(x) &= \frac{(1-x)\frac{d}{dx}(x-2) - (x-2)\frac{d}{dx}(1-x)}{(1-x)^2} \\
&= \frac{(1-x)(1) - (x-2)(-1)}{(1-x)^2} \\
&= \frac{1-x+x-2}{(1-x)^2} \\
&= -\frac{1}{(1-x)^2}
\end{aligned}$$

e) $N(t) = \dfrac{t-t^2}{2-t}$

$$\begin{aligned}
N'(t) &= \frac{(2-t)\frac{d}{dt}(t-t^2) - (t-t^2)\frac{d}{dt}(2-t)}{(2-t)^2} \\
&= \frac{(2-t)(1-2t) - (t-t^2)(-1)}{(2-t)^2} \\
&= \frac{2-4t-t+2t^2+t-t^2}{(2-t)^2} \\
&= \frac{2-4t+t^2}{(2-t)^2}
\end{aligned}$$

f) $f(x) = \dfrac{\sqrt{x}+3}{\sqrt{x}}$

$$\begin{aligned}
f'(x) &= \frac{(\sqrt{x})\frac{d}{dx}(\sqrt{x}+3) - (\sqrt{x}+3)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2} \\
&= \frac{(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x}+3)\left(\frac{1}{2\sqrt{x}}\right)}{x} \\
&= \frac{\frac{1}{2} - \frac{1}{2} - \frac{3}{2\sqrt{x}}}{x} \\
&= -\frac{3}{2x\sqrt{x}}
\end{aligned}$$

Section Review Page 255 Question 19

a)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1-x)\frac{d}{dx}(2) - (2)\frac{d}{dx}(1-x)}{(1-x)^2} \\
&= \frac{(1-x)(0) - 2(-1)}{(1-x)^2} \\
&= \frac{2}{(1-x)^2} \\
\frac{dy}{dx}_{|x=2} &= \frac{2}{(1-2)^2} \\
&= \frac{2}{1} \\
&= 2
\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - (-2) &= 2(x-2) \\
y + 2 &= 2x - 4 \\
y &= 2x - 6
\end{aligned}$$

The equation of the tangent is $y = 2x - 6$.

b)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(2-x^2)\frac{d}{dx}(x)-(x)\frac{d}{dx}(2-x^2)}{(2-x^2)^2}\\
&= \frac{(2-x^2)(1)-x(-2x)}{(2-x^2)^2}\\
&= \frac{2-x^2+2x^2}{(2-x^2)^2}\\
&= \frac{2+x^2}{(2-x^2)^2}\\
\frac{dy}{dx}_{|x=2} &= \frac{2+2^2}{(2-2^2)^2}\\
&= \frac{6}{(2-4)^2}\\
&= \frac{6}{4}\\
&= \frac{3}{2}
\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}
y-y_1 &= m(x-x_1)\\
y-(-1) &= \frac{3}{2}(x-2)\\
y+1 &= \frac{3}{2}x-3\\
y &= \frac{3}{2}x-4
\end{aligned}$$

The equation of the tangent is $y=\frac{3}{2}x-4$.

c)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(2x)\frac{d}{dx}(x^2+4)-(x^2+4)\frac{d}{dx}(2x)}{(2x)^2}\\
&= \frac{(2x)(2x)-(x^2+4)(2)}{4x^2}\\
&= \frac{4x^2-2x^2-8}{4x^2}\\
&= \frac{x^2-4}{2x^2}\\
\frac{dy}{dx}_{|x=2} &= \frac{(2)^2-4}{2(2)^2}\\
&= \frac{0}{8}\\
&= 0
\end{aligned}$$

The tangent is a horizontal line passing through $(2, 2)$.
The equation of the tangent is $y=2$.

Section Review Page 255 Question 20

a)

$$\begin{aligned}
f(x) &= \frac{g(x)}{\sqrt{x}}\\
f'(x) &= \frac{(\sqrt{x})\frac{d}{dx}(g(x))-g(x)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}\\
&= \frac{(\sqrt{x})g'(x)-g(x)\left(\frac{1}{2\sqrt{x}}\right)}{x}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}\\
&= \frac{2xg'(x)-g(x)}{2x\sqrt{x}}
\end{aligned}$$

b)

$$\begin{aligned}
f(x) &= \frac{g(x)}{x}\\
f'(x) &= \frac{(x)\frac{d}{dx}(g(x))-g(x)\frac{d}{dx}(x)}{x^2}\\
&= \frac{xg'(x)-g(x)(1)}{x^2}\\
&= \frac{xg'(x)-g(x)}{x^2}
\end{aligned}$$

c) $f(x) = \dfrac{xg(x)}{x+1}$

$$f'(x) = \frac{(x+1)\dfrac{d}{dx}(xg(x)) - xg(x)\dfrac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)\left[g(x)\dfrac{d}{dx}(x) + (x)\dfrac{d}{dx}g(x)\right] - xg(x)\dfrac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)\left[g(x)(1) + xg'(x)\right] - xg(x)(1)}{(x+1)^2}$$

$$= \frac{(x+1)g(x) + x(x+1)g'(x) - xg(x)}{(x+1)^2}$$

$$= \frac{xg(x) + g(x) + x(x+1)g'(x) - xg(x)}{(x+1)^2}$$

$$= \frac{g(x) + x(x+1)g'(x)}{(x+1)^2}$$

Section Review Page 255 Question 21

a) $P(t) = \dfrac{500t^2}{1+t^3} + 100$

$$P'(t) = \frac{(1+t^3)\dfrac{d}{dt}(500t^2) - (500t^2)\dfrac{d}{dt}(1+t^3)}{(1+t^3)^2} + \frac{d(100)}{dt}$$

$$= \frac{(1+t^3)(1000t) - (500t^2)(3t^2)}{(1+t^3)^2} + 0$$

$$= \frac{500t(2 + 2t^3 - 3t^3)}{(1+t^3)^2}$$

$$= \frac{500t(2 - t^3)}{(1+t^3)^2}$$

$P'(t)$ defines the rate of change of population in hundreds of people per year.

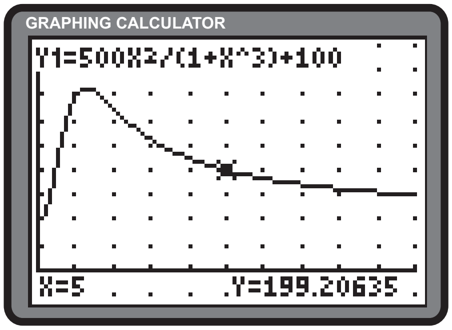
b)

$$\begin{aligned} P'(t) &= 0 \\ \frac{500t(2-t^3)}{(1+t^3)^2} &= 0 \\ 500t(2-t^3) &= 0 \\ t(2-t^3) &= 0 \\ t &= 0 \\ 2 - t^3 &= 0 \\ t^3 &= 2 \\ t &= \sqrt[3]{2} \end{aligned}$$

For $t > 0$, $P'(t)$ is zero when $t = \sqrt[3]{2}$. This indicates the point at which the population stops increasing.

c)

$$P(t) = \frac{500t^2}{1+t^3} + 100$$

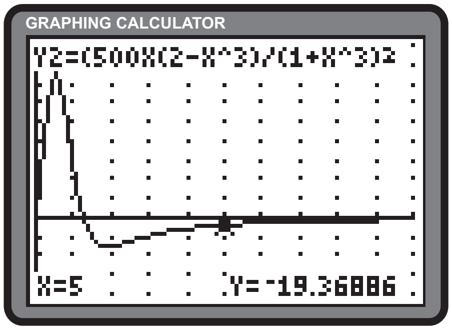


Y1=500X²/(1+X^3)+100

X=5 Y=199.20635

The population increases dramatically early on, then decreases, gradually levelling off.

$$P'(t) = \frac{500t(2-t^3)}{(1+t^3)^2}$$



Y2=(500X(2-X^3)/(1+X^3)²

X=5 Y=⁻19.36886

Early on, $P'(t)$ is positive, reflecting an increasing population. At $t = \sqrt[3]{2}$, the rate of increase is zero, reflecting the point at which the population achieves a maximum. $P'(t) < 0$ for $t > \sqrt[3]{2}$, reflecting a decreasing population.

d) Answers will vary.

4.5 Derivatives of Derivatives

Section Review Page 256 Question 22

a)

$$\begin{aligned} y &= x^2 - 3x + 2 \\ \frac{dy}{dx} &= 2x - 3 \\ \frac{d^2y}{dx^2} &= 2 \end{aligned}$$

b)

$$\begin{aligned} y &= 4x^3 - 2\sqrt{x} \\ &= 4x^3 - 2x^{\frac{1}{2}} \\ \frac{dy}{dx} &= 12x^2 - x^{-\frac{1}{2}} \\ \frac{d^2y}{dx^2} &= 24x + \frac{1}{2}x^{-\frac{3}{2}} \\ &= 24x + \frac{1}{2x^{\frac{3}{2}}} \\ &= 24x + \frac{1}{2x\sqrt{x}} \end{aligned}$$

c)

$$\begin{aligned} y &= \sqrt{x}(x^2 + 1) \\ &= x^{\frac{5}{2}} + x^{\frac{1}{2}} \\ \frac{dy}{dx} &= \frac{5}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}} \\ \frac{d^2y}{dx^2} &= \frac{15}{4}x^{\frac{1}{2}} - \frac{1}{4}x^{-\frac{3}{2}} \\ &= \frac{15}{4}\sqrt{x} - \frac{1}{4x\sqrt{x}} \end{aligned}$$

d)

$$\begin{aligned} y &= (x-2)(x^2 + 3x) \\ &= x^3 + 3x^2 - 2x^2 - 6x \\ &= x^3 + x^2 - 6x \\ \frac{dy}{dx} &= 3x^2 + 2x - 6 \\ \frac{d^2y}{dx^2} &= 6x + 2 \end{aligned}$$

e)

$$\begin{aligned} y &= \frac{x-1}{x+1} \\ \frac{dy}{dx} &= \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2} \\ &= \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} \\ &= \frac{x+1-x+1}{(x+1)^2} \\ &= \frac{2}{(x+1)^2} \\ \frac{d^2y}{dx^2} &= \frac{(x+1)^2\frac{d}{dx}(2) - (2)\frac{d}{dx}(x^2+2x+1)}{((x+1)^2)^2} \\ &= \frac{(x+1)^2(0) - 2(2x+2)}{(x+1)^4} \\ &= \frac{-4}{(x+1)^3} \end{aligned}$$

f)

$$\begin{aligned} y &= \frac{x^2}{x+2} \\ \frac{dy}{dx} &= \frac{(x+2)\frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(x+2)}{(x+2)^2} \\ &= \frac{(x+2)(2x) - (x^2)(1)}{(x+2)^2} \\ &= \frac{2x^2 + 4x - x^2}{(x+2)^2} \\ &= \frac{x^2 + 4x}{(x+2)^2} \\ \frac{d^2y}{dx^2} &= \frac{(x+2)^2\frac{d}{dx}(x^2+4x) - (x^2+4x)\frac{d}{dx}(x^2+4x+4)}{((x+2)^2)^2} \\ &= \frac{(x+2)^2(2x+4) - (x^2+4x)(2x+4)}{(x+2)^4} \\ &= \frac{(x+2)(2x+4) - 2(x^2+4x)}{(x+2)^3} \\ &= \frac{2x^2 + 8x + 8 - 2x^2 - 8x}{(x+2)^3} \\ &= \frac{8}{(x+2)^3} \end{aligned}$$

Section Review Page 256 Question 23

$$\begin{aligned} f(x) &= x^3 - 2x^2 + 3x \\ f'(x) &= 3x^2 - 4x + 3 \\ f''(x) &= 6x - 4 \\ f''(2) &= 6(2) - 4 \\ &= 12 - 4 \\ &= 8 \end{aligned}$$

The slope of the tangent to $f'(x)$ at $x = 2$ is 8.

Section Review Page 256 Question 24

a)

$$\begin{aligned} s(t) &= \frac{Mt}{C+t} \\ &= \frac{Mt + MC - MC}{C+t} \\ &= \frac{Mt + MC}{C+t} - \frac{MC}{C+t} \\ &= \frac{M(C+t)}{C+t} - \frac{MC}{C+t} \\ &= M - \frac{MC}{C+t} \end{aligned}$$

After a very long time,

$$\begin{aligned} \lim_{t\to\infty} s(t) &= \lim_{t\to\infty} M - \frac{MC}{C+t} \\ &= M^- \end{aligned}$$

The mass that collects on the string approaches, but never reaches, a limit of M.

b)

$$\begin{aligned} s(t) &= M - \frac{MC}{C+t} \\ s'(t) &= \frac{d}{dt}\left(M - \frac{MC}{C+t}\right) \end{aligned}$$

Since $d(t) = d(C+t)$,

$$\begin{aligned} &= \frac{d(M)}{dt} - MC\frac{d(C+t)^{-1}}{d(C+t)} \\ &= 0 - MC(-1)(C+t)^{-2} \\ &= \frac{MC}{(C+t)^2} \\ s'(0) &= \frac{MC}{(C+0)^2} \\ &= \frac{MC}{C^2} \\ &= \frac{M}{C} \end{aligned}$$

The initial growth rate is $\frac{M}{C}$ g/week.

c)

$$\begin{aligned} s'(t) &= \frac{MC}{(C+t)^2} \\ &= MC(C+t)^{-2} \\ s''(t) &= MC \cdot \frac{d}{dt}(C+t)^{-2} \end{aligned}$$

Since $d(t) = d(C+t)$,

$$\begin{aligned} &= MC \cdot \frac{d(C+t)^{-2}}{d(C+t)} \\ &= MC \cdot (-2)(C+t)^{-3} \\ &= -\frac{2MC}{(C+t)^3} \end{aligned}$$

$s''(t)$ represents the rate at which the growth rate is changing.

d)

$$\begin{aligned} s(t) &= \frac{M}{2} \\ \frac{Mt}{C+t} &= \frac{M}{2} \\ \frac{t}{C+t} &= \frac{1}{2} \\ 2t &= C + t \\ t &= C \end{aligned}$$

It will take C weeks for half the mass to accumulate on the string.

$$\begin{aligned} s''(C) &= -\frac{2MC}{(C+C)^3} \\ &= -\frac{2MC}{8C^3} \\ &= -\frac{M}{4C^2} \end{aligned}$$

4.6 Velocity and Acceleration

Section Review Page 256 Question 25

a)

$$\begin{aligned} s(t) &= 2t^3 + 4t^2 - t \\ v(t) &= 6t^2 + 8t - 1 \\ a(t) &= 12t + 8 \end{aligned}$$

b)

$$\begin{aligned} v(4) &= 6(4)^2 + 8(4) - 1 \\ &= 96 + 32 - 1 \\ &= 127 \end{aligned}$$

The velocity after 4 s is 127 m/s.

$$\begin{aligned} a(4) &= 12(4) + 8 \\ &= 48 + 8 \\ &= 56 \end{aligned}$$

The acceleration after 4 s is 56 m/s^2.

Section Review Page 256 Question 26

Determine the system of functions defining the motion of the particle.

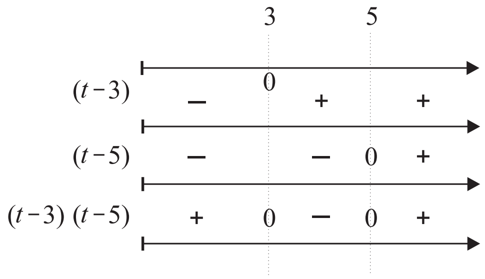
$$\begin{aligned} s(t) &= t^3 - 12t^2 + 45t + 3,\ t \geq 0 \\ v(t) &= 3t^2 - 24t + 45 \\ a(t) &= 6t - 24 \end{aligned}$$

a)

$$\begin{aligned} v(t) &= 0 \\ 3t^2 - 24t + 45 &= 0 \\ t^2 - 8t + 15 &= 0 \\ (t - 3)(t - 5) &= 0 \\ t &= 3 \text{ or } 5 \end{aligned}$$

The particle is at rest at 3 s and 5 s.

b)

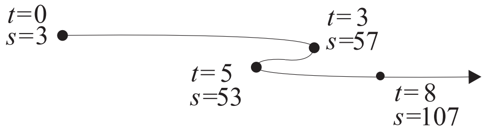


The velocity is positive over the intervals $t \in [0, 3)$ and $t \in (5, \infty)$. The velocity is negative over the interval $t \in (3, 5)$.

c) The acceleration is positive when $6t - 24 > 0$ or $t \in (4, \infty)$. The acceleration is negative when $t \in [0, 4)$.

d) The acceleration is zero at $t = 4$. At this time $v(4) = 3(4)^2 - 24(4) + 45$ or -3 m/s.

e)



f) In the first 3 s, the particle travels $57 - 3$ or 54 m. Over the interval $t \in [3, 5]$ the particle travels $57 - 53$ or 4 m. Over the interval $t \in [5, 8]$, the particle travels $107-53$ or 54 m. The total distance travelled by the particle is $54 + 4 + 54$ or 112 m.

Section Review Page 256 Question 27

Determine the system of functions defining the motion of the tool.

$$\begin{aligned} s(t) &= 6.5t - 0.83t^2 \\ v(t) &= 6.5 - 1.66t \\ a(t) &= -1.66 \end{aligned}$$

a)
$$\begin{aligned} v(1) &= 6.5 - 1.66(1) \\ &= 6.5 - 1.66 \\ &= 4.84 \end{aligned}$$

The velocity of the tool after 1 s is 4.84 m/s.

b) The acceleration of the tool is −1.66 m/s^2.

c) The tool will hit the moon when $s = 0$.

$$\begin{aligned} s(t) &= 0 \\ 6.5t - 0.83t^2 &= 0 \\ t(6.5 - 0.83t) &= 0 \\ t &= 0 \\ 6.5 - 0.83t &= 0 \\ t &= \frac{6.5}{0.83} \\ &\doteq 7.83 \end{aligned}$$

The tool will hit the moon at approximately 7.83 s.

d)
$$\begin{aligned} v(7.83) &= 6.5 - 1.66(7.83) \\ &\doteq -6.5 \end{aligned}$$

The tool will hit the moon at approximately −6.5 m/s.

e) These events take a longer time than on Earth.

4.7 Rates of Change in the Social Sciences

Section Review Page 256 Question 28

a)
$$\begin{aligned} C(x) &= 15x + 0.07x^2 \\ C'(x) &= 15 + 0.14x \end{aligned}$$

The marginal cost function is $C'(x) = 15 + 0.14x$.

b)
$$\begin{aligned} C'(250) &= 15 + 0.14(250) \\ &= 15 + 35 \\ &= 50 \end{aligned}$$

At the 250-unit production level, the marginal cost is \$50/unit.

c)
$$\begin{aligned} C(251) - C(250) &= 15(251) + 0.07(251)^2 - (15(250) + 0.07(250)^2) \\ &= 50.07 \end{aligned}$$

The cost of producing the 251st item is \$50.07.

d) The cost of producing the 251st item is very close to $C'(250)$.

Section Review Page 256 Question 29

a) Let $n'(t)$ represent the rate of change in the population per year.

$$\begin{aligned} n(t) &= 1200 - 170t + 18t^2 \\ n'(t) &= -170 + 36t \end{aligned}$$

- **i)** After 4 years, $n'(4) = -170 + 36(4)$ or −26 people/year.
- **ii)** After 5 years, $n'(5) = -170 + 36(5)$ or 10 people/year.
- **iii)** After 10 years, $n'(10) = -170 + 36(10)$ or 190 people/year.

b) The growth rate increases.

Section Review Page 257 Question 30

a) Let the profit function be $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 0.78x + 0.0003x^2 - (480 - 0.32x + 0.0005x^2) \\ &= 0.78x + 0.0003x^2 - 480 + 0.32x - 0.0005x^2 \\ &= 1.1x - 0.0002x^2 - 480 \end{aligned}$$

The profit function is $P(x) = 1.1x - 0.0002x^2 - 480$.

b) The marginal profit function is $P'(x)$.

$$\begin{aligned} P(x) &= 1.1x - 0.0002x^2 - 480 \\ P'(x) &= 1.1 - 0.0004x \end{aligned}$$

The marginal profit function is $P'(x) = 1.1 - 0.0004x$.

c) **i)** The profit when 300 bottles are sold is $P(300) = 1.1(300) - 0.0002(300)^2 - 480$ or $-\$168$. The marginal profit is $P'(300) = 1.1 - 0.0004(300)$ or \$0.98/bottle.

ii) The profit when 500 bottles are sold is $P(500) = 1.1(500) - 0.0002(500)^2 - 480$ or \$20. The marginal profit is $P'(500) = 1.1 - 0.0004(500)$ or \$0.90/bottle.

iii) The profit when 700 bottles are sold is $P(700) = 1.1(700) - 0.0002(700)^2 - 480$ or \$192. The marginal profit is $P'(700) = 1.1 - 0.0004(700)$ or \$0.82/bottle.

Section Review Page 257 Question 31

a) Let $C'(x)$ represent the marginal cost function.

$$\begin{aligned} C(x) &= 61\ 000 + 8x + 0.009x^2 \\ C'(x) &= 8 + 0.018x \end{aligned}$$

The marginal cost of producing x boxes is $C'(x) = 8 + 0.018x$.

b) Let $R(x)$ represent the revenue function.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(90 - 0.03x) \\ &= 90x - 0.03x^2 \\ R'(x) &= 90 - 0.06x \end{aligned}$$

The marginal revenue from selling x boxes is $R'(x) = 90 - 0.06x^2$.

c) Let $P(x)$ represent the profit function.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 90x - 0.03x^2 - (61\ 000 + 8x + 0.009x^2) \\ &= 90x - 0.03x^2 - 61\ 000 - 8x - 0.009x^2 \\ &= 82x - 0.039x^2 - 61\ 000 \\ P'(x) &= 82 - 0.078x \end{aligned}$$

The marginal profit from selling x boxes is $P'(x) = 82 - 0.078x$.

d)

$$\begin{aligned} P'(x) &= 82 - 0.078x \\ P'(200) &= 82 - 0.078(200) \\ &= 66.4 \end{aligned}$$

The marginal profit at the 200-box production level is \$66.40/box.

Section Review Page 257 Question 32

a) Let $N'(d)$ represent the rate of change of highway usage.

$$\begin{aligned} N(d) &= 58.6\sqrt{d} \\ N'(d) &= 58.6\left(\frac{1}{2\sqrt{d}}\right) \\ &= \frac{29.3}{\sqrt{d}} \end{aligned}$$

The rate of change of highway usage is $N'(d) = \frac{29.3}{\sqrt{d}}$ users per kilometre.

b) **i)** At a driving distance of 50 km, the rate of change of the number of users is $N'(50) = \frac{29.3}{\sqrt{50}}$ or approximately 4.14 users/km.

ii) At a driving distance of 100 km, the rate of change of the number of users is $N'(100) = \frac{29.3}{\sqrt{100}}$ or 2.93 users/km.

Chapter Test

Section Chapter Test Page 258 Question 1

i) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

ii) a)
$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{(x+h)^2-3(x+h)+2-(x^2-3x+2)}{h}\\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-3x-3h+2-x^2+3x-2}{h}\\
&= \lim_{h\to 0}\frac{2xh+h^2-3h}{h}\\
&= \lim_{h\to 0}(2x+h-3)\\
&= 2x-3\\
f'(x) &= 2x-3
\end{aligned}$$

b)
$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\dfrac{1}{x+h+2}-\dfrac{1}{x+2}}{h}\\
&= \lim_{h\to 0}\frac{x+2-(x+h+2)}{h(x+h+2)(x+2)}\\
&= \lim_{h\to 0}\frac{x+2-x-h-2}{h(x+h+2)(x+2)}\\
&= \lim_{h\to 0}\frac{-h}{h(x+h+2)(x+2)}\\
&= \lim_{h\to 0}\frac{-1}{(x+h+2)(x+2)}\\
&= -\frac{1}{(x+2)^2}
\end{aligned}$$

$$f'(x) = -\frac{1}{(x+2)^2}$$

Section Chapter Test Page 258 Question 2

a)
$$\begin{aligned}
f(x) &= x^3-2x^2-4x^{-1}\\
f'(x) &= \frac{d}{dx}(x^3-2x^2-4x^{-1})\\
&= 3x^2-4x+4x^{-2}
\end{aligned}$$

b)
$$\begin{aligned}
g(x) &= \sqrt[4]{x^3}\\
&= x^{\frac{3}{4}}\\
g'(x) &= \frac{d}{dx}\left(x^{\frac{3}{4}}\right)\\
&= \frac{3}{4}x^{-\frac{1}{4}}\\
&= \frac{3}{4\sqrt[4]{x}}
\end{aligned}$$

c)
$$\begin{aligned}
s(t) &= (2t+1)(t^2-3t)\\
s'(t) &= (t^2-3t)\frac{d}{dt}(2t+1)+(2t+1)\frac{d}{dt}(t^2-3t)\\
&= (t^2-3t)(2)+(2t+1)(2t-3)\\
&= 2t^2-6t+4t^2-6t+2t-3\\
&= 6t^2-10t-3
\end{aligned}$$

d)
$$\begin{aligned}
h(x) &= \sqrt{x}(2x^4-\sqrt{x})\\
&= x^{\frac{1}{2}}(2x^4-x^{\frac{1}{2}})\\
h'(x) &= (2x^4-x^{\frac{1}{2}})\frac{d}{dx}(x^{\frac{1}{2}})+(x^{\frac{1}{2}})\frac{d}{dx}(2x^4-x^{\frac{1}{2}})\\
&= (2x^4-x^{\frac{1}{2}})\left(\frac{1}{2}x^{-\frac{1}{2}}\right)+(x^{\frac{1}{2}})\left(8x^3-\frac{1}{2}x^{-\frac{1}{2}}\right)\\
&= x^{\frac{7}{2}}-\frac{1}{2}+8x^{\frac{7}{2}}-\frac{1}{2}\\
&= 9x^{\frac{7}{2}}-1
\end{aligned}$$

e)
$$\begin{aligned}
f(x) &= \frac{x^2}{x-4}\\
f'(x) &= \frac{(x-4)\frac{d}{dx}(x^2)-(x^2)\frac{d}{dx}(x-4)}{(x-4)^2}\\
&= \frac{(x-4)(2x)-(x^2)(1)}{(x-4)^2}\\
&= \frac{2x^2-8x-x^2}{(x-4)^2}\\
&= \frac{x^2-8x}{(x-4)^2}
\end{aligned}$$

f)
$$\begin{aligned}
A(x) &= \frac{3-x}{x^2+2}\\
A'(x) &= \frac{(x^2+2)\frac{d}{dx}(3-x)-(3-x)\frac{d}{dx}(x^2+2)}{(x^2+2)^2}\\
&= \frac{(x^2+2)(-1)-(3-x)(2x)}{(x^2+2)^2}\\
&= \frac{-x^2-2-6x+2x^2}{(x^2+2)^2}\\
&= \frac{x^2-6x-2}{(x^2+2)^2}
\end{aligned}$$

Section Chapter Test Page 258 Question 3

a)

$$
\begin{aligned}
y &= (x^3 - 3)(5 - 2x) \\
y &= 5x^3 - 2x^4 - 15 + 6x \\
\frac{dy}{dx} &= 15x^2 - 8x^3 + 6 \\
\frac{dy}{dx}_{|x=2} &= 15(2)^2 - 8(2)^3 + 6 \\
&= 60 - 64 + 6 \\
&= 2
\end{aligned}
$$

Determine the equation of the tangent.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 5 &= 2(x - 2) \\
y - 5 &= 2x - 4 \\
y &= 2x + 1
\end{aligned}
$$

The equation of the tangent is $y = 2x + 1$.

b)

$$
\begin{aligned}
y &= \frac{4x+3}{x^2+x} \\
\frac{dy}{dx} &= \frac{(x^2+x)\frac{d}{dx}(4x+3) - (4x+3)\frac{d}{dx}(x^2+x)}{(x^2+x)^2} \\
&= \frac{(x^2+x)(4) - (4x+3)(2x+1)}{(x^2+x)^2} \\
&= \frac{4x^2 + 4x - 8x^2 - 10x - 3}{(x^2+x)^2} \\
&= -\frac{4x^2 + 6x + 3}{(x^2+x)^2} \\
\frac{dy}{dx}_{|x=-3} &= -\frac{4(-3)^2 + 6(-3) + 3}{((-3)^2 + (-3))^2} \\
&= -\frac{36 - 18 + 3}{(9-3)^2} \\
&= -\frac{7}{12}
\end{aligned}
$$

Determine the equation of the tangent.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \left(-\frac{3}{2}\right) &= -\frac{7}{12}(x - (-3)) \\
y + \frac{3}{2} &= -\frac{7}{12}(x + 3) \\
y + \frac{3}{2} &= -\frac{7}{12}x - \frac{7}{4} \\
y &= -\frac{7}{12}x - \frac{13}{4}
\end{aligned}
$$

The equation of the tangent is $y = -\frac{7}{12}x - \frac{13}{4}$.

c)

$$
\begin{aligned}
y &= \frac{x^3 - 5x - 7}{x} \\
\frac{dy}{dx} &= \frac{(x)\frac{d}{dx}(x^3 - 5x - 7) - (x^3 - 5x - 7)\frac{d}{dx}(x)}{x^2} \\
&= \frac{x(3x^2 - 5) - (x^3 - 5x - 7)(1)}{x^2} \\
&= \frac{3x^3 - 5x - x^3 + 5x + 7}{x^2} \\
&= \frac{2x^3 + 7}{x^2} \\
\frac{dy}{dx}_{|x=-2} &= \frac{2(-2)^3 + 7}{(-2)^2} \\
&= \frac{-16 + 7}{4} \\
&= -\frac{9}{4}
\end{aligned}
$$

Determine the y-coordinate at $x = -2$.

$$
\begin{aligned}
y(-2) &= \frac{(-2)^3 - 5(-2) - 7}{(-2)} \\
&= \frac{-8 + 10 - 7}{-2} \\
&= \frac{5}{2}
\end{aligned}
$$

Determine the equation of the tangent.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \frac{5}{2} &= -\frac{9}{4}(x - (-2)) \\
y - \frac{5}{2} &= -\frac{9}{4}(x + 2) \\
y - \frac{5}{2} &= -\frac{9}{4}x - \frac{9}{2} \\
y &= -\frac{9}{4}x - 2
\end{aligned}
$$

The equation of the tangent is $y = -\frac{9}{4}x - 2$.

Section Chapter Test Page 258 Question 4

a)

$$\begin{aligned} y &= 6x^3 - 5x^2 + 8x - 1 \\ \frac{dy}{dx} &= 18x^2 - 10x + 8 \\ \frac{d^2y}{dx^2} &= 36x - 10 \\ \frac{d^2y}{dx^2}_{|x=2} &= 36(2) - 10 \\ &= 72 - 10 \\ &= 62 \end{aligned}$$

c)

$$\begin{aligned} y &= (x-3)^{-2} \\ \frac{dy}{dx} &= -2(x-3)^{-3} \\ \frac{d^2y}{dx^2} &= 6(x-3)^{-4} \\ \frac{d^2y}{dx^2}_{|x=2} &= 6(2-3)^{-4} \\ &= 6(1) \\ &= 6 \end{aligned}$$

b)

$$\begin{aligned} y &= \sqrt{x}(x^2 - 2) \\ &= x^{\frac{1}{2}}(x^2 - 2) \\ &= x^{\frac{5}{2}} - 2x^{\frac{1}{2}} \\ \frac{dy}{dx} &= \frac{5}{2}x^{\frac{3}{2}} - x^{-\frac{1}{2}} \\ \frac{d^2y}{dx^2} &= \frac{15}{4}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} \\ &= \frac{15\sqrt{x}}{4} + \frac{1}{2\sqrt{x^3}} \\ \frac{d^2y}{dx^2}_{|x=2} &= \frac{15\sqrt{2}}{4} + \frac{1}{2\sqrt{2^3}} \\ &= \frac{15\sqrt{2}}{4} + \frac{1}{4\sqrt{2}} \\ &= \frac{15\sqrt{2}}{4} \cdot \frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{4\sqrt{2}} \\ &= \frac{30+1}{4\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{31\sqrt{2}}{8} \end{aligned}$$

Section Chapter Test Page 258 Question 5

a) The slope of the line $3x + y = 2$ is -3. Parallel lines have the same slopes.

$$\begin{aligned} \frac{d}{dx}(x^2 - 4x + 3) &= -3 \\ 2x - 4 &= -3 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

At $x = \frac{1}{2}$,

$$\begin{aligned} y &= \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 3 \\ &= \frac{1}{4} - \frac{8}{4} + \frac{12}{4} \\ &= \frac{5}{4} \end{aligned}$$

The point on the parabola $y = x^2 - 4x + 3$ at which the slope of the tangent is parallel to $3x + y = 2$ is $\left(\frac{1}{2}, \frac{5}{4}\right)$.

b) Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \frac{5}{4} &= -3\left(x - \frac{1}{2}\right) \\ y - \frac{5}{4} &= -3x + \frac{3}{2} \\ y &= -3x + \frac{11}{4} \end{aligned}$$

The equation of the tangent is $y = -3x + \frac{11}{4}$.

Section Chapter Test Page 258 Question 6

a)

$$\begin{aligned}(fg)'(x) &= g(x)f'(x) + f(x)g'(x)\\ (fg)'(4) &= g(4)f'(4) + f(4)g'(4)\\ &= 7(-2) + 3(5)\\ &= -14 + 15\\ &= 1\end{aligned}$$

b)

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}\\ \left(\frac{f}{g}\right)'(4) &= \frac{g(4)f'(4) - f(4)g'(4)}{[g(4)]^2}\\ &= \frac{7(-2) - 3(5)}{(7)^2}\\ &= \frac{-14 - 15}{49}\\ &= -\frac{29}{49}\end{aligned}$$

c)

$$\begin{aligned}(g^2)'(x) &= (gg)'(x)\\ &= g(x)g'(x) + g(x)g'(x)\\ &= 2g(x)g'(x)\\ (g^2)'(4) &= 2g(4)g'(4)\\ &= 2(7)(5)\\ &= 70\end{aligned}$$

d)

$$\begin{aligned}\left(\frac{1}{f}\right)'(x) &= \frac{f(x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(f(x))}{[f(x)]^2}\\ &= \frac{f(x)(0) - (1)f'(x)}{[f(x)]^2}\\ &= -\frac{f'(x)}{[f(x)]^2}\\ \left(\frac{1}{f}\right)'(4) &= -\frac{f'(4)}{[f(4)]^2}\\ &= -\frac{-2}{3^2}\\ &= \frac{2}{9}\end{aligned}$$

Section Chapter Test Page 258 Question 7

Determine the power, P, in terms of t, through substitution.

$$\begin{aligned}P &= I^2R\\ &= (5 - 0.01t)^2(10 + 0.08t)\\ &= (25 - 0.1t + 0.0001t^2)(10 + 0.08t)\\ &= 250 + 2t - t - 0.008t^2 + 0.001t^2 + 0.000\,008t^3\\ &= 250 + t - 0.007t^2 + 0.000\,008t^3\end{aligned}$$

a)

$$\begin{aligned}P'(t) &= \frac{d}{dt}(250 + t - 0.007t^2 + 0.000\,008t^3)\\ &= 1 - 0.014t + 0.000\,024t^2\end{aligned}$$

After t seconds, the power is $P(t) = 1 - 0.014t + 0.000\,024t^2$ watts per second.

b)

$$\begin{aligned}P'(10) &= 1 - 0.014(10) + 0.000\,024(10)^2\\ &= 1 - 0.14 + 0.0024\\ &= 0.8624\end{aligned}$$

After 10 s, the power is 0.8624 W/s.

Section Chapter Test   Page 258   Question 8

Determine the system of functions defining the motion of the particle.

$$\begin{aligned} s(t) &= t^3 - 6t^2 + 9t + 1,\ t \geq 0 \\ v(t) &= 3t^2 - 12t + 9 \\ a(t) &= 6t - 12 \end{aligned}$$

a)

$$\begin{aligned} v(4) &= 3(4)^2 - 12(4) + 9 \\ &= 48 - 48 + 9 \\ &= 9 \end{aligned}$$

The velocity after 4 s is 9 m/s.

b)

$$\begin{aligned} a(4) &= 6(4) - 12 \\ &= 24 - 12 \\ &= 12 \end{aligned}$$

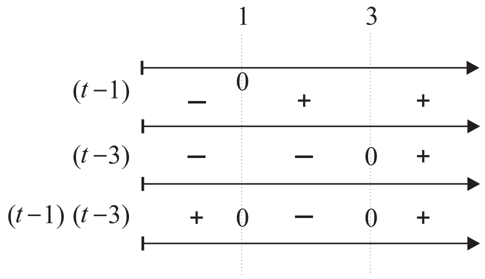
The acceleration after 4 s is 12 m/s^2.

c)

$$\begin{aligned} v(t) &= 0 \\ 3t^2 - 12t + 9 &= 0 \\ t^2 - 4t + 3 &= 0 \\ (t - 1)(t - 3) &= 0 \\ t &= 1 \text{ or } 3 \end{aligned}$$

The particle is at rest at 1 s and 3 s.

d)



The velocity is positive over the intervals $t \in [0, 1)$ and $t > 3$ s.

e)

$$\begin{aligned} a(t) &= 0 \\ 6t - 12 &= 0 \\ 6t &= 12 \\ t &= 2 \\ v(2) &= 3(2)^2 - 12(2) + 9 \\ &= 12 - 24 + 9 \\ &= -3 \end{aligned}$$

When the acceleration is zero, the velocity is −3 m/s.

f) The position at 0 s is $0^3 - 6(0)^2 + 9(0) + 1$ or 1 m. The position at 1 s is $1^3 - 6(1)^2 + 9(1) + 1$ or 5 m. The position at 3 s is $3^3 - 6(3)^2 + 9(3) + 1$ or 1 m. The position at 4 s is $4^3 - 6(4)^2 + 9(4) + 1$ or 5 m. The total distance travelled over the first 4 s is $|5 - 1| + |1 - 5| + |5 - 1|$ or 12 m.

Section Chapter Test Page 258 Question 9

a) Let $n'(t)$ represent the rate of change in the population per hour.

$$\begin{aligned} n(t) &= 750 + 120t + 13t^2 + 2t^3 \\ n'(t) &= 120 + 26t + 6t^2 \end{aligned}$$

i) At 10 min, or $\frac{1}{6}$ h,

$$\begin{aligned} n'\left(\frac{1}{6}\right) &= 120 + 26\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)^2 \\ &= \frac{720}{6} + \frac{26}{6} + \frac{1}{6} \\ &= \frac{747}{6} \\ &= 124.5 \end{aligned}$$

The growth rate of the population after 10 min is 124.5 bacteria/h.

ii) At 3 h,

$$\begin{aligned} n'(3) &= 120 + 26(3) + 6(3)^2 \\ &= 120 + 78 + 54 \\ &= 252 \end{aligned}$$

The growth rate of the population after 3 h is 252 bacteria/h.

iii) At 8 h,

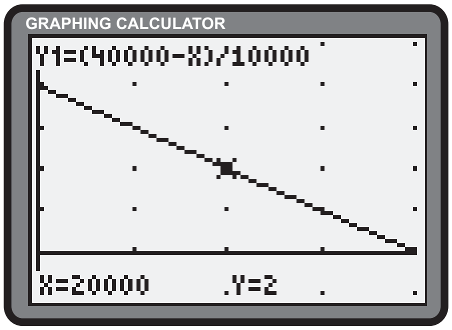
$$\begin{aligned} n'(8) &= 120 + 26(8) + 6(8)^2 \\ &= 120 + 208 + 384 \\ &= 712 \end{aligned}$$

The growth rate of the population after 8 h is 712 bacteria/h.

b) Answers will vary.

Section Chapter Test Page 259 Question 10

a)



b)

$p(x)$	x
0	40 000
\$1.00	30 000
\$2.00	20 000
\$3.50	5 000
\$4.00	0

As demand increases, the price decreases. As demand decreases, the price increases.

c)

$$\begin{aligned} R(x) &= xp(x) \\ &= x\left(\frac{40\,000 - x}{10\,000}\right) \\ &= 4x - \frac{x^2}{10\,000} \\ &= 4x - 0.0001x^2 \end{aligned}$$

The revenue function is $R(x) = 4x - 0.0001x^2$.

d)

$$\begin{aligned} R'(x) &= \frac{d}{dx}(4x - 0.0001x^2) \\ &= 4 - 0.0002x \end{aligned}$$

The marginal revenue function is $R'(x) = 4 - 0.0002x$.

e)

$$\begin{aligned} R'(10\,000) &= 4 - 0.0002(10\,000) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The marginal revenue when $x = 10\,000$ is \$2/unit.

f)

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 4x - 0.0001x^2 - (2500 + 0.04x) \\ &= 4x - 0.0001x^2 - 2500 - 0.04x \\ &= 3.96x - 0.0001x^2 - 2500 \end{aligned}$$

The profit function is $P(x) = 3.96x - 0.0001x^2 - 2500$.

g)

$$\begin{aligned} P'(x) &= \frac{d}{dx}(3.96x - 0.0001x^2 - 2500) \\ &= 3.96 - 0.0002x \end{aligned}$$

The marginal profit function is $P'(x) = 3.96 - 0.0002x$.

h)

$$\begin{aligned} P'(30\,000) &= 3.96 - 0.0002(30\,000) \\ &= -2.04 \end{aligned}$$

The marginal profit when $x = 30\,000$ is −\$2.04/unit.

Section Chapter Test Page 259 Question 11

$$\begin{aligned} M(x) &= 0.4\sqrt{x} \\ M'(x) &= \frac{d}{dx}(0.4\sqrt{x}) \\ &= 0.4\left(\frac{1}{2\sqrt{x}}\right) \\ &= \frac{0.2}{\sqrt{x}} \\ M'(3) &= \frac{0.2}{\sqrt{3}} \\ &\doteq 0.1155 \end{aligned}$$

The linear density of the rod is approximately 0.1155 kg/m.

Section Chapter Test Page 259 Question 12

a) Determine the velocity.

$$\begin{aligned} h(t) &= -4.9t^2 + 500 \\ v(t) &= \frac{d}{dt}(-4.9t^2 + 500) \\ &= -9.8t \end{aligned}$$

Determine the time to reach a speed of 56 m/s.

$$\begin{aligned} |v(t)| &= 56 \\ |-9.8t| &= 56 \\ 9.8t &= 56 \\ t &= \frac{56}{9.8} \\ &= \frac{40}{7} \end{aligned}$$

It takes $\frac{40}{7}$ s to reach a speed of 56 m/s.

b)

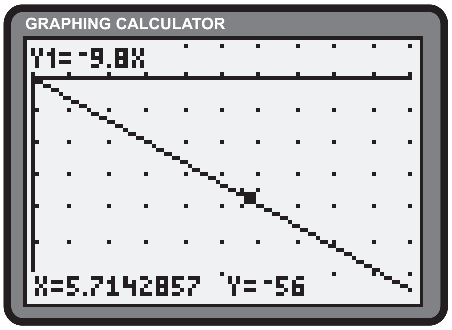
$$\begin{aligned} h\left(\frac{40}{7}\right) &= -4.9\left(\frac{40}{7}\right)^2 + 500 \\ &= \frac{-4.9(1600)}{49} + 500 \\ &= -160 + 500 \\ &= 340 \end{aligned}$$

The speed is 56 m/s at a height of 340 m.

c)

$$v(t) = \begin{cases} -9.8t & ; 0 \leq t \leq \dfrac{40}{7} \\ -56 & ; t > \dfrac{40}{7} \end{cases}$$

d)



e) The graph would curve to approach $v = -56$ as an asymptote. It would take longer to reach an adequate approximation of −56, and would result in a longer fall.

Section Chapter Test Page 259 Question 13

Let $h(x)$ represent the height of the ball, in metres, thrown upward from the base of the CN tower after t seconds. A system of functions describing the motion of the ball can be developed.

$$a(t) = -9.8 \tag{1}$$
$$v(t) = -9.8t + v_0 \tag{2}$$
$$h(t) = -4.9t^2 + v_0 t \tag{3}$$

where v_0 is the initial velocity imparted to the ball, in metres per second. The time taken to achieve a maximum height can be determined from (2).

$$\begin{aligned} v(t) &= 0 \\ -9.8t + v_0 &= 0 \\ t &= \frac{v_0}{9.8} \end{aligned} \tag{4}$$

A value for v_0 can be obtained for a maximum height of 550 m to be achieved.

$$\begin{aligned} h(t) &= 550 \\ -4.9t^2 + v_0 t &= 550 \end{aligned} \tag{5}$$

Substitute (4) into (5).

$$\begin{aligned} -4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) &= 550 \\ -\frac{v_0^2}{19.6} + \frac{v_0^2}{9.8} &= 550 \\ -v_0^2 + 2v_0 &= 550(19.6) \\ v_0^2 &= 10\,780 \\ v_0 &\doteq 103.83 \end{aligned}$$

To reach the top of the CN tower, the person would have to throw the ball at approximately 103.83 m/s, which is impossible.

Challenge Problems

Section Challenge Problems Page 260 Question 1

Since the slope of the tangent is 2, c can be obtained.

$$\begin{aligned} \frac{dy}{dx}_{|x=1} &= 2 \\ 2cx_{|x=1} &= 2 \\ 2c(1) &= 2 \\ c &= 1 \end{aligned}$$

From $y = 2x + 3$, the coordinates of the point of tangency are determined to be $(1, 2(1) + 3)$ or $(1, 5)$.

The value for d can be determined.

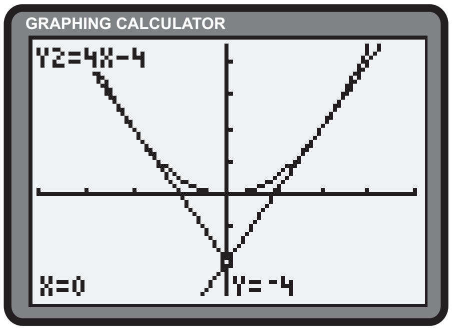
$$\begin{aligned} y(1) &= 5 \\ 1(1)^2 + d &= 5 \\ 1 + d &= 5 \\ d &= 4 \end{aligned}$$

The solution is $(c, d) = (1, 4)$.

Section Challenge Problems Page 260 Question 2

The coordinates of the general point, P, on $y = x^2$ are (x, x^2). Define $(0, -4)$ as Q.

$$\begin{aligned} \frac{d}{dx}(x^2) &= \text{Slope of PQ} \\ 2x &= \frac{x^2 - (-4)}{x - 0} \\ 2x^2 &= x^2 + 4 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$



When $x = -2$, $y = 4$. The slope of the tangent is $2(-2)$ or -4.

$$\begin{aligned} y - 4 &= -4(x - (-2)) \\ y &= -4x - 4 \end{aligned}$$

When $x = 2$, $y = 4$. The slope of the tangent is $2(2)$ or 4.

$$\begin{aligned} y - 4 &= 4(x - 2) \\ y &= 4x - 4 \end{aligned}$$

The equations of the tangents are $y = \pm 4x - 4$.

Section Challenge Problems Page 260 Question 3

Let the coordinates of the point of tangency be $P(a, a^3)$. The slope of the tangent at P is $y'(a)$ or $3a^2$. Determine the general equation of the tangent at P.

$$\begin{aligned} y - a^3 &= 3a^2(x - a) \\ y &= 3a^2x - 3a^3 + a^3 \\ y &= 3a^2x - 2a^3 \end{aligned}$$

Determine the coordinates of Q, the point of intersection of the tangent and $y = x^3$.

$$\begin{aligned} x^3 &= 3a^2x - 2a^3 \\ x^3 - 3a^2x + 2a^3 &= 0 \qquad (1) \end{aligned}$$

Since $x - a$ is a multiple factor of (1), synthetic division is used to reveal the other factor of $x + 2a$. The x-coordinate of Q is $-2a$. The slope of the tangent at $x = -2a$ is $3(-2a)^2$ or $12a^2$. This result can be expressed as $4(3a^2)$ or four times the slope of the tangent at P.

Section Challenge Problems Page 260 Question 4

Show true for $n = 1$.

$$\begin{aligned} \frac{d}{dx}x^1 &= 1 \\ &= 1 \cdot x^0 \end{aligned}$$

The rule is true for $n = 1$. Assume true for $n = k$, or

$$\frac{d}{dx}x^k = k \cdot x^{k-1} \tag{1}$$

Prove true for $n = k + 1$ using the product rule.

$$\begin{aligned} \frac{d}{dx}x^{k+1} &= \frac{d}{dx}\left(x \cdot x^k\right) \\ &= (x^k)\frac{d}{dx}(x) + (x)\frac{d}{dx}(x^k) \\ &= (x^k)(1) + (x)\frac{d}{dx}(x^k) \\ &= x^k + (x)\frac{d}{dx}(x^k) \end{aligned} \tag{2}$$

Substitute (1) into (2).

$$\begin{aligned} &= x^k + (x)k \cdot x^{k-1} \\ &= x^k + k \cdot x^k \\ &= x^k(1 + k) \\ &= (k + 1)x^k \end{aligned}$$

Hence, $\frac{d}{dx}x^n = nx^{n-1}$.

Section Challenge Problems Page 260 Question 5

Determine the slope of the tangent to $y = \sqrt{x}$ at $(16, 4)$.

$$\begin{aligned} \frac{dy}{dx}\sqrt{x}_{|x=16} &= \frac{1}{2\sqrt{x}}_{|x=16} \\ &= \frac{1}{2\sqrt{16}} \\ &= \frac{1}{8} \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - 4 &= \frac{1}{8}(x - 16) \\ y &= \frac{1}{8}x - 2 + 4 \\ y &= \frac{1}{8}x + 2 \end{aligned}$$

The coordinates of the x-intercept of the tangent are $Q(-16, 0)$. The coordinates of the y-intercept are $M(0, 2)$. The coordinates of the midpoint of PQ are $\left(\frac{-16 + 16}{2}, \frac{0 + 4}{2}\right)$ or $(0, 2)$, or M.

Section Challenge Problems Page 260 Question 6

Let the coordinates of the point of tangency be $P(a, 5a^2 - 4a^3)$. The slope of the tangent at P must be the same as the slope of OP.

$$\begin{aligned}10a - 12a^2 &= \frac{5a^2 - 4a^3 - 0}{a - 0}\\10a^2 - 12a^3 &= 5a^2 - 4a^3\\5a^2 - 8a^3 &= 0\\a^2(5 - 8a) &= 0\\a &= 0 \text{ or } \frac{5}{8}\end{aligned}$$

The tangents pass through the origin at $(0, 0)$ and $\left(\frac{5}{8}, 5\left(\frac{5}{8}\right)^2 - 4\left(\frac{5}{8}\right)^3\right)$ or $\left(\frac{5}{8}, \frac{125}{128}\right)$.

Section Challenge Problems Page 260 Question 7

Determine the slope of the tangent at P.

$$\begin{aligned}\frac{dy}{dx}_{|x=\frac{1}{2}} &= 4x^3 - 24x^2 - 4x + 24_{|x=\frac{1}{2}}\\&= \frac{4}{8} - \frac{24}{4} - 2 + 24\\&= \frac{1}{2} + 16\\&= \frac{33}{2}\end{aligned}$$

Determine the equation of the tangent at P.

$$\begin{aligned}y - \left(-\frac{55}{16}\right) &= \frac{33}{2}\left(x - \frac{1}{2}\right)\\y &= \frac{33}{2}x - \frac{33}{4} - \frac{55}{16}\\y &= \frac{33}{2}x - \frac{187}{16} \qquad (1)\end{aligned}$$

Equate (1) with the given quartic to determine the quadratic solution through A and B.

$$\begin{aligned}x^4 - 8x^3 - 2x^2 + 24x - 14 &= \frac{33}{2}x - \frac{187}{16}\\16x^4 - 128x^3 - 32x^2 + 384x - 224 &= 264x - 187\\16x^4 - 128x^3 - 32x^2 + 120x - 37 &= 0 \qquad (2)\end{aligned}$$

Since $2x - 1$ is a multiple root of (2) $\left(\text{due to tangency at } x = \frac{1}{2}\right)$, polynomial division yields the required quotient leading to the quadratic solution $y = 4x^2 - 28x - 37$.

Section Challenge Problems Page 260 Question 8

List the given results.

$$\begin{aligned}f(x + y) &= 2f(x)f(y) \qquad (1)\\f(x) &= \frac{1}{2} + xg(x) \qquad (2)\\\lim_{x\to 0} g(x) &= 1 \qquad (3)\end{aligned}$$

Define the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} \qquad (4)$$

Use (1) to re-express the numerator in (4).

$$\begin{aligned}&= \lim_{h\to 0} \frac{2f(x)f(h) - f(x)}{h}\\&= \lim_{h\to 0} \frac{f(x)\left[2f(h) - 1\right]}{h} \qquad (5)\end{aligned}$$

Using $x = h$ in (2), substitute (2) into (5).

$$\begin{aligned}
&= \lim_{h\to 0} \frac{f(x)\left[2\left(\frac{1}{2} + hg(h)\right) - 1\right]}{h} \\
&= \lim_{h\to 0} \frac{f(x)\left[1 + 2hg(h) - 1\right]}{h} \\
&= \lim_{h\to 0} f(x)\left(2g(h)\right) \\
&= 2f(x)\lim_{h\to 0} g(h) \qquad (6)
\end{aligned}$$

Substitute (3) into (6).

$$\begin{aligned}
&= 2f(x)\cdot 1 \\
&= 2f(x)
\end{aligned}$$

Section Challenge Problems Page 260 Question 9

a) $$\begin{aligned}
Q(t) &= 0.09\sqrt{8+t^2}(t-7)^2 \\
Q(7) &= 0.09\sqrt{8+49}(7-7)^2 \\
&= 0.09\sqrt{57}(0)^2 \\
&= 0
\end{aligned}$$

The urn is empty after 7 min.

b) $$\begin{aligned}
\frac{Q(7)-Q(0)}{7-0} &= \frac{0.09\sqrt{8+7^2}(7-7)^2 - 0.09\sqrt{8+0^2}(0-7)^2}{7} \\
&= \frac{-0.09\sqrt{8}(49)}{7} \\
&= 7(-0.09)\sqrt{8} \\
&\doteq -1.7819
\end{aligned}$$

The average rate of delivery over the first 7 min is -1.7819 cups/min.

c) The derivative of $Q(t)$ is $Q'(t) = 0.09(t-7)^2\dfrac{t}{\sqrt{8+t^2}} + 0.09\sqrt{8+t^2}(2t-24)$.

$$\begin{aligned}
Q'(2) &= 0.09(25)\frac{2}{\sqrt{12}} + 0.09\sqrt{12}(-10) \\
&\doteq -1.8187
\end{aligned}$$

The rate of delivery after 2 min is approximately -1.8187 cups/min.

Section Challenge Problems Page 260 Question 10

The symmetry of the function $y = x^2$ dictates that the intersection point of the three normals occur on the y-axis. Although there is an infinite number of sets of three points, symmetry requires the origin $B(0, 0)$ to be one of the points, and the other two are symmetric about the y-axis. If a is the x-coordinate of the point A, then $-a$ will be the x-coordinate of C. The sum of the x-coordinates of A, B, and C is $a + 0 + (-a)$ or 0.

Section Challenge Problems Page 260 Question 11

The y-coordinates at $x = -1$, $x = 0$, and $x = 1$, are 2, 4, and 2 respectively. The derivative of the function $y = \frac{4}{x^2+1}$ is $\frac{dy}{dx} = \frac{-8x}{(x^2+1)^2}$. Determine the slopes of the tangents at the indicated points.

$$\begin{aligned}\frac{dy}{dx}_{|x=-1} &= \frac{-8(-1)}{((-1)^2+1)^2}\\ &= \frac{8}{2^2}\\ &= 2\\ \frac{dy}{dx}_{|x=0} &= \frac{-8(0)}{(0^2+1)^2}\\ &= 0\\ \frac{dy}{dx}_{|x=1} &= \frac{-8(1)}{(1^2+1)^2}\\ &= \frac{-8}{2^2}\\ &= -2\end{aligned}$$

The slopes of the normals at $x = -1$, $x = 0$, and $x = 1$ are $-\frac{1}{2}$, undefined, and $\frac{1}{2}$ respectively.

Determine the line of support at $(-1, 2)$.

$$\begin{aligned}y - 2 &= -\frac{1}{2}(x - (-1))\\ y - 2 &= -\frac{1}{2}x - \frac{1}{2}\\ y &= -\frac{1}{2}x + \frac{3}{2}\end{aligned}$$

The line of support at $(0, 4)$ is $x = 0$.

Determine the line of support at $(1, 2)$.

$$\begin{aligned}y - 2 &= \frac{1}{2}(x - 1)\\ y - 2 &= \frac{1}{2}x - \frac{1}{2}\\ y &= \frac{1}{2}x + \frac{3}{2}\end{aligned}$$

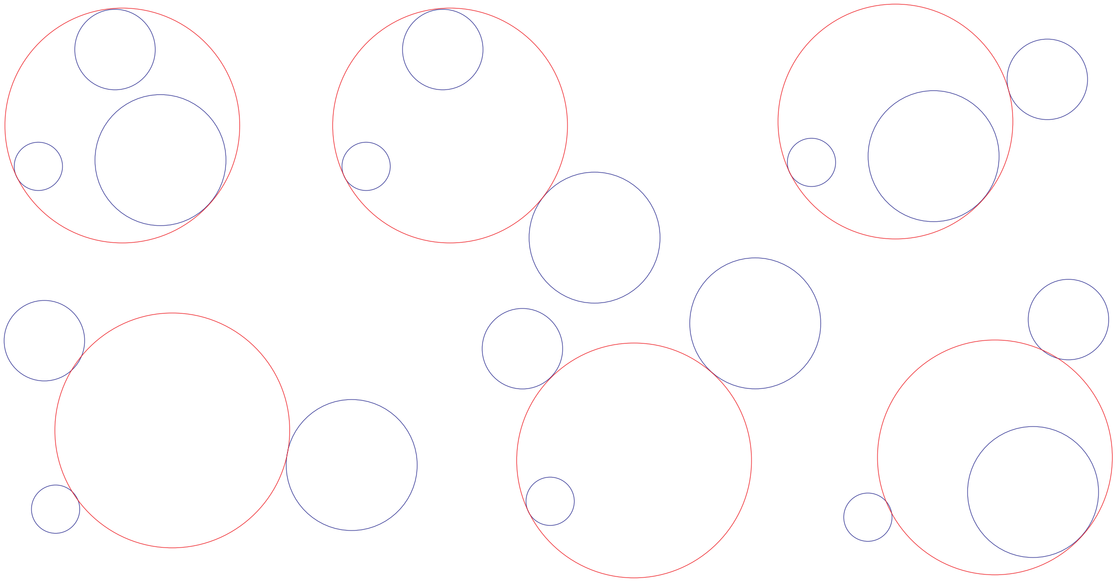
The lines of support are $y = -\frac{1}{2}x + \frac{3}{2}$, $x = 0$, and $y = \frac{1}{2}x + \frac{3}{2}$.

Using the Strategies

Section Problem Solving Page 263 Question 1

Since 757 is a prime number, there is no multiple of any integer that equals it. Only the 756th student will change the state of the 757th locker. Since the locker was initially open, it will be closed at the end of the process.

Section Problem Solving Page 263 Question 2



Section Problem Solving Page 263 Question 3

(**Note.** The expression n mod m is the remainder when n is divided by m.)

From the equality, $3^1, 3^2, 3^3, 3^4, 3^5, \ldots = \underline{3}, \underline{9}, 2\underline{7}, 8\underline{1}, 24\underline{3}, 72\underline{9}, \ldots$, the sequence of last digits is determined to be $3, 9, 7, 1$ and so on. Since 142 mod $4 \equiv 2$, the last digit in the expansion of 3^{142} is 9.

Section Problem Solving Page 263 Question 4

1 L: Fill 9-L, pour 4 L into smaller, dump 4-L, repeat step 2, then have 1 L in 9-L bottle.

2 L: Fill 9-L, pour 4 L into smaller and dump, twice; 1 L into 4-L; fill 9-L, 3 L into 4-L bottle, dump 4-L, 4 L into smaller, then have 2 L left in 9-L bottle.

3 L: Fill 4-L, pour into 9-L, twice; fill 4-L, pour 1 L into 9-L, then have 3 L left in 4-L bottle.

4 L: Fill 4-L bottle

5 L: Fill 9-L bottle, pour 4 L into 4-L bottle, then have 5 L left in 9-L bottle.

6 L: Same as 2 L but omit last step, then have 6 L in 9-L bottle.

7 L: As 3 L, then pour 3 L into empty 9-L, fill 4-L and pour into 9-L, then have 7 L in 9-L bottle.

8 L: Fill 4-L, pour into 9-L, twice.

9 L: Fill 9-L.

10 L: As 1 L, then pour into empty 4-L and fill 9-L, then have 1 L in small and 9 L in large.

11 L: As 2 L, then pour into empty 4-L, fill 9-L, then have 2 L in small and 9 L in large.

12 L: Fill 4-L and pour into 9-L twice, fill 4-L, then have 4 L in small and 8 L in large.

13 L: Fill both.

Section Problem Solving Page 263 Question 5

$$\begin{array}{r} 9\ 703 \\ +9\ 703 \\ \hline 19\ 406 \end{array}$$

HALF is 9703; WHOLE is 19 406.

Section Problem Solving Page 263 Question 6

a)

17	24	1		15
		7	14	
4		13		22
	12	19	21	3
11	18		2	

b) Yes.

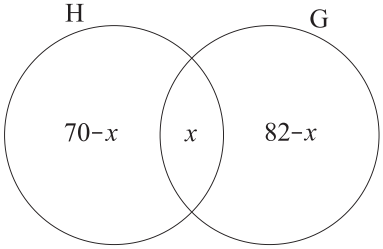
c) Pairs are in opposite positions relative to the centre.

Section Problem Solving Page 263 Question 7

Let H be the number of students that have taken at least one history course.

Let G be the number of students that have taken at least one geography course.

Let x be the number of students that have taken at least one history course and one geography course.



$$\begin{aligned} (70 - x) + x + (82 - x) &= 100 - 10 \\ -x + 152 &= 90 \\ x &= 62 \end{aligned}$$

A total of 62 applicants have taken at least one course in history and at least one course in geography.

Section Problem Solving Page 263 Question 8

Answers may vary. A possible 6-step solution is FLOUR, FLOOR, FLOOD, BLOOD, BROOD, BROAD, and BREAD.

CHAPTER 5 The Chain Rule and Its Applications

5.1 Composite Functions

Practise

Section 5.1 Page 274 Question 1

a)
$$\begin{aligned} g(f(2)) &= g(4) \\ &= 3 \end{aligned}$$

b)
$$\begin{aligned} (f \circ g)(4) &= f(g(4)) \\ &= f(3) \\ &= 6 \end{aligned}$$

c)
$$\begin{aligned} f(g(3)) &= f(4) \\ &= 8 \end{aligned}$$

d)
$$\begin{aligned} (g \circ f)(6) &= g(f(6)) \\ &= g(12) \end{aligned}$$

This cannot be evaluated since 12 is not in the domain of $g(x)$.

e)
$$\begin{aligned} (g \circ f)(1) &= g(f(1)) \\ &= g(2) \\ &= 5 \end{aligned}$$

f)
$$f(g(0)) = f(7)$$

This cannot be evaluated since 7 is not in the domain of $f(x)$.

Section 5.1 Page 274 Question 3

a)
$$\begin{aligned} f(12) &= 12^2 + 1 \\ &= 144 + 1 \\ &= 145 \end{aligned}$$

b)
$$\begin{aligned} g(9) &= 9 - 3 \\ &= 6 \end{aligned}$$

c)
$$\begin{aligned} f(8r - 6) &= (8r - 6)^2 + 1 \\ &= 64r^2 - 96r + 36 + 1 \\ &= 64r^2 - 96r + 37 \end{aligned}$$

d)
$$\begin{aligned} f(g(x)) &= f(x - 3) \\ &= (x - 3)^2 + 1 \\ &= x^2 - 6x + 9 + 1 \\ &= x^2 - 6x + 10 \end{aligned}$$

e)
$$\begin{aligned} g(g(x)) &= g(x - 3) \\ &= (x - 3) - 3 \\ &= x - 6 \end{aligned}$$

f)
$$\begin{aligned} (f \circ g)(-2) &= f(g(-2)) \\ &= f(-5) \\ &= (-5)^2 + 1 \\ &= 26 \end{aligned}$$

g)
$$\begin{aligned} (g \circ f)(x^2) &= g(f(x^2)) \\ &= g((x^2)^2 + 1) \\ &= g(x^4 + 1) \\ &= x^4 + 1 - 3 \\ &= x^4 - 2 \end{aligned}$$

h)
$$\begin{aligned} f(f(0)) &= f(1) \\ &= 1^2 + 1 \\ &= 2 \end{aligned}$$

i)
$$\begin{aligned} g(f(3x - 2)) &= g((3x - 2)^2 + 1) \\ &= g(9x^2 - 12x + 5) \\ &= 9x^2 - 12x + 5 - 3 \\ &= 9x^2 - 12x + 2 \end{aligned}$$

Section 5.1 Page 274 Question 5

a)
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(4x + 3) \\ &= \frac{1}{4x + 3} \end{aligned}$$

The domain of $f \circ g$ includes all real numbers except $x = -\frac{3}{4}$.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g\left(\frac{1}{x}\right) \\ &= 4\left(\frac{1}{x}\right) + 3 \\ &= \frac{4}{x} + 3 \end{aligned}$$

The domain of $g \circ f$ includes all real numbers except $x = 0$.

b)
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(x + 6) \\ &= 2(x + 6) - 3 \\ &= 2x + 9 \end{aligned}$$

The domain of $f \circ g$ includes all real numbers.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g\,(2x - 3) \\ &= (2x - 3) + 6 \\ &= 2x + 3 \end{aligned}$$

The domain of $g \circ f$ includes all real numbers.

c)
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(x-5)\\ &= \sqrt{x-5}\end{aligned}$$

$f \circ g$ is defined for $x \geq 5$.

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(\sqrt{x}\right)\\ &= \sqrt{x}-5\end{aligned}$$

$g \circ f$ is defined for $x \geq 0$.

d)
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(4x+1)\\ &= \sqrt{(4x+1)+8}\\ &= \sqrt{4x+9}\end{aligned}$$

$f \circ g$ is defined for $x \geq -\frac{9}{4}$.

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(\sqrt{x+8}\right)\\ &= 4\sqrt{x+8}+1\end{aligned}$$

$g \circ f$ is defined for $x \geq -8$.

e)
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(x+2)\\ &= (x+2)^2\\ &= x^2+4x+4\end{aligned}$$

$f \circ g$ is defined for all real numbers.

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(x^2\right)\\ &= x^2+2\end{aligned}$$

$g \circ f$ is defined for all real numbers.

f)
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(2x+5)\\ &= (2x+5)^3\\ &= 8x^3+60x^2+150x+125\end{aligned}$$

$f \circ g$ is defined for all real numbers.

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(x^3\right)\\ &= 2x^3+5\end{aligned}$$

$g \circ f$ is defined for all real numbers.

g)
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(x^2)\\ &= (x^2)^3-x^2\\ &= x^6-x^2\end{aligned}$$

$f \circ g$ is defined for all real numbers.

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(x^3-x\right)\\ &= (x^3-x)^2\\ &= x^6-2x^4+x^2\end{aligned}$$

$g \circ f$ is defined for all real numbers.

h)
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(x^4)\\ &= \sqrt{(x^4)^2+49}\\ &= \sqrt{x^8+49}\end{aligned}$$

$f \circ g$ is defined for all real numbers.

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(\sqrt{x^2+49}\right)\\ &= \left(\sqrt{x^2+49}\right)^4\\ &= (x^2+49)^2\\ &= x^4+98x^2+2401\end{aligned}$$

$g \circ f$ is defined for all real numbers.

Apply, Solve, Communicate

Section 5.1 Page 275 Question 7

a)
$$\begin{aligned} f(g(x)) &= f(x+4) \\ &= \sqrt{x+4} \end{aligned}$$

b)
$$\begin{aligned} g(x) &\geq 0 \\ x+4 &\geq 0 \\ x &\geq -4 \end{aligned}$$

The domain of $f(g(x))$ is $x \geq -4$.

c) The range of $f(g(x))$ is $\Re$.

d)
$$\begin{aligned} g(f(x)) &= g(\sqrt{x}) \\ &= \sqrt{x} + 4 \end{aligned}$$

e) The domain of $g(f(x))$ is $x \geq 0$.

f) The range of $g(f(x))$ is $y \geq 4$.

Section 5.1 Page 275 Question 8

a)
$$\begin{aligned} (K \circ C)(F) &= K(C(F)) \\ &= K\left(\frac{5}{9}(F-32)\right) \\ &= \frac{5}{9}(F-32) + 273.15 \end{aligned}$$

b)
$$\begin{aligned} K(25) &= \frac{5}{9}(25-32) + 273.15 \\ &\doteq 269.26 \end{aligned}$$

25°F is equivalent to approximately 269.26°K.

Section 5.1 Page 275 Question 9

a) From the information given it is determined that $A(t) = 500t$.

b)
$$\begin{aligned} A &= \pi r^2 \\ r^2 &= \frac{A}{\pi} \\ r &= \pm\sqrt{\frac{A}{\pi}} \end{aligned}$$

Since $r > 0$, $r(A) = \sqrt{\dfrac{A}{\pi}}$.

c)
$$\begin{aligned} r(t) &= (r \circ A)(t) \\ &= r(A(t)) \\ &= r(500t) \\ &= \sqrt{\frac{500t}{\pi}} \end{aligned}$$

$r(t)$ is the radius, in metres, of the oil spill, after t minutes.

d)
$$\begin{aligned} r(60) &= \sqrt{\frac{500(60)}{\pi}} \\ &\doteq 97.72 \end{aligned}$$

The radius of the oil spill after 1 h is 97.72 m.

Section 5.1 Page 275 Question 10

a) From the information given it is determined that $V(t) = 50t$.

Determine an expression for the radius, r, as a function of volume, V.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ r^3 &= \frac{3V}{4\pi} \\ r &= \sqrt[3]{\frac{3V}{4\pi}} \end{aligned}$$

Determine an expression for $r(t)$, in centimetres.

$$\begin{aligned} r(t) &= (r \circ V)(t) \\ &= r(V(t)) \\ &= r(50t) \\ &= \sqrt[3]{\frac{3(50t)}{4\pi}} \\ &= \sqrt[3]{\frac{75t}{2\pi}} \end{aligned}$$

The radius of the melon can be expressed as $V(t) = \sqrt[3]{\dfrac{75t}{2\pi}}$.

b) At eight weeks, $t = 56$. The radius is $\sqrt[3]{\dfrac{75(56)}{2\pi}}$ or approximately 8.74 cm.

Section 5.1 Page 275 Question 11

a) Delivery persons: one per 45 subscribers plus one replacement; supervisors: one per 12 delivery persons.

b)

$$\begin{aligned} s(x) &= (s \circ p)(x) \\ &= s(p(x)) \\ &= s\left(\frac{x}{45} + 1\right) \\ &= \frac{1}{12}\left(\frac{x}{45} + 1\right) \\ &= \frac{x + 45}{540} \end{aligned}$$

c)

$$\begin{aligned} s(5000) &= \frac{5000 + 45}{540} \\ &\doteq 9.34 \end{aligned}$$

Ten supervisors are required for 5000 subscribers (you need more than 9).

Section 5.1 Page 275 Question 12

a) $a(s) = 5s$

b) $w(a) = 0.05a + 200$

c)

$$\begin{aligned} w(s) &= (w \circ a)(s) \\ &= w(a(s)) \\ &= w(5s) \\ &= 0.05(5s) + 200 \\ &= 0.25s + 200 \end{aligned}$$

Her gross wages are $w(s) = 0.25s + 200$.

d)

$$\begin{aligned} w(s) &= 0.25s + 200 \\ w(2500) &= 0.25(2500) + 200 \\ &= 825 \end{aligned}$$

Her gross wages are \$825.

e) Let e be her extra wages.

$$\begin{aligned} e &= 0.07(3000 + 1500) \\ &= 315 \end{aligned}$$

Her extra wages are \$315.

Section 5.1 Page 275 Question 13

a) The radius is $r = 3000t + 1\ 000\ 000$.

b)

$$\begin{aligned} V(r) &= \frac{4}{3}\pi r^3 \\ V(t) &= (V \circ r)(t) \\ &= V(r(t)) \\ &= V(3000t + 1\ 000\ 000) \\ &= \frac{4}{3}\pi(3000t + 1\ 000\ 000)^3 \end{aligned}$$

$V \circ r$ represents the volume, in cubic kilometres, after t seconds.

c)

$$\begin{aligned} V(15) &= \frac{4}{3}\pi(3000(15) + 1\ 000\ 000)^3 \\ &\doteq 4.780 \times 10^{18} \end{aligned}$$

The volume is approximately 4.780×10^{18} km^3 after 15 s.

Section 5.1 Page 276 Question 14

The radius of the fire can be defined as $r(t) = 600t$, where t is measured in days. Determine a definition for $A(t)$, the area burned.

$$\begin{aligned} A(r) &= \pi r^2 \\ A(t) &= (A \circ r)(t) \\ &= A(r(t)) \\ &= A(600t) \\ &= \pi(600t)^2 \\ &= 360\ 000\pi t^2 \end{aligned}$$

At two weeks, $t = 14$. The total area burned is $360\ 000\pi(14)^2$ m^2 or approximately 221.7 km^2.

Section 5.1 Page 276 Question 15

a) The distance, p, can be expressed as $p = f(c) = \sqrt{c^2 + 225}$.

b) The distance travelled by the cyclist, c, in metres, can be expressed as $c = g(t) = \frac{15(1000)}{36}t$, or $c = \frac{25}{6}t$, where t is measured in seconds.

c)

$$\begin{aligned}(f \circ g)(t) &= f(g(t)) \\ &= f\left(\frac{25}{6}t\right) \\ &= \sqrt{\left(\frac{25}{6}t\right)^2 + 225}\end{aligned}$$

The function $(f \circ g)(t) = \sqrt{\left(\frac{25}{6}t\right)^2 + 225}$ represents the distance between the cyclist and the lock after 1:00 p.m., in seconds.

d) The cyclist will be $\sqrt{\left(\frac{250}{6}\right)^2 + 225}$ or approximately 44.28 m from the lock.

Section 5.1 Page 276 Question 16

Answers may vary.

	$F(x) = (f \circ g)(x)$	$f(x)$	$g(x)$
a)	$32x^2 + 3$	$32x + 3$	x^2
b)	$\sqrt{6x^2 + 7}$	$\sqrt{x}$	$6x^2 + 7$
c)	$\sqrt{\frac{1}{x}}$	$\sqrt{x}$	$\frac{1}{x}$
d)	$(3x^2 - 5x^3)^{-5}$	x^{-5}	$3x^2 - 5x^3$

Section 5.1 Page 276 Question 17

$$\begin{aligned}f \circ g &= h \\ f(g(x)) &= h(x) \\ 4g(x) - 3 &= 4x^2 - 21 \\ 4g(x) &= 4x^2 - 18 \\ g(x) &= x^2 - \frac{9}{2}\end{aligned}$$

Section 5.1 Page 276 Question 18

$$\begin{aligned}f \circ g &= h \\ f(g(x)) &= h(x) \\ f(8x + 6) &= \frac{1}{8x + 9} \\ f(8x + 6) &= \frac{1}{(8x + 6) + 3} \qquad (1)\end{aligned}$$

By comparison of both sides of (1), $f(x) = \frac{1}{x + 3}$.

5.2 The Chain Rule

Practise

Section 5.2 Page 282 Question 1

$F(x) = f(g(x))$	$f(x)$	$g(x)$	$g'(x)$	$f'(g(x))$	$F'(x) = f'(g(x))g'(x)$
$(x^7+3)^5$	x^5	x^7+3	$7x^6$	$5(x^7+3)^4$	$5(x^7+3)^4(7x^6)$
$(x^3-2x^2)^{-4}$	x^{-4}	x^3-2x^2	$3x^2-4x$	$-4(x^3-2x^2)^{-5}$	$-4(x^3-2x^2)^{-5}(3x^2-4x)$
$(x^4+5)^{\frac{1}{2}}$	$x^{\frac{1}{2}}$	x^4+5	$4x^3$	$\frac{1}{2}(x^4+5)^{-\frac{1}{2}}$	$\frac{1}{2}(x^4+5)^{-\frac{1}{2}}(4x^3)$
$\frac{1}{x^2+2x}$	$\frac{1}{x}$	x^2+2x	$2x+2$	$-\frac{1}{(x^2+2x)^2}$	$\frac{-1}{(x^2+2x)^2}(2x+2)$
$\sqrt{2x-1}$	$\sqrt{x}$	$2x-1$	2	$\frac{1}{2\sqrt{2x-1}}$	$\frac{1}{2\sqrt{2x-1}}(2)$
$(x^2+5x-8)^4$	x^4	x^2+5x-8	$2x+5$	$4(x^2+5x-8)^3$	$4(x^2+5x-8)^3(2x+5)$

Section 5.2 Page 282 Question 3

a)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(2x+7)^{\frac{1}{2}}}{d\,2x+7}\cdot\frac{d\,2x+7}{dx}\\ &= \frac{1}{2}(2x+7)^{-\frac{1}{2}}\cdot 2\\ &= \frac{1}{\sqrt{2x+7}}\end{aligned}$$

b)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2+6)^3}{d\,x^2+6}\cdot\frac{d\,x^2+6}{dx}\\ &= 3(x^2+6)^2\cdot 2x\\ &= 6x(x^2+6)^2\end{aligned}$$

c)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(3x-1)^{\frac{1}{3}}}{d\,3x-1}\cdot\frac{d\,3x-1}{dx}\\ &= \frac{1}{3}(3x-1)^{-\frac{2}{3}}\cdot 3\\ &= \frac{1}{\sqrt[3]{(3x-1)^2}}\end{aligned}$$

d)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(2x-2)^2}{d\,2x-2}\cdot\frac{d\,2x-2}{dx}\\ &= 2(2x-2)\cdot 2\\ &= 8(x-1)\end{aligned}$$

e)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2+6)^{\frac{1}{2}}}{d\,x^2+6}\cdot\frac{d\,x^2+6}{dx}\\ &= \frac{1}{2}(x^2+6)^{-\frac{1}{2}}\cdot 2x\\ &= \frac{x}{\sqrt{x^2+6}}\end{aligned}$$

f)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2+3x-8)^{-1}}{d\,x^2+3x-8}\cdot\frac{d\,x^2+3x-8}{dx}\\ &= -1(x^2+3x-8)^{-2}\cdot(2x+3)\\ &= -\frac{2x+3}{(x^2+3x-8)^2}\end{aligned}$$

Apply, Solve, Communicate

Section 5.2 Page 282 Question 5

a) $\frac{dV}{dt} = 15$ L/min; $\frac{dV}{dh} = \frac{1}{0.2}$ L/cm.

b)
$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh}\cdot\frac{dh}{dt}\\ 15 &= \frac{1}{0.2}\cdot\frac{dh}{dt}\\ \frac{dh}{dt} &= 3\end{aligned}$$

The water level is rising at 3 cm/min.

Section 5.2 Page 282 Question 6

From the given data, $\frac{dr}{dt} = 0.002$ km/s. Determine an expression for how fast the area of the polluted region is changing with respect to time.

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr}\cdot\frac{dr}{dt}\\ &= \frac{d\pi r^2}{dr}\cdot\frac{dr}{dt}\\ &= 2\pi r\cdot 0.002\\ &= 0.004\pi r\end{aligned}$$

a) When $r = 8$ km , $\frac{dA}{dt} = 0.004\pi(8)$ or approximately 0.101 km^2/s.

b) When $r = 4$ km, $\frac{dA}{dt} = 0.004\pi(4)$ or approximately 0.050 km^2/s.

c) When $r = 0.1$ km, $\frac{dA}{dt} = 0.004\pi(0.1)$ or approximately 0.001 km^2/s.

Section 5.2 Page 282 Question 7

a)

$$\begin{aligned}G'(x) &= \frac{d\left(\frac{x-9}{x+4}\right)^3}{d\left(\frac{x-9}{x+4}\right)}\cdot\frac{d\left(\frac{x-9}{x+4}\right)}{dx}\\ &= \frac{d\left(\frac{x-9}{x+4}\right)^3}{d\left(\frac{x-9}{x+4}\right)}\cdot\frac{(x+4)\frac{d}{dx}(x-9)-(x-9)\frac{d}{dx}(x+4)}{(x+4)^2}\\ &= 3\left(\frac{x-9}{x+4}\right)^2\cdot\frac{(x+4)(1)-(x-9)(1)}{(x+4)^2}\\ &= \left(\frac{x-9}{x+4}\right)^2\cdot\frac{39}{(x+4)^2}\\ &= \frac{39(x-9)^2}{(x+4)^4}\end{aligned}$$

b)

$$\begin{aligned}h'(x) &= \frac{d\left(3+2\sqrt{x}\right)^2}{d\left(3+2\sqrt{x}\right)}\cdot\frac{d3+2x^{\frac{1}{2}}}{dx}\\ &= 2\left(3+2\sqrt{x}\right)\cdot x^{-\frac{1}{2}}\\ &= \frac{2\left(3+2\sqrt{x}\right)}{\sqrt{x}}\\ &= \frac{6}{\sqrt{x}}+4\end{aligned}$$

c)

$$\begin{aligned}F'(x) &= \frac{d\left(4x^2+x\right)^{\frac{1}{2}}}{d\left(4x^2+x\right)}\cdot\frac{d(4x^2+x)}{dx}\\ &= \frac{1}{2}\left(4x^2+x\right)^{-\frac{1}{2}}\cdot(8x+1)\\ &= \frac{8x+1}{2\sqrt{4x^2+x}}\end{aligned}$$

d)

$$\begin{aligned}f'(x) &= (2x+1)\frac{d(4x-1)^5}{d(4x-1)}\cdot\frac{d(4x-1)}{dx}+(4x-1)^5\cdot\frac{d(2x+1)}{dx}\\ &= (2x+1)[5(4x-1)^4(4)]+(4x-1)^5(2)\\ &= 2(4x-1)^4[10(2x+1)+(4x-1)]\\ &= 2(4x-1)^4(20x+10+4x-1)\\ &= 2(4x-1)^4(24x+9)\end{aligned}$$

e)

$$\begin{aligned}H'(x) &= (x^2-4)^3\frac{d\,3-5x}{dx}+(3-5x)\cdot\frac{d(x^2-4)^3}{d\,x^2-4}\cdot\frac{d\,x^2-4}{dx}\\ &= (x^2-4)^3(-5)+(3-5x)[3(x^2-4)^2(2x)]\\ &= (x^2-4)^2[-5(x^2-4)+6x(3-5x)]\\ &= (x^2-4)^2(-5x^2+20+18x-30x^2)\\ &= -(x^2-4)^2(35x^2-18x-20)\end{aligned}$$

Section 5.2 Page 282 Question 8

From the given data, when $r = 30$ cm, $\frac{dr}{dt} = -5$ cm/min.

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\ &= \frac{d\,\frac{4}{3}\pi r^3}{dr} \cdot \frac{dr}{dt} \\ &= 4\pi r^2 \cdot \frac{dr}{dt} \qquad (1)\end{aligned}$$

Substitute $r = 30$ and $\frac{dr}{dt} = -5$ into (1).

$$\begin{aligned}\frac{dV}{dt} &= 4\pi(30)^2(-5) \\ &= -18\ 000\pi\end{aligned}$$

The volume is decreasing at a rate of 18 000π cm^3/min.

Section 5.2 Page 282 Question 9

From the given data, when $r = 70$ cm, $\frac{dr}{dt} = 3$ m/day.

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr} \cdot \frac{dr}{dt} \\ &= \frac{d\,\pi r^2}{dr} \cdot \frac{dr}{dt} \\ &= 2\pi r \cdot \frac{dr}{dt} \qquad (1)\end{aligned}$$

Substitute $r = 70$ and $\frac{dr}{dt} = 3$ into (1).

$$\begin{aligned}\frac{dA}{dt} &= 2\pi(70)(3) \\ &= 420\pi\end{aligned}$$

The area is increasing at a rate of 420π m^2/day.

Section 5.2 Page 283 Question 10

The volume of a cone with base radius, r, and height, h, is $V = \frac{1}{3}\pi r^2 h$. Since $h = \frac{r}{2}$, the formula for volume can be simplified to $V = \frac{1}{3}\pi r^2\left(\frac{r}{2}\right)$ or $V = \frac{\pi r^3}{6}$.

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\ &= \frac{d\frac{\pi r^3}{6}}{dr} \cdot \frac{dr}{dt} \\ \frac{dV}{dt} &= \frac{\pi r^2}{2} \cdot \frac{dr}{dt} \qquad (1)\end{aligned}$$

Substitute $\frac{dV}{dt} = 4$ and $r = 7$ into (1).

$$\begin{aligned}4 &= \frac{\pi(7)^2}{2} \cdot \frac{dr}{dt} \\ 8 &= 49\pi \cdot \frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{8}{49\pi}\end{aligned}$$

The radius is increasing at the rate of $\frac{8}{49\pi}$ m/min.

Section 5.2 Page 283 Question 11

$$\begin{aligned}F'(x) &= f'(g(x))(g'(x)) \\ F'(2) &= f'(g(2))(g'(2)) \\ &= f'(6)(4) \\ &= 108(4) \\ &= 432\end{aligned}$$

Section 5.2 Page 283 Question 12

a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2-4)^{\frac{1}{3}}}{dx^2-4}\cdot\frac{dx^2-4}{dx}\\ &= \frac{1}{3}(x^2-4)^{-\frac{2}{3}}(2x)\\ &= \frac{2x}{3\sqrt[3]{(x^2-4)^2}}\end{aligned}$$

b)

$$\begin{aligned}g'(x) &= \frac{d(x^3+1)^{-5}}{dx^3+1}\cdot\frac{dx^3+1}{dx}\\ &= -5(x^3+1)^{-6}(3x^2)\\ &= \frac{-15x^2}{(x^3+1)^6}\end{aligned}$$

c)

$$\begin{aligned}h'(x) &= \frac{d\left(\frac{x^2+1}{x+1}\right)^8}{d\left(\frac{x^2+1}{x+1}\right)}\cdot\frac{d\left(\frac{x^2+1}{x+1}\right)}{dx}\\ &= \frac{d\left(\frac{x^2+1}{x+1}\right)^8}{d\left(\frac{x^2+1}{x+1}\right)}\cdot\frac{(x+1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x+1)}{(x+1)^2}\\ &= 8\left(\frac{x^2+1}{x+1}\right)^7\cdot\frac{(x+1)(2x)-(x^2+1)(1)}{(x+1)^2}\\ &= 8\left(\frac{x^2+1}{x+1}\right)^7\cdot\frac{x^2+2x-1}{(x+1)^2}\end{aligned}$$

d)

$$\begin{aligned}F'(x) &= \frac{\sqrt{3x+2}\cdot\frac{dx}{dx}-x\cdot\frac{d(3x+2)^{\frac{1}{2}}}{d3x+2}\cdot\frac{d3x+2}{dx}}{\left(\sqrt{3x+2}\right)^2}\\ &= \frac{\sqrt{3x+2}-x\cdot\frac{1}{2}(3x+2)^{-\frac{1}{2}}(3)}{3x+2}\\ &= \frac{(3x+2)^{-\frac{1}{2}}(3x+2-\frac{3}{2}x)}{3x+2}\\ &= \frac{3x+4}{2(3x+2)^{\frac{3}{2}}}\end{aligned}$$

e)

$$\begin{aligned}G'(x) &= (3x-2)^3\cdot\frac{d(2x^2+5)^4}{d2x^2+5}\cdot\frac{d2x^2+5}{dx}+\left(2x^2+5\right)^4\cdot\frac{d(3x-2)^3}{d3x-2}\cdot\frac{d3x-2}{dx}\\ &= (3x-2)^3[4(2x^2+5)^3(4x)]+(2x^2+5)^4[3(3x-2)^2(3)]\\ &= 16x(3x-2)^3(2x^2+5)^3+9(2x^2+5)^4(3x-2)^2\\ &= (3x-2)^2(2x^2+5)^3[16x(3x-2)+9(2x^2+5)]\\ &= (3x-2)^2(2x^2+5)^3(48x^2-32x+18x^2+45)\\ &= (3x-2)^2(2x^2+5)^3(66x^2-32x+45)\end{aligned}$$

f)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\left(\frac{2x-1}{x^2+1}\right)^3}{d\left(\frac{2x-1}{x^2+1}\right)}\cdot\frac{d\left(\frac{2x-1}{x^2+1}\right)}{dx}\\ &= \frac{d\left(\frac{2x-1}{x^2+1}\right)^3}{d\left(\frac{2x-1}{x^2+1}\right)}\cdot\frac{(x^2+1)\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\\ &= 3\left(\frac{2x-1}{x^2+1}\right)^2\cdot\frac{(x^2+1)(2)-(2x-1)(2x)}{(x^2+1)^2}\\ &= \left(\frac{2x-1}{x^2+1}\right)^2\cdot\frac{-6(x^2-x-1)}{(x^2+1)^2}\\ &= \frac{-6(2x-1)(x^2-x-1)}{(x^2+1)^4}\end{aligned}$$

g)

$$\begin{aligned}
f'(x) &= \frac{d\left(\frac{x^3-1}{x^3+1}\right)^{\frac{1}{4}}}{d\left(\frac{x^3-1}{x^3+1}\right)} \cdot \frac{d\left(\frac{x^3-1}{x^3+1}\right)}{dx} \\
&= \frac{d\left(\frac{x^3-1}{x^3+1}\right)^{\frac{1}{4}}}{d\left(\frac{x^3-1}{x^3+1}\right)} \cdot \frac{(x^3+1)\frac{d}{dx}(x^3-1)-(x^3-1)\frac{d}{dx}(x^3+1)}{(x^3+1)^2} \\
&= \frac{1}{4}\left(\frac{x^3-1}{x^3+1}\right)^{-\frac{3}{4}} \cdot \frac{(x^3+1)(3x^2)-(x^3-1)(3x^2)}{(x^3+1)^2} \\
&= \left(\frac{x^3-1}{x^3+1}\right)^{-\frac{3}{4}} \cdot \frac{3x^2(x^3+1-x^3+1)}{4(x^3+1)^2} \\
&= \left(\frac{x^3-1}{x^3+1}\right)^{-\frac{3}{4}} \cdot \frac{3x^2}{2(x^3+1)^2}
\end{aligned}$$

h)

$$\begin{aligned}
H'(x) &= \frac{\sqrt[3]{x+2}\cdot\frac{d}{dx}(x-2)^{\frac{1}{2}}-\sqrt{x-2}\cdot\frac{d}{dx}(x+2)^{\frac{1}{3}}}{\left(\sqrt[3]{x+2}\right)^2} \\
&= \frac{\sqrt[3]{x+2}\cdot\frac{1}{2}(x-2)^{-\frac{1}{2}}-\sqrt{x-2}\cdot\frac{1}{3}(x+2)^{-\frac{2}{3}}}{\left(\sqrt[3]{x+2}\right)^2} \\
&= \frac{(x+2)^{-\frac{2}{3}}(x-2)^{-\frac{1}{2}}\left[\frac{1}{2}(x+2)-\frac{1}{3}(x+2)\right]}{(x+2)^{\frac{2}{3}}} \\
&= \frac{\frac{1}{6}x+\frac{5}{3}}{(x+2)^{\frac{4}{3}}(x-2)^{\frac{1}{2}}} \\
&= \frac{x+10}{6(x+2)^{\frac{4}{3}}(x-2)^{\frac{1}{2}}}
\end{aligned}$$

i)

$$\begin{aligned}
g'(x) &= \frac{d(x^2-3)^{-\frac{1}{2}}}{dx} \\
&= \frac{d(x^2-3)^{-\frac{1}{2}}}{dx^2-3} \cdot \frac{dx^2-3}{dx} \\
&= -\frac{1}{2}(x^2-3)^{-\frac{3}{2}} \cdot 2x \\
&= -\frac{x}{\sqrt{(x^2-3)^3}}
\end{aligned}$$

Section 5.2 Page 283 Question 13

a)
$$\begin{aligned}\frac{dy}{dx}&=\frac{dh(g(f(x)))}{dg(f(x))}\cdot\frac{dg(f(x))}{df(x)}\cdot\frac{df(x)}{dx}\\&=h'(g(f(x))(g'(f(x))f'(x)\end{aligned}$$

b)
$$\begin{aligned}\frac{dy}{dx}&=\frac{d[g(h(x))]^2}{dg(h(x))}\cdot\frac{dg(h(x))}{dh(x)}\cdot\frac{dh(x)}{dx}\\&=2g(h(x))g'(h(x))h'(x)\end{aligned}$$

c)
$$\begin{aligned}\frac{dy}{dx}&=\frac{d\,g([h(3x-2)]^2)}{d[h(3x-2)]^2}\cdot\frac{d[h(3x-2)]^2}{dh(3x-2)}\cdot\frac{dh(3x-2)}{d3x-2}\cdot\frac{d3x}{dx}\\&=g'([h(3x-2)]^2)(2)h(3x-2)h'(3x-2)(3)\\&=6g'([h(3x-2)]^2)h(3x-2)h'(3x-2)\end{aligned}$$

d)
$$\begin{aligned}\frac{dy}{dx}&=\frac{d\,g\left(f\left((x+1)^{-\frac{1}{2}}\right)\right)}{d\,f\left((x+1)^{-\frac{1}{2}}\right)}\cdot\frac{d\,f\left((x+1)^{-\frac{1}{2}}\right)}{d\,(x+1)^{-\frac{1}{2}}}\cdot\frac{d\,(x+1)^{-\frac{1}{2}}}{dx}\\&=g'\left(f\left((x+1)^{-\frac{1}{2}}\right)\right)f'\left((x+1)^{-\frac{1}{2}}\right)\left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}}\end{aligned}$$

e)
$$\begin{aligned}\frac{dy}{dx}&=\frac{d\,f\left((x+g(x))^{-1}\right)}{d\,(x+g(x))^{-1}}\cdot\frac{d\,(x+g(x))^{-1}}{d\,(x+g(x))}\cdot\frac{d(x+g(x))}{dx}\\&=f'\left((x+g(x))^{-1}\right)(-1)\,(x+g(x))^{-2}\left[1+g'(x)\right]\\&=-f'\left((x+g(x))^{-1}\right)(x+g(x))^{-2}\left[1+g'(x)\right]\end{aligned}$$

Section 5.2 Page 283 Question 14

a) Prove that if $F=f\circ g$ and $g(x)=c$ is a constant function, then $F'(x)=f'(g(x))g'(x)$.

$$\begin{aligned}F(x)&=f(g(x))\\&=f(c)\\F'(x)&=f'(c)\\&=0\\f'(g(x))g'(x)&=f'(c)\cdot 0\\&=0\end{aligned}$$

Thus, $F'(x)=f'(g(x))g'(x)$ if $g(x)=c$.

b)
$$\begin{aligned}F'(a)&=\lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a}\\&=\lim_{x\to a}\left[\frac{f(g(x))-f(g(a))}{g(x)-g(a)}\cdot\frac{g(x)-g(a)}{x-a}\right]\\&=\lim_{x\to a}\frac{f(g(x))-f(g(a))}{g(x)-g(a)}\cdot\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\qquad(1)\end{aligned}$$

c) As $x\to a$, $g(x)\to g(a)$, so we can write the first term of the product on the right-hand side of (1) as

$$\lim_{g(x)\to g(a)}\frac{f(g(x))-f(g(a))}{g(x)-g(a)}$$

Thus,

$$\begin{aligned}F'(a)&=\lim_{g(x)\to g(a)}\frac{f(g(x))-f(g(a))}{g(x)-g(a)}\cdot\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\\&=f'(g(a))g'(a)\qquad(2)\end{aligned}$$

Replace *a* with *x* in (2) to obtain

$$F'(x)=f'(g(x))g'(x)$$

5.3 Implicit Differentiation

Practise

Section 5.3 Page 289 Question 1

a) i) Solve for y.

$$\begin{aligned}3x^2 - y^2 &= 23\\ y^2 &= 3x^2 - 23\\ y &= \pm\sqrt{3x^2-23}\end{aligned}$$

Since $(x, y) = (4, 5)$, use $y = \sqrt{3x^2 - 23}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(3x^2-23)^{\frac{1}{2}}}{d\,3x^2-23}\cdot\frac{d\,3x^2-23}{dx}\\ &= \frac{1}{2}(3x^2-23)^{-\frac{1}{2}}\cdot 6x\\ &= \frac{3x}{\sqrt{3x^2-23}} \qquad (1)\end{aligned}$$

Substitute $x = 4$ into (1).

$$\begin{aligned}\frac{dy}{dx} &= \frac{3(4)}{\sqrt{3(4)^2-23}}\\ &= \frac{12}{\sqrt{25}}\\ &= \frac{12}{5}\end{aligned}$$

The slope of the tangent at $(4, 5)$ is $\frac{12}{5}$.

ii) Differentiate implicitly.

$$\begin{aligned}\frac{d}{dx}(3x^2-y^2) &= \frac{d}{dx}23\\ \frac{d3x^2}{dx} - \frac{dy^2}{dy}\cdot\frac{dy}{dx} &= \frac{d23}{dx}\\ 6x - 2y\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{-6x}{-2y}\\ \frac{dy}{dx} &= \frac{3x}{y} \qquad (1)\end{aligned}$$

Substitute $(x, y) = (4, 5)$ into (1).

$$\begin{aligned}\frac{dy}{dx} &= \frac{3(4)}{5}\\ &= \frac{12}{5}\end{aligned}$$

The slope of the tangent at $(4, 5)$ is $\frac{12}{5}$.

b) Answers will vary.

Section 5.3 Page 289 Question 3

a)

$$\begin{aligned}\frac{d}{dx}\left(x^2+9y^2\right) &= \frac{d}{dx}37\\ \frac{d\,x^2}{dx} + \frac{d\,9y^2}{dy}\cdot\frac{dy}{dx} &= \frac{d\,37}{dx}\\ 2x + 18y\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{-2x}{18y}\\ \frac{dy}{dx} &= -\frac{x}{9y} \qquad (1)\end{aligned}$$

Substitute $(x, y) = (1, 2)$ into (1).

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{9(2)}\\ &= -\frac{1}{18}\end{aligned}$$

The slope of the tangent at $(1, 2)$ is $-\frac{1}{18}$. Determine the equation of the tangent.

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - 2 &= -\frac{1}{18}(x-1)\\ 18y - 36 &= -x + 1\\ x + 18y - 37 &= 0\end{aligned}$$

The equation of the tangent is $x + 18y - 37 = 0$.

b)

$$\begin{aligned}\frac{d}{dx}(xy) &= \frac{d}{dx}36\\ y\frac{dx}{dx} + x\frac{dy}{dx} &= \frac{d\,36}{dx}\\ y + x\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{y}{x} \qquad (1)\end{aligned}$$

Substitute $(x, y) = (9, 4)$ into (1).

$$\frac{dy}{dx} = -\frac{4}{9}$$

The slope of the tangent at $(9, 4)$ is $-\frac{4}{9}$. Determine the equation of the tangent.

$$\begin{aligned}y - 4 &= -\frac{4}{9}(x-9)\\ 9y - 36 &= -4x + 36\\ 4x + 9y - 72 &= 0\end{aligned}$$

The equation of the tangent is $4x + 9y - 72 = 0$.

c)

$$\begin{aligned}
\frac{d}{dx}\left(x^2y^2+xy\right) &= \frac{d}{dx}(30)\\
y^2\frac{d\,x^2}{dx}+x^2\frac{d\,y^2}{dy}\cdot\frac{dy}{dx}+y\frac{dx}{dx}+x\frac{dy}{dx} &= \frac{d\,30}{dx}\\
y^2(2x)+x^2(2y)\frac{dy}{dx}+y+x\frac{dy}{dx} &= 0\\
\frac{dy}{dx}(2x^2y+x) &= -2xy^2-y\\
\frac{dy}{dx} &= -\frac{2xy^2+y}{2x^2y+x}\\
&= -\frac{y(2xy+1)}{x(2xy+1)}\\
&= -\frac{y}{x} \qquad (1)
\end{aligned}$$

Substitute $(x, y) = (-3, 2)$ into (1).

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{2}{-3}\\
&= \frac{2}{3}
\end{aligned}$$

The slope of the tangent at $(-3, 2)$ is $\frac{2}{3}$. Determine the equation of the tangent.

$$\begin{aligned}
y-2 &= \frac{2}{3}(x-(-3))\\
3y-6 &= 2x+6\\
2x-3y+12 &= 0
\end{aligned}$$

The equation of the tangent is $2x-3y+12=0$.

d)

$$\begin{aligned}
\frac{d}{dx}\left(y^4+x^2y^3\right) &= \frac{d}{dx}(5)\\
\frac{d\,y^4}{dy}\cdot\frac{dy}{dx}+y^3\frac{d\,x^2}{dx}+x^2\frac{d\,y^3}{dy}\cdot\frac{dy}{dx} &= \frac{d\,5}{dx}\\
4y^3\frac{dy}{dx}+y^3(2x)+x^2(3y^2)\frac{dy}{dx} &= 0\\
\frac{dy}{dx}(4y^3+3x^2y^2) &= -2xy^3\\
\frac{dy}{dx} &= -\frac{2xy^3}{4y^3+3x^2y^2}\\
&= -\frac{2xy}{4y+3x^2} \qquad (1)
\end{aligned}$$

Substitute $(x, y) = (2, 1)$ into (1).

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{2(2)(1)}{4(1)+3(2)^2}\\
&= -\frac{4}{4+12}\\
&= -\frac{1}{4}
\end{aligned}$$

The slope of the tangent at $(2, 1)$ is $-\frac{1}{4}$. Determine the equation of the tangent.

$$\begin{aligned}
y-1 &= -\frac{1}{4}(x-2)\\
4y-4 &= -x+2\\
x+4y-6 &= 0
\end{aligned}$$

The equation of the tangent is $x+4y-6=0$.

Apply, Solve, Communicate

Section 5.3 Page 289 Question 4

a) i)

$$\begin{aligned}\frac{d}{dx}\left((x-3)^2+(y+1)^2\right) &= \frac{d}{dx}(16)\\ \frac{d\,(x-3)^2}{dx}+\frac{d\,(y+1)^2}{dy}\cdot\frac{dy}{dx} &= \frac{d\,16}{dx}\\ 2(x-3)+2(y+1)\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{x-3}{y+1} \qquad (1)\end{aligned}$$

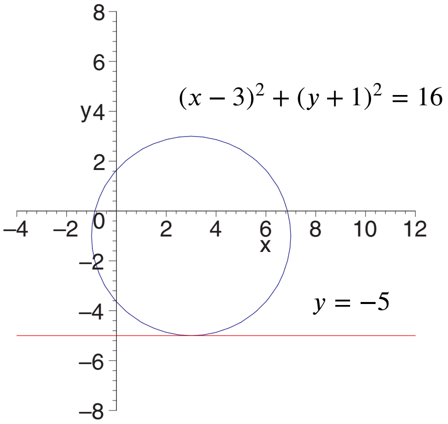
Substitute $(x, y) = (3, -5)$ into (1).

$$\begin{aligned}\frac{dy}{dx} &= -\frac{3-3}{-5+1}\\ &= 0\end{aligned}$$

The slope of the tangent at $(3, -5)$ is 0.

ii) The equation of the tangent is $y = -5$.

iii)



b) i)

$$\begin{aligned}\frac{d}{dx}\left(y^2-2xy\right) &= \frac{d}{dx}(11)\\ \frac{dy^2}{dy}\cdot\frac{dy}{dx}-2\left(y\frac{dx}{dx}+x\frac{dy}{dx}\right) &= \frac{d\,11}{dx}\\ 2y\frac{dy}{dx}-2y-2x\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{y}{y-x} \qquad (1)\end{aligned}$$

Substitute $(x, y) = (5, -1)$ into (1).

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{-1-5}\\ &= \frac{1}{6}\end{aligned}$$

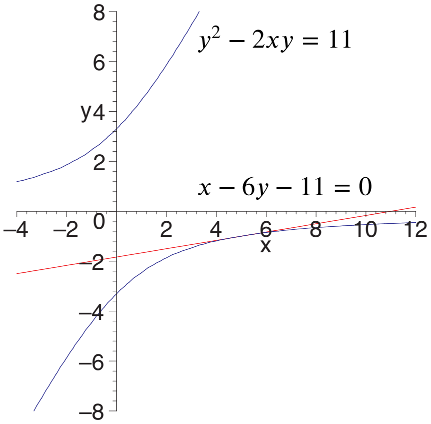
The slope of the tangent at $(5, -1)$ is $\frac{1}{6}$.

ii) Determine the equation of the tangent.

$$\begin{aligned}y-(-1) &= \frac{1}{6}(x-5)\\ 6y+6 &= x-5\\ x-6y-11 &= 0\end{aligned}$$

The equation of the tangent is $x - 6y - 11 = 0$.

iii)



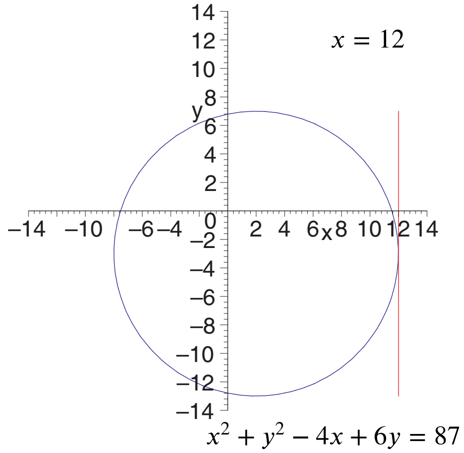
c) i)

$$
\begin{aligned}
\frac{d}{dx}\left(x^2+y^2-4x+6y\right) &= \frac{d}{dx}(87)\\
\frac{dx^2}{dx}+\frac{dy^2}{dy}\cdot\frac{dy}{dx}-4\frac{dx}{dx}+6\frac{dy}{dx} &= \frac{d\,87}{dx}\\
2x+2y\frac{dy}{dx}-4+6\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= \frac{2-x}{3+y} \qquad (1)
\end{aligned}
$$

Substitution of $(x, y) = (12, -3)$ into (1) yields division by zero; hence, the slope of the tangent is undefined.

ii) The tangent is a vertical line through $(12, -3)$. The equation of the tangent is $x = 12$.

iii)



d) i)

$$
\begin{aligned}
\frac{d}{dx}\left(x^2-xy+y^3\right) &= \frac{d}{dx}(3)\\
\frac{dx^2}{dx}-\left(y\frac{dx}{dx}+x\frac{dy}{dx}\right)+\frac{dy^3}{dy}\cdot\frac{dy}{dx} &= \frac{d\,3}{dx}\\
2x-y-x\frac{dy}{dx}+3y^2\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= \frac{y-2x}{3y^2-x} \qquad (1)
\end{aligned}
$$

Substitute $(x, y) = (-1, 1)$ into (1).

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1-2(-1)}{3(1)^2-(-1)}\\
&= \frac{3}{4}
\end{aligned}
$$

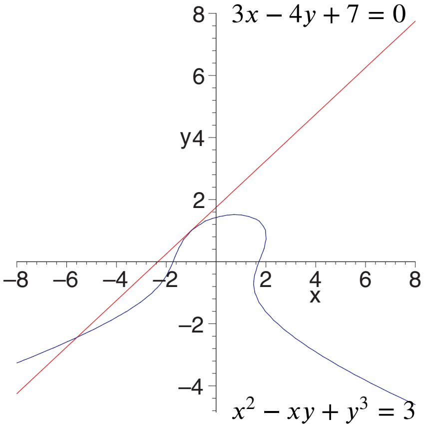
The slope of the tangent at $(-1, 1)$ is $\frac{3}{4}$.

ii) Determine the equation of the tangent.

$$
\begin{aligned}
y-1 &= \frac{3}{4}(x-(-1))\\
4y-4 &= 3x+3\\
3x-4y+7 &= 0
\end{aligned}
$$

The equation of the tangent is $3x - 4y + 7 = 0$.

iii)



Section 5.3 Page 289 Question 5

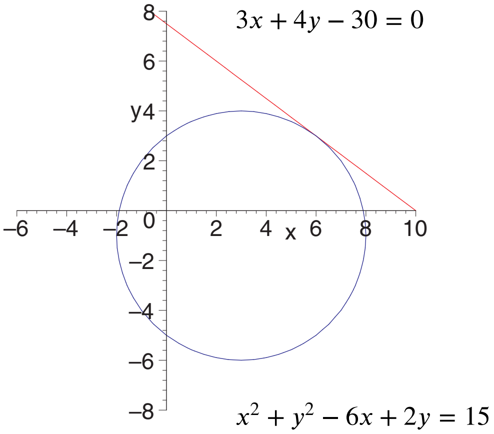
a)

$$\begin{aligned}\frac{d}{dx}\left(x^2+y^2-6x+2y\right) &= \frac{d}{dx}(15)\\ \frac{dx^2}{dx}+\frac{dy^2}{dy}\cdot\frac{dy}{dx}-6\frac{dx}{dx}+2\frac{dy}{dx} &= \frac{d\,15}{dx}\\ 2x+2y\frac{dy}{dx}-6+2\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{3-x}{1+y} \qquad (1)\end{aligned}$$

Substitution of $(x, y) = (6, 3)$ into (1) yields a slope of $-\frac{3}{4}$. Determine the equation of the tangent.

$$\begin{aligned}y-3 &= -\frac{3}{4}(x-6)\\ 4y-12 &= -3x+18\\ 3x+4y-30 &= 0\end{aligned}$$

b)



Section 5.3 Page 289 Question 6

a) Determine the coordinates of the centre, C.

$$\begin{aligned}x^2+y^2+4x-12y &= 60\\ x^2+4x+4+y^2-12y+36 &= 60+4+36\\ (x+2)^2+(y-6)^2 &= 100\end{aligned}$$

The centre of the circle is C$(-2, 6)$. The slope of the radius, CP, is $\frac{-2-6}{4-(-2)}$ or $-\frac{4}{3}$. Determine the slope of the tangent at P.

$$\begin{aligned}\frac{d}{dx}\left(x^2+y^2+4x-12y\right) &= \frac{d}{dx}(60)\\ \frac{dx^2}{dx}+\frac{dy^2}{dy}\cdot\frac{dy}{dx}+4\frac{dx}{dx}-12\frac{dy}{dx} &= \frac{d\,60}{dx}\\ 2x+2y\frac{dy}{dx}+4-12\frac{dy}{dx} &= 0\\ x+y\frac{dy}{dx}+2-6\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{-x-2}{y-6} \qquad (1)\end{aligned}$$

Substitution of $(x, y) = (4, -2)$ into (1) yields a slope of $\frac{-4-2}{-2-6}$ or $\frac{3}{4}$. Since the slope of the radius, CP, and the slope of the tangent at P are negative reciprocals of one another, they are perpendicular to each other.

b) Since translation preserves slope, the general case can be developed using a circle of radius r, centred at the origin. The equation of this circle is $x^2+y^2 = r^2$. Consider a point P on the circle with coordinates (x, y). The slope of the radius CP is given by

$$\frac{y}{x} \qquad (1)$$

Determine the slope of the tangent to the circle at P.

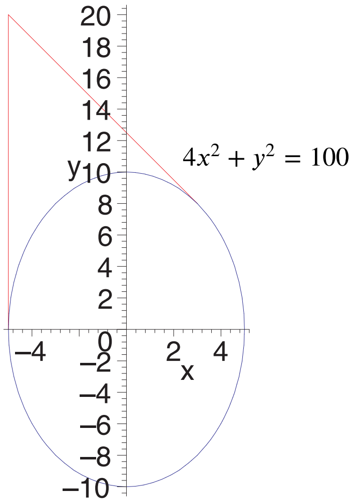
$$\begin{aligned}\frac{d}{dx}(x^2+y^2) &= \frac{d}{dx}(r^2)\\ \frac{d\,x^2}{dx}+\frac{d\,y^2}{dy}\cdot\frac{dy}{dx} &= \frac{d\,r^2}{dx}\\ 2x+2y\frac{dy}{dx} &= 0\\ x+y\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{x}{y} \qquad (2)\end{aligned}$$

Since (1) and (2) are negative reciprocals of one another, the slope of the tangent at point P to a circle with centre C is perpendicular to the radius CP.

Section 5.3 Page 290 Question 7

Determine the slope of the tangent to the orbit.

$$\begin{aligned}\frac{d}{dx}\left(4x^2+y^2\right) &= \frac{d}{dx}(100)\\ \frac{d\,4x^2}{dx}+\frac{dy^2}{dy}\cdot\frac{dy}{dx} &= \frac{d\,100}{dx}\\ 8x+2y\frac{dy}{dx} &= 0\\ 4x+y\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{4x}{y} \qquad (1)\end{aligned}$$



From the data given, the general point, P, on the ellipse can be assigned the coordinates $P\left(x, \sqrt{100-4x^2}\right)$. The coordinates of the moon are $M(-5, 20)$. To receive the signals, the slope of PM and the slope of the tangent at P must be equal.

$$\begin{aligned}-\frac{4x}{y} &= \frac{y-20}{x-(-5)}\\ \frac{-4x}{\sqrt{100-4x^2}} &= \frac{\sqrt{100-4x^2}-20}{x-(-5)}\\ -4x(x+5) &= \sqrt{100-4x^2}\left(\sqrt{100-4x^2}-20\right)\\ -4x^2-20x &= 100-4x^2-20\left(\sqrt{100-4x^2}\right)\\ \sqrt{100-4x^2} &= x+5\\ 100-4x^2 &= x^2+10x+25\\ 5x^2+10x-75 &= 0\\ x^2+2x-15 &= 0\\ (x+5)(x-3) &= 0\\ x &= -5 \text{ or } 3\end{aligned}$$

The satellite must be at $\left(3, \sqrt{100-4(3)^2}\right) = (3, 8)$ or $\left(-5, \sqrt{100-4(-5)^2}\right) = (-5, 0)$.

Section 5.3 Page 290 Question 8

a)

$$\begin{aligned}\frac{d}{dV}(PV) &= \frac{d}{dV}(kT)\\ V\cdot\frac{dP}{dV}+P\cdot\frac{dV}{dV} &= k\cdot\frac{dT}{dV}\\ V\cdot\frac{dP}{dV}+P &= k\cdot\frac{dT}{dV}\end{aligned}$$

b) k must be positive, because P, V, and T are all positive in the equation $PV = kT$ (T is measured in Kelvins).

c)

$$\begin{aligned}\frac{d}{dT}(PV) &= \frac{d}{dT}(kT)\\ V\cdot\frac{dP}{dT}+P\cdot\frac{dV}{dT} &= k\cdot\frac{dT}{dT}\\ V\cdot\frac{dP}{dT}+P\cdot\frac{dV}{dT} &= k\end{aligned}$$

k is positive since $\frac{dP}{dt}$ is positive and $\frac{dV}{dt} \doteq 0$.

Section 5.3 Page 290 Question 9

a)
$$\begin{aligned}
\frac{d}{dx}\left(2(x^2+y^2)^2\right) &= \frac{d}{dx}\left(25(x^2-y^2)\right)\\
2\cdot\frac{d(x^2+y^2)^2}{d(x^2+y^2)}\cdot\left(\frac{d\,x^2}{dx}+\frac{d\,y^2}{dy}\cdot\frac{dy}{dx}\right) &= 25\left(\frac{d\,x^2}{dx}-\frac{d\,y^2}{dy}\cdot\frac{dy}{dx}\right)\\
2(2)(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) &= 25\left(2x-2y\cdot\frac{dy}{dx}\right)\\
4(x^2+y^2)\left(x+y\frac{dy}{dx}\right) &= 25\left(x-y\frac{dy}{dx}\right)\\
4x(x^2+y^2)+4y(x^2+y^2)\frac{dy}{dx} &= 25x-25y\frac{dy}{dx}\\
\frac{dy}{dx}\left(4y(x^2+y^2)+25y\right) &= 25x-4x(x^2+y^2)\\
\frac{dy}{dx} &= \frac{x\left(25-4(x^2+y^2)\right)}{y\left(4(x^2+y^2)+25\right)}
\end{aligned}$$

b) Substitute the coordinates $(-3, 1)$ into the result from part a).

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-3\left(25-4((-3)^2+1^2)\right)}{1\left(4((-3)^2+1^2)+25\right)}\\
&= \frac{-3\,(-15)}{65}\\
&= \frac{9}{13}
\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}
y-1 &= \frac{9}{13}(x-(-3))\\
13y-13 &= 9x+27\\
9x-13y+40 &= 0
\end{aligned}$$

The equation of the tangent is $9x-13y+40=0$.

c) The slope is horizontal when $\frac{dy}{dx}=0$. Thus,

$$\begin{aligned}
\frac{x(25-4(x^2+y^2))}{y(4(x^2+y^2)+25)} &= 0\\
25-4(x^2+y^2) &= 0,\ x\neq 0, y\neq 0\\
x^2+y^2 &= \frac{25}{4}
\end{aligned}$$

Any point on the circle $x^2+y^2=\frac{25}{4}$ that is also on the lemniscate will have a horizontal tangent. Find the points of intersection of the circle and the lemniscate.

$$x^2+y^2=\frac{25}{4} \qquad (1)$$
$$2(x^2+y^2)=25(x^2-y^2) \qquad (2)$$

Substitute (1) into (2).

$$\begin{aligned}
2\left(\frac{25}{4}\right)^2 &= 25\left(x^2-\left(\frac{25}{4}-x^2\right)\right)\\
\frac{625}{8}+\frac{625}{4} &= 50x^2\\
x^2 &= \frac{1875}{400}
\end{aligned}$$

$$\begin{aligned}
&= \frac{75}{16} \\
x &= \pm\frac{5\sqrt{3}}{4} \\
y^2 &= \frac{25}{4} - x^2 \\
&= \frac{25}{4} - \frac{75}{16} \\
&= \frac{25}{16} \\
y &= \pm\frac{5}{4}
\end{aligned}$$

The points on the lemniscate with horizontal tangents are $\left(\pm\frac{5\sqrt{3}}{4}, \frac{5}{4}\right)$ and $\left(\pm\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$.

Section 5.3 Page 290 Question 10

a)

$$\begin{aligned}
\frac{d}{dx}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right) &= \frac{d}{dx}(1) \\
\frac{d\,x^{\frac{2}{3}}}{dx} + \frac{d\,y^{\frac{2}{3}}}{dy}\cdot\frac{dy}{dx} &= \frac{d\,1}{dx} \\
\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\cdot\frac{dy}{dx} &= 0 \\
\frac{1}{\sqrt[3]{x}} + \frac{1}{\sqrt[3]{y}}\cdot\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\sqrt[3]{\frac{y}{x}}
\end{aligned}$$

b) Substitute the coordinates $\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$ into the result from part a).

$$\begin{aligned}
\frac{dy}{dx} &= -\sqrt[3]{\frac{\frac{3\sqrt{3}}{8}}{\frac{1}{8}}} \\
&= -\sqrt[3]{3\sqrt{3}} \\
&= -\sqrt{3}
\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}
y - \frac{3\sqrt{3}}{8} &= -\sqrt{3}\left(x - \frac{1}{8}\right) \\
8y - 3\sqrt{3} &= -8\sqrt{3}x + \sqrt{3} \\
8\sqrt{3}x + 8y - 4\sqrt{3} &= 0 \\
2\sqrt{3}x + 2y - \sqrt{3} &= 0
\end{aligned}$$

The equation of the tangent is $2\sqrt{3}x + 2y - \sqrt{3} = 0$.

c)

$$\begin{aligned}
\frac{dy}{dx} &= 1 \\
-\sqrt[3]{\frac{y}{x}} &= 1 \\
\frac{y}{x} &= -1 \\
y &= -x \qquad (1)
\end{aligned}$$

Substitute (1) into $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ and solve for x.

$$\begin{aligned} x^{\frac{2}{3}} + (-x)^{\frac{2}{3}} &= 1 \\ 2x^{\frac{2}{3}} &= 1 \\ x^{\frac{2}{3}} &= \frac{1}{2} \\ x &= \pm\sqrt{\frac{1}{8}} \\ x &= \pm\frac{1}{2\sqrt{2}} \qquad (2) \end{aligned}$$

Substitution of (2) into (1) yields the points with coordinates $\left(\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$ and $\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$.

Section 5.3 Page 290 Question 11

$$\begin{aligned} x^2y^2 + xy &= 2 \qquad (1) \\ \frac{d}{dx}\left(x^2y^2 + xy\right) &= \frac{d}{dx}(2) \\ y^2\frac{dx^2}{dx} + x^2\frac{dy^2}{dy}\cdot\frac{dy}{dx} + y\frac{dx}{dx} + x\frac{dy}{dx} &= \frac{d2}{dx} \\ y^2(2x) + x^2(2y)\frac{dy}{dx} + y + x\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}\left(2x^2y + x\right) &= -2xy^2 - y \\ \frac{dy}{dx} &= -\frac{y(2xy+1)}{x(2xy+1)} \\ &= -\frac{y}{x} \end{aligned}$$

Determine points where the slope of the tangent is −1.

$$\begin{aligned} \frac{dy}{dx} &= -1 \\ -\frac{y}{x} &= -1 \\ y &= x \qquad (2) \end{aligned}$$

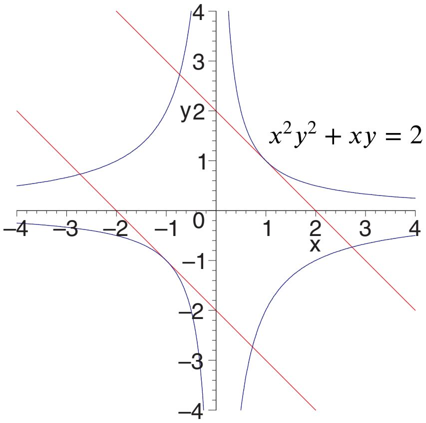
Substitute (2) into (1) and solve for x.

$$\begin{aligned} x^2(x)^2 + x(x) &= 2 \\ x^4 + x^2 - 2 &= 0 \\ (x^2 - 1)(x^2 + 2) &= 0 \\ x^2 - 1 &= 0 \\ x &= \pm 1 \qquad (3) \end{aligned}$$

Substitute (3) into (2) and solve for y.

$$y = \pm 1$$

The slope of the tangent to $x^2y^2 + xy = 2$ is -1 at $(-1, -1)$ and $(1, 1)$.



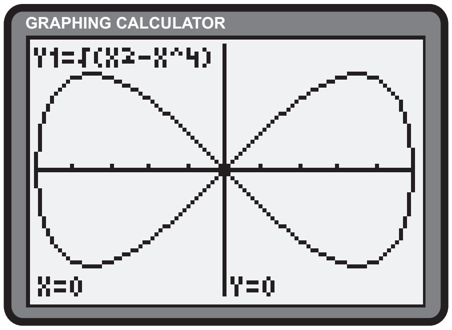
Section 5.3 Page 290 Question 12

$$\begin{aligned}
x^2 - y^2 &= A && (1)\\
\frac{d}{dx}\left(x^2 - y^2\right) &= \frac{d}{dx}(A)\\
2x - 2y\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= \frac{x}{y} && (2)
\end{aligned}$$

$$\begin{aligned}
xy &= B && (3)\\
\frac{d}{dx}(xy) &= \frac{d}{dx}(B)\\
y\frac{dx}{dx} + x\frac{dy}{dx} &= 0\\
y + x\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= -\frac{x}{y} && (4)
\end{aligned}$$

Since the product of the derivative results in (2) and (4) yield -1, the families of curves defined in (1) and (3) are orthogonal.

Section 5.3 Page 290 Question 13



The relation has two solutions, $y = \pm\sqrt{x^2 - x^4}$, neither of which is differentiable at the origin. The slope of the tangent can be determined by considering only right-hand differentiability at $(0, 0)$ for $y = \sqrt{x^2 - x^4}$ from first principles. Let $\mathrm{P}\left(x, \sqrt{x^2 - x^4}\right)$, be a point on the function $y = \sqrt{x^2 - x^4}$. Let m be the slope of the tangent at the origin, O.

$$\begin{aligned}
m &= \lim_{x\to 0^+} \text{slope of OP}\\
&= \lim_{x\to 0^+} \frac{\sqrt{x^2 - x^4} - 0}{x - 0}\\
&= \lim_{x\to 0^+} \frac{|x|\sqrt{1 - x^2}}{x}\\
&= \lim_{x\to 0^+} \frac{x\sqrt{1 - x^2}}{x}\\
&= \lim_{x\to 0^+} \sqrt{1 - x^2},\ x \neq 0\\
&= 1
\end{aligned}$$

The slope of the tangent at the origin between the first and third quadrants is 1. A similar approach reveals the slope of the tangent between the second and third quadrants is -1.

5.4 Related Rates

Practise

Section 5.4 Page 298 Question 1

$$\begin{aligned}\frac{dy}{dt} &= \frac{dx^2}{dx}\cdot\frac{dx}{dt}+\frac{d4x}{dx}\cdot\frac{dx}{dt}\\ &= 2x\frac{dx}{dt}+4\frac{dx}{dt}\\ &= \frac{dx}{dt}(2x+4) \qquad (1)\end{aligned}$$

Substitute $\frac{dx}{dt} = 10$ and $x = 5$ into (1).

$$\begin{aligned}\frac{dy}{dt} &= 10[2(5)+4]\\ &= 140\end{aligned}$$

Section 5.4 Page 298 Question 3

$$V = \pi r^2 h \qquad (1)$$

Substitute $r = h$ into (1).

$$\begin{aligned}V &= \pi(h)^2 h\\ V &= \pi h^3 \qquad (2)\end{aligned}$$

Differentiate (2) with respect to time.

$$\begin{aligned}\frac{dV}{dt} &= \frac{d\pi h^3}{dh}\cdot\frac{dh}{dt}\\ &= 3\pi h^2\frac{dh}{dt} \qquad (3)\end{aligned}$$

Substitute $\frac{dV}{dt} = -5$ and $h = 2$ into (3).

$$\begin{aligned}-5 &= 3\pi(2)^2\frac{dh}{dt}\\ \frac{dh}{dt} &= -\frac{5}{12\pi}\end{aligned}$$

Apply, Solve, Communicate

Section 5.4 Page 298 Question 4

Let A be the area of a square with side length x, in metres.

$$A = x^2 \qquad (1)$$

Differentiate both sides of (1) with respect to time, t.

$$\begin{aligned}\frac{dA}{dt} &= \frac{dx^2}{dx}\cdot\frac{dx}{dt}\\ &= 2x\frac{dx}{dt} \qquad (2)\end{aligned}$$

Substitute $x = 6$ and $\frac{dx}{dt} = 2$ in (2).

$$\begin{aligned}\frac{dA}{dt} &= 2(6)(2)\\ &= 24\end{aligned}$$

The area of the square is increasing at the rate of 24 m^2/min.

Section 5.4 Page 298 Question 5

This is similar to Example 1 on page 292. The baseball will be ejected with the same speed as a point on the circumference of either wheel. Thus,

$$\frac{dx}{dt} = 2\pi r\frac{dN}{dt} \qquad (1)$$

where $\frac{dx}{dt}$ is the speed of the ball, in metres per second, and $\frac{dN}{dt}$ is the rate of turning of the wheels, in revolutions per second. The speed of the ball is 60 km/h, or $\frac{50}{3}$ m/s, and the radius of the wheels is 60 cm, or 0.6 m. Substitute this information into (1).

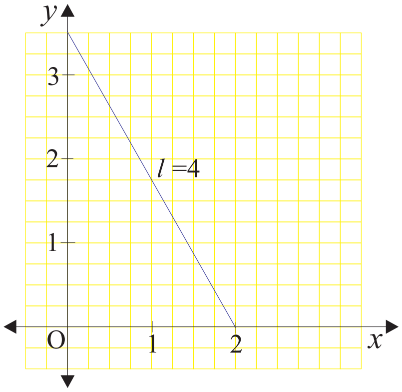
$$\begin{aligned}\frac{50}{3} &= 2\pi(0.6)\frac{dN}{dt}\\ \frac{dN}{dt} &= \frac{50}{3.6\pi}\\ &\doteq 4.42\end{aligned}$$

The wheels must turn at approximately 4.42 revolutions per second for the ball to be ejected at 60 km/h.

Section 5.4 Page 298 Question 6

The lumber, wall, and floor form a right triangle defined by the relation $x^2 + y^2 = 16$.

$$\begin{aligned}\frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}(16)\\ \frac{dx^2}{dx}\cdot\frac{dx}{dt} + \frac{dy^2}{dy}\cdot\frac{dy}{dt} &= \frac{d16}{dt}\\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0\\ \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt} \qquad (1)\end{aligned}$$



When $x = 2$, $y = \sqrt{16 - 2^2}$ or $2\sqrt{3}$. Substitute $x = 2$, $y = 2\sqrt{3}$, and $\frac{dx}{dt} = \frac{1}{4}$ into (1).

$$\begin{aligned}\frac{dy}{dt} &= -\frac{2}{2\sqrt{3}}\left(\frac{1}{4}\right)\\ &= -\frac{1}{4\sqrt{3}}\end{aligned}$$

The top of the ladder is sliding down the wall at a rate of $\frac{1}{4\sqrt{3}}$ or approximately 0.144 m/s.

Section 5.4 Page 298 Question 7

a) Since the surface area of the sun is increasing at the rate of 5000 km^2/year, we have

$$5000t = \Delta S \qquad (1)$$

To achieve a 10% increase in the radius of the sun, (1) becomes

$$\begin{aligned}5000t &= 4\pi(r + 0.1r)^2 - 4\pi r^2\\ &= 4\pi(1.1r)^2 - 4\pi r^2\\ &= 0.84\pi r^2\\ t &= \frac{0.84\pi r^2}{5000} \qquad (2)\end{aligned}$$

Substitute the radius of the sun today, $r = 700\,000$, into (2).

$$\begin{aligned}t &= \frac{0.84\pi(700\,000)^2}{5000}\\ &\doteq 258\,615\,907\end{aligned}$$

The sun's radius will increase by 10% in approximately 258 615 907 years.

b)

$$\begin{aligned}\frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)\\ \frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt}\end{aligned}$$

From Example 2, when $r = 1\,000\,000$ km, $\frac{dr}{dt} = \frac{5}{8000\pi}$ km/year .

$$\begin{aligned}\frac{dV}{dt} &= 4\pi(1\,000\,000)^2\left(\frac{5}{8000\pi}\right)\\ &= 2.5 \times 10^9\end{aligned}$$

The sun's volume is increasing at a rate of 2.5×10^9 km^3/year when $r = 1\,000\,000$ km.

c) First, determine the volume, V, of the sun today.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(700\,000)^3 \\ &\doteq 1.437 \times 10^{18} \end{aligned} \tag{1}$$

Let d be the density of the sun and m be mass of the sun.

$$d = \frac{m}{V} \tag{2}$$

Substitute $m = 2 \times 10^{30}$ and (1) into (2).

$$\begin{aligned} d &= \frac{2 \times 10^{30}}{1.437 \times 10^{18}} \\ &\doteq 1.39 \times 10^{12} \end{aligned}$$

The density of the sun is approximately 1.39×10^{12} kg/km^3.

d) So as not to confuse it with the differentiation symbol, let ρ represent density.

$$\begin{aligned} \frac{d}{dt}(\rho) &= \frac{d}{dt}\left(\frac{m}{V}\right) \\ \frac{d\rho}{dt} &= m \cdot \frac{dV^{-1}}{dV} \cdot \frac{dV}{dt} \\ \rho &= m \cdot (-1)V^{-2} \cdot \frac{dV}{dt} \\ \frac{d\rho}{dt} &= -\frac{m}{V^2} \cdot \frac{dV}{dt} \end{aligned} \tag{1}$$

Substitute $m = 2 \times 10^{30}$, $V = 1.437 \times 10^{18}$, and $\frac{dV}{dt} = 2.5 \times 10^9$ into (1).

$$\begin{aligned} \frac{d\rho}{dt} &= -\frac{2 \times 10^{30}}{\left(1.437 \times 10^{18}\right)^2} \cdot (2.5 \times 10^9) \\ &\doteq -2421 \end{aligned}$$

The density of the sun is decreasing at a rate of approximately 2421 $\frac{\text{kg}}{\text{km}^3 \cdot \text{year}}$.

Section 5.4 Page 298 Question 8

a) The volume, V, of water can be expressed as

$$V = \frac{1}{3}\pi r^2 h \qquad (1)$$

Simplify (1). Since,

$$\begin{aligned} \frac{r}{h} &= \frac{3}{10} \\ r &= \frac{3h}{10} \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h \\ &= \frac{3\pi}{100}h^3 \qquad (3) \end{aligned}$$

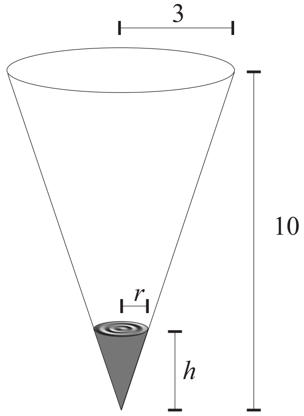
Differentiate (3) with respect to time, t.

$$\begin{aligned} \frac{dV}{dt} &= \frac{3\pi}{100}\cdot\frac{dh^3}{dh}\cdot\frac{dh}{dt} \\ &= \frac{9\pi}{100}\cdot h^2\cdot\frac{dh}{dt} \qquad (4) \end{aligned}$$

Substitute $\frac{dV}{dt} = -1$ m^3/min and $h = 2$ m into (4).

$$\begin{aligned} -1 &= \frac{9\pi}{100}\cdot(2)^2\cdot\frac{dh}{dt} \\ -1 &= \frac{9\pi}{25}\cdot\frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{25}{9\pi} \end{aligned}$$

The level of the water is decreasing at the rate of $\frac{25}{9\pi}$ m/min.



b) From (4),

$$\frac{dV}{dt} = \frac{9\pi}{100}\cdot h^2\cdot\frac{dh}{dt} \qquad (4)$$

Substitute $\frac{dV}{dt} = 1.5$ m^3/min and $h = 7$ m into (4).

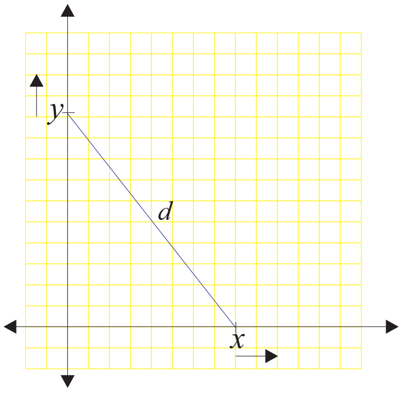
$$\begin{aligned} 1.5 &= \frac{9\pi}{100}\cdot(7)^2\cdot\frac{dh}{dt} \\ 1.5 &= \frac{441\pi}{100}\cdot\frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{50}{147\pi} \end{aligned}$$

The water level is rising at $\frac{50}{147\pi}$ m/min.

Section 5.4 Page 299 Question 9

The positions of Coffee World, Kruno, and Zarko form a right triangle defined by the relation $x^2 + y^2 = s^2$.

$$\begin{aligned} \frac{d}{dt}(x^2+y^2) &= \frac{d}{dt}(s^2) \\ \frac{dx^2}{dx}\cdot\frac{dx}{dt} + \frac{dy^2}{dy}\cdot\frac{dy}{dt} &= \frac{ds^2}{ds}\cdot\frac{ds}{dt} \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2s\frac{ds}{dt} \\ x\frac{dx}{dt} + y\frac{dy}{dt} &= s\frac{ds}{dt} \qquad (1) \end{aligned}$$



Two minutes after they leave Coffee World, Kruno has bladed 40 m and Zarko has cycled 52 m. At this instant, the distance, s, between them is $\sqrt{40^2 + 52^2}$ or $\sqrt{4304}$ m. Perform the respective substitutions into (1).

$$\begin{aligned} 40(20) + 52(26) &= \sqrt{4304}\frac{ds}{dt} \\ \frac{ds}{dt} &= \frac{2152}{\sqrt{4304}} \\ &\doteq 32.8 \end{aligned}$$

The distance between Kruno and Zarko is increasing at a rate of approximately 32.8 m/min.

Section 5.4 Page 299 Question 10

The area, A, of a circle of radius r is given by $A = \pi r^2$.

$$
\begin{aligned}
A &= \pi r^2 \qquad (1)\\
\frac{d}{dt}(A) &= \frac{d}{dt}(\pi r^2)\\
\frac{dA}{dt} &= \pi \cdot \frac{dr^2}{dr} \cdot \frac{dr}{dt}\\
&= 2\pi r \frac{dr}{dt} \qquad (2)
\end{aligned}
$$

After 4 s, the radius of the circle is 4 × 3 or 12 cm. Substitute $\frac{dr}{dt} = 3$ and $r = 12$ into (2).

$$
\begin{aligned}
\frac{dA}{dt} &= 2\pi(12)(3)\\
&= 72\pi
\end{aligned}
$$

The area within the circle is increasing at a rate of 72π cm^2/s after 4 s.

Section 5.4 Page 299 Question 11

For parts a), b), and c), use the model $x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$ developed in Example 4. All distances are measured in kilometres.

a) Given $x = 0.45$ and $y = 0.6$, determine z.

$$
\begin{aligned}
z^2 &= x^2 + y^2\\
z^2 &= 0.45^2 + 0.6^2\\
&= 0.5625\\
z &= \pm 0.75
\end{aligned}
$$

Distances are positive, so $z = 0.75$. Solve for $\frac{dz}{dt}$.

$$
\begin{aligned}
(0.45)(-45) + (0.6)(-60) &= (0.75)\frac{dz}{dt}\\
\frac{dz}{dt} &= -75
\end{aligned}
$$

The distance between the two cars is decreasing at a rate of 75 km/h.

b) Given $x = 0.3$ and $y = 0.4$, determine z.

$$
\begin{aligned}
z^2 &= x^2 + y^2\\
z^2 &= 0.3^2 + 0.4^2\\
&= 0.25\\
z &= \pm 0.5
\end{aligned}
$$

Distances are positive, so $z = 0.5$. Solve for $\frac{dz}{dt}$.

$$
\begin{aligned}
(0.3)(-45) + (0.4)(-60) &= (0.5)\frac{dz}{dt}\\
\frac{dz}{dt} &= -75
\end{aligned}
$$

The distance between the two cars is decreasing at a rate of 75 km/h.

c) Given $x = 0.15$ and $y = 0.2$, determine z.

$$
\begin{aligned}
z^2 &= x^2 + y^2\\
z^2 &= 0.15^2 + 0.2^2\\
&= 0.0625\\
z &= \pm 0.25
\end{aligned}
$$

Distances are positive, so $z = 0.25$.

Solve for $\frac{dz}{dt}$.

$$
\begin{aligned}
(0.15)(-45) + (0.2)(-60) &= (0.25)\frac{dz}{dt}\\
\frac{dz}{dt} &= -75
\end{aligned}
$$

The distance between the two cars is decreasing at a rate of 75 km/h.

d) The results are equal because the initial positions of the cars are in the ratio 3 : 4 for all three parts.

e) Answers will vary.

f) Let the starting position of A be $(x_0, 0)$ and the starting position of B be $(0, y_0)$. In Example 4 and in parts a), b), and c), $\frac{x_0}{y_0} = \frac{3}{4}$, or $y_0 = \frac{4}{3}x_0$. Since vehicle A is travelling toward the origin at 45 km/h, $x = x_0 - 45t$. Since vehicle B is travelling toward the origin at 60 km/h, $y = y_0 - 60t$.

$$\begin{aligned} z^2 &= x^2 + y^2 \\ &= (x_0 - 45t)^2 + (y_0 - 60t)^2 \\ &= (x_0 - 45t)^2 + \left(\frac{4}{3}x_0 - 60t\right)^2 \\ &= {x_0}^2 - 90x_0t + 2025t^2 + \frac{16}{9}{x_0}^2 - 160x_0t + 3600t^2 \\ &= \frac{25}{9}{x_0}^2 - 250x_0t + 5625t^2 \\ &= \frac{25}{9}({x_0}^2 - 90x_0t + 2025) \\ &= \frac{25}{9}(x_0 - 45t)^2 \\ z &= \frac{5}{3}(x_0 - 45t) \\ \frac{dz}{dt} &= \frac{5}{3}(-45) \\ &= -75 \end{aligned}$$

Thus, if the starting positions are in the ratio 3 : 4, the distance between the cars is decreasing at a rate of 75 km/h.

Section 5.4 Page 299 Question 12

a) The volume, V, of a cube of side length x is given by $V = x^3$.

$$\begin{aligned} V &= x^3 && (1) \\ \frac{d}{dt}(V) &= \frac{d}{dt}(x^3) \\ \frac{dV}{dt} &= \frac{dx^3}{dx} \cdot \frac{dx}{dt} \\ \frac{dV}{dt} &= (3x^2)\frac{dx}{dt} && (2) \end{aligned}$$

Substitute $x = 2.3$ and $\frac{dx}{dt} = 0.01$ into (2).

$$\begin{aligned} \frac{dV}{dt} &= 3(2.3)^2(0.01) \\ &= 0.1587 && (3) \end{aligned}$$

The volume of the cube is increasing at a rate of 0.1587 mm^3/s.

b) Let v represent the value of the crystal, in dollars.

$$\begin{aligned} v &= 4V \\ \frac{d}{dt}(v) &= \frac{d}{dt}(4V) \\ \frac{dv}{dt} &= (4)\frac{dV}{dt} && (4) \end{aligned}$$

Substitute (2) into (4).

$$\begin{aligned} \frac{dv}{dt} &= 4(3x^2)\frac{dx}{dt} \\ &= 12x^2\frac{dx}{dt} && (5) \end{aligned}$$

Substitute $x = 2.7$ and $\frac{dx}{dt} = 0.01$ into (5).

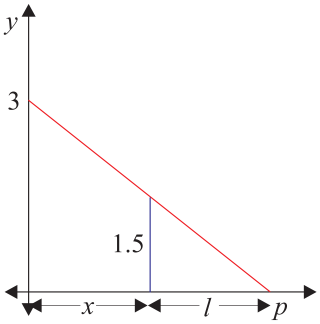
$$\begin{aligned} \frac{dv}{dt} &= 12(2.7)^2(0.01) \\ &\doteq 0.87 \end{aligned}$$

The value of the crystal is increasing at the rate of \$0.87.

Section 5.4 Page 299 Question 13

Let l be the length of Faye's shadow. Let x be the distance from Faye to the base of the sensor light. Faye's walking rate can be expressed as $\frac{dx}{dt} = 0.5$ m/s. Use properties of similar triangles to obtain $l(x)$.

$$\frac{l}{1.5} = \frac{l+x}{3}$$
$$3l = 1.5(l+x)$$
$$2l = l + x$$
$$l = x \qquad (1)$$



Let p be the position of the tip of Faye's shadow.

$$p = x + l \qquad (2)$$

Substitute (1) into (2).

$$p = x + x$$
$$p = 2x \qquad (3)$$

Differentiate (3) with respect to time, t.

$$\frac{dp}{dt} = (2)\frac{dx}{dt} \qquad (4)$$

Substitute $\frac{dx}{dt} = 0.5$ into (4).

$$\frac{dp}{dt} = 2(0.5)$$
$$= 1$$

The tip of Faye's shadow is moving at a rate of 1 m/s.

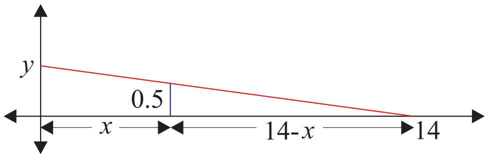
Section 5.4 Page 299 Question 14

Let x be the distance from the dog to the building. The dog's running rate can be expressed as $\frac{dx}{dt} = -1$ m/s. Let y be the height of the dog's shadow on the building. Use properties of similar triangles to obtain $y(x)$.

$$\frac{y}{14} = \frac{0.5}{14-x}$$
$$y = \frac{7}{14-x} \qquad (1)$$

Differentiate (1) with respect to time, t.

$$\frac{dy}{dt} = 7 \cdot \frac{d(14-x)^{-1}}{d14-x} \cdot \frac{d14-x}{dx} \cdot \frac{dx}{dt}$$
$$= -7(14-x)^{-2}(-1) \cdot \frac{dx}{dt}$$
$$= \frac{7}{(14-x)^2} \cdot \frac{dx}{dt} \qquad (2)$$



Substitute $x = 5$ and $\frac{dx}{dt} = -1$ into (2).

$$\frac{dy}{dt} = \frac{7}{(14-5)^2}(-1)$$
$$= -\frac{7}{81}$$

The height of the dog's shadow on the building is decreasing at the rate of $\frac{7}{81}$ m/s when the dog is 5 m from the building.

Section 5.4 Page 299 Question 15

a) The volume of the volcano is given by $V = \frac{1}{3}\pi r^2 h$. But the height is 50% greater than the base radius, so $h = 1.5r$, or $r = \frac{2}{3}h$. Thus,

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h \\ &= \frac{4}{27}\pi h^3 \qquad (1) \end{aligned}$$

Differentiate (1) with respect to time, t.

$$\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt} \qquad (2)$$

Substitute $h = 700$ and $\frac{dh}{dt} = 0.01$ into (2).

$$\begin{aligned} \frac{dV}{dt} &= \frac{4}{9}\pi(700)^2(0.01) \\ &= \frac{19\ 600\pi}{9} \end{aligned}$$

Since lava erupts from the volcano at 20 000 m^3/s, not all of it is staying on the slopes,

$$20\ 000 - \frac{19\ 600\pi}{9} \doteq 13\ 158.3$$

When the height is 700 m, the rate of lava spreading beyond the slopes is approximately 13 158.3 m^3/s.

b) The volume of lava around the slope is given by the thickness times the surface area of the lava. The surface area is $\pi R^2 - \pi r^2$, where R is the radius of the lava beyond the slopes, and r is the base radius of the volcano. Thus,

$$\begin{aligned} V &= 0.2\pi(R^2 - r^2) \\ &= 0.2\pi\left(R^2 - \left(\frac{2}{3}h\right)^2\right) \\ &= 0.2\pi\left(R^2 - \frac{4}{9}h^2\right) \qquad (3) \end{aligned}$$

Differentiate both sides of (3) with respect to time, t.

$$\frac{dV}{dt} = 0.2\pi\left(2R\frac{dh}{dt} - \frac{8}{9}h\frac{dh}{dt}\right) \qquad (4)$$

Substitute $\frac{dV}{dt} = 13\ 158.3$, $R = 500$, $h = 700$, and $\frac{dh}{dt} = 0.01$ into (4).

$$\begin{aligned} 13\ 158.3 &= 0.2\pi\left(2(500)\frac{dR}{dt} - \frac{8}{9}(700)(0.01)\right) \\ \frac{dR}{dt} &\doteq 21.0 \end{aligned}$$

The lava flow is approaching the villages at a rate of 21.0 m/s.

Review of Key Concepts

5.1 Composite Functions

Section Review Page 301 Question 1

a)
$$\begin{aligned} f(2) &= \frac{2+4}{2} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

b)
$$\begin{aligned} (g \circ f)(2) &= g(f(2)) \\ &= g(3) \\ &= 2(3) + 3 \\ &= 6 + 3 \\ &= 9 \end{aligned}$$

c)
$$\begin{aligned} g(-4) &= 2(-4) + 3 \\ &= -8 + 3 \\ &= -5 \end{aligned}$$

d)
$$\begin{aligned} (f \circ g)(-4) &= f(g(-4)) \\ &= f(-5) \\ &= \frac{-5+4}{-5} \\ &= \frac{1}{5} \end{aligned}$$

e)
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f\,(2x+3) \\ &= \frac{(2x+3)+4}{2x+3} \\ &= \frac{2x+7}{2x+3} \end{aligned}$$

f)
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g\left(\frac{x+4}{x}\right) \\ &= 2\left(\frac{x+4}{x}\right) + \frac{3x}{x} \\ &= \frac{5x+8}{x} \end{aligned}$$

g)
$$\begin{aligned} (f \circ f)(x) &= f(f(x)) \\ &= f\left(\frac{x+4}{x}\right) \\ &= \frac{\frac{x+4}{x} + 4}{\frac{x+4}{x}} \\ &= \frac{\frac{x+4+4x}{x}}{\frac{x+4}{x}} \\ &= \frac{5x+4}{x+4},\ x \neq 0 \end{aligned}$$

h)
$$\begin{aligned} (g \circ g)(-11) &= g(g(-11)) \\ &= g\,(2(-11) + 3) \\ &= g(-19) \\ &= 2(-19) + 3 \\ &= -38 + 3 \\ &= -35 \end{aligned}$$

Section Review Page 301 Question 2

$g(-4) = -3$, and -3 is not in the domain of $f(x)$.

Section Review Page 301 Question 3

Answers may vary.

If $g(x) = x + 12$ and $f(x) = \sqrt[3]{x}$, then

$$\begin{aligned} F(x) &= (f \circ g)(x) \\ &= f(g(x)) \\ &= f(x+12) \\ &= \sqrt[3]{x+12} \end{aligned}$$

Section Review Page 301 Question 4

$$\begin{aligned} H(x) &= \frac{1}{3x+2} \\ f(g(x)) &= \frac{1}{3x+2} \\ f(3x+2) &= \frac{1}{3x+2} \qquad (1) \end{aligned}$$

By comparison of both sides of (1), $f(x) = \frac{1}{x}$.

Section Review Page 301 Question 5

a) The radius can be expressed as $r = 0.2t$ or $r = \dfrac{t}{5}$.

b) The volume of the hailstone can be expressed as $V = \dfrac{4}{3}\pi r^3$.

c)

$$\begin{aligned}(V \circ r)(t) &= V(r(t))\\ &= V\left(\frac{t}{5}\right)\\ &= \frac{4}{3}\pi\left(\frac{t}{5}\right)^3\\ &= \frac{4\pi t^3}{375}\end{aligned}$$

$V(t)$ represents the volume, in cubic millimetres, after t minutes.

d)

$$\begin{aligned}(V \circ r)(60) &= \frac{4\pi(60)^3}{375}\\ &\doteq 7238\end{aligned}$$

The volume of the hailstone after 1 h is approximately 7238 mm^3.

Section Review Page 301 Question 6

a) Determine $f^{-1}(x)$.

$$\begin{aligned}f(x) &= x^2 + 5,\ x \in [0, +\infty)\\ y &= x^2 + 5,\ x \in [0, +\infty)\end{aligned}$$

Interchange x and y.

$$x = y^2 + 5,\ y \in [0, +\infty)$$

Solve for y.

$$\begin{aligned}y^2 &= x - 5,\ y \in [0, +\infty)\\ y &= \sqrt{x-5}\\ f^{-1}(x) &= \sqrt{x-5}\end{aligned}$$

Determine $g^{-1}(x)$.

$$\begin{aligned}g(x) &= -3x + 1\\ y &= -3x + 1\end{aligned}$$

Interchange x and y.

$$x = -3y + 1$$

Solve for y.

$$\begin{aligned}3y &= -x + 1\\ y &= \frac{-x+1}{3}\\ g^{-1}(x) &= \frac{-x+1}{3}\end{aligned}$$

i)

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x))\\ &= f\left(\sqrt{x-5}\right)\\ &= \left(\sqrt{x-5}\right)^2 + 5\\ &= x - 5 + 5\\ &= x\end{aligned}$$

ii)

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x))\\ &= f^{-1}\left(x^2 + 5\right)\\ &= \sqrt{(x^2 - 5) + 5}\\ &= \sqrt{x^2}\\ &= x,\ x \in [0, +\infty)\end{aligned}$$

iii)

$$\begin{aligned}(g \circ g^{-1})(x) &= g(g^{-1}(x))\\ &= g\left(\frac{-x+1}{3}\right)\\ &= -3\left(\frac{-x+1}{3}\right) + 1\\ &= x - 1 + 1\\ &= x\end{aligned}$$

iv)

$$\begin{aligned}(g^{-1} \circ g)(x) &= g^{-1}(g(x))\\ &= g^{-1}(-3x + 1)\\ &= \frac{-(-3x+1)+1}{3}\\ &= \frac{3x}{3}\\ &= x\end{aligned}$$

b) The composition of a function with its inverse results in x.

c) Arguments will vary.

5.2 The Chain Rule

Section Review Page 301 Question 7

a)

$$\begin{aligned} y &= (x^2 - 5x)^3 \\ y &= (x^2)^3 + 3(x^2)^2(-5x) + 3(x^2)(-5x)^2 + (-5x)^3 \\ y &= x^6 - 15x^5 + 75x^4 - 125x^3 \\ \frac{dy}{dx} &= \frac{d}{dx}(x^6 - 15x^5 + 75x^4 - 125x^3) \\ &= 6x^5 - 75x^4 + 300x^3 - 375x^2 \end{aligned}$$

b)

$$\begin{aligned} y &= (x^2 - 5x)^3 \\ \frac{dy}{dx} &= \frac{d}{dx}((x^2 - 5x)^3) \\ &= \frac{d(x^2 - 5x)^3}{d(x^2 - 5x)} \cdot \frac{d(x^2 - 5x)}{dx} \\ &= 3(x^2 - 5x)^2(2x - 5) \end{aligned}$$

c) The answer in part b) expands to the answer in part a). Explanations will vary.

Section Review Page 301 Question 8

a)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(2x+8)^2}{d\,2x+8} \cdot \frac{d\,2x+8}{dx} \\ &= 2(2x+8) \cdot 2 \\ &= 8(x+4) \end{aligned}$$

b)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^2+7x)^{\frac{1}{2}}}{d\,x^2+7x} \cdot \frac{d\,x^2+7x}{dx} \\ &= \frac{1}{2}(x^2+7x)^{-\frac{1}{2}} \cdot (2x+7) \\ &= \frac{2x+7}{2\sqrt{x^2+7x}} \end{aligned}$$

c)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^4+5x)^{-3}}{d\,x^4+5x} \cdot \frac{dx^4+5x}{dx} \\ &= -3(x^4+5x)^{-4} \cdot (4x^3+5) \\ &= -\frac{4x^3+5}{3(x^4+5x)^4} \end{aligned}$$

d)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(2x^8-2)^{\frac{1}{3}}}{d\,2x^8-2} \cdot \frac{d\,2x^8-2}{dx} \\ &= \frac{1}{3}(2x^8-2)^{-\frac{2}{3}} \cdot 16x^7 \\ &= \frac{16x^7}{3\sqrt[3]{(2x^8-2)^2}} \end{aligned}$$

e)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(6x^2+4x)^{\frac{1}{4}}}{d\,6x^2+4x} \cdot \frac{d\,6x^2+4x}{dx} \\ &= \frac{1}{4}(6x^2+4x)^{-\frac{3}{4}} \cdot (12x+4) \\ &= \frac{3x+1}{\sqrt[4]{(6x^2+4x)^3}} \end{aligned}$$

Section Review Page 301 Question 9

a)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^4-2x^3)^2}{d\,x^4-2x^3} \cdot \frac{d\,x^4-2x^3}{dx} \\ &= 2(x^4-2x^3)(4x^3-6x^2) \\ &= 4x^5(x-2)(2x-3) \end{aligned}$$

b)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^2+8x-6)^{\frac{1}{2}}}{d\,x^2+8x-6} \cdot \frac{d\,x^2+8x-6}{dx} \\ &= \frac{1}{2}(x^2+8x-6)^{-\frac{1}{2}} \cdot (2x+8) \\ &= \frac{x+4}{\sqrt{x^2+8x-6}} \end{aligned}$$

c)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^3+9x)^5}{d\,x^3+9x} \cdot \frac{d\,x^3+9x}{dx} \\ &= 5(x^3+9x)^4(3x^2+9) \end{aligned}$$

d)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^2+x-10)^3}{d\,x^2+x-10} \cdot \frac{d\,x^2+x-10}{dx} \\ &= 3(x^2+x-10)^2(2x+1) \end{aligned}$$

e)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2+12)^{\frac{1}{5}}}{dx^2+12}\cdot\frac{dx^2+12}{dx}\\ &= \frac{1}{5}(x^2+12)^{-\frac{4}{5}}\cdot(2x)\\ &= \frac{2x}{5\sqrt[5]{(x^2+12)^4}}\end{aligned}$$

f)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^3-4x)^{-1}}{dx^3-4x}\cdot\frac{dx^3-4x}{dx}\\ &= -1(x^3-4x)^{-2}\cdot(3x^2-4)\\ &= -\frac{3x^2-4}{(x^3-4x)^2}\end{aligned}$$

g)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(5x-2)^{-\frac{1}{2}}}{d5x-2}\cdot\frac{d5x-2}{dx}\\ &= -\frac{1}{2}(5x-2)^{-\frac{3}{2}}\cdot(5)\\ &= \frac{-5}{2\sqrt{(5x-2)^3}}\end{aligned}$$

h)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^3+6x)^{\frac{1}{3}}}{d\,x^3+6x}\cdot\frac{d\,x^3+6x}{dx}\\ &= \frac{1}{3}(x^3+6x)^{-\frac{2}{3}}\cdot(3x^2+6)\\ &= \frac{x^2+2}{\sqrt[3]{(x^3+6x)^2}}\end{aligned}$$

Section Review Page 301 Question 10

a)
$$\begin{aligned}A &= \pi r^2\\ \frac{dA}{dt} &= \frac{d\pi r^2}{dr}\cdot\frac{dr}{dt}\\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt} \qquad (1)\end{aligned}$$

Substitute $r = 40$ and $\frac{dr}{dt} = 0.003$ into (1).

$$\begin{aligned}\frac{dA}{dt} &= 2\pi(40)(0.003)\\ &= 0.24\pi \qquad (2)\end{aligned}$$

When $r = 40$ m, the area is increasing at the rate of 0.24π m^2/min or approximately 0.754 m^2/min.

b) Determine the radius of the circular bloom, five hours later.

$$\begin{aligned}A &= \pi(40)^2 + 5(60)(0.24\pi)\\ &= 1672\pi\end{aligned}$$

Determine the radius of the bloom at this time.

$$\begin{aligned}\pi r^2 &= 1672\pi\\ r &= \sqrt{1672} \qquad (3)\end{aligned}$$

Substitute (2) and (3) into (1).

$$\begin{aligned}0.24\pi &= 2\pi\sqrt{1672}\frac{dr}{dt}\\ \frac{dr}{dt} &= \frac{0.12}{\sqrt{1672}}\\ &\doteq 0.002\,93\end{aligned}$$

Five hours later, the radius is increasing at the rate of approximately 2.93 mm/min.

5.3 Implicit Differentiation

Section Review Page 301 Question 11

a) i)
$$\begin{aligned}9x^2 + y^2 &= 36\\ y^2 &= 36 - 9x^2\\ y &= \pm\sqrt{36-9x^2}\\ \frac{dy}{dx} &= \pm\frac{d(36-9x^2)^{\frac{1}{2}}}{d36-9x^2}\cdot\frac{d36-9x^2}{dx}\\ &= \pm\frac{1}{2}(36-9x^2)^{-\frac{1}{2}}(-18x)\\ &= \mp\frac{9x}{\sqrt{36-9x^2}}\end{aligned}$$

ii)
$$\begin{aligned}\frac{d}{dx}(9x^2+y^2) &= \frac{d}{dx}(36)\\ \frac{d9x^2}{dx} + \frac{dy^2}{dy}\cdot\frac{dy}{dx} &= \frac{d36}{dx}\\ 18x + 2y\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{-18x}{2y}\\ \frac{dy}{dx} &= -\frac{9x}{y}\end{aligned}$$

b) If the relation cannot be expressed explicitly in terms of x, then implicit differentiation is the only possible way to find $\frac{dy}{dx}$, without using technology.

Section Review Page 302 Question 12

a)

$$\begin{aligned}\frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}25\\ \frac{dx^2}{dx}+\frac{dy^2}{dy}\cdot\frac{dy}{dx} &= \frac{d25}{dx}\\ 2x+2y\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{x}{y}\end{aligned}$$

b)

$$\begin{aligned}\frac{d}{dx}\left(x^3y^2+4x^2y\right) &= \frac{d}{dx}(12)\\ y^2\frac{d\,x^3}{dx}+x^3\frac{d\,y^2}{dy}\cdot\frac{dy}{dx}+y\frac{d4x^2}{dx}+4x^2\frac{dy}{dx} &= \frac{d\,12}{dx}\\ y^2(3x^2)+x^3(2y)\frac{dy}{dx}+y(8x)+4x^2\frac{dy}{dx} &= 0\\ \frac{dy}{dx}(2x^3y+4x^2) &= -3x^2y^2-8xy\\ \frac{dy}{dx} &= -\frac{y(3xy+8)}{2x(xy+2)},\ x\neq 0\end{aligned}$$

c)

$$\begin{aligned}\frac{d}{dx}\left(x^2y^3+2xy\right) &= \frac{d}{dx}(20)\\ y^3\frac{d\,x^2}{dx}+x^2\frac{d\,y^3}{dy}\cdot\frac{dy}{dx}+y\frac{d2x}{dx}+2x\frac{dy}{dx} &= \frac{d\,20}{dx}\\ y^3(2x)+x^2(3y^2)\frac{dy}{dx}+y(2)+2x\frac{dy}{dx} &= 0\\ \frac{dy}{dx}(3x^2y^2+2x) &= -2xy^3-2y\\ \frac{dy}{dx} &= -\frac{2y(xy^2+1)}{x(3xy^2+2)}\end{aligned}$$

d)

$$\begin{aligned}\frac{d}{dx}\left(2y^3+x^2y^2\right) &= \frac{d}{dx}(29)\\ \frac{d2y^3}{dy}\cdot\frac{dy}{dx}+y^2\frac{dx^2}{dx}+x^2\frac{dy^2}{dy}\cdot\frac{dy}{dx} &= \frac{d29}{dx}\\ 6y^2\frac{dy}{dx}+y^2(2x)+x^2(2y)\frac{dy}{dx} &= 0\\ \frac{dy}{dx}(6y^2+2x^2y) &= -2xy^2\\ \frac{dy}{dx} &= -\frac{xy}{3y+x^2}\end{aligned}$$

e)

$$\begin{aligned}\frac{d}{dx}(x-2)^2+(y+7)^3 &= \frac{d}{dx}(64)\\ \frac{d(x-2)^2}{dx}+\frac{d(y+7)^3}{dy}\cdot\frac{dy}{dx} &= \frac{d64}{dx}\\ 2(x-2)+3(y+7)^2\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{2(x-2)}{3(y+7)^2}\end{aligned}$$

f)

$$\begin{aligned}\frac{d}{dx}(x^2+y^3x-6x+8y) &= \frac{d}{dx}(9)\\ \frac{dx^2}{dx}+y^3\frac{dx}{dx}+x\frac{dy^3}{dy}\cdot\frac{dy}{dx}-\frac{d6x}{dx}+\frac{d8y}{dy}\cdot\frac{dy}{dx} &= \frac{d9}{dx}\\ 2x+y^3(1)+x(3y^2)\frac{dy}{dx}-6+8\frac{dy}{dx} &= 0\\ \frac{dy}{dx}(3xy^2+8) &= -2x-y^3+6\\ \frac{dy}{dx} &= -\frac{2x+y^3-6}{3xy^2+8}\end{aligned}$$

Section Review Page 302 Question 13

a) i)

$$\frac{d}{dx}\left((x+1)^2+(y+4)^2\right)=\frac{d}{dx}(13)$$

$$\frac{d\,(x+1)^2}{dx}+\frac{d\,(y+4)^2}{dy}\cdot\frac{dy}{dx}=\frac{d\,13}{dx}$$

$$2(x+1)+2(y+4)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x+1}{y+4} \qquad (1)$$

Substitute $(x, y) = (2, -2)$ into (1).

$$\frac{dy}{dx}=-\frac{2+1}{-2+4}=-\frac{3}{2}$$

The slope of the tangent at $(2, -2)$ is $-\frac{3}{2}$.

ii) Determine the equation of the tangent.

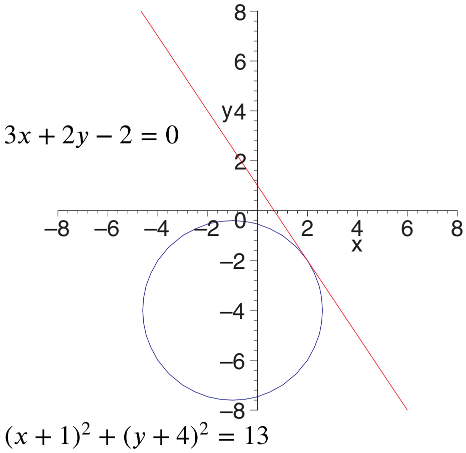
$$y-(-2)=-\frac{3}{2}(x-2)$$

$$2y+4=-3x+6$$

$$3x+2y-2=0$$

The equation of the tangent is $3x+2y-2=0$.

iii)



b) i)

$$\frac{d}{dx}\left(y^3+2xy\right)=\frac{d}{dx}(20)$$

$$\frac{dy^3}{dy}\cdot\frac{dy}{dx}+2\left(y\frac{dx}{dx}+x\frac{dy}{dx}\right)=\frac{d\,20}{dx}$$

$$3y^2\frac{dy}{dx}+2y+2x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2y}{3y^2+2x} \qquad (1)$$

Substitute $(x, y) = (3, 2)$ into (1).

$$\frac{dy}{dx}=\frac{-4}{12+6}=-\frac{2}{9}$$

The slope of the tangent at $(3, 2)$ is $-\frac{2}{9}$.

ii) Determine the equation of the tangent.

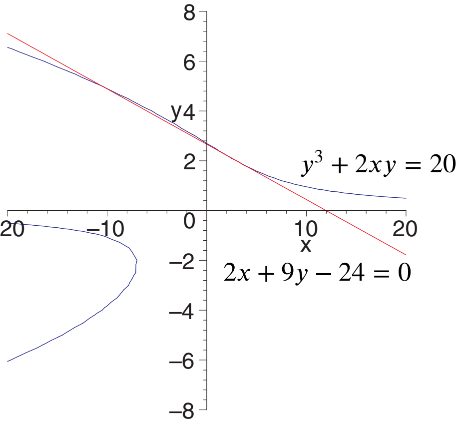
$$y-2=-\frac{2}{9}(x-3)$$

$$9y-18=-2x+6$$

$$2x+9y-24=0$$

The equation of the tangent is $2x+9y-24=0$.

iii)



c) i)

$$\frac{d}{dx}\left(x^3y - y^3\right) = \frac{d}{dx}(60)$$
$$y\frac{dx^3}{dx} + x^3\frac{dy}{dx} - \frac{dy^3}{dy}\cdot\frac{dy}{dx} = \frac{d\,60}{dx}$$
$$3x^2y + x^3\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{3x^2y}{3y^2 - x^3} \qquad (1)$$

Substitute $(x, y) = (1, -4)$ into (1).

$$\frac{dy}{dx} = \frac{3(1)^2(-4)}{3(-4)^2 - 1^3}$$
$$= \frac{-12}{47}$$

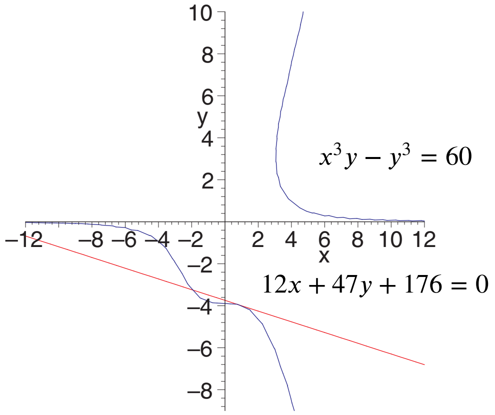
The slope of the tangent at $(1, -4)$ is $-\frac{12}{47}$.

ii) Determine the equation of the tangent.

$$y - (-4) = -\frac{12}{47}(x - 1)$$
$$47y + 188 = -12x + 12$$
$$12x + 47y + 176 = 0$$

The equation of the tangent is $12x+47y+176 = 0$.

iii)



d) i)

$$\frac{d}{dx}\left(x^2y^3 + xy + 30\right) = \frac{d}{dx}(0)$$
$$y^3\frac{dx^2}{dx} + x^2\frac{dy^3}{dy}\cdot\frac{dy}{dx} + y\frac{dx}{dx} + x\frac{dy}{dx} = \frac{d(-30)}{dx}$$
$$2xy^3 + 3x^2y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2xy^3 - y}{3x^2y^2 + x} \qquad (1)$$

Substitute $(x, y) = (-6, -1)$ into (1).

$$\frac{dy}{dx} = \frac{-2(-6)(-1)^3 - (-1)}{3(-6)^2(-1)^2 + (-6)}$$
$$= -\frac{11}{102}$$

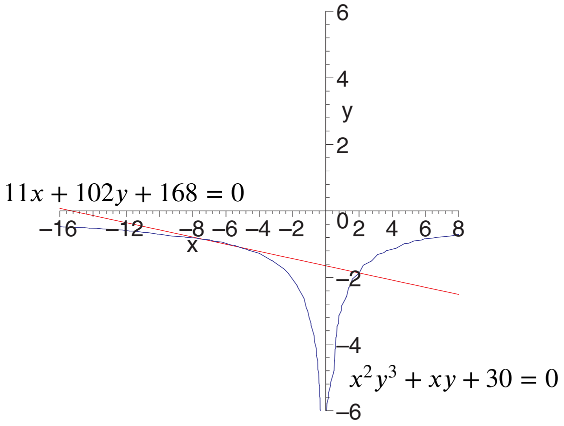
The slope of the tangent at $(-6, -1)$ is $-\frac{11}{102}$.

ii) Determine the equation of the tangent.

$$y - (-1) = -\frac{11}{102}(x - (-6))$$
$$102y + 102 = -11x - 66$$
$$11x + 102y + 168 = 0$$

The equation of the tangent is $11x + 102y + 168 = 0$.

iii)



5.4 Related Rates

Section Review Page 302 Question 14

$$V = \frac{1}{3}\pi r^2 h \qquad (1)$$

Substitute $r = h$ into (1).

$$V = \frac{1}{3}\pi(h)^2 h$$

$$V = \frac{1}{3}\pi h^3 \qquad (2)$$

Differentiate (2) with respect to time.

$$\frac{dV}{dt} = \frac{1}{3}\frac{d\pi h^3}{dh} \cdot \frac{dh}{dt}$$

$$= \pi h^2 \frac{dh}{dt} \qquad (3)$$

Substitute $\frac{dV}{dt} = 4$ and $h = 6$ into (3).

$$4 = \pi(6)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{9\pi}$$

Section Review Page 302 Question 15

Let x be the distance from the man to the building. The man's walking rate can be expressed as $\frac{dx}{dt} = -0.75$ m/s. Let y be the height of the man's shadow on the building. Use properties of similar triangles to obtain $y(x)$.

$$\frac{y}{12} = \frac{2}{12 - x}$$

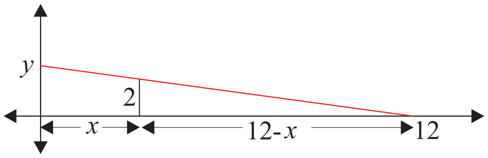
$$y = \frac{24}{12 - x} \qquad (1)$$

Differentiate (3) with respect to time, t.

$$\frac{dy}{dt} = 24 \cdot \frac{d(12 - x)^{-1}}{d12 - x} \cdot \frac{d12 - x}{dx} \cdot \frac{dx}{dt}$$

$$= -24(12 - x)^{-2}(-1) \cdot \frac{dx}{dt}$$

$$= \frac{24}{(12 - x)^2} \cdot \frac{dx}{dt} \qquad (2)$$



Substitute $x = 4$ and $\frac{dx}{dt} = -0.75$ into (2).

$$\frac{dy}{dt} = \frac{24}{(12 - 4)^2}(-0.75)$$

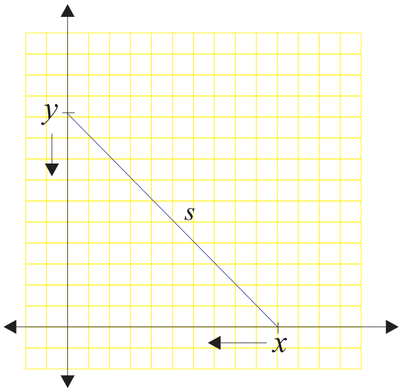
$$= -\frac{9}{32}$$

The height of the man's shadow on the building is decreasing at the rate of $\frac{9}{32}$ m/s when the man is 4 m from the building.

Section Review Page 302 Question 16

The positions of the park, Cassandra, and Marissa form a right triangle defined by the relation $x^2 + y^2 = s^2$.

$$\begin{aligned}\frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}(s^2)\\ \frac{dx^2}{dx} \cdot \frac{dx}{dt} + \frac{dy^2}{dy} \cdot \frac{dy}{dt} &= \frac{ds^2}{ds} \cdot \frac{ds}{dt}\\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2s\frac{ds}{dt}\\ x\frac{dx}{dt} + y\frac{dy}{dt} &= d\frac{ds}{dt} \qquad (1)\end{aligned}$$



Cassandra's position and velocity can be interpreted as $x = 1800$ m and $\frac{dx}{dt} = -32$ m/min, respectively. Marissa's position and velocity can be interpreted as $y = 1200$ m and $\frac{dx}{dt} = -26$ m/min, respectively. At the required instant, the distance, d, between them is $\sqrt{1800^2 + 1200^2}$ or $600\sqrt{13}$ m. Perform the respective substitutions into (1).

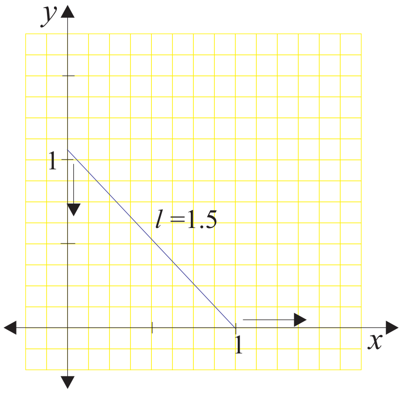
$$\begin{aligned}1800(-32) + 1200(-26) &= 600\sqrt{13}\frac{ds}{dt}\\ \frac{ds}{dt} &= -\frac{148}{\sqrt{13}}\\ &\doteq -41.05\end{aligned}$$

The distance between the girls is decreasing at the rate of approximately 41.05 m/min.

Section Review Page 302 Question 17

The sled, wall, and ground form a right triangle defined by the relation $x^2 + y^2 = 1.5^2$.

$$\begin{aligned}\frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}(1.5^2)\\ \frac{dx^2}{dx} \cdot \frac{dx}{dt} + \frac{dy^2}{dy} \cdot \frac{dy}{dt} &= \frac{d1.5^2}{dt}\\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0\\ \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt} \qquad (1)\end{aligned}$$



When $x = 1$, $y = \sqrt{1.5^5 - 1^2}$ or $\sqrt{1.25}$. Substitute $x = 1$, $y = \sqrt{1.25}$, and $\frac{dx}{dt} = 0.15$ into (1).

$$\begin{aligned}\frac{dy}{dt} &= -\frac{1}{\sqrt{1.25}}(0.15)\\ &\doteq -0.1342\end{aligned}$$

The top of the sled is sliding down the wall at a rate of approximately 13.42 cm/s.

Section Review Page 302 Question 18

a)

$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad (1)$$

Substitute $r = 1$ and $\frac{dr}{dt} = 0.15$ into (1).

$$\begin{aligned}\frac{dV}{dt} &= 4\pi(1)^2(0.15)\\ &= 0.6\pi\end{aligned}$$

Air should be pumped into the balloon at the rate of 0.6π mm^3/s or approximately 1.885 mm^3/s.

b)

$$V = \pi r^2 h \qquad (1)$$

Substitute $h = 10$ mm into (1).

$$V = \pi r^2(10)$$

$$V = 10\pi r^2 \qquad (2)$$

Differentiate both sides of (2) with respect to time, t.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(10\pi r^2\right)$$

$$\frac{dV}{dt} = 20\pi r \frac{dr}{dt} \qquad (3)$$

Substitute $r = 1$ and $\frac{dr}{dt} = 0.15$ into (3).

$$\begin{aligned}\frac{dV}{dt} &= 20\pi(1)(0.15)\\ &= 3\pi\end{aligned}$$

Air should be pumped into the balloon at the rate of 3π mm^3/s or approximately 9.425 mm^3/s.

Section Review Page 302 Question 19

$$PV = C$$

$$V = CP^{-1} \qquad (1)$$

Differentiate both sides of (1) with respect to time, t.

$$\begin{aligned}\frac{d}{dt}(V) &= \frac{d}{dt}\left(CP^{-1}\right)\\ \frac{dV}{dt} &= C\frac{dP^{-1}}{dP} \cdot \frac{dP}{dt}\\ &= -CP^{-2} \cdot \frac{dP}{dt}\\ &= -\frac{C}{P^2} \cdot \frac{dP}{dt} \qquad (2)\end{aligned}$$

When $V = 450$ and $P = 150$, $C = 450(150)$ or 67 500. Substitute $C = 67\,500$, $P = 150$, and $\frac{dP}{dt} = 15$ into (2).

$$\begin{aligned}\frac{dV}{dt} &= -\frac{67\,500}{(150)^2}(15)\\ &= -45\end{aligned}$$

At this instant, the volume is decreasing at the rate of 45 cm^3/min.

Chapter Test

Section Chapter Test Page 303 Question 1

a)
$$\begin{aligned} f(3) &= 3^2 - 3 \\ &= 9 - 3 \\ &= 6 \end{aligned}$$

b)
$$\begin{aligned} f(g(-2)) &= f(-6(-2) + 5) \\ &= f(12 + 5) \\ &= f(17) \\ &= 17^2 - 3 \\ &= 286 \end{aligned}$$

c)
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(-6x + 5) \\ &= (-6x + 5)^2 - 3 \\ &= 36x^2 - 60x + 25 - 3 \\ &= 36x^2 - 60x + 22 \end{aligned}$$

d)
$$\begin{aligned} g(f(x+2)) &= g((x+2)^2 - 3) \\ &= g(x^2 + 4x + 1) \\ &= -6(x^2 + 4x + 1) + 5 \\ &= -6x^2 - 24x - 6 + 5 \\ &= -6x^2 - 24x - 1 \end{aligned}$$

e)
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g\left(x^2 - 3\right) \\ &= -6(x^2 - 3) + 5 \\ &= -6x^2 + 18 + 5 \\ &= -6x^2 + 23 \end{aligned}$$

f)
$$\begin{aligned} (g \circ g)(-3) &= g(g(-3)) \\ &= g(-6(-3) + 5) \\ &= g(23) \\ &= -6(23) + 5 \\ &= -133 \end{aligned}$$

Section Chapter Test Page 303 Question 2

No. Examples will vary.

Section Chapter Test Page 303 Question 3

$$\begin{aligned} H(x) &= \sqrt{7x - 2} \\ (f \circ g)(x) &= \sqrt{7x - 2} \\ f(g(x)) &= \sqrt{7x - 2} \\ \sqrt{g(x)} &= \sqrt{7x - 2} \qquad (1) \end{aligned}$$

Comparing both sides of (1) suggests $g(x) = 7x - 2$.

Section Chapter Test Page 303 Question 4

a)
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(2x - 3) \\ &= \frac{1}{2x - 3} \end{aligned}$$

b)
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g\left(\frac{1}{x}\right) \\ &= \frac{2}{x} - 3 \end{aligned}$$

c) No. $g(1.5) = 0$ and $f(0)$ is not defined.

d) $D_f = \{x \in \Re \mid x \neq 0\}$, $R_f = \{y \in \Re \mid y \neq 0\}$; $D_g = \Re$, $R_g = \Re$

e) $D_{f\circ g} = \{x \in \Re \mid x \neq 1.5\}$, $R_{f\circ g} = \{y \in \Re \mid y \neq 0\}$; $D_{g\circ f} = \{x \in \Re \mid x \neq 0\}$, $R_{g\circ f} = \{y \in \Re \mid y \neq -3\}$

Section Chapter Test Page 303 Question 5

a)
$$\begin{aligned} f(x) &= 3x - 7 \\ y &= 3x - 7 \end{aligned}$$

Interchange x and y.

$$x = 3y - 7$$

Solve for y.

$$\begin{aligned} 3y &= x + 7 \\ y &= \frac{x + 7}{3} \\ f^{-1}(x) &= \frac{x + 7}{3} \end{aligned}$$

b)
$$\begin{aligned} (f \circ f^{-1})(x) &= f(f(x)) \\ &= f\left(\frac{x + 7}{3}\right) \\ &= 3\left(\frac{x + 7}{3}\right) - 7 \\ &= x + 7 - 7 \\ (f \circ f^{-1})(x) &= x \end{aligned}$$

c) Yes. Composites of mutual inverses always yield the identity line.

Section Chapter Test Page 303 Question 6

a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2+2x^4)^3}{d(x^2+2x^4)} \cdot \frac{d(x^2+2x^4)}{dx}\\ &= 3(x^2+2x^4)^2(2x+8x^3)\end{aligned}$$

b)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2+x-3)^{-1}}{dx^2+x-3} \cdot \frac{dx^2+x-3}{dx}\\ &= -1(x^2+x-3)^{-2} \cdot (2x+1)\\ &= -\frac{2x+1}{(x^2+x-3)^2}\end{aligned}$$

c)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^4+3x)^{\frac{1}{2}}}{d\ x^4+3x} \cdot \frac{d\ x^4+3x}{dx}\\ &= \frac{1}{2}(x^4+3x)^{-\frac{1}{2}} \cdot (4x^3+3)\\ &= \frac{4x^3+3}{2\sqrt{x^4+3x}}\end{aligned}$$

d)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^3+6x)^{-\frac{1}{2}}}{dx^3+6x} \cdot \frac{dx^3+6x}{dx}\\ &= -\frac{1}{2}(x^3+6x)^{-\frac{3}{2}} \cdot (3x^2+6)\\ &= -\frac{3(x^2+2)}{2\sqrt{(x^3+6x)^3}}\end{aligned}$$

e)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^3+2x^2-5)^{\frac{1}{4}}}{dx^3+2x^2-5} \cdot \frac{dx^3+2x^2-5}{dx}\\ &= \frac{1}{4}(x^3+2x^2-5)^{-\frac{3}{4}} \cdot (3x^2+4x)\\ &= \frac{3x^2+4x}{4\sqrt[4]{(x^3+2x^2-5)^3}}\end{aligned}$$

f)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^5+6x^3-x)^4}{d(x^5+6x^3-x)} \cdot \frac{d(x^5+6x^3-x)}{dx}\\ &= 4(x^5+6x^3-x)^3(5x^4+18x^2-1)\end{aligned}$$

g)

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2-2)\dfrac{dx^2+2}{dx} - (x^2+2)\dfrac{dx^2-2}{dx}}{(x^2-2)^2}\\ &= \frac{(x^2-2)(2x)-(x^2+2)(2x)}{(x^2-2)^2}\\ &= \frac{2x(x^2-2-x^2-2)}{(x^2-2)^2}\\ &= \frac{-8x}{(x^2-2)^2}\end{aligned}$$

h)

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sqrt{x^2-1}\dfrac{d\sqrt{x^3-2}}{dx} - \sqrt{x^3-2}\dfrac{d\sqrt{x^2-1}}{dx}}{\left(\sqrt{x^2-1}\right)^2}\\ &= \frac{\frac{1}{2}\sqrt{x^2-1}(x^3-2)^{-\frac{1}{2}}(3x^2) - \frac{1}{2}\sqrt{x^3-2}(x^2-1)^{-\frac{1}{2}}(2x)}{x^2-1}\\ &= \frac{x}{\sqrt{x^2-1}\sqrt{x^3-2}} \cdot \frac{\frac{3x(x^2-1)}{2}-(x^3-2)}{x^2-1}\\ &= \frac{x}{2(x^2-1)^{\frac{3}{2}}(x^3-2)^{\frac{1}{2}}} \cdot (3x^3-3x-2x^3+4)\\ &= \frac{x(x^3-3x+4)}{2(x^2-1)^{\frac{3}{2}}(x^3-2)^{\frac{1}{2}}}\end{aligned}$$

Section Chapter Test Page 303 Question 7

$$\begin{aligned}V &= \frac{4}{3}\pi r^3\\ \frac{dV}{dt} &= \frac{4\pi}{3} \cdot \frac{dr^3}{dr} \cdot \frac{dr}{dt}\\ \frac{dV}{dt} &= 4\pi r^2 \cdot \frac{dr}{dt} \qquad (1)\end{aligned}$$

Substitute $r = 12$ and $\frac{dV}{dt} = -3$ into (1).

$$\begin{aligned}-3 &= 4\pi(12)^2\frac{dr}{dt}\\ -3 &= 576\pi\frac{dr}{dt}\\ \frac{dr}{dt} &= -\frac{1}{192\pi}\end{aligned}$$

The radius is decreasing at the rate of $\frac{1}{192\pi}$ m/min.

Section Chapter Test Page 303 Question 8

a) Determine $\frac{dy}{dx}$.

$$\frac{d(x+2)^2}{dx} + \frac{d(y-3)^2}{dy-3} \cdot \frac{dy-3}{dx} = \frac{d2}{dx}$$
$$2(x+2) + 2(y-3)\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x+2}{y-3} \qquad (1)$$

Substitute $(x, y) = (-1, 2)$ into (1).

$$\frac{dy}{dx} = -\frac{-1+2}{2-3}$$
$$m = 1$$

Determine the equation of the tangent.

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 1(x - (-1))$$
$$y - 2 = x + 1$$
$$x - y + 3 = 0$$

The equation of the tangent is $x - y + 3 = 0$.

b) Determine $\frac{dy}{dx}$.

$$y^3\frac{d2x^2}{dx} + 2x^2\frac{dy^3}{dx} + y\frac{d3x}{dx} + 3x\frac{dy}{dx} + \frac{d5}{dx} = \frac{d0}{dx}$$
$$y^3(4x) + 2x^2(3y^2)\frac{dy}{dx} + y(3) + 3x\frac{dy}{dx} + 0 = 0$$
$$\frac{dy}{dx}(6x^2y^2 + 3x) = -4xy^3 - 3y$$
$$\frac{dy}{dx} = \frac{-4xy^3 - 3y}{6x^2y^2 + 3x} \qquad (1)$$

Substitute $(x, y) = (1, -1)$ into (1).

$$\frac{dy}{dx} = \frac{-4(1)(-1)^3 - 3(-1)}{6(1)^2(-1)^2 + 3(1)}$$
$$m = \frac{7}{9}$$

Determine the equation of the tangent.

$$y - y_1 = m(x - x_1)$$
$$y - (-1) = \frac{7}{9}(x - 1)$$
$$9y + 9 = 7x - 7$$
$$7x - 9y - 16 = 0$$

The equation of the tangent is $7x - 9y - 16 = 0$.

c) Determine $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dx^3}{dx} + \frac{dy^3}{dy} \cdot \frac{dy}{dx} &= y\frac{d2x}{dx} + 2x\frac{dy}{dx} \\ 3x^2 + 3y^2\frac{dy}{dx} &= y(2) + (2x)\frac{dy}{dx} \\ \frac{dy}{dx}(-2x + 3y^2) &= 2y - 3x^2 \\ \frac{dy}{dx} &= \frac{2y - 3x^2}{3y^2 - 2x} \qquad (1)\end{aligned}$$

Substitute $(x, y) = (1, 1)$ into (1).

$$\begin{aligned}\frac{dy}{dx} &= \frac{2(1) - 3(1)^2}{3(1)^2 - 2(1)} \\ m &= -1\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ y - 1 &= -1(x - 1) \\ y - 1 &= -x + 1 \\ x + y - 2 &= 0\end{aligned}$$

The equation of the tangent is $x + y - 2 = 0$.

d) Determine $\frac{dy}{dx}$.

$$\begin{aligned}y^3\frac{dx}{dx} + x\frac{dy^3}{dx} + \frac{d4y}{dx} &= \frac{d16}{dx} \\ y^3(1) + x(3y^2)\frac{dy}{dx} + 4\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(3xy^2 + 4) &= -y^3 \\ \frac{dy}{dx} &= \frac{-y^3}{3xy^2 + 4} \qquad (1)\end{aligned}$$

Substitute $(x, y) = (-3, -2)$ into (1).

$$\begin{aligned}\frac{dy}{dx} &= \frac{-(-2)^3}{3(-3)(-2)^2 + 4} \\ m &= \frac{8}{-32} \\ &= -\frac{1}{4}\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ y - (-2) &= -\frac{1}{4}(x - (-3)) \\ 4y + 8 &= -x - 3 \\ x + 4y + 11 &= 0\end{aligned}$$

The equation of the tangent is $x + 4y + 11 = 0$.

Section Chapter Test   Page 303   Question 9

Let S be the surface area of the spherical comet, in metres.

$$\begin{aligned} S &= 4\pi r^2 \\ \frac{dS}{dt} &= 4\pi \frac{dr^2}{dt} \\ &= 8\pi r \frac{dr}{dt} \end{aligned} \tag{1}$$

Substitute $r = 5000$ and $\frac{dS}{dt} = -250$ into (1).

$$\begin{aligned} -250 &= 8\pi(5000)\frac{dr}{dt} \\ \frac{dr}{dt} &= -\frac{1}{160\pi} \end{aligned}$$

The radius is decreasing at the rate of $\frac{1}{160\pi}$ m/min or approximately 0.001 99 m/min.

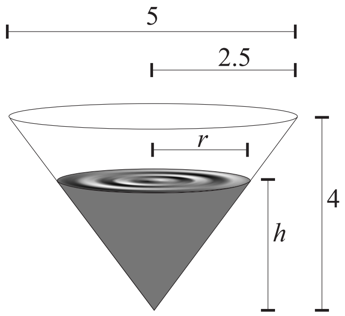
Section Chapter Test   Page 303   Question 10

The liquid maintains a conical shape throughout the entire filling activity such that the volume can be expressed as

$$V = \frac{1}{3}\pi r^2 h \tag{1}$$

From the dimensions of the tank,

$$\begin{aligned} \frac{2r}{h} &= \frac{5}{4} \\ r &= \frac{5h}{8} \end{aligned} \tag{2}$$



Substitute (2) into (1) and simplify.

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{5h}{8}\right)^2 h \\ &= \frac{1}{3}\pi\left(\frac{25h^2}{64}\right) h \\ &= \frac{25\pi}{192}h^3 \end{aligned} \tag{3}$$

Differentiate (3) with respect to time, t.

$$\begin{aligned} \frac{dV}{dt} &= \frac{25\pi}{192} \cdot \frac{dh^3}{dt} \\ &= \frac{25\pi}{192}(3h^2)\frac{dh}{dt} \\ &= \frac{25\pi h^2}{64} \cdot \frac{dh}{dt} \end{aligned} \tag{4}$$

Substitute $h = 3$ and $\frac{dV}{dt} = 1.2$ into (4).

$$\begin{aligned} 1.2 &= \frac{25\pi(3)^2}{64} \cdot \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{1.2(64)}{225\pi} \\ &\doteq 0.109 \end{aligned}$$

The level of the water is rising at the rate of approximately 0.109 m/min.

Challenge Problems

Section Challenge Problems Page 304 Question 1

$$\begin{aligned} f(g(x)) &= h(x) \\ 4(g(x)) - 3 &= 4x^2 - 21 \\ 4g(x) &= 4x^2 - 18 \\ 4g(x) &= 4\left(x^2 - \frac{9}{2}\right) \\ g(x) &= x^2 - \frac{9}{2} \end{aligned}$$

Section Challenge Problems Page 304 Question 3

a)
$$\begin{aligned} F(x) &= (f \circ g)(x) \\ F(x) &= f(g(x)) \\ F'(x) &= f'(g(x))g'(x) \qquad (1) \end{aligned}$$

Perform the required substitutions in (1) and simplify.

$$\begin{aligned} F'(1) &= f'(g(1))g'(1) \\ &= f'(2)(4) \\ &= 12(4) \\ &= 48 \end{aligned}$$

Section Challenge Problems Page 304 Question 4

a) Determine $\frac{dy}{dx}$ for $2x^2 + y^2 = 3$.

$$\begin{aligned} \frac{d2x^2}{dx} + \frac{dy^2}{dx} &= \frac{d3}{dx} \\ 4x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2x}{y} \qquad (1) \end{aligned}$$

Determine the point(s) of intersection of the two relations.

$$\begin{aligned} 2x^2 + y^2 &= 3 \qquad (3) \\ x &= y^2,\ x \ge 0 \qquad (4) \end{aligned}$$

Substitute (4) into (3).

$$\begin{aligned} 2x^2 + x &= 3 \\ 2x^2 + x - 3 &= 0 \\ (2x + 3)(x - 1) &= 0 \end{aligned}$$

Since $x \ge 0$, there is only one solution.

$$x = 1 \qquad (5)$$

Substituting (5) into (4) yields $(1, 1)$ and $(1, -1)$ as points of intersection.

Determine $\frac{dy}{dx}$ for $x = y^2$.

$$\begin{aligned} \frac{dx}{dx} &= \frac{dy^2}{dx} \\ 1 &= 2y\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{2y} \qquad (2) \end{aligned}$$

Substituting $(1, 1)$ into (1) and (2) yields slopes of -2 and $\frac{1}{2}$, respectively. Since $-2 \times \frac{1}{2} = -1$, the curves are orthogonal at $(1, 1)$.

Substituting $(1, -1)$ into (1) and (2) yields slopes of 2 and $-\frac{1}{2}$, respectively. Since $2 \times \left(-\frac{1}{2}\right) = -1$, the curves are orthogonal at $(1, -1)$.

Section Challenge Problems Page 304 Question 2

$$\begin{aligned} (f \circ g)(x) &= h(x) \\ f(g(x)) &= h(x) \\ f(8x + 6) &= \frac{1}{8x + 9} \\ f(8x + 6) &= \frac{1}{(8x + 6) + 3} \qquad (1) \end{aligned}$$

By comparing both sides of (1), $f(x) = \frac{1}{x + 3}$.

b)
$$\begin{aligned} H(x) &= (g \circ f)(x) \\ H(x) &= g(f(x)) \\ H'(x) &= g'(f(x))f'(x) \qquad (1) \end{aligned}$$

Perform the required substitutions in (1) and simplify.

$$\begin{aligned} H'(1) &= g'(f(1))f'(1) \\ &= g'(f(1))(3) \end{aligned}$$

To continue, a value for $g'(f(1))$ is required.

b) Determine $f'(x)$ for $f(x) = \dfrac{x-4}{x}$.

$$\begin{aligned} f'(x) &= \frac{x\dfrac{dx-4}{dx} - (x-4)\dfrac{dx}{dx}}{x^2} \\ &= \frac{x - x + 4}{x^2} \\ &= \frac{4}{x^2} \end{aligned} \tag{1}$$

Determine $g'(x)$ for $g(x) = \dfrac{x+2}{x^2-4x}$.

$$\begin{aligned} g'(x) &= \frac{(x^2-4x)\dfrac{dx+2}{dx} - (x+2)\dfrac{d(x^2-4x)}{dx}}{(x^2-4x)^2} \\ &= \frac{(x^2-4x)(1) - (x+2)(2x-4)}{(x^2-4x)^2} \\ &= \frac{x^2-4x-2(x^2-4)}{(x^2-4x)^2} \\ &= \frac{x^2-4x-2x^2+8}{(x^2-4x)^2} \\ &= -\frac{x^2+4x-8}{(x^2-4x)^2} \end{aligned} \tag{2}$$

Substituting $x = 2$ into (1) and (2) yields slopes of 1 and $-\dfrac{1}{4}$, respectively. Since $1 \times \left(-\dfrac{1}{4}\right) \neq -1$, the curves are not orthogonal at $(2, -1)$. Determine any other points of intersection of the two relations.

$$\begin{aligned} \frac{x-4}{x} &= \frac{x+2}{x^2-4x} \\ (x-4)(x^2-4x) &= x(x+2) \\ x(x-4)(x-4) &= x(x+2) \\ x^2 - 8x + 16 &= x + 2,\ x \neq 0 \\ x^2 - 9x + 14 &= 0 \\ (x-2)(x-7) &= 0 \\ x &= 2 \text{ or } 7 \end{aligned}$$

The curves also intersect at $(7, f(7))$ or $\left(7, \dfrac{3}{7}\right)$. Substituting $x = 7$ into (1) and (2) yields slopes of $\dfrac{4}{49}$ and $-\dfrac{23}{147}$. Since $\dfrac{4}{49} \times \left(-\dfrac{23}{147}\right) \neq 1$, the curves are not orthogonal at $\left(7, \dfrac{3}{7}\right)$.

Section Challenge Problems Page 304 Question 5

Let A be the area of the triangle. Let h be the altitude of the triangle. Let b be the base of the triangle. Determine b, for the given instant.

$$A = \frac{1}{2}bh \tag{1}$$

$$\begin{aligned} 100 &= \frac{1}{2}b(10) \\ b &= 20 \end{aligned} \tag{2}$$

Differentiate (1) with respect to time, t.

$$\frac{dA}{dt} = \frac{1}{2}\left(h\frac{db}{dt} + b\frac{dh}{dt}\right) \tag{3}$$

Substitute (2) and the given information into (3).

$$2 = \frac{1}{2}\left(10\frac{db}{dt} + (20)(1)\right) \tag{3}$$

$$\begin{aligned} \frac{db}{dt} &= \frac{4-20}{10} \\ &= -1.6 \end{aligned}$$

The base of the triangle is decreasing at the rate of 1.6 cm/min at the given instant.

Section Challenge Problems Page 304 Question 6

$$\begin{aligned} v &= \frac{P}{4\eta L}(R^2 - r^2) \\ \frac{dv}{dr} &= \frac{P}{4\eta L} \cdot \frac{d(R^2 - r^2)}{dr} \\ &= \frac{P}{4\eta L}(-2r) \\ &= -\frac{Pr}{2\eta L} \end{aligned} \tag{1}$$

Substitute the given information into (1).

$$\begin{aligned} \frac{dv}{dr} &= -\frac{400(0.003)}{2(0.004)(2)} \\ &\doteq -75 \end{aligned}$$

The velocity gradient is approximately −75 cm/s/cm.

Section Challenge Problems Page 304 Question 7

Determine $\frac{dy}{dx}$ for $y = \frac{1}{x^2+1}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^2+1)^{-1}}{dx^2+1} \cdot \frac{dx^2+1}{dx} \\ &= -1(x^2+1)^{-2}(2x) \\ &= \frac{-2x}{(x^2+1)^2} \end{aligned}$$

Let $\mathrm{P}\left(x, \frac{1}{x^2+1}\right)$ be the general coordinates of each point on the given function. Let Q$(6, -2)$ be the remote point.

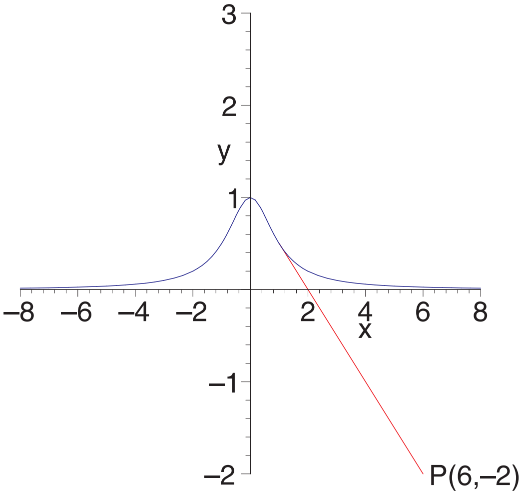
Slope of tangent at P = slope of segment PQ

$$\begin{aligned} \frac{-2x}{(x^2+1)^2} &= \frac{\frac{1}{x^2+1} - (-2)}{x-6} \\ \frac{-2x}{(x^2+1)^2} &= \frac{1 + 2(x^2+1)}{(x^2+1)(x-6)} \\ -2x(x-6) &= (x^2+1)(2x^2+3) \\ -2x^2 + 12x &= 2x^4 + 5x^2 + 3 \\ 2x^4 + 7x^2 - 12x + 3 &= 0 \\ (x-1)(2x^3 + 2x^2 + 9x - 3) &= 0 \end{aligned}$$

Determine the roots.

$$x = 1 \tag{1}$$

$$2x^3 + 2x^2 + 9x - 3 = 0 \tag{2}$$



From (1), one solution is the point $\left(1, \frac{1}{2}\right)$. Using a graphing calculator to solve (2) suggests a second solution at approximately $(x, y) = (0.306, 0.914)$.

Section Challenge Problems Page 304 Question 8

$$\begin{aligned} f(g(x)) &= 4x^2 - 12x + 6 \\ f(2x-3) &= 4x^2 - 12x + 9 - 3 \\ &= (2x-3)^2 - 3 \qquad (1) \end{aligned}$$

By comparing both sides of (1), $f(x) = x^2 - 3$ and, hence, $f(2) = 2^2 - 3$ or 1.

Section Challenge Problems Page 304 Question 9

The relation $y^2 = x^2 - x^4$ has both even and odd symmetry. First consider the function $y = \sqrt{x^2 - x^4}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{3}} \\ \frac{d\sqrt{x^2-x^4}}{dx^2-x^4} \cdot \frac{dx^2-x^4}{dx} &= \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{x^2-x^4}} \cdot (2x-4x^3) &= \frac{1}{\sqrt{3}} \\ \frac{1-2x^2}{\sqrt{1-x^2}} &= \frac{1}{\sqrt{3}} \\ \sqrt{3}(1-2x^2) &= \sqrt{1-x^2} \\ 3(1-2x^2)^2 &= 1-x^2 \\ 3(1-4x^2+4x^4) &= 1-x^2 \\ 3-12x^2+12x^4 &= 1-x^2 \\ 12x^4-11x^2+2 &= 0 \\ (3x^2-2)(4x^2-1) &= 0 \\ 3x^2-2 &= 0 \\ x^2 &= \frac{2}{3} \qquad (1) \\ x &= \pm\sqrt{\frac{2}{3}} \qquad (2) \\ 4x^2-1 &= 0 \\ x^2 &= \frac{1}{4} \qquad (3) \\ x &= \pm\frac{1}{2} \qquad (4) \end{aligned}$$

There are four solutions. Substituting (1) into the equation of the curve yields $(x, y) = \left(\pm\sqrt{\frac{2}{3}}, \mp\frac{\sqrt{2}}{3}\right)$. Substituting (3) into the equation of the curve yields $(x, y) = \left(\pm\frac{1}{2}, \mp\frac{\sqrt{3}}{4}\right)$.

CHAPTER 6 Extreme Values: Curve Sketching and Optimization Problems

6.1 Increasing and Decreasing Functions

Practise

Section 6.1 Page 317 Question 1

a)

$$\begin{aligned} f'(x) &= 0 \\ 2 - x &= 0 \\ x &= 2 \end{aligned}$$

Since $f'(x)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(-\infty, 2)$ and $(2, \infty)$.

b)

$$\begin{aligned} f'(x) &= 0 \\ -3(3 + x) &= 0 \\ 3 + x &= 0 \\ x &= -3 \end{aligned}$$

Since $f'(x)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(-\infty, -3)$ and $(-3, \infty)$.

c)

$$\begin{aligned} f'(x) &= 0 \\ (x + 3)(x - 3) &= 0 \\ x &= \pm 3 \end{aligned}$$

Since $f'(x)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$.

d)

$$\begin{aligned} f'(x) &= 0 \\ -3x - 3 &= 0 \\ -3x &= 3 \\ x &= -1 \end{aligned}$$

Since $f'(x)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(-\infty, -1)$ and $(-1, \infty)$.

e)

$$\begin{aligned} f'(x) &= 0 \\ x(x + 2)(x + 3) &= 0 \\ x &= 0, -2, \text{ or } -3 \end{aligned}$$

Since $f'(x)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(-\infty, -3)$, $(-3, -2)$, $(-2, 0)$, and $(0, \infty)$.

f)

$$\begin{aligned} f'(x) &= 0 \\ -x^2(x - 1)(x - 2) &= 0 \\ x^2(x - 1)(x - 2) &= 0 \\ x &= 0, 1, \text{ or } 2 \end{aligned}$$

Since $f'(x)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$.

g)

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 + 12x + 9 &= 0 \\ x^2 + 4x + 3 &= 0 \\ (x + 1)(x + 3) &= 0 \\ x &= -1 \text{ or } -3 \end{aligned}$$

Since $f'(x)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(-\infty, -3)$, $(-3, -1)$, and $(-1, \infty)$.

h)

$$\begin{aligned} f'(x) &= 0 \\ 4x^3 - 12x^2 - 16x &= 0 \\ x^3 - 3x^2 - 4x &= 0 \\ x(x^2 - 3x - 4) &= 0 \\ x(x + 1)(x - 4) &= 0 \\ x &= 0, -1, \text{ or } 4 \end{aligned}$$

Since $f'(x)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(-\infty, -1)$, $(-1, 0)$, $(0, 4)$, and $(4, \infty)$.

i)

$$\begin{aligned} f'(x) &= 0 \\ \frac{x+2}{x} &= 0 \\ x+2 &= 0 \\ x &= -2 \end{aligned}$$

Since $f'(x)$ is undefined at $x = 0$, the intervals that need to be tested are $(-\infty, -2)$, $(-2, 0)$, and $(0, \infty)$.

j)

$$\begin{aligned} f'(x) &= 0 \\ \frac{(x+1)(x-2)}{x^2} &= 0 \\ (x+1)(x-2) &= 0 \\ x &= -1 \text{ or } 2 \end{aligned}$$

Since $f'(x)$ is undefined at $x = 0$, the intervals that need to be tested are $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$, and $(2, \infty)$.

Apply, Solve, Communicate

Section 6.1 Page 317 Question 3

$$\begin{aligned} T'(t) &= 0 \\ t^2 - 6t + 8 &= 0 \\ (t-2)(t-4) &= 0 \\ t &= 2 \text{ or } 4 \end{aligned}$$

Since $T(t)$ is a polynomial, there are no discontinuities. The intervals that need to be tested are $(0, 2)$, $(2, 4)$, and $(4, 5)$. Using test values within each interval reveals that the temperature is falling ($T'(t) < 0$) over the interval $(2, 4)$.

Section 6.1 Page 317 Question 4

a) The domain of $f(x) = x^3$ is $\Re$. For this function, $f(x_2) > f(x_1)$ whenever $x_2 > x_1$ in the interval $(-\infty, \infty)$. This result suggests $f(x)$ is increasing over all real numbers, despite the property that $f'(x) = 3x^2$ is zero at $x = 0$ and positive for all other real numbers. Another function with a similar property is $g(x) = x^5$.

b) No. The test only says that if $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b). It is not stating that the converse must be true.

Section 6.1 Page 318 Question 5

a) For this application, assume that $D_0 > 0$ and $t > 0$.

$$\begin{aligned} D'(t) &= 0 \\ -\frac{D_0}{(1+t^2)^2} \cdot 2t &= 0 \\ \frac{2tD_0}{(1+t^2)^2} &= 0 \\ t &= 0 \end{aligned}$$

Since $D(t)$ has no discontinuities, the single test interval is $(0, \infty)$. Since $D'(t) < 0$ for $x \in (0, \infty)$, the amount of medication is decreasing on this interval.

b) No. $D'(t) < 0$ over $(0, \infty)$.

c) Since $D(t) > 0$, for $x \in (0, \infty)$, the amount of medication is never zero.

d) No. The amount of medication in the bloodstream will eventually be zero.

Section 6.1 Page 318 Question 6

a)

$$\begin{aligned} h'(t) &= 0 \\ -9.8t + 8 &= 0 \\ t &= \frac{8}{9.8} \\ &= \frac{40}{49} \end{aligned}$$

$h(t)$ has no discontinuities. The two test intervals are $\left(0, \frac{40}{49}\right)$ and $\left(\frac{40}{49}, \infty\right)$. Since $h'(0.5) = 3.1$, the stone is rising on $\left(0, \frac{40}{49}\right)$.

b) The stone will hit the ground when $h = 0$.

$$\begin{aligned} -4.9t^2 + 8t + 10 &= 0 \\ t &= \frac{-(8) \pm \sqrt{(8)^2 - 4(-4.9)(10)}}{2(-4.9)} \\ &\doteq -0.829 \text{ or } 2.462 \end{aligned}$$

Since $t \geq 0$, the stone hits the ground after approximately 2.462 s.

c) The stone is falling for $2.462 - \frac{40}{49}$ or approximately 1.645 s.

Section 6.1 Page 318 Question 7

a)

$$\begin{aligned} h(x) &= f(x) + g(x) \\ &= x^2 - 3x + 2 + x + 1 \\ &= x^2 - 2x + 3 \\ h'(x) &= 0 \\ 2x - 2 &= 0 \\ x &= 1 \end{aligned}$$

Since $h(x)$ is a polynomial, there are no discontinuities. The two test intervals are $(-\infty, 1)$ and $(1, \infty)$. Test values reveal $h(x)$ is decreasing on the interval $(-\infty, 1)$ and increasing on the interval $(1, \infty)$.

b)

$$\begin{aligned} h(x) &= g(f(x)) \\ &= g(x^2 - 3x + 2) \\ &= x^2 - 3x + 2 + 1 \\ &= x^2 - 3x + 3 \\ h'(x) &= 0 \\ 2x - 3 &= 0 \\ x &= \frac{3}{2} \end{aligned}$$

Since $h(x)$ is a polynomial, there are no discontinuities. The test intervals are $\left(-\infty, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, \infty\right)$. Test values reveal $h(x)$ is decreasing on the interval $\left(-\infty, \frac{3}{2}\right)$ and increasing on the interval $\left(\frac{3}{2}, \infty\right)$.

c)

$$\begin{aligned} h(x) &= f(x)g(x) \\ &= (x^2 - 3x + 2)(x + 1) \\ &= x^3 - 3x^2 + 2x + x^2 - 3x + 2 \\ &= x^3 - 2x^2 - x + 2 \\ h'(x) &= 0 \\ 3x^2 - 4x - 1 &= 0 \\ x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-1)}}{2(3)} \\ &= \frac{4 \pm \sqrt{28}}{6} \\ &= \frac{2 \pm \sqrt{7}}{3} \end{aligned}$$

Since $h(x)$ is a polynomial, there are no discontinuities. The test intervals are $\left(-\infty, \frac{2 - \sqrt{7}}{3}\right)$, $\left(\frac{2 - \sqrt{7}}{2}, \frac{2 + \sqrt{7}}{3}\right)$, and $\left(\frac{2 + \sqrt{7}}{3}, \infty\right)$. Test values reveal $h(x)$ is increasing on the intervals $\left(-\infty, \frac{2 - \sqrt{7}}{3}\right)$ and $\left(\frac{2 + \sqrt{7}}{3}, \infty\right)$. The function is decreasing on the interval $\left(\frac{2 - \sqrt{7}}{2}, \frac{2 + \sqrt{7}}{3}\right)$.

d)

$$\begin{aligned} h(x) &= \frac{f(x)}{g(x)} \\ &= \frac{x^2 - 3x + 2}{x + 1} \\ h'(x) &= 0 \\ \frac{(x + 1)(2x - 3) - (x^2 - 3x + 2)(1)}{(x + 1)^2} &= 0 \\ \frac{2x^2 - x - 3 - x^2 + 3x - 2}{(x + 1)^2} &= 0 \\ \frac{x^2 + 2x - 5}{(x + 1)^2} &= 0 \\ x &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-5)}}{2(1)} \\ &= -1 \pm \sqrt{6} \end{aligned}$$

$h'(x)$ is not defined for $x = -1$. The test intervals are $(-\infty, -1 - \sqrt{6})$, $(1 - \sqrt{6}, -1)$, $(-1, -1 + \sqrt{6})$, and $(-1 + \sqrt{6}, \infty)$. Test values reveal $h(x)$ is increasing on the intervals $(-\infty, -1 - \sqrt{6})$ and $(-1 + \sqrt{6}, \infty)$. The function is decreasing on the intervals $(-1 - \sqrt{6}, -1)$ and $(-1, -1 + \sqrt{6})$.

Section 6.1 Page 318 Question 8

$$\begin{aligned} p'(x) &= 0 \\ 400 \cdot \frac{1}{2\sqrt{12x - x^2}} \cdot (12 - 2x) &= 0 \\ \frac{400(6 - x)}{\sqrt{12x - x^2}} &= 0 \\ 400(6 - x) &= 0 \\ x &= 6 \end{aligned}$$

$p'(x)$ is undefined at $x = 0$ and $x = 12$. The test intervals are $(0, 6)$ and $(6, 12]$.

a) Since $p'(1) > 0$, the profit, $p(x)$, is increasing on the interval $(0, 6)$.

b) Since $p'(10) < 0$, the profit, $p(x)$, is decreasing on the interval $(6, 12]$.

Section 6.1 Page 318 Question 9

$$\begin{aligned} h'(t) &= 0 \\ 12t^2 - 100t &= 0 \\ 3t^2 - 25t &= 0 \\ t(3t - 25) &= 0 \\ t &= 0 \text{ or } \frac{25}{3} \end{aligned}$$

Since the domain is restricted to $t \in [0, 15]$, the test intervals are $\left[0, \frac{25}{3}\right)$ and $\left(\frac{25}{3}, 15\right]$.

a) Since $h'(1) < 0$, the airplane is descending on the interval $\left(0, \frac{25}{3}\right)$ and ascending on the interval $\left(\frac{25}{3}, 15\right]$.

b) The plane is diving for the first 8.3 s and then climbing for 6.7 s.

Section 6.1 Page 318 Question 10

$$\begin{aligned} P'(R) &= 0 \\ \frac{(R + 0.5)^2(120) - 120R(2)(R + 0.5)(1)}{(R + 0.5)^4} &= 0 \\ \frac{(R + 0.5)(120) - 120R(2)}{(R + 0.5)^3} &= 0 \\ \frac{120R + 60 - 240R}{(R + 0.5)^3} &= 0 \\ \frac{60 - 120R}{(R + 0.5)^3} &= 0 \\ 60 - 120R &= 0 \\ R &= 0.5 \end{aligned}$$

Since $R \geq 0$, the test intervals are $[0, 0.5)$ and $(0.5, \infty)$.

a) Since $P'(0.25) > 0$, the power is increasing on the interval $[0, 0.5)$.

b) Since $P'(1) < 0$, the power is decreasing on the interval $(0.5, \infty)$.

Section 6.1   Page 318   Question 11

$$\begin{aligned} f(x) &= kx^3 - 3x + 3 \\ f'(x) &= 3kx^2 - 3 \end{aligned}$$

$f(x)$ is decreasing when $f'(x) < 0$.

$$\begin{aligned} 3kx^2 - 3 &< 0 \\ kx^2 &< 1 \\ x^2 &< \frac{1}{k} \\ |x| &< \frac{1}{\sqrt{k}} \end{aligned}$$

For f to be decreasing on $(0, 3)$ (which might be part of a wider interval on which $f(x)$ is decreasing),

$$\begin{aligned} \frac{1}{\sqrt{k}} &\geq 3 \\ \sqrt{k} &\leq \frac{1}{3} \\ k &\leq \frac{1}{9} \end{aligned}$$

Section 6.1   Page 318   Question 12

Answers will vary.

Section 6.1   Page 318   Question 13

$f(n) \leq \lim\limits_{x \to n^-} f(x)$ and $f(n) \leq \lim\limits_{x \to n^+} f(x)$ must both be satisfied.

Section 6.1   Page 318   Question 14

The domain of $f(x)$ is restricted to $1 + x \geq 0$ or $[-1, \infty)$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{5}{2}(1 + x)^{\frac{3}{2}} - \frac{5}{2} &= 0 \\ (1 + x)^{\frac{3}{2}} - 1 &= 0 \\ (1 + x)^{\frac{3}{2}} &= 1 \\ 1 + x &= 1 \\ x &= 0 \end{aligned}$$

The test intervals are $(-1, 0)$ and $(0, \infty)$

a) Since $f'(-0.5) < 0$, $f(x)$ is decreasing on the interval $(-1, 0)$. Since $f'(3) > 0$, $f(x)$ is increasing on the interval $(0, \infty)$.

b) On the interval $x \in (0, \infty)$, since $f(0) = 0$ and $f(x)$ is increasing,

$$\begin{aligned} f(x) &> 0 \\ (1 + x)^{\frac{5}{2}} - \frac{5}{2}x - 1 &> 0 \\ (1 + x)^{\frac{5}{2}} &> 1 + \frac{5}{2}x \end{aligned}$$

6.2 Maximum and Minimum Values

Practise

Section 6.2 Page 327 Question 1

a) The coordinates of the left endpoint are $(0, 12)$. The coordinates of the right endpoint are $(10, 6)$. The coordinates of the critical points are $(2, 2)$, $(4, 4)$, and $(6, -6)$. Comparison of these results yields an absolute maximum value of 12 and an absolute minimum value of -6.

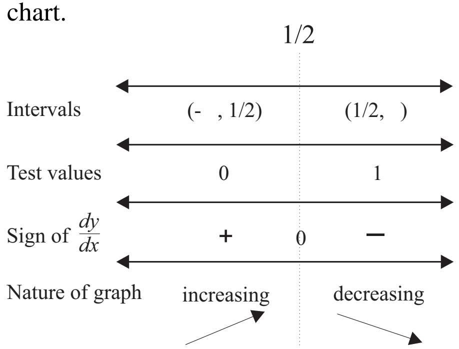
b) The coordinates of the critical points are $(-1.5, 3)$, $(0, -6)$, $(1, 0)$, $(2.2, -13)$, and $(3, 0)$. As $|x| \to \infty,\ y \to -\infty$. Comparison of these results yields an absolute maximum value of 3 and no absolute minimum value.

Section 6.2 Page 328 Question 3

a)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 1 - 2x &= 0\\ x &= \frac{1}{2}\end{aligned}$$

Since the function is differentiable over $\Re$, the only critical number is $x = \frac{1}{2}$. Consider the following interval chart.



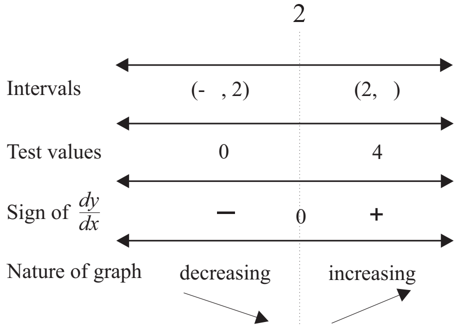
	1/2	
Intervals	(- , 1/2)	(1/2,)
Test values	0	1
Sign of $\frac{dy}{dx}$	+	−
Nature of graph	increasing	decreasing

By the first derivative test, the function has a local maximum at $x = \frac{1}{2}$. Its value is $y = \frac{1}{2} - \left(\frac{1}{2}\right)^2$ or $\frac{1}{4}$.

b)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 2(x - 2) &= 0\\ x &= 2\end{aligned}$$

Since the function is differentiable over $\Re$, the only critical number is $x = 2$. Consider the following interval chart.



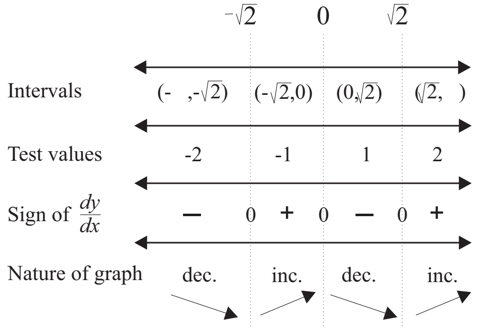
	2	
Intervals	(- , 2)	(2,)
Test values	0	4
Sign of $\frac{dy}{dx}$	−	+
Nature of graph	decreasing	increasing

By the first derivative test, the function has a local minimum at $x = 2$. Its value is $y = (2 - 2)^2$ or 0.

c)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 2(x^2 - 2) \cdot 2x &= 0\\ x &= 0, \pm\sqrt{2}\end{aligned}$$

Since the function is differentiable over $\Re$, the critical numbers are $x = 0, \pm\sqrt{2}$. Consider the following interval chart.



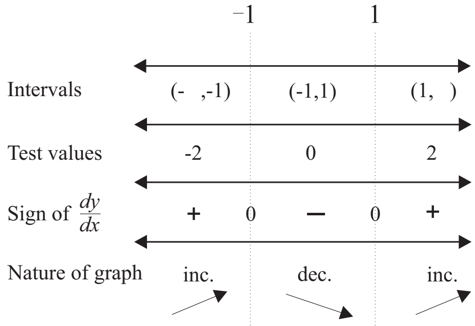
	−√2	0	√2	
Intervals	(- ,-√2)	(-√2,0)	(0,√2)	(√2,)
Test values	-2	-1	1	2
Sign of $\frac{dy}{dx}$	−	+	−	+
Nature of graph	dec.	inc.	dec.	inc.

By the first derivative test, the function has local minima at $x = \pm\sqrt{2}$. Their value is $y = \left(\left(\pm\sqrt{2}\right)^2 - 2\right)^2$ or 0. The local maximum is at $(0, (0^2 - 2)^2)$ or $(0, 4)$.

d)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 6x^2 - 6 &= 0\\ x &= \pm 1\end{aligned}$$

Since the function is differentiable over $\Re$, the critical numbers are $x = \pm 1$. Consider the following interval chart.



	−1	1	
Intervals	(- ,-1)	(-1,1)	(1,)
Test values	-2	0	2
Sign of $\frac{dy}{dx}$	+	−	+
Nature of graph	inc.	dec.	inc.

By the first derivative test, the function has a local maximum at $x = -1$. Its value is $y = 2(-1)^3 - 6(-1) + 1$ or 5. The function has a local minimum at $x = 1$. Its value is $y = 2(1)^3 - 6(1) + 1$ or -3.

e)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 1-\frac{1}{2\sqrt{x}} &= 0\\ 2\sqrt{x} &= 1\\ x &= \frac{1}{4}\end{aligned}$$

The domain is $[0, \infty)$. Since the function is not differentiable at $x = 0$, the critical numbers are $x = \frac{1}{4}$ and 0. Consider the following interval chart.

		1/4	
Intervals	(0, 1/4)		(1/4,)
Test values	1/16		1
Sign of $\frac{dy}{dx}$	−	0	+
Nature of graph	decreasing ↘		increasing ↗

By the first derivative test, the function has a local minimum at $\left(\frac{1}{4}, -\frac{1}{4}\right)$. There are no local maxima.

f)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 6x-8 &= 0\\ x &= \frac{4}{3}\end{aligned}$$

Since the function is differentiable over $\Re$, the only critical number is $x = \frac{4}{3}$. Consider the following interval chart.

		4/3	
Intervals	(- , 4/3)		(4/3,)
Test values	0		2
Sign of $\frac{dy}{dx}$	−	0	+
Nature of graph	decreasing ↘		increasing ↗

By the first derivative test, the function has a local minimum at $\left(\frac{4}{3}, -\frac{4}{3}\right)$. There are no local maxima.

g)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ x^{-\frac{2}{3}} &= 0\\ \frac{1}{\sqrt[3]{x^2}} &= 0\end{aligned}$$

Since the function is not differentiable at the origin, the only critical number is $x = 0$. Consider the following interval chart.

		0	
Intervals	(- ,0)		(0,)
Test values	-1		1
Sign of $\frac{dy}{dx}$	+		+
Nature of graph	increasing ↗		increasing ↗

By the first derivative test, the function has no local extrema.

h)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 8x^3-32x &= 0\\ x^3-4x &= 0\\ x(x-2)(x+2) &= 0\\ x &= 0, \pm 2\end{aligned}$$

Since the function is differentiable over $\Re$, the critical numbers are $x = 0, \pm 2$. Consider the following interval chart.

		−2		0		2	
Intervals	(- ,-2)		(-2,0)		(0,2)		(2,)
Test values	-3		-1		1		3
Sign of $\frac{dy}{dx}$	−	0	+	0	−	0	+
Nature of graph	dec. ↘		inc. ↗		dec. ↘		inc. ↗

By the first derivative test, the function has local minima at $(\pm 2, -20)$ and a local maximum at $(0, 12)$.

i)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 5x^4 - 15x^2 &= 0 \\ x^4 - 3x^2 &= 0 \\ x^2(x^2 - 3) &= 0 \\ x &= 0, 0, \pm\sqrt{3} \end{aligned}$$

Since the function is differentiable over $\Re$, the only critical numbers are $x = 0, \pm\sqrt{3}$. Consider the following interval chart.

		$-\sqrt{3}$		0		$\sqrt{3}$	
Intervals	(- ,-√3)		(-√3,0)		(0,√3)		(√3,)
Test values	-2		-1		1		2
Sign of $\frac{dy}{dx}$	+	0	−	0	−	0	+
Nature of graph	inc. ↗		dec. ↘		dec. ↘		inc. ↗

By the first derivative test, the function has a local maximum at $(-\sqrt{3}, 6\sqrt{3})$ and a local minimum at $(\sqrt{3}, -6\sqrt{3})$.

j)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 3\,|x+2|^2 \cdot \frac{x+2}{|x+2|} &= 0 \\ \frac{|x+2|^2\,(x+2)}{|x+2|} &= 0 \end{aligned}$$

The only critical number is $x = -2$. Consider the following interval chart.

		-2	
Intervals	(- ,-2)		(-2,)
Test values	-3		0
Sign of $\frac{dy}{dx}$	−		+
Nature of graph	decreasing ↘		increasing ↗

By the first derivative test, the function has a local minimum at $(-2, 0)$ and no local maxima.

k)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ (x-1)^2\left(\frac{2}{3}x^{-\frac{1}{3}}\right) + x^{\frac{2}{3}}(2)(x-1) &= 0 \\ \frac{(x-1)^2}{\sqrt[3]{x}} + \frac{3x(x-1)}{\sqrt[3]{x}} &= 0 \\ \frac{(x-1)(x-1+3x)}{\sqrt[3]{x}} &= 0 \\ \frac{(x-1)(4x-1)}{\sqrt[3]{x}} &= 0 \\ x &= 1 \text{ or } \frac{1}{4} \end{aligned}$$

Since the function is not differentiable at $x = 0$, the critical numbers are $x = 0, \frac{1}{4}$, and 1. Consider the following interval chart.

		0		1/4		1	
Intervals	(- ,0)		(0,1/4)		(1/4,1)		(1,)
Test values	-1		1/8		3/8		8
Sign of $\frac{dy}{dx}$	−		+	0	−	0	+
Nature of graph	dec. ↘		inc. ↗		dec. ↘		inc. ↗

The function has a local minima at $(0, 0)$ and $(1, 0)$ and a local maximum at $\left(\frac{1}{4}, \frac{9}{32\sqrt[3]{2}}\right)$.

l)

$$\begin{aligned} y &= x^2 - 4x,\ x \neq 0 \\ \frac{dy}{dx} &= 0 \\ 2x - 4 &= 0 \\ x &= 2,\ x \neq 0 \end{aligned}$$

The function is not differentiable at $x = 0$. The critical numbers are 0 and 2. Consider the following interval chart.

		0		2	
Intervals	(- ,0)		(0,2)		(2,)
Test values	-1		1		3
Sign of $\frac{dy}{dx}$	−		−	0	+
Nature of graph	dec. ↘		dec. ↘		inc. ↗

By the first derivative test, the function has a local minimum at $(2, -4)$ and no local maxima.

Apply, Solve, Communicate

Section 6.2 Page 328 Question 5

Determine the critical numbers.

$$\begin{aligned}\frac{dV}{dx} &= 0\\ 120-92x+12x^2 &= 0\\ 3x^2-23x+30 &= 0\\ (3x-5)(x-6) &= 0\\ x &= \frac{5}{3} \text{ or } 6\end{aligned}$$

	0	5/3	4
Intervals	[0,5/3)		(5/3,4]
Test values	1		2
Sign of $\frac{dV}{dx}$	+	0	−
Nature of graph	inc. ↗		dec. ↘

Since $6 \notin [0,4]$, the only critical number is $\frac{5}{3}$. An interval chart is used to organize the information. Testing the endpoints yields $V(0) = V(4) = 0$. The volume of the shed is an absolute maximum if $x = \frac{5}{3}$ m.

Section 6.2 Page 328 Question 6

a) The height of the flare is $h(0) = 2$ m.

b) Determine the critical number, t.

$$\begin{aligned}\frac{dh}{dt} &= 0\\ -9.8t+44 &= 0\\ t &= \frac{44}{9.8}\\ &= \frac{220}{49}\end{aligned}$$

Determine the height of the flare at $t = \frac{220}{49}$.

$$\begin{aligned}h\left(\frac{220}{49}\right) &= -4.9\left(\frac{220}{49}\right)^2 + 44\left(\frac{220}{49}\right) + 2\\ &= -\frac{4840}{49} + \frac{9680}{49} + \frac{98}{49}\\ &= \frac{4938}{49}\end{aligned}$$

The maximum height of the flare is $\frac{4938}{49}$ m.

c) Determine t when $h = 0$.

$$\begin{aligned}h(t) &= 0\\ -4.9t^2+44t+2 &= 0\\ t &= \frac{-(44) \pm \sqrt{(44)^2 - 4(-4.9)(2)}}{2(-4.9)}\\ &\doteq -0.045 \text{ or } 9.025\end{aligned}$$

Since $-0.045 < 0$, the flare hits the ground after approximately 9 s.

Section 6.2 Page 328 Question 7

Determine the critical numbers.

$$\begin{aligned}\frac{dW}{dv} &= 0\\ -0.51v^2+36.8v-584 &= 0\\ v &= \frac{-(36.8) \pm \sqrt{(36.8)^2 - 4(-0.51)(-584)}}{2(-0.51)}\\ &\doteq 23.57 \text{ or } 48.59\end{aligned}$$

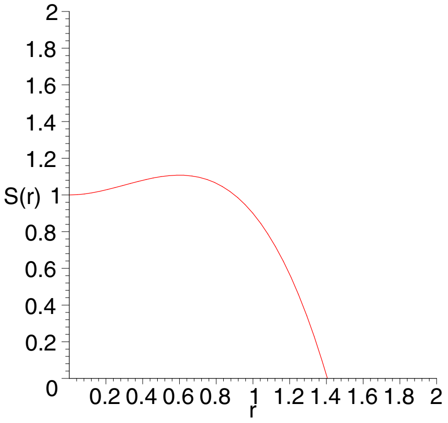
	0	23.57	45
Intervals	[0,23.57)		(23.57,45]
Test values	1		30
Sign of $\frac{dW}{dv}$	−	0	+
Nature of graph	dec. ↘		inc. ↗

Since $48.59 \notin [0,45]$, the only critical number is 23.57, which yields $W(23.57) = -6007.87$. An interval chart is used to organize the information. Testing the endpoints yields $W(0) = -239$ and $W(45) \doteq -4750.25$. The wind-chill factor is a minimum if $v \doteq 23.57$ m/s.

Section 6.2 Page 328 Question 8

Determine the critical numbers.

$$\begin{aligned}\frac{dS}{dr} &= 0\\ F(1.8r - 3r^2) &= 0\\ r^2 - 0.6r &= 0\\ r(r - 0.6) &= 0\\ r &= 0 \text{ or } 0.6\end{aligned}$$

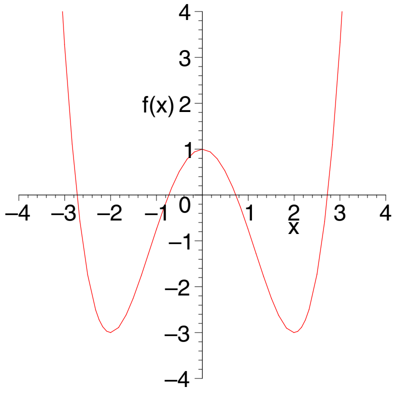


Since $S'(r) > 0$ on the interval $(0, 0.6)$ and $S'(r) < 0$ on the interval $(0.6, \infty)$, the maximum speed occurs at the critical number $r = 0.6$. The maximum speed of airflow is $S(0.6) = 1.108F$ cm/s.

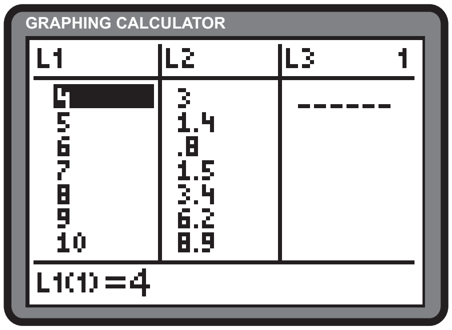
Section 6.2 Page 328 Question 9

a) The sign change of the first derivative must exist for an arbitrarily small interval containing the critical number.

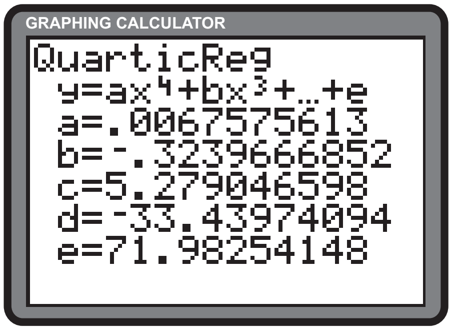
b) Lawrie would have eventually discovered the error if ever smaller intervals around $x = 0$ were tested.



Section 6.2 Page 329 Question 10



```
L1     L2     L3     1
4      3      ------
5      1.4
6      .8
7      1.5
8      3.4
9      6.2
10     8.9
L1(1)=4
```

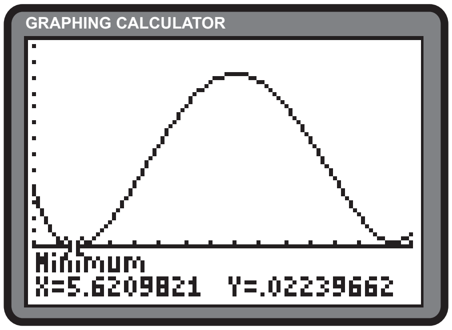


```
QuarticReg
 y=ax⁴+bx³+…+e
 a=.0067575613
 b=-.3239666852
 c=5.279046598
 d=-33.43974094
 e=71.98254148
```

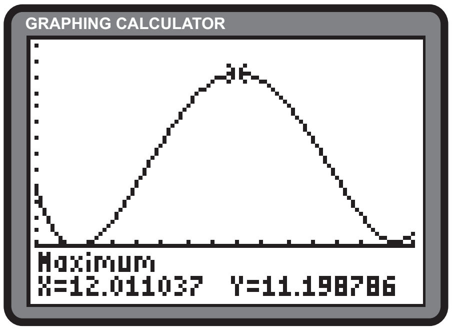
a) Let h be the height of the tide, in metres. Let t be the time, in hours. The data can be approximated by the quartic function $h(t) = 0.0068t^4 - 0.3240t^3 + 5.2790t^2 - 33.4397t + 71.9825$.

b) Through the use of the minimum and maximum features of the graphing calculator, the critical numbers of $h(t)$ are revealed. Conversion of these results to approximate times leads to increasing intervals from 5:37 am to 12:01 p.m. and 6:19 p.m. to 7:00 p.m. Decreasing intervals include 4:00 a.m. to 5:37 a.m. and 12:01 p.m. to 6:19 p.m. The tide is neither increasing nor decreasing at 5:37 a.m., 12:01 p.m., and 6:19 p.m.

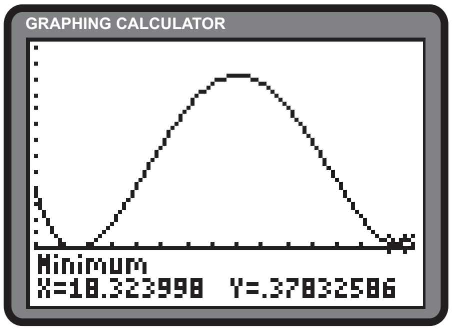
c) The tide is a maximum at 12:01 p.m. Comparing the two minimum values yields an absolute minimum at 5:37 a.m.



```
Minimum
X=5.6209821  Y=.02239662
```



```
Maximum
X=12.011037  Y=11.198786
```



```
Minimum
X=18.323998  Y=.37832586
```

Section 6.2 Page 329 Question 11

a) Determine the critical numbers.

$$\frac{dP}{dx} = 0$$

$$\frac{x}{200} - \frac{1}{10} = 0$$

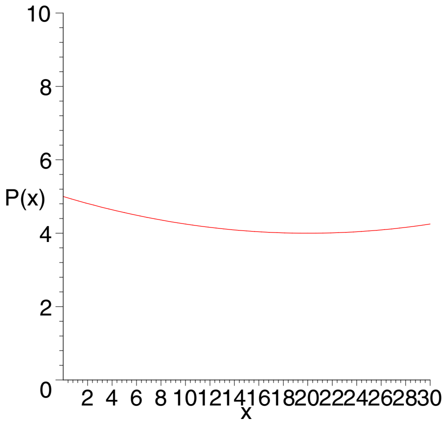
$$x - 20 = 0$$

$$x = 20$$

Since $P'(10) < 0$, the population is decreasing on the interval $[0, 20)$. Since $P'(25) > 0$, the population is increasing on the interval $(20, 30]$.

b) The population reaches a minimum at $t = 20$ years. The minimum population is $P(20)$ or 4 million people.

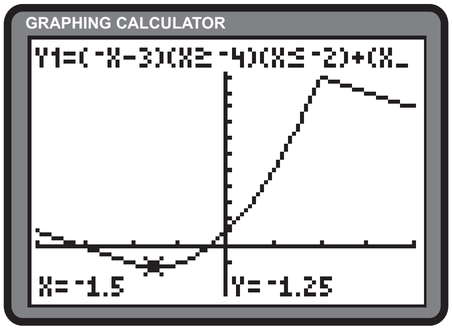
c)



Section 6.2 Page 329 Question 12

Tests of the points $x = \pm 2$ reveals that $f(x)$ is continuous at these transition points. The first derivative of the middle piece, $x^2 + 3x + 1$, yields a critical number at $x = -\frac{3}{2}$.

Since the first derivatives of the pieces defined by $-x - 3$ and $13 - x$ yield no zeros, the critical numbers exist solely at the endpoints of the intervals for which they are defined. The critical numbers of the function are, therefore, $x = -4, -2, -\frac{3}{2}, 2,$ and 4. Comparison of the function values at these critical numbers yields an absolute minimum of $-\frac{5}{4}$ at $x = -\frac{3}{2}$ and an absolute maximum of 11 at $x = 2$.

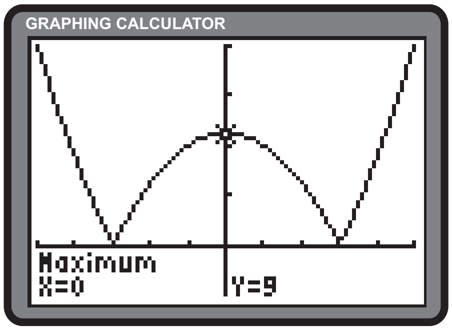


```
Y1=(⁻X-3)(X≥⁻4)(X≤⁻2)+(X_
X=⁻1.5  Y=⁻1.25
```

Section 6.2 Page 329 Question 13

Express $g(x) = \left|x^2 - 9\right|$ as $g(x) = \sqrt{(x^2 - 9)^2}$. Determine the critical numbers.

$$\begin{aligned} g'(x) &= 0 \\ \frac{1}{2\sqrt{(x^2-9)^2}} \cdot 2(x^2 - 9) \cdot 2x &= 0 \\ \frac{x^2 - 9}{\left|x^2 - 9\right|} \cdot 2x &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$



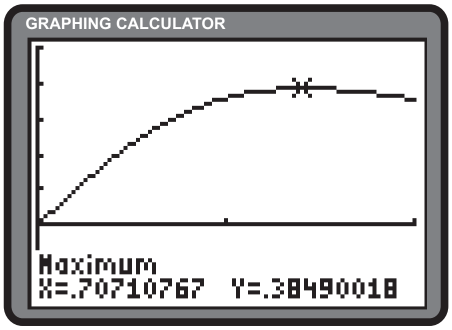
```
Maximum
X=0          Y=9
```

In addition to the critical number $x = 0$, $g(x)$ is not differentiable at $x = \pm 3$ and ± 5. Evaluating and comparing the function values at the five critical numbers reveals an absolute minimum of 0 at $x = \pm 3$ and an absolute maximum of 16 at $x = \pm 5$.

Section 6.2 Page 329 Question 14

Express $y = x(r^2 + x^2)^{-\frac{3}{2}}$ as $y = \dfrac{x}{(r^2 + x^2)^{\frac{3}{2}}}$. Determine the critical numbers.

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{(r^2 + x^2)^{\frac{3}{2}}(1) - x \cdot \frac{3}{2}(r^2 + x^2)^{\frac{1}{2}}(2x)}{(r^2 + x^2)^3} &= 0 \\ (r^2 + x^2)^{\frac{3}{2}} - 3x^2(r^2 + x^2)^{\frac{1}{2}} &= 0 \\ 3x^2 &= r^2 + x^2 \\ 2x^2 &= r^2 \\ x &= \frac{r}{\sqrt{2}}, \ x \in [0, r] \end{aligned}$$



```
Maximum
X=.70710767  Y=.38490018
```

$r = 1$

The three critical numbers are $x = 0$, $\dfrac{r}{\sqrt{2}}$, and r. Evaluating the function at these critical numbers yields $y(0) = 0$, $y\left(\dfrac{r}{\sqrt{2}}\right) = \dfrac{2}{3\sqrt{3}r^2}$ and $y(r) = \dfrac{1}{2\sqrt{2}r^2}$. Comparing these results reveals an absolute maximum of $\dfrac{2}{3\sqrt{3}r^2}$.

Section 6.2 Page 329 Question 15

a) Yes. For example, $y = \dfrac{x}{1 + x^2}$ has an absolute maximum at $\left(1, \dfrac{1}{2}\right)$ and an absolute minimum at $\left(-1, -\dfrac{1}{2}\right)$.

b) Yes. The absolute maximum must either occur at one of the endpoints of the closed interval or a critical number within the interval. However, if the interval were open, as in the function $y = x^2$ over $x \in (0, 1)$, there would *not* be an absolute maximum.

Section 6.2 Page 329 Question 16

a) Yes. Since $f(x)$ is differentiable and $f'(x)$ changes sign on the interval $(1, 2)$, there must exist a critical number within the interval. Since the sign change is from positive to negative, the first derivative test declares that $f(x)$ has a local maximum within the interval.

b) No. For example, $y = \dfrac{3}{4(2x - 3)^2}$ has an infinite discontinuity at $x = \dfrac{3}{2}$.

6.3 Concavity and the Second Derivative Test

Practise

Section 6.3 Page 338 Question 1

a) The function is concave upward on the intervals $(-6, -3)$, $(1, 4)$, and $(12, 14)$. The function is concave downward on the intervals $(-10, -6)$, $(-3, 1)$, $(4, 7)$, and $(7, 12)$.

b) The coordinates of the points of inflection are $(-6, -1)$, $(-3, 0)$, $(1, 4)$, $(4, 0.8)$, and $(12, 2.5)$.

Section 6.3 Page 338 Question 3

a) f is increasing on the intervals where $f'(x) > 0$, that is, on the intervals $(-\infty, 0)$, $(2, 4)$, and $(6, \infty)$. f is decreasing on the intervals where $f'(x) < 0$, that is, on the intervals $(0, 2)$ and $(4, 6)$.

b) f is concave upward on the intervals where f' is increasing, that is, on the intervals $(1, 3)$, $(5, 7)$, and $(8, \infty)$. f is concave downward on the intervals where f' is decreasing, that is, on the intervals $(-\infty, 1)$, $(3, 5)$, and $(7, 8)$.

c) f has a local maximum at $x = c$, where $f'(c) = 0$ and $f''(c) < 0$, that is, at $x = 0$ and $x = 4$. f has a local minimum at $x = c$, where $f'(c) = 0$ and $f''(c) > 0$, that is, at $x = 2$ and $x = 6$.

d) f has a point of inflection at $x = d$ if $f''(d) = 0$ and f'' changes its sign in the arbitrarily small interval of the domain around d. The x-coordinates of the points of inflection of f are $1, 3, 5, 7$, and 8.

Section 6.3 Page 339 Question 5

a)

$$\begin{aligned} f(x) &= 2x^2 - 3x + 5 \\ f'(x) &= 4x - 3 \\ f''(x) &= 4 \end{aligned}$$

Since $f''(x) > 0$ for all $\Re$, f is concave upward on $(-\infty, \infty)$. Since f'' never changes its sign, there are no points of inflection.

b)

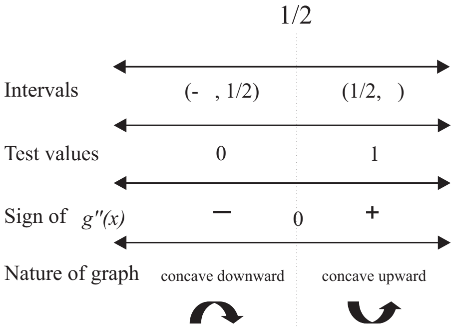
$$\begin{aligned} y &= 6 + 5x - 3x^2 \\ \frac{dy}{dx} &= 5 - 6x \\ \frac{d^2y}{dx^2} &= -6 \end{aligned}$$

Since $\frac{d^2y}{dx^2} < 0$ for all $\Re$, y is concave downward on $(-\infty, \infty)$. Since $\frac{d^2y}{dx^2}$ never changes its sign, there are no points of inflection.

c)

$$\begin{aligned} g(x) &= 2x^3 - 3x^2 - 12x + 6 \\ g'(x) &= 6x^2 - 6x - 12 \\ g''(x) &= 12x - 6 \\ &= 6(2x - 1) \end{aligned}$$

$g''(x) = 0$ at $x = \frac{1}{2}$. An interval chart can be used to determine intervals of concavity. g is concave upward on $\left(\frac{1}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{1}{2}\right)$. g has a point of inflection at $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

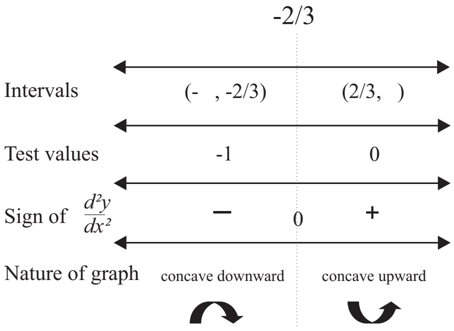


		1/2	
Intervals	(- , 1/2)		(1/2,)
Test values	0		1
Sign of $g''(x)$	−	0	+
Nature of graph	concave downward		concave upward

d)

$$\begin{aligned} y &= x^3 + 2x^2 - 4x + 3 \\ \frac{dy}{dx} &= 3x^2 + 4x - 4 \\ \frac{d^2y}{dx^2} &= 6x + 4 \\ &= 2(3x + 2) \end{aligned}$$

$\frac{d^2y}{dx^2} = 0$ at $x = -\frac{2}{3}$. An interval chart can be used to determine intervals of concavity. y is concave upward on $\left(-\frac{2}{3}, \infty\right)$ and concave downward on $\left(-\infty, -\frac{2}{3}\right)$. y has a point of inflection at $\left(-\frac{2}{3}, \frac{169}{27}\right)$.

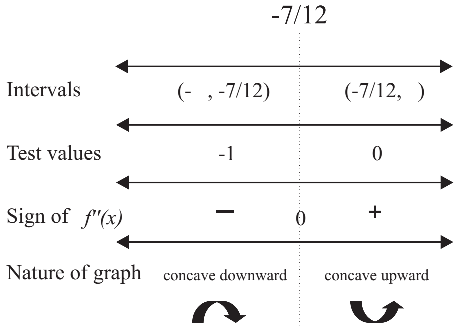


		-2/3	
Intervals	(- , -2/3)		(2/3,)
Test values	-1		0
Sign of $\frac{d^2y}{dx^2}$	−	0	+
Nature of graph	concave downward		concave upward

e)

$$\begin{aligned} f(x) &= 16 - 3x + 3.5x^2 + 2x^3 \\ f'(x) &= -3 + 7x + 6x^2 \\ f''(x) &= 12x + 7 \end{aligned}$$

$f''(x) = 0$ at $x = -\frac{7}{12}$. An interval chart can be used to determine intervals of concavity. f is concave upward on $\left(-\frac{7}{12}, \infty\right)$ and concave downward on $\left(-\infty, -\frac{7}{12}\right)$. f has a point of inflection at $\left(-\frac{7}{12}, \frac{8011}{432}\right)$.



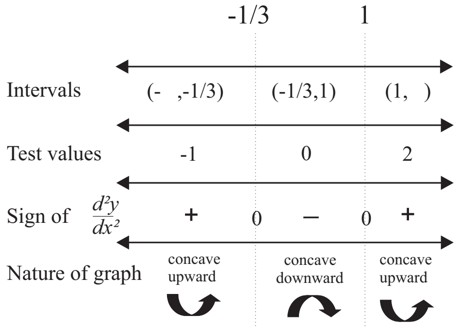
		-7/12	
Intervals	(- , -7/12)		(-7/12,)
Test values	-1		0
Sign of $f''(x)$	−	0	+
Nature of graph	concave downward		concave upward

f)

$$\begin{aligned} y &= 3x^4 - 4x^3 - 6x^2 + 2 \\ \frac{dy}{dx} &= 12x^3 - 12x^2 - 12x \\ \frac{d^2y}{dx^2} &= 36x^2 - 24x - 12 \end{aligned}$$

Determine the roots of $\frac{d^2y}{dx^2}$.

$$\begin{aligned} 36x^2 - 24x - 12 &= 0 \\ 3x^2 - 2x - 1 &= 0 \\ (3x + 1)(x - 1) &= 0 \\ x &= -\frac{1}{3} \text{ or } 1 \end{aligned}$$



		-1/3		1	
Intervals	(- ,-1/3)		(-1/3,1)		(1,)
Test values	-1		0		2
Sign of $\frac{d^2y}{dx^2}$	+	0	−	0	+
Nature of graph	concave upward		concave downward		concave upward

An interval chart can be used to determine intervals of concavity. y is concave upward on $\left(-\infty, -\frac{1}{3}\right)$ and $(1, \infty)$. y is concave downward on $\left(-\frac{1}{3}, 1\right)$. y has points of inflection at $\left(-\frac{1}{3}, \frac{41}{27}\right)$ and $(1, -5)$.

g)

$$\begin{aligned} h(x) &= 3 + 6x - x^6 \\ h'(x) &= 6 - 6x^5 \\ h''(x) &= -30x^4 \end{aligned}$$

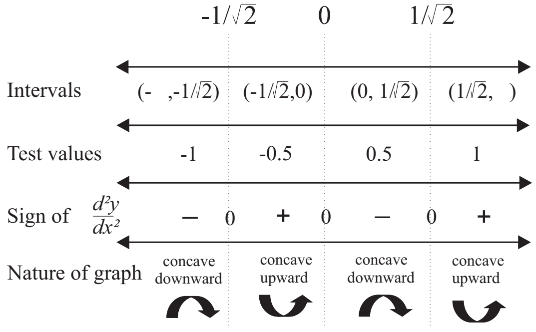
$h''(x) < 0$ for all $\Re$. h is never concave upward. h is concave downward on the interval $(-\infty, \infty)$. Since h'' never changes its sign, there are no points of inflection.

h)

$$\begin{aligned} y &= 3x^5 - 5x^3 + 2 \\ \frac{dy}{dx} &= 15x^4 - 15x^2 \\ \frac{d^2y}{dx^2} &= 60x^3 - 30x \\ &= 30x(2x^2 - 1) \end{aligned}$$

Determine the roots of $\frac{d^2y}{dx^2}$.

$$\begin{aligned} 30x(2x^2 - 1) &= 0 \\ x &= 0 \text{ or } \pm\frac{1}{\sqrt{2}} \end{aligned}$$



		-1/√2		0		1/√2	
Intervals	(- ,-1/√2)		(-1/√2,0)		(0, 1/√2)		(1/√2,)
Test values	-1		-0.5		0.5		1
Sign of $\frac{d^2y}{dx^2}$	−	0	+	0	−	0	+
Nature of graph	concave downward		concave upward		concave downward		concave upward

An interval chart can be used to determine intervals of concavity. y is concave upward on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. y is concave downward on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$. y has points of inflection at $\left(-\frac{1}{\sqrt{2}}, 2 + \frac{7}{4\sqrt{2}}\right)$, $(0, 2)$, and $\left(\frac{1}{\sqrt{2}}, 2 - \frac{7}{4\sqrt{2}}\right)$.

i)

$$\begin{aligned} f(x) &= \frac{x-1}{3-x} \\ f'(x) &= \frac{(3-x)(1)-(x-1)(-1)}{(3-x)^2} \\ &= \frac{2}{(3-x)^2} \\ &= 2(3-x)^{-2} \\ f''(x) &= -4(3-x)^{-3}(-1) \\ &= \frac{4}{(3-x)^3} \end{aligned}$$

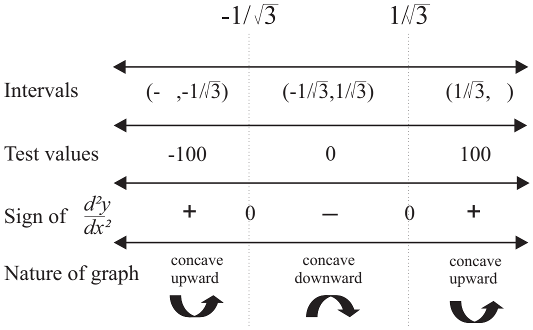
Since f'' has no zeros, there are no points of inflection. Since $f''(x) > 0$ for $x < 3$, f is concave upward on the interval $(-\infty, 3)$. Since $f''(x) < 0$ for $x > 3$, f is concave downward on the interval $(3, \infty)$.

j)

$$\begin{aligned} g(x) &= (x^2+1)^{-1} \\ g'(x) &= -1(x^2+1)^{-2}(2x) \\ &= -2x(x^2+1)^{-2} \\ g''(x) &= (x^2+1)^{-2}(-2) + (-2x)(-2)(x^2+1)^{-3}(2x) \\ &= \frac{2(3x^2-1)}{(x^2+1)^3} \end{aligned}$$

Determine the roots of $g''(x)$.

$$\begin{aligned} 3x^2 - 1 &= 0 \\ x^2 &= \frac{1}{3} \\ x &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$



		-1/√3		1/√3	
Intervals	(- ,-1/√3)		(-1/√3,1/√3)		(1/√3,)
Test values	-100		0		100
Sign of $\frac{d^2y}{dx^2}$	+	0	–	0	+
Nature of graph	concave upward		concave downward		concave upward

An interval chart can be used to determine intervals of concavity. g is concave upward on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$. g is concave downward on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. g has points of inflection at $\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$.

k)

$$\begin{aligned} y &= \frac{1-x^2}{x} \\ &= \frac{1}{x} - x \\ \frac{dy}{dx} &= -\frac{1}{x^2} - 1 \\ \frac{d^2y}{dx^2} &= \frac{2}{x^3} \end{aligned}$$

Since $\frac{d^2y}{dx^2}$ has no zeros, there are no points of inflection. Since $\frac{d^2y}{dx^2} < 0$ for $x < 0$, y is concave downward on the interval $(-\infty, 0)$. Since $\frac{d^2y}{dx^2} > 0$ for $x > 0$, y is concave upward on the interval $(0, \infty)$.

Section 6.3 Page 339 Question 7

a)

$$\begin{aligned} g'(x) &= 0 \\ 4x^3 - 12x &= 0 \\ x(x^2 - 3) &= 0 \\ x &= 0, \pm\sqrt{3} \\ g''(x) &= 12x^2 - 12 \end{aligned}$$

Since $g''(0) = -12 < 0$, g has a local maximum of $0^4 - 6(0)^2 + 10$ or 10 at $x = 0$. Since $g''(-\sqrt{3}) = 24 > 0$, g has a local minimum of $\left(-\sqrt{3}\right)^4 - 6\left(-\sqrt{3}\right)^2 + 10$ or 1 at $x = -\sqrt{3}$. Since $g''(\sqrt{3}) = 24 > 0$, g has a local minimum of $\left(\sqrt{3}\right)^4 - 6\left(\sqrt{3}\right)^2 + 10$ or 1 at $x = \sqrt{3}$.

b)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{(x-1)^2(1) - x(2)(x-1)}{(x-1)^4} &= 0 \\ \frac{-x-1}{(x-1)^3} &= 0 \\ -x-1 &= 0 \\ x &= -1 \\ \frac{d^2y}{dx^2} &= \frac{(x-1)^3(-1) - (-x-1)(3)(x-1)^2}{(x-1)^6} \\ &= \frac{2x+4}{(x-1)^6} \end{aligned}$$

Since $\frac{d^2y}{dx^2}\bigg|_{x=-1} = \frac{1}{32} > 0$, y has a local minimum of $\frac{-1}{(-1-1)^2}$ or $-\frac{1}{4}$ at $x = -1$. y has no local maxima.

c)

$$\begin{aligned} h'(x) &= 0 \\ \frac{(2x+3)(2x) - x^2(2)}{(2x+3)^2} &= 0 \\ \frac{2x^2+6x}{(2x+3)^2} &= 0 \\ 2x^2 + 6x &= 0 \\ x(x+3) &= 0 \\ x &= 0 \text{ or } -3 \\ h''(x) &= \frac{(2x+3)^2(4x+6) - (2x^2+6x)(2)(2x+3)(2)}{(2x+3)^4} \\ &= \frac{18}{(2x+3)^3} \end{aligned}$$

Since $h''(-3) = -\frac{2}{3} < 0$, h has a local maximum of $\frac{(-3)^2}{2(-3)+3}$ or -3 at $x = -3$. Since $h''(0) = \frac{2}{3} > 0$, h has a local minimum of $\frac{0^2}{2(0)+3}$ or 0 at $x = 0$.

d)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ -\frac{9}{x^2} - 2x &= 0 \\ x^3 &= -\frac{9}{2} \\ x &= -\sqrt[3]{\frac{9}{2}} \\ \frac{d^2y}{dx^2} &= \frac{18}{x^3} - 2 \end{aligned}$$

Since $\frac{d^2y}{dx^2} = -6 < 0$ at $x = -\sqrt[3]{\frac{9}{2}}$, y has a local maximum of $\frac{9}{-\sqrt[3]{\frac{9}{2}}} - \left(-\sqrt[3]{\frac{9}{2}}\right)^2$ or $-\frac{27\sqrt[3]{2}}{2\sqrt[3]{9}}$ at $x = -\sqrt[3]{\frac{9}{2}}$. y has no local minima.

Apply, Solve, Communicate

Section 6.3 Page 339 Question 9

a)

Function	Negative	Zero	Positive
$f(x)$	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$(-\infty, -3)$	$-3, 0$	$(-3, 0), (0, \infty)$
$f''(x)$	$(-\infty, -5), \left(-\frac{3}{2}, 0\right)$	$-5, -\frac{3}{2}, 0$	$\left(-5, -\frac{3}{2}\right), (0, \infty)$

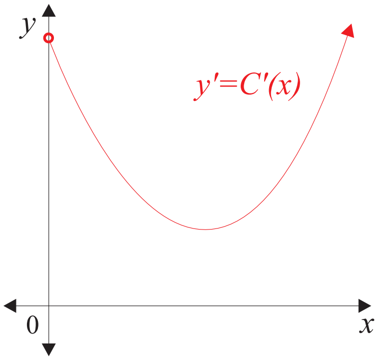
b) Yes. $\left(-\frac{3}{2}, 0\right)$.

Section 6.3 Page 340 Question 10

a) Manufacturers have fixed costs before a single item is produced.

b) The point of inflection defines the production level for which the the cost per item manufactured is neither decreasing nor increasing. It is where the marginal cost is maximized.

c)



Section 6.3 Page 340 Question 11

a) The prey population function is concave upward on the interval $(100, \infty)$. It appears concave downward in the interval $(0, 100)$.

b) The coordinates of the point of inflection are approximately $(100, 200)$.

c) The time of the maximum value for the prey population and the point of inflection for the predator function are the same. The time of the point of inflection of the prey model and the maximum value for the predator model are the same. The concave downward interval of the prey population coincides with the interval of increase for the predator population. The concave upward interval of the prey population coincides with the interval of decrease for the predator population.

Section 6.3 Page 340 Question 12

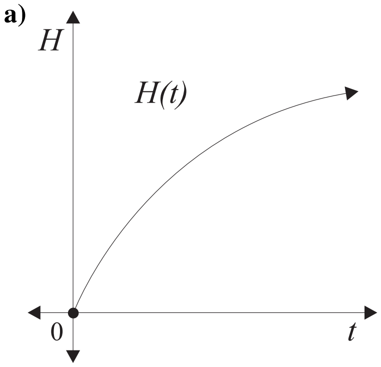
a)

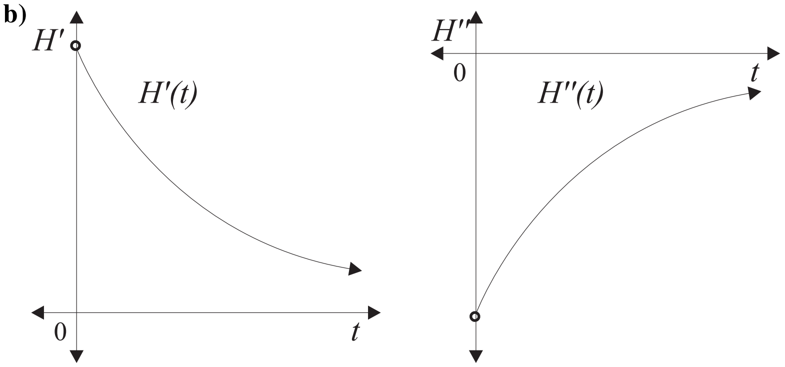
$$\begin{aligned} N'(x) &= 0 \\ -2x + 200 &= 0 \\ x &= 100 \end{aligned}$$

The critical numbers also include 0 and 150, the endpoints of the interval. Comparison of the function values at these points leads to the following results. N has an absolute minimum value of 12 at $x = 0$. Since $N''(10) = -2 < 0$, N has an absolute maximum of $-100^2 + 200(100) + 12$ or 10 012 at $x = 100$.

b) Answers will vary.

Section 6.3 Page 340 Question 13





c) As the height of the vase increases, so does the marginal volume. This is to say, ever-increasing amounts of water are required to increase the height by the same amount. With water pouring in a constant rate, although $H(t)$ will always be increasing, it will increase at a slower rate as time evolves. This positive, but lower, rate is reflected in the shape of $H'(t)$. The slowing down of $H'(t)$ is reflected in the shape of $H''(t)$. Its function values are negative (falling), but increasing.

Section 6.3 Page 340 Question 14

The function can be rewritten as $f(x) = \sqrt[3]{x^3 + 4x^2 + 4x}$.

$$f'(x) = 0$$

$$\frac{3x^2 + 8x + 4}{3\sqrt[3]{(x^3 + 4x^2 + 4x)^2}} = 0$$

$$\frac{(3x + 2)(x + 2)}{3\sqrt[3]{(x(x + 2)^2)^2}} = 0$$

$$x = -\frac{2}{3};\ x \neq 0 \text{ or } -2$$

$$f''(x) = 0$$

$$\frac{-8}{9\sqrt[3]{x^5(x + 2)^4}} = 0$$

$$x = \text{no roots}$$

The critical numbers of both $f'(x)$ and $f''(x)$ are −2 and 0. Interval charts are used to determine the properties of $f(x)$.

-2 -2/3 0

Intervals	(- , -2)	(-2,-2/3)	(-2/3,0)	(0,)
Test values	-3	-1	-0.5	1
Sign of $f'(x)$	+	−	0 +	+
Nature of graph	increasing ↗	decreasing ↘	increasing ↗	increasing ↗

-2 0

Intervals	(- , -2)	(-2,0)	(0,)
Test values	-3	-1	1
Sign of $f''(x)$	+	+	−
Nature of graph	concave upward ∪	concave upward ∪	concave downward ∩

a) f is increasing on the intervals $(-\infty, -2)$ and $\left(-\frac{2}{3}, \infty\right)$. f is decreasing on the interval $\left(-2, -\frac{2}{3}\right)$.

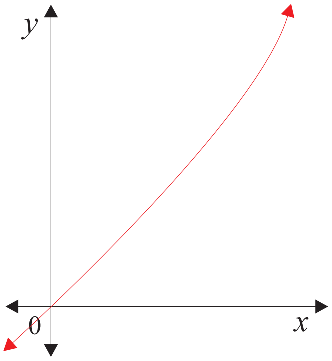
b) Comparing function values at the three critical numbers, −2, $-\frac{2}{3}$, and 0, reveals a local maximum value of 0 at $x = -2$ and a local minimum value of $-\frac{2}{3}\sqrt[3]{4}$ at $x = -\frac{2}{3}$.

c) f is concave downward on the interval $(0, \infty)$. f is concave upward on the intervals $(-\infty, -2)$ and $(-2, 0)$.

d) The lone point of inflection is at $(0, 0)$.

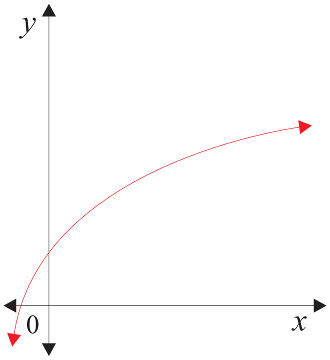
Section 6.3 Page 340 Question 15

a)

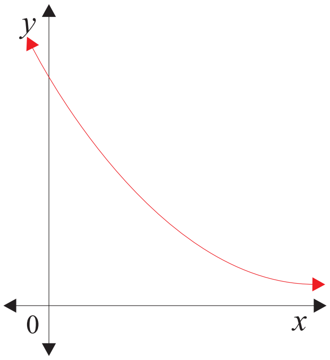


b) This function is not possible. If the first derivative is negative, it is decreasing. Eventually, the function would cross the x-axis and the function values would become negative.

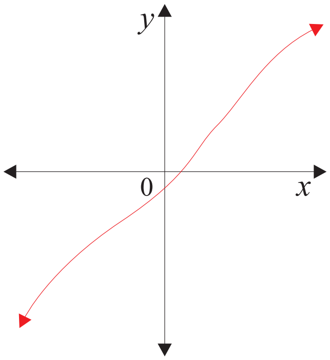
c)



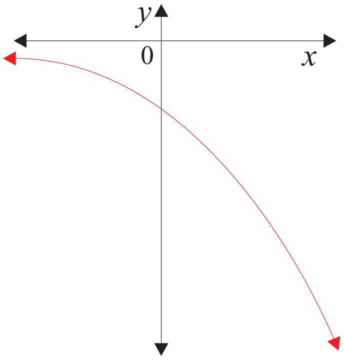
d)



e)



f)

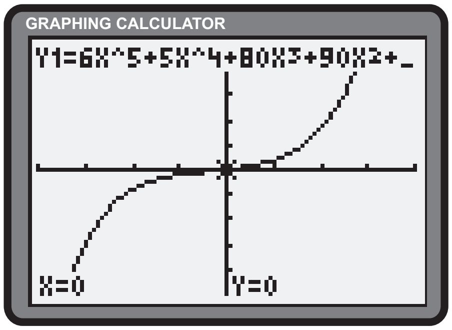


Section 6.3 Page 341 Question 16

a) **i)** The graph of f appears at the right.

ii) f has no extrema.

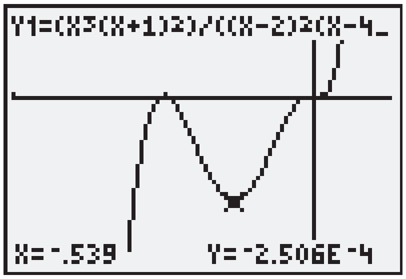
iii) f has a point of inflection at $(0, 0)$.



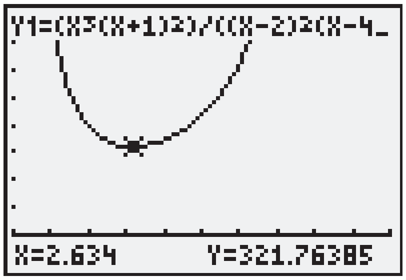
```
Y1=6X^5+5X^4+80X³+90X²+_
X=0                Y=0
```

$x \in [-4, 4], \; y \in [-5000, 5000]$

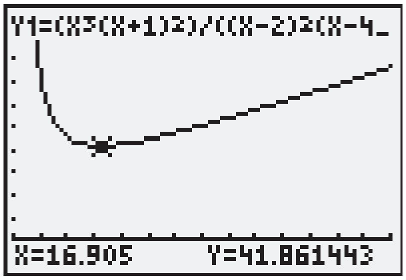
b) **i)** Three views of f appear below.



$x \in [-2, 0.5],\ y \in [-0.0004, 0.0002]$



$x \in [2, 4],\ y \in [-100, 800]$



$x \in [4, 60],\ y \in [-10, 100]$

ii) f has a local maximum at $(-1, 0)$. Local minima are located at $(-0.539, -0.000\,25)$, $(2.634, 321.764)$, and $(16.905, 41.861)$.

iii) f has points of inflection at approximately $(-0.8, -0.000\,11)$, $(-0.287, -0.000\,13)$, and $(0, 0)$.

Section 6.3 Page 341 Question 17

No. Extrema of the function will occur where there is a sign change at the zeros of the second derivative. Since extrema of f at $x = c$ can also occur where $f'(c)$ does not exist, it is not possible to determine *all* of the x-coordinates of the maximum and minimum points.

Section 6.3 Page 341 Question 18

$$y = x^3 + cx^2 + x + d \tag{1}$$
$$\frac{dy}{dx} = 3x^2 + 2cx + 1$$
$$\frac{d^2y}{dx^2} = 6x + 2c \tag{2}$$

Substitute $x = 4$ into (2), set $\frac{d^2y}{dx^2} = 0$, and solve for c.

$$\begin{aligned} 6(4) + 2c &= 0 \\ 2c &= -24 \\ c &= -12 \end{aligned}$$

Substitute $x = 4$, $c = -12$ and $y = 3$ into (1) and solve for d.

$$\begin{aligned} 3 &= 4^3 - 12(4)^2 + 4 + d \\ 3 &= 64 - 192 + 4 + d \\ d &= 127 \end{aligned}$$

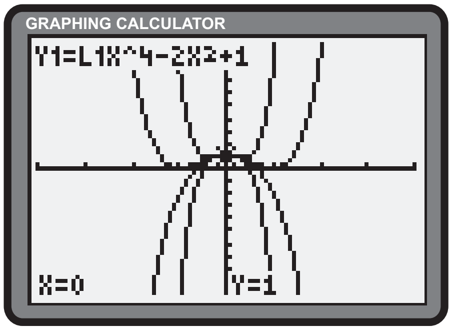
If $c = -12$ and $d = 127$, the cubic curve $y = x^3 - 12x^2 + x + 127$ has a point of inflection at $(4, 3)$.

Section 6.3 Page 341 Question 19

a) Determine the critical numbers.

$$\begin{aligned} f'(x) &= 0 \\ 4cx^3 - 4x &= 0 \\ x(cx^2 - 1) &= 0 \\ x &= 0, \pm\frac{1}{\sqrt{c}} \end{aligned}$$

$$\begin{aligned} f''(x) &= 12cx^2 - 4 \\ f''(0) &= -4 < 0 \\ f''\left(\pm\frac{1}{\sqrt{c}}\right) &= 8 > 0 \end{aligned}$$



If $c > 0$, three critical numbers exist. Results from the second derivative test reveal a maximum value of 1 at $x = 0$ and a minimum of $\frac{c-1}{c}$ at $x = \pm\frac{1}{\sqrt{c}}$. If $c \le 0$, one critical number exists at $x = 0$. Results from the second derivative test reveal a maximum value of 1 at $x = 0$.

b) For $c > 0$, the maximum and minimum points are $(0, 1)$, $\left(\frac{1}{\sqrt{c}}, \frac{c-1}{\sqrt{c}}\right)$, and $\left(-\frac{1}{\sqrt{c}}, \frac{c-1}{c}\right)$. Show that these points lie on the curve, $y = 1 - x^2$.

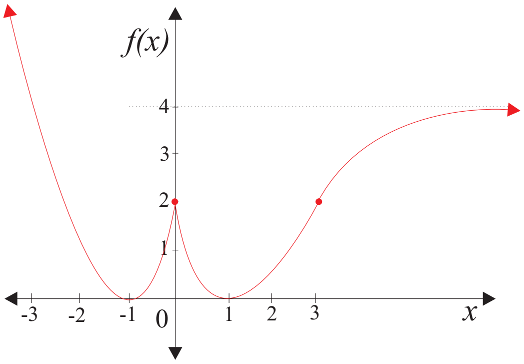
$$\begin{aligned} y(0) &= 1 - 0^2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} y\left(\frac{1}{\sqrt{c}}\right) &= 1 - \left(\frac{1}{\sqrt{c}}\right)^2 \\ &= 1 - \frac{1}{c} \\ &= \frac{c-1}{c} \end{aligned}$$

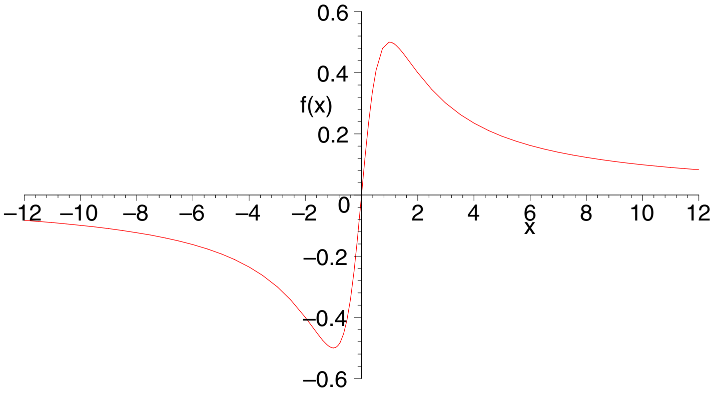
$$\begin{aligned} y\left(-\frac{1}{\sqrt{c}}\right) &= 1 - \left(-\frac{1}{\sqrt{c}}\right)^2 \\ &= 1 - \frac{1}{c} \\ &= \frac{c-1}{c} \end{aligned}$$

Thus, all three extrema lie on the curve $y = 1 - x^2$. For $c \leq 0$ the maximum point is $(0, 1)$, which lies on the curve $y = 1 - x^2$.

Section 6.3 Page 341 Question 20



Section 6.3 Page 341 Question 21

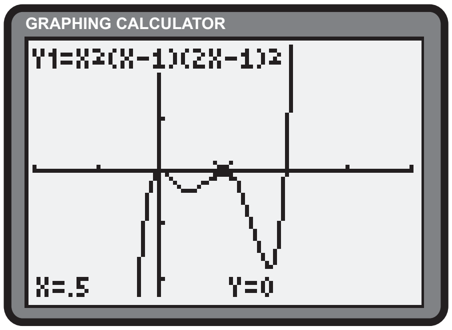


Section 6.3 Page 341 Question 22

a) The roots of f are $0, -\frac{1}{2}$, and 1. Since the first two are double roots, f is tangent to the x-axis at these locations. Using this information, local maxima will occur at $(0, 0)$ and $\left(\frac{1}{2}, 0\right)$. Local minima are anticipated on the intervals $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$. The product rule can be used to show that

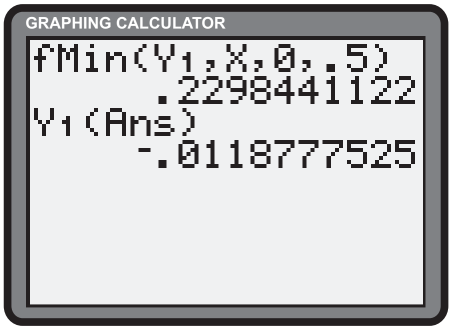
$$(fgh)' = f'gh + fg'h + fgh'$$

This is used to simplify the differentiation process.

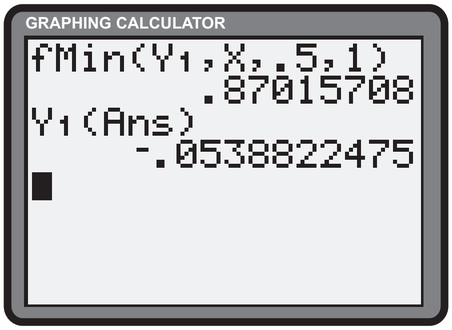


$$\begin{aligned} f'(x) &= 0 \\ 2x(x-1)(2x-1)^2 + x^2(1)(2x-1)^2 + x^2(x-1)(2(2x-1)(2)) &= 0 \\ x(2x-1)(10x^2 - 11 + 2) &= 0 \\ x &= 0, -\frac{1}{2}, \frac{11 \pm \sqrt{41}}{20} \quad (1) \end{aligned}$$

The first two critical numbers confirm the two maxima. The latter two critical numbers are confirmed through the graphing calculator activities below.



```
fMin(Y1,X,0,.5)
       .2298441122
Y1(Ans)
      -.0118777525
```

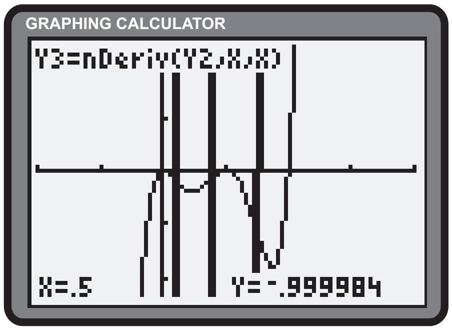


```
fMin(Y1,X,.5,1)
         .87015708
Y1(Ans)
      -.0538822475
█
```

The local minima are stated approximately as $(0.2298, -0.0119)$ and $(0.8702, -0.0539)$. For the points of inflection, the graphing calculator can be used to determine the zeros of the second derivative. The points of inflection are $(0.0913, -0.005\ 06)$, $(0.3712, -0.005\ 75)$, and $(0.7375, -0.0322)$.

b) No. The given window gives no indication of activity on the interval $(0, 1)$.

c) A suitable window is $x \in [-1, 2]$, $y \in [-0.07, 0.07]$.



6.4 Vertical Asymptotes

Practise

Section 6.4 Page 347 Question 1

a) The equations of the vertical asymptotes are $x = -8$, $x = -5$, $x = -1$, $x = 4$, and $x = 9$.

b) **i)** $\lim\limits_{x \to -5^-} f(x) = \infty$

ii) $\lim\limits_{x \to -5^+} f(x) = \infty$

iii) $\lim\limits_{x \to -1^-} f(x) = -\infty$

iv) $\lim\limits_{x \to -1^+} f(x) = -\infty$

v) $\lim\limits_{x \to 4^-} f(x) = \infty$

vi) $\lim\limits_{x \to 4^+} f(x) = \infty$

vii) $\lim\limits_{x \to 9^-} f(x) = -\infty$

viii) $\lim\limits_{x \to 9^+} f(x) = -\infty$

c) For each of the following, the limit does not exist. Since the function values approach the same infinity from both sides, a more specific result is provided.

i) $\lim\limits_{x \to -5} f(x) = \infty$

ii) $\lim\limits_{x \to -1} f(x) = -\infty$

iii) $\lim\limits_{x \to 4} f(x) = \infty$

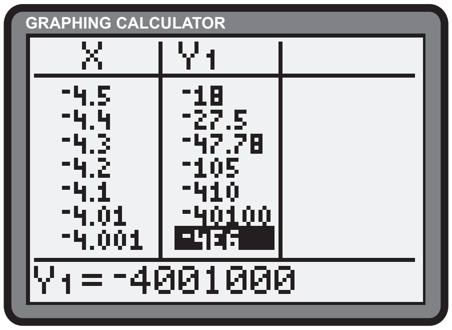
iv) $\lim\limits_{x \to 9} f(x) = -\infty$

Section 6.4 Page 347 Question 3

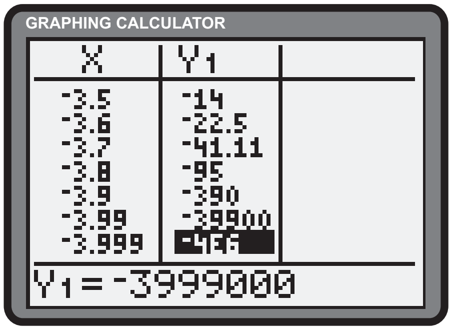
a) As x approaches 5 from both sides, $(x - 5)^2$ becomes smaller, but positive. The result is that $\frac{1}{(x-5)^2}$ increases positively, without bound. $\lim\limits_{x \to 5} \frac{1}{(x-5)^2} = \infty$.

b) As x approaches -6 from both sides, $(x + 6)^2$ becomes smaller, but positive. The result is that $\frac{-3}{(x+6)^2}$ increases negatively, without bound. $\lim\limits_{x \to -6} \frac{-3}{(x+6)^2} = -\infty$.

c) The **TABLE feature** of the graphing calculator can be used to determine the behaviour of the function from both sides of -4. $\lim\limits_{x \to -4} \frac{x}{(x+4)^2} = -\infty$.

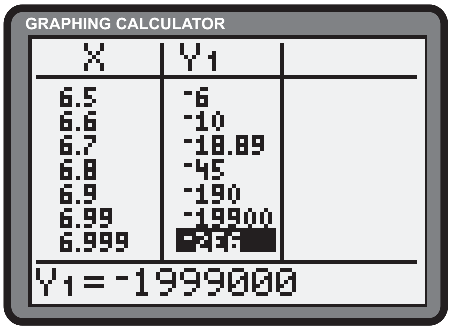

GRAPHING CALCULATOR

```
  X     | Y1
-4.5    | -18
-4.4    | -27.5
-4.3    | -47.78
-4.2    | -105
-4.1    | -410
-4.01   | -40100
-4.001  | -4E6
Y1=-4001000
```

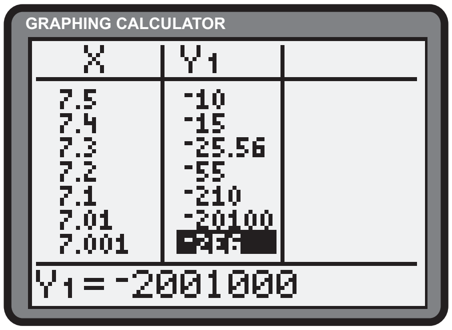

GRAPHING CALCULATOR

```
  X     | Y1
-3.5    | -14
-3.6    | -22.5
-3.7    | -41.11
-3.8    | -95
-3.9    | -390
-3.99   | -39900
-3.999  | -4E6
Y1=-3999000
```

d) The **TABLE feature** of the graphing calculator can be used to determine the behaviour of the function from both sides of 7. $\lim\limits_{x \to 7} \frac{5-x}{(x-7)^2} = -\infty$.

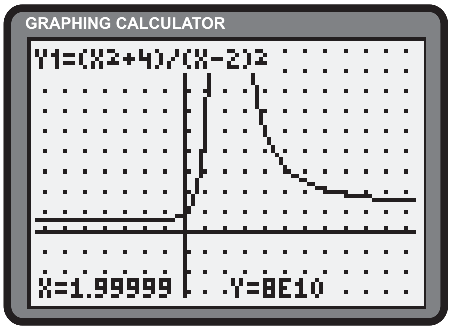

GRAPHING CALCULATOR

```
  X     | Y1
6.5     | -6
6.6     | -10
6.7     | -18.89
6.8     | -45
6.9     | -190
6.99    | -19900
6.999   | -2E6
Y1=-1999000
```

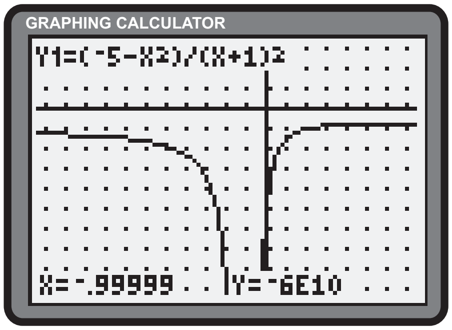

GRAPHING CALCULATOR

```
  X     | Y1
7.5     | -10
7.4     | -15
7.3     | -25.56
7.2     | -55
7.1     | -210
7.01    | -20100
7.001   | -2E6
Y1=-2001000
```

e) The **TRACE feature** of the graphing calculator suggests $\lim\limits_{x \to 2} \dfrac{x^2 + 4}{(x - 2)^2} = \infty$.

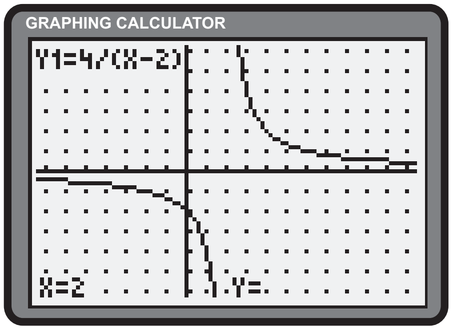


f) The **TRACE feature** of the graphing calculator suggests $\lim\limits_{x \to -1} \dfrac{-5 - x^2}{(x + 1)^2} = -\infty$.

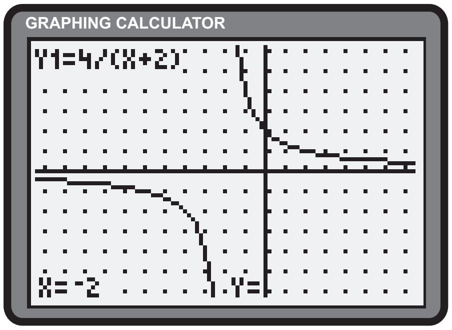


Section 6.4 Page 348 Question 5

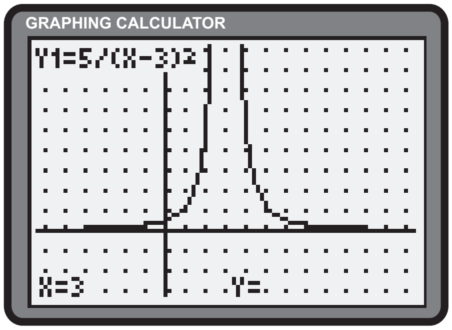
a) Since 2 is a root of the denominator and *not* a root of the numerator, $x = 2$ is a vertical asymptote of f.



b) Since -2 is a root of the denominator and *not* a root of the numerator, $x = -2$ is a vertical asymptote of y.



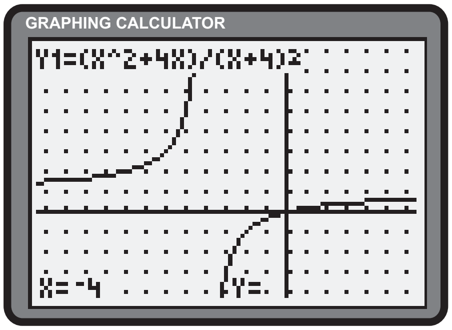
c) Since 3 is a root of the denominator and *not* a root of the numerator, $x = 3$ is a vertical asymptote of k.



d)

$$\begin{aligned} g(x) &= \frac{x(x+4)}{(x+4)^2} \\ &= \frac{x}{x+4} \end{aligned}$$

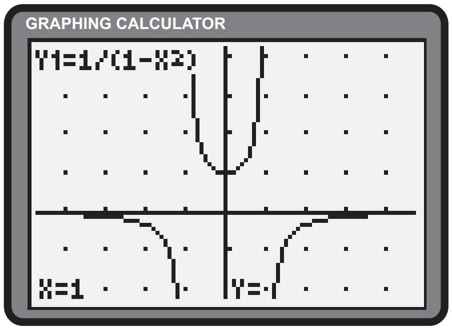
Since -4 is a root of the denominator and *not* a root of the numerator, $x = -4$ is a vertical asymptote of g.



e)

$$\begin{aligned} y &= \frac{1}{1-x^2} \\ &= \frac{1}{(1-x)(1+x)} \end{aligned}$$

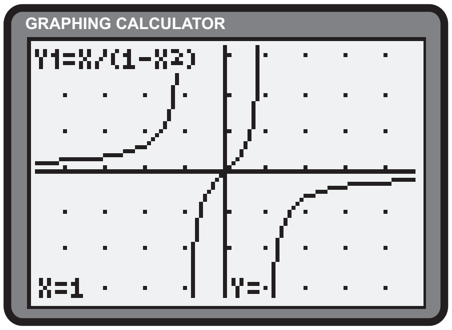
Since ±1 are roots of the denominator and *not* roots of the numerator, $x = \pm 1$ are vertical asymptotes of y.



f)

$$\begin{aligned} y &= \frac{x}{1-x^2} \\ &= \frac{x}{(1-x)(1+x)} \end{aligned}$$

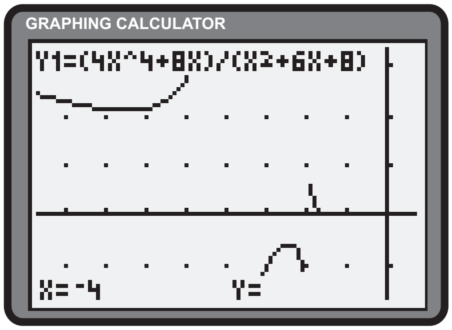
Since ±1 are roots of the denominator and *not* roots of the numerator, $x = \pm 1$ are vertical asymptotes of y.



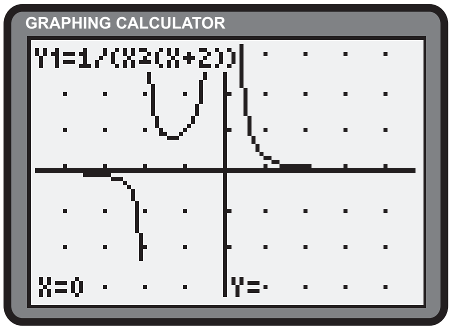
g)

$$\begin{aligned} h(x) &= \frac{4x^4 + 8x}{(x+2)(x+4)} \\ &= \frac{4x(x^3+2)}{(x+2)(x+4)} \end{aligned}$$

Since −2 and −4 are roots of the denominator and *not* roots of the numerator, $x = -2$ and $x = -4$ are vertical asymptotes of h.



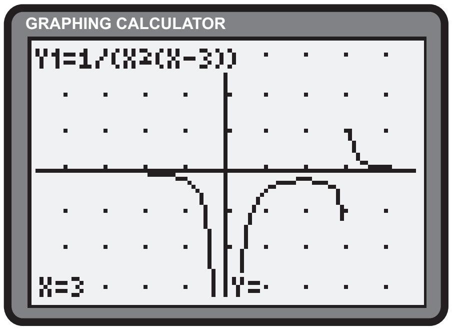
h) Since 0 and −2 are roots of the denominator and *not* roots of the numerator, $x = 0$ and $x = -2$ are vertical asymptotes of y.



i)

$$\begin{aligned} y &= \frac{x+3}{x^2(x-3)(x+3)} \\ &= \frac{1}{x^2(x-3)};\ x \neq -3 \end{aligned}$$

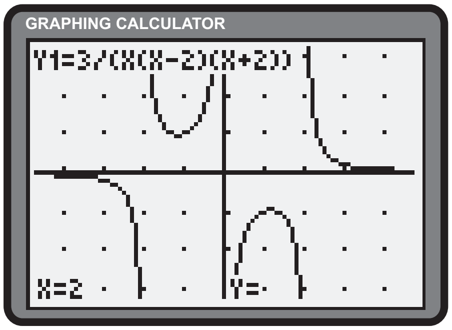
Since 0 and 3 are roots of the denominator and *not* roots of the numerator, $x = 0$ and $x = 3$ are vertical asymptotes of y.



j)

$$\begin{aligned} q(x) &= \frac{3}{x(x^2-4)} \\ &= \frac{3}{x(x-2)(x+2)} \end{aligned}$$

Since 0, −2 and 2 are roots of the denominator and *not* roots of the numerator, $x = 0$ and $x = \pm 2$ are vertical asymptotes of q.



Apply, Solve, Communicate

Section 6.4 Page 348 Question 7

a)

$$\begin{aligned} C(50) &= \frac{50\,000}{100-50} \\ &= \frac{50\,000}{50} \\ &= 1000 \end{aligned}$$

The cost of removing half (50%) of the pollutants is \$1000.

$$\begin{aligned} C(90) &= \frac{50\,000}{100-90} \\ &= \frac{50\,000}{10} \\ &= 5000 \end{aligned}$$

The cost of removing 90% of the pollutants is \$5000.

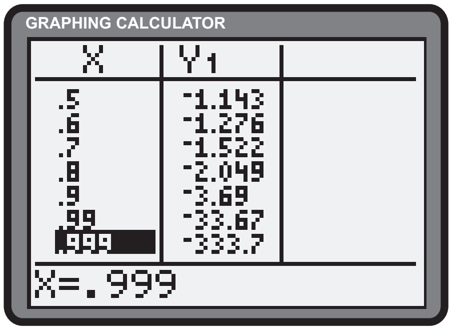
b) As $p \to 100^-$, the denominator, $100 - p$, approaches 0^+. The result is that $\lim\limits_{x\to 100^-} C(p) = \infty$.

c) No. The costs would be prohibitive.

d) No.

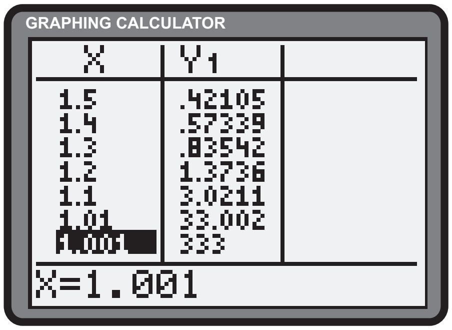
Section 6.4 Page 348 Question 8

a) As $x \to 1^-$, the denominator, $x^3 - 1$, approaches 0^-. The result is that $\lim\limits_{x\to 1^-} \frac{1}{x^3-1} = -\infty$. As $x \to 1^+$, the denominator, $x^3 - 1$, approaches 0^+. The result is that $\lim\limits_{x\to 1^+} \frac{1}{x^3-1} = \infty$.



X	Y1
.5	-1.143
.6	-1.276
.7	-1.522
.8	-2.049
.9	-3.69
.99	-33.67
.999	-333.7

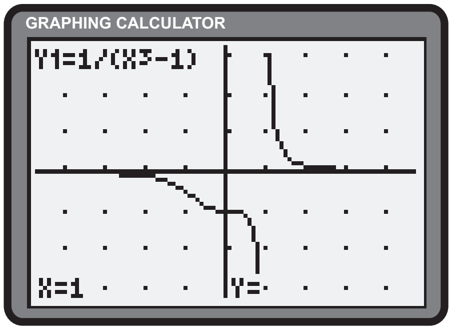
X=.999



X	Y1
1.5	.42105
1.4	.57339
1.3	.83542
1.2	1.3736
1.1	3.0211
1.01	33.002
1.001	333

X=1.001

b) The graphing calculator confirms the results $\lim\limits_{x\to 1^-} \frac{1}{x^3-1} = -\infty$ and $\lim\limits_{x\to 1^+} \frac{1}{x^3-1} = \infty$.



Section 6.4 Page 348 Question 9

a) Since d defines the distance between the masses, $d > 0$. As $d \to 0^+$, the denominator, d^2, approaches 0^+. The result is that $\lim\limits_{d\to 0^+} \frac{Gm_1m_2}{d^2} = \infty$.

b) The masses of normal size objects are so small the force of gravity is negligible.

c) Answers will vary.

Section 6.4 Page 349 Question 10

$$\begin{aligned}\frac{1}{x^4} &> 100\,000\,000\\ x^4 &< \frac{1}{100\,000\,000}\\ |x| &< \frac{1}{100}\end{aligned}$$

For $\frac{1}{x^4} < 100\,000\,000$, $|x| < 0.01$.

Section 6.4 Page 349 Question 11

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{5}{x}-\frac{2}{x^3}\right) &= \lim_{x\to 0^+}\left(\frac{5x^2}{x^3}-\frac{2}{x^3}\right)\\ &= \lim_{x\to 0^+}\left(\frac{5x^2-2}{x^3}\right)\end{aligned}$$

As $x \to 0^+$, the numerator approaches -2^+. The denominator, x^3, approaches 0^+. The result is that $\lim\limits_{x\to 0^+}\left(\frac{5}{x}-\frac{2}{x^3}\right) = -\infty$.

Section 6.4 Page 349 Question 12

a) As $v \to c^-$, the expression $\frac{v^2}{c^2}$ approaches 1^-. In turn, the expression $1-\frac{v^2}{c^2}$ approaches 0^+. As a result, the denominator approaches 0^+. The net result is that $m = \lim\limits_{v\to c^-}\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ or ∞. To summarize, as the velocity of the object approaches the speed of light, its mass increases without bound.

b) If the particle reaches a speed equal to or greater than the speed of light, the mass of the particle will be undefined.

Section 6.4 Page 349 Question 13

a) $\lim\limits_{x\to a^-} f(x) = \infty$ if n is even and $p(a) > 0$ or n is odd and $p(a) < 0$.

$\lim\limits_{x\to a^-} f(x) = -\infty$ if n is odd and $p(a) > 0$ or n is even and $p(a) < 0$.

b) $\lim\limits_{x\to a^+} f(x) = \infty$ if $p(a) > 0$. $\lim\limits_{x\to a^+} f(x) = -\infty$ if $p(a) < 0$.

Section 6.4 Page 349 Question 14

a) Since ∞ is not a real number, it is not bound to the result $a - a = 0$. In isolation, the expression $\infty - \infty$ has no meaning, hence no numerical value.

b) For the given functions, as $x \to 3$, both f and g tend to ∞. This would suggest the following,

$$\begin{aligned}\lim_{x\to 3}\left(\frac{x^2}{(x-3)^2}-\frac{6x-9}{(x-3)^2}\right) &= \lim_{x\to 3}\frac{x^2}{(x-3)^2} - \lim_{x\to 3}\frac{6x-9}{(x-3)^2}\\ &= \infty - \infty\end{aligned}$$

However, simplifying the expression before evaluating the limit provides further insight.

$$\begin{aligned}\lim_{x\to 3}\left(\frac{x^2}{(x-3)^2}-\frac{6x-9}{(x-3)^2}\right) &= \lim_{x\to 3}\frac{x^2-6x+9}{(x-3)^2}\\ &= \lim_{x\to 3}\frac{(x-3)^2}{(x-3)^2}\\ &= \lim_{x\to 3} 1\\ &= 1\end{aligned}$$

Under the influence of limits, this $\infty - \infty$ leads to a result of 1. Using different functions may lead to a result other than 1. In conclusion, when the expressions $\infty \pm \infty$ arise in the context of limits, the results are said to be indeterminate. This is to say, they require further investigation.

6.5 Horizontal and Oblique Asymptotes

Practise

Section 6.5 Page 359 Question 1

a) horizontal asymptotes: $y = 2,\ y = -1$; vertical asymptotes: $x = -1,\ x = 4$.
b) horizontal asymptotes: $y = 2,\ y = -2$; vertical asymptotes: $x = -4,\ x = 4$.

Section 6.5 Page 359 Question 3

a) $\lim_{x\to\infty} \frac{3}{x} = 0^+$

b) $\lim_{x\to-\infty} \frac{12}{x} = 0^-$

c)
$$\begin{aligned}\lim_{x\to\infty} \frac{4x}{x^6} &= \lim_{x\to\infty} \frac{4}{x^5},\ x \neq 0\\ &= 0^+\end{aligned}$$

d) $\lim_{x\to-\infty} \frac{6}{x^3} = 0^-$

e)
$$\begin{aligned}\lim_{x\to-\infty} \frac{-5}{x^5} &= -\lim_{x\to-\infty} \frac{5}{x^5}\\ &= -(0^-)\\ &= 0^+\end{aligned}$$

f)
$$\begin{aligned}\lim_{x\to\infty} \frac{3x+2}{x-1} &= \lim_{x\to\infty} \frac{\frac{3x+2}{x}}{\frac{x-1}{x}}\\ &= \lim_{x\to\infty} \frac{3+\frac{2}{x}}{1-\frac{1}{x}}\\ &= \frac{3+0}{1-0}\\ &= 3\end{aligned}$$

g)
$$\begin{aligned}\lim_{x\to-\infty} \frac{3x+2}{x-1} &= \lim_{x\to-\infty} \frac{\frac{3x+2}{x}}{\frac{x-1}{x}}\\ &= \lim_{x\to-\infty} \frac{3+\frac{2}{x}}{1-\frac{1}{x}}\\ &= \frac{3+0}{1-0}\\ &= 3\end{aligned}$$

h)
$$\begin{aligned}\lim_{x\to\infty} \frac{4-x^3}{3+2x^3} &= \lim_{x\to\infty} \frac{\frac{4-x^3}{x^3}}{\frac{3+2x^3}{x^3}}\\ &= \lim_{x\to\infty} \frac{\frac{4}{x^3}-1}{\frac{3}{x^3}+2}\\ &= \frac{0-1}{0+2}\\ &= -\frac{1}{2}\end{aligned}$$

i)
$$\begin{aligned}\lim_{x\to\infty} \frac{x+7}{x^2-7x+5} &= \lim_{x\to\infty} \frac{\frac{x+7}{x^2}}{\frac{x^2-7x+5}{x^2}}\\ &= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{7}{x^2}}{1-\frac{7}{x}+\frac{5}{x^2}}\\ &= \frac{0+0}{1-0+0}\\ &= 0\end{aligned}$$

Apply, Solve, Communicate

Section 6.5 Page 360 Question 5

a)

$$\begin{aligned}\lim_{x\to\infty}\frac{2x-3}{5-x} &= \lim_{x\to\infty}\frac{\frac{2x-3}{x}}{\frac{5-x}{x}}\\ &= \lim_{x\to\infty}\frac{2-\frac{3}{x}}{\frac{5}{x}-1}\\ &= \frac{2-0}{0-1}\\ &= -2\end{aligned}$$

A similar result occurs as $x \to -\infty$. The equation of the horizontal asymptote is $y = -2$.

b)

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{x^2-4} &= \lim_{x\to\infty}\frac{\frac{x}{x^2}}{\frac{x^2-4}{x^2}}\\ &= \lim_{x\to\infty}\frac{\frac{1}{x}}{1-\frac{4}{x^2}}\\ &= \frac{0}{1-0}\\ &= 0\end{aligned}$$

A similar result occurs as $x \to -\infty$. The equation of the horizontal asymptote is $y = 0$.

c)

$$\begin{aligned}\lim_{x\to\infty}\frac{3-8x}{2x+5} &= \lim_{x\to\infty}\frac{\frac{3-8x}{x}}{\frac{2x+5}{x}}\\ &= \lim_{x\to\infty}\frac{\frac{3}{x}-8}{2+\frac{5}{x}}\\ &= \frac{0-8}{2+0}\\ &= -4\end{aligned}$$

A similar result occurs as $x \to -\infty$. The equation of the horizontal asymptote is $y = -4$.

d)

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2+1}{x^2-1} &= \lim_{x\to\infty}\frac{\frac{x^2+1}{x^2}}{\frac{x^2-1}{x^2}}\\ &= \lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}\\ &= \frac{1+0}{1-0}\\ &= 1\end{aligned}$$

A similar result occurs as $x \to -\infty$. The equation of the horizontal asymptote is $y = 1$.

e)

$$\begin{aligned}\lim_{x\to\infty}\left(1-\frac{x}{x^2-9}\right) &= \lim_{x\to\infty}\left(1-\frac{\frac{x}{x^2}}{\frac{x^2-9}{x^2}}\right)\\ &= \lim_{x\to\infty}\left(1-\frac{\frac{1}{x}}{1-\frac{9}{x^2}}\right)\\ &= 1-\frac{0}{1-0}\\ &= 1\end{aligned}$$

A similar result occurs as $x \to -\infty$. The equation of the horizontal asymptote is $y = 1$.

f)

$$\begin{aligned}\lim_{x\to\infty}\frac{6x^2+4x+1}{5-3x^2} &= \lim_{x\to\infty}\frac{\frac{6x^2+4x+1}{x^2}}{\frac{5-3x^2}{x^2}}\\ &= \lim_{x\to\infty}\frac{6+\frac{4}{x}+\frac{1}{x^2}}{\frac{5}{x^2}-3}\\ &= \frac{6+0+0}{0-3}\\ &= -2\end{aligned}$$

A similar result occurs as $x \to -\infty$. The equation of the horizontal asymptote is $y = -2$.

Section 6.5 Page 360 Question 6

a) Rewrite y.

$$\begin{aligned} y &= \frac{3x^2 - 4x + 5}{x} \\ &= 3x - 4 + \frac{5}{x} \qquad (1) \end{aligned}$$

As $|x| \to \infty$, $\frac{5}{x} \to 0$. As a consequence, (1) approximates the expression $3x - 4$. The equation of the oblique asymptote is $y = 3x - 4$.

b) Rewrite $h(x)$.

$$\begin{aligned} h(x) &= \frac{x^3 - 4}{x^2} \\ &= x - \frac{4}{x^2} \qquad (1) \end{aligned}$$

As $|x| \to \infty$, $\frac{4}{x^2} \to 0$. As a consequence, (1) approximates the expression x. The equation of the oblique asymptote is $y = x$.

c) Rewrite y.

$$\begin{aligned} y &= \frac{2x^2 + 4x + 1}{x + 1} \\ &= \frac{2x^2 + 2x + 2x + 1}{x + 1} \\ &= \frac{2x(x + 1) + 2x + 1}{x + 1} \\ &= 2x + \frac{2x + 1}{x + 1} \qquad (1) \end{aligned}$$

As $|x| \to \infty$, $\frac{2x + 1}{x + 1} \to 2$. As a consequence, (1) approximates the expression $2x + 2$. The equation of the oblique asymptote is $y = 2x + 2$.

d) Rewrite y.

$$\begin{aligned} y &= \frac{6x^2}{3x - 2} \\ &= \frac{6x^2 - 4x + 4x}{3x - 2} \\ &= \frac{2x(3x - 2) + 4x}{3x - 2} \\ &= 2x + \frac{4x}{3x - 2} \qquad (1) \end{aligned}$$

As $|x| \to \infty$, $\frac{4x}{3x - 2} \to \frac{4}{3}$. As a consequence, (1) approximates the expression $2x + \frac{4}{3}$. The equation of the oblique asymptote is $y = 2x + \frac{4}{3}$.

e) Rewrite $f(x)$.

$$\begin{aligned} f(x) &= \frac{x^3 + 5x^2 + 3x + 10}{x^2 + 2} \\ &= \frac{x^3 + 2x + 5x^2 + x + 10}{x^2 + 2} \\ &= \frac{x(x^2 + 2) + 5x^2 + x + 10}{x^2 + 2} \\ &= x + \frac{5x^2 + x + 10}{x^2 + 2} \qquad (1) \end{aligned}$$

As $|x| \to \infty$, $\frac{5x^2 + x + 10}{x^2 + 2} \to 5$. As a consequence, (1) approximates the expression $x + 5$. The equation of the oblique asymptote is $y = x + 5$.

f) Rewrite $g(x)$.

$$\begin{aligned} g(x) &= \frac{2x - x^2 - x^4}{x^3 - 2} \\ &= \frac{-x^2 - x^4 + 2x}{x^3 - 2} \\ &= \frac{-x^2 - x(x^3 - 2)}{x^3 - 2} \\ &= \frac{-x^2}{x^3 - 2} - x \qquad (1) \end{aligned}$$

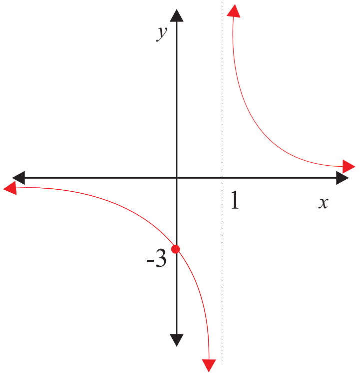
As $|x| \to \infty$, $\frac{-x^2}{x^3 - 2} \to 0$. As a consequence, (1) approximates the expression $-x$. The equation of the oblique asymptote is $y = -x$.

Section 6.5 Page 360 Question 7

a) Since the numerator has no roots, there are no x-intercepts. The y-intercept is $g(0) = \frac{3}{0-1}$ or -3. Since 1 is a root of the denominator, $x = 1$ is a vertical asymptote. Since $g(x) \to 0$ as $|x| \to \infty$, $y = 0$ is a horizontal asymptote. An interval chart is used to determine the signs of the range of the function.

1

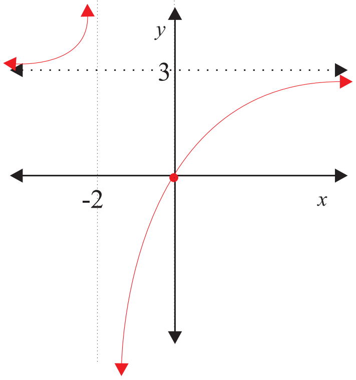
Intervals	(- , 1)	(1,)
Test values	0	2
Sign of *g(x)*	−	+



b) Since 0 is a root of the numerator, the x-intercept is 0. The y-intercept is $y(0) = \frac{3(0)}{0+2}$ or 0. Since -2 is a root of the denominator, $x = -2$ is a vertical asymptote. Since $y \to 3$ as $|x| \to \infty$, $y = 3$ is a horizontal asymptote. An interval chart is used to determine the signs of the range of the function.

-2 0

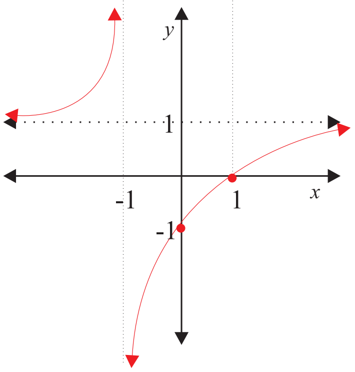
Intervals	(- , -2)	(-2,0)	(0,)
Test values	-3	-1	1
Sign of *y*	+	−	+



c) Since 1 is a root of the numerator, the x-intercept is 1. The y-intercept is $y(0) = \frac{0-1}{0+1}$ or -1. Since -1 is a root of the denominator, $x = -1$ is a vertical asymptote. Since $y \to 1$ as $|x| \to \infty$, $y = 1$ is a horizontal asymptote. An interval chart is used to determine the signs of the range of the function.

-1 1

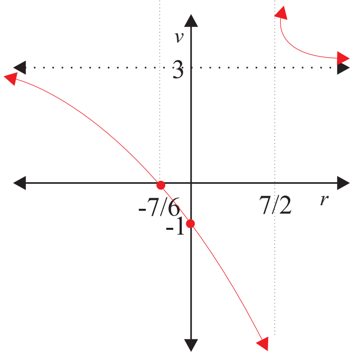
Intervals	(- , -1)	(-1,1)	(1,)
Test values	-2	0	2
Sign of *y*	+	−	+



d) Since $-\frac{7}{6}$ is a root of the numerator, the r-intercept is $-\frac{7}{6}$. The v-intercept is $v(0) = \frac{6(0)+7}{2(0)-7}$ or -1. Since $\frac{7}{2}$ is a root of the denominator, $r = \frac{7}{2}$ is a vertical asymptote. Since $v \to 3$ as $|r| \to \infty$, $v = 3$ is a horizontal asymptote. An interval chart is used to determine the signs of the range of the function.

-7/6 7/2

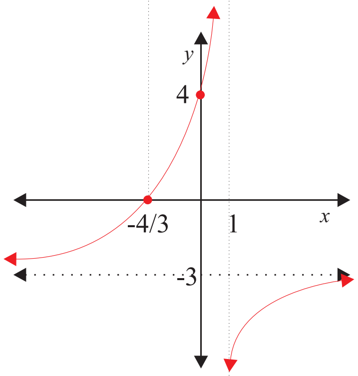
Intervals	(- , -7/6)	(-7/6,7/2)	(7/2,)
Test values	-2	0	4
Sign of *v(r)*	+	−	+



e) Since $-\frac{4}{3}$ is a root of the numerator, the x-intercept is $-\frac{4}{3}$. The y-intercept is $y(0) = \frac{3(0)+4}{1-0}$ or 4. Since 1 is a root of the denominator, $x = 1$ is a vertical asymptote. Since $y \to -3$ as $|x| \to \infty$, $y = -3$ is a horizontal asymptote. An interval chart is used to determine the signs of the range of the function.

-4/3 1

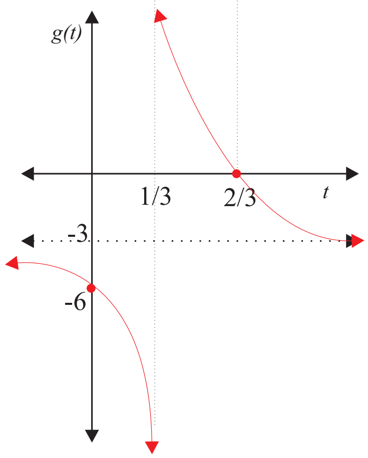
Intervals	(- , -4/3)	(-4/3,1)	(1,)
Test values	-2	0	2
Sign of y	–	+	–



f) Since $\frac{2}{3}$ is a root of the numerator, the t-intercept is $\frac{2}{3}$. The g-intercept is $g(0) = \frac{9(0)-6}{1-3(0)}$ or -6. Since $\frac{1}{3}$ is a root of the denominator, $t = \frac{1}{3}$ is a vertical asymptote. Since $g \to -3$ as $|t| \to \infty$, $y = 3$ is a horizontal asymptote. An interval chart is used to determine the signs of the range of the function.

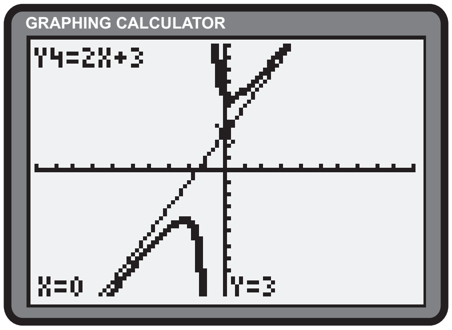
1/3 2/3

Intervals	(- ,1/3)	(1/3,2/3)	(2/3,)
Test values	0	1/2	1
Sign of $g(t)$	–	+	–

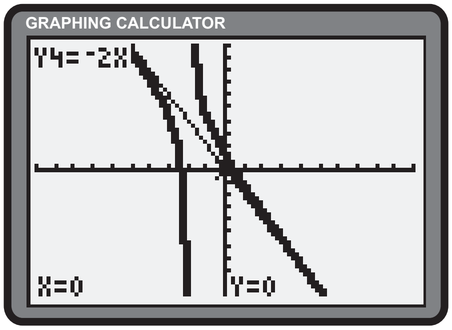


Section 6.5 Page 360 Question 8

a) As $|x| \to \infty$, $\frac{3}{x+1} \to 0$. As a consequence, the function approximates $2x + 3$. The equation of the oblique asymptote is $y = 2x + 3$.



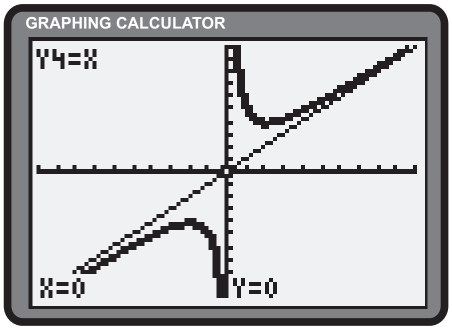
b) As $|x| \to \infty$, $\frac{2}{x+2} \to 0$. As a consequence, the function approximates $-2x$. The equation of the oblique asymptote is $y = -2x$.



c) Rewrite y.

$$\begin{aligned} y &= \frac{x^2+4}{x} \\ &= x + \frac{4}{x} \qquad (1) \end{aligned}$$

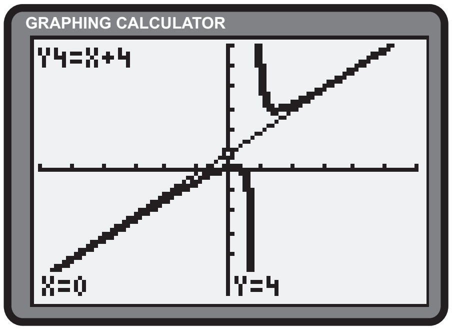
As $|x| \to \infty$, $\frac{4}{x} \to 0$. As a consequence, the function approximates x. The equation of the oblique asymptote is $y = x$.



d) Rewrite y.

$$\begin{aligned} y &= \frac{x^2-16+12}{x-4} \\ &= x + 4 + \frac{12}{x-4} \qquad (1) \end{aligned}$$

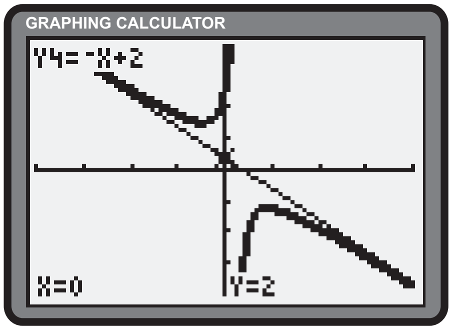
As $|x| \to \infty$, $\frac{12}{x-4} \to 0$. As a consequence, the function approximates $x + 4$. The equation of the oblique asymptote is $y = x + 4$.



e) Rewrite $s(t)$.

$$\begin{aligned} s(t) &= \frac{t^2-3t+2-12}{1-t} \\ &= \frac{(t-1)(t-2)-12}{1-t} \\ &= -t + 2 - \frac{12}{1-t} \qquad (1) \end{aligned}$$

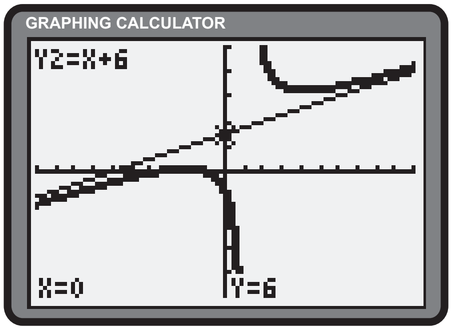
As $|t| \to \infty$, $\frac{12}{1-t} \to 0$. As a consequence, the function approximates $-t + 2$. The equation of the oblique asymptote is $y = -t + 2$.



f) Rewrite y.

$$\begin{aligned} y &= \frac{x^2+5x-6+10}{x-1} \\ &= \frac{(x-1)(x+6)+10}{x-1} \\ &= x + 6 + \frac{10}{x-1} \qquad (1) \end{aligned}$$

As $|x| \to \infty$, $\frac{10}{x-1} \to 0$. As a consequence, the function approximates $x + 6$. The equation of the oblique asymptote is $y = x + 6$.



Section 6.5 Page 360 Question 9

a) i)

$$\begin{aligned} V(1) &= 5000 - \frac{2000(1)^2}{(1+2)^2} \\ &= 5000 - \frac{2000}{9} \\ &\doteq 4777.78 \end{aligned}$$

After 1 month, the value of the machinery is \$4777.78.

ii)

$$\begin{aligned} V(6) &= 5000 - \frac{2000(6)^2}{(6+2)^2} \\ &= 5000 - \frac{72\,000}{64} \\ &\doteq 3875 \end{aligned}$$

After 6 months, the value is \$3875.00.

iii)
$$\begin{aligned}V(12) &= 5000 - \frac{2000(12)^2}{(12+2)^2}\\ &= 5000 - \frac{288\ 000}{196}\\ &\doteq 3530.61\end{aligned}$$

After 1 year, the value of the machinery is \$3530.61.

iv)
$$\begin{aligned}V(120) &= 5000 - \frac{2000(120)^2}{(120+2)^2}\\ &= 5000 - \frac{28\ 800\ 000}{14\ 884}\\ &\doteq 3065.04\end{aligned}$$

After 10 years, the value of the machinery is \$3065.04.

b) The value of most machinery typically declines continuously with use. Hence the machinery is likely worth the most at the time of purchase ($t = 0$). Over the interval $[0, \infty)$ there should be no extrema.

c)
$$\begin{aligned}\lim_{t\to\infty} V(t) &= \lim_{t\to\infty}\left[5000 - \frac{2000t^2}{(t+2)^2}\right]\\ &= \lim_{t\to\infty}\left[5000 - \frac{\frac{2000t^2}{t^2}}{\frac{t^2+4t+4}{t^2}}\right]\\ &= \lim_{t\to\infty}\left[5000 - \frac{2000}{1+\frac{4}{t}+\frac{4}{t^2}}\right]\\ &= 5000 - 2000\\ &= 3000\end{aligned}$$

The limit of the value of the machinery over time is \$3000.

d) No.

e) No.

Section 6.5 Page 360 Question 10

a)
$$\begin{aligned}\lim_{n\to\infty} S(n) &= \lim_{n\to\infty}\frac{20n^2+2n+4}{17n+4}\\ &= \lim_{n\to\infty}\frac{\frac{20n^2+2n+4}{n}}{\frac{17n+4}{n}}\\ &= \lim_{n\to\infty}\frac{20n+2+\frac{4}{n}}{17+\frac{4}{n}}\\ &= \infty\end{aligned}$$

The result suggests sales continue to increase over time.

b) For an average rate of inflation of 3.3% (per year), the real value of sales is

$$S_R = \frac{20n^2+2n+4}{(17n+4)(1.033)^n}$$

The growth rate is given by $S_R'(n)$. the long term growth rate is given by $\lim_{n\to\infty} S_R'(n)$. Use graphing technology to determine that this limit is equal to 0. Over the long term, the rate of inflation takes over the growth rate is sales.

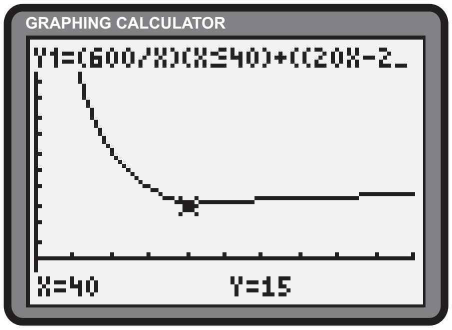
Section 6.5 Page 360 Question 11

a)
$$c(x) = \begin{cases}\frac{600}{x} & , x \le 40\\ \frac{20x-200}{x} & , x > 40\end{cases}$$

b)
$$\begin{aligned}\lim_{x\to\infty} c(x) &= \lim_{x\to\infty}\frac{20x-200}{x}\\ &= \lim_{x\to\infty}\left(20 - \frac{200}{x}\right)\\ &= 20\end{aligned}$$

As the area to be carpeted increases, the average cost per square metre approaches \$20/m^2.

c)



Y1=(600/X)(X≤40)+((20X-2_

X=40 Y=15

d) No. For very small areas, the average cost per square metre is prohibitively expensive.

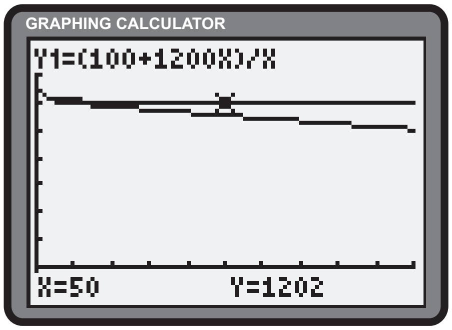
Section 6.5 Page 360 Question 12

a) The current pricing formula is $A(x) = \dfrac{100 + 1200x}{x}$.

b) $C(x)$ provides a better pricing formula since the price decreases without limit as the number of computers ordered increases.

c)
$$\begin{aligned}\lim_{x\to\infty} A(x) &= \lim_{x\to\infty} \frac{100 + 1200x}{x}\\ &= \lim_{x\to\infty} \frac{100}{x} + 1200\\ &= 1200\end{aligned}$$

As the number of computers ordered increases, the price approaches \$1200 per computer.



d)
$$\begin{aligned}\lim_{x\to\infty} C(x) &= \lim_{x\to\infty} \frac{100 + 1200x - 2x^2}{x}\\ &= \lim_{x\to\infty} \frac{100}{x} + 1200 - 2x\\ &= -\infty\end{aligned}$$

This result suggests the price per computer ordered declines without bound. This is unrealistic.

e) Yes.

Section 6.5 Page 361 Question 13

For $x < 0$, $|x| = -x$.

$$\begin{aligned}\lim_{x\to-\infty} \frac{2x}{-x+1} &= \lim_{x\to-\infty} \frac{\frac{2x}{x}}{\frac{-x+1}{x}}\\ &= \lim_{x\to-\infty} \frac{2}{-1 + \frac{1}{x}}\\ &= -2\end{aligned}$$

As $x \to -\infty$, the function approaches a horizontal asymptote of $y = -2$.

For $x \geq 0$, $|x| = x$.

$$\begin{aligned}\lim_{x\to\infty} \frac{2x}{-x+1} &= \lim_{x\to\infty} \frac{\frac{2x}{x}}{\frac{x+1}{x}}\\ &= \lim_{x\to\infty} \frac{2}{1 + \frac{1}{x}}\\ &= 2\end{aligned}$$

As $x \to \infty$, the function approaches a horizontal asymptote of $y = 2$.

Section 6.5 Page 361 Question 14

a)
$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{4x^2+5}}{2x-1} &= \lim_{x\to\infty} \frac{\frac{\sqrt{4x^2+5}}{\sqrt{x^2}}}{\frac{2x-1}{x}}\\ &= \lim_{x\to\infty} \frac{\sqrt{4 + \frac{5}{x^2}}}{2 - \frac{1}{x}}\\ &= \frac{\sqrt{4}}{2}\\ &= 1\end{aligned}$$

b)

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2+3x+5}-x &= \lim_{x\to\infty}\left[\left(\sqrt{x^2+3x+5}-x\right)\cdot\frac{\sqrt{x^2+3x+5}+x}{\sqrt{x^2+3x+5}+x}\right]\\ &= \lim_{x\to\infty}\frac{x^2+3x+5-x^2}{\sqrt{x^2+3x+5}+x}\\ &= \lim_{x\to\infty}\frac{\frac{3x+5}{x}}{\frac{\sqrt{x^2+3x+5}}{\sqrt{x^2}}+\frac{x}{x}}\\ &= \lim_{x\to\infty}\frac{3+\frac{5}{x}}{\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}+1}\\ &= \frac{3+0}{\sqrt{1+0+0}+1}\\ &= \frac{3}{2}\end{aligned}$$

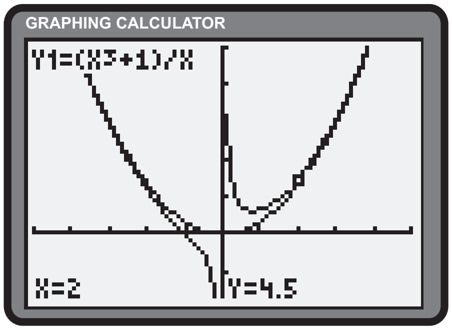
Section 6.5 Page 361 Question 15

a)

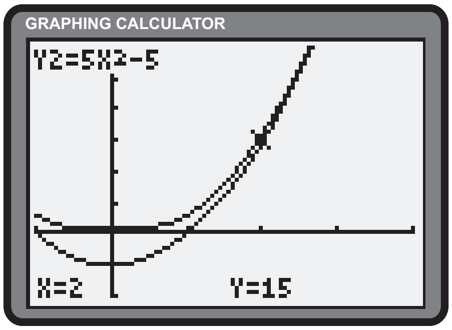
$$\begin{aligned}\lim_{x\to\infty}\left[f(x)-x^2\right] &= \lim_{x\to\infty}\left[\frac{x^3+1}{x}-x^2\right]\\ &= \lim_{x\to\infty}\frac{x^3+1-x^3}{x}\\ &= \lim_{x\to\infty}\frac{1}{x}\\ &= 0\end{aligned}$$

$f(x)$ is asymptotic to $g(x) = x^2$.

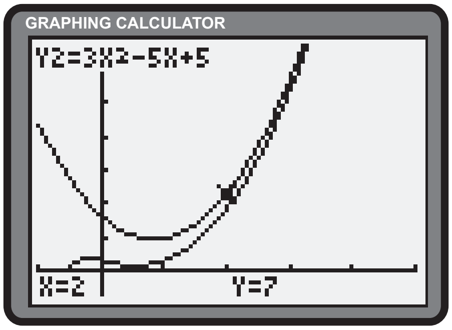
b)



c) **i)** Dividing $5x^4 + 1$ by $x^2 + 1$ yields $5x^2 - 5 + \frac{5}{x^2+1}$. As $x \to \infty$, $\frac{5}{x^2+1} \to 0$. The result is that $g(x)$ is asymptotic to $h(x) = 5x^2 - 5$.



ii) Dividing $3x^3-2x^2+1$ by $x+1$ yields $3x^2 - 5x + 5 - \frac{4}{x+1}$. As $x \to \infty$, $\frac{4}{x+1} \to 0$. The result is that $h(x)$ is asymptotic to $k(x) = 3x^2 - 5x + 5$.



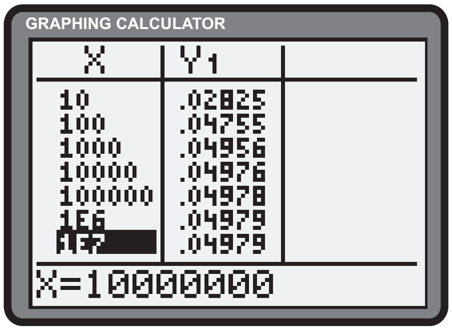
d) Denote a rational function by $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials of degree $n+k$ and n respectively. n is a whole number and k is a natural number. p_i and q_i are the coefficients of the ith term of $P(x)$ and $Q(x)$, respectively.

$$
\begin{aligned}
\lim_{|x|\to\infty}\frac{P(x)}{Q(x)} &= \lim_{|x|\to\infty}\frac{\sum\limits_{i=0}^{n+k} p_i x^i}{\sum\limits_{i=0}^{n} q_i x^i} \\
&= \lim_{|x|\to\infty}\frac{\dfrac{\sum\limits_{i=0}^{n+k} p_i x^i}{x^n}}{\dfrac{\sum\limits_{i=0}^{n} q_i x^i}{x^n}} \\
&= \lim_{|x|\to\infty}\frac{\sum\limits_{i=0}^{n+k} p_i x^{i-n}}{q_n + \sum\limits_{i=0}^{n-1} q_i \dfrac{1}{x^i}} \\
&= \frac{\lim\limits_{|x|\to\infty}\sum\limits_{i=0}^{n+k} p_i x^{i-n}}{\lim\limits_{|x|\to\infty}\left(q_n + \sum\limits_{i=0}^{n-1} q_i \dfrac{1}{x^i}\right)} \\
&= \frac{\lim\limits_{|x|\to\infty}\sum\limits_{i=0}^{n+k} p_i x^{i-n}}{q_n + 0} \\
&= \lim_{|x|\to\infty}\sum_{i=0}^{n+k}\frac{p_i}{q_n}x^{i-n} \qquad (1)
\end{aligned}
$$

As $|x| \to \infty$, $\dfrac{P(x)}{Q(x)}$ behaves as (1); a polynomial of degree k.

Section 6.5 Page 361 Question 16

a) Define $Y_1 = \left(1 - \dfrac{3}{x}\right)^x$ and use the **TABLE feature** of the graphing calculator to approximate a limit of 0.05.



X	Y1
10	.02825
100	.04755
1000	.04956
10000	.04976
100000	.04978
1E6	.04979
1E7	.04979

X=10000000

b) Use the **TABLE feature** of the graphing calculator to approximate a limit of 0.0498, correct to four decimal places.

6.6 Curve Sketching

Practise

Section 6.6 Page 370 Question 1

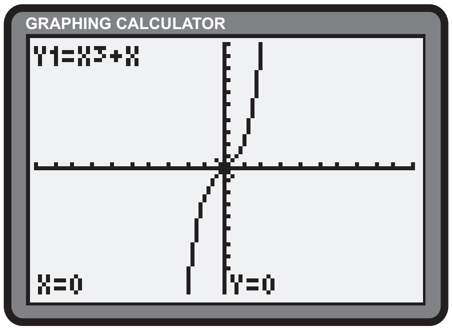
a) vi. f has a y-intercept of $-\frac{1}{2}$, a vertical asymptote of $x = 2$, and a horizontal asymptote of $y = 0$.
b) viii. f has a y-intercept of 0, a vertical asymptote of $x = 2$, and a horizontal asymptote of $y = 1$.
c) iii. The function has a y-intercept of $-\frac{1}{4}$, vertical asymptotes of $x = \pm 2$, and a horizontal asymptote of $y = 0$.
d) ix. f has a y-intercept of $\frac{1}{4}$, a vertical asymptote of $x = 2$, and a horizontal asymptote of $y = 0$.
e) v. The function has a y-intercept of $\frac{1}{2}$, a vertical asymptote of $x = 2$, and a horizontal asymptote of $y = 0$.
f) xi. q has a y-intercept of 0, a vertical asymptote of $x = 2$, and an oblique asymptote of $y = x + 2$.
g) iv. f has a y-intercept of 0, vertical asymptotes of $x = \pm 2$, and a horizontal asymptote of $y = 0$.
h) i. The function has a y-intercept of 0, a vertical asymptote of $x = 2$, and a horizontal asymptote of $y = 0$.
i) x. g has a y-intercept of 0, a vertical asymptote of $x = 2$, and an oblique asymptote of $y = -x - 2$.
j) ii. k has a y-intercept of 0, vertical asymptotes of $x = \pm 2$, and a horizontal asymptote of $y = 1$.
k) vii. The function has a y-intercept of $-\frac{1}{4}$, a vertical asymptote of $x = 2$, and a horizontal asymptote of $y = 0$.
l) xii. f has a y-intercept of 0, vertical asymptotes of $x = \pm 2$, and an oblique asymptote of $y = x$.

Apply, Solve, Communicate

Section 6.6 Page 372 Question 3

a)

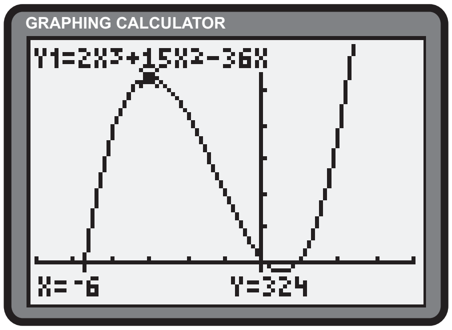
$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 3x^2 + 1 &= 0 \\ \frac{d^2y}{dx^2} &= 0 \\ 6x &= 0 \\ x &= 0 \end{aligned}$$



Since there are no critical numbers, there are no extrema. There is a point of inflection at $(0, 0)$.

b)

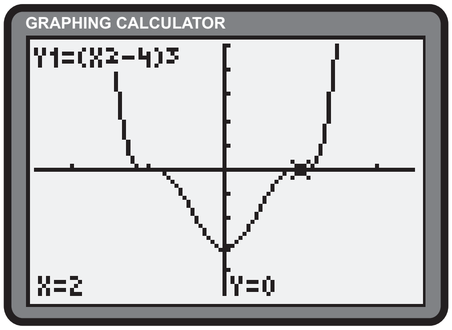
$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 6x^2 + 30x - 36 &= 0 \\ (x + 6)(x - 1) &= 0 \\ x &= -6 \text{ or } 1 \\ \frac{d^2y}{dx^2} &= 0 \\ 12x + 30 &= 0 \\ 2x + 5 &= 0 \\ x &= -\frac{5}{2} \end{aligned}$$



There is a local maximum at $(-6, 324)$ and a local minimum at $(1, -19)$. A point of inflection occurs at $\left(-\frac{5}{2}, \frac{305}{2}\right)$.

c)

$$\begin{aligned} g'(x) &= 0 \\ 3(x^2-4)^2(2x) &= 0 \\ 2x(x-2)^2(x+2)^2 &= 0 \\ x &= 0, \pm 2 \\ g''(x) &= 0 \\ (x^2-4)^2(6) + 6x(2)(x^2-4)(2x) &= 0 \\ 6(x^2-4)(5x^2-4) &= 0 \\ x &= \pm 2, \pm\frac{2}{\sqrt{5}} \end{aligned}$$

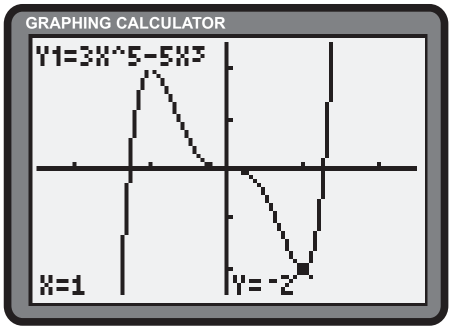


There are no local maxima. A local minimum occurs at $(0, -64)$.

Points of inflection occur at $(\pm 2, 0)$ and $\left(\pm\frac{2}{\sqrt{5}}, \frac{-4096}{125}\right)$.

d)

$$\begin{aligned} f'(x) &= 0 \\ 15x^4 - 15x^2 &= 0 \\ x^2(x^2-1) &= 0 \\ x &= 0, \pm 1 \\ f''(x) &= 0 \\ 60x^3 - 30x &= 0 \\ x(2x^2-1) &= 0 \\ x &= 0, \pm\frac{1}{\sqrt{2}} \end{aligned}$$

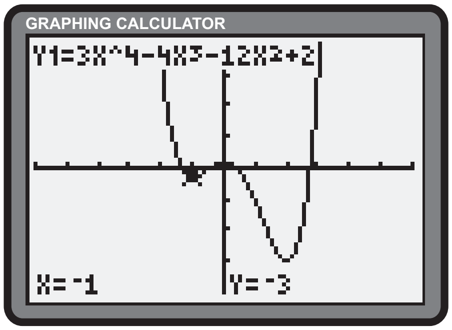


A local maximum occurs at $(-1, 2)$. A local minimum occurs at $(1, -2)$.

Points of inflection occur at $(0, 0)$, $\left(-\frac{1}{\sqrt{2}}, \frac{7}{4\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}}, -\frac{7}{4\sqrt{2}}\right)$.

e)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 12x^3 - 12x^2 - 24x &= 0 \\ x(x^2 - x - 2) &= 0 \\ x(x+1)(x-2) &= 0 \\ x &= 0, -1, \text{ or } 2 \\ \frac{d^2y}{dx^2} &= 0 \\ 36x^2 - 24x - 24 &= 0 \\ 3x^2 - 2x - 2 &= 0 \\ x &= \frac{2 \pm \sqrt{4 + 4(6)}}{6} \\ x &= \frac{1 \pm \sqrt{7}}{3} \end{aligned}$$

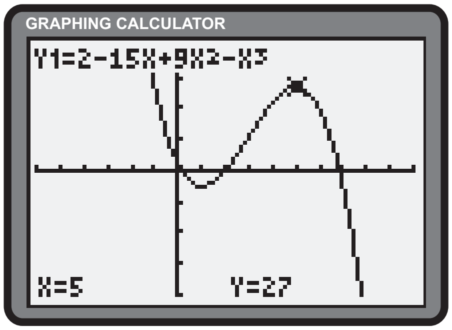


A local maximum occurs at $(0, 2)$. Local minima occur at $(-1, -3)$ and $(2, -30)$.

Points of inflection occur at $\left(\frac{1+\sqrt{7}}{3}, \frac{-230-80\sqrt{7}}{27}\right)$ and $\left(\frac{1-\sqrt{7}}{3}, \frac{-230+80\sqrt{7}}{27}\right)$.

f)

$$\begin{aligned} h'(x) &= 0 \\ -15 + 18x - 3x^2 &= 0 \\ x^2 - 6x + 5 &= 0 \\ (x-1)(x-5) &= 0 \\ x &= 1 \text{ or } 5 \\ h''(x) &= 0 \\ 18 - 6x &= 0 \\ x &= 3 \end{aligned}$$



A local maximum occurs at $(5, 27)$. A local minimum occurs at $(1, -5)$. A point of inflection occurs at $(3, 11)$.

Section 6.6 Page 372 Question 4

a) • Frame the curve. Since -2 is a root of the denominator, the function has a vertical asymptote of $x = -2$. Since the degrees of the numerator and denominator are the same, the function has horizontal asymptote of $y = \frac{1}{1}$ or $y = 1$. $y > 0$ on $x \in (-\infty, -2)$ and $(2, \infty)$. $y < 0$ on $x \in (-2, 2)$.

• Find important points. The y-intercept is $\frac{0-2}{0+2}$ or -1. The x-intercept is 2.

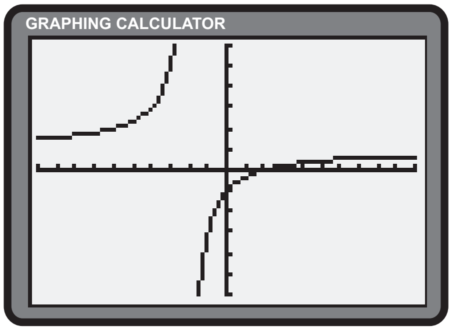
$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{(x+2)(1) - (x-2)(1)}{(x+2)^2} &= 0 \\ \frac{4}{(x+2)^2} &= 0 \end{aligned}$$

no critical numbers

$$\begin{aligned} \frac{d^2y}{dx^2} &= 0 \\ -\frac{8}{(x+2)^3} &= 0 \end{aligned}$$

no roots

• Sketch the curve.



There are no extrema and no points of inflection.

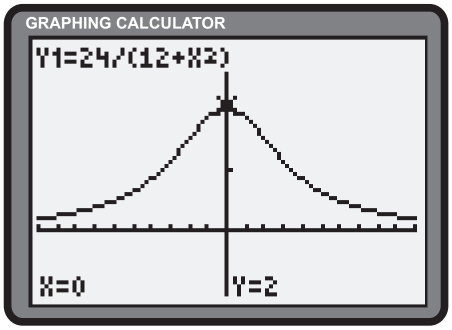
• Add details. No symmetry. It is not necessary to determine intervals of concavity.

b) • Frame the curve. Since there are no roots of the denominator, there are no vertical asymptotes. Since the degree of the numerator is less than the degree of the denominator, the function has horizontal asymptote of $y = 0$. Since both the numerator and denominator are positive for all real numbers, $f(t) > 0$ for all $\Re$.

• Find important points. The y-intercept is $\frac{24}{12 + 0^2}$ or 2. There is no t-intercept.

$$\begin{aligned} f'(t) &= 0 \\ \frac{-48t}{(12+t^2)^2} &= 0 \\ t &= 0 \\ f''(t) &= 0 \\ \frac{48(3t^2 - 12)}{(12+t^2)^3} &= 0 \\ 3t^2 - 12 &= 0 \\ t^2 - 4 &= 0 \\ t &= \pm 2 \end{aligned}$$

• Sketch the curve.



Since $f''(0) < 0$, a local maximum exists at $(0, 2)$. Points of inflection exist at $\left(\pm 2, \frac{3}{2}\right)$.

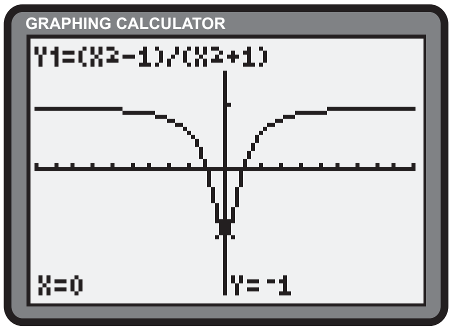
• Add details. Since $f(t) = f(-t)$, the function possesses even symmetry. It is not necessary to determine intervals of concavity.

c) • Frame the curve. Since there are no roots of the denominator, there are no vertical asymptotes. Since the degrees of the numerator and denominator are the same, the function has horizontal asymptote of $y = \frac{1}{1}$ or $y = 1$. Since the denominator is positive for all real numbers, the sign of the function is determined by the numerator. $y > 0$ on $|x| > 1$ and $y < 0$ on $|x| < 1$.

• Find important points. The y-intercept is $\frac{0^2 - 1}{0^2 + 1}$ or -1. The x-intercepts are $x^2 - 1 = 0$ or $x = \pm 1$.

$$\begin{aligned}
\frac{dy}{dx} &= 0\\
\frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} &= 0\\
\frac{4x}{(x^2+1)^2} &= 0\\
x &= 0\\
\frac{d^2y}{dx^2} &= 0\\
\frac{(x^2+1)^2(4) - 4x(2)(x^2+1)(2x)}{(x^2+1)^4} &= 0\\
\frac{4(1-3x^2)}{(x^2+1)^3} &= 0\\
x &= \pm\frac{1}{\sqrt{3}}
\end{aligned}$$

• Sketch the curve.



Since $f''(0) > 0$, a local minimum exists at $(0, -1)$. Points of inflection exist at $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{2}\right)$.

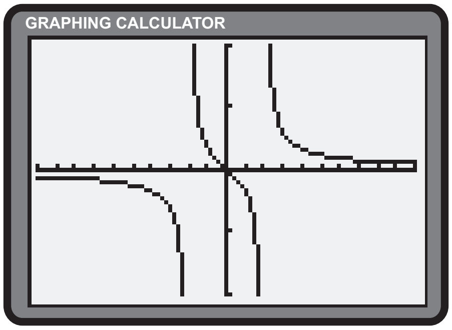
• Add details. Since $y(x) = y(-x)$, the function possesses even symmetry. It is not necessary to determine intervals of concavity.

d) • Frame the curve. Since ± 2 are roots of the denominator, the function has a vertical asymptotes of $x = \pm 2$. Since the degree of the numerator is less than the degree of the denominator, the function is asymptotic to the x-axis. Comparison of the root of the numerator and the two roots of the denominator reveals $y < 0$ on $x \in (-\infty, -2)$ and $(0, 2)$. $y > 0$ on $x \in (-2, 0)$ and $(2, \infty)$.

• Find important points. The y-intercept is $\frac{0}{0^2 - 4}$ or 0. The x-intercept is also 0.

$$\begin{aligned}
\frac{dy}{dx} &= 0\\
\frac{(x^2-4)(1) - x(2x)}{(x^2-4)^2} &= 0\\
\frac{-4-x^2}{(x^2-4)^2} &= 0\\
&\text{no roots}\\
\frac{d^2y}{dx^2} &= 0\\
\frac{(x^2-4)^2(-2x) - (-4-x^2)(2)(x^2-4)(2x)}{(x^2-4)^4} &= 0\\
\frac{2x(x^2+12)}{(x^2-4)^3} &= 0\\
x &= 0
\end{aligned}$$

• Sketch the curve.



There are no local extrema. A point of inflection exists at $(0, 0)$.

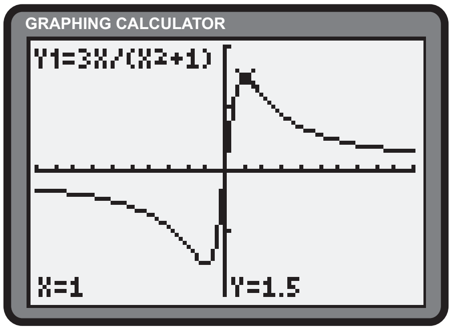
• Add details. Since $y(x) = -y(-x)$, the function possesses odd symmetry. It is not necessary to determine intervals of concavity.

e) • Frame the curve. Since there are no roots of the denominator, there are no vertical asymptotes. Since the degree of the numerator is less than the degree of the denominator, the function is asymptotic to the x-axis. Since the denominator is positive for all real numbers, the sign of the function is determined by the numerator. $y < 0$ on $x < 0$ and $y > 0$ on $x > 0$.

• Find important points. The y-intercept is $\frac{3(0)}{0^2+1}$ or 0. The x-intercept is also 0.

$$\begin{aligned}
\frac{dy}{dx} &= 0 \\
\frac{(x^2+1)(3)-3x(2x)}{(x^2+1)^2} &= 0 \\
\frac{3-3x^2}{(x^2+1)^2} &= 0 \\
x &= \pm 1 \\
\frac{d^2y}{dx^2} &= 0 \\
\frac{(x^2+1)^2(-6x)-(3-3x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} &= 0 \\
\frac{6x(x^2-3)}{(x^2+1)^3} &= 0 \\
x &= 0, \pm\sqrt{3}
\end{aligned}$$

• Sketch the curve.



Since $f''(1) < 0$, a local maximum exists at $\left(1, \frac{3}{2}\right)$. A local minimum exists at $\left(-1, -\frac{3}{2}\right)$.

Points of inflection exist at $(0, 0)$, $\left(-\sqrt{3}, -\frac{3\sqrt{3}}{4}\right)$, and $\left(\sqrt{3}, \frac{3\sqrt{3}}{4}\right)$.

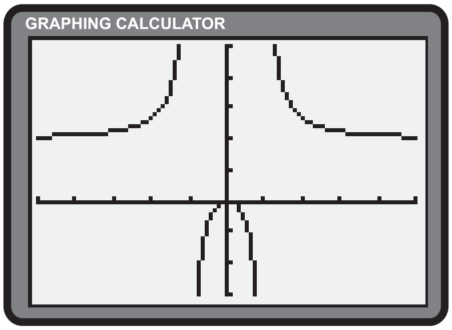
• Add details. Since $y(x) = -y(-x)$, the function possesses odd symmetry. It is not necessary to determine intervals of concavity.

f) • Frame the curve. The roots of the denominator define vertical asymptotes at $x = \pm 1$. Since the degrees of the numerator and denominator are equal, the function is asymptotic to $y = \frac{2}{1}$ or $y = 2$. Since the numerator is non-negative for all real numbers, the sign of the function is determined by the denominator. $y < 0$ on $|x| < 1$ and $y > 0$ on $|x| > 1$.

• Find important points. The y-intercept is $\frac{2(0)^2}{0^2-1}$ or 0. The x-intercept is also 0.

$$\begin{aligned}
\frac{dy}{dx} &= 0 \\
\frac{(x^2-1)(4x)-2x^2(2x)}{(x^2-1)^2} &= 0 \\
\frac{-4x}{(x^2-1)^2} &= 0 \\
x &= 0 \\
\frac{d^2y}{dx^2} &= 0 \\
\frac{(x^2-1)^2(-4)-(-4x)(2)(x^2-1)(2x)}{(x^2-1)^4} &= 0 \\
\frac{4(3x^2+1)}{(x^2-1)^3} &= 0 \\
&\text{no roots}
\end{aligned}$$

• Sketch the curve.



Since $f''(0) < 0$, a local maximum exists at $(0, 0)$. There are no local minima. There are no points of inflection.

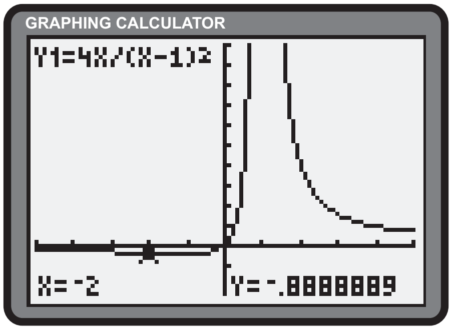
• Add details. Since $g(x) = g(-x)$, the function possesses even symmetry. It is not necessary to determine intervals of concavity.

g) • Frame the curve. The root of the denominator defines a vertical asymptote at $x = 1$. Since the degree of the numerator is less than the degree of the denominator, the function is asymptotic to the x-axis. Since the denominator is non-negative for all real numbers, the sign of the function is determined by the numerator. $y < 0$ on $x < 0$ and $y > 0$ on $x > 0$.

• Find important points. The y-intercept is $\frac{4(0)^2}{(0-1)^2}$ or 0. The x-intercept is also 0.

$$\begin{aligned}
\frac{dy}{dx} &= 0 \\
\frac{(x-1)^2(4) - 4x(2)(x-1)}{(x-1)^4} &= 0 \\
\frac{-4(x+1)}{(x-1)^3} &= 0 \\
x &= -1 \\
\frac{d^2y}{dx^2} &= 0 \\
\frac{(x^2-1)^2(-4) - (-4x)(2)(x^2-1)(2x)}{(x^2-1)^4} &= 0 \\
\frac{8(x+2)}{(x-1)^4} &= 0 \\
x &= -2
\end{aligned}$$

• Sketch the curve.



Y1=4X/(X-1)²

X=-2 Y=-.8888889

There are no local maxima. Since $f''(-1) > 0$, a local minimum exists at $(-1, -1)$.

A point of inflection exists at $\left(-2, -\frac{8}{9}\right)$.

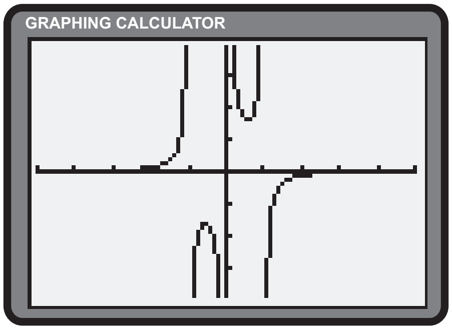
• Add details. No symmetry. It is not necessary to determine intervals of concavity.

h) • Frame the curve. The denominator can be factored as $x - x^3 = x(1 + x)(1 - x)$. The roots of the denominator define vertical asymptotes at $x = 0$, $x = -1$, and $x = 1$. Since the degree of the numerator is less than the degree of the denominator, the function is asymptotic to the x-axis. Since the numerator is positive for all real numbers, the sign of the function is determined by the denominator. $y < 0$ for $x \in (-1, 0)$ and $(1, \infty)$ and $y > 0$ for $x \in (-\infty, -1)$ and $(0, 1)$.

• Find important points. There are no x-intercepts and no y-intercept.

$$\begin{aligned}
\frac{dy}{dx} &= 0 \\
\frac{3(3x^2-1)}{(x-x^3)^2} &= 0 \\
x &= \pm\frac{1}{\sqrt{3}} \\
\frac{d^2y}{dx^2} &= 0 \\
\frac{(x-x^3)^2(18x) - (9x^2-3)(2)(x-x^3)(1-3x^2)}{(x-x^3)^4} &= 0 \\
\frac{6(6x^4-3x^2+1)}{(x-x^3)^3} &= 0 \\
&\text{no roots}
\end{aligned}$$

• Sketch the curve.



Since $f''\left(-\frac{1}{\sqrt{3}}\right) < 0$, a local maximum exists at $\left(-\frac{1}{\sqrt{3}}, -\frac{9\sqrt{3}}{2}\right)$.

Since $f''\left(\frac{1}{\sqrt{3}}\right) > 0$, a local minimum exists at $\left(\frac{1}{\sqrt{3}}, \frac{9\sqrt{3}}{2}\right)$. There are no points of inflection.

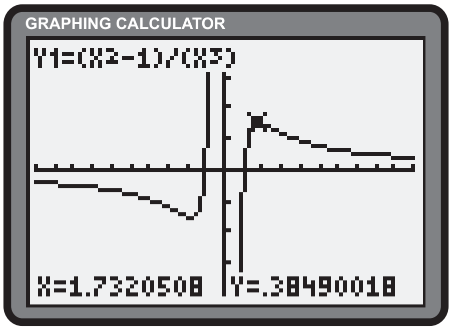
• Add details. Since $h(x) = -h(-x)$, the function possesses odd symmetry. It is not necessary to determine intervals of concavity.

i) • Frame the curve. The root of the denominator defines a vertical asymptote at $x = 0$. Since the degree of the numerator is less than the degree of the denominator, the function is asymptotic to the x-axis. Comparing the roots of the numerator and denominator with an interval chart yields $y < 0$ for $x \in (-\infty, -1)$ and $(0, 1)$ and $y > 0$ for $x \in (-1, 0)$ and $(1, \infty)$.

• Find important points. There are x-intercepts at $x = \pm 1$. There is no y-intercept.

$$\begin{aligned}\frac{dy}{dx} &= 0\\ \frac{x^3(2x) - (x^2 - 1)3x^2}{x^6} &= 0\\ \frac{3 - x^2}{x^4} &= 0\\ x &= \pm\sqrt{3}\\ \frac{d^2y}{dx^2} &= 0\\ \frac{x^4(-2x) - (3 - x^2)(4x^3)}{x^8} &= 0\\ \frac{2(x^2 - 6)}{x^5} &= 0\\ x &= \pm\sqrt{6}\end{aligned}$$

• Sketch the curve.



Since $f''(\sqrt{3}) < 0$, a local maximum exists at $\left(\sqrt{3}, \frac{2}{3\sqrt{3}}\right)$.

Since $f''(-\sqrt{3}) > 0$, a local minimum exists at $\left(-\sqrt{3}, -\frac{2}{3\sqrt{3}}\right)$.

Points of inflection exist at $\left(-\sqrt{6}, -\frac{5}{6\sqrt{6}}\right)$ and $\left(\sqrt{6}, \frac{5}{6\sqrt{6}}\right)$.

• Add details. Since $y(x) = -y(-x)$, the function possesses odd symmetry. It is not necessary to determine intervals of concavity.

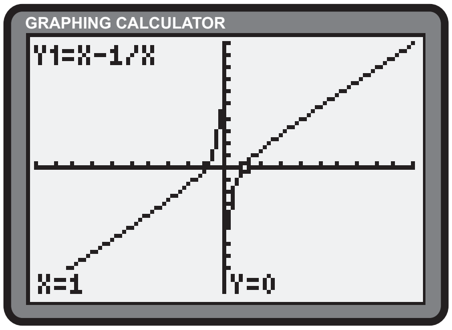
Section 6.6 Page 372 Question 5

a) • Frame the curve. The root of the denominator defines a vertical asymptote at $x = 0$. As $|x| \to \infty$, $\frac{1}{x} \to 0$, thus leaving $f(x)$ to approximate its oblique asymptote $y = x$.

• Find important points. Solving $f(x) = 0$ reveals x-intercepts of ± 1. There is no y-intercept.

$$\begin{aligned}f'(x) &= 0\\ 1 + \frac{1}{x^2} &= 0\\ x^2 &= -1\\ &\text{no roots}\\ f''(x) &= 0\\ -\frac{2}{x^3} &= 0\\ &\text{no roots}\end{aligned}$$

• Sketch the curve.



There are no local extrema and no points of inflection.

• Add details. Since $f(x) = -f(-x)$, the function possesses odd symmetry.

b) • Frame the curve. The root of the denominator defines a vertical asymptote at $x = 1$. As $|x| \to \infty$, $\frac{x}{x-1} \to 1$, leaving $f(x)$ to approximate its oblique asymptote $y = 1 + 2x + 1$ or $y = 2x + 2$.

• Find important points.

$$\begin{aligned} f'(x) &= 0 \\ \frac{2(x-1)^2 - 1}{(x-1)^2} &= 0 \\ 2(x-1)^2 - 1 &= 0 \\ x &= 1 \pm \frac{1}{\sqrt{2}} \end{aligned}$$

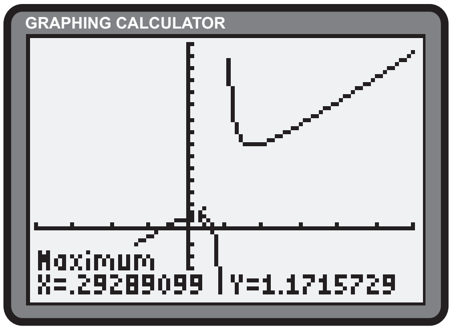
A local maximum exists at $\left(1 - \frac{1}{\sqrt{2}}, 4 - 2\sqrt{2}\right)$.

A local minimum exists at $\left(1 + \frac{1}{\sqrt{2}}, 4 + 2\sqrt{2}\right)$.

There are no points of inflection.

• Add details. There is no symmetry.

• Sketch the curve.



Maximum
X=.29289099 Y=1.1715729

c) • Frame the curve. The roots of the denominator define vertical asymptote at $x = \pm 1$. As $|x| \to \infty$, $\frac{x}{x^2-1} \to 0$, leaving y to approximate its oblique asymptote $y = 1 + 2x$.

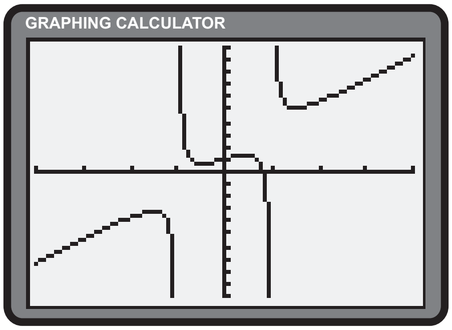
• Find important points. The y-intercept is $1 + 0 + \frac{0}{0^2 - 1}$ or 1. Determine the x-intercept.

$$\begin{aligned} y &= 0 \\ 1 + 2x + \frac{x}{x^2-1} &= 0 \\ x^2 - 1 + 2x^3 - 2x + x &= 0 \\ 2x^3 + x^2 - x - 1 &= 0 \end{aligned}$$

Technology reveals an approximate x-intercept of 0.829

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 2 + \frac{(x^2-1)(1) - x(2x)}{(x^2-1)^2} &= 0 \\ \frac{2x^4 - 5x^2 + 1}{(x^2-1)^2} &= 0 \\ x &= \pm\frac{\sqrt{5 \pm \sqrt{17}}}{2} \end{aligned}$$

• Sketch the curve.



$$\begin{aligned} \frac{d^2y}{dx^2} &= 0 \\ \frac{(x^2-1)^2(8x^3 - 10x) - (2x^4 - 5x^2 + 1)(2)(x^2-1)(2x)}{(x^2-1)^4} &= 0 \\ \frac{2x(x^2+3)}{(x^2-1)^3} &= 0 \\ x &= 0 \end{aligned}$$

Local maxima exist at $\left(-\frac{\sqrt{5+\sqrt{17}}}{2}, -3.2\right)$ and $\left(\frac{\sqrt{5-\sqrt{17}}}{2}, 1.3\right)$.

Local minima exist at $\left(\frac{\sqrt{5+\sqrt{17}}}{2}, 5.2\right)$ and $\left(-\frac{\sqrt{5-\sqrt{17}}}{2}, 0.7\right)$.

A point of inflection exists at $(0, 1)$.

• Add details. There is no symmetry.

d) • Frame the curve. The root of the denominator defines a vertical asymptote at $x = 0$. The function can be rewritten as $y = x + \frac{4}{x}$. As $|x| \to \infty$, $\frac{4}{x} \to 0$ and the function eventually approximates its oblique asymptote $y = x$.

• Find important points. There is no x- or y-intercept.

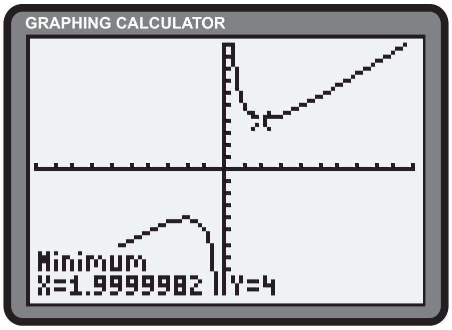
$$\frac{dy}{dx} = 0$$
$$\frac{x(2x) - (x^2 + 4)(1)}{x^2} = 0$$
$$\frac{x^2 - 4}{x^2} = 0$$
$$x = \pm 2$$

Since $f''(-2) < 0$, a local maximum exists at $(-2, -4)$.

Since $f''(2) > 0$, a local minimum exists at $(2, 4)$.

There is no point of inflection.

• Sketch the curve.



• Add details. Since $y(x) = -y(-x)$, the function has odd symmetry.

e) • Frame the curve. The root of the denominator defines a vertical asymptote at $x = 0$. The function can be rewritten as $y = x - 2 - \frac{3}{x}$. As $|x| \to \infty$, $\frac{3}{x} \to 0$ and the function eventually approximates its oblique asymptote $y = x - 2$.

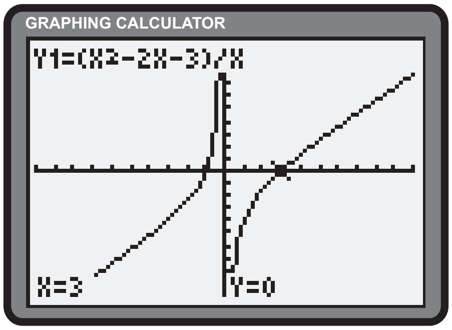
• Find important points. Setting $y = 0$ and solving reveals x-intercepts of 3 and -1.

$$\frac{dy}{dx} = 0$$
$$\frac{x(2x - 2) - (x^2 - 2x - 3)(1)}{x^2} = 0$$
$$\frac{x^2 + 3}{x^2} = 0$$
$$\text{no roots}$$

There are no local extrema.

Since $f''(x) = -\frac{6}{x^3}$ has no roots, there are no points of inflection.

• Sketch the curve.



• Add details. The function has neither even nor odd symmetry.

f) • Frame the curve. The roots of the denominator define vertical asymptotes of $t = \pm 1$. The function can be rewritten as $h(t) = t + \frac{t}{t^2 - 1}$. As $|t| \to \infty$, $\frac{t}{t^2 - 1} \to 0$ and the function eventually approximates its oblique asymptote $y = t$.

• Find important points. The curve passes through the origin.

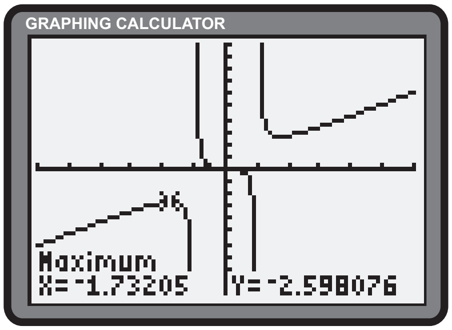
$$\begin{aligned} h'(t) &= 0 \\ \frac{(t^2-1)(3t^2) - t^3(2t)}{(t^2-1)^2} &= 0 \\ \frac{t^4 - 3t^2}{(t^2-1)^2} &= 0 \\ \frac{t^2(t^2-3)}{(t^2-1)^2} &= 0 \\ t &= 0, \pm\sqrt{3} \end{aligned}$$

Since $h''(-\sqrt{3}) < 0$, a local maximum exists at $\left(-\sqrt{3}, -\frac{3\sqrt{3}}{2}\right)$.

Since $h''(\sqrt{3}) > 0$, a local minimum exists at $\left(\sqrt{3}, \frac{3\sqrt{3}}{2}\right)$.

Since $h''(0) = 0$, a point of inflection exists at $(0, 0)$.

• Sketch the curve.



Maximum
X=-1.73205 Y=-2.598076

• Add details. Since $h(t) = -h(-t)$, the function possesses odd symmetry.

g) • Frame the curve. The root of the denominator defines a vertical asymptote of $x = 0$. The function can be rewritten as $y = -2x + 6 - \frac{2(3x-1)}{x^2}$. As $|x| \to \infty$, $\frac{2(3x-1)}{x^2} \to 0$ and the function eventually approximates its oblique asymptote, $y = -2x + 6$.

• Find important points. There is no y-intercept. The x-intercept is 1.

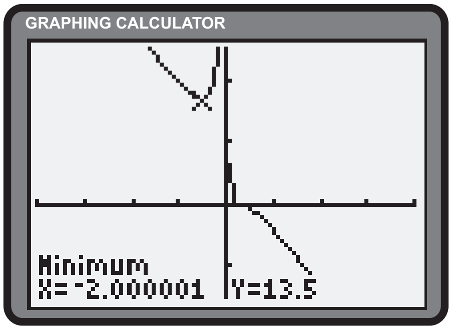
$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{x^2(-6)(x-1)^2 - (-2)(x-1)^3(2x)}{x^4} &= 0 \\ \frac{(x-1)^2(x+2)}{x^3} &= 0 \\ x &= 1 \text{ or } -2 \end{aligned}$$

Since $f''(-2) < 0$, a local minimum exists at $(-2, 13.5)$.

A point of inflection exists at $(1, 0)$.

• Add details. The function has neither even nor odd symmetry.

• Sketch the curve.



Minimum
X=-2.000001 Y=13.5

h) • Frame the curve. The root of the denominator defines a vertical asymptote of $x = -2$. The function can be rewritten as $y = x - 2 + \frac{4}{x+2}$. As $|x| \to \infty$, $\frac{4}{x+2} \to 0$ and the function eventually approximates its oblique asymptote $y = x - 2$.

- Find important points. The curve passes through the origin.

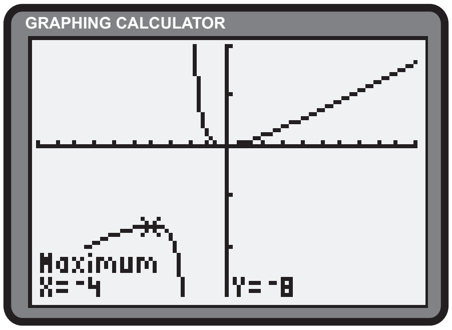
$$\begin{aligned} f'(x) &= 0 \\ \frac{(x+2)(2x) - x^2(1)}{(x+2)^2} &= 0 \\ \frac{x(x+4)}{(x+2)^2} &= 0 \\ x &= 0 \text{ or } -4 \end{aligned}$$

Since $f''(-4) < 0$, a local maximum exists at $(-4, -8)$.

Since $f''(0) > 0$, a local minimum exists at $(0, 0)$.

- Add details. The function has neither even nor odd symmetry.

- Sketch the curve.



Section 6.6 Page 372 Question 6

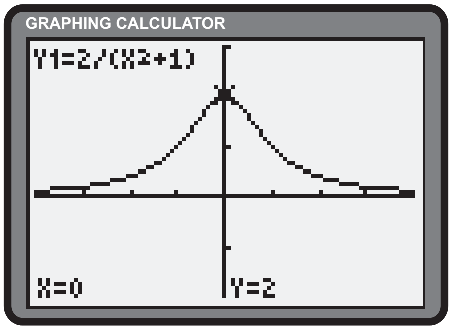
- Frame the curve. As $|x| \to \infty$, $\frac{2}{x^2+1} \to 0$ and the function eventually approximates the x-axis. Since both the numerator and denominator are positive, the function is positive for all $\Re$.
- Find important points. The y-intercept is 2.

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{-4x}{(x^2+1)^2} &= 0 \\ x &= 0 \end{aligned}$$

Since $\frac{d^2y}{dx^2} < 0$ at $x = 0$, a local maximum at $(0, 2)$.

There are no local minima.

- Sketch the curve.



Since $\frac{d^2y}{dx^2} = 0$ at $x = \pm\frac{1}{\sqrt{3}}$, points of inflection exist at $\left(\pm\frac{1}{\sqrt{3}}, \frac{3}{2}\right)$.

- Add details. Since $y(x) = y(-x)$, the function has even symmetry.

Section 6.6 Page 372 Question 7

a) • Frame the curve. The root of the denominator defines a vertical asymptote of $r = 0$. As $r \to \infty$, $h \to 0$ and the function eventually approximates the r-axis. Since both the numerator and denominator are positive, the function is positive for all $\Re$.

• Find important points.

$$h'(r) = 0$$
$$-\frac{80\,000}{\pi r^3} = 0$$

no roots

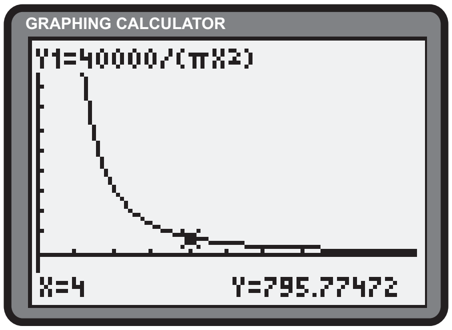
$$h''(r) = 0$$
$$\frac{240\,000}{\pi r^4} = 0$$

no roots

There are no local extrema or points of inflection.

• Add details. Due to the reciprocal nature of the function, as r increases, h decreases.

• Sketch the curve.



b) • Frame the curve. The root of the denominator defines a vertical asymptote of $r = 0$. As $r \to \infty$, $\frac{80\,000}{r} \to 0$ and the function eventually approximates the function $f(r) = 2\pi r^2$. The function is positive for all positive real numbers.

• Find important points.

$$S'(r) = 0$$
$$4\pi r - \frac{80\,000}{r^2} = 0$$
$$r = \sqrt[3]{\frac{20\,000}{\pi}}$$

$$S''(r) = 0$$
$$4\pi + \frac{160\,000}{r^3} = 0$$

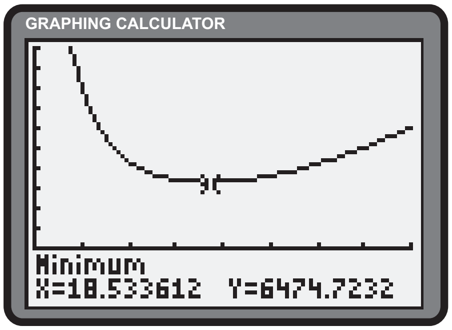
no roots

There is a local minimum at $r = \sqrt[3]{\frac{20\,000}{\pi}}$.

There are no local minima or points of inflection.

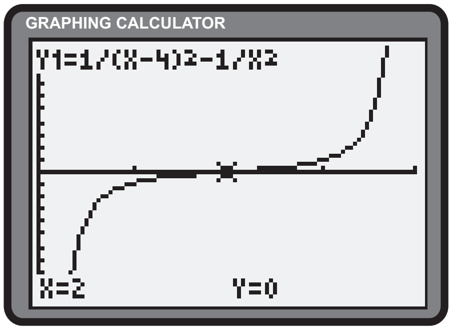
• Add details. None.

• Sketch the curve.



Section 6.6 Page 372 Question 8

a)

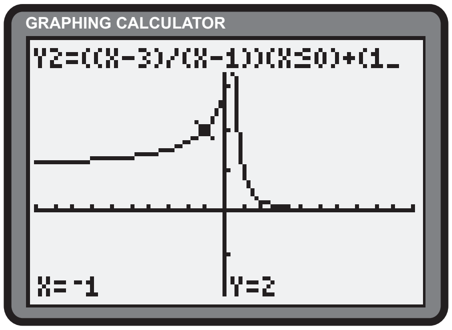


b) The force is undefined at $x = 0$ and $x = 4$ due to division by zero.

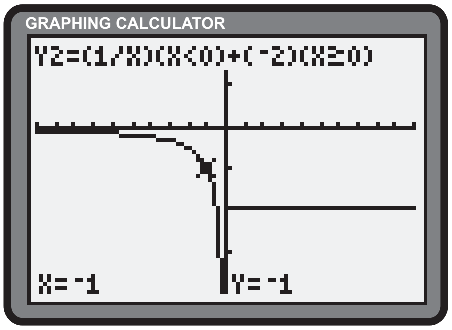
c) Since $F(2) = 0$, the force is zero at $x = 2$.

Section 6.6 Page 373 Question 9

a)

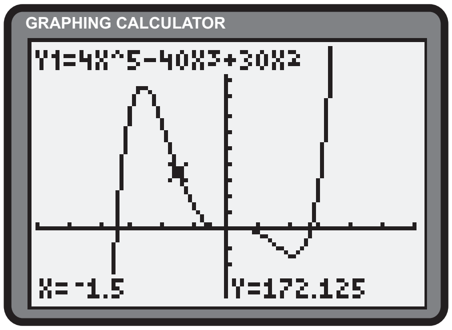


b)



Section 6.6 Page 373 Question 10

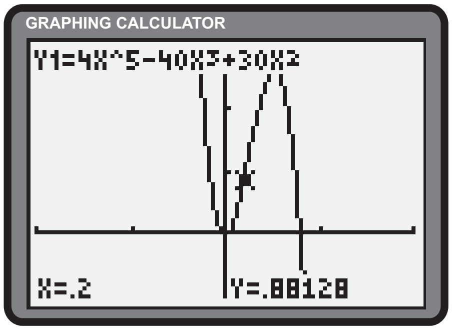
a) i) Two views of g appear below.



Y1=4X^5-40X^3+30X^2

X=-1.5 Y=172.125

$x \in [-6, 6],\ y \in [-150, 550]$



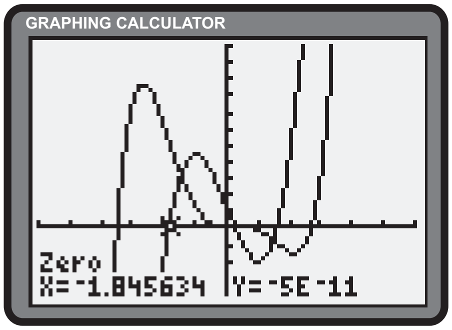
Y1=4X^5-40X^3+30X^2

X=.2 Y=.88128

$x \in [-2, 2],\ y \in [-1, 3]$

ii) Estimates may vary.

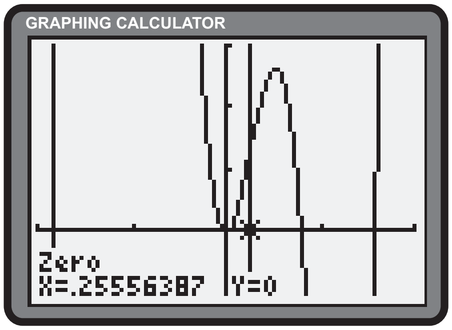
iii) Enter $g''(x) = 80x^3 - 240x + 60$ into the graphing calculator and use the **Zero operation**. Concave downward: $(-\infty, -1.846),\ (0.256, 1.590)$; Concave upward: $(-1.846, 0.256),\ (1.590, \infty)$; Points of inflection: $(-1.846, 268.0)$, $(0.256, 1.296)$, and $(1.590, -44.30)$.



Zero

X=-1.845634 Y=-5E-11

$x \in [-6, 6],\ y \in [-150, 550]$

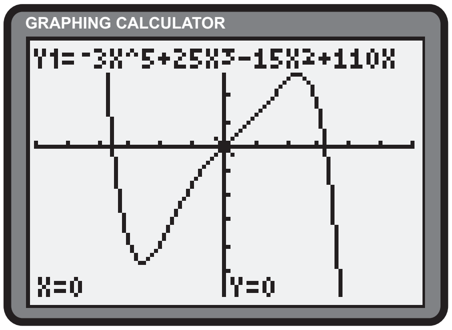


Zero

X=.25556387 Y=0

$x \in [-2, 2],\ y \in [-1, 3]$

b) i) $g(x) = -3x^5 + 25x^3 - 15x^2 + 110x$ is graphed below.



Y1=-3X^5+25X^3-15X^2+110X

X=0 Y=0

$x \in [-6, 6],\ y \in [-600, 400]$

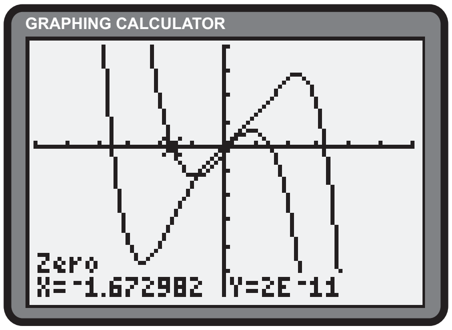
ii) Estimates may vary.

iii) Enter $g''(x) = -60x^3 + 150x - 30$ into the graphing calculator and use the **Zero operation**.

Concave downward: $(-1.673, 0.203)$ and $(1.470, \infty)$;

Concave upward: $(-\infty, -1.673)$ and $(0.203, 1.470)$;

Points of inflection: $(-1.673, -303.756)$, $(0.203, 21.959)$, and $(1.470, 188.047)$.

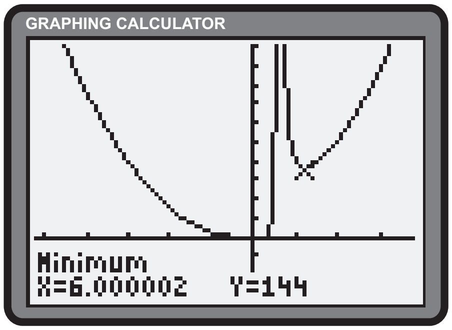


Zero

X=-1.672982 Y=2E-11

Section 6.6 Page 373 Question 11

a) **i)** $x \in [-26, 19]$, $y \in [-120, 400]$



```
Minimum
X=6.000002    Y=144
```

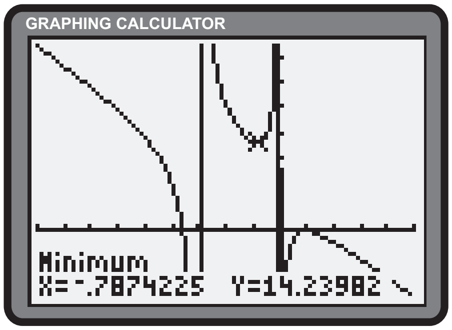
ii) Estimates may vary.

iii) Determine the critical numbers.

$$\begin{aligned} h'(x) &= 0 \\ \frac{(x-3)^2(4x^3) - 2x^4(x-3)}{(x-3)^4} &= 0 \\ \frac{2x^3(x-6)}{(x-3)^3} &= 0 \\ x &= 0 \text{ or } 6 \end{aligned}$$

Local minima exist at (0, 0) and (6, 144).

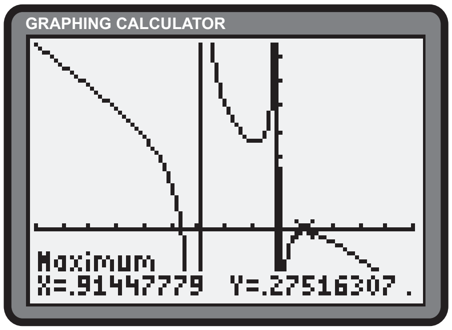
b) **i)** $x \in [-9, 5]$, $y \in [-10, 30]$



```
Minimum
X=-.7874225  Y=14.23982
```

ii) Estimates may vary.

iii) A local minimum exists at $(-0.787, 14.240)$. A local maximum exists at $(0.914, 0.275)$.



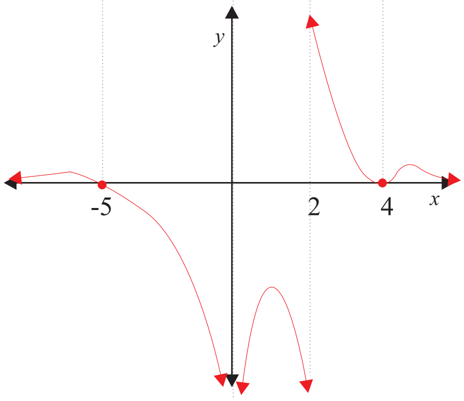
```
Maximum
X=.91447779  Y=.27516307
```

Section 6.6 Page 373 Question 12

a) The roots of the numerator define x-intercepts at -5 and 4. There is no y-intercept. The roots of the denominator define vertical asymptotes of $x = 0$ and $x = 2$. Since the degree of the numerator is one less than the degree of the denominator, the function is asymptotic to the x-axis.

-5 0 2 4

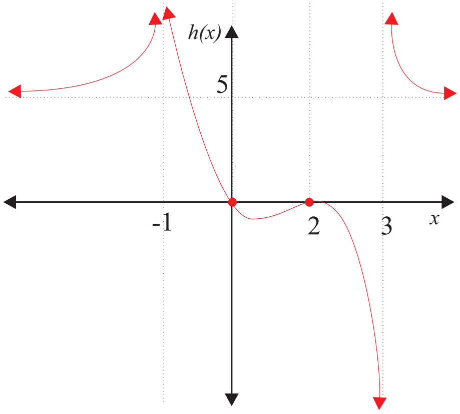
Intervals	(- , -5)	(-5,0)	(0,2)	(2,4)	(4,)
Test values	-6	-2	1	3	5
Sign of y	+	–	–	+	+



b) The roots of the numerator define x-intercepts at 0 and 2. The y-intercept is 0. The roots of the denominator define vertical asymptotes of $x = -1$ and $x = 3$. Since the degrees of the numerator and denominator are equal, as $|x| \to \infty$, $h(x) \to 5$. The function has a horizontal asymptote of $y = 5$.

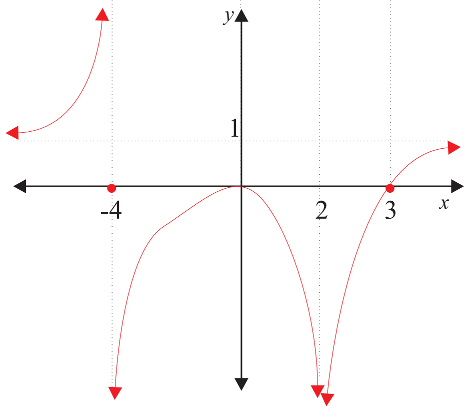
-1 0 2 3

Intervals	(- , -1)	(-1,0)	(0,2)	(2,3)	(3,)
Test values	-2	-1/2	1	2.5	5
Sign of $h(x)$	+	+	–	–	+



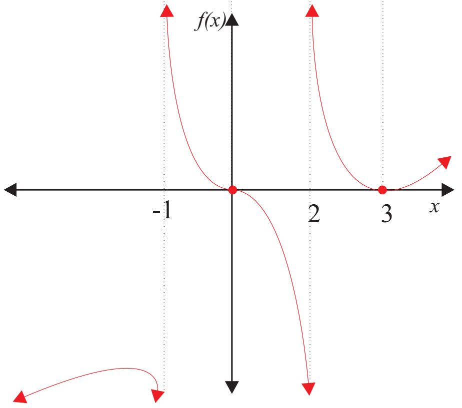
c) The roots of the numerator define x-intercepts at 0 and 3. The y-intercept is 0. The roots of the denominator define vertical asymptotes of $x = -4$ and $x = 2$. Since the degrees of the numerator and denominator are equal, as $|x| \to \infty$, $y \to 1$. The function has a horizontal asymptote of $y = 1$.

	-4		0		2		3	
Intervals	(- , -4)		(-4,0)		(0,2)	(2,3)		(3,)
Test values	-6		-2		1	2.5		5
Sign of y	+		–		–	–		+



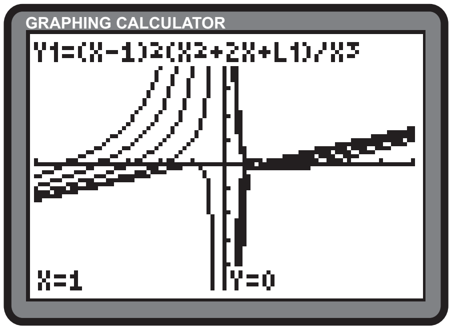
d) The roots of the numerator define x-intercepts at 0 and 3. The y-intercept is 0. The roots of the denominator define vertical asymptotes of $x = -1$ and $x = 2$. Since the degree of the numerator is one more than the degree of the denominator, the function has an oblique linear asymptote. Long division reveals its equation as $y = x - 7$.

	-1		0		2		3	
Intervals	(- , -1)		(-1,0)		(0,2)	(2,3)		(3,)
Test values	-2		-1/2		1	2.5		5
Sign of $f(x)$	–		+		–	+		+



Section 6.6 Page 373 Question 13

a) Answers may vary. The family of curves below uses $k = -15, -8, -3, 0$, and 1. All members have a vertical asymptote at $x = 0$, an oblique asymptote of $y = x$, and an x-intercept of 1.



$Y1=(X-1)^2(X^2+2X+L1)/X^3$

X=1 Y=0

b) Expanding the numerator and simplifying the expression leads to

$$y = x + \frac{(k-3)x^2 + (2-2k)x + k}{x^3} \qquad (1)$$

As $|x| \to \infty$, $\dfrac{(k-3)x^2 + (2-2k)x + k}{x^3} \to 0$ and the function approximates the line $y = x$.

c) When the quadratic numerator of (1) is 0, the function will intersect its oblique asymptote. Determine when its discriminant is greater than or equal to zero.

$$\begin{aligned}((2-2k)^2 - 4(k-3)k &\geq 0\\ 4 - 8k + 4k^2 - 4k^2 + 12k &\geq 0\\ 4k + 4 &\geq 0\\ k &\geq -1\end{aligned}$$

The function will intersect $y = x$ if $k \geq -1$.

d) There will be two x-intercepts if $x^2 + 2x + k$ is a perfect square. Thus, $k = 1$, and the x-intercepts are ±1. It will also have two intercepts if $x - 1$ is a factor of $x^2 + 2x + k$.

$$\begin{aligned}1^2 + 2(1) + k &= 0\\ k &= -3\end{aligned}$$

The x-intercepts are 1 and -3.

Section 6.6 Page 373 Question 14

a)

$$\begin{aligned} g'(x) &= 0 \\ \frac{df(x^2)}{dx^2} \cdot \frac{dx^2}{dx} &= 0 \\ \frac{df(x^2)}{dx^2} \cdot 2x &= 0 \\ x &= 0 \end{aligned}$$

Since $f'(x) > 0$, so too will $\frac{df(x^2)}{dx^2} > 0$. $x = 0$ is the single critical number of g.

b) The transformation $x \to x^2$ maps all real numbers except 0 to positive real numbers. Since f is concave upward over the positive real numbers, g will be concave upward for negative and positive numbers. g is concave upward at $x = 0$, since it is concave upward on either side of 0 and g is differentiable at $x = 0$.

Section 6.6 Page 373 Question 15

$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \\ f'(x) &= 3ax^2 + 2bx + c \end{aligned}$$

Substitute given coordinates.

$$f(-3) = 3$$

$$-27a + 9b - 3c + d = 3 \tag{1}$$

$$f'(-3) = 0$$

$$27a - 6b + c = 0 \tag{2}$$

$$f(2) = 0$$

$$8a + 4b + 2c + d = 0 \tag{3}$$

$$f'(2) = 0$$

$$12a + 4b + c = 0 \tag{4}$$

Reduce system of equations.

$$35a - 5b + 5c = -3 \tag*{(3) − (1) = (5)}$$

$$15a - 10b = 0 \tag*{(2) − (4)}$$

$$3a - 2b = 0 \tag{6}$$

$$25a + 25b = 3 \tag*{5 × (4) − (5) = (7)}$$

$$125a = 6 \tag*{25 × (6) + 2 × (7)}$$

$$a = \frac{6}{125} \tag{8}$$

Perform back substitution.

$$3\left(\frac{6}{125}\right) - 2b = 0 \tag*{substitute (8) into (6)}$$

$$b = \frac{9}{125} \tag{9}$$

$$12\left(\frac{6}{125}\right) + 4\left(\frac{9}{125}\right) + c = 0 \tag*{substitute (8) and (9) into (4)}$$

$$c = -\frac{108}{125} \tag{10}$$

$$8\left(\frac{6}{125}\right) + 4\left(\frac{9}{125}\right) + 2\left(\frac{-108}{125}\right) + d = 0 \tag*{substitute (8), (9), and (10) into (3)}$$

$$d = \frac{132}{125}$$

The cubic function $f(x) = \frac{6}{125}x^3 + \frac{9}{125}x^2 - \frac{108}{125}x + \frac{132}{125}$ satisfies the requirements.

Section 6.6 Page 373 Question 16

a) Let $f(x) = ax^3 + bx^2 + cx + d$ be the general cubic function.

$$\begin{aligned} f'(x) &= 3ax^2 + 2bx + c \\ f''(x) &= 0 \\ 6ax + 2b &= 0 \\ x &= -\frac{b}{3a} \end{aligned}$$

The general cubic function has a single point of inflection at $x = -\frac{b}{3a}$.

b) The general cubic function with x-intercepts p, q, and r is given by

$$\begin{aligned} f(x) &= k(x-p)(x-q)(x-r) \\ &= k(x^3 - (p+q+r)x^2 + (pr+qr+pq)x - pqr) \qquad (1) \end{aligned}$$

Compare (1) to $f(x) = ax^3 + bx^2 + cx + d$.

$$\begin{aligned} x &= -\frac{b}{3a} \\ &= \frac{k(p+q+r)}{3k} \\ &= \frac{p+q+r}{3} \end{aligned}$$

Section 6.6 Page 373 Question 17

a)

$$\begin{aligned} Q'(x) &= 4x^3 + 3px^2 + 2x \\ Q''(x) &= 0 \\ 12x^2 + 6px + 2 &= 0 \\ 6x^2 + 3px + 1 &= 0 \end{aligned}$$

The discriminant of the above quadratic is $9p^2 - 24$.

i) $Q(x)$ will have exactly two points of inflection if $9p^2 - 24 \geq 0$ or $|p| > \frac{2\sqrt{6}}{3}$.

ii) $Q(x)$ will have exactly one point of inflection if $9p^2 - 24 = 0$ or $p = \pm\frac{2\sqrt{6}}{3}$.

iii) $Q(x)$ will have no points of inflection if $9p^2 - 24 \leq 0$ or $|p| < \frac{2\sqrt{6}}{3}$.

b) Sketches will vary.

Section 6.6 Page 374 Question 18

a)

$$\begin{aligned} f'(x) &= 3ax^2 + 2bx + c \\ f''(x) &= 6ax + 2b \\ f''\left(-\frac{b}{3a}\right) &= 6a\left(-\frac{b}{3a}\right) + 2b \\ &= -2b + 2b \\ &= 0 \end{aligned}$$

b) Since $f''\left(-\frac{b}{3a}\right) = 0$ and f'' changes sign on either side of $-\frac{b}{3a}$, f has an inflection point at $\left(-\frac{b}{3a}, f\left(-\frac{b}{3a}\right)\right)$ or $\left(-\frac{b}{3a}, d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)$.

c) The translation defined by $(x, y) \to \left(x + \frac{b}{3a}, y - \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)\right) = (X, Y)$ will map the point of inflection to the origin. Hence $X = x + \frac{b}{3a}$ and $Y = y - \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)$.

d) Substitute $X - \frac{b}{3a}$ and $Y + d + \frac{2b^3}{27a^2} - \frac{bc}{3a}$ for x and y in $y = ax^3 + bx^2 + cx + d$.

$$\begin{aligned}
Y + d + \frac{2b^3}{27a^2} - \frac{bc}{3a} &= a\left(X - \frac{b}{3a}\right)^3 + b\left(X - \frac{b}{3a}\right)^2 + c\left(X - \frac{b}{3a}\right) + d \\
Y + \frac{2b^3}{27a^2} - \frac{bc}{3a} &= \left(X - \frac{b}{3a}\right)\left[a\left(X - \frac{b}{3a}\right)^2 + b\left(X - \frac{b}{3a}\right) + c\right] \\
Y + \frac{2b^3}{27a^2} - \frac{bc}{3a} &= \left(X - \frac{b}{3a}\right)\left[aX^2 - \frac{2b}{3}X + \frac{b^2}{9a} + bX - \frac{b^2}{3a} + c\right] \\
Y + \frac{2b^3}{27a^2} - \frac{bc}{3a} &= \left(X - \frac{b}{3a}\right)\left[aX^2 + \frac{b}{3}X - \frac{2b^2}{9a} + c\right] \\
Y + \frac{2b^3}{27a^2} - \frac{bc}{3a} &= aX^3 + \frac{b}{3}X^2 - \frac{2b^2}{9a}X + cX - \frac{b}{3}X^2 - \frac{b^2}{9a}X + \frac{2b^3}{27} - \frac{bc}{3a} \\
Y &= aX^3 - \frac{3b^2}{9a}X + cX \\
F(X) &= aX^3 + \left(c - \frac{3b^2}{9a}\right)X \\
&= aX^3 + \left(c - \frac{b^2}{3a}\right)X
\end{aligned}$$

e) Since $F(X) = -F(-X)$, $F(X)$ is an odd function.

f) All cubic functions possess odd symmetry with respect to their point of inflection.

Section 6.6 Page 374 Question 19

Every quadratic function has even symmetry with respect to its axis of symmetry.

Section 6.6 Page 374 Question 20

Answers will vary.

6.7 Introducing Optimization Problems

Practise

Section 6.7 Page 382 Question 1

From $mn = 1000$, the constraint $n = \dfrac{1000}{m}$ can be obtained. Substitute this relationship into $R = m + n$ and optimize.

$$\begin{aligned} R &= m + \frac{1000}{m} \\ R'(m) &= 0 \\ 1 - \frac{1000}{m^2} &= 0 \\ m^2 &= 1000 \\ m &= \pm 10\sqrt{10} \\ R''(m) &= \frac{2000}{m^3} \end{aligned}$$

Since $R''(10\sqrt{10}) > 0$, a minimum value for R occurs when $m = 10\sqrt{10}$.

Section 6.7 Page 382 Question 3

From $g^2+4gh = 2700$, the constraint $h = \dfrac{2700 - g^2}{4g}$ can be obtained. Substitute this relationship into $W = g^2h$ and optimize.

$$\begin{aligned} W &= g^2\left(\frac{2700 - g^2}{4g}\right) \\ &= \frac{2700g - g^3}{4} \\ W'(g) &= 0 \\ \frac{2700 - 3g^2}{4} &= 0 \\ g^2 &= 900 \\ g &= 30,\ g > 0 \\ W''(g) &= -\frac{3g}{2} \end{aligned}$$

Since $W''(30) < 0$, a maximum value for W occurs when $g = 30$. Substituting $g = 30$ into the constraint yields $h = \dfrac{2700 - 900}{4(30)}$ or 15.

Apply, Solve, Communicate

Section 6.7 Page 382 Question 4

Let the dimensions of the playpen be x and y, in metres. Let A be the area of the playpen. The perimeter of the playpen provides the constraint $2x + 2y = 16$ or $y = 8 - x$. Substitute the constraint into the area model and optimize.

$$\begin{aligned} A &= xy \\ &= x(8 - x) \\ &= 8x - x^2 \\ A'(x) &= 0 \\ 8 - 2x &= 0 \\ x &= 4 \\ A''(x) &= -2 \end{aligned}$$

Since $A''(4) < 0$, $A(4)$ is a maximum. Substitute $x = 4$ into the constraint to reveal $y = 8 - 4$ or 4. To achieve maximum area for the playpen, the dimensions should be 4 m by 4 m.

Section 6.7 Page 382 Question 5

Let x and y be the dimensions of the corral, in metres. Let A be the area.

$$A = xy \qquad (1)$$

Since the perimeter P, remains constant at 40 m, we have

$$\begin{aligned} P &= 40 \\ 2x + y &= 40 \\ y &= 40 - 2x \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} A &= x(40 - 2x) \\ &= 40x - 2x^2 \\ A'(x) &= 0 \\ 40 - 4x &= 0 \\ x &= 10 \qquad (3) \\ A''(x) &= -4 \end{aligned}$$

Since $A''(x) < 0$, $x = 10$ provides a maximum result. Substitute (3) into (2).

$$\begin{aligned} y &= 40 - 2(10) \\ &= 20 \end{aligned}$$

The dimensions producing maximum area are 10 m by 20 m.

Section 6.7 Page 382 Question 6

Let x and y be the dimensions of the garden, in metres. Let P be the perimeter.

$$P = 2x + 2y \qquad (1)$$

Since the area, A, remains constant at 32 m^2, we have

$$\begin{aligned} A &= 32 \\ xy &= 32 \\ y &= \frac{32}{x} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} P &= 2x + 2\left(\frac{32}{x}\right) \\ &= 2x + \frac{64}{x} \\ P'(x) &= 0 \\ 2 - \frac{64}{x^2} &= 0 \\ x^2 &= 32 \\ x &= 4\sqrt{2} \qquad (3) \\ P''(x) &= \frac{128}{x^3} \end{aligned}$$

Since $P''(x) > 0$, $x = 4\sqrt{2}$ provides a minimum result. Substitute (3) into (2).

$$\begin{aligned} y &= \frac{32}{4\sqrt{2}} \\ &= 4\sqrt{2} \end{aligned}$$

The dimensions requiring the least amount of fencing are $4\sqrt{2}$ m by $4\sqrt{2}$ m.

Section 6.7 Page 382 Question 7

a) Let x and y be the dimensions of each of the 12 pens, in metres. Let A be the total area.

$$\begin{aligned} A &= 3x(3y) \\ &= 9xy && (1) \end{aligned}$$

Since the perimeter P, remains constant at 100 m, we have

$$\begin{aligned} P &= 100 \\ 16x + 15y &= 100 \\ y &= \frac{100 - 16x}{15} && (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

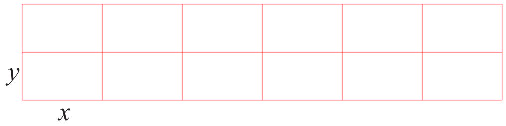
$$\begin{aligned} A &= 9x\left(\frac{100 - 16x}{15}\right) \\ &= \frac{3}{5}(100x - 16x^2) \\ A'(x) &= 0 \\ \frac{3}{5}(100 - 32x) &= 0 \\ x &= \frac{25}{8} && (3) \end{aligned}$$

Since $A''(x) < 0$, $x = \frac{25}{8}$ provides a maximum result. Substitute (3) into (2).

$$\begin{aligned} y &= \frac{100 - 50}{15} \\ &= \frac{10}{3} \end{aligned}$$

The dimensions of each pen producing maximum area are $\frac{25}{8}$ m by $\frac{10}{3}$ m.

b) For a two-by-six arrangement, the perimeter constraint is $P = 18x + 14y$.



$$\begin{aligned} P &= 100 \\ 18x + 14y &= 100 \\ y &= \frac{50 - 9x}{7} && (4) \end{aligned}$$

Substitute (4) into (1) and optimize.

$$\begin{aligned} A &= 9x\left(\frac{50 - 9x}{7}\right) \\ &= \frac{9}{7}(50x - 9x^2) \\ A'(x) &= 0 \\ \frac{9}{7}(50 - 18x) &= 0 \\ x &= \frac{25}{9} && (5) \end{aligned}$$

Substitute (5) into (4).

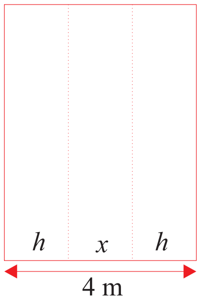
$$\begin{aligned} y &= \frac{50 - 25}{7} \\ &= \frac{25}{7} \end{aligned}$$

The dimensions of each pen producing maximum area are $\frac{25}{9}$ m by $\frac{25}{3}$ m.

c) The three-by-four grid encloses greater area.

Section 6.7 Page 382 Question 8

a) Let x and h be the width and height of the tunnel, respectively, in metres. Let A be the cross-sectional area.



$$A = hx \qquad (1)$$

Since the width, w, of the cardboard, remains constant at 4 m,

$$\begin{aligned} w &= 4 \\ x + 2h &= 4 \\ x &= 4 - 2h && (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} A &= h(4 - 2h) \\ &= 4h - 2h^2 \\ A'(h) &= 0 \\ 4 - 4h &= 0 \\ h &= 1 && (3) \end{aligned}$$

Since $A''(h) < 0$, $h = 1$ provides a maximum result.

To maximize the cross-sectional area, the fold should be made 1 m from each edge.

b) Substitute (3) into (2).

$$\begin{aligned} x &= 4 - 2(1) \\ &= 2 \end{aligned}$$

The dimensions of the tunnel are 1 m by 2 m. Sufficient dimensions depend on the size of the child.

Section 6.7 Page 383 Question 9

a) Let x and h be the width and height of the battery, respectively, in centimetres. Let L be the total wall length.

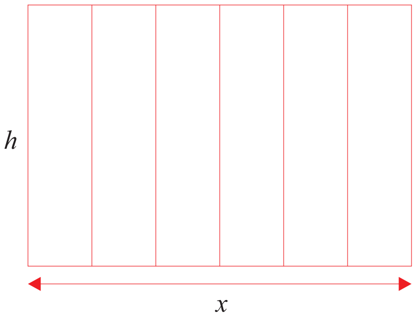
$$L = 2x + 7h \qquad (1)$$

Since the total top area remains constant,

$$\begin{aligned} xh &= 6(65) \\ h &= \frac{390}{x} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} L &= 2x + 7\left(\frac{390}{x}\right) \\ &= 2x + \frac{2730}{x} \\ L'(x) &= 0 \\ 2 - \frac{2730}{x^2} &= 0 \\ x^2 &= 1365 \\ x &= \sqrt{1365},\ x > 0 \qquad (3) \end{aligned}$$



Since $L''(\sqrt{1365}) > 0$, $x = \sqrt{1365}$ provides a minimum result. Substituting (3) into (2) yields a height of $h = \frac{390}{\sqrt{1365}}$ or $\frac{2\sqrt{1365}}{7}$ cm. For a minimum total wall length, the dimensions should be $\sqrt{1365}$ cm by $\frac{2\sqrt{1365}}{7}$ cm.

b) The area of each cell of this battery is 17.5×3.75 or 65.625 cm^2. Substitution of this value in the above calculation leads to an optimal top width of $\frac{105}{2\sqrt{2}}$ cm. Since this more than 22.5 cm, the design has not used the optimum dimensions in the manufacturing of the battery.

Section 6.7 Page 383 Question 10

a) Let w and h be the width and height of the rectangular window, in metres. Let P be the perimeter.

$$P = 2w + 2h \qquad (1)$$

Since the area, A, remains constant at 12 m^2,

$$\begin{aligned} A &= 12 \\ wh &= 12 \\ h &= \frac{12}{w} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} P &= 2w + 2\left(\frac{12}{w}\right) \\ &= 2w + \frac{24}{w} \\ P'(w) &= 0 \\ 2 - \frac{24}{w^2} &= 0 \\ w^2 &= 12 \\ w &= 2\sqrt{3},\ w > 0 \qquad (3) \end{aligned}$$

Since $P''(2\sqrt{3}) > 0$, $w = 2\sqrt{3}$ provides a minimum result. Substitute (3) into (2).

$$\begin{aligned} h &= \frac{12}{2\sqrt{3}} \\ &= 2\sqrt{3} \end{aligned}$$

The dimensions of the rectangle producing minimum perimeter are $2\sqrt{3}$ m by $2\sqrt{3}$ m.

b) Let b and h be the base and altitude of the isosceles triangular window, in metres. Let P be the perimeter.

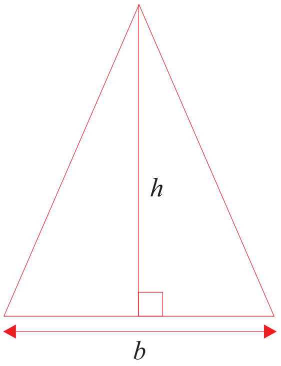
$$\begin{aligned} P &= b + 2\sqrt{h^2 + \left(\frac{b}{2}\right)^2} \\ &= b + \sqrt{4h^2 + b^2} \qquad (1) \end{aligned}$$

Since the area, A, remains constant at 12 m^2,

$$\begin{aligned} A &= 12 \\ \frac{bh}{2} &= 12 \\ h &= \frac{24}{b} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} P &= b + \sqrt{4\left(\frac{24}{b}\right)^2 + b^2} \\ &= b + \sqrt{\frac{2304 + b^4}{b^2}} \\ &= \frac{b^2 + \sqrt{2304 + b^4}}{b} \end{aligned}$$



$$\begin{aligned} P'(b) &= 0 \\ b\left(2b + \frac{4b^3}{2\sqrt{2304 + b^4}}\right) - b^2 - \sqrt{2304 + b^4} &= 0 \\ 2b^2 + \frac{2b^4}{\sqrt{2304 + b^4}} - b^2 - \sqrt{2304 + b^4} &= 0 \\ b^2\sqrt{2304 + b^4} &= 2304 - b^4 \\ 2304b^4 + b^8 &= 2304^2 - 2(2304)b^4 + b^8 \\ b^4 &= \frac{2304}{3} \\ &= 768 \\ b &= 4\sqrt[4]{3},\ b > 0 \qquad (3) \end{aligned}$$

Substitute (3) into (2).

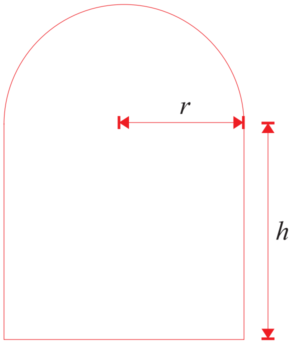
$$\begin{aligned} h &= \frac{24}{4\sqrt[4]{3}} \\ &= \frac{6}{\sqrt[4]{3}} \cdot \frac{\sqrt[4]{27}}{\sqrt[4]{27}} \\ &= \frac{6\sqrt[4]{27}}{3} \\ &= 2\sqrt[4]{27} \end{aligned}$$

The isosceles triangle yielding minimum perimeter has a base of $4\sqrt[4]{3}$ m and an altitude of $2\sqrt[4]{27}$ m.

c) Let r and h be the radius of the semicircle and the height of the window, respectively, in metres. Let P be the perimeter.

$$\begin{aligned} P &= \pi r + 2r + 2h \\ &= (\pi + 2)r + 2h \qquad (1) \end{aligned}$$

Since the area, A, remains constant at 12 m^2,



$$\begin{aligned} A &= 12 \\ \frac{\pi r^2}{2} + 2rh &= 12 \\ \pi r^2 + 4rh &= 24 \\ h &= \frac{24 - \pi r^2}{4r} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} P &= (\pi + 2)r + 2\left(\frac{24 - \pi r^2}{4r}\right) \\ &= (\pi + 2)r + \frac{12}{r} - \frac{\pi}{2}r \\ &= \left(\frac{\pi}{2} + 2\right)r + \frac{12}{r} \\ P'(r) &= 0 \\ \frac{\pi}{2} + 2 - \frac{12}{r^2} &= 0 \\ r^2 &= \frac{12}{\frac{\pi}{2} + 2} \\ &= \frac{24}{\pi + 4} \qquad (3) \\ r &= 2\sqrt{\frac{6}{\pi + 4}},\ r > 0 \qquad (4) \end{aligned}$$

Substitute (3) and (4) into (2).

$$\begin{aligned} h &= \frac{24 - \pi\left(\frac{24}{\pi + 4}\right)}{4\left(2\sqrt{\frac{6}{\pi + 4}}\right)} \\ &= \frac{\frac{12}{\pi + 4}}{\sqrt{\frac{6}{\pi + 4}}} \\ &= 2\sqrt{\frac{6}{\pi + 4}} \end{aligned}$$

The dimensions of the window yielding minimum perimeter are $2\sqrt{\frac{6}{\pi + 4}}$ m by $2\sqrt{\frac{6}{\pi + 4}}$ m.

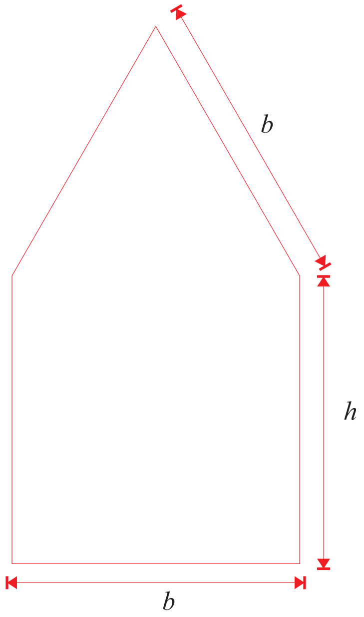
d) Let b and h be the base and height of the rectangular component of the window, in metres. Let P be the perimeter.

$$P = 3b + 2h \qquad (1)$$

Since the area, A, remains constant at 12 m^2,

$$\begin{aligned} A &= 12 \\ bh + \frac{\sqrt{3}}{4}b^2 &= 12 \\ h &= \frac{12 - \frac{\sqrt{3}}{4}b^2}{b} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.



$$\begin{aligned} P &= 3b + 2\left(\frac{12 - \frac{\sqrt{3}}{4}b^2}{b}\right) \\ &= 3b + \frac{24}{b} - \frac{\sqrt{3}}{2}b \\ &= \left(\frac{6-\sqrt{3}}{2}\right)b + \frac{24}{b} \\ P'(b) &= 0 \\ \frac{6-\sqrt{3}}{2} - \frac{24}{b^2} &= 0 \\ b^2 &= \frac{48}{6-\sqrt{3}} \qquad (3) \\ &= \frac{48}{6-\sqrt{3}} \cdot \frac{6+\sqrt{3}}{6+\sqrt{3}} \\ &= \frac{48(6+\sqrt{3})}{33} \\ &= \frac{16(6+\sqrt{3})}{11} \\ b &= \frac{4\sqrt{6+\sqrt{3}}}{\sqrt{11}};\ b > 0 \qquad (4) \end{aligned}$$

Substitute (3) and (4) into (2).

$$\begin{aligned} h &= \frac{12 - \frac{\sqrt{3}}{4} \cdot \frac{48}{6-\sqrt{3}}}{\frac{4\sqrt{6+\sqrt{3}}}{\sqrt{11}}} \\ &= \frac{3\left(1 - \frac{\sqrt{3}}{6-\sqrt{3}}\right)}{\frac{\sqrt{6+\sqrt{3}}}{\sqrt{11}}} \\ &= \frac{\left(\frac{30-6\sqrt{3}}{11}\right)}{\frac{\sqrt{6+\sqrt{3}}}{\sqrt{11}}} \\ &= \frac{30-6\sqrt{3}}{\sqrt{66+11\sqrt{3}}} \end{aligned}$$

The dimensions of the window yielding minimum perimeter are $b \times h = \dfrac{4\sqrt{6+\sqrt{3}}}{\sqrt{11}}$ m $\times$ $\dfrac{30-6\sqrt{3}}{\sqrt{66+11\sqrt{3}}}$ m.

Section 6.7 Page 383 Question 11

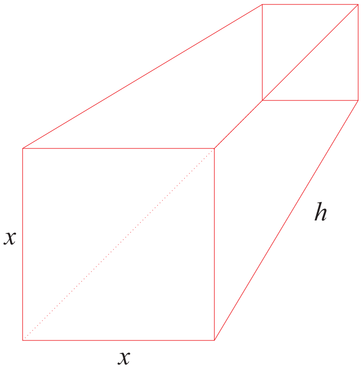
a) Let x and h be the side length of the base and height of the package, respectively, in centimetres. Let A be the surface area.

$$A = 2x^2 + 4xh \qquad (1)$$

Since the volume, V, remains constant at 1000 cm^3,

$$\begin{aligned} V &= 1000 \\ x^2h &= 1000 \\ h &= \frac{1000}{x^2} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.



$$\begin{aligned} A &= 2x^2 + 4x\left(\frac{1000}{x^2}\right) \\ &= 2x^2 + \frac{4000}{x} \\ A'(x) &= 0 \\ 4x - \frac{4000}{x^2} &= 0 \\ x^3 &= 1000 \\ x &= 10 \qquad (3) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} h &= \frac{1000}{10^2} \\ &= 10 \end{aligned}$$

The dimensions of the package yielding minimum surface area are 10 cm×10 cm×10 cm.

b) Answers may vary.

Section 6.7 Page 383 Question 12

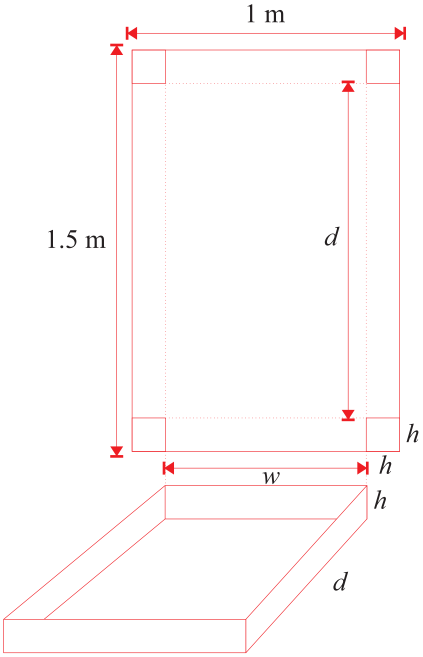
Let h be the side length, in metres, of the square cut from each corner. From the dimensions of the sheet metal, $0 < h < 0.5$. Let w and d be the width and depth of the box, respectively, in metres. Let V be the volume of the box.

$$V = hwd \qquad (1)$$

The dimensions of the sheet metal provide constraints on the width and depth.

$$w = 1 - 2h \qquad (2)$$

$$d = 1.5 - 2h \qquad (3)$$



Substitute (2) and (3) into (1) and optimize.

$$\begin{aligned} V &= h(1 - 2h)(1.5 - 2h) \qquad (4) \\ &= h(1.5 - 5h + 4h^2) \\ &= 4h^3 - 5h^2 + 1.5h \\ V'(h) &= 0 \\ 12h^2 - 10h + 1.5 &= 0 \\ h &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(12)(1.5)}}{2(12)} \\ &= \frac{5 - \sqrt{7}}{12},\ 0 < h < 0.5 \qquad (5) \end{aligned}$$

Substitute (5) into (4).

$$\begin{aligned} V &= \left(\frac{5 - \sqrt{7}}{12}\right)\left(1 - 2\left(\frac{5 - \sqrt{7}}{12}\right)\right)\left(1.5 - 2\left(\frac{5 - \sqrt{7}}{12}\right)\right) \\ &= \left(\frac{5 - \sqrt{7}}{12}\right)\left(\frac{1 + \sqrt{7}}{6}\right)\left(\frac{4 + \sqrt{7}}{6}\right) \\ &= \frac{20 + 14\sqrt{7}}{432} \\ &= \frac{10 + 7\sqrt{7}}{216} \end{aligned}$$

The maximum capacity of the box is $\frac{10 + 7\sqrt{7}}{216}$ m^3.

Section 6.7 Page 383 Question 13

Let h, w, and d be the height, width and depth of the case, respectively, in metres. From the dimensions of the sheet metal, $0 < h < 1$. Let V be the volume of the case.

$$V = hwd \qquad (1)$$

The dimensions of the acrylic sheet provide constraints on the variables.

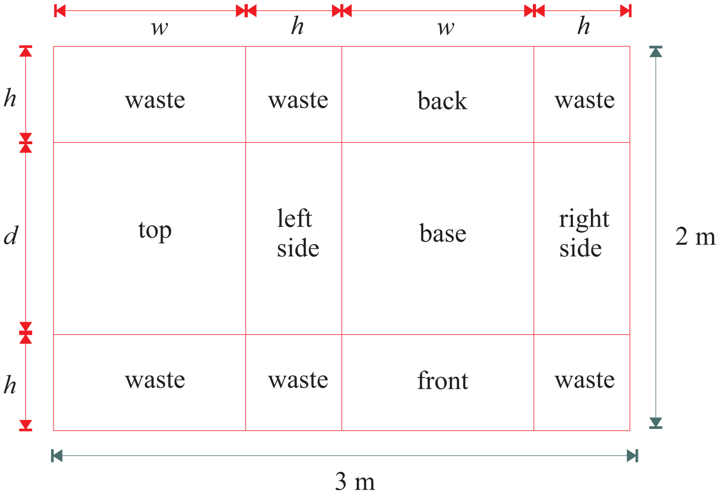
$$\begin{aligned} 2w + 2h &= 3 \\ w &= \frac{3-2h}{2} \qquad (2) \\ d + 2h &= 2 \\ d &= 2 - 2h \qquad (3) \end{aligned}$$

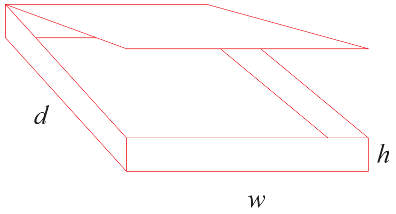
Substitute (2) and (3) into (1) and optimize.

$$\begin{aligned} V &= h\left(\frac{3-2h}{2}\right)(2-2h) \\ &= h(3-2h)(1-h) \\ &= h(3-5h+2h^2) \\ &= 2h^3 - 5h^2 + 3h \\ V'(h) &= 0 \\ 6h^2 - 10h + 3 &= 0 \\ h &= \frac{10 \pm \sqrt{100-72}}{12} \\ &= \frac{5-\sqrt{7}}{6},\ 0 < h < 1 \qquad (4) \end{aligned}$$

Substitute (4) into (2) and (3).

$$\begin{aligned} w &= \frac{3 - 2\left(\frac{5-\sqrt{7}}{6}\right)}{2} \\ &= \frac{4+\sqrt{7}}{6} \\ d &= 2 - 2\left(\frac{5-\sqrt{7}}{6}\right) \\ &= \frac{1+\sqrt{7}}{3} \end{aligned}$$

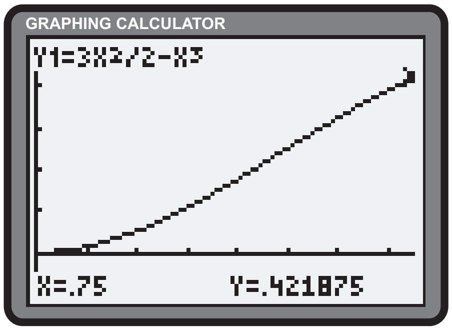




The dimensions of the case yielding maximum capacity are $w \times d \times h = \frac{4+\sqrt{7}}{6}$ m $\times \frac{1+\sqrt{7}}{3}$ m $\times \frac{5-\sqrt{7}}{6}$ m.

Section 6.7 Page 383 Question 14

Let w be the side length of the square end of the package and l be the length of the package. The dimensions are in metres. Let $V = w^2 l$ be the capacity of the package. Since the distance around must be no more than 3 m, a constraint of $2w + 2l = 3$ or $l = 1.5 - w$ is used to simplify V. Therefore $V = w^2(1.5 - w)$. The graph of this function appears at the right. Since the distance around also constrains the square end, $w \leq 0.75$. Therefore, the domain of V is limited to $w \in [0, 0.75]$. The maximum volume over this interval occurs at the right endpoint, $w = 0.75$. Substitution yields $l = 0.75$. Thus, the dimensions of the rectangular box of maximum capacity are 0.75 m × 0.75 m × 0.75 m (a cube).

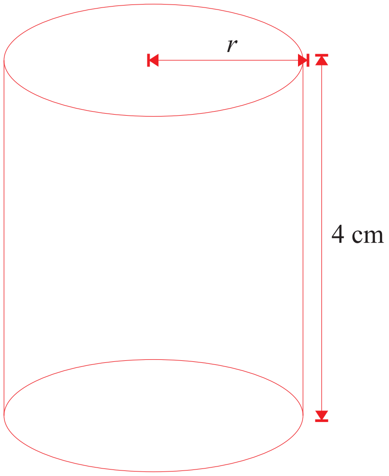


Y1=3X²/2-X³

X=.75 Y=.421875

Section 6.7 Page 384 Question 15

Right Circular Cylinder. Let r be the radius of the ends of the can, in centimetres. Since each unit is to contain 355 cm^3, the radius can be determined.



$$\begin{aligned} V &= 355 \\ \pi r^2(4) &= 355 \\ r^2 &= \frac{355}{4\pi} \\ r &= \sqrt{\frac{355}{4\pi}} \\ &\doteq 5.32 \qquad (1) \end{aligned}$$

The surface area, A, can be determined.

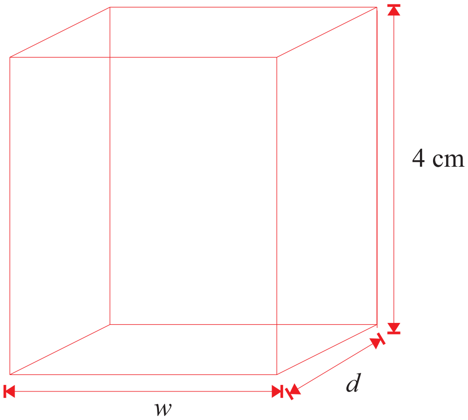
$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r+h) \qquad (2) \end{aligned}$$

Substitute $h = 4$ and (1) into (2).

$$\begin{aligned} &= 2\pi(5.32)(5.32+4) \\ &\doteq 311.5 \end{aligned}$$

The can requires approximately 311.5 cm^2 in materials.

Rectangular Prism. Let w and d be the width and depth of the base respectively, in centimetres. Let A be the surface area of the juice box.



$$\begin{aligned} A &= 2wd + 2(4d) + 2(4w) \\ &= 2(wd + 4d + 4w) \qquad (1) \end{aligned}$$

Since the volume V, remains constant at 355 cm^3,

$$\begin{aligned} V &= 355 \\ 4wd &= 355 \\ d &= \frac{355}{4w} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} A &= 2\left(w\left(\frac{355}{4w}\right) + 4\left(\frac{355}{4w}\right) + 4w\right) \\ A &= 2\left(\frac{355}{4} + \frac{355}{w} + 4w\right) \qquad (3) \\ A'(w) &= 0 \\ 2\left(-\frac{355}{w^2} + 4\right) &= 0 \\ w^2 &= \frac{355}{4} \\ w &\doteq 9.42 \qquad (4) \end{aligned}$$

Substitute (4) into (3).

$$\begin{aligned} A &= 2\left(\frac{355}{4} + \frac{355}{9.42} + 4(9.42)\right) \\ &\doteq 328.2 \end{aligned}$$

The juice box requires approximately 328.2 cm^2 in materials.

The cylindrical can requires the minimum amount of materials.

Section 6.7 Page 384 Question 16

Let b and l be the equilateral side length and length, respectively, of the package, in centimetres. Let A be the area of the package.

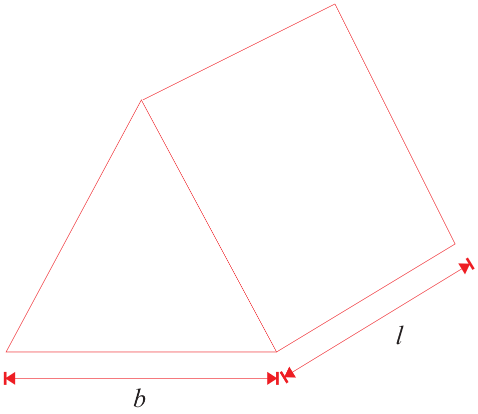
$$\begin{aligned} A &= 2\left(\frac{b}{2}\right)\left(\frac{b\sqrt{3}}{2}\right) + 3bl \\ &= \frac{\sqrt{3}}{2}b^2 + 3bl \qquad (1) \end{aligned}$$

The volume, V, of the package provides a constraint on the variables.

$$\begin{aligned} V &= 400 \\ \frac{\sqrt{3}}{4}b^2 l &= 400 \\ l &= \frac{1600}{\sqrt{3}b^2} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} A &= \frac{\sqrt{3}}{2}b^2 + 3b\left(\frac{1600}{\sqrt{3}b^2}\right) \\ &= \frac{\sqrt{3}}{2}b^2 + \frac{1600\sqrt{3}}{b} \\ A'(b) &= 0 \\ \sqrt{3}b - \frac{1600\sqrt{3}}{b^2} &= 0 \\ b^3 &= 1600 \\ b &= 4\sqrt[3]{25} \qquad (3) \end{aligned}$$



Substitute (3) into (2).

$$\begin{aligned} l &= \frac{1600}{\sqrt{3}\left(4\sqrt[3]{25}\right)^2} \\ &= \frac{100}{\sqrt{3}\sqrt[3]{625}} \\ &= \frac{20}{\sqrt{3}\sqrt[3]{5}} \\ &= \frac{4\sqrt[3]{25}}{\sqrt{3}} \end{aligned}$$

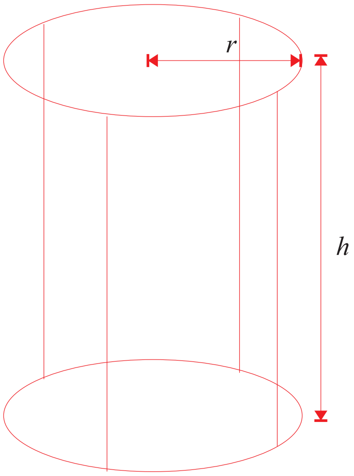
The dimensions of the package yielding the minimum amount of materials are an equilateral side length of $4\sqrt[3]{25}$ cm and a length of $\frac{4\sqrt[3]{25}}{\sqrt{3}}$ cm.

Section 6.7 Page 384 Question 17

Let r and h be the radius of the circular ends and the height, respectively, of the kite, in metres. Let V be the volume of the kite.

$$V = \pi r^2 h \qquad (1)$$

The frame length, L, of the kite provides a constraint on the variables.



$$\begin{aligned} L &= 4 \\ 2(2\pi r) + 4h &= 4 \\ \pi r + h &= 1 \\ h &= 1 - \pi r \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} V &= \pi r^2(1 - \pi r) \\ &= \pi(r^2 - \pi r^3) \\ V'(r) &= 0 \\ \pi(2r - 3\pi r^2) &= 0 \\ r(2 - 3\pi r) &= 0 \\ r &= \frac{2}{3\pi},\ r > 0 \qquad (3) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} h &= 1 - \pi\left(\frac{2}{3\pi}\right) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

For the circular ends, the lengths must be $2\pi\left(\frac{2}{3\pi}\right)$ or $\frac{4}{3}$ cm. The four straight rods should be $\frac{1}{3}$ cm in length.

Section 6.7 Page 384 Question 18

Let r and h be the radius of the ends of the open-topped vase and the height, respectively, in centimetres. Let A be the surface area of the vase.

$$A = \pi r^2 + 2\pi rh \qquad (1)$$

The volume, V, of the vase provides a constraint on the variables.

$$\begin{aligned} V &= 1000 \\ \pi r^2 h &= 1000 \\ h &= \frac{1000}{\pi r^2} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} A &= \pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \\ &= \pi r^2 + \frac{2000}{r} \\ A'(r) &= 0 \\ 2\pi r - \frac{2000}{r^2} &= 0 \\ r^3 &= \frac{1000}{\pi} \\ r &= \frac{10}{\sqrt[3]{\pi}} \qquad (3) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} h &= \frac{1000}{\pi\left(\frac{10}{\sqrt[3]{\pi}}\right)^2} \\ &= \frac{10}{\sqrt[3]{\pi}} \end{aligned}$$

The dimensions of the vase that minimize the amount of glass are $r \times h = \frac{10}{\sqrt[3]{\pi}}$ cm $\times$ $\frac{10}{\sqrt[3]{\pi}}$ cm. This would be a wide, shallow base, not suitable for large bouquets of flowers.

Section 6.7 Page 384 Question 19

Let w be the width of the trough, in centimetres. The height of the triangular ends is $\sqrt{30^2 - \left(\frac{w}{2}\right)^2}$ or $\frac{\sqrt{3600 - w^2}}{2}$. Let V be the volume of the trough, in cubic centimetres.

$$\begin{aligned} V &= \frac{w}{2}\left(\frac{\sqrt{3600 - w^2}}{2}\right)(300) \\ &= 75w\sqrt{3600 - w^2} \qquad (1) \end{aligned}$$

Determine the critical number of (1).

$$\begin{aligned} V'(w) &= 0 \\ 75\sqrt{3600 - w^2} + 75w \cdot \frac{1}{2\sqrt{3600 - w^2}} \cdot -2w &= 0 \\ \sqrt{3600 - w^2} - \frac{w^2}{\sqrt{3600 - w^2}} &= 0 \\ 3600 - w^2 &= w^2 \\ 2w^2 &= 3600 \\ w^2 &= 1800 \\ w &= 30\sqrt{2} \end{aligned}$$

The trough should be $30\sqrt{2}$ cm wide in order to maximize its capacity.

Section 6.7 Page 384 Question 20

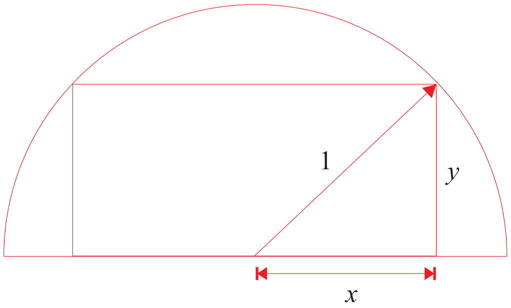
Let $2x$ and y be the dimensions of the rectangular accent, in metres. Let A be the area of the accent.

$$A = 2xy \qquad (1)$$

The semicircular window provides a constraint on the variables.

$$\begin{aligned} x^2 + y^2 &= 1 \\ y &= \sqrt{1 - x^2} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.



$$\begin{aligned} A &= 2x\sqrt{1 - x^2} \\ A'(x) &= 0 \\ 2\sqrt{1 - x^2} + 2x \cdot \frac{1}{2\sqrt{1 - x^2}} \cdot -2x &= 0 \\ \sqrt{1 - x^2} - \cdot \frac{x^2}{\sqrt{1 - x^2}} &= 0 \\ 1 - x^2 &= x^2 \\ x^2 &= \frac{1}{2} \qquad (3) \\ x &= \frac{1}{\sqrt{2}} \end{aligned}$$

The width of the accent is $2\left(\frac{1}{\sqrt{2}}\right)$ or $\sqrt{2}$ m. Substitute (3) into (2).

$$\begin{aligned} y &= \sqrt{1 - \frac{1}{2}} \\ &= \sqrt{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

The dimensions that maximize the area of the stained glass accent are $\sqrt{2}$ m by $\frac{1}{\sqrt{2}}$ m.

Section 6.7 Page 384 Question 21

Let x and y be the dimensions of the rose garden, in metres. Let A be the area of the rose garden and the lawn combined.

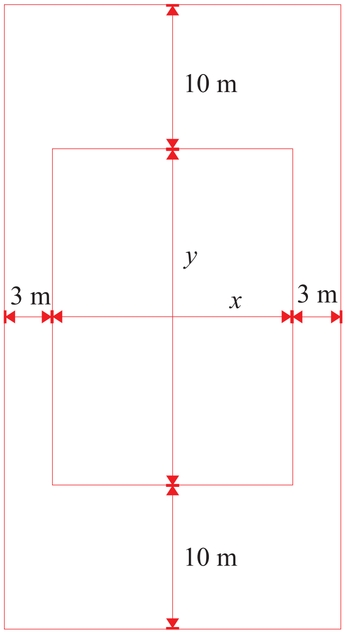
$$A = (x+6)(y+20) \qquad (1)$$

The area of the rose garden provides a constraint on the variables.

$$\begin{aligned} xy &= 60 \\ y &= \frac{60}{x} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} A &= (x+6)\left(\frac{60}{x} + 20\right) \\ &= 60 + 20x + \frac{360}{x} + 120 \\ &= 180 + 20x + \frac{360}{x} \\ A'(x) &= 0 \\ 20 - \frac{360}{x^2} &= 0 \\ x^2 &= 18 \\ x &= 3\sqrt{2},\ x > 0 \qquad (3) \end{aligned}$$



Substitute (3) into (2).

$$\begin{aligned} y &= \frac{60}{3\sqrt{2}} \\ &= 10\sqrt{2} \end{aligned}$$

The dimensions of the rose garden that minimize the total area are $x \times y = 10\sqrt{2}$ m × $3\sqrt{2}$ m.

Section 6.7 Page 384 Question 22

a) $$F(r) = -8r^4 + 12r^3$$

Determine the critical numbers of F.

$$\begin{aligned} F'(r) &= 0 \\ -32r^3 + 36r^2 &= 0 \\ r^2(-8r + 9) &= 0 \\ r &= 0 \text{ or } \frac{9}{8} \end{aligned}$$

Since $F''\left(\frac{9}{8}\right) < 0$, the maximum occurs at $x = \frac{9}{8}$.

Determine the maximum flow rate.

$$\begin{aligned} F\left(\frac{9}{8}\right) &= -8\left(\frac{9}{8}\right)^4 + 12\left(\frac{9}{8}\right)^3 \\ &= \frac{-9^4 + 12(9^3)}{8^3} \\ &= \frac{2187}{512} \end{aligned}$$

When $k = 8$ and $c = 12$, the maximum flow rate is $\frac{2187}{512}$.

b) $$F(r) = -kr^4 + cr^3$$

Determine the critical numbers of F.

$$\begin{aligned} F'(r) &= 0 \\ -4kr^3 + 3cr^2 &= 0 \\ r^2(-4kr + 3c) &= 0 \\ r &= 0 \text{ or } \frac{3c}{4k} \end{aligned}$$

Determine the maximum flow rate.

$$\begin{aligned} F\left(\frac{3c}{4k}\right) &= -k\left(\frac{3c}{4k}\right)^4 + c\left(\frac{3c}{4k}\right)^3 \\ &= -\frac{81c^4}{256k^3} + \frac{27c^4}{64k^3} \\ &= \frac{-81c^4 + 108c^4}{256k^3} \\ &= \frac{27c^4}{256k^3} \end{aligned}$$

The maximum flow rate in terms of k and c is $\frac{27c^4}{256k^3}$.

Section 6.7 Page 385 Question 23

Although a strategy involving derivatives exists, often a simpler approach exists. Consider the following solution. The shortest distance from a point to a line is the perpendicular distance. The point at which the plane is closest to the tower must, therefore, lie on the line $y = 2x + 3$ *and* the line perpendicular to it through the origin. Equating $2x + 3$ and $-\frac{1}{2}x$ yields a result of $x = -\frac{6}{5}$. Substituting $x = -\frac{6}{5}$ into either line reveals $y = \frac{3}{5}$. Hence, the point $\left(-\frac{6}{5}, \frac{3}{5}\right)$ represents the closest the plane gets to the tower.

Section 6.7 Page 385 Question 24

The growth rate equation defines a downward-opening parabola with roots of 0 and 48. The maximum of such a parabola also lies on the parabola's axis of symmetry. Since the axis of symmetry passes through the midpoint of the roots, the equation of the axis of symmetry is $x = 24$. The maximum growth rate, the y-value of the vertex, is $48(24) - 24^2$ or 576. Hence, the maximum growth rate is 576 and it is achieved when $t = 24$ h.

Section 6.7 Page 385 Question 25

Determine the critical number(s) of P.

$$\begin{aligned}P'(g) &= 0\\ 0.1 - \frac{1000}{(1+g)^2} &= 0\\ (1+g)^2 &= 10\,000\\ 1 + g &= 100,\ g > 0\\ g &= 99\end{aligned}$$

A gait of 99 minimizes the required power for an animal to run.

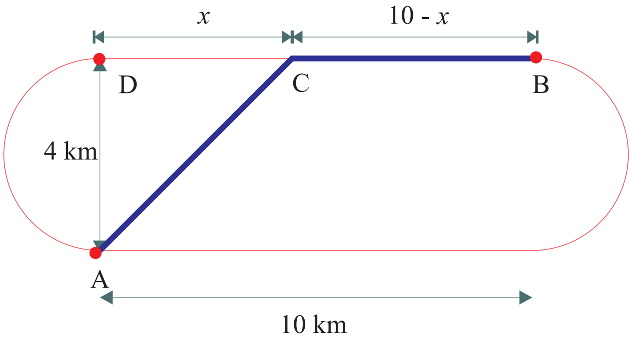
Section 6.7 Page 385 Question 26

Let x be the distance from C to D, in kilometres. The distance from A to C can be expressed as $\sqrt{x^2 + 16}$. Let T be the total time taken for the race, in hours.

$$\begin{aligned}T &= \frac{\sqrt{x^2+16}}{2} + \frac{10-x}{10}\\ &= \frac{\sqrt{x^2+16}}{2} + 1 - \frac{x}{10}\end{aligned}$$

Determine the critical numbers of T.

$$\begin{aligned}T'(x) &= 0\\ \frac{1}{2}\cdot\frac{2x}{2\sqrt{x^2+16}} - \frac{1}{10} &= 0\\ \frac{x}{\sqrt{x^2+16}} - \frac{1}{5} &= 0\\ 5x &= \sqrt{x^2+16}\\ 25x^2 &= x^2 + 16\\ 24x^2 &= 16\\ x^2 &= \frac{2}{3}\\ x &= \sqrt{\frac{2}{3}}\end{aligned}$$



Point C should be taken $10 - \sqrt{\frac{2}{3}}$ or approximately 9.18 km from B.

Section 6.7 Page 385 Question 27

Determine the critical number(s) of D.

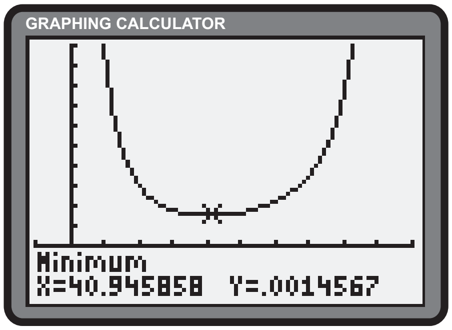
$$\begin{aligned} D'(v) &= 0 \\ 60v + \frac{150^2}{25}\cdot\left(-\frac{2}{v^3}\right) &= 0 \\ 60v - \frac{1800}{v^3} &= 0 \\ v^4 &= 30 \\ v &= \sqrt[4]{30},\ v > 0 \end{aligned}$$

A speed of $\sqrt[4]{30}$ m/s minimizes the drag.

Section 6.7 Page 385 Question 28

Let l be the volume of the quieter band, where l is a positive constant. Let d be the distance, in metres, from the quieter band. Let I be the total intensity of the two bands.

$$\begin{aligned} I &= \frac{l}{d^2} + \frac{3l}{(100-d)^2} \\ &= l\left(\frac{1}{d^2} + \frac{3}{(100-d)^2}\right) \end{aligned}$$



```
Minimum
X=40.945858   Y=.0014567
```

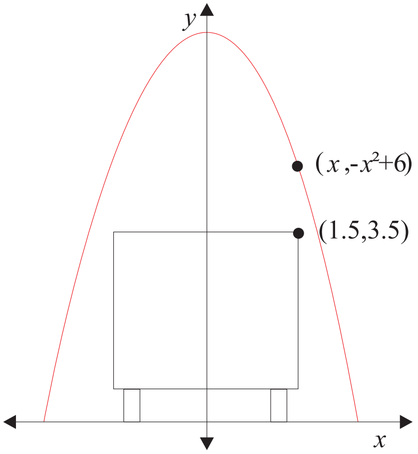
Determine the critical number(s) of I.

$$\begin{aligned} I'(d) &= 0 \\ l\left(-\frac{2}{d^3} + \frac{-6}{(100-d)^3}\cdot(-1)\right) &= 0 \\ -\frac{2}{d^3} + \frac{6}{(100-d)^3} &= 0 \\ 3d^3 &= (100-d)^3 \\ \sqrt[3]{3}d &= 100 - d \\ d(1+\sqrt[3]{3}) &= 100 \\ d &= \frac{100}{1+\sqrt[3]{3}} \end{aligned}$$

The quietest location is $\frac{100}{1+\sqrt[3]{3}}$ or approximately 40.95 m from the quieter band.

Section 6.7 Page 385 Question 29

Determine the distance, d, between the corner of the truck and a point on the bridge.



$$\begin{aligned} d &= \sqrt{(1.5-x)^2 + (3.5+x^2-6)^2} \\ &= \sqrt{x^4 - 4x^2 - 3x + 8.5} \end{aligned}$$

Determine the critical number(s) of d.

$$\begin{aligned} d'(x) &= 0 \\ \frac{4x^3 - 8x - 3}{2\sqrt{x^4 - 4x^2 - 3x + 8.5}} &= 0 \\ 4x^3 - 8x - 3 &= 0 \\ x &\doteq 1.574 \\ d(1.574) &\doteq 0.077 \end{aligned}$$

The maximum clearance is approximately 0.077 m.

Section 6.7 Page 385 Question 30

Let x be the distance to the less massive planet.

$$\begin{aligned} F(x) &= G\frac{Mm}{x^2} + G\frac{2Mm}{(1-x)^2} \\ &= GMm\left(\frac{1}{x^2} + \frac{2}{(1-x)^2}\right) \end{aligned}$$

Determine the critical number(s) of F.

$$\begin{aligned} F'(x) &= 0 \\ GMm\left(-\frac{2}{x^3} + \frac{4}{(1-x)^3}\right) &= 0 \\ 2x^3 &= (1-x)^3 \\ \sqrt[3]{2}x &= 1-x \\ x(1+\sqrt[3]{2}) &= 1 \\ x &= \frac{1}{1+\sqrt[3]{2}} \end{aligned}$$

The vessel should be located $\frac{1}{1+\sqrt[3]{2}}$ or 44.25% of the distance between the planets, from the smaller planet.

Section 6.7 Page 385 Question 31

$$\begin{aligned} P(M) &= M^2\left(k - \frac{M}{3}\right) \\ &= kM^2 - \frac{M^3}{3} \end{aligned}$$

Determine the critical number(s) of P.

$$\begin{aligned} P'(M) &= 0 \\ 2kM - M^2 &= 0 \\ M(2k - M) &= 0 \\ M &= 0 \text{ or } 2k \end{aligned}$$

Since $P''(2k) < 0$, the body is most sensitive when $M = 2k$.

Section 6.7 Page 386 Question 32

a) Let S be the strength of the source of light from each standard. The distance from A to P is $\sqrt{x^2 + k^2}$. The distance from B to P is $\sqrt{(20 - x)^2 + k^2}$.

$$\begin{aligned} I(x) &= \frac{S}{\left(\sqrt{x^2 + k^2}\right)^2} + \frac{S}{\left(\sqrt{(20 - x)^2 + k^2}\right)^2} \\ &= S\left(\frac{1}{x^2 + k^2} + \frac{1}{(20 - x)^2 + k^2}\right) \end{aligned}$$

b) Determine $I'(x)$.

$$\begin{aligned} I'(x) &= S\left(\frac{-2x}{(x^2 + k^2)^2} + \frac{-2(20 - x)(-1)}{((20 - x)^2 + k^2)^2}\right) \\ &= -2S\left(\frac{x}{(x^2 + k^2)^2} - \frac{20 - x}{((20 - x)^2 + k^2)^2}\right) \qquad (1) \end{aligned}$$

Substitute $k = 5$ into $I'(x)$.

$$I'(x) = -2S\left(\frac{x}{(x^2 + 25)^2} - \frac{20 - x}{((20 - x)^2 + 25)^2}\right)$$

P is at the midpoint of l when $x = 10$. Determine $I'(10)$.

$$\begin{aligned} I'(10) &= -2S\left(\frac{10}{(100 + 25)^2} - \frac{10}{(100 + 25)^2}\right) \\ &= 0 \end{aligned}$$

Since $I'(10) = 0$, 10 is a critical number of I. Since $I''(10) > 0$, a minimum value of I is achieved at the midpoint of l when $k = 5$.

c) Substitute $k = 20$ into (1).

$$\begin{aligned} I'(x) &= -2S\left(\frac{x}{(x^2 + 400)^2} - \frac{20 - x}{((20 - x)^2 + 400)^2}\right) \\ I'(10) &= -2S\left(\frac{10}{(100 + 400)^2} - \frac{10}{(100 + 400)^2}\right) \\ &= 0 \end{aligned}$$

Since $I'(10) = 0$, 10 is a critical number of I. Since $I''(10) < 0$, a maximum value of I is achieved at the midpoint of l when $k = 20$.

d) From (1), $I''(x)$ is determined to be,

$$I''(x) = -2S\left[\frac{k^2 - 3x^2}{(x^2 + k^2)^3} + \frac{k^2 - 3(20 - x)^2}{((20 - x)^2 + k^2)^3}\right]$$

The position of the minimum illumination point changes when $x = 10$ no longer provides the minimum value. This happens when $I''(10)$ changes sign from positive to negative.

$$\begin{aligned} I''(10) &= 0 \\ -2S\left[\frac{k^2 - 300}{(100 + k^2)^3} + \frac{k^2 - 300}{(100 + k^2)^3}\right] &= 0 \\ \frac{k^2 - 300}{(100 + k^2)^3} &= 0 \\ k^2 &= 300 \\ k &= 10\sqrt{3},\ k \geq 0 \end{aligned}$$

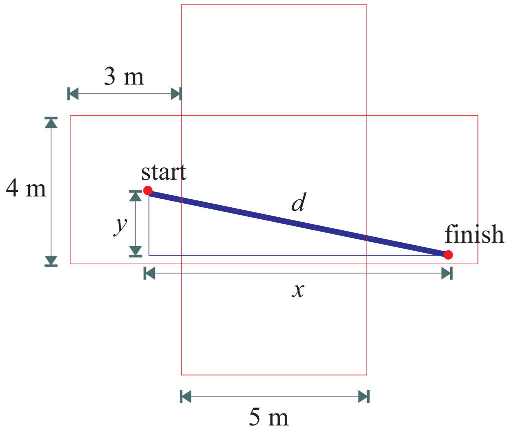
When $k < 10\sqrt{3}$, $I''(10) > 0$. When $k > 10\sqrt{3}$, $I''(10) < 0$. The minimum illumination point changes abruptly when $k = 10\sqrt{3}$.

e) The minimum illumination point will lie on the perpendicular bisector of AB, for $k \leq 10\sqrt{3}$. When $k = 10\sqrt{3}$, $\triangle$ABP is equilateral. For $k > 10\sqrt{3}$, the minimum illumination point occurs at the endpoints of the interval, either $x = 0$ m or $x = 20$ m.

f) Answers may vary.

Section 6.7 Page 386 Question 33

The shortest distance is the straight line distance on the net of the room. Let d be the distance the spider must travel, in metres.



$$\begin{aligned} d &= \sqrt{x^2 + y^2} \\ &= \sqrt{(1+5+2)^2 + (2-0.2)^2} \\ &= \sqrt{8^2 + 1.8^2} \\ &= 8.2 \end{aligned}$$

The minimum distance the spider must walk is 8.2 m.

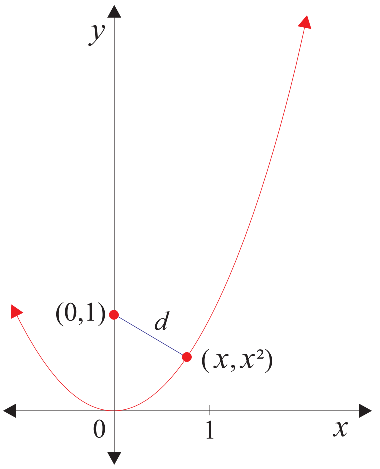
Section 6.7 Page 386 Question 34

Since $x \in \left[0, \dfrac{1}{\sqrt{k}}\right]$, $x \geq 0$. Let d be the distance from the point $(0, 1)$ to the general point on the arc, (x, x^2).

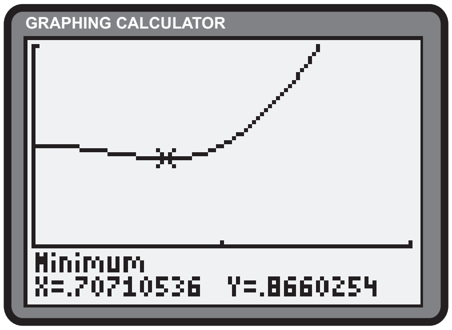
$$\begin{aligned} d &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(x-0)^2 + \left(x^2 - 1\right)^2} \\ &= \sqrt{x^2 + x^4 - 2x^2 + 1} \\ &= \sqrt{x^4 - x^2 + 1} \end{aligned}$$

Determine the critical numbers(s) of d.

$$\begin{aligned} d'(x) &= 0 \\ \frac{4x^3 - 2x}{2\sqrt{x^4 - x^2 + 1}} &= 0 \\ \frac{x(2x^2 - 1)}{\sqrt{x^4 - x^2 + 1}} &= 0 \\ x(2x^2 - 1) &= 0 \\ x &= 0 \text{ or } \frac{1}{\sqrt{2}},\ x > 0 \end{aligned}$$



Since $d''\left(\dfrac{1}{\sqrt{2}}\right) > 0$, the point $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$ is the closest point over the interval $x \geq 0$. Over the restricted domain, $x \in \left[0, \dfrac{1}{\sqrt{k}}\right]$, for $k > 2$, the closest point is the right endpoint of the interval, $\left(\dfrac{1}{\sqrt{k}}, \dfrac{1}{k}\right)$. For $k \leq 2$, the closest point remains $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$. If $k < 1$, the most distant points are endpoints. If $k > 1$, the most distant point is $(0, 0)$. If $k = 1$, endpoints and $(0, 0)$ are all 1 unit away.



```
Minimum
X=.70710536  Y=.8660254
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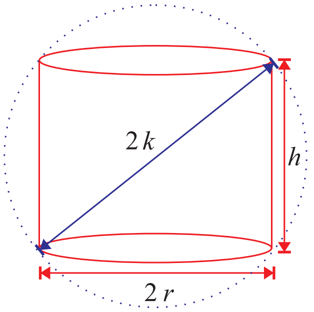
Section 6.7 Page 386 Question 35

a) Let r and h be the base radius and height of the cylinder, respectively. Let V be the volume of the cylinder.



$$V = \pi r^2 h \tag{1}$$

The sphere provides a constraint on r and h.

$$\begin{aligned} (2r)^2 + h^2 &= (2k)^2 \\ 4r^2 + h^2 &= 4k^2 \\ r^2 &= \frac{4k^2 - h^2}{4} \end{aligned} \tag{2}$$

Substitute (2) into (1).

$$\begin{aligned} V &= \pi \left(\frac{4k^2 - h^2}{4} \right) h \\ &= \frac{\pi}{4}(4k^2 h - h^3) \end{aligned}$$

Determine the critical number(s) of V.

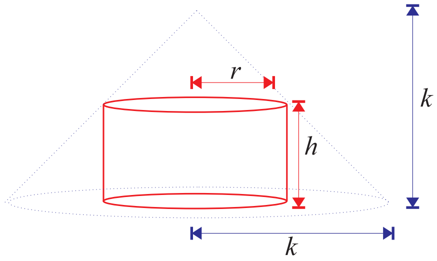
$$\begin{aligned} V'(h) &= 0 \\ \frac{\pi}{4}(4k^2 - 3h^2) &= 0 \\ 3h^2 &= 4k^2 \\ h^2 &= \frac{4k^2}{3} \qquad (3) \\ h &= \frac{2k}{\sqrt{3}}, \ h > 0 \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} r^2 &= \frac{4k^2 - \frac{4k^2}{3}}{4} \\ &= k^2 - \frac{k^2}{3} \\ &= \frac{2k^2}{3} \\ r &= \frac{\sqrt{2}k}{\sqrt{3}}, \ r > 0 \end{aligned}$$

For maximum volume, $(r, h) = \left(\frac{\sqrt{2}k}{\sqrt{3}}, \frac{2k}{\sqrt{3}} \right)$.

b) Let r and h be the base radius and height of the cylinder, respectively. Let V be the volume of the cylinder.



$$V = \pi r^2 h \tag{1}$$

The cone provides a constraint on r and h.

$$h = k - r \tag{2}$$

Substitute (2) into (1).

$$\begin{aligned} V &= \pi r^2 (k - r) \\ &= \pi(kr^2 - r^3) \end{aligned}$$

Determine the critical number(s) of V.

$$\begin{aligned} V'(r) &= 0 \\ \pi(2kr - 3r^2) &= 0 \\ r(2k - 3r) &= 0 \\ r &= \frac{2k}{3} \quad r > 0 \end{aligned} \tag{3}$$

Substitute (3) into (2).

$$\begin{aligned} h &= k - \frac{2k}{3} \\ &= \frac{k}{3} \end{aligned}$$

For maximum volume, $(r, h) = \left(\frac{2k}{3}, \frac{k}{3} \right)$.

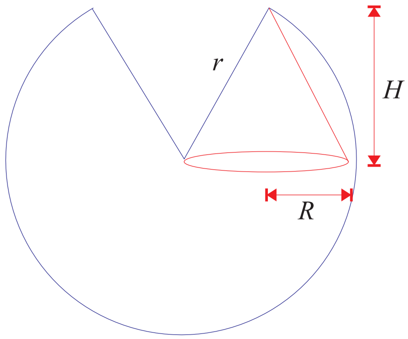
Section 6.7 Page 386 Question 36

Let R and H be the base radius and height of the cup, respectively. Let V be the volume of the cup.

$$V = \frac{\pi}{3}R^2H \qquad (1)$$

The sector of the disk provides a constraint on R and H.

$$\begin{aligned} H^2 + R^2 &= r^2 \\ R^2 &= r^2 - H^2 \qquad (2) \end{aligned}$$



Substitute (2) into (1).

$$\begin{aligned} V &= \frac{\pi}{3}(r^2 - H^2)H \qquad (3) \\ &= \frac{\pi}{3}(r^2H - H^3) \end{aligned}$$

Determine the critical number(s) of V.

$$\begin{aligned} V'(H) &= 0 \\ \frac{\pi}{3}(r^2 - 3H^2) &= 0 \\ H^2 &= \frac{r^2}{3} \qquad (4) \\ H &= \frac{r}{\sqrt{3}};\ H > 0 \qquad (5) \end{aligned}$$

Substitute (4) and (5) into (3).

$$\begin{aligned} V &= \frac{\pi}{3}\left(r^2 - \frac{r^2}{3}\right)\left(\frac{r}{\sqrt{3}}\right) \\ &= \frac{\pi}{3}\left(\frac{2r^2}{3}\right)\left(\frac{r}{\sqrt{3}}\right) \\ &= \frac{2\pi r^3}{9\sqrt{3}} \end{aligned}$$

The maximum capacity of the cup is $\dfrac{2\pi r^3}{9\sqrt{3}}$ cubic units.

Section 6.7 Page 386 Question 37

Assume $y = P(x)$ and $y = Q(x)$ are continuous and differentiable.

a)

$$\begin{aligned} y &= Q(x) \\ y' &= Q'(x) \\ y'' &= Q''(x) \end{aligned}$$

$$\begin{aligned} y &= Q^2(x) \\ y' &= 2Q(x)Q'(x) \\ y'' &= 2(Q'(x)Q'(x) + Q(x)Q''(x) \\ &= 2\left[Q'(x)\right]^2 + 2Q(x)Q''(x) \end{aligned}$$

If $y = Q(x)$ has a critical number at c, then $Q'(c) = 0$. Thus, $2Q(c)Q'(c) = 0$, so $y = Q^2(x)$ has a critical number at c.

If $Q(c)$ is a minimum, then $Q''(c) > 0$. Thus for $y = Q^2(x)$,

$$\begin{aligned} y''(c) &= 2\left[Q'(c)\right]^2 + 2Q(c)Q''(c) \\ &= 2(0)^2 + 2Q(c)Q''(c) \\ &= 2Q(c)Q''(c) \end{aligned}$$

Since $Q''(c) > 0$ for $y = Q^2(x)$ to have a minimum at c, $Q(c) > 0$.

If $Q(c)$ is a maximum, then $Q''(c) < 0$. Thus for $y = Q^2(x)$,

$$y''(c) = 2Q(c)Q''(c)$$

Since $Q''(c) < 0$ for $y = Q^2(x)$ to have a maximum at c, $Q(c) > 0$.

Thus, for $y = Q(x)$ and $y = Q^2(x)$ to have the same type of extremum at c, $Q(c) > 0$.

b)

$$y = Q(x) \qquad y = P(Q(x))$$

$$y' = Q'(x) \qquad y' = P'(Q(x))Q'(x)$$

$$y'' = Q''(x) \qquad y'' = Q'(x)P''(Q(x))Q'(x) + P'(Q(x))Q''(x)$$

$$= \left[Q'(x)\right]^2 P''(Q(x)) + P'(Q(x))Q''(x)$$

If $y = Q(x)$ has a critical number at c, then $Q'(c) = 0$. Determine whether $y = P(Q(x))$ has a critical number at c.

$$\begin{aligned} P'(Q(c))Q'(c) &= P'(Q(c))(0) \\ &= 0 \end{aligned}$$

Thus, if $y = Q(x)$ has a critical number at $x = c$, then $y = P(Q(x))$ has a critical number at c.

c) If $y = P(x)$ has a minimum at c, then $Q''(c) > 0$. If $y = P(x)$ has a maximum at c, then $Q''(c) < 0$. Determine what type of extremum $y = P(Q(x))$ has at c.

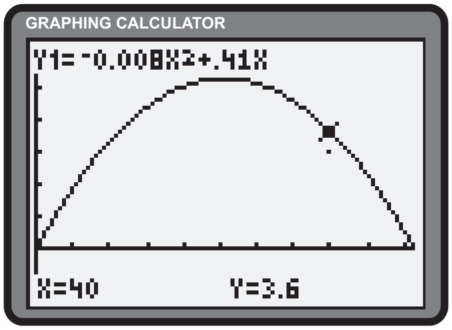
$$\begin{aligned} \left[Q'(c)\right]^2 P''(Q(c)) + P'(Q(c))Q''(c) &= (0)P''(Q(c) + P'(Q(c))Q''(c) \\ &= P'(Q(c))Q''(c) \end{aligned}$$

For $y = P(Q(x))$ to have a minimum at c, since $Q''(c) > 0$, $P'(Q(c)) > 0$. For $y = P(Q(x))$ to have a maximum at c, since $Q''(c) < 0$, $P'(Q(c)) > 0$.

Thus, for $y = P(x)$ and $y = P(Q(x))$ to have the same type of extremum at their critical number c, $P'Q(c)) > 0$.

Section 6.7 Page 386 Question 38

a) Since $y(40) = -0.008(40)^2 + 0.41(40)$ or 3.6, the football clears the horizontal bar.

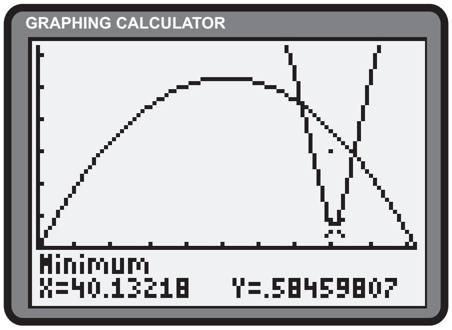


```
Y1=-0.008X²+.41X
X=40        Y=3.6
```

b) Let d be the distance from the bar at $(40, 3)$ and the general point on $(x, -0.008x^2 + 0.41x)$. The distance model can be expressed as

$$d = \sqrt{(x - 40)^2 + (-0.008x^2 + 0.41x - 3)^2}$$

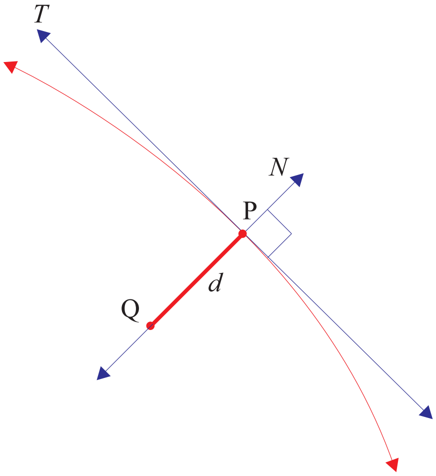
The **Minimum operation** of the graphing calculator determines the smallest distance between the ball and the bar to be approximately 0.585 m.



```
Minimum
X=40.13218   Y=.58459807
```

Section 6.7 Page 387 Question 39

a) Let T be the tangent drawn to a function at P. The shortest distance, d, from a remote point, Q, is the length of the segment of the line, N, perpendicular to T, containing P and Q.



b) Let P$(x, -0.008x^2+0.41x)$ be the coordinates of the general point on $y = -0.008x^2 + 0.41x$. The coordinates of Q are $(40, 3)$. Let m be the slope of PQ. Since the tangent, T, is perpendicular to PQ, the slopes of the lines are negative reciprocals of one another.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{m} \\ \frac{dy}{dx} \times m &= -1 \\ -0.016x + 0.41 \left(\frac{-0.008x^2 + 0.41x - 3}{x - 40} \right) &= -1 \qquad (1) \end{aligned}$$

Substitution of $x = 40.132\ 18$, the result obtained in question 38, verifies (1).

6.8 Optimization Problems in Business and Economics

Practise

Section 6.8 Page 392 Question 1

a) Formulate the profit function, P.

$$\begin{aligned}P(x) &= R(x) - C(x)\\ &= xp(x) - C(x)\\ &= x(50 - 0.5x) - (10 + 4x)\\ &= 50x - 0.5x^2 - 10 - 4x\\ &= -0.5x^2 + 46x - 10\end{aligned}$$

Determine the critical number(s) of P.

$$\begin{aligned}P'(x) &= 0\\ -x + 46 &= 0\\ x &= 46\end{aligned}$$

Since $P''(46) < 0$, profit will be a maximum for $x = 46$.

b) Formulate the profit function, P.

$$\begin{aligned}P(x) &= R(x) - C(x)\\ &= xp(x) - C(x)\\ &= x(10) - (500 + 5x + 0.01x^2)\\ &= 10x - 500 - 5x - 0.01x^2\\ &= -0.01x^2 + 5x - 500\end{aligned}$$

Determine the critical number(s) of P.

$$\begin{aligned}P'(x) &= 0\\ -0.02x + 5 &= 0\\ x &= 250\end{aligned}$$

Since $P''(250) < 0$, profit will be a maximum for $x = 250$.

c) Formulate the profit function, P.

$$\begin{aligned}P(x) &= R(x) - C(x)\\ &= xp(x) - C(x)\\ &= x(10 - 0.002x) - (500 + 5x + 0.01x^2)\\ &= 10x - 0.002x^2 - 500 - 5x - 0.01x^2\\ &= -0.012x^2 + 5x - 500\end{aligned}$$

Determine the critical number(s) of P.

$$\begin{aligned}P'(x) &= 0\\ -0.024x + 5 &= 0\\ x &\doteq 208\end{aligned}$$

Since $P''(208) < 0$, profit will be a maximum for $x = 208$.

d) Formulate the profit function, P.

$$\begin{aligned}P(x) &= R(x) - C(x)\\ &= xp(x) - C(x)\\ &= x(50 - 0.01x) - (1000 + 20x + x^2 + 0.0001x^3)\\ &= 50x - 0.01x^2 - 1000 - 20x - x^2 - 0.0001x^3\\ &= -0.0001x^3 - 1.01x^2 + 30x - 1000\end{aligned}$$

Determine the critical number(s) of P.

$$\begin{aligned}P'(x) &= 0\\ -0.0003x^2 - 2.02x + 30 &= 0\\ x &= \frac{2.02 \pm \sqrt{(-2.02)^2 + 4(0.0003)(30)}}{-0.0006}\\ &\doteq -6748 \text{ or } 15\end{aligned}$$

Since $P''(15) < 0$, profit will be a maximum for approximately $x = 15$. (Note that this product will never make a profit.)

e) Formulate the profit function, P.

$$\begin{aligned}P(x) &= R(x) - C(x)\\ &= xp(x) - C(x)\\ &= x(9 - 2x) - (1 + 4x - 3x^2 + x^3)\\ &= 9x - 2x^2 - 1 - 4x + 3x^2 - x^3\\ &= -x^3 + x^2 + 5x - 1\end{aligned}$$

Determine the critical number(s) of P.

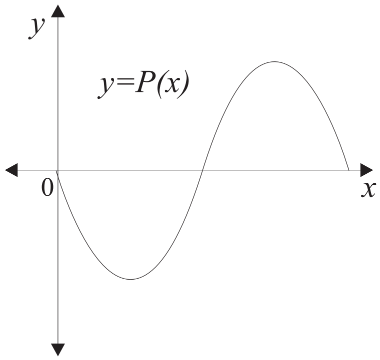
$$\begin{aligned}P'(x) &= 0\\ -3x^2 + 2x + 5 &= 0\\ 3x^2 - 2x - 5 &= 0\\ (3x - 5)(x + 1) &= 0\\ x &= -1 \text{ or } \frac{5}{3}\end{aligned}$$

Since $P''\left(\frac{5}{3}\right) < 0$, profit will be a maximum for approximately $x = 2$.

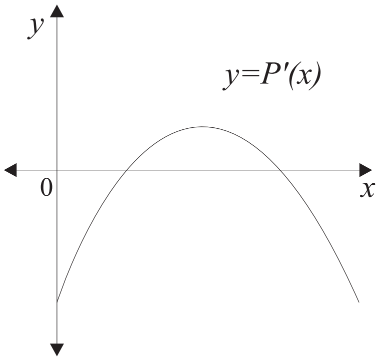
Section 6.8 Page 392 Question 2

a) Maximum profit is realized at the x-value with the longest vertical line segment from $R(x)$ down to $C(x)$.

b)



c)



d) When $P'(x) > 0$, increasing production will increase profits.

Section 6.8 Page 392 Question 3

a) The quadratic regression feature of the graphing calculator suggests $C(x) \doteq 0.017x^2 + 301.897x + 70\,906.212$.

b) Using the regression equation, $C(3000) \doteq 1\,128\,031$.

c)

$$\begin{aligned} R(x) &= xp(x) \\ &= x(3000 - 0.01x) \\ &= 3000x - 0.01x^2 \\ R(3000) &= 3000(3000) - 0.01(3000)^2 \\ &= 8\,910\,000 \end{aligned}$$

Painting 3000 cars will earn revenues of \$8 910 000.

d)

$$\begin{aligned} R'(x) &= 0 \\ 3000 - 0.02x &= 0 \\ 0.02x &= 3000 \\ x &= 150\,000 \end{aligned}$$

Revenue will be maximized at $x = 150\,000$.

e)

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 3000x - 0.01x^2 - (0.017x^2 + 301.897x + 70\,906.212) \\ &= -0.027x^2 + 2698.103x - 70\,906.212 \end{aligned}$$

Determine the critical number of P.

$$\begin{aligned} P'(x) &= 0 \\ -0.054x + 2698.103 &= 0 \\ x &\doteq 49\,964.87 \end{aligned}$$

Profit will be maximized at approximately $x = 50\,000$.

Section 6.8 Page 392 Question 4

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= xp(x) - C(x) \\ &= x(16 - 0.03x) - (100 + 8x - 0.1x^2 + 0.001x^3) \\ &= 16x - 0.03x^2 - 100 - 8x + 0.1x^2 - 0.001x^3 \\ &= -0.001x^3 + 0.07x^2 + 8x - 100 \end{aligned}$$

Determine the critical number of P.

$$\begin{aligned} P'(x) &= 0 \\ -0.003x^2 + 0.14x + 8 &= 0 \\ x &\doteq 80 \end{aligned}$$

Profit will be maximized at approximately $x = 80$ m.

Section 6.8 Page 392 Question 5

a) From the information given, $p(x)$ is a linear function with a slope of $\frac{-10}{5}$ or -2.

$$\begin{aligned} p(x) - 850 &= -2(x - 120) \\ p(x) &= -2x + 240 + 850 \\ &= 1090 - 2x \end{aligned}$$

The price function is $p(x) = 1090 - 2x$.

b)

$$\begin{aligned} R(x) &= xp(x) \\ &= x(1090 - 2x) \\ &= -2x^2 + 1090x \end{aligned}$$

Determine the critical number of R.

$$\begin{aligned} R'(x) &= 0 \\ -4x + 1090 &= 0 \\ x &= 272.5 \end{aligned}$$

Since $R''(272.5) < 0$, maximum revenue is realized at $x = 272.5$. The cameras should be sold at $p(272.5) = 1090 - 2(272.5)$ or \$545.

c) To maximize revenue, the retailer is offering a $\frac{850 - 545}{850}$ or approximately 35.9% discount.

Section 6.8 Page 393 Question 6

From the information given, $p(x)$ is a linear function with a slope of $\frac{-5}{2000}$ or -0.0025.

$$\begin{aligned} p(x) - 60 &= -0.0025(x - 18\ 000) \\ p(x) &= -0.0025x + 45 + 60 \\ &= 105 - 0.0025x \end{aligned}$$

Determine the revenue function, $R(x)$.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(105 - 0.0025x) \\ &= 105x - 0.0025x^2 \end{aligned}$$

Determine the critical number of R.

$$\begin{aligned} R'(x) &= 0 \\ 105 - 0.005x &= 0 \\ x &= 21\ 000 \end{aligned}$$

Since $R''(21\ 000) < 0$, maximum revenue is realized at $x = 21\ 000$. The ticket price should be set at $p(21\ 000) = 105 - 0.0025(21\ 000)$ or \$52.50.

Section 6.8 Page 393 Question 7

If between 18 and 30 people sign up, the maximum revenue is 30 × \$45 or \$1350. Consider the case for more than 30 people. From the information given, $p(x)$ is a linear function with a slope of $\frac{-1}{1}$ or -1.

$$\begin{aligned} p(x) - 45 &= -1(x - 30) \\ p(x) &= -x + 30 + 45 \\ &= 75 - x \end{aligned}$$

Determine the revenue function, $R(x)$.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(75 - x) \\ &= 75x - x^2 \end{aligned}$$

Determine the critical number of R.

$$\begin{aligned} R'(x) &= 0 \\ 75 - 2x &= 0 \\ x &= 37.5 \end{aligned}$$

For more than 30 people, maximum revenue is realized at $x = 37.5$. This revenue is $R(37.5) = 75(37.5) - (37.5)^2$ or \$1406.25. Maximum revenue is realized when 37 or 38 passengers sign up.

Section 6.8 Page 393 Question 8

a) From the information given, $p(x)$ is a linear function with a slope of $\frac{-0.1}{20}$ or -0.005.

$$\begin{aligned} p(x) - 1.2 &= -0.005(x - 500) \\ p(x) &= -0.005x + 2.5 + 1.2 \\ &= -0.005x + 3.7 \end{aligned}$$

Determine the revenue function, $R(x)$.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(-0.005x + 3.7) \\ &= -0.005x^2 + 3.7x \end{aligned}$$

Determine the critical number of R.

$$\begin{aligned} R'(x) &= 0 \\ -0.01x + 3.7 &= 0 \\ x &= 370 \end{aligned}$$

Since $R''(370) < 0$, maximum revenue is realized at $x = 370$. The price should be set at $p(370) = -0.005(370) + 3.7$ or \$1.85.

b) Formulate the profit function, P.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.005x^2 + 3.7x - (25 + 0.12x + 0.001x^2) \\ &= -0.005x^2 + 3.7x - 25 - 0.12x - 0.001x^2 \\ &= -0.006x^2 + 3.58x - 25 \end{aligned}$$

Determine the critical number of P.

$$\begin{aligned} P'(x) &= 0 \\ -0.012x + 3.58 &= 0 \\ x &\doteq 298 \end{aligned}$$

Since $P''(298) < 0$, maximum profit is realized at $x = 298$. The price should be set at $p(298) = -0.005(298) + 3.7$ or \$2.21.

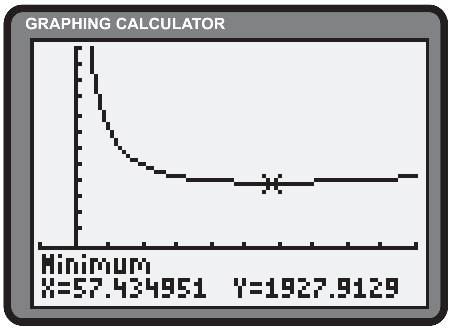
Section 6.8 Page 393 Question 9

Let C be the total cost for the trip.

$$\begin{aligned} C(v) &= \text{running costs} + \text{driver's wages} \\ &= 1500\left(0.85 + 0.0004v^{\frac{3}{2}}\right) + \left(\frac{1500}{v} \times 15\right) \\ &= 1500\left(0.85 + 0.0004v^{\frac{3}{2}} + \frac{15}{v}\right) \end{aligned}$$

Determine the critical numbers of C.

$$\begin{aligned} C'(v) &= 0 \\ 1500\left(0.0006v^{\frac{1}{2}} - \frac{15}{v^2}\right) &= 0 \\ 0.0006v^{\frac{1}{2}} &= \frac{15}{v^2} \\ 3.6 \times 10^{-7}v &= \frac{225}{v^4} \\ v^5 &= \frac{225}{3.6 \times 10^{-7}} \\ v &\doteq 57.4 \end{aligned}$$



```
Minimum
X=57.434951  Y=1927.9129
```

To minimize total costs the truck should be driven at approximately 57.4 km/h.

Section 6.8 Page 393 Question 10

It will be assumed that $p(x)$ is a linear function with a slope of $\frac{-15}{10}$ or -1.5.

$$\begin{aligned} p(x) - 175 &= -1.5(x - 150) \\ p(x) &= -1.5x + 225 + 175 \\ &= 400 - 1.5x \end{aligned}$$

Determine the revenue function, $R(x)$.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(400 - 1.5x) \\ &= 400x - 1.5x^2 \end{aligned}$$

Determine the critical number of R.

$$\begin{aligned} R'(x) &= 0 \\ 400 - 3x &= 0 \\ x &\doteq 133 \end{aligned}$$

Maximum revenue is realized at $x = 133$ rooms. The price should be set at $p(133) = 400 - 1.5(133)$ or \$200.50.

Section 6.8 Page 393 Question 11

It is evident that that maximum yield occurs over the interval $[30, 50]$. Determine the critical number of Y.

$$\begin{aligned} Y'(t) &= 0 \\ 80 - 2t &= 0 \\ t &= 40 \end{aligned}$$

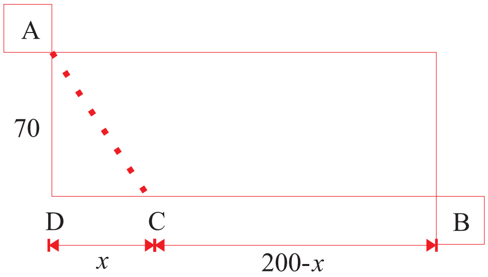
Since $Y'(t) < 0$, a maximum yield of $Y(40) = 80(40) - 40^2 - 1500$ or 100 t is realized on the 40th day.

Section 6.8 Page 393 Question 12

a) Let x be the distance from D to C. Let T be the total cost in dollars.

$$\begin{aligned} T(x) &= 1000\sqrt{70^2 + x^2} + 500(200 - x) \\ &= 500\left(2\sqrt{4900 + x^2} + 200 - x\right) \end{aligned}$$

Determine the critical number(s) of T.



$$\begin{aligned} T'(x) &= 0 \\ 500\left(\frac{2(2x)}{2\sqrt{4900 + x^2}} - 1\right) &= 0 \\ \frac{2x}{\sqrt{4900 + x^2}} - 1 &= 0 \\ 2x &= \sqrt{4900 + x^2} \\ 4x^2 &= 4900 + x^2 \\ 3x^2 &= 4900 \\ x &= \sqrt{\frac{4900}{3}} \\ &\doteq 40.4 \end{aligned}$$

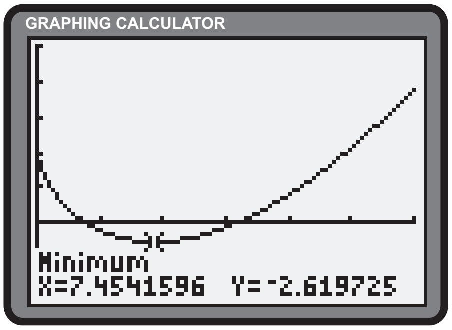
Since $T''(x) > 0$, to minimize the total cost, C should be chosen approximately $200 - 40.4$ or 159.6 m from B.

b) Answers will vary.

Section 6.8 Page 393 Question 13

Determine the critical number(s) of r.

$$\begin{aligned} r'(x) &= 0 \\ \frac{3}{4}x^{\frac{1}{2}} - 4x^{-\frac{1}{3}} &= 0 \\ \frac{3}{4}x^{\frac{1}{2}} &= 4x^{-\frac{1}{3}} \\ x^{\frac{5}{6}} &= \frac{16}{3} \\ x &= \left(\frac{16}{3}\right)^{\frac{6}{5}} \\ &\doteq 7.45 \end{aligned}$$

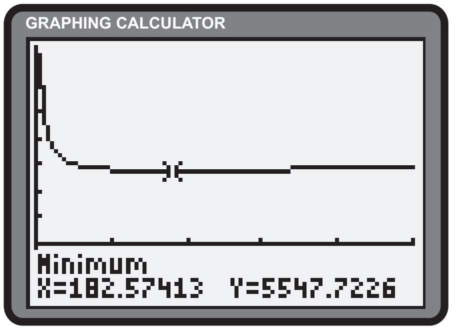


```
Minimum
X=7.4541596  Y=-2.619725
```

Since $r''(7.45) > 0$, the worst time to invest is month 7. Since there is only a single critical number, an evaluation of the endpoints of the interval is required. Since $r(24) > r(0) > r(7.45)$, the best time to invest is month 24.

Section 6.8 Page 394 Question 14

Let $C(x)$ be the cost of producing x steering wheels per day.



```
Minimum
X=182.57413  Y=5547.7226
```

$$C(x) = 5000 + 1.5x + \frac{50\,000}{x}$$

Determine the critical number(s) of C.

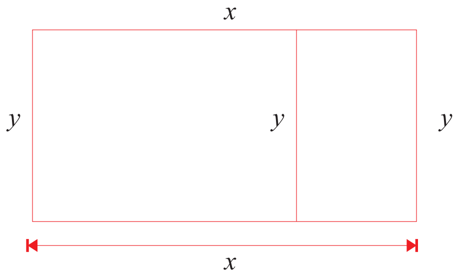
$$\begin{aligned} C'(x) &= 0 \\ 1.5 - \frac{50\,000}{x^2} &= 0 \\ x^2 &= \frac{50\,000}{1.5} \\ x &\doteq 183 \end{aligned}$$

Since $C''(183) > 0$, the factory should produce approximately 183 steering wheels per day to minimize costs.

Section 6.8 Page 394 Question 15

Let C be the cost function of the fencing.

$$\begin{aligned} C(x) &= 10(2x + 2y) + 4y \\ &= 20x + 24y \\ &= 4(5x + 6y) \qquad (1) \end{aligned}$$



The required area constrains the variables.

$$\begin{aligned} xy &= 5000 \\ y &= \frac{5000}{x} \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} C &= 4\left(5x + 6\left(\frac{5000}{x}\right)\right) \\ &= 4\left(5x + \frac{30\,000}{x}\right) \end{aligned}$$

Determine the critical number of C.

$$\begin{aligned} C'(x) &= 0 \\ 4\left(5 - \frac{30\,000}{x^2}\right) &= 0 \\ x^2 &= 6000 \\ x &= 20\sqrt{15}, ;\ x > 0 \qquad (3) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} y &= \frac{5000}{20\sqrt{15}} \\ &= \frac{50\sqrt{15}}{3} \end{aligned}$$

To minimize the cost of fencing, the dimensions of the field should be $x \times y = 20\sqrt{15}$ m $\times$ $\frac{50\sqrt{15}}{3}$ m.

Section 6.8 Page 394 Question 16

Determine the revenue function, $R(x)$.

$$\begin{aligned}R(x) &= xp(x)\\ &= x\sqrt{8000 - x^2}\end{aligned}$$

Determine the critical number of R.

$$\begin{aligned}R'(x) &= 0\\ \sqrt{8000 - x^2}(1) + x\left(\frac{-2x}{2\sqrt{8000 - x^2}}\right) &= 0\\ \frac{8000 - 2x^2}{\sqrt{8000 - x^2}} &= 0\\ x^2 &= 4000\\ x &\doteq 63.2,\ x > 0\end{aligned}$$

To maximize revenue, the artist should produce approximately 63 prints.

Section 6.8 Page 394 Question 17

a) Let A be the average cost.

$$A(x) = \frac{C(x)}{x} \qquad (1)$$

Determine the critical number of $A(x)$.

$$\begin{aligned}A'(x) &= 0\\ \frac{xC'(x) - C(x)(1)}{x^2} &= 0\\ xC'(x) - C(x) &= 0\\ x &= \frac{C(x)}{C'(x)} \qquad (2)\end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned}A(x) &= \frac{C(x)}{\frac{C(x)}{C'(x)}}\\ A(x) &= C'(x)\end{aligned}$$

The smallest average cost is achieved when average cost equals marginal cost.

b)

$$\begin{aligned}A(x) &= \frac{C(x)}{x}\\ &= \frac{x^3 - 2x^2 + 4x}{x}\\ &= x^2 - 2x + 4\end{aligned}$$

Determine the critical number of A.

$$\begin{aligned}A'(x) &= 0\\ 2x - 2 &= 0\\ x &= 1\end{aligned}$$

A production level of 10 000 units will yield a minimal average cost.

Review of Key Concepts

6.1 Increasing and Decreasing Functions

Section Review Page 399 Question 1

a)

$$\begin{aligned} f'(x) &= 0 \\ 1 - 2x &= 0 \\ x &= \frac{1}{2} \end{aligned}$$

Since $f'(0) > 0$, f is increasing on $\left(-\infty, \frac{1}{2}\right)$. Since $f'(1) < 0$, f is decreasing on $\left(\frac{1}{2}, \infty\right)$.

b)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 3x^2 &= 0 \\ x &= 0 \end{aligned}$$

Since $y' > 0$ for $x < 0$ and $x > 0$, y is increasing on $(-\infty, \infty)$.

c)

$$\begin{aligned} g'(x) &= 0 \\ 3x^2 - 4 &= 0 \\ x^2 &= \frac{4}{3} \\ x &= \pm\frac{2}{\sqrt{3}} \end{aligned}$$

g is increasing on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$. g is decreasing on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

		−2/√3		2/√3	
Intervals	(- ,-2/√3)		(-2/√3,2/√3)		(2/√3,)
Test values	-2		0		2
Sign of *g′(x)*	+	0	—	0	+
Nature of graph	inc. ↗		dec. ↘		inc. ↗

d)

$$\begin{aligned} h'(x) &= 0 \\ 2x + 4x^3 &= 0 \\ x(2x^2 + 1) &= 0 \\ x &= 0 \end{aligned}$$

Since $h'(-1) < 0$, h is decreasing on $(-\infty, 0)$. Since $h'(1) > 0$, h is increasing on $(0, \infty)$.

e)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 3x^2 + 12x + 9 &= 0 \\ x^2 + 4x + 3 &= 0 \\ (x + 3)(x + 1) &= 0 \\ x &= -3 \text{ or } -1 \end{aligned}$$

y is increasing on $(-\infty, -3)$ and $(-1, \infty)$.

y is decreasing on $(-3, -1)$.

		−3		−1	
Intervals	(- ,-3)		(-3,-1)		(-1,)
Test values	-4		-2		0
Sign of $\frac{dy}{dx}$	+	0	—	0	+
Nature of graph	inc. ↗		dec. ↘		inc. ↗

f)

$$\begin{aligned} f'(x) &= 0 \\ 4x^3 - 12x^2 - 16x &= 0 \\ x(x^2 - 3x - 4) &= 0 \\ x(x - 4)(x + 1) &= 0 \\ x &= -1, 0, \text{ or } 4 \end{aligned}$$

f is decreasing on $(-\infty, -1)$ and $(0, 4)$.

f is increasing on $(-1, 0)$ and $(4, \infty)$.

		−1		0		4	
Intervals	(- ,-1)		(-1,0)		(0,4)		(4,)
Test values	-3		-1/2		1		5
Sign of *f′(x)*	—	0	+	0	—	0	+
Nature of graph	dec. ↘		inc. ↗		dec. ↘		inc. ↗

Section Review Page 399 Question 2

Determine when the ball is in flight, $t > 0$.

$$\begin{aligned} h(t) &= 0 \\ 1 + 20t - 5t^2 &= 0 \\ t &= \frac{-(20) \pm \sqrt{(20)^2 - 4(-5)(1)}}{2(-5)} \\ &\doteq -0.05 \text{ or } 4.05 \end{aligned}$$

The ball is in the air on the interval $(0, 4.05)$.

Determine the critical numbers of h.

$$\begin{aligned} h'(t) &= 0 \\ 20 - 10t &= 0 \\ t &= 2 \end{aligned}$$

Since $h'(1) > 0$, h is increasing on $(0, 2)$.

Since $h'(3) < 0$, h is decreasing on $(2, 4.05)$.

6.2 Maximum and Minimum Values

Section Review Page 399 Question 3

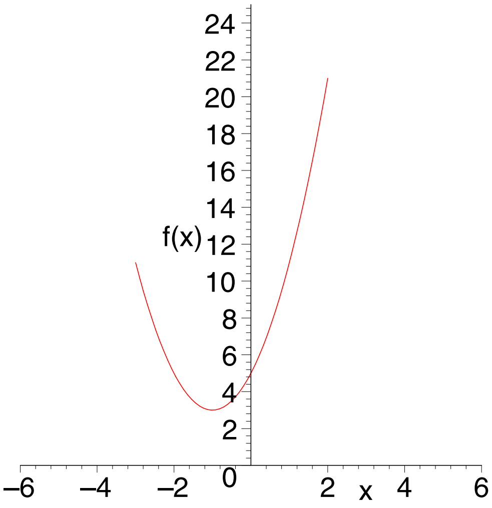
a)

$$\begin{aligned} f'(x) &= 0 \\ 4(x + 1) &= 0 \\ x &= -1 \end{aligned}$$

Determine the function values.

$$\begin{aligned} f(-3) &= 3 + 2(-3 + 1)^2 \\ &= 3 + 2(4) \\ &= 11 \\ f(-1) &= 3 + (-1 + 1)^2 \\ &= 3 \\ f(2) &= 3 + 2(2 + 1)^2 \\ &= 3 + 2(9) \\ &= 21 \end{aligned}$$

Comparison of the function values reveals an absolute maximum of 21 and an absolute minimum of 3.



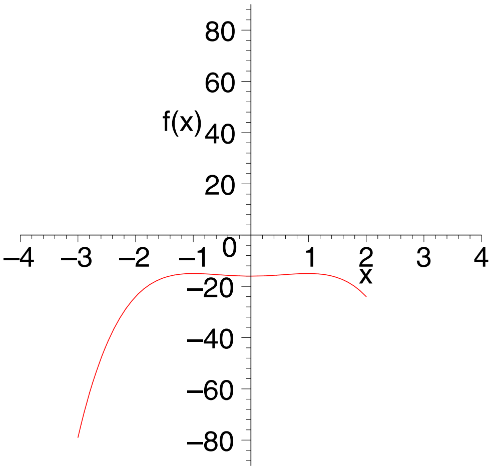
b)

$$\begin{aligned} f'(x) &= 0 \\ 4x - 4x^3 &= 0 \\ x(1 - x^2) &= 0 \\ x &= 0, \pm 1 \end{aligned}$$

Determine the function values.

$$\begin{aligned} f(-3) &= 2(-3)^2 - (-3)^4 - 16 \\ &= -79 \\ f(-1) &= 2(-1)^2 - (-1)^4 - 16 \\ &= -15 \\ f(0) &= 2(0)^2 - (0)^4 - 16 \\ &= -16 \\ f(1) &= 2(1)^2 - (1)^4 - 16 \\ &= -15 \\ f(2) &= 2(2)^2 - (2)^4 - 16 \\ &= -24 \end{aligned}$$

Comparison of the function values reveals an absolute maximum of -15 and an absolute minimum of -79.

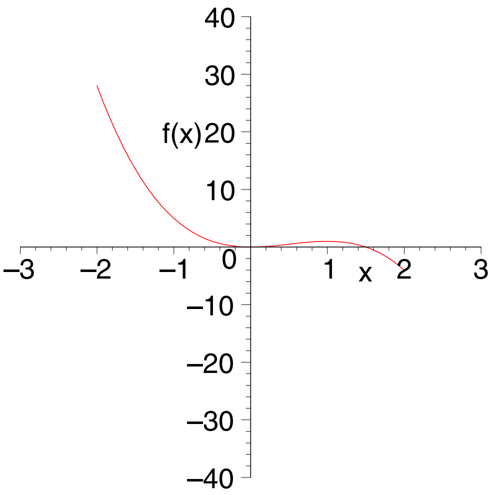


c)
$$\begin{aligned} f'(x) &= 0 \\ -6x^2 + 6x &= 0 \\ x(x-1) &= 0 \\ x &= 0 \text{ or } 1 \end{aligned}$$

Determine the function values.

$$\begin{aligned} f(-2) &= -2(-2)^3 + 3(-2)^2 \\ &= 28 \\ f(0) &= -2(0)^3 + 3(0)^2 \\ &= 0 \\ f(1) &= -2(1)^3 + 3(1)^2 \\ &= 1 \\ f(2) &= -2(2)^3 + 3(2)^2 \\ &= -4 \end{aligned}$$

Comparison of the function values reveals an absolute maximum of 28 and an absolute minimum of −4.

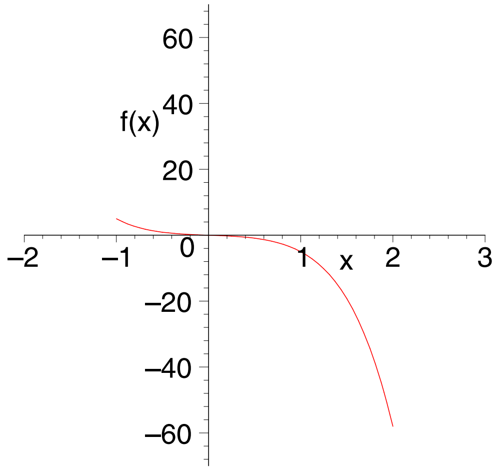


d)
$$\begin{aligned} f'(x) &= 0 \\ -(5x^4 + 9x^2 + 1) &= 0 \\ 5x^4 + 9x^2 + 1 &= 0 \end{aligned}$$

Since f has no critical numbers, determine the function values at the endpoints of the interval.

$$\begin{aligned} f(-1) &= -((-1)^5 + 3(-1)^3 + (-1)) \\ &= 5 \\ f(2) &= -(2^5 + 3(2)^3 + 2) \\ &= -58 \end{aligned}$$

Comparison of the function values reveals an absolute maximum of 5 and an absolute minimum of −58.



Section Review Page 399 Question 4

a) i) f has an absolute maximum of 9 and an absolute minimum of −5.

ii) f has local maxima of 1, 1.5 and 9. f has local minima of −5 and −0.2.

b) i) f has an absolute maximum of 1000 and an absolute minimum of −2550.

ii) f has a local maximum of approximately 600. f has local minima of −1850 and −2550.

Section Review Page 399 Question 5

$$\begin{aligned} P'(n) &= 0 \\ -10n + 500 &= 0 \\ n &= 50 \end{aligned}$$

Since $P'(40) > 0$ and $P'(60) < 0$, a local maximum occurs at $n = 50$. The company should manufacture 50 speakers to maximize profits.

6.3 Concavity and the Second Derivative Test

Section Review Page 399 Question 6

a) **i)** f is concave upward on the intervals $(-2, 1)$ and $(5, \infty)$. f is concave downward on the intervals $(-\infty, -6)$, $(-6, -2)$, and $(1, 5)$.

ii) Points of inflection occur at $(-2, 0)$, $(1, 0)$, and $(5, 1)$.

b) **i)** f is concave upward on the intervals $(0, 6)$ and $(8, \infty)$. f is concave downward on the intervals $(-\infty, 0)$ and $(6, 8)$.

ii) Points of inflection occur at $(0, 0)$, $(6, 5)$, and $(8, 2)$.

Section Review Page 400 Question 7

a)

$$\begin{aligned} f(x) &= 3 + 4x - 2x^2 \\ f'(x) &= 4 - 4x \\ f''(x) &= -4 \end{aligned}$$

Since $f'' < 0$ for all $\Re$, f is concave downward on $(-\infty, \infty)$. Since f'' never changes its sign, there are no points of inflection.

b)

$$\begin{aligned} y &= 5x^3 + 12x^2 - 3x + 2 \\ \frac{dy}{dx} &= 15x^2 + 24x - 3 \\ \frac{d^2y}{dx^2} &= 0 \\ 30x + 24 &= 0 \\ 5x + 4 &= 0 \\ x &= -\frac{4}{5} \end{aligned}$$

		-4/5	
Intervals	(- , -4/5)		(-4/5,)
Test values	-1		0
Sign of $\frac{d^2y}{dx^2}$	−	0	+
Nature of graph	concave downward		concave upward

An interval chart can be used to determine intervals of concavity. y is concave upward on $\left(-\frac{4}{5}, \infty\right)$ and concave downward on $\left(-\infty, -\frac{4}{5}\right)$. y has a point of inflection at $\left(-\frac{4}{5}, \frac{238}{25}\right)$.

c)

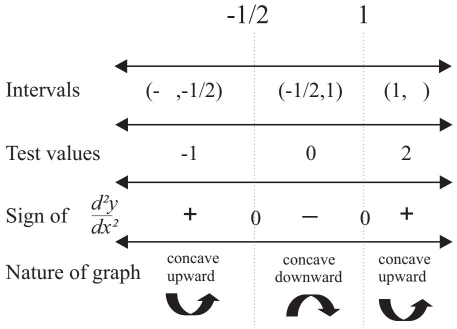
$$\begin{aligned} y &= 16 + 4x + x^2 - x^3 \\ \frac{dy}{dx} &= 4 + 2x - 3x^2 \\ \frac{d^2y}{dx^2} &= 0 \\ 2 - 6x &= 0 \\ x &= \frac{1}{3} \end{aligned}$$

		1/3	
Intervals	(- , 1/3)		(1/3,)
Test values	0		1
Sign of $\frac{d^2y}{dx^2}$	+	0	−
Nature of graph	concave upward		concave downward

An interval chart can be used to determine intervals of concavity. y is concave upward on $\left(-\infty, \frac{1}{3}\right)$ and concave downward on $\left(\frac{1}{3}, \infty\right)$. y has a point of inflection at $\left(\frac{1}{3}, \frac{470}{27}\right)$.

d)

$$\begin{aligned}
y &= x^4 - x^3 - 3x^2 + 5x - 12\\
\frac{dy}{dx} &= 4x^3 - 3x^2 - 6x + 5\\
\frac{d^2y}{dx^2} &= 0\\
12x^2 - 6x - 6 &= 0\\
2x^2 - x - 1 &= 0\\
(2x+1)(x-1) &= 0\\
x &= -\frac{1}{2} \text{ or } 1
\end{aligned}$$

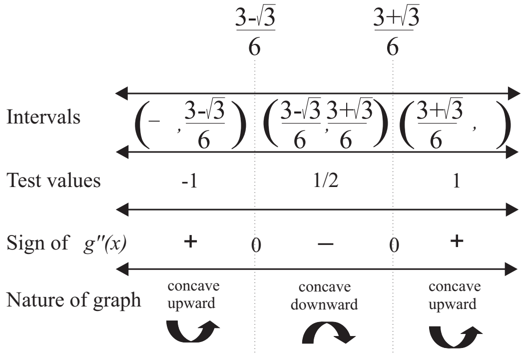


	-1/2		1	
Intervals	(- ,-1/2)	(-1/2,1)	(1,)	
Test values	-1	0	2	
Sign of $\frac{d^2y}{dx^2}$	+ 0	− 0	+	
Nature of graph	concave upward	concave downward	concave upward	

An interval chart can be used to determine intervals of concavity. y is concave upward on $\left(-\infty, -\frac{1}{2}\right)$ and $(1, \infty)$. y is concave downward on $\left(-\frac{1}{2}, 1\right)$. y has points of inflection at $\left(-\frac{1}{2}, -\frac{241}{16}\right)$ and $(1, -10)$.

e)

$$\begin{aligned}
g(x) &= x^4 - 2x^3 + x^2 - 2\\
g'(x) &= 4x^3 - 6x^2 + 2x\\
g''(x) &= 0\\
12x^2 - 12x + 2 &= 0\\
6x^2 - 6x + 1 &= 0\\
x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(6)(1)}}{2(6)}\\
&= \frac{6 \pm 2\sqrt{3}}{12}\\
x &= \frac{3 \pm \sqrt{6}}{6}
\end{aligned}$$



	$\frac{3-\sqrt{3}}{6}$		$\frac{3+\sqrt{3}}{6}$	
Intervals	$\left(- , \frac{3-\sqrt{3}}{6}\right)$	$\left(\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\right)$	$\left(\frac{3+\sqrt{3}}{6}, \right)$	
Test values	-1	1/2	1	
Sign of $g''(x)$	+ 0	− 0	+	
Nature of graph	concave upward	concave downward	concave upward	

An interval chart can be used to determine intervals of concavity. g is concave upward on $\left(-\infty, \frac{3-\sqrt{3}}{6}\right)$ and $\left(\frac{3+\sqrt{3}}{6}, \infty\right)$. g is concave downward on $\left(\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\right)$. g has points of inflection at $\left(\frac{3-\sqrt{3}}{6}, -\frac{71}{36}\right)$ and $\left(\frac{3+\sqrt{3}}{6}, -\frac{71}{36}\right)$.

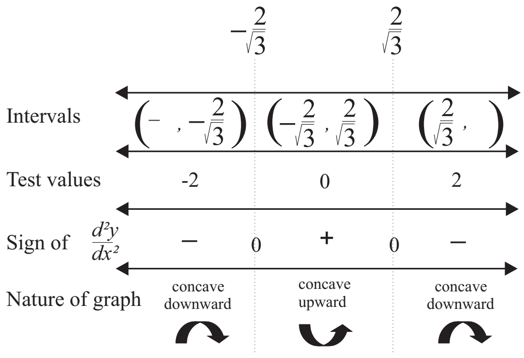
f)

$$\begin{aligned}
y &= \frac{x-2}{5-x}\\
\frac{dy}{dx} &= \frac{(5-x)(1) - (x-2)(-1)}{(5-x)^2}\\
&= \frac{3}{(5-x)^2}\\
&= 3(5-x)^{-2}\\
\frac{d^2y}{dx^2} &= 0\\
-6(5-x)^{-3}(-1) &= 0\\
\frac{6}{(5-x)^3} &= 0
\end{aligned}$$

Since $\frac{d^2y}{dx^2}$ has no zeros, there are no points of inflection. Since $\frac{d^2y}{dx^2} > 0$ for $x < 5$, y is concave upward on the interval $(-\infty, 5)$. Since $\frac{d^2y}{dx^2} < 0$ for $x > 5$, y is concave downward on the interval $(5, \infty)$.

g)

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+4)(2x)-x^2(2x)}{(x^2+4)^2}\\ &= \frac{2x(4)}{(x^2+4)^2}\\ &= \frac{8x}{(x^2+4)^2}\\ \frac{d^2y}{dx^2} &= 0\\ 8\left(\frac{(x^2+4)^2(1)-x(2)(x^2+4)(2x)}{(x^2+4)^4}\right) &= 0\\ \frac{4-3x^2}{(x^2+4)^3} &= 0\\ x &= \pm\frac{2}{\sqrt{3}}\end{aligned}$$

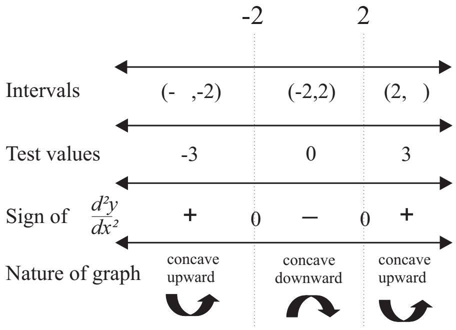


	$-\frac{2}{\sqrt3}$		$\frac{2}{\sqrt3}$	
Intervals	$\left(- \ , -\frac{2}{\sqrt3}\right)$		$\left(-\frac{2}{\sqrt3}, \frac{2}{\sqrt3}\right)$		$\left(\frac{2}{\sqrt3}, \ \right)$
Test values	-2		0		2
Sign of $\frac{d^2y}{dx^2}$	−	0	+	0	−
Nature of graph	concave downward ↷		concave upward ↺		concave downward ↷

An interval chart can be used to determine intervals of concavity. y is concave upward on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. y is concave downward on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$. y has points of inflection at $\left(\pm\frac{2}{\sqrt{3}}, \frac{1}{4}\right)$.

h)

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2-4)(2x)-x^2(2x)}{(x^2-4)^2}\\ &= \frac{2x(-4)}{(x^2-4)^2}\\ &= \frac{-8x}{(x^2-4)^2}\\ \frac{d^2y}{dx^2} &= -8\left(\frac{(x^2-4)^2(1)-x(2)(x^2-4)(2x)}{(x^2-4)^4}\right)\\ &= \frac{8(4+3x^2)}{(x^2-4)^3} \neq 0\end{aligned}$$



	-2		2	
Intervals	(- ,-2)		(-2,2)		(2,)
Test values	-3		0		3
Sign of $\frac{d^2y}{dx^2}$	+	0	−	0	+
Nature of graph	concave upward ↺		concave downward ↷		concave upward ↺

The zeroes of the denominator of $\frac{d^2y}{dx^2}$ are ±2. An interval chart can be used to determine intervals of concavity. y is concave upward on $(-\infty, -2)$ and $(2, \infty)$. y is concave downward on $(-2, 2)$. Since the domain does not include ±2, there are no points of inflection.

i)

$$\begin{aligned}h'(x) &= 1-\frac{1}{x^2}\\ h''(x) &= \frac{2}{x^3}\end{aligned}$$

Since $h'' < 0$ for $x < 0$, h is concave downward on $(-\infty, 0)$. Since $h'' > 0$ for $x > 0$, h is concave upward on $(0, \infty)$. Since the domain of h excludes $x = 0$, there are no points of inflection.

Section Review Page 400 Question 8

a)

$$\begin{aligned} f'(x) &= 0 \\ 2x - 3x^2 &= 0 \\ x(2 - 3x) &= 0 \\ x &= 0, \frac{2}{3} \\ g''(x) &= 2 - 6x \end{aligned}$$

Since $g''(0) > 0$, g has a local minimum value of $0^2 - 0^3$ or 0. Since $g''\left(\frac{2}{3}\right) < 0$, g has a local maximum value of $\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$ or $\frac{4}{27}$.

b)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 4x - 8 &= 0 \\ x &= 2 \\ \frac{d^2y}{dx^2} &= 4 \end{aligned}$$

Since $\frac{d^2y}{dx^2} > 0$ for all $\Re$, y has a local minimum value of $2(2)^2 - 8(2) + 3$ or -5. y has no local maxima.

c)

$$\begin{aligned} k'(x) &= 0 \\ 6x - 24 &= 0 \\ x &= 4 \\ k''(x) &= 6 \end{aligned}$$

Since $k'' > 0$ for all $\Re$, k has a local minimum value of at $3(4)^2 - 24(4) + 15$ or -33. k has no local maxima.

d)

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} &= 0 \\ \frac{x^2 - 2x}{(x-1)^2} &= 0 \\ \frac{x(x-2)}{(x-1)^2} &= 0 \\ x &= 0 \text{ or } 2 \\ \frac{d^2y}{dx^2} &= \frac{(x-1)^2(2x-2) - (x^2-2x)(2)(x-1)}{(x-1)^4} \\ &= \frac{2}{(x-1)^3} \end{aligned}$$

Since $\frac{d^2y}{dx^2}_{|x=0} < 0$, y has a local maximum value of $\frac{0^2}{0-1}$ or 0. Since $\frac{d^2y}{dx^2}_{|x=2} > 0$, y has a local minimum value of $\frac{2^2}{2-1}$ or 4.

e)

$$\begin{aligned} g'(x) &= 0 \\ 2x - \frac{16}{x^2} &= 0 \\ x^3 &= 8 \\ x &= 2 \\ g''(x) &= 2 + \frac{32}{x^3} \end{aligned}$$

Since $g''(2) > 0$, g has a local minimum value of $2^2 + \frac{16}{2}$ or 12. g has no local maximum value.

f)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 4x^3 - 16x &= 0\\ x(x^2 - 4) &= 0\\ x &= 0, \pm 2\\ \frac{d^2y}{dx^2} &= 12x^2 - 16\end{aligned}$$

Since $\frac{d^2y}{dx^2}_{|x=-2} > 0$, y has a local minimum value of $(-2)^4 - 8(-2)^2 + 5$ or -11. Since $\frac{d^2y}{dx^2}_{|x=0} < 0$, y has a local maximum value of $0^4 - 8(0)^2 + 5$ or 5. Since $\frac{d^2y}{dx^2}_{|x=2} > 0$, y has another local minimum value of $2^4 - 8(2)^2 + 5$ or -11.

Section Review Page 400 Question 9

$$\begin{aligned}P'(t) &= \frac{(3t+2)(5) - (5t+1)(3)}{(3t+2)^2}\\ &= \frac{15t + 10 - 15t - 3}{(3t+2)^2}\\ &= \frac{7}{(3t+2)^2}\\ &= 7(3t+2)^{-2}\\ P''(t) &= -14(3t+2)^{-3}(3)\\ &= \frac{-42}{(3t+2)^3}\end{aligned}$$

Since $P''(t) < 0$ for $t \in [0, 20]$, $P'(t)$ is decreasing over the same interval.

6.4 Vertical Asymptotes

Section Review Page 400 Question 10

a) The equations of the vertical asymptotes are $x = -2$, $x = 1$, and $x = 6$.

b) **i)** $\lim\limits_{x \to 6^-} f(x) = -\infty$ **ii)** $\lim\limits_{x \to 6^+} f(x) = -\infty$ **iii)** $\lim\limits_{x \to -2^-} f(x) = \infty$

iv) $\lim\limits_{x \to -2^+} f(x) = -\infty$ **v)** $\lim\limits_{x \to -2} f(x)$ does not exist **vi)** $\lim\limits_{x \to 1^-} f(x) = \infty$

vii) $\lim\limits_{x \to 1^+} f(x) = \infty$ **viii)** $\lim\limits_{x \to 4^-} f(x) = 3^+$ **ix)** $\lim\limits_{x \to 4^+} f(x) = 1^-$

Section Review Page 400 Question 11

a) As $x \to 3^-$, $3 - x \to 0^+$, resulting in $\lim\limits_{x \to 3^-} \frac{5}{3 - x} = \infty$.

b) As $x \to 3^+$, $3 - x \to 0^-$ resulting in $\lim\limits_{x \to 3^+} \frac{5}{3 - x} = -\infty$.

c) As $x \to -4^-$, $x + 4 \to 0^-$ resulting in $\lim\limits_{x \to -4^-} \frac{-3}{x + 4} = \infty$.

d) As $x \to -4^+$, $x + 4 \to 0^+$ resulting in $\lim\limits_{x \to -4^+} \frac{-3}{x + 4} = -\infty$.

e) As $x \to 1$, $(x - 1)^2 \to 0^+$ resulting in $\lim\limits_{x \to 1} \frac{2}{(x - 1)^2} = \infty$.

f) As $x \to -6^-$, $(x + 6)^2 \to 0^+$ resulting in $\lim\limits_{x \to -6^-} \frac{1}{(x + 6)^2} = \infty$.

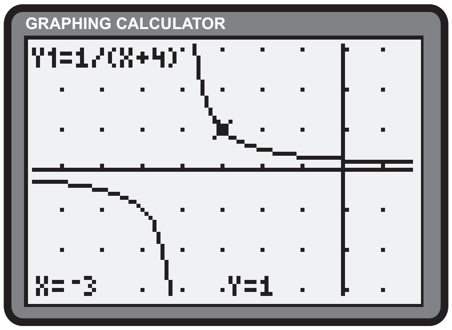
g) As $x \to -3^-$, $(x + 3)^2 \to 0^+$ resulting in $\lim\limits_{x \to -3^-} \frac{x}{(x + 3)^2} = -\infty$.

h) As $x \to -3^+$, $(x+3)^2 \to 0^+$ resulting in $\lim\limits_{x\to -3^+} \dfrac{x}{(x+3)^2} = -\infty$.

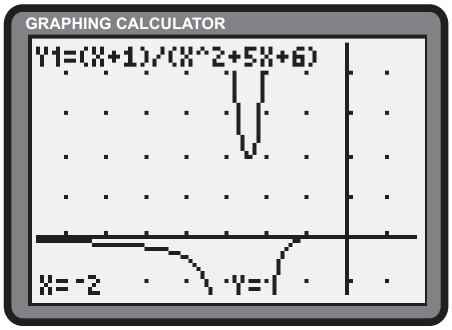
i) $\lim\limits_{x\to -4^-} \dfrac{x+2}{x^2+5x+4} = \lim\limits_{x\to -4^-} \dfrac{x+2}{(x+1)(x+4)}$. As $x \to -4^-$, $x+2 \to -2^-$, $x+1 \to 3^-$, and $x+4 \to 0^-$, resulting in $\lim\limits_{x\to -4^-} \dfrac{x+2}{(x+1)(x+4)} = -\infty$.

Section Review Page 400 Question 12

a) Since −4 is a root of the denominator and *not* a root of the numerator, $x = -4$ is a vertical asymptote of f.



b) Since $x+3$ and $x+2$ are factors of the denominator and *not* of the numerator, $x = -3$ and $x = -2$ are vertical asymptotes of y.



6.5 Horizontal and Oblique Asymptotes

Section Review Page 400 Question 13

a) The equation of the horizontal asymptote is $y = 2$. The equations of the vertical asymptotes are $x = -2$ and $x = 1$.

b) The equations of the horizontal asymptotes are $y = -2$ and $y = 1$. The equations of the vertical asymptotes are $x = -1$ and $x = 4$.

Section Review Page 401 Question 14

a) $\lim\limits_{x\to\infty} \dfrac{5}{x} = 0^+$

b) $\lim\limits_{x\to-\infty} \dfrac{5}{x} = 0^-$

c)
$$\begin{aligned}\lim_{x\to\infty}\frac{3-2x}{x+4} &= \lim_{x\to\infty}\frac{\frac{3-2x}{x}}{\frac{x+4}{x}}\\ &= \lim_{x\to\infty}\frac{\frac{3}{x}-2}{1+\frac{4}{x}}\\ &= \frac{-2}{1}\\ &= -2\end{aligned}$$

d)
$$\begin{aligned}\lim_{x\to-\infty}\frac{3-2x}{x+4} &= \lim_{x\to-\infty}\frac{\frac{3-2x}{x}}{\frac{x+4}{x}}\\ &= \lim_{x\to-\infty}\frac{\frac{3}{x}-2}{1+\frac{4}{x}}\\ &= \frac{-2}{1}\\ &= -2\end{aligned}$$

e)
$$\begin{aligned}\lim_{x\to\infty}\frac{4-x^2}{2x^2-3} &= \lim_{x\to\infty}\frac{\frac{4-x^2}{x^2}}{\frac{2x^2-3}{x^2}}\\ &= \lim_{x\to\infty}\frac{\frac{4}{x^2}-1}{2-\frac{3}{x^2}}\\ &= \frac{0-1}{2-0}\\ &= -\frac{1}{2}\end{aligned}$$

f)
$$\begin{aligned}\lim_{x\to\infty}\frac{3x^2-4x+2}{x^2-3x+5} &= \lim_{x\to\infty}\frac{\frac{3x^2-4x+2}{x^2}}{\frac{x^2-3x+5}{x^2}}\\ &= \lim_{x\to\infty}\frac{3-\frac{4}{x}-\frac{2}{x^2}}{1-\frac{3}{x}+\frac{5}{x^2}}\\ &= \frac{3-0-0}{1-0-0}\\ &= 3\end{aligned}$$

g)

$$\begin{aligned}\lim_{x\to-\infty}\frac{5x^3-2x^2}{5-x^3} &= \lim_{x\to-\infty}\frac{\dfrac{5x^3-2x^2}{x^3}}{\dfrac{5-x^3}{x^3}}\\ &= \lim_{x\to-\infty}\frac{5-\dfrac{2}{x}}{\dfrac{5}{x}-1}\\ &= \frac{5-0}{0-1}\\ &= -5\end{aligned}$$

h) As $x \to \infty$, the difference between x^4 and $6x^2$ continues to increase. As a result, $\lim_{x\to\infty}(x^4 - 6x^2) = \infty$.

i) For $x > 0$, $|x| = x$.

$$\begin{aligned}\lim_{x\to\infty}|x| &= \lim_{x\to\infty} x\\ &= \infty\end{aligned}$$

Section Review Page 401 Question 15

a) Since 2 is a root of the denominator, $x = 2$ is a vertical asymptote.

$$\begin{aligned}\lim_{x\to\infty}\frac{4x-3}{2-x} &= \lim_{x\to\infty}\frac{\dfrac{4x-3}{x}}{\dfrac{2-x}{x}}\\ &= \lim_{x\to\infty}\frac{4-\dfrac{3}{x}}{\dfrac{2}{x}-1}\\ &= \frac{4}{-1}\\ &= -4\end{aligned}$$

The function behaves similarly for $x \to -\infty$. The equation of the horizontal asymptote is $y = -4$.

b) Since -4 is a root of the denominator, $x = -4$ is a vertical asymptote.

$$\begin{aligned}\lim_{x\to\infty}\frac{x-5}{x+4} &= \lim_{x\to\infty}\frac{\dfrac{x-5}{x}}{\dfrac{x+4}{x}}\\ &= \lim_{x\to\infty}\frac{1-\dfrac{5}{x}}{1+\dfrac{4}{x}}\\ &= \frac{1}{1}\\ &= 1\end{aligned}$$

The function behaves similarly for $x \to -\infty$. The equation of the horizontal asymptote is $y = 1$.

c) The denominator can be expressed as $(x-5)(x+3)$. $x = 5$ and $x = -3$ are vertical asymptotes of the function. Since the degree of the denominator exceeds the degree of the numerator, $y \to 0$ as $|x| \to \infty$. The function has a horizontal asymptote of $y = 0$.

d) The denominator can be expressed as $(3x-2)(x-1)$. $x = \frac{2}{3}$ and $x = 1$ are vertical asymptotes of the function. Since the degree of the denominator exceeds the degree of the numerator, $y \to 0$ as $|x| \to \infty$. The function has a horizontal asymptote of $y = 0$.

e) The denominator can be expressed as $(2x+1)(x-3)$. $x = -\frac{1}{2}$ and $x = 3$ are vertical asymptotes of the function. Since the degrees of the numerator and denominator are equal, the function has a horizontal asymptote of $y = \frac{6}{2}$ or $y = 3$.

f) Rewriting the denominator yields $(x-1)(x^2+x+1)$. $x = 1$ is a vertical asymptote of the function. Since the degrees of the numerator and denominator are equal, the function has a horizontal asymptote of $y = \frac{1}{1}$ or $y = 1$.

Section Review Page 401 Question 16

a) Rewrite y.

$$\begin{aligned}y &= \frac{3x-2x^2+6}{x}\\ &= 3-2x+\frac{6}{x} \qquad (1)\end{aligned}$$

As $|x| \to \infty$, $\frac{6}{x} \to 0$. As a consequence, (1) approximates the expression $3 - 2x$. The equation of the oblique asymptote is $y = 3 - 2x$.

b) Rewrite y.

$$\begin{aligned}y &= \frac{2x^3-5}{2x^2}\\ &= x-\frac{5}{2x^2} \qquad (1)\end{aligned}$$

As $|x| \to \infty$, $\frac{5}{2x^2} \to 0$. As a consequence, (1) approximates the expression x. The equation of the oblique asymptote is $y = x$.

c) Division of the numerator by the denominator yields $5x + 8 + \frac{6}{x-1}$. As $|x| \to \infty$, $\frac{6}{x-1} \to 0$. As a consequence, the function has an oblique asymptote of $y = 5x + 8$.

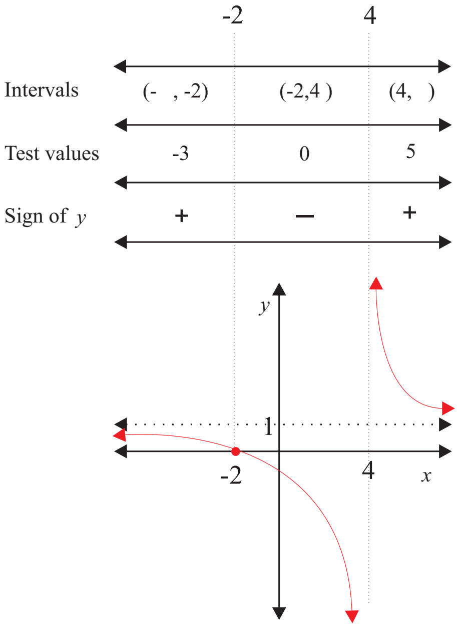
d) Division of the numerator by the denominator yields $2x - \frac{2}{3} - \frac{\frac{13}{3}}{3x+1}$. As $|x| \to \infty$, $\frac{\frac{13}{3}}{3x+1} \to 0$. As a consequence, the function has an oblique asymptote of $y = 2x - \frac{2}{3}$.

e) Division of the numerator by the denominator yields $x + 4 + \frac{x}{x^2+4}$. As $|x| \to \infty$, $\frac{x}{x^2+4} \to 0$. As a consequence, the function has an oblique asymptote of $y = x + 4$.

f) Division of the numerator by the denominator yields $-x + \frac{x^2}{x^3-1}$. As $|x| \to \infty$, $\frac{x^2}{x^3-1} \to 0$. As a consequence, the function has an oblique asymptote of $y = -x$.

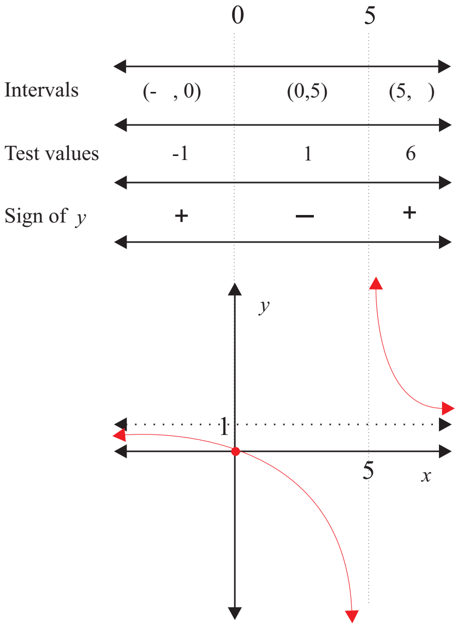
Section Review Page 401 Question 17

a) Since -2 is a root of the numerator, the x-intercept is -2. The y-intercept is $y(0) = \frac{0+2}{0-4}$ or $-\frac{1}{2}$. Since 4 is a root of the denominator, $x = 4$ is a vertical asymptote. Since $y \to 1$ as $|x| \to \infty$, $y = 1$ is a horizontal asymptote. An interval chart is used to determine the signs of the range of the function.



	-2		4	
Intervals	(- , -2)		(-2,4)	(4,)
Test values	-3		0	5
Sign of y	+		–	+

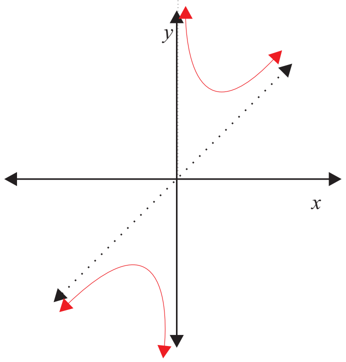
b) Since 0 is a root of the numerator, the x-intercept is 0. The y-intercept is $y(0) = \frac{0}{0-5}$ or 0. Since 5 is a root of the denominator, $x = 5$ is a vertical asymptote. Since $y \to 1$ as $|x| \to \infty$, $y = 1$ is a horizontal asymptote. An interval chart is used to determine the signs of the range of the function.



	0	5	
Intervals	(- , 0)	(0,5)	(5,)
Test values	-1	1	6
Sign of y	+	–	+

c) Since the numerator has no roots, there are no x-intercepts. There is no y-intercept. Since 0 is a root of the denominator, $x = 0$ is a vertical asymptote. The function can be rewritten as $y = x + \frac{2}{x}$. As $|x| \to \infty$, $\frac{2}{x} \to 0$ and the function approximates its oblique asymptote $y = x$. An interval chart is used to determine the signs of the range of the function.

		0	
Intervals	(- , 0)		(0,)
Test values	-1		1
Sign of y	−		+



Section Review Page 401 Question 18

$$\begin{aligned}
\lim_{x\to\infty} C(x) &= \lim_{x\to\infty} \frac{\sqrt{2x^2+20}}{x} \\
&= \lim_{x\to\infty} \sqrt{\frac{2x^2+20}{x^2}} \\
&= \lim_{x\to\infty} \sqrt{2 + \frac{20}{x^2}} \\
&= \sqrt{2} \\
&\doteq 1.41
\end{aligned}$$

The long-term cost is approximately \$1.41 per pair.

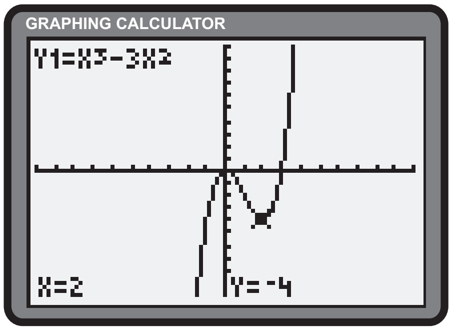
6.6 Curve Sketching

Section Review Page 401 Question 19

a)

$$\begin{aligned}
y &= 0 \\
x^3 - 3x^2 &= 0 \\
x^2(x-3) &= 0 \\
x &= 0, 0, \text{ or } 3 \\
\frac{dy}{dx} &= 0 \\
3x^2 - 6x &= 0 \\
x(x-2) &= 0 \\
x &= 0 \text{ or } 2 \\
\frac{d^2y}{dx^2} &= 0 \\
6x - 6 &= 0 \\
x - 1 &= 0 \\
x &= 1
\end{aligned}$$

Since y is a polynomial, the domain is $\Re$ and there are no asymptotes. x-intercepts are at 0 and 3. The y-intercept is 0. Since $\frac{d^2y}{dx^2}_{|x=0} < 0$, a local maximum exists at $(0, 0)$. Since $\frac{d^2y}{dx^2}_{|x=2} > 0$, a local minimum exists at $(2, -4)$. A point of inflection exists at $(1, -2)$. The function has neither odd nor even symmetry.



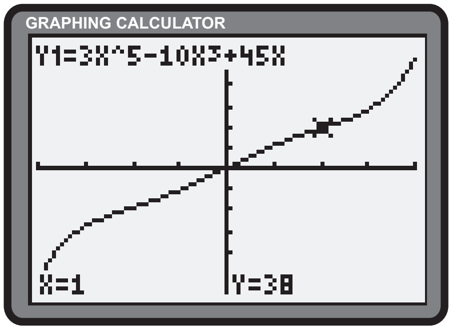
Y1=X³-3X²

X=2 Y=-4

b)

$$\begin{aligned} y &= 0 \\ 3x^5 - 10x^3 + 45x &= 0 \\ x(3x^4 - 10x^2 + 45) &= 0 \\ x &= 0 \\ \frac{dy}{dx} &= 0 \\ 15x^4 - 30x^2 + 45 &= 0 \\ x^4 - 2x^2 + 3 &= 0 \\ &\text{no roots} \\ \frac{d^2y}{dx^2} &= 0 \\ 60x^3 - 60x &= 0 \\ x(x^2 - 1) &= 0 \\ x &= 0, \pm 1 \end{aligned}$$

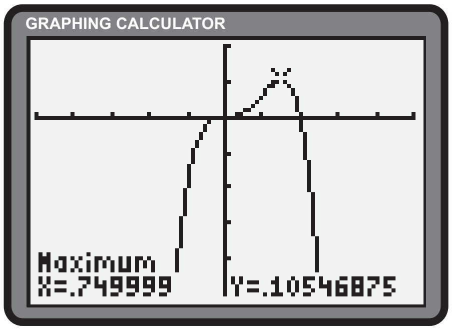
Since y is a polynomial, the domain is $\Re$ and there are no asymptotes. The function passes through the origin. There are no local extrema. Points of inflection exist at $(-1, -38)$, $(0, 0)$, and $(1, 38)$. Since $y(x) = -y(-x)$, the function has odd symmetry.



c)

$$\begin{aligned} y &= 0 \\ x^3 - x^4 &= 0 \\ x^3(1 - x) &= 0 \\ x &= 0 \text{ or } 1 \\ \frac{dy}{dx} &= 0 \\ 3x^2 - 4x^3 &= 0 \\ x^2(3 - 4x) &= 0 \\ x &= 0 \text{ or } \frac{3}{4} \\ \frac{d^2y}{dx^2} &= 0 \\ 6x - 12x^2 &= 0 \\ x(1 - 2x) &= 0 \\ x &= 0 \text{ or } \frac{1}{2} \end{aligned}$$

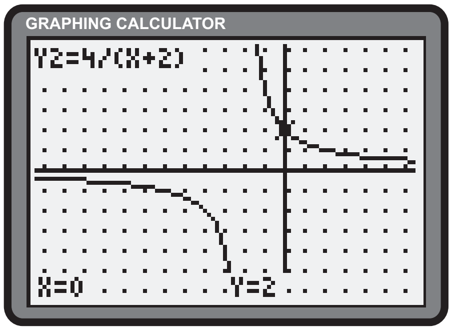
Since y is a polynomial, the domain is $\Re$ and there are no asymptotes. The x-intercepts are 0 and 1. The y-intercept is 0. Since $\frac{d^2y}{dx^2}\big|_{x=\frac{3}{4}} < 0$, a local maximum exists at $\left(\frac{3}{4}, \frac{27}{256}\right)$. Points of inflection exist at $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{16}\right)$. The function has neither odd nor even symmetry.



d)

$$\begin{aligned} y &= 0 \\ \frac{4}{2 + x} &= 0 \\ &\text{no roots} \\ \frac{dy}{dx} &= 0 \\ \frac{-4}{(2 + x)^2} &= 0 \\ &\text{no roots} \\ \frac{d^2y}{dx^2} &= 0 \\ \frac{8}{(2 + x)^3} &= 0 \\ &\text{no roots} \end{aligned}$$

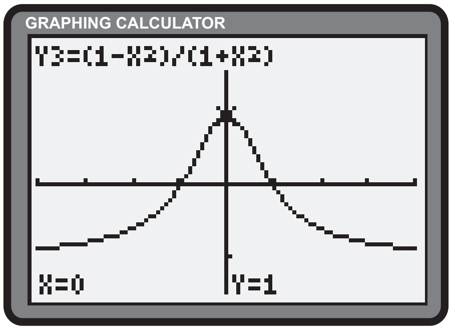
The domain is $\{x \in \Re \mid x \neq -2\}$. When $x < -2$, $y < 0$. When $x > -2$, $y > 0$. There are no x-intercepts. The y-intercept is 2. y has a vertical asymptote of $x = -2$ and a horizontal asymptote of $y = 0$. There are no local extrema and no points of inflection. The function has neither odd nor even symmetry.



e)

$$\begin{aligned} y &= 0 \\ \frac{1-x^2}{1+x^2} &= 0 \\ 1-x^2 &= 0 \\ x &= \pm 1 \\ \frac{dy}{dx} &= 0 \\ \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2} &= 0 \\ \frac{-4x}{(1+x^2)^2} &= 0 \\ x &= 0 \\ \frac{d^2y}{dx^2} &= 0 \\ \frac{(1+x^2)^2(-4)-(-4x)(2)(1+x^2)(2x)}{(1+x^2)^4} &= 0 \\ \frac{4(3x^2-1)}{(1+x^2)^3} &= 0 \\ x &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

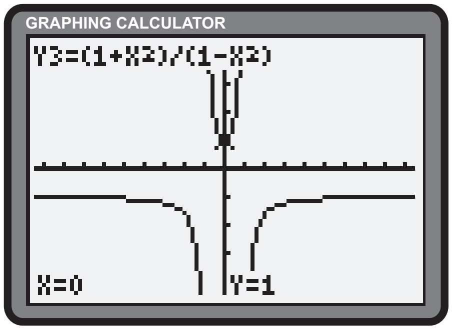
The domain is $\Re$. The x-intercepts are ± 1. The y-intercept is 1. y has a horizontal asymptote of $y = \frac{-1}{1}$ or $y = -1$. Since $\frac{d^2y}{dx^2}_{|x=0} < 0$, there is a local maximum at $(0, 1)$. There are points of inflection at $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{2}\right)$. Since $y(x) = y(-x)$, the function has even symmetry.



f)

$$\begin{aligned} y &= 0 \\ \frac{1+x^2}{1-x^2} &= 0 \\ &\text{no roots} \\ \frac{dy}{dx} &= 0 \\ \frac{(1-x^2)(2x)-(1+x^2)(-2x)}{(1-x^2)^2} &= 0 \\ \frac{4x}{(1-x^2)^2} &= 0 \\ x &= 0 \\ \frac{d^2y}{dx^2} &= 0 \\ \frac{(1-x^2)^2(4)-(4x)(2)(1-x^2)(-2x)}{(1-x^2)^4} &= 0 \\ \frac{4(3x^2+1)}{(1-x^2)^3} &= 0 \\ &\text{no roots} \end{aligned}$$

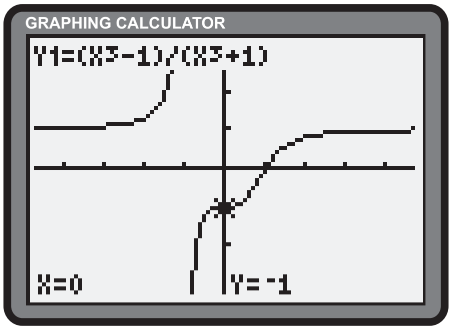
The domain is $\{x \in \Re \mid x \neq \pm 1\}$. There are no x-intercepts. The y-intercept is 1. There are vertical asymptotes at $x = \pm 1$. y has a horizontal asymptote of $y = \frac{1}{-1}$ or $y = -1$. Since $\frac{d^2y}{dx^2}_{|x=0} > 0$, there is a local minimum at $(0, 1)$. There are no points of inflection. Since $y(x) = y(-x)$, the function has even symmetry.



g)

$$
\begin{aligned}
y &= 0\\
\frac{x^3-1}{x^3+1} &= 0\\
\frac{(x-1)(x^2+x+1)}{x^3+1} &= 0\\
x &= 1\\
\frac{dy}{dx} &= 0\\
\frac{(x^3+1)(3x^2)-(x^3-1)(3x^2)}{(x^3+1)^2} &= 0\\
\frac{6x^2}{(x^3+1)^2} &= 0\\
x &= 0\\
\frac{d^2y}{dx^2} &= 0\\
\frac{(x^3+1)^2(12x)-(6x^2)(2)(x^3+1)(3x^2)}{(x^3+1)^4} &= 0\\
\frac{12x(1-2x^3)}{(x^3+1)^3} &= 0\\
x &= 0 \text{ or } \frac{1}{\sqrt[3]{2}}
\end{aligned}
$$

The domain is $\{x \in \Re \mid x \neq -1\}$. The x-intercept is 1. The y-intercept is -1. There is a vertical asymptote of $x = -1$. y has a horizontal asymptote of $y = \frac{1}{1}$ or $y = 1$. There are no local extrema. There are points of inflection at $(0, -1)$ and $\left(\frac{1}{\sqrt[3]{2}}, -\frac{1}{3}\right)$.



GRAPHING CALCULATOR

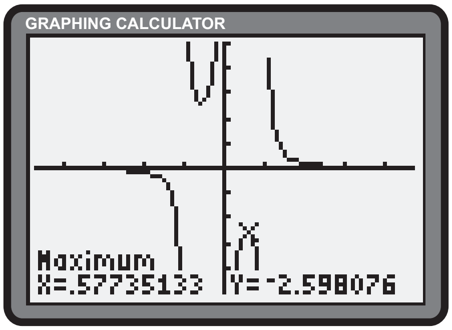
Y1=(X³-1)/(X³+1)

X=0 Y=-1

h)

$$
\begin{aligned}
y &= 0\\
\frac{1}{x^3-x} &= 0\\
&\text{no roots}\\
\frac{dy}{dx} &= 0\\
\frac{-(3x^2-1)}{(x^3-x)^2} &= 0\\
\frac{1-3x^2}{(x^3-x)^2} &= 0\\
x &= \pm\frac{1}{\sqrt{3}}\\
\frac{d^2y}{dx^2} &= 0\\
\frac{(x^3-x)^2(-6x)-(1-3x^2)(2)(x^3-x)(3x^2-1)}{(x^3-x)^4} &= 0\\
\frac{2(6x^4-3x^2+1)}{(x^3-x)^3} &= 0\\
&\text{no roots}
\end{aligned}
$$

The domain is $\{x \in \Re \mid x \neq 0, \pm 1\}$. There are no x- or y-intercepts. There are vertical asymptotes of $x = \pm 1$ and $x = 0$. As $|x| \to \infty$, $y \to 0$. The function is asymptotic to the x-axis. A local minimum is confirmed at $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{2}\right)$. A local maximum is confirmed at $\left(\frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{2}\right)$. There are no points of inflection. Since $y(x) = -y(-x)$, the function has odd symmetry.



GRAPHING CALCULATOR

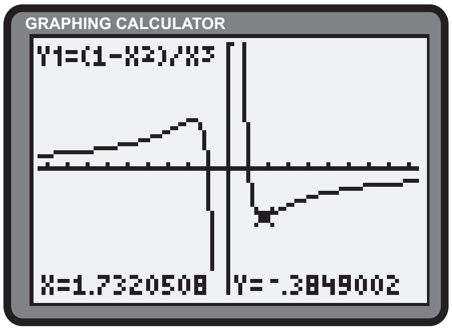
Maximum

X=.57735133 Y=-2.598076

i)

$$\begin{aligned}
y &= 0\\
\frac{1-x^2}{x^3} &= 0\\
x &= \pm 1\\
\frac{dy}{dx} &= 0\\
\frac{x^3(-2x)-(1-x^2)(3x^2)}{x^6} &= 0\\
\frac{x^2-3}{x^4} &= 0\\
x &= \pm\sqrt{3}\\
\frac{d^2y}{dx^2} &= 0\\
\frac{x^4(2x)-(x^2-3)4x^3}{x^8} &= 0\\
\frac{2(6-x^2)}{x^5} &= 0\\
x &= \pm\sqrt{6}
\end{aligned}$$

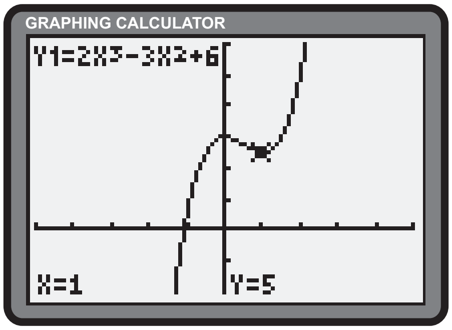
The domain is $\{x \in \Re \,|x \neq 0\}$. There are x-intercepts of ± 1. There is a vertical asymptote of $x = 0$. As $|x| \to \infty$, $y \to 0$. The function is asymptotic to the x-axis. A local minimum is confirmed at $\left(\sqrt{3}, -\frac{2}{3\sqrt{3}}\right)$. A local maximum is confirmed at $\left(-\sqrt{3}, \frac{2}{3\sqrt{3}}\right)$. Points of inflection are at $\left(\sqrt{6}, -\frac{5}{6\sqrt{6}}\right)$ and $\left(-\sqrt{6}, \frac{5}{6\sqrt{6}}\right)$. Since $y(x) = -y(-x)$, the function has odd symmetry.



Section Review Page 401 Question 20

a)

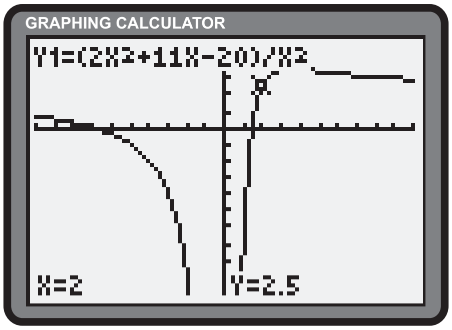
$$\begin{aligned}
\frac{dy}{dx} &= 0\\
6x^2-6x &= 0\\
6x(x-1) &= 0\\
x &= 0 \text{ or } 1\\
\frac{d^2y}{dx^2} &= 0\\
12x-6 &= 0\\
x &= \frac{1}{2}
\end{aligned}$$



Estimates will vary. The function is increasing on the intervals $(-\infty, 0)$ and $(1, \infty)$. y is decreasing on $(0, 1)$. A local maximum is confirmed at $(0, 6)$. A local minimum is confirmed at $(1, 5)$. The curve is concave downward on the interval $\left(-\infty, \frac{1}{2}\right)$ and concave upward on the interval $\left(\frac{1}{2}, \infty\right)$. The point of inflection is $\left(\frac{1}{2}, \frac{11}{2}\right)$.

b)

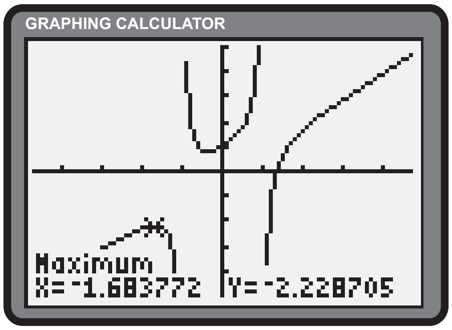
$$\begin{aligned}
\frac{dy}{dx} &= 0\\
\frac{x^2(2x+11)-(x^2+11x-20)(2x)}{x^4} &= 0\\
\frac{40-11x}{x^3} &= 0\\
x &= \frac{40}{11}\\
\frac{d^2y}{dx^2} &= 0\\
\frac{x^3(-11)-(40-11x)(3x^2)}{x^6} &= 0\\
\frac{11x-60}{x^4} &= 0\\
x &= \frac{60}{11}
\end{aligned}$$

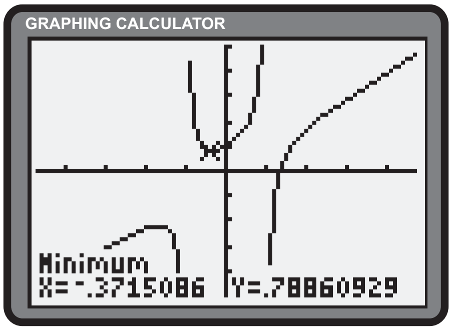


Estimates will vary. The function is decreasing on the intervals $(-\infty, 0)$ and $\left(\frac{40}{11}, \infty\right)$. y is increasing on $\left(0, \frac{40}{11}\right)$. A local maximum is confirmed at $\left(\frac{40}{11}, \frac{201}{80}\right)$. There are no local minima. The curve is concave downward on the intervals $(-\infty, 0)$ and $\left(0, \frac{60}{11}\right)$. The function is concave upward on the interval $\left(\frac{60}{11}, \infty\right)$. The point of inflection is $\left(\frac{60}{11}, \frac{211}{90}\right)$.

c)

$$\begin{aligned}
\frac{dy}{dx} &= 0\\
1+\frac{2x}{(x^2-1)^2} &= 0\\
\frac{x^4-2x^2+2x+1}{(x^2-1)^2} &= 0\\
\frac{d^2y}{dx^2} &= 0\\
\frac{(x^2-1)^2(2)-2x(2)(x^2-1)(2x)}{(x^2-1)^4} &= 0\\
\frac{-1-3x^2}{(x^2-1)^3} &= 0\\
&\text{no roots}
\end{aligned}$$





Estimates will vary and exact quantities cannot be determined. The function is increasing on the intervals, $(-\infty, -1.684)$, $(-0.372, 1)$, and $(1, \infty)$. y is decreasing on $(-1.684, -1)$ and $(-1, -0.372)$. A local maximum occurs at $(-1.684, -2.229)$. A local minimum occurs at $(-0.372, 0.789)$. The curve is concave upward on the interval $(-1, -1)$. The function is concave downward on the intervals $(-\infty, -1)$ and $(1, \infty)$. There are no points of inflection.

6.7 Introducing Optimization Problems

Section Review Page 401 Question 21

Let x and y be the width and height of the central rectangular area, in centimetres. Let A be the area of the entire canvas.

$$\begin{aligned} A &= (x+8)(y+12) \\ &= xy + 12x + 8y + 96 \qquad (1) \end{aligned}$$

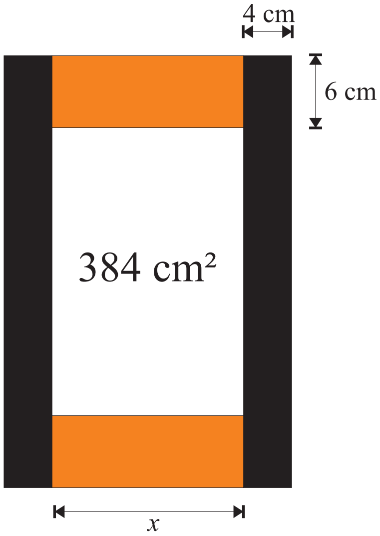
The inner area constrains the variables.

$$xy = 384 \qquad (2)$$

$$y = \frac{384}{x} \qquad (3)$$

Substitute (2) and (3) into (1).

$$\begin{aligned} A &= 384 + 12x + 8\left(\frac{384}{x}\right) + 96 \\ &= 12x + \frac{3072}{x} + 480 \end{aligned}$$



Determine the critical numbers of A.

$$\begin{aligned} A'(x) &= 0 \\ 12 - \frac{3072}{x^2} &= 0 \\ x^2 &= 256 \\ x &= 16,\ x > 0 \qquad (4) \end{aligned}$$

Substitute (4) into (3).

$$\begin{aligned} y &= \frac{384}{16} \\ &= 24 \end{aligned}$$

The dimensions of the canvas that provide the smallest area are 16 + 8 or 24 cm wide and 24 + 12 or 36 cm high.

Section Review Page 401 Question 22

Let x and y be the dimensions of the bin, in metres. Since the height of the bin is fixed at 1 m, only the area of the top needs to be optimized. Let A be the area of the top of the bin.

$$A = xy \qquad (1)$$

The total length constrains the variables.

$$\begin{aligned} x + y &= 4 \\ y &= 4 - x \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} A &= x(4-x) \\ &= 4x - x^2 \end{aligned}$$

Determine the critical numbers of A.

$$\begin{aligned} A'(x) &= 0 \\ 4 - 2x &= 0 \\ x &= 2 \qquad (3) \end{aligned}$$

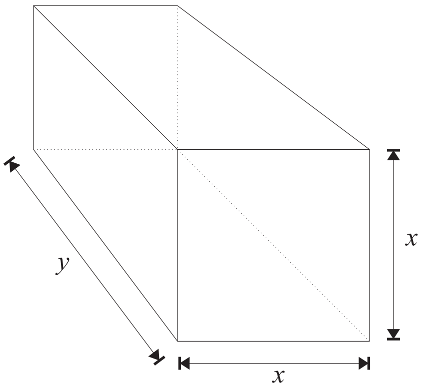
Substitute (3) into (2).

$$\begin{aligned} y &= 4 - 2 \\ &= 2 \end{aligned}$$

The capacity of the bin will be maximized if the dimensions of the top are 2 m by 2 m.

Section Review Page 402 Question 23

Let x be the side length of the square ends, in metres. Let y be the width of the cedar chest, in metres.



a) Let C be the cost of the chest.

$$\begin{aligned} C &= 8(2x^2 + 2xy) + 4(2xy) \\ &= 16x^2 + 24xy \qquad (1) \end{aligned}$$

The capacity of the chest constrains the variables.

$$\begin{aligned} x^2y &= 2 \\ y &= \frac{2}{x^2} \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} C &= 16x^2 + 24x\left(\frac{2}{x^2}\right) \\ &= 16x^2 + \frac{48}{x} \end{aligned}$$

Determine the critical number(s) of C.

$$\begin{aligned} C'(x) &= 0 \\ 32x - \frac{48}{x^2} &= 0 \\ x^3 &= \frac{3}{2} \\ x &= \sqrt[3]{\frac{3}{2}} \qquad (3) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} y &= \frac{2}{\left(\sqrt[3]{\frac{3}{2}}\right)^2} \\ &= 2\sqrt[3]{\frac{4}{9}} \end{aligned}$$

To minimize the cost of the chest, the dimensions should be $\sqrt[3]{\frac{3}{2}}$ m by $\sqrt[3]{\frac{3}{2}}$ m by $2\sqrt[3]{\frac{4}{9}}$ m.

b) Let V be the volume of the chest.

$$V = x^2y \qquad (1)$$

The cost of the chest constrains the variables.

$$\begin{aligned} 16x^2 + 24xy &= 1200 \\ 2x^2 + 3xy &= 150 \\ y &= \frac{150 - 2x^2}{3x} \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} V &= x^2\left(\frac{150 - 2x^2}{3x}\right) \\ &= \frac{150x - 2x^3}{3} \end{aligned}$$

Determine the critical number(s) of V.

$$\begin{aligned} V'(x) &= 0 \\ \frac{1}{3}(150 - 6x^2) &= 0 \\ x^2 &= 25 \\ x &= 5,\ x > 0 \qquad (3) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} y &= \frac{150 - 2(5^2)}{3(5)} \\ &= \frac{100}{15} \\ &= \frac{20}{3} \end{aligned}$$

To maximize the capacity of the chest, the dimensions should be $\frac{20}{3}$ m by 5 m by 5 m.

Section Review Page 402 Question 24

Since the depth of the attic is fixed, only the area of the face needs to be optimized. Let x be the width of the face and y be the height, in metres. Let A be the area of the face.

$$A = xy \qquad (1)$$

The roof line constrains the variables.

$$\begin{aligned} 3x + 4y &= 12 \\ y &= \frac{12 - 3x}{4} \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} A &= x\left(\frac{12 - 3x}{4}\right) \\ &= \frac{12x - 3x^2}{4} \end{aligned}$$

Determine the critical number(s) of A.

$$\begin{aligned} A'(x) &= 0 \\ \frac{1}{4}(12 - 6x) &= 0 \\ x &= 2 \qquad (3) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} y &= \frac{12 - 3(2)}{4} \\ &= 1.5 \end{aligned}$$

To maximize the capacity of the storage area, the width should be 2 m and the height should be 1.5 m.

Section Review Page 402 Question 25

Let h and b be the height and base of the isosceles cross section, in centimetres. Let S be the strength of the rod.

$$S = bh^2 \qquad (1)$$

The diameter of cylinder constrains the variables.

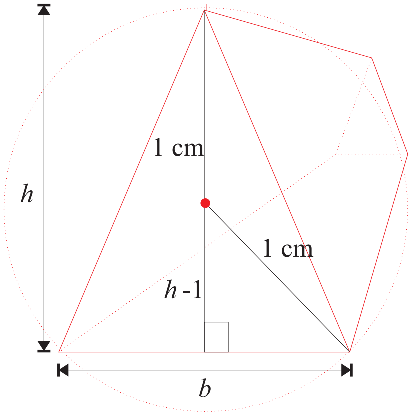
$$\begin{aligned} \frac{b}{2} &= \sqrt{1 - (h - 1)^2} \\ b &= 2\sqrt{2h - h^2} \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$S = 2h^2\sqrt{2h - h^2}$$

Determine the critical numbers of S.

$$\begin{aligned} S'(h) &= 0 \\ 2\left(2h\sqrt{2h - h^2} + h^2\left(\frac{2 - 2h}{2\sqrt{2h - h^2}}\right)\right) &= 0 \\ 2\sqrt{2h - h^2} - \frac{h^2 - h}{\sqrt{2h - h^2}} &= 0 \\ 4h - 2h^2 &= h^2 - h \\ h(3h - 5) &= 0 \\ h &= \frac{5}{3},\ h > 0 \qquad (3) \end{aligned}$$



Substitute (3) into (2).

$$\begin{aligned} b &= 2\sqrt{\frac{10}{3} - \frac{25}{9}} \\ &= \frac{2\sqrt{5}}{3} \end{aligned}$$

To maximize the strength of the rod, the height should be $\frac{5}{3}$ cm and the width should be $\frac{2\sqrt{5}}{3}$ cm.

Section Review Page 402 Question 26

a)
$$\begin{aligned} P(t) &= \frac{0.2t}{t^2+4} \\ P(0.5) &= \frac{0.2(0.5)}{0.5^2+4} \\ &\doteq 0.0235 \end{aligned}$$

After 30 min, the concentration in the bloodstream is approximately 2.35%.

b)
$$\begin{aligned} P'(t) &= 0 \\ \frac{(t^2+4)(0.2)-0.2t(2t)}{(t^2+4)^2} &= 0 \\ 0.8-0.2t^2 &= 0 \\ t^2 &= 4 \\ t &= 2,\ t > 0 \end{aligned}$$

A maximum concentration of $\frac{0.2(2)}{2^2+4}$ or 5% exists in the bloodstream 2 h after administration.

Section Review Page 402 Question 27

Let d be the distance, in kilometres, from the factory emitting the greater amount of particulate. The distance from the other factory is $20 - d$ kilometres. Let C be the concentration d kilometres from the first factory.

$$C = \frac{4}{d^2} + \frac{1}{(20-d)^2}$$

Determine the critical number(s) of C.

$$\begin{aligned} C'(x) &= 0 \\ \frac{-8}{d^3} - \frac{-2}{(20-d)^3} &= 0 \\ \frac{1}{(20-d)^3} &= \frac{4}{d^3} \\ \left(\frac{20-d}{d}\right)^3 &= \frac{1}{4} \\ \frac{20}{d} - 1 &= \frac{1}{\sqrt[3]{4}} \\ \frac{20}{d} &= 1 + \frac{1}{\sqrt[3]{4}} \\ d &= \frac{20}{1+\frac{1}{\sqrt[3]{4}}} \\ &\doteq 12.27 \end{aligned}$$

The concentration will be the least approximately 12.27 km from the plant with the greater emission.

6.8 Optimization Problems in Business and Economics

Section Review Page 402 Question 28

a)
$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= xp(x) - C(x) \\ &= x(25-0.01x) - (300\,000 + 10x + 0.5x^2) \\ &= 25x - 0.01x^2 - 300\,000 - 10x - 0.5x^2 \\ &= -0.51x^2 + 15x - 300\,000 \end{aligned}$$

Determine the critical number of P.

$$\begin{aligned} P'(x) &= 0 \\ -1.02x + 15 &= 0 \\ x &\doteq 14.7 \end{aligned}$$

For maximum profit, the production level should be set to approximately 14.7 units.

b)
$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= xp(x) - C(x) \\ &= x(2-0.001x) - (6000 + 0.1x + 0.01x^2) \\ &= 2x - 0.001x^2 - 6000 - 0.1x - 0.01x^2 \\ &= -0.011x^2 + 1.9x - 6000 \end{aligned}$$

Determine the critical number of P.

$$\begin{aligned} P'(x) &= 0 \\ -0.022x + 1.9 &= 0 \\ x &\doteq 86.4 \end{aligned}$$

For maximum profit, the production level should be set to approximately 86.4 units.

Section Review Page 402 Question 29

From the information given, $p(x)$ is a linear function with a slope of $\frac{-50}{2}$ or -25.

$$\begin{aligned} p(x) - 1300 &= -25(x - 20) \\ p(x) &= -25x + 500 + 1300 \\ &= 1800 - 25x \end{aligned}$$

The price function is $p(x) = 1800 - 25x$. Determine the revenue function, $R(x)$.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(1800 - 25x) \\ &= -25x^2 + 1800x \end{aligned}$$

Determine the critical number of R.

$$\begin{aligned} R'(x) &= 0 \\ -50x + 1800 &= 0 \\ x &= 36 \end{aligned}$$

Maximum revenue is realized at $x = 36$. The golf sets should be sold at $p(36) = 1800 - 25(36)$ or \$900.

Section Review Page 402 Question 30

From the information given, $p(x)$ is a linear function with a slope of $\frac{-0.1}{100}$ or -0.001.

$$\begin{aligned} p(x) - 3 &= -0.001(x - 1000) \\ p(x) &= -0.001x + 1 + 3 \\ &= 4 - 0.001x \end{aligned}$$

The price function is $p(x) = 4 - 0.001x$. Determine the profit function, $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= xp(x) - C(x) \\ &= x(4 - 0.001x) - (2000 + 2.4x + 0.0008x^2) \\ &= 4x - 0.001x^2 - 2000 - 2.4x - 0.0008x^2 \\ &= -0.0018x^2 + 1.6x - 2000 \end{aligned}$$

Determine the critical number of P.

$$\begin{aligned} P'(x) &= 0 \\ -0.0036x + 1.6 &= 0 \\ x &\doteq 444 \end{aligned}$$

Maximum profit is realized at approximately $x = 444$ pencils. The pencils should be sold at $4 - 0.001(444)$ or approximately \$3.56 each.

Chapter Test

Section Chapter Test Page 403 Question 1

a) As $x \to -3^-$, $x + 2 \to -1^-$ and $x^2 - 9 \to 0^+$, resulting in $\lim\limits_{x \to -3^-} \dfrac{x+2}{x^2-9} = -\infty$.

b) As $x \to -3^+$, $x + 2 \to -1^+$ and $x^2 - 9 \to 0^-$, resulting in $\lim\limits_{x \to -3^-} \dfrac{x+2}{x^2-9} = \infty$.

d) Since the numerator tends to ∞ at a greater rate than the denominator, due to their respective degrees, $\lim\limits_{x \to -\infty} \dfrac{2x^2 - 3x^3}{x^2 - 4} \to \infty$.

c)
$$\begin{aligned}
\lim_{x \to -\infty} \frac{4x^2 - 5x + 2}{2x^2 + 3x - 7} &= \lim_{x \to -\infty} \frac{\dfrac{4x^2 - 5x + 2}{x^2}}{\dfrac{2x^2 + 3x - 7}{x^2}} \\
&= \lim_{x \to -\infty} \frac{4 - \dfrac{5}{x} + \dfrac{2}{x^2}}{2 + \dfrac{3}{x} - \dfrac{7}{x^2}} \\
&= \frac{4 - 0 + 0}{2 + 0 - 0} \\
&= 2
\end{aligned}$$

Section Chapter Test Page 403 Question 2

a) Since $-\dfrac{7}{2}$ is a root of the denominator, $x = -\dfrac{7}{2}$ is a vertical asymptote. Since the degrees of the numerator and denominator are equal, the function has a horizontal asymptote of $y = \dfrac{-4}{2}$ or $y = -2$.

b) Since ±2 are roots of the denominator, the function has vertical asymptotes of $x = \pm 2$. Division of the numerator by the denominator yields $x + \dfrac{4x-9}{x^2-4}$. As $|x| \to \infty$, $\dfrac{4x-9}{x^2-4} \to 0$. As a consequence, the function has an oblique asymptote of $y = x$.

Section Chapter Test Page 403 Question 3

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x+1)^2(1) - 2x(x+1)}{(x+1)^4} \\
&= \frac{1-x}{(x+1)^3} \\
\frac{d^2y}{dx^2} &= 0 \\
\frac{(x+1)^3(-1) - (1-x)(3)(x+1)^2}{(x+1)^6} &= 0 \\
\frac{2(x-2)}{(x+1)^4} &= 0 \\
x &= 2
\end{aligned}$$

a) Since $\dfrac{d^2y}{dx^2} < 0$ for $x \in (-\infty, -1)$ and $(-1, 2)$, the function is concave downward on these intervals. Since $\dfrac{d^2y}{dx^2} > 0$ for $x \in (2, \infty)$, the function is concave upward on this interval.

b) Since $\dfrac{d^2y}{dx^2}$ changes its sign at $x = 2$, the function has a point of inflection at $\left(2, \dfrac{2}{9}\right)$.

Section Chapter Test Page 403 Question 4

$$\begin{aligned}
\frac{dy}{dx} &= 0 \\
-12 + 18x - 6x^2 &= 0 \\
x^2 - 3x + 2 &= 0 \\
(x-1)(x-2) &= 0 \\
x &= 1 \text{ or } 2 \\
\frac{d^2y}{dx^2} &= 0 \\
-12x + 18 &= 0 \\
x &= 1.5
\end{aligned}$$

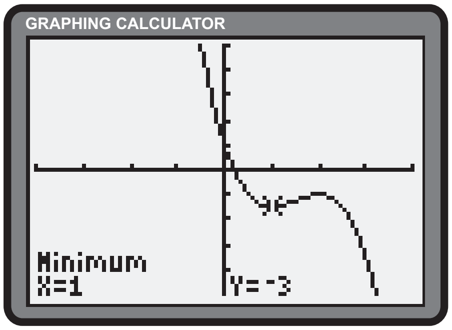
a) Since $\dfrac{dy}{dx} < 0$ for $x \in (-\infty, 1)$ and $(2, \infty)$, the function is decreasing on these intervals. The function is increasing on the interval, $(1, 2)$.

b) A local minimum exists at $(1, -3)$. A local maximum exists at $(2, -2)$.

c) The curve is concave upward on $(-\infty, 1.5)$ and concave downward on $(1.5, \infty)$.

d) A point of inflection exists at $(1.5, -2.5)$.

e)



Minimum
X=1 Y=-3

Section Chapter Test Page 403 Question 5

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 9x + 2 \\ f'(x) &= 0 \\ 3x^2 - 12x + 9 &= 0 \\ x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ x &= 1 \text{ or } 3 \end{aligned}$$

The critical numbers are 1 and 3. Evaluation of the function yields the following results: $f(0.5) = 5.125$, $f(1) = 6$, $f(3) = 2$, and $f(4.5) = 12.125$. The function has an absolute maximum on the interval of 12.125 and an absolute minimum of 2.

Section Chapter Test Page 403 Question 6

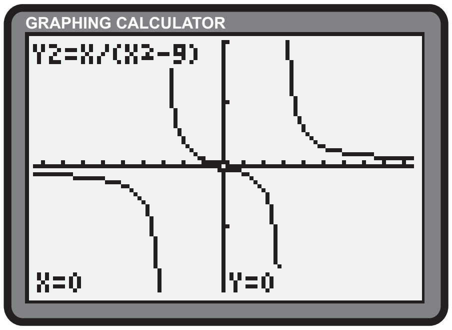
$$\begin{aligned} f'(x) &= 0 \\ \frac{x(2x) - (x^2+1)(1)}{x^2} &= 0 \\ \frac{x^2 - 1}{x^2} &= 0 \\ x &= \pm 1 \end{aligned}$$

Since $f'(x)$ changes its sign from positive to negative on either side of $x = -1$, a local maximum exists at $(-1, -2)$. Since $f'(x)$ changes its sign from negative to positive on either side of $x = 1$, a local minimum exists at $(1, 2)$.

Section Chapter Test Page 403 Question 7

$$\begin{aligned} f(x) &= 0 \\ \frac{x}{x^2 - 9} &= 0 \\ x &= 0 \\ f'(x) &= 0 \\ \frac{(x^2-9)(1) - x(2x)}{(x^2-9)^2} &= 0 \\ \frac{-x^2 - 9}{(x^2-9)^2} &= 0 \\ &\text{no roots} \\ f''(x) &= 0 \\ \frac{(x^2-9)^2(-2x) - (-x^2-9)(2)(x^2-9)(2x)}{(x^2-9)^4} &= 0 \\ \frac{2x(x^2+27)}{(x^2-9)^3} &= 0 \\ x &= 0 \end{aligned}$$

The domain is $\{x \in \Re \mid x \neq \pm 3\}$. The function passes through the origin. There are vertical asymptotes at $x = \pm 3$. As $x \to -\infty$, $f(x) \to 0^-$. As $x \to \infty$, $f(x) \to 0^+$. The function is asymptotic to the x-axis. There are no local extrema. There is a point of inflection at $(0, 0)$. Since $y(x) = -y(-x)$, the function has odd symmetry.



Section Chapter Test Page 403 Question 8

Let x be the side length of the square base and y be the height of the box, in centimetres. Let V be the volume of the box.

$$V = x^2 y \tag{1}$$

The area of the material constrains the variables.

$$\begin{aligned} x^2 + 4xy &= 2400 \\ y &= \frac{2400 - x^2}{4x} \end{aligned} \tag{2}$$

Substitute (2) into (1).

$$\begin{aligned} V &= x^2\left(\frac{2400 - x^2}{4x}\right) \\ &= \frac{2400x - x^3}{4} \end{aligned}$$

Determine the critical numbers of V.

$$\begin{aligned} V'(x) &= 0 \\ \frac{1}{4}(2400 - 3x^2) &= 0 \\ x^2 &= 800 \quad (3) \\ x &= 20\sqrt{2},\ x > 0 \quad (4) \end{aligned}$$

Substitute (3) and (4) into (2).

$$\begin{aligned} y &= \frac{2400 - 800}{4(20\sqrt{2})} \\ &= \frac{20}{\sqrt{2}} \\ &= 10\sqrt{2} \end{aligned}$$

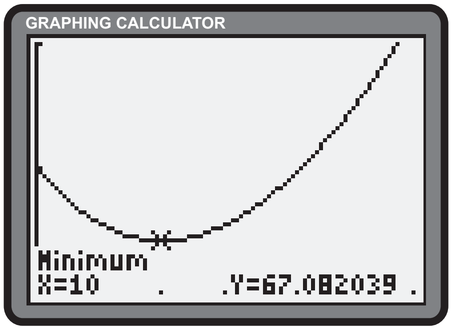
To maximize the volume, the dimensions should be $20\sqrt{2}$ cm by $20\sqrt{2}$ cm by $10\sqrt{2}$ cm.

Section Chapter Test Page 403 Question 9

Let x be the distance, in metres, from the junction box to where the line from the closer cottage meets the power line. Let d be the total distance, in metres, from the junction box to the cottages.

$$\begin{aligned} d &= \sqrt{x^2 + 20^2} + \sqrt{(30 - x)^2 + 40^2} \\ &= \sqrt{x^2 + 400} + \sqrt{x^2 - 60x + 2500} \end{aligned}$$

Determine the critical numbers of d.



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Minimum
X=10          Y=67.082039
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$$\begin{aligned} d'(x) &= 0 \\ \frac{x}{\sqrt{x^2 + 400}} + \frac{x - 30}{\sqrt{x^2 - 60x + 2500}} &= 0 \\ \left(\frac{x}{\sqrt{x^2 + 400}}\right)^2 &= \left(-\frac{x - 30}{\sqrt{x^2 - 60x + 2500}}\right)^2 \\ \frac{x^2}{x^2 + 400} &= \frac{x^2 - 60x + 900}{x^2 - 60x + 2500} \\ x^4 - 60x^3 + 2500x^2 &= x^4 - 60x^3 + 1300x^2 - 24\,000x + 360\,000 \\ 1200x^2 + 24\,000x - 360\,000 &= 0 \\ x^2 + 20x - 300 &= 0 \\ (x - 10)(x + 30) &= 0 \\ x &= 10,\ x \geq 0 \end{aligned}$$

a) The junction box should be located 10 m from where the line from the closer cottage meets the power line.
b) Answers may vary.

Section Chapter Test Page 403 Question 10

a) From the information given, $p(x)$ is a linear function with a slope of $\frac{-0.1}{20}$ or -0.005.

$$\begin{aligned} p(x) - 6 &= -0.005(x - 100) \\ p(x) &= -0.005x + 0.5 + 6 \\ &= 6.5 - 0.005x \end{aligned}$$

The price function is $p(x) = 6.5 - 0.005x$. Determine the revenue function, $R(x)$.

$$\begin{aligned} R(x) &= xp(x) \\ &= x(6.5 - 0.005x) \\ &= 6.5x - 0.005x^2 \end{aligned}$$

Determine the critical number of R.

$$\begin{aligned} R'(x) &= 0 \\ 6.5 - 0.01x &= 0 \\ x &= 650 \end{aligned}$$

Since $R''(x) < 0$, maximum revenue is realized at $x = 650$. The muffins should be sold at $p(650) = 6.5 - 0.005(650)$ or \$3.25 per dozen.

b)

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 6.5x - 0.005x^2 - (300 + x + 0.01x^2) \\ &= 6.5x - 0.005x^2 - 300 - x - 0.01x^2 \\ &= -0.015x^2 + 5.5x - 300 \end{aligned}$$

Determine the critical numbers of P.

$$\begin{aligned} P'(x) &= 0 \\ -0.03x + 5.5 &= 0 \\ x &\doteq 183.3 \end{aligned}$$

Since $P''(x) < 0$, maximum profit occurs at a production level of approximately 183 dozen.

Section Chapter Test Page 404 Question 11

Let S be the strength of the lumber.

$$S = clw^2 \qquad (1)$$

The diameter of the tree constrains the variables.

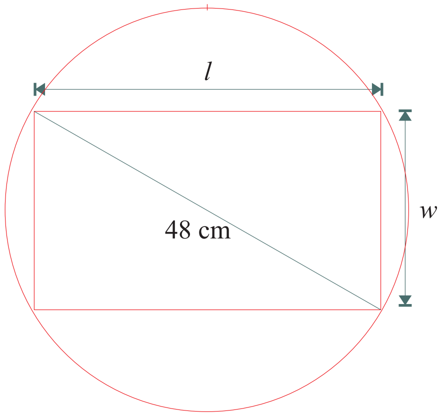
$$\begin{aligned} l^2 + w^2 &= 48^2 \\ w^2 &= 2304 - l^2 \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} S &= cl(2304 - l^2) \\ &= c(2304l - l^3) \end{aligned}$$

Determine the critical numbers of S.

$$\begin{aligned} S'(l) &= 0 \\ c(2304 - 3l^2) &= 0 \\ l^2 &= 768 \qquad (3) \\ l &\doteq 27.7 \end{aligned}$$



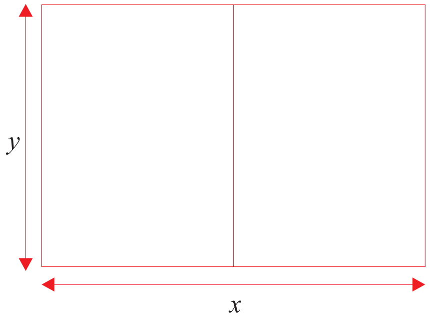
Substitute (3) into (2).

$$\begin{aligned} w^2 &= 2304 - 768 \\ w &= \sqrt{1536} \\ &\doteq 39.2 \end{aligned}$$

To maximize the strength of the lumber, the dimensions should be approximately 39.2 cm by 27.7 cm.

Section Chapter Test Page 404 Question 12

Let x and y be the dimensions of the floor, in metres, as shown in the diagram. Let L be the total wall length.



$$L = 2x + 3y \qquad (1)$$

The total floor space constrains the variables.

$$\begin{aligned} xy &= 100 \\ y &= \frac{100}{x} \qquad (2) \end{aligned}$$

Substitute (2) into (1) and optimize.

$$\begin{aligned} L &= 2x + 3\left(\frac{100}{x}\right) \\ &= 2x + \frac{300}{x} \\ L'(x) &= 0 \\ 2 - \frac{300}{x^2} &= 0 \\ x^2 &= 150 \\ x &= 5\sqrt{6},\ x > 0 \qquad (3) \end{aligned}$$

Substituting (3) into (2) yields $y = \frac{10\sqrt{6}}{3}$ m. To minimize the total wall length, the dimensions of the floor should be $5\sqrt{6}$ m by $\frac{10\sqrt{6}}{3}$ m.

Section Chapter Test Page 404 Question 13

a)

$$\begin{aligned} C'(t) &= 0 \\ \frac{(t^2+2t+2)(0.12) - 0.12t(2t+2)}{(t^2+2t+2)^2} &= 0 \\ \frac{2-t^2}{(t^2+2t+2)^2} &= 0 \\ t^2 &= 2 \\ t &= \sqrt{2},\ t > 0 \\ &\doteq 1.41 \end{aligned}$$

The concentration of the drug in the bloodstream will be at a maximum after approximately 1.41 h.

b) The concentration is $C(\sqrt{2}) = \frac{0.12\sqrt{2}}{2 + 2\sqrt{2} + 2}$ or approximately 0.0249 mg/cm^3.

Section Chapter Test Page 404 Question 14

Determine the revenue function, $R(x)$.

$$\begin{aligned} R(x) &= xp(x) \\ &= \frac{32x}{x^2 - 32x + 320} \end{aligned}$$

Determine the critical numbers of R.

$$\begin{aligned} R'(x) &= 0 \\ \frac{(x^2-32x+320)(32) - 32x(2x-32)}{(x^2-32x+320)^2} &= 0 \\ \frac{320-x^2}{(x^2-32x+320)^2} &= 0 \\ x^2 &= 320 \\ &\doteq 17.9 \end{aligned}$$

To maximum revenue, approximately 18 000 candies should be sold.

Section Chapter Test Page 404 Question 15

$$\begin{aligned}
y &= 200\sqrt{x+5} - x^{\frac{3}{2}} \\
\frac{dy}{dx} &= 0 \\
\frac{200}{2\sqrt{x+5}} - \frac{3}{2}x^{\frac{1}{2}} &= 0 \\
\frac{100}{\sqrt{x+5}} - \frac{3}{2}\sqrt{x} &= 0 \\
3\sqrt{x^2+5x} &= 200 \\
9(x^2+5x) &= 40\ 000 \\
9x^2 + 45x - 40\ 000 &= 0 \\
x &= \frac{-(45) \pm \sqrt{(45)^2 - 4(9)(-40\ 000)}}{2(9)} \\
&\doteq 64.2,\ x > 0
\end{aligned}$$

Property tax revenue will be maximized with approximately 64 new houses.

Challenge Problems

Section Challenge Problems Page 405 Question 1

Let x and y be the dimensions of the rectangle. Let A be the area of the rectangle.

$$A = xy \qquad (1)$$

The equation of the parabola constrains the variables.

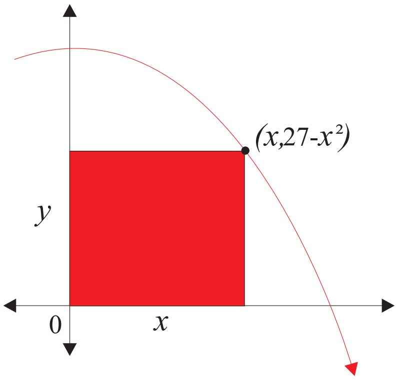
$$y = 27 - x^2 \qquad (2)$$

Substitute (2) into (1).

$$\begin{aligned} A &= x(27 - x^2) \\ &= 27x - x^3 \end{aligned}$$

Determine the critical numbers of A.

$$\begin{aligned} A'(x) &= 0 \\ 27 - 3x^2 &= 0 \\ x^2 &= 9 \\ x &= 3,\ x > 0 \end{aligned}$$



Since $A''(3) < 0$, $x = 3$ defines a maximum value for area of $A(3) = 27(3) - 3^3$ or 54 square units.

Section Challenge Problems Page 405 Question 2

Let x be the positive number. Let E be the defined function.

$$\begin{aligned} E &= x - \frac{1}{x} + 5\left(\frac{1}{x}\right)^2 \\ &= x - \frac{1}{x} + \frac{5}{x^2} \qquad (1) \end{aligned}$$

Determine the critical numbers of E.

$$\begin{aligned} E'(x) &= 0 \\ 1 + \frac{1}{x^2} - \frac{10}{x^3} &= 0 \\ \frac{x^3 + x - 10}{x^3} &= 0 \\ x^3 + x - 10 &= 0 \\ (x - 2)(x^2 + 2x + 5) &= 0 \\ x &= 2 \end{aligned}$$

Since $E''(2) > 0$, a minimum value of $E(2) = 2 - \frac{1}{2} + \frac{5}{2^2}$ or $\frac{11}{4}$ exists at $x = 2$.

Section Challenge Problems Page 405 Question 3

$$\begin{aligned} f'(x) &= 0 \\ (1 - x)^q \left(px^{p-1}\right) + x^p q(1 - x)^{q-1}(-1) &= 0 \\ x^{p-1}(1 - x)^{q-1}\left[(1 - x)p - qx\right] &= 0 \\ x^{p-1}(1 - x)^{q-1}\left[p - (p + q)x\right] &= 0 \\ x &= 0, 1, \text{ or } \frac{p}{p + q}, \quad \text{where } p \geq 2,\ q \geq 2, \text{ and } p \neq -q \end{aligned}$$

Section Challenge Problems Page 405 Question 4

$$\begin{aligned}
\frac{dy}{dx} &= 0 \\
(r^2+x^2)^{-\frac{2}{3}}(1) + x\left(-\frac{2}{3}\right)(r^2+x^2)^{-\frac{5}{3}}(2x) &= 0 \\
(r^2+x^2)^{-\frac{5}{3}}\left(r^2+x^2-\frac{4}{3}x^2\right) &= 0 \\
(r^2+x^2)^{-\frac{5}{3}}\left(r^2-\frac{1}{3}x^2\right) &= 0 \\
x^2 &= 3r^2 \\
x &= r\sqrt{3};\ x \geq 0
\end{aligned}$$

Since $r\sqrt{3} > r$, a check of the endpoints of the interval is required.

$$\begin{aligned}
y(0) &= 0 \\
y(r) &= r(r^2+r^2)^{-\frac{2}{3}} \\
&= \frac{r}{\sqrt[3]{4r^4}} \\
&= \frac{1}{\sqrt[3]{4r}}
\end{aligned}$$

The absolute maximum of $y = x(r^2+x^2)^{-\frac{2}{3}}$ over the interval $x \in [0, r]$ is $\frac{1}{\sqrt[3]{4r}}$.

Section Challenge Problems Page 405 Question 5

a) Develop and solve a system of equations.

$$f'(x) = 3x^2 + 2ax + b$$

Given that extrema exist at $x = \pm 1$,

$$\begin{aligned}
f'(1) &= 0 \\
3 + 2a + b &= 0 \qquad (1) \\
f'(-1) &= 0 \\
3 - 2a + b &= 0 \qquad (2)
\end{aligned}$$

Subtract (2) from (1).

$$\begin{aligned}
4a &= 0 \\
a &= 0 \qquad (3)
\end{aligned}$$

Substitute (3) into (1).

$$\begin{aligned}
3 + b &= 0 \\
b &= -3
\end{aligned}$$

To meet the requirements, $a = 0$ and $b = -3$. Check that f has a minimum at $x = 1$.

$$\begin{aligned}
f''(x) &= 6x + 2a \\
&= 6x \\
f''(1) &= 6 > 0
\end{aligned}$$

Check that f has a maximum at $x = -1$.

$$f''(-1) = -6 < 0$$

b) Develop and solve a system of equations.

$$\begin{aligned}
f(1) &= 1 \\
a + b + c + d &= 1 \qquad (1) \\
f(3) &= 0 \\
27a + 9b + 3c + d &= 0
\end{aligned}$$

Subtract (1) from (2).

$$\begin{aligned}
26a + 8b + 2c &= -1 \qquad (3) \\
f'(x) &= 3ax^2 + 2bx + c
\end{aligned}$$

Given that extrema exist at $x = 1$ and $x = 3$,

$$\begin{aligned}
f'(1) &= 0 \\
3a + 2b + c &= 0 \qquad (4) \\
f'(3) &= 0 \\
27a + 6b + c &= 0 \qquad (5)
\end{aligned}$$

Subtract 2×(4) from (3).

$$20a + 4b = -1 \qquad (6)$$

Subtract (4) from (5).

$$24a + 4b = 0 \qquad (7)$$

Substitute (6) into (7).

$$\begin{aligned}
4a &= 1 \\
a &= \frac{1}{4} \qquad (8)
\end{aligned}$$

Back substitution yields the remaining values. The required cubic equation is $f(x) = \frac{1}{4}x^3 - \frac{3}{2}x^2 + \frac{9}{4}x$.

Section Challenge Problems Page 405 Question 6

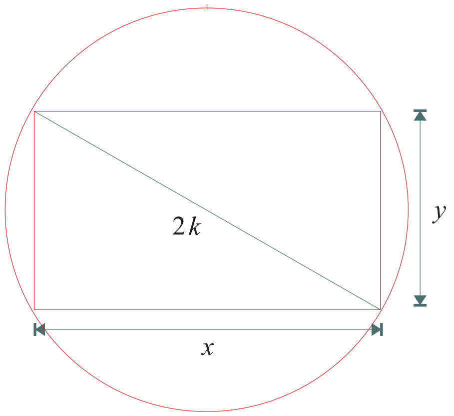
For the given function, $f'(x) = 13x^{12} + 182x^6 + 637$. By inspection, the smallest value the derivative can attain is 637 at $x = 0$. Since the given polynomial function is differentiable over all real numbers, and there are no critical numbers, the function has no local extrema.

Section Challenge Problems   Page 405   Question 7

$g(x)$ is a reflection of $f(x)$ in the x-axis. Thus, any maximum value of f will correspond to a minimum value of g at the same x-coordinate.

Section Challenge Problems   Page 405   Question 8

a) Let x and y be the dimensions of the rectangle. Let A be the area of the rectangle.



$$A = xy \qquad (1)$$

The diameter of the circle constrains the variables.

$$x^2 + y^2 = (2k)^2$$
$$y = \sqrt{4k^2 - x^2} \qquad (2)$$

Substitute (2) into (1).

$$A = x\sqrt{4k^2 - x^2}$$

Determine the critical number of A.

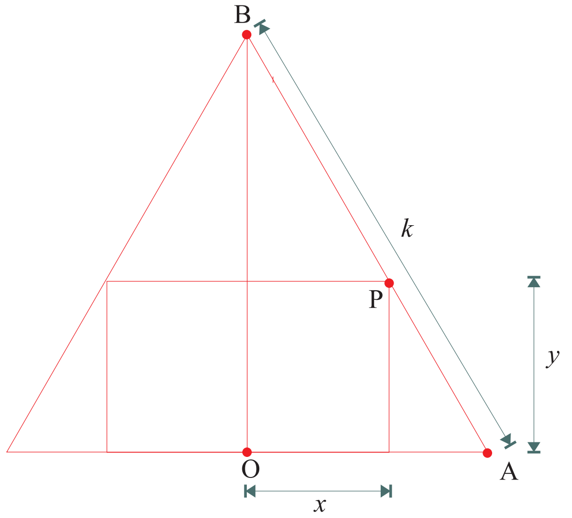
$$\begin{aligned} A'(x) &= 0 \\ \sqrt{4k^2 - x^2}(1) + x\left(\frac{-x}{\sqrt{4k^2 - x^2}}\right) &= 0 \\ 4k^2 - x^2 &= x^2 \\ x^2 &= 2k^2 \qquad (3) \\ x &= k\sqrt{2},\ x > 0 \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} y &= \sqrt{4k^2 - 2k^2} \\ &= k\sqrt{2} \end{aligned}$$

To maximize the area of the rectangle, the dimensions should be $k\sqrt{2}$ by $k\sqrt{2}$.

b) Let x and y be the dimensions shown on the rectangle. Let A be the area of the rectangle.



$$A = 2xy \qquad (1)$$

The equilateral triangle constrains the variables. If point O is situated at the origin, the line containing points A and B can be expressed as

$$y = -\sqrt{3}x + \frac{k\sqrt{3}}{2} \qquad (2)$$

Substitute (2) into (1).

$$\begin{aligned} A &= 2x\left(-\sqrt{3}x + \frac{k\sqrt{3}}{2}\right) \\ &= -2\sqrt{3}x^2 + k\sqrt{3}x \end{aligned}$$

Determine the critical number of A.

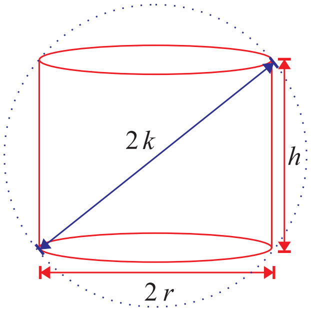
$$\begin{aligned} A'(x) &= 0 \\ -4\sqrt{3}x + k\sqrt{3} &= 0 \\ x &= \frac{k}{4} \qquad (3) \\ 2x &= \frac{k}{2} \qquad (4) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} y &= -\sqrt{3}\left(\frac{k}{4}\right) + \frac{k\sqrt{3}}{2} \\ &= \frac{k\sqrt{3}}{4} \end{aligned}$$

To maximize the area of the rectangle, the dimensions should be $\frac{k}{2}$ (along the side) by $\frac{k\sqrt{3}}{4}$.

c) Let r and h be the base radius and height of the cylinder, respectively. Let V be the volume of the cylinder.



$$V = \pi r^2 h \tag{1}$$

The sphere provides a constraint on r and h.

$$\begin{aligned}(2r)^2 + h^2 &= (2k)^2\\ 4r^2 + h^2 &= 4k^2\\ r^2 &= \frac{4k^2 - h^2}{4} \qquad (2)\end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned}V &= \pi\left(\frac{4k^2 - h^2}{4}\right)h\\ &= \frac{\pi}{4}(4k^2h - h^3)\end{aligned}$$

Determine the critical number(s) of V.

$$\begin{aligned}V'(h) &= 0\\ \frac{\pi}{4}(4k^2 - 3h^2) &= 0\\ 3h^2 &= 4k^2\\ h^2 &= \frac{4k^2}{3} \qquad (3)\\ h &= \frac{2k}{\sqrt{3}},\ h > 0\end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned}r^2 &= \frac{4k^2 - \frac{4k^2}{3}}{4}\\ &= k^2 - \frac{k^2}{3}\\ &= \frac{2k^2}{3}\\ r &= \frac{\sqrt{2}k}{\sqrt{3}},\ r > 0\end{aligned}$$

For maximum volume, $(r, h) = \left(\frac{\sqrt{2}k}{\sqrt{3}}, \frac{2k}{\sqrt{3}}\right)$.

Section Challenge Problems Page 405 Question 9

Let r and h be the base radius and height of the cylinder, respectively. Let C be the cost of the juice can, in cents. Let the volume be k cm^3.

$$\begin{aligned} C &= 0.25(2\pi r^2) + 0.5(2\pi r h) \\ &= 0.5\pi r^2 + \pi r h \qquad (1) \end{aligned}$$

The volume of the can constrains the variables.

$$\begin{aligned} \pi r^2 h &= k \\ h &= \frac{k}{\pi r^2} \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} C &= 0.5\pi r^2 + \pi r\left(\frac{k}{\pi r^2}\right) \\ &= 0.5\pi r^2 + \frac{k}{r} \end{aligned}$$

Determine the critical number of C.

$$\begin{aligned} C'(r) &= 0 \\ \pi r - \frac{k}{r^2} &= 0 \\ r^3 &= \frac{k}{\pi} \qquad (3) \end{aligned}$$

Determine the required ratio, R.

$$R = \frac{h}{r} \qquad (4)$$

Substitute (2) into (4).

$$\begin{aligned} R &= \frac{\frac{k}{\pi r^2}}{r} \\ &= \frac{k}{\pi r^3} \qquad (5) \end{aligned}$$

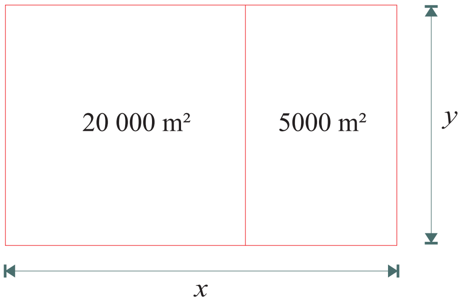
Substitute (3) into (5).

$$\begin{aligned} R &= \frac{k}{\pi\left(\frac{k}{\pi}\right)} \\ &= 1 \end{aligned}$$

For minimum cost, the ratio of height to radius should be 1 : 1.

Section Challenge Problems Page 405 Question 10

Let x and y be the dimensions of the rectangular floor. Let C be the cost of the walls, in dollars.



$$\begin{aligned} C &= 300(2x + y) + 150y + 500y \\ &= 600x + 300y + 650y \\ &= 600x + 950y \qquad (1) \end{aligned}$$

The area constrains the variables.

$$\begin{aligned} xy &= 25\ 000 \\ y &= \frac{25\ 000}{x} \qquad (2) \end{aligned}$$

Substitute (2) into (1).

$$\begin{aligned} C &= 600x + 950\left(\frac{25\ 000}{x}\right) \\ &= 600x + \frac{237\ 500}{x} \end{aligned}$$

Determine the critical number of C.

$$\begin{aligned} C'(x) &= 0 \\ 600 - \frac{23\ 750\ 000}{x^2} &= 0 \\ x^2 &= \frac{237\ 500}{6} \\ x &= 50\sqrt{\frac{95}{6}} \qquad (3) \end{aligned}$$

Substitute (3) into (2).

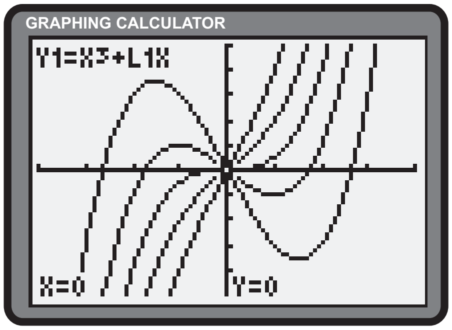
$$\begin{aligned} y &= \frac{25\ 000}{50\sqrt{\frac{95}{6}}} \\ &= 100\sqrt{\frac{30}{19}} \end{aligned}$$

To minimize the cost of construction, the front should be $100\sqrt{\frac{30}{19}}$ m and the sides should be $50\sqrt{\frac{95}{6}}$ m.

Section Challenge Problems Page 405 Question 11

a)

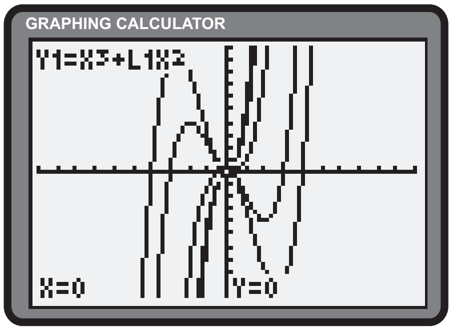
$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 3x^2 + c &= 0 \\ x^2 &= \frac{-c}{3} \\ x &= \pm\sqrt{\frac{-c}{3}} \\ \frac{d^2y}{dx^2} &= 0 \\ 6x &= 0 \\ x &= 0 \end{aligned}$$



Answers may vary. For $c = 0$, a point of inflection exists at $(0, 0)$. For $c \geq 0$, the only intercept is at $(0, 0)$; as c increases, the slope at the origin increases. For $c < 0$, x-intercepts at $-\sqrt{-c}, 0, \sqrt{-c}$; spreads and gets larger at maximum, smaller at minimum as $|c|$ increases.

b)

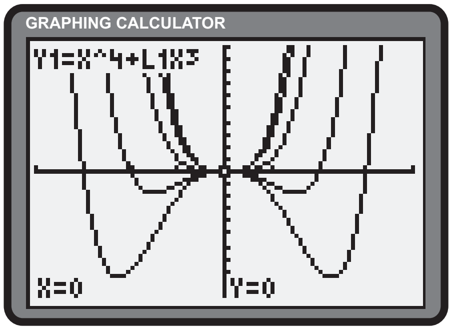
$$\begin{aligned} f'(x) &= 0 \\ 3x^2 + 2cx &= 0 \\ x(3x + 2c) &= 0 \\ x &= 0 \text{ or } -\frac{2c}{3} \\ f''(x) &= 0 \\ 6x + 2c &= 0 \\ x &= -\frac{c}{3} \end{aligned}$$



Answers may vary. x-intercepts: $-c, 0$; for $c > 0$, minimum at $(0, 0)$ and maximum at $\left(-\frac{2}{3}c, \frac{4}{27}c^3\right)$; for $c < 0$, minimum at $\left(-\frac{2}{3}c, \frac{4}{27}c^3\right)$ and maximum at $(0, 0)$.

c)

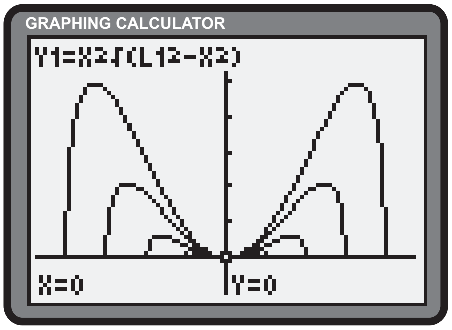
$$\begin{aligned} f'(t) &= 0 \\ 4t^3 + 3ct^2 &= 0 \\ t^2(4t + 3c) &= 0 \\ t &= 0 \text{ or } -\frac{3c}{4} \\ f''(t) &= 0 \\ 12t^2 + 6ct &= 0 \\ t(2t + c) &= 0 \\ t &= 0 \text{ or } -\frac{c}{2} \end{aligned}$$



Answers may vary. x-intercepts: $-c, 0$; for $c = 0$, minimum at $(0, 0)$; for $c \neq 0$, points of inflection at $(0, 0)$ and $\left(-\frac{1}{2}c, -\frac{1}{16}c^4\right)$; minimum at $\left(-\frac{3}{4}c, -\frac{27}{256}c^4\right)$

d)

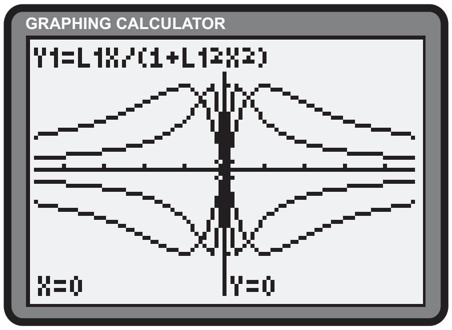
$$g'(x) = 0$$
$$\sqrt{c^2 - x^2}(2x) + x^2\left(\frac{-x}{\sqrt{c^2 - x^2}}\right) = 0$$
$$c^2 - x^2 = \frac{x^2}{2} \text{ or } x = 0$$
$$x^2 = \frac{2}{3}c^2$$
$$x = \pm\sqrt{\frac{2}{3}}c$$



Answers may vary. x-intercepts: $0, \pm c$; domain: $x \in [-c, c]$, minimum at $(0, 0)$; maxima at $\left(\pm\sqrt{\frac{2}{3}}c, \frac{2c^3}{3\sqrt{3}}\right)$

e)

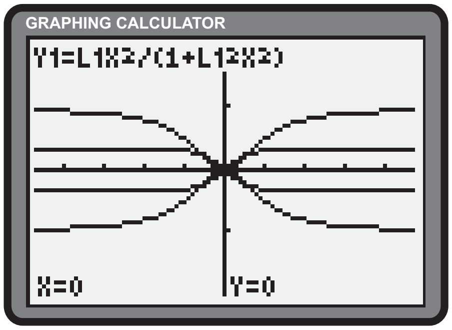
$$\frac{dy}{dx} = 0$$
$$\frac{(1 + c^2x^2)(c) - cx(2c^2x)}{(1 + c^2x^2)^2} = 0$$
$$\frac{1 - c^3x^2}{(1 + c^2x^2)^2} = 0$$
$$c^3x^2 = 1$$
$$x^2 = \frac{1}{c^3}$$
$$x = \pm\frac{1}{c\sqrt{c}}$$



Answers may vary. x-intercept: 0; asymptote: $y = 0$; for $c > 0$, minimum at $\left(-\frac{1}{c}, -\frac{1}{2}\right)$ and maximum at $\left(\frac{1}{c}, \frac{1}{2}\right)$; for $c < 0$, minimum at $\left(\frac{1}{c}, \frac{1}{2}\right)$ and maximum at $\left(-\frac{1}{c}, -\frac{1}{2}\right)$; points of inflection: $(0, 0)$, $\left(\frac{\sqrt{3}}{c}, \frac{\sqrt{3}}{4}\right)$, $\left(-\frac{\sqrt{3}}{c}, -\frac{\sqrt{3}}{4}\right)$

f)

$$h'(x) = 0$$
$$\frac{(1 + c^2x^2)(2cx) - cx^2(2c^2x)}{(1 + c^2x^2)^2} = 0$$
$$\frac{2cx}{(1 + c^2x^2)^2} = 0$$
$$x = 0$$



Answers may vary. x-intercept: 0; asymptote: $y = \frac{1}{c}$; minimum at $(0,)$ if $c > 0$; maximum at $(0, 0)$ if $c < 0$; points of inflection: $\left(\pm\frac{1}{\sqrt{3}c}, \frac{1}{4c}\right)$

Using the Strategies

Section Problem Solving Page 408 Question 1

a)–d) Answers may vary.
e) The sum is $2 + 6 + 13 + 20 + 24$ or 65.
f) The sum is 65.
g) The sum is 15.
h) The sum is 34.
i) Answers will vary.

a)

~~1~~	②	~~3~~	~~4~~	~~5~~
⑥	~~7~~	~~8~~	~~9~~	~~10~~
~~11~~	~~12~~	⑬	~~14~~	~~15~~
~~16~~	~~17~~	~~18~~	~~19~~	⑳
~~21~~	~~22~~	~~23~~	㉔	~~25~~

Section Problem Solving Page 408 Question 2

Let S_3 be the sum of three consecutive numbers, n, $n + 1$, and $n + 2$.

$$\begin{aligned} S_3 &= n + n + 1 + n + 2 \\ &= 3n + 3 \\ &= 3(n + 1) \end{aligned}$$

Since S_3 contains a factor of 3, it is divisible by 3.

Let S_5 be the sum of five consecutive numbers, n, $n + 1$, $n + 2$, $n + 3$, and $n + 4$.

$$\begin{aligned} S_5 &= n + n + 1 + n + 2 + n + 3 + n + 4 \\ &= 5n + 10 \\ &= 5(n + 2) \end{aligned}$$

Since S_5 contains a factor of 5, it is divisible by 5.

Let S_4 be the sum of four consecutive numbers, n, $n + 1$, $n + 2$, and $n + 3$.

$$\begin{aligned} S_4 &= n + n + 1 + n + 2 + n + 3 \\ &= 4n + 6 \\ &= 2(n + 3) \end{aligned}$$

Since S_4 does not contain a factor of 4, it is not divisible by 4.

Section Problem Solving Page 408 Question 3

Start both timers. When 5-min timer is done, start cooking; use 4 min left in 9-min timer, and then restart 9-min timer. When it is done, you are done.

Section Problem Solving Page 408 Question 4

Let r be the radius of the circle. The diagonal of the square is $4r$. Use the Pythagorean theorem.

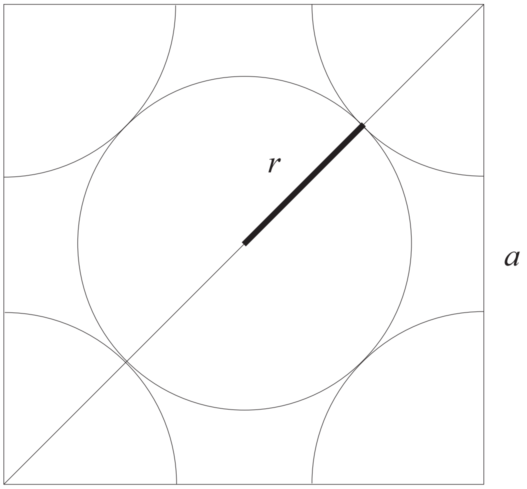
$$\begin{aligned} (4r)^2 &= a^2 + a^2 \\ 16r^2 &= 2a^2 \\ r^2 &= \frac{a^2}{8} \qquad (1) \end{aligned}$$

Let A be the area of the shaded region.

$$\begin{aligned} A &= A_{\text{square}} - A_{\text{circles}} \\ &= a^2 - \left(\pi r^2 + 4 \times \frac{\pi r^2}{4}\right) \\ &= a^2 - \left(\pi r^2 + \pi r^2\right) \\ &= a^2 - 2\pi r^2 \qquad (2) \end{aligned}$$

Substitute (1) into (2).

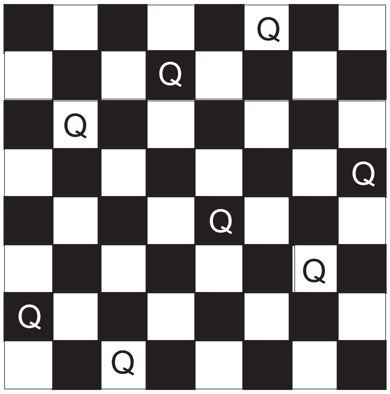
$$\begin{aligned} &= a^2 - 2\pi \times \frac{a^2}{8} \\ &= a^2 - \frac{\pi a^2}{4} \\ &= a^2\left(1 - \frac{\pi}{4}\right) \end{aligned}$$



The area of the shaded region is $a^2\left(1 - \frac{\pi}{4}\right)$ square units.

Section Problem Solving Page 408 Question 5

Answers may vary.



Section Problem Solving Page 408 Question 6

Let h be the length of the hypotenuse. Let x and y be the lengths of the two shorter sides of the right triangle.

First, establish the primary relations.

$$\begin{aligned} A &= \frac{xy}{2} \\ xy &= 2A && (1) \\ P &= x + y + h \\ x + y &= P - h && (2) \end{aligned}$$

Determine h in terms of A and P.

$$\begin{aligned} h^2 &= x^2 + y^2 \\ h^2 &= (x + y)^2 - 2xy && (3) \end{aligned}$$

Substitute (1) and (2) into (3).

$$\begin{aligned} h^2 &= (P - h)^2 - 2(2A) \\ &= P^2 - 2Ph + h^2 - 4A \\ 2Ph &= P^2 - 4A \\ h &= \frac{P}{2} - \frac{2A}{P} \end{aligned}$$

The length of the hypotenuse can be expressed as $h = \frac{P}{2} - \frac{2A}{P}$.

Section Problem Solving Page 408 Question 7

Construct QR through P, parallel to the sides of the rectangle.

In △APQ,

$$a^2 = x^2 + (\text{PQ})^2 \qquad (1)$$

In △BPQ,

$$b^2 = y^2 + (\text{PQ})^2 \qquad (2)$$

Subtract (2) from (1).

$$a^2 - b^2 = x^2 - y^2 \qquad (3)$$

In △DPR,

$$d^2 = x^2 + (\text{PR})^2 \qquad (4)$$

In △CPR,

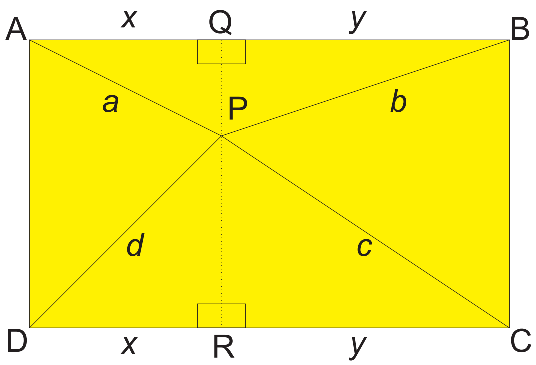
$$c^2 = y^2 + (\text{PR})^2 \qquad (5)$$

Subtract (5) from (4).

$$d^2 - c^2 = x^2 - y^2 \qquad (6)$$

Substitute (6) into (3).

$$\begin{aligned} a^2 - b^2 &= d^2 - c^2 \\ a^2 + c^2 &= b^2 + d^2 \end{aligned}$$



Section Problem Solving   Page 408   Question 8

Of the 937 valid responses, since 426 said yes to at least two items and all respondents said yes to at least one, a total of 937 – 426 or 511 households purchased exactly one item.

Of the 937 valid responses, 314 respondents purchased at least three and 511 purchased exactly one. As a result, 937 – 314 – 511 or 112 households purchased exactly two items.

Of the 937 valid responses, 282 purchased all four items, 511 purchased one item, and 112 purchased exactly two items. A total of 937 – 282 – 511 – 112 or 32 households purchased exactly three items.

Section Problem Solving   Page 409   Question 9

The hypotenuse of the right triangle is $\sqrt{6^2 + 8^2}$ or 10 cm.

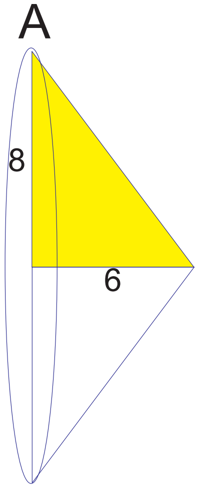


Fig. A is a right circular cone of base radius 8 cm and height 6 cm.

$$\begin{aligned} V_{\mathrm{A}} &= \frac{1}{3}\pi(8)^2(6) \\ &= 128\pi \end{aligned}$$

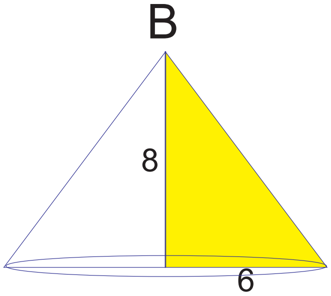


Fig. B is a right circular cone of base radius 6 cm and height 8 cm.

$$\begin{aligned} V_{\mathrm{B}} &= \frac{1}{3}\pi(6)^2(8) \\ &= 96\pi \end{aligned}$$

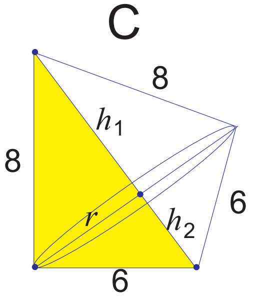


Fig. C consists of two right circular cones of common radius and heights that total 10 cm. Let r, h_1, and h_2 stand for the lengths between the indicated points. The value for r can be determined from similar triangles.

$$\begin{aligned} \frac{r}{6} &= \frac{8}{10} \\ r &= 4.8 \end{aligned}$$

$$\begin{aligned} V_{\mathrm{C}} &= \frac{1}{3}\pi r^2 h_1 + \frac{1}{3}\pi r^2 h_2 \\ &= \frac{1}{3}\pi r^2(h_1 + h_2) \\ &= \frac{1}{3}\pi 4.8^2(10) \\ &= 76.8\pi \end{aligned}$$

The maximum possible volume is 128π cm^3.

Section Problem Solving   Page 409   Question 10

$$\begin{aligned} n &= 5^x + 5^x + 5^x + 5^x + 5^x \\ &= 5(5^x) \\ &= 5^{x+1} \\ n^5 &= \left(5^{x+1}\right)^5 \\ &= 5^{5x+5} \end{aligned}$$

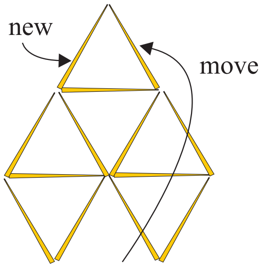
Section Problem Solving Page 409 Question 11

Since the product of five consecutive integers is 95 040, it is reasonable to assume that the middle factor is approximately $\sqrt[5]{95\,040}$. The result, $\sqrt[5]{95\,040} \doteq 9.8$, suggests the middle factor could be 10. Since n is the largest of the five integers, we test $n = 12$.

$$\begin{aligned}\textbf{L.S.} &= 12(12-1)(12-2)(12-3)(12-4)\\ &= 12(11)(10)(9)(8)\\ &= 95\,040\\ &= \textbf{R.S.}\end{aligned}$$

The value of n is 12.

Section Problem Solving Page 409 Question 12



Section Problem Solving Page 409 Question 13

There are three ways to write 81 as the sum of consecutive whole numbers.

$$\begin{aligned}81 &= 40 + 41\\ &= 26 + 27 + 28\\ &= 11 + 12 + 13 + 14 + 15 + 16\end{aligned}$$

Section Problem Solving Page 409 Question 14

Answers may vary.

Section Problem Solving Page 409 Question 15

Answers may vary.

Section Problem Solving Page 409 Question 16

Answers may vary.

Section Problem Solving Page 409 Question 17

Answers may vary.

Section Problem Solving Page 409 Question 18

Answers may vary.

Section Problem Solving Page 409 Question 19

Answers may vary.

Section Problem Solving Page 409 Question 20

Answers may vary.

CHAPTER 7 Exponential and Logarithmic Functions

7.1 Exponential Functions

Practise

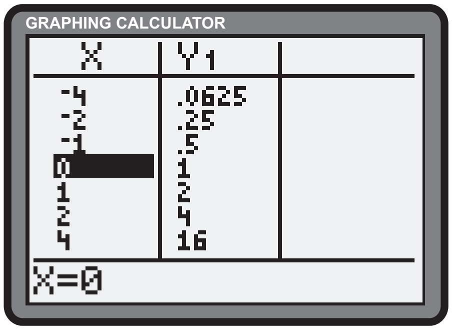
Section 7.1 Page 419 Question 1

a) All pass through $(0, 1)$ and have the x-axis as an asymptote as $x \to -\infty$. $y = 9^x$ approaches 0^+ faster than $y = 2^x$ as $x \to -\infty$ and increases more rapidly than $y = 2^x$ as $x \to \infty$. $y = 6^x$ lies between the other two functions.

b) All pass through $(0, 1)$ and have the x-axis as an asymptote as $x \to \infty$. $y = \left(\frac{1}{9}\right)^x$ approaches 0^+ faster than $y = \left(\frac{1}{2}\right)^x$ as $x \to \infty$ and increases more rapidly than $y = \left(\frac{1}{2}\right)^x$ as $x \to -\infty$. $y = \left(\frac{1}{6}\right)^x$ lies between the other two functions.

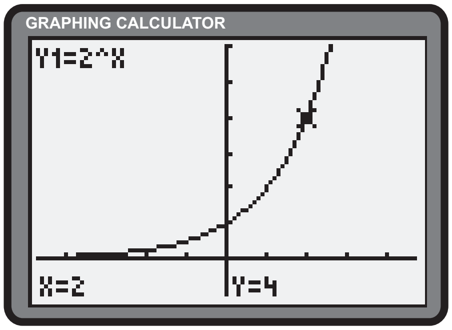
Section 7.1 Page 419 Question 3

a) The ordered pairs appearing below left are used to construct the graph of $y = 2^x$ appearing below right.



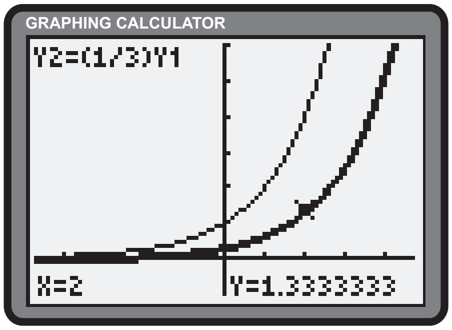
X	Y1
-4	.0625
-2	.25
-1	.5
0	1
1	2
2	4
4	16

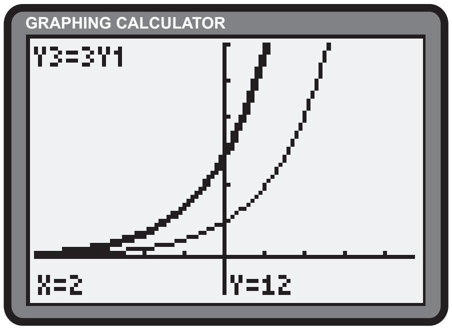
X=0



b) To obtain the graph of $g(x)$, apply a vertical compression of factor $\frac{1}{3}$ to the graph of $f(x)$. To obtain the graph of $h(x)$, apply a vertical stretch of factor 3 to the graph of $f(x)$.

c)





d) The domain of all three functions is $x \in (-\infty, \infty)$. The range is $y \in (0, \infty)$.

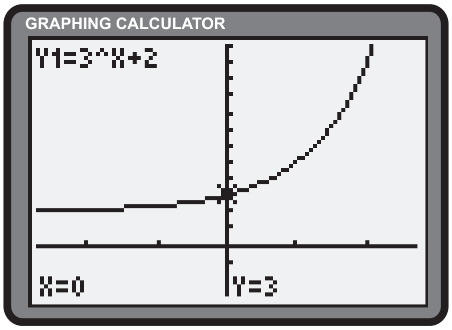
Section 7.1 Page 420 Question 5

a) i) y-intercept: 3

ii) domain: $\Re$; range: $y > 2$

iii) horizontal asymptote: $y = 2$

iv)

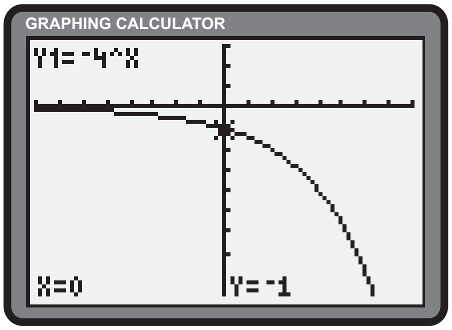


b) i) y-intercept: -1

ii) domain: $\Re$; range: $y < 0$

iii) horizontal asymptote: $y = 0$

iv)

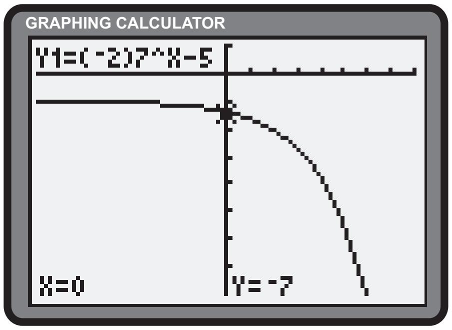


c) **i)** y-intercept: -7

ii) domain: $\Re$; range: $y < -5$

iii) horizontal asymptote: $y = -5$

iv)

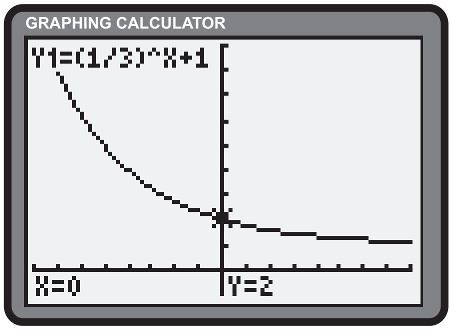


d) **i)** y-intercept: 2

ii) domain: $\Re$; range: $y > 1$

iii) horizontal asymptote: $y = 1$

iv)

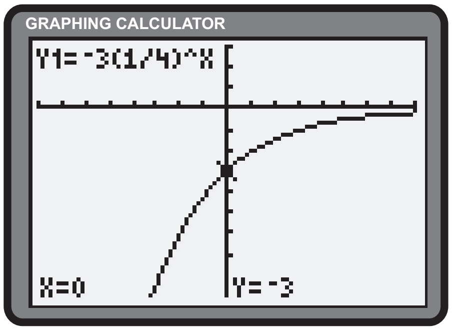


e) **i)** y-intercept: -3

ii) domain: $\Re$; range: $y < 0$

iii) horizontal asymptote: $y = 0$

iv)

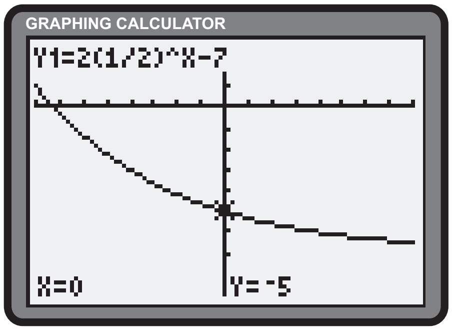


f) **i)** y-intercept: -5

ii) domain: $\Re$; range: $y > -7$

iii) horizontal asymptote: $y = -7$

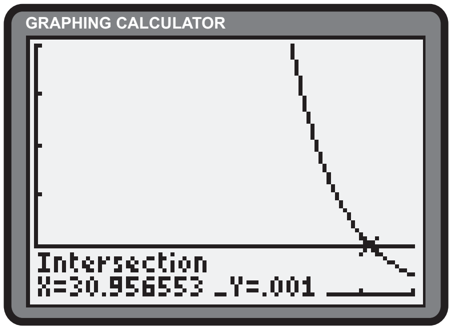
iv)



Apply, Solve, Communicate

Section 7.1 Page 420 Question 7

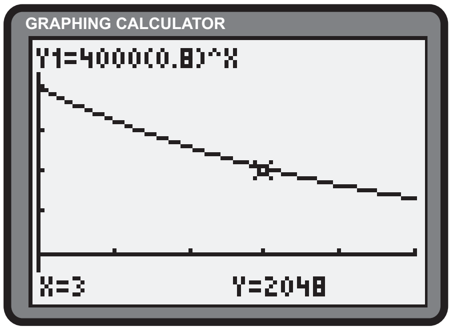
To solve the problem, the equation $0.8^d = 0.001$ must be solved. Using the **Intersect operation** of the graphing calculator, it is determined that 0.1% of the sunlight remains at a depth of approximately 31 m.



Section 7.1 Page 420 Question 8

a) A depreciation rate of 20% each year means that 80% of its value remains at the end of each year. The value, V, in dollars, of the computer system can be modelled by $V = 4000(0.8)^t$, where t is measured in years.

b)



c) The value of the computer system after 3 years is \$2048.

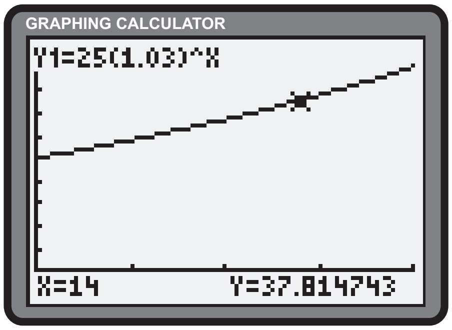
d)

$$\begin{aligned} V(t) &= 4000(0.8)^t \\ V(3) &= 4000(0.8)^3 \\ &= 4000(.512) \\ &= 2048 \end{aligned}$$

The value of \$2048 is confirmed.

Section 7.1 Page 420 Question 9

a)



```
Y1=25(1.03)^X
X=14          Y=37.814743
```

b) The year 2015 corresponds to $t = 14$.

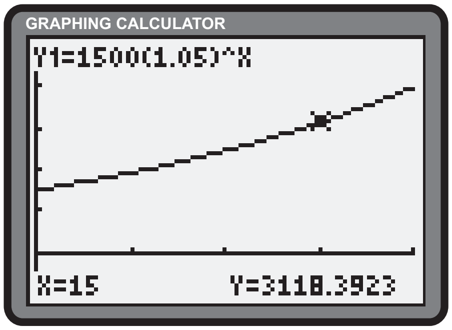
$$\begin{aligned} y(14) &= 25(1.03)^{14} \\ &\doteq 37.8 \end{aligned}$$

In 2015, the slide rule is estimated to be worth approximately \$38.

c) The **TRACE feature** of the graphing calculator suggests the slide rule will be worth \$50 in the 23rd year of ownership, or in the year 2024.

Section 7.1 Page 421 Question 10

a)



```
Y1=1500(1.05)^X
X=15          Y=3118.3923
```

b) Answers may vary. The **Value operation** of the graphing calculator suggests the value of the bond at maturity will be approximately \$3118.

Section 7.1 Page 421 Question 11

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{4^{x+h}-4^x}{h} \\ &= \frac{4^x(4^h-1)}{h} \\ &= 4^x\left(\frac{4^h-1}{h}\right) \end{aligned}$$

Section 7.1 Page 421 Question 12

a)

$$\begin{aligned} P(24) &= 1\ 000\ 000(2^{-\frac{24}{8}}) \\ &= 125\ 000 \end{aligned}$$

125 000 atoms remain after 24 days.

b)

$$\begin{aligned} P(80) &= 1\ 000\ 000(2^{-\frac{80}{8}}) \\ &\doteq 977 \end{aligned}$$

Approximately 977 atoms remain after 80 days.

c)

$$\begin{aligned} P(360) &= 1\ 000\ 000(2^{-\frac{360}{8}}) \\ &\doteq 0 \end{aligned}$$

No atoms remain after 360 days.

d) Answers will vary. It doesn't make sense because atoms cannot be split in half indefinitely.

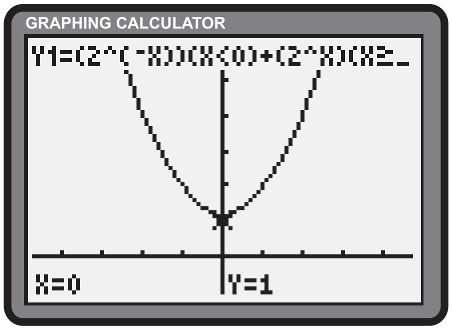
e) Answers will vary.

Section 7.1 Page 421 Question 13

Recall the definition $|x| = \begin{cases} -x & , x < 0 \\ x & , x \geq 0 \end{cases}$

a) $y = 2^{|x|} = \begin{cases} 2^{-x} & , x < 0 \\ 2^{x} & , x \geq 0 \end{cases}$

To graph $y = 2^{|x|}$, graph $y = 2^{-x}$ for $x < 0$ and $y = 2^{x}$ for $x \geq 0$.



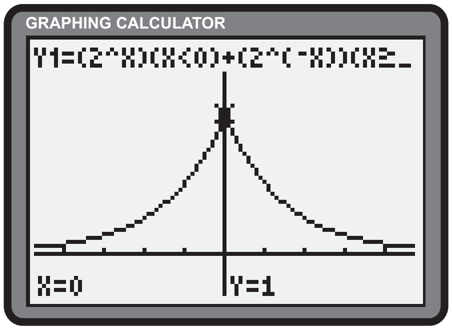
Y1=(2^(-X))(X<0)+(2^X)(X≥_

X=0 Y=1

domain: ℜ; range: $y \in [1, \infty)$

b) $y = 2^{-|x|} = \begin{cases} 2^{-(-x)} & , x < 0 \\ 2^{-(x)} & , x \geq 0 \end{cases} = \begin{cases} 2^{x} & , x < 0 \\ 2^{-x} & , x \geq 0 \end{cases}$

To graph $y = 2^{-|x|}$, graph $y = 2^{x}$ for $x < 0$ and $y = 2^{-x}$ for $x \geq 0$.



Y1=(2^X)(X<0)+(2^(-X))(X≥_

X=0 Y=1

domain: ℜ; range: $y \in (0, 1]$

Section 7.1 Page 421 Question 14

a)
$$\begin{aligned} \lim_{x\to\infty} 5^{|x|} &= \lim_{x\to\infty} 5^{x} \\ &= \infty \end{aligned}$$

$\lim_{x\to\infty} 5^{|x|}$ does not exist.

b)
$$\begin{aligned} \lim_{x\to-\infty} 5^{|x|} &= \lim_{x\to-\infty} 5^{-x} \\ &= \infty \end{aligned}$$

$\lim_{x\to-\infty} 5^{|x|}$ does not exist.

c)
$$\begin{aligned} \lim_{x\to\infty} 3^{-|x|} &= \lim_{x\to\infty} 3^{-x} \\ &= \lim_{x\to\infty} \frac{1}{3^{x}} \\ &= 0 \end{aligned}$$

$\lim_{x\to\infty} 3^{-|x|} = 0$

d)
$$\begin{aligned} \lim_{x\to-\infty} 3^{-|x|} &= \lim_{x\to-\infty} 3^{x} \\ &= 0 \end{aligned}$$

$\lim_{x\to-\infty} 3^{-|x|} = 0$

Section 7.1 Page 421 Question 15

a) $5^{\frac{3}{2}}$ can be expressed as $\left(\sqrt{5}\right)^{3}$.

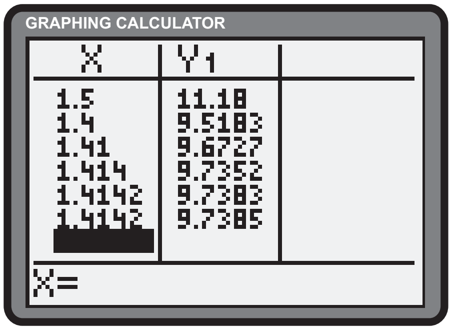
b) Since $1.4 = \frac{14}{10}$, $5^{1.4}$ can be expressed as $\left(\sqrt[10]{5}\right)^{14}$.

c) Since $1.41 = \frac{141}{100}$, $5^{1.41}$ can be expressed as $\left(\sqrt[100]{5}\right)^{141}$.

d) Since $1.414 = \frac{1414}{1000}$, $5^{1.414}$ can be expressed as $\left(\sqrt[1000]{5}\right)^{1414}$.

e) Since $1.4142 = \frac{14\,142}{10\,000}$, $5^{1.4142}$ can be expressed as $\left(\sqrt[10\,000]{5}\right)^{14\,142}$.

f) An irrational number can be thought of as the limit of a sequence of rational approximations. Hence, $5^{\sqrt{2}} = \lim_{x\to\sqrt{2}} 5^{x}$.



X	Y1
1.5	11.18
1.4	9.5183
1.41	9.6727
1.414	9.7352
1.4142	9.7383
1.4142	9.7385

X=

7.2 Logarithmic Functions

Practise

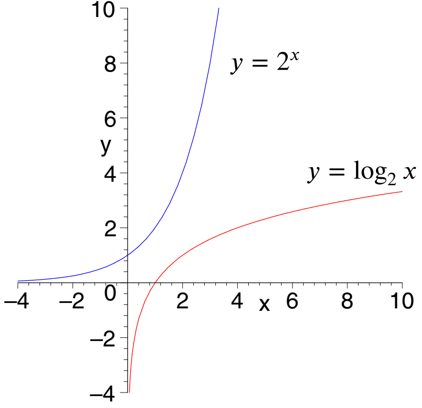
Section 7.2 Page 428 Question 1

Logarithmic Form	Exponential Form
$\log_{10} 100 = 2$	$10^2 = 100$
$\log_2 64 = 6$	$2^6 = 64$
$\log_2 8 = 3$	$2^3 = 8$
$\log_3 9 = 2$	$3^2 = 9$
$\log_5 \frac{1}{25} = -2$	$5^{-2} = \frac{1}{25}$
$\log_3 \frac{1}{9} = -2$	$3^{-2} = \frac{1}{9}$
$\log_7 1 = 0$	$7^0 = 1$
$\log_4 2 = \frac{1}{2}$	$4^{\frac{1}{2}} = 2$
$\log_4 0.125 = -\frac{3}{2}$	$4^{-\frac{3}{2}} = 0.125$
$\log_{36} 6 = \frac{1}{2}$	$36^{\frac{1}{2}} = 6$

Section 7.2 Page 428 Question 3

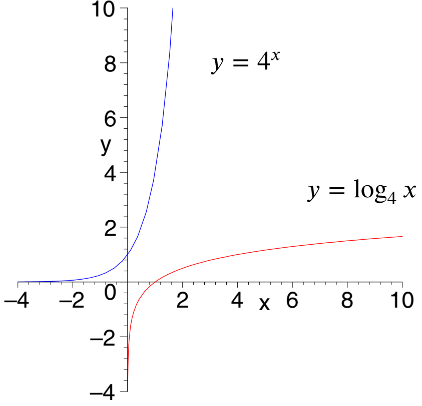
a) $y = 2^x$: domain: $\Re$; range: $y > 0$

$y = \log_2 x$: domain: $x > 0$; range: $\Re$



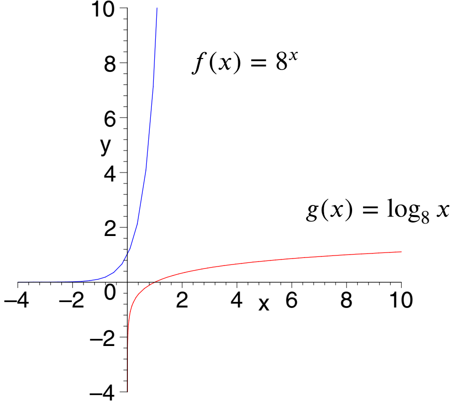
b) $y = 4^x$: domain: $\Re$; range: $y > 0$

$y = \log_4 x$: domain: $x > 0$; range: $\Re$



c) $f(x) = 8^x$: domain: $\Re$; range: $y > 0$

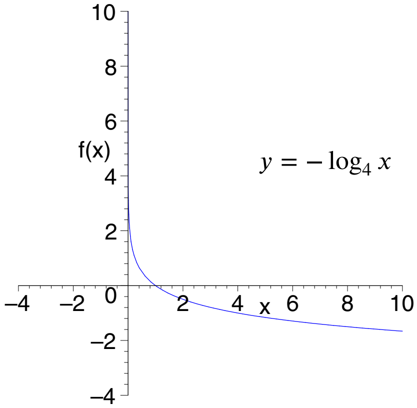
$g(x) = \log_8 x$: domain: $x > 0$; range: $\Re$



Section 7.2 Page 428 Question 5

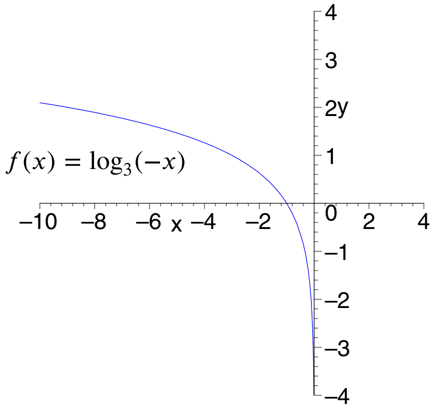
a) Reflect the graph of $y = \log_4 x$ in the x-axis.

domain: $x > 0$; range: $\Re$; vertical asymptote: $x = 0$



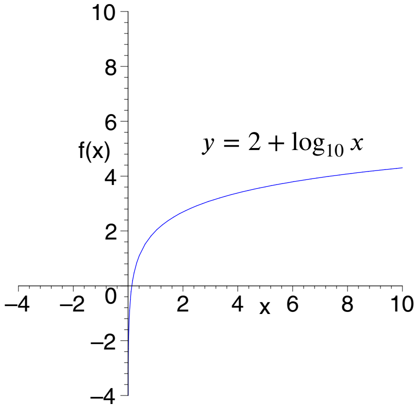
b) Reflect the graph of $y = \log_3 x$ in the y-axis.

domain: $x < 0$; range: $\Re$; vertical asymptote: $x = 0$



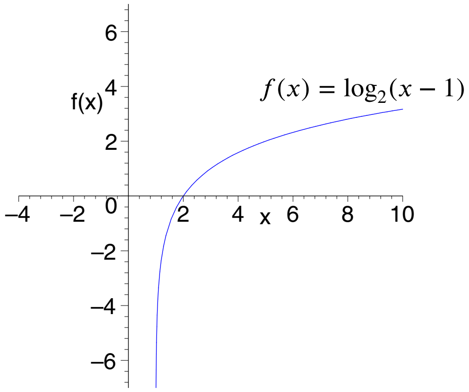
c) Translate the graph of $y = \log_{10} x$ upward 2 units.

domain: $x > 0$; range: $\Re$; vertical asymptote: $x = 0$



d) Translate $f(x) = \log_2 x$ to the right 1 unit.

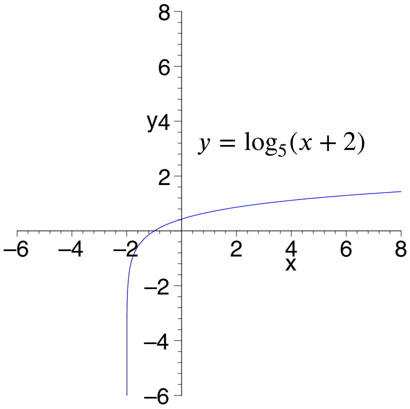
domain: $x > 1$; range: $\Re$; vertical asymptote: $x = 1$



e) Translate the graph of $y = \log_5 x$ to the left 2 units.

domain: $x > -2$; range: $\Re$;

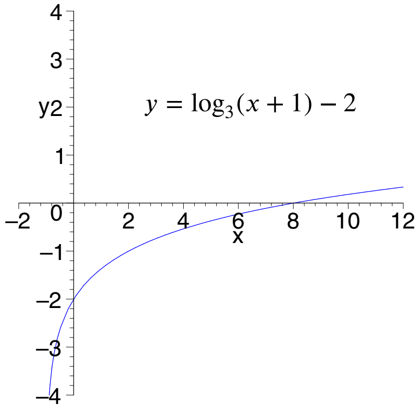
vertical asymptote: $x = -2$



f) Translate $y = \log_3 x$ left 1 unit and down 2 units.

domain: $x > -1$; range: $\Re$;

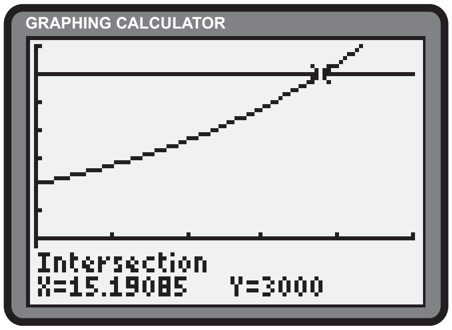
vertical asymptote: $x = -1$



Apply, Solve, Communicate

Section 7.2 Page 428 Question 7

The answer requires a solution to the equation $1000(1.075)^x = 3000$, where x is the time, in years. The **Intersect operation** of the graphing calculator yields an answer of approximately 15.2 years.

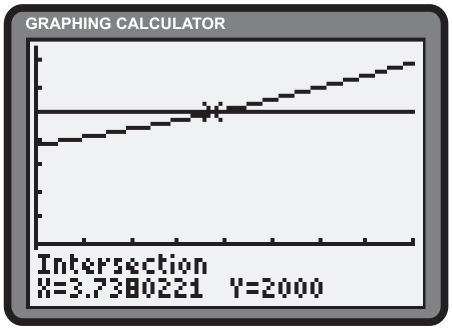


```
Intersection
X=15.19085   Y=3000
```

Section 7.2 Page 428 Question 8

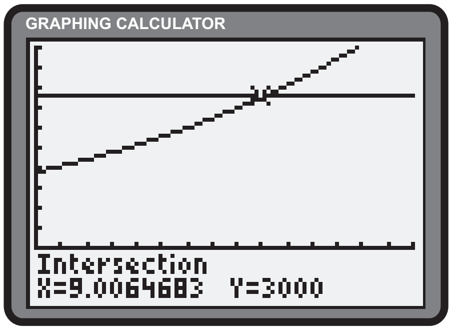
a) Let A be the accumulated amount after t years. $A = 1500(1.08)^t$.

i) The **Intersect operation** of the graphing calculator yields an answer of approximately 3.74 years.



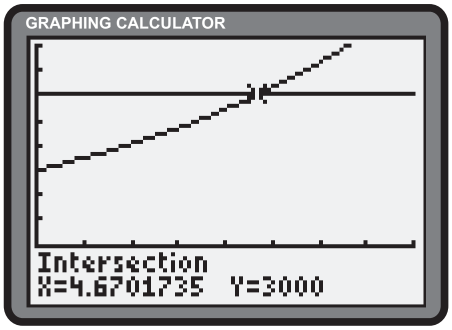
```
Intersection
X=3.7380221  Y=2000
```

ii) The **Intersect operation** of the graphing calculator yields an answer of approximately 9.01 years.



```
Intersection
X=9.0064683  Y=3000
```

b) If the interest rate is doubled, the doubling time of the investment decreases, since interest accumulates faster. The **Intersect operation** of the graphing calculator suggests it will take approximately 4.67 years.

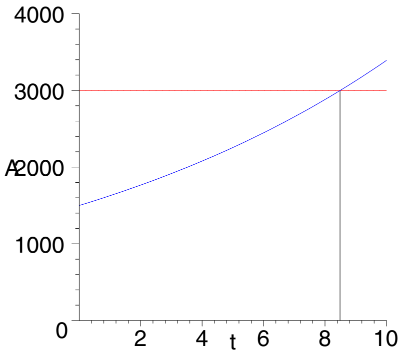


```
Intersection
X=4.6701735  Y=3000
```

Section 7.2 Page 428 Question 9

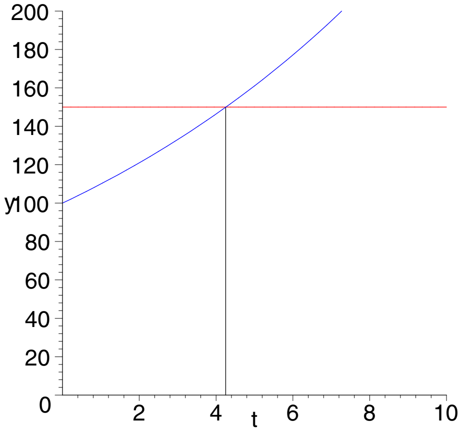
Let $A(t)$ be the amount of the investment after t years. The value of Wyatt's investment can be expressed as $A(t) = 1500(1 + 0.85)^t$. A graph suggests an intersection point with $y = 3000$ of approximately $(8.5, 3000)$.

Wyatt's age will be 18 + 8.5 or 26.5 years when the investment has doubled.

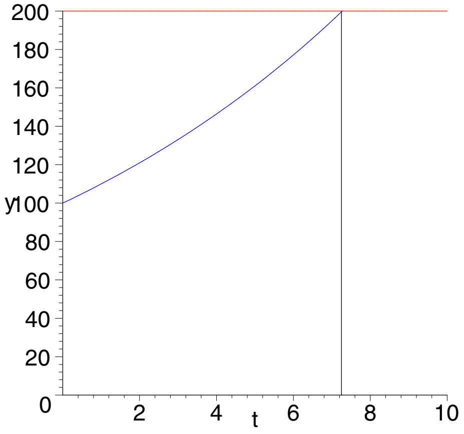


Section 7.2 Page 428 Question 10

a) i) Answers will vary. The population will increase to 150 in approximately 4.25 years.



ii) Answers will vary. The population will double in approximately 7.27 years.



b) i) The model suggests there will be $A(1000) = 100(1.1)^{1000}$ or approximately 2.47×10^{43} gophers in 1000 years. This is not realistic.

ii) Answers will vary.

iii) Answers will vary. Factors would include predators, food supply, and loss of habitat.

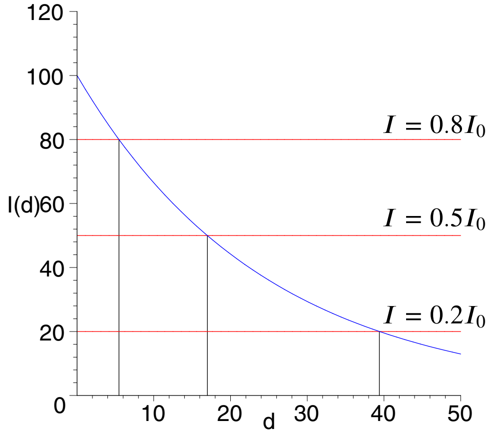
Section 7.2 Page 428 Question 11

Answers may vary.

a) The graph suggests the light will be reduced to $0.8I_0$ at a depth of approximately 5.47 m.

b) The graph suggests the light will be reduced to $0.5I_0$ at a depth of approximately 17.0 m.

c) The graph suggests the light will be reduced to $0.2I_0$ at a depth of approximately 39.4 m.



Section 7.2 Page 429 Question 12

This behaviour can be modelled by the function

$$I(d) = I_0(0.95)^d$$

where I is the intensity of light, in lumens, at a depth of d metres.

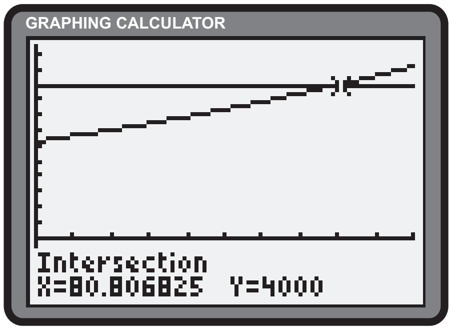
a) $I(4) = I_0(0.95)^4$ or approximately 81.45% of the original intensity remains at a depth of 4 m.

b) Trial and error reveals that $0.95^d = 0.40$ when d is approximately 17.86 m.

c) Answers may vary.

Section 7.2 Page 429 Question 13

The amount, A, of the investment can be modelled by the function $A(t) = 2500\left(1 + \frac{0.07}{12}\right)^t$, where t is in months. The graphing calculator suggests the investment will be worth $4000 in approximately 81 months or 6 years and 9 months.



Intersection
X=80.806825 Y=4000

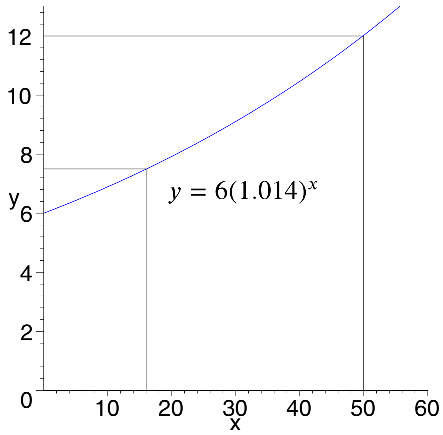
Section 7.2 Page 429 Question 14

a) The data suggest the population can be modelled by the function $y = 6(1.014)^x$, where y is measured in billions and x is measured in years.

c) Answers will vary. The graph suggests that in the year 2015 ($x = 2015 - 1999$ or 16), the population of the world will be approximately 7.5 billion.

d) The graph suggests it will take 50 years for the world's population to double. This event would occur in 2049.

b)

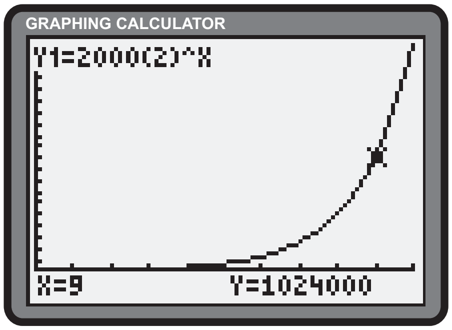


Section 7.2 Page 429 Question 15

The formula returns the number of digits in x.

Section 7.2 Page 429 Question 16

a)



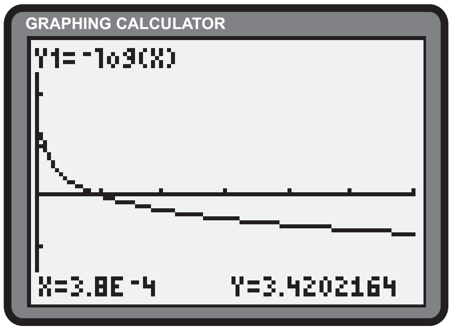
Y1=2000(2)^X

X=9 Y=1024000

b) The formula ceases to be valid when the culture's environment is unable to sustain its population.

Section 7.2 Page 429 Question 17

a)



Y1=⁻log(X)

X=3.8E⁻4 Y=3.4202164

b) The pH value is $-\log_{10} 0.000\,38$ or approximately 3.4202.

c) The logarithmic equation $-\log_{10} x = 7$ can be expressed as an exponential equation, yielding the result $x = 10^{-7}$.

Section 7.2 Page 429 Question 18

a) The valid domain of $\log_2 x$ is $x > 0$. The domain of $\log_{10} x$ that yields a range of $y > 0$ is $x > 1$. Hence, the domain of $f(x) = \log_2(\log_{10} x)$ is $x > 1$.

b)

$$f(x) = \log_2(\log_{10} x)$$

Replace $f(x)$ with y.

$$y = \log_2(\log_{10} x)$$

Express in exponential form.

$$2^y = \log_{10} x$$

Interchange x and y to formulate the inverse relation.

$$2^x = \log_{10} y$$

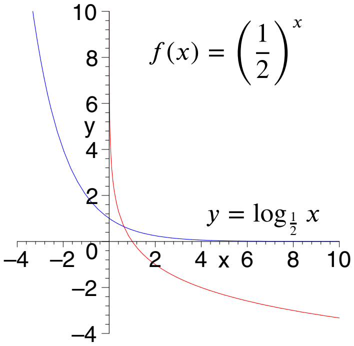
$$10^{2^x} = y$$

$$f^{-1}(x) = 10^{2^x}$$

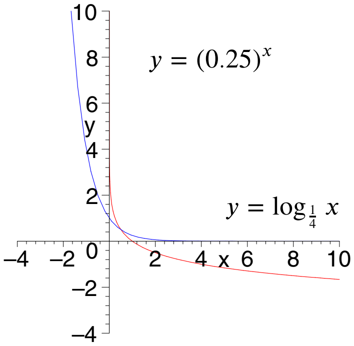
Section 7.2 Page 429 Question 19

Logarithms can have only positive bases: $y = \log_a x$ is defined for $a, x > 0$.

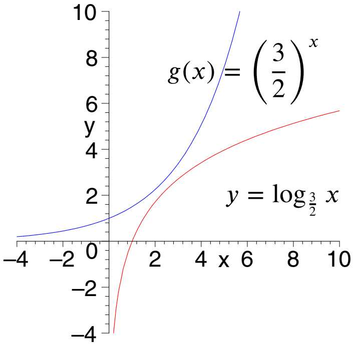
a)



b)



c)



The inverse functions are continuous.

7.3 Laws of Logarithms

Practise

Section 7.3 Page 434 Question 1

Single Logarithm	Sum or Difference of Logarithms
$\log_2(12 \times 5)$	$\log_2 12 + \log_2 5$
$\log_4 22$	$\log_4 2 + \log_4 11$
$\log_6(kg)$	$\log_6 k + \log_6 g$
$\log_8 \frac{14}{3}$	$\log_8 14 - \log_8 3$
$\log_{13} \frac{h^2}{f}$	$2\log_{13} h - \log_{13} f$
$\log_3 \frac{\pi}{5}$	$\log_3 \pi - \log_3 5$
$\log_{10} \frac{1}{7}$	$\log_{10} 1 - \log_{10} 7$
$\log_{11} x^8$	$2\log_{11} x + 6\log_{11} x$
$\log_{12} \frac{20}{3}$	$\log_{12} \frac{5}{3} + \log_{12} 4$

Section 7.3 Page 434 Question 3

a) $$\begin{aligned}\log_3 5 + \log_3 8 + \log_3 15 &= \log_3(5 \times 8 \times 15)\\ &= \log_3 600\end{aligned}$$

b) $$\begin{aligned}\log_4 8 - \log_4 10 + \log_4 3 &= \log_4 \frac{8 \times 3}{10}\\ &= \log_4 \frac{12}{5}\end{aligned}$$

c) $$\begin{aligned}\log_2 19 + \log_2 4 - \log_2 31 &= \log_2 \frac{19 \times 4}{31}\\ &= \log_2 \frac{76}{31}\end{aligned}$$

d) $$\begin{aligned}\frac{1}{2}\log 17 - \log 5 &= \log \sqrt{17} - \log 5\\ &= \log \frac{\sqrt{17}}{5}\end{aligned}$$

e) $$\log(a + b) + \log a^3 = \log \left[a^3(a + b)\right]$$

f) $$\log(x + y) - \log(x - y) = \log \frac{x + y}{x - y}$$

f) $$\begin{aligned}4\log x - 3\log y &= \log x^4 - \log y^3\\ &= \log \frac{x^4}{y^3}\end{aligned}$$

g) $$\log_3 ab + \log_3 bc = \log_3(ab^2c)$$

Section 7.3 Page 434 Question 5

a) $$\begin{aligned}\log_8 32 + \log_8 2 &= \log_8 64\\ &= 2\end{aligned}$$

b) $$\begin{aligned}\log_2 72 - \log_2 9 &= \log_2 8\\ &= 3\end{aligned}$$

c) $$\begin{aligned}\log_4 192 - \log_4 3 &= \log_4 64\\ &= 3\end{aligned}$$

d) $$\begin{aligned}\log_{12} 9 + \log_{12} 16 &= \log_{12} 144\\ &= 2\end{aligned}$$

e) $$\begin{aligned}\log_2 6 + \log_2 8 - \log_2 3 &= \log_2 16\\ &= 4\end{aligned}$$

f) $$\begin{aligned}\log_3 108 - \log_3 4 &= \log_3 27\\ &= 3\end{aligned}$$

g) $$\begin{aligned}\log_8 6 - \log_8 3 + \log_8 4 &= \log_8 8\\ &= 1\end{aligned}$$

h) $$\begin{aligned}\log_2 80 - \log_2 5 &= \log_2 16\\ &= 4\end{aligned}$$

i) $$\begin{aligned}\log 1.25 + \log 80 &= \log 100\\ &= 2\end{aligned}$$

j) $$\begin{aligned}\log_2 8^{27} &= 27\log_2 8\\ &= 81\end{aligned}$$

Apply, Solve, Communicate

Section 7.3 Page 435 Question 7

a)

$$\begin{aligned} d &\doteq -166.67 \log \frac{I(d)}{125} \\ -\frac{d}{166.67} &= \log \frac{I(d)}{125} \\ 10^{-\frac{d}{166.67}} &= \frac{I(d)}{125} \\ I(d) &= 125 \times 10^{-\frac{d}{166.67}} \end{aligned}$$

b)

$$\begin{aligned} d &\doteq -166.67 \log \frac{40}{125} \\ &\doteq 82.5 \end{aligned}$$

The approaching car is approximately 82.5 m away.

Section 7.3 Page 435 Question 8

a) i) $C_2 = 2C_1$

$$\begin{aligned} E &= 1.4 \log \frac{C_1}{2C_1} \\ &= 1.4 \log \frac{1}{2} \\ &\doteq -0.421 \end{aligned}$$

The energy required is −0.421 kilocalories per gram molecule.

ii) $C_2 = 3C_1$

$$\begin{aligned} E &= 1.4 \log \frac{C_1}{3C_1} \\ &= 1.4 \log \frac{1}{3} \\ &\doteq -0.668 \end{aligned}$$

The energy required is −0.668 kilocalories per gram molecule.

b) If $C_1 < C_2$, the sign of E is negative. A negative value for E means the cell *gains* energy.

Section 7.3 Page 435 Question 9

$\log_6 7 > \log_8 9$.

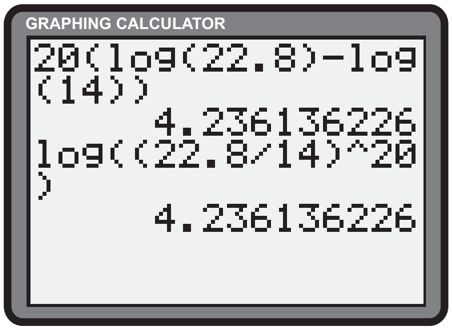
The change of base formula yields $\log_6 7 = \frac{\log 7}{\log 6}$ or approximately 1.086. Applying the formula to $\log_8 9$ yields $\frac{\log 9}{\log 8}$ or approximately 1.057.

Section 7.3 Page 435 Question 10

a)

$$\begin{aligned} A_v &= 20\,(\log V_o - \log V_i) \\ &= 20 \log \frac{V_o}{V_i} \\ &= \log \left(\frac{V_o}{V_i}\right)^{20} \end{aligned}$$

b) The gain is the same, approximately 4.24 V.



```
20(log(22.8)-log
(14))
        4.236136226
log((22.8/14)^20
)
        4.236136226
```

Section 7.3 Page 435 Question 11

a) $0.434\mu\theta = \log \frac{T_L}{T_S}$ can be rewritten as $0.434\mu\theta = \log T_L - \log T_S$.

b)

$$\begin{aligned} 0.434\mu\theta &= \log T_L - \log T_S \\ \mu &= \frac{\log T_L - \log T_S}{0.434\theta} \end{aligned}$$

Substitute the given data.

$$\begin{aligned} &= \frac{\log 250 - \log 200}{0.434\pi} \\ &\doteq 0.0711 \end{aligned}$$

The friction coefficient is approximately 0.071.

c) Wrapping the rope 2.5 times around the object corresponds to $\theta = 2.5 \times 2\pi$ or 5π.

$$\begin{aligned} 0.434\mu\theta &= \log \frac{T_L}{T_S} \\ 10^{0.434\mu\theta} &= \frac{T_L}{T_S} \\ T_S &= \frac{T_L}{10^{0.434\mu\theta}} \end{aligned}$$

Substitute the data.

$$\begin{aligned} T_S &= \frac{250}{10^{0.434(0.071)5\pi}} \\ &\doteq 82.0 \end{aligned}$$

A force of approximately 82.0 N is required to balance the 250 N force.

Section 7.3 Page 435 Question 12

$$\begin{aligned} (\log_b a)(\log_y b) &= c^2 \\ \left(\frac{\log a}{\log b}\right)\left(\frac{\log b}{\log y}\right) &= c^2 \\ \frac{\log a}{\log y} &= c^2 \\ \frac{\log y}{\log a} &= c^{-2} \\ \log_a y &= c^{-2} \end{aligned}$$

Section 7.3 Page 435 Question 13

$$\begin{aligned} \log_a \frac{p}{q} &= \log_a \left(pq^{-1}\right) \\ &= \log_a p + \log_a q^{-1} \\ &= \log_a p - \log_a q \end{aligned}$$

Section 7.3 Page 435 Question 14

$$\begin{aligned} \log_a (p^c) &= \log_a \left(\underbrace{p \times p \times p \times \ldots \times p}_{c \text{ times}} \right) \\ &= \underbrace{\log_a p + \log_a p + \ldots + \log_a p}_{c \text{ times}} \\ &= c \log_a p \end{aligned}$$

Section 7.3 Page 435 Question 15

$$\begin{aligned} y &= \log_a x \\ a^y &= x \\ \log_b a^y &= \log_b x \\ y \log_b a &= \log_b x \\ y &= \frac{\log_b x}{\log_b a} \end{aligned}$$

Section 7.3 Page 435 Question 16

The error is in the opening. Since $\log_3 0.1 < 0$, the statement should be written $\log 0.1 > 2 \log_3 0.1$.

7.4 Exponential and Logarithmic Equations

Practise

Section 7.4 Page 441 Question 1

Since $4^{2(0.6)} \doteq 5.28$, 0.6 is not a root of the equation $4^{2x} = 5$.

Section 7.4 Page 441 Question 3

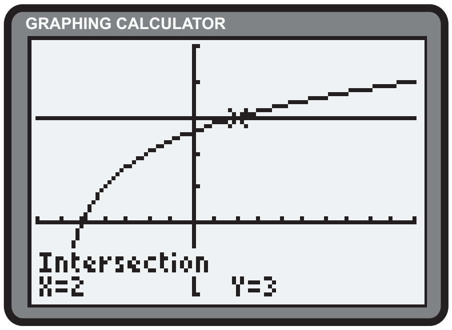
a) i)

$$\begin{aligned} \log_2(x+6) &= 3 \\ x+6 &= 2^3 \\ x+6 &= 8 \\ x &= 2 \end{aligned}$$

ii)

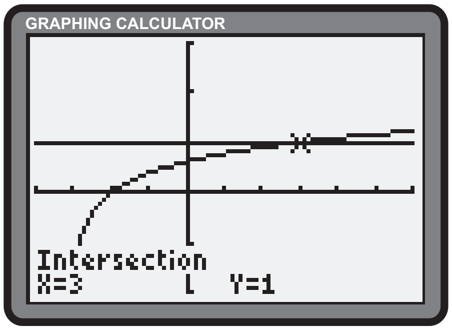
$$\begin{aligned} \log_6(x+3) &= 1 \\ x+3 &= 6^1 \\ x+3 &= 6 \\ x &= 3 \end{aligned}$$

b) i)



Intersection
X=2 Y=3

ii)



Intersection
X=3 Y=1

Section 7.4 Page 441 Question 5

a)

$$\begin{aligned} 4\log_5 x &= \log_5 625 \\ 4\log_5 x &= 4 \\ \log_5 x &= 1 \\ x &= 5 \end{aligned}$$

Check.

$$\begin{aligned} \textbf{L.S.} &= 4\log_5 5 \\ &= \log_5 5^4 \\ &= \log_5 625 \\ &= \textbf{R.S.} \end{aligned}$$

b)

$$\begin{aligned} -\log_3 1 &= \log_3 7 - \log_3 x \\ 0 &= \log_3 \frac{7}{x} \\ \frac{7}{x} &= 1 \\ x &= 7 \end{aligned}$$

Check.

$$\begin{aligned} \textbf{L.S.} &= -\log_3 1 \\ &= 0 \\ \textbf{R.S.} &= \log_3 7 - \log_3 7 \\ &= 0 \\ &= \textbf{L.S.} \end{aligned}$$

c)

$$\begin{aligned} \log_6 n &= \frac{3}{4}\log_6 16 \\ \log_6 n &= \log_6 16^{\frac{3}{4}} \\ \log_6 n &= \log_6 8 \\ n &= 8 \end{aligned}$$

Check.

$$\begin{aligned} \textbf{L.S.} &= \log_6 8 \\ \textbf{R.S.} &= \frac{3}{4}\log_6 16 \\ &= \log_6 16^{\frac{3}{4}} \\ &= \log_6 8 \\ &= \textbf{L.S.} \end{aligned}$$

d)

$$\begin{aligned} -\log_2 x - \log_2 3 &= \log_2 12 \\ \log_2 \frac{1}{3x} &= \log_2 12 \\ \frac{1}{3x} &= 12 \\ x &= \frac{1}{36} \end{aligned}$$

Check.

$$\begin{aligned} \textbf{L.S.} &= -\log_2 \frac{1}{36} - \log_2 3 \\ &= \log_2 \frac{36}{3} \\ &= \log_2 12 \\ \textbf{R.S.} &= \log_2 12 \\ &= \textbf{L.S.} \end{aligned}$$

e)

$$\begin{aligned}\log 12 &= \log 8 - \log x\\ \log 12 &= \log \frac{8}{x}\\ 12 &= \frac{8}{x}\\ x &= \frac{2}{3}\end{aligned}$$

Check.

$$\begin{aligned}\textbf{L.S.} &= \log 12\\ \textbf{R.S.} &= \log 8 - \log \frac{2}{3}\\ &= \log \left(8 \div \frac{2}{3}\right)\\ &= \log 12\\ &= \textbf{L.S.}\end{aligned}$$

f)

$$\begin{aligned}\log 2^{3x} &= \log 35\\ x \log 8 &= \log 35\\ x &= \frac{\log 35}{\log 8}\\ x &\doteq 1.7098\end{aligned}$$

Check.

$$\begin{aligned}\textbf{L.S.} &= \log \left[2^{\frac{3\log 35}{\log 8}}\right]\\ &= \log \left[2^{\frac{3\log 35}{3\log 2}}\right]\\ &= \log \left(2^{\log_2 35}\right)\\ &= \log 35\\ \textbf{R.S.} &= \log 35\\ &= \textbf{L.S.}\end{aligned}$$

g)

$$\begin{aligned}4\log_6 x &= \log_6 25\\ \log_6 x^4 &= \log_6 25\\ x^4 &= 25\\ x^2 &= 5\\ x &= \sqrt{5},\ x > 0\end{aligned}$$

Check.

$$\begin{aligned}\textbf{L.S.} &= 4\log \sqrt{5}\\ &= \log \left(\sqrt{5}\right)^4\\ &= \log 25\\ &= \textbf{R.S.}\end{aligned}$$

h)

$$\begin{aligned}2\log_7 x &= \log_7 81\\ \log_7 x^2 &= \log_7 81\\ x^2 &= 81\\ x &= 9,\ x > 0\end{aligned}$$

Check.

$$\begin{aligned}\textbf{L.S.} &= 2\log_7 9\\ &= \log_7 81\\ &= \textbf{R.S.}\end{aligned}$$

Section 7.4 Page 441 Question 7

a)

$$\begin{aligned}\textbf{L.S.} &= 5^{2\log_5 4} + 5^{\log_5 4} - 20\\ &= 5^{\log_5 16} + 5^{\log_5 4} - 20\\ &= 16 + 4 - 20\\ &= 0\\ &= \textbf{R.S.}\end{aligned}$$

b)

$$\begin{aligned}5^{2x} + 5^x - 20 &= 0\\ (5^x)^2 + (5^x) - 20 &= 0\\ (5^x - 4)(5^x + 5) &= 0\\ 5^x - 4 &= 0\\ 5^x &= 4\\ x &= \log_5 4\\ 5^x + 5 &= 0\end{aligned}$$

There is no solution to $5^x + 5 = 0$, so there are no other roots.

Apply, Solve, Communicate

Section 7.4 Page 442 Question 9

a)

$$\begin{aligned} A &= 1226.23 \\ 1000(1.06)^n &= 1226.23 \\ (1.06)^n &= 1.226\ 23 \\ n\log 1.06 &= \log 1.226\ 23 \\ n &= \frac{\log 1.226\ 23}{\log 1.06} \\ &\doteq 3.5 \end{aligned}$$

It will take approximately 3.5 years.

b)

$$\begin{aligned} A &= 1664.08 \\ 1000(1.06)^n &= 1664.08 \\ (1.06)^n &= 1.664\ 08 \\ n\log 1.06 &= \log 1.664\ 08 \\ n &= \frac{\log 1.664\ 08}{\log 1.06} \\ &\doteq 8.74 \end{aligned}$$

It will take approximately 8.74 years.

c)

$$\begin{aligned} A &= 2000 \\ 1000(1.06)^n &= 2000 \\ (1.06)^n &= 2000 \\ n\log 1.06 &= \log 2 \\ n &= \frac{\log 2}{\log 1.06} \\ &\doteq 11.90 \end{aligned}$$

It will take approximately 11.9 years.

d)

$$\begin{aligned} A &= 5000 \\ 1000(1.06)^n &= 5000 \\ (1.06)^n &= 5 \\ n\log 1.06 &= \log 5 \\ n &= \frac{\log 5}{\log 1.06} \\ &\doteq 27.62 \end{aligned}$$

It will take approximately 27.62 years.

Section 7.4 Page 442 Question 10

a)

$$\begin{aligned} I(t) &= 0.5I_0 \\ I_0(0.97)^x &= 0.5I_0 \\ (0.97)^x &= 0.5 \\ x &= \log_{0.97} 0.5 \\ &= \frac{\log 0.5}{\log 0.97} \\ &\doteq 22.8 \end{aligned}$$

A thickness of approximately 22.8 cm is required.

b) If the thickness of the glass were doubled, the intensity of the light would be reduced by the same factor, yet again.

Section 7.4 Page 442 Question 11

a)

$$\begin{aligned} S &= 1.5(25\ 000) \\ 25\ 000(1.05)^n &= 1.5(25\ 000) \\ (1.05)^n &= 1.5 \\ n &= \log_{1.05} 1.5 \\ &= \frac{\log 1.5}{\log 1.05} \\ &\doteq 8.31 \end{aligned}$$

It would take approximately 8.31 years for the salary to increase by 50%.

b)

$$\begin{aligned} S &= 1.5(35\ 000) \\ 35\ 000(1.05)^n &= 1.5(35\ 000) \\ (1.05)^n &= 1.5 \\ n &= \log_{1.05} 1.5 \\ &= \frac{\log 1.5}{\log 1.05} \\ &\doteq 8.31 \end{aligned}$$

It would take approximately 8.31 years for the salary to increase by 50%. (The length of time is independent of the starting salary.)

Section 7.4 Page 442 Question 12

The value of this investment, A, can be expressed as $A = 600(1.055)^n$, where n is measured in years.

a)

$$\begin{aligned} A &= 1200 \\ 600(1.055)^n &= 1200 \\ 1.055^n &= 2 \\ n \log 1.055 &= \log 2 \\ n &= \frac{\log 2}{\log 1.055} \\ &\doteq 12.95 \end{aligned}$$

It will take approximately 12.95 years or 12 years 11 months for the investment to double.

b)

$$\begin{aligned} A &= 1800 \\ 600(1.055)^n &= 1800 \\ 1.055^n &= 3 \\ n \log 1.055 &= \log 3 \\ n &= \frac{\log 3}{\log 1.055} \\ &\doteq 20.52 \end{aligned}$$

It will take approximately 20.52 years or 20 years 6 months for the investment to triple.

c)

$$\begin{aligned} A &= 900 \\ 600(1.055)^n &= 900 \\ 1.055^n &= 1.5 \\ n \log 1.055 &= \log 1.5 \\ n &= \frac{\log 1.5}{\log 1.055} \\ &\doteq 7.57 \end{aligned}$$

It will take approximately 7.57 years or 7 years 7 months for the investment to accumulate to \$900.

Section 7.4 Page 442 Question 13

$$\begin{aligned} v &= 13 \\ 65(10)^{-0.23t} &= 13 \\ (10)^{-0.23t} &= 0.2 \\ -0.23t &= \log 0.2 \\ t &= \frac{\log 0.2}{-0.23} \\ &\doteq 3.04 \end{aligned}$$

It will take the skier approximately 3.04 s to slow to a speed of 13 km/h.

Section 7.4 Page 442 Question 14

a)

$$\begin{aligned} N &= 32 \\ 40 - 24(0.74)^t &= 32 \\ 24(0.74)^t &= 8 \\ (0.74)^t &= \frac{1}{3} \\ t &= \log_{0.74} \frac{1}{3} \\ &= \frac{\log \frac{1}{3}}{\log 0.74} \\ &\doteq 3.65 \end{aligned}$$

It will take the trainee approximately 3.65 days.

b) After 15 days, the trainee can inspect $40 - 24(0.74)^{15}$ or approximately 39.74 items. This figure is $45 - 39.74$ or 5.26 items less than the experienced employee.

Section 7.4 Page 442 Question 15

a)

$$\begin{aligned} H(0) &= 140(10)^{-0.034(0)} \\ &= 140(10)^0 \\ &= 140(1) \\ &= 140 \end{aligned}$$

At 0°C, the cheese will stay safe for 140 h.

b)

$$\begin{aligned} H(16) &= 140(10)^{-0.034(16)} \\ &\doteq 40.0 \end{aligned}$$

At 16°C, the cheese will stay safe for approximately 40 h.

c)

$$\begin{aligned} H(25) &= 140(10)^{-0.034(25)} \\ &\doteq 19.78 \end{aligned}$$

At 25°C, the cheese will stay safe for approximately 20 h.

Section 7.4 Page 442 Question 16

$$\begin{aligned} P &= 95 \\ 101.3(1.133)^{-x} &= 95 \\ (1.133)^{-x} &= \frac{95}{101.3} \\ -x &= \log_{1.133}\left(\frac{95}{101.3}\right) \\ x &= -\log_{1.133}\left(\frac{95}{101.3}\right) \\ &= -\frac{\log\left(\frac{95}{101.3}\right)}{\log 1.133} \\ &\doteq 0.514 \end{aligned}$$

The mountain climber is approximately 0.514 km above sea level.

Section 7.4 Page 442 Question 17

At sea level, the atmospheric pressure is $101.3(1.133)^0$ or 101.3 kPa. At an altitude of 10 km, the atmospheric pressure is $101.3(1.133)^{-10}$ or approximately 29.06 kPa. The designers must control the range from 29.06 kPa to 101.3 kPa.

Section 7.4 Page 442 Question 18

$$\begin{aligned}
I(d) &= I_0(1-0.046)^d \\
\frac{I(d)}{I_0} &= (0.954)^d \\
\log\frac{I(d)}{I_0} &= \log(0.954)^d \\
\log\frac{I(d)}{I_0} &= d \times \log(0.954) \\
d &= \frac{1}{\log 0.954} \times \log\frac{I(d)}{I_0} \\
&\doteq -48.90 \log\frac{I(d)}{I_0}
\end{aligned}$$

The depth can be expressed as $d \doteq -48.9 \times \log\frac{I(d)}{I_0}$ metres.

Section 7.4 Page 442 Question 19

a) Since 4096 is the lowest common power of bases 2, 4, 8, and 16, use the change of base formula in the base 4096.

$$\begin{aligned}
\log_2 x + \log_4 x + \log_8 x + \log 16x &= 25 \\
\frac{\log_{4096} x}{\log_{4096} 2} + \frac{\log_{4096} x}{\log_{4096} 4} + \frac{\log_{4096} x}{\log_{4096} 8} + \frac{\log_{4096} x}{\log_{4096} 16} &= 25 \\
\log_{4096} x\left(\frac{1}{\log_{4096} 2} + \frac{1}{\log_{4096} 4} + \frac{1}{\log_{4096} 8} + \frac{1}{\log_{4096} 16}\right) &= 25 \\
\log_{4096} x\left(\frac{1}{\frac{1}{12}} + \frac{1}{\frac{1}{6}} + \frac{1}{\frac{1}{4}} + \frac{1}{\frac{1}{3}}\right) &= 25 \\
\log_{4096} x\,(12+6+4+3) &= 25 \\
25\log_{4096} x &= 25 \\
\log_{4096} x &= 1 \\
x &= 4096
\end{aligned}$$

Check.

$$\begin{aligned}
\textbf{L.S.} &= \log_2 4096 + \log_4 4096 + \log_8 4096 + \log_{16} 4096 \\
&= 12+6+4+3 \\
&= 25 \\
&= \textbf{R.S.}
\end{aligned}$$

b)

$$2\left(5^{6x}\right)-9\left(5^{4x}\right)+13\left(5^{2x}\right)-6=0$$

$$2\left(5^{2x}\right)^3-9\left(5^{2x}\right)^2+13\left(5^{2x}\right)-6=0 \qquad (1)$$

Substitute $z=5^{2x}$ in (1).

$$2z^3-9z^2+13z-6=0 \qquad (2)$$

$$\begin{aligned}2(1)^3-9(1)^2+13(1)-6&=2-9+13-6\\&=0\end{aligned}$$

Since 1 is a root of (2), $z-1$ is a factor. Division reveals the other factors.

$$\begin{aligned}(z-1)(2z^2-7z+6)&=0\\(z-1)(2z-3)(z-2)&=0\\z&=1,\frac{3}{2},\text{ or }2\end{aligned}$$

Substitute each of the roots in (2) into $z=5^{2x}$ and solve for x.

$$\begin{aligned}5^{2x}&=1\\2x&=0\\x&=0 &(3)\\5^{2x}&=\frac{3}{2}\\2x&=\frac{\log\frac{3}{2}}{\log 5}\\x&=\frac{\log\frac{3}{2}}{2\log 5}\\&\doteq 0.1260 &(4)\\5^{2x}&=2\\2x&=\frac{\log 2}{\log 5}\\x&=\frac{\log 2}{2\log 5}\\&\doteq 0.2153 &(5)\end{aligned}$$

The solutions are 0, 0.1260, and 0.2153. Check $x=0$.

$$\begin{aligned}\textbf{L.S.}&=2\left(5^{6(0)}\right)-9\left(5^{4(0)}\right)+13\left(5^{2(0)}\right)-6\\&=2-9+13-6\\&=0\\&=\textbf{R.S.}\end{aligned}$$

Check $x=0.1260$.

$$\begin{aligned}\textbf{L.S.}&=2\left(5^{6(0.1260)}\right)-9\left(5^{4(0.1260)}\right)+13\left(5^{2(0.1260)}\right)-6\\&\doteq 6.7588-20.2676+19.5085-6\\&\doteq 0\\&=\textbf{R.S.}\end{aligned}$$

Check $x=0.2153$.

$$\begin{aligned}\textbf{L.S.}&=2\left(5^{6(0.2153)}\right)-9\left(5^{4(0.2153)}\right)+13\left(5^{2(0.2153)}\right)-6\\&\doteq 15.9941-35.9911+25.9968-6\\&\doteq 0\\&=\textbf{R.S.}\end{aligned}$$

7.5 Logarithmic Scales

Apply, Solve, Communicate

Section 7.5 Page 446 Question 1

For each of the following, let I_1 be the intensity of Japan's earthquake and let M be the desired magnitude.

a)

$$\begin{aligned} M &= \log \frac{I}{I_0} \\ &= \log \frac{2I_1}{I_0} \\ &= \log 2 + \log \frac{I_1}{I_0} \\ &= \log 2 + 7.2 \\ &\doteq 7.5 \end{aligned}$$

An earthquake twice as intense as Japan's earthquake has a magnitude of 7.5.

b)

$$\begin{aligned} M &= \log \frac{I}{I_0} \\ &= \log \frac{1.5I_1}{I_0} \\ &= \log 1.5 + \log \frac{I_1}{I_0} \\ &= \log 1.5 + 7.2 \\ &\doteq 7.4 \end{aligned}$$

An earthquake 1.5 times as intense as Japan's earthquake has a magnitude of 7.4.

c)

$$\begin{aligned} M &= \log \frac{I}{I_0} \\ &= \log \frac{3I_1}{I_0} \\ &= \log 3 + \log \frac{I_1}{I_0} \\ &= \log 3 + 7.2 \\ &\doteq 7.7 \end{aligned}$$

An earthquake 3 times as intense as Japan's earthquake has a magnitude of 7.7.

Section 7.5 Page 446 Question 2

$$\begin{aligned} 7.45 > \text{pH} &> 7.35 \\ -7.45 < -\text{pH} &< -7.35 \\ 10^{-7.45} < 10^{-\text{pH}} &< 10^{-7.35} \\ 3.548 \times 10^{-8} < [\text{H}^+] &< 4.467 \times 10^{-8} \end{aligned}$$

The corresponding range is 3.548×10^{-8} to 4.467×10^{-8}.

Section 7.5 Page 446 Question 4

Let I_1 be the intensity of the Mackenzie Region's earthquake. Let M be the magnitude of the earthquake on Vancouver Island.

$$\begin{aligned} M &= \log \frac{I}{I_0} \\ &= \log \frac{2.5I_1}{I_0} \\ &= \log 2.5 + \log \frac{I_1}{I_0} \\ &= \log 2.5 + 6.9 \\ &\doteq 7.3 \end{aligned}$$

The magnitude of the earthquake on Vancouver Island was 7.3. This was strong enough to cause metal buildings to collapse.

Section 7.5 Page 446 Question 3

a) Let L_f and L_s be the loudness of the first and second sounds, respectively. Let I_f and I_s be the intensity of the first and second sounds, respectively.

$$\begin{aligned} L_f - L_s &= 10 \\ 10 \log \frac{I_f}{I_0} - 10 \log \frac{I_s}{I_0} &= 10 \\ \log \frac{I_f}{I_0} - \log \frac{I_s}{I_0} &= 1 \\ \log \left[\frac{I_f}{I_0} \div \frac{I_s}{I_0} \right] &= 1 \\ \log \frac{I_f}{I_s} &= 1 \\ \frac{I_f}{I_s} &= 10 \\ I_f &= 10I_s \end{aligned}$$

The first sound is 10 times as intense as the second sound.

b) Let I_1 be the sound intensity level of the hair dryer and I_2 be the sound intensity level of the air conditioner.

$$\begin{aligned} 10 \log \frac{I_1}{I_0} &= 70 \\ 10 \log \frac{I_2}{I_0} &= 50 \\ 10 \left[\log \frac{I_1}{I_0} - \log \frac{I_2}{I_0} \right] &= 20 \\ 10 \log \frac{I_1}{I_2} &= 20 \\ \log \frac{I_1}{I_2} &= 2 \\ I_1 &= 100I_2 \end{aligned}$$

The sound of the hair dryer is 100 times as intense as the sound of the air conditioner.

Section 7.5 Page 447 Question 5

a) For the sun, $L = L_0$.

$$\begin{aligned} M &= 4.72 - \log \frac{L_0}{L_0} \\ &= 4.72 - \log 1 \\ &= 4.72 \end{aligned}$$

The absolute magnitude of the sun is 4.72.

b)

$$\begin{aligned} M &= 4.72 - \log \frac{L}{L_0} \\ 1.41 &= 4.72 - \log \frac{L}{L_0} \\ 3.31 &= \log \frac{L}{L_0} \\ \frac{L}{L_0} &= 10^{3.31} \\ L &= 10^{3.31} L_0 \\ &\doteq 2042 L_0 \end{aligned}$$

Sirius is more luminous than the sun by a factor of approximately 2042.

c)

$$\begin{aligned} M &= 4.72 - \log \frac{L}{L_0} \\ -4.7 &= 4.72 - \log \frac{L}{L_0} \\ 9.42 &= \log \frac{L}{L_0} \\ \frac{L}{L_0} &= 10^{9.42} \\ L &= 10^{9.42} L_0 \\ &\doteq 2.63 \times 10^9 L_0 \end{aligned}$$

Canopus is more luminous than the sun by a factor of approximately 2.63×10^9.

d)

$$\begin{aligned} M &= 4.72 - \log \frac{L}{L_0} \\ M &= 4.72 - \log \frac{10^{38}}{4 \times 10^{26}} \\ &\doteq -6.68 \end{aligned}$$

The absolute magnitude of the quasar is approximately −6.68.

Section 7.5 Page 447 Question 6

a)

$$\begin{aligned} h &= 18\,400 \log \frac{P_0}{P} \\ h &= 18\,400 \log \frac{102}{32.5} \\ &\doteq 9140 \end{aligned}$$

The height of the aircraft is approximately 9140 m.

c) Answers may vary.

b)

$$\begin{aligned} h &= 18\,400 \log \frac{P_0}{P} \\ h &= 18\,400 \log \frac{102}{0.5(102)} \\ &\doteq 5539 \end{aligned}$$

The height of the aircraft would need to be approximately 5539 m for the air pressure to be half of the air pressure at ground level.

Section 7.5 Page 447 Question 7

a) Use a figure of 8850 m for the height of Mount Everest.

$$\begin{aligned} h &= 18\,400 \log \frac{P_0}{P} \\ 8850 &= 18\,400 \log \frac{102}{P} \\ \frac{8850}{18\,400} &= \log \frac{102}{P} \\ 10^{\frac{8850}{18\,400}} &= \frac{102}{P} \\ P &= \frac{102}{10^{\frac{8850}{18\,400}}} \\ &\doteq 33.7 \end{aligned}$$

The air pressure at the top of Mount Everest is approximately 33.7 kPa.

b) Answers may vary.

c) Answers may vary.

Section 7.5 Page 447 Question 8

a)

$$\begin{aligned}\text{shade \#} &= \frac{-7\log T}{3}+1\\ 14 &= \frac{-7\log T}{3}+1\\ \frac{3(14-1)}{-7} &= \log T\\ T &= 10^{-\frac{39}{7}}\\ &\doteq 2.68\times 10^{-6}\end{aligned}$$

The fraction of light that #14 welding glasses transmit is approximately 2.68×10^{-6}.

b)

$$\begin{aligned}\text{shade \#} &= \frac{-7\log T}{3}+1\\ 2 &= \frac{-7\log T}{3}+1\\ \frac{3(2-1)}{-7} &= \log T\\ T &= 10^{-\frac{3}{7}}\\ &\doteq 0.373\end{aligned}$$

The fraction of light that #2 welding glasses transmit is approximately 0.373.

c) Let R be the required ratio.

$$\begin{aligned}R &= \frac{10^{-\frac{3}{7}}}{10^{-\frac{39}{7}}}\\ &\doteq 139\,000\end{aligned}$$

#2 glasses transmit approximately 139 000 times as much visible light as #14 glasses do.

Section 7.5 Page 447 Question 9

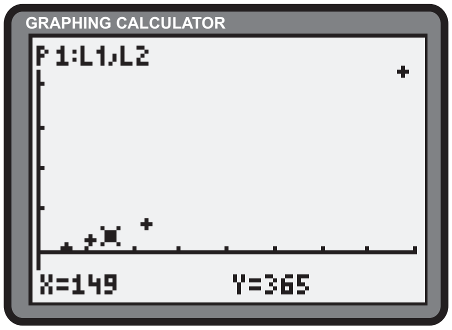
Let S be the required shade number of the glasses.

$$\begin{aligned}S &= \frac{-7\log T}{3}+1\\ S &= \frac{-7\log\left(5.1795\times 10^{-5}\right)}{3}+1\\ &\doteq 11\end{aligned}$$

The electric welder should use #11 welding glasses.

Section 7.5 Page 447 Question 10

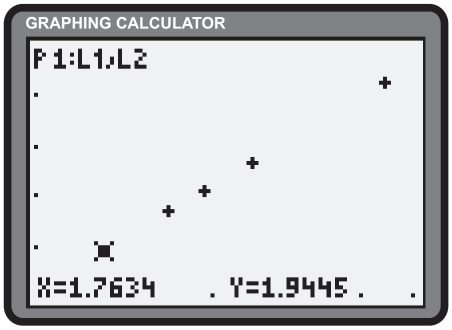
a)



b)

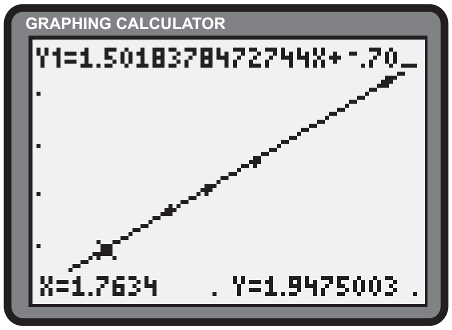
Planet	log *d*	log *t*
Mercury	1.7634	1.9445
Venus	2.0294	2.3522
Earth	2.1732	2.5623
Mars	2.3560	2.8363
Jupiter	2.8882	3.6364

c)



A linear function would model these data best.

d)



The linear regression feature of the graphing calculator yields the model $\log t \doteq 1.5019 \log d - 0.7010$.

e)

$$\begin{aligned}\log t &= 1.5019\log d - 0.7010\\ &= \log d^{1.5019} - 0.7010\\ 0.7010 &= \log d^{1.5019} - \log t\\ &= \log\frac{d^{1.5019}}{t}\\ 10^{0.7010} &= \frac{d^{1.5019}}{t}\\ t &= \frac{d^{1.5019}}{10^{0.7010}}\\ &\doteq 0.1991d^{1.5019}\end{aligned}$$

The model defining t as a function of d can be expressed as $t = 0.1991d^{1.5019}$.

f)

$$\begin{aligned}t &= 0.1991d^{1.5019}\\ t &= 0.1991(1430)^{1.5019}\\ &\doteq 10\,916\end{aligned}$$

Saturn orbits the sun every 10 916 days.

Section 7.5 Page 448 Question 11

a) For each of the following, x and y are values determined from the given tables. Let r be the desired result.

i)

$$\begin{aligned}r &= 2469 \times 491\\ &= 2.469 \times 10^3 \times 4.91 \times 10^2\\ &= 2.469 \times 4.91 \times 10^5\\ \log r &= \log(2.469 \times 4.91 \times 10^5)\\ &= \log 2.469 + \log 4.91 + 5\\ &= x + y + 5\\ r &= 10^{x+y+5}\\ &= 10^{x+y} \times 10^5\\ &= \text{antilog}(x + y) \times 10^5\end{aligned}$$

ii)

$$\begin{aligned}r &= \sqrt[3]{181}\\ \log r &= \frac{1}{3}\log(1.81 \times 10^2)\\ &= \frac{1}{3}(2 + \log 1.81)\\ &= \frac{2 + x}{3}\\ r &= 10^{\frac{2+x}{3}}\\ &= \text{antilog}\left(\frac{2 + x}{3}\right)\end{aligned}$$

iii)

$$\begin{aligned}r &= 4830 \div 21.73\\ &= 4.830 \times 10^3 \div (2.173 \times 10)\\ &= 4.830 \div 2.173 \times 10^2\\ \log r &= \log(4.830 \div 2.173 \times 10^2)\\ &= \log 4.830 - \log 2.173 + 2\\ &= x - y + 2\\ r &= 10^{x-y+2}\\ &= 10^{x-y} \times 10^2\\ &= \text{antilog}(x - y) \times 10^2\end{aligned}$$

b) i)

$$\begin{aligned}r &= 10^{\log 2.469+\log 4.91+5}\\ &= 1\,212\,279\\ &= 2469 \times 491\end{aligned}$$

ii)

$$\begin{aligned}r &= 10^{\frac{2+\log 1.81}{3}}\\ &\doteq 5.657\\ &= \sqrt[3]{181}\end{aligned}$$

iii)

$$\begin{aligned}r &= 10^{\log 4.830-\log 2.173+2}\\ &\doteq 222.273\\ &= 4830 \div 21.73\end{aligned}$$

c) Answers may vary.

Section 7.5 Page 448 Question 12

a) Answers may vary.
b) Answers may vary.

7.6 Derivatives of Exponential Functions

Practise

Section 7.6 Page 459 Question 1

a) Since $e \in (1, \infty)$, $\lim\limits_{x \to -\infty} e^x = 0$.

b) Since $e \in (1, \infty)$, $\lim\limits_{x \to -\infty} e^{-x} = \infty$.

c) Since $e \in (1, \infty)$, $\lim\limits_{x \to \infty} e^{2x} = \infty$.

d) Since $e \in (1, \infty)$, $\lim\limits_{x \to \infty} e^{-3x} = 0$.

Section 7.6 Page 459 Question 3

a)

$$\begin{aligned} 2^x &= e^{kx} \\ \ln 2^x &= \ln e^{kx} \\ x \ln 2 &= kx \\ k &= \ln 2 \end{aligned}$$

The expression 2^x can be written as $e^{x \ln 2}$.

b)

$$\begin{aligned} y &= 2^x \\ &= e^{x \ln 2} \\ \frac{dy}{dx} &= \frac{de^{x \ln 2}}{dx \ln 2} \cdot \frac{dx \ln 2}{dx} \\ &= e^{x \ln 2} \cdot \ln 2 \\ &= 2^x \ln 2 \end{aligned}$$

c)

$$\begin{aligned} y &= a^x \\ &= e^{x \ln a} \\ \frac{dy}{dx} &= \frac{de^{x \ln a}}{dx \ln a} \cdot \frac{dx \ln a}{dx} \\ &= e^{x \ln a} \cdot \ln a \\ &= a^x \ln a \end{aligned}$$

Section 7.6 Page 459 Question 5

$$\begin{aligned} e^{xy} &= x + y \\ \frac{de^{xy}}{dxy} \cdot \left(y\frac{dx}{dx} + x\frac{dy}{dx} \right) &= \frac{dx}{dx} + \frac{dy}{dx} \\ e^{xy}\left(y + xy'\right) &= 1 + y' \\ ye^{xy} + xe^{xy}y' - y' &= 1 \\ y'(xe^{xy} - 1) &= 1 - ye^{xy} \\ y' &= \frac{1 - ye^{xy}}{xe^{xy} - 1} \end{aligned}$$

Section 7.6 Page 459 Question 7

a)

$$\begin{aligned} g(x) &= xe^{-4x} \\ g'(x) &= 0 \\ e^{-4x}\frac{dx}{dx} + x\frac{de^{-4x}}{d(-4x)} \cdot \frac{d(-4x)}{dx} &= 0 \\ e^{-4x} + xe^{-4x}(-4) &= 0 \\ e^{-4x}(1 - 4x) &= 0 \\ x &= \frac{1}{4} \end{aligned}$$

The domain of g is $\Re$. There are two test intervals for g'. Since $g'(0) > 0$, g is increasing on the interval $\left(-\infty, \frac{1}{4}\right)$. Since $g'(1) < 0$, g is decreasing on the interval $\left(\frac{1}{4}, \infty\right)$.

b)

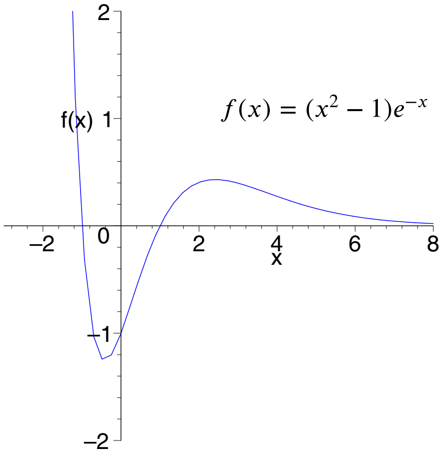
$$\begin{aligned} h(x) &= xe^{3x} \\ h'(x) &= 0 \\ e^{3x}\frac{dx}{dx} + x\frac{de^{3x}}{d(3x)} \cdot \frac{d(3x)}{dx} &= 0 \\ e^{3x} + xe^{3x}(3) &= 0 \\ e^{3x}(1 + 3x) &= 0 \\ x &= -\frac{1}{3} \end{aligned}$$

The domain of h is $\Re$. There are two test intervals for h'. Since $h'(-1) < 0$, h is decreasing on the interval $\left(-\infty, -\frac{1}{3}\right)$. Since $h'(0) > 0$, h is increasing on the interval $\left(-\frac{1}{3}, \infty\right)$.

Section 7.6 Page 459 Question 9

a)

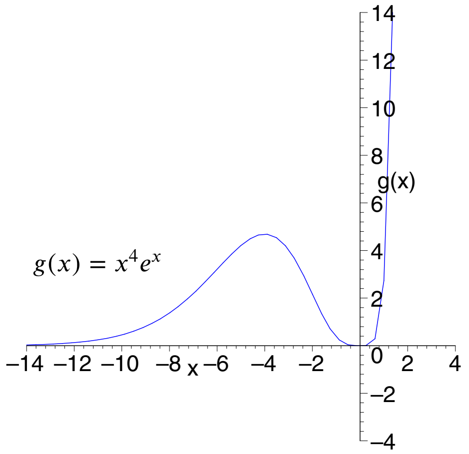
$$\begin{aligned}
f(x) &= (x^2-1)e^{-x}\\
f'(x) &= 0\\
e^{-x}(2x)+(x^2-1)(-e^{-x}) &= 0\\
e^{-x}(x^2-2x-1) &= 0\\
x &= \frac{2 \pm \sqrt{(-2)^2-4(1)(-1)}}{2}\\
&= 1 \pm \sqrt{2}\\
f''(x) &= 0\\
(x^2-2x-1)(-e^{-x})+e^{-x}(2x-2) &= 0\\
e^{-x}(x^2-4x+1) &= 0\\
x &= \frac{4 \pm \sqrt{(-4)^2-4(1)(1)}}{2}\\
&= 2 \pm \sqrt{3}
\end{aligned}$$



Property	Result
Domain	$\Re$
Range	$\left((2-2\sqrt{2})e^{-1+\sqrt{2}}, \infty\right)$
Intercepts	$x: \pm 1;\ y: -1$
Symmetry	none
Asymptote	$y = 0$
Increase	$(1-\sqrt{2}, 1+\sqrt{2})$
Decrease	$(-\infty, 1-\sqrt{2})$, $(1+\sqrt{2}, \infty)$
Extrema	local min.: $\left(1-\sqrt{2}, (2-2\sqrt{2})e^{-1+\sqrt{2}}\right)$; local max.: $\left(1+\sqrt{2}, (2+2\sqrt{2})e^{-1-\sqrt{2}}\right)$
Concavity	upward: $(-\infty, 2-\sqrt{3})$, $(2+\sqrt{3}, \infty)$; downward: $(2-\sqrt{3}, 2+\sqrt{3})$
Points of inflection	$\left(2-\sqrt{3}, (6-4\sqrt{3})e^{-2+\sqrt{3}}\right)$, $\left(2+\sqrt{3}, (6+4\sqrt{3})e^{-2-\sqrt{3}}\right)$

b)

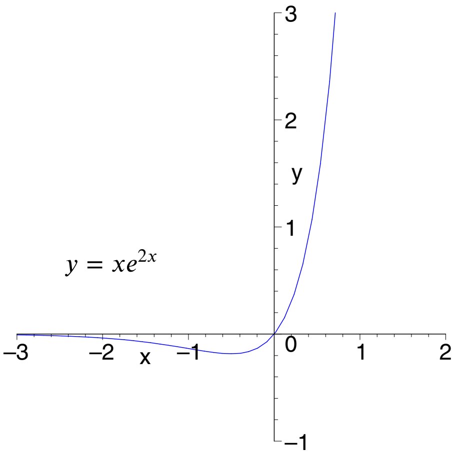
$$\begin{aligned}
g(x) &= x^4e^x \\
g'(x) &= 0 \\
e^x(4x^3) + x^4e^x &= 0 \\
x^3e^x(x+4) &= 0 \\
x &= 0, -4 \\
g''(x) &= 0 \\
3x^2e^x(x+4) + x^3e^x(x+4) + x^3e^x(1) &= 0 \\
x^2e^x\,(3(x+4) + x(x+4) + x) &= 0 \\
x^2e^x(x^2+8x+12) &= 0 \\
x^2e^x(x+6)(x+2) &= 0 \\
x &= 0, -6, -2
\end{aligned}$$



Property	Result
Domain	$\Re$
Range	$[0, \infty)$
Intercepts	x: 0, y: 0
Symmetry	none
Asymptote	$y = 0$
Increase	$(-\infty, -4)$, $(0, \infty)$
Decrease	$(-4, 0)$
Local maximum	$\left(-4, \dfrac{256}{e^4}\right)$
Local minimum	$(0, 0)$
Concave upward	$(-\infty, -6)$, $(-2, \infty)$
Concave downward	$(-6, -2)$
Points of inflection	$\left(-6, \dfrac{1296}{e^6}\right)$, $\left(-2, \dfrac{16}{e^2}\right)$

c)

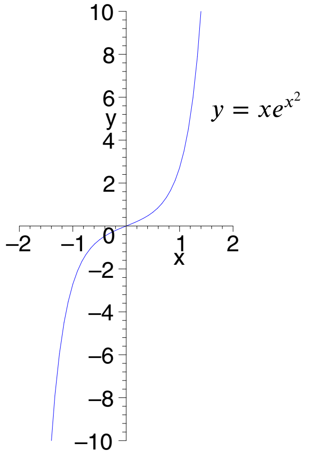
$$\begin{aligned}
y &= xe^{2x} \\
\frac{dy}{dx} &= 0 \\
e^{2x}(1) + xe^{2x}(2) &= 0 \\
e^{2x}(2x+1) &= 0 \\
x &= -\frac{1}{2} \\
\frac{d^2y}{dx^2} &= 0 \\
(2x+1)e^{2x}(2) + e^{2x}(2) &= 0 \\
e^{2x}(4x+4) &= 0 \\
x &= -1
\end{aligned}$$



Property	Result
Domain	$\Re$
Range	$\left[-\dfrac{1}{2e}, \infty\right)$
Intercepts	x: 0, y: 0
Symmetry	none
Asymptote	$y = 0$
Increase	$\left(-\dfrac{1}{2}, \infty\right)$
Decrease	$\left(-\infty, -\dfrac{1}{2}\right)$
Local minimum	$\left(-\dfrac{1}{2}, -\dfrac{1}{2e}\right)$
Concave upward	$(-1, \infty)$
Concave downward	$(-\infty, -1)$
Point of inflection	$\left(-1, -\dfrac{1}{e^2}\right)$

d)

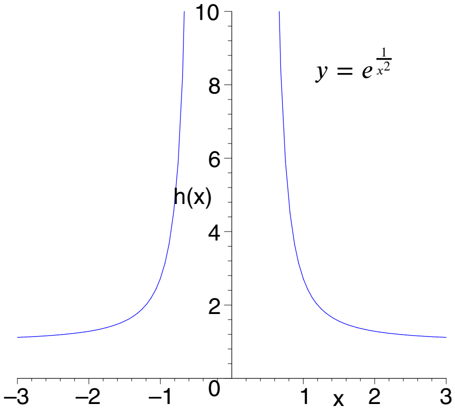
$$
\begin{aligned}
y &= xe^{x^2}\\
\frac{dy}{dx} &= 0\\
e^{x^2}(1) + xe^{x^2}(2x) &= 0\\
e^{x^2}(2x^2+1) &= 0\\
&\text{no roots}\\
\frac{d^2y}{dx^2} &= 0\\
(2x^2+1)e^{x^2}(2x) + e^{x^2}(4x) &= 0\\
2xe^{x^2}(2x^2+3) &= 0\\
x &= 0
\end{aligned}
$$



Property	Result
Domain	ℜ
Range	ℜ
Intercepts	x: 0, y: 0
Symmetry	odd
Asymptote	none
Increase	ℜ
Decrease	none
Extrema	none
Concave upward	$(0, \infty)$
Concave downward	$(-\infty, 0)$
Point of inflection	$(0, 0)$

e)

$$
\begin{aligned}
h(x) &= e^{\frac{1}{x^2}}\\
h'(x) &= 0\\
-\frac{2e^{\frac{1}{x^2}}}{x^3} &= 0\\
&\text{no roots}\\
h''(x) &= 0\\
-2\left(\frac{x^3e^{\frac{1}{x^2}}\left(-\frac{2}{x^3}\right) - e^{\frac{1}{x^2}}(3x^2)}{x^6}\right) &= 0\\
\frac{2e^{\frac{1}{x^2}}(2+3x^2)}{x^6} &= 0\\
&\text{no roots}
\end{aligned}
$$



Property	Result
Domain	$\{x \in \Re \mid x \neq 0\}$
Range	$(1, \infty)$
Intercepts	none
Symmetry	even
Horizontal asymptote	$y = 1$
Vertical asymptote	$x = 0$
Increase	$(-\infty, 0)$
Decrease	$(0, \infty)$
Extrema	none
Concave upward	$(-\infty, 0)$, $(0, \infty)$
Concave downward	none
Point of inflection	none

f)

$$y = \frac{e^x}{x^2}$$

$$\frac{dy}{dx} = 0$$

$$\frac{x^2e^x - e^x(2x)}{x^4} = 0$$

$$\frac{e^x(x-2)}{x^3} = 0$$

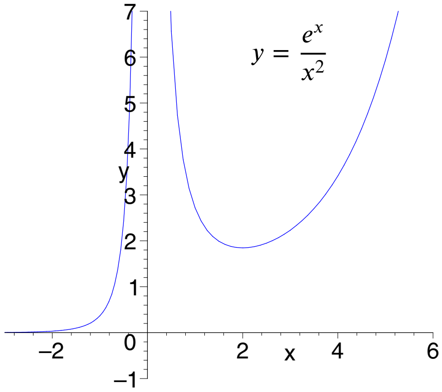
$$x = 2$$

$$\frac{d^2y}{dx^2} = 0$$

$$\frac{x^3\left((x-2)e^x + e^x(1)\right) - e^x(x-2)(3x^2)}{x^6} = 0$$

$$\frac{e^x(x^2-4x+6)}{x^4} = 0$$

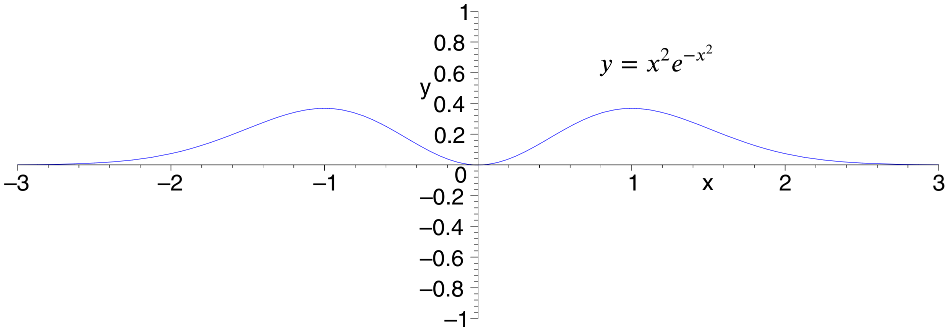
no roots



Property	Result
Domain	$\{x \in \Re \mid x \neq 0\}$
Range	$(0, \infty)$
Intercepts	none
Symmetry	none
Horizontal asymptote	$y = 0$
Vertical asymptote	$x = 0$
Increase	$(-\infty, 0),\ (2, \infty)$
Decrease	$(0, 2)$
Local minimum	$\left(2, \frac{e^2}{4}\right)$
Concave upward	$(-\infty, 0)\ ,\ (0, \infty)$
Point of inflection	none

g)

$$
\begin{aligned}
y &= x^2e^{-x^2}\\
\frac{dy}{dx} &= 0\\
e^{-x^2}(2x) + x^2e^{-x^2}(-2x) &= 0\\
2xe^{-x^2}(1-x^2) &= 0\\
x &= 0, \pm 1\\
\frac{d^2y}{dx^2} &= 0\\
2\left(e^{-x^2}(1-x^2) + xe^{-x^2}(-2x)(1-x^2) + xe^{-x^2}(-2x)\right) &= 0\\
e^{-x^2}(2x^4 - 5x^2 + 1) &= 0\\
x^2 &= \frac{5 \pm \sqrt{(-5)^2 - 4(2)(1)}}{4}\\
&= \frac{5 \pm \sqrt{17}}{4}\\
x &= \pm\frac{\sqrt{5 \pm \sqrt{17}}}{2}
\end{aligned}
$$

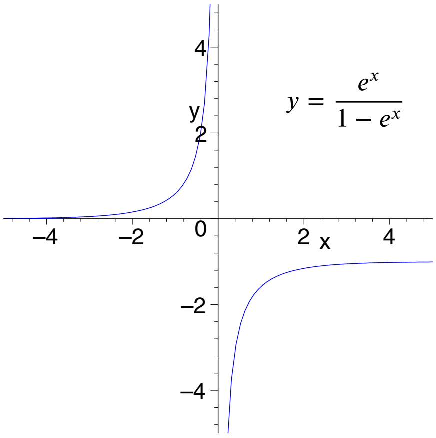


Property	Result
Domain	$\Re$
Range	$\left[0, \frac{1}{e}\right]$
Intercepts	x: 0, y: 0
Symmetry	even
Horizontal asymptote	$y = 0$
Increase	$(-\infty, 1),\ (0, 1)$
Decrease	$(-1, 0),\ (1, \infty)$
Local minimum	$(0, 0)$
Local maxima	$\left(\pm 1, \frac{1}{e}\right)$

Property	Result
Concave upward	$\left(-\infty, -\frac{\sqrt{5+\sqrt{17}}}{2}\right)$
	$\left(-\frac{\sqrt{5-\sqrt{17}}}{2}, \frac{\sqrt{5-\sqrt{17}}}{2}\right)$
	$\left(\frac{\sqrt{5+\sqrt{17}}}{2}, \infty\right)$
Concave downward	$\left(-\frac{\sqrt{5+\sqrt{17}}}{2}, -\frac{\sqrt{5-\sqrt{17}}}{2}\right)$
	$\left(\frac{\sqrt{5-\sqrt{17}}}{2}, \frac{\sqrt{5+\sqrt{17}}}{2}\right)$
Points of inflection	$\left(\pm\frac{\sqrt{5+\sqrt{17}}}{2}, \frac{5+\sqrt{17}}{4e^{\frac{5+\sqrt{17}}{4}}}\right)$
	$\left(\pm\frac{\sqrt{5-\sqrt{17}}}{2}, \frac{5-\sqrt{17}}{4e^{\frac{5-\sqrt{17}}{4}}}\right)$

h)

$$\begin{aligned} y &= \frac{e^x}{1-e^x} \\ \frac{dy}{dx} &= 0 \\ \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} &= 0 \\ \frac{1}{(1-e^x)^2} &= 0 \\ &\text{no roots} \\ \frac{d^2y}{dx^2} &= 0 \\ \frac{-2}{(1-e^x)^3}\cdot -e^x &= 0 \\ \frac{e^x}{(1-e^x)^3} &= 0 \\ &\text{no roots} \end{aligned}$$



Property	Result
Domain	$\{x \in \Re \mid x \neq 0\}$
Range	$(-\infty, -1), (0, \infty)$
Intercepts	none
Symmetry	none
Horizontal asymptote	$y = 0,\ y = -1$
Vertical asymptote	$x = 0$
Increase	$(-\infty, 0),\ (0, \infty)$
Decrease	none
Local minima	none
Local maxima	none
Concave upward	$(-\infty, 0)$
Concave downward	$(0, \infty)$
Point of inflection	none

Section 7.6 Page 459 Question 11

For each of the following limits, recall that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

a) $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$

b) $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2$

c) $\lim_{n\to\infty}\left(1+\frac{6x}{n}\right)^n = e^{6x}$

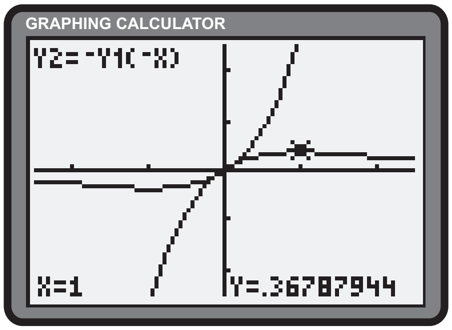
Apply, Solve, Communicate

Section 7.6 Page 459 Question 12

The function $y = xe^{-x}$ can be obtained by reflecting the function $f(x) = xe^x$ in the y-axis and then in the x-axis.

$$\begin{aligned} f(x) &= xe^x \\ -f(-x) &= -(-xe^{-x}) \\ &= xe^{-x} \end{aligned}$$

So, all the important points and information can be reflected in the same way.



Y2=-Y1(-X)

X=1 Y=.36787944

Section 7.6 Page 459 Question 13

a)

$$\begin{aligned} P(t) &= 250e^{0.04t} \\ P(0) &= 250e^{0.04(0)} \\ &= 250e^{0} \\ &= 250 \end{aligned}$$

At time $t = 0$, there are 250 bacteria.

b) i)

$$\begin{aligned} P(t) &= 2(250) \\ 250e^{0.04t} &= 2(250) \\ e^{0.04t} &= 2 \\ 0.04t &= \ln 2 \\ t &= \frac{\ln 2}{0.04} \\ &\doteq 17.33 \end{aligned}$$

It will take 17.33 h for the bacteria population to double.

ii)

$$\begin{aligned} P(t) &= 2(250) \\ 250e^{0.04t} &= 3(250) \\ e^{0.04t} &= 3 \\ 0.04t &= \ln 3 \\ t &= \frac{\ln 3}{0.04} \\ &\doteq 27.47 \end{aligned}$$

It will take 27.47 h for the bacteria population to triple.

iii)

$$\begin{aligned} P(t) &= 20\,000 \\ 250e^{0.04t} &= 20\,000 \\ e^{0.04t} &= 80 \\ 0.04t &= \ln 80 \\ t &= \frac{\ln 80}{0.04} \\ &\doteq 109.55 \end{aligned}$$

It will take 109.55 h for the bacteria population to reach 20 000.

c)

$$\begin{aligned} P'(t) &= 250e^{0.04t}(0.04) \\ &= 10e^{0.04t} \\ P'(10) &= 10e^{0.04(10)} \\ &\doteq 14.92 \end{aligned}$$

After 10 h, the population is growing at a rate of 14.92 bacteria/h.

$$\begin{aligned} P'(20) &= 10e^{0.04(20)} \\ &\doteq 22.26 \end{aligned}$$

After 20 h, the population is growing at a rate of 22.26 bacteria/h.

d) Answers may vary.

Section 7.6 Page 460 Question 14

a) Let P be the population of rats. $P(t) = 400e^{0.018t}$.

b) i)

$$\begin{aligned} P(t) &= 2(400) \\ 400e^{0.018t} &= 2(400) \\ e^{0.018t} &= 2 \\ 0.018t &= \ln 2 \\ t &= \frac{\ln 2}{0.018} \\ &\doteq 38.51 \end{aligned}$$

It will take 38.51 years for the population to double.

ii)

$$\begin{aligned} P(t) &= 3(400) \\ 400e^{0.018t} &= 3(400) \\ e^{0.018t} &= 3 \\ 0.018t &= \ln 3 \\ t &= \frac{\ln 3}{0.018} \\ &\doteq 61.03 \end{aligned}$$

It will take 61.03 years for the population to triple.

c)

$$\begin{aligned} P'(t) &= 400e^{0.018t}(0.018) \\ &= 7.2e^{0.018t} \\ P'(10) &= 7.2e^{0.018(10)} \\ &\doteq 8.62 \end{aligned}$$

After 10 years, the population is growing at a rate of 8.62 rats/year.

$$\begin{aligned} P'(25) &= 7.2e^{0.018(25)} \\ &\doteq 11.29 \end{aligned}$$

After 25 years, the population is growing at a rate of 11.29 rats/year.

d) Answers may vary.

Section 7.6 Page 460 Question 15

a) Let V be the value of the computer system. $V(t) = 5000e^{-0.11t}$.

b) i)

$$\begin{aligned} V(t) &= 2500 \\ 5000e^{-0.11t} &= 2500 \\ e^{-0.11t} &= 0.5 \\ -0.11t &= \ln 0.5 \\ t &= \frac{\ln 0.5}{-0.11} \\ &\doteq 6.3 \end{aligned}$$

It will take 6.3 years to decrease to \$2500.

ii)

$$\begin{aligned} V(t) &= 0.1(5000) \\ 5000e^{-0.11t} &= 0.1(5000) \\ e^{-0.11t} &= 0.1 \\ -0.11t &= \ln 0.1 \\ t &= \frac{\ln 0.1}{-0.11} \\ &\doteq 20.9 \end{aligned}$$

It will take 20.9 years to decrease to 10% of the original value.

Section 7.6 Page 460 Question 16

a) For annual compounding, the formula $A = P(1 + r)^t$ is used.

i)

$$\begin{aligned} A &= 2(2000) \\ 2000\,(1 + 0.07)^t &= 2(2000) \\ (1 + 0.07)^t &= 2 \\ t \ln 1.07 &= \ln 2 \\ t &= \frac{\ln 2}{\ln 1.07} \\ &\doteq 10.245 \end{aligned}$$

It will take 10.245 years for the investment to double.

ii)

$$\begin{aligned} A &= 3(2000) \\ 2000\,(1 + 0.07)^t &= 3(2000) \\ (1 + 0.07)^t &= 3 \\ t \ln 1.07 &= \ln 3 \\ t &= \frac{\ln 3}{\ln 1.07} \\ &\doteq 16.238 \end{aligned}$$

It will take 16.238 years for the investment to triple.

b) For continuous compounding the formula $A = Pe^{rt}$ is used.

i)

$$\begin{aligned} A &= 2(2000) \\ 2000e^{0.07t} &= 2(2000) \\ e^{0.07t} &= 2 \\ 0.07t &= \ln 2 \\ t &= \frac{\ln 2}{0.07} \\ &\doteq 9.902 \end{aligned}$$

Her estimate was 10.245−9.902 or 0.343 years sooner.

ii)

$$\begin{aligned} A &= 3(2000) \\ 2000e^{0.07t} &= 3(2000) \\ e^{0.07t} &= 3 \\ 0.07t &= \ln 3 \\ t &= \frac{\ln 3}{0.07} \\ &\doteq 15.694 \end{aligned}$$

Her estimate was 16.238 − 15.694 or 0.544 years sooner.

Section 7.6 Page 460 Question 17

a) For periodic compounding, the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ is used.

i)

$$\begin{aligned} A &= 2(2000) \\ 2000\left(1 + \frac{0.07}{2}\right)^{2t} &= 2(2000) \\ \left(1 + \frac{0.07}{2}\right)^{2t} &= 2 \\ 2t \ln\left(1 + \frac{0.07}{2}\right) &= \ln 2 \\ t &= \frac{\ln 2}{2 \ln\left(1 + \frac{0.07}{2}\right)} \\ &\doteq 10.074 \end{aligned}$$

It will take 10.074 years for the investment to double.

ii)

$$\begin{aligned} A &= 2(2000) \\ 2000\left(1 + \frac{0.07}{12}\right)^{12t} &= 2(2000) \\ \left(1 + \frac{0.07}{12}\right)^{12t} &= 2 \\ 12t \ln\left(1 + \frac{0.07}{12}\right) &= \ln 2 \\ t &= \frac{\ln 2}{12 \ln\left(1 + \frac{0.07}{12}\right)} \\ &\doteq 9.931 \end{aligned}$$

It will take 9.931 years for the investment to double.

iii)

$$\begin{aligned} A &= 2(2000) \\ 2000\left(1 + \frac{0.07}{365}\right)^{365t} &= 2(2000) \\ \left(1 + \frac{0.07}{365}\right)^{365t} &= 2 \\ 365t \ln\left(1 + \frac{0.07}{365}\right) &= \ln 2 \\ t &= \frac{\ln 2}{365 \ln\left(1 + \frac{0.07}{365}\right)} \\ &\doteq 9.903 \end{aligned}$$

It will take 9.903 years for the investment to double.

b) Answers may vary.

Section 7.6 Page 460 Question 18

a)

$$\begin{aligned} y &= 20t + 400e^{-0.05t} \\ v &= \frac{dy}{dt} \\ &= 20 + 400e^{-0.05t}(-0.05) \\ &= 20 - 20e^{-0.05t} \end{aligned}$$

The velocity is $v = 20 - 20e^{-0.05t}$ mm/s.

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= 0 - 20e^{-0.05t}(-0.05) \\ &= e^{-0.05t} \end{aligned}$$

The acceleration is $a = e^{-0.05t}$ mm/s^2.

b)

$$\begin{aligned} y(5) &= 20(5) + 400e^{-0.05(5)} \\ &\doteq 411.5 \end{aligned}$$

After 5 s, the position is 411.5 mm.

$$\begin{aligned} v(5) &= 20 - 20e^{-0.05(5)} \\ &\doteq 4.42 \end{aligned}$$

After 5 s, the velocity is 4.42 mm/s.

$$\begin{aligned} a(5) &= e^{-0.05(5)} \\ &\doteq 0.78 \end{aligned}$$

After 5 s, the acceleration is 0.78 mm/s^2.

Section 7.6 Page 460 Question 19

a)

$$\begin{aligned} V(t) &= 50\,000e^{-0.4t} \\ V(5) &= 50\,000e^{-0.4(5)} \\ &\doteq 6766.76 \end{aligned}$$

After 5 years, the value of the machine is $6766.76.

$$\begin{aligned} V(10) &= 50\,000e^{-0.4(10)} \\ &\doteq 915.78 \end{aligned}$$

After 10 years, the value of the machine is $915.78.

b)

$$\begin{aligned} \frac{dV}{dt} &= 50\,000e^{-0.4t}(-0.4) \\ &= -20\,000e^{-0.4t} \\ \frac{dV}{dt}_{|t=5} &= -20\,000e^{-0.4(5)} \\ &= -2706.71 \end{aligned}$$

After 5 years, the rate of change of the machine's value is −$2706.71 per year.

$$\begin{aligned} \frac{dV}{dt}_{|t=10} &= -20\,000e^{-0.4(10)} \\ &= -366.31 \end{aligned}$$

After 10 years, the rate of change of the machine's value is −$366.31 per year.

Section 7.6 Page 460 Question 20

a)

$$\begin{aligned} \lim_{t\to\infty} I &= \lim_{t\to\infty} 0.6(1 - e^{-0.1t}) \\ &= 0.6(1 - 0) \\ &= 0.6 \end{aligned}$$

As t increases without bound, I approaches 0.6 A.

The rate of change of current with respect to time is $\frac{dI}{dt} = 0.6(-e^{-0.1t}(-0.1))$ or $0.06e^{-0.1t}$ A/s.

b)

$$\begin{aligned} \frac{dI}{dt}_{|t=40} &= 0.06e^{-0.1(40)} \\ &\doteq 0.001 \end{aligned}$$

After 40 s, the rate is 0.001 A/s.

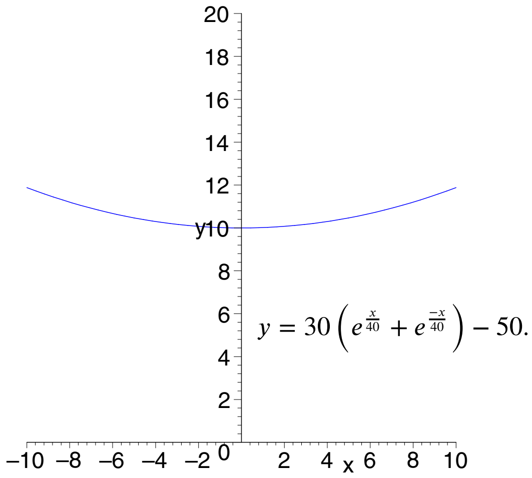
c)

$$\begin{aligned} \frac{dI}{dt}_{|t=60} &= 0.06e^{-0.1(60)} \\ &\doteq 0.0001 \end{aligned}$$

After 60 s, the rate is 0.0001 A/s.

Section 7.6 Page 460 Question 21

a) The domain is $x \in [-10, 10]$.



b) Let s be the amount the wires sag between the poles.

$$\begin{aligned} s &= y(10) - y(0) \\ &= 30\left(e^{\frac{10}{40}} + e^{\frac{-10}{40}}\right) - 50 - \left(30\left(e^{\frac{0}{40}} + e^{\frac{-0}{40}}\right) - 50\right) \\ &= 30\left(e^{\frac{10}{40}} + e^{\frac{-10}{40}} - 2\right) \\ &\doteq 1.885 \end{aligned}$$

The wires sag 1.885 m between the support poles.

Section 7.6 Page 460 Question 22

a)

$$\begin{aligned} A &= 2P \\ Pe^{rt} &= 2P \\ e^{rt} &= 2 \\ rt &= \ln 2 \\ t &= \frac{\ln 2}{r} \\ &\doteq \frac{0.7}{r} \end{aligned}$$

The above result suggests that the doubling time of continuous compounding is approximately $\frac{70}{r\%}$.

b) i) At 5%, the doubling time is $\frac{70}{5}$ or 14 years.

ii) At 14%, the doubling time is $\frac{70}{14}$ or 5 years.

iii) At 8%, the doubling time is $\frac{70}{8}$ or 8.75 years.

c) i)

$$\begin{aligned} A(t) &= 2P \\ P(1 + 0.05)^t &= 2P \\ (1.05)^t &= 2 \\ t\ln(1.05) &= \ln 2 \\ t &= \frac{\ln 2}{\ln 1.05} \\ &\doteq 14.2 \end{aligned}$$

The doubling time is 14.2 years.

ii)

$$\begin{aligned} A(t) &= 2P \\ P(1 + 0.14)^t &= 2P \\ (1.14)^t &= 2 \\ t\ln(1.14) &= \ln 2 \\ t &= \frac{\ln 2}{\ln 1.14} \\ &\doteq 5.29 \end{aligned}$$

The doubling time is 5.29 years.

iii)

$$\begin{aligned} A(t) &= 2P \\ P(1 + 0.08)^t &= 2P \\ (1.08)^t &= 2 \\ t\ln(1.08) &= \ln 2 \\ t &= \frac{\ln 2}{\ln 1.08} \\ &\doteq 9.01 \end{aligned}$$

The doubling time is 9.01 years.

Comments on accuracy may vary.

7.7 Derivatives of Logarithmic Functions

Practise

Section 7.7 Page 469 Question 1

a)

$$\begin{aligned} y &= \ln(9x-2) \\ \frac{dy}{dx} &= \frac{d\ln(9x-2)}{d(9x-2)} \cdot \frac{d(9x-2)}{dx} \\ &= \frac{1}{9x-2} \cdot 9 \\ &= \frac{9}{9x-2} \end{aligned}$$

b)

$$\begin{aligned} y &= \log_5(9x-7) \\ 5^y &= 9x-7 \\ y\ln 5 &= \ln(9x-7) \\ \ln 5 \cdot \frac{dy}{dx} &= \frac{1}{9x-7} \cdot 9 \\ \frac{dy}{dx} &= \frac{9}{(9x-7)\ln 5} \end{aligned}$$

c)

$$\begin{aligned} y &= 3^x \\ \ln y &= x\ln 3 \\ \frac{1}{y} \cdot \frac{dy}{dx} &= \ln 3 \\ \frac{dy}{dx} &= y\ln 3 \\ &= 3^x \ln 3 \end{aligned}$$

d)

$$\begin{aligned} y &= \log_4(x^2+3) \\ 4^y &= x^2+3 \\ y\ln 4 &= \ln(x^2+3) \\ \ln 4 \cdot \frac{dy}{dx} &= \frac{1}{x^2+3} \cdot 2x \\ \frac{dy}{dx} &= \frac{2x}{(x^2+3)\ln 4} \end{aligned}$$

e)

$$\begin{aligned} h(x) &= \ln 2x^2 \\ &= \ln 2 + 2\ln x \\ h'(x) &= 2 \cdot \frac{1}{x} \\ &= \frac{2}{x} \end{aligned}$$

f)

$$\begin{aligned} f(x) &= -x\ln x \\ f'(x) &= -\left(\ln x \cdot 1 + x \cdot \frac{1}{x}\right) \\ &= -\ln x - 1 \end{aligned}$$

g)

$$\begin{aligned} g(x) &= x^2 5^x \\ \ln g(x) &= \ln(x^2 5^x) \\ &= 2\ln x + x\ln 5 \\ \frac{d\ln g(x)}{d\,g(x)} \cdot \frac{d\,g(x)}{dx} &= \frac{d2\ln x}{dx} + \frac{dx\ln 5}{dx} \\ \frac{1}{g(x)} \cdot g'(x) &= 2 \cdot \frac{1}{x} + \ln 5 \\ g'(x) &= g(x)\left(\frac{2}{x} + \ln 5\right) \\ &= x^2 5^x \left(\frac{2}{x} + \ln 5\right) \\ &= x5^x(2 + x\ln 5) \end{aligned}$$

h)

$$\begin{aligned} h(x) &= 4^x \ln x \\ \ln h(x) &= \ln(4^x \ln x) \\ &= x\ln 4 + \ln(\ln x) \\ \frac{d\ln h(x)}{d\,h(x)} \cdot \frac{d\,h(x)}{dx} &= \frac{dx\ln 4}{dx} + \frac{dx\ln(\ln x)}{d\ln x} \cdot \frac{d\,\ln x}{dx} \\ \frac{1}{h(x)} \cdot h'(x) &= \ln 4 + \frac{1}{\ln x} \cdot \frac{1}{x} \\ h'(x) &= h(x)\left(\ln 4 + \frac{1}{x\ln x}\right) \\ &= 4^x \ln x \left(\ln 4 + \frac{1}{x\ln x}\right) \\ &= 4^x \left((\ln 4)(\ln x) + \frac{1}{x}\right) \end{aligned}$$

i)

$$\begin{aligned} y &= 2^{x^2} \\ \ln y &= x^2 \ln 2 \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 2x\ln 2 \\ \frac{dy}{dx} &= y(2x\ln 2) \\ &= 2^{x^2}(2x\ln 2) \\ &= x2^{x^2+1}\ln 2 \end{aligned}$$

Section 7.7 Page 469 Question 3

a)

$$\begin{aligned} f(x) &= \frac{\ln x}{x} \\ f'(x) &= 0 \\ \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} &= 0 \\ \frac{1 - \ln x}{x^2} &= 0 \\ \ln x &= 1 \\ x &= e \end{aligned}$$

Since $f'(1) > 0$ and $f'(3) < 0$, a local maximum exists at $\left(e, \frac{1}{e}\right)$.

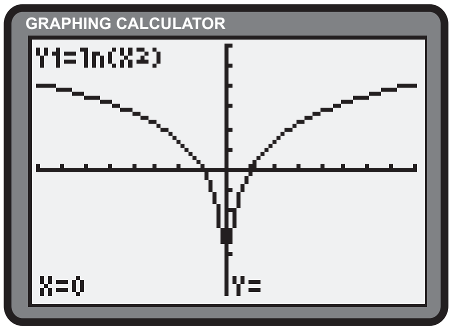
b)

$$\begin{aligned} y &= x \ln x \\ \frac{dy}{dx} &= 0 \\ \ln x \cdot 1 + x \cdot \frac{1}{x} &= 0 \\ \ln x + 1 &= 0 \\ \ln x &= -1 \\ x &= \frac{1}{e} \end{aligned}$$

Since $f'\left(\frac{1}{3}\right) < 0$ and $f'\left(\frac{1}{2}\right) > 0$, a local minimum exists at $\left(\frac{1}{e}, -\frac{1}{e}\right)$.

Section 7.7 Page 469 Question 5

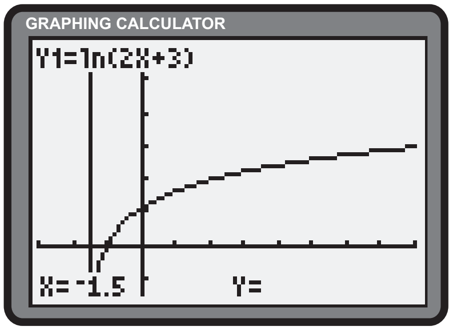
a) $y = \ln x^2$



```
Y1=ln(X2)
X=0          Y=
```

$x \in [-8, 8],\ y \in [-6, 6]$

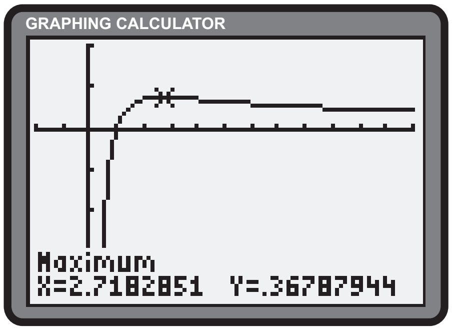
b) $g(x) = \ln(2x + 3)$



```
Y1=ln(2X+3)
X=⁻1.5       Y=
```

$x \in [-3, 8],\ y \in [-2, 6]$

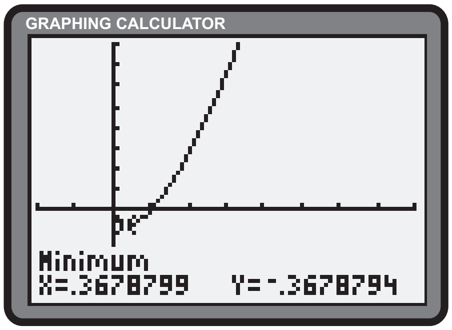
c) $y = \frac{\ln x}{x}$



```
Maximum
X=2.7182851  Y=.36787944
```

$x \in [-2, 12],\ y \in [-2, 1]$

d) $y = x \ln x$



```
Minimum
X=.3678799   Y=⁻.3678794
```

$x \in [-2, 8],\ y \in [-2, 4]$

Section 7.7 Page 469 Question 7

a)

$$\begin{aligned} y &= \ln x \\ y' &= \frac{1}{x} \\ y'' &= -\frac{1}{x^2} \\ y''' &= \frac{2}{x^3} \\ y^{(4)} &= -\frac{6}{x^4} \end{aligned}$$

b) It can be concluded that the nth derivative of $y = \ln x$ can be expressed as

$$y^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Apply, Solve, Communicate

Section 7.7 Page 469 Question 8

a) Since $P = 200$ at $t = 0$, $P_0 = 200$.

$$\begin{aligned} P(3) &= 800 \\ 200a^3 &= 800 \\ a^3 &= 4 \\ a &= \sqrt[3]{4} \end{aligned}$$

The value of a is $\sqrt[3]{4}$.

b)

$$\begin{aligned} P &= 2(200) \\ 200\left(\sqrt[3]{4}\right)^t &= 2(200) \\ 4^{\frac{t}{3}} &= 2 \\ 2^{\frac{2t}{3}} &= 2^1 \\ \frac{2t}{3} &= 1 \\ t &= \frac{3}{2} \\ &= 1.5 \end{aligned}$$

It will take the population 1.5 h to double.

c) This solution uses the result $\frac{da^x}{dx} = a^x \ln a,\ a \in \Re^+$.

$$\begin{aligned} P(t) &= 200\left(\sqrt[3]{4}\right)^t \\ P'(t) &= 200\left(\sqrt[3]{4}\right)^t \cdot \ln \sqrt[3]{4} \\ &= \frac{200 \ln 4}{3}\left(\sqrt[3]{4}\right)^t \end{aligned}$$

The rate of change of the algae population is $\frac{200 \ln 4}{3}\left(\sqrt[3]{4}\right)^t$.

i)

$$\begin{aligned} P'(1) &= \frac{200 \ln 4}{3}\left(\sqrt[3]{4}\right)^1 \\ &\doteq 146.7 \end{aligned}$$

After 1 h, the rate is 146.7 algae/h.

ii)

$$\begin{aligned} P'(6) &= \frac{200 \ln 4}{3}\left(\sqrt[3]{4}\right)^6 \\ &\doteq 1478.7 \end{aligned}$$

After 6 h, the rate is 1478.7 algae/h.

d) Answers may vary.

Section 7.7 Page 469 Question 9

a) Let $P(t)$ be the number of bacteria at time t, in hours.

$$P(t) = P_0 a^t \qquad (1)$$

Substitute $P_0 = 1000$ into (1).

$$P(t) = 1000a^t$$

At $t = 2$, $P = 10\,000$.

$$\begin{aligned} P(2) &= 10\,000 \\ 1000a^2 &= 10\,000 \\ a^2 &= 10 \\ a &= 10^{\frac{1}{2}} \end{aligned}$$

The growth equation is $P(t) = 1000\left(10^{\frac{t}{2}}\right)$.

b) This solution uses the result $\frac{da^x}{dx} = a^x \ln a,\ a \in \Re^+$.

$$\begin{aligned} P(t) &= 1000\left(10^{\frac{t}{2}}\right) \\ P'(t) &= 1000 \cdot \frac{d\left(10^{\frac{t}{2}}\right)}{d\frac{t}{2}} \cdot \frac{d\frac{t}{2}}{dt} \\ &= 1000\left(10^{\frac{t}{2}}\right)\ln 10 \cdot \frac{1}{2} \\ &= 500\left(10^{\frac{t}{2}}\right)\ln 10 \\ P'(5) &= 500\left(10^{\frac{5}{2}}\right)\ln 10 \\ &\doteq 364\,071 \end{aligned}$$

After 5 h, the growth rate is 364 071 bacteria/h.

Section 7.7 Page 469 Question 10

a) The initial mass is $m(0) = 2(2)^{-\frac{0}{15}}$ or 2 g.

b)

$$\begin{aligned} m &= 1 \\ 2(2)^{-\frac{t}{15}} &= 1 \\ (2)^{-\frac{t}{15}} &= 0.5 \\ -\frac{t}{15}\ln 2 &= \ln 0.5 \\ t &= -\frac{15(-\ln 2)}{\ln 2} \\ &= 15 \end{aligned}$$

The half-life of the substance is 15 h.

c) This solution uses the result $\dfrac{da^x}{dx} = a^x \ln a,\ a \in \Re^+$.

$$\begin{aligned} m(t) &= 2(2)^{-\frac{t}{15}} \\ m'(t) &= 2 \cdot \frac{d(2)^{-\frac{t}{15}}}{d\left(-\frac{t}{15}\right)} \cdot \frac{d\left(-\frac{t}{15}\right)}{dt} \\ &= 2(2)^{-\frac{t}{15}} \ln 2 \cdot \left(-\frac{1}{15}\right) \\ &= -\frac{\ln 4}{15}(2)^{-\frac{t}{15}} \\ m'(5) &= -\frac{\ln 4}{15}(2)^{-\frac{5}{15}} \\ &\doteq -0.734 \end{aligned}$$

After 5 h, the growth rate is −0.0734 g/h.

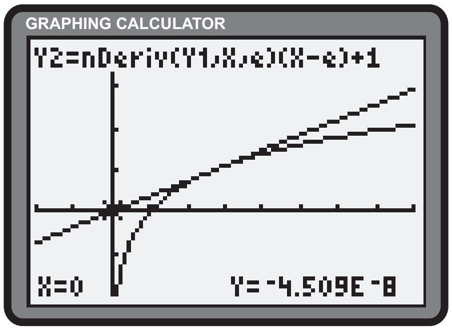
Section 7.7 Page 470 Question 11

For this solution, a value of -4.509×10^{-8} will be interpreted as sufficiently close to 0.

The slope of the tangent to $y = \ln x$ at $x = e$ is $\frac{1}{e}$. The bf nDeriv function of the graphing calculator can be used. The equation of the tangent can be developed as follows.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{e}(x - e) \\ y &= \frac{1}{e}x \end{aligned}$$

At $x = e$, $y = 0$.



GRAPHING CALCULATOR

```
Y2=nDeriv(Y1,X,e)(X-e)+1
X=0                Y=-4.509E-8
```

Section 7.7 Page 470 Question 12

a) Determine the critical numbers of $P(t) = \dfrac{50\ln(t+1)}{t+1} + 40$.

$$\begin{aligned} P'(t) &= 0 \\ 50\left(\frac{(t+1)\cdot\frac{1}{t+1} - \ln(t+1)\cdot 1}{(t+1)^2}\right) &= 0 \\ \frac{1-\ln(t+1)}{(t+1)^2} &= 0 \\ \ln(t+1) &= 1 \\ t+1 &= e \\ t &= e-1 \\ P'(1) &> 0 \\ P'(2) &< 0 \end{aligned}$$

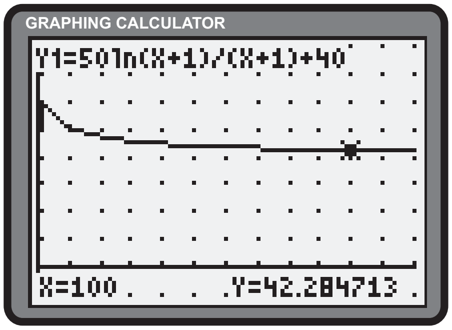
Thus, by the first derivative test, a local maximum occurs at $t = e - 1$.

$$\begin{aligned} P(e-1) &= \frac{50\ln(e-1+1)}{e-1+1} + 40 \\ &= \frac{50}{e} + 40 \\ &\doteq 58 \end{aligned}$$

The maximum number of foxes is approximately 58.

b) As $t \to \infty$, $(t+1)$ increases at a faster rate than $50\ln(t+1)$. As a result $\dfrac{50\ln(t+1)}{t+1}$ approaches 0, leaving the long-term stable fox population to approach 40.

$$\begin{aligned} \lim_{t\to\infty} \frac{50\ln(t+1)}{t+1} + 40 &= 0 + 40 \\ &= 40 \end{aligned}$$



GRAPHING CALCULATOR

```
Y1=50ln(X+1)/(X+1)+40
X=100          Y=42.284713
```

Section 7.7 Page 470 Question 13

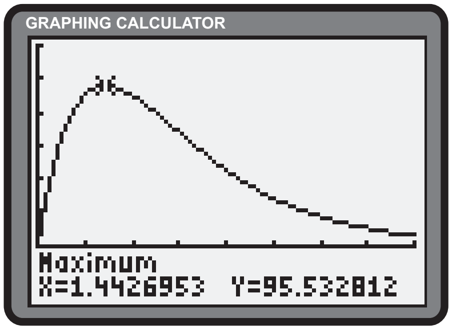
a) This solution uses the result $\frac{da^x}{dx} = a^x \ln a,\ a \in \Re^+$.

$$\begin{aligned} a(t) &= 180t(2)^{-t} \\ a'(t) &= 180\left((2)^{-t}(1) + t(2)^{-t}\ln 2(-1)\right) \\ &= 180(2)^{-t}(1 - t\ln 2) \end{aligned}$$

The speed at which the angle is changing is given by $a'(t) = 180(2)^{-t}(1 - t\ln 2)$.

c) After 5 s, the door is closing at $180(2)^{-5}(5\ln 2 - 1)$ or approximately 13.9°/s.

b) A maximum speed of 95.5°/s is achieved at 1.4 s.



```
Maximum
X=1.4426953  Y=95.532812
```

Section 7.7 Page 470 Question 14

a) This solution uses the result $\frac{da^x}{dx} = a^x \ln a,\ a \in \Re^+$.

$$\begin{aligned} s(t) &= 12(2 - 0.8^t) \\ s'(t) &= 12\left(-0.8^t \ln 0.8\right) \\ &= -12\ln 0.8(0.8)^t \qquad (1) \end{aligned}$$

The speed at which the leading edge of the lava is flowing is $s'(t) = -12\ln 0.8(0.8)^t$.

b)

$$\begin{aligned} s'(1) &= -12\ln 0.8(0.8)^1 \\ &\doteq 2.1 \end{aligned}$$

After 1 h, the lava is flowing at 2.1 km/h.

$$\begin{aligned} s'(4) &= -12\ln 0.8(0.8)^4 \\ &\doteq 1.1 \end{aligned}$$

After 4 h, the lava is flowing at 1.1 km/h.

c)

$$\begin{aligned} s(t) &= 12(2 - 0.8^t) \\ \frac{s(t)}{12} &= 2 - (0.8)^t \\ (0.8)^t &= 2 - \frac{s(t)}{12} \\ t\ln 0.8 &= \ln\left(2 - \frac{s(t)}{12}\right) \\ t &= \frac{\ln\left(2 - \frac{s(t)}{12}\right)}{\ln 0.8} \end{aligned}$$

d)

$$\begin{aligned} t &= \frac{\ln\left(2 - \frac{s}{12}\right)}{\ln 0.8} \\ \frac{dt}{ds} &= \frac{1}{\ln 0.8}\left[\frac{1}{2 - \frac{s}{12}} \cdot \left(-\frac{1}{12}\right)\right] \\ &= -\frac{1}{(24 - s)\ln 0.8} \end{aligned}$$

$\frac{dt}{ds}$ represents the time it takes for the leading edge of the lava to advance 1 km.

e) When the leading edge of the lava flow is 5 km from the crater, it takes $-\frac{1}{(24 - 5)\ln 0.8}$ or approximately 0.24 h for the leading edge of the lava to advance 1 km.

Section 7.7 Page 470 Question 15

a)

$$\begin{aligned} y &= a^x \\ \ln y &= x\ln a \\ \frac{1}{y} \cdot \frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y\ln a \end{aligned}$$

Substitute $y = a^x$.

$$\frac{da^x}{dx} = a^x \ln a$$

b)

$$\begin{aligned} y &= \ln x^n \\ y &= n\ln x \\ \frac{dy}{dx} &= n \cdot \frac{1}{x} \\ &= \frac{n}{x} \end{aligned}$$

Substitute $y = \ln x^n$.

$$\frac{d\ln x^n}{dx} = \frac{n}{x}$$

c)

$$\begin{aligned} y &= \ln nx \\ y &= \ln n + \ln x \\ \frac{dy}{dx} &= \frac{d\ln n}{dx} + \frac{d\ln x}{dx} \\ \frac{dy}{dx} &= \frac{d\ln x}{dx} \end{aligned}$$

Substitute $y = \ln nx$.

$$\frac{d\ln nx}{dx} = \frac{d\ln x}{dx}$$

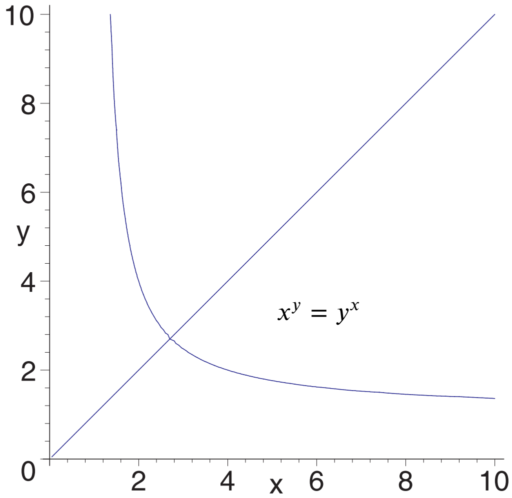
Section 7.7 Page 470 Question 16

$$\begin{aligned} y &= x^{\ln x} \\ \ln y &= \ln x(\ln x) \\ \frac{1}{y}\cdot\frac{dy}{dx} &= \ln x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{x} \\ \frac{dy}{dx} &= 2\left(\frac{y}{x}\right)\ln x \end{aligned}$$

Substitute $y = x^{\ln x}$.

$$\begin{aligned} \frac{dy}{dx} &= 2\left(\frac{x^{\ln x}}{x}\right)\ln x \\ &= 2\left(x^{\ln x - 1}\right)\ln x \end{aligned}$$

Section 7.7 Page 470 Question 17



The locus of points includes the functions $f(x) = x$ and $g(x) = \frac{1}{x}$.

Section 7.7 Page 470 Question 18

a) Determine the condition for which $e^{kx} > x^m$, where $k, m, \in \Re^+$.

$$e^{kx} > x^m \tag{1}$$

Take the natural logarithm of both sides of (1).

$$\begin{aligned} kx &> m\ln x \\ \frac{x}{\ln x} &> \frac{m}{k} \end{aligned} \tag{2}$$

As $x \to \infty$, $\frac{x}{\ln x} \to \infty$, thereby exceeding the constant value, $\frac{m}{k}$.

b) For the given functions, $f'(x) = ke^{kx}$ and $g'(x) = mx^{m-1}$. Determine the condition for which $f'(x) > g'(x)$, where $k, m, \in \Re^+$.

$$\begin{aligned} ke^{kx} &> mx^{m-1} \\ e^{kx} &> \frac{m}{k}x^{m-1} \end{aligned} \tag{1}$$

Take the natural logarithm of both sides of (1).

$$kx > \ln\frac{m}{k} + (m-1)\ln x$$

$$kx - (m-1)\ln x > \ln\frac{m}{k} \tag{2}$$

Differentiation of the left side of (2) reveals that the expression is increasing without bound for $x > \frac{m-1}{k}$. Hence, as $x \to \infty$, $kx - (m-1)\ln x \to \infty$, thereby exceeding the constant value, $\ln\frac{m}{k}$.

c) Yes. Explanations may vary.

7.8 Applications of Exponential and Logarithmic Functions

Practise

Section 7.8 Page 475 Question 1

a)

$$\begin{aligned}
5000 &= 4000(1+0.075)^n\\
(1.075)^n &= 1.25\\
n\ln 1.075 &= \ln 1.25\\
n &= \frac{\ln 1.25}{\ln 1.075}\\
&\doteq 3.0855
\end{aligned}$$

b)

$$\begin{aligned}
2600 &= 1900(1+0.0575)^n\\
(1.0575)^n &= \frac{26}{19}\\
n\ln 1.0575 &= \ln\frac{26}{19}\\
n &= \frac{\ln\frac{26}{19}}{\ln 1.0575}\\
&\doteq 5.6103
\end{aligned}$$

c)

$$\begin{aligned}
8500 &= 5800\left(1+\frac{0.065}{2}\right)^{2n}\\
(1+.0325)^{2n} &= \frac{85}{58}\\
2n\ln 1.0325 &= \ln\frac{85}{58}\\
n &= \frac{\ln\frac{85}{58}}{2\ln 1.0325}\\
&\doteq 5.9752
\end{aligned}$$

d)

$$\begin{aligned}
3000 &= 1700\left(1+\frac{0.0875}{4}\right)^{4n}\\
(1+.021\,875)^{4n} &= \frac{30}{17}\\
4n\ln(1.021\,875) &= \ln\frac{30}{17}\\
n &= \frac{\ln\frac{30}{17}}{4\ln 1.021\,875}\\
&\doteq 6.5620
\end{aligned}$$

e)

$$\begin{aligned}
10\,000 &= 2500e^{0.08t}\\
e^{0.08t} &= 4\\
0.08t &= \ln 4\\
t &= \frac{\ln 4}{0.08}\\
&\doteq 17.3287
\end{aligned}$$

f)

$$\begin{aligned}
12\,000 &= 9000e^{0.09t}\\
e^{0.09t} &= \frac{4}{3}\\
0.09t &= \ln\frac{4}{3}\\
t &= \frac{\ln\frac{4}{3}}{0.09}\\
&\doteq 3.1965
\end{aligned}$$

g)

$$\begin{aligned}
7000 &= 1200e^{12k}\\
e^{12k} &= \frac{35}{6}\\
12k &= \ln\frac{35}{6}\\
k &= \frac{\ln\frac{35}{6}}{12}\\
&\doteq 0.1470
\end{aligned}$$

h)

$$\begin{aligned}
2400 &= 400e^{10k}\\
e^{10k} &= 6\\
10k &= \ln 6\\
k &= \frac{\ln 6}{10}\\
&\doteq 0.1792
\end{aligned}$$

Apply, Solve, Communicate

Section 7.8 Page 475 Question 2

From the given information, $T = 65°\text{C}$, $T_S = 21°\text{C}$, $T_0 = 86°\text{C}$, and $t = 15$ min.

$$\begin{aligned}
T - T_S &= (T_0 - T_S)e^{kt} \\
65 - 21 &= (86 - 21)e^{15k} \\
44 &= 65e^{15k} \\
\frac{44}{65} &= e^{15k} \\
15k &= \ln\frac{44}{65} \\
k &= \frac{1}{15}\ln\frac{44}{65} \\
&\doteq -0.0260
\end{aligned}$$

The cooling constant, k, is -0.0260.

Section 7.8 Page 475 Question 3

Determine the model defining the growth of the population. From the given information, $P = 3600$, $P_0 = 1800$, and $t = 0.035$.

$$\begin{aligned}
P &= P_0e^{kt} \\
3600 &= 1800e^{0.035k} \\
2 &= e^{0.035k} \\
0.035k &= \ln 2 \\
k &= \frac{\ln 2}{0.035} \\
&\doteq 19.8042
\end{aligned}$$

The population model is $P = 1800e^{19.8042t}$. Determine t for $P = 5400$.

$$\begin{aligned}
P &= 5400 \\
1800e^{19.8042t} &= 5400 \\
e^{19.8042t} &= 3 \\
19.8042t &= \ln 3 \\
t &= \frac{\ln 3}{19.8042} \\
&\doteq 0.05547
\end{aligned}$$

The population will triple in 0.055 days.

Section 7.8 Page 475 Question 4

a) Use $P = P_0(1 + r)^n$ with $r = 0.09$.

$$\begin{aligned}
P &= 13\,000 \\
6500(1 + 0.09)^n &= 13\,000 \\
1.09^n &= 2 \\
n\ln 1.09 &= \ln 2 \\
n &= \frac{\ln 2}{\ln 1.09} \\
&\doteq 8.043
\end{aligned}$$

It will take 8.043 years for the amount to double.

b) Use $P = P_0(1+r)^{4n}$ with $r = \frac{0.0875}{4}$ or 0.021 875.

$$\begin{aligned}
P &= 13\,000 \\
6500(1 + 0.021\,875)^{4n} &= 13\,000 \\
1.021\,875^{4n} &= 2 \\
4n\ln 1.021\,875 &= \ln 2 \\
n &= \frac{\ln 2}{4\ln 1.021\,875} \\
&\doteq 8.008
\end{aligned}$$

It will take 8.008 years for the amount to double.

c) Use $P = P_0(1 + r)^{2n}$ with $r = \frac{0.065}{2}$ or 0.0325.

$$\begin{aligned}
P &= 13\,000 \\
6500(1 + 0.0325)^{2n} &= 13\,000 \\
1.0325^{2n} &= 2 \\
2n\ln 1.0325 &= \ln 2 \\
n &= \frac{\ln 2}{2\ln 1.0325} \\
&\doteq 10.836
\end{aligned}$$

It will take 10.836 years for the amount to double.

d) Use $P = P_0(1 + r)^{12n}$ with $r = \frac{0.06}{12}$ or 0.005.

$$\begin{aligned}
P &= 13\,000 \\
6500(1 + 0.005)^{12n} &= 13\,000 \\
1.005^{12n} &= 2 \\
12n\ln 1.005 &= \ln 2 \\
n &= \frac{\ln 2}{12\ln 1.005} \\
&\doteq 11.581
\end{aligned}$$

It will take 11.581 years for the amount to double.

e) Use $P = P_0e^{rt}$ with $r = 0.08$.

$$\begin{aligned} P &= 13\,000 \\ 6500e^{0.08t} &= 13\,000 \\ e^{0.08t} &= 2 \\ 0.08t &= \ln 2 \\ t &= \frac{\ln 2}{0.08} \\ &\doteq 8.664 \end{aligned}$$

It will take 8.664 years for the amount to double.

f) Use $P = P_0e^{rt}$ with $r = 0.095$.

$$\begin{aligned} P &= 13\,000 \\ 6500e^{0.095t} &= 13\,000 \\ e^{0.095t} &= 2 \\ 0.095t &= \ln 2 \\ t &= \frac{\ln 2}{0.095} \\ &\doteq 7.296 \end{aligned}$$

It will take 7.296 years for the amount to double.

Section 7.8 Page 475 Question 5

a) Use $P = 3000$, $P_0 = 1500$, and $t = 8$.

$$\begin{aligned} P &= P_0e^{kt} \\ 3000 &= 1500e^{8k} \\ 2 &= e^{8k} \\ 8k &= \ln 2 \\ k &= \frac{\ln 2}{8} \end{aligned}$$

The population model is $P = 1500e^{\frac{t\ln 2}{8}}$.

b) Determine t for $P = 25\,000$.

$$\begin{aligned} P &= 25\,000 \\ 1500e^{\frac{t\ln 2}{8}} &= 25\,000 \\ e^{\frac{t\ln 2}{8}} &= \frac{50}{3} \\ \frac{\ln 2}{8}t &= \ln \frac{50}{3} \\ t &= \frac{8\ln \frac{50}{3}}{\ln 2} \\ &\doteq 32.471 \end{aligned}$$

The population will reach 25 000 in 32.471 h.

Section 7.8 Page 475 Question 6

a) Use $P = P_0(1 + r)^n$, $P = 11\,000$, $P_0 = 4000$, and $n = 12$.

$$\begin{aligned} P &= 11\,000 \\ 4000(1 + r)^{12} &= 11\,000 \\ (1 + r)^{12} &= \frac{11}{4} \\ 1 + r &= \sqrt[12]{\frac{11}{4}} \\ r &= \sqrt[12]{\frac{11}{4}} - 1 \\ &\doteq 0.087\,955\,3 \end{aligned}$$

The required interest rate is 8.795 53%.

b) Use $P = P_0(1 + r)^n$, $P = 11\,000$, $P_0 = 4000$, and $n = 15$.

$$\begin{aligned} P &= 11\,000 \\ 4000(1 + r)^{15} &= 11\,000 \\ (1 + r)^{15} &= \frac{11}{4} \\ 1 + r &= \sqrt[15]{\frac{11}{4}} \\ r &= \sqrt[15]{\frac{11}{4}} - 1 \\ &\doteq 0.069\,766\,1 \end{aligned}$$

The required interest rate is 6.976 61%.

Section 7.8 Page 475 Question 7

a) From the given information, $T_S = 30°\text{C}$, $T_0 = 1250°\text{C}$, and $k = -0.014$.

$$\begin{aligned} T - T_S &= (T_0 - T_S)e^{kt} \\ T - 30 &= (1250 - 30)e^{-0.014t} \\ T &= 1220e^{-0.014t} + 30 \end{aligned}$$

The temperature model is $T = 1220e^{-0.014t} + 30$.

c) Determine a new temperature model using $T_S = 15°\text{C}$.

$$\begin{aligned} T - T_S &= (T_0 - T_S)e^{kt} \\ T - 15 &= (1250 - 15)e^{-0.014t} \\ T &= 1235e^{-0.014t} + 15 \end{aligned}$$

Determine the new value for t.

$$\begin{aligned} 1235e^{-0.014t} + 15 &= 1100 \\ 1235e^{-0.014t} &= 1085 \\ e^{-0.014t} &= \frac{217}{247} \\ -0.014t &= \ln\frac{217}{247} \\ t &= \frac{\ln\frac{217}{247}}{-0.014} \\ &\doteq 9.25 \end{aligned}$$

The steel can be worked for 9.25 min.

This is 9.37 – 9.25 or 0.12 min less.

b) Use $T = 1100°\text{C}$.

$$\begin{aligned} T &= 1100 \\ 1220e^{-0.014t} + 30 &= 1100 \\ 1220e^{-0.014t} &= 1070 \\ e^{-0.014t} &= \frac{107}{122} \\ -0.014t &= \ln\frac{107}{122} \\ t &= \frac{\ln\frac{107}{122}}{-0.014} \\ &\doteq 9.37 \end{aligned}$$

The steel can be worked for 9.37 min.

d) Use $T = 30°\text{C}$.

$$\begin{aligned} 1220e^{-0.014t} + 30 &= 30 \\ 1220e^{-0.014t} &= 0 \\ e^{-0.014t} &= 0 \qquad (1) \end{aligned}$$

The left side of (1) approaches 0 as $t \to \infty$. Thus, the steel would never reach the temperature of the forge shop. This is not realistic.

Section 7.8 Page 475 Question 8

From the given information, $T = 150°\text{C}$, $T_S = 28°\text{C}$, $T_0 = 190°\text{C}$, and $t = 5$ min. Determine k.

$$\begin{aligned} T - T_S &= (T_0 - T_S)e^{kt} \\ 150 - 28 &= (190 - 28)e^{5k} \\ 122 &= 162e^{5k} \\ e^{5k} &= \frac{61}{81} \\ 5k &= \ln\frac{61}{81} \\ k &= \frac{1}{5}\ln\frac{61}{81} \end{aligned}$$

The temperature model is $T = 162e^{\frac{t}{5}\ln\frac{61}{81}} + 28$. Determine t when $T = 80°\text{C}$.

$$\begin{aligned} T &= 80 \\ 162e^{\frac{t}{5}\ln\frac{61}{81}} + 28 &= 80 \\ 162e^{\frac{t}{5}\ln\frac{61}{81}} &= 52 \\ e^{\frac{t}{5}\ln\frac{61}{81}} &= \frac{26}{81} \\ \frac{t}{5}\ln\frac{61}{81} &= \ln\frac{26}{81} \\ t &= \frac{5\ln\frac{26}{81}}{\ln\frac{61}{81}} \\ &\doteq 20.0 \end{aligned}$$

It will take 20 min for the engine to cool to 80°C.

Section 7.8 Page 475 Question 9

a) Use $P = 2100$, $P_0 = 700$, and $t = 30$ min.

$$\begin{aligned} P &= P_0e^{kt} \\ 2100 &= 700e^{30k} \\ 3 &= e^{30k} \\ 30k &= \ln 3 \\ k &= \frac{\ln 3}{30} \end{aligned}$$

The population model is $P = 700e^{\frac{t\ln 3}{30}}$.

b) Determine t for $P = 18\ 000$.

$$\begin{aligned} P &= 18\ 000 \\ 700e^{\frac{t\ln 3}{30}} &= 18\ 000 \\ e^{\frac{t\ln 3}{30}} &= \frac{180}{7} \\ \frac{\ln 3}{30}t &= \ln\frac{180}{7} \\ t &= \frac{30\ln\frac{180}{7}}{\ln 3} \\ &\doteq 88.67 \end{aligned}$$

The population will reach 18 000 in 88.67 min.

Section 7.8 Page 475 Question 10

a) Determine the value of k using the half-life. Let $A(t)$ be the amount of thorium-234, in grams, after t days. When $t = 25$, $A = 0.5A_0$.

$$\begin{aligned} A &= 0.5A_0 \\ A_0e^{25k} &= 0.5A_0 \\ e^{25k} &= 0.5 \\ 25k &= \ln 0.5 \\ k &= \frac{\ln 0.5}{25} \end{aligned}$$

The model is $A = A_0e^{\frac{t\ln 0.5}{25}}$.

Determine t for $A = 500$ g and $A_0 = 30\ 000$ g.

$$\begin{aligned} A &= 500 \\ 30\ 000e^{\frac{t\ln 0.5}{25}} &= 500 \\ e^{\frac{t\ln 0.5}{25}} &= \frac{1}{60} \\ \frac{t\ln 0.5}{25} &= \ln\frac{1}{60} \\ t &= \frac{-25\ln 60}{\ln 0.5} \\ &\doteq 147.67 \end{aligned}$$

It will take 147.67 days for 30 000 g to decay to 500 g.

b) Using this model, $A \to 0$ as $t \to \infty$. Theoretically, there will always be some amount left.

Section 7.8 Page 476 Question 11

For continuous compounding, use $A = A_0e^{rt}$. From the given information, $A_0 = \$12\ 000$, $A = \$30\ 000$, $r = 0.07$. Solve for t.

$$\begin{aligned} 12\ 000e^{0.07t} &= 30\ 000 \\ e^{0.07t} &= 2.5 \\ 0.07t &= \ln 2.5 \\ t &= \frac{\ln 2.5}{0.07} \\ &\doteq 13.09 \end{aligned}$$

Continuous compounding requires 13.09 years to achieve the goal.

For quarterly compounding, use $A = A_0(1 + r)^{4t}$. From the given information, $A_0 = \$12\ 000$, $A = \$30\ 000$, $r = \dfrac{0.0725}{4}$. Solve for t.

$$\begin{aligned} 12\ 000\left(1 + \frac{0.0725}{4}\right)^{4t} &= 30\ 000 \\ (1 + 0.018\ 125)^{4t} &= 2.5 \\ 4t\ln 1.018\ 125 &= \ln 2.5 \\ t &= \frac{\ln 2.5}{4\ln 1.018\ 125} \\ &\doteq 12.75 \end{aligned}$$

Quarterly compounding requires 12.75 years to achieve the goal.

Richard will achieve his goal faster with the quarterly compounding strategy.

Section 7.8 Page 476 Question 12

Determine the population model. Use $P = 2P_0$ and $t = 25$ years.

$$\begin{aligned} P &= 2P_0 \\ P_0e^{25k} &= 2P_0 \\ e^{25k} &= 2 \\ 25k &= \ln 2 \\ k &= \frac{\ln 2}{25} \end{aligned}$$

The population model is $P = P_0e^{\frac{t\ln 2}{25}}$.

Determine t for $P = 3P_0$.

$$\begin{aligned} P &= 3P_0 \\ P_0e^{\frac{t\ln 2}{25}} &= 3P_0 \\ e^{\frac{t\ln 2}{25}} &= 3 \\ \frac{t\ln 2}{25} &= \ln 3 \\ t &= \frac{25\ln 3}{\ln 2} \\ &\doteq 39.624 \end{aligned}$$

It will take 39.624 years for the population to triple.

Section 7.8 Page 476 Question 13

a) Determine the population model for Mississauga. Use $P_0 = 33\ 310$, $P = 315\ 056$, and $t = 30$ years to determine k.

$$\begin{aligned} P &= 315\ 056 \\ 33\ 310e^{30k} &= 315\ 056 \\ 30k &= \ln\frac{315\ 056}{33\ 310} \\ k &= \frac{1}{30}\ln\frac{315\ 056}{33\ 310} \end{aligned}$$

The population model is $P = 33\ 310e^{\frac{t}{30}\ln\frac{315\ 056}{33\ 310}}$.

Determine P for $t = 2021 - 1951$ or 70.

$$\begin{aligned} P &= 33\ 310e^{\frac{70}{30}\ln\frac{315\ 056}{33\ 310}} \\ &\doteq 6\ 301\ 906 \end{aligned}$$

The model suggests a population of 6 301 906 in 2021. This is not likely.

b) Determine the population model for Caledon. Use $P_0 = 8767$, $P = 26\ 645$, and $t = 30$ years to determine k.

$$\begin{aligned} P &= 26\ 645 \\ 8767e^{30k} &= 26\ 645 \\ 30k &= \ln\frac{26\ 645}{8767} \\ k &= \frac{1}{30}\ln\frac{26\ 645}{8767} \end{aligned}$$

The population model is $P = 8767e^{\frac{t}{30}\ln\frac{26\ 645}{8767}}$.

Determine P for $t = 2031 - 1951$ or 80.

$$\begin{aligned} P &= 8767e^{\frac{80}{30}\ln\frac{26\ 645}{8767}} \\ &\doteq 169\ 912 \end{aligned}$$

The model suggests a population of 169 912 in 2031. This figure is possible.

Section 7.8 Page 476 Question 14

a)

$$\begin{aligned} T - T_S &= (T_0 - T_S)e^{kt} \\ T - 20 &= (350 - 20)e^{-0.2(10)} \\ T &= 330e^{-2} + 20 \\ &\doteq 64.66 \end{aligned}$$

After 10 min, the temperature is 64.66°C.

b)

$$\begin{aligned} T &= 75 \\ 330e^{-0.2t} + 20 &= 75 \\ 330e^{-0.2t} &= 55 \\ e^{-0.2t} &= \frac{1}{6} \\ -0.2t &= -\ln 6 \\ t &= 5\ln 6 \\ &\doteq 8.96 \end{aligned}$$

It will take the bricks 8.96 min to cool to 75°C.

Section 7.8 Page 476 Question 15

Determine the value of k using the half-life. Let $A(t)$ be the amount of radium, in grams, after t years. When $t = 1656$, $A = 0.5A_0$.

$$\begin{aligned} A &= 0.5A_0 \\ A_0e^{1656k} &= 0.5A_0 \\ e^{1656k} &= 0.5 \\ 1656k &= \ln 0.5 \\ k &= \frac{\ln 0.5}{1656} \end{aligned}$$

The model is $A = A_0e^{\frac{t\ln 0.5}{1656}}$.

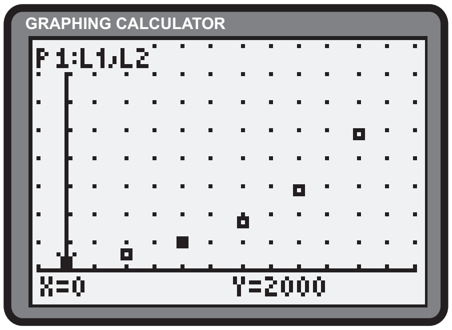
Determine t for $A = 12$ g and $A_0 = 50$ g.

$$\begin{aligned} A &= 12 \\ 50e^{\frac{t\ln 0.5}{1656}} &= 12 \\ e^{\frac{t\ln 0.5}{1656}} &= \frac{6}{25} \\ \frac{t\ln 0.5}{1656} &= \ln\frac{6}{25} \\ t &= \frac{1656\ln\frac{6}{25}}{\ln 0.5} \\ &\doteq 3410 \end{aligned}$$

It will take 3410 years for 50 g to decay to 12 g.

Section 7.8 Page 476 Question 16

a)



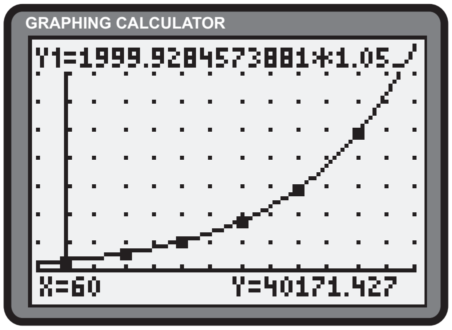
Estimates may vary.

b) The doubling time is estimated to be 14 min.

c) The tripling time is estimated to be 22 min.

d) After 60 min, the number of cells is estimated to be 40 000.

e) The exponential regression feature of the graphing calculator suggests the model $y \doteq 1999.93(1.0513)^x$.



f) Use $P_0 = 2000$, $P = 3297$, and $t = 10$ to find k.

$$\begin{aligned} P &= 3297 \\ 2000e^{10k} &= 3297 \\ e^{10k} &= \frac{3297}{2000} \\ 10k &= \ln\frac{3297}{2000} \\ k &= \frac{1}{10}\ln\frac{3297}{2000} \\ &\doteq 0.05 \end{aligned}$$

The data is represented by the model $P = 2000e^{0.05t}$. The table belows confirms the model is accurate.

Time (min)	Number of Bacteria	$P = 2000e^{0.05t}$
0	2 000	2 000
10	3 297	3 297
20	5 437	5 437
30	8 963	8 963
40	14 778	14 778
50	24 365	24 365

Review of Key Concepts

7.1 Exponential Functions

Section Review Page 478 Question 1

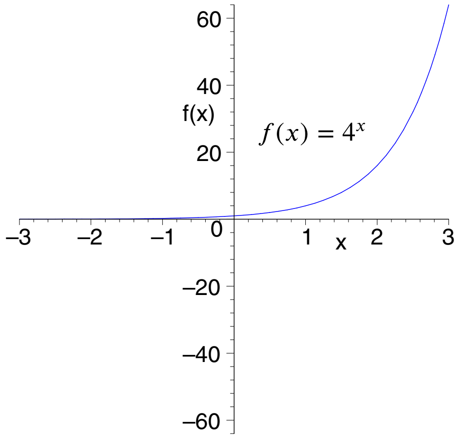
a) Answers may vary. The domain ($x \in \Re$) and range ($y > 0$) of both functions are the same. The y-intercept of both functions is 1. Both functions are increasing throughout their entire domain. As $x \to -\infty$, both functions approach the x-axis, with $y = 5^x$ approaching more rapidly. As $x \to \infty$, both functions increase without bound, with $y = 5^x$ increasing more rapidly.

b) Answers may vary. The domain ($x \in \Re$) and range ($y > 0$) of both functions are the same. The y-intercept of both functions is 1. Both functions are decreasing throughout their entire domain. As $x \to -\infty$, both functions increase without bound, with $y = \left(\frac{1}{5}\right)^x$ increasing more rapidly. As $x \to \infty$, both functions approach the x-axis, with $y = \left(\frac{1}{5}\right)^x$ decreasing more rapidly.

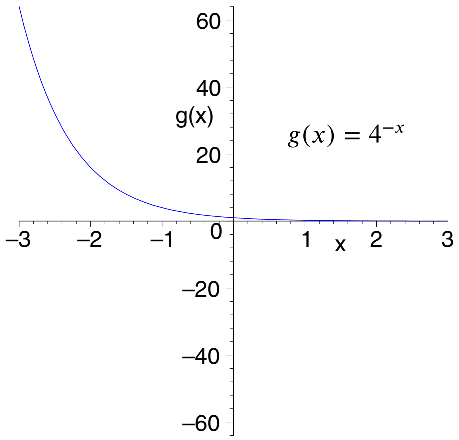
Section Review Page 478 Question 2

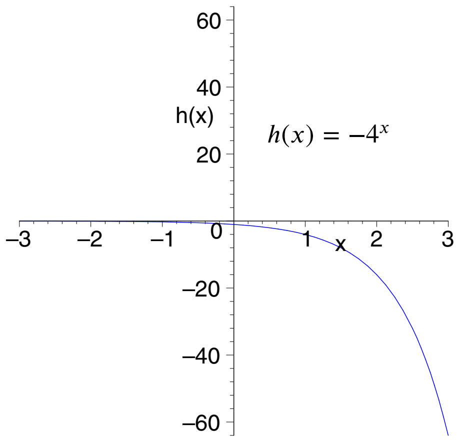
a)

x	$y = 4^x$
−3	$y = 4^{-3} = \frac{1}{64}$
−2	$y = 4^{-2} = \frac{1}{16}$
−1	$y = 4^{-1} = \frac{1}{4}$
0	$y = 4^0 = 1$
1	$y = 4^1 = 4$
2	$y = 4^2 = 16$
3	$y = 4^3 = 64$



b)





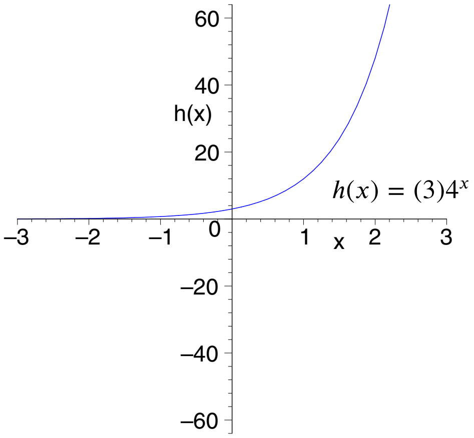
c) $y = 4^x$: y-intercept: 1; domain: $\Re$; range: $y > 0$; horizontal asymptote: $y = 0$.

$y = 4^{-x}$: y-intercept: 1; domain: $\Re$; range: $y > 0$; horizontal asymptote: $y = 0$.

$y = -4^x$: y-intercept: −1; domain: $\Re$; range: $y < 0$; horizontal asymptote: $y = 0$.

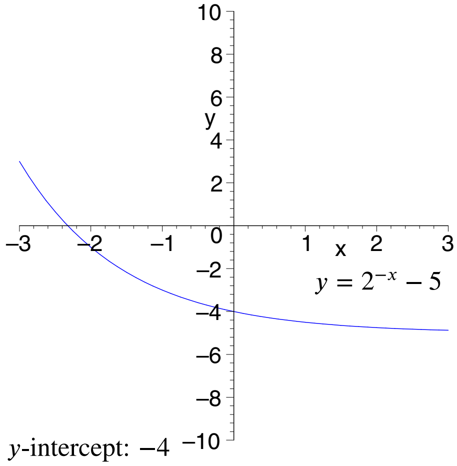
Section Review Page 478 Question 3

a) i)



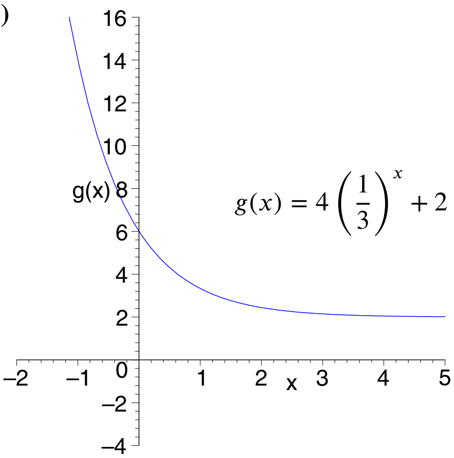
ii) y-intercept: 3
iii) domain: $\Re$; range: $y > 0$
iv) horizontal asymptote: $y = 0$

b) i)



ii) y-intercept: -4
iii) domain: $\Re$; range: $y > -5$
iv) horizontal asymptote: $y = -5$

c) i)



ii) y-intercept: 6
iii) domain: $\Re$; range: $y > 2$
iv) horizontal asymptote: $y = 2$

Section Review Page 478 Question 4

a) As $x \to -\infty$, $5^x \to 0$. Thus, $\lim\limits_{x\to\infty} 5^{-x} = 0$.

b) As $x \to 0^+$, $-\frac{1}{x} \to -\infty$. Hence $7^{-\frac{1}{x}} \to 0$. Thus, $\lim\limits_{x\to 0^+} 7^{-\frac{1}{x}} = 0$.

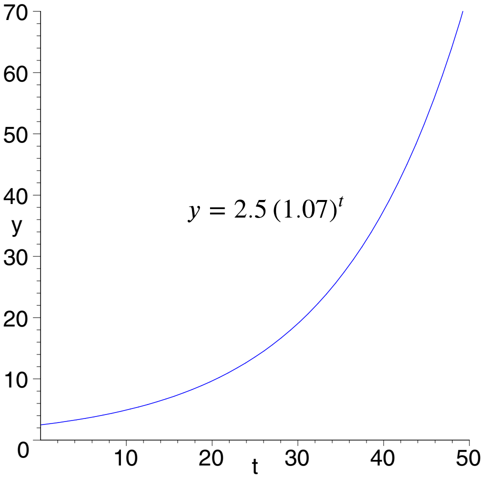
c) As $x \to \infty$, $\frac{1}{x} \to 0$. Hence $6^{\frac{1}{x}} \to 1$. Thus, $\lim\limits_{x\to\infty} 6^{\frac{1}{x}} = 1$.

d) As $x \to -3^+$, $\frac{1}{x+3} \to \infty$. Hence $4^{\frac{1}{x+3}} \to \infty$.

Thus, $\lim\limits_{x\to -3^+} 4^{\frac{1}{x+3}} = \infty$.

Section Review Page 478 Question 5

a)



b) For the year 2007, $t = 2007 - 2001$ or 6.

$$\begin{aligned} y &= 2.5(1.07)^6 \\ &\doteq 3.75 \end{aligned}$$

The approximate value of the card in 2007 will be \$3.75.

c)

$$\begin{aligned} y &= 70 \\ 2.5(1.07)^t &= 70 \\ 1.07^t &= 28 \\ t \ln 1.07 &= \ln 28 \\ t &= \frac{\ln 28}{\ln 1.07} \\ &\doteq 49 \end{aligned}$$

The card will be worth \$50 in approximately 49 years or in 2050.

7.2 Logarithmic Functions

Section Review Page 478 Question 6

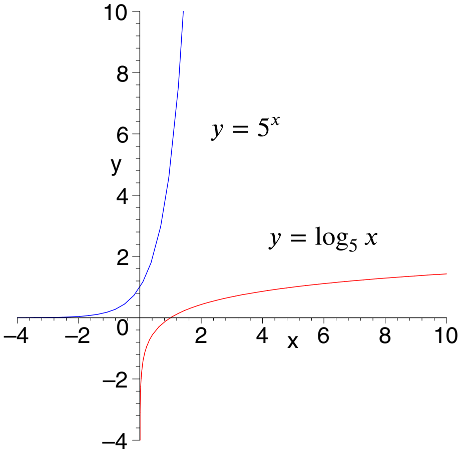
Exponential Form	**Logarithmic Form**
$5^2 = 25$	$\log_5 25 = 2$
$3^4 = 81$	$\log_3 81 = 4$
$3^x = 18$	$\log_3 18 = x$
$10^9 = x - 6$	$\log(x - 6) = 9$

Section Review Page 478 Question 7

a) $\log_4 64 = 3$

b) $\log_8 1 = 0$

Section Review Page 478 Question 8



$y = 5^x$: domain: $\Re$; range: $y > 0$

$y = \log_5 x$: domain: $x > 0$; range: $\Re$

Section Review Page 478 Question 9

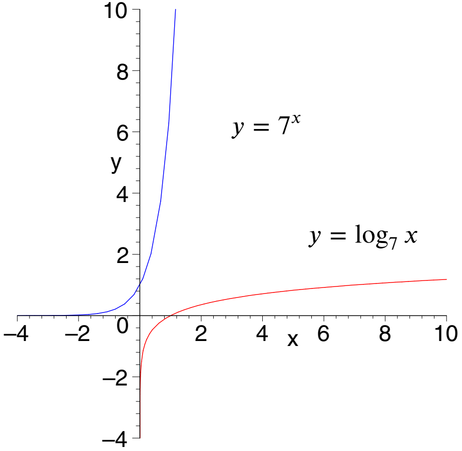
a)

$$y = \log_7 x$$
$$7^y = x$$

Interchange x and y.

$$y = 7^x$$

The inverse of $y = \log_7 x$ is $y = 7^x$.



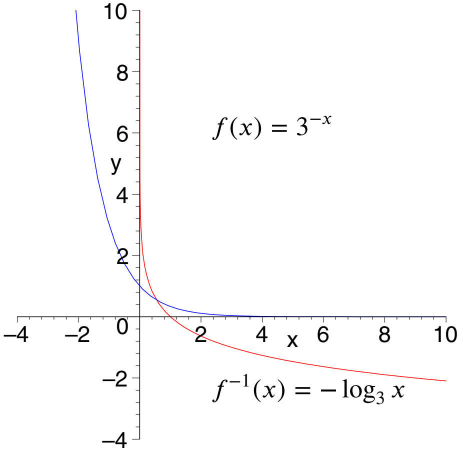
b)

$$y = 3^{-x}$$
$$\log_3 y = -x$$

Interchange x and y.

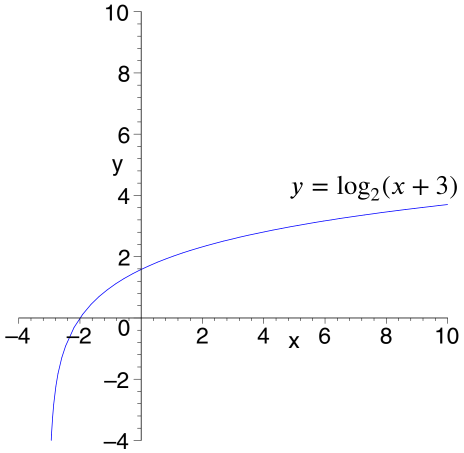
$$y = -\log_3 x$$

The inverse of $f(x) = 3^{-x}$ is $f^{-1}(x) = -\log_3 x$.



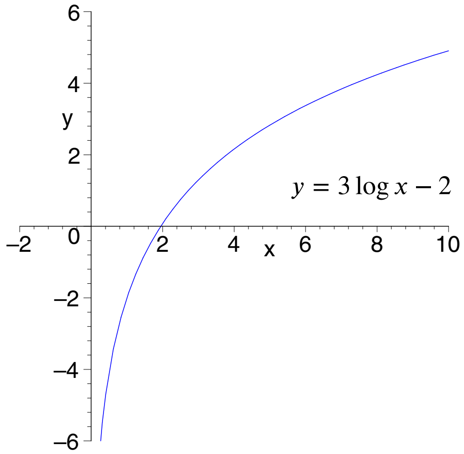
Section Review Page 478 Question 10

a)



domain: $x > -3$; range: $\Re$; asymptote: $x = -3$

b)



domain: $x > 0$; range: $\Re$; asymptote: $x = 0$

Section Review Page 478 Question 11

a)

$$\begin{aligned} x^2 &> 0 \\ |x| &> 0 \\ x &\neq 0 \end{aligned}$$

The domain of $y = \log_3 x^2$ is $x \neq 0$.

b)

$$\begin{aligned} \frac{1}{x} &> 0 \\ x &> 0 \end{aligned}$$

The domain of $y = \log\left(\frac{1}{x}\right)$ is $x > 0$.

c)

$$\begin{aligned} 1 - x^2 &> 0 \\ x^2 &< 1 \\ |x| &< 1 \end{aligned}$$

The domain of $y = \log_4(1 - x^2)$ is $x \in (-1, 1)$.

Section Review Page 478 Question 12

$$\begin{aligned} A &= 2(500) \\ 500\left(1 + \frac{0.12}{4}\right)^{4n} &= 2(500) \\ (1 + 0.03)^{4n} &= 2 \\ 4n \ln 1.03 &= \ln 2 \\ n &= \frac{\ln 2}{4 \ln 1.03} \\ &\doteq 5.862 \end{aligned}$$

It will take 5.862 years to double the investment.

7.3 Laws of Logarithms

Section Review Page 478 Question 13

a) $\log_3 xy = \log_3 x + \log_3 y$

b) $\log_7((x - 1)(x + 5)) = \log_7(x - 1) + \log_7(x + 5)$

c)

$$\begin{aligned} \log\left(\frac{x}{2y}\right) &= \log x - \log 2y \\ &= \log x - \log 2 - \log y \end{aligned}$$

d)

$$\begin{aligned} \log_4\left(\frac{a^2}{3}\right) &= \log_4 a^2 - \log_4 3 \\ &= 2\log_4 |a| - \log_4 3,\ a \neq 0 \end{aligned}$$

Section Review Page 478 Question 14

a)

$$\begin{aligned} \log_5 21 - \log_5 7 + \log_5 2 &= \log_5 \frac{21(2)}{7} \\ &= \log_5 6 \end{aligned}$$

b)

$$\begin{aligned} 5\log m + 6\log n &= \log m^5 + \log n^6 \\ &= \log(m^5 n^6) \end{aligned}$$

c) $\frac{1}{3}\log_2 27 - \log_2 9 = \frac{1}{3}\log_2 3^3 - \log_2 3^2$
$= \log_2 3 - 2\log_2 3$
$= -\log_2 3$

d) $\log_7(x-2) + 2\log_7(x+2) = \log_7(x-2) + \log_7(x+2)^2$
$= \log_7\left[(x-2)(x+2)^2\right]$

Section Review Page 479 Question 15

a) $\log_4 32 - \log_4 2 = \log_4 \frac{32}{2}$
$= \log_4 16$
$= 2$

b) $\log_6 27 + \log_6 8 = \log_6(27 \times 8)$
$= \log_6 216$
$= 3$

c) $\log_3 \sqrt{3} = \log_3 3^{\frac{1}{2}}$
$= \frac{1}{2}$

d) $\log_8 4 = \log_8 8^{\frac{2}{3}}$
$= \frac{2}{3}$

Section Review Page 479 Question 16

a) $\log_2 5 = \frac{\log 5}{\log 2}$
$\doteq 2.3219$

b) $\log 7 \doteq 0.8451$

c) $\log_3 8 = \frac{\log 8}{\log 3}$
$\doteq 1.8928$

Section Review Page 479 Question 17

No. Since $\log_2 5 = \frac{\log 5}{\log 2}$ and $\log_5 2 = \frac{\log 2}{\log 5}$, these values are reciprocals of one another.

Section Review Page 479 Question 18

a)

$$\begin{aligned}\log P &= \frac{1}{2}(\log k + 3\log R)\\ \frac{1}{2}\log k + \frac{3}{2}\log R - \log P &= 0\\ \log k^{\frac{1}{2}} + \log R^{\frac{3}{2}} - \log P &= 0\\ \log\left(\frac{k^{\frac{1}{2}}R^{\frac{3}{2}}}{P}\right) &= 0\end{aligned}$$

b)

$$\begin{aligned}\log\left(\frac{k^{\frac{1}{2}}R^{\frac{3}{2}}}{P}\right) &= 0\\ 10^0 &= \frac{k^{\frac{1}{2}}R^{\frac{3}{2}}}{P}\end{aligned}$$

c) $P = k^{\frac{1}{2}}R^{\frac{3}{2}}$

7.4 Exponential and Logarithmic Equations

Section Review Page 479 Question 19

a)

$$\begin{aligned}2\log_4 x &= \log_4 64\\ \log_4 x^2 &= 3\\ x^2 &= 64\\ x &= 8,\ x > 0\end{aligned}$$

Check.

$$\begin{aligned}\textbf{L.S.} &= 2\log_4 8\\ &= \log_4 8^2\\ &= \log_4 64\\ &= \textbf{R.S.}\end{aligned}$$

b)

$$\begin{aligned}7^{3x} &= 8^{2x}\\ 3x\log 7 &= 2x\log 8\\ x(3\log 7 - 2\log 8) &= 0\\ x &= 0\end{aligned}$$

Check.

$$\begin{aligned}\textbf{L.S.} &= 7^{3(0)}\\ &= 1\\ \textbf{R.S.} &= 8^{2(0)}\\ &= 1\\ &= \textbf{L.S.}\end{aligned}$$

c)

$$\begin{aligned}
3(4)^{6x+5} &= 25\\
(4)^{6x}(4)^5 &= \frac{25}{3}\\
(4)^{6x} &= \frac{25}{3072}\\
6x\ln 4 &= \ln\frac{25}{3072}\\
x &= \frac{1}{6\ln 4}\ln\frac{25}{3072}\\
x &\doteq -0.5784
\end{aligned}$$

Check.

$$\begin{aligned}
\textbf{L.S.} &= 3(4)^{6(-0.5784)+5}\\
&\doteq 25\\
&= \textbf{R.S.}
\end{aligned}$$

d)

$$\begin{aligned}
\log(x+8)+\log(x-1) &= 1\\
\log(x^2+7x-8) &= 1\\
x^2+7x-8 &= 10\\
x^2+7x-18 &= 0\\
(x-2)(x+9) &= 0\\
x &= 2,\ x > -8
\end{aligned}$$

Check.

$$\begin{aligned}
\textbf{L.S.} &= \log(2+8)+\log(2-1)\\
&= \log 10 + \log 1\\
&= 1\\
&= \textbf{R.S.}
\end{aligned}$$

e)

$$\begin{aligned}
(5)^{2x}-(5)^x-20 &= 0\\
(5^x)^2-(5^x)-20 &= 0\\
(5^x-5)(5^x+4) &= 0\\
5^x-5 &= 0\\
x &= 1
\end{aligned}$$

Check.

$$\begin{aligned}
\textbf{L.S.} &= (5)^{2(1)}-(5)^1-20\\
&= 25-5-20\\
&= 0\\
&= \textbf{R.S.}
\end{aligned}$$

f)

$$\begin{aligned}
\log_7 x + \log_7 3 &= \log_7 24\\
\log_7 3x &= \log_7 24\\
3x &= 24\\
x &= 8
\end{aligned}$$

Check.

$$\begin{aligned}
\textbf{L.S.} &= \log_7 8 + \log_7 3\\
&= \log_7 24\\
&= \textbf{R.S.}
\end{aligned}$$

Section Review Page 479 Question 20

The value of this investment, A, can be expressed as $A = 1000(1.0375)^{2n}$, where n is measured in years.

a)

$$\begin{aligned}
A &= 1500\\
1000(1.0375)^{2n} &= 1500\\
1.0375^{2n} &= 1.5\\
2n\log 1.0375 &= \log 1.5\\
n &= \frac{\log 1.5}{2\log 1.0375}\\
&\doteq 5.5
\end{aligned}$$

It will take approximately 5 years 6 months for the investment to accumulate to \$1500.

b)

$$\begin{aligned}
A &= 2000\\
1000(1.0375)^{2n} &= 2000\\
1.0375^{2n} &= 2\\
2n\log 1.0375 &= \log 2\\
n &= \frac{\log 2}{2\log 1.0375}\\
&\doteq 9.4
\end{aligned}$$

It will take approximately 9 years 5 months for the investment to double.

c)

$$\begin{aligned}
A &= 3000\\
1000(1.0375)^{2n} &= 3000\\
1.0375^{2n} &= 3\\
2n\log 1.0375 &= \log 3\\
n &= \frac{\log 3}{2\log 1.0375}\\
&\doteq 14.9
\end{aligned}$$

It will take approximately 14 years 11 months for the investment to triple.

Section Review Page 479 Question 21

a) Let P be Canada's population.

$$\begin{aligned}
P &= P_0(1+r)^{x-1971}\\
P &= 21\ 962\ 082(1+0.0117)^{x-1971}\\
&= 21\ 962\ 082(1.0117)^{x-1971}
\end{aligned}$$

Canada's population can be modelled as $P = 21\ 962\ 082(1.0117)^{x-1971}$.

$$\begin{aligned} P(2001) &= 21\ 962\ 082(1.0117)^{2001-1971} \\ &\doteq 31\ 133\ 361 \end{aligned}$$

The model suggests Canada's population in the year 2001 was 31 133 361.

c)

$$\begin{aligned} P &= 21\ 962\ 082(1.0117)^{x-1971} \\ \frac{P}{21\ 962\ 082} &= (1.0117)^{x-1971} \\ \log\frac{P}{21\ 962\ 082} &= x - 1971 \log 1.0117 \\ x &= \frac{\log\frac{P}{21\ 962\ 082}}{\log 1.0117} + 1971 \end{aligned}$$

The model can be expressed as $x = \dfrac{\log\frac{P}{21\ 962\ 082}}{\log 1.0117} + 1971$.

d)

$$\begin{aligned} x(P) &= \frac{\log\frac{P}{21\ 962\ 082}}{\log 1.0117} + 1971 \\ x(50\ 000\ 000) &= \frac{\log\frac{50\ 000\ 000}{21\ 962\ 082}}{\log 1.0117} + 1971 \\ &\doteq 2041.7 \end{aligned}$$

Canada's population should reach 50 000 000 in the year 2041.

Section Review Page 479 Question 22

a)

$$\begin{aligned} N(t) &= 5000(2)^{\frac{t}{2}} \\ N(12) &= 5000(2)^{\frac{12}{2}} \\ &= 5000(2)^6 \\ &= 320\ 000 \end{aligned}$$

After 12 h, there are 320 000 bacteria.

b) Determine t for $N = 1\ 000\ 000$.

$$\begin{aligned} N &= 1\ 000\ 000 \\ 5000(2)^{\frac{t}{2}} &= 1\ 000\ 000 \\ 2^{\frac{t}{2}} &= 200 \\ \frac{t}{2}\ln 2 &= \ln 200 \\ t &= \frac{2\ln 200}{\ln 2} \\ &\doteq 15.3 \end{aligned}$$

The population will reach 1 000 000 in 15.3 h.

Section Review Page 479 Question 23

a) The formula can be expressed as $P(t) = P_0\left(1 + \dfrac{r}{100}\right)^t$.

b)

$$\begin{aligned} P(t) &= P_0\left(1 + \frac{r}{100}\right)^t \\ \frac{P(t)}{P_0} &= \left(1 + \frac{r}{100}\right)^t \\ \log\frac{P(t)}{P_0} &= t\log\left(1 + \frac{r}{100}\right) \\ t &= \frac{\log\frac{P(t)}{P_0}}{\log\left(1 + \frac{r}{100}\right)} \end{aligned}$$

Section Review Page 479 Question 24

a)

$$\begin{aligned} I &= 0.8(1 - 10^{-0.0434t}) \\ I(5) &= 0.8(1 - 10^{-0.0434(5)}) \\ &\doteq 0.31 \end{aligned}$$

After 5 s, the current in the circuit is 0.31 A.

b)

$$\begin{aligned} I &= 0.8\left(1 - 10^{-0.0434t}\right) \\ \frac{I}{0.8} &= 1 - 10^{-0.0434t} \\ 10^{-0.0434t} &= 1 - \frac{I}{0.8} \\ -0.0434t &= \log\left(1 - \frac{I}{0.8}\right) \\ t &= \frac{\log\left(1 - \frac{I}{0.8}\right)}{-0.0434} \end{aligned}$$

c)
$$\begin{aligned} t(I) &= \frac{\log\left(1 - \frac{I}{0.8}\right)}{-0.0434} \\ t(0.5) &= \frac{\log\left(1 - \frac{0.5}{0.8}\right)}{-0.0434} \\ &\doteq 9.81 \end{aligned}$$

It takes 9.81 s for the current to reach 0.5 A.

7.5 Logarithmic Scales

Section Review Page 479 Question 25

a)
$$\begin{aligned} \text{pH} &= -\log\left[\text{H}^+\right] \\ &= -\log\left(3.3 \times 10^{-4}\right) \\ &\doteq 3.5 \end{aligned}$$

The cheese has a pH level of 3.5.

b) Since 3.5 < 7.0, the cheese is acidic.

Section Review Page 479 Question 26

Let L_t and L_c be the loudness of the train and the conversation, respectively. Let I_t and I_c be the intensity of the train and the conversation, respectively.

$$\begin{aligned} L_t - L_c &= 100 - 50 \\ 10\log\frac{I_t}{I_0} - 10\log\frac{I_c}{I_0} &= 50 \\ \log\frac{I_t}{I_0} - \log\frac{I_c}{I_0} &= 5 \\ \log I_t - \log I_0 - \log I_c + \log I_0 &= 5 \\ \log\left(\frac{I_t}{I_0} \div \frac{I_c}{I_0}\right) &= 5 \\ \log\frac{I_t}{I_c} &= 5 \\ \frac{I_t}{I_c} &= 10^5 \\ I_t &= 100\,000 I_c \end{aligned}$$

The sound of the train is 100 000 times as intense as the sound of the conversation.

Section Review Page 479 Question 27

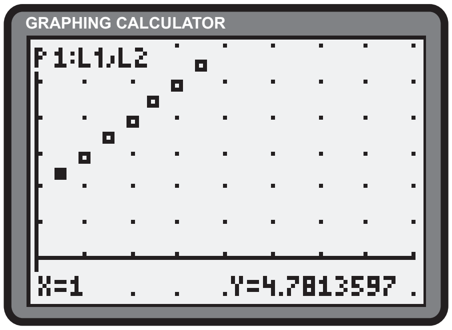
Let $M_1 = 7.2$ be the magnitude of the 1995 earthquake. Let $M_2 = 4.8$ be the magnitude of the 1906 earthquake.

$$\begin{aligned} \frac{I_1}{I_2} &= \frac{I_0 \times 10^{M_1}}{I_0 \times 10^{M_2}} \\ &= 10^{M_1 - M_2} \\ &= 10^{7.2-4.8} \\ &= 10^{2.4} \\ &\doteq 251 \end{aligned}$$

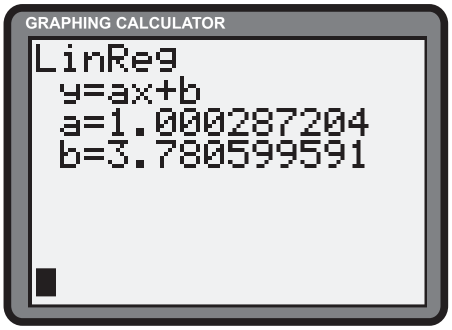
The earthquake in 1995 was 251 times as intense as the 1906 earthquake.

Section Review Page 480 Question 28

a)



b) Linear regression yields the model $\log_2 f = x + 3.78$ or $\log_2 f = x + \log_2 13.75$.



```
LinReg
 y=ax+b
 a=1.000287204
 b=3.780599591
```

c)

$$\begin{aligned}\log_2 f &= x + \log_2 13.75\\ 2^{\log_2 f} &= 2^{x+\log_2 13.75}\\ f &= 13.75(2)^x\end{aligned}$$

The equation can be expressed as $f = 13.75(2)^x$.

d)

$$\begin{aligned}f &= 13.75(2)^x\\ f(8) &= 13.75(2)^8\\ &= 3520\end{aligned}$$

The frequency of the 8th A note is 3520 Hz.

7.6 Derivatives of Exponential Functions

Section Review Page 480 Question 29

a)

$$\begin{aligned}y &= e^x\\ \frac{dy}{dx} &= \frac{de^x}{dx}\\ &= e^x\end{aligned}$$

b)

$$\begin{aligned}y &= x^3e^{-x}\\ \frac{dy}{dx} &= e^{-x}\frac{dx^3}{dx} + x^3\frac{de^{-x}}{d(-x)}\cdot\frac{d(-x)}{dx}\\ &= e^{-x}3x^2 + x^3e^{-x}(-1)\\ &= x^2e^{-x}(3-x)\end{aligned}$$

c)

$$\begin{aligned}f(x) &= x^2e^x\\ f'(x) &= e^x\frac{dx^2}{dx} + x^2\frac{de^x}{dx}\\ &= e^x2x + x^2e^x\\ &= xe^x(2+x)\end{aligned}$$

d)

$$\begin{aligned}y &= \frac{e^x}{x}\\ \frac{dy}{dx} &= \frac{x\frac{de^x}{dx} - e^x\frac{dx}{dx}}{x^2}\\ &= \frac{xe^x - e^x(1)}{x^2}\\ &= \frac{e^x(x-1)}{x^2}\end{aligned}$$

e)

$$\begin{aligned}h(x) &= e^{2x}\\ h'(x) &= \frac{de^{2x}}{d(2x)}\cdot\frac{d(2x)}{dx}\\ &= e^{2x}\cdot 2\\ &= 2e^{2x}\end{aligned}$$

f)

$$\begin{aligned}g(x) &= \frac{e^x}{1+e^{-3x}}\\ g'(x) &= \frac{\left(1+e^{-3x}\right)\frac{de^x}{dx} - e^x\left(\frac{d1}{dx} + \frac{de^{-3x}}{d(-3x)}\cdot\frac{d(-3x)}{dx}\right)}{(1+e^{-3x})^2}\\ &= \frac{\left(1+e^{-3x}\right)e^x - e^x\left(e^{-3x}(-3)\right)}{(1+e^{-3x})^2}\\ &= \frac{e^x\left(1+4e^{-3x}\right)}{(1+e^{-3x})^2}\end{aligned}$$

g)

$$\begin{aligned}y &= \frac{2}{e^x}\\ &= 2e^{-x}\\ \frac{dy}{dx} &= 2\cdot\frac{de^{-x}}{d(-x)}\cdot\frac{d(-x)}{dx}\\ &= 2e^{-x}(-1)\\ &= -\frac{2}{e^x}\end{aligned}$$

h)

$$\begin{aligned}f(x) &= (1-e^{2x})^2\\ f'(x) &= \frac{d(1-e^{2x})^2}{d(1-e^{2x})}\cdot\left(\frac{d1}{dx} - \frac{de^{2x}}{d(2x)}\cdot\frac{d(2x)}{dx}\right)\\ &= 2(1-e^{2x})(-e^{2x}(2))\\ &= -4e^{2x}(1-e^{2x})\end{aligned}$$

i)

$$\begin{aligned}k(x) &= \frac{e^{\sqrt{x}}}{e^{-1}}\\ &= e^{\sqrt{x}+1}\\ k'(x) &= \frac{de^{\sqrt{x}+1}}{d(\sqrt{x}+1)}\cdot\frac{d(\sqrt{x}+1)}{dx}\\ &= e^{\sqrt{x}+1}\cdot\frac{1}{2\sqrt{x}}\\ &= \frac{e^{\sqrt{x}+1}}{2\sqrt{x}}\end{aligned}$$

j)

$$\begin{aligned}y &= xe^{\sqrt{x}}\\ \frac{dy}{dx} &= e^{\sqrt{x}}\cdot\frac{dx}{dx}+x\cdot\frac{de^{\sqrt{x}}}{d\sqrt{x}}\cdot\frac{d\sqrt{x}}{dx}\\ &= e^{\sqrt{x}}\cdot 1+xe^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\\ &= e^{\sqrt{x}}\left(1+\frac{\sqrt{x}}{2}\right)\end{aligned}$$

Section Review Page 480 Question 30

When $x = 1$, $y = e$. Determine the slope, m, of the tangent.

$$\begin{aligned}m &= \frac{de^x}{dx}_{|x=1}\\ &= e^x{}_{|x=1}\\ &= e\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}y-y_1 &= m(x-x_1)\\ y-e &= e(x-1)\\ y &= ex\end{aligned}$$

The equation of the tangent to $y = e^x$ at $(1, e)$ is $y = ex$.

Section Review Page 480 Question 31

$$\begin{aligned}p &= 70\\ 101.3e^{-0.125x} &= 70\\ e^{-0.125x} &= \frac{70}{101.3}\\ -0.125x &= \ln\frac{70}{101.3}\\ x &= \frac{\ln\frac{70}{101.3}}{-0.125}\\ &\doteq 2.957\end{aligned}$$

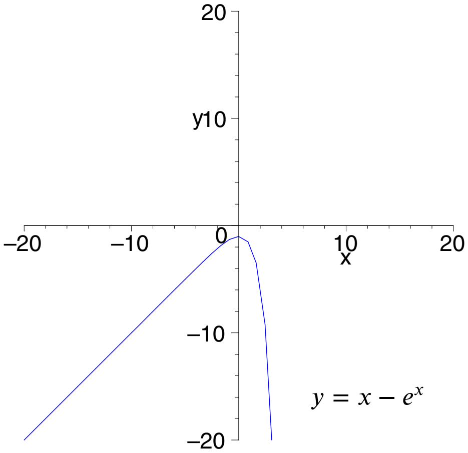
The atmospheric pressure is 70 kPa at 2.957 km.

Section Review Page 480 Question 32

a)

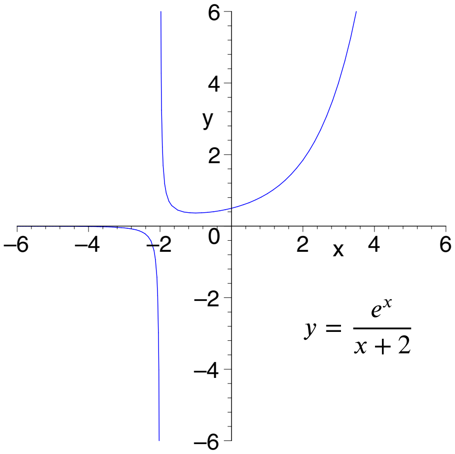
$$\begin{aligned}y &= x-e^x\\ \frac{dy}{dx} &= 0\\ 1-e^x &= 0\\ e^x &= 1\\ x &= 0\\ \frac{d^2y}{dx^2} &= 0\\ -e^x &= 0\\ &\text{no roots}\end{aligned}$$

Since $\frac{d^2y}{dx^2}_{|x=0} < 0$, a local maximum is confirmed at $(0, -1)$. As $x \to -\infty$, $e^x \to 0$. Hence $x - e^x \to x$. The function has an oblique asymptote of $y = x$. There are no inflection points.



b)

$$\begin{aligned}
y &= \frac{e^x}{x+2}\\
\frac{dy}{dx} &= 0\\
\frac{(x+2)e^x - e^x(1)}{(x+2)^2} &= 0\\
\frac{e^x(x+1)}{(x+2)^2} &= 0\\
x &= -1\\
\frac{d^2y}{dx^2} &= 0\\
\frac{(x+2)^2((x+1)e^x + e^x(1)) - e^x(x+1)2(x+2)}{(x+2)^4} &= 0\\
\frac{e^x(x^2+2x+2)}{(x+2)^3} &= 0\\
&\text{no roots}
\end{aligned}$$



Since $\frac{d^2y}{dx^2}_{|x=-1} > 0$, a local minimum is confirmed at $(-1, e^{-1})$. The function has a vertical asymptote of $x = -2$. As $x \to -\infty$, $e^x \to 0$. Hence $\frac{e^x}{x+2} \to 0$. The function has a horizontal asymptote of $y = 0$. There are no inflection points.

Section Review Page 480 Question 33

a)

$$\begin{aligned}
P(t) &= 50e^{-0.004t}\\
P'(t) &= 50 \cdot \frac{de^{-0.004t}}{d(-0.004t)} \cdot \frac{d(-0.004t)}{dt}\\
&= 50e^{-0.004t}(-0.004)\\
&= -0.2e^{-0.004t}
\end{aligned}$$

The rate of change of power output is $P'(t) = -0.2e^{-0.004t}$.

b)

$$\begin{aligned}
P'(100) &= -0.2e^{-0.004(100)}\\
&= -0.2e^{-0.4}\\
&\doteq -0.13
\end{aligned}$$

The power output is decreasing at a rate of 0.13 W/day.

Section Review Page 480 Question 34

a)

$$\begin{aligned}
C(x) &= 10x - 75x^2e^{-x} + 1500\\
C'(x) &= 10 - 75(e^{-x}(2x) + x^2(-e^{-x}))\\
&= 10 + 75xe^{-x}(x-2)
\end{aligned}$$

The marginal cost is $C'(x) = 10 + 75xe^{-x}(x-2)$.

b)

$$\begin{aligned}
C'(100) &= 10 + 75xe^{-100}(100-2)\\
&\doteq 10
\end{aligned}$$

The marginal cost of producing 100 CDs is \$10/unit.

c)

$$\begin{aligned}
C'(1000) &= 10 + 75xe^{-1000}(1000-2)\\
&\doteq 10
\end{aligned}$$

The marginal cost of producing 1000 CDs is \$10/unit.

7.7 Derivatives of Logarithmic Functions

Section Review Page 480 Question 35

a)

$$\begin{aligned}
g(x) &= \ln x^7\\
&= 7\ln x\\
g'(x) &= 7 \cdot \frac{d\ln x}{dx}\\
&= 7 \cdot \frac{1}{x}\\
&= \frac{7}{x}
\end{aligned}$$

b)

$$\begin{aligned}
y &= 8^x\\
\ln y &= x\ln 8\\
\frac{1}{y} \cdot \frac{dy}{dx} &= \ln 8\\
\frac{dy}{dx} &= y\ln 8\\
&= 8^x\ln 8
\end{aligned}$$

c)

$$\begin{aligned} h(x) &= \ln(5x+1) \\ h'(x) &= \frac{d\ln(5x+1)}{d(5x+1)} \cdot \frac{d(5x+1)}{dx} \\ &= \frac{1}{5x+1} \cdot 5 \\ &= \frac{5}{5x+1} \end{aligned}$$

d)

$$\begin{aligned} g(x) &= \log(4x+15) \\ &= \frac{\ln(4x+15)}{\ln 10} \\ g'(x) &= \frac{1}{\ln 10} \cdot \frac{d\ln(4x+15)}{d(4x+15)} \cdot \frac{d(4x+15)}{dx} \\ &= \frac{1}{\ln 10} \cdot \frac{1}{4x+15} \cdot 4 \\ &= \frac{4}{(4x+15)\ln 10} \end{aligned}$$

e)

$$\begin{aligned} y &= 2^x \log_2(x-8) \\ &= 2^x \cdot \frac{\ln(x-8)}{\ln 2} \\ \frac{dy}{dx} &= \frac{1}{\ln 2}\left(\ln(x-8) \cdot 2^x \ln 2 + 2^x \cdot \frac{1}{x-8}\right) \\ &= 2^x\left(\ln(x-8) + \frac{1}{(x-8)\ln 2}\right) \end{aligned}$$

f)

$$\begin{aligned} y &= \log_3(7x^2+2) \\ &= \frac{\ln(7x^2+2)}{\ln 3} \\ \frac{dy}{dx} &= \frac{1}{\ln 3} \cdot \frac{d\ln(7x^2+2)}{d(7x^2+2)} \cdot \frac{d(7x^2+2)}{dx} \\ &= \frac{1}{\ln 3} \cdot \frac{1}{7x^2+2} \cdot 14x \\ &= \frac{14x}{(7x^2+2)\ln 3} \end{aligned}$$

g)

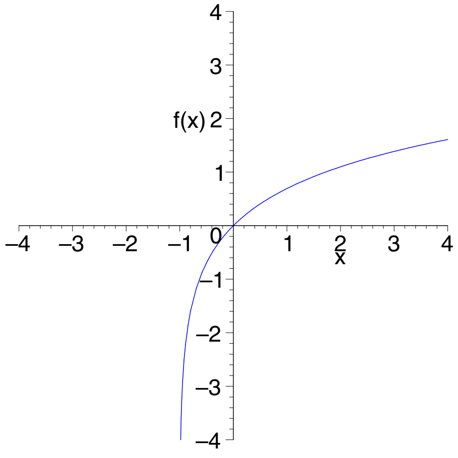
$$\begin{aligned} f(x) &= \frac{4^x}{1+x} \\ f'(x) &= \frac{(1+x)\dfrac{d(4^x)}{dx} - 4^x \cdot \dfrac{d(x+1)}{dx}}{(1+x)^2} \\ &= \frac{(1+x)4^x \ln 4 - 4^x(1)}{(1+x)^2} \\ &= \frac{4^x(x\ln 4 + \ln 4 - 1)}{(1+x)^2} \end{aligned}$$

h)

$$\begin{aligned} y &= \frac{1}{x\ln x} \\ &= (x\ln x)^{-1} \\ \frac{dy}{dx} &= \frac{d(x\ln x)^{-1}}{d(x\ln x)} \cdot \left(\ln x \cdot \frac{dx}{dx} + x \cdot \frac{d\ln x}{dx}\right) \\ &= -1(x\ln x)^{-2}\left(\ln x(1) + x \cdot \frac{1}{x}\right) \\ &= -\frac{\ln x + 1}{(x\ln x)^2} \end{aligned}$$

Section Review Page 480 Question 36

The graph of $y = \ln(x+1)$ can be obtained by translating the graph of $y = \ln x$ to the left 1 unit.



Section Review Page 480 Question 37

a) When $x = 5$, $y = 5^5$. Determine the slope, m, of the tangent.

$$\begin{aligned} m &= \frac{d5^x}{dx}_{|x=5} \\ &= 5^x \ln 5_{|x=5} \\ &= 5^5 \ln 5 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - 5^5 &= 5^5 \ln 5(x - 5) \\ y &= 5^5 x \ln 5 - 5^6 \ln 5 + 5^5 \end{aligned}$$

The equation of the tangent to $y = 5^x$ at $(5, 5^5)$ is $y = 5^5 x \ln 5 - 5^6 \ln 5 + 5^5$.

c) Determine the slope, m, of the tangent.

$$\begin{aligned} m &= \frac{d\left(\frac{1}{2}\right)^x}{dx}_{|x=-3} \\ &= \left(\frac{1}{2}\right)^x \ln\left(\frac{1}{2}\right)_{|x=-3} \\ &= -\left(\frac{1}{2}\right)^x \ln 2_{|x=-3} \\ &= -8 \ln 2 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - 8 &= -8 \ln 2(x - (-3)) \\ y &= -8x \ln 2 - 24 \ln 2 + 8 \end{aligned}$$

The equation of the tangent to $y = \left(\frac{1}{2}\right)^x$ at $(-3, 8)$ is $y = -8x \ln 2 - 24 \ln 2 + 8$.

b) Determine the slope, m, of the tangent.

$$\begin{aligned} m &= \frac{d3^x}{dx}_{|x=3} \\ &= 3^x \ln 3_{|x=3} \\ &= 27 \ln 3 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - 27 &= 27 \ln 3(x - 3) \\ y &= 27x \ln 3 - 81 \ln 3 + 27 \end{aligned}$$

The equation of the tangent to $y = 3^x$ at $(3, 27)$ is $y = 27x \ln 3 - 81 \ln 3 + 27$.

d) Determine the slope, m, of the tangent.

$$\begin{aligned} m &= \frac{d7^x}{dx}_{|x=2} \\ &= 7^x \ln 7_{|x=2} \\ &= 49 \ln 7 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - 49 &= 49 \ln 7(x - 2) \\ y &= 49x \ln 7 - 98 \ln 7 + 49 \end{aligned}$$

The equation of the tangent to $y = 7^x$ at $(2, 49)$ is $y = 49x \ln 7 - 98 \ln 7 + 49$.

Section Review Page 480 Question 38

From the given information, $(1.5, 6400)$ and $(2, 12\,800)$ are two points on the function $P = P_0 a^t$. Determine P_0 and a.

$$\begin{aligned} P_0 a^{1.5} &= 6400 && (1) \\ P_0 a^2 &= 12\,800 && (2) \end{aligned}$$

Divide (2) by (1).

$$\begin{aligned} a^{0.5} &= 2 \\ a &= 4 && (3) \end{aligned}$$

Substitute (3) into (2).

$$\begin{aligned} P_0(4)^2 &= 12\,800 \\ P_0 &= 800 \end{aligned}$$

The growth model is $P = 800(4)^t$.

Determine the rate of growth.

$$\begin{aligned} P(t) &= 800(4)^t \\ P'(t) &= 800(4)^t \ln 4 \end{aligned}$$

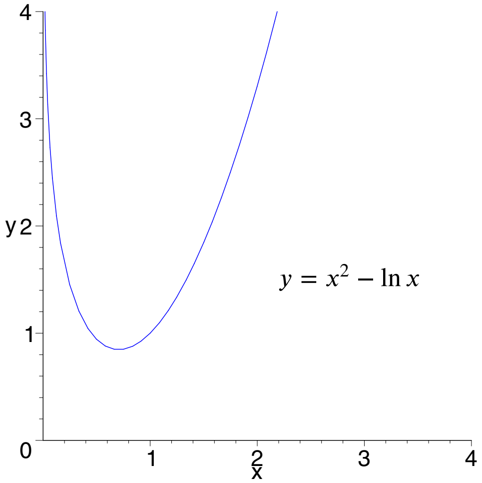
Determine the growth rate after 90 min.

$$\begin{aligned} P'(1.5) &= 800(4)^{1.5} \ln 4 \\ &= 6400 \ln 4 \end{aligned}$$

After 90 min, the growth rate is $6400 \ln 4$ bacteria/h.

Section Review Page 481 Question 39

$$\begin{aligned}
y &= x^2 - \ln x \\
\frac{dy}{dx} &= 0 \\
2x - \frac{1}{x} &= 0 \\
2x &= \frac{1}{x} \\
x^2 &= \frac{1}{2} \\
x &= \frac{1}{\sqrt{2}},\ x > 0 \\
\frac{d^2y}{dx^2} &= 0 \\
2 + \frac{1}{x^2} &= 0 \\
&\text{no roots}
\end{aligned}$$



Since $\frac{d^2y}{dx^2} > 0$ for all $\Re$, a local minimum is confirmed at $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}(1 + \ln 2)\right)$. The function has a vertical asymptote at $x = 0$. There are no inflection points.

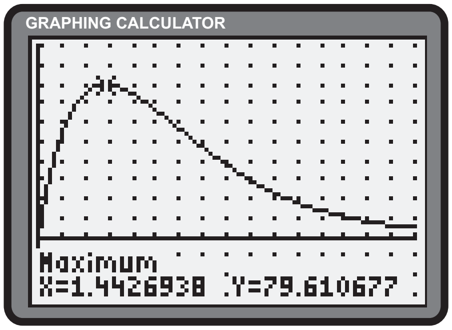
Section Review Page 481 Question 40

a) Estimates may vary. Determine $C'(t)$.

$$\begin{aligned}
C'(t) &= 150(0.5^t(1) + t(0.5)^t \ln 0.5) \\
&= 150(0.5)^t(1 - t\ln 2) \qquad (1)
\end{aligned}$$

Determine the critical number of C.

$$\begin{aligned}
C'(t) &= 0 \\
150(0.5)^t(1 - t\ln 2) &= 0 \\
t\ln 2 &= 1 \\
t &= \frac{1}{\ln 2} \\
&\doteq 1.44
\end{aligned}$$



```
Maximum
X=1.4426938  Y=79.610677
```

The highest concentration is $C(1.44) = 150(1.44)(0.5)^{1.44}$ or 79.6 ppm.

b) From (1), since $C'(2) = 150(0.5)^2(1 - 2\ln 2)$ or -14.5, the concentration is decreasing at the rate of 14.5 ppm/h after 2 h.

7.8 Applications of Exponential and Logarithmic Functions

Section Review Page 481 Question 41

a) Use $P = P_0(1 + r)^{2n}$ with $r = \frac{0.085}{2}$ or 0.0425.

$$\begin{aligned}
P &= 3(7000) \\
7000(1 + 0.0425)^{2n} &= 3(7000) \\
(1.0425)^{2n} &= 3 \\
2n \ln 1.0425 &= \ln 3 \\
n &= \frac{\ln 3}{2\ln 1.0425} \\
&\doteq 13.198
\end{aligned}$$

It will take 13.198 years for the amount to triple.

b) Use $P = P_0e^{rt}$ with $r = 0.09$.

$$\begin{aligned}
P &= 3(7000) \\
7000e^{0.09t} &= 3(7000) \\
e^{0.09t} &= 3 \\
0.09t &= \ln 3 \\
t &= \frac{\ln 3}{0.09} \\
&\doteq 12.207
\end{aligned}$$

It will take 12.207 years for the amount to triple.

Section Review Page 481 Question 42

From the given information, $(0, 51\,003)$ and $(20, 149\,030)$ are two points on the function $P = P_0e^{kt}$. Determine the growth model.

$$\begin{aligned}
P(20) &= 149\,030\\
51\,003e^{20k} &= 149\,030\\
e^{20k} &= \frac{149\,030}{51\,003}\\
20k &= \ln\frac{149\,030}{51\,003}\\
k &= \frac{1}{20}\ln\frac{149\,030}{51\,003}
\end{aligned}$$

The growth model is $P = 51\,003e^{\frac{t}{20}\ln\frac{149\,030}{51\,003}}$.

b) Results may vary.

c) Answers may vary.

Determine t for $P = 310\,792$.

$$\begin{aligned}
P &= 310\,792\\
51\,003e^{\frac{t}{20}\ln\frac{149\,030}{51\,003}} &= 310\,792\\
e^{\frac{t}{20}\ln\frac{149\,030}{51\,003}} &= \frac{310\,792}{51\,003}\\
\frac{t}{20}\ln\frac{149\,030}{51\,003} &= \ln\frac{310\,792}{51\,003}\\
t &= \frac{20\ln\frac{310\,792}{51\,003}}{\ln\frac{149\,030}{51\,003}}\\
&\doteq 33.71
\end{aligned}$$

The population was 51 003 in the year 1966 (33.71 years before 2000).

Section Review Page 481 Question 43

From the given information, $(0, 1070)$ and $(1, 850)$ are two points on the function $T = (T_0 - T_S)e^{kt} + T_S$, where $T_S = 27$°C. Determine the temperature model.

$$\begin{aligned}
T(1) &= 850\\
(1070 - 27)e^{k} + 27 &= 850\\
1043e^{k} &= 823\\
e^{k} &= \frac{823}{1043}\\
k &= \ln\frac{823}{1043}
\end{aligned}$$

The temperature model is $T = 1043e^{t\ln\frac{823}{1043}} + 27$.

Determine t for $T = 38$°C.

$$\begin{aligned}
T &= 38\\
1043e^{t\ln\frac{823}{1043}} + 27 &= 38\\
1043e^{t\ln\frac{823}{1043}} &= 11\\
e^{t\ln\frac{823}{1043}} &= \frac{11}{1043}\\
t\ln\frac{823}{1043} &= \ln\frac{11}{1043}\\
t &= \frac{\ln\frac{11}{1043}}{\ln\frac{823}{1043}}\\
&\doteq 19.2
\end{aligned}$$

The steel should be removed after 19.2 min.

Section Review Page 481 Question 44

Use the model $P = P_0 2^{\frac{t}{d}}$, where $P_0 = 600$ and $d = 3$. Determine t for $P = 40\,000$.

$$\begin{aligned}
P &= 40\,000\\
600(2)^{\frac{t}{3}} &= 40\,000\\
(2)^{\frac{t}{3}} &= \frac{200}{3}\\
\frac{t}{3}\ln 2 &= \ln\frac{200}{3}\\
t &= \frac{3\ln\frac{200}{3}}{\ln 2}\\
&\doteq 18.177
\end{aligned}$$

It will take 18.177 h for the population to reach 40 000.

Section Review Page 481 Question 45

Use the model $P(t) = P_0e^{kt}$, where P is measured in billions of people and t is measured in years. From the given information, $P_0 = 2.3$ and $(10, 3)$ is a point on the function. Determine the population model.

$$\begin{aligned} P(10) &= 3 \\ 2.3e^{10k} &= 3 \\ e^{10k} &= \frac{3}{2.3} \\ 10k &= \ln\frac{3}{2.3} \\ k &\doteq 0.026\,57 \end{aligned}$$

The population model is $P = 2.3e^{0.026\,57t}$.

The year 2019 corresponds to $t = 2019 - 1989$ or 30.

$$\begin{aligned} P(30) &= 2.3e^{0.026\,57(30)} \\ &\doteq 5.1 \end{aligned}$$

The model suggests an urban population of 5.1 billion in the year 2019.

Section Review Page 481 Question 46

a) Use the model $P(x) = P_0e^{kx}$, where P is the atmospheric pressure and x is the altitude. From the given information, $P_0 = 100$ and $(2, 75.73)$ is a point on the function. Determine the pressure model.

$$\begin{aligned} P(2) &= 75.73 \\ 100e^{2k} &= 75.73 \\ e^{2k} &= 0.7573 \\ 2k &= \ln 0.7573 \\ k &= \frac{1}{2}\ln 0.7573 \\ &\doteq -0.139 \end{aligned}$$

The pressure model is $P(x) = 100e^{-0.139x}$.

b)

$$\begin{aligned} P(5) &= 100e^{-0.139(5)} \\ &\doteq 49.9 \end{aligned}$$

At an altitude of 5 km, the atmospheric pressure is 49.9 kPa.

c)

$$\begin{aligned} P(10) &= 100e^{-0.139(10)} \\ &\doteq 24.9 \end{aligned}$$

The model ceases to be accurate at an atmospheric pressure of 24.9 kPa.

Section Review Page 481 Question 47

a) Use the model $I(x) = I_0e^{kx}$, where I is the intensity of light and x is the thickness of the glass. From the given information, $I_0 = 2$ lumens and $(5, 0.736)$ is a point on the function. Determine the intensity model.

$$\begin{aligned} I(5) &= 0.736 \\ 2e^{5k} &= 0.736 \\ e^{5k} &= 0.368 \\ 5k &= \ln 0.368 \\ k &= \frac{1}{5}\ln 0.368 \\ &\doteq -0.1999 \end{aligned}$$

The intensity model is $I(x) = 2e^{-0.1999x}$.

b) The general model can be expressed as

$$I(x) = I_0e^{-0.1999x}$$

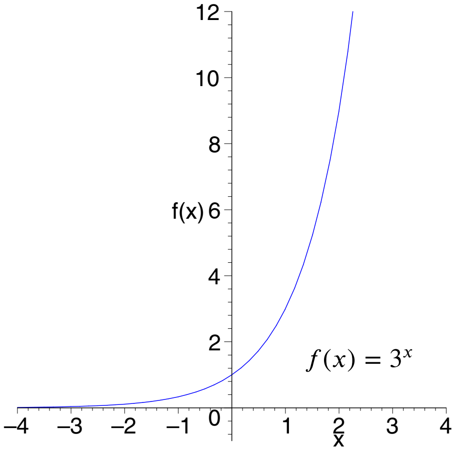
c) Doubling the thickness of the glass corresponds to $x = 10$.

$$\begin{aligned} \frac{I(10)}{I(5)} &= \frac{I_0e^{-0.1999(10)}}{I_0e^{-0.1999(5)}} \\ &= \frac{e^{-0.1999(10)}}{e^{-0.1999(5)}} \\ &\doteq 0.368 \end{aligned}$$

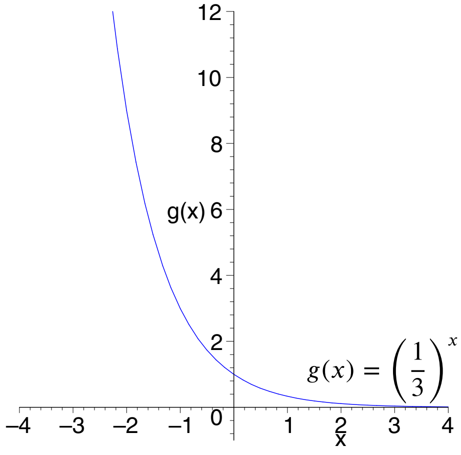
Doubling the thickness of the glass further reduces the intensity by 36.8% compared to 5 mm.

Chapter Test

Section Chapter Test Page 482 Question 1



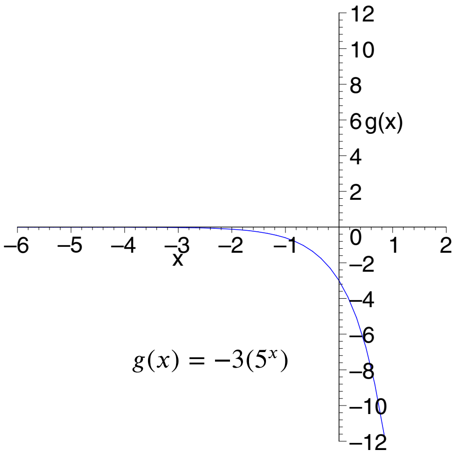
b)



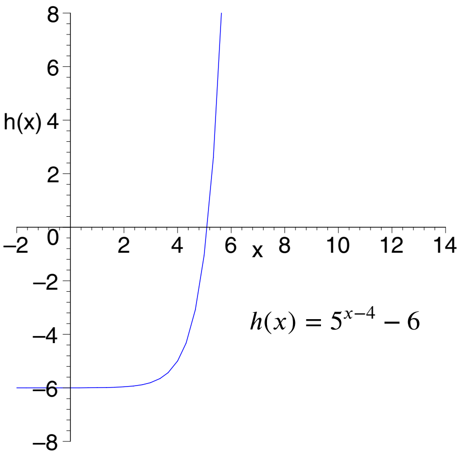
a) f are g are reflections of one another in the y-axis.

Section Chapter Test Page 482 Question 2

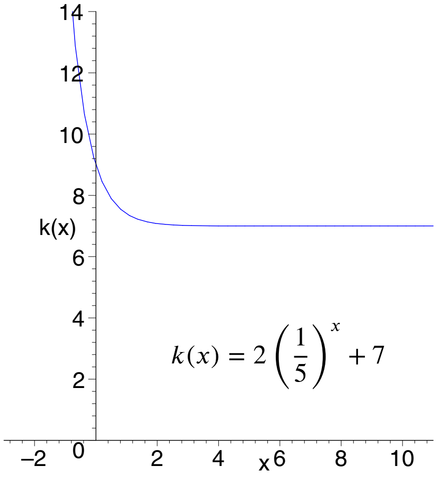
a) Stretch the graph of $f(x) = 5^x$ vertically by a factor of 3 and then reflect it in the x-axis.



b) Translate the graph of $f(x) = 5^x$ horizontally to the right 4 units, then translate it downward 6 units.



c) Reflect the graph of $f(x) = 5^x$ in the y-axis, stretch vertically by a factor of 2, then translate upward 7 units.



Section Chapter Test   Page 482   Question 3

a) As $x \to \infty$, $3x \to \infty$. Therefore, $\lim\limits_{x\to\infty} 2^{3x} = \infty$.

b) As $x \to \infty$, $-x \to -\infty$. Therefore, $\lim\limits_{x\to\infty} 4^{-x} = 0$.

c)
$$\begin{aligned}\log_2 32 &= \log_2 2^5\\ &= 5\log_2 2\\ &= 5(1)\\ &= 5\end{aligned}$$

d)
$$\begin{aligned}\log_7 12 &= \frac{\log 12}{\log 7}\\ &\doteq 1.2770\end{aligned}$$

e)
$$\begin{aligned}\log_6 9 + \log_6 4 &= \log_6(9 \times 4)\\ &= \log_6 36\\ &= 2\end{aligned}$$

f)
$$\begin{aligned}\frac{1}{2}\log_5 25 &= \frac{1}{2} \times 2\\ &= 1\end{aligned}$$

Section Chapter Test   Page 482   Question 4

a)
$$\begin{aligned}3^{x-2} &= 27\\ 3^{x-2} &= 3^3\\ x-2 &= 3\\ x &= 5\end{aligned}$$

Check.

$$\begin{aligned}\textbf{L.S.} &= 3^{5-2}\\ &= 3^3\\ &= 27\\ &= \textbf{R.S.}\end{aligned}$$

b)
$$\begin{aligned}\log_3(x+7) &= \log_3(x+2) + 1\\ \log_3(x+7) &= \log_3(x+2) + \log_3 3\\ \log_3(x+7) &= \log_3[3(x+2)]\\ x+7 &= 3x+6\\ 2x &= 1\\ x &= \frac{1}{2}\end{aligned}$$

Check.

$$\begin{aligned}\textbf{L.S.} &= \log_3\left(\frac{1}{2}+7\right)\\ &= \log_3\left(\frac{15}{2}\right)\\ \textbf{R.S.} &= \log_3\left(\frac{1}{2}+2\right) + \log_3 3\\ &= \log_3\left(\frac{5}{2}\right) + \log_3 3\\ &= \log_3\left(\frac{15}{2}\right)\\ &= \textbf{L.S.}\end{aligned}$$

c)

$$\begin{aligned} 2^{2x} - 3(2^x) + 2 &= 0 \\ (2^x)^2 - 3(2^x) + 2 &= 0 \\ (2^x - 1)(2^x - 2) &= 0 \\ 2^x - 1 &= 0 \\ x &= 0 \\ 2^x - 2 &= 0 \\ x &= 1 \end{aligned}$$

Check $x = 0$.

$$\begin{aligned} \textbf{L.S.} &= 2^0 - 3(2^0) + 2 \\ &= 1 - 3 + 2 \\ &= 0 \\ &= \textbf{R.S.} \end{aligned}$$

Check $x = 1$.

$$\begin{aligned} \textbf{L.S.} &= 2^2 - 3(2^1) + 2 \\ &= 4 - 6 + 2 \\ &= 0 \\ &= \textbf{R.S.} \end{aligned}$$

d)

$$\begin{aligned} 2\log_4(x+1) + \log_4(x-2) &= 2 \\ \log_4[(x+1)^2(x-2)] &= 2 \\ (x+1)^2(x-2) &= 16 \\ (x^2 + 2x + 1)(x-2) &= 16 \\ x^3 - 3x - 2 &= 16 \\ x^3 - 3x - 18 &= 0 \\ (x-3)(x^2 + 3x + 6) &= 0 \\ x &= 3 \end{aligned}$$

Check.

$$\begin{aligned} \textbf{L.S.} &= 2\log_4(3+1) + \log_4(3-2) \\ &= 2\log_4 4 + \log_4 1 \\ &= 2(1) + 0 \\ &= 2 \\ &= \textbf{R.S.} \end{aligned}$$

Section Chapter Test Page 482 Question 5

a)

$$\begin{aligned} f(x) &= \frac{e^{-x}}{x} \\ f'(x) &= \frac{x\dfrac{de^{-x}}{d(-x)} \cdot \dfrac{d(-x)}{dx} - e^{-x}\dfrac{dx}{dx}}{x^2} \\ &= \frac{-xe^{-x} - e^{-x}(1)}{x^2} \\ &= \frac{-e^{-x}(x+1)}{x^2} \end{aligned}$$

b)

$$\begin{aligned} y &= xe^x \\ \frac{dy}{dx} &= e^x\frac{dx}{dx} + x\frac{de^x}{dx} \\ &= e^x(1) + xe^x \\ &= e^x(1+x) \end{aligned}$$

c)

$$\begin{aligned} y &= e^x x^{-5} \\ \frac{dy}{dx} &= x^{-5}\frac{de^x}{dx} + e^x\frac{dx^{-5}}{dx} \\ &= x^{-5}e^x + e^x(-5)x^{-6} \\ &= e^x x^{-6}(x-5) \end{aligned}$$

d)

$$\begin{aligned} g(x) &= \ln x^5 \\ &= 5\ln x \\ \frac{dy}{dx} &= 5\frac{d\ln x}{dx} \\ &= 5 \cdot \frac{1}{x} \\ &= \frac{5}{x} \end{aligned}$$

e)

$$\begin{aligned} h(x) &= \log(8x+7) \\ &= \frac{\ln(8x+7)}{\ln 10} \\ h'(x) &= \frac{1}{\ln 10} \cdot \frac{d\ln(8x+7)}{d(8x+7)} \cdot \frac{d(8x+7)}{dx} \\ &= \frac{1}{\ln 10} \cdot \frac{1}{8x+7} \cdot 8 \\ &= \frac{8}{(8x+7)\ln 10} \end{aligned}$$

f)

$$\begin{aligned} y &= \log_4(x-2) \\ &= \frac{\ln(x-2)}{\ln 4} \\ \frac{dy}{dx} &= \frac{1}{\ln 4} \cdot \frac{d\ln(x-2)}{d(x-2)} \cdot \frac{d(x-2)}{dx} \\ &= \frac{1}{\ln 4} \cdot \frac{1}{x-2} \cdot 1 \\ &= \frac{1}{(x-2)\ln 4} \end{aligned}$$

Section Chapter Test Page 482 Question 6

a) Determine the slope, m, of the tangent.

$$\begin{aligned} m &= \frac{d4^x}{dx}_{|x=2} \\ &= 4^x \ln 4_{|x=2} \\ &= 16 \ln 4 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 16 &= 16 \ln 4(x - 2) \\ y &= (16 \ln 4)x - 32 \ln 4 + 16 \end{aligned}$$

The equation of the tangent to $y = 4^x$ at $(2, 16)$ is $y = (16 \ln 4)x - 32 \ln 4 + 16$.

b) $f(2) = e^2$. Determine the slope, m, of the tangent.

$$\begin{aligned} m &= \frac{de^x}{dx}_{|x=2} \\ &= e^x{}_{|x=2} \\ &= e^2 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - e^2 &= e^2(x - 2) \\ y &= e^2 x - e^2 \end{aligned}$$

The equation of the tangent to $y = e^x$ at $(2, e^2)$ is $y = e^2 x - e^2$.

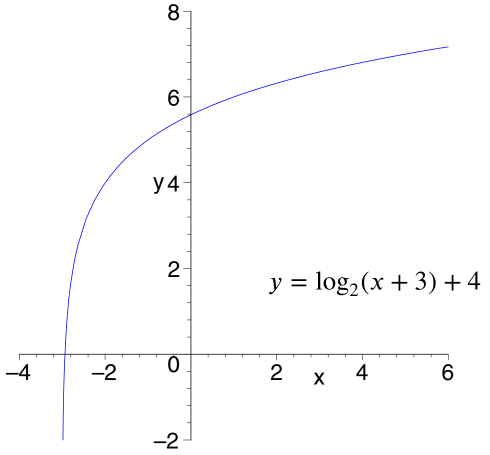
Section Chapter Test Page 482 Question 7

$$\begin{aligned} p &= 59 \\ 101.3e^{-0.125x} &= 59 \\ e^{-0.125x} &= \frac{59}{101.3} \\ -0.125x &= \ln \frac{59}{101.3} \\ x &= -8 \ln \frac{59}{101.3} \\ &\doteq 4.324 \end{aligned}$$

The atmospheric pressure is 59 kPa at an altitude of 4.324 km.

Section Chapter Test Page 482 Question 8

a)



b) domain: $x > -3$; range: $\Re$; asymptote: $x = -3$

Section Chapter Test Page 482 Question 9

a) Use $P = P_0(1 + r)^n$ with $r = 0.08$.

$$\begin{aligned} P &= 2(500) \\ 500(1 + 0.08)^n &= 2(500) \\ (1.08)^n &= 2 \\ n \ln 1.08 &= \ln 2 \\ n &= \frac{\ln 2}{\ln 1.08} \\ &\doteq 9.01 \end{aligned}$$

It will take 9.01 years for the amount to double.

b) Use $P = P_0 e^{rt}$ with $r = 0.08$.

$$\begin{aligned} P &= 2(500) \\ 500e^{0.08t} &= 2(500) \\ e^{0.08t} &= 2 \\ 0.08t &= \ln 2 \\ t &= \frac{\ln 2}{0.08} \\ &\doteq 8.66 \end{aligned}$$

It will take 8.66 years for the amount to double.

Section Chapter Test Page 482 Question 10

Determine the value of k using the half-life. Let $A(t)$ be the amount of iodine-135, in kilograms, after t years. When $t = 8$, $A = 0.5A_0$.

$$\begin{aligned} A &= 0.5A_0 \\ A_0e^{8k} &= 0.5A_0 \\ e^{8k} &= 0.5 \\ 8k &= \ln 0.5 \\ k &= \frac{\ln 0.5}{8} \end{aligned}$$

The model is $A = A_0e^{\frac{t\ln 0.5}{8}}$.

Determine t for $A = 0.2$ kg and $A_0 = 28$ kg.

$$\begin{aligned} A &= 0.2 \\ 28e^{\frac{t\ln 0.5}{8}} &= 0.2 \\ e^{\frac{t\ln 0.5}{8}} &= \frac{1}{140} \\ \frac{t\ln 0.5}{8} &= \ln\frac{1}{140} \\ t &= \frac{8\ln\frac{1}{140}}{\ln 0.5} \\ &\doteq 57.03 \end{aligned}$$

It will take 57.03 days for 28 kg to decay to 200 g.

Section Chapter Test Page 482 Question 11

If I is the intensity of the 1994 earthquake, $\log\frac{I}{I_0} = 6.8$. Let M be the magnitude of the more powerful earthquake.

$$\begin{aligned} M &= \log\frac{1.8I}{I_0} \\ &= \log 1.8 + \log\frac{I}{I_0} \\ &= \log 1.8 + 6.8 \\ &\doteq 7.1 \end{aligned}$$

An earthquake 1.8 times as intense as a magnitude 6.8 earthquake registers 7.1 on the Richter scale.

Section Chapter Test Page 482 Question 12

Let L_O and L_I be the loudness of the output and input sounds, respectively, in decibels.

$$\begin{aligned} L_O - L_I &= 55 \\ 10\log\frac{I_O}{I_0} - 10\log\frac{I_I}{I_0} &= 55 \\ \log\frac{I_O}{I_0} - \log\frac{I_I}{I_0} &= 5.5 \\ \log\left(\frac{I_O}{I_0} \div \frac{I_I}{I_0}\right) &= 5.5 \\ \log\frac{I_O}{I_I} &= 5.5 \\ \frac{I_O}{I_I} &= 10^{5.5} \\ &\doteq 316\,228 \end{aligned}$$

The amplifier increases the power of the signal by a factor of 316 228.

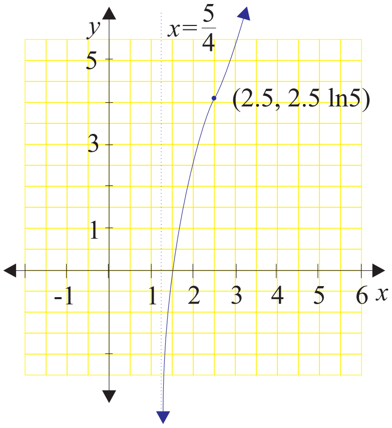
Section Chapter Test Page 482 Question 13

$$y = x\ln(4x - 5)$$

Determine the critical numbers.

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \ln(4x-5)\cdot\frac{dx}{dx} + x\cdot\frac{d\ln(4x-5)}{d(4x-5)}\cdot\frac{d(4x-5)}{dx} &= 0 \\ \ln(4x-5)\cdot 1 + x\cdot\frac{4}{4x-5} &= 0 \\ \ln(4x-5) + \frac{4x}{4x-5} &= 0 \qquad (1) \\ \frac{(4x-5)\ln(4x-5) + 4x}{4x-5} &= 0 \end{aligned}$$

no critical numbers



Determine the roots of $\frac{d^2y}{dx^2}$ from (1).

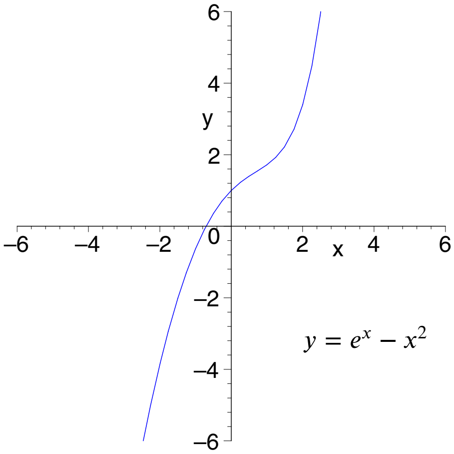
$$\begin{aligned}
\frac{d^2y}{dx^2} &= 0 \\
\frac{d\ln(4x-5)}{d(4x-5)}\cdot\frac{d(4x-5)}{dx} + \frac{(4x-5)\cdot\frac{d(4x)}{dx} - 4x\cdot\frac{d(4x-5)}{dx}}{(4x-5)^2} &= 0 \\
\frac{1}{4x-5}\cdot 4 + \frac{(4x-5)\cdot 4 - 4x\cdot 4}{(4x-5)^2} &= 0 \\
\frac{4}{4x-5} - \frac{20}{(4x-5)^2} &= 0 \\
\frac{4x-5-5}{(4x-5)^2} &= 0 \\
\frac{2x-5}{(4x-5)^2} &= 0 \\
x &= 2.5
\end{aligned}$$

A point of inflection exists at $(2.5, 2.5\ln 5)$. A vertical asymptote exists at $x = \frac{5}{4}$.

Section Chapter Test Page 482 Question 14

a)

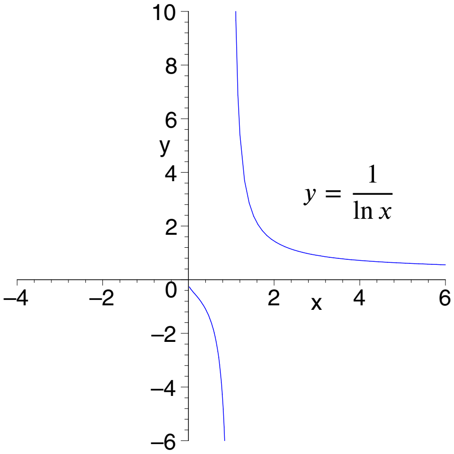
$$\begin{aligned}
y &= e^x - x^2 \\
\frac{dy}{dx} &= 0 \\
e^x - 2x &= 0 \\
e^x &= 2x \\
&\text{no roots} \\
\frac{d^2y}{dx^2} &= 0 \\
e^x - 2 &= 0 \\
e^x &= 2 \\
x &= \ln 2
\end{aligned}$$



There are no extrema. There are no linear asymptotes. There is a point of inflection at $(\ln 2, 2 - (\ln 2)^2)$.

b)

$$\begin{aligned}
y &= \frac{1}{\ln x} \\
&= (\ln x)^{-1} \\
\frac{dy}{dx} &= 0 \\
-1\cdot(\ln x)^{-2}\cdot\frac{1}{x} &= 0 \\
-\frac{1}{x(\ln x)^2} &= 0 \\
&\text{no roots} \\
\frac{d^2y}{dx^2} &= 0 \\
\frac{1}{(x(\ln x)^2)^2}\cdot\left((\ln x)^2\cdot 1 + x\cdot 2\ln x\cdot\frac{1}{x}\right) &= 0 \\
\frac{\ln x + 2}{x^2(\ln x)^3} &= 0 \\
\ln x + 2 &= 0 \\
\ln x &= -2 \\
x &= e^{-2}
\end{aligned}$$



There are no extrema. There is a vertical asymptote of $x = 1$ and a horizontal asymptote of $y = 0$. There is a point of inflection at $\left(e^{-2}, -\frac{1}{2}\right)$.

Section Chapter Test Page 483 Question 15

Let $A(t) = A_0e^{kt}$ be the amount of strontium-87 after t hours. Use $A_0 = 100$ and $A(1) = 78.1$. Determine the value of k.

$$\begin{aligned} A(1) &= 78.1 \\ 100e^k &= 78.1 \\ e^k &= 0.781 \\ k &= \ln 0.781 \end{aligned}$$

The model is $A = 100e^{t\ln 0.781}$.

Determine t for $A(t) = 50$ mg.

$$\begin{aligned} A(t) &= 50 \\ 100e^{t\ln 0.781} &= 50 \\ e^{t\ln 0.781} &= 0.5 \\ t\ln 0.781 &= \ln 0.5 \\ t &= \frac{\ln 0.5}{\ln 0.781} \\ &\doteq 2.8 \end{aligned}$$

The half-life of strontium-87 is 2.8 h.

Section Chapter Test Page 483 Question 16

Let $I(d)$ be the intensity of the sunlight at a depth of d metres. From the given information, $I(d) = I_0(1 - 0.046)^d$ or $I(d) = I_0(0.954)^d$. Solve the model for d.

$$\begin{aligned} I(d) &= I_0(0.954)^d \\ \frac{I(d)}{I_0} &= (0.954)^d \\ \log\frac{I(d)}{I_0} &= d\log(0.954) \\ d &= \frac{1}{\log 0.954}\log\frac{I(d)}{I_0} \\ &\doteq -48.9\log\frac{I(d)}{I_0} \end{aligned}$$

Section Chapter Test Page 483 Question 17

a)

$$\begin{aligned} C(x) &= 200.2\sqrt{\ln(10x+1)} \\ C'(x) &= 200.2\cdot\frac{1}{2\sqrt{\ln(10x+1)}}\cdot\frac{1}{10x+1}\cdot 10 \\ &= \frac{1001}{(10x+1)\sqrt{\ln(10x+1)}} \end{aligned}$$

The marginal cost is $C'(x) = \dfrac{1001}{(10x+1)\sqrt{\ln(10x+1)}}$.

b) $C(1000) = 200.2\sqrt{\ln(10(1000)+1)}$

$\doteq 607.58$

The total cost is \$607.58.

$$\begin{aligned} C'(1000) &= \frac{1001}{(10(1000)+1)\sqrt{\ln(10(1000)+1)}} \\ &\doteq 0.033 \end{aligned}$$

The marginal cost is \$0.033/L.

Section Chapter Test Page 483 Question 18

a)

$$\begin{aligned} M(t) &= 20(1.5 + e^{-0.02t}) \\ \frac{dM}{dt} &= 20(-0.02e^{-0.02t}) \\ &= -0.4e^{-0.02t} \end{aligned}$$

The rate of change model is $\dfrac{dM}{dt} = -0.4e^{-0.02t}$.

b)

$$\begin{aligned} \left.\frac{dM}{dt}\right|_{t=5} &= -0.4e^{-0.02(5)} \\ &= -0.4e^{-0.1} \\ &\doteq -0.362 \end{aligned}$$

The rate of change of the mass of the salt after 5 min is −0.362 kg/min.

Section Chapter Test   Page 483   Question 19

a) Determine the cooling constant, k.

$$\begin{aligned} T - T_S &= (T_0 - T_S)e^{kt} \\ 12 - 5 &= (20 - 5)e^{k(1)} \\ 7 &= 15e^k \\ e^k &= \frac{7}{15} \\ k &= \ln\frac{7}{15} \end{aligned}$$

The temperature model is $T(t) = 15e^{t \ln \frac{7}{15}} + 5$. One more hour corresponds to $t = 2$.

$$\begin{aligned} T(2) &= 15e^{2 \ln \frac{7}{15}} + 5 \\ &\doteq 8.3 \end{aligned}$$

After two hours, the temperature will be 8.3°C.

b) Two more hours corresponds to $t = 3$.

$$\begin{aligned} T(3) &= 15e^{3 \ln \frac{7}{15}} + 5 \\ &\doteq 6.5 \end{aligned}$$

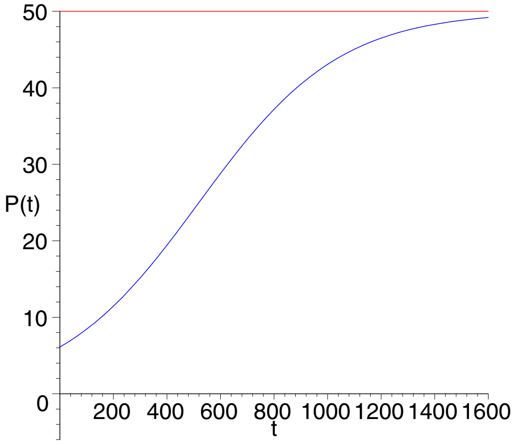
After two more hours, the temperature will be 6.5°C.

c)

$$\begin{aligned} T(t) &= 7 \\ 15e^{t \ln \frac{7}{15}} &= 2 \\ e^{t \ln \frac{7}{15}} &= \frac{2}{15} \\ t \ln\frac{7}{15} &= \ln\frac{2}{15} \\ t &= \frac{\ln\frac{2}{15}}{\ln\frac{7}{15}} \\ &\doteq 2.6 \end{aligned}$$

The temperature will be 7°C after 2.6 h.

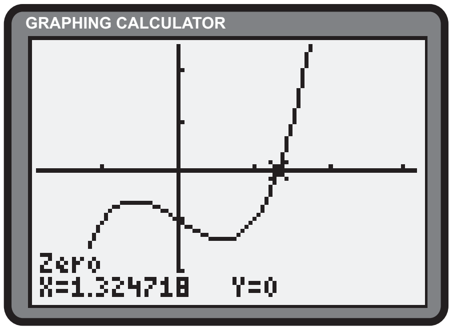
Section Chapter Test   Page 483   Question 20



The logistic model has a horizontal asymptote indicating a maximum sustainable population of 50 billion, rather than unbounded growth reflected in the exponential model.

Challenge Problems

Section Challenge Problems Page 484 Question 1



Zero
X=1.324718 Y=0

Let $x = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\ldots}}}}$.

Then,

$$\begin{aligned} x &= \sqrt[3]{1+x} \\ x^3 &= 1+x \\ x^3 - x - 1 &= 0 \qquad (1) \end{aligned}$$

The graphing calculator reveals a solution to (1) of approximately 1.32.

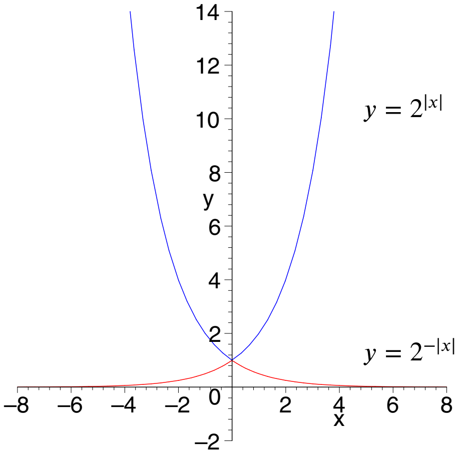
Section Challenge Problems Page 484 Question 2

$$\begin{aligned} f(7.5) &= (1-0.4)f(3) \\ 10e^{-7.5k} &= (0.6)10^{-3k} \\ 10^{-4.5k} &= 0.6 \\ -4.5k &= \log 0.6 \\ k &= -\frac{1}{4.5}\log 0.6 \\ &\doteq 0.0493 \end{aligned}$$

A value of $k = 0.0493$ satisfies the requirements.

Section Challenge Problems Page 484 Question 3

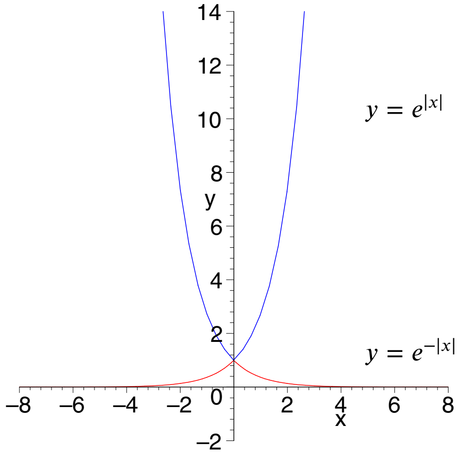
a)



$y = 2^{|x|}$: domain: $\Re$; range: $y \in [1, \infty)$.

$y = 2^{-|x|}$: domain: $\Re$; range: $y \in (0, 1]$.

b)



$y = e^{|x|}$: domain: $\Re$; range: $y \in [1, \infty)$.

$y = e^{-|x|}$: domain: $\Re$; range: $y \in (0, 1]$.

Section Challenge Problems Page 484 Question 4

Let $A(t) = A_0(10)^{-kt}$ be the amount of material after t hours. Determine the value of k.

$$\begin{aligned} A(5) &= 0.35A_0 \\ A_0(10)^{-5k} &= 0.35A_0 \\ 10^{-5k} &= 0.35 \\ -5k &= \log 0.35 \\ k &= -\frac{\log 0.35}{5} \end{aligned}$$

The model is $A = A_0(10)^{\frac{t\log 0.35}{5}}$.

Determine t for $A(t) = 0.5A_0$.

$$\begin{aligned} A(t) &= 0.5A_0 \\ A_0(10)^{\frac{t\log 0.35}{5}} &= 0.5A_0 \\ 10^{\frac{t\log 0.35}{5}} &= 0.5 \\ \frac{t\log 0.35}{5} &= \log 0.5 \\ t &= \frac{5\log 0.5}{\log 0.35} \\ &\doteq 3.3 \end{aligned}$$

The half-life of the material is 3.3 h.

Section Challenge Problems Page 484 Question 5

$$\begin{aligned} 3a + 2b &= 3\log_7(11 - 6\sqrt{2}) + 2\log_7(45 + 29\sqrt{2}) \\ &= \log_7(11 - 6\sqrt{2})^3 + \log_7(45 + 29\sqrt{2})^2 \\ &= \log_7\left((11 - 6\sqrt{2})^3(45 + 29\sqrt{2})^2\right) \\ &= \log_7\left((3707 - 2610\sqrt{2})(3707 + 2610\sqrt{2})\right) \\ &= \log_7(13\,741\,849 - 13\,624\,200)) \\ &= \log_7(117\,649) \\ &= \log_7\left(7^6\right) \\ &= 6 \end{aligned}$$

$3a + 2b = 6$

Section Challenge Problems Page 484 Question 6

$$\begin{aligned} y &= e^{-x^2} \\ \frac{dy}{dx} &= e^{-x^2} \cdot -2x \\ &= -2xe^{-x^2} \end{aligned}$$

Determine the roots of $\frac{d^2y}{dx^2}$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= 0 \\ -2(e^{-x^2} \cdot 1 + x \cdot e^{-x^2} \cdot -2x) &= 0 \\ -2e^{-x^2}(1 - 2x^2) &= 0 \\ 1 - 2x^2 &= 0 \\ x &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

There are three test intervals: $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. Since $\frac{d^2y}{dx^2} < 0$ for $x = 0$, the function is concave downward on the interval $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Section Challenge Problems Page 484 Question 7

The nth derivative of $g(x) = e^x$ is $g^{(n)} = e^x$, and $g^{(n)}(0) = 1$.

Determine the best quadratic approximation.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ f'(x) &= 2ax + b \\ f''(x) &= 2a \end{aligned}$$

Determine a.

$$\begin{aligned} f''(0) &= g''(0) \\ 2a &= 1 \\ a &= \frac{1}{2} \end{aligned}$$

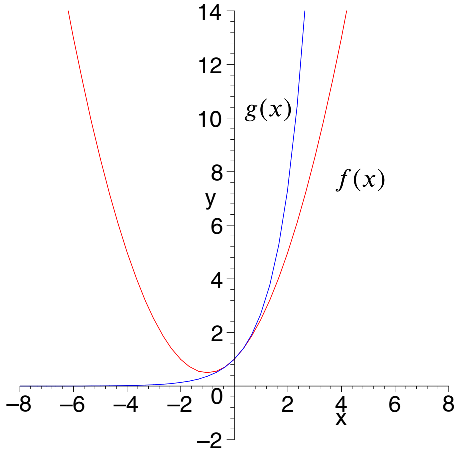
Determine b.

$$\begin{aligned} f'(0) &= g'(0) \\ 2a(0) + b &= e^0 \\ b &= 1 \end{aligned}$$

Determine c.

$$\begin{aligned} f(0) &= g(0) \\ 2a(0)^2 + b(0) + c &= e^0 \\ c &= 1 \end{aligned}$$

The quadratic polynomial that best approximates $g(x) = e^x$ at $x = 0$ is $f(x) = \frac{1}{2}x^2 + x + 1$.



Observations may vary.

Determine the best cubic approximation.

$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \\ f'(x) &= 3ax^2 + 2bx + c \\ f''(x) &= 6ax + 2b \\ f'''(x) &= 6a \end{aligned}$$

Determine a.

$$\begin{aligned} f'''(0) &= g'''(0) \\ 6a &= 1 \\ a &= \frac{1}{6} \end{aligned} \tag{4}$$

Determine b.

$$\begin{aligned} f''(0) &= g''(0) \\ 6a(0) + 2b &= e^0 \\ b &= \frac{1}{2} \end{aligned}$$

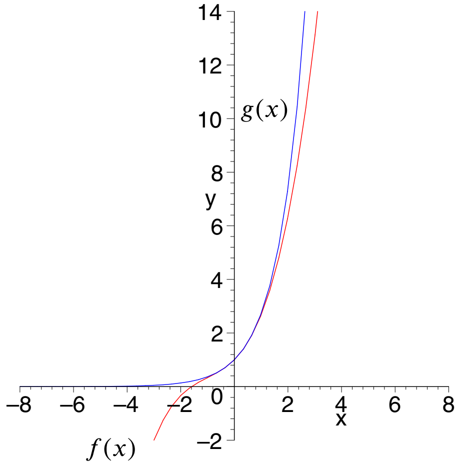
Determine c.

$$\begin{aligned} f'(0) &= g'(0) \\ 3a(0)^2 + 2b(0) + c &= e^0 \\ c &= 1 \end{aligned}$$

Determine d.

$$\begin{aligned} f(0) &= g(0) \\ a(0)^3 + b(0)^2 + c(0) + d &= e^0 \\ d &= 1 \end{aligned}$$

The cubic polynomial that best approximates $g(x) = e^x$ at $x = 0$ is $f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$.



Observations may vary.

Section Challenge Problems Page 484 Question 8

a) From problem 7, the quadratic function that approximates $g(x) = e^x$ near $x = 0$ is

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + x + 1 \\ &= \frac{1}{2!}x^2 + \frac{1}{1!}x + \frac{1}{0!} \end{aligned}$$

and the cubic function is

$$\begin{aligned} f(x) &= \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1 \\ &= \frac{1}{3!}x^3 + \frac{1}{2!}x^2 + \frac{1}{1!}x + \frac{1}{0!} \end{aligned}$$

This pattern implies that an nth-degree polynomial function that approximates $g(x) = e^x$ near $x = 0$ is

$$f(x) = \frac{1}{n!}x^n + \frac{1}{(n-1)!}x^{n-1} + \ldots + \frac{1}{2!}x^2 + \frac{1}{1!}x + \frac{1}{0!}$$

b) $S_n = \sum_{i=0}^{n} x^i$ is a geometric series with $a = 1$ and $r = x$. Thus,

$$\begin{aligned} S_n &= \frac{a(1 - r^{n+1})}{1 - r} \\ &= \frac{1(1 - x^{n+1})}{1 - x} \\ &= \frac{1 - x^{n+1}}{1 - x} \\ S(x) &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} \end{aligned}$$

As $n \to \infty$, $x^{n+1} \to 0$ if $-1 < x < 1$. Thus,

$$\begin{aligned} S(x) &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} \\ &= \lim_{n\to\infty} \frac{1}{1 - x} - \lim_{n\to\infty} \frac{x^{n+1}}{1 - x} \\ &= \frac{1}{1 - x} - \frac{0}{1 - x} \\ &= \frac{1}{1 - x} \end{aligned}$$

c)

$$\begin{aligned}
\left|\frac{x^r}{r!}\right| &= \left|\frac{x^k(x^{r-k})}{r!}\right| \\
&= \frac{|x|^k\,|x|^{r-k}}{r!} \\
&< \frac{|x|^k\,|x|^{r-k}}{k!} && \text{(since } k < r,\ k! < r!\text{, which makes the fraction bigger} \\
&< \frac{|x|^k\,|x|^{r-k}}{k!k^{r-k}} && \text{(since } k < r\text{, dividing by a positive power of } k \text{ makes the fraction bigger} \\
&< \frac{|x|^k}{k!}\left(\frac{|x|}{k}\right)^{r-k}
\end{aligned}$$

d)

$$\begin{aligned}
f_n(x) &= \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \frac{x^{n-2}}{(n-2)!} + \ldots + \frac{x^3}{3!} + \frac{x^2}{2!} + \frac{x^1}{1!} + \frac{x^0}{0!} \\
&< \frac{|x|^k}{k!}\left(\frac{|x|}{k}\right)^{n-k} + \frac{|x|^k}{k!}\left(\frac{|x|}{k}\right)^{n-1-k} + \frac{|x|^k}{k!}\left(\frac{|x|}{k}\right)^{n-2-k} + \ldots \\
&\quad \ldots + \frac{|x|^k}{k!}\left(\frac{|x|}{k}\right)^{3-k} + \frac{|x|^k}{k!}\left(\frac{|x|}{k}\right)^{2-k} + \frac{|x|^k}{k!}\left(\frac{|x|}{k}\right)^{1-k} + \frac{|x|^k}{k!}\left(\frac{|x|}{k}\right)^{-k} \\
&= \frac{|x|^k}{k!}\left[\left(\frac{|x|}{k}\right)^{n-k} + \left(\frac{|x|}{k}\right)^{n-1-k} + \left(\frac{|x|}{k}\right)^{n-2-k} + \ldots + \left(\frac{|x|}{k}\right)^{3-k} + \left(\frac{|x|}{k}\right)^{2-k} + \left(\frac{|x|}{k}\right)^{1-k} + \left(\frac{|x|}{k}\right)^{-k}\right] \\
&= \frac{|x|^k}{k!}\sum_{i=0}^{n}\left(\frac{|x|}{k}\right)^{i-k}
\end{aligned}$$

This is a geometric series with common ratio $s = \frac{|x|}{k}$. Since $-1 < x < 1$ and $k > x$, $-1 < s < 1$.

e) $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

Section Challenge Problems Page 484 Question 9

a)

$$\begin{aligned}
f_n(x) &= \left(1 + \frac{x}{n}\right)^n \\
f_n{}'(x) &= \frac{d\left(1 + \frac{x}{n}\right)^n}{d\left(1 + \frac{x}{n}\right)} \cdot \frac{d\left(1 + \frac{x}{n}\right)}{dx} \\
&= n\left(1 + \frac{x}{n}\right)^{n-1} \cdot \frac{1}{n} \\
&= \left(1 + \frac{x}{n}\right)^{n-1}
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \\
&= \lim_{n\to\infty}\left[\left(1 + \frac{x}{n}\right)^{n-1} \cdot \left(1 + \frac{x}{n}\right)\right] \\
&= \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^{n-1} \cdot \lim_{n\to\infty}\left(1 + \frac{x}{n}\right) \\
&= \lim_{n\to\infty} f_n{}'(x) \cdot 1 \\
&= \lim_{n\to\infty} f_n{}'(x)
\end{aligned}$$

c) For $g(x) = e^x$, $g(x) = g'(x)$. Thus, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n{}'(x)$ is evidence that $\lim_{n\to\infty} f_n(x) = e^x$.

CHAPTER 8 Trigonometric Functions and Their Derivatives

8.1 Addition and Subtraction Formulas

Throughout this section, all angle measures are in radians unless otherwise specified.

Practise

Section 8.1 Page 497 Question 1

a) $$\begin{aligned}\sin 45° \cos 15° - \cos 45° \sin 15° &= \sin(45° - 15°)\\ &= \sin 30°\\ &= \frac{1}{2}\end{aligned}$$

b) $$\begin{aligned}\sin 45° \cos 15° + \cos 45° \sin 15° &= \sin(45° + 15°)\\ &= \sin 60°\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

c) $$\begin{aligned}\cos 45° \cos 15° - \sin 45° \sin 15° &= \cos(45° + 15°)\\ &= \cos 60°\\ &= \frac{1}{2}\end{aligned}$$

d) $$\begin{aligned}\cos 45° \cos 15° + \sin 45° \sin 15° &= \cos(45° - 15°)\\ &= \cos 30°\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

e) $$\begin{aligned}\frac{\tan\frac{\pi}{12} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{12}\tan\frac{\pi}{6}} &= \tan\left(\frac{\pi}{12} + \frac{\pi}{6}\right)\\ &= \tan\frac{3\pi}{12}\\ &= \tan\frac{\pi}{4}\\ &= 1\end{aligned}$$

Section 8.1 Page 497 Question 3

a) $$\begin{aligned}\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{\pi}{3}\\ &= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

b) $$\begin{aligned}\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) &= \cos\frac{\pi}{2}\cos\frac{\pi}{3} + \sin\frac{\pi}{2}\sin\frac{\pi}{3}\\ &= 0\cdot\frac{1}{2} + 1\cdot\frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

c)

$$\begin{aligned}\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}}\\ &= \frac{1+\frac{1}{\sqrt{3}}}{1-1\cdot\frac{1}{\sqrt{3}}}\\ &= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}\\ &= \frac{3+2\sqrt{3}+1}{3-1}\\ &= \frac{4+2\sqrt{3}}{2}\\ &= 2+\sqrt{3}\end{aligned}$$

Apply, Solve, Communicate

Section 8.1 Page 497 Question 4

a)

$$\begin{aligned}\sin x\cos 2+\cos x\sin 2 &= -0.5\\ \sin(x+2) &= -0.5\\ x+2 &= \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}\\ x &= \frac{7\pi}{6}-2 \text{ or } \frac{11\pi}{6}-2\end{aligned}$$

b)

$$\begin{aligned}\cos x\cos 1-\sin x\sin 1 &= 0.4\\ \cos(x+1) &= 0.4\\ x+1 &\doteq 1.1593 \text{ or } 5.1239\\ x &\doteq 0.1593 \text{ or } 4.1239\end{aligned}$$

c)

$$\begin{aligned}\cos 3x\cos x-\sin 3x\sin x &= 0\\ \cos(3x+x) &= 0\\ \cos(4x) &= 0\end{aligned}$$

If $x \in [0, 2\pi]$, then $4x \in [0, 8\pi]$.

$$\begin{aligned}4x &= \frac{k\pi}{2},\ k \in \{1, 3, 5, 7, 9, 11, 13, 15\}\\ x &= \frac{k\pi}{8},\ k \in \{1, 3, 5, 7, 9, 11, 13, 15\}\end{aligned}$$

d)

$$\begin{aligned}\sin 2x\cos x-\cos 2x\sin x &= 0\\ \sin(2x-x) &= 0\\ \sin x &= 0\\ x &= 0, \pi, \text{ or } 2\pi\end{aligned}$$

e)

$$\begin{aligned}2\cos(2+x)-\sqrt{3} &= 0\\ \cos(2+x) &= \frac{\sqrt{3}}{2}\\ 2+x &= \frac{11\pi}{6} \text{ or } \frac{13\pi}{6}\\ x &= \frac{11\pi}{6}-2 \text{ or } \frac{13\pi}{6}-2\end{aligned}$$

f)

$$\begin{aligned}\sin 2x\cos 3x &= -1-\cos 2x\sin 3x\\ \sin 2x\cos 3x+\cos 2x\sin 3x &= -1\\ \sin(2x+3x) &= -1\\ \sin(5x) &= -1\end{aligned}$$

If $x \in [0, 2\pi]$, then $5x \in [0, 10\pi]$.

$$\begin{aligned}5x &= \frac{k\pi}{2},\ k \in \{3, 7, 11, 15, 19\}\\ x &= \frac{k\pi}{10},\ k \in \{3, 7, 11, 15, 19\}\end{aligned}$$

g)

$$\begin{aligned}\cos 2x\cos x &= \sin 2x \sin x\\ \cos 2x\cos x - \sin 2x\sin x &= 0\\ \cos(2x+x) &= 0\\ \cos 3x &= 0\end{aligned}$$

If $x \in [0, 2\pi]$, then $3x \in [0, 6\pi]$.

$$\begin{aligned}3x &= \frac{k\pi}{2},\ k \in \{1,3,5,7,9,11\}\\ x &= \frac{k\pi}{6},\ k \in \{1,3,5,7,9,11\}\end{aligned}$$

h)

$$\begin{aligned}\frac{\tan x + \tan\frac{\pi}{3}}{1-\tan x\tan\frac{\pi}{3}} &= \sqrt{3}\\ \tan\left(x+\frac{\pi}{3}\right) &= \sqrt{3}\\ x+\frac{\pi}{3} &= \frac{\pi}{3}, \frac{4\pi}{3},\ \text{or}\ \frac{7\pi}{3}\\ x &= 0, \pi,\ \text{or}\ 2\pi\end{aligned}$$

i)

$$\begin{aligned}\tan 3x - \tan x &= 3(1+\tan 3x\tan x)\\ \frac{\tan 3x - \tan x}{1-\tan 3x\tan x} &= 3\\ \tan(3x-x) &= 3\\ \tan 2x &= 3\\ 2x &\doteq 1.2490, 4.3906, 7.5322,\ \text{or}\ 10.6738\\ x &= 0.6245, 2.1953, 3.7661,\ \text{or}\ 5.3369\end{aligned}$$

j)

$$\begin{aligned}\cos 5x\cos x &= \sin 5x\sin x - 0.5\\ \cos 5x\cos x - \sin 5x\sin x &= -0.5\\ \cos(5x+x) &= -0.5\\ \cos(6x) &= -0.5\end{aligned}$$

If $x \in [0, 2\pi]$, then $6x \in [0, 12\pi]$.

$$\begin{aligned}6x &= \frac{k\pi}{3},\ k \in \{2,4,8,10,14,16,20,22,26,28,32,34\}\\ x &= \frac{k\pi}{18},\ k \in \{2,4,8,10,14,16,20,22,26,28,32,34\}\\ &= \frac{k\pi}{9},\ k \in \{1,2,4,5,7,8,10,11,13,14,16,17\}\end{aligned}$$

k)

$$\begin{aligned}4\sin 2x\cos x &= 4\cos 2x\sin x + 2\\ 4(\sin 2x\cos x - \cos 2x\sin x) &= 2\\ \sin(2x-x) &= 0.5\\ \sin x &= 0.5\\ x &= \frac{\pi}{6}\ \text{or}\ \frac{5\pi}{6}\end{aligned}$$

l)

$$\begin{aligned}\frac{3(\tan x + \tan 1)}{4(1-\tan x\tan 1)} &= 1\\ \frac{\tan x + \tan 1}{1-\tan x\tan 1} &= \frac{4}{3}\\ \tan(x+1) &= \frac{4}{3}\end{aligned}$$

If $x \in [0, 2\pi]$, then $x+1 \in [1, 2\pi+1]$.

$$\begin{aligned}x+1 &\doteq 4.0689\ \text{or}\ 7.2105\\ x &\doteq 3.0689\ \text{or}\ 6.2105\end{aligned}$$

m)

$$\begin{aligned}6\sin 5x\cos 3x &= 3 + 6\cos 5x\sin 3x\\ 6(\sin 5x\cos 3x - \cos 5x\sin 3x) &= 3\\ \sin(5x-3x) &= 0.5\\ \sin(2x) &= 0.5\end{aligned}$$

If $x \in [0, 2\pi]$, then $2x \in [0, 4\pi]$.

$$\begin{aligned}2x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6},\ \text{or}\ \frac{17\pi}{6}\\ x &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12},\ \text{or}\ \frac{17\pi}{12}\end{aligned}$$

Section 8.1 Page 497 Question 5

a) They are equal ($\tan\alpha = 2$ and $\tan\beta = 3$).

b)

$$\begin{aligned}\theta &= \beta - \alpha\\ \tan\theta &= \tan(\beta-\alpha)\\ &= \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha}\\ &= \frac{3-2}{1+(3)(2)}\end{aligned}$$

c)

$$\begin{aligned}\tan\alpha &= 2\\ \alpha &\doteq 1.1071\\ \tan\beta &= 3\\ \beta &\doteq 1.2490\\ \tan\theta &= \frac{3-2}{1+(3)(2)}\\ &= \frac{1}{7}\\ \theta &\doteq 0.1419\end{aligned}$$

Section 8.1 Page 497 Question 6

Determine $\angle$RUQ. In $\triangle$TUP, $\angle\text{TUP} = \frac{\pi}{2} - a$. Since $\angle\text{TUP} + \angle\text{RUQ} = \frac{\pi}{2}$, it follows that $\angle\text{RUQ} = a$.
In $\triangle$TRU,

$$\begin{aligned}\sin b &= \frac{\text{RU}}{\text{RT}}\\ &= \text{RU} \quad (\text{since RT} = 1)\end{aligned} \qquad \begin{aligned}\cos b &= \frac{\text{TU}}{\text{TR}}\\ &= \text{TU} \quad (\text{since TR} = 1)\end{aligned}$$

In $\triangle$TUP,

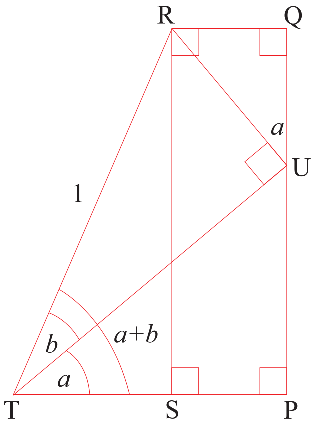
$$\begin{aligned}\cos a &= \frac{\text{TP}}{\text{TU}}\\ &= \frac{\text{TP}}{\cos b}\\ \text{TP} &= \cos a\cos b\end{aligned}$$

In $\triangle$RQU,

$$\begin{aligned}\sin a &= \frac{\text{RQ}}{\text{RU}}\\ &= \frac{\text{RQ}}{\sin b}\\ \text{RQ} &= \sin a\sin b\end{aligned}$$

In $\triangle$TRS,

$$\begin{aligned}\cos(a+b) &= \frac{\text{TS}}{\text{RT}}\\ &= \text{TS} \quad (\text{since TR} = 1)\\ &= \text{TP} - \text{SP}\\ &= \text{TP} - \text{RQ}\\ &= \cos a\cos b - \sin a\sin b\end{aligned}$$



Section 8.1 Page 497 Question 7

a)

$$\begin{aligned}\cos(\pi+x) &= \cos\pi\cos x - \sin\pi\sin x\\ &= -1\cdot\cos x - 0\cdot\sin x\\ &= -\cos x\end{aligned}$$

b)

$$\begin{aligned}\sin(\pi+x) &= \sin\pi\cos x + \cos\pi\sin x\\ &= 0\cdot\cos x + (-1)\cdot\sin x\\ &= -\sin x\end{aligned}$$

c)

$$\begin{aligned}\cos(\pi-x) &= \cos\pi\cos x + \sin\pi\sin x\\ &= -1\cdot\cos x + 0\cdot\sin x\\ &= -\cos x\end{aligned}$$

d)

$$\begin{aligned}\sin(\pi-x) &= \sin\pi\cos x - \cos\pi\sin x\\ &= 0\cdot\cos x - (-1)\cdot\sin x\\ &= \sin x\end{aligned}$$

e)

$$\begin{aligned}\tan(\pi+x) &= \frac{\tan\pi + \tan x}{1-\tan\pi\tan x}\\ &= \frac{0+\tan x}{1-0\cdot\tan x}\\ &= \tan x\end{aligned}$$

f)

$$\begin{aligned}\tan(\pi-x) &= \frac{\tan\pi - \tan x}{1+\tan\pi\tan x}\\ &= \frac{0-\tan x}{1+0\cdot\tan x}\\ &= -\tan x\end{aligned}$$

g)
$$\begin{aligned}\sin\left(\frac{3\pi}{2}+x\right) &= \sin\frac{3\pi}{2}\cos x + \cos\frac{3\pi}{2}\sin x\\ &= -1\cdot\cos x + 0\cdot\sin x\\ &= -\cos x\end{aligned}$$

h)
$$\begin{aligned}\cos\left(-\frac{\pi}{2}-x\right) &= \cos\left(-\frac{\pi}{2}\right)\cos x + \sin\left(-\frac{\pi}{2}\right)\sin x\\ &= 0\cdot\cos x + (-1)\cdot\sin x\\ &= -\sin x\end{aligned}$$

Section 8.1 Page 498 Question 8

a)
$$\begin{aligned}\frac{\sin(x-30^\circ)+\cos(60^\circ-x)}{\sin x} &= \frac{\sin x\cos 30^\circ - \cos x\sin 30^\circ + \cos 60^\circ\cos x + \sin 60^\circ\sin x}{\sin x}\\ &= \frac{\sin x\cos 30^\circ - \cos x\sin 30^\circ + \sin 30^\circ\cos x + \sin 60^\circ\sin x}{\sin x},\ (\cos 60^\circ = \sin 30^\circ)\\ &= \frac{\sin x\cos 30^\circ + \sin 60^\circ\sin x}{\sin x}\\ &= \cos 30^\circ + \sin 60^\circ,\ \sin x \neq 0\\ &= \frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\\ &= \sqrt{3}\end{aligned}$$

b)
$$\begin{aligned}\frac{\tan\left(\frac{\pi}{4}-x\right)-\tan\left(\frac{\pi}{4}+x\right)}{\tan x} &= \frac{1}{\tan x}\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}-\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right)\\ &= \frac{1}{\tan x}\left(\frac{1-\tan x}{1+\tan x}-\frac{1+\tan x}{1-\tan x}\right)\\ &= \frac{1}{\tan x}\cdot\frac{(1-\tan x)^2-(1+\tan x)^2}{(1-\tan x)(1+\tan x)}\\ &= \frac{1}{\tan x}\cdot\frac{1-2\tan x+\tan^2 x-1-2\tan x-\tan^2 x}{1-\tan^2 x}\\ &= \frac{1}{\tan x}\cdot\frac{-4\tan x}{1-\tan^2 x}\\ &= \frac{-4}{1-\tan^2 x}\end{aligned}$$

Section 8.1 Page 498 Question 9

a) Recall the identity $\sin^2\alpha+\cos^2\alpha = 1$.

$$\begin{aligned}\sin^2 x+\cos^2 x &= 1\\ \cos^2 x &= 1-\sin^2 x\\ \cos x &= \sqrt{1-\sin^2 x},\ \cos x > 0 \qquad (1)\end{aligned}$$

Substitute $\sin x = \frac{4}{5}$ into (1).

$$\begin{aligned}\cos x &= \sqrt{1-\left(\frac{4}{5}\right)^2}\\ &= \sqrt{1-\frac{16}{25}}\\ &= \sqrt{\frac{9}{25}}\\ &= \frac{3}{5}\end{aligned}$$

$$\begin{aligned}\sin^2 y+\cos^2 y &= 1\\ \sin^2 y &= 1-\cos^2 y\\ \sin y &= \sqrt{1-\cos^2 y},\ \sin y > 0 \qquad (2)\end{aligned}$$

Substitute $\cos y = \frac{5}{13}$ into (2).

$$\begin{aligned}\sin y &= \sqrt{1-\left(\frac{5}{13}\right)^2}\\ &= \sqrt{1-\frac{25}{169}}\\ &= \sqrt{\frac{144}{169}}\\ &= \frac{12}{13}\end{aligned}$$

b) i) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{12}{13}\right)$$
$$= \frac{20}{65} + \frac{36}{65}$$
$$= \frac{56}{65}$$

ii) $\cos(x - y) = \cos x \cos y + \sin x \sin y$

$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{12}{13}\right)$$
$$= \frac{15}{65} + \frac{48}{65}$$
$$= \frac{63}{65}$$

iii) First, determine $\tan x$ and $\tan y$.

$$\tan x = \frac{\sin x}{\cos x}$$
$$= \frac{\frac{4}{5}}{\frac{3}{5}}$$
$$= \frac{4}{3} \qquad (3)$$
$$\tan y = \frac{\sin y}{\cos y}$$
$$= \frac{\frac{12}{13}}{\frac{5}{13}}$$
$$= \frac{12}{5} \qquad (4)$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad (5)$$

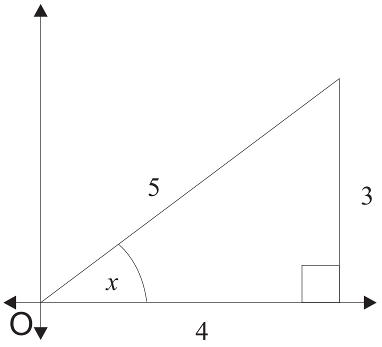
Substitute (3) and (4) into (5).

$$= \frac{\frac{4}{3} + \frac{12}{5}}{1 - \left(\frac{4}{3}\right)\left(\frac{12}{5}\right)}$$
$$= \frac{\frac{56}{15}}{1 - \frac{48}{15}}$$
$$= \frac{\frac{56}{15}}{-\frac{33}{15}}$$
$$= -\frac{56}{33}$$

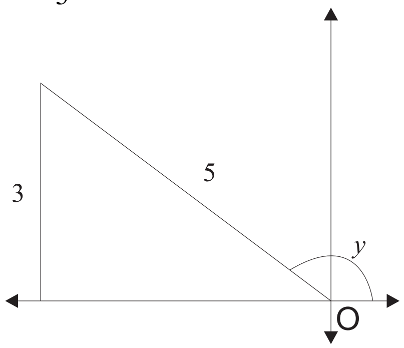
Section 8.1 Page 498 Question 10

Recall the Pythagorean triple 3, 4, 5.

From $\sin x = \frac{3}{5},\ x \in \left[0, \frac{\pi}{2}\right]$, it follows that $\cos x = \frac{4}{5}$ and $\tan x = \frac{3}{4}$.



From $\tan y = -\frac{3}{4},\ y \in \left[\frac{\pi}{2}, \pi\right]$, it follows that $\sin y = \frac{3}{5}$ and $\cos y = -\frac{4}{5}$.



a) $\sin(x - y) = \sin x \cos y - \cos x \sin y$

$$= \left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$$
$$= -\frac{12}{25} - \frac{12}{25}$$
$$= -\frac{24}{25}$$

b) $\cos(x + y) = \cos x \cos y - \sin x \sin y$

$$= \left(\frac{4}{5}\right)\left(-\frac{4}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)$$
$$= -\frac{16}{25} - \frac{9}{25}$$
$$= -1$$

c)

$$\begin{aligned}\tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y}\\ &= \frac{\frac{3}{4} - \left(-\frac{3}{4}\right)}{1 + \left(\frac{3}{4}\right)\left(-\frac{3}{4}\right)}\\ &= \frac{\frac{6}{4}}{1 - \frac{9}{16}}\\ &= \frac{\frac{24}{16}}{\frac{7}{16}}\\ &= \frac{24}{7}\end{aligned}$$

Section 8.1 Page 498 Question 11

a)

$$\begin{aligned}\frac{\cos(x+h) - \cos x}{h} &= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}\\ &= \frac{\cos x \cos h - \cos x - \sin x \sin h}{h}\\ &= \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}\\ &= \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)\end{aligned}$$

b)

$$\begin{aligned}\frac{\sin(x+h) - \sin x}{h} &= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}\\ &= \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}\\ &= \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}\\ &= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\end{aligned}$$

Section 8.1 Page 498 Question 12

a)

$$\begin{aligned}\sec(a+b) &= \frac{1}{\cos(a+b)}\\ &= \frac{1}{\cos a \cos b - \sin a \sin b}\end{aligned}$$

Divide the numerator and denominator by cos *a* cos *b*.

$$\begin{aligned}&= \frac{\frac{1}{\cos a \cos b}}{\frac{\cos a \cos b}{\cos a \cos b} - \frac{\sin a \sin b}{\cos a \cos b}}\\ &= \frac{\sec a \sec b}{1 - \tan a \tan b}\end{aligned}$$

b)

$$\begin{aligned}\sec(a-b) &= \frac{1}{\cos(a-b)}\\ &= \frac{1}{\cos a \cos b + \sin a \sin b}\end{aligned}$$

Divide the numerator and denominator by cos *a* cos *b*.

$$\begin{aligned}&= \frac{\frac{1}{\cos a \cos b}}{\frac{\cos a \cos b}{\cos a \cos b} + \frac{\sin a \sin b}{\cos a \cos b}}\\ &= \frac{\sec a \sec b}{1 + \tan a \tan b}\end{aligned}$$

Section 8.1 Page 498 Question 13

a)

$$\begin{aligned}\csc(a+b) &= \frac{1}{\sin(a+b)}\\ &= \frac{1}{\sin a\cos b+\cos a\sin b}\end{aligned}$$

Divide the numerator and denominator by $\sin a \sin b$.

$$\begin{aligned}&= \frac{\frac{1}{\sin a\sin b}}{\frac{\sin a\cos b}{\sin a\sin b}+\frac{\cos a\sin b}{\sin a\sin b}}\\ &= \frac{\csc a\csc b}{\cot b+\cot a}\end{aligned}$$

b)

$$\begin{aligned}\csc(a-b) &= \frac{1}{\sin(a-b)}\\ &= \frac{1}{\sin a\cos b-\cos a\sin b}\end{aligned}$$

Divide the numerator and denominator by $\sin a \sin b$.

$$\begin{aligned}&= \frac{\frac{1}{\sin a\sin b}}{\frac{\sin a\cos b}{\sin a\sin b}-\frac{\cos a\sin b}{\sin a\sin b}}\\ &= \frac{\csc a\csc b}{\cot b-\cot a}\end{aligned}$$

Section 8.1 Page 498 Question 14

a)

$$\begin{aligned}\cot(a+b) &= \frac{\cos(a+b)}{\sin(a+b)}\\ &= \frac{\cos a\cos b-\sin a\sin b}{\sin a\cos b+\cos a\sin b}\end{aligned}$$

Divide the numerator and denominator by $\sin a \sin b$.

$$\begin{aligned}&= \frac{\frac{\cos a\cos b}{\sin a\sin b}-\frac{\sin a\sin b}{\sin a\sin b}}{\frac{\sin a\cos b}{\sin a\sin b}+\frac{\cos a\sin b}{\sin a\sin b}}\\ &= \frac{\cot a\cot b-1}{\cot a+\cot b}\end{aligned}$$

b)

$$\begin{aligned}\cot(a-b) &= \frac{\cos(a-b)}{\sin(a-b)}\\ &= \frac{\cos a\cos b+\sin a\sin b}{\sin a\cos b-\cos a\sin b}\end{aligned}$$

Divide the numerator and denominator by $\sin a \sin b$.

$$\begin{aligned}&= \frac{\frac{\cos a\cos b}{\sin a\sin b}+\frac{\sin a\sin b}{\sin a\sin b}}{\frac{\sin a\cos b}{\sin a\sin b}-\frac{\cos a\sin b}{\sin a\sin b}}\\ &= \frac{\cot a\cot b+1}{\cot b-\cot a}\end{aligned}$$

Section 8.1 Page 498 Question 15

a)

$$\begin{aligned}\sin(x+y) &= \sin(x-y)\\ \sin x\cos y+\cos x\sin y &= \sin x\cos y-\cos x\sin y\\ 2\cos x\sin y &= 0\\ \cos x &= 0\\ \sin y &= 0\end{aligned}$$

The equation is true for x such that $\cos x = 0$ or $\sin y = 0$.

b)

$$\begin{aligned}\cos(x+y) &= \cos(x-y)\\ \cos x\cos y-\sin x\sin y &= \cos x\cos y+\sin x\sin y\\ 2\sin x\sin y &= 0\\ \sin x &= 0\\ \sin y &= 0\end{aligned}$$

The equation is true for x such that $\sin x = 0$ or $\sin y = 0$.

c)

$$\sin x + \sin y = \cos x + \cos y$$
$$2\cos\frac{x-y}{2}\sin\frac{x+y}{2} = 2\cos\frac{x-y}{2}\cos\frac{x+y}{2}$$

$$\begin{aligned}\cos x - y &= 0\\ \frac{x-y}{2} &= (2k+1)\frac{\pi}{2},\ k \text{ any integer}\\ x-y &= (2k+1)\pi\end{aligned}$$

$$\begin{aligned}\sin x + y &= \cos\frac{x+y}{2}\\ \tan\frac{x+y}{2} &= 1\\ \frac{x+y}{2} &= \frac{\pi}{4}+\pi k,\ k \text{ any integer}\\ x+y &= \frac{\pi}{2}+2\pi k\end{aligned}$$

Section 8.1 Page 498 Question 16

a)

$$\begin{aligned}\tan x + \tan y &= \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}\\ &= \frac{\sin x\cos y + \sin y\cos x}{\cos x\cos y}\\ &= \frac{\sin(x+y)}{\cos x\cos y}\end{aligned}$$

b)

$$\begin{aligned}\tan x - \tan y &= \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}\\ &= \frac{\sin x\cos y - \sin y\cos x}{\cos x\cos y}\\ &= \frac{\sin(x-y)}{\cos x\cos y}\end{aligned}$$

8.2 Double-Angle Formulas

Practise

Section 8.2 Page 503 Question 1

a)
$$\begin{aligned} 10\sin x\cos x &= 5(2\sin x\cos x) \\ &= 5\sin 2x \end{aligned}$$

b)
$$\begin{aligned} 5\sin(2x)\cos(2x) &= 5\left(\frac{1}{2}\sin(2(2x))\right) \\ &= \frac{5}{2}\sin 4x \end{aligned}$$

c)
$$\begin{aligned} \sin(6x)\cos(6x) &= \frac{1}{2}\sin 2(6x) \\ &= \frac{1}{2}\sin 12x \end{aligned}$$

d)
$$\begin{aligned} 4\sin\frac{x}{2}\cos\frac{x}{2} &= 2\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right) \\ &= 2\sin 2\left(\frac{x}{2}\right) \\ &= 2\sin x \end{aligned}$$

e)
$$\begin{aligned} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} &= \cos 2\left(\frac{\theta}{2}\right) \\ &= \cos\theta \end{aligned}$$

f)
$$\begin{aligned} 2\cos^2 5\theta - 1 &= \cos 2(5\theta) \\ &= \cos 10\theta \end{aligned}$$

g)
$$\begin{aligned} 1 - 2\sin^2\frac{2\theta}{3} &= \cos 2\left(\frac{2\theta}{3}\right) \\ &= \cos\frac{4\theta}{3} \end{aligned}$$

h)
$$\begin{aligned} 2\cos^2(3\theta - 2) - 1 &= \cos 2(3\theta - 2) \\ &= \cos(6\theta - 4) \end{aligned}$$

Apply, Solve, Communicate

Section 8.2 Page 503 Question 3

a)
$$\begin{aligned} 3\sin 4x &= 3\sin 2(2x) \\ &= 3(2\sin 2x\cos 2x) \\ &= 6\sin 2x\cos 2x \end{aligned}$$

b)
$$6\cos 6x = 6\cos 2(3x)$$

Expression 1:
$$\begin{aligned} &= 6(2\cos^2 3x - 1) \\ &= 12\cos^2 3x - 6 \end{aligned}$$

Expression 2:
$$\begin{aligned} &= 6(1 - 2\sin^2 3x) \\ &= 6 - 12\sin^2 3x \end{aligned}$$

Expression 3:
$$\begin{aligned} &= 6(\cos^2 3x - \sin^2 3x) \\ &= 6\cos^2 3x - 6\sin^2 3x \end{aligned}$$

c)
$$\begin{aligned} 1 - \cos 8x &= 1 - \cos 2(4x) \\ &= 1 - (1 - 2\sin^2 4x \\ &= 2\sin^2 4x \end{aligned}$$

d)
$$\begin{aligned} \tan 4x &= \tan 2(2x) \\ &= \frac{2\tan 2x}{1 - \tan^2 2x} \end{aligned}$$

e)
$$\begin{aligned} \cos 2x - \frac{\sin 2x}{\sin x} &= 2\cos^2 x - 1 - \frac{2\sin x\cos x}{\sin x} \\ &= 2\cos^2 x - 1 - 2\cos x \end{aligned}$$

Section 8.2 Page 503 Question 4

a)
$$\begin{aligned} \sin x &= \sin 2\left(\frac{x}{2}\right) \\ &= 2\sin\frac{x}{2}\cos\frac{x}{2} \end{aligned}$$

b)
$$\begin{aligned} \cos x &= \cos 2\left(\frac{x}{2}\right) \\ &= 1 - 2\sin^2\left(\frac{x}{2}\right) \end{aligned}$$

Section 8.2 Page 503 Question 5

a)

$$\begin{aligned}
\cos 2x + \cos x + 1 &= 0\\
2\cos^2 x - 1 + \cos x + 1 &= 0\\
\cos x(2\cos x + 1) &= 0\\
\cos x &= 0\\
x &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2}\\
2\cos x + 1 &= 0\\
\cos x &= -\frac{1}{2}\\
x &= \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}
\end{aligned}$$

The equation is true for $x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$.

b)

$$\begin{aligned}
\cos 2x &= \sin x\\
1 - 2\sin^2 x &= \sin x\\
2\sin^2 x + \sin x - 1 &= 0\\
(2\sin x - 1)(\sin x + 1) &= 0\\
2\sin x - 1 &= 0\\
\sin x &= \frac{1}{2}\\
x &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6}\\
\sin x + 1 &= 0\\
\sin x &= -1\\
x &= \frac{3\pi}{2}
\end{aligned}$$

The equation is true for $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

c)

$$\begin{aligned}
3\tan x &= \tan 2x\\
3\tan x &= \frac{2\tan x}{1 - \tan^2 x}\\
\tan x &= 0\\
x &= 0, \pi, \text{ or } 2\pi\\
3 &= \frac{2}{1 - \tan^2 x}\\
1 - \tan^2 x &= \frac{2}{3}\\
\tan^2 x &= \frac{1}{3}\\
\tan x &= \pm\frac{1}{\sqrt{3}}\\
x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}
\end{aligned}$$

The equation is true for $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$.

d)

$$\begin{aligned}
\sin x &= 6\sin 2x\\
\sin x &= 6(2\sin x\cos x)\\
\sin x &= 12\sin x\cos x\\
\sin x &= 0\\
x &= 0, \pi, \text{ or } 2\pi\\
1 &= 12\cos x\\
\cos x &= \frac{1}{12}\\
x &\doteq 1.4874 \text{ or } 4.7958
\end{aligned}$$

The equation is true for $x = 0, 1.4874, \pi, 4.7958, 2\pi$.

e)

$$\begin{aligned}
\sin 2x\cos x + \sin^2 x &= 1\\
2\sin x\cos x\cos x + (\sin^2 x - 1) &= 0\\
2\sin x\cos^2 x - \cos^2 x &= 0\\
\cos^2 x(2\sin x - 1) &= 0\\
\cos x &= 0\\
x &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2}\\
2\sin x - 1 &= 0\\
\sin x &= \frac{1}{2}\\
x &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6}
\end{aligned}$$

The equation is true for $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

f)

$$\begin{aligned}
\sin 2x + \sin x &= 0\\
2\sin x\cos x + \sin x &= 0\\
\sin x(2\cos x + 1) &= 0\\
\sin x &= 0\\
x &= 0, \pi, \text{ or } 2\pi\\
2\cos x + 1 &= 0\\
\cos x &= -\frac{1}{2}\\
x &= \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}
\end{aligned}$$

The equation is true for $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$.

g)
$$\begin{aligned} \sin 2x - \cos 2x &= 0 \\ \sin 2x &= \cos 2x \\ \frac{\sin 2x}{\cos 2x} &= 1 \\ \tan 2x &= 1 \\ 2x &= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \text{ or } \frac{13\pi}{4} \\ x &= \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \text{ or } \frac{13\pi}{8} \end{aligned}$$

The equation is true for $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \text{ or } \frac{13\pi}{8}$.

h)
$$\begin{aligned} 3\cos 2x + 2 + \cos x &= 0 \\ 3(2\cos^2 x - 1) + 2 + \cos x &= 0 \\ 6\cos^2 x - 3 + 2 + \cos x &= 0 \\ 6\cos^2 x + \cos x - 1 &= 0 \\ (2\cos x + 1)(3\cos x - 1) &= 0 \\ 2\cos x + 1 &= 0 \\ \cos x &= -\frac{1}{2} \\ x &= \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \\ 3\cos x - 1 &= 0 \\ \cos x &= \frac{1}{3} \\ x &\doteq 1.2310 \text{ or } 5.0522 \end{aligned}$$

The equation is true for $x = 1.2310, \frac{2\pi}{3}, \frac{4\pi}{3}, 5.0522$.

i)
$$\begin{aligned} \sin 2x &= \tan x \\ 2\sin x\cos x &= \frac{\sin x}{\cos x} \\ \sin x &= 0 \\ x &= 0, \pi, \text{ or } 2\pi \\ 2\cos^2 x &= 1 \\ \cos^2 x &= \frac{1}{2} \\ \cos x &= \pm\frac{1}{\sqrt{2}} \\ x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4} \end{aligned}$$

The equation is true for $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$.

j)
$$\begin{aligned} 2\sin x\cos x &= \cos 2x \\ \sin 2x &= \cos 2x \\ \frac{\sin 2x}{\cos 2x} &= 1 \\ \tan 2x &= 1 \\ 2x &= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \text{ or } \frac{13\pi}{4} \\ x &= \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \text{ or } \frac{13\pi}{8} \end{aligned}$$

The equation is true for $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \text{ or } \frac{13\pi}{8}$.

k)
$$\begin{aligned} 3\sin 2x - \cos x &= 0 \\ 3(2\sin x\cos x) - \cos x &= 0 \\ 6\sin x\cos x - \cos x &= 0 \\ \cos x(6\sin x - 1) &= 0 \\ \cos x &= 0 \\ x &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\ 6\sin x - 1 &= 0 \\ \sin x &= \frac{1}{6} \\ x &\doteq 0.1674 \text{ or } 2.9741 \end{aligned}$$

The equation is true for $x = 0.1674, \frac{\pi}{2}, 2.9741, \frac{3\pi}{2}$.

l)
$$\begin{aligned} 3\sin x + \cos 2x &= 2 \\ 3\sin x + 1 - 2\sin^2 x &= 2 \\ 2\sin^2 x - 3\sin x + 1 &= 0 \\ (2\sin x - 1)(\sin x - 1) & \\ 2\sin x - 1 &= 0 \\ \sin x &= \frac{1}{2} \\ x &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \\ \sin x - 1 &= 0 \\ \sin x &= 1 \\ x &= \frac{\pi}{2} \end{aligned}$$

The equation is true for $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

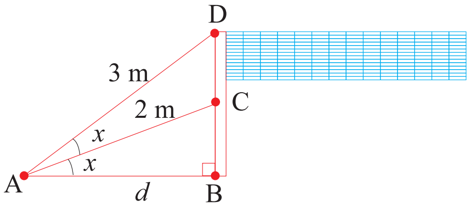
m)

$$\begin{aligned}
5 - 13\sin x &= 2\cos 2x \\
5 - 13\sin x &= 2(1 - 2\sin^2 x) \\
5 - 13\sin x &= 2 - 4\sin^2 x \\
4\sin^2 x - 13\sin x + 3 &= 0 \\
(\sin x - 3)(4\sin x - 1) &= 0 \\
\sin x - 3 &\neq 0 \\
4\sin x - 1 &= 0 \\
\sin x &= \frac{1}{4} \\
x &\doteq 0.2527 \text{ or } 2.8889
\end{aligned}$$

The equation is true for $x = 0.2527, 2.8889$.

Section 8.2 Page 503 Question 6

Let d be the distance from the base of the pole to the anchor point of the ropes. In △ABC,



$$\begin{aligned}
\cos x &= \frac{d}{2} \\
d &= 2\cos x \qquad (1)
\end{aligned}$$

In △ABD,

$$\begin{aligned}
\cos 2x &= \frac{d}{3} \\
d &= 3\cos 2x \qquad (2)
\end{aligned}$$

Equate (1) and (2) and solve for $\cos x$.

$$\begin{aligned}
2\cos x &= 3\cos 2x \\
&= 3(2\cos^2 x - 1) \\
&= 6\cos^2 x - 3 \\
6\cos^2 x - 2\cos x - 3 &= 0 \\
\cos x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(6)(-3)}}{2(6)} \\
&= \frac{2 \pm \sqrt{76}}{12} \\
&= \frac{1 + \sqrt{19}}{6}, \ \cos x > 0 \qquad (3)
\end{aligned}$$

Substitute (3) into (1).

$$\begin{aligned}
d &= 2 \cdot \frac{1 + \sqrt{19}}{6} \\
&\doteq 1.79
\end{aligned}$$

The distance is approximately 1.79 m.

Section 8.2 Page 503 Question 7

a)
$$\begin{aligned}\sec 2\theta &= \frac{1}{\cos 2\theta}\\ &= \frac{1}{\cos^2\theta - \sin^2\theta}\\ &= \frac{\frac{1}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}}\\ &= \frac{\sec^2\theta}{1-\tan^2\theta}\end{aligned}$$

b)
$$\begin{aligned}\csc 2\theta &= \frac{1}{\sin 2\theta}\\ &= \frac{1}{2\sin\theta\cos\theta}\\ &= \frac{1}{2}\cdot\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\\ &= \frac{1}{2}\csc\theta\sec\theta\end{aligned}$$

Section 8.2 Page 503 Question 8

a)
$$\begin{aligned}\sin 3\theta &= \sin(\theta + 2\theta)\\ &= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta\\ &= \sin\theta(1-2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)\\ &= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta\\ &= \sin\theta - 2\sin^3\theta + 2\sin\theta(1-\sin^2\theta)\\ &= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

b)
$$\begin{aligned}\cos 3\theta &= \cos(\theta + 2\theta)\\ &= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta\\ &= \cos\theta(2\cos^2\theta - 1) - \sin\theta(2\sin\theta\cos\theta)\\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

c)
$$\begin{aligned}\tan 3\theta &= \tan(\theta + 2\theta)\\ &= \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}\\ &= \frac{\tan\theta + \frac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}\\ &= \frac{\frac{\tan\theta(1-\tan^2\theta) + 2\tan\theta}{1-\tan^2\theta}}{\frac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}}\\ &= \frac{\tan\theta - \tan^3\theta + 2\tan\theta}{1-3\tan^2\theta}\\ &= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\end{aligned}$$

Section 8.2 Page 503 Question 9

a)

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= \frac{2\sin\theta}{\sec\theta}\\ &= \frac{2\sin\theta}{\sec\theta}\times\frac{\sec\theta}{\sec\theta}\\ &= \frac{2\tan\theta}{\sec^2\theta}\\ &= \frac{2\tan\theta}{1+\tan^2\theta}\end{aligned}$$

Section 8.2 Page 503 Question 10

$$\begin{aligned}\sin\theta+\cos\theta &= \frac{1}{2}\\ (\sin\theta+\cos\theta)^2 &= \left(\frac{1}{2}\right)^2\\ \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta &= \frac{1}{4}\\ (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta &= \frac{1}{4}\\ (1)+\sin 2\theta &= \frac{1}{4}\\ \sin 2\theta &= -\frac{3}{4}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta-\sin^2\theta\\ &= (\cos^2\theta-\sin^2\theta)\times\frac{\sec^2\theta}{\sec^2\theta}\\ &= \frac{\cos^2\theta\sec^2\theta-\sin^2\theta\sec^2\theta}{\sec^2\theta}\\ &= \frac{1-\tan^2\theta}{\sec^2\theta}\\ &= \frac{1-\tan^2\theta}{1+\tan^2\theta}\end{aligned}$$

Section 8.2 Page 503 Question 11

$$\begin{aligned}\cos 4\theta-\cos 2\theta &= \frac{7}{8}\\ 2\cos^2 2\theta-1-\cos 2\theta &= \frac{7}{8}\\ 2\cos^2 2\theta-\cos 2\theta-\frac{15}{8} &= 0\\ 16\cos^2\theta-8\cos 2\theta-15 &= 0\\ \cos 2\theta &= \frac{-(-8)\pm\sqrt{(-8)^2-4(16)(-15)}}{2(16)}\\ &= \frac{8\pm\sqrt{1024}}{32}\\ &= \frac{8-32}{32},\ |\cos 2\theta|\le 1\\ &= -\frac{3}{4}\\ 2\theta &\doteq 2.4189, 3.8643, 8.7020 \text{ or } 10.1475\\ \theta &\doteq 1.2094, 1.9322, 4.3510 \text{ or } 5.0738\\ \cos\theta &= \pm 0.3536\end{aligned}$$

Section 8.2 Page 503 Question 12

a) Label the diagram as shown. Let r and h be the radius and height of the cone, in centimetres, respectively. Since CD is 30 cm, △CBD is isosceles and the measure of ∠CBD is also x. Since ∠ACB is an exterior angle of △CBD, its measure is $2x$. The length of AC can be expressed as $h-30$ centimetres.

In △CBA,

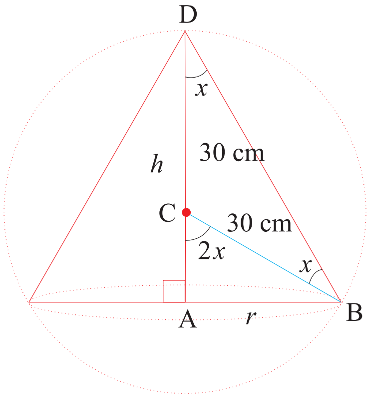
$$\begin{aligned}\sin 2x &= \frac{r}{30}\\ r &= 30\sin 2x && (1)\\ \cos 2x &= \frac{h-30}{30}\\ h &= 30\cos 2x+30\\ &= 30(2\cos^2 x-1)+30\\ &= 60\cos^2 x && (2)\end{aligned}$$

Let V be the volume of the cone.

$$V = \frac{1}{3}\pi r^2 h \qquad (3)$$

Substitute (1) and (2) into (3).

$$\begin{aligned}V &= \frac{1}{3}\pi(30\sin 2x)^2(60\cos^2 x)\\ &= 18\,000\pi\sin^2 2x\cos^2 x\end{aligned}$$

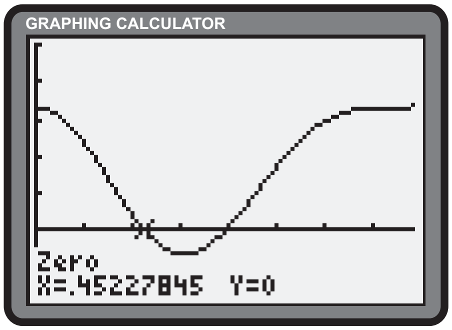


The volume of the cone is $V = 18\,000\pi\sin^2 2x\cos^2 x$ cm^3.

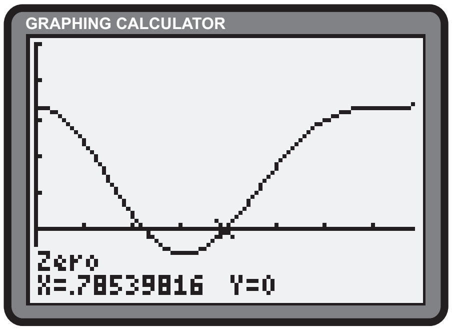
b)

$$\begin{aligned}
V &= 9000\pi \\
18\,000\pi \sin^2 2x \cos^2 x &= 9000\pi \\
2 \sin^2 2x \cos^2 x &= 1 \\
2(2 \sin x \cos x)^2 \cos^2 x &= 1 \\
8 \sin^2 x \cos^2 x \cos^2 x &= 1 \\
8 \cos^4 x(1 - \cos^2 x) &= 1 \\
8 \cos^4 x - 8 \cos^6 x &= 1 \\
8 \cos^6 x - 8 \cos^4 x + 1 &= 0
\end{aligned}$$

c) The measures of the semivertical angles that yield a cone of volume 9000π cm^3 are $x = 0.4523$ and $x = 0.7854$.



Zero
X=.45227845 Y=0



Zero
X=.78539816 Y=0

8.3 Limits of Trigonometric Functions

A number of solutions that follow make use the of the result $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = 1$. It is confirmed as follows,

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x}{\cos x}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}\\ &= 1\cdot 1\\ &= 1\end{aligned}$$

Practise

Section 8.3 Page 508 Question 1

a)

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

b)

$$\lim_{x\to 0}\frac{\sin(x-1)}{x} = \frac{\sin -1}{0}$$

does not exist

c)

$$\begin{aligned}\lim_{x\to 1}\frac{\sin x}{x} &= \frac{\sin 1}{1}\\ &= \sin 1\end{aligned}$$

d)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{3} &= \frac{0}{3}\\ &= 0\end{aligned}$$

e)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{3x} &= \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x}\\ &= \frac{1}{3}(1)\\ &= \frac{1}{3}\end{aligned}$$

f)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{x} &= \lim_{3x\to 0}\frac{3\sin 3x}{3x}\\ &= 3(1)\\ &= 3\end{aligned}$$

g)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{4x} &= \frac{1}{4}\lim_{x\to 0}\frac{\sin 3x}{x}\cdot\frac{3}{3}\\ &= \frac{1}{4}\cdot 3\cdot\lim_{3x\to 0}\frac{\sin 3x}{3x}\\ &= \frac{3}{4}\end{aligned}$$

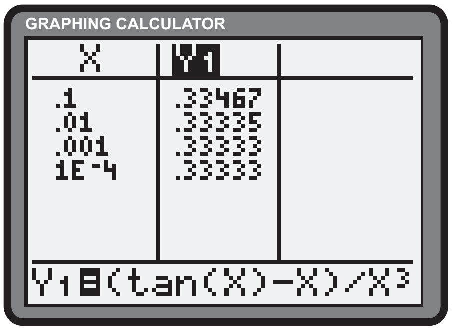
h)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 5x}{2x} &= \frac{1}{2}\lim_{x\to 0}\frac{\sin 5x}{x}\cdot\frac{5}{5}\\ &= \frac{1}{2}\cdot 5\cdot\lim_{5x\to 0}\frac{\sin 5x}{5x}\\ &= \frac{5}{2}\end{aligned}$$

Apply, Solve, Communicate

Section 8.3 Page 509 Question 3

a)

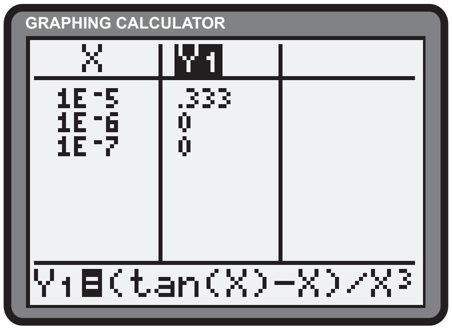


```
  X     Y1
.1      .33467
.01     .33335
.001    .33333
1E-4    .33333

Y1=(tan(X)-X)/X³
```

b) The results from part a) suggest $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^3} = \dfrac{1}{3}$.

c)



```
  X     Y1
1E-5    .333
1E-6    0
1E-7    0

Y1=(tan(X)-X)/X³
```

Values of x close to zero may lead to roundoff error.

Section 8.3 Page 509 Question 4

a)
$$\begin{aligned}\lim_{x\to0}\frac{\tan 6x}{2x} &= \frac{1}{2}\lim_{x\to0}\frac{\tan 6x}{x}\cdot\frac{6}{6}\\ &= \frac{1}{2}\cdot 6\cdot\lim_{6x\to0}\frac{\tan 6x}{6x}\\ &= 3\end{aligned}$$

b) Explanations may vary.

Section 8.3 Page 509 Question 5

a)
$$\begin{aligned}\lim_{x\to0}\frac{\cos 2x-1}{x^2} &= \lim_{x\to0}\frac{1-2\sin^2 x-1}{x^2}\\ &= \lim_{x\to0}\frac{-2\sin^2 x}{x^2}\\ &= -2\cdot\lim_{x\to0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{x}\right)\\ &= -2\cdot 1\cdot 1\\ &= -2\end{aligned}$$

b) Explanations may vary.

Section 8.3 Page 509 Question 6

$$\begin{aligned}\lim_{h\to0}\frac{\sin(a+h)-\sin a}{h} &= \lim_{h\to0}\frac{\sin a\cos h+\cos a\sin h-\sin a}{h}\\ &= \lim_{h\to0}\frac{\sin a\cos h-\sin a+\cos a\sin h}{h}\\ &= \lim_{h\to0}\frac{\sin a(\cos h-1)+\cos a\sin h}{h}\\ &= \sin a\cdot\lim_{h\to0}\frac{\cos h-1}{h}+\cos a\cdot\lim_{h\to0}\frac{\sin h}{h}\\ &= \sin a\cdot 0+\cos a\cdot 1\\ &= \cos a\end{aligned}$$

Section 8.3 Page 509 Question 7

$$\begin{aligned}\lim_{h\to0}\frac{\cos(a+h)-\cos a}{h} &= \lim_{h\to0}\frac{\cos a\cos h-\sin a\sin h-\cos a}{h}\\ &= \lim_{h\to0}\frac{\cos a\cos h-\cos a-\sin a\sin h}{h}\\ &= \lim_{h\to0}\frac{\cos a(\cos h-1)-\sin a\sin h}{h}\\ &= \cos a\cdot\lim_{h\to0}\frac{\cos h-1}{h}-\sin a\cdot\lim_{h\to0}\frac{\sin h}{h}\\ &= \cos a\cdot 0-\sin a\cdot 1\\ &= -\sin a\end{aligned}$$

Section 8.3 Page 509 Question 8

$$\lim_{x\to0}\frac{\sin x}{|x|}=\begin{cases}\displaystyle\lim_{x\to0^+}\frac{\sin x}{x}=1\\ \displaystyle\lim_{x\to0^-}\frac{\sin x}{-x}=-1\end{cases}$$

Since the limit from the positive side of 0 is not equal to the limit from the negative side of 0, the limit does not exist.

Section 8.3 Page 509 Question 9

$$\begin{aligned}\lim_{x\to0}\frac{\sin(\cos x)}{\frac{1}{\cos x}} &= \frac{\sin(\cos 0)}{\frac{1}{\cos 0}}\\ &= \frac{\sin 1}{\frac{1}{1}}\\ &= \sin 1\end{aligned}$$

Section 8.3 Page 509 Question 10

a)
$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-1}{x^2}&=\lim_{x\to 0}\frac{\cos x-1}{x^2}\cdot\frac{\cos x+1}{\cos x+1}\\&=\lim_{x\to 0}\frac{\cos^2 x-1}{x^2(\cos x+1)}\\&=\lim_{x\to 0}\frac{-\sin^2 x}{x^2(\cos x+1)}\\&=-1\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{\cos x+1}\right)\\&=-1\cdot 1\cdot 1\cdot\frac{1}{1+1}\\&=-\frac{1}{2}\end{aligned}$$

b)
$$\begin{aligned}\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right)&=\lim_{\frac{1}{x}\to 0}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\\&=1\end{aligned}$$

c)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin(\tan x)}{\sin x}&=\lim_{x\to 0}\frac{\frac{\sin(\tan x)}{\tan x}}{\frac{\sin x}{\tan x}}\\&=\frac{\lim\limits_{x\to 0}\frac{\sin(\tan x)}{\tan x}}{\lim\limits_{x\to 0}\cos x}\\&=\frac{\lim\limits_{\tan x\to 0}\frac{\sin(\tan x)}{\tan x}}{\lim\limits_{x\to 0}\cos x}\\&=\frac{1}{1}\\&=1\end{aligned}$$

Section 8.3 Page 509 Question 11

a)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 5x}{\sin 2x}&=\lim_{x\to 0}\frac{\frac{\sin 5x}{x}}{\frac{\sin 2x}{x}}\\&=\frac{\lim\limits_{x\to 0}\frac{\sin 5x}{x}\cdot\frac{5}{5}}{\lim\limits_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{2}{2}}\\&=\frac{5\cdot\lim\limits_{5x\to 0}\frac{\sin 5x}{5x}}{2\cdot\lim\limits_{2x\to 0}\frac{\sin 2x}{2x}}\\&=\frac{5\cdot 1}{2\cdot 1}\\&=\frac{5}{2}\end{aligned}$$

b)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 5x}{\sin^2 2x}&=\lim_{x\to 0}\frac{\frac{\sin^2 5x}{x^2}}{\frac{\sin^2 2x}{x^2}}\\&=\frac{\lim\limits_{x\to 0}\frac{\sin^2 5x}{x^2}\cdot\frac{25}{25}}{\lim\limits_{x\to 0}\frac{\sin^2 2x}{x^2}\cdot\frac{4}{4}}\\&=\frac{25\cdot\lim\limits_{5x\to 0}\left(\frac{\sin 5x}{5x}\cdot\frac{\sin 5x}{5x}\right)}{4\cdot\lim\limits_{2x\to 0}\left(\frac{\sin 2x}{2x}\cdot\frac{\sin 2x}{2x}\right)}\\&=\frac{25\cdot 1\cdot 1}{4\cdot 1\cdot 1}\\&=\frac{25}{4}\end{aligned}$$

c)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 3x}{\sin 4x} &= \lim_{x\to 0}\frac{\dfrac{\sin^2 3x}{x}}{\dfrac{\sin 4x}{x}}\\
&= \frac{\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{x}\cdot\dfrac{3}{3}}{\lim\limits_{x\to 0}\dfrac{\sin 4x}{x}\cdot\dfrac{4}{4}}\\
&= \frac{3\cdot\lim\limits_{3x\to 0}\left(\sin 3x\cdot\dfrac{\sin 3x}{3x}\right)}{4\cdot\lim\limits_{4x\to 0}\dfrac{\sin 4x}{4x}}\\
&= \frac{3\cdot 0\cdot 1}{4\cdot 1}\\
&= 0\end{aligned}$$

d)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{\tan 4x} &= \lim_{x\to 0}\frac{\dfrac{\sin 3x}{x}}{\dfrac{\tan 4x}{x}}\\
&= \frac{\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}\cdot\dfrac{3}{3}}{\lim\limits_{x\to 0}\dfrac{\tan 4x}{x}\cdot\dfrac{4}{4}}\\
&= \frac{3\cdot\lim\limits_{3x\to 0}\dfrac{\sin 3x}{3x}}{4\cdot\lim\limits_{4x\to 0}\dfrac{\tan 4x}{4x}}\\
&= \frac{3\cdot 1}{4\cdot 1}\\
&= \frac{3}{4}\end{aligned}$$

e)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 3x}{\tan^2 4x} &= \lim_{x\to 0}\frac{\dfrac{\sin^2 3x}{x^2}}{\dfrac{\tan^2 4x}{x^2}}\\
&= \frac{\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{x^2}\cdot\dfrac{9}{9}}{\lim\limits_{x\to 0}\dfrac{\tan^2 4x}{x^2}\cdot\dfrac{16}{16}}\\
&= \frac{9\cdot\lim\limits_{3x\to 0}\left(\dfrac{\sin 3x}{3x}\cdot\dfrac{\sin 3x}{3x}\right)}{16\cdot\lim\limits_{4x\to 0}\left(\dfrac{\tan 4x}{4x}\cdot\dfrac{\tan 4x}{4x}\right)}\\
&= \frac{9\cdot 1\cdot 1}{16\cdot 1\cdot 1}\\
&= \frac{9}{16}\end{aligned}$$

f)
$$\begin{aligned}\lim_{x\to 0}\frac{\tan 3x}{\tan 4x} &= \lim_{x\to 0}\frac{\dfrac{\tan 3x}{x}}{\dfrac{\tan 4x}{x}}\\
&= \frac{\lim\limits_{x\to 0}\dfrac{\tan 3x}{x}\cdot\dfrac{3}{3}}{\lim\limits_{x\to 0}\dfrac{\tan 4x}{x}\cdot\dfrac{4}{4}}\\
&= \frac{3\cdot\lim\limits_{3x\to 0}\dfrac{\tan 3x}{3x}}{4\cdot\lim\limits_{4x\to 0}\dfrac{\tan 4x}{4x}}\\
&= \frac{3\cdot 1}{4\cdot 1}\\
&= \frac{3}{4}\end{aligned}$$

8.4 Derivatives of the Sine, Cosine, and Tangent Functions

Practise

Section 8.4 Page 518 Question 1

a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin(4x+7)}{d(4x+7)} \cdot \frac{d(4x+7)}{dx} \\ &= \cos(4x+7) \cdot 4 \\ &= 4\cos(4x+7)\end{aligned}$$

b)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin(x^2+3)}{d(x^2+3)} \cdot \frac{d(x^2+3)}{dx} \\ &= \cos(x^2+3) \cdot 2x \\ &= 2x\cos(x^2+3)\end{aligned}$$

c)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\cos(5x+3)}{d(5x+3)} \cdot \frac{d(5x+3)}{dx} \\ &= -\sin(5x+3) \cdot 5 \\ &= -5\sin(5x+3)\end{aligned}$$

d)

$$\begin{aligned}\frac{dy}{dx} &= 3 \cdot \frac{d\tan(2x+3)}{d(2x+3)} \cdot \frac{d(2x+3)}{dx} \\ &= 3\sec^2(2x+3) \cdot 2 \\ &= 6\sec^2(2x+3)\end{aligned}$$

e)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\cos(3x^2-1)}{d(3x^2-1)} \cdot \frac{d(3x^2-1)}{dx} \\ &= -\sin(3x^2-1) \cdot 6x \\ &= -6x\sin(3x^2-1)\end{aligned}$$

f)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\tan(5x^2+1)}{d(5x^2+1)} \cdot \frac{d(5x^2+1)}{dx} \\ &= \sec^2(5x^2+1) \cdot 10x \\ &= 10x\sec^2(5x^2+1)\end{aligned}$$

g)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin(\cos^2 x)}{d\cos^2 x} \cdot \frac{d\cos^2 x}{d\cos x} \cdot \frac{d\cos x}{dx} \\ &= \cos(\cos^2 x) \cdot 2\cos x \cdot (-\sin x) \\ &= -2\cos x\sin x\cos(\cos^2 x)\end{aligned}$$

h)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin(\cos x)}{d\cos x} \cdot \frac{d\cos x}{dx} \\ &= \cos(\cos x) \cdot (-\sin x) \\ &= -\sin x\cos(\cos x)\end{aligned}$$

Section 8.4 Page 518 Question 3

a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin^3(\cos^2 x)}{d\sin(\cos^2 x)} \cdot \frac{d\sin(\cos^2 x)}{d\cos^2 x} \cdot \frac{d\cos^2 x}{d\cos x} \cdot \frac{d\cos x}{dx} \\ &= 3\sin^2(\cos^2 x) \cdot \cos(\cos^2 x) \cdot 2\cos x \cdot (-\sin x) \\ &= -6\cos x\sin x\sin^2(\cos^2 x)\cos(\cos^2 x)\end{aligned}$$

b)

$$\begin{aligned}f'(x) &= \frac{d\cos^4(\sin x^3)}{d\cos(\sin x^3)} \cdot \frac{d\cos(\sin x^3)}{d\sin x^3} \cdot \frac{d\sin x^3}{dx^3} \cdot \frac{dx^3}{dx} \\ &= 4\cos^3(\sin x^3) \cdot -\sin(\sin x^3) \cdot \cos x^3 \cdot 3x^2 \\ &= -12x^2\cos x^3\cos^3(\sin x^3)\sin(\sin x^3)\end{aligned}$$

c)

$$\begin{aligned}\frac{dy}{dx} &= \cos(\tan x) \cdot \frac{dx}{dx} + x \cdot \frac{d\cos(\tan x)}{d\tan x} \cdot \frac{d\tan x}{dx} \\ &= \cos(\tan x) \cdot 1 + x \cdot -\sin(\tan x) \cdot \sec^2 x \\ &= \cos(\tan x) - x\sin(\tan x)\sec^2 x\end{aligned}$$

d)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin(e^x)}{de^x} \cdot \frac{de^x}{dx} \\ &= \cos(e^x) \cdot e^x \\ &= e^x\cos(e^x)\end{aligned}$$

e)

$$\begin{aligned}h'(x) &= \frac{d\cos(\ln x)}{d\ln x} \cdot \frac{d\ln x}{dx} \\ &= -\sin(\ln x) \cdot \frac{1}{x} \\ &= -\frac{\sin(\ln x)}{x}\end{aligned}$$

f)

$$\begin{aligned}g'(x) &= \frac{d\ln(\sin(e^x))}{d\sin(e^x)} \cdot \frac{d\sin(e^x)}{de^x} \cdot \frac{de^x}{dx} \\ &= \frac{1}{\sin(e^x)} \cdot \cos(e^x) \cdot e^x \\ &= e^x\cot(e^x)\end{aligned}$$

g)

$$\begin{aligned} y &= \sin^2 x + \cos^2 x \\ &= 1 \\ \frac{dy}{dx} &= 0 \end{aligned}$$

h)

$$\begin{aligned} f'(x) &= \frac{(1+\cos x)\cdot \dfrac{d \sin x}{dx} - \sin x \cdot \dfrac{d(1+\cos x)}{dx}}{(1+\cos x)^2} \\ &= \frac{(1+\cos x)\cdot \cos x - \sin x \cdot (-\sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x(1+\cos x) + \sin^2 x}{(1+\cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \\ &= \frac{\cos x + 1}{(1+\cos x)^2} \\ &= \frac{1}{1+\cos x} \end{aligned}$$

Apply, Solve, Communicate

Section 8.4 Page 518 Question 4

a)

$$\begin{aligned} \frac{dy}{dx} &= \sin 2x \cdot 1 + x \cdot \cos 2x \cdot 2 \\ &= \sin 2x + 2x \cos 2x \\ \frac{dy}{dx}_{|x=\frac{\pi}{4}} &= \sin 2\left(\frac{\pi}{4}\right) + 2 \cdot \frac{\pi}{4} \cdot \cos 2\left(\frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{2}\right) + 2 \cdot \frac{\pi}{4} \cdot \cos\left(\frac{\pi}{2}\right) \\ &= 1 + 2 \cdot \frac{\pi}{4} \cdot 0 \\ &= 1 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \frac{\pi}{4} &= 1\left(x - \frac{\pi}{4}\right) \\ y &= x \end{aligned}$$

The equation of the tangent is $y = x$.

c)

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{(\tan^2 x)^2} \cdot 2\tan x \cdot (1 + \tan^2 x) \\ &= -\frac{2(1+\tan^2 x)}{\tan^3 x} \\ \frac{dy}{dx}_{|x=\frac{\pi}{4}} &= -\frac{2\left(1 + \tan^2\left(\frac{\pi}{4}\right)\right)}{\tan^3\left(\frac{\pi}{4}\right)} \\ &= -\frac{2(1+1^2)}{1^3} \\ &= -4 \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -4\left(x - \frac{\pi}{4}\right) \\ y &= -4x + \pi + 1 \end{aligned}$$

The equation of the tangent is $y = -4x + \pi + 1$.

b)

$$\begin{aligned} \frac{dy}{dx} &= 2\cos x \cdot (-\sin x) \\ &= -2 \sin x \cos x \\ &= -\sin 2x \\ \frac{dy}{dx}_{|x=\frac{\pi}{3}} &= -\sin 2\left(\frac{\pi}{3}\right) \\ &= -\sin\left(\frac{2\pi}{3}\right) \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \frac{1}{4} &= -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) \\ y &= -\frac{\sqrt{3}}{2}x + \frac{\pi\sqrt{3}}{6} + \frac{1}{4} \end{aligned}$$

The equation of the tangent is $y = -\frac{\sqrt{3}}{2}x + \frac{\pi\sqrt{3}}{6} + \frac{1}{4}$.

Section 8.4 Page 518 Question 5

a)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ -\sin x - \cos x &= 0\\ -\sin x &= \cos x\\ \tan x &= -1\\ x &= -\frac{\pi}{4} \text{ or } \frac{3\pi}{4}\end{aligned}$$

Since $\frac{d^2y}{dx^2} < 0$ at $x = -\frac{\pi}{4}$, a local maximum exists at $\left(-\frac{\pi}{4}, \sqrt{2}\right)$. Since $\frac{d^2y}{dx^2} > 0$ at $x = \frac{3\pi}{4}$, a local minimum exists at $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$.

b)

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 2\sin x\cos x - \cos x &= 0\\ \cos x(2\sin x - 1) &= 0\\ \cos x &= 0\\ x &= \pm\frac{\pi}{2}\\ \sin x &= \frac{1}{2}\\ x &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6}\end{aligned}$$

Since $\frac{d^2y}{dx^2} > 0$ at $x = \frac{\pi}{6}$, a local minimum exists at $\left(\frac{\pi}{6}, -\frac{1}{4}\right)$. Since $\frac{d^2y}{dx^2} < 0$ at $x = \frac{\pi}{2}$, a local maximum exists at $\left(\frac{\pi}{2}, 0\right)$. Since $\frac{d^2y}{dx^2} > 0$ at $x = \frac{5\pi}{6}$, a local minimum exists at $\left(\frac{5\pi}{6}, -\frac{1}{4}\right)$. Since $\frac{d^2y}{dx^2} < 0$ at $x = \frac{-\pi}{2}$, a local maximum exists at $\left(-\frac{\pi}{2}, 2\right)$.

c)

$$\begin{aligned}y &= 2\cos x - \cos 2x\\ &= 2\cos x - (2\cos^2 x - 1)\\ &= -2\cos^2 x + 2\cos x + 1\end{aligned}$$

Determine the critical number(s).

$$\begin{aligned}\frac{dy}{dx} &= 0\\ -4\cos x(-\sin x) - 2\sin x &= 0\\ 2\sin x(2\cos x - 1) &= 0\\ \sin x &= 0\\ x &= \pi\\ 2\cos x - 1 &= 0\\ &\text{no solution in } \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\end{aligned}$$

Since $\frac{d^2y}{dx^2} > 0$ at $x = \pi$, a local minimum exists at $(\pi, -3)$.

d)

$$\begin{aligned}y &= \frac{1}{\cos x} + \tan x\\ &= \frac{1 + \sin x}{\cos x}\end{aligned}$$

Determine the critical number(s).

$$\begin{aligned}\frac{dy}{dx} &= 0\\ \frac{\cos x \cdot \cos x - (-\sin x)(1 + \sin x)}{\cos^2 x} &= 0\\ \frac{\cos^2 x + \sin^2 x + \sin x}{\cos^2 x} &= 0\\ \frac{1 + \sin x}{\cos^2 x} &= 0\\ 1 + \sin x &= 0\\ \sin x &= -1\\ x &= -\frac{\pi}{2}\end{aligned}$$

Since y is undefined at $x = -\frac{\pi}{2}$, there are no extrema.

Section 8.4 Page 518 Question 6

a)

$$\begin{aligned} y &= 2\cos x + \sin 2x \\ \frac{dy}{dx} &= -2\sin x + 2\cos 2x \\ \frac{d^2y}{dx^2} &= 0 \\ -2\cos x - 4\sin 2x &= 0 \\ -2\cos x - 8\sin x\cos x &= 0 \\ \cos x(1 + 4\sin x) &= 0 \\ \cos x &= 0 \\ x &= \frac{\pi}{2} \end{aligned}$$

There is no solution to $1 + \sin x = 0$ on $[0, \pi]$. Since $\frac{d^2y}{dx^2}$ changes sign at $x = \frac{\pi}{2}$, a point of inflection exists at $\left(\frac{\pi}{2}, 0\right)$.

b)

$$\begin{aligned} y &= 2\sin^2 x - 1 \\ &= -\cos 2x \\ \frac{dy}{dx} &= -(-\sin 2x)(2) \\ &= 2\sin 2x \\ \frac{d^2y}{dx^2} &= 0 \\ 2(\cos 2x)(2) &= 0 \\ \cos 2x &= 0 \\ 2x &= -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \text{ or } \frac{3\pi}{2} \\ x &= -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \text{ or } \frac{3\pi}{4} \end{aligned}$$

Since $\frac{d^2y}{dx^2}$ changes sign at each of its zeros, points of inflection exist at $\left(\pm\frac{3\pi}{4}, 0\right)$ and $\left(\pm\frac{\pi}{4}, 0\right)$.

c)

$$\begin{aligned} y &= \sin x - \tan x \\ \frac{dy}{dx} &= \cos x - \sec^2 x \\ \frac{d^2y}{dx^2} &= 0 \\ -\sin x - 2\sec x \cdot \sec x \tan x &= 0 \\ \sin x + 2\sec^2 x \tan x &= 0 \\ \sin x + \frac{2\sin x}{\cos^3 x} &= 0 \\ \sin x\left(1 + \frac{2}{\cos^3 x}\right) &= 0 \\ \sin x &= 0 \\ x &= 0 \text{ or } \pi \\ \left(1 + \frac{2}{\cos^3 x}\right) &= 0 \\ \cos^3 x &\neq -2 \end{aligned}$$

Since $\frac{d^2y}{dx^2}$ changes sign at $x = 0$ and $x = \pi$, points of inflection exist at $(0, 0)$ and $(\pi, 0)$.

Section 8.4 Page 518 Question 7

a)

$$\begin{aligned} \frac{dy}{dx} &= \cos x & \frac{d^5y}{dx^5} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x & \frac{d^6y}{dx^6} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x & \frac{d^7y}{dx^7} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x & \frac{d^8y}{dx^8} &= \sin x \end{aligned}$$

Every fourth derivative is the same.

Note. Parts b), c), and d) makes use of the **mod** operator. The expression $m \bmod n \equiv p$, states that p is the remainder when m is divided by n, where m, n, and p are whole numbers.

b) Since $87 \bmod 4 \equiv 3$, $\frac{d^{87}y}{dx^{87}} = \frac{d^3y}{dx^3}$ or $-\cos x$.

Since $138 \bmod 4 \equiv 2$, $\frac{d^{138}y}{dx^{138}} = \frac{d^2y}{dx^2}$ or $-\sin x$.

c)

$$\frac{d^n \sin x}{dx^n} = \begin{cases} \sin x, & n \bmod 4 \equiv 0 \\ \cos x, & n \bmod 4 \equiv 1 \\ -\sin x, & n \bmod 4 \equiv 2 \\ -\cos x, & n \bmod 4 \equiv 3 \end{cases}$$

d)

$$\frac{d^n \cos x}{dx^n} = \begin{cases} \cos x, & n \bmod 4 \equiv 0 \\ -\sin x, & n \bmod 4 \equiv 1 \\ -\cos x, & n \bmod 4 \equiv 2 \\ \sin x, & n \bmod 4 \equiv 3 \end{cases}$$

Section 8.4 Page 518 Question 8

a)

$$\begin{aligned}y &= A\sin kx + B\cos kx\\ y' &= A\cos kx\cdot k + B\cdot(-\sin kx)\cdot k\\ &= k(A\cos kx - B\sin kx)\\ y'' &= k(A\cdot(-\sin kx)\cdot k - B\cdot\cos kx\cdot k)\\ &= -k^2(A\sin kx + B\cos kx)\\ &= -k^2y\end{aligned}$$

$$y'' + k^2y = 0$$

b)

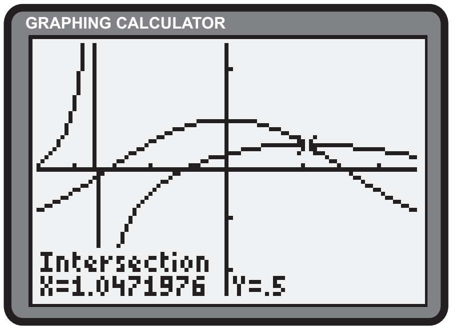
$$\begin{aligned}y &= C\sin(kx + D)\\ y' &= C\cos(kx + D)\cdot k\\ &= kC\cos(kx + D)\\ y'' &= kC\cdot -\sin(kx + D)\cdot k\\ &= -k^2C\sin(kx + D)\\ &= -k^2y\end{aligned}$$

$$y'' + k^2y = 0$$

Section 8.4 Page 518 Question 9

Let $Q\left(-\sqrt{3}, -\frac{\pi}{6}\right)$ be the remote point. Let $P(x, \sin x)$ be point of tangency on $y = \sin x$. Determine x.

$$\begin{aligned}\frac{dy}{dx} &= \text{Slope of PQ}\\ \cos x &= \frac{\sin x - \left(-\frac{\pi}{6}\right)}{x - (-\sqrt{3})}\\ \cos x &= \frac{\sin x + \frac{\pi}{6}}{x + \sqrt{3}} \qquad (1)\end{aligned}$$



```
Intersection
X=1.0471976  Y=.5
```

The **Intersect operation** of the graphing calculator suggests a solution to (1) of $x \doteq 1.047$ or $\frac{\pi}{3}$. Determine the equation of the tangent line.

$$\begin{aligned}y - \sin\frac{\pi}{3} &= \cos\frac{\pi}{3}\left(x - \frac{\pi}{3}\right)\\ y - \frac{\sqrt{3}}{2} &= \frac{1}{2}\left(x - \frac{\pi}{3}\right)\\ y &= \frac{1}{2}x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}\end{aligned}$$

The solution is $y = \frac{1}{2}x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$.

Section 8.4 Page 518 Question 10

a)

$$\begin{aligned}\theta(A) &= A \times \frac{\pi}{180}\\ \frac{d\theta}{dA} &= \frac{\pi}{180}\end{aligned}$$

c) The exact value of 0.17 . . . is $\frac{\pi}{180}$.

b)

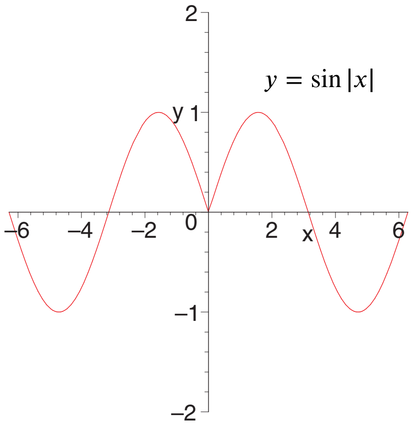
$$\begin{aligned}y &= \sin\theta\\ \frac{dy}{dA} &= \frac{dy}{d\theta}\cdot\frac{d\theta}{dA}\\ &= \cos\theta\cdot\frac{\pi}{180}\\ &= \frac{\pi}{180}\cos\theta\end{aligned}$$

d)

$$\begin{aligned}y &= \cos\theta\\ \frac{dy}{dA} &= \frac{dy}{d\theta}\cdot\frac{d\theta}{dA}\\ &= -\sin\theta\cdot\frac{\pi}{180}\\ &= -\frac{\pi}{180}\sin\theta\end{aligned}$$

Section 8.4 Page 519 Question 11

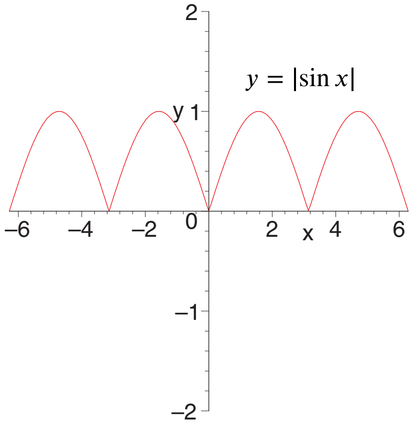
a)



$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d\sin|x|}{dx}\\
&= \frac{d\sin\sqrt{x^2}}{dx}\\
&= \frac{d\sin\sqrt{x^2}}{d\sqrt{x^2}}\cdot\frac{d\sqrt{x^2}}{d\,x^2}\cdot\frac{d\,x^2}{d\,x}\\
&= \cos\sqrt{x^2}\cdot\frac{1}{2\sqrt{x^2}}\cdot 2x\\
&= \frac{x\cos|x|}{|x|}
\end{aligned}
$$

$\sin|x|$ is not differentiable for $|x| = 0$ or $x = 0$.

b)



$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d\,|\sin x|}{dx}\\
&= \frac{d\sqrt{(\sin x)^2}}{dx}\\
&= \frac{d\sqrt{(\sin x)^2}}{d(\sin x)^2}\cdot\frac{d(\sin x)^2}{d(\sin x)}\cdot\frac{d\sin x}{dx}\\
&= \frac{1}{2\sqrt{(\sin x)^2}}\cdot 2\sin x\cdot\cos x\\
&= \frac{\sin x\cos x}{|\sin x|}
\end{aligned}
$$

$|\sin x|$ is not differentiable for $\sin x = 0$ or $x = k\pi,\ k \in I$.

Section 8.4 Page 519 Question 12

$$
\begin{aligned}
f'(x) &= 0\\
-\sin\left(x+\frac{1}{x}\right)\cdot\left(1-\frac{1}{x^2}\right) &= 0\\
1-\frac{1}{x^2} &= 0\\
x^2 &= 1\\
x &= \pm 1 \qquad (1)\\
\sin\left(x+\frac{1}{x}\right) &= 0\\
x+\frac{1}{x} &= k\pi,\ k \in I\\
x^2+1 &= k\pi x\\
x^2-k\pi x+1 &= 0\\
x &= \frac{k\pi \pm \sqrt{(k\pi)^2-4(1)(1)}}{2}\\
&= \frac{k\pi \pm \sqrt{k^2\pi^2-4}}{2} \qquad (2)
\end{aligned}
$$

The zeros of $f'(x)$ are $x = \pm 1$ and $x = \dfrac{k\pi \pm \sqrt{k^2\pi^2-4}}{2}$.

Section 8.4 Page 519 Question 13

The expression

$$\lim_{h\to 0}\frac{\tan\left(\frac{\pi}{4}+h\right)-1}{h} \qquad (2)$$

can be rewritten as

$$\lim_{h\to 0}\frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\frac{\pi}{4}}{h} \qquad (2)$$

Expression (2) defines the slope of the tangent to $f(x) = \tan x$ at $x = \dfrac{\pi}{4}$.

$$
\begin{aligned}
f(x) &= \tan x\\
f'(x) &= \sec^2 x\\
\lim_{h\to 0}\frac{\tan\left(\frac{\pi}{4}+h\right)-1}{h} &= f'\left(\frac{\pi}{4}\right)\\
&= \sec^2\left(\frac{\pi}{4}\right)\\
&= \left(\sqrt{2}\right)^2\\
&= 2
\end{aligned}
$$

Section 8.4 Page 519 Question 14

a) Since $\dfrac{d\tan x}{dx} = 1 + \tan^2 x$, to achieve a derivative of $\tan^2 x$, use $y = \tan x - x$.

b) Since $\tan x = \dfrac{1}{\cos x} \cdot \sin x$, use $y = -\ln(\cos x)$ or $y = \ln(\sec x)$.

c) $y = \ln(\sec x + \tan x)$.

8.5 Modelling with Trigonometric Functions

Apply, Solve, Communicate

Section 8.5 Page 529 Question 1

a)

$$\begin{aligned} F &= G\frac{Mm}{x^2} \\ &= 6.67 \times 10^{-11} \times \frac{2.0 \times 10^{30} \times 5.98 \times 10^{24}}{(1.5 \times 10^{11})^2} \\ &= 3.5455 \times 10^{22} \end{aligned}$$

The sun exerts a force of 3.5455×10^{22} N on Earth.

b)

$$\begin{aligned} F &= G\frac{Mm}{x^2} \\ &= 6.67 \times 10^{-11} \times \frac{7.4 \times 10^{22} \times 5.98 \times 10^{24}}{(3.84 \times 10^{8})^2} \\ &= 2.0017 \times 10^{20} \end{aligned}$$

The moon exerts a force of 2.0017×10^{20} N on Earth.

c)

$$\begin{aligned} \frac{F_{\text{sun}}}{F_{\text{moon}}} &= \frac{3.5455 \times 10^{22}}{2.0017 \times 10^{20}} \\ &= 1.7712 \times 10^{2} \\ &= 177.12 \end{aligned}$$

d) Both ratios are close to 2.

Section 8.5 Page 530 Question 2

The model for tidal force is $F'(x) = -2G\dfrac{Mm}{x^3}$.

a)

$$\begin{aligned} F'(7.0 \times 10^{8}) &= \frac{-2(6.67 \times 10^{-11})(2.0 \times 10^{30})(1.0 \times 10^{4})}{(7.0 \times 10^{8})^3} \\ &= -0.0777\,84 \times 10^{-1} \\ &= -7.7784 \times 10^{-3} \end{aligned}$$

The sun would exert a force of -7.7784×10^{-3} N on the spacecraft.

b)

$$\begin{aligned} F'(1.0 \times 10^{6}) &= \frac{-2(6.67 \times 10^{-11})(2.0 \times 10^{30})(1.0 \times 10^{4})}{(1.0 \times 10^{6})^3} \\ &= -26.68 \times 10^{5} \\ &= -2.668 \times 10^{6} \end{aligned}$$

The white dwarf star would exert a force of -2.668×10^{6} N on the spacecraft.

c)

$$\begin{aligned} F'(1.0 \times 10^{4}) &= \frac{-2(6.67 \times 10^{-11})(2.0 \times 10^{30})(1.0 \times 10^{4})}{(1.0 \times 10^{4})^3} \\ &= -26.68 \times 10^{11} \\ &= -2.668 \times 10^{12} \end{aligned}$$

The neutron star would exert a force of -2.668×10^{12} N on the spacecraft.

Section 8.5 Page 530 Question 3

a) The length of the remaining side can be given by $\sqrt{(10\sqrt{2})^2 - x^2}$ or $\sqrt{200 - x^2}$ centimetres. Let A be the area of the triangle.

$$A(x) = \frac{1}{2}x\sqrt{200 - x^2}$$

Determine the critical numbers of A.

$$\begin{aligned} A'(x) &= 0 \\ \frac{1}{2}\left(\sqrt{200 - x^2} \cdot 1 + x \cdot \frac{1}{2\sqrt{200 - x^2}} \cdot (-2x)\right) &= 0 \\ \sqrt{200 - x^2} - \frac{x^2}{\sqrt{200 - x^2}} &= 0 \\ \sqrt{200 - x^2} &= \frac{x^2}{\sqrt{200 - x^2}} \\ 200 - x^2 &= x^2 \\ x^2 &= 100 \\ x &= 10,\ x > 0 \end{aligned}$$

The maximum area of the triangle is $A(10)$ or 50 cm^2.

b) Comparisons may vary.

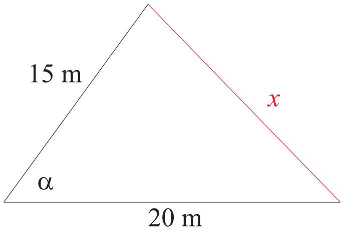
Section 8.5 Page 530 Question 4

Let α be the contained angle, in radians. Let x represent the length of the third side, in metres. Let t be the time, in seconds.

$$\frac{d\alpha}{dt} = \frac{\pi}{90} \qquad (1)$$

Use the cosine law to determine a value for x at the required instant.

$$\begin{aligned} x^2 &= 15^2 + 20^2 - 2(15)(20)\cos\alpha \\ &= 625 - 600\cos\alpha \qquad (2) \end{aligned}$$



Substitute $\alpha = \frac{\pi}{3}$ into (2) to determine x.

$$\begin{aligned} x^2 &= 625 - 600\cos\frac{\pi}{3} \\ &= 625 - 600 \cdot \frac{1}{2} \\ &= 625 - 300 \\ &= 325 \\ x &= 5\sqrt{13},\ x > 0 \qquad (3) \end{aligned}$$

Differentiate (2) with respect to t.

$$\begin{aligned} \frac{dx^2}{dx} \cdot \frac{dx}{dt} &= \frac{d625 - 600\cos\alpha}{d\alpha} \cdot \frac{d\alpha}{dt} \\ 2x \cdot \frac{dx}{dt} &= 600\sin\alpha \cdot \frac{d\alpha}{dt} \\ \frac{dx}{dt} &= \frac{300}{x}\sin\alpha \cdot \frac{d\alpha}{dt} \qquad (4) \end{aligned}$$

Substitute (1), (3), and $\alpha = \frac{\pi}{3}$ into (4).

$$\begin{aligned} \frac{dx}{dt} &= \frac{300}{5\sqrt{13}}\sin\frac{\pi}{3} \cdot \frac{\pi}{90} \\ &\doteq 0.503 \end{aligned}$$

The third side is increasing at a rate of 0.503 m/s.

Section 8.5 Page 530 Question 5

a)

$$\begin{aligned} x &= 0.05\cos(880\pi t)\\ v &= x'(t)\\ &= 0.05\cdot -\sin(880\pi t)\cdot 880\pi\\ &= -44\pi\sin(880\pi t)\\ a &= v'(t)\\ &= -44\pi\cdot\cos(880\pi t)\cdot 880\pi\\ &= -38\ 720\pi^2\cos(880\pi t)\end{aligned}$$

The velocity is $-44\pi\sin(880\pi t)$ cm/s. The acceleration is $-38\ 720\pi^2\cos(880\pi t)$ cm/s^2.

b) Since the maximum value of $-\sin(880\pi t)$ is 1, the maximum value of $-44\sin(880\pi t)$ is 44π. The maximum velocity is 44π cm/s.

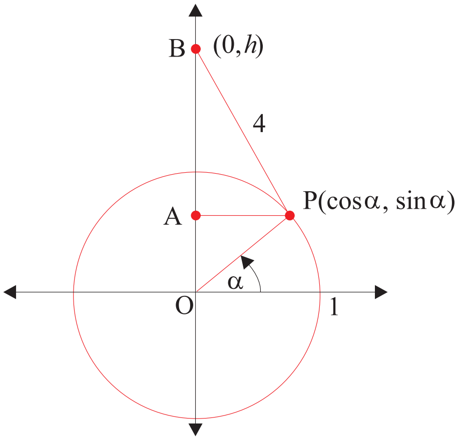
c)

$$\begin{aligned}\frac{d^2x}{dt}+(880\pi)^2x &= -38\ 720\pi^2\cos(880\pi t)+(880\pi)^2(0.05\cos(880\pi t))\\ &= -38\ 720\pi^2\cos(880\pi t)+38\ 720\pi^2\cos(880\pi t)\\ &= 0\end{aligned}$$

Section 8.5 Page 530 Question 6

a) Let P be a point on the unit circle centred at the origin. Given the central angle, θ, the coordinates of P can be expressed as $(\cos\alpha, \sin\alpha)$.

$$\begin{aligned} h &= \text{OA}+\text{AB}\\ &= \sin\alpha+\sqrt{4^2-(\cos\alpha)^2}\\ &= \sin\alpha+\sqrt{16-\cos^2\alpha}\end{aligned}$$



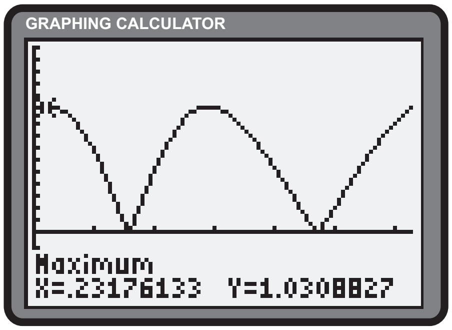
b)

$$\begin{aligned}\frac{dh}{dt} &= \cos\alpha\cdot\frac{d\alpha}{dt}+\frac{1}{2\sqrt{16-\cos^2\alpha}}\cdot -2\cos\alpha(-\sin\alpha)\cdot\frac{d\alpha}{dt}\\ v &= \left[\cos\alpha+\frac{\cos\alpha\sin\alpha}{\sqrt{16-\cos^2\alpha}}\right]\frac{d\alpha}{dt}\end{aligned}$$

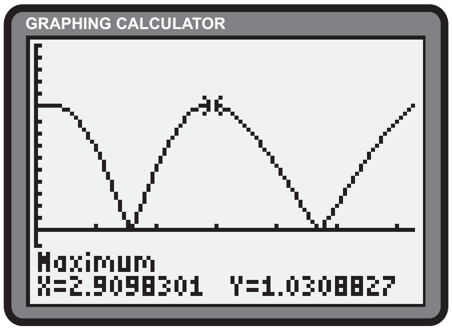
The velocity of the piston is $\left[\cos\alpha+\dfrac{\cos\alpha\sin\alpha}{\sqrt{16-\cos^2\alpha}}\right]\dfrac{d\alpha}{dt}$ m/s.

c) Observation of the diagram reveals that the height will not be changing when α is any odd integer multiple of $\frac{\pi}{2}$ rad. Determining the roots of v confirms this observation.

d) For maximum speed, the maxima of $|v|$ must be sought.



Maximum
X=.23176133 Y=1.0308827



Maximum
X=2.9098301 Y=1.0308827

The maximum speed of the piston is achieved at $\alpha = 0.2318 + 2k\pi$ and $\alpha = 2.9098 + 2k\pi$, where k is any integer.

Section 8.5 Page 530 Question 7

a) The period of $V = 120\cos 120\pi t$ is $\frac{2\pi}{120\pi}$ or $\frac{1}{60}$. If each cycle takes $\frac{1}{60}$ s, there are 60 cycles per second.

b) The current changes direction 2 × 60 or 120 times per second.

c)

$$\begin{aligned} f(t) &= -\frac{dV}{dt} \\ &= -120 \cdot -\sin 120\pi t \cdot 120\pi \\ &= 14\,400\pi \sin 120\pi t \end{aligned}$$

The electric field $f(t) = 14\,400\pi \sin 120\pi t$.

e)

$$\begin{aligned} f(t) &= 0 \\ 14\,400\pi \sin 120\pi t &= 0 \\ \sin 120\pi t &= 0 \\ t &= \frac{k\pi}{120},\ k \text{ is any integer} \end{aligned}$$

The electric field is zero at $t = \frac{k\pi}{120}$, where k is any integer. The value of the voltage at these times is ±120.

d) The maximum occurs when $\sin 120\pi t = 1$. Thus,

$$\begin{aligned} 120\pi t &= \frac{\pi}{2} + 2k\pi \\ t &= \frac{1}{240} + \frac{k}{60} \\ &= \frac{4k+1}{240} \end{aligned}$$

Maxima occur at $t = \frac{4k+1}{240}$, where k is any integer. The value of the voltage at these times is 0.

Section 8.5 Page 530 Question 8

Let x be one of the shorter sides of the triangle, in centimetres. The length of the remaining side can be given by $\sqrt{(10\sqrt{2})^2 - x^2}$ or $\sqrt{200 - x^2}$ centimetres. Let P be the perimeter of the triangle.

$$P(x) = x + \sqrt{200 - x^2} + 10\sqrt{2}$$

Determine the critical numbers of P.

$$\begin{aligned} P'(x) &= 0 \\ 1 + \frac{1}{2\sqrt{200 - x^2}} \cdot -2x &= 0 \\ 1 - \frac{x}{\sqrt{200 - x^2}} &= 0 \\ x &= \sqrt{200 - x^2} \\ x^2 &= 200 - x^2 \\ 2x^2 &= 200 \\ x^2 &= 100 \\ x &= 10,\ x > 0 \end{aligned}$$

The maximum perimeter is $P(10) = 10 + \sqrt{200 - 10^2} + 10\sqrt{2}$ or $20 + 10\sqrt{2}$ cm.

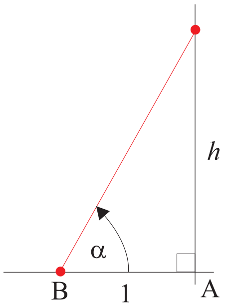
Section 8.5 Page 531 Question 9

a) Let $h(t)$ be the height of the weather balloon, in kilometres, after t hours.

$$\begin{aligned} h &= 10t && (1) \\ \tan\alpha &= \frac{h}{1} \\ \tan\alpha &= h && (2) \end{aligned}$$

Substitute (1) into (2).

$$\begin{aligned} \tan\alpha &= 10t && (3) \\ \alpha &= \tan^{-1} 10t \end{aligned}$$



The angle of inclination can be expressed as $\alpha = \tan^{-1} 10t$ rad.

b) From (3),

$$\begin{aligned} \frac{d\tan\alpha}{dt} &= \frac{d10t}{dt} \\ \frac{d\tan\alpha}{d\alpha}\cdot\frac{d\alpha}{dt} &= \frac{d10t}{dt} \\ \sec^2\alpha\cdot\frac{d\alpha}{dt} &= 10 \\ \frac{d\alpha}{dt} &= 10\cos^2\alpha && (4) \end{aligned}$$

Substitute $\alpha = \frac{\pi}{6}$ into (4).

$$\begin{aligned} \frac{d\alpha}{dt} &= 10\cos^2\left(\frac{\pi}{6}\right) \\ &= 10\left(\frac{\sqrt{3}}{2}\right)^2 \\ &= 7.5 \end{aligned}$$

The weather balloon is rising at 7.5 rad/h.

Section 8.5 Page 531 Question 10

A central angle of a circle has twice the measure of the angle inscribed in the circle that is subtended by the same arc. Hence, ∠ ACB $= \theta$. It follows that ∠ BCD $= \pi - \theta$. Use of the Cosine Law in △BCD, leads to the following,

$$\begin{aligned} x^2 &= r^2 + r^2 - 2r^2\cos(\pi - \theta) \\ &= 2r^2(1 + \cos\theta) \qquad (1) \end{aligned}$$

Let A be the area of isosceles △BDE.

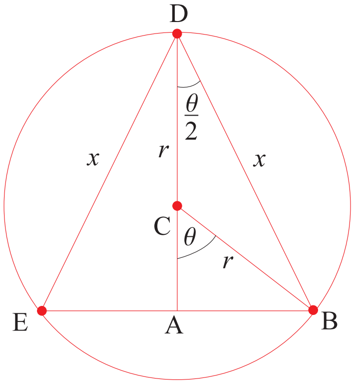
$$A = \frac{1}{2}x^2\sin\theta \qquad (2)$$

Substitute (1) into (2).

$$\begin{aligned} A &= \frac{1}{2}\cdot 2r^2(1 + \cos\theta)\cdot\sin\theta \\ &= r^2\sin\theta(1 + \cos\theta) \end{aligned}$$

Determine the critical number(s) of A.

$$\begin{aligned} \frac{dA}{d\theta} &= 0 \\ r^2\left((1 + \cos\theta)\cos\theta + \sin\theta(-\sin\theta)\right) &= 0 \\ (1 + \cos\theta)\cos\theta - \sin^2\theta &= 0 \\ (1 + \cos\theta)\cos\theta - (1 - \cos^2\theta) &= 0 \\ (1 + \cos\theta)(\cos\theta - (1 - \cos\theta)) &= 0 \\ (1 + \cos\theta)(2\cos\theta - 1) &= 0 \\ 2\cos\theta - 1 &= 0 \\ \cos\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3} \end{aligned}$$



The maximum area of the isosceles triangle is achieved if $\theta = \frac{\pi}{3}$.

Section 8.5 Page 531 Question 11

a)

$$R(\theta) = \left(\frac{v^2}{g}\right)\sin 2\theta$$

Determine the critical number(s) of R.

$$\begin{aligned} R'(\theta) &= 0 \\ \left(\frac{v^2}{g}\right)\cdot\cos 2\theta\cdot 2 &= 0 \\ \cos 2\theta &= 0 \\ 2\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

The maximum range will be achieved if $\theta = \frac{\pi}{4}$ rad.

b)

$$\begin{aligned} R(\theta) &= \left(\frac{2v^2}{g}\right)\left(\frac{\sin(\theta-\phi)\cos\theta}{\cos^2\phi}\right) \\ &= \left(\frac{2v^2}{g\cos^2\phi}\right)(\sin(\theta-\phi)\cos\theta) \end{aligned}$$

Determine the critical number(s) of R.

$$\begin{aligned} R'(\theta) &= 0 \\ \left(\frac{2v^2}{g\cos^2\phi}\right)(\cos\theta\cos(\theta-\phi)+\sin(\theta-\phi)\cdot(-\sin\theta)) &= 0 \\ \cos\theta\cos(\theta-\phi)-\sin(\theta-\phi)\sin\theta &= 0 \\ \cos(\theta+\theta-\phi) &= 0 \\ \cos(2\theta-\phi) &= 0 \\ 2\theta-\phi &= \frac{\pi}{2} \\ 2\theta &= \frac{\pi}{2}+\phi \\ \theta &= \frac{\pi}{4}+\frac{\phi}{2} \end{aligned}$$

The maximum range will be achieved if $\theta = \frac{\pi}{4}+\frac{\phi}{2}$ rad.

c) In part b), if $\phi = 0$, the result is $\theta = \frac{\pi}{4}$ rad. This is the same result as in part a).

Section 8.5 Page 531 Question 12

$$\theta = 0.15\cos 6t$$

Determine the angular speed.

$$\begin{aligned} \frac{d\theta}{dt} &= 0.15\cdot(-\sin 6t)\cdot 6 \\ &= -0.9\sin 6t \end{aligned}$$

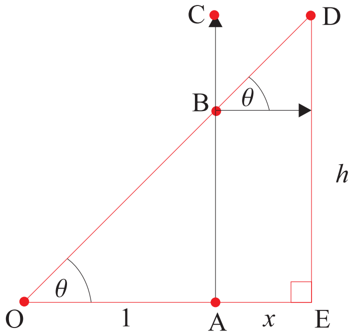
The maximum angular speed occurs when $-\sin 6t = 1$. Thus, the maximum angular speed is 1(0.9) or 0.9 rad/s.

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= -0.9\cos 6t\cdot 6 \\ &= -5.4\cos 6t \end{aligned}$$

The maximum angular acceleration occurs when $-\cos 6t = 1$. Thus, the maximum angular acceleration is (5.4)(1) or 5.4 rad/s^2.

Section 8.5 Page 531 Question 13

a) Let O be the position of the observer and θ be the line of sight, in radians. Let B be the position of the balloon. In $\triangle$DOE,



$$h = 10t \qquad (1)$$

$$x = 5t \qquad (2)$$

$$\tan\theta = \frac{h}{1+x} \qquad (3)$$

Substitute (1) and (2) into (3).

$$\tan\theta = \frac{10t}{1+5t} \qquad (4)$$

$$\theta = \tan^{-1}\left(\frac{10t}{1+5t}\right)$$

The angle of the line of sight can be expressed as $\theta = \tan^{-1}\left(\frac{10t}{1+5t}\right)$ rad.

b) Determine t for $\theta = \frac{\pi}{6}$. From (4),

$$\begin{aligned}\tan\frac{\pi}{6} &= \frac{10t}{1+5t}\\ \frac{1}{\sqrt{3}} &= \frac{10t}{1+5t}\\ 1+5t &= 10\sqrt{3}t\\ (10\sqrt{3}-5)t &= 1\\ t &= \frac{1}{10\sqrt{3}-5} \qquad (5)\end{aligned}$$

The angle of the line of sight is $\frac{\pi}{6}$ after $\frac{1}{10\sqrt{3}-5}$ h. Determine $\frac{d\theta}{dt}$ from (4).

$$\begin{aligned}\frac{d\tan\theta}{d\theta}\cdot\frac{d\theta}{dt} &= \frac{d}{dt}\left(\frac{10t}{1+5t}\right)\\ \sec^2\theta\cdot\frac{d\theta}{dt} &= \frac{(1+5t)\cdot 10 - 10t\cdot 5}{(1+5t)^2}\\ \frac{d\theta}{dt} &= \cos^2\theta\cdot\frac{10}{(1+5t)^2} \qquad (6)\end{aligned}$$

Substitute $\theta = \frac{\pi}{6}$ and (5) into (6).

$$\begin{aligned}\frac{d\theta}{dt} &= \cos^2\left(\frac{\pi}{6}\right)\cdot\frac{10}{\left(1+5\left(\frac{1}{10\sqrt{3}-5}\right)\right)^2}\\ &= \left(\frac{\sqrt{3}}{2}\right)^2\cdot\frac{10}{\left(1+\frac{1}{2\sqrt{3}-1}\right)^2}\\ &= \frac{3}{4}\cdot\frac{10}{\left(\frac{2\sqrt{3}}{2\sqrt{3}-1}\right)^2}\\ &= \frac{3}{4}\cdot\frac{10}{\frac{12}{13-4\sqrt{3}}}\\ &= \frac{5}{8}(13-4\sqrt{3})\end{aligned}$$

When $\theta = \frac{\pi}{6}$ rad, θ is increasing at the rate of $\frac{5}{8}(13-4\sqrt{3})$ rad/h.

Section 8.5 Page 531 Question 14

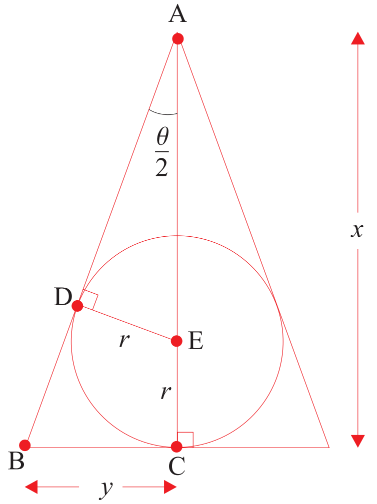
a) Let A be the area of the isosceles triangle. Let x be the height and y be half the base of the triangle.

$$A(x, y) = xy \qquad (1)$$

$$\text{AE} = x - r \qquad (2)$$

Since ADE is a right triangle,

$$\begin{aligned}\text{AD} &= \sqrt{(x-r)^2 - r^2}\\ &= \sqrt{x^2 - 2xr}\end{aligned}$$



Since $\triangle\text{ADE} \sim \triangle\text{ACB}$,

$$\frac{\text{AD}}{\text{DE}} = \frac{\text{AC}}{\text{CB}}$$

$$\frac{\sqrt{x^2-2xr}}{r} = \frac{x}{y}$$

$$y = \frac{rx}{\sqrt{x^2-2xr}} \qquad (3)$$

Substitute (3) into (1).

$$A(x) = \frac{rx^2}{\sqrt{x^2-2xr}}$$

Determine the critical number(s) of A.

$$A'(x) = 0$$

$$r\left(\frac{\sqrt{x^2-2rx}(2x) - \frac{1}{2\sqrt{x^2-2xr}}\cdot(2x-2r)\cdot x^2}{x^2-2xr}\right) = 0$$

$$\frac{(x^2-2rx)(2x)-(x^3-rx^2)}{\sqrt{x^2-2xr}} = 0$$

$$2x^3 - 4rx^2 - x^3 + rx^2 = 0$$

$$x^3 - 3rx^2 = 0$$

$$x^2(x-3r) = 0$$

$$x = 3r \qquad (4)$$

Substitute (4) into (2).

$$\begin{aligned}\text{AE} &= 3r - r\\ &= 2r\end{aligned}$$

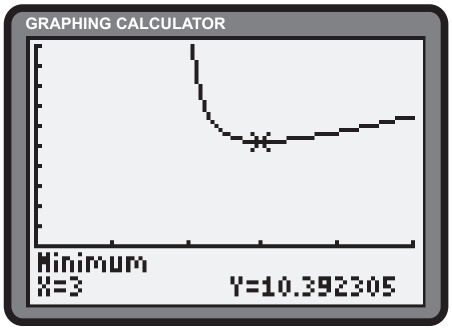
In $\triangle$ADE, since $\frac{\theta}{2} = \sin^{-1}\left(\frac{r}{2r}\right)$ or $\sin^{-1}\left(\frac{1}{2}\right)$, $\frac{\theta}{2} = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$.

b) Let P be the perimeter of the triangle.

$$P(x, y) = 2\left(y + \sqrt{x^2+y^2}\right) \qquad (5)$$

Substitute (3) into (5).

$$P(x) = 2\left(\frac{rx}{\sqrt{x^2-2xr}} + \sqrt{x^2 + \left(\frac{rx}{\sqrt{x^2-2xr}}\right)^2}\right) \qquad (6)$$



Minimum
X=3 Y=10.392305

Using the **Minimum operation** of the graphing calculator and $r = 1$ yields $x = 3$ as the critical number. As a result, $\theta = \frac{\pi}{3}$ rad.

c) Answers may vary.

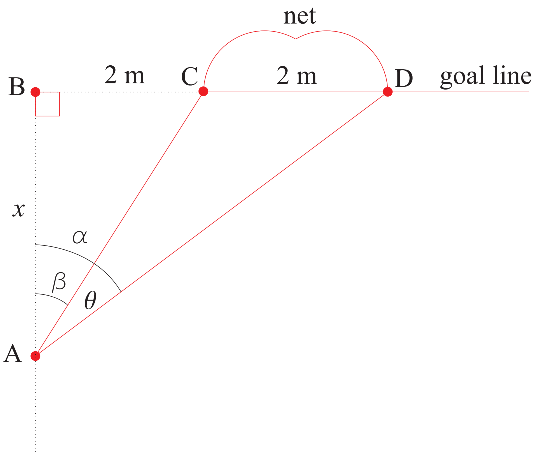
Section 8.5 Page 531 Question 15

Let x be the distance from the goal line, in metres. In △ABD,

$$\tan \alpha = \frac{4}{x} \qquad (1)$$

In △ABC,

$$\tan \beta = \frac{2}{x} \qquad (2)$$



Determine $\theta(x)$.

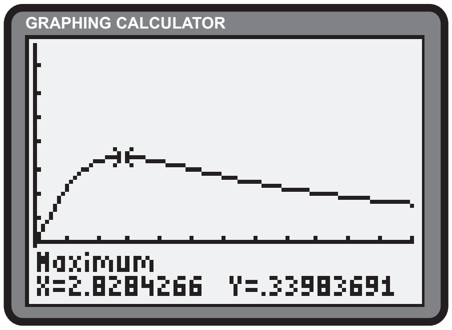
$$\begin{aligned} \theta &= \alpha - \beta \\ \tan \theta &= \tan(\alpha - \beta) \\ &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \qquad (3) \end{aligned}$$

Substitute (1) and (2) into (3).

$$\begin{aligned} \tan \theta &= \frac{\frac{4}{x} - \frac{2}{x}}{1 + \left(\frac{4}{x}\right)\left(\frac{2}{x}\right)} \\ &= \frac{\frac{2}{x}}{1 + \frac{8}{x^2}} \\ \tan \theta &= \frac{2x}{x^2 + 8} \\ \theta(x) &= \tan^{-1}\left(\frac{2x}{x^2 + 8}\right) \end{aligned}$$

Determine the critical number(s) of $\theta(x)$.

$$\begin{aligned} \frac{d\theta}{dx} &= 0 \\ \frac{1}{1 + \left(\frac{2x}{x^2+8}\right)} \cdot \left(\frac{(x^2 + 8)(2) - 2x(2x)}{(x^2 + 8)^2}\right) &= 0 \\ 2x^2 + 16 - 4x^2 &= 0 \\ 16 - 2x^2 &= 0 \\ x^2 &= 8 \\ x &= 2\sqrt{2};\ x \geq 0 \end{aligned}$$



Maximum
X=2.8284266 Y=.33983691

The player should shoot the puck $2\sqrt{2}$ m from the goal line.

Review of Key Concepts

8.1 Addition and Subtraction Formulas

Section Review Page 534 Question 1

a)

$$\begin{aligned}\sin 64^\circ \cos 4^\circ - \cos 64^\circ \sin 4^\circ &= \sin(64^\circ - 4^\circ)\\ &= \sin 60^\circ\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

b)

$$\begin{aligned}\sin 32^\circ \cos 13^\circ + \cos 32^\circ \sin 13^\circ &= \sin(32^\circ + 13^\circ)\\ &= \sin 45^\circ\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

c)

$$\begin{aligned}\cos 45^\circ \cos 15^\circ - \sin 45^\circ \sin 15^\circ &= \cos(45^\circ + 15^\circ)\\ &= \cos 60^\circ\\ &= \frac{1}{2}\end{aligned}$$

d)

$$\begin{aligned}\cos 45^\circ \cos 15^\circ + \sin 45^\circ \sin 15^\circ &= \cos(45^\circ - 15^\circ)\\ &= \cos 30^\circ\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

e)

$$\begin{aligned}\frac{\tan 35^\circ + \tan 10^\circ}{1 - \tan 35^\circ \tan 10^\circ} &= \tan(35^\circ + 10^\circ)\\ &= \tan 45^\circ\\ &= 1\end{aligned}$$

8.2 Double-Angle Formulas

Section Review Page 534 Question 2

a)

$$\begin{aligned}50 \sin x \cos x &= 25(2 \sin x \cos x)\\ &= 25 \sin 2x\end{aligned}$$

b)

$$\begin{aligned}15 \sin 3x \cos 3x &= 15\left(\frac{1}{2}\sin(2(3x))\right)\\ &= \frac{15}{2}\sin 6x\end{aligned}$$

c)

$$\begin{aligned}2\cos^2(3\theta + 2) - 1 &= \cos 2(3\theta + 2)\\ &= \cos(6\theta + 4)\end{aligned}$$

d)

$$\begin{aligned}8 \sin \frac{x}{2} \cos \frac{x}{2} &= 8\left(\frac{1}{2}\sin 2\left(\frac{x}{2}\right)\right)\\ &= 4 \sin x\end{aligned}$$

e)

$$\begin{aligned}\cos^2 2x - \sin^2 2x &= \cos 2(2x)\\ &= \cos 4x\end{aligned}$$

f)

$$\begin{aligned}2\cos^2 10\theta - 1 &= \cos 2(10\theta)\\ &= \cos 20\theta\end{aligned}$$

g)

$$\begin{aligned}1 - 2\sin^2 \frac{2\theta}{5} &= \cos 2\left(\frac{2\theta}{5}\right)\\ &= \cos \frac{4\theta}{5}\end{aligned}$$

h)

$$\begin{aligned}\sin 7x \cos 7x &= \frac{1}{2}\sin 2(7x)\\ &= \frac{1}{2}\sin 14x\end{aligned}$$

Section Review Page 534 Question 3

a) If $x \in [0, 2\pi]$, then $x + 3 \in [3, 2\pi + 3]$.

$$\begin{aligned}\sin 3 \cos x + \cos 3 \sin x &= 0.5\\ \sin(3 + x) &= 0.5\\ 3 + x &= \frac{13\pi}{6} \text{ or } \frac{17\pi}{6}\\ x &= \frac{13\pi}{6} - 3 \text{ or } \frac{17\pi}{6} - 3\end{aligned}$$

The equation is true for $x = \frac{13\pi}{6} - 3$ or $\frac{17\pi}{6} - 3$.

b) If $x \in [0, 2\pi]$, then $x + \pi \in [\pi, 3\pi]$.

$$\begin{aligned}\cos x \cos \pi - \sin x \sin \pi &= 0\\ \cos(x + \pi) &= 0\\ x + \pi &= \frac{3\pi}{2} \text{ or } \frac{5\pi}{2}\\ x &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2}\end{aligned}$$

The equation is true for $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

c) If $x \in [0, 2\pi]$, then $3x \in [0, 6\pi]$.

$$\begin{aligned}
\cos 2x \cos x - \sin 2x \sin x &= 0\\
\cos(2x + x) &= 0\\
\cos 3x &= 0\\
3x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \text{ or } \frac{11\pi}{2}\\
x &= \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \text{ or } \frac{11\pi}{6}
\end{aligned}$$

Solutions include $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \text{ or } \frac{11\pi}{6}$.

d) If $x \in [0, 2\pi]$, then $3x \in [0, 6\pi]$.

$$\begin{aligned}
\sin 2x \cos x + \cos 2x \sin x &= 1\\
\sin(2x + x) &= 1\\
\sin 3x &= 1\\
3x &= \frac{\pi}{2}, \frac{5\pi}{2}, \text{ or } \frac{9\pi}{2}\\
x &= \frac{\pi}{6}, \frac{5\pi}{6}, \text{ or } \frac{3\pi}{2}
\end{aligned}$$

Solutions include $x = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ or } \frac{3\pi}{2}$.

e) If $x \in [0, 2\pi]$, then $2x \in [0, 4\pi]$.

$$\begin{aligned}
\sin x \cos x &= -1 - \cos x \sin x\\
\sin x \cos x + \cos x \sin x &= -1\\
\sin(x + x) &= -1\\
\sin 2x &= -1\\
2x &= \frac{3\pi}{2} \text{ or } \frac{7\pi}{2}\\
x &= \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}
\end{aligned}$$

The equation is true for $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

f) Since $\tan \frac{\pi}{2}$ is undefined, there are no solutions.

g)
$$\begin{aligned}
3 \sin 2x \cos x + 3 \sin^2 x &= 0\\
\sin 2x \cos x + \sin^2 x &= 0\\
2 \sin x \cos^2 x + \sin^2 x &= 0\\
2 \sin x(1 - \sin^2 x) + \sin^2 x &= 0\\
\sin x(2 - 2 \sin^2 x + \sin x) &= 0\\
\sin x &= 0\\
x &= 0, \pi, \text{ or } 2\pi\\
2 - 2 \sin^2 x + \sin x &= 0\\
2 \sin^2 x - \sin x - 2 &= 0\\
\sin x &= \frac{1 - \sqrt{17}}{4}, \ \sin x \in [-1, 1]\\
x &= 4.0375 \text{ or } 5.3873
\end{aligned}$$

Solutions include $x = 0, \pi, 4.0375, 5.3873, \text{ or } 2\pi$.

h)
$$\begin{aligned}
\sin 2x - \sin x &= 0\\
2 \sin x \cos x - \sin x &= 0\\
\sin x(2 \cos x - 1) &= 0\\
\sin x &= 0\\
x &= 0, \pi, \text{ or } 2\pi\\
2 \cos x - 1 &= 0\\
\cos x &= \frac{1}{2}\\
x &= \frac{\pi}{3} \text{ or } \frac{5\pi}{3}
\end{aligned}$$

The equation is true for $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$.

i)
$$\begin{aligned}
\sin x - \cos x &= 0\\
\sin x &= \cos x\\
\tan x &= 1\\
x &= \frac{\pi}{4} \text{ or } \frac{5\pi}{4}
\end{aligned}$$

The equation is true for $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

j) If $x \in [0, 2\pi]$, then $2x \in [0, 4\pi]$.

$$\begin{aligned}
3 \cos 2x + 2 &= 0\\
\cos 2x &= -\frac{2}{3}\\
2x &= 2.3005, 3.9827, 8.5837 \text{ or } 10.2658\\
x &= 1.1503, 1.9913, 4.2919 \text{ or } 5.1329
\end{aligned}$$

Solutions include $x = 1.1503, 1.9913, 4.2919$ or 5.1329.

k)

$$\begin{aligned}
\sin 2x - \tan x &= 0 \\
2\sin x\cos x - \frac{\sin x}{\cos x} &= 0 \\
\sin x\left(2\cos x - \frac{1}{\cos x}\right) &= 0 \\
\sin x &= 0 \\
x &= 0, \pi, \text{ or } 2\pi \\
2\cos x - \frac{1}{\cos x} &= 0 \\
2\cos x &= \frac{1}{\cos x} \\
\cos^2 x &= \frac{1}{2} \\
\cos x &= \pm\frac{1}{\sqrt{2}} \\
x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}
\end{aligned}$$

Solutions include $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$ or 2π.

l) If $x \in [0, 2\pi]$, then $2x \in [0, 4\pi]$.

$$\begin{aligned}
\sin 2x - \cos 2x &= 0 \\
\sin 2x &= \cos 2x \\
\tan 2x &= 1 \\
2x &= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \text{ or } \frac{13\pi}{4} \\
x &= \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \text{ or } \frac{13\pi}{8}
\end{aligned}$$

The equation is true for $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \text{ or } \frac{13\pi}{8}$.

m)

$$\begin{aligned}
2\sin 2x &= \cos x \\
4\sin x\cos x - \cos x &= 0 \\
\cos x(4\sin x - 1) &= 0 \\
\cos x &= 0 \\
x &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\
4\sin x - 1 &= 0 \\
\sin x &= \frac{1}{4} \\
x &\doteq 0.2527 \text{ or } 2.8889
\end{aligned}$$

Solutions include $x = 0.2527, \frac{\pi}{2}, 2.8889, \text{ or } \frac{3\pi}{2}$.

n)

$$\begin{aligned}
3\sin x &= 2 - \cos 2x \\
3\sin x &= 2 - (1 - 2\sin^2 x) \\
3\sin x &= 1 + 2\sin^2 x \\
2\sin^2 x - 3\sin x + 1 &= 0 \\
(2\sin x - 1)(\sin x - 1) &= 0 \\
2\sin x - 1 &= 0 \\
\sin x &= \frac{1}{2} \\
x &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \\
\sin x &= 1 \\
x &= \frac{\pi}{2}
\end{aligned}$$

The equation is true for $x = \frac{\pi}{6}, \frac{\pi}{2}, \text{ or } \frac{5\pi}{6}$.

Section Review Page 534 Question 4

Using the 3 : 4 : 5 Pythagorean triple and the information given, it is determined that $\cos x = \frac{4}{5}$, $\tan x = \frac{3}{4}$, $\sin y = \frac{3}{5}$, and $\cos y = -\frac{4}{5}$.

a)

$$\begin{aligned}
\sin 2(x - y) &= 2\sin(x - y)\cos(x - y) \\
&= 2(\sin x\cos y - \cos x\sin y)(\cos x\cos y + \sin x\sin y) \\
&= 2\left[\frac{3}{5}\cdot\left(-\frac{4}{5}\right) - \frac{4}{5}\cdot\frac{3}{5}\right]\left[\frac{4}{5}\cdot\left(-\frac{4}{5}\right) + \frac{3}{5}\cdot\frac{3}{5}\right] \\
&= 2\left(-\frac{24}{25}\right)\left(-\frac{7}{25}\right) \\
&= \frac{336}{625}
\end{aligned}$$

b)

$$\begin{aligned}\cos 2(x+y) &= 2\cos^2(x+y) - 1\\ &= 2(\cos x\cos y - \sin x\sin y)^2 - 1\\ &= 2\left[\frac{4}{5}\cdot\left(-\frac{4}{5}\right) - \frac{3}{5}\cdot\frac{3}{5}\right]^2 - 1\\ &= 2\,(-1)^2 - 1\\ &= 2 - 1\\ &= 1\end{aligned}$$

c) Determine a value for $\tan(x-y)$.

$$\begin{aligned}\tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x\tan y}\\ &= \frac{\frac{3}{4} - \left(-\frac{3}{4}\right)}{1 + \frac{3}{4}\cdot\left(-\frac{3}{4}\right)}\\ &= \frac{\frac{6}{4}}{\frac{7}{16}}\\ &= \frac{24}{7} \qquad (1)\end{aligned}$$

Determine $\tan 2(x-y)$.

$$\tan 2(x-y) = \frac{2\tan(x-y)}{1 - \tan^2(x-y)} \qquad (2)$$

Substitute (1) into (2).

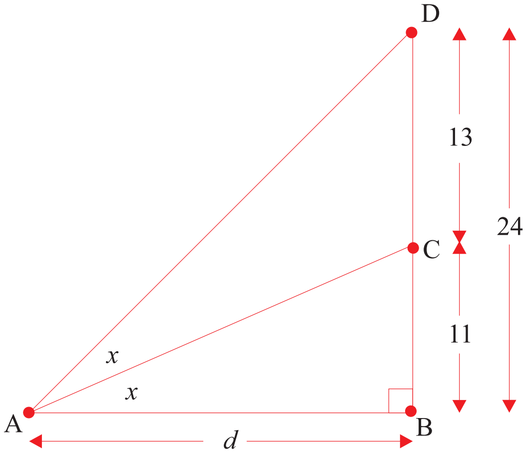
$$\begin{aligned}\tan 2(x-y) &= \frac{2\cdot\frac{24}{7}}{1 - \left(\frac{24}{7}\right)^2}\\ &= \frac{\frac{48}{7}}{-\frac{527}{49}}\\ &= -\frac{336}{527}\end{aligned}$$

d)

$$\begin{aligned}\sin 2x + \cos 2x &= 2\sin x\cos x + \cos^2 x - \sin^2 x\\ &= 2\cdot\frac{3}{5}\cdot\frac{4}{5} + \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2\\ &= \frac{24}{25} + \frac{16}{25} - \frac{9}{25}\\ &= \frac{31}{25}\end{aligned}$$

Section Review Page 534 Question 5

Let d be the required distance, in metres. In △ABC, $\tan x = \frac{11}{d}$. In △ABD,



$$\begin{aligned}\tan 2x &= \frac{24}{d}\\ \frac{2\tan x}{1-\tan^2 x} &= \frac{24}{d} \qquad (1)\end{aligned}$$

Substitute $\tan x = \frac{11}{d}$ in (1).

$$\begin{aligned}\frac{2\cdot\frac{11}{d}}{1-\left(\frac{11}{d}\right)^2} &= \frac{24}{d}\\ 22 &= 24\left(1-\frac{121}{d^2}\right)\\ 11d^2 &= 12d^2 - 1452\\ d^2 &= 1452\\ d &\doteq 38.1\end{aligned}$$

Marion is approximately 38.1 m from the base of the cliff.

Section Review Page 534 Question 6

Using the 5 : 12 : 13 and 3 : 4 : 5 Pythagorean triples and the information given, it is determined that $\cos x = \frac{12}{13}$ and $\sin y = \frac{3}{5}$.

$$\begin{aligned}\sin 2(x-y) &= 2\sin(x-y)\cos(x-y)\\ &= 2(\sin x\cos y - \cos x\sin y)(\cos x\cos y + \sin x\sin y)\\ &= 2\left[\frac{5}{13}\cdot\left(-\frac{4}{5}\right) - \frac{12}{13}\cdot\frac{3}{5}\right]\left[\frac{12}{13}\cdot\left(-\frac{4}{5}\right) + \frac{5}{13}\cdot\frac{3}{5}\right]\\ &= 2\left(-\frac{56}{65}\right)\left(-\frac{33}{65}\right)\\ &= \frac{3696}{4225}\end{aligned}$$

8.3 Limits of Trigonometric Functions

A number of solutions that follow make use the of the result $\lim_{x\to 0}\frac{\tan x}{x} = 1$. It is confirmed as follows,

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x}{\cos x}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}\\ &= 1\cdot 1\\ &= 1\end{aligned}$$

Section Review Page 534 Question 7

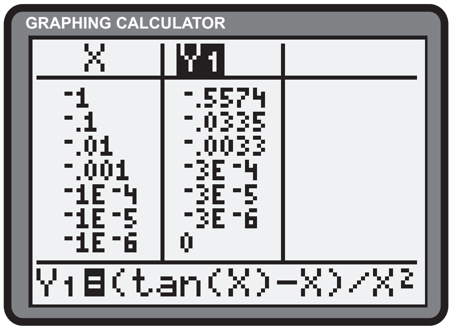
a)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{4x} &= \frac{1}{4}\lim_{x\to 0}\frac{\sin 3x}{x}\cdot\frac{3}{3}\\ &= \frac{1}{4}\cdot 3\cdot\lim_{3x\to 0}\frac{\sin 3x}{3x}\\ &= \frac{3}{4}\cdot 1\\ &= \frac{3}{4}\end{aligned}$$

b)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin\frac{1}{2}x}{x} &= \lim_{x\to 0}\frac{\sin\frac{1}{2}x}{x}\cdot\frac{\frac{1}{2}}{\frac{1}{2}}\\ &= \frac{1}{2}\cdot\lim_{\frac{1}{2}x\to 0}\frac{\sin\frac{1}{2}x}{\frac{1}{2}x}\\ &= \frac{1}{2}\cdot 1\\ &= \frac{1}{2}\end{aligned}$$

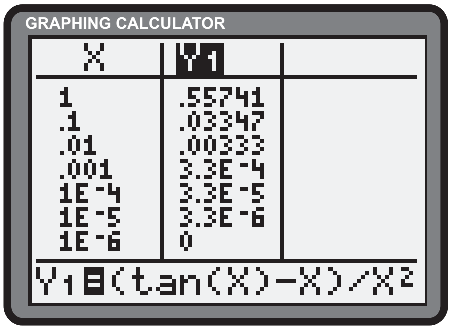
c)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin x\cos x}{x} &= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\cos x\right)\\ &= 1\cdot 1\\ &= 1\end{aligned}$$

d)
$$\begin{aligned}\lim_{x\to 0}2\frac{\tan^2 x}{x^2} &= 2\cdot\lim_{x\to 0}\left(\frac{\tan x}{x}\cdot\frac{\tan x}{x}\right)\\ &= 2\cdot 1\cdot 1\\ &= 2\end{aligned}$$

e) The calculator suggests $\lim_{x\to 0}\frac{\tan x - x}{x^2} = 0$.



```
X        Y1
-1       -.5574
-.1      -.0335
-.01     -.0033
-.001    -3E-4
-1E-4    -3E-5
-1E-5    -3E-6
-1E-6    0
Y1=(tan(X)-X)/X²
```



```
X        Y1
1        .55741
.1       .03347
.01      .00333
.001     3.3E-4
1E-4     3.3E-5
1E-5     3.3E-6
1E-6     0
Y1=(tan(X)-X)/X²
```

8.4 Derivatives of the Sine, Cosine, and Tangent Functions

Section Review Page 534 Question 8

a)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin(3x^2+5)}{d(3x^2+5)}\cdot\frac{d(3x^2+5)}{dx}\\ &= \cos(3x^2+5)\cdot 6x\\ &= 6x\cos(3x^2+5)\end{aligned}$$

b)
$$\begin{aligned}\frac{dy}{dx} &= \sin^2(6x^2-2)\cdot\frac{d4x^3}{dx}+4x^3\cdot\frac{d\sin^2(6x^2-2)}{d\sin(6x^2-2)}\cdot\frac{d\sin(6x^2-2)}{d(6x^2-2)}\cdot\frac{d(6x^2-2)}{dx}\\ &= \sin^2(6x^2-2)\cdot 12x^2+4x^3\cdot 2\sin(6x^2-2)\cdot\cos(6x^2-2)\cdot 12x\\ &= 12x^2\sin^2(6x^2-2)+96x^4\sin(6x^2-2)\cos(6x^2-2)\end{aligned}$$

c)
$$\begin{aligned}\frac{dy}{dx} &= \cos(4x^2+7)\cdot\frac{dx^2}{dx}+x^2\cdot\frac{d\cos(4x^2+7)}{d(4x^2+7)}\cdot\frac{d(4x^2+7)}{dx}\\ &= \cos(4x^2+7)\cdot 2x+x^2\cdot -\sin(4x^2+7)\cdot 8x\\ &= 2x\cos(4x^2+7)-8x^3\sin(4x^2+7)\end{aligned}$$

d)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin^2(\cos^3 x+\tan x)}{d\sin(\cos^3 x+\tan x)}\cdot\frac{d\sin(\cos^3 x+\tan x)}{d(\cos^3 x+\tan x)}\cdot\left(\frac{d\cos^3 x}{d\cos x}\cdot\frac{d\cos x}{dx}+\frac{d\tan x}{dx}\right)\\ &= 2\sin(\cos^3 x+\tan x)\cdot\cos(\cos^3 x+\tan x)\cdot\left(3\cos^2 x\cdot -\sin x+\sec^2 x\right)\\ &= 2\sin(\cos^3 x+\tan x)\cos(\cos^3 x+\tan x)\left(\sec^2 x-3\cos^2 x\sin x\right)\end{aligned}$$

e)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sin^3(\cos x^3)}{d\sin(\cos x^3)}\cdot\frac{d\sin(\cos x^3)}{d\cos x^3}\cdot\frac{d\cos x^3}{dx^3}\cdot\frac{dx^3}{dx}\\ &= 3\sin^2(\cos x^3)\cdot\cos(\cos x^3)\cdot(-\sin x^3)\cdot 3x^2\\ &= -9x^2\sin^2(\cos x^3)\cos(\cos x^3)\sin x^3\end{aligned}$$

f)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d\cos(e^x)}{de^x}\cdot\frac{de^x}{dx}\\ &= -\sin(e^x)\cdot e^x\\ &= -e^x\sin(e^x)\end{aligned}$$

g)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d\cos(\ln(\tan x))}{d\ln(\tan x)}\cdot\frac{d\ln(\tan x)}{d\tan x}\cdot\frac{d\tan x}{dx}\\ &= -\sin(\ln(\tan x))\cdot\frac{1}{\tan x}\cdot\sec^2 x\\ &= -\sec^2 x\cot x\sin(\ln(\tan x))\end{aligned}$$

h)
$$\begin{aligned}\frac{dy}{dx} &= \frac{d\ln(\sin(e^x))}{d\sin(e^x)}\cdot\frac{d\sin(e^x)}{e^x}\cdot\frac{de^x}{dx}\\ &= \frac{1}{\sin(e^x)}\cdot\cos(e^x)\cdot e^x\\ &= e^x\cot(e^x)\end{aligned}$$

i)
$$\begin{aligned}\frac{dy}{dx} &= \frac{x\cdot\dfrac{d\sin x}{dx}-\sin x\cdot\dfrac{dx}{dx}}{x^2}\\ &= \frac{x\cos x-\sin x}{x^2}\end{aligned}$$

j)
$$\begin{aligned}y &= \sin^2 x+\cos^2 x\\ &= 1\\ \frac{dy}{dx} &= 0\end{aligned}$$

Section Review Page 534 Question 9

$$y = 2\sin x+\sin^2 x$$

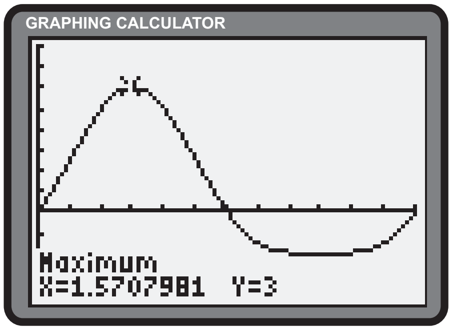
Determine the critical number(s).

$$\begin{aligned}\frac{dy}{dx} &= 0\\ 2\cos x+2\sin x\cos x &= 0\\ \cos x(1+\sin x) &= 0\\ x &= \frac{\pi}{2}\text{ or }\frac{3\pi}{2}\end{aligned}$$

Determine the points of inflection.

$$\begin{aligned}\frac{d^2y}{dx^2} &= 0\\ -2\sin x+2(\cos^2 x-\sin^2 x) &= 0\\ \sin x-1+2\sin^2 x &= 0\\ 2\sin^2 x+\sin x-1 &= 0\\ (2\sin x-1)(\sin x+1) &= 0\\ x &= \frac{\pi}{6},\ \frac{5\pi}{6}\text{ or }\frac{3\pi}{2}\end{aligned}$$

Points of inflection occur at $\left(\frac{\pi}{6}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{6}, \frac{5}{4}\right)$.



```
Maximum
X=1.5707981  Y=3
```

A maximum value of 3 occurs at $x = \frac{\pi}{2}$. A minimum value of -1 occurs at $x = \frac{3\pi}{2}$.

Section Review Page 534 Question 10

$$\begin{aligned}y &= 2\sin x+\sin^2 x\\ \frac{dy}{dx} &= 2\cos x+2\sin x\cos x\end{aligned}$$

Determine the slope, m, of the tangent.

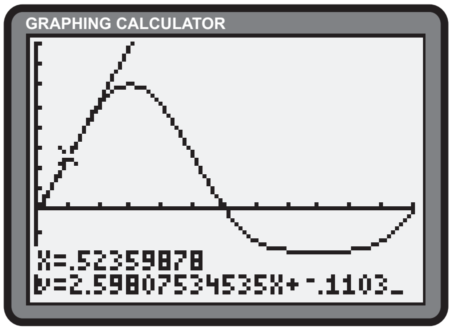
$$\begin{aligned}m &= \frac{dy}{dx}_{|x=\frac{\pi}{6}}\\ &= 2\cos\frac{\pi}{6}+2\sin\frac{\pi}{6}\cos\frac{\pi}{6}\\ &= 2\cdot\frac{\sqrt{3}}{2}+2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\\ &= \sqrt{3}+\frac{\sqrt{3}}{2}\\ &= \frac{3\sqrt{3}}{2}\end{aligned}$$

Determine the equation of the tangent.

$$\begin{aligned}y-y_1 &= m(x-x_1)\\ y-\frac{5}{4} &= \frac{3\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)\\ y &= \frac{3\sqrt{3}}{2}x-\frac{\pi\sqrt{3}}{4}+\frac{5}{4}\end{aligned}$$

The equation of the tangent is,

$$y = \frac{3\sqrt{3}}{2}x-\frac{\pi\sqrt{3}}{4}+\frac{5}{4}$$



```
X=.52359878
y=2.59807534535X+⁻.1103_
```

8.5 Modelling with Trigonometric Functions

Section Review Page 535 Question 11

a) Models may vary. At two pulses per rotation, the period is $\frac{0.033}{2}$ s.

$$y = \frac{A}{2}\sin\left(\frac{2\pi}{\frac{0.033}{2}}t\right)$$
$$y = \frac{A}{2}\sin\left(\frac{4\pi t}{0.033}\right) + 1$$

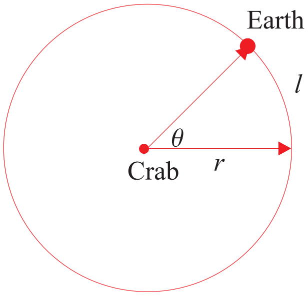
b) Assuming the rate decreases linearly, 0.033 can be replaced with $0.033\left(2500 - \frac{t}{k}\right)$, where k is the number of seconds in a year (3.154×10^7). This expression will ensure that the rotational rate will be 0 after 2500 years. The updated model is

$$y = \frac{A}{2}\sin\left(\frac{4\pi t}{0.033\left(2500 - \frac{t}{k}\right)}\right) + 1$$

c)
$$l = r\theta$$
$$\frac{dl}{dt} = r \cdot \frac{d\theta}{dt}$$

Substitute $r = 6 \times 10^{16}$ and $\frac{d\theta}{dt} = \frac{2\pi}{0.033}$.

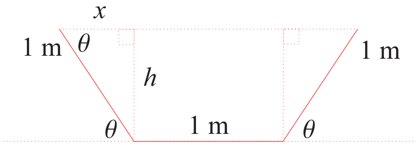
$$\begin{aligned}\frac{dl}{dt} &= 6 \times 10^{16} \cdot \frac{2\pi}{0.033}\\ &\doteq 1142.4 \times 10^{16}\\ &= 1.1424 \times 10^{19}\end{aligned}$$



The pulsar sweeps the surface of Earth at an approximate rate of 1.1424×10^{19} km/s.

Section Review Page 535 Question 12

Let A be the area of a cross-section of the trough.



$$\begin{aligned}A(x,h) &= h + 2\left(\frac{1}{2}xh\right)\\ &= h(1+x) \quad (1)\\ x &= \cos\theta \quad (2)\\ h &= \sin\theta \quad (3)\end{aligned}$$

Substitute (2) and (3) into (1).

$$A(\theta) = \sin\theta(1+\cos\theta)$$

Determine the critical number(s) of A.

$$\begin{aligned}A'(\theta) &= 0\\ (1+\cos\theta)\cos\theta + \sin\theta(-\sin\theta) &= 0\\ (1+\cos\theta)\cos\theta - (1-\cos^2\theta) &= 0\\ (1+\cos\theta)(\cos\theta - (1-\cos\theta)) &= 0\\ (1+\cos\theta)(2\cos\theta - 1) &= 0\\ 2\cos\theta - 1 &= 0\\ \cos\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{3}\end{aligned}$$

A value of $\theta = \frac{\pi}{3}$ or 60° will maximize the area of the trough.

Section Review Page 535 Question 13

Let A be the area of the triangle in square metres. Let θ be the contained angle between the two sides measuring 15 m and 20 m.

$$\begin{aligned} A(\theta) &= \frac{1}{2}(15)(20)\sin\theta \\ &= 150\sin\theta \qquad (1) \end{aligned}$$

Differentiate both sides of (1) with respect to time, t, in seconds.

$$\begin{aligned} \frac{dA}{dt} &= \frac{d150\sin\theta}{d\theta}\cdot\frac{d\theta}{dt} \\ &= 150\cos\theta\cdot\frac{d\theta}{dt} \qquad (2) \end{aligned}$$

Substitute $\theta = \frac{\pi}{3}$ and $\frac{d\theta}{dt} = \frac{\pi}{90}$ into (2).

$$\begin{aligned} \frac{dA}{dt} &= 150\cos\frac{\pi}{3}\cdot\frac{\pi}{90} \\ &= 150\cdot\frac{1}{2}\cdot\frac{\pi}{90} \\ &= \frac{5\pi}{6} \\ &\doteq 2.62 \end{aligned}$$

The area is changing at the rate of approximately 2.62 m^2/s.

Section Review Page 535 Question 14

a)

$$\begin{aligned} T &= 1 \\ 2\pi\sqrt{\frac{L}{g}} &= 1 \\ L &= \frac{g}{4\pi^2} \\ &= \frac{9.8}{4\pi^2} \\ &\doteq 24.824 \end{aligned}$$

The length of the pendulum is approximately 24.824 cm.

b) Answers may vary.

c) Let a be the amplitude. $y = a\sin(2\pi t + k)$.

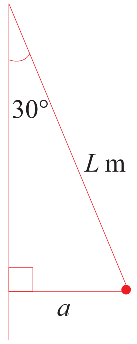
d)

$$\begin{aligned} \sin 30^\circ &= \frac{a}{L} \\ a &= \frac{L}{2} \end{aligned}$$

Substitute $L = 24.824$ cm.

$$\begin{aligned} a &= \frac{24.824}{2} \\ &= 12.412 \end{aligned}$$

The amplitude is 12.412 cm.



e) The measure of the central angle, θ, is defined by the model, $\theta = 12.412\sin(2\pi t + k)$. Determine the angular velocity.

$$\begin{aligned} \frac{d\theta}{dt} &= 12.412\cos(2\pi t + k)\cdot 2\pi \\ &= 24.824\pi\cos(2\pi t + k) \qquad (1) \end{aligned}$$

Determine the critical number(s) of (1).

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= 0 \\ 24.842\pi\cdot -\sin(2\pi t + k)\cdot 2\pi &= 0 \\ 24.842\pi\cdot -\sin(2\pi t + k)\cdot 2\pi &= 0 \\ -49.684\pi^2\sin(2\pi t + k) &= 0 \qquad (2) \\ \sin(2\pi t + k) &= 0 \\ 2\pi t + k &= n\pi;\ n \in I \\ t &= \frac{n}{2} - \frac{k}{2\pi} \qquad (3) \end{aligned}$$

Substitution of (2) in (1) yields a maximum angular speed of $|24.824\pi|$ or approximately 77.99 rad/s. The maximum angular speed occurs at $t = \frac{n}{2} - \frac{k}{2\pi}$ s, where n is any integer

f) Recall the angular acceleration from (2) in part e).

$$\frac{d^2\theta}{dt^2} = -49.684\pi^2 \sin(2\pi t + k) \tag{4}$$

Determine the critical number(s) of (4).

$$\begin{aligned}
\frac{d^3\theta}{dt^3} &= 0 \\
-49.684\pi^2 \cdot \cos(2\pi t + k) \cdot 2\pi &= 0 \\
-99.368\pi^3 \cos(2\pi t + k) &= 0 \\
\cos(2\pi t + k) &= 0 \\
2\pi t + k &= \frac{(2n+1)\pi}{2};\ n \in I \\
t &= \frac{2n+1}{4} - \frac{k}{2\pi} \\
&= \frac{n}{2} + \frac{1}{4} - \frac{k}{2\pi} \qquad (5)
\end{aligned}$$

Substitution of (5) in (4) yields a maximum angular acceleration of $\left|-49.684\pi^2\right|$ or approximately 490.4 rad/s^2. The maximum angular acceleration occurs at $t = \frac{n}{2} + \frac{1}{4} - \frac{k}{2\pi}$ s, where n is any integer.

Section Review Page 535 Question 15

$$S(\theta) = 6ab + \frac{3}{2}a^2\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

Determine the critical number(s) of S.

$$\begin{aligned}
S'(\theta) &= 0 \\
\frac{3}{2}a^2\left(\frac{\sin\theta \cdot \sin\theta - (\sqrt{3} - \cos\theta)\cos\theta}{\sin^2\theta}\right) &= 0 \\
\sin^2\theta - \sqrt{3}\cos\theta + \cos^2\theta &= 0 \\
1 - \sqrt{3}\cos\theta &= 0 \\
\cos\theta &= \frac{1}{\sqrt{3}} \\
\theta &= 0.9553
\end{aligned}$$

An angle of approximately 0.9553 rad or 54.74° minimizes the surface area.

Chapter Test

Section Chapter Test Page 536 Question 1

a)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{\sin x} &= \lim_{x\to 0}\frac{\frac{\sin 2x}{x}}{\frac{\sin x}{x}}\\ &= \frac{\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{2}{2}}{\lim_{x\to 0}\frac{\sin x}{x}}\\ &= \frac{2\cdot\lim_{2x\to 0}\frac{\sin 2x}{2x}}{\lim_{x\to 0}\frac{\sin x}{x}}\\ &= \frac{2\cdot 1}{1}\\ &= 2\end{aligned}$$

b)
$$\begin{aligned}\lim_{x\to 0}\frac{\tan 20x}{x} &= \lim_{x\to 0}\frac{\tan 20x}{x}\cdot\frac{20}{20}\\ &= 20\cdot\lim_{20x\to 0}\frac{\tan 20x}{20x}\\ &= 20\cdot 1\\ &= 20\end{aligned}$$

Section Chapter Test Page 536 Question 2

a)
$$\begin{aligned}y &= 2\sin 3x - 3\cos 4x\\ \frac{dy}{dx} &= 2\cdot\frac{d\sin 3x}{d3x}\cdot\frac{d3x}{dx} - 3\cdot\frac{d\cos 4x}{d4x}\cdot\frac{d4x}{dx}\\ &= 2\cdot\cos 3x\cdot 3 - 3\cdot -\sin 4x\cdot 4\\ &= 6\cos 3x + 12\sin 4x\\ \frac{d^2y}{dx^2} &= 6\cdot\frac{d\cos 3x}{d3x}\cdot\frac{d3x}{dx} + 12\cdot\frac{d\sin 4x}{d4x}\cdot\frac{d4x}{dx}\\ &= 6\cdot -\sin 3x\cdot 3 + 12\cdot\cos 4x\cdot 4\\ &= -18\sin 3x + 48\cos 4x\end{aligned}$$

b)
$$\begin{aligned}y &= \frac{\sin x}{1+\tan x}\\ \frac{dy}{dx} &= \frac{(1+\tan x)\cdot\cos x - \sin x\cdot\sec^2 x}{(1+\tan x)^2}\\ &= \frac{\cos x + \sin x - \sec^2 x\sin x}{(1+\tan x)^2}\\ \frac{d^2y}{dx^2} &= \frac{(1+\tan x)(\sin x+\cos x - 2\sin x\sec^2 x\tan x - \sec^2 x\cos x)) - 2\sec^2 x(\cos x+\sin x-\sec^2 x\sin x)}{(1+\tan x)^3}\end{aligned}$$

Simplifications will vary.

Section Chapter Test Page 536 Question 3

a) One complete cycle takes $\frac{2\pi}{6}$ s. There are $\frac{60}{\frac{2\pi}{6}}$ or approximately 57 heartbeats in one minute.

b) The range of $20\sin 6t$ is $[-20, 20]$. The range of P is, therefore, $[80, 120]$. The maximum pressure is 120. The minimum pressure is 80. These are healthy values for blood pressure.

c) All constants increase.

d) All constants decrease.

Section Chapter Test Page 536 Question 4

$$\begin{aligned}
f(x) &= \sin x + \cos x \\
f'(x) &= \cos x - \sin x \\
f''(x) &= -\sin x - \cos x \\
f'''(x) &= -\cos x + \sin x \\
f^{(4)}(x) &= \sin x + \cos x
\end{aligned}$$

$f(x) = f^{(n)}(x)$ for $n = 4k$, where k is a natural number.

Section Chapter Test Page 536 Question 5

The capacity of the trough will be maximized when the area of the cross section is maximized. Let s be the constant side length of the triangular end of the trough. Let A be the area of the triangle.

$$A(\theta) = \frac{1}{2}s^2 \sin\theta$$

Determine the critical number(s) of A.

$$\begin{aligned}
A'(\theta) &= 0 \\
\frac{1}{2}s^2 \cos\theta &= 0 \\
\cos\theta &= 0 \\
\theta &= \frac{\pi}{2}
\end{aligned}$$

The capacity of the trough is maximized when the vertex angle is $\frac{\pi}{2}$ rad or 90°.

Section Chapter Test Page 536 Question 6

a) Let A be the area of $\triangle$OAB. Let x and y be the base and height of $\triangle$OAC respectively. In $\triangle$OPA, OP $= p$ and PA $= \sqrt{p^2 - 1}$. Develop an expression for $A(p)$.

$$A(x, y) = xy \tag{1}$$

Since $\triangle$OAC $\sim$ $\triangle$OPA,

$$\frac{x}{1} = \frac{1}{p} \tag{2}$$

$$\frac{y}{1} = \frac{\sqrt{p^2 - 1}}{p} \tag{3}$$

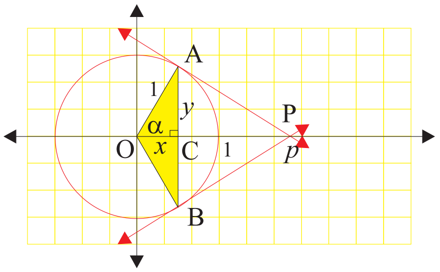
Substitute (2) and (3) into (1).

$$A(p) = \frac{\sqrt{p^2 - 1}}{p^2} \tag{4}$$

Determine the critical number(s) of A.

$$\begin{aligned}
A'(p) &= 0 \\
\frac{p^2 \cdot \dfrac{2p}{2\sqrt{p^2 - 1}} - \sqrt{p^2 - 1} \cdot 2p}{p^4} &= 0 \\
\frac{p^2}{\sqrt{p^2 - 1}} - 2\sqrt{p^2 - 1} &= 0 \\
p^2 &= 2(p^2 - 1) \\
p^2 &= 2p^2 - 2 \\
p^2 &= 2 \\
p &= \pm\sqrt{2}
\end{aligned}$$

The maximum area is achieved when $p = \pm\sqrt{2}$.



b) Let α be the angle between OA and the x-axis.

$$\cos\alpha = \frac{1}{p} \tag{5}$$

Substitute $p = \sqrt{2}$ in (5).

$$\begin{aligned}
\cos\alpha &= \frac{1}{\sqrt{2}} \\
\alpha &= \frac{\pi}{4}
\end{aligned}$$

When the area of $\triangle$OAB is a maximum, $\alpha = \frac{\pi}{4}$.

c) From (4),

$$\begin{aligned}
A\left(\pm\sqrt{2}\right) &= \frac{\sqrt{\left(\sqrt{2}\right)^2 - 1}}{\left(\sqrt{2}\right)^2} \\
&= \frac{1}{2}
\end{aligned}$$

The maximum area is $\frac{1}{2}$ square units.

Challenge Problems

Section Challenge Problems Page 537 Question 1

a)

$$\begin{aligned} 4\sin x + 3\sin x &= A\sin(x+\theta) \\ &= A\sin x\cos\theta + A\cos x\sin\theta \end{aligned}$$

Comparison of the left and right sides reveals,

$$\begin{aligned} A\cos\theta &= 4 \\ A\sin\theta &= 3 \\ \tan\theta &= \frac{3}{4} \end{aligned}$$

If θ is in the first quadrant, $\theta \in \left(0, \frac{\pi}{2}\right)$.

$$\begin{aligned} \cos\theta &= \frac{4}{5} \\ A\cos\theta &= 4 \\ A\left(\frac{4}{5}\right) &= 4 \\ A &= 5 \end{aligned}$$

Thus,

$$\begin{aligned} 4\sin x + 3\cos x &= 5\sin\left(x + \tan^{-1}\left(\frac{3}{4}\right)\right) \\ &\doteq 5\sin(x + 0.644) \end{aligned}$$

If θ is in the third quadrant, $\theta \in \left(\pi, \frac{3\pi}{2}\right)$.

$$\begin{aligned} \cos\theta &= -\frac{4}{5} \\ A\cos\theta &= 4 \\ A\left(-\frac{4}{5}\right) &= 4 \\ A &= -5 \end{aligned}$$

Thus,

$$\begin{aligned} 4\sin x + 3\cos x &= -5\sin\left(x + \tan^{-1}\left(\frac{3}{4}\right) + \pi\right) \\ &\doteq -5\sin(x + 3.785) \end{aligned}$$

b)

$$\begin{aligned} a\sin x + b\sin x &= A\sin(x+\theta) \\ &= A\sin x\cos\theta + A\cos x\sin\theta \qquad (1) \end{aligned}$$

Comparison of the left and right sides of (1) reveals,

$$\begin{aligned} A\cos\theta &= a \qquad (2) \\ A\sin\theta &= b \end{aligned}$$

Thus,

$$\tan\theta = \frac{b}{a} \qquad (2)$$

Therefore,

$$\cos\theta = \pm\frac{a}{\sqrt{a^2+b^2}} \qquad (4)$$

Substitute (4) into (2).

$$\begin{aligned} A\left(\pm\frac{a}{\sqrt{a^2+b^2}}\right) &= a \\ A &= \pm\sqrt{a^2+b^2} \end{aligned}$$

$A = \pm\sqrt{a^2+b^2}$ and, from (3), $\theta = \tan^{-1}\left(\frac{a}{b}\right)$.

Section Challenge Problems   Page 537   Question 2

a)

$$4\sin x + 3\sin x = A\cos(x+\theta)$$

$$= A\cos x\cos\theta - A\sin x\sin\theta \tag{1}$$

Comparison of the left and right sides of (1) reveals,

$$-A\sin\theta = 4 \tag{2}$$

$$A\cos\theta = 3$$

$$\tan\theta = -\frac{4}{3}$$

$$\theta = \tan^{-1}\left(-\frac{4}{3}\right) \tag{3}$$

$$\doteq -0.927 \text{ or } 2.124$$

$$\sin\theta = \pm\frac{4}{5} \tag{4}$$

Substitute (4) into (2).

$$-A\left(\pm\frac{4}{5}\right) = 4$$

$$A = \pm 5$$

Thus, $4\sin x + 3\cos x = \pm 5\cos\left(x + \tan^{-1}\left(-\frac{4}{3}\right)\right)$.

b)

$$a\sin x + b\sin x = A\cos(x+\theta)$$

$$= A\cos x\cos\theta - A\sin x\sin\theta \tag{1}$$

Comparison of the left and right sides of (1) reveals,

$$-A\sin\theta = a \tag{2}$$

$$A\cos\theta = b$$

$$\tan\theta = -\frac{a}{b}$$

$$\theta = \tan^{-1}\left(-\frac{a}{b}\right) \tag{3}$$

$$\sin\theta = \pm\frac{a}{\sqrt{a^2+b^2}} \tag{4}$$

Substitute (4) into (2).

$$-A\left(\pm\frac{a}{\sqrt{a^2+b^2}}\right) = a$$

$$A = \pm\sqrt{a^2+b^2}$$

Thus, $a\sin x + b\cos x = \pm\sqrt{a^2+b^2}\cos\left(x + \tan^{-1}\left(-\frac{a}{b}\right)\right)$.

Section Challenge Problems Page 537 Question 3

$$\begin{aligned}\frac{\sin x+\sin 2x+\sin 3x}{\cos x+\cos 2x+\cos 3x} &= \frac{\sin x+\sin 2x+(\sin x\cos 2x+\cos x\sin 2x)}{\cos x+\cos 2x+(\cos 2x\cos x-\sin 2x\sin x)}\\ &= \frac{\sin x+\sin 2x+\sin x(2\cos^2 x-1)+2\cos^2 x\sin x}{\cos x+\cos 2x+(2\cos^2 x-1)\cos x-\sin 2x\sin x}\\ &= \frac{\sin x+\sin 2x+4\cos^2 x\sin x-\sin x}{\cos x+\cos 2x+2\cos^3 x-\cos x-2\sin^2 x\cos x}\\ &= \frac{\sin 2x+2\sin x\cos x(2\cos x)}{\cos 2x+2\cos x(\cos^2 x-\sin^2 x)}\\ &= \frac{\sin 2x+\sin 2x(2\cos x)}{\cos 2x+\cos 2x(2\cos x)}\\ &= \frac{\sin 2x(1+2\cos x)}{\cos 2x(1+2\cos x)}\\ &= \tan 2x\end{aligned}$$

Section Challenge Problems Page 537 Question 4

Let θ be the acute angle of intersection between the two lines. The slopes of the two lines are $m_1 = 2$ and $m_2 = -4$.

$$\begin{aligned}\tan\theta &= \frac{m_1-m_2}{1+m_1m_2}\\ &= \frac{2-(-4)}{1+(-8)}\\ &= -\frac{6}{7}\\ \theta &= \tan^{-1}\left(-\frac{6}{7}\right)\\ &\doteq 40.6^\circ\end{aligned}$$

The angle of intersection between the two lines is approximately 40.6°.

Section Challenge Problems Page 537 Question 5

Equating the curves reveals a point of intersection at $(2, 8)$. The slope of $y = x^3$ at $x = 2$ is $m_1 = 3(2)^2$ or 12. The slope of $y = x^2 + 4$ at $x = 2$ is $m_2 = 2(2)$ or 4. Let θ be the acute angle of intersection between the two lines.

$$\begin{aligned}\tan\theta &= \frac{m_1-m_2}{1+m_1m_2}\\ &= \frac{12-4}{1+(12)(4)}\\ &= \frac{8}{49}\\ \theta &= \tan^{-1}\left(\frac{8}{49}\right)\\ &\doteq 9.27^\circ\end{aligned}$$

The angle of intersection between the two curves is approximately 9.27°.

Section Challenge Problems Page 537 Question 6

$$\tan(A+B+C) = \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

Since $A + B + C = 180°$,

$$\begin{aligned}\tan 180^\circ &= \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}\\ 0 &= \tan(A+B)+\tan C\\ 0 &= \frac{\tan A+\tan B}{1-\tan A\tan B}+\tan C\\ -\tan C &= \frac{\tan A+\tan B}{1-\tan A\tan B}\\ -\tan C(1-\tan A\tan B) &= \tan A+\tan B\\ -\tan C+\tan A\tan B\tan C &= \tan A+\tan B\\ \tan A\tan B\tan C &= \tan A+\tan B+\tan C\end{aligned}$$

Section Challenge Problems Page 537 Question 7

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} &= \lim_{x\to 0}\left[\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}\times\frac{\sqrt{1+\tan x}+\sqrt{1+\sin x}}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}\right]\\
&= \lim_{x\to 0}\frac{1+\tan x-1-\sin x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\\
&= \lim_{x\to 0}\frac{\dfrac{\sin x}{\cos x}-\sin x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\\
&= \lim_{x\to 0}\frac{\sin x\left(\dfrac{1}{\cos x}-1\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\\
&= \lim_{x\to 0}\frac{\sin x\left(\dfrac{1-\cos x}{\cos x}\right)\left(\dfrac{1+\cos x}{1+\cos x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\\
&= \lim_{x\to 0}\frac{\dfrac{\sin x(1-\cos^2 x)}{\cos x(1+\cos x)}}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\\
&= \lim_{x\to 0}\left[\frac{\sin^3 x}{x^3}\times\frac{1}{\cos x(1+\cos x)}\times\frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}\right]\\
&= 1\times\frac{1}{2}\times\frac{1}{2}\\
&= \frac{1}{4}
\end{aligned}$$

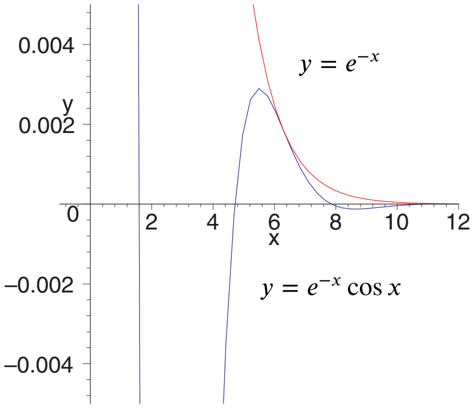
Section Challenge Problems Page 537 Question 8

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sin\dfrac{2}{x}}{\sin\dfrac{1}{x}} &= \lim_{x\to\infty}\frac{\sin\dfrac{2}{x}}{\sin\dfrac{1}{x}}\cdot\frac{x}{x}\\
&= \frac{\lim\limits_{x\to\infty} x\cdot\sin\dfrac{2}{x}}{\lim\limits_{x\to\infty} x\cdot\sin\dfrac{1}{x}}\\
&= \frac{\lim\limits_{\frac{1}{x}\to 0}\dfrac{\sin\frac{2}{x}}{\frac{1}{x}}\cdot\dfrac{2}{2}}{\lim\limits_{\frac{1}{x}\to 0}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}}\\
&= \frac{2\lim\limits_{\frac{2}{x}\to 0}\dfrac{\sin\frac{2}{x}}{\frac{2}{x}}}{\lim\limits_{\frac{1}{x}\to 0}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}}\\
&= \frac{2\cdot 1}{1}\\
&= 2
\end{aligned}$$

Section Challenge Problems Page 537 Question 9

Determine where the curves intersect.

$$\begin{aligned} e^{-x} &= e^{-x}\cos x \\ e^{-x}(1-\cos x) &= 0 \\ \cos x &= 1 \\ x &= 2n\pi;\ n \in I \end{aligned}$$



The curves intersect at $x = 2n\pi$, where $n \in I$. Determine the slopes of the curves at the intersection points.

$$\frac{de^{-x}}{dx} = -e^{-x} \tag{1}$$

$$\begin{aligned} \frac{de^{-x}\cos x}{dx} &= \cos x \cdot -e^{-x} + e^{-x} \cdot -\sin x \\ &= -e^{-x}(\sin x + \cos x) \end{aligned} \tag{2}$$

Since $\sin x + \cos x = 1$ for $x = 2n\pi$, (1) and (2) are equal and the curves are tangent to one another at their intersection points.

Section Challenge Problems Page 537 Question 10

$$E = I\frac{dI}{dt} + rI \tag{1}$$

Substitute $I = I_0\cos(\omega t + b)$ into (1).

$$\begin{aligned} E &= I_0\cos(\omega t + b) \cdot \frac{dI_0\cos(\omega t + b)}{dt} + rI_0\cos(\omega t + b) \\ &= I_0\cos(\omega t + b) \cdot (-I_0\sin(\omega t + b) \cdot \omega + rI_0\cos(\omega t + b) \\ &= I_0\cos(\omega t + b)(r - I_0\omega\sin(\omega t + b)) \end{aligned}$$