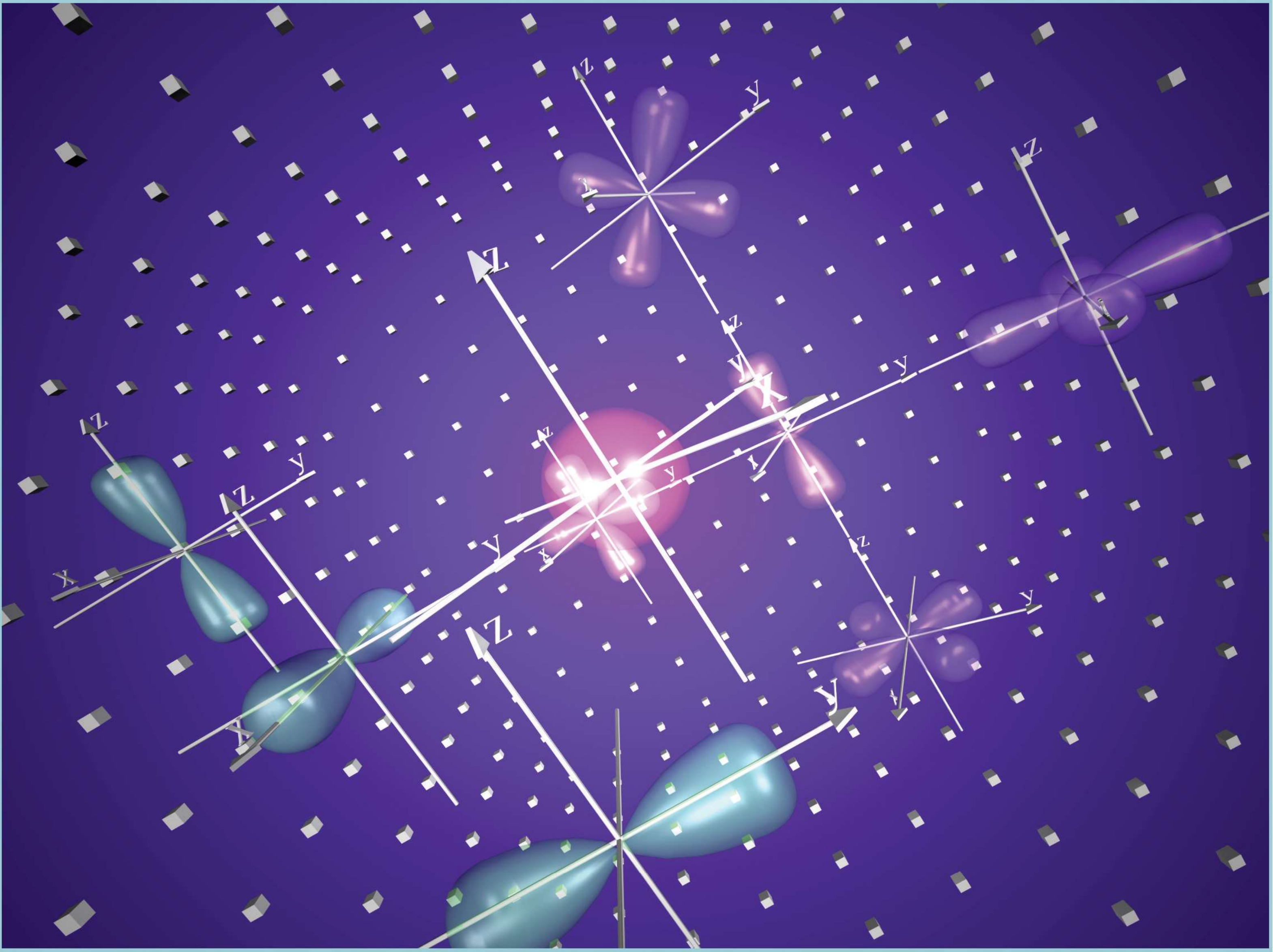
Math-Ordinate 3D



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Math-Ordinate 3D

Vector Analysis

and

3 Dimensions Analytical Geometry

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Dedicated to the Students

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Authors

Vector Analysis

Chapter 1

Vector quantities, Addition and substation, Definition, representation; like and unlike vectors, unit vectors, null vector, collinear vectors, coplanar vector, co-initial vector. Scalar multiplication of vectors, addition of vector sum is commutative $l - m$ theorem Solved examples.

Chapter 2

Resultant of vector quantities - Resultant of vector in two directions. Resultant of vector having same origin. Examples. Resolution of vector in three non-coplanar vectors, examples. Linear combination of vectors. Linearly, independent vectors, collinear co-planer/ Examples.

Chapter 3

Product of vectors – Dot product of two vectors. Dot product is commutative. If distributes over addition of vectors. Examples based on it. Vector of vectors base on it. Vector product or cross product of vectors. If is not commutative, but distributes over addition of vectors, area of parallelogram, triangle etc. Moment of a force about a point, about a straight line. Moment of a couple. Lagranges identity . Examples properties. Its geometrical representation examples, vector triple product illustrations and examples.

Chapter 4

Vector equations of Straight lines and planes. Equation of a straight line passing through a point and parallel to a given vector $\rightarrow$ passing through two points. Equation of angle bisector of an angle. Example. Equation of a plane through a point and parallel to two non-parallel planes. Equation of plane with three points. Equation of straight line and plane is product form. Distance of a plane from origin. Distance of a point from a given plane.

Chapter 5

Formulae and Concepts at a glance

3 Dimensions Analytical Geometry

Chapter 6

Co-ordinates of point in space Distance between two points, section formula. Direction cosines of a straight line, relation between them. Directing ratios. Projection of a line segment upon a straight line. Angle between two lines straight line through two points. Equation of angle bisector. Length of perpendicular from a points on a straight line Shortest distance between two straight lines. Equation of straight line through point of intersection of two straight lines.

Chapter 7

Plane : General equation of plane. One point form, Normal form; intercept form, equations of plane. Reduction of general equation of plane to these forms. Angle between two planes Equation of plane through intersection of two planes. Distance of a point from a plane. Distance between two parallel planes. Equation of angle bisectors of angle between two planes. Angle between straight line and plane. Condition for a line to lie in a plane. Line of greatest slope. Coplaner lines.

Chapter 8

Sphere : Equation of sphere, equation of sphere when ends of diameter are given. Section of sphere cut by a plane. Equation of sphere through a given circle Equation of sphere through the points of intersection of two spheres. Angle of intersection of two spheres. Orthogonal cutting of two spheres.

Chapter 9

Miscellaneous

Chapter 10

Miscellaneous Exercise

Examples – Exercise

1. Vector Quantities: Addition, Subtraction

1.1 Scalar and Vector Quantities:

In our daily life we come across two kinds of physical quantities. One is whose quantitative value is sufficient to specify them, while the other type need more explanation.

'Volume', 'area', 'mass', 'temperature' … need only quantative specification. Their magnitude is sufficient to specify them. But displacement, 'force', 'velocity'… need some thing more for complete specification. They need both magnitude and direction for complete specification. These are called vectors, while others are called scalar.

Scalar – a physical quantity having magnitude only is called a scalar.

Vector – a physical quantity which has magnitude as well as direction is called a vector.

1.2 A vector is represented by a directed line segment

In the fig. vector $\overrightarrow{AB}$ is represented by the directed line segment AB. Length of line segment AB gives magnitude of the vector (on specified scale) and arrow shows the direction.

A is the initial point of vector $\overrightarrow{AB}$ while B is the head or terminal point of vector $\overrightarrow{AB}$. The vector from A to B is written as $\overrightarrow{AB}$ or $\overline{AB}$ i.e. an arrow or bar is placed over it $|\overrightarrow{AB}|$ indicates magnitude of the vector. Here $|\overline{AB}|$ = AB; Vector are generally denoted by small English alphabets written in bold writing, like $\vec{a},\vec{b},\vec{c},\vec{d}$…. etc. Greek letters α,β,γ are also used to denote vectors.

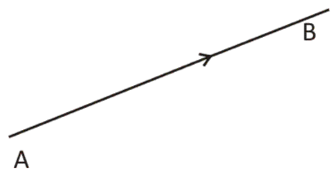


Fig 1

1.3 Modulus of a vector

The positive quantity that gives quantitative value of vector is called modulus of that vector and written as $|\overline{AB}|$ read as mod $\overline{AB}$.

1.4 Type of vectors

Vectors are of two types

(a) Free vectors (b) Localised vectors

a. **Free vectors :** These vectors are not related to a point. These can be transferred to other places i.e. points, without causing any change in their direction and magnitude.

b. **Localised Vectors :** These vectors act at definite points and if these are transferred to some other point their effect also changes. Force acting at a point is a localized vector.

1.5 Definitions

(a) **Equal vectors** : If magnitude of two vectors be equal and direction is same or parallel, then these are equal vectors.

(b) **Like Vectors** : Vectors having same direction or same support are like vectors. They may be equal or unequal $\overline{AB}$ and $\overline{CD}$ are like vectors.

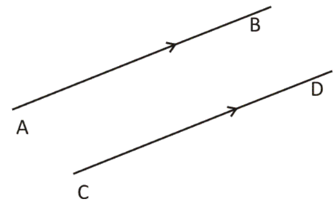


Fig 2

(c) **Unlike Vectors**: Vector having opposite direction or support are unlike vector. In the fig. $\overline{PQ}$ and $\overline{MN}$ are unlike vectors.

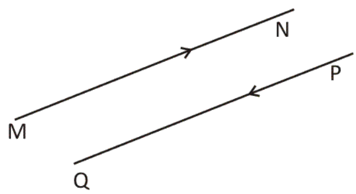


Fig 3

(d) **Unit Vector** : If the modulus of a vector is unity or one it is called unit vector. If $|\vec{a}| = a$ then $\frac{|\vec{a}|}{a}$ = unit vector, denoted by $\hat{a}$. Sign ^ is placed over unit vector.

(e) **Null vector or Zero vector** : A vector whose magnitude is zero and whose direction cannot be determined is null vector, denoted by $\vec{O}$.

(f) **Negative Vector** : This terms is applied when we examine a vector with reference to another vector. In fig. $\vec{a}$ and $\vec{b}$ are two unlike vectors. If we take the direction of $\vec{a}$ positive then $\vec{b}$ is negative vector and if direction of $\vec{b}$ is taken positive then $\vec{a}$ is a negative vector. If modulus of both $\vec{a}$ and $\vec{b}$ be equal then $\vec{a} = -\vec{b}$ or $\vec{b} = -\vec{a}$

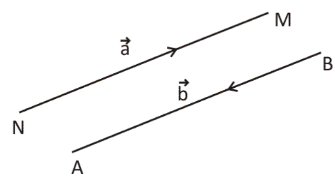


Fig 4

(g) **Collinear Vectors**: Vectors parallel to a straight line or having the same support or parallel support are collinear vectors. Their magnitude may be different.

(h) **Coplanar vector** ; All vectors that lie in a plane or are parallel to the same plane are called coplanar vectors.

(i) **Co-initial vectors** : Vectors having same initial point are called co-initial vectors. In the fig,. $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are co-initial vectors.

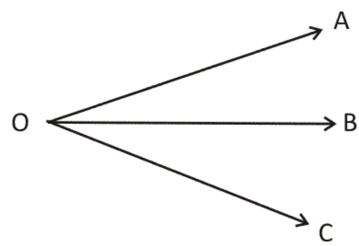


Fig 5

(j) **Displacement vector** : If a particle moves from point position A to point position B, then $\overrightarrow{AB}$ is called displacement vector.

1.6 Multiplication of Vectors by Scalar

Let $\vec{a}$ be a vector and λ (positive) a scalar then $\lambda\,\vec{a}$ is a vector whose magnitude is λ times that of $\vec{a}$ and its direction is same as of $\vec{a}$.

In case λ is negative integer then modulus of $\lambda\,\vec{a}$ is l times that of $\vec{a}$ i.e. $|\lambda\,\vec{a}| = \lambda\,|\vec{a}|$ but direction $\lambda\,\vec{a}$ shall be parallel to $\vec{a}$ but opposite in direction of $\vec{a}$. Figures shall clarify the concept.

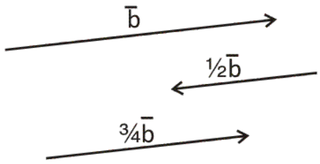


Fig 6

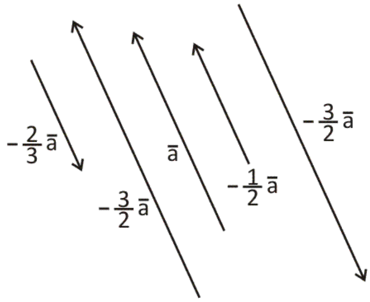


Fig 7

and $\frac{3}{2}\bar{a} = \left(1+\frac{1}{2}\right)\bar{a}$, $\quad \frac{3}{4}\bar{b} = \left(1-\frac{1}{4}\right)\bar{b}$

$\therefore \qquad (\lambda+\mu)\bar{a} = \lambda\bar{a} + \mu\bar{a}$ and $(\lambda-\mu)\bar{b} = \lambda\bar{b} - \mu\bar{b}$

and just as $\mu\,\bar{a}$ represents a like vector whose modulus is μ times of $\bar{a}$, similarly $\lambda(\mu\bar{a})$ shall represents a like vector whose modulus is λ times the modulus of $\mu\,\bar{a}$; and therefore $\lambda(\mu\,\bar{a}) = (\lambda\mu)\bar{a}$.

1.7 Addition of vectors

Just as resultant of two forces acting at a point is determined by law of parallelogram of forces or by triangle law in the same way the resultant of two vectors is determined. In the fig. $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{CD} = \vec{b}$ are two vectors.

Now OM is drawn parallel to AB and $= AB$. So that $|\overrightarrow{OM}| = \overrightarrow{AB}| = |\vec{a}|$. It represents vector $\vec{a}$. Through M, MN is drawn parallel to CD and MN = CD and as $|\overrightarrow{CD}| = |\overrightarrow{MN}| = \vec{b}$

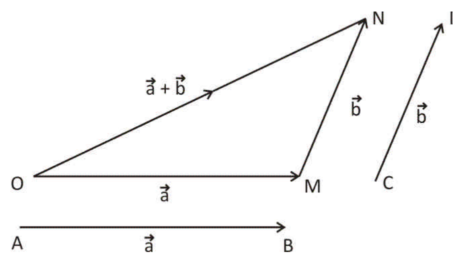


Fig 8

MN represents vector $\vec{b}$. Join ON. $\overrightarrow{ON}$ represents the sum $\vec{a}+\vec{b}$. This law of addition of vectors is called triangle law and as a side of a triangle is always less than the sum of the two remaining sides. Therefore $|\overline{ON}| < |\overline{OM}| + |\overline{MN}|$ unless they all lie in the same direction.

1.8 Vector Theorems

(a) Vector sum is commutative. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$. In the fig OABC is a paralleogram; $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{AB}=\vec{b}$. OC is parallel and = AB $\therefore$ $\overrightarrow{OC}=\vec{b}$. Similarly $\overrightarrow{CB}=\vec{a}$

from ΔOAB, $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$

i.e. $\vec{a}+\vec{b}=\overrightarrow{OB}$

from ΔOCB, $\overrightarrow{OC}+\overrightarrow{CB}=\overrightarrow{OB} \Rightarrow \vec{b}+\vec{a}=\overrightarrow{OB}$

$\therefore\ \vec{a}+\vec{b}=\vec{b}+\vec{a}$

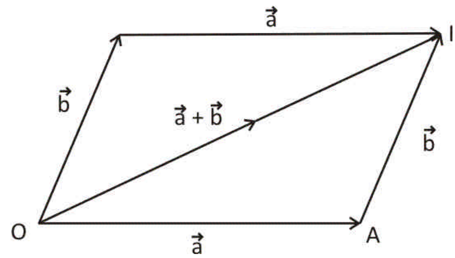


Fig 9

(b) Vector sum is associative : i.e. $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$

Proof – In the Fig. 10, ABCD is a quadrilateral. $\overrightarrow{AB}=\vec{a}$

$\overrightarrow{BC}=\vec{b}$ and $\overrightarrow{CD}=\vec{c}$

from ΔABC, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\therefore \overrightarrow{AC}=\vec{a}+\vec{b}$ ....(i)

and from ΔDBC, $\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BD}$

i.e., $\overrightarrow{BD}=\vec{b}+\vec{c}$ ....(ii)

Now from ΔACD, $\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$

i.e. $\left(\vec{a}+\vec{b}\right)+\vec{c}=\overrightarrow{AD}$ ....(iii)

and from ΔABD, $\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$ i.e.
$\vec{a}+\left(\vec{b}+\vec{c}\right)=\overrightarrow{AB}$

from 3 and 4, i.e.
$\vec{a}+\left(\vec{b}+\vec{c}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$

$\Rightarrow$ Addition of vector is associative

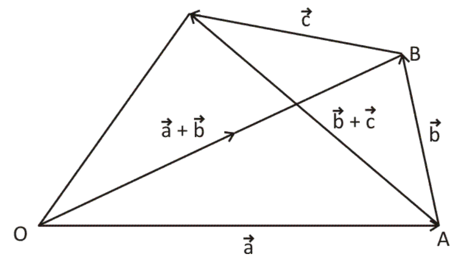


Fig 10

(c) Zero vector $\vec{O}$ is additive identity of vectors addition:

$\vec{a}+\vec{o}=\vec{o}+\vec{a}=\vec{a}$

(d) Additive inverse of vector $\vec{a}$ is $-\vec{a}$

$\because \vec{a}+\left(-\vec{a}\right)=\vec{o}=\left(-\vec{a}\right)+\left(\vec{a}\right)$

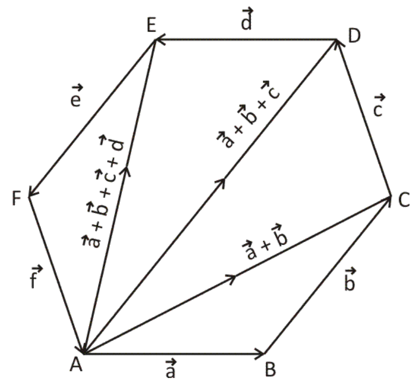


Fig 11

(e) Sum of the vectors represented by three sides of a triangle taken in order is zero.

(f) Sum of the vectors acting along the sides of a polygon and represented by the sides taken in order is zero.

From ΔAEF

$\overrightarrow{AE} + \overrightarrow{EF} = \overrightarrow{AF}$

$\Rightarrow \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} = AF = \vec{f}$

and $\overrightarrow{AF} + \overrightarrow{FA} = -\vec{f} + \vec{f} = \vec{o}$

DISTRIBUTION LAW FOR VECTORS

(g) K is a scalar, $\vec{a}$ and $\vec{b}$ are vectors; then $k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$,

i.e. Distribution law for scalar multiplication of vectors is true.

Proof : OAB and PQR as similar triangles.

$\therefore \frac{PQ}{OA} = \frac{QR}{AB} = \frac{RP}{BO}$ = k (say) scalar

If $\overrightarrow{OA} = \vec{a}, \overrightarrow{AB} = \vec{b}, OB = \vec{c}$

Then $\overrightarrow{PQ} = k\vec{a}$, $QR = k\vec{b}$ and $\overrightarrow{PR} = k\vec{c}$

And from triangle law. $\vec{a} + \vec{b} = \vec{c}$

And from ΔPCR $k\vec{a} + k\vec{b} = k\vec{c} \Rightarrow k(\vec{a}+\vec{b}) = k\vec{c}$

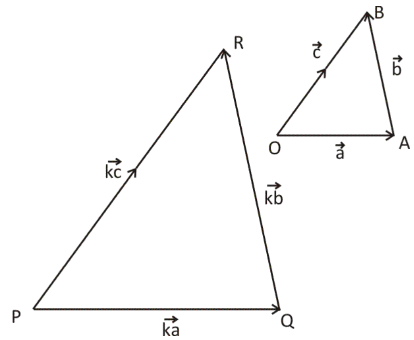


Fig 12

1.9 Position Vector

Suppose O is a fixed point, P, Q,R are any points, then vector $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ are called position vector of P, Q, and R with reference to origin O.

1.10 Subtraction of two vectors

Let vector $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$

$\therefore \overrightarrow{BO} = -\vec{b}$

from triangle BOA

$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$

$= -\vec{b} + \vec{a} \quad = \vec{a} - \vec{b}$

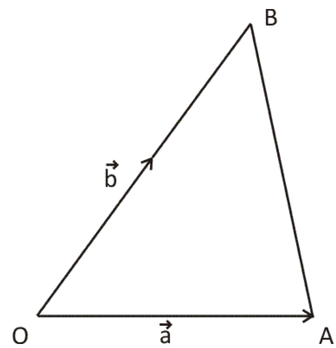


Fig 13

and $\overrightarrow{AB} = -\left(\vec{a} - \vec{b}\right) = \vec{b} - \vec{a}$

Now if we take O as origin then position vectors of A and B are $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$

And $\overrightarrow{AB} = \vec{b} - \vec{a}$ $\quad = \text{P.V. of B} - \text{P.V. of A}$

and $\overrightarrow{BA} = \vec{a} - \vec{b} = \text{P. vector of A} - \text{P. vector of B}$. It can be expressed in the following way also

$\overrightarrow{AB} = \text{P. vector of Head} - \text{P. vector of Tail}$.

1.11 Theorem I; $\lambda - \mu$ Theorem

The resultant of two vectors. $\lambda.\overrightarrow{OA} + \mu.\overrightarrow{OB}$ is $(\lambda + \mu).\ \overrightarrow{OG}$ where G is a point on AB such that AG : GB = $\mu : \lambda$.

Proof : In fig. 14; vectors $\lambda.\overrightarrow{OA}$ and $\mu\overrightarrow{OB}$ act along OA and OB respectively. G divides AB in the ratio of $\mu : \lambda$.

From ΔGOA; $\quad \lambda.\overrightarrow{OA} = \lambda.\overrightarrow{OG} + \lambda.\overrightarrow{GA}$

From ΔBOG; $\mu.\overrightarrow{OB} = \mu.\overrightarrow{OG} + \mu.\overrightarrow{GB}$

$\therefore \lambda.\overrightarrow{OA} + \mu.\overrightarrow{OB} = (\lambda + \mu)\overrightarrow{OG} + \lambda.\overrightarrow{GA} + \mu.\overrightarrow{GB}$

$\overrightarrow{GA}$ and $\overrightarrow{GB}$ are vectors in opposite directions of each other and as $\lambda\overrightarrow{OA} + \lambda\overrightarrow{OB} = (\lambda + \mu)\overline{OG}$

$\Rightarrow \lambda.\overrightarrow{GA} + \mu\overrightarrow{GB} = 0$

Which is possible only when $\lambda.|\overrightarrow{GA}| = \mu|\overrightarrow{GB}|$

i.e. $\dfrac{GA}{GB} = \dfrac{\mu}{\lambda} \Rightarrow$ G divides AB in $\mu : \lambda$

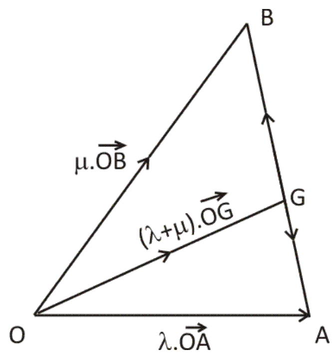


Fig 14

Corollary I : The resultant of two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ shall be $(1+1)\overrightarrow{OC}$ where c is mid point of AB.

Corollary II : $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B. If a point P is on Abdivides AB in the ratio of p : q then the position of P is $\dfrac{q\vec{a} + p\vec{b}}{a + b}$. Origin being the same.

Proof : Fig. (15) has been drawn. Let $\overline{v}$ be the position vector of P.

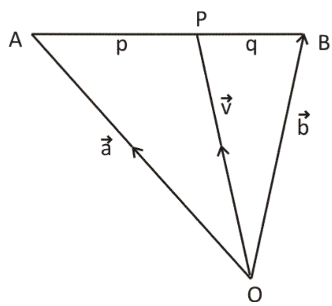


Fig 15

Then $\bar{v} = \vec{a} + \overrightarrow{AP} \Rightarrow \overrightarrow{AP} = \vec{v} - \vec{a}$

Similarly $BP = \bar{v} - \bar{b}$

But $\frac{AP}{PB} = \frac{p}{q} \Rightarrow q.\overrightarrow{AP} = p.\overrightarrow{PB} \qquad \Rightarrow q(\vec{v} - \vec{a}) = p.(\vec{b} - \vec{v})$

$\Rightarrow (q+p)\vec{v} = q\vec{a} + p.\bar{b} \Rightarrow \vec{v} = \frac{q\vec{a} + p\vec{b}}{p+q}$

$\therefore$ The p.v. of a point P which divides internally the line segment joining A(p.v. $\bar{a}$) and B(p.v. $\vec{b}$) in the ratio of p : q is $(q\vec{a} + p\vec{b})/(p+q)$

(b) Similarly if P divides AB externally in the ratio of p : q then p.v. of P is $\frac{p\vec{b} - q\vec{a}}{p-q}$, $\bar{a}$ and $\bar{b}$ are p. vectors of A and B.

Note : See similarity with co-ordinate geometry.

Conclusions (1) If $\vec{a}$ and $\vec{b}$ be p.v. of point A and B then $\frac{\vec{a}+\vec{b}}{2}$ is p.v. of mid point of AB.

(2) We have seen that p.v. of point P in AB

$$\vec{r} = \frac{q.\vec{a} + p\vec{b}}{p+q} = \left(\frac{q}{p+q}\right)\vec{a} + \frac{p}{(p+q)}.\vec{b} = \lambda\vec{a} + \mu\vec{b}$$

where $\lambda + \mu = \frac{q}{p+q} + \frac{p}{p+q} = 1$.

$\therefore$ Position vector of any point on AB can be taken as $\lambda\vec{a} + \mu\vec{b}$ where $\lambda + \mu = 1$

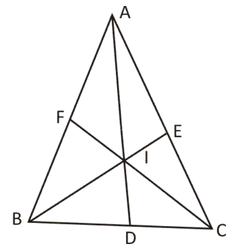


Fig 16(a)

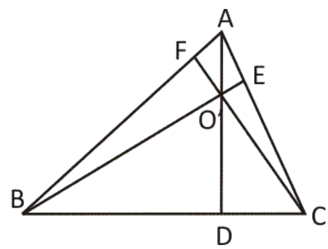


Fig 16(b)

1.12 Some important Properties of Triangle

(a) In the Fig (16 a), I is in center of ΔABC i.e. the internal bisector of angles. A, B, and C meet at I. Remember. $\frac{BD}{DC} = \frac{c}{b}$ and $\frac{AI}{ID} = \frac{b+c}{a}$.

(b) The points of intersection of altitudes of a triangle is orthocentre O, In fig. 16(b) AD, BE and CF are altitudes,

Here $\frac{BD}{DC} = \frac{\tan c}{\tan B}$ and $\frac{AO'}{OD} = \frac{\tan B + \tan C}{\tan A}$

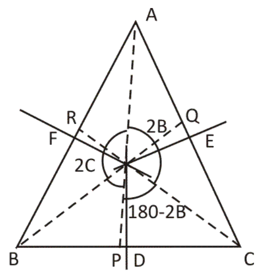


Fig 16(c)

(c) The right bisectors of the sides of a triangle are concurrent. These meet in point O, called the circumcentre of the triangle ABC, AO, BO, and CO on extension meets sides BC, AC, AB in P, Q, R.

Here BP/PC = $\frac{\sin 2c}{\sin 2B}$

And AO/OP = $\frac{\sin 2B + \sin 2C}{\sin 2A}$.

OA = OB = OC = radius of the circumcircle of triangle ABC.

Solved Example

Example 1. Prove that the straight line segment joining mid points of diagonals of a trapezium is parallel to parallel sides and equal to half of their difference.

Sol. : In the Fig. ABCD is a trapezium. DC is parallel to AB. P and Q are mid points of diagonals. AC and BD.

Let position vectors of A, B, C and D with reference to origin O be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $DC = \vec{c} - \vec{d}$

From ΔAOC, $\overrightarrow{OP} = \frac{1}{2}(\vec{a} + \vec{c})$ and from ΔABD, $\overrightarrow{OQ} = \frac{1}{2}(\vec{b} + \vec{d})$

$\overrightarrow{PQ} = \text{P.V. of } Q - \text{P.V. of } P = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \frac{1}{2}(b + d) - \frac{1}{2}(\vec{a} + \vec{c})$$

$$= \frac{1}{2}(\vec{b} - \vec{a}) - \frac{1}{2}(\vec{c} - \vec{d})$$

$$= \frac{1}{2}\left[\overrightarrow{AB} - \overrightarrow{DC}\right]$$

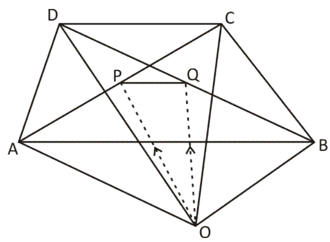


Fig 17

and as AB||DC, (AB – DC) is ||AB.

$\therefore$ PQ || AB and = $\frac{1}{2}(AB - DC)$

Example 2. The mid points of sides of a quadrilateral are joined in order. Prove that the figure so formed is a parallelograms.

Sol. In the fig. ABCD is a quadrilateral and P, Q, R and S are mid points of sides AB, BC, CD and DA. Let position vectors of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$.

Then $\overrightarrow{OP} = \frac{1}{2}(\vec{a} + \vec{b})$, $\overrightarrow{OQ} = \frac{1}{2}(\vec{b} + \vec{c})$

$\overrightarrow{OR} = \frac{1}{2}(\vec{c} + \vec{d})$ $OS = \frac{1}{2}(\vec{d} + \vec{a})$

Now $\overrightarrow{PQ} = \text{P.V. of } Q - \text{P.V. of } P = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \frac{1}{2}(\vec{b} + \vec{c}) - \frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2}(\vec{c} - \vec{a})$$

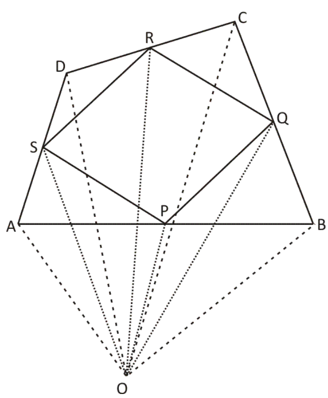


Fig 18

$\overrightarrow{SR} = \text{P.V. of R- P.V. of S} = \overrightarrow{OR} - \overrightarrow{OS}$

$$=\frac{1}{2}(\vec{c}+\vec{d})-\frac{1}{2}(\vec{d}+\vec{a})=\frac{1}{2}(\vec{c}-\vec{a})$$

$\therefore PQ \,||\, SR$ and $|\overrightarrow{PQ}|=|\overrightarrow{SR}| \Rightarrow PQ = SR$

$\therefore$ PQRS is a parallelogram.

Example 3 : AD, B E and CF are medians of a triangle. Prove that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=0$

Sol. Figure 19 has been drawn.

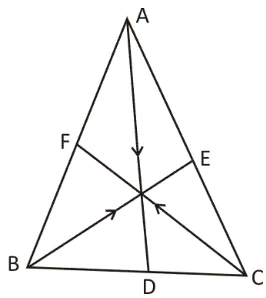


Fig 19

From ΔABD

$$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$$

But $\overrightarrow{BD}=\frac{1}{2}\overrightarrow{BC}$

$\therefore \overrightarrow{AD}=\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}$

from ΔEBC, $\overrightarrow{BE}=\overrightarrow{BC}+\overrightarrow{CE}=\overrightarrow{BC}+\frac{1}{2}\overrightarrow{CA}$

from ΔAFC, $\overrightarrow{CF}=\overrightarrow{CA}+\overrightarrow{AF}=\overrightarrow{CA}+\frac{1}{2}\overrightarrow{AB}$

$\therefore \overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\frac{3}{2}(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA})$

$$=\frac{3}{2}(\overrightarrow{AC}+\overrightarrow{CA})=\frac{3}{2}(\overrightarrow{AC}-\overrightarrow{AC})=0$$

Example 4 : ABCD is a quadrilateral. Prove $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}=4\overrightarrow{PQ}$ where P and Q are mid points of diagonal. AC and BD.

Sol. Quadrilateral has been drawn. Join mid point Q of BD with A and C

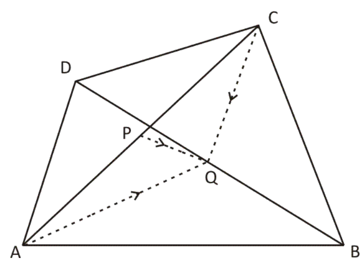


Fig 20

from ΔDAB $\overrightarrow{AD}+\overrightarrow{AB}=2\overrightarrow{AQ}$

from ΔCDB, $\overrightarrow{CB}+\overrightarrow{CD}=2\overrightarrow{CQ}$

from ΔAQC

$=2\overrightarrow{AQ}+2\overrightarrow{CQ}=(2+2)\overrightarrow{PQ}=4\overrightarrow{PQ}$

Examples 5 : O is circumcentre and O'orthocentre of triangle ABC, then prove.

(a) $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OO'}$

(b) $\overrightarrow{O'A}+\overrightarrow{O'B}+\overrightarrow{O'C}=2\overrightarrow{O'O}$

(c) $\overrightarrow{AO'}+\overrightarrow{O'B}+\overrightarrow{O'C}=AP$ where AP is diameter of circumcircles.

Sol. In Fig. 21 a point S has been taken inside the triangle PQR., at S $\overrightarrow{SP},\overrightarrow{SQ}$ and $\overrightarrow{SR}$ act. M is the mid point of QR, (Fig. 21(ii))

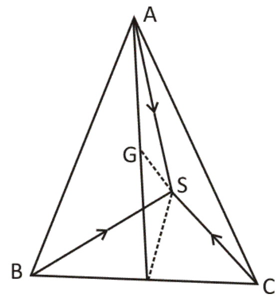


Fig 21(i)

$\therefore \overrightarrow{SQ}+\overrightarrow{SR}=2\overrightarrow{SM}$

Now consider Δ PSM; $\overrightarrow{SP}+2\overrightarrow{SM}=(1+2)SG'$

Where G' shall divided PM internally in the ratio of 2 : 1; and as PM is the median $\therefore$ G' is centroid of triangle

$\therefore \overrightarrow{SP}+\overrightarrow{SQ}+\overrightarrow{SR}=3SG$

Now if we replace S by circumcentre O, then

$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=3\overrightarrow{OG}$

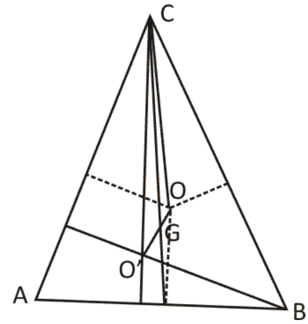


Fig 21(ii)

and as orthocentre O, centroid G and circumcentre O are collinear i.e. are in the same straight line and

$O'G:GO=2:1 \quad \Rightarrow OG=\frac{1}{3}OO'$

$\therefore \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=3\overrightarrow{OG}=\overrightarrow{OO'}$

and similarly $O'A+O'B+O'C=3\overrightarrow{O'G}=3\left(\frac{2}{3}3\overrightarrow{O'O}\right)=2\overrightarrow{OO'}$

(c) $AO'+O'B+O'C=2AO'+(O'A+O'B+O'C)$

$=2AO'+2O'O$

$=2(AO'+O'O)$

$= 2\ AO \quad (\Delta O'AO)$

and AO is radius of circumcircle of Δ

$\therefore \overrightarrow{AO'}+\overrightarrow{O'B}+\overrightarrow{O'C}=\overrightarrow{AP}$ (diameter of circumcircle)

Example 6 : P.V. of A is $\vec{a}+2\vec{b}$ and P is $\vec{a}$, and P divides AB internally in the ratio of 2 : 3. Find P.V. of B.

Sol. Let P.V. of P be $\bar{r}$; P divides AB in the ratio 2 : 3

$\therefore$ P.V. of P, $\vec{a}=\dfrac{2.\vec{r}+3(\vec{a}+2\vec{b})}{2+3}$

$\Rightarrow 5\vec{a}=2\vec{r}+3\vec{a}+6\vec{b}$

$\Rightarrow \vec{r}=\bar{a}-3\bar{b}$

Example 7 : A, B, C, D, E. F are vertices of a regular hexagon. Prove that $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}=6\overrightarrow{AG}$ where G is center of hexagon.

Sol. ABCDEF is a regular hexagon. Let its side be AC = AE = 2 a cos 30 = $\sqrt{3}$ a

$AD=2a$ and $GA=a$

Where G is mid point of AD and center of hexagon. FB meets AD in P, P is mid point of FD. Similarly Q is mid point of EC.

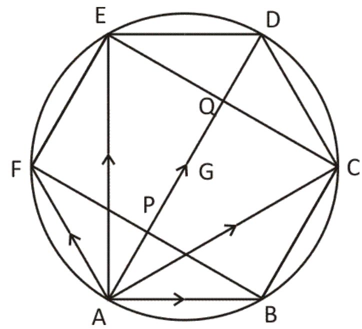


Fig 22

AP = AB sin 30 = $\frac{a}{2}=QD\Rightarrow AQ=\frac{3}{2}a$.

From ΔFAB, $\overrightarrow{AB}+\overrightarrow{AF}=2\overrightarrow{AP}$

From ΔEAC, $\overrightarrow{AC}+\overrightarrow{AE}=2AQ$

$\therefore$

$$\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}=\left(\overrightarrow{AB}+\overrightarrow{AF}\right)+\left(\overrightarrow{AC}+\overrightarrow{AE}\right)+\overrightarrow{AD}$$

$$=2\overrightarrow{AP}+2\overrightarrow{AQ}+2\overrightarrow{AG}=\overrightarrow{AG}+3\overrightarrow{AG}+2\overrightarrow{AG}$$

$$=6\overrightarrow{AG}$$

Example 9 : $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are position vectors of A, B, C and D. $2\bar{a}+3\bar{b}=3\bar{c}+2\bar{d}$. Prove that AB and CD intersect.

Sol. Given $2\bar{a}+3\bar{b}=3\bar{c}+2\bar{d}$ and $2+3=3+2\Rightarrow\frac{2\bar{a}+3\bar{b}}{5}=\frac{3\bar{c}+2\bar{d}}{5}$

$$\Rightarrow\frac{2\bar{a}+3\bar{b}}{3+2}=\frac{3\bar{c}+2\bar{d}}{2+3}=\bar{e}$$

and $\bar{e}$, divides line segment joining $\bar{a}$ and $\bar{b}$ in the ratio of 3 : 2.

$\bar{e}$ also divides line segment joining $\bar{c}$ and $\bar{d}$ in the ratio of 2 : 3.

$\therefore$ AB and CD intersect.

Examples 9 : Scalars p, q, r are such that p + q + r = 0, not all equal to zero, and $p\vec{a}+q\vec{b}+r\bar{c}=0$, where $\bar{a},\bar{b},\bar{c}$ are .. position vector of A, B and C. Prove A, B, C are collinear.

Sol. P + q + r = 0 $\Rightarrow r=-p-q\Rightarrow p\vec{a}+q\vec{b}+r\bar{c}=p\vec{a}+q\vec{b}-p\bar{c}-q\bar{c}=0$

$$\Rightarrow(p+q)\bar{c}=p\bar{a}+q\bar{b}$$

$$\Rightarrow\bar{c}=\frac{p\bar{a}+q\bar{b}}{p+q}$$

$\Rightarrow$ C divides AB in the ratio of q : p internally.

$\therefore$ A,B, C are collinear, c lies on line segment AB.

Example 10 : Forces $\alpha.\overrightarrow{AB}, \beta\overrightarrow{CB}, \gamma\overrightarrow{CD}$ and $\delta\overrightarrow{AD}$ act along sides AB, CB, CD and AD respectively of a quadrilateral ABCD and are in equilibrium. Prove $\alpha\gamma = \beta\delta$.

Sol.: Suppose $\alpha\overrightarrow{AB}$ and $\delta.\overrightarrow{AD}$, acting along AB and AD respectively = $(\alpha+\delta)$. $\overrightarrow{AP}$ where P is point in BD such that BP : PD = $\delta : \alpha$ and similarly $\beta\overrightarrow{CB} + \gamma\overrightarrow{CD} = (\beta+\gamma)\overrightarrow{CQ}$ where Q is a point in DB such that $BQ : QD = \gamma : \beta$

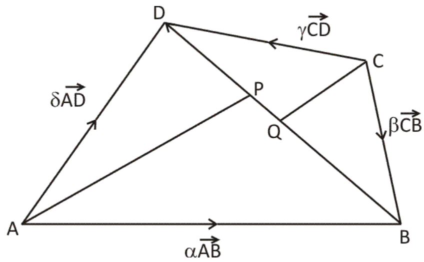


Fig 23

But forces are in equilibrium.

$\therefore$ P and Q should coincide, APC should be one line.

And $\therefore$ BQ : QD = BP : PD

$$\Rightarrow \gamma : \beta = \delta : \alpha \Rightarrow \frac{\gamma}{\beta} = \frac{\delta}{\alpha}$$

$\therefore \delta\gamma = \beta\delta$.

Examples 11 : Two chords AB and CD of a circle intersect at right angle at P. Show that $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 2\overline{PO}$ where O is the center of circle.

Sol. : AB and CD chords of circle whose center is O intersect at right angles at P.

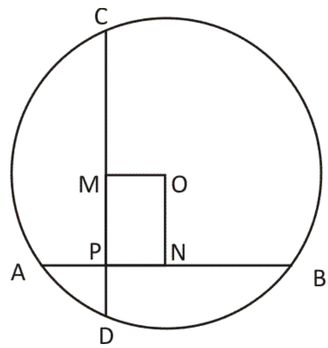


Fig 24

ON and OM are $\perp$ from O and AB and CD

$\Rightarrow$ N and M are midpoints of AB and CD.

Let $\overline{a}, \overline{b}, \overline{c}, \overline{d}.\overline{m}, \overline{n}$ and $\overline{p}$ be p.v. of A, B, C, D, M, N, P with respect to origin O

$\therefore \overline{n} = (\overline{a} + \overline{b})/2, \quad \overline{m} = (\overline{c} + \overline{d})/2$ .....(i)

OMPN is a rectangle $\therefore OM = NP$.

$\therefore \overrightarrow{ON} + \overrightarrow{OM} = \overrightarrow{ON} + \overrightarrow{NP} = \overline{OP}$

$\therefore \vec{n} + \vec{m} = \overrightarrow{OP} \Rightarrow 2\overline{n} + 2\overline{m} + = \overline{a} + \overline{b} + \overline{c} + \overline{d} = 2\overrightarrow{OP}$

Now $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = (\overline{a} - \overline{p}) + (\overline{b} - \overline{p}) + (\overline{c} - \overline{p}) + (\overline{d} - \overline{p})$

$= (\overline{a} + \overline{b}) + (\overline{c} + \overline{d}) - 4\overline{p}$

But $2\overline{OP} = 2\overline{p} = \overline{a} + \overline{b} + \overline{c} + \overline{d}$

$\therefore \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 2\overline{p} - 4\overline{p} = -2\overline{p}$

$= -2\overrightarrow{OP} = 2\overrightarrow{PO}$

Example 12: Position vectors of points A, B, C and D are $\overline{a}, \overline{b}, \overline{c}$ and $\overline{d}$. If $\overline{a} - \overline{b} = 2(\overline{d} - \overline{c})$ then prove. AC and BD trisect each other.

Sol. : $\overline{a} - \overline{b} = 2(\overline{d} - \overline{c})$

$\Rightarrow \overline{a} + 2\overline{c} = \overline{b} + 2\overline{d}$

$\Rightarrow \frac{a+2c}{2+1} = \frac{b+2d}{2+1} = \bar{r}$

$\therefore$ vector r is position vector of point of intersection of AC and BD and it divides line segment AC and BD internally in the ratio of 2 : 1

Practice Worksheet (Foundation Level) – I

1) Prove that the line segment joining mid points of two sides of a triangle is parallel to the third side and equal to half of it.

2) K is any point outside the triangle ABC. The centroid of triangle is G. Prove $\overrightarrow{KA}+\overrightarrow{KB}+\overrightarrow{KC}=3\overrightarrow{KG}$

3) Prove that medians of a triangle divide each other in the ratio of 2 : 1.

4) Position vectors of points A, B, C, D are $\overline{a},\overline{b}$, $2\overline{a}+\overline{b}$ and $\overline{a}-2\overline{b}$ respectively. Find AD, $\overline{AD},\overline{AC},\overline{AB},\overline{BC}$ and $\overline{DB}$.

5) Diagonals of a parallelogram intersect at O; k is any point insider or outside of parallelogram. Prove $\overrightarrow{KA}+\overrightarrow{KB}+\overrightarrow{KC}+\overrightarrow{KD}=4\overrightarrow{KO}$.

6) Prove that line segments joining mid points of opposite sides of a quadrilateral bisected each other.

7) In quadrilateral ABCD prove that $\overline{BA}+\overline{BC}+\overline{CD}+\overline{DA}=2\overrightarrow{BA}$.

8) D and E are mid points of sides AB and AC of triangle ABC. Prove $\overrightarrow{BE}+\overrightarrow{DC}=\frac{3}{2}\overrightarrow{BC}$.

9) A, B, C, D are tour points in a plane and four vectors are $\overrightarrow{AB},\overrightarrow{BD},\overrightarrow{DC}$ and $\overrightarrow{CA}$. Prove that $\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}+\overrightarrow{CA}=\overline{O}$.

10) A, B, C, D and E are five points in a plane. At A vectors $\overline{AC},\overline{AD}$ and $\overline{AE}$ act and at B vectors $\overrightarrow{CB},\overrightarrow{DB}$ and $\overrightarrow{EB}$ act. Prove that the resultant of these is 3 $\overrightarrow{AB}$.

11) ABCDEF is a regular hexagon forces $2\overline{AB},3\overline{AC},2\overline{AD},3\overline{AE}$ and 2AF act at A. Prove that their resultant is $\frac{15}{2}\overline{AD}$.

12) ABCDEF is a regular hexagon, $\overrightarrow{AB}=\overline{a},\overline{BC}=\overline{b}$ find $\overrightarrow{CD},\overline{DE},\overline{EF}$ and $\overline{FA}$.

13) Position vector of vertices A, B, C and D of a quadrilateral is $\overline{a},\overline{b},\overline{c}$ and $\overline{d}$ and $\overline{b}-\overline{a}$ = $\overline{c}-\overline{d}$. Prove ABCD is a parallelogram.

14) Forces $2\overline{AB}$ and $3\overline{AC}$ act at A along AB and AC sides of a triangle. Resultant lies a long AP where P is a point in BC, find P and resultant.

15) AT O, forces act and are represented by the sides AB, 2BC, 2CD, DB and DA both in magnitude and direction of square ABCD. Prove that forces are in equilibrium i.e. resultant is zero.

16) Position vector of A and B w.r. to origin O are $\overline{a}$ and $\overline{b}$ and p.v. of C and D are $\frac{5\overline{a}+3\overline{b}}{8}$ and $2\overline{a}-\overline{b}$ respectively. Prove that A, B, C, D are collinear.

17) $\overline{a},\overline{b},\overline{c},\overline{d}$ are p.v. of points, A, B, C,D respectively and 4a + 5b = 6 $\overline{c}$ +3 $\overline{d}$. Prove AB and CD intersect each other.

18) $\overline{a},\overline{b},\overline{c},\overline{d}$ are p.v. of points A,B,C and D respectively such that $5\overline{c}=3\overline{a}+2\overline{b}$ and $3\overline{d}=5\overline{a}-2\overline{b}$. Prove A,B,C,D are collinear.

19) AB and CD are two equal and parallel chords of a circle, T is a point on circumference of circle such that AT = TB, show that :

$\overline{TA}+\overline{TB}+\overline{TC}+\overline{TD}=2TL$ where TL is diameter of circle.

20) P.V. of A is $\overline{a}-3\overline{b}$ and of P is $\overline{a}-\overline{b}$ where P divides line segment. AB in the ratio of 2 : 3 internally. Find P.V. of B.

21) A, B, C are points in a line and 2AB = 3BC P.V. of A,B,C are $\overline{a},\overline{b},\overline{c}$, then

(a) $\overline{b}=\frac{3\overline{c}+1\overline{a}}{5}$ (b) $c=\frac{5\overline{b}-2\overline{a}}{3}$

(c) $\overline{a}=\frac{5\overline{c}-2\overline{b}}{3}$ (d) $\overline{a}=\frac{3\overline{c}-2\overline{b}}{3}$

22) D,E,F are mid points of sides BC, CA, AB of triangle ABC, O is any point then =

(a) zero (b) $\overline{OG}$ where G is centroid

(c) 2 $\overline{OG}$ (d) $\overline{OA}+\overline{OB}+\overline{OC}$

23) O is circumcentre, O' orthocentre of a triangle ABC; G is centroid then

(a) zero (b) $\overline{OO'}$ (c) $3\overline{O'O}$ (d) $3\overline{OO'}$

24) AB and CD are two parallel chords of a x < y circle distance x and y from center O lying on opposite sides of O. M and N their mid points. If $\overline{OA}+\overline{OB}+\overline{OC}+\overline{OD}=\overline{OP}$ then P divides MN in the ratio of :

(a) y : x (b) x : y (c) x – y : y (d) none of these

25) Two points P and Q divides AB in the ratio of x :1 (x > 1) internally and externally respectively. If and be position vector of P and Q then p.v. of A is:

(a) $\frac{1}{2x}\left[(x+1)\overline{p}+(x-1)\overline{q}\right]$ (b) $\frac{1}{2}\left[\overline{p}(x+1)-(x-1)\overline{q}\right]$

(c) $\frac{1}{2x}\left(\overline{p}-x\overline{q}\right)$ (d) $\frac{1}{2}\left[x\overline{p}-\overline{q}\right]$

2. Resolution of Vector Quantities

2.1 Resolution of a vector in two given direction

$\overline{PQ} = \overline{r}$ Is a vector OX and OY are two directions.

Vector $\overline{r}$ is to be resolved in these two directions.

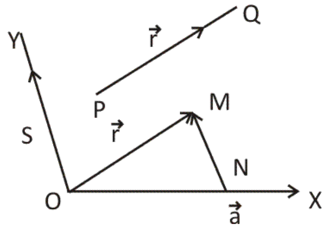


Fig 1

From O drawn OM parallel to PQ and OM = PQ. Now from M draw MN parallel to OY meeting OX in N.

$$\overrightarrow{OM} = \overrightarrow{PQ} = \overline{r} \text{ (by construction)}$$

And from triangle ONM, $\overrightarrow{ON} + \overrightarrow{NM} = \overline{OM} = \overline{r} \Rightarrow \overline{ON}$ is the resolved part in direction OX and $\overrightarrow{NM}$ is the resolved part of $\overline{r}$ in direction OY.

Now if point S be taken on OY such that OS = NM then OS || NM and equal to NM.

$\therefore$ $\overrightarrow{OS}$ is the resolved part of $\overline{r}$ along OY.

For convenience OX and OY are taken perpendicular to each other just as in co-ordinate geometry. O is origin, OX and OY are axes.

P is a point in x-y plane. PN is perpendicular from P on OX. PN || OY.

ON = distance of P from OY

= a

PN = distance of P from OX

= b

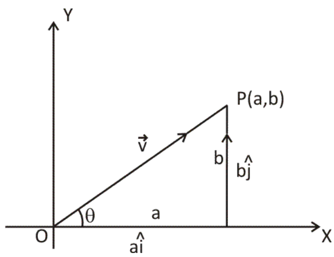


Fig 2

Point P is (a, b) ; (a, b) is an ordered pair. If $\overrightarrow{OP} = \vec{v}$ then, $\overrightarrow{ON} + \overrightarrow{NP} = \overline{v}$

If $\hat{i}$ be unit vector along OX and $\hat{j}$ unit vectors along OY then $\overrightarrow{ON} = a\hat{i}$ and $\overrightarrow{NP} = b\hat{j}$

And $\vec{v} = a\hat{i} + b\hat{j}$

Modulus $|\overrightarrow{OP}| = |\vec{v}| = OP = \sqrt{a^2 + b^2}$

(ii) If $\angle POX = \theta$ then

ON = OP cos θ and NP = OP sin θ

$\therefore$ $\overrightarrow{ON}$ = OP cos θ = $\overline{v}$ cos θ = a $\hat{i}$

NP = OP sin θ = $\overline{v}$ sin θ = b $\hat{j}$

$$\left[\overline{OP}\cos\theta = \sqrt{a^2+b^2}\,.\frac{a}{\sqrt{a^2+b^2}}.\hat{i} = a\hat{i}\right]$$

2.2 To find resultant of vectors having same origin and write reference to two perpendicular directions OX and OY i.e. by resolving then along two perpendicular directions.

In figure (3) OX and OY are two perpendicular directions.

Vectors $\overline{OA}, \overline{OB}, \overline{OC}$ have initial point O and are inclined at θ_1, θ_2, θ_3 with OX. Resolved part of these vectors along OX are $\overrightarrow{OA}\cos\theta_1, \overrightarrow{OB}\cos\theta_2, \overrightarrow{OC}\cos\theta_3$

Resolved parts of these vectors along OY are $\overrightarrow{OA}\sin\theta_1, \overrightarrow{OB}\sin\theta_2, \overrightarrow{OC}\sin\theta_3$.

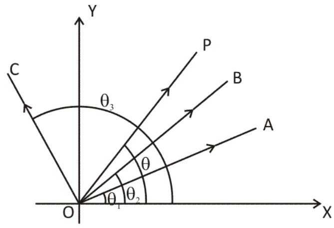


Fig 3

If OP be the resultant of these vectors and be inclined at θ with x-axis, then its resolved parts are

P cos θ along OX, P sin θ, along OY

$\therefore$ $OP\cos\theta = \overline{OA}\cos\theta_1 + \overline{OB}\cos\theta_2 + \overline{OC}\cos\theta_3$

and $\overline{OP}\sin\theta = \overline{OA}\sin\theta_1 + \overline{OB}\sin\theta_2 + \overline{OC}\sin\theta_3$

i.e. $\overline{OP}\cos\theta = \sum\overline{OA}\cos\theta_1$

$\overline{OP}\sin\theta = \sum\overline{OA}\sin\theta_1$

$\therefore$ $OP = \sqrt{\left(\sum OA\cos\theta_1\right)^2 + \sum\left(\{OA\sin\theta_1\right)^2} = \overline{r}$

And tan θ = $\dfrac{\sum\overline{OA'}\sin\theta_1}{\sum\overline{OA'}\cos\theta_1}$

Solved Example

Example 1: 5, $4\sqrt{2}$ and 10 N forces act along AB, AC and AD sides and diagonal of a square, find resultant force and its inclination.

Sol. Let i be the unit vector along AB, j unit vector along AD, then

$\overline{AB} = 5i + 0 \cdot j$

$4\sqrt{2}$ N acting along OC

$\Rightarrow 4\sqrt{2}\cos 45^o + 4\sqrt{2}\sin 45^o$

$= 4\,i + 4j$

10 N force acting along AD = $10 \cdot 0\hat{i} + 10 \cdot \hat{j}$

$\therefore$ Resultant force $\overline{F} = 5\hat{i} + 4\hat{i} + 4\hat{j} + 10\hat{j} = 9\hat{i} + 14\hat{j}$

$\therefore |\overline{F}| = \sqrt{81+196} = \sqrt{277}$ N and $\tan\theta = \frac{14}{9}$

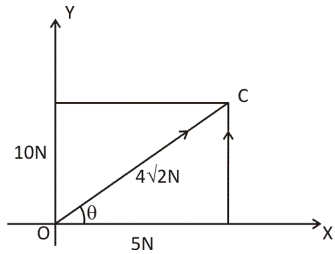


Fig 4

Example 2: Co-ordinate of points A, B and C are (–3, 4), (12, –5) and (3, 4). Forces of magnitude 8, 10, 13 and 15 N act along OY, OA, OB and OC respectively (as shown in figure). Find their resultant and its direction.

Sol. OX and OY are two perpendicular directions and $\hat{i}$ and $\hat{j}$ are unit vectors along OX and OY.

In figure $\overline{OA}$ is inclined at (180 – θ) with OX, OB is inclined at –φ with OY and OC is inclined at ψ with OX.

$\cos(180^o - \theta) = -\frac{3}{5}$, $\cos(-\phi) = \frac{12}{13}$, $\cos\psi = \frac{3}{5}$

$\sin(180^o - \theta) = \frac{4}{5}$, $\sin(-\phi) = --\frac{5}{13}$, $\cos\psi = \frac{4}{5}$

Sum of resolved parts of forces along OX is

$= (8\cos 90^o + 10\cos(180^o - \theta) + 13\cos(-\theta) + 15\cos\psi)\,\hat{i}$

$= (0 - 10 \cdot \frac{3}{5}. + 13 \cdot \left(\frac{12}{13}\right) + 15 \cdot \frac{3}{5})\,\hat{i} = 15\,\hat{i}$

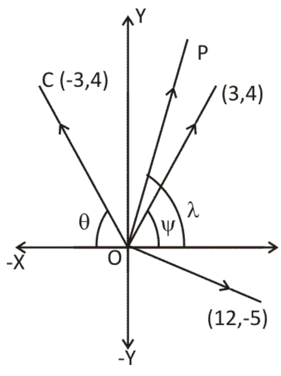


Fig 5

Sum of resolved parts along OY is

$= (8\sin 90^o + 10\sin(180^o - \theta) + 13\sin(-\theta) + 15\sin\psi)\,\hat{j}$

$= (8 + 10 \cdot \frac{4}{5} + 13\left(\frac{-5}{13}\right) + 15 \cdot \frac{4}{5})\,\hat{j} = 23\,\hat{j}$

If resultant force $\overline{P}$ acts along OP and $\angle POX = \alpha$, then

$i\overline{OP}\cos\alpha = 15\,i$; $j\,OP\sin\alpha = 23j$

$\therefore\ |OP| = \sqrt{15^2 + 23^2}N = \sqrt{754}N$ and $\tan\alpha = \frac{23}{15}$

$\therefore$ Resultant is $\sqrt{754}N$ and acts along $\tan^{-1}\frac{23}{15}$

Example 3: ABCDEF is a regular hexagon forces equal to $\overline{AB}, 2\overline{AC}, 3\overline{AD}, 2\overline{AE}$ and $\overline{AF}$ act along AB, AC, AD, AE and AF. Find resultant force and its direction.

Sol. ABCDEF is a regular hexagon. AE is $\perp$ on AB. Let $\hat{i}$ be the unit vector along AB and $\hat{j}$ be unit vector along AE.

If AB = a, then AC = $\sqrt{3}$ a, AD = 2a, AE = $\sqrt{3}$ a

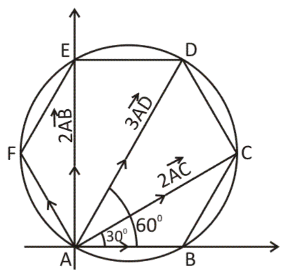


Fig 6

Let $\overline{AB} = \overline{P}$ and $|\overline{P}| = P$.

$\angle CAB = 30^o$, $\angle DAB = 60^o$, $\angle EAB = 90^o$, $\angle FAB = 120^o$.

Sum of resolved parts of forces along AB is

= (P cos 0^o + $2\sqrt{3}$ cos 30^o + 6 cos 60^o + $2\sqrt{3}$ cos 90^o + P cos 120^o) $\hat{i}$

$$= P\left(1 + 2\sqrt{3}\cdot\frac{\sqrt{3}}{2} + 6P\cdot\frac{1}{2} + 2\sqrt{3}\times 0 - \frac{1}{2}\right)\hat{i} = \frac{13\sqrt{3}}{2}P\hat{i}$$

And sum of resolved parts of forces along AE

= P(0 + $2\sqrt{3}$ sin 30^o + 6 sin 60^o + $2\sqrt{3}$ sin 90^o + sin 120^o) $\hat{j}$

$$= P\left(2\sqrt{3}\cdot\frac{1}{2} + 6\frac{\sqrt{3}}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)\hat{j} = \frac{13\sqrt{3}}{2}\hat{j}$$

$\therefore$ Resultant force = $\sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{13}{2}\right)^2}P$ $= \frac{1}{2}\sqrt{13^2 + 13^2\cdot 3P} = 13P$

Its inclination with AB, $\tan\theta = \frac{13\sqrt{3}\cdot 2}{2\cdot 13} = \sqrt{3} \Rightarrow \theta = 60^o$

$\therefore$ Resultant is 13AB and it acts along AD.

Example 4: P vectors of vertices A, B and C of triangle ABC are $\hat{i}+\hat{j}$, $-2\hat{i}-3\hat{j}$ and $4\hat{i}+5\hat{j}$. What type of triangle ABC.

Sol. $\overrightarrow{AB} = (-2\hat{i} - 3\hat{j}) - (\hat{i} + \hat{j}) = -3\hat{i} - 4\hat{j} \Rightarrow |\overrightarrow{AB}| = 5$

$\overrightarrow{BC} = (4\hat{i} + 5\hat{j}) - (-2\hat{i} - 3\hat{j}) = 6\hat{i} + 8\hat{j} \Rightarrow |\overrightarrow{BC}| = 10$

$\overrightarrow{AC} = (4\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j}) = 3\hat{i} + 4\hat{j} \Rightarrow |\overrightarrow{AC}| = 5$

AB + AC = 5 + 5 = 10 = BC, Δ does not exists.

A is mid point of BC $\Rightarrow$ A, B, C are collinear

Example 5: P.V. of A, B, C and D are $2\hat{i} + 7\hat{j}, -\hat{i} + 3\hat{j}$, $4\hat{i} - 3\hat{j}$ and $7\hat{i} + \hat{j}$. Prove ABCD is parallelogram.

Sol. ABCD shall be parallelogram if AB = CD and AB || CD and AC $\neq$ BD.

$\overline{AB} = (-\hat{i} + 3\hat{j}) - (2\hat{i} + 7\hat{j}) = -3\hat{j} - 4\hat{j} \Rightarrow |\overline{AB}| = 5$

$\overline{DC} = (4\hat{i} - 3\hat{j}) - (7\hat{i} + \hat{j}) = -3\hat{j} - 4\hat{j} \Rightarrow |\overline{DC}| = 5$

$\overrightarrow{AC} = (4\hat{i} - 3\hat{j}) - (2\hat{i} + 7\hat{j}) = 2\hat{i} - 10\hat{j}, \overrightarrow{BD} = 7\hat{i} + \hat{j} + \hat{i} - 3\hat{j} = 8\hat{i} - 2\hat{j}$

AC $\neq$ BD

And AB = and || DC $\Rightarrow$

$\therefore$ ABCD is a parallelogram.

Example 6: The position vectors of points A and B are $a\hat{i} + b\hat{j}$ and $2a\hat{i} - 3b\hat{j}$.

Find the position vector of point P which divides AB (i) Internally in the ratio of 2 : 3 (ii) Externally in the ratio of 4 : 3.

Sol. P. vectors of A $(a\hat{i} + b\hat{j})$, B($2a\hat{i} - 3b\hat{j}$)

(i) P divides internally AB in the ratio of 2 : 3

$\therefore$ P. vector of P is $\dfrac{2(2a\hat{i} - 3b\hat{j}) + 3(a\hat{i} + b\hat{j})}{5}$

i.e. $\dfrac{1}{5}(7a\hat{i} - 3b\hat{j})$

(ii) P divides AB externally in the ratio of 4 : 3

$\therefore$ P. vector of P is $\dfrac{4(2a\hat{i} - 3b\hat{j}) - 3(a\hat{i} + b\hat{j})}{4 - 3}$

i.e. $5a\hat{i} - 15b\hat{j}$

Resolution of a vector in three non-coplanar vectors

For resolution of a vector in three non-coplanar compounds in space , the space has to be divided by three non-coplanar straight lines X'OX, YOY' and ZOZ, each perpendicular to the other two and O is their common intersecting points. These straight lines are called axes, X'OX is x-axis, YOY' is y-axis and ZOZ' is z-axis, O is origin. Planes YOX and XOZ meet in OX, planes YOX and ZOY meet in OY, planes YOZ and XOZ meet in z-axis. This space is called three dimensional Euclidean space.

In the figure 6; OX, OY OZ are three mutually perpendicular lines. $\overrightarrow{AB}$ is a vector. We shall resolve this vector along OX, OY and OZ.

From O draw OP parallel to AB and equal to AB. Thus $\overrightarrow{OP}=\overrightarrow{AB}$. Now from P draw PN parallel to OZ which meets XOY plane in N. Now from N draw NM parallel to OY which meets OX in M.

Now in XOY plane $\overrightarrow{OM}+\overrightarrow{MN}=\overrightarrow{ON}$

and from triangle PON $\overrightarrow{ON}+\overrightarrow{NP}=\overrightarrow{OP}$

$\therefore\ \overrightarrow{OP}=\overrightarrow{OM}+\overrightarrow{MN}+\overrightarrow{NP}$

from rectangle OMNT in XOY plane and from rectangle LONP

$\therefore\ \overrightarrow{OT}=\overrightarrow{MN}$ and $\overrightarrow{NP}=\overrightarrow{OL}$

$\therefore\ \overrightarrow{OP}=\overrightarrow{ON}+\overrightarrow{OT}+\overrightarrow{OL}$

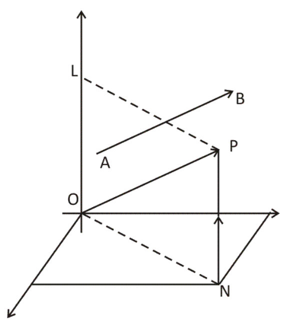


Fig 7

If point P is (x, y, z) then

x = distance of point from YOZ plane

y = distance of point from ZOX plane.

z = distance of point from XOY plane.

$\therefore$ OM = x, OT = y, OL = z

If $\hat{i}, \hat{j}, \hat{k}$ be unit vector along OX, OY and OZ and $\overrightarrow{OP}=\hat{r}$ then $\vec{r}=\overrightarrow{OM}+\overrightarrow{OT}+\overrightarrow{OL}$

$$=x\hat{i}+y\hat{j}+z\hat{k}$$

(x, y, z) are co-ordinates of the point P in three dimensional space.

From rectangle OMNT, $(ON)^2 = (OM)^2 + (MN)^2$

$\Rightarrow (ON)^2 = (OM)^2 + (OT)^2 = x^2 + y^2$

From right angled triangle ONP, $(OP)^2 = (ON)^2 + (PN)^2$

$\therefore (OP)^2 = x^2 + y^2 + z^2$

$\therefore$ Modulus of vector $\overrightarrow{OP} = \vec{x} = \sqrt{x^2 + y^2 + z^2}$

$\therefore$ $\overrightarrow{AB} = \overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

Note: In two dimension i.e. in a plane, we take axes X'OX and YOY' and each point in the plane is represented by an ordered pair (x, y). Similarly in three dimensions each point in space is represented by an ordered triad (x, y, z).

In two dimension if P is the point (a, b) the component of $\overrightarrow{OP}$ in OX direction is $\overline{a}$ and in OY direction is $\overline{b}$. Similarly if P is a point (a, b, c) in three dimensions, then $\overline{a}, \overline{b}, \overline{c}$ are compounds (resolved parts) of $\overrightarrow{OP}$, along OX, OY and OZ respectively.

Theorem 1: If two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are equation then their resolved parts shall be equal.

Proof: $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} = \vec{b} \Rightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\Rightarrow (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k} = 0$

$\Rightarrow 0 \cdot \hat{i} + 0 \cdot \hat{j} + 0 \cdot \hat{k}$

$\Rightarrow a_1 - b_1 = 0, a_2 - b_2 = 0, a_3 - b_3 = 0$

$\therefore a_1 = b_1, a_2 = b_2, a_3 = b_3$

Theorem 2: If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then $\overline{a} + \overline{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ i.e. the resolved parts of the sum of two vectors is equal to the sum of resolved parts of these vectors.

Proof: Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then $\overline{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\overline{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and as scalar product of vectors is distributive

$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

$= (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Theorem 3: If $\overline{a} = (a_1, a_2, a_3)$ and t be a scalar then $t\vec{a} = (a_1t, a_2t, a_3t)$

Proof: $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$t\vec{a} = ta_1\hat{i} + ta_1\hat{j} + ta_3\hat{k}$

$\Rightarrow \quad t\vec{a} = [ta_1, ta_2, ta_3]$

and if t = –1 then

$-\vec{a} = [-a_1, -a_2, -a_3]$

Example 7: The position vectors of A, B, and C are $2\hat{i}+3\hat{j}-\hat{k}, \hat{i}+5\hat{j}-2\hat{k}$ and $-3\hat{i}-2\hat{j}+3\hat{k}$ then find $\overrightarrow{AB}, \overrightarrow{BC}$ an $\overrightarrow{CA}$

Sol. $\overrightarrow{AB}$ = P. vector of B – P.V. of A

$= \hat{i}+5\hat{j}-2\hat{k}-2\hat{i}-3\hat{j}+\hat{k}$

$= -\hat{i}+2\hat{j}-\hat{k}$

$\overrightarrow{BC}$ = P.V. of C – P.V. of B

$= -3\hat{i}-2\hat{j}+3\hat{k}-\hat{i}-5\hat{j}+2\hat{k}=-4\hat{i}-7\hat{j}+5\hat{k}$

$\overrightarrow{CA}$ = P.V. of A – P.V. of C

$= 2\hat{i}+3\hat{j}-\hat{k}+3\hat{i}+2\hat{j}-3\hat{k}=5\hat{i}+5\hat{j}-4\hat{k}$

Example 8: $\vec{a}=4i-j+k, \vec{b}=2i+j-k$ and $\vec{c}=-i-2j+3k$. Find $2\vec{a}+3\vec{b}-4\vec{c}$

Sol. $\vec{a}=4\hat{i}-\hat{j}+\hat{k} \Rightarrow 2\vec{a}=8\hat{i}-2\hat{j}+2\hat{k}$

$\vec{b}=2\hat{i}+\hat{j}-\hat{k} \Rightarrow 3\overline{b}=6\hat{i}+3\hat{j}-3\hat{k}$

$\overline{c}=-\hat{i}-2\hat{j}+3\hat{k} \Rightarrow 4\vec{c}=-4\hat{i}-8\hat{j}+12\hat{k}$

$\therefore\ 2\vec{a}+3\vec{b}-4\vec{c}=(8\hat{i}-2\hat{j}+2\hat{k})+(6\hat{i}+3\hat{j}-3\hat{k})-(-4\hat{i}-8\hat{j}+12\hat{k})$

$=18\hat{i}+9\hat{j}-13\hat{k}$

Example 9: Position vector of points A, B, C and D are $3\hat{i}-4\hat{j}+5\hat{k}, 4\hat{i}-2\hat{j}+3\hat{k}, 2\hat{i}+5\hat{j}-3\hat{k}$ and $\hat{i}+3\hat{j}-\hat{k}$ respectively. Prove that ABCD is a parallelogram

Sol. A, $(3\hat{i}-4\hat{j}+5\hat{k})$, B $(4\hat{i}-2\hat{j}+3\hat{k})$

C, $(2\hat{i}+5\hat{j}-3\hat{k})$, D $(\hat{i}+3\hat{j}-\hat{k})$

$\overrightarrow{AB}$ = P.V. of B – P.V. of A = $4\hat{i}-2\hat{j}+3\hat{k}-3\hat{i}+4\hat{j}-5\hat{k}=\hat{i}+2\hat{j}-2\hat{k}$

$\overrightarrow{DC}$ = P.V. of C – P.V. of D = $(2\hat{i}+5\hat{j}-3\hat{k})-(\hat{i}+3\hat{j}-\hat{k})$

$= \hat{i}+2\hat{j}-2\hat{k}$

AB || DC and AB = DC

$\overrightarrow{BC}$ = P.V. of C – P.V. of B = $2\hat{i}+5\hat{j}-3\hat{k}-4\hat{i}+2\hat{j}-3\hat{k}$

$= -2\hat{i}+7\hat{j}-6\hat{k}$

$\overrightarrow{AD}$ = P.V. of D – P.V. of A = $\hat{i}+3\hat{j}-\hat{k}-3\hat{i}+4\hat{j}-5\hat{k}$

$= -2\hat{i}+7\hat{j}-6\hat{k}$

BC || AD and BC = AD

$\therefore$ ABCD is a rectangle or parallelogram

$\overrightarrow{AC}$ = P.V. of C – P.V. of A = $2\hat{i}+5\hat{j}-3\hat{k}-3\hat{i}+4\hat{j}-5\hat{k}$

$= -\hat{i}+9\hat{j}-8\hat{k}$

$\overrightarrow{BD}$ = P.V. of D – P.V. of B = $\hat{i}+3\hat{j}-\hat{k}-4\hat{i}+2\hat{j}-3\hat{k}$

$= -3\hat{i}+5\hat{j}-4\hat{k}$

$|\overline{AC}| \neq |\overline{BD}|$ $\therefore$ ABCD is a parallelogram

Example 10: P.V. of A and B are 2i + 3j – k and 3i – 2j + 3k. Find P.V. point P which divides AB internally in the ratio of 3 : 2.

Sol. P.V. of A is $2\hat{i}+3\hat{j}-\hat{k}$ and B is $3\hat{i}-2\hat{j}+3\hat{k}$ point P divides AB in the ratio of 3 : 2.

$\therefore$ P.V. of P is $\dfrac{3(3\hat{i}-2\hat{j}+3\hat{k})+2(2\hat{i}+3\hat{j}-\hat{k})}{3+2}$

i.e. $\dfrac{1}{5}[13i+0\cdot j+7k]=\dfrac{1}{5}[13i+7k]$

Example 11: P.V. of A, B, C and D, vertices of a quadrilateral are $\hat{i}+2\hat{j}+\hat{k};\ 4\hat{i}+2\hat{j}+5\hat{k};\ \hat{i}-2\hat{j}-2\hat{k}$ and $-3\hat{i}-\hat{j}+\hat{k}$ respectively. Three forces of 5N, 8N and 2N act at vertex A in directions AB, AC, and AD respectively. Find their resultant.

Sol. $\overrightarrow{AB}$ = P.V. of B – P.V. of A

$= (4\hat{i}+2\hat{j}+5\hat{k})-(\hat{i}+2\hat{j}+\hat{k}) = 3\hat{i}+4\hat{k}$

$\overrightarrow{AC}$ = P.V. of C – P.V. of A

$= (\hat{i}-2\hat{j}-2\hat{k})-(\hat{i}+2\hat{j}+\hat{k}) = -4\hat{j}-3\hat{k}$

$\overrightarrow{AD}$ = P.V. of D – P.V. of A

$= (-3\hat{i}-\hat{j}+\hat{k})-(\hat{i}+2\hat{j}+\hat{k}) = -4\hat{i}-3\hat{k}$

$|\overrightarrow{AB}|=5$, $|\overrightarrow{AC}|=5$, $|\overrightarrow{AD}|=5$ $\Rightarrow$ Unit vectors

along AB, Ac and AD are $\dfrac{3\hat{i}+4\hat{k}}{5}, \dfrac{-4\hat{j}-3\hat{k}}{5}$ and $\dfrac{-4\hat{i}-3\hat{j}}{5}$

$\therefore$ Forces acting along AB, AC, and AD are

$\frac{5}{5}(3\hat{i}+4\hat{k})$; $\frac{8}{5}(-4\hat{j}-3\hat{k})$ and $\frac{2}{5}(-4\hat{i}-3\hat{j})$

Resultant $= \frac{1}{5}\left[15\hat{i}+20\hat{k}-32\hat{j}-24\hat{k}-8\hat{i}-6\hat{j}\right]$

$= \frac{1}{5}\left[7\hat{i}-38\hat{j}-4\hat{k}\right]$

Example 12: Vertices of triangle ABC are A (2, –1, 1), B (1, –3, –5) and C, (3, –4, –4). prove by vector method that triangle is right angled.

Sol. P.V. of A, B, and C are

$2\hat{i}-\hat{j}+\hat{k};\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively

$\therefore\ \overrightarrow{AB} = (\hat{i}-3\hat{j}-5\hat{k})-(2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}$

$\overrightarrow{BC} = (3\hat{i}-4\hat{j}-4\hat{k})-(\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{CA} = (2\hat{i}-\hat{j}+\hat{k})-(3\hat{i}-4\hat{j}-4\hat{k}) = -\hat{i}+3\hat{j}+5\hat{k}$

$|\overline{AB}| = \sqrt{1+4+36} = \sqrt{41}, |\overline{BC}| = \sqrt{6}, |\overline{CA}| = \sqrt{35}$

$AB^2 = 41 = (BC)^2 + (CA)^2 = 6 + 35$

$\therefore$ Triangle is right angled triangle.

Example 13: The adjacent sides of a parallelogram is parallel to $3\hat{i}+2\hat{j}-\hat{k}$ and $\hat{i}+2\hat{j}-3\hat{k}$. Find unit vectors parallel to diagonals

Sol. AB and AD are adjacent sides of parallelogram ABCD,

$\overrightarrow{AB} = 3\hat{i}+2\hat{j}-\hat{k}; \overrightarrow{AD} = \hat{i}+2\hat{j}-3\hat{k}$

BC is parallel to AD

$\therefore\ \overrightarrow{BC} = \hat{i}+2\hat{j}-3\hat{k}$

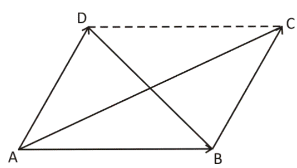


Fig 8

From ΔABD; $\overrightarrow{AB}+\overrightarrow{BD} = \overrightarrow{AD} \Rightarrow \overrightarrow{BD} = \overrightarrow{AD}-\overrightarrow{AB}$

$\therefore\ \overrightarrow{BD} = \hat{i}+2\hat{j}-3\hat{k}-3\hat{i}-2\hat{j}+\hat{k} = -2\hat{i}-2\hat{k}$

$\therefore$ Unit vector along $\overrightarrow{BD}$ is $\frac{-\hat{i}-\hat{k}}{\sqrt{2}}$

From Δ ABC, $\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC} = 3\hat{i}+2j-\hat{k}+\hat{i}+2\hat{j}-3\hat{k}$

$= 4\hat{i} + 4\hat{j} - 4\hat{k}$

$\therefore$ Unit vector along $AC = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} - \hat{k})$

Example 14 : At one corner of a cube, forces of 2, 3 and 4 kg act along the diagonals of the three faces of the cube. Take this corner of cube as origin and find the resultant.

Sol: OA, OB, OC are three conterminous edges of cube, OA, OB and OC are along axes OX, OY and OZ. The diagonal of faces are inclined at 45^o with sides.

Diagonals are OM, ON and OT, forces, $\overrightarrow{OM} = (4\hat{i} + 4\hat{j})\frac{1}{\sqrt{2}}$; $\overrightarrow{ON} = (2\hat{j} + 2\hat{k})\frac{1}{\sqrt{2}}$;

$\overrightarrow{OT} = (3\hat{i} + 3\hat{k})\frac{1}{\sqrt{2}}$

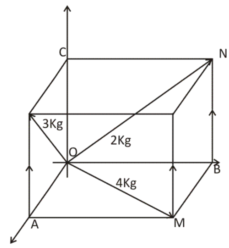


Fig 9

$\therefore$ Resultant $= \frac{1}{\sqrt{2}}\left[4\hat{i} + 4\hat{j} + 2\hat{j} + 2\hat{k} + 3\hat{i} + 3\hat{k}\right]$

$= \frac{1}{\sqrt{2}}\left[7\hat{i} + 6\hat{j} + 5\hat{k}\right]$

Magnitude = Modulus of resultant = $\sqrt{55}$ kg wt.

Example 15: The position vectors of vertices of a triangle A, B, C are $\bar{a}$, $\bar{b}$ and $\bar{c}$ and $\bar{h}$ is p.v. of orthocentre; Prove

(a) $\bar{h} = \frac{\bar{a}\tan A + \bar{b}\tan B + \bar{c}\tan C}{\tan A + \tan B + \tan C}$

(b) $\overline{HA}\tan A + \overline{HB}\tan B + \overline{HC}\tan C = \overline{O}$

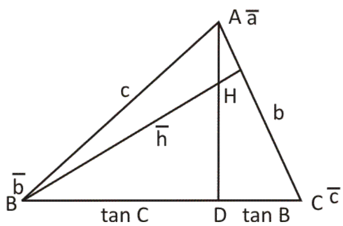


Fig 10

Sol. (a) AD is $\perp$ BC $\therefore \frac{BD}{DC} = \frac{\tan C}{\tan B}$

$\therefore$ If $\bar{d}$ p.v. of D, then

$d = \frac{\bar{c}\tan c + \bar{b}\tan B}{\tan B + \tan C}$

$\therefore$ $\bar{d}(\tan B + \tan C) = \bar{b}\tan B + \bar{c}\tan C$ ... (1)

Orthocentre H lies on AD and divides it in the ratio of $\tan B + \tan C$ = tan A

$\therefore$ $h = \frac{\bar{d}(\tan B + \tan C) + \bar{a}\tan A}{\tan A + \tan B + \tan C}$

$= \frac{(\bar{b}\tan B + \bar{c}\tan C) + \bar{a}\tan A}{\tan A + \tan B + \tan C}$ from (1)

$$= \frac{\bar{a}\tan A + \bar{b}\tan B + \bar{c}\tan C}{\tan A + \tan B + \tan C} \qquad \dots (2)$$

(b) $\overline{HA}\tan A = (\bar{a} - \bar{h})\tan A$; and

$\overline{HB}\tan B = (\bar{b} - \bar{h})\tan B$ and

$\overline{HC}\tan C = (\bar{c} - \bar{h})\tan C$

adding, $= \bar{a}\tan A + \bar{b}\tan B + \bar{c}\tan c - \bar{h}(\tan A + \tan B + \tan C)$

$= (\bar{a}\tan A + \bar{b}\tan B + \bar{c}\tan C) - (\bar{a}\tan A + \bar{b}\tan B + \bar{c}\tan C)$

$= \bar{O}$ from (2)

Example 16: $\bar{a}, \bar{b}, \bar{c}$ are p.v. of vertices A, B and C of triangle ABC. I is centre of triangle and its p.v. is I, prove. (a) $\frac{a\bar{a} + b\bar{b} + c\bar{c}}{a+b+c}$ (b) $a \cdot \overline{IA} + b \cdot \overline{IB} + c \cdot \overline{IC} = 0$

Sol. (a) AD is the angle bisector of $\angle A$ and it meets BC in D

$\therefore$ BD : DC = AB : AC = c : b

If $\bar{d}$ is p.v. of D then

$d = \frac{b\bar{b} + c\bar{c}}{b+c} \Rightarrow \bar{d}(b+c) = b\bar{b} + c\bar{c}$

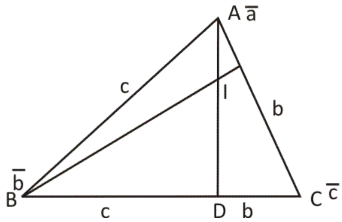


Fig 11

Incentre I lies on AD and divides it in the ratio of (b + c) : a

$\therefore \bar{I} = \frac{\bar{d}(b+c) + a \cdot \bar{a}}{a+b+c} = \frac{a\bar{a} + b\bar{b} + c\bar{c}}{a+b+c}$

= P.V. of incentre (3)

(b) $a \cdot \overline{IA} + b \cdot \overline{IB} + c \cdot \overline{IC}$

$= a(\bar{a} - \bar{I}) + b \cdot (\bar{b} - \bar{I}) + c \cdot (\bar{c} - \bar{I})$

$= a\bar{a} + b\bar{b} + c\bar{c} - (a+b+c)I = \bar{O}$ from (3)

Example 17: $\bar{a}, \bar{b}, \bar{c}$ are position vectors of vertices of triangle; A, B and C. O is circumcentre of triangle. Prove that

(a) $\bar{O} = \frac{a\sin 2A + b\sin 2B + c\sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

(b) $\overline{OA}\sin 2A + \overline{OB}\sin 2B + \overline{OC}\sin 2C = \bar{O}$

Sol. (a) In the figure OM and ON are right bisector of sides BC and AB · O is circumcentre OA = OB = OC = radius of circumcircle. $\angle$AOB = 2c, $\angle$AOC = 2B.

AO is produced to meet BC in D; $\angle$ BOD =180 – 2C and $\angle$DOC = 180 – 2B

$$\frac{BD}{DC}=\frac{\sin 2C}{\sin 2B}; \quad \text{and} \quad \frac{AO}{OD}=\frac{\sin 2B+\sin 2C}{\sin 2A}$$

If $\overline{d}$ is p.v. of D, then

$$\overline{d}=\frac{\overline{c}\cdot\sin 2C+\overline{b}\cdot\sin 2B}{\sin 2B+\sin 2C}\Rightarrow \overline{d}(\sin 2B+\sin 2C)=\overline{b}\sin 2B+\overline{c}\sin 2C$$

O divides AD in the ratio of (sin 2B + sin 2C) : sin 2A

$$\therefore\ O=\frac{\overline{d}(\sin 2B+\sin 2C)+\overline{a}\sin 2A}{\sin 2A+\sin 2B+\sin 2C}$$

$$=\frac{\overline{a}\sin 2A+\overline{b}\sin 2B+\overline{c}\sin 2C}{\sin 2A+\sin 2B+\sin 2C} \quad \ldots(\alpha)$$

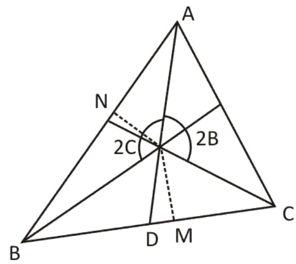


Fig 12

(b) $\overline{OA}\sin 2A+\overline{OB}\sin 2B+\overline{OC}\sin 2C$

$$=(\overline{a}-\overline{O})\sin 2A+(\overline{b}-\overline{O})\sin 2B+(\overline{c}-\overline{O})\sin 2C$$
$$=\overline{a}\sin 2A+\overline{b}\sin 2B+\overline{c}\sin 2C-(\sin 2A+\sin 2B+\sin 2C)\cdot\overline{O}$$

$$=\overline{a}\sin 2A+\overline{b}\sin 2B+\overline{c}\sin 2C-(\overline{a}\sin 2A+\overline{b}\sin 2B+\overline{c}\sin 2C)$$

= 0 from (α)

Practice Worksheet (Foundation Level) –2(a)

1. A system of coplanar forces acting on a rigid body is represented in magnitude and direction and sense by the sides of a polygon taken in order, then the system is equivalent to

   (a) single force

   (b) a couple whose moment is equal to area of polygon

   (c) zero force

   (d) a couple whose moment is equal to thrice the area of polygon

2. The position vectors of points A, B, C and D are $-2\hat{i}-\hat{j};\hat{i}-\hat{j};2\hat{i}+\hat{j}$ and $-\hat{i}+\hat{j}$ ABCD is

   (a) rectangle (b) parallelogram (c) square (d) rhombus

3. Points A, B, C and D are $3i+4j$, $-3ai-bj$, $6i-2\hat{j}$ and $3ai-4bj$. AC and BD meets in E, E divides AC in ratio of 1 : 2 and E divides BD in the ratio of 2 : 1 then

   (a) a = 3, b = 1 (b) a = 6, b = $-\frac{2}{3}$ (c) a = 4, $b=\frac{2}{3}$ (d) a = 4, $b=-\frac{2}{3}$

4. ABCD is a rectangle whose adjacent AB and BC are 4 cm and 3 cm. A body is placed at A and on it forces of 6, 10 and 8 kg weight acts along AB, AC and AD. Their resultant is

   (a) $14\sqrt{2}$ kg wt. at 45^o with AB

   (b) $14\sqrt{2}$ kg wt. at $\tan^{-1}\frac{3}{4}$

   (c) $28\sqrt{2}$ kg wt. at 45^o with AB

   (d) $7\sqrt{2}$ kg wt. $\tan^{-1}\frac{3}{4}$

5. At a point O forces 10, 8, 4 and 2 N act along OA, OB, OC, OD and $\angle$AOB = $\angle$BOC = $\angle$COD = 30^o. Find the resultant and the angle it makes with OA.

   (a) $2(\sqrt{3}+1)\sqrt{15},\tan^{-1}\frac{1}{2}$

   (b) $2(\sqrt{3}-1)\sqrt{15},\tan^{-1}2$

   (c) $\sqrt{272+88\sqrt{3}},\tan^{-1}\frac{1}{3}$

   (d) none of these

6. Unit vector along 3i – j + 5k is

   (a) $\frac{1}{\sqrt{33}}(3i-j+5k)$

   (b) $\frac{1}{\sqrt{34}}(3i-j+5k)$

   (c) $\frac{1}{\sqrt{35}}(3i-j+5k)$

   (d) none of these

7. Position vectors of point A, B, C and D are $\hat{i}+\hat{j}+\hat{k}$, $3\hat{i}-\hat{j}+5\hat{k}$, $a\hat{i}+b\hat{j}+c\hat{k}$, $5\hat{i}-2\hat{j}+3\hat{k}$ AB intersect CD at E. If E is the point of bisection of both of them, then

(a) a = –1, b = –3, c = 2 (b) a = –1, b = 2, c = 3

(c) a = –1, b = 2, c = –3 (d) a = 1, b = –2, c = –3

8. The modulus of the sum of vectors $3\hat{i}-4\hat{j}+2\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$ and $a\hat{i}+5\hat{j}+3a\hat{k}$ is $\sqrt{53}$ then a is

(a) $1,\frac{4}{5}$ (b) $-1,\frac{4}{5}$ (c) $1,-\frac{4}{5}$ (d) $-1,-\frac{4}{5}$

9. $\vec{a}=3\hat{i}+2\hat{j}-6\hat{k}$ and $\bar{b}=\sqrt{2}\hat{i}+\sqrt{7}\hat{j}-4\hat{k}$ then $5|\bar{a}|-12|\bar{a}|\cdot|\bar{b}|+4|\bar{b}|^2$ is

(a) 75 (b) –65 (c) –85 (d) –75

10. $\vec{a}=3\hat{i}-\hat{j}+2\hat{k},\vec{b}=-5\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{c}=2\hat{i}-3\hat{j}+4\hat{k}$. Modulus of $2\vec{a}-\vec{b}-3\vec{c}$ is

(a) $5\sqrt{3}$ (b) $5\sqrt{2}$ (c) 5 (d) $5\sqrt{6}$

11. Points P and Q are (1, 1, –1) and (3, –2, –1) then $\overrightarrow{PQ}$ is parallel to

(a) y – z plane (b) z – x plane (c) x – y plane (d) none of these

12. $\vec{a}=3\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{b}=m\hat{i}+2\hat{j}+2\hat{k}$ if $|\bar{a}-\bar{b}|=7$. Then m is

(a) 3, 9 (b) –3, –9 (c) –3, 9 (d) 3, –9

13. The co-ordinates of vertices A, B, C, D of a quadrilateral are (2, 3, 1), (2, –5, 1), (2, 5, 7) and (1, –5, –2) respectively, then unit vectors along diagonals and relation between the two diagonals.

(a) $\frac{1}{\sqrt{10}}(j+3k)$ and $\frac{1}{\sqrt{10}}(i-3k), AC=2BD$ (b) $\frac{2}{\sqrt{10}}(j+3k),\frac{1}{\sqrt{10}}(i-3k), AC=2BD$

(c) $\frac{1}{\sqrt{10}}(j+3k),\frac{2}{\sqrt{10}}(-i+3k), AC=\frac{1}{2}BD$ (d) None of these

14. The vectors of magnitude $\vec{a}$, $2\vec{a}$, $3\vec{a}$ meet at a vertex O of cube and their directions are along the diagonals of adjacent faces of a cube their resultant is

(a) $\frac{a}{\sqrt{2}}(3\hat{i}+4\hat{j}+5\hat{k})$ (b) $\frac{a}{\sqrt{2}}(4i+5j+3k)$

(c) $\frac{a}{\sqrt{2}}(5\hat{i}+4\hat{j}+3\hat{k})$ (d) $\frac{a}{\sqrt{2}}(4\hat{i}+3\hat{j}+5\hat{k})$

15. The adjacent sides of a parallelogram are parallel to $4\hat{i}+3\hat{j}-2\hat{k}$ and $\hat{i}+2\hat{j}-3\hat{k}$ then unit vectors parallel to diagonals are

(a) $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}-\hat{k}),\frac{1}{\sqrt{11}}(3\hat{i}+\hat{j}+\hat{k})$

(b) $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}-\hat{k}),\frac{1}{\sqrt{11}}(3\hat{i}+\hat{j}+\hat{k})$

(c) $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k}),\frac{-1}{\sqrt{11}}(3\hat{i}+\hat{j}+\hat{k})$

(d) $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}-\hat{k}),-\frac{1}{\sqrt{11}}(3\hat{i}+\hat{j}+\hat{k})$

16. Forces of 4, 6, 6, 8 and 4 Newton act at the vertex A along AB, AC, AD, AE and AF of regular hexagon ABCDEF. The resultant is

(a) $(2+3\sqrt{3})\sqrt{8}$

(b) $(3+2\sqrt{2})\sqrt{6}$

(c) $2\sqrt{62+35\sqrt{3}}$

(d) $2\sqrt{64+35\sqrt{3}}$

17. ABC is a triangle. O is any point inside or outside the triangle, prove that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$ where G is centroid of triangle.

18. The position vectors of points A, B, C and D are $2\hat{i}+4\hat{k}, 5\hat{i}+3\sqrt{3}\hat{j}+4\hat{k}, -2\sqrt{3}\hat{j}+\hat{k}$ and $2\hat{i}+\hat{k}$. Prove that $\overrightarrow{CD}$ is parallel $\overrightarrow{AB}$ and equal to two third of $\overrightarrow{AB}$.

19. The vertices of a quadrilateral ABCD are A (2, 1, –1); B (–2, 3, 4); C (4, 1, –5) and D (2, –1, 3). At the vertex A forces of 3, 2 and 4 Newton act along AB, AC and AD respectively. Show that the resultant force is $\frac{1}{\sqrt{5}}\left[-2\hat{i}-2\hat{j}+9\hat{k}\right]$

20. The diagonals of a parallelogram ABCD, intersect at O. Show that sum of position vectors of the vertices with respect to any origin is four times position vector of O with respect to that origin.

2.4 Linear combination of vectors

$\bar{a}_1, \bar{a}_2, \bar{a}_3, ..., \bar{a}_n$ are vectors $\ell_1, \ell_2, \ell_3, ..., \ell_n$ are scalars, then relation

$\ell_1\bar{a}_1 + \ell_2\bar{a}_2 + \ell_3\bar{a}_3 + ... + \ell_n\bar{a}_n$

is called a linear combination of vectors. You have seen that in three dimension, position vector of pt. (3, −1, 2) is $3\hat{i} - \hat{j} + 2\hat{k}$ which is a linear combination of three non-coplanar vectors $\hat{i}, \hat{j}$ and $\hat{k}$.

(a) If $\bar{a}$ and $\bar{b}$ are non-collinear vectors and $\bar{r}$ is a vector in plane of $\bar{a}$ and $\bar{b}$, then

$$\bar{r} = \lambda\bar{a} + \mu\bar{b} \qquad (\lambda, \mu \text{scalars} \neq 0)$$

is a unique linear combination. In figure 12, O is the initial point and A the final point of vector $\overline{OA} = r$; it lies in the plane of two non-collinear vectors $\bar{a}$ and $\bar{b}$. By drawing OM parallel to $\bar{a}$ from O, and drawing AM parallel to $\bar{b}$ from A, we get triangle OAM and $\overline{OA} = \overline{OM} + \overline{MA}$.

But OM || $\bar{a} \Rightarrow \overline{OM} = \lambda\bar{a}$, MA || $\bar{b} \Rightarrow \overline{MA} = \mu b$

$\therefore\ \bar{r} = \lambda\bar{a} + \mu\bar{b}$, unique linear combination.

(b) If it is not unique, then

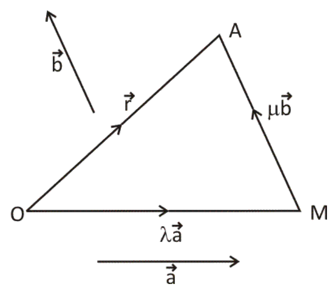


Fig 13

Let $\bar{r} = \lambda\bar{a} + \mu\bar{b}$ and

$\bar{r} = \lambda_1\bar{a} + \mu_1\bar{b}$

$\lambda \neq \lambda_1$, μ & μ_1

then $\lambda\bar{a} + \mu\bar{b} = \lambda_1\bar{a} + \mu_1\bar{b}$

$\Rightarrow (\lambda - \lambda_1)\bar{a} = (\mu_1 - \mu)\bar{b}$

$\Rightarrow \bar{a} = \left(\dfrac{\mu_1 - \mu}{\lambda - \lambda_1}\right)\bar{b}$ where $\dfrac{\mu_1 - \mu}{\lambda - \lambda_1}$ is scalar.

This shows that $\bar{a}, \bar{b}$ are collinear. Which is contradiction.

$\therefore$ This is unique linear combination.

Note: (a) $\bar{r}, \lambda\bar{a}$ and $\mu\bar{b}$ are coplanar vectors and $\bar{r} = \lambda\bar{a} + \mu\bar{b}$ goes to prove if three vectors $\bar{r}, \bar{a}$ and $\bar{b}$ are coplanar then each can be expressed as the combination of other two in a linear combination.

(b) In three dimensions if P is a point (ℓ, m, n) then position vector of P is $\ell\hat{i} + m\hat{j} + n\hat{k}$

here $\hat{i}, \hat{j}, \hat{k}$ are non-coplanar vectors. In figure (13) P (a, b, c) and $\overrightarrow{OP} = \vec{r} = \overrightarrow{OA} + \overrightarrow{AM} + \overrightarrow{MP}$

$\Rightarrow \bar{r} = a\hat{i} + b\hat{j} + c\hat{k}$

This is unique linear combination.

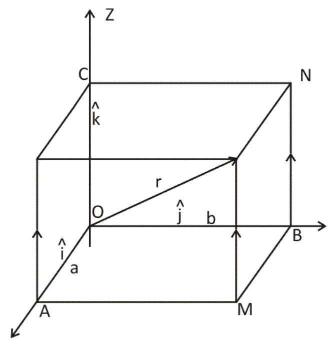


Fig 14

Proof: Suppose it is not linear unique combination and

$\bar{r} = a\hat{i} + b\hat{j} + c\hat{k}$ and $r = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

then $a\hat{i} + b\hat{j} + c\hat{k} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$\Rightarrow (a - a_1)\hat{i} = (b_1 - b)\hat{j} + (c_1 - c)\hat{k}$

$\Rightarrow \hat{i} = \left(\frac{b_1 - b}{a - a_1}\right)\hat{j} + \left(\frac{c_1 - c}{a - a_1}\right)\hat{k}; \frac{b - b}{a - a_1}, \frac{c_1 - c}{a - a_1}$ are scalars

$\therefore\ i = \lambda\hat{j} + \mu\hat{k} \Rightarrow \hat{i}, \hat{j}$ and $\hat{k}$ are coplanar

It is contradiction of what we have assumed.

$\therefore\ \bar{r} = a\hat{i} + b\hat{j} + c\hat{k}$ is a unique linear combination

2.5 Linearly dependent combination of vectors

(a) If a relation of type

$\ell_1\bar{a}_1 + \ell_2\bar{a}_2 + \ell_3\bar{a}_3 + ...\ell_n\bar{a}_n = 0$

exists between vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, ...\bar{a}_n$ = 0 and scalars $\ell_1, \ell_2, \ell_3 ... \ell_n$ are (not all equal to zero); then this relation is called linearly dependent combination of vectors.

(b) But if $\ell_1 = 0, \ell_2 = 0, ...\ell_n = 0$ i.e. all scalars are equal to zero then no such relation exists between vectors, and vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, ..., \bar{a}_n$ are called Linearly Independent Vectors. Here $\ell_1\bar{a}_1 + \ell_2\bar{a}_2 + \ell_3\bar{a}_3 + ... + \ell_n\bar{a}_n = 0 \Rightarrow \ell_1 = \ell_2 = \ell_3 = -\ell_n = 0$

2.6 Collinear vectors:

You have seen that if A, B, C are points in a straight line and their P. vectors are $\bar{a}, \bar{b}, \bar{c}$, then $\bar{b}$ may divide the distance between A and C in the ratio of m : n internally and so $\bar{b} = \lambda\bar{c} + \mu\bar{a}$ and λ + μ = 1 as $\bar{b} = \frac{m\bar{c} + n\bar{a}}{m + n} = \frac{m}{m + n} \cdot \bar{c} + \frac{n}{m + n} \cdot \bar{a}$

(ii) If $\bar{b}$ divides AB externally in the ratio of p : q

$$\bar{c} = \left(\frac{m}{m-n}\right)\bar{b} + \left(\frac{-n}{m-n}\right)\bar{a} = \lambda_1\bar{b} + \mu_1\bar{a}$$

and here too $\lambda_1 + \mu_1 = 1$

∴ If $\bar{a}, \bar{b}, \bar{c}$ are three collinear vectors, then each can be expressed as a linear combination of the other two, like $\bar{a} = \lambda\bar{b} + \mu\bar{c}$ and $\lambda + \mu = 1$

∴ If $\bar{a} + \ell\bar{b} + m\bar{c} = 0$ and $1 + \ell + m = 0$

then vectors $\bar{a}, \bar{b}, \bar{c}$ are collinear.

Note: Given position vectors of points A, B an C find $\overline{AB}$ and $\overline{BC}$. Points shall be collinear if $\frac{\overline{AB}}{\overline{BC}} = \lambda$ (integer or rational).

2.7 Condensation for four points shall be coplanar

If $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ are position vectors of four points A, B, C and D, find $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$; put

$$\overline{AB} = \lambda\overline{BC} + \mu\overline{CA} \quad \ldots (\alpha)$$

If vectors $\bar{a}, \bar{b}, \bar{c}$ are linear combination of three non-coplanar vectors $\hat{i}, \hat{j}$ and $\hat{k}$, then from (α) equate co-efficients of i, j and k. You get three equation in λ and μ. Solve any two, for vectors to be coplanar, third equation should be satisfied with values of λ and μ already calculated.

Note: Collinear vectors are coplanar.

Example 18: $\overline{a}, \overline{b}, \overline{c}$ are any three non-zero, non-coplanar vectors. Show that vectors, $\overline{a}+\overline{b}+\overline{c}$, $\overline{a}+\overline{b}-2\overline{c}$, $\overline{a}-3\overline{b}-\overline{c}$ and 2a – b – c are linearly dependent.

Sol. Let $\overline{a}+\overline{b}+\overline{c}+\lambda_1(\overline{a}+\overline{b}-2\overline{c})+\lambda_2(\overline{a}-3\overline{b}-\overline{c})+\lambda_3(2\overline{a}-\overline{b}-\overline{c})=0$

$\Rightarrow (1+\lambda_1+\lambda_2+2\lambda_3)\overline{a}+(1+\lambda_1-3\lambda_2-\lambda_3)\overline{b}+(1-2\lambda_1-\lambda_2-\lambda_3)\overline{c} = 0$

$\overline{a}, \overline{b}, \overline{c}$ are non-coplanar

$\therefore$ $1 + \lambda_1 + \lambda_2 + 2\lambda_3 = 0$ ... (i)

$1 + \lambda_1 - 3\lambda_2 - \lambda_3 = 0$ ... (ii)

$1 - 2\lambda_1 - \lambda_2 - \lambda_3 = 0$ ... (iii)

Adding (i), (ii), (iii)

$3 - \lambda_2 = 0$

$\therefore \lambda_2 = 3$

(i) – (ii) $4\lambda_2 + 3\lambda_3 = 0 \Rightarrow \lambda_3 = - -\frac{4}{3}\lambda_2 = -4$

and from (i) $\lambda_1 = 4$

$\lambda_1, \lambda_2, \lambda_3$ are not equal to zero

$\therefore$ These vectors are linearly dependent.

Example 19: Express $\overline{p}=5\hat{i}+4\hat{j}-\hat{k}$ as a linear combination of vectors $\overline{a}=\hat{i}, \overline{b}=\hat{i}+\hat{j}$ and $\overline{c}=\hat{i}+\hat{j}+\hat{k}$.

Sol. Let $\overline{p}=\ell\overline{a}+m\overline{b}+n\overline{c}$

where ℓ, m, n are scalars.

$\therefore\ 5\hat{i}+4\hat{j}-\hat{k}=\ell(\hat{i})+m(\hat{i}+\hat{j})+n(\hat{i}+\hat{j}+\hat{k})$

Equating co-efficients of i, j, k separately.

$\therefore 5 = \ell + m + n$

$4 = m + n$

$-1 = n$

$\Rightarrow m = 5, \ell = 1$

$n = -1$

$\therefore\ \overline{p}=\overline{a}+5\overline{b}-\overline{c}$

Example 20: $\overline{a}, \overline{b}, \overline{c}$ are three non-coplanar vectors. Prove that points whose p.v. are $\overline{a} - 2\overline{b} + 3\overline{c}$, $-2\overline{a} + 3\overline{b} - \overline{c}$ and $4\overline{a} - 7\overline{b} + 7\overline{c}$ are collinear.

Sol. If the given vectors are collinear,

then a – 2b + 3c + λ(–2a + 3b – c) + μ $(4\overline{a} - 7\overline{b} + 7\overline{c})$ = 0 … (α)

and 1 + λ + μ = 0

From (α) (1 – 2λ + 4μ) $\overline{a}$ + (–2 + 3λ – 7μ) $\overline{b}$ + (3 – λ + 7μ) $\overline{c}$ = 0

$\overline{a}, \overline{b}, \overline{c}$ are non-coplanar.

$\therefore$ 1 – 2λ + 4μ = 0

–2 + 3 λ – 7μ = 0 & $\lambda = -\frac{1}{2}$ ⇒3 – λ + 7μ = 0

$\Rightarrow \mu = -\frac{1}{2}$

and $\therefore$ 1 + λ + μ = 0, points are collinear.

Example 21: Show that vectors $\hat{i} - 3\hat{j} + 2\hat{k}, 2\hat{i} - 4\hat{j} - 4\hat{k}$ and $3\hat{i} + 2\hat{j} - \hat{k}$ are linearly independent.

Sol. Let a(i – 3j + 2k) + b(2i – 4j – 4k) + c (3i + 2j – k) = 0

where a, b and c are scalars

⇒ (a + 2b + 3c) $\hat{i}$ + (–3a – 4b + 2c) $\hat{j}$ + (2a – 4b – c) $\hat{k}$ = 0

i, j and k are non-coplanar.

$\therefore$ a + 2b + 3c = 0

–3a – 4b + 2c = 0

2a – 4b – c = 0

This is set of homogeneous equations.

$$\therefore\ |A| = \begin{vmatrix} 1 & 2 & 3 \\ -3 & -4 & 2 \\ 2 & -4 & 1 \end{vmatrix} = 1(4 + 8) - 2\,(3 - 4) + 3(12 + 8) \neq 0 = 12 + 2 + 60 \neq 0$$

$\therefore$ The system of equation has only trivial solution a = b = c = 0

hence given set of vectors are linearly independent. (all scalars are equal to zero)

Example 22: Show that the vectors $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 3\hat{j} - \hat{k}$ and $-\hat{i} - 2\hat{j} + 2\hat{k}$ are linearly dependent.

Sol. Let a, b, c be the scalars such that $a(\hat{i}+\hat{j}+\hat{k})+b(2\hat{i}+3\hat{j}-\hat{k})+c(-\hat{i}-2\hat{j}+2\hat{k})=0$

$\Rightarrow (a+2b-c)\hat{i}+(a+3b-2c)\hat{j}+(a-b+2c)\hat{k}=0$

$\hat{i},\hat{j},\hat{k}$ are not co-planar.

$$\therefore \left.\begin{array}{l} a+2b-c=0 \\ a+3b-2c=0 \\ a-b+2c=0 \end{array}\right\} \text{This is set of homogeneous equation}$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 1(6-2)-2(2+2)-1(-1-3) = 4 - 8 + 4 = 0$$

System has infinitely many solutions.

$\therefore$ a, b, c can assume any non-zero value

$\therefore$ Given vectors are linearly dependent.

Example 23: For what value of a are the vectors $10\hat{i}+3\hat{j}, 12\hat{i}-5\hat{j}$ and $a\hat{i}+11\hat{j}$ collinear.

Sol. Let P.V. of A, B and C be

$10i + 3j, 12i - 5j$ and $ai + 11j$

$\overline{AB} = 2i - 8j$ $\quad \overline{BC} = (a + 12)i + 16j$

Since B is common point.

$\therefore \dfrac{a-12}{2}=\dfrac{16}{-8}\Rightarrow a=8$

Alitor – Let $ai + 11j = \lambda(10i + 3j) + \mu(12i - 5j)$

$\Rightarrow a = 10\lambda + 12\mu, \quad 11 = 3\lambda - 5\mu$

Solving $\lambda=\dfrac{132+5a}{86}, \quad \mu=\dfrac{3a-110}{86}$

But $\mu + \lambda = 1 \Rightarrow 22 + 8a = 86 \Rightarrow a = 8$

Example 24: $\overline{a}=\hat{i}+\hat{j}+\hat{k}, \overline{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\overline{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are three linearly dependent vectors and $|\overline{c}|=\sqrt{3}$ then

(a) $\alpha = 1, \beta = \pm 1$ (b) $\alpha = -1, \beta \pm 1$ (c) $\alpha = 1, \beta = -1$ (d) $\alpha = \pm 1, \beta = 1$

Sol. Let $\hat{i}+\alpha\hat{j}+\beta\hat{k}=\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(4\hat{i}+3\hat{j}+4\hat{k})$

$\lambda + 4\mu = 1$ ……(i) $\quad \lambda + 3\mu = \alpha$ ….(ii) $\quad \lambda + 4\mu = \beta$ ….(iii)

From (i) and (iii) $\beta = 1$

$|\bar{c}| = \sqrt{3} \Rightarrow \quad 1 + \alpha^2 + \beta^2 = |\vec{c}|^2 = 3 \quad \Rightarrow \quad \alpha^2 = 3 - 2$

$\Rightarrow \quad \alpha = \pm 1, \beta = 1$ therefore (d) is correct.

Practice Worksheet (Foundation Level) – 2(b)

1. Express $\overline{p}=5\hat{i}+3\hat{j}+4\hat{k}$ as a linear combination of vectors $\overline{a}=\hat{i}, \overline{b}=\hat{i}+\hat{j}, \overline{c}=\hat{i}+\hat{j}+\hat{k}$

2. Express $\overline{r}=5\hat{i}+7\hat{j}-2\hat{k}$ as a linear combination of vectors $\overline{a}=\hat{i}+\hat{j}-\hat{k}$, $\overline{b}=\hat{j}+\hat{k}-\hat{i}$, $\overline{c}=\hat{k}+\hat{i}-\hat{j}$.

3. Express $\overline{q}=3\hat{i}+4\hat{j}+\hat{k}$ as a linear combination of vectors $\overline{a}=2\hat{i}-\hat{j}+\hat{k}, \overline{b}=\hat{i}+2\hat{j}-\hat{k}$, $\overline{c}=2\hat{k}-\hat{i}+\hat{j}$.

4. Prove that following vectors are co-planar.

   (a) $5\overline{p}+6\overline{q}+7\overline{r}; 7\overline{p}-8\overline{q}+9\overline{r}; 3\overline{p}+20\overline{q}+5\overline{r}$

   (b) $\overline{a}+\overline{b}+\overline{c}; 5\overline{a}-5\overline{b}-5\overline{c}; -\overline{a}+7\overline{b}+7\overline{c}$

5. Prove that points whose p.v. are $\overline{a}-2\overline{b}+3\overline{c}$; $2\overline{a}+3\overline{b}-4\overline{c}$ and $-7\overline{b}+10\overline{c}$ are collinear.

6. $\overline{a}, \overline{b}$ and $\overline{c}$ are non-coplanar vectors. Prove that $4\overline{a}+11\overline{b}+7\overline{c}$; $6\overline{a}-7\overline{b}+9\overline{c}$ and $3\overline{a}+20\overline{b}+6\overline{c}$ are coplanar

7. $\overline{a}, \overline{b}$ and $\overline{c}$ are non-coplanar vectors. Prove $2\overline{a}+3\overline{b}-\overline{c}, \overline{a}+2\overline{b}+3\overline{c}$ and $3\overline{a}-\overline{b}+4\overline{c}$ are linearly independent.

8. If $a\hat{i}-\hat{j}+4\hat{k}; -3\hat{i}+9\hat{j}+8\hat{k}$ and $\hat{i}-6\hat{j}+2\hat{k}$ are co-planar than a is = ...

9. Prove that $2\hat{i}+3\hat{j}+4\hat{k}; \hat{i}-3\hat{j}+5\hat{k}$ and $-9\hat{j}+6\hat{k}$ are collinear.

10. Vectors $10\hat{i}+3\hat{j}+2\hat{k}; 11\hat{i}-a\hat{j}-5\hat{k}$ and $a\hat{i}+15\hat{j}+a\hat{k}$ are collinear. Find value of a.

11. Show that vectors $\overline{a}-2\overline{b}+3\overline{c}, 2\overline{a}-3\overline{b}+4\overline{c}$ and $-\overline{b}+2\overline{c}$ are coplanar.

12. Show that vectors $\overline{a}+\overline{b}+3\overline{c}; 2\overline{a}+3\overline{b}+4\overline{c}$ and $-\overline{a}-3\overline{b}+\overline{c}$ are coplanar.

13. $\overline{a}, \overline{b}, \overline{c}$ are three non-zero, non-coplanar vectors. Show that $6\overline{a}-4\overline{b}+10\overline{c}$; $4\overline{a}+3\overline{b}-10\overline{c}$, $4\overline{a}-6\overline{b}-10\overline{c}$ and $2\overline{b}+10\overline{c}$ are coplanar.

14. Points whose P . vectors are $\hat{i}-4\hat{j}-2\hat{k}$; $3\hat{i}+4\hat{j}+3\hat{k}$; $a\hat{i}+2\hat{j}+b\hat{k}$ are collinear, find a and b.

15. $\overline{a}, \overline{b}, \overline{c}$ are non-zero, non coplanar vectors. If vectors $5\overline{a}+6\overline{b}+7\overline{c}$, $p\overline{a}-8\overline{b}+9\overline{c}$ and $3\overline{a}+20\overline{b}+5\overline{c}$ be coplanar then find p.

16. Show that $\overline{a}+2\overline{b}+4\overline{c}, \overline{a}+3\overline{b}+9\overline{c}, \overline{a}+\overline{b}+\overline{c}$ and $\overline{a}+4\overline{b}+16\overline{c}$ are linearly dependent.

17. A, B, C, D are points, such that $\overline{AB}=p(3\hat{i}-4\hat{j}-5\hat{k})$; $\overline{BC}=9\hat{i}+5\hat{j}+\hat{k}$; and $\overline{CD}=q(2\hat{i}+3\hat{j}+2\hat{k})$. Find condition on p and q so that AB intersect CD.

18. $\bar{a} = 2\hat{i} + \hat{j} + 2\hat{k}$, $\bar{b} = 5\hat{i} + 4\hat{j} + 5\hat{k}$ and $\bar{c} = 2\hat{i} + \alpha\hat{j} + \beta\hat{k}$, are linearly dependent vectors. If $c = 2\sqrt{3}$ then find α and β.

19. Find a, if vectors $a\hat{i} + 2\hat{j} + 3\hat{k}$, $4\hat{i} - a\hat{j} - 2\hat{k}$ and $3\hat{i} + \hat{j} + a\hat{k}$ are linearly independent.

20. Find p, if vectors pi + 2j + 7k; 4i – pj – 2k and 5i + 3pj + 23k are linearly dependent.

3. Product of Vector

3.1 Product of Vector

The product of two vectors is quite different from the product of two scalars, because of the direction and the sense they carry. The product of two vectors is done in two ways – Dot product and Vector product. The result of dot product of two vectors is a scalar quantity, while the result of vector product of two vectors is a vector quantity.

3.2 Dot product of two vectors

The dot product of two vectors $\vec{a}$ and $\vec{b}$ is abcos θ where $|\vec{a}| = a, |\vec{b}| = b$ and acute angle between the two is θ. It is written as $\overline{a}.\overline{b}$

$$\therefore\ \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \qquad \dots (1)$$

In the figure $\overrightarrow{AB} = \vec{a}, \overrightarrow{AC} = \vec{b}$

$$AB = |\vec{a}| \text{ and } AC = |\overline{b}|$$

$$AN = AC \cos\theta$$

= Resolved part of vector $\vec{b}$ in the direction of vector $\vec{a}$

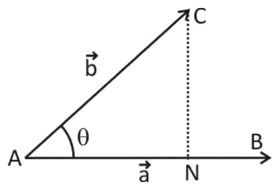


Fig 1

Now (1) can be written as

$$\vec{a}.\vec{b} = (\overline{b}\cos\theta).a$$

= $|\overline{a}|$ (resolved part of $\overline{b}$ in the direction of $\overline{a}$)

$$\therefore \text{ Resolved part of vector } \vec{b} \text{ in the direction of vector } \overline{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|} \qquad \dots (2)$$

The restult abcosθ is a scalar quantity, therefore dot product is also called scalar product of two vectors.

3.3 Properties of dot (scalar) product

(a) Scalar product is commutative.

$$\vec{a}.\vec{b} = \vec{b}.\vec{a}$$

$$\vec{a}.\vec{b} = ab\cos\theta = \vec{b}.a\cos\theta = |\vec{b}||\vec{a}|\cos\theta = \vec{b}.\vec{a}$$

(b) $(n\vec{a})\cdot(m\cdot\overline{b}) = nm(\vec{a}.\overline{b}) = nmab\cos\theta$

= (ma) (nb) cos θ

= $(m\overline{a})\cdot(n\vec{b})$

(c) Dot product in distributive over the sum of vectors

$\overline{a}.(\overline{b}+\overline{c})=\overline{a}\cdot\overline{b}+\overline{a}\cdot\overline{c}$

In the figure $\overrightarrow{OA}=\vec{a},\overrightarrow{OB}=\vec{b}$

and $\overrightarrow{BC}=\overline{c}$

$\therefore\ \overrightarrow{OC}=\vec{b}+\vec{c}$

BM and CN are perpendicular from B and C on OA.

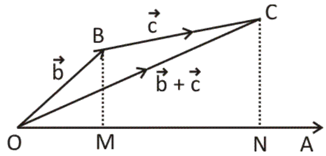


Fig 2

$OM=$ projection of $\vec{b}$ on $\vec{a}$

$ON=$ projection of $(\vec{b}+\vec{c})$ on $\vec{a}$

$MN=$ projection of $\vec{c}$ on $\vec{a}$

and, $a\cdot(\vec{b}+\vec{c})=a$(projection of $(\vec{b}+\vec{c})$ on $\vec{a}$)

= a (ON)

= a (OM) + a(MN)

= a (projection of $\overline{b}$ on $\overline{a}$) + a(projection of $\overline{c}$ on $\overline{a}$)

= $\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

$\therefore\ \overline{a}\cdot(\overline{b}+\overline{c})=\overline{a}\cdot\overline{b}+\overline{a}\cdot\overline{c}$

(d) $\vec{a}.\vec{a}=|\overline{a}||\overline{a}|\cos\theta=a^2\cdot 1=a^2$

(e) If $\vec{a}\cdot\vec{b}=0\Rightarrow\cos\theta=\cos 90^o=0$ or $\vec{a}=0$ or $\vec{b}=c$

$\therefore$ If angle between two vectors is 90^o, their dot product is 0, conversely if the dot product of two non zero vectors is zero, then angle between is 90^o.

(f) If $a=x\hat{i}+y\hat{j}+z\hat{k}$ then you know $|\overline{a}|=\sqrt{x^2+y^2+z^2}$

$\hat{i},\hat{j},\hat{k}$ are unit vectors and each is perpendicular to the other two.

$\therefore\ \hat{i}\cdot\hat{j}=\hat{i}\cdot\hat{k}=\hat{j}\cdot\hat{k}=0$ and $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}$ = 1 and if $\overline{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

then $\vec{a}\cdot\vec{b}=|\overline{a}||\overline{b}|\cos\theta$

$\Rightarrow\ (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})=\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}\cdot\cos\theta$

$$\Rightarrow \cos\theta = \frac{(a_1b_1 + a_2b_2 + a_3b_3)}{(\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^3 + b_3^2})}$$

None if $a_1b_1 + a_2b_2 + a_3b_3 = 0$

then $\cos \theta = 0 \Rightarrow \theta = 90^o$

This is the necessary and sufficient condition for the two vectors to be perpendicular to each other.

(g) If vectors $\overline{a}$ and $\overline{b}$ are parallel then each can be expressed in terms of the other i.e. $\overline{a} = t\overline{b}$

$\overline{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \overline{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\overline{a} - t\overline{b} = 0 \Rightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} - t(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$\Rightarrow (a_1 - tb_1)\hat{i} + (a_2 - tb_2)\hat{j} + (a_3 - tb_3)\hat{k} = 0$

$\hat{i}, \hat{j}$ and $\hat{k}$ are non coplanar

$\therefore a_1 - tb_1 = 0, a_2 - tb_2 = 0, a_3 - tb_3 = 0$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = t$$

$\therefore$ The two vectors $\overline{a}$ and $\overline{b}$ are parallel

If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

3.4 Application

(a) We know, force $\times$ displacement = work done

Force and displacement are both vector quantities but work is a scalar quantity.

$\therefore$ If $\overline{F}$ is force and $\overline{d}$ is displacement then work done = $\overline{F} \cdot \overline{d}$.

If $\overline{F}$ is in Newton and $\overline{d}$ in meters then work done is in joule.

(b) We can prove many geometrical theorems or properties of triangles, squares etc. with the help of dot product.

1. To prove angle in semi-circle is right angle

In the figure C is any point on the semicircle, BA is diameter, O centre

$\therefore$ OA = OB = OC = radius

Let O be origin, OA = $\overline{a}$

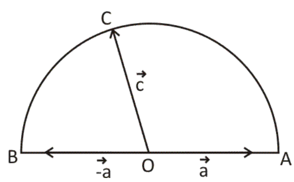


Fig 3

$\therefore\ \overrightarrow{OB} = -\bar{a}$, and let $\overrightarrow{OC} = \bar{c}$

and $|\bar{a}| = |\bar{c}|$

$\overrightarrow{BC} = \vec{c} - (-\vec{a}) = \vec{c} + \vec{a}$

$\overrightarrow{AC} = \vec{c} - \vec{a}$ $\quad\therefore\ \overrightarrow{AC} \cdot \overrightarrow{BC} = (\bar{c} - \bar{a}) \cdot (\bar{c} + \bar{a}) = |\bar{c}| - |\bar{a}| = 0$

$\therefore\ BC \perp AC \Rightarrow \angle BCA = 90^{o}$

2. Prove that perpendicular dropped from vertices on the opposite sides of a triangle are concurrent

In the figure AD and BE are perpendicular from A and B an opposite sides and these meet in O. CO is joined and produced to meet AB in F.

To prove CF is perpendicular to AB.

Let O be the origin and

$\overrightarrow{OA} = \bar{a}, \overrightarrow{OB} = \bar{b}$ and $\overrightarrow{OC} = \bar{c}$

$\overrightarrow{AB} = \bar{b} - \bar{a}, \overrightarrow{BC} = \bar{c} - \bar{b}$ and $\overrightarrow{CA} = \bar{a} - \bar{c}$

AOD is a straight line $\therefore\ \overrightarrow{DA} = \lambda\vec{a}$

and similarly $\overrightarrow{EB} = \mu \cdot \vec{b}$

DA is $\perp$ on BC $\therefore$ $\quad \lambda\bar{a} \cdot (\bar{c} - \bar{b}) = 0$

$\Rightarrow\ \bar{a} \cdot \bar{c} = \bar{a} \cdot \bar{b}$ ... (i)

EB is $\perp$ on AC $\quad \Rightarrow\ \mu \cdot \bar{b}(\bar{c} - \bar{a}) = 0$

$\Rightarrow\ \bar{b} \cdot \bar{c} = \bar{b} \cdot \bar{a}$ ... (ii)

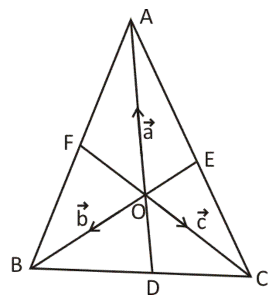


Fig 4

from (i) and (ii) $\ \bar{a} \cdot \bar{c} = \bar{b} \cdot \bar{c}$

$\Rightarrow\ (\bar{a} - \bar{b}) \cdot \bar{c} = 0$, $\quad \Rightarrow\ c \cdot (a - b) = 0 \quad t\vec{c} \cdot (\vec{a} - \vec{b}) = 0$

$\therefore\ CF \perp AB$

3. Prove that the right bisectors of sides of a triangle are concurrent

In the figure 5, the right bisectors of BC, OD and of AC, OE meet in O. F is mid point of AB. OF is joined to prove OF is perpendicular on AB.

Let O be the origin $\overrightarrow{OA} = \bar{a}, \overrightarrow{OB} = \bar{b}$

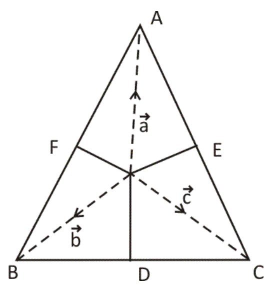


Fig 5

and $\overrightarrow{OC} = \bar{c}$

$\overrightarrow{OD} = \frac{1}{2}(\bar{b} + \bar{c})$ and $\overrightarrow{OE} = \frac{1}{2}(\bar{a} + \bar{c})$

$\overrightarrow{OF} = \frac{1}{2}(\bar{a} + \bar{b})$

$\overrightarrow{OD} \perp \overrightarrow{BC}$ $\therefore\ \frac{1}{2}(\bar{b} + \bar{c}) \cdot (\bar{c} - \bar{b}) = 0$

$\Rightarrow\ \frac{1}{2}\left[(c)^2 - (b)^2\right] = 0$

$\Rightarrow\ (\bar{c})^2 = (\bar{b})^2$ ... (i)

OE $\perp$ AC $\therefore\ \frac{1}{2}(\bar{a} + \bar{c}) \cdot (\bar{a} - \bar{c}) = 0$

$\Rightarrow\ \frac{1}{2}\left[(\bar{a})^2 - (\bar{c})^2\right] = 0 \Rightarrow (\bar{a})^2 = (\bar{c})^2$ ... (ii)

From (i) and (ii) $(\bar{a})^2 = (\bar{b})^2$

$\Rightarrow\ (\bar{a})^2 - (\bar{b})^2 = (\bar{a} + \bar{b})(\bar{a} - \bar{b}) = 0$

$\Rightarrow\ \frac{1}{2}(\bar{a} + \bar{b})(\bar{a} - \bar{b}) = 0$

$\Rightarrow\ \overrightarrow{OF} \cdot \overrightarrow{BA} = 0 = \overrightarrow{OF} \cdot \overrightarrow{BA}$

$\therefore$ OF is perpendicular on AB.

3.5 Trigonometry formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ can also be proved with the help of vectors

In the triangle ABC, let A be the origin and

$\overrightarrow{AC} = \bar{b}$ and $\overrightarrow{AB} = \bar{c}$

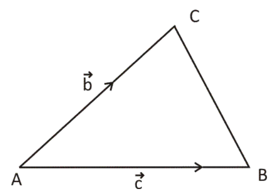


Fig 6

then $\overrightarrow{BC} = \bar{c} - \bar{b}$

and $(AC)^2 = |\bar{b}|^2 = b^2$

$(AB)^2 = |\bar{c}|^2 = c^2$

and $(BC)^2 = a^2 = (\bar{c} - \bar{b})^2$

$\Rightarrow\ a^2 = (\bar{c} - \bar{b}) \cdot (\bar{c} - \bar{b}) = \bar{c} \cdot \bar{c} - \bar{c} \cdot \bar{b} - \bar{b} \cdot \bar{c} + \bar{b} \cdot \bar{b}$

$\Rightarrow a^2 = c^2 - cb \cos A - bc \cos A + b^2$

$\Rightarrow 2\ bc \cos A = c^2 + b^2 - a^2$

$$\Rightarrow \cos A = \frac{c^2 + b^2 - a^2}{2bc}$$

Solved Example

Example 1: For what value of a the vectors $a\hat{i}-2a\hat{j}+3\hat{k}$ and $-2\hat{i}-3\hat{j}+4\hat{k}$ are perpendicular to each other.

Sol. For the vectors to be perpendicular to each other, their dot product should be equal to zero.

$\therefore\ (a\hat{i}-2a\hat{j}+3\hat{k})\cdot(-2\hat{i}+3\hat{j}+4\hat{k})=0$

$\Rightarrow -2a + 6a + 12 = 0 \Rightarrow 4a + 12 = 0$

$\Rightarrow a = -3$

Example 2: The vertices of a triangle are $7\hat{j}+10\hat{k}$, $-\hat{i}+6\hat{j}+6\hat{k}$ and $-4\hat{i}+9\hat{j}+6\hat{k}$, prove that triangle is right angled

Sol. P.V. of A is $7\hat{j}+10\hat{k}$, of B is $(-\hat{i}+6\hat{j}+6\hat{k})$ and of c is $(-4\hat{i}+\hat{j}+6\hat{k})$.

$\therefore\ \overrightarrow{AB}=(-\hat{i}+6\hat{j}+6\hat{k})-(7\hat{j}+10\hat{k})=-\hat{i}-\hat{j}-4\hat{k}$

$\overrightarrow{BC}=(-4\hat{i}+\hat{j}+6\hat{k})-(-\hat{i}+6\hat{j}+6\hat{k})=-3\hat{i}+3\hat{j}+0\hat{k}$

$\overrightarrow{AB}\cdot\overrightarrow{BC}=(-\hat{i}-\hat{j}-4\hat{k})\cdot(-3\hat{i}+3\hat{j}+0\cdot\hat{k})$

$= 3 - 3 + 0 = 0 \quad \Rightarrow \cos B = 0$

$\therefore \angle B = 90^o$, $\quad\therefore$ Triangle is right angled

Example 3: Force $F=3\hat{i}+\hat{j}+6\hat{k}$ N, displaces a body through $2\hat{i}+\hat{j}-\hat{k}$. Find work done.

Sol. Work done $=F\cdot d=(3\hat{i}+\hat{j}+6\hat{k})\cdot(2\hat{i}+\hat{j}-\hat{k})$

$= 6 + 1 - 6 = 1$ joule

Example 4: For any vector $\overline{a}$ prove $\overline{a}=(\overline{a}\cdot i)i+(\overline{a}\cdot j)j+(\overline{a}\cdot k)k$

Sol. Let $\overline{a}=xi+yj+2k$

$\therefore\ \overline{a}\cdot i=xi\cdot i+yi\cdot j+zi\cdot k=x+0+0$

and $\therefore\ \overline{a}\cdot j=y, \overline{a}\cdot k=z$

$\therefore\ (\overline{a}\cdot i)i+(\overline{a}\cdot j)j+(\overline{a}\cdot k)k$

$= xi + yj + zk = \overline{a}$

Example 5: Angle between unit vectors, r_1 and r_2 is 60^o, prove $r_1 - r_2$ is a unit vectors.

Sol. $[r_1 - r_2]^2 \quad = (r_1 - r_2)\cdot(r_1 - r_2)$

$= r_1 \cdot r_2 - r_1 \cdot r_2 - r_2 \cdot r_1 + r_2 \cdot r_2$

$= |r_1|^2 + |r_2|^2 - 2 \cdot |r_1|\,|r_2| \cos 60^o$

$= 1 + 1 - 2 \cdot 1 \cdot 1 \cdot \frac{1}{2} = 1$

$\therefore\ |r_1 - r_2| = 1$ unit vector.

Example 6: Angle between unit vectors $\bar{r}_1$ and $\bar{r}_1$ is θ. Prove $\cos\frac{\theta}{2} = \frac{1}{2}|r_1 + r_2|$

Sol. $|\bar{r}_1 + \bar{r}_2|^2 = (\bar{r}_1 + \bar{r}_2)\cdot(\bar{r}_1 + \bar{r}_2{}^2)$

$= \bar{r}_1 \cdot \bar{r}_2 + \bar{r}_2 \cdot \bar{r}_2 + 2 \cdot \bar{r}_1 \cdot \bar{r}_2$

$= 1 + 1 + 2 \cdot 1 \cdot 1 \cos\theta$

$= 2 + 2\cos\theta = 2(1 + \cos\theta) - 2 \cdot 2\cos^2\frac{\theta}{2}$

$\therefore\ |\bar{r}_1 + \bar{r}_2| = 2\cdot\cos\frac{\theta}{2} \Rightarrow \cos\frac{\theta}{2} = \frac{1}{2}|\bar{r}_1 + \bar{r}_2|$

Example 7: If $\bar{p} + \bar{q} = \bar{r}$ and $|\bar{p}| = 3, |\bar{q}| = 5$ and $|\bar{r}| = 7$. Find angle between $\bar{p}$ and $\bar{q}$.

Sol.: Let θ be angle between $\bar{p}$ and $\bar{q}$, then

$\bar{p}\cdot\bar{q} = |\bar{p}|\cdot|\bar{q}|\cos\theta = 3\cdot 5\cdot\cos\theta = 15\cos\theta$

Given p + q = r $\Rightarrow$ (p + q) (p + q) = $\bar{r}.\bar{r}$

$\Rightarrow \bar{p}\cdot\bar{p} + \bar{p}\cdot\bar{q} + \bar{q}\cdot\bar{p} + \bar{q}\cdot\bar{q} = \bar{r}\cdot\bar{r}$

$\Rightarrow |\bar{p}^2| + |\bar{q}|^2 + 2|\bar{p}|\cdot|\bar{q}|\cos\theta = |\bar{r}|\cdot|\bar{r}|$

$\Rightarrow 9 + 25 + 2 \cdot 3 \cdot 5 \cos\theta = 49$

$\Rightarrow 30\cos\theta = 15 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o = \frac{\pi}{3}$

Note: If angle between $\bar{p}$ and $\bar{r}$ is required then put $\bar{q} = \bar{p} - \bar{r}$ and then proceed as above.

Example 8: Show that projection vector of $\bar{a}$ on vector $\bar{b}$ is $\left[\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}\right]\cdot\bar{b}$

Sol. Magnitude of projection of vector $\bar{a}$ on $\bar{b}$ = $\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$; this is in direction of $\bar{b}$

Unit vector along $\bar{b}$ is $\frac{\bar{b}}{|\bar{b}|}$.

$\therefore$ Projection vector of $\overline{a}$ and $\overline{b}$ = $\left(\frac{\overline{a}\cdot\overline{b}}{|\overline{b}|}\right)\left(\frac{\overline{b}}{|\overline{b}|}\right)=\left(\frac{\overline{a}\cdot\overline{b}}{|\overline{b}|^2}\right)\overline{b}$

Example 9: Three forces of magnitude 5, 8 and 4 N act along $2\hat{i}-3\hat{j}+2\sqrt{3}\hat{k}, 3\hat{i}+2\hat{j}+\sqrt{3}\hat{k}$ and $5\hat{i}+6\hat{j}-\sqrt{3}\hat{k}$ respectively. These displace body from point (3, –1, –1) to (5, 1, –1). Find work done.

Sol. Unit vector along $2\hat{i}-3\hat{j}+2\sqrt{3}\hat{k}$ is $\frac{1}{5}(2\hat{i}-3\hat{j}+2\sqrt{3}\hat{k})$ and unit vector along $3\hat{i}+2\hat{j}+\sqrt{3}\hat{k}$ is $\frac{1}{4}(3\hat{i}+2\hat{j}+\sqrt{3}\hat{k})$ and unit vector along $5\hat{i}+6\hat{j}-\sqrt{3}\hat{k}$ is $\frac{1}{8}(5\hat{i}+6\hat{j}-\sqrt{3}\hat{k})$

$\therefore$ Total force $= \frac{5}{5}(2\hat{i}-3\hat{j}+2\sqrt{3}\hat{k})+\frac{8}{4}(3\hat{i}+2\hat{j}+\sqrt{3}\hat{k})+\frac{4}{8}(5\hat{i}+6\hat{j}-\sqrt{3}\hat{k})$

$= 2\hat{i}-3\hat{j}+2\sqrt{3}\hat{k}+6\hat{i}+4\hat{j}+2\sqrt{3}\hat{k}+\frac{5}{2}\hat{i}+3\hat{j}-\frac{\sqrt{3}}{2}\hat{k}$

$= \frac{1}{2}\left[21\hat{i}+8\hat{j}+7\sqrt{3}\hat{k}\right]$

Displacement AB = $(5\hat{i}+\hat{j}-\hat{k})-(3\hat{i}-\hat{j}-\hat{k})=2\hat{i}+2\hat{j}$

Work done $= \frac{1}{2}\left[21\hat{i}+8\hat{j}-7\sqrt{3}\hat{k}\right]\cdot\left[2\hat{i}+2\hat{j}\right]$

= 21 + 8 + 0 = 29 joule

Example 10: Find resolved parts of vector $2\hat{i}-3\hat{j}+4\hat{k}$ along $2\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to it.

Sol. Let $\overrightarrow{AB}=2\hat{i}-3\hat{j}+4\hat{k}$ and $\overrightarrow{AC}=2\hat{i}+2\hat{j}+\hat{k}$

Resolved part of $\overrightarrow{AB}$ along $\overrightarrow{AC}$ is

$=\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AC}|}$ in magnitude

$=\frac{(2i-3j+4k)\cdot(2i+2j+k)}{3}=\frac{4-6+4}{3}=\frac{2}{3}$

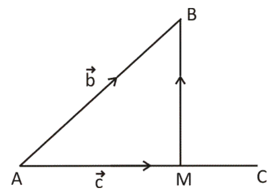


Fig 7

Unit vector along $\overrightarrow{AC}=\frac{1}{3}(2\hat{i}+2\hat{j}+\hat{k})$

$\therefore$ Resolved part along $AC = \frac{2}{3}\cdot\frac{1}{3}(2\hat{i}+2\hat{j}+\hat{k}) = \overrightarrow{AM}$

Resolved part of $\overline{AB}$, perpendicular to AC is $\overline{MB}$

and $\overline{MB} = \overline{AB} - \overline{AM} = 2\hat{i} - 3\hat{j} + 4\hat{k} = \frac{2}{9}(2\hat{i}+2\hat{j}+\hat{k})$

$= \frac{1}{9}(14\hat{i} - 31\hat{j} + 34\hat{k})$

Example 11: Vectors $\overline{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\overline{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\overline{c} = \hat{i} + \hat{j} - 2\hat{k}$, what is that vector which lies in the plane of $\overline{b}$ and $\overline{c}$ and whose projection on $\overline{a}$ has magnitude $\sqrt{2/3}$.

Sol. Any vector lying in the plane of $\overline{b}$ and $\hat{c}$ is

$\overline{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k})$

$= (1+\lambda)\hat{i} + (2+\lambda)\hat{j} - (1+2\lambda)\hat{k}$

Its projection on $\overline{a} = \frac{\overline{a}\cdot\overline{r}}{|\overline{a}|}$

$$= \frac{(2\hat{i} - \hat{j} - \hat{k})\cdot\left[(1+\lambda)\hat{i} + (2+\lambda)\hat{j} - (1+2\lambda)\right]\hat{k}}{\sqrt{1+1+4}} = \sqrt{\frac{2}{3}}$$

$$= \frac{2(1+\lambda) - (2+\lambda) + (1+2k)}{\sqrt{6}} = \sqrt{\frac{2}{3}}, \text{solving } \lambda = -3$$

$\therefore\ \overline{r} = (1-3)i + (2-3)j - (1-6)k = -2\hat{i} - \hat{j} + 5\hat{k}$

Example 12: $\overline{a}$ and $\overline{b}$ are two non-zero vectors and $|\overline{a}+\overline{b}| = |\overline{a}-\overline{b}|$, prove $\overline{a}$ and $\overline{b}$ are perpendicular.

Sol. $|\overline{a}+\overline{b}| = |\overline{a}-\overline{b}| \Rightarrow |\overline{a}+\overline{b}|^2 = |\overline{a}-\overline{b}|^2$

$\Rightarrow (\overline{a}+\overline{b})\cdot(\overline{a}+\overline{b}) = (\overline{a}-\overline{b})\cdot(\overline{a}-\overline{b})$

$\Rightarrow \overline{a}\cdot\overline{a} + \overline{a}\cdot\overline{b} + \overline{b}\cdot\overline{a} + \overline{b}\cdot\overline{b} = \overline{a}\cdot\overline{a} - \overline{a}\cdot\overline{b} - \overline{b}\cdot\overline{a} + b\cdot\overline{a}$

$\Rightarrow 2\overline{a}\cdot\overline{b} = -2\overline{a}\cdot\overline{b} \Rightarrow 4\overline{a}\cdot\overline{b} = 0$

$\Rightarrow \overline{a}\cdot\overline{b} = 0$ $\therefore\ \overline{a}$ is perpendicular to $\overline{b}$.

Example 13: If two pairs of opposite sides of a tetrahedron are perpendicular to each other then prove third pair is also perpendicular to each other.

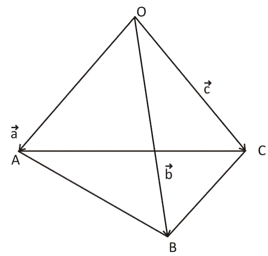


Fig 8

Sol. Figure (8); OABC is the tetrahedron. Let O be the origin and p.v. of A, B and C be $\overline{a}, \overline{b}$ and $\overline{c}$, respectively.

(i) OA and BC are perpendicular $\Rightarrow \overline{a}\cdot(\overline{c}-\overline{b})=0$

(ii) OC and AB are perpendicular $\Rightarrow \overline{c}\cdot(\overline{b}-\overline{a})=0$ adding both $\overline{a}\cdot\overline{c}-\overline{a}\cdot\overline{b}+\overline{c}\cdot\overline{b}-\overline{c}\cdot\overline{a}=0$

$\Rightarrow \overline{c}\cdot\overline{b}-\overline{a}\cdot\overline{b}=(\overline{c}-\overline{a})\cdot\overline{b}=0$

$\Rightarrow AC \perp OB$

Example 14: vector $\beta=3\hat{j}+4\hat{k}$ is to be written as sum of vector β_1 parallel to $\overline{A}=\hat{i}+\hat{j}$ and of β_2, perpendicular to $\overline{A}$. Find β_1

Sol. Vector perpendicular to i + j is i + j

$\therefore$ 3j + 4k = λ(i + j) + μ(i – j)

$\therefore \lambda + \mu = 0, \lambda - \mu = 3 \Rightarrow \lambda = \frac{3}{2}, \mu = 0$

$\therefore \quad \beta_1 = \frac{3}{2}(i+j)$

Example 15: Adjacent sides of a parallelograms are $3\hat{i}-2\hat{j}+6\hat{k}$ and $2\hat{i}+2\hat{j}-\hat{k}$. Find length of diagonals and acute angle between them.

Sol. Let $\overline{AB}=3\hat{i}-2\hat{j}+6\hat{k}$

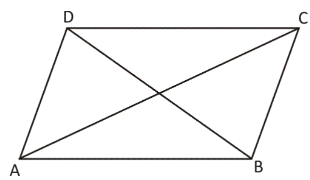


Fig 9

$\overline{AD}=2\hat{i}+2\hat{j}-\hat{k}$

BC || AD $\Rightarrow \overline{BC}=2\hat{i}+2\hat{j}-\hat{k}$

Diagonal $\overline{AC}=\overline{AB}+\overline{BC}$

$= (3\hat{i}-2\hat{j}+6\hat{k})+(2\hat{i}+2\hat{j}-\hat{k})$

= 5i + 5k

and $\overline{AB}+\overline{BD}=\overline{AD} \Rightarrow \overline{BD}=\overline{AD}-\overline{AB}$

$\overline{BD}=(2\hat{i}+2\hat{j}-\hat{k})-(3\hat{i}-2\hat{j}+6\hat{k})$

$=-\hat{i}+4\hat{j}-7\hat{k}$

$|AC|=\sqrt{25+25}=5\sqrt{2}; \overline{BD}=\sqrt{1+16+49}=\sqrt{66}$

angle between diagonal

$$\cos\theta = \frac{(5\hat{i}+5\hat{k})\cdot(-\hat{i}+4\hat{j}-7\hat{k})}{5\cdot 2\cdot\sqrt{33}} = \frac{-40}{10\sqrt{33}}$$

$$\cos\theta = \frac{-4}{\sqrt{33}},\ \text{acute angle } \theta = \cos^{-1}\left(\frac{4}{\sqrt{33}}\right)$$

Practice Worksheet (Foundation Level) – 3(a)

1. Find angle between vectors $2\hat{i}+2\hat{j}+\hat{k}$ and $3\hat{i}-2\hat{j}+2\sqrt{3}\hat{k}$

2. $\bar{a}=\hat{i}-2\hat{j}-2\hat{k}, \bar{b}=2\hat{i}-3\hat{j}+6\hat{k}$, angle between them is

3. Sides of triangle ABC are $\overline{BC}=2\hat{i}-\hat{j}+\hat{k}, \overline{CA}=\hat{i}-3\hat{j}+5\hat{k}$ and $\overline{AB}=3\hat{i}-4\hat{j}-4\hat{k}$. prove that ABC is right angled triangle.

4. Find λ, if vectors $\lambda\hat{i}+3\hat{j}-4\hat{k}$ and $\lambda\hat{i}-4\hat{j}-\lambda\hat{k}$ include an angle of $\frac{\pi}{2}$.

5. Points A, B, C and D are (1, 3, –2), (–1, 2, 4), (0, 1, 3) and (2, –2, 1). Projection of AB on CD is …

6. Point A, B, C and D are (1, 3, –2), (–1, 6, 4), (0, 5, 5) and ($2\sqrt{3}$, 3, 2). Find angle ABC and BCD.

7. Vectors $\bar{a}+\bar{b}=\bar{c}$ and $|\bar{a}|=5, |\bar{b}|=4, |\bar{c}|=6$ then angle between $\bar{a}$ and $\bar{c}$ is …

8. Vectors $\bar{a}+\bar{b}=\bar{c}$ and $\bar{a}\perp\bar{b}$. If θ is angle between $\bar{a}$ and $\bar{c}$, then prove cosθ=$\frac{|a|}{|c|}$.

9. Find projection of vector $2\hat{i}-\hat{j}+\hat{k}$ in direction of vector $3\hat{i}-2\hat{j}+6\hat{k}$.

10. $\frac{7}{2}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{2}\hat{k}$ and $\frac{1}{2}\hat{i}-\frac{5}{2}\hat{j}+\frac{5}{2}\hat{k}$ are adjacent sides of a parallelogram. Find angle between diagonals.

11. Find resolved part of vector $8\hat{i}+\hat{j}$ in the directions perpendicular to $\hat{i}+2\hat{j}-2\hat{k}$.

12. Forces of magnitude 5 N and 3 N acting along 6i + 2j + 3k and 3i – 2j + 6k respectively acting on a body simultaneously and displaces it from (2, 2, –1) to (4, 3, 1). Find work done.

13. Vectors $\bar{a}=2\hat{i}-3\hat{j}+2\sqrt{3}\hat{k}$; $\bar{b}=2\hat{i}+4\hat{j}+\sqrt{3}\hat{k}$ and $\bar{c}=5\hat{i}-4\hat{j}+\sqrt{3}\hat{k}$. A vector lies in the plane of vectors $\bar{b}$ and $\bar{c}$ and its projection on vector $\bar{a}$ has magnitude 3. Find this vector.

14. prove that two non-zero vectors cannot be parallel and perpendicular simultaneously.

15. Two adjacent sides of a parallelogram are $5\hat{i}-5\hat{j}+\hat{k}$ and $\hat{i}+5\hat{j}-5\hat{k}$. Prove that diagonals are perpendicular and have magnitude $2\sqrt{13}$ and $2\sqrt{38}$.

16. $\hat{i}+a\hat{j}+a\hat{k}$ and $a\hat{i}+a\hat{j}-\hat{k}$ are two adjacent sides of a parallelogram. Find angle between diagonals.

17. $\bar{a}+\bar{b}+\bar{c}=0$ and $\bar{a}\perp\bar{b}$, ($\bar{a}\neq 0$), prove $|\bar{a}|+|\bar{c}|\cos\theta=0$, where θ is angle between $\bar{a}$ and $\bar{c}$

18. θ is the angle between two unit vectors $\bar{r}_1$ and $\bar{r}_2$ prove $\sin\frac{\theta}{2}=\frac{1}{2}|r_1-r_2|$.

19. Vector $\overrightarrow{OA}=\bar{a},\overrightarrow{OB}=\vec{b}$, D is mid point of AB. Prove that $OA^2 + OB^2 = 2(OD^2 + AD^2)$

20. Prove by vectors method – (a) angle in semicircle is right angle, (b) the altitudes of a triangle are concurrent.

21. The acute angle between x – y plane and vector $2\hat{i}+2\hat{j}-\hat{k}$ is

(a) $\sin^{-1}\left(\frac{1}{3}\right)$ (b) $\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$ (c) $\cos^{-1}\left(\frac{2}{5}\right)$ (d) $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

22. Vectors $\bar{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\bar{b}=3\hat{i}-\hat{j}+\lambda\hat{k}$ include angle $\frac{\pi}{3}$ then λ is

(a) –2 (b) 3 (c) –3 (d) 2

23. Angle between vectors $2\hat{i}+\lambda\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$ is 60^o, then λ is =

(a) ± 1 (b) 1, –71 (c) 71, 1 (d) –71, –1

24. Vectors $\bar{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\bar{b}=3\hat{i}+2\hat{j}+6\hat{k}$, projection of $\bar{a}$ on $\bar{b}$ is

(a) $-\frac{11}{7}$ (b) $\frac{17}{7}$ (c) $-\frac{17}{7}$ (d) $-\frac{11}{7}$

25. The scalar product of vector $\hat{i}+\hat{j}+\hat{k}$ with unit vector parallel to sum of vectors. $2\hat{i}+4\hat{j}-5\hat{k}$ and $b\hat{i}+2\hat{j}+3\hat{k}$ is unity. then b is =

(a) –2 (b) –1 (c) 1 (d) 2

26. Three forces of magnitude 5, 3 and 1 N, acting along $6\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{i}-2\hat{j}+6\hat{k}$ and $2\hat{i}-3\hat{j}-6\hat{k}$ respectively act on a body and displaces it from A(2, –1, –3) to B (5, –1, 1). Work done by forces is

(a) 33 j (b) 44 j (c) 63 j (d) 55 j

27. Vectors, $\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{i}-2\hat{j}+\hat{k}$ and $-2\hat{i}+3\hat{j}-\hat{k}$ are

(a) Coplanar (b) Linearly dependent

(c) linearly independent (d) Collinear

28. A force of P dynes acting along $3\hat{i}+2\hat{j}-6\hat{k}$ displaces a particle from (5, 3, 1) to (8, –1, –1) and does 34 ergs of work, then P is

(a) 28 dynes (b) 35 dynes (c) 21 dynes (d) 14 dynes

29. If angle between $a\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-a\hat{k}$ is double of the angle between $3\hat{i}+\sqrt{2}\hat{j}+\hat{k}$ and $3\hat{i}+\sqrt{2}\hat{j}-2\hat{k}$ then a =

(a) 2 (b) ± 4 (c) $\pm\sqrt{6}$ (d) ± 3

30. θ is angle between $\sqrt{3}\hat{i}+\sqrt{2}\hat{j}+\sqrt{3}\hat{k}$ and $\sqrt{2}\hat{i}-\sqrt{3}\hat{j}+\sqrt{3}\hat{k}$ and ϕ is the angle between $2\hat{i}+\hat{j}-\sqrt{3}\hat{k}$ and $\sqrt{3}\hat{i}-\hat{j}+2\hat{k}$, then cos ($\phi-\theta$) =

(a) $\frac{3\sqrt{385}-3}{64}$ (b) $\frac{3(1+\sqrt{55})}{64}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

3.6 Vector Product or Cross Product

The vector product of two vectors $\bar{a}$ and $\bar{b}$ is a vector, which is perpendicular to both $\bar{a}$ and $\bar{b}$ i.e. perpendicular to plane containing $\bar{a}$ and $\bar{b}$. Its modulus is ab sin θ, where $|\bar{a}|=a, |\bar{b}|=b$ and θ is acute angle between $\bar{a}$ and $\bar{b}$. It is written as

$$\bar{a}\times\bar{b}=|a||b|\hat{\eta}\sin\theta$$

η is unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ the direction of η is determined by right angle and screw rule.

1. (i) $\bar{a}\times\bar{b}$ (ii) $\bar{b}\times\bar{a}$

   It is clear that $\bar{b}\times\bar{a}=-(\bar{a}\times\bar{b})$

   Cross Product is not Commutative

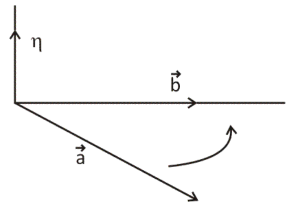


Fig 10a

2. $(p\bar{a})\times(q\times\bar{b})=pq(\bar{a}\times\bar{b})$

   $=qp(\bar{a}\times\bar{b})\Rightarrow(q\bar{a})\times(p\bar{b})$

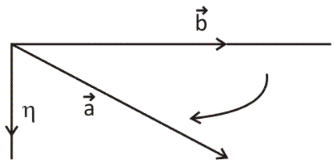


Fig 10b

3. $\hat{i},\hat{j},\hat{k}$ are unit vectors along OX, OY, OZ and each is perpendicular to the other two.

   $\therefore\ \hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=1\cdot1\cdot\sin0=0$

   and from the figure (11),

   $\hat{i}\times\hat{j}=\hat{k},\hat{j}\times\hat{k}=\hat{i},$

   and $\hat{k}\times\hat{i}=\hat{j}$

   and $\hat{j}\times\hat{i}=-\hat{k},\hat{k}\times\hat{j}=-\hat{i}$

   $\hat{i}\times\hat{k}=-\hat{j}$

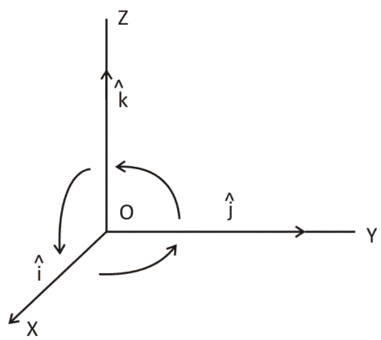


Fig 11

4. Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

   and $\bar{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

   then $\bar{a}\times\bar{b}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$

   $=a_1b_1\ \hat{i}\times\hat{i}+a_2b_2\ \hat{j}\times\hat{j}+a_3b_3\ \hat{k}\times\hat{k}+a_1b_2\ \hat{i}\times\hat{j}+a_1b_3\ \hat{i}\times\hat{k}+a_2b_1\ \hat{j}\times\hat{i}+a_2b_3\ \hat{j}\times\hat{k}$
   $+a_3b_1\ \hat{k}\times\hat{i}+a_3b_2\ \hat{k}\times\hat{j}$

   $=0+0+0+a_1b_2\hat{k}-a_1b_3\hat{j}-a_2b_1\hat{k}+a_2b_3\hat{i}+a_3b_1\hat{j}+a_3b_2\hat{i}$

   $=\hat{i}(a_2b_3-a_3b_2)-\hat{j}(a_1b_3-a_3b_1)+\hat{k}(a_1b_2-b_1a_2)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. While evaluating $\bar{a}\times\bar{b}$ in (4) we have assumed that vectors product is distributive over sum, i.e. $(\bar{a}\times(\bar{b}+\bar{c}))=\bar{a}\times\bar{b}+\bar{a}\times\bar{c}$. This is true. Vector product is distributive over sum of vectors.

3.7 To prove $\bar{a}\times(\bar{b}+\bar{c})=\bar{a}\times\bar{b}+\bar{a}\times\bar{c}$

In figure 12, vectors $\bar{a},\bar{b}$ and $\bar{c}$ lie in a vertical plane and S is the plane which is perpendicular to vectors A, $\overline{OA}=\bar{a},\overline{OB}=\bar{b},\overline{BC}=\bar{c},\bar{b}+\bar{c}=\overline{OC}$.

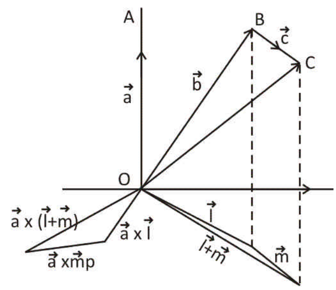


Fig 12

Projections vectors of $\overline{OB},\overline{BC}$ on plane S are ℓ and m

$\therefore\ OC' = \bar{\ell}+\bar{m}$; $OB' = |\overline{OB}|\sin\theta = b\sin\theta$

where θ is angle between vectors a and b

By definition of vector product, $\bar{a}\times\bar{b}=(ab\sin\theta)\hat{\eta}$

where η is unit vector perpendicular to both $\bar{a}$ and $\bar{b}$

vectors $\bar{a},\bar{b},\bar{c}$ lie in same vertical plane, $\therefore$ η which, is perpendicular to them lies in plane S.

$\therefore\ \bar{a}\times\bar{b}=(ab\sin\theta)\eta=a(b\sin\theta)\eta=a\ell n=\bar{a}\times\bar{\ell}$ ... (1)

arguing in same way $\bar{a}\times\bar{c}=\bar{a}\times\bar{m}$ ... (2)

OC` is vector projection of $(\bar{b}+\bar{c})$ on plane S

$\therefore\ \bar{a}\times(\bar{b}+\bar{c})=\bar{a}\times(\bar{\ell}+\bar{m})$ ... (3)

Now, $\bar{\ell}$ is perpendicular to $\bar{a}$ and $a\times\ell$ is perpendicular to both $\bar{a}$ and $\bar{\ell}$, this means $\bar{a}\times\bar{\ell}$ lie in plane S and can be obtained by rotating a times vector $\bar{\ell}$, i.e. a$\bar{\ell}$ is in positive direction about OA through angle 90^o in the plane S. In figure it is OP. Similarly by rotating in positive direction m times about OA through 90^o we get $\bar{a}\times\bar{m}$, which is PQ in the figure 12.

$\therefore\ \bar{a}\times\bar{\ell}+\bar{a}\times\bar{m}$ is equal to the sum of those two vectors which are obtained by rotating $\bar{a}\times\bar{\ell}$ and $\bar{a}\times\bar{m}$ which are obtained by rotating $\bar{a}\times\bar{\ell}$ and $\bar{a}\times\bar{m}$ through 90^o in positive direction in plane S. It is $\overline{OQ}$.

If $\bar{a}\times(\bar{\ell}+\bar{m})$ was rotated about OA in positive direction in plane S through 90^o, then we get $\bar{a}\times(\bar{\ell}+\bar{m})$ which is also represented by OQ.

$\therefore\ \overline{a}\times(\overline{\ell}+\overline{m}) = \overline{a}\times\overline{\ell}+\overline{a}\times\overline{m}$ ... (4)

from (1), (2) and (3)

$$\overline{a}\times(\overline{b}+\overline{c})=\overline{a}\times\overline{b}+\overline{a}\times\overline{c}$$

3.8 Area of a parallelogram

In figure 13, ABCD is parallelogram; $\angle ABC = \theta$

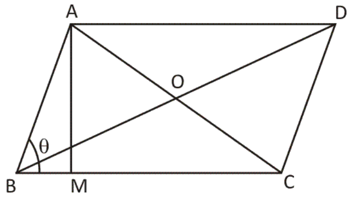


Fig 13

1. Area of parallelogram = BC $\times$ AM = BC . AB sin θ

   If $\overline{BC}=\overline{p}$ and $\overline{BA}=\overline{q}$ then $|\overline{p}|=BC, |\overline{q}|=AB$

   $\overline{p}\times\overline{q}$ = (BC $\cdot$ AB sin θ) $\eta \Rightarrow |\overline{p}\times\overline{q}|=BC\cdot AB\sin\theta$

   $\therefore$ If $\overline{p}$ and $\overline{q}$ be vectors of two adjacent sides of a parallelogram, then its area $|\overline{p}\times\overline{q}|$

2. Area of ΔABC $= \frac{1}{2}$ area of parallelogram

   $= \frac{1}{2}|\overline{p}\times\overline{q}|$

   Where $BC=\overline{p}$, and $BA=\overline{q}$

3. The diagonals of a parallelogram bisect each other. If C is point of intersection then area of Δ OAD = $\frac{1}{4}$ area of parallelogram

   $\therefore$ Area of parallelogram = 4 (area of ΔOAD)

   Now if $\overline{CA}=\overline{\alpha}$, and $\overline{BD}=\overline{\beta}$ then $\overline{OA}=\frac{1}{2}\overline{\alpha}$

   $\overline{OD}=\frac{1}{2}\overline{\beta}$. Area of parallelogram = $4\left[\frac{1}{2}\cdot\frac{1}{2}\overline{\alpha}\times\frac{1}{2}\overline{\beta}\right]$

   $$=\frac{1}{2}|\overline{\alpha}\times\overline{\beta}|$$

3.9 Moment of a force about a point

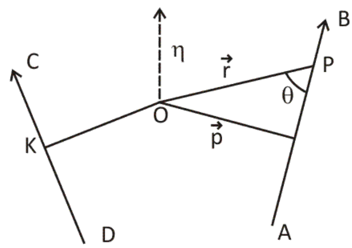


Fig 14

In figure 14, line segment AB represents force F in direction and magnitude O is a point in plane containing $\overline{AB}$, OL is perpendicular from O on AB

Let OL = p

Moment of force F about O is P·F. In figure 14, moment pf is positive as tendency of movement is anti-clockwise. Moment of F_1 is OK. F and it is negative as tendency of movement is clock-wise

Now if P be any point on the line of action of F and $\angle$ OPL = θ then OL = OP sin θ. You know $\overline{a}\times\overline{b} = ab\sin\theta\cdot\hat{\eta}$ therefore if force is $\overline{F}$ and $\overline{OP} = \overline{r}$ then

$$\overline{r}\times\overline{F} = OP\cdot AB\sin\theta\cdot\hat{\eta}$$

$$= AB\cdot(OP\sin\theta)\cdot\hat{\eta}$$

$$= AB\cdot(OP\sin\theta)\cdot\hat{\eta}$$

$$= AB\cdot p\hat{\eta}$$

$$= \text{moment of } \overline{F} \text{ about O}$$

It is perpendicular to the plane containing OP and line of action of F.

$\therefore$ Moment of a force F about a point O is $\overline{r}\times\overline{F}$ where $\overline{r}$ is the P.V. of any point on the line of action of force with respect to origin O

3.10 Lagrange's Identity:

$(\overline{a}\times\overline{b})^2 = a^2b^2 - (\overline{a}\cdot\overline{b})^2$

where $|\overline{a}| = a, |\overline{b}| = b$ is known as Lagrange's identity.

Proof: $(\overline{a}\times\overline{b})^2 = (\overline{a}\times\overline{b})\cdot(\overline{a}\times\overline{b}) = |\overline{a}\times\overline{b}|^2$

$\therefore$ $(\overline{a}\times\overline{b})^2 = |\overline{a}\times\overline{b}|^2 = (ab\sin\theta\cdot\eta)^2$

$$= a^2b^2\sin^2\theta = a^2b^2(1-\cos^2\theta)$$

$$= a^2b^2 - a^2b^2\cos^2\theta = a^2b^2 - (\overline{a}\cdot\overline{b})^2$$

Hence, $(\overline{a}\times\overline{b})^2 = a^2b^2 - (\overline{a}\cdot\overline{b})^2$

Solved Example

Example 16: Find unit vector perpendicular to vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}+2\hat{k}$.

Sol. Let $\bar{a}=4\hat{i}-\hat{j}+3\hat{k}$ and $\bar{b}=-2\hat{i}+\hat{j}+2\hat{k}$ vector perpendicular to $\bar{a}$ and $\bar{b}$ is $\bar{a}\times\bar{b}$ and $\bar{b}\times\bar{a}$ is also perpendicular. The direction of unit perpendicular vector is just opposite in the two cases. If $\bar{a}\times\bar{b}$ is upwards the plane of $\bar{a}$ and $\bar{b}$ then $\bar{b}\times\bar{a}$ is downwards this plane.

$$\bar{a}\times\bar{b}=\begin{vmatrix} i & j & k \\ 4 & -1 & 3 \\ -2 & 1 & 2 \end{vmatrix}=\hat{i}(-5)-\hat{j}(14)+\hat{k}(2)$$

$$=-5\hat{i}-14\hat{j}+2\hat{k}$$

$$|\bar{a}\times\bar{b}|=\sqrt{25+196+4}=\sqrt{225}=15$$

$\therefore$ Unit vector is $-\frac{1}{3}\hat{i}-\frac{14}{15}\hat{j}+\frac{2}{5}\hat{k}$

and $\frac{1}{3}\hat{i}+\frac{14}{15}\hat{j}-\frac{2}{15}\hat{k}$

Example 17: Show that two non-zero vectors cannot be simultaneously parallel and perpendicular.

Sol. Let $\bar{a}$ and $\bar{b}$ be two vectors, non-zero put parallel and perpendicular simultaneously.

$\bar{a}$ is parallel to $\bar{b}$ $\quad\therefore\ \bar{a}=\lambda\bar{b}$ where λ is an non-zero scalar.

$\bar{a}$ is perpendicular to $\bar{b}=\bar{a}\cdot\bar{b}=0$ and as $\bar{a}=\lambda\bar{b}$

$\therefore\ \lambda\ \bar{b}\cdot\bar{b}=0=\lambda|\bar{b}|^2=0 \qquad \because \lambda\neq 0 \qquad \therefore\ |\bar{b}|=0$

But we have supposed that $|\bar{b}|\neq 0$.

$\therefore$ $\bar{a}$ and $\bar{b}$ can not be simultaneously parallel and perpendicular.

Example 18: The adjacent sides of a parallelogram are $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$. Find its area.

Sol. If $\overline{AB}$ and $\overline{AD}$ are adjacent sides of a parallelogram then its area is $|\bar{a}\times\bar{b}|$

$\therefore$ Area = $(\hat{i}+2\hat{j}+3\hat{k})\times(3\hat{i}-2\hat{j}+\hat{k})$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 3 & -2 & 1\end{vmatrix}=8\hat{i}+8\hat{j}-8\hat{k}$$

$\therefore$ Area = $\left|8\hat{i}+8\hat{j}-8\hat{k}\right|=8\sqrt{3}$ sq. units

Example 19: The diagonals of a parallelogram are $\bar{a}=3\hat{i}+4\hat{j}+5\hat{k}$ and $\bar{b}=3\hat{i}+2\hat{j}+3\hat{k}$. Find its area

Sol. $\bar{a}=3\hat{i}+4\hat{j}+5\hat{k}$ and $b=3\hat{i}+2\hat{j}+3\hat{k}$

$$\text{area} = \frac{1}{2}|\bar{a}\times\bar{b}|=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & 5\\ 3 & +2 & 3\end{vmatrix}=\frac{1}{2}\left|2\hat{i}+6\hat{j}-6\hat{k}\right|$$

$$=\frac{1}{2}\sqrt{4+36+36}=\frac{2}{2}\cdot\sqrt{19}=\sqrt{19}\,\text{sq.units}$$

Example 20: The force $\bar{F}=\hat{i}+2\hat{j}+3\hat{k}$, goes through point $-2\hat{i}+3\hat{j}+\hat{k}$. Find its moment about the point $\hat{i}+\hat{j}+\hat{k}$.

Sol. Let O be the point - $\hat{i}+\hat{j}+\hat{k}$

and P the point - $2\hat{i}+3\hat{j}+\hat{k}$

$\therefore\ \bar{r}=\overline{OP}=(-2\hat{i}+3\hat{j}+\hat{k})-(\hat{i}+\hat{j}+\hat{k})=-3\hat{i}+2\hat{j}+0\cdot\hat{k}$

Moment about O = $\overrightarrow{OP}\times\vec{F}$

$$=(-3\hat{i}+2\hat{j})\times(\hat{i}+2\hat{j}+3\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 2 & 0\\ 1 & 2 & 3\end{vmatrix}$$

$$=6\hat{i}+9\hat{j}-8\hat{k}$$

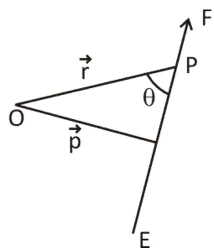


Fig 15

Example 21: $\hat{i}\times(\bar{A}\times\hat{i})+\hat{j}\times(\bar{A}\times\hat{j})+\hat{k}\times(\bar{A}\times\hat{k})$ is equal to

(a) $\bar{A}$ (b) $2\bar{A}$ (c) $3\bar{A}$ (d) 0

Sol. Let $\bar{A}=x\hat{i}+y\hat{j}+z\hat{k}$

$\therefore\ \bar{A}\times\hat{i}=(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}$

$=0-y\hat{k}+z\hat{j}$

$\hat{i}\times(\overline{A}\times\hat{i})=\hat{i}\times(-y\hat{k}+z\hat{j})$

$= y\hat{j}+z\hat{k}$

and $\therefore$ $\hat{j}\times(A\times\hat{j})=z\hat{k}+x\hat{i}$

and $\hat{k}\times(A\times\hat{k})=x\hat{i}+y\hat{j}$

$\therefore$ $\hat{i}\times(\overline{A}\times\hat{i})+\hat{j}\times(\overline{A}\times\hat{j})+\hat{k}\times(\overline{A}\times\hat{k})$

$=y\hat{j}+z\hat{k}+z\hat{k}+x\hat{i}+x\hat{i}+y\hat{j}$

$=2(x\hat{i}+y\hat{j}+z\hat{k})=2\overline{A}$

Example 22: $\overline{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\overline{b}=-2\hat{i}+\hat{j}+\hat{k}$ are two vectors. Find a vector of magnitude λ normal to the plane containing $\overline{a}$ and $\overline{b}$.

Sol. Unit vector perpendicular to the plane

Containing $\overline{a}$ and $\overline{b}$ is $\dfrac{\overline{a}\times\overline{b}}{|\overline{a}\times\overline{b}|}$

$\overline{a}\times\overline{b}=(2\hat{i}+3\hat{j}-\hat{k})\times(-2\hat{i}+\hat{j}+\hat{k})$

$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -1\\ 2 & 1 & 1\end{vmatrix}= 4\hat{i}-0\hat{j}+8\hat{k}=4\hat{i}+8\hat{k}$ and $|\overline{a}\times\overline{b}|=\sqrt{16+64}=4\sqrt{5}$

$\therefore$ Required vector is $\dfrac{\lambda(4\hat{i}+8\hat{k})}{4\sqrt{5}}$

Example 23: $\overline{a}\times(\overline{b}+\overline{c})+\overline{b}\times(\overline{c}+\overline{a})+\overline{c}\times(\overline{a}+\overline{b})$ is $=$

(a) 1 (b) 0 (c) $\overline{a}+\overline{b}+\overline{c}$ (d) none of these

Sol. $\overline{a}\times(\overline{b}+\overline{c})+\overline{b}\times(\overline{c}+\overline{a})+\overline{c}\times(\overline{a}+\overline{b})$

$=\overline{a}\times\overline{b}+\overline{a}\times\overline{c}+\overline{b}\times\overline{c}+\overline{b}\times\overline{a}+\overline{c}\times\overline{a}+\overline{c}\times\overline{b}$

$=(\overline{a}\times\overline{b}+\overline{b}\times\overline{a})+(\overline{a}\times\overline{c}+\overline{c}\times\overline{a})+(\overline{b}\times\overline{c}+\overline{c}\times\overline{b})$

$=(\overline{a}\times\overline{b}-\overline{a}\times\overline{b})+(\overline{a}+\overline{c}-\overline{a}\times\overline{c})+(\overline{b}\times\overline{c}-\overline{b}\times\overline{c})$

$=\overline{0}+\overline{0}+\overline{0}=0$

Example 24: $\overline{a},\overline{b},\overline{c}$ are position vectors of vertices A, B and C of triangle ABC. Prove that

(a) area of $\Delta ABC = \frac{1}{2}\left[\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}\right]$

(b) Find length of perpendicular from C on AB

Sol. (a) Given $\overline{AB}=\bar{b}-\bar{a},\overline{AC}=\bar{c}-\bar{a}$

area of $\Delta = \frac{1}{2}(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})$

$$=\frac{1}{2}(\bar{b}\times\bar{c}-\bar{b}\times\bar{a}-\bar{a}\times\bar{c}+\bar{a}\times\bar{a})$$

$$=\frac{1}{2}(\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}+0)$$

$$=\frac{1}{2}(\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b})$$

(b) If CM is perpendicular from C on AB.

$$\text{Area } \Delta ABC = \frac{1}{2}\cdot AB\cdot CM=\frac{1}{2}\left[\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}\right]$$

$$\therefore\ CM=\frac{\left[b\times c+c\times a+a\times\bar{b}\right]}{|b-a|}$$

Example 25: $\bar{a},\bar{b},\bar{c}$ are three non-zero vectors such that $\bar{a}\times\bar{b}=\bar{c}$, $\bar{b}\times\bar{c}=\bar{a}$; prove that $\bar{a},\bar{b}$ and $\bar{c}$ are mutually perpendicular and $|\bar{b}|=1$ and $|\bar{c}|=|\bar{a}|$

Sol. (i) $\bar{a}\times\bar{b}=\bar{c}\Rightarrow\bar{c}\perp\bar{a}\,\text{and}\,\bar{c}\perp\bar{b}$

$\bar{b}\times\bar{c}=\bar{a}\Rightarrow\bar{b}\perp\bar{a}\,\text{and}\,\bar{a}\perp\bar{c}$

$\therefore$ $\bar{a},\bar{b}$ and $\bar{c}$ are mutually perpendicular vectors

(ii) $\bar{a}\times\bar{b}=\bar{c}\Rightarrow|\bar{a}\times\bar{b}|=|\bar{c}|\Rightarrow a\cdot b=c$

similarly $|\bar{b}\times\bar{c}|=\bar{a}\Rightarrow bc=a$

$\therefore$ $b=\frac{c}{a}$ and $b=\frac{a}{c}$ = b^2 = 1 $\Rightarrow$ $|\bar{b}|=1$

$\because|\bar{a}|=a,|\bar{b}|=1\quad \sin 90^{o}=1$

Example 26: let Ar (r = 1, 2, 3, 4) be area of faces of a tetrahedron. Let η r be the outward normals drawn to the respective faces with magnitude equal to corresponding and

Prove that $\eta_1+\eta_2+\eta_3+\eta_4=0$

Sol. Let OABC be the tetrahedron and $\overline{OA} = \overline{a}$, $\overline{OB} = b$, $\overline{OC} = \overline{c}$

η_1 = vector area of ΔOAB

face $= \frac{1}{2} \cdot \overline{OA} \times \overline{OB}$

$= \frac{1}{2} \hat{a} \times \hat{b}$

For face OBC, $\eta_2 = \frac{1}{2} \overline{OB} \times \overline{OC} = \frac{1}{2} \overline{b} \times \overline{c}$

For face OCA, $\eta_3 = \frac{1}{2} \overline{OC} \times \overline{OA} = \frac{1}{2} \overline{c} \times \overline{a}$

For face ABC, $\eta_4 = \frac{1}{2} \overline{AC} \times \overline{AB}$

$\Rightarrow \quad \eta_4 = \frac{1}{2} (\overline{c} - \overline{a}) \times (\overline{b} - \overline{a})$

$= \frac{1}{2} (\overline{c} \times \overline{b} - \overline{c} \times \overline{a} - \overline{a} \times \overline{b} + 0)$

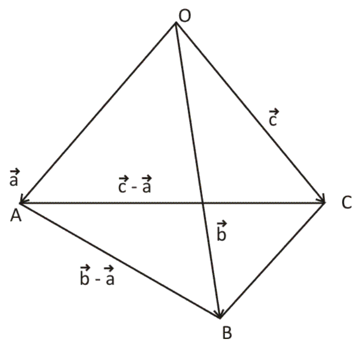


Fig 16

$\therefore \quad \eta_1 + \eta_2 + \eta_3 + \eta_4 = \frac{1}{2} \left[\overline{a} \times \overline{b} + \overline{b} \times \overline{c} + \overline{c} \times \overline{a} + \overline{c} \times \overline{b} - \overline{c} \times \overline{a} - \overline{a} \times \overline{b} \right]$

$= \frac{1}{2} \left[\overline{b} \times \overline{c} + \overline{c} \times \overline{b} \right] = \frac{1}{2} \left[\overline{b} \times \overline{c} - \overline{b} \times \overline{c} \right]$

$= 0$

Example 27: $\overline{a} \times \overline{b} = \overline{c} \times \overline{d}$ and $\overline{a} \times \overline{c} = \overline{b} \times \overline{d}$ show that $\overline{a} - \overline{d}$ is parallel to $\overline{b} - \overline{c}$; $\overline{a} \neq \overline{d}$, $\overline{b} \neq \overline{c}$

Sol. $(\overline{a} - \overline{d})$ shall be parallel to $\overline{b} - \overline{c}$

If $(\overline{a} - \overline{d}) \times (\overline{b} - \overline{c}) = 0$

$(\overline{a} - \overline{d}) \times (\overline{b} - \overline{c}) = \overline{a} \times \overline{b} - \overline{a} \times \overline{c} - \overline{d} \times \overline{b} + \overline{d} \times \overline{c}$

$= (\overline{a} \times \overline{b} - \overline{c} \times \overline{d}) + (\overline{b} \times \overline{d} - \overline{a} \times \overline{c})$

$= 0 + 0 = 0$

$\therefore$ parallel

Example 28: Find all possible vectors $\overline{A}$ and $\overline{B}$ such that $\overline{A} \times \overline{B} = 2\hat{i} + 2\hat{j} - \hat{k}, \overline{A} + \overline{B} = \hat{i} - 3\hat{j} - 4\hat{k}$

Sol. $\therefore$ Let $\overline{A} = a_1\hat{i} + b_1\hat{j} + c_2\hat{k},\ \overline{B} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

$\therefore\ \overline{A} + \overline{B} = (a_1 + a_2)\hat{i} + (b_1 + b_2)\hat{j} + (c_1 + c_2)\hat{k}$

But $\overline{A} + \overline{B} = \hat{i} - 3\hat{j} - 4\hat{k}$

$$\left.\begin{aligned} \therefore a_1 + a_2 &= 1 \ \Rightarrow a_2 = 1 - a_1 & \text{....(i)} \\ b_1 + b_2 &= -4 \ \Rightarrow b_2 = -3 - b_1 & \text{....(ii)} \\ c_1 + c_2 &= -4 \ \Rightarrow c_2 = -4 - c_1 & \text{....(iii)} \end{aligned}\right\} (\alpha)$$

and, $\overline{A} \times \overline{B} = (b_1c_2 - c_1b_2)\hat{i} + (a_2c_1 - a_1c_2)\hat{j} + (a_1b_2 - b_1a_2)\hat{k}$

Given $\overline{A} \times \overline{B} = 2\hat{i} + 2\hat{j} - \hat{k}$

$\therefore\ b_1c_2 - c_1b_2 = 2,\ a_2c_1 - a_1c_2 = 2,\ a_1b_2 - b_1a_2 = -1$

from (α) $\therefore\ b_1(-4 - c_1) - c_1(-3 - b_1) = -4b_1 + 3c_1 = 2$ ... (1)

$c_1(1 - a_1) - a_1(-4 - c_1) = c_1 + 4a_2 = 2$ ... (2)

$a_1(-3 - b_1) - (1 - a_1) \cdot b_1 = 3a_1 + b_1 = 1$ ... (3)

$$\left.\begin{aligned} \text{from (1) and (2)} \quad & 2a_1 + 2b_1 = c \\ \text{from (3)} \quad & b_1 = 1 - 3a_1 \end{aligned}\right\} (\beta)$$

By giving any value to a_1 in β value of b_1 and c_1 calculated and from corresponding values of a_2, b_2, c_2 can be calculated. So any $\overline{A}$ and corresponding $\overline{B}$ can be calculated.

If $a_1 = 1,\ b_1 = -2,\ c_1 = -2$

$a_2 = 0,\ b_2 = -1,\ c_2 = -2$

$\overline{A} = \hat{i} - 2\hat{j} - 2\hat{k},\ \overline{B} = -\hat{j} - 2\hat{k}$

Example 29: Prove by vector method, that medians of a triangle are concurrent.

Sol. D, E and mid points of BC and AC. Medians AD and BE meet in O. Co is joined and produced to meet AB in F. We shall prove that F is mid point of AB.

Let O, be origin and $\overline{OA} = \overline{a}, \overline{OB} = \overline{b}$ and $\overline{OC} = \overline{c}$

$\therefore\ OD = \frac{1}{2}(\overline{b} + \overline{c})$ and $OE = \frac{1}{2}(\overline{c} + \overline{a})$

$\overrightarrow{OD}$ and $\overrightarrow{OA}$ are in opposite direction. $\angle AOD = 180^o$

$\therefore\ \overline{OA} \times \overline{OD} = 0 \ \Rightarrow\ a \times \frac{1}{2}(\overline{b} + \overline{c}) = 0$ .... (i)

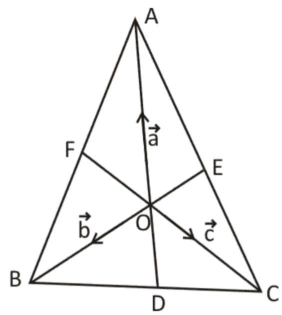


Fig 17

Similarly $\overline{OB}\times\overline{OE}=0\Rightarrow\overline{b}\times\frac{1}{2}(\overline{a}+\overline{c})=0$ .... (ii)

$$\left.\begin{array}{l}\text{from (i) } \overline{a}\times\overline{b}+\overline{a}\times\overline{c}=0\\ \text{from (ii) } \overline{b}\times\overline{a}+\overline{b}\times\overline{c}=0\end{array}\right\}\Rightarrow\overline{a}\times\overline{b}+\overline{a}\times\overline{c}+(-\overline{a}\times\overline{b})+\overline{b}\times\overline{c}=0$$

$\Rightarrow\overline{a}\times\overline{c}+\overline{b}\times\overline{c}=0\Rightarrow(\overline{a}+\overline{b})\times\overline{c}=0$

and it proves $\frac{1}{2}(\overline{a}+\overline{b})$ and $\overline{c}$ are parallel.

But $\frac{1}{2}(\overline{a}+\overline{b})$ is p.v. of mid point of AB

$\therefore$ CO meets AB is mid point F medians are concurrent

Example 30: A rigid body is rotating about an axis through point (3, –1, –2). If particle at point (4, 1, 0) has velocity $4\hat{i}-4\hat{j}+2\hat{k}$ and the particle at point (3, 2, 1) has velocity $6\hat{i}-4\hat{j}+4\hat{k}$. Calculate magnitude and direction of angular velocity of the body.

Sol. A is (3, –1, –2) P is (4, 1, 0), Q (3, 2, 1) r_1 = $\overline{AP}=\hat{i}+2\hat{j}+2\hat{k}$

$r_2=\overline{AQ}=3\hat{j}+3\hat{k}$

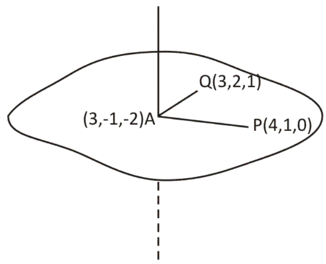


Fig 18

Let $\overline{\omega}=a\hat{i}+b\hat{j}+c\hat{k}$ be angular velocity.

$\overline{v}_1=\overline{r}_1\times\overline{\omega}$

$\Rightarrow(4\hat{i}-4\hat{j}+2\hat{k})=(\hat{i}+2\hat{j}+2\hat{k})\times(a\hat{i}+b\hat{j}+c\hat{k})$

$\Rightarrow 4\hat{i}-4\hat{j}+2\hat{k}=2(c-b)\hat{i}+(2a-c)\hat{j}+(b-2a)\hat{k}$

$\therefore$ c – b = 2, c – 2a = 4, b – 2a = 2 ... (α)

Here c-b = 2, c-2a =4 $\Rightarrow$ $\overline{b}-2\overline{a}=2$

and v_2 = 6i – 4j + 4k = $(3\hat{j}+3\hat{k})\times(a\hat{i}+b\hat{j}+c\hat{k})$

= $3(c-b)\hat{i}+3a\hat{j}-3a\hat{k}$

$\therefore$ c – b = 2, 3a = –4, 3a = –4 ... (β)

$\therefore$ a = $-\frac{4}{3}$; and from (α) b = 2 + 2a = $-\frac{2}{3}$

from (β) c = 2 + b = 2 – $\frac{2}{3}$ = $\frac{4}{3}$

$\therefore$ angular velocity $\overline{\omega} = -\frac{4}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{4}{3}\hat{k}$

and $|\overline{\omega}| = \frac{1}{3}\sqrt{36} = 2$

3.11 (a) Moment about a line

The moment of a force F acting at a point P, about a line L is given by

$$(\bar{r}\times\bar{F})\cdot\hat{a}$$

where $\hat{a}$ is unit vector in the discussion of line and $\overline{OP}=r$, where O is any point on the line.

3.11 (b) Moment of a couple

Two equal, unlike parallel faces from a couple a straight line BN drawn perpendicular to the two lines of action of the forces is called arm of the couple. In figure BN is arm of the couple, F, –F are forces.

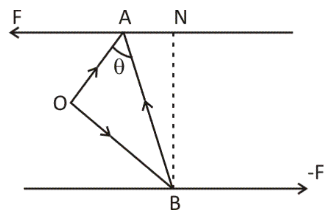


Fig 19

If (BN) is also called axis of the couple.

Let A be a point on the lines of action of F and B on the line of action of –F

O is any point in the plane.

Sum of moment of $\vec{F}$ and $-\vec{F}$ (same direction)

about O is given by

$$\overrightarrow{OA}\times F+\overrightarrow{OB}\times(-F)$$

$$=(\overrightarrow{OA}-\overrightarrow{OB})\times F=(\overrightarrow{OA}+\overrightarrow{BO})\times F$$

$$=(\overrightarrow{BO}+\overrightarrow{OA})\times F=\overrightarrow{BA}\times F=\vec{M}$$

Thus the moment of the forces of a couple is independent of O. Vector $\vec{M}$ is called the moment of couple

$$\vec{M}=|\overrightarrow{BA}\times F|=|\overrightarrow{BA}|\,|F|\sin\theta$$

$$=(BA\sin\theta)F=(BN)F$$

$$\therefore\ M=\pm\,|\vec{F}|\ \text{(arm of the couple)}$$

It is a anticlockwise, it is clockwise –ve

Solved Example

Example 31: Find the moment about a line through origin having direction $2\hat{i}+2\hat{j}-\hat{k}$ due to 3.5 kg force acting at (–4, 2, 5) in the direction of $6\hat{i}-2\hat{j}+3\hat{k}$.

Sol. Force $\vec{F}=35\frac{\left[6\hat{i}-2\hat{j}+3\hat{k}\right]}{\sqrt{6^2+4+9}}=30\hat{i}-10\hat{j}+10\hat{k}$

It acts at P; $-4\hat{i}+2\hat{j}+5\hat{k}$

Line goes through origin $\therefore$ O is (0, 0, 0)

$\vec{r}=\overrightarrow{OP}=-4\hat{i}+2\hat{j}+5\hat{k}$.

Direction of line $2\hat{i}+2\hat{j}-\hat{k} \Rightarrow \hat{a}=\frac{1}{3}(2\hat{i}+2\hat{j}-\hat{k})$

Moment of force about line is

$$(\vec{r}\times\vec{F})\cdot\hat{a}=\left[\left(-4\hat{i}+2\hat{j}+5\hat{k}\right)\times\left(3a\hat{i}-10\hat{j}+15\hat{k}\right)\right]\cdot\frac{1}{3}(2\hat{i}+2\hat{j}-\hat{k})$$

$$=(80\hat{i}+210\hat{j}+20\hat{k})\cdot\frac{1}{3}(2\hat{i}+2\hat{j}+\hat{k})$$

$$=\frac{1}{3}[160+420-20]=\frac{560}{3}$$

Example 32: Find the moment of the couple consisting of force $F=3\hat{i}+2\hat{j}-\hat{k}$ acting through $\hat{i}-2\hat{j}+3\hat{k}$ and force $-F$ acting through $2\hat{i}+3\hat{j}-\hat{k}$.

Sol. $\vec{r}=BA=(\hat{i}-2\hat{j}+3\hat{k})-(2\hat{i}+3\hat{j}-\hat{k})=-\hat{i}-5\hat{j}+4\hat{k}$

$$\overline{M}=\bar{r}\times\overline{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & -5 & 4\\ 3 & 2 & -1\end{vmatrix}=-3\hat{i}+11\hat{j}+13\hat{k}$$

Moment of couple = $|\overline{M}|=\sqrt{9+121+169}=\sqrt{299}$

Practice Worksheet (Foundation Level) – 3(b)

1. $\bar{a}\times(\bar{b}+\bar{c})+\bar{b}\times(\bar{c}+\bar{a})+\bar{c}\times(\bar{a}+\bar{b})=...$

2. $\bar{a}=\hat{i}+2\hat{j}+3\hat{k},\ \bar{b}=-\hat{i}+2\hat{j}+\hat{k}, \bar{c}=3\hat{i}+\hat{j}$ unit vector along their resultant is ...

3. Unit vector perpendicular to $\bar{a}=2i-j+k$ and $\bar{b}=-i+2j-3k$ is ...

4. $\bar{a},\bar{b}$ and $\bar{c}$ are three vectors such that $\bar{a}+\bar{b}+\bar{c}=0$ then prove

   $\bar{a}\times\bar{b}=\bar{b}\times\bar{c}=\bar{c}\times\bar{a}$

5. Prove that normal to the plane containing three points whose position vectors are $\bar{a},\bar{b}$ and $\bar{c}$ lies along $\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}$.

6. If $|\bar{a}|=3,|\bar{b}|=4$ and $|\bar{c}|=5$ are such that each is perpendicular to the sum of the other two. Prove $|\bar{a}+\bar{b}+\bar{c}|=5\sqrt{2}$.

   (Hint $\bar{a}\cdot(\bar{a}+\bar{b}+\bar{c})=\bar{a}\cdot\bar{a}=|\bar{a}|^2=9...(\bar{a}+\bar{b}+\bar{c})\cdot(\bar{a}+\bar{b}+\bar{c})=9+16+25=50$)

7. Position vectors of vertices A, B, C of triangle ABC are (1, 2, 3), (–1, 2, 1) and (0, 4, 2) than $\angle ABC$ is ...

8. $\bar{a}=2\hat{i}+5\hat{j}+3\hat{k},\bar{b}=3\hat{i}+3\hat{j}+6\hat{k}$ and $\bar{c}=2\hat{i}+2\hat{j}+4\hat{k}$ then $(\bar{a}-\bar{b})\times(\bar{c}-\bar{b})$ = ...

9. Two adjacent sides of a parallelogram are $4\hat{i}+2\hat{j}-5\hat{k}$ and $-3\hat{i}+2\hat{j}+5\hat{k}$. Its area is ...

10. The diagonals of a square are $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$. Its area is ...

11. The area of triangle whose vertices are (1, 1, 2), (2, 3, 5) and (1, 3, 6) is ...

12. A force 5i – 2j + 3k acts through (1, 2, 3) Its moment about the point (3, 2, 1) is ...

13. A force of 6 N acting along the line joining A(2, 1, 0) and B (3, –1, 2). Its moment about point C, (1, 1, –2) is ...

14. A force of 12 N acting along vector 9i + 6j – 2k through (4, –1, 7). The moment of the force about (–1, –8, 2) is ...

15. $\bar{A}$ and $\bar{B}$ vectors are such that $\bar{A}+\bar{B}=5i+3j-2k$ and $\bar{A}-\bar{B}$ is i + 5j – 8k then $\bar{A}\times\bar{B}$ is ...

16. Prove that $\begin{vmatrix}\bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{b}\\ \bar{a}\cdot\bar{b} & \bar{b}\cdot\bar{b}\end{vmatrix}=(\bar{a}\times\bar{b})^2$

17. P.V. of vertices of a triangles are $2\hat{i}+\hat{j}-\hat{k},\hat{i}-2\hat{j}+\hat{k}$ and $-5\hat{i}+2\hat{j}-2\hat{k}$. Find (i) vectors representing sides (ii) length of sides (iii) area of triangle.

18. If $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$ are four vectors in space then prove

$(\overline{a}-\overline{d})\cdot(\overline{b}-\overline{c})+(\overline{b}-\overline{d})\cdot(\overline{c}-\overline{a})+(\overline{c}-\overline{d})(\overline{a}-\overline{b})=0$

19. $\overline{a}$ and $\overline{b}$ are two non-zero non-collinear vectors then (i) a vector in plane of $\overline{a}$ and $\overline{b}$ is ... (ii) a vector $\perp$ to the plane containing g vector $\overline{b}$ and $\overline{c}$ is ... (iii) Vector $\perp$ to $\overline{a}+\overline{b}$ and $\overline{a}-\overline{b}$ is ...

20. $\overline{a},\overline{b}$ and $\overline{c}$ are three vectors such that $\overline{a}\neq 0$. If $\overline{a}\times\overline{b}=\overline{a}\times\overline{c}$ and $\overline{a}\cdot\overline{b}=\overline{a}\cdot\overline{c}$ then prove $\overline{b}=\overline{c}$.

21. $\overline{a},\overline{b}$ and $\overline{c}$ are non-zero vectors of which no two are collinear. If $\overline{a}+2\overline{b}$ is collinear with $\overline{c}$ and $\overline{b}+3\overline{c}$ collinear with $\overline{a}$ then $\overline{a}+2\overline{b}+6\overline{c}$ is

(a) $\lambda\overline{a}$ (b) $\lambda\overline{b}$ (c) $\lambda\overline{c}$ (d) 0

22. If $\overline{a}=(1,1,1)$ and $\overline{b}=(1,1,-1)$ and $\overline{p},\overline{q}$ are two vectors such that $\overline{a}=2\overline{p}+\overline{q}$ and $\overline{b}=\overline{p}+2\overline{q}$ then angle between $\overline{p}$ and $\overline{q}$ is

(a) $\cos^{-1}\left(\frac{7}{99}\right)$ (b) $\cos^{-1}\left(\frac{7}{11}\right)$ (c) $\cos^{-1}\left(\frac{6\sqrt{2}}{11}\right)$ (d) $\cos^{-1}\left(-\frac{7}{11}\right)$

23. The vectors of two adjacent sides of a parallelogram are $\overline{a}$ and $\overline{b}$, then vector which is altitude of parallelogram and perpendicular to $\overline{a}$ is

(a) $\frac{(\overline{a}\cdot\overline{b})}{|\overline{a}|^2}\overline{a}-\overline{b}$ (b) $\frac{1}{|\overline{a}|^2}\left[|\overline{a}|^2\,\overline{b}-(\overline{a}\cdot\overline{b})\overline{a}\right]$

(c) $\frac{\overline{a}\times(\overline{a}\times\overline{b})}{|\overline{a}|^2}$ (d) $\frac{\overline{a}\times(\overline{b}\times\overline{a})}{|\overline{b}|^2}$

24. The vectors of two adjacent sides of a parallelogram are $\overline{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\overline{b}=3\hat{i}+2\hat{j}+2\hat{k}$ the altitude vector of parallelogram $\perp$ to $\overline{a}$ is

(a) $\frac{1}{3}(5\hat{i}+8\hat{j}+4\hat{k})$ (b) $\frac{1}{3}(5\hat{i}-8\hat{j}+4\hat{k})$ (c) $\hat{i}+\hat{j}+3\hat{k}$ (d) $\hat{i}+\hat{j}-3\hat{k}$

25. ABC is a triangle and $\overline{BC},\overline{CA}$ and $\overline{AB}$ are vectors $\overline{a},\overline{b}$ and $\overline{c}$, then $\overline{b}\times\overline{c}=$

(a) $\overline{a}\times\overline{c}$ (b) $\overline{a}\times\overline{b}$ (c) $\overline{c}\times\overline{a}$ (d) $\overline{b}\times\overline{a}$

26. A vector $\overline{a}$ is expressed as sum of two vectors α and β along and perpendicular to the given vector $\overline{b}$ then β is equal to

(a) $\frac{(\overline{a}\times\overline{b})\times\overline{b}}{|\overline{b}|^2}$ (b) $\frac{(\overline{b}\cdot\overline{b})\overline{a}-(\overline{a}\cdot\overline{b})\overline{b}}{|\overline{b}|^2}$ (c) $\frac{\overline{b}\times(\overline{a}\times\overline{b})}{|\overline{b}|^2}$ (d) $\left(\frac{\overline{a}\cdot\overline{b}}{|\overline{b}|^2}\right)\overline{b}$

27. The adjacent sides of a parallelogram are $4\bar{a}+2\bar{b}$ and $\bar{a}-3\bar{b}$ and angle between $\bar{a}$ and $\bar{b}$ is 45^{o}, $|\bar{a}|=1$ and $|b|=\sqrt{2}$, then length of larger diagonal is

(a) $\sqrt{17}$ (b) $\sqrt{103}$ (c) both equal (d) $\sqrt{89}$

28. ABCDEF is regular hexagon $\overline{AB}=\bar{a}$, $\overline{BC}=\bar{b}$ then $\overline{CE}=$

(a) $\bar{b}-\bar{a}$ (b) $2\bar{b}-\bar{a}$ (c) $\bar{b}-2\bar{a}$ (d) $-\bar{b}$

29. D, E and F are mid points of sides BC, CA and AB of a triangle then,

$\overrightarrow{AD}+\overrightarrow{BF}+\overrightarrow{CF}=$

(a) $\bar{0}$ (b) 0 (c) 2 (d) 1

30. The vectors $2\hat{i}+3\hat{j}$, $5\hat{i}+6\hat{j}$ and $8\hat{i}+9\hat{j}$ have their initial point at (1, –1). The final points A, B and C of vectors lie on a straight line and B divides AC in the ratio of λ : 1 then λ is

(a) 2 (b) $\frac{1}{2}$ (c) 3 (d) 1

31. A line goes through origin and its direction is $3\hat{i}-4\hat{j}+5\hat{k}$. Force of 35 N act through point $2\hat{i}-3\hat{j}+4\hat{k}$ in the direction of $3i+2j+6\hat{k}$. Its moment about line is

(a) $\frac{143}{\sqrt{2}}$ (b) $\frac{13\sqrt{2}}{2}$ (c) $-\frac{13\sqrt{2}}{2}$ (d) $-13\sqrt{2}$

32. Forces F and –F form a couple and act through $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-3\hat{k}$ respectively. If F = $2\hat{i}+\hat{j}+2\hat{k}$, then arm of couple is

(a) $\frac{5\sqrt{3}}{3}$ (b) $\frac{5\sqrt{2}}{3}$ (c) $\frac{5}{3}$ (d) $\frac{5\sqrt{5}}{3}$

33. Vector $\bar{a}=$

(a) $(\bar{a}\cdot\hat{j})\hat{j}+(\bar{a}\cdot\hat{k})\hat{k}+(\bar{a}\cdot\hat{i})\hat{i}$ (b) $(\hat{a}\cdot\hat{j})\hat{i}+(\hat{a}\cdot\hat{k})\hat{j}+(\hat{a}\cdot\hat{i})\hat{k}$

(c) $(\hat{a}\cdot\hat{j})\hat{k}+(\hat{a}\cdot\hat{k})\hat{i}+(\hat{a}\cdot\hat{i})\hat{j}$ (d) None of these

34. Angle between vectors $\bar{a}$ and $\bar{b}$ is θ and $\bar{a}\cdot\bar{b}\geq 0$ then

(a) $0\leq\theta\leq\pi$ (b) $\frac{\pi}{2}\leq\theta\leq\pi$ (c) $0\leq\theta\leq\frac{\pi}{2}$ (d) $0<\theta<\frac{\pi}{2}$

35. $\bar{a},\bar{b}$ and $\bar{c}$ are vertices of an equilateral triangle whose orthocentre is at origin, then

(a) $\bar{a}+\bar{b}+\bar{c}=0$ (b) $\bar{a}^2+\bar{b}^2+\bar{c}^2=0$

(c) $\bar{a}+\bar{b}=\bar{c}$ (d) $\bar{b}+\bar{c}=\bar{a}$

36. $\bar{a}, \bar{b}$ and $\bar{c}$ are three mutually perpendicular vectors of equal magnitude, $(\bar{a}+\bar{b}+\bar{c})$ makes angle θ with any of them, then θ is

(a) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{1}{3}\right)$ (c) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (d) $\cos^{-1}\left(\frac{2}{3}\right)$

[Hint: Let $|\bar{a}|=|\bar{b}|=|\bar{c}|=\lambda$ then $(\bar{a}+\bar{b}+\bar{c}) \cdot (\bar{a}+\bar{b}+\bar{c})=3\lambda^2 \Rightarrow |\bar{a}+\bar{b}+\bar{c}| = \sqrt{3}\lambda$]

37. The sides of a parallelogram are $2\hat{i}+4\hat{j}-4\hat{k}$ and $\hat{i}+2\hat{j}+4\hat{k}$, then

(A) its area is

(a) 60 sq. u. (b) $12\sqrt{5}$ sq. u. (c) 24 sq. u. (d) none of these

(B) One of its diagonal is

(a) $\frac{1}{\sqrt{5}}(\hat{i}-2\hat{j})$ (b) $\frac{1}{\sqrt{5}}(\hat{i}+2\hat{j})$ (c) $\frac{1}{\sqrt{5}}(2\hat{i}-\hat{j})$ (d) $\frac{1}{\sqrt{5}}(2\hat{i}+\hat{j})$

38. If θ is the angle between two unit vectors $\bar{\eta}_1$ and $\bar{\eta}_2$ then, $|\bar{\eta}_1+\bar{\eta}_2|$ is =

(a) $\cos\frac{\theta}{2}$ (b) $2\cos\frac{\theta}{2}$ (c) $\sin\frac{\theta}{2}$ (d) $2\sin\frac{\theta}{2}$

39. Two vectors each of magnitude 3, and which are perpendicular to vectors $\hat{i}+2\hat{j}+\hat{k}$ and $3\hat{i}+\hat{j}-2\hat{k}$ are

(a) $\pm\left[-\hat{i}+\hat{j}-\hat{k}\right]$ (b) $\pm\sqrt{3}\left(\hat{i}-\hat{j}+\hat{k}\right)$

(c) $\pm\sqrt{3}\left(\hat{i}+\hat{j}-\hat{k}\right)$ (d) $\pm\sqrt{3}\left[\hat{i}-\hat{j}-\hat{k}\right]$

40. $(\bar{a}\cdot\bar{b})\times(\bar{a}+\bar{b})=...$

(a) $2\bar{a}\times\bar{b}$ (b) $4\bar{a}\times\bar{b}$ (c) $2\bar{b}\times\bar{a}$ (d) 2 ab sin θ

41. Prove by vector method that in a triangle

(a) $\cos A=\frac{b^2+c^2-a^2}{2bc}$ (b) $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

3.12 Scalar Triple Product

$\overline{a}\cdot\overline{b}$ is a scalar quantity. It is dot product of two vectors. $\overline{a}\times\overline{b}$ is cross product of two vectors. It is a vector quantity.

$\therefore$ $c\cdot(a\times b)$ which is dot product of vector $\overline{c}$ and vector $(\overline{a}\times\overline{b})$ is a scalar quantity. It is called scalar Triple product, triple, because three vectors are involved.

$\therefore$ $\overline{a}\cdot(\overline{b}\times\overline{c})$ is scalar Triple product.

Now if $a=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\overline{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$$

and $\overline{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

then $\hat{b}\times\hat{c}=(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\times(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})$

$$=(b_2c_3-b_3c_2)\hat{i}+(b_3b_1-c_3b_1)\hat{j}+(b_1c_2-c_1b_2)\hat{k}$$

and $\therefore$ $\overline{a}\cdot(\overline{b}\times\overline{c})$

$$=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(\overline{b}\times\overline{c})$$

$$=a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-c_3b_1)+a_3(b_1c_2-c_1b_2)$$

$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3.12 Properties of scalar triple product

(a) Dot and cross can be interchanged

i.e. $\overline{a}\cdot(\overline{b}\times\overline{c})=(\overline{a}\times\overline{b})\cdot\overline{c}$

the scalar value remains unchanged

$$(\overline{a}\times\overline{b})\cdot\overline{c}=\left[(a_2b_3-b_2a_3)\hat{i}+(b_1a_3-a_1b_3)\hat{j}+(a_1b_2-b_1a_2)\hat{k}\right]\cdot\left[c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right]$$

$= (a_2b_3 - b_2a_3)c_1 + (b_1a_3 - a_1b_3)c_2 + (a_1b_2 - b_1a_2)c_3$

$$=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=-\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=\overline{a}\cdot(\overline{b}\times\overline{c})$$

$$(a\times b)\cdot c=a\cdot(b\times c)=\left[\overline{a}\,\overline{b}\,\overline{c}\right]$$

(Dot) · and x can be interchanged

hence $(\bar{a}\times\bar{b})\cdot\bar{c};\bar{a}\cdot(\bar{b}\times\bar{c})$ is denoted as $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ is called Box-product of $\bar{a},\bar{b},\bar{c}$

(b) The value of $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ remains unchanged. If $\bar{a},\bar{b},\bar{c}$ are taken in cyclic order

i.e. $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=\left[\bar{b}\ \bar{c}\ \bar{a}\right]=\left[\bar{c}\ \bar{a}\ \bar{b}\right]$

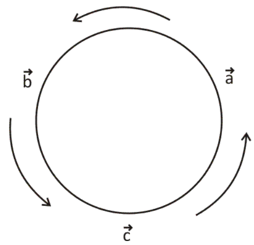


Fig 20

$$[\bar{b}\ \bar{c}\ \bar{a}]=\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}=-\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=[\bar{a}\ \bar{b}\ \bar{c}]$$

It can be proved as below also

$[\bar{a}\ \bar{b}\ \bar{c}]=(\bar{a}\times\bar{b})\cdot\bar{c}=-(\bar{b}\times\bar{a})\cdot\bar{c}$

$=-\left[\bar{b}\cdot(\bar{a}\times\bar{c})\right]=-\left[-\bar{b}\cdot(\bar{c}\times\bar{a})\right]=\left[\bar{b}\ \bar{c}\ \bar{a}\right]$

(c) If one of the vectors $\bar{a}$ or $\bar{b}$ or $\bar{c}$ is a zero vector then $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$

(d) If $\bar{a},\bar{b},\bar{c}$ are coplanar then $[\bar{a}\ \bar{b}\ \bar{c}]=0$

Proof: $\bar{a}\times\bar{b}$ is a vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$. If $\bar{a},\bar{b},\bar{c}$ are coplanar then $\bar{a}\times\bar{b}$ is also perpendicular to $\bar{c}$

Hence $(\bar{a}\times\bar{b})\cdot\bar{c}=0\Rightarrow\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$

(e) If of the three vectors $\bar{a},\bar{b},\bar{c}$ any two are same say $\left[\bar{a},\bar{b},\bar{b}\right]$ or $\left[\bar{a},\bar{a},\bar{c}\right]$ then scalar triple product is zero as two rows of the determinant becomes identical.

(f) For any 4 vectors $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$

$\left[\bar{a}\ \bar{b}\,(\bar{c}+\bar{d})\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right]+\left[\bar{a}\ \bar{b}\ \bar{d}\right]$

Proof: $\left[\bar{a}\ \bar{b}\ (\bar{c}+\bar{d})\right]=(\bar{a}\times\bar{b})\cdot(\bar{c}+\bar{d})$

$=(\bar{a}\times\bar{b})\cdot\bar{c}+(\bar{a}\times\bar{b})\cdot\bar{d}$

$=\left[\bar{a}\ \bar{b}\ \bar{c}\right]+\left[\bar{a}\ \bar{b}\ \bar{d}\right]$

3.13 Geometrical interpretation of scalar Triple

Product: IN figure 20; OA, OB and OC are the conterminas edges of a parallelopiped. OA is along OX, OB along OY and OC is inclined at angle ϕ with z axis

Let $\overline{OA}=\bar{a},\overline{OB}=\bar{b}$ and $\overline{OC}=\bar{c}$

$\overline{a}\times\overline{b}$ is a vector perpendicular to the plane containing vectors $\overline{a}$ and $\overline{b}$ and is inclined at ϕ with $\overline{OC}$.

$\therefore\ (\overline{a}\times\overline{b})\cdot\overline{c} = |\overline{a}\times\overline{b}||\overline{c}|\cos\phi$

= (area parallelogram OAPB) | c | cos θ

and | c | cos θ is the distance of C from the plane of parallelogram OAPB i.e. it is the perpendicular distance of c from parallelogram OAPB.

$\therefore\ (\overline{a}\times\overline{b})\cdot\overline{c}$ = Volume of parallelopiped

whose contiminous edges are OA, OB and OC

Note: (i) If ϕ is acute angle then $v = \overline{a}\cdot(\overline{b}\times\overline{c})$

(ii) If ϕ is obtuse angle then $\overline{a}\cdot(\overline{b}\times\overline{c}) = -v$

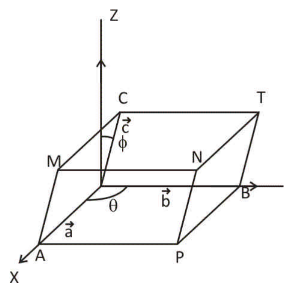


Fig 21

Solved Example

Example 33: The edges of a parallelopiped at one corner are represented by vectors. $\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{i}+2\hat{j}-4\hat{k}$ and $\hat{i}+5\hat{j}+3\hat{k}$. Find volume of parallelopiped.

Sol. $\bar{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\bar{b}=3\hat{i}+2\hat{j}-4\hat{k}$ and $\bar{c}=\hat{i}+5\hat{j}+3\hat{k}$.

$$\therefore \text{ Volume } [a\ b\ c]=\begin{vmatrix}1 & 2 & 3\\ 3 & 2 & -4\\ 1 & 5 & 3\end{vmatrix}$$

= 1 (6 + 20) – 2 (9 + 4) + 3(15 – 2) = 39 sq. u.

Example 34: Prove that $[\bar{a}+\bar{b}\ \bar{b}+\bar{c}\ \bar{c}+\bar{a}]=2[\bar{a}\ \bar{b}\ \bar{c}]$

Sol. $[\bar{a}+\bar{b}\ \bar{b}+\bar{c}\ \bar{c}+\bar{a}] \quad =(\bar{a}+\bar{b})\cdot[(\bar{b}+\bar{c})\times(\bar{c}+\bar{a})]$

$$=(\bar{a}+\bar{b})\cdot[\bar{b}\times\bar{c}+\bar{b}\times\bar{a}+0+\bar{c}\times\bar{a}]$$

$$=\bar{a}\cdot(\bar{b}\times\bar{c})+\bar{a}\cdot(\bar{b}\times\bar{a})+\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{b}\cdot(\bar{b}\times\bar{c})+\bar{b}\cdot(\bar{b}\times\bar{a})+\bar{b}\cdot(\bar{c}\times\bar{a})$$

$$=[\bar{a}\ \bar{b}\ \bar{c}]+0+0+0+0+[\bar{b}\ \bar{c}\ \bar{a}]$$

$$=[\bar{a}\ \bar{b}\ \bar{c}]+[\bar{a}\ \bar{b}\ \bar{c}]=2[\bar{a}\ \bar{b}\ \bar{c}]$$

Example 35: $\bar{a}=3i+pj+6k$, $\bar{b}=3i-4j-12k$, $\bar{c}=i+3j+5k$. Find p if $[\bar{a}\ \bar{b}\ \bar{c}]=0$

Sol. $$[\bar{a}\ \bar{b}\ \bar{c}]=\begin{vmatrix}3 & p & 6\\ 3 & -4 & -12\\ 1 & 3 & 5\end{vmatrix}=3(-20+36)-p(15+12)+6(9+4)$$

$[\bar{a}\ \bar{b}\ \bar{c}]=0 \qquad \therefore$ 48 – 27p + 78 = 0 $\Rightarrow p=\dfrac{14}{3}$

Example 36: Prove that four points whose p.v. are (2i + 3j –k), (i – 2j + 3k), (3i + 4j – 2k) and (i – 6j + 6k) are coplanar.

Sol. Let A be $(2\bar{i}+3\bar{j}-\bar{k})$, $B(\hat{i}-2\hat{j}+3\hat{k})$, $C(3\hat{i}+4\hat{j}-2\hat{k})$ and $D(\bar{i}-6\bar{j}+6\bar{k})$

$\therefore \ \overline{AB}=-\bar{i}-5\bar{j}+4\bar{k}, \overline{BC}=2\bar{i}+6\bar{j}-5\bar{k}$, $\overline{CD}=-2\bar{i}-10\bar{j}+8\bar{k}$

These vectors shall be coplanar

$$\text{If } \overline{AB}\cdot(\overline{BC}\times\overline{CD})=0\Rightarrow\begin{vmatrix}-1 & 5 & 4\\ 2 & 6 & -5\\ -2 & -10 & 8\end{vmatrix}$$

= –1 (48 – 50) + 5(16 – 10) + (–20 + 12) = 0

$\therefore$ points are coplanar.

Example 37: Points O, (0, 0, 0); A(1, 3, 5), B(2, 4, 3) and C (x, y, z) are coplanar, then prove 11x – 7y + 2z = 0.

Sol. $\overline{OA} = (1,3,5)$; $\overline{OB}(2,4,3)$, $\overline{OC}(x,y,z)$

points are coplanar $\Rightarrow \left[\overline{OA}\ \overline{OB}\ \overline{OC}\right] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 3 \\ x & y & z \end{vmatrix} = 0 = \begin{vmatrix} x & y & z \\ 1 & 3 & 5 \\ 2 & 4 & 3 \end{vmatrix} = 0$$

= x(9 – 20) – y(3 – 10) + z(4 – 6) = 0 $\Rightarrow$ 11x – 7y + 2z = 0

Example 38: (i) If $\bar{c} = 2\bar{a} + 3\bar{b}$ then prove that $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

(ii) If angle between $\bar{a}$ and $\bar{b}$ = $\frac{\pi}{4}$, prove that $c^2 = 4a^2 + 6\sqrt{2}ab + 9b^2$, when $|\bar{a}| = a, |\bar{b}| = b$

Sol. (i) $\bar{c} = 2\bar{a} + 3\bar{b}$ i.e. $\bar{c}$ has been expressed as a linear combination of $\bar{a}$ and $\bar{b}$ $\Rightarrow \bar{c}$ lies in the plane of $\bar{a}$ and $\bar{b}$ i.e. $\bar{a}, \bar{b}, \bar{c}$ are coplanar $\therefore \left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$

(ii) $\bar{c} = 2\bar{a} + 3\bar{b} \Rightarrow \bar{c}\cdot\bar{c} = (2\bar{a} + 3\bar{b})\cdot(2\bar{a} + 3\bar{b})$

$\Rightarrow \bar{c}^2 = 4\cdot\bar{a}\cdot\bar{a} + 6\bar{a}\cdot\bar{b} + 6\bar{b}\cdot\bar{a} + 9\bar{b}\cdot\bar{b}$

$= 4a + 12|\bar{a}|\cdot|\bar{b}|\cdot\cos 45^o + 9b^2$

$= 4a^2 + 12ab\cdot\frac{1}{\sqrt{2}} + 9b^2$

$= 4a + 6\sqrt{2}ab + 9b^2$

Example 39: Vectors $\ell i + j + k$, $i + mj + k$ and $i + j + nk$ are coplanar, $\ell \neq 1, m \neq 1$, $n \neq 1$, prove that $\frac{1}{1-\ell} + \frac{1}{1-m} + \frac{1}{1-n} = -1$

Sol. Given vectors

$\ell i + j + k$, i + mj + k and i + j + nk

are co-plane = $\begin{vmatrix} \ell & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & n \end{vmatrix} = 0$

= $R_1 - R_3$ and $R_2 = R_2 - R_3$

$$\begin{vmatrix} \ell-1 & 0 & 1-n \\ 0 & m-1 & 1-n \\ 1 & 1 & n \end{vmatrix} = 0$$

$$\Rightarrow (\ell-1)(m-1)(1-n)\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \frac{1}{\ell-1} & \frac{1}{m-1} & \frac{n}{1-n} \end{vmatrix} = 0$$

$$\Rightarrow \left[\frac{n}{1-n} - \frac{1}{m-1}\right] + 0\left[-\frac{1}{\ell-1}\right] = 0$$

$$\Rightarrow \frac{1}{\ell-1} + \frac{1}{m-1} - \frac{n}{1-n} = 0 \Rightarrow \frac{1}{\ell-1} + \frac{2}{m-1} - \frac{(1-n)-1}{1-n} = 0$$

$$\Rightarrow \frac{1}{\ell-1} + \frac{1}{m-1} + 1 + \frac{-1}{1-n} = 0 \Rightarrow \frac{1}{\ell-1} + \frac{1}{m-1} + \frac{-1}{1-n} = -1$$

Example 40 : The vector $\overline{OP} = 2\hat{i} + 2\hat{j} + \hat{k}$ turns through an angle $\frac{\pi}{2}$ about O through the positive side of $\hat{j}$ axis. Find new position OQ of the vector.

Sol. Let $\overline{OQ} = x\hat{i} + y\hat{j} + z\hat{k}, \overline{OP} = 2\hat{i} + 2\hat{j} + \hat{k}$

$|OQ| = |OP| \quad \therefore x^2 + y^2 + z^2 = 9$ ... (1)

(ii) $\overline{OP}, \hat{j}$ and $\overline{OQ}$ are coplanar

$$\therefore \begin{vmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = 0 \Rightarrow 2z - x = 0$$

$\therefore z = \frac{x}{2}$ ... (2)

(iii) OP turns through $\frac{\pi}{2} \Rightarrow \overline{OP} \cdot \overline{OQ} = 0$

$\therefore 2x + 2y + z = 0 \Rightarrow 2x + 2y + \frac{x}{2} = 0$

$\therefore y = \frac{5}{4}x$

$\therefore$ from (1) $x^2 + \frac{25}{16}x^2 + \frac{x^2}{4} = 9$

$\Rightarrow 45x^2 = 144 \Rightarrow x = \pm\dfrac{4}{\sqrt{5}}$

$\therefore\ OQ = \pm\left(\dfrac{4}{\sqrt{5}}\hat{i} - \sqrt{5}\hat{j} + \dfrac{2}{\sqrt{5}}\hat{k}\right)$

Example 41: Find λ, if volume of the tetrahedron whose vertices are A, $(\hat{i} - 6\hat{j} + 10\hat{k})$; $B(-\hat{i} - 3\hat{j} + 7\hat{k})$, $C(5\hat{i} - \hat{j} + \lambda\hat{k})$ and $D(7\hat{i} - 4\hat{j} + 7\hat{k})$ is 11 cubic units.

Sol. $\overrightarrow{AB} = (-i - 3j + 7k) - (i - 6j + 10k) = -2i + 3j - 3k$

$\overrightarrow{AC} = (5i - j + \lambda k) - (i - 6j + 10k) = 4i + 5j + (\lambda - 10)k$

$\overrightarrow{AD} = (7i - 4j + 7k) - (i - 6j + 10k) = 6i + 2j - 3k$

Volume of tetrahedron = $\dfrac{1}{6}$ volume of parallelopiped

$$= \frac{1}{6}\begin{vmatrix} -2 & 3 & -3 \\ 4 & 5 & (\lambda - 10) \\ 6 & 2 & -3 \end{vmatrix} \Rightarrow [30 + 4(\lambda - 10)] + 3(6\lambda - 60 + 12) + 3.22 = 6 \times 11$$

$\therefore\ 22\lambda - 10 - 144 + 66 = 66$

$\Rightarrow 22\lambda = 154 \Rightarrow \lambda = 7$

Practice Worksheet (Foundation Level) – 3 (c)

1. The p.v. of points A, B, C and D are i + j + k, 2i + j + 3k, 3i + 2j + 2k and 3i + 3j + 4k. Find the volume of the parallelopiped whose concurrent edges are AB, AC, AD.

2. Find the volume of tetrahedron whose vertices are (1, 2, –2), (3, 4, –5), (–2, 1, 3) and (–1, –2, 3)

3. Find value of a if vectors 2i + 3j – 7k, i – 4j + 2k and 5i – aj – 4k are coplanar

4. Find the value of $\left[\bar{a}-\bar{b}\ \bar{b}-\bar{c}\ \bar{c}-\bar{a}\right]$.

5. Find a linear expression between x, y, z if vectors 3i – 4j + 5k, 2i + j – k and xi + yj + zk are coplanar

6. If points (0, 0, 0), A (x, 1, –1), B (0, y, 2) and C (2, 3, z) be coplanar then prove x y z = 6x – 2y – 4.

7. Find condition that vectors $\ell\hat{i}-\hat{j}+\ell\hat{k}$, $\hat{i}+m\hat{j}-m\hat{k}$ and $-n\hat{i}+n\hat{j}+\hat{k}$ be coplanar.

8. Find p, if vectors $p\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-p\hat{j}+4\hat{k}$ and $3\hat{i}-2\hat{j}+p\hat{k}$ are coplanar.

9. If $c=2\bar{a}+5\bar{b}$, show that $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$ and if $|\bar{a}|=a, |\bar{b}|=b$ and angle between $\bar{a}$ and $\bar{b}$ be 60^o, then calculate value of c^2 in terms of a and b.

10. $\bar{a}=\hat{i}+\hat{j}+\hat{k}, \bar{b}=\hat{i}+\hat{j}-\hat{k}, \bar{c}=\hat{i}-\hat{j}+\hat{k}$ evaluate $\left[\bar{a}+2\bar{b}\ \bar{b}+2\bar{c}\ \bar{c}+2\bar{a}\right]$

11. Vectors $\frac{1}{a}\hat{i}+\hat{j}+\hat{k}, \hat{i}+\frac{1}{b}\hat{j}+\hat{k}$, and $\hat{i}+\hat{j}+\frac{1}{c}\hat{k}$ are coplanar, prove ab + bc + ca = 1 + 2 abc

12. Vector $\overline{OA}=i+2j-2k$ is rotated about O through 90^o in such a way that it goes through positive side of z axis while rotating about it. Find new position OB of vector.

13. $\bar{a},\bar{b},\bar{c}$ are three non coplanar vectors if $p=\frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]}$, $\bar{q}=\frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}$, $\bar{r}=\frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$. Calculate value of $(\bar{a}+\bar{b})\cdot\bar{p}+(\bar{b}+\bar{c})\cdot\bar{q}+(\bar{c}+\bar{a})\cdot\bar{r}$

14. A vector $\bar{a}$ is coplanar with vectors $3\hat{i}+2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$ and it is also coplanar with vectors $2\hat{i}-3\hat{j}-2\hat{k}$ and $-3\hat{i}+\hat{j}+\hat{k}$. Find $\bar{a}$.

15. For any vectors $\bar{p},\bar{q}$ and $\bar{r}$ prove that $\bar{p}-\bar{q}, \bar{q}-\bar{r}$ and $\bar{r}-\bar{p}$ are always coplanar.

16. D is the mid-point of sides BC of triangle ABC prove that $(AB)^2 + (AC)^2 = 2\ (AD)^2 + 2(BD)^2$ by vector method

17. $\bar{a}+\bar{b}+\bar{c}=0$ and $|\bar{a}|=3, |\bar{b}|=5, |\bar{c}|=7$, then angle between $\bar{a}$ and $\bar{b}$ is ….

[Hint: $\bar{c}\cdot\bar{c}=(\bar{a}+\bar{b})\cdot(\bar{a}+\bar{b})(-1)^2$]

18. Value of $\left[\bar{a}-\bar{b}\ \ \bar{b}-\bar{c}\ \ \bar{c}-\bar{a}\right]$ when $|\bar{a}|=3$, $|\bar{b}|=5$, $|\bar{c}|=1$ is ...

19. Unit vector perpendicular to $(\hat{i}-2\hat{j})$ and $(2\hat{i}+\hat{j})$ is ...

20. In a right angled triangle ABC, ∠B = 90° and AC = p then $\overline{AB}\cdot\overline{AC}-\overline{BC}\cdot\overline{CA}+\overline{BA}\cdot\overline{BC}=...$

21. $\bar{x}=3\hat{i}+6\hat{j}-\hat{k}$, $\bar{y}=\hat{i}-4\hat{j}+3\hat{k}$ and $\bar{z}=3\hat{i}+4\hat{j}-12\hat{k}$ then magnitude of projection of $\bar{x}\times\bar{y}$ on z is

22. $(\bar{a}+\bar{b}+\bar{c})\cdot(\bar{b}\times\bar{c})+(2\bar{a}-3\bar{b}+4\bar{c})\cdot(\bar{c}\times\bar{a})$

(a) $[\bar{a}\ \bar{b}\ \bar{c}]$ (b) $2[\bar{a}\ \bar{b}\ \bar{c}]$ (c) $-2[\bar{a}\ \bar{b}\ \bar{c}]$ (d) 0

23. A $(\hat{i}+\hat{j}-\hat{k})$, B $(\hat{i}-4\hat{j}+\hat{k})$, C $(2\hat{i}-\hat{j}+2\hat{k})$ and D $(3\hat{i}+3\hat{j}+a\hat{k})$ are vertices of a tetrahedron, whose volume is 2 cu. units, then a is

(a) –1 (b) 2 (c) –2 (d) 5

24. $\bar{a}=4\hat{i}+3\hat{j}+2\hat{k},\bar{b}=2\hat{i}-\hat{j}-\hat{k},\bar{c}=2\hat{i}+3\hat{j}-2\hat{k}$. The diagonals of a parallelogram are $\bar{a}-\bar{b}+\bar{c}$ and $\bar{a}+\bar{b}-\bar{c}$. Its area is

(a) $\sqrt{393}$ sq.u. (b) $\frac{1}{2}\sqrt{393}$ sq.u. (c) $\frac{1}{2}\sqrt{373}$ (d) $\sqrt{373}$

25. $\bar{a}=\hat{i}-\hat{j},\bar{b}=\hat{j}-\hat{k}$ and $\bar{c}=\hat{k}-\hat{i}$. If $\bar{d}$ is a unit vector such that $\bar{a}\cdot\bar{d}=0$ = 0 and $[\bar{b}\ \bar{c}\ \bar{d}]$ = 0, then d is ...

(a) $\frac{1}{\sqrt{6}}(-\hat{i}-\hat{j}+2\hat{k})$ (b) $\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}-2\hat{k})$

(c) $\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$ (d) $\frac{1}{\sqrt{6}}(\hat{i}-\hat{j}+2\hat{k})$

26. The value of $[\bar{a}\times\bar{p}\ \ \bar{b}\times\bar{q}\ \ \bar{c}\times\bar{r}]+[\bar{a}\times\bar{q}\ \ \bar{b}\times\bar{r}\ \ \bar{c}\times\bar{q}]$ + $[\bar{a}\times\bar{r}\ \ \bar{b}\times\bar{p}\ \ \bar{c}\times\bar{q}]$ is ...

(a) 1 (b) –1 (c) 0 (d) undetermind

27. Vector $\bar{a}=3\hat{i}-2\hat{j}+6\hat{k},\bar{b}=5\hat{i}+3\hat{j}+2\hat{k}$ and $\bar{p}$ = projection of $\bar{b}$ and $\bar{a}$. Then $(\bar{p}\times\bar{a})\cdot\bar{b}=$

(a) 3 (b) 9 (c) 0 (d) 6

28. $i\times(j\times k)$ is

(a) 0 (b) i (c) j (d) k

3.15 Vector triple Product

Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three vectors $(\bar{b}\times\bar{c})$ is a vector $\Rightarrow$ $\bar{a}\times(\bar{b}\times\bar{c})$ shall be vector $\therefore$ $\bar{a}\times(\bar{b}\times\bar{c})$ is vector triple product

(a) $(\bar{b}\times\bar{c})$ is a vector perpendicular to plane containing $\bar{b}$ and $\bar{c}$. Let this vector be η, then $\bar{a}\times\bar{\eta}$ will be a vector perpendicular to $\bar{a}$ and $\bar{\eta}$ which means it lies ion the plane of $\bar{b}$ and $\bar{c}$

$\therefore$ $\bar{a}\times\bar{\eta} = p\bar{b} + q\bar{c}$ where p and q are scalars.

Now we shall drive the formula and find p and q.

(b) Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$\therefore$ $\bar{b}\times\bar{c} = (b_2c_3 - c_2b_3)\hat{i} + (c_1b_3 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}$

$\therefore$ $\bar{a}\times(\bar{b}\times\bar{c})$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times\left[(b_2c_3 - c_2b_3)\hat{i} + (c_1b_3 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}\right]$

and as $\hat{i}\times\hat{j} = 0, \hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

$$\bar{a}\times(\bar{b}\times\bar{c}) = a_1(c_1b_3 - b_1c_3)\hat{k} - a_1(b_1c_2 - b_2c_1)\hat{j} - a_2(b_2c_3 - c_2b_3)\hat{k} + a_2(b_1c_2 - b_2c_1)\hat{i}$$
$$- a_2(b_2c_3 - c_2b_3)\hat{k} + a_2(b_1c_2 - b_2c_1)\hat{i} + a_3(b_2c_3 - c_2b_3)\hat{j} - a_3(c_1b_3 - b_1c_3)\hat{i}$$

$$= a_1c_1(b_3\hat{k}) + a_2c_2(b_3\hat{k}) + a_3c_3(b_2\hat{j}) + a_1c_1(b_2\hat{j}) + a_2c_2(b_1\hat{i}) + a_3c_3(b_1\hat{i})$$
$$- [a_1b_1(c_3\hat{k}) + a_1b_1(c_2\hat{j}) + a_2b_2(c_3\hat{k}) + a_2b_2(c_1\hat{i}) + a_3b_3(c_2\hat{j}) + a_3b_3(c_1\hat{i})]$$

$$= a_1c_1(b_3\hat{k}) + a_2c_2(b_3\hat{k}) + a_3c_3(b_1\hat{j}) + a_1c_1(b_2\hat{j}) - [(a_2b_2 + a_3b_3)c_i\hat{i} + (a_3b_3 + a_1b_1)c_2\hat{j} + (a_1b_1 + a_2b_2)c_3\hat{k}]$$

Now adding $a_1c_1(b_1\hat{i}) + a_2c_2(b_2\hat{j}) + a_3c_3(b_3\hat{k})$ and subtracting the same, we have

$$(a_1c_1 + a_2c_2 + a_3c_3)b_1\hat{i} + (a_1c_1 + a_2c_2 + a_3c_3)b_2\hat{j} + (a_1c_1 + a_2c_2 + a_3c_3)b_3\hat{k} - [a_1b_1 + a_2b_2 + a_3b_3]c_1\hat{i}$$
$$+ (a_1b_1 + a_2b_2 + a_3b_3)c_2\hat{j} + (a_1b_1 + a_2b_2 + a_3b_3)c_3\hat{k}]$$

$$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1i + b_2j + b_3k) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1i + c_2j + c_3k)$$

$$= (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c} \qquad \begin{bmatrix} p = \bar{a}\cdot\bar{c} \\ q = -(\bar{a}\cdot\bar{b}) \end{bmatrix}$$

$\therefore$ $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$ and $(\bar{a}\times\bar{b})\times\bar{c} = -\bar{c}\times(\bar{a}\times\bar{b})$

$$= -\left[(\bar{c}\cdot\bar{b})\bar{a} - (\bar{c}\cdot\bar{a})\bar{b}\right] \qquad = (\bar{c}\cdot\bar{a})\bar{b} - (\bar{c}\cdot\bar{b})\bar{a}$$

and as $\bar{c}\cdot\bar{a} = \bar{a}\cdot\bar{c}$ and $\bar{c}\cdot\bar{b} = \bar{b}\cdot\bar{c}$ $\therefore$ $(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{b}\cdot\bar{c})\bar{a}$

Solved Examples

Example 42 : (a): Prove $\bar{a}\times(\bar{b}\times\bar{c})+\bar{b}\times(\bar{c}\times\bar{a})+\bar{c}\times(\bar{a}\times\bar{b})=0$

Sol. $\bar{a}\times(\bar{b}\times\bar{c})=(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}$

$\bar{b}\times(\bar{c}\times\bar{a})=(\bar{b}\cdot\bar{a})\bar{c}-(\bar{b}\cdot\bar{c})\bar{a}=(\bar{a}\cdot\bar{b})\bar{c}-(\bar{b}\cdot\bar{c})\bar{a}$

$\bar{c}\times(\bar{a}\times\bar{b})=(\bar{c}\cdot\bar{b})\bar{a}-(\bar{c}\cdot\bar{a})\bar{b}=(\bar{b}\cdot\bar{c})\bar{a}-(\bar{a}\cdot\bar{c})\bar{b}$

adding all three

$\bar{a}\times(\bar{b}\times\bar{c})+\bar{b}\times(\bar{c}\times\bar{a})+\bar{c}\times(\bar{a}\times\bar{b})=0$

Example 42 (b): Prove $[\bar{a}\times\bar{b}\ \bar{a}\times\bar{c}\ \bar{d}]=(\bar{a}\cdot\bar{d})/(\bar{a}\ \bar{b}\ \bar{c})$

Sol. $\left[\bar{a}\times\bar{b}\ \bar{a}\times\bar{c}\ \bar{d}\right]=(\bar{a}\times\bar{b})\cdot\left[(\bar{a}\times\bar{c})\times\bar{d}\right]$

$=(\bar{a}\times\bar{b})\cdot\left[(\bar{a}\cdot\bar{d})\bar{c}-(\bar{c}\cdot\bar{d})\bar{a}\right]$

$=(\bar{a}\cdot\bar{d})\left[(\bar{a}\times\bar{b})\cdot\bar{c}\right]-(\bar{c}\cdot\bar{d})\left[(\bar{a}\times\bar{b})\cdot\bar{a}\right]$

$=(\bar{a}\cdot\bar{d})[\bar{a}\ \bar{b}\ \bar{c}]-(\bar{c}\cdot\bar{d})\ [\bar{a}\ \bar{b}\ \bar{a}]$

$=(\bar{a}\cdot\bar{d})[\bar{a}\ \bar{b}\ \bar{c}]-0=(\bar{a}\cdot\bar{d})\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

Example 43: Prove $[\bar{a}\times\bar{b}\ \bar{b}\times\bar{c}\ \bar{c}\times\bar{a}]=[\bar{a}\ \bar{b}\ \bar{c}]^2$

Sol. $[\bar{a}\times\bar{b}\ \bar{b}\times\bar{c}\ \bar{c}\times\bar{a}]=[\bar{a}\times\bar{b}\ \bar{m}\ \bar{c}\times\bar{a}]$

where $\bar{m}=\bar{b}\times\bar{c}$

$\therefore$ Exp. $=(\bar{a}\times\bar{b})\cdot\left[\bar{m}\times(\bar{c}\times\bar{a})\right]$

$=(\bar{a}\times\bar{b})\cdot\left[(\bar{m}\cdot\bar{a})\bar{c}-(\bar{m}\cdot\bar{c})\bar{a}\right]$

$=(\bar{m}\cdot\bar{a})(\bar{a}\times\bar{b})\cdot\bar{c}-(\bar{m}\cdot\bar{c})(\bar{a}\times\bar{b})\cdot\bar{a}$

$=(\bar{m}\cdot\bar{a})\left[\bar{a}\ \bar{b}\ \bar{c}\right]-(\bar{m}\cdot\bar{c})[\bar{a}\ \bar{b}\ \bar{a}]$

$=(\bar{m}\cdot\bar{a})[\bar{a}\ \bar{b}\ \bar{c}]=0$

$=[b\ c\ a][a\ b\ c]=[a\ b\ c]$

Example 44: $\bar{a},\bar{b},\bar{c}$ are three non-parallel vectors, having magnitude 1, 1, 2 respectively if $\bar{a}\times(\bar{b}\times\bar{c})+b=0$, then find acute angle between $\bar{a}$ & $\bar{c}$.
(I.I.T.)

Sol. Given $|\bar{a}|=1,|\bar{b}|=1,|\bar{c}|=2$

$\bar{a}\times(\bar{b}\times\bar{c})+\bar{b}=(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}+\bar{b}=[\bar{a}\cdot\bar{c}+1]\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}$

$\bar{a}, \bar{b}$ and $\bar{c}$ are non-parallel $\Rightarrow [\bar{a}\cdot\bar{c}+1]\bar{b}=0$

$\Rightarrow \bar{a}\cdot\bar{c}+1=0$

$\Rightarrow |\bar{a}||\bar{c}|\cos\theta=-1 \Rightarrow 1\cdot 2\cdot\cos\theta=-1$

$\therefore \cos\theta = -\frac{1}{2}$; and acute angle is $\cos\theta=\frac{1}{2}, \theta=\frac{\pi}{3}$

Example 45: Let $\bar{a}=2\hat{i}+\hat{j}+\hat{k}, \bar{b}=\hat{i}+2\hat{j}-\hat{k}$ and a unit vector $\bar{c}$ be coplanar. If $\bar{c}$ is perpendicular to $\bar{a}$, then find c.

Sol. Let $\bar{c}=x\hat{i}+y\hat{j}+z\hat{k}$

$|\bar{c}|=1 \quad \therefore x^2 + y^2 + z^2 = 1$ ... (1)

$\bar{c}$ is $\perp$ $\bar{a}$ $\therefore$ (xi + yj + zk) · (2i + j + k) = 0

$\Rightarrow 2x + y + z = 0$ ... (2)

$\bar{a}, \bar{b}, \bar{c}$ are coplanar $\Rightarrow \bar{c}\cdot(\bar{a}\times\bar{b})=0$

$$\bar{a}\times\bar{b}=(2\hat{i}+\hat{j}+\hat{k})\times(\hat{i}+2\hat{j}-\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 1\\ 1 & 2 & -1\end{vmatrix}=-3\hat{i}+3\hat{j}+3\hat{k}$$

and $\therefore (x\hat{i}+y\hat{j}+z\hat{k})\cdot(-3\hat{i}+3\hat{j}+3\hat{k})=0$

$\Rightarrow -x + y + z = 0$ ... (3)

from (2) – (3) $\quad 3y + 3z = 0 \Rightarrow y = -z$ and $x = 0$

$\therefore$ from (1) $0 + y^2 + (-y)^2 = 1 \quad \Rightarrow y=\pm\frac{1}{\sqrt{2}}$

$\therefore z=\mp\frac{1}{\sqrt{2}}$, $\bar{c}=\frac{1}{\sqrt{2}}(\hat{j}-\hat{k}), \bar{c}=\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$

Example 46: Let $\bar{a}=2\hat{i}+\hat{j}-2\hat{k}, \bar{b}=\hat{i}+\hat{j}$. If $\bar{c}$ be vector such that $\bar{a}\cdot\bar{c}=|\bar{c}|$ and $|\bar{c}-\bar{a}|=2\sqrt{2}$ and angle between $\bar{a}\times\bar{b}$ and $\bar{c}$ be 30. Then find value of $|(\bar{a}\times\bar{b})\times\bar{c}|$.

Sol. Given $\bar{a}\cdot\bar{c}=|\bar{c}|$ and $|\bar{c}-\bar{a}|=2\sqrt{2}$

$\therefore |\bar{c}-\bar{a}|^2=8 \Rightarrow |\bar{c}|^2+|\bar{a}|-2\bar{a}\cdot\bar{c}=8$

$|\bar{a}|=3 \quad \therefore |\bar{c}|^2+9-2|\bar{c}|=8 \quad (\because \bar{a}\cdot\bar{c}=|\bar{c}|)$

$\Rightarrow |\bar{c}|^2-2|\bar{c}|+1=(|\bar{c}|-1)^2=0$

$\therefore\ |c|=1$

$\therefore\ (\overline{a}\times\overline{b})\times\overline{c}=|\overline{a}\times\overline{b}|\cdot|\overline{c}|\cdot\sin 30\cdot\eta$

and $\overline{a}\times\overline{b}=(2\hat{i}+\hat{j}-2\hat{k})\times(\hat{i}+\hat{j})=2\hat{i}-2\hat{j}+\hat{k}$

$\therefore\ |\overline{a}\times\overline{b}|=3 \text{ and} \therefore \left|(\overline{a}\times\overline{b})\times\overline{c}\right|=3\cdot 1\cdot\frac{1}{2}=\frac{3}{2}$

Example 47: $\overline{r}\times\overline{p}=\overline{q}$ and $\overline{p}$ is $\perp$ $\overline{q}$, then prove $\overline{r}=x\overline{p}+\frac{\overline{p}\times\overline{q}}{|p|^2}$

Sol. $\overline{p}\perp\overline{q}$ and $\overline{q}$ is $\perp$ to $\overline{r}$ and also $\perp$ to $\overline{p}$

$\therefore$ vectors $\overline{p}$, $\overline{q}$ and $\overline{r}$ are non coplanar.

$\therefore\ \overline{r}=x\overline{p}+y\overline{q}+z(\overline{p}\times\overline{q})$

where x, y and z are scalars.

$q=\overline{r}x\overline{p}=[x\overline{p}+y\overline{q}+z(\overline{p}x\overline{q})x\overline{p}$ $\quad =0+y(\overline{q}\times\overline{p})-z[(\overline{p}\times\overline{q})]\times p$

$=y(\overline{q}\times\overline{p})-z[\overline{p}\times(\overline{p}\times\overline{q})]$ $\quad =y(\overline{q}\times\overline{p})-z[(\overline{p}\cdot\overline{q})\overline{p}-(\overline{p}\cdot\overline{p})\overline{q}]$

$=y(\overline{q}\times\overline{p})-z[0-(\overline{p}\cdot\overline{p})\overline{q}]$ $\quad \overline{q}=y(\overline{q}\times\overline{p})+z\cdot(\overline{p}\cdot\overline{p})\overline{q}$

Comparing co-efficient of this $y(\overline{q}\times\overline{p})=0$

$\Rightarrow$ y = 0, and as $\quad z(\overline{p}\cdot\overline{p})=1$

$$z=\frac{1}{(\overline{p}\cdot\overline{p})}=\frac{1}{|\overline{p}|^2}$$

$$\therefore\ \overline{r}=x\overline{p}+\frac{\overline{p}\times\overline{q}}{|p|^2}$$

Example 48: $\overline{a}$ and $\overline{b}$ are two non-collinear unit vectors. If $\overline{u}=\overline{a}-(\overline{a}\cdot\overline{b})\overline{b}$ and $\overline{v}=\overline{a}\times\overline{b}$ then evaluate v.

Sol. $\overline{u}=\overline{a}-(\overline{a}\cdot\overline{b})\cdot\overline{b}=|\overline{b}|^2\ \overline{a}-(\overline{a}\cdot\overline{b})\cdot\overline{b}$

(as $|\overline{b}|$ = 1, unit vector)

$\therefore\ \overline{u}=(\overline{b}\cdot\overline{b})\overline{a}-(\overline{a}\cdot\overline{b})\overline{b}=\overline{b}\times(\overline{a}\times\overline{b})$ ... (1)

and $\overline{v}=\overline{a}\times\overline{b}=\overline{v}$ is $\perp$ $\overline{a}$ and $\perp$ $\overline{b}$

from (1) $\overline{u}=\overline{b}\times\overline{v}=|\overline{b}||\overline{v}|\sin\theta\cdot\eta$

again $\overline{u}=\overline{b}\times(\overline{a}\times\overline{b})\Rightarrow\overline{u}\perp\overline{b}$ and $\perp(\overline{a}\times\overline{b})$

$\therefore\ \overline{u}\cdot\overline{b}=0$

$\therefore\ \overline{u}+\overline{u}\cdot b=\overline{u}+0=\overline{u}$

$\therefore\ \overline{v}=\overline{u}$

Example 49: The P.V. of P and Q are $\overline{p}$ and $\overline{q}$ and $|\overline{p}|=p, |\overline{q}|=q$. Points R and S divide PQ internally and externally in the ratio of 2 : 3. If $\overline{OR}$ and $\overline{OS}$ be perpendicular (0, origin) then prove. $4q^2 = 9p^2$.

Sol. R divides $P(\overline{p})$ and $Q(\overline{q})$ internally in ratio 2 : 3.

$$OR=\frac{2\overline{q}+3\overline{p}}{5}$$

S divides PQ externally in the ratio of 2 : 3

$$\therefore\ \overline{OS}=\frac{2\overline{q}-3\overline{p}}{-1}$$

Given $\overline{OR}\perp\overline{OS}$ $\therefore$ $\left(\frac{2q+3p}{5}\right)\cdot\left(\frac{3p-2q}{1}\right)=0$

$\Rightarrow 9p^2 - 4q^2 = 0 \Rightarrow 4q^2 = 9p^2$

Example 50: Express vector product of four vectors and show that any vector can be expressed as a linear combination of other three.

Sol. Let $\overline{a},\overline{b},\overline{c}$ and $\overline{d}$ be four vectors.

(i) $(\overline{a}\times\overline{b})\times(\overline{c}\times\overline{d})=(\overline{a}\times\overline{b})\times\overline{n}=-\overline{n}\times(\overline{a}\times\overline{b})$

$$=-[(\overline{n}\cdot\overline{b})\overline{a}-(\overline{n}\cdot\overline{a})\overline{b}]$$

$$=(\overline{n}\cdot\overline{a})\overline{b}-(\overline{n}\cdot\overline{b})\overline{a}$$

$$=[(\overline{c}\times\overline{d}\cdot\overline{a})]\overline{b}-[(\overline{c}\times\overline{d})\cdot\overline{b}]\overline{a}$$

$$=[\overline{c}\ \overline{d}\ \overline{a}]\overline{b}-(\overline{c}\ \overline{d}\ \overline{b})\overline{a}$$

(ii) $(\overline{a}\times\overline{b})\times(\overline{c}\times\overline{d})=\overline{m}\times(\overline{c}\times\overline{d})$

$$=(\overline{m}\cdot\overline{d})\overline{c}-(\overline{m}\cdot\overline{c})\overline{d}$$

$$=\left[(\overline{a}\times\overline{b})\cdot\overline{d}\right]\overline{c}-\left[(\overline{a}\times\overline{b})\cdot\overline{c}\right]\overline{d}$$

$$=\left[\overline{a}\ \overline{b}\ \overline{d}\right]\overline{c}-[\overline{a}\ \overline{b}\ \overline{c}]\overline{d}$$

from (i) and (ii)

$$[\overline{a}\ \overline{b}\ \overline{d}]\overline{c}-[\overline{a}\ \overline{b}\ \overline{c}]\overline{d}=(\overline{c}\ \overline{d}\ \overline{a})\overline{b}-(\overline{c}\ \overline{d}\ \overline{b})\overline{a}$$

$\therefore\ (\bar{c}\ \bar{d}\ \bar{b})\bar{a} = [\bar{c}\ \bar{d}\ \bar{a}]\bar{b} - [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} + [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

and as [c d a] · [c d b] · [a b d] · [a b c] are all scalers.

Vector A has been expressed as a linear combination of the other three vectors.

Example 51: Let $r_1, r_2, r_3, \ldots r_n$ be the P. vectors of $A_1, A_2\ A_3, \ldots A_n$ relative to origin O. Show that if vectors equation

$\bar{a}_1\bar{r}_1 + \bar{a}_2\bar{r}_2 + \bar{a}_3\bar{r}_3 + \ldots\bar{a}_n\bar{r}_n$ holds good, then a similar equation shall hold good with respect to any origin P. If $\bar{a}_1 + \bar{a}_2 + \bar{a}_3 + \ldots + \bar{a}_n = 0$.

Sol. In figure $\overline{OA_1} = r_1, \overline{OA_2} = r_2$, P is a point $\overline{OP} = \beta$.

If P is taken as origin then $\overrightarrow{PA_1} = \vec{r}_1 - \vec{\beta}, \overrightarrow{PA_2} = \vec{r}_2 - \vec{\beta}, \overrightarrow{PA_3} = \vec{r}_3 - \vec{\beta} \ldots \overrightarrow{PA_n} = \vec{r}_n - \vec{\beta}$

Given relation is

$\bar{a}_1\bar{r}_1 + \bar{a}_2\bar{r}_2 + \bar{a}_3\bar{r}_3 + \ldots + \bar{a}_n\bar{r}_n$. It becomes $\bar{a}_1(\bar{r}_1 - \bar{\beta}) + \bar{a}_2(\bar{r}_2 - \bar{\beta}) + \bar{a}_3(\bar{r}_3 - \bar{\beta}) \ldots \bar{a}_n(\bar{r}_n - \bar{\beta})$

$= \bar{a}_1\bar{r}_1 + \bar{a}_2\bar{r}_2 + \bar{a}_3\bar{r}_3 + \ldots + \bar{a}_n\bar{r}_n - \bar{\beta}(\bar{a}_1 + \bar{a}_2 + \bar{a}_3 + \ldots + \bar{a}_n)$

$= \sum \bar{a}_1\bar{r}_1 - \bar{\beta}\sum \bar{a} = \sum \bar{a}_1\bar{r}_1 - 0 = \sum \bar{a}_1\bar{r}_1$

Example 52: For vectors $\bar{u}$ and $\bar{v}$, prove, (a) $(\bar{u}\cdot\bar{v})^2 + (\bar{u}\times\bar{v})^2 = |\bar{u}|^2|\bar{v}|^2$

(b) $\left(1+|\bar{u}|^2\right)\left(1+|\bar{v}|^2\right) = \left(1-\bar{u}\bar{v}\right)^2\left(\bar{u}+\bar{v}+\bar{u}\times\bar{v}\right)$ (I.I.T)

Sol. (a) $\left(\bar{u}\cdot\bar{v}\right)^2 + \left(\bar{u}\times\bar{v}\right)^2$

$= |u|^2|v|^2\cos^2\theta + |u^2||v^2|\sin^2\theta$

$= |\bar{u}|^2|\bar{v}|^2(\cos^2\theta + \sin^2\theta) = |\bar{u}|^2|\bar{v}|^2$

(b) $\left(1-\bar{u}\cdot\bar{v}\right)^2 + \left(\bar{u}+\bar{v}+\bar{u}\times\bar{v}\right)^2$

$= \left[1 + (\bar{u}\cdot\bar{v})^2 - 2\bar{u}\cdot\bar{v}\right] + |\bar{u}|^2 + |\bar{v}|^2 + (\bar{u}\times\bar{v})^2 + 2\bar{u}\cdot(\bar{u}\times\bar{v}) + 2\bar{v}\cdot(\bar{u}\times\bar{v}) + 2(\bar{u}\cdot\bar{v})$

$\bar{u}\cdot(\bar{u}\times\bar{v}) = 0, \bar{v}\cdot(\bar{u}\times\bar{v}) = 0$

$\therefore$ $= 1 + u^2v^2\cos^2\alpha - 2uv\cos\alpha + u^2 + v^2 + u^2v^2\sin^2\alpha + 2uv\cos\alpha$

$= 1 + u^2v^2(\cos^2\alpha + \sin^2\alpha) + u^2 + v^2$

$= 1 + u^2 + v^2 + u^2v^2 = (1 + u^2)(1 + v^2)$

$= (1 + |\bar{u}|^2)(1 + |\bar{v}|^2)$

Example 53: Find scalars α, β if $\bar{a}\times(\bar{b}\times\bar{c}) + (\bar{a}\cdot\bar{b})\bar{b} = (4 - 2\beta - \sin\alpha)\bar{b} + (\beta^2 - 1)\bar{c}$ and $(\bar{c}\cdot\bar{c})\bar{a} = \bar{c}$

while $\bar{b}$ and $\bar{c}$ are noncollinear.

Sol. Given $(\bar{c}\cdot\bar{c})\bar{a}=\bar{c}$

$\Rightarrow (\bar{c}\cdot\bar{c})\bar{a}\cdot\bar{c}=\bar{c}\cdot\bar{c}\Rightarrow \bar{a}\cdot\bar{c}=1$ ... (1)

and $\bar{a}\times(\bar{b}\times\bar{c})+(\bar{a}\cdot\bar{b})\bar{b}=(4-2\beta-\sin\alpha)\bar{b}+(\beta^2-1)\bar{c}$

$\Rightarrow (\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}+(\bar{a}\cdot\bar{b})\bar{b}=(4-2\beta-\sin\alpha)\bar{b}+(\beta^2-1)\bar{c}$

$\bar{b}$ and $\bar{c}$ are non collinear

$$\left.\begin{aligned}\bar{a}\cdot\bar{c}+\bar{a}\cdot\bar{b}&=4-2\beta-\sin\alpha\\ \bar{a}\cdot\bar{b}&=1-\beta^2\end{aligned}\right\}\quad\begin{aligned}&...(2)\\&...(3)\end{aligned}$$

From (1), (2) becomes 1+ $(1-\beta^2) = 4-2\beta-\sin\alpha$

$\Rightarrow 1+\sin\alpha = 3-2\beta+\beta^2$

$\Rightarrow -1+\sin\alpha = 1-2\beta+\beta^2 = (1-\beta)^2$

$\Rightarrow (\beta-1)^2 = \sin\alpha-1$

$(\beta-1)^2$ is +ve $\therefore$ $\sin\alpha-1$ should be positive.

$\therefore \sin\alpha \le 1$ $\sin\alpha = 1 \Rightarrow \alpha = \dfrac{\pi}{2}$

$\therefore (\beta-1)^2 = 0 \Rightarrow \beta = 1$

Example 54: Vectors $\bar{c}$, $\bar{a}=x\hat{i}+y\hat{j}+z\hat{k}, \bar{b}=\hat{j}$ form a right handed system, find C

Sol. $\bar{c}$, $\bar{a}$, $\bar{b}$ form a right handed system

$$\therefore\ \bar{c}=\lambda(\bar{b}\times\bar{a})=\lambda\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\0&1&0\\x&y&2\end{vmatrix}=\lambda(i\hat{z}-x\hat{k})$$

If $\lambda = 1$ then $\bar{c}=z\hat{i}-x\hat{k}$

Example 55: Find vector which makes equal angles with vectors $2\hat{i}-\hat{j}+2\hat{k}$, $-3\hat{i}+4\hat{k}$ and $2\hat{i}-3\hat{j}+6\hat{k}$.

Sol. Let $\bar{a}=x\hat{i}+y\hat{j}+z\hat{k}$

$\bar{a}\cdot(2\hat{i}-\hat{j}+2\hat{k})=2x-y+2z=3|\bar{a}|\cos\theta$ ... (i)

$a\cdot(-3i+4k)=-3x+4z=5|a|\cos\theta$ ... (ii)

$\bar{a}\cdot(2i-3j+6k)=2x-3y+6z=7|\bar{a}|\cos\theta$ ... (iii)

$\therefore \ \frac{2x-y+2z}{3}=\frac{-3x+4z}{5} \Rightarrow 19x-5y-2z=0$ ... (iv)

$\frac{2x-y+2z}{3}=\frac{2x-3y+6z}{7} \Rightarrow 4x+y-2z=0$ ... (v)

from (iv) and (v) $\frac{x}{12}=\frac{y}{30}=\frac{z}{39} \Rightarrow \frac{x}{4}=\frac{y}{10}=\frac{z}{13}=\mu$

$\Rightarrow$ x = 4μ, y = 10μ and z = 13 μ $\therefore$ vector is $\mu(4\hat{i}+14\hat{j}+13\hat{k})$

Practice Worksheet (Foundation Level) – 3(d)

1. If $\bar{a}=\hat{i}-2\hat{j}+2\hat{k}$, $\bar{b}=3\hat{i}+6\hat{j}-2\hat{k}$, $\bar{c}=\hat{i}-\hat{j}-\hat{k}$, then calculate $\bar{a}\times(\bar{b}\times\bar{c})$ and $\bar{b}\cdot(\bar{a}\times\bar{c})$ and $\bar{b}\cdot(\bar{a}\times\bar{c})$.

2. $\bar{a}=3\hat{i}-2\hat{j}+6\hat{k}, \bar{b}=2\hat{i}-2\hat{j}-\hat{k}, \bar{c}=4\hat{i}-3\hat{j}-2\hat{k}$, calculate $\left|\frac{\bar{a}\times(\bar{b}\times\bar{c})}{\bar{b}\times(\bar{a}\times\bar{c})}\right|$.

3. $\bar{a}=3\hat{i}-2\hat{j}+6\hat{k}, \bar{b}=2\hat{i}+2\hat{j}-\hat{k}, \bar{c}=3\hat{i}+4\hat{j}$, and $\bar{d}=4\hat{j}-3\hat{k}$. Calculate $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})$

4. For vectors given in Q. 3, calculate $\bar{a}\cdot\left[\bar{b}\times(\bar{c}\times\bar{d})\right]$

5. $\bar{a}=\hat{i}+\hat{j}+\hat{k}, \bar{b}=\hat{i}-2\hat{j}+3\hat{k}, \bar{c}=3\hat{i}+\hat{j}-2\hat{k}$ and $\bar{d}=2\hat{i}-3\hat{j}+\hat{k}$, calculate $\left[\bar{a}\times(\bar{b}\times\bar{c})\right]\times\bar{d}$ and write also its unit vector.

6. $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors, but not mutually perpendicular, then prove $\bar{a}-2\bar{b}+3\bar{c}, -2\bar{a}+3\bar{b}-4\bar{c}$ and $\bar{a}-3\bar{b}+5\bar{c}$ are coplanar.

7. $\bar{\lambda}$ is projection vector of vector $\bar{a}=2\hat{i}+3\hat{j}-5\hat{k}$ on vector $3\hat{i}+2\hat{j}-6\hat{k}$, calculate $\bar{\lambda}\times\bar{a}$.

8. $\bar{a}, \bar{b}, \bar{c}$ are position vector of vertices of triangle ABC. Show that area of triangle is $\frac{1}{2}\left[\bar{b}-\bar{c} \ \ \bar{c}-\bar{a} \ \ \bar{a}-\bar{b}\right]$.

9. Prove that $\bar{p}\times(\bar{q}\times\bar{r})+\bar{q}\times(\bar{r}\times\bar{p})+\bar{r}\times(\bar{p}\times\bar{q})=0$

10. Prove $[\bar{d} \ \ \bar{a}\times\bar{b} \ \ \bar{a}\times\bar{c}]=[\bar{a}\cdot\bar{d}][\bar{a} \ \bar{b} \ \bar{c}]$.

11. $\bar{p}, \bar{q}, \bar{r}$ are three non-parallel vectors of magnitude 1, 2, 2 respectively. If $\bar{p}\times(\bar{q}\times\bar{r})+\bar{q}=0$, then calculate acute angle between $\bar{p}$ and $\bar{r}$.

12. $\bar{p}=2\hat{i}+2\hat{j}+\hat{k}, \bar{q}=\hat{i}-2\hat{j}+2\hat{k}$, a vector $\bar{r}$ is coplanar with $\bar{p}$ and $\bar{q}$ and vector $\bar{r}$ is also perpendicular to $\bar{q}$. Calculate $\bar{r}$ if $|\bar{r}|=\sqrt{17}$.

13. $\bar{a} = 2\hat{i} + 2\hat{j} - \hat{k}, \bar{b} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\bar{c}$ is a vector such that $\bar{b} \cdot \bar{c} = |\bar{c}|$ and $|\bar{c} - \bar{b}| = 2\sqrt{2}$. If angle between $(\bar{a} \times \bar{b})$ and $\bar{c}$ is 30^o, find $[(\bar{a} \times \bar{b}) \times \bar{c}]$.

14. If $\bar{r} \times \bar{a} = \bar{b}$ and $\bar{b}$ is perpendicular to $\bar{a}$ then prove $\bar{r} = x\bar{a} + \dfrac{(\bar{a} \times \bar{b})}{|\bar{a}|^2}$.

15. The P. vectors of P and Q are $\bar{p}$ and $\bar{q}$, $|\bar{p}|$ = 5 and $|\bar{q}| = \lambda$. Points R and S divide PQ, internally and externally in the ratio of 4 : 3 if OR and OS, (origin, O) are perpendicular, the find $|\bar{q}|$.

16. Vector $\bar{c} = \hat{i}, \bar{a} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\bar{b}$ form a right handed system. Find $\bar{b}$.

17. Find vector which makes equal angles with vectors $3\hat{i} + 4\hat{j}, 2\hat{i} - \hat{j} + 2\hat{k}$ and $3\hat{i} + 2\hat{j} + 6\hat{k}$.

18. In a right angled triangle ABC, $\overline{AB} = \bar{p}$ and $\angle c$ = 90^o. Evaluate $\overline{AB} \cdot \overline{AC} + \overline{BC} \cdot \overline{BA} + \overline{CA} \cdot \overline{CB}$.

19. Find a vector of magnitude $\sqrt{51}$ which makes equal angles with vectors.

$$\bar{a} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}),\ \bar{b} = \frac{1}{5}(-4\hat{i} - 3\hat{k}), \text{ and } \bar{c} = \hat{j}$$

20. Find vectors $\bar{a}$ and $\bar{b}$ when $\bar{a} + \bar{b} = 5\hat{i} + 4\hat{j} + 3\hat{k}$, $\bar{a} \times \bar{b} = -14\hat{i} + 15\hat{j} + 2\hat{k}$, $|\bar{a}| : |\bar{b}| = 7 : 3$.

Solved Example

Example 56: Let $\overline{p},\overline{q},\overline{r}$ be three mutually perpendicular vectors of same magnitude. Find vector x, if satisfies the equation. $\overline{p}\times[(\overline{x}-\overline{q})\times\overline{p}]+\overline{q}\times[(\overline{x}-\overline{r})\times\overline{q}]+\overline{r}\times[(\overline{x}-\overline{p})\times\overline{r}]=0$

Sol. $\overline{p}\times[(\overline{x}-\overline{q})\times\overline{p}]+\overline{q}\times[(\overline{x}-\overline{r})\times\overline{q}]+\overline{r}\times[(\overline{x}-\overline{p})\times\overline{r}]=0$

$\Rightarrow(\overline{p}\cdot\overline{p})(\overline{x}-\overline{q})-[\overline{p}\cdot(\overline{x}-\overline{q})]\overline{p}+\overline{q}\cdot\overline{q}(\overline{x}-\overline{r})-[\overline{q}\cdot(\overline{x}-\overline{r})]\overline{q}+(\overline{r}\cdot\overline{r})(\overline{x}-\overline{p})-[\overline{r}\cdot(\overline{x}-\overline{p})]\overline{r}=0$

$\Rightarrow(|\overline{p}|^2+|\overline{q}|^2+|\overline{r}|^2)\overline{x}-[|\overline{p}|^2\ \overline{q}+|\overline{q}|^2\ \overline{r}+|\overline{r}|^2\ \overline{p}]-[(\overline{p}\cdot\overline{x})\overline{p}+(\overline{q}\cdot\overline{x})\overline{q}+(\overline{r}\cdot\overline{x})\overline{r}]$

$+[(\overline{p}\cdot\overline{q})\overline{p}+(\overline{q}\cdot\overline{r})\overline{q}+(\overline{r}\cdot\overline{p})\overline{r}]=0$

Let $|\overline{p}|=|\overline{q}|=|\overline{r}|=\lambda$ (magnitude)

$\therefore \text{Exp}=3\lambda^2\overline{x}-\lambda^2(\overline{p}+\overline{q}+\overline{r})-[(\overline{p}\cdot\overline{x})\overline{p}+(\overline{q}\cdot\overline{x})\overline{q}+(\overline{r}\cdot\overline{x})\cdot\overline{r}]\times 0=0$

$(\overline{p}.\overline{q}=\overline{q}\cdot\overline{r}=\overline{r}\cdot\overline{p}=0)$ ... (1)

Taking dot product with $\overline{p}$

$3\lambda^2(\overline{x}\cdot\overline{p})-\lambda^2(\overline{p}\cdot\overline{p}+0+0)-[(\overline{p}\cdot\overline{x})|\overline{p}|^2+0+0]=0$

$\Rightarrow\ 3\lambda^2(\overline{x}\cdot\overline{p})-\lambda^4-(\overline{p}\cdot\overline{x})\lambda^2=0$

$\Rightarrow\ 2\lambda^2(x\cdot p)=\lambda^4\Rightarrow(x\cdot p)=\frac{1}{2}\lambda^2$

Similarly taking dot product of (1) with $\overline{q}$ and then with $\overline{r}$.

$\overline{q}\cdot\overline{x}=\frac{\lambda^2}{2},\overline{r}\cdot\overline{x}=\frac{\lambda^2}{2}$

Putting these values in (1)

$3\lambda^2\overline{x}-\lambda^2(\overline{p}+\overline{q}+\overline{r})-\frac{1}{2}\lambda^2(\overline{p}+\overline{q}+\overline{r})=0$

$\therefore\ x=\frac{1}{2}(\overline{p}+\overline{q}+\overline{r})$

Example 57: $\overline{a},\overline{b},\overline{c}$ are non-coplanar unit vectors equally inclined to one another at angle θ. If $\overline{a}\times\overline{b}+\overline{b}\times\overline{c}=p\overline{a}+q\overline{b}+r\overline{c}$, then find scalars p, q, r in term of θ.
(I.I.T.)

Sol. $\overline{a}\times\overline{b}+\overline{b}\times\overline{c}=p\overline{a}+q\overline{b}+r\overline{c}$ ... (1)

and $\overline{a}\cdot\overline{b}=\overline{b}\cdot\overline{c}=\overline{c}\cdot\overline{a}=1\cdot 1\cos\theta$

$\therefore\ \bar{a}\cdot\left[\bar{a}\times\bar{b}+\bar{b}\times\bar{c}\right]=p\cdot|\bar{a}|^2+q\cdot\bar{a}\cdot\bar{b}+r\cdot\bar{a}\cdot\bar{c}$

$[a\ \ a\ \ b] + [a\ \ b\ \ c] = p + q\cos\theta + r\cos\theta$

$\therefore\ p + q\cos\theta + r\cos\theta = [a\ \ b\ \ c]$ ... (2)

Dot product of (1) by $\bar{b}$ shall give

$p\cos\theta+q+r\cos\theta=[\bar{b}\ \bar{a}\ \bar{b}]+[\bar{b}\ \bar{b}\ \bar{c}]=0$ ... (3)

$p\cos\theta+q\cos\theta+r=[\bar{c}\ \bar{a}\ \bar{b}]=[\bar{a}\ \bar{b}\ \bar{c}]$ ... (4)

and by $\bar{r}$, (4) – (2) $p(\cos\theta-1)+r(1-\cos\theta)=0$ ... (5)

i.e. $\left.\begin{array}{c} p(\cos\theta-1)+0\cdot q+r(1-\cos\theta)=0 \\ p\cos\theta+q+r\cos\theta=0 \end{array}\right\}$

from (3)

$$\therefore\ \frac{p}{\cos\theta-1}=\frac{q}{2\cos\theta(1-\cos\theta)}=\frac{r}{\cos\theta-1}=k$$

$$\Rightarrow \frac{\bar{p}}{1}=\frac{\bar{q}}{-2\cos\theta}=\frac{\bar{r}}{1}=k$$

$\therefore\ p = k,\ q = -2k\cos\theta,\ r = k$

Putting these values of p, q, r in (2)

$k - 2k\cos^2\theta + k\cos\theta = [\bar{a}\ \bar{b}\ \bar{c}]$

$$\therefore\ k=\frac{[\bar{a}\ \bar{b}\ \bar{c}]}{1+\cos\theta-2\cos^2\theta}=\frac{[\bar{a}\ \bar{b}\ \bar{c}]}{(1-\cos\theta)(1+2\cos\theta)}$$

$$\therefore\ p-r=\frac{[\bar{a}\ \bar{b}\ \bar{c}]}{(1-\cos\theta)(1+2\cos\theta)},q=\frac{-2\cos\theta[a\ b\ c]}{(1-\cos\theta)(1+2\cos\theta)}$$

Example 58: Four vectors acting at a point are in equilibrium. Show that magnitude of each is proportional to the scalar product of the other three.

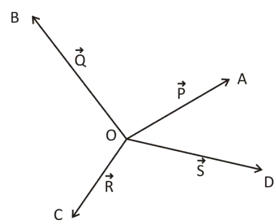


Fig 22

Sol. Forces $\bar{P}, \bar{Q}, \bar{R}$ and $\bar{S}$ act at O (as shown in figure) and are in equilibrium.

$\Rightarrow$ P is equal to the magnitude of resultant of $\bar{Q}, \bar{R}$ and $\bar{S}$ but opposite in direction.

Forces $\bar{Q}, \bar{R}$ and $\bar{S}$ act along OB, OC and OD and these can be taken as conterminous edges of a parallelopiped and resultant of these forces $\bar{Q}, \bar{R}, \bar{S}$ is equal to the volume of parallelopiped which is scalar product of these forces. This is true for other forces too.

Example 59: $\overline{b}, \overline{c}, \overline{d}$ are non-coplanar vectors prove $(\overline{a}\times\overline{b})\times(\overline{c}\times\overline{d})+(\overline{a}\times\overline{c})\times(\overline{d}\times\overline{b})+(\overline{a}\times\overline{d})\times(\overline{b}\times\overline{c})$ is parallel to $\overline{a}$.

Sol. $(\overline{a}\times\overline{b})\times(\overline{c}\times\overline{d})+(\overline{a}\times\overline{c})\times(\overline{d}\times\overline{b})+(\overline{a}\times\overline{d})\times(\overline{b}\times\overline{c})$

$=(\overline{a}\times\overline{b})\times\overline{m}+(\overline{a}+\overline{c})\times\overline{n}+(\overline{a}\times\overline{d})\times\overline{p}$

$=(\overline{a}\cdot\overline{m})\overline{b}-(\overline{b}\cdot\overline{m})\overline{a}+(\overline{a}\cdot\overline{n})\overline{c}-(\overline{c}\cdot\overline{n})\overline{a}+(\overline{a}\cdot\overline{p})\overline{d}-(\overline{d}\cdot\overline{p})\overline{a}$

$=\left[\overline{a}\cdot(\overline{c}\times\overline{d})\right]\overline{b}-\left[\overline{b}\cdot(\overline{c}\times\overline{d})\right]\overline{a}+\left[\overline{a}\cdot(\overline{d}\times\overline{b})\right]\overline{c}-\left[\overline{c}\cdot(\overline{d}\times\overline{b})\right]\overline{a}+\left[\overline{a}\cdot(\overline{b}\times\overline{c})\right]\overline{d}-\left[\overline{d}\cdot(\overline{b}\times\overline{c})\right]\overline{a}$

$=\left[\overline{a}\ \overline{c}\ \overline{d}\right]\overline{b}-\left[\overline{b}\ \overline{c}\ \overline{d}\right]\overline{a}+\left[\overline{a}\ \overline{d}\ \overline{b}\right]\overline{c}-\left[\overline{c}\ \overline{d}\ \overline{b}\right]\overline{a}+\left[\overline{a}\ \overline{b}\ \overline{c}\right]\overline{d}-\left[\overline{d}\ \overline{b}\ \overline{c}\right]\overline{a}$

$\left[\overline{b}\ \overline{c}\ \overline{d}\right],\left[\overline{c}\ \overline{d}\ \overline{b}\right],\left[\overline{d}, \overline{b}, \overline{c}\right]$ are scalars = μ

$\therefore$ Given exp = $\left[\overline{a}\ \overline{c}\ \overline{d}\right]\overline{b}+\left[\overline{a}\ \overline{d}\ \overline{b}\right]\overline{c}+\left[\overline{a}\ \overline{b}\ \overline{c}\right]\overline{d}-3\mu\overline{a}$

and $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ are non coplanar then $\overline{a}$ can be expressed as a linear combination of $\overline{b}, \overline{c}$ and $\overline{d}$

$\therefore$ $\left[\overline{a}\ \overline{c}\ \overline{d}\right]\overline{b}+\left[\overline{a}\ \overline{d}\ \overline{b}\right]\overline{c}+\left[\overline{a}\ \overline{b}\ \overline{c}\right]\overline{d}$ is parallel to $\overline{a}$.

$\therefore$ Given expression is parallel to $\overline{a}$.

Example 60: $\hat{x}, \hat{y}, \hat{z}$ are unit vectors such that $\hat{x}+\hat{y}+\hat{z}=\overline{a}, \hat{x}\times(\hat{y}\times\hat{z})=\overline{b}, (\hat{x}\times\hat{y})\times\hat{z}=\overline{c}$ and $\overline{a}\cdot\hat{x}=\frac{3}{2}, \overline{a}\cdot\hat{y}=\frac{7}{4}, |\overline{a}|=2$, find $\hat{x}, \hat{y}, \hat{z}$

Sol. $\hat{x}\cdot\overline{a}=\hat{x}\cdot(\hat{x}+\hat{y}+\hat{z})=1+\hat{x}\cdot\hat{y}+\hat{x}\cdot\hat{z}=\frac{3}{2}$ ... (i)

$\hat{y}\cdot\hat{a}=\hat{y}\cdot(\hat{x}+\hat{y}+\hat{z})=1+\hat{y}\cdot\hat{x}+\hat{y}\cdot\hat{z}=\frac{7}{4}$ ... (ii)

from (i) $\hat{x}\cdot\hat{y}+\hat{x}\cdot\hat{z}=\frac{1}{2}$ ... (1)

from (ii) $\hat{y}.\hat{x}+\hat{y}.\hat{z}=\frac{3}{4}$ ... (2)

Now $a\cdot\left[\hat{x}+\hat{y}+\hat{z}\right]=\overline{a}\cdot\overline{a}=\overline{a}\cdot\hat{x}+\overline{a}\cdot\hat{y}+\overline{a}\cdot\hat{z}$

$\Rightarrow \overline{a}\cdot\hat{z}=|\overline{a}|^2-\overline{a}\cdot\hat{x}-\overline{a}\cdot\hat{y}=4-\frac{3}{2}-\frac{7}{4}=\frac{3}{4}$

and $\hat{z}\cdot\bar{a}=\hat{z}(\hat{x}+\hat{y}+\hat{z})=\hat{z}\cdot\hat{x}+\hat{z}\cdot\hat{y}+1=\dfrac{3}{4}$

$\therefore\ \hat{z}\cdot\hat{x}+\hat{z}\cdot\hat{y}=-\dfrac{1}{4}$ ... (3)

adding (1) (2) and (3)

$\hat{z}\cdot\hat{x}+\hat{x}\cdot\hat{y}+\hat{y}\cdot\hat{z}=\dfrac{1}{2}\left[\dfrac{1}{2}+\dfrac{3}{4}-\dfrac{1}{4}\right]=\dfrac{1}{2}$ ... (4)

Subtracting from (4), (1), (2) and (3) turn by turn

$\hat{y}\cdot\hat{z}=0,\hat{z}\cdot\hat{x}=-\dfrac{1}{4}\quad \hat{x}\cdot\hat{y}=\dfrac{3}{4}$

Now $\hat{x}\times(\hat{y}\times\hat{z})=(\hat{x}\cdot\hat{z})\hat{y}-(\hat{x}\cdot\hat{y})\hat{z}=\bar{b}$

$\therefore\ -\dfrac{1}{4}\hat{y}-\dfrac{3}{4}\hat{z}=\bar{b}\Rightarrow\hat{y}+3\hat{z}=-4\bar{b}$ ... (5)

$(\hat{x}\times\hat{y})\times\hat{z}=(\hat{x}\cdot\hat{z})\hat{y}-(\hat{y}\cdot\hat{z})\hat{x}=\bar{c}$

$\Rightarrow-\dfrac{1}{4}\hat{y}-0=\bar{c}\Rightarrow\hat{y}=-4\bar{c}$

and $\hat{x}=\bar{a}-\hat{y}-\hat{z}=\bar{a}+4\bar{c}+\dfrac{4}{3}(\bar{b}-\bar{c})=\dfrac{1}{3}\left[3\bar{a}+4\bar{b}-8\bar{c}\right]$

$\hat{y}=-4\bar{c},\ \ \hat{z}=\dfrac{4}{3}(\bar{b}-\bar{c})$

Example 61: The vectors $\bar{x},\bar{y},\bar{z}$, each of magnitude $\sqrt{2}$, make angle 60^{o} with each other. If $\bar{x}\times(\bar{y}\times\bar{z})=\bar{a},\ \bar{y}=(\bar{z}\times\bar{x})=\bar{b}$ and $\bar{x}+\bar{y}=\bar{c}$, then find $\bar{x},\bar{y},\bar{z}$ in terms of $\bar{a},\bar{b},\bar{c}$.

Sol. $\bar{x}\cdot\bar{y}=\bar{y}\cdot\bar{z}=\bar{z}\cdot\bar{x}=\sqrt{2}\cdot\sqrt{2}\cdot\dfrac{1}{2}=1$

$\bar{x}\times(\bar{y}\times\bar{z})=(\bar{x}\cdot\bar{z})\bar{y}-(\bar{x}\cdot\bar{y})\bar{z}=\bar{y}-\bar{z}=\bar{a}$ ... (1)

$\bar{y}\times(\bar{z}\times\bar{x})=(\bar{y}\cdot\bar{x})\bar{z}-(\bar{y}\cdot\bar{z})\bar{x}=\bar{z}-\bar{x}=\bar{b}$ ... (2)

$\bar{z}\times(\bar{x}\times\bar{y})=(\bar{z}\cdot\bar{y})\bar{x}-(\bar{z}\cdot\bar{x})\bar{y}=\bar{x}-\bar{y}\Rightarrow\bar{z}\times\bar{c}=\bar{x}-\bar{y}$ ... (3)

from (1) $\bar{z}\times(\bar{y}-\bar{z})=\bar{z}\times\bar{a}\Rightarrow\bar{z}\times\bar{y}=\bar{z}\times\bar{a}\Rightarrow\bar{y}=\bar{a}$

from (2) $\bar{x}\times(\bar{z}-\bar{x})=\bar{x}\times\bar{b}\Rightarrow\bar{x}\times\bar{z}=\bar{x}\times\bar{b}$

$\therefore\ \bar{z}=\bar{b}$, putting these values in 3.

$\bar{x}-\bar{y}=\bar{z}\times\bar{c}\Rightarrow\bar{x}-\bar{a}=\bar{b}\times\bar{c}\Rightarrow\bar{x}=\bar{a}+\bar{b}\times\bar{c}$

$\therefore\ \bar{x} = \bar{a} + (\bar{b} \times \bar{c})\ \bar{y} = \bar{a},\ \bar{z} = \bar{b}$

Example 62 : $\bar{r} \times \bar{a} = \bar{b} \times \bar{a}$ and $\bar{r} \times \bar{b} = \bar{a} \times \bar{b}$ $\bar{a} \neq 0, \bar{b} \neq 0, \bar{a} \neq \lambda\bar{b}$ and $\bar{a}$ not perpendicular to $\bar{b}$, then find $\bar{r}$.

Sol. $\bar{r} \times \bar{a} = \bar{b} \times \bar{a} \Rightarrow (\bar{x} - \bar{b}) \times \bar{a} = 0$

$\therefore\ \bar{r} - \bar{b}$ is parallel to $\bar{a}$ and $\bar{r} \times \bar{b} - \bar{a} \times \bar{b} = (\bar{r} - \bar{a}) \times \bar{b} = 0$

$\bar{r} - \bar{a}$ is parallel to $\bar{b}$

$\therefore\ \bar{r} = \bar{b} + \bar{a}\lambda$ and $\bar{r} = \bar{a} + \mu\bar{b}$ and $\bar{b} + \bar{a}\lambda = \bar{a} + \mu\bar{b} \Rightarrow \mu = 1, \lambda = 1$

$\therefore\ \bar{r} = \bar{a} + \bar{b}$

Example 63: Let $\bar{a}$ be a unit vector and $\bar{b}$ a non-zero vector not parallel to $\bar{a}$. Find angles of the triangle, two sides of which are $\sqrt{3}(a \times b)$ and $\bar{b} - (\bar{a}.\bar{b})\bar{a}$

Sol. Let $\overline{AB} = \sqrt{3}(\hat{a} \times \bar{b})$ and $\overline{AC} = \bar{b} - (\hat{a} \cdot \bar{b})\hat{a}$

$\therefore\ \overline{AB} \cdot \overline{AC} = \sqrt{3}\left[(\hat{a} \times \bar{b}) \cdot \bar{b} - (\hat{a} \times \bar{b}) \cdot (\hat{a} \cdot \bar{b})\hat{a}\right]$

$= \sqrt{3}\left[(\hat{a}\ \bar{b}\ \bar{b}) - (\hat{a}\ \bar{b}\ \hat{a})(\hat{a} \cdot \bar{b})\right] = \sqrt{3}[0 - 0] = 0$

$\therefore$ AB $\perp$ AC, $\angle A = 90^o$

and $(AC)^2 = \left[\bar{b} - (\hat{a} \cdot \bar{b})\hat{a}\right]^2 = (\bar{b}) + (\hat{a} \cdot \bar{b})(\hat{a}) - 2(\bar{b} \cdot \hat{a})(\hat{a} \cdot \bar{b})$

$= \bar{b}^2 + (\hat{a} \cdot \bar{b})^2 - 2(\hat{a}.\hat{b}) = \bar{b}^2 - (\hat{a} \cdot \bar{b})^2$

$= b^2 + 1.b^2 \cos^2\theta = b^2 \sin^2\theta$

$= 1^2.b^2 \sin^2\theta = (\hat{a})^2 (\bar{b})^2 \sin^2\theta$

$= (\hat{a} \times \bar{b})^2 = \frac{1}{3} AB^2$

$\therefore\ \sqrt{3} \cdot AC = AB \Rightarrow \frac{AB}{AC} = \sqrt{3} = \tan 60^o, \therefore C = 60^o, B = 30^o$

Example 64: Let β = 4i + 3 and γ be, two vectors perpendicular to each other in x-y plane, all the vectors in the same plane having projection 1 and 2 along β and γ respectively are given by … **(I.I.T)**

Sol. Let $\beta = 4\hat{i} + 3\hat{j}, \gamma = \lambda(3\hat{i} - 4\hat{j})$, coplanar and perpendicular to each other.

Let $\bar{\alpha} = pi + qj$ $\therefore\ \bar{\alpha} \cdot \bar{\beta} = 4p + 3q$

Projection of α on $\beta = = \frac{\bar{\alpha} \cdot \bar{\beta}}{|\bar{\beta}|} = \frac{4p + 3q}{\pm 5} = 1$

Projection of α on $\gamma = \frac{\overline{\alpha}\cdot\overline{\gamma}}{|\overline{\gamma}|} = \frac{\lambda(3p-4q)}{\pm 5\lambda} = 2$

$\therefore$ 4p + 3q = 5, 3p – 4q = 10; 4p + 3q = 5, 3p – 4q = –10

and 4p + 3q = –5, 3p – 4q = –10; 4p + 3q = –5, 3p – 4q = 10

solving, $\alpha = 2\hat{i} - \hat{j}, \alpha = \frac{2}{5}\hat{i} - \frac{11}{5}\hat{j}$

$\alpha = -\frac{2}{5}\hat{i} + \frac{11}{5}\hat{j}, \alpha = -2\hat{i} + \hat{j}$

Example 65: The vector sum of $\overline{a}$ and $\overline{b}$ trisects the angle between them. If $|\overline{a}| = \overline{a}$ and $|\overline{b}| = \overline{b}$ and a > b. Then prove the angle between two vectors $\overline{a}$ and $\overline{b}$ is 3 $\cos^{-1}\left(\frac{a}{2b}\right)$ and the sum vector $\overline{a}+\overline{b}$ has magnitude $\frac{a^2 - b^2}{b}$.

Sol. Let 3θ be the angle between $\overline{a}$ and $\overline{b}$ a > b. Then resultant shall be nearer to $\overline{a}$

$\therefore$ angle between $\overline{a}+\overline{b}$ and $\overline{a}$ shall be θ and angle between $\overline{a}+\overline{b}$ and $\overline{b}$ shall be 2θ.

$\therefore\ (\overline{a}+\overline{b})\cdot\overline{a} = |\overline{a}+\overline{b}||\overline{a}|\cos\theta$ ... (i)

and $(\overline{a}+\overline{b})\cdot\overline{b} = |\overline{a}+\overline{b}||\overline{b}|\cos 2\theta$ ... (ii)

from (i) $\overline{a}\cdot\overline{a} + \overline{a}\cdot\overline{b} = |\overline{a}+\overline{b}||\overline{a}|\cos 2\theta \Rightarrow a^2 + ab\cos 3\theta = |\overline{a}+\overline{b}||\overline{a}|\cos 2\theta$ ... (1)

and $\overline{a}\cdot\overline{b} + \overline{b}\cdot\overline{b} = |\overline{a}+\overline{b}||\overline{b}|\cos 2\theta \Rightarrow ab\cos 3\theta + b^2 = |\overline{a}+\overline{b}||\overline{b}|\cos 2\theta$ ... (2)

$\therefore\ \frac{(1)}{(2)} \Rightarrow \frac{a^2 + ab\cos 3\theta}{ab\cos 3\theta + b^2} = \frac{a\cos\theta}{b\cos 2\theta}$

$\therefore \Rightarrow$ a cos 2θ + b cos 3θ · cos 2θ = a cos θ cos 3θ + b cos θ

$\Rightarrow$ a (cos 2θ – cos θ cos 3θ) = b (cos θ – cos 3θ cos 2θ)

$\Rightarrow$ a [2 cos 2θ – cos 4θ – cos 2θ] = b[2 cos θ – cos 5θ – cos θ]

$\Rightarrow$ a · [cos 2θ – cos 4θ] = b [cos θ – cos 5θ]

$\Rightarrow$ 2 · a sin 3θ·sin θ = b sin 3θ · sin 2θ

$\Rightarrow$ a sin θ = b sin 2θ $\Rightarrow$ a = 2 b cos θ

$\therefore\ \cos^{-1}\left(\frac{a}{2b}\right)$

$\therefore$ angle between $\bar{a}$ and $\bar{b}$ = 3θ = $3\cos^{-1}\left(\frac{\bar{a}}{2\bar{b}}\right)$

(ii) $|\bar{a}+\bar{b}|^2 = |\bar{a}|^2 + |\bar{b}|^2 + 2|\bar{a}||\bar{b}|\cos 3\theta = a^2 + b^2 + 2ab\cos 3\theta$

$= a^2 + b^2 + 2ab\left[4.\frac{a^3}{8b^3} - \frac{3a}{2b}\right]$

$= a^2 + b^2 + \frac{a^4}{b^2} - 3a^2 = \frac{b^2a^2 + b^4 + a^4 - 3a^2b^2}{b^2}$

$= \frac{(a^2-b^2)^2}{b^2} \Rightarrow |\bar{a}+\bar{b}| = (a^2-b^2)/b$

Example 66: If $\bar{a}$ is perpendicular to $\bar{b}$ then prove solution of $\bar{r}\times\bar{a} = \bar{b}$ is given by $\bar{r} = x\bar{a} + \frac{1}{|\bar{a}|^2}(\bar{a}x\bar{b})$ where x is a scalar.

Sol : $\bar{a}$ **,** $\bar{b}$ and $\bar{a}X\bar{b}$ are non coplaner.

$\therefore$ Let $\bar{r} = \bar{a}x + \bar{b}y + (\bar{a}\times\bar{b})z$ for same scalars x,y and z.

Now $\bar{b} = \bar{r}\times\bar{a} = [x\bar{a} + y\bar{b} + z(\bar{a}\times\bar{b})]x\bar{a}$

$= y(\bar{b}\times\bar{a}) + z(\bar{a}\times\bar{b})x\bar{a}$

$= y(\bar{b}\times\bar{a}) - z[\bar{a}\times(\bar{a}\times\bar{b})]$

$= y(\bar{b}\times\bar{a}) - z[(\bar{a}.\bar{b})\bar{a} - (\bar{a}.\bar{a})\bar{b}]$

$= y(\bar{b}\times\bar{a}) + z(\bar{a}.\bar{a})\bar{b}$ $[\bar{a}.\bar{b} = 0]$

Comparing the coefficients we get

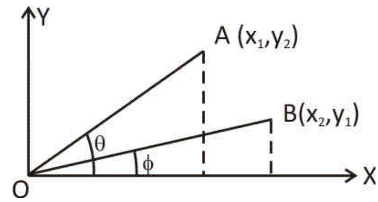
y = 0, z = $\frac{1}{\bar{a}.\bar{a}} = \frac{1}{|\bar{a}|^2}$

$\therefore\ \bar{r} = x\bar{a} + \frac{1}{|\bar{a}|^2}(\bar{a}\times\bar{b})$

SUBJECTIVE

1. Prove by vector method that $\sin(\theta-\phi)=\sin\theta.\cos\phi-\cos\theta\sin\phi$.

Sol: Let r_1, r_2 be the position vectors of the points $A(x_1 y_1)$ and $B(x_2 y_2)$ in the x y - plane, making angles θ and ϕ respectively with the x-axis $(\theta>\phi)$.



Hence $x_1=r_1\cos\theta$, $y_1=r_1\sin\theta$, ……… (i)

$x_2=r_2\cos\phi$, $y_2=r_2\sin\phi$.

Now $r_2\times r_1=r_2 r_1\sin(\theta-\phi)\hat{n}$ ……… (ii)

But $r_2=x_2 i+y_2 j$ and $r_1=x_1 i+y_1 j$

= $r_2\times r_1=(x_2 i+y_2 j)\times(x_1 i+y_1 j)$

= $(x_2 y_1-x_1 y_2)\hat{n}$ ……… (iii)

(as $i\times j=-j\times i=-\hat{n}$)and $i\times j=j\times i=0$.

From (ii) and (iii),

$r_1 r_2\sin(\theta-\phi)\hat{n}=(x_2 y_1-x_1 y_2)\hat{n}$

so that $r_1 r_2\sin(\theta-\phi)\hat{n}=(x_2 y_1-x_1 y_2)\hat{n}$

= $r_1 r_2[\sin\theta\cos\phi-\sin\phi\cos\theta]$

$\Rightarrow\sin(\theta-\phi)=\sin\theta\cos\phi-\sin\phi\cos\theta$.

2. Show that the solution of the equation $k\,\vec{r}+\vec{r}\times\vec{a}=\vec{b}$ where *k* is a non-zero scalar and $\vec{a}$ and $\vec{b}$ are two non- collinear vectors, is of the form $\vec{r}=\frac{1}{(k^2+a^2)}\left(\frac{\vec{a}.\vec{b}}{k}\vec{a}+k\vec{b}+\vec{a}\times\vec{b}\right)$.

Sol: The vectors $\vec{a}$, $\vec{b}$ are non- collinear and therefore $\vec{r}$ can be expressed as

$\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$ ……… (i)

Where x, y, z are some scalars.

Substituting this in the given equation

$\Rightarrow kx\vec{a}+ky\vec{b}+kz(\vec{a}\times\vec{b})+(x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b}))\times\vec{a}=\vec{b}$

$\Rightarrow kx\vec{a}+ky\vec{b}+kz(\vec{a}\times\vec{b})+0+y(\vec{b}\times\vec{a})+z(\vec{a}\times\vec{b})\times\vec{a}=\vec{b}$

$\Rightarrow kx\vec{a}+ky\vec{b}+kz(\vec{a}\times\vec{b})-y(\vec{a}\times\vec{b})+z[(\vec{a}.\vec{a})\vec{b}-(\vec{a}.\vec{b})\vec{a}]=\vec{b}$

$\Rightarrow (kx-(\vec{a}.\vec{b})z)\vec{a}+(ky+z(\vec{a}.\vec{a}))\vec{b}+(\vec{a}\times\vec{b})(kz-y)=\vec{b}$

Comparing we get, $x=\dfrac{\vec{a}.\vec{b}}{k(k^2+a^2)}, y=\dfrac{k}{k^2+a^2}\ \&\ z=\dfrac{1}{k^2+a^2}$

Putting these values in (i), we get

$$\vec{r}=\frac{1}{(k^2+a^2)}\left[\left(\frac{\vec{a}.\vec{b}}{k}\right)\vec{a}+k\vec{b}+\vec{a}\times\vec{b}\right]$$

3. Unit vectors $\hat{a}$ and $\hat{b}$ are perpendicular to each other and the unit vector $\vec{c}$ is inclined at angle θ to both $\vec{a}$ and $\vec{b}$. If $\hat{c}=m(\hat{a}+\hat{b})+n(\hat{a}+\hat{b})$ and m, n are real, prove that $\pi/4 \le \theta \le 3\pi/4$.

Sol: $\hat{a}$ is perpendicular $\hat{b} \Rightarrow \hat{a}.\hat{b}=0$ and $|\hat{a}\times\hat{b}|=1$

Hence $\hat{c}=m(\hat{a}+\hat{b})+n(\hat{a}+\hat{b}) \Rightarrow \hat{a}.\hat{c}=m\hat{a}.\hat{a}=ma^2=m \Rightarrow \cos\theta.$

Also $\hat{c}.\hat{c}=[m(\hat{a}+\hat{b})+n(\hat{a}\times\ddot{b})].[m(\hat{a}\times\hat{b})]=2m^2+n^2$

$\Rightarrow n^2=1-2m^2=1-2\cos^2\theta=-\cos 2\theta \ge 0$

$\Rightarrow \pi/4 \le \theta \le 3\pi/4.$

4. If $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$, prove that $|\vec{a}|=|\vec{c}|$.

Sol: Here $a.(\vec{a}\times\vec{b})=\vec{a}.\vec{c}=0$ Also $\vec{b}.(\vec{a}\times\vec{b})=\vec{b}.\vec{c}=0$, and $\vec{b}.(\vec{b}\times\vec{c})=\vec{a}.\vec{b}=0$

$\Rightarrow \vec{a}.\vec{b}.\vec{c}$ are mutually perpendicular vectors.

$\Rightarrow |\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|=|\vec{c}|$ and $|\vec{b}||\vec{c}|=|\vec{a}| \Rightarrow |\vec{b}|=1 \Rightarrow |\vec{a}|=|\vec{c}|$.

5. Find $\vec{X}$ such that $\vec{A}\times\vec{X}=c, \vec{A}\times\vec{X}=\vec{B}, c\ne 0.$

Sol: We note that $\vec{A}\times\vec{B}=\vec{B}$.

If we take vector product of both sides with $\vec{A}$, we get

L.H.S = $\vec{A}\times(\vec{A}\times\vec{B})=(\vec{A}.\vec{X})\vec{A}-(\vec{A}.\vec{A})\vec{X}$

Therefore $c\vec{A}-|\vec{A}|^2\vec{X}=\vec{A}\times\vec{B} \Rightarrow \dfrac{c\vec{A}-\vec{A}\times\vec{B}}{|A|^2}.$

6. Two points P and Q are given in the rectangular cartesian coordinates on the $y=2^{x+2}$, such that $\overrightarrow{OP}.\hat{i}=-1$ and $\overrightarrow{OQ}.\hat{i}=2$, where $\hat{i}$ is a unit vector along the x-axis. Find the magnitude of $\overrightarrow{OQ}-4\overrightarrow{OP}.$

Sol: Let $P(x_1, y_1), Q(x_2, y_2)$ be the given points on $y = 2^{x+2}$,

$\overrightarrow{OP}.\hat{i} =$ projection of $\overrightarrow{OQ}$ on the x-axis = $x_1 = -1$

$\Rightarrow y_1 = 2,$

and $\overrightarrow{OQ}.\hat{i} =$ projection of $\overrightarrow{OQ}$ on the x-axis = $x_2 = 2 \Rightarrow y_2 = 16$

If $\hat{j}$ is a unit vector along the y-axis, then $\overrightarrow{OP} = -\hat{i} + 2\hat{j}, \overrightarrow{OQ} = 2\hat{i} + 16\hat{j}$

$\Rightarrow \overrightarrow{OQ} - 4\overrightarrow{OP} = 6\hat{i} + 8\hat{j} \Rightarrow \left|\overrightarrow{OQ} - 4\overrightarrow{OP}\right| = \sqrt{36+64} = 10.$

7. Consider the vectors:

$\hat{i} + \cos(\beta - \alpha)\hat{j} + \cos(\gamma - \alpha)\hat{k}, \cos(\alpha - \beta)\hat{i} + \hat{j} + \cos(\gamma - \beta)\hat{k}$ and

$\cos(\alpha - \gamma)\hat{i} + \cos(\beta - \gamma)\hat{j} + a\hat{k},$

where α, β and γ are different angles.

If these vectors are coplanar, show that a is independent of a , b and g .

Sol: Since the three vectors are coplanar,

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \text{coa}(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$$

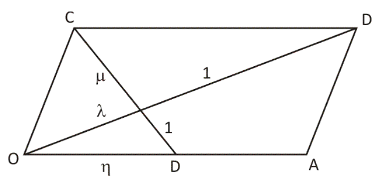
$$\Rightarrow \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & a-1 \end{vmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix} = 0$$

$\Rightarrow a = 1.$

8. In a parallelogram OABC with $\overrightarrow{OA} = \vec{a}, \overrightarrow{OC} = \vec{c}$, point D divides OA in the ratio n: 1 and CD and OB intersect in point E. Find the ratio CE/ED.

Sol: Let O be the initial point.

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OC} = \vec{a} + \vec{c}$



Also, $\overrightarrow{OD} = \dfrac{n}{n+1}\vec{a}$

Suppose OE: EB = $\lambda : 1$

and CE:ED = $\mu : 1$

Then $\overrightarrow{OE} = \frac{\lambda}{\lambda+1}.\overrightarrow{OB} = \frac{\mu\overrightarrow{OD}+\overrightarrow{OC}}{\mu+1}$

$\Rightarrow \frac{\lambda}{\lambda+1}(\vec{a}+\vec{c}) = \frac{\mu}{\mu+1}.\frac{n}{n+1}.\vec{a}+\frac{1}{\mu+1}.\vec{c}$

or $\left(\frac{\lambda}{\lambda+1}-\frac{\mu}{\mu+1}.\frac{n}{n+1}\right)\vec{a} = \left(\frac{1}{\mu+1}-\frac{\lambda}{\lambda+1}\right)\vec{c}$

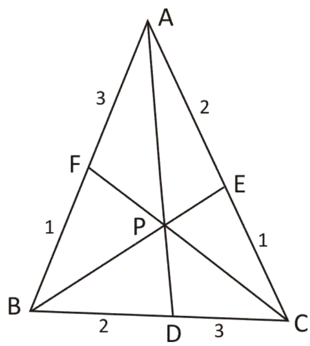
$\Rightarrow \frac{\lambda}{\lambda+1}-\frac{\mu}{\mu+1}.\frac{n}{n+1} = 0$ and $\frac{1}{\mu+1}-\frac{\lambda}{\lambda+1} = 0$

(Since $\vec{a}$ and $\vec{c}$ are non-collinear)

Hence solving for λ, μ, we find that $\frac{CE}{ED} = \mu = \frac{n+1}{n}$.

9. ABC is a triangle and D, E, F are three points on the sides BC, CA and AB respectively such that BD: DC = 2:3, CE: EA = 1:2, AF: FB = 3:1. Using vector method prove that AD, BE and CF are concurrent.

Sol: Let B be the initial point. Let the position vectors of A and C be $\vec{a}$ and $\vec{c}$. The position vectors of D, E and F are then $\frac{2\vec{c}}{5}, \frac{\vec{a}+2\vec{c}}{3}, \frac{\vec{a}}{4}$ respectively. Equation of line AD is



$\vec{r} = \vec{a}+t\left(\frac{2\vec{c}}{5}-\vec{a}\right)$ and that of line CF $\vec{r} = \vec{c}+s\left(\frac{\vec{a}}{4}=\vec{c}\right)$. If these line intersect at P,

then $\vec{a}+t\left(\frac{2\vec{c}}{5}-\vec{a}\right) = \vec{c}+s\left(\frac{\vec{a}}{4}=\vec{c}\right)$

$\Rightarrow t = \frac{5}{6}, s = \frac{2}{3}$, $\Rightarrow$ position vector of P is $\frac{\vec{a}+2\vec{c}}{6}$.

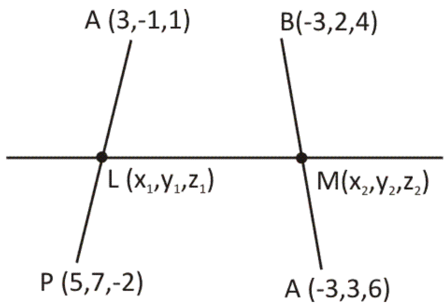
Equation of the line BE is $\vec{r} = k\frac{\vec{a}+2\vec{c}}{6}$.

Its intersection with CF gives k = 1/2, s = 2/3

$\Rightarrow$ Point of intersection of BE and CF is $\frac{\vec{a}+2\vec{c}}{6}$. which is the same as P. Hence the lines AD, BE, CF are concurrent.

10. The position vectors of the points P and Q are $5\hat{i}+7\hat{j}-2\hat{k}$ and $-3\hat{i}+3\hat{j}+6\hat{k}$ respectively. The vector $\vec{A}=3\hat{i}-\hat{j}+\hat{k}$ passes through the point P and the vector $\vec{B}=-3\hat{i}+2\hat{j}+4\hat{k}$ passes through the point Q.

A third vector $2\hat{i}+7\hat{j}-5\hat{k}$ intersects vectors $\vec{A}$ and $\vec{B}$. Find the position vector of the points of intersection.



Sol: Let L and M be the points of intersection.
Since P, L, A are collinear.

$\overrightarrow{PL}=\lambda_1\vec{A}$

$\Rightarrow (x_1-5)\hat{i}+(y_1-7)\hat{j}+(z_1+2)\hat{k}$
$=\lambda_1(3\hat{i}-\hat{j}+\hat{k})$

$$\Rightarrow \frac{x_1-5}{3}=\frac{y_1-7}{-1}=\frac{z_1+2}{1}=\lambda_1$$

$\Rightarrow L=(3\lambda_1+5, 7-\lambda_1, \lambda_1-2)$.

Similarly Q, M, B the collinear $\Rightarrow \overrightarrow{QM}=\lambda_2\vec{B}$

$\Rightarrow (x_2+3)\hat{i}+(y_2-3)\hat{j}+(z_2-6)\hat{k}=\lambda_2(-3\hat{i}+2\hat{j}+4\hat{k})$

$$\Rightarrow \frac{x_2+3}{-3}=\frac{y_2-3}{2}=\frac{z_2-6}{4}=\lambda_2$$

$\Rightarrow M=(-3\lambda_2-3, 2\lambda_2+3, 4\lambda_2+6)$. Again L and M are collinear with (2, 7 -5)

$$\Rightarrow \frac{x_2-x_1}{2}=\frac{y_2-y_1}{7}=\frac{z_2-z_1}{-5}=\lambda_3$$

$$\Rightarrow \frac{-3\lambda_2-3\lambda_1-8}{2}=\frac{2\lambda_2-\lambda_1-4}{7}=\frac{4\lambda_2-\lambda_1+8}{-5}=\lambda_3$$

$\Rightarrow 3\lambda_1+3\lambda_2+2\lambda_3=-8$ .... (i)

$\lambda_1+2\lambda_2-7\lambda_3=4$ ...... (ii)

and $-\lambda_1+4\lambda_2-5\lambda_3=-8$ ...... (iii)

Solving (i), (ii) and (iii), we get $\lambda_1=\lambda_2=\lambda_3=-1$

Hence L= (2, 8, -3) or $2\hat{i}+8\hat{j}-3\hat{k}$ and M = (0, 1. 2) or $\hat{j}+2\hat{k}$

Alternative:

Any point L on the vector A has position vector

$(5+3t)\hat{i}+(7-t)\hat{j}+(-2+t)\hat{k}$

Any point M on the vector B has position vector

$(-3-3m)\hat{i}+(3+3m)\hat{j}+(6+4m)\hat{k}$

Since $\overrightarrow{LM}$ is along $2\hat{i}+7\hat{j}-5\hat{k}$,

$\overrightarrow{LM}=\lambda\left(2\hat{i}+7\hat{j}-5\hat{k}\right)$

Equating the coefficient of $\hat{i},\hat{j},\hat{k}$ we find that m = t =-1.

OBJECTIVE EXAMPLE

1. $\vec{A},\vec{B},\vec{C}$ are three vectors respectively given by $2\hat{i}+\hat{k},\hat{i}+\hat{j}+\hat{k}$ and $4\hat{i}-3\hat{j}+7\hat{k}$. Then vector $\vec{R}$. Which satisfies the relation $\vec{R}\times\vec{B}=\vec{c}\times\vec{B}$ and $\vec{R}.\vec{A}=0$, is

(a) $2i-5\hat{j}+2\hat{k}$ (b) $-\hat{i}+4\hat{j}+2\hat{k}$

(c) $-\hat{i}-8\hat{j}+2\hat{k}$ (d) none of these

Sol: We have $\vec{R}\times\vec{B}=\vec{c}\times\vec{B}$ and $\vec{R}.\vec{A}=0$

$$\Rightarrow \vec{A}\times(\vec{R}\times\vec{B})=\vec{A}\times(\vec{C}\times\vec{B})$$

$$\Rightarrow (\vec{A}.\vec{B})\vec{R}=(\vec{A}.\vec{R})\vec{B}=(\vec{A}.\vec{B})\vec{C}-(\vec{A}.\vec{C})\vec{B}$$

$$\Rightarrow (2+1)\vec{R}=3\vec{C}-(8+7)\vec{B}$$

$$\Rightarrow \vec{R}=\vec{C}-5\vec{B}=-\hat{i}-8\hat{j}+2\hat{k}$$

Hence (C) is correct answer.

2. The points with position vectors $60\hat{i}+3\hat{j},40\hat{i}-8\hat{j}$ and $a\hat{i}-52\hat{j}$ are collinear if

(a) a = -40 (b) a = 40 (c) a = 20 (d) none of these

Sol: Three points are collinear $\Rightarrow \lambda(60\hat{i}+3\hat{j})+\mu(40\hat{i}-8j)+\nu(a\hat{i}-52\hat{j})$ = 0 with

$\lambda+\mu+\gamma=0\Rightarrow 60\lambda+40\mu+\nu a=0,3\lambda-8\mu-52\nu=0,\lambda+\mu+\nu=0$

For non-zero set (λ,μ,ν), $\begin{vmatrix}60 & 40 & a\\ 3 & -8 & -52\\ 1 & 1 & 1\end{vmatrix}=0\Rightarrow a=-40$

Hence (A) is correct answer

3. Let a, b, c be distinct and non- negative. If vectors $a\hat{i}+a\hat{j}+c\hat{k},\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is

(a) A.M of a and b (b) G.M. of a and b

(c) H.M. of a and b (d) equal to zero

Sol: Since these vectors are coplanar. $\begin{vmatrix}a & a & c\\ 1 & 0 & 1\\ c & c & b\end{vmatrix}=0$

$$\Rightarrow -ac-a(b-c)+c^2=0\Rightarrow c^2=ab$$

Hence (B) is correct answer

4. The number of vectors of unit length perpendicular to vectors $a \equiv (1,1,0)$ and $b \equiv (0,1,1)$ is

(a) 1 (b) 2 (c) 3 (d) 4

Sol: The vector of unit length perpendicular to given vectors

$$= \pm\left(\frac{a \times b}{|a||b|}\right)$$

Hence there are two such vectors

Hence (B) is correct answer.

5. The scalar $\vec{A}\{(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})\}$equals

(a) 0 (b) $[\vec{A}\vec{B}\vec{C}][\vec{B}\vec{C}\vec{A}]$ (c) $[\vec{A}\vec{B}\vec{C}]$ (d) none of these

Sol: $(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C}) = \vec{B}\times\vec{A}+\vec{B}\times\vec{C}+\vec{C}\times\vec{A}+\vec{C}\times\vec{B} = \vec{B}\times\vec{A}+\vec{C}\times\vec{A}$

$$\Rightarrow \vec{A}\{(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+)\}$$

Hence (A) is correct answer.

6. Let $\vec{a} \equiv 2\hat{i}-\hat{j}+\hat{k}, \vec{b} \equiv \hat{i}+2\hat{j}-\hat{k}$ and $\vec{c} = \hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on a is of magnitude $\sqrt{\frac{2}{3}}$ is

(a) $2\hat{i}-3\hat{j}-3\hat{k}, -2\hat{i}-\hat{j}+5\hat{k}$ (b) $2\hat{i}+3\hat{j}+3\hat{k}, -2\hat{i}-\hat{j}+5\hat{k}$

(c) $-2\hat{i}-\hat{j}+5\hat{k}\ \ 2\hat{i}-3\hat{j}-3\hat{k}$ (d) $2\hat{i}+\hat{j}+5\hat{k}\ \ 2\hat{i}-3\hat{j}-3\hat{k}$

Sol: Let $\vec{R}$ be a vector in the plane of b and c.

$$\Rightarrow \vec{R} = (\hat{i}+2\hat{j}-\hat{k})+\mu(\hat{i}+\hat{j}-2\hat{k})$$

its projection on $\vec{a} = \frac{\vec{a}\vec{R}}{|\vec{a}|} = \frac{1}{\sqrt{6}}[2+2\mu-2-\mu-1-2\mu] = \frac{-(1+\mu)}{\sqrt{6}}$

$$\frac{1+\mu}{\sqrt{6}} = \pm\sqrt{\frac{2}{3}} \Rightarrow -(1+\mu,-3) = \pm 2 \Rightarrow \mu = 1,-3$$

$$\Rightarrow \vec{R} \equiv 2\hat{i}+3\hat{j}-3\hat{k} \text{ and } -2\hat{i}-\hat{j}+5\hat{k}$$

Hence **(A)** is correct answer.

7. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector. Then the angle between $\vec{a}$ and $\vec{b}$ is

(a) 30^0 (b) 60^0 (c) 90^0 (d) 120^0

Sol: Since $\vec{a}$ and $\vec{b}$ is a unit vector $(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=1$

$\Rightarrow \vec{a}\ \ \vec{a}+\vec{b}\ \ \vec{b}+2\vec{a}\ \ \vec{b}=1$

$\Rightarrow 1+1+2\vec{a}\ \ \vec{b}=1 \Rightarrow \vec{a}\ \ \vec{b}=-1/2$

Hence the angle θ between $\vec{a}$ and $\vec{b}$ is given by

$\cos\theta=-1/2 \Rightarrow \theta=120^0$

Hence (D) is correct answer.

8. It is given that $\bar{t}_1=2\hat{i}-\hat{j}+\hat{k}$ and $\bar{t}_2=\hat{i}+3\hat{j}-2\hat{k}, \bar{t}_3=-2\hat{i}+\hat{j}-3\hat{k}$ and $\bar{t}_4=3\hat{i}+2\hat{j}-5\hat{k}$

If $\bar{t}_4=p\bar{t}_1+q\bar{t}_2+r\bar{t}_3$, , then

(a) $p=qr$ (b) $p=q+r$

(c) $p=q=r$ (d) $q=\dfrac{2pr}{p+r}$

Sol: We have $3\hat{i}+2\hat{j}-5\hat{k}=p(2\hat{i}-\hat{j}+\hat{k})+q(\hat{i}+3\hat{j}-2\hat{k})+r(-2\hat{i}+\hat{j}-3\hat{k})$

$\Rightarrow 2p+q-2r=3,$ ...... (1)

$-p+3q+r=2,$

$p-2q-3r=-5$

Adding all the equations, we get

2p + 2q = 4r i.e. p + q = 2r.

Equation (1), gives $p=3 \Rightarrow q=1, r=2$

With these values of q, p, r,

Hence (B) is correct answer.

9. If r, a, b, c are non null vectors such that $\vec{r}.\vec{a}=\vec{r}.\vec{b}=\vec{r}.\vec{c}=0$, then $[\vec{a}\vec{b}\vec{c}]$

(a) is equal to 1 (b) cannot be evaluated

(c) is equal to zero (d) none of these

Sol: Since $\vec{r}.\vec{a}=0, \vec{r}.\vec{a}=0$ and $\vec{r}.\vec{c}=0$, must be perpendicular to all the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Hence (C) is correct answer.

10. A parallelogram is constructed on $5\vec{a}+2\vec{b}$ and $\vec{a}-3\vec{b}$ where $|a|=2\sqrt{2}$ and $|b|=3$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$, then the length of the longer diagonal is

(a) $\sqrt{473}$ (b) $\sqrt{593}$ (c) $\sqrt{474}$ (d) $\sqrt{594}$

Sol: The vector representing one of the diagonals is $5\vec{a}+2\vec{b}+\vec{a}-3\vec{b}=6\vec{a}-\vec{b}$

Hence the length of the diagonal = $\sqrt{(6\vec{a}-\vec{b}).(6\vec{a}-\vec{b})}$

= $\sqrt{36|\vec{a}|^2+|\vec{b}|^2-12\vec{a}.\vec{b}}=\sqrt{36\times 8+9-12\times 3\times 2}=15.$

The other diagonal is $5\vec{a}+2\vec{b}-\vec{a}+3\vec{b}=4\vec{a}+5\vec{b}$

Its length $\sqrt{16|a|^2+25|\vec{b}|^2+40\vec{a}.\vec{b}}$

= $\sqrt{128+225+40\times 2\times 3}=\sqrt{593}$

Hence (B) is correct answer.

11. A non- zero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by the vectors $\hat{i}, \hat{i}+\hat{j}$ and the plane determined by vectors $\hat{i}-\hat{j}, \hat{j}+\hat{k}$. The angle between $\vec{a}$ and the vector $\hat{i}-2\hat{j}+2\hat{k}$ is.

(a) $\pi/4$ (b) $3\pi/4$ (c) $\pi/6$ (d) $\pi/3$

Sol: Since $\vec{a}$ is parallel to the line intersection of the two planes, it is parallel to both the planes

$\Rightarrow \vec{a}$ is perpendicular to the normals f both the planes.

Hence $\vec{a}.[\hat{i}\times(\hat{i}\times\hat{j})]=0$ and $\vec{a}.[(\hat{i}-\hat{j})\times(\hat{j}+\hat{k})]=0$

$\Rightarrow \vec{a}.\vec{k}=0$ and $\vec{a}.(\hat{i}+\hat{j})=0$ so that $\vec{a}$ is in the direction of $\hat{i}-\hat{j}$.

Hence $\hat{a}=\pm\frac{1}{\sqrt{2}}(\hat{i}-\hat{j})$. If θ is the angle between the given vector and $\vec{a}$,

$$\cos\theta=\pm\frac{1}{\sqrt{2}}(\hat{i}-\hat{j}).\frac{(\hat{i}-2\hat{j}+2\hat{k})}{3}=\pm\frac{1}{\sqrt{2}}\Rightarrow\theta=\frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Hence (A) and (B) is correct answer.

12. If $\vec{A},\vec{B}$ and $\vec{C}$ are vectors such that $|\vec{B}|=|\vec{C}|$, then $\left[\left(\vec{A}+\vec{B}\right)\times\left(\vec{A}+\vec{C}\right)\right]\times\left(\vec{B}\times\vec{C}\right).\left(\vec{B}+\vec{C}\right)$ is equal to

(a) 0 (b)1 (c) $\left[\vec{A}\vec{B}\vec{C}\right]$ (d) −1

Sol: We have $\left(\vec{A}+\vec{B}\right)\times\left(\vec{A}+\vec{C}\right)=\vec{A}\times\vec{C}+\vec{B}\times\vec{A}+\vec{B}\times\vec{C}$

$$\Rightarrow \left[\left(\vec{A}+\vec{B}\right)\times\left(\vec{A}+\vec{C}\right)\right]\times\left(\vec{B}\times\vec{C}\right)=\left[\vec{A}.\left(\vec{B}\times\vec{C}\right)\right]\left(\vec{C}-\vec{B}\right)$$

$$\Rightarrow \left[\left(\vec{A}+\vec{B}\right)\times\left(\vec{A}\times\vec{C}\right)\right]\times\left(\vec{B}\times\vec{C}\right).\left(\vec{B}+\vec{C}\right)=\vec{A}.\left(\vec{B}\times\vec{C}\right)\left[\vec{C}-\vec{B}\right].\left[\vec{B}+\vec{C}\right]$$

$$=\vec{A}.\left(\vec{B}\times\vec{C}\right)\left[\left|\vec{C}\right|^2-\left|\vec{B}\right|^2\right]=0 .$$ (Given that $|\vec{B}|=|\vec{C}|$)

Hence (A) is correct answer.

Practice Worksheet (Foundation Level) – 3(e)

1. If $\overline{a}=\hat{i}-2\hat{j}+2\hat{k}$ and $\overline{b}=3\hat{i}+6\hat{j}-2\hat{k}$, then vector parallel to $\overline{a}$, having magnitude equal to $\overline{b}$ is ………

2. $\overline{a}$, $\overline{b}$ and $\overline{c}$ are unit vectors –

   i) Calculate $\overline{a}.(\overline{b}\times\overline{c})$ when $\overline{a}$ is $\perp$ to $\overline{b}$.

   ii) Calculate $\overline{a}\times(\overline{b}\times\overline{c})$ when $\overline{a}$ is $\perp$ to $\overline{b}$ and inclined at 60^0 with $\overline{c}$.

   iii) Calculate $(\overline{a}-\overline{b})^2+(\overline{a}-\overline{c})^2$.

   iv) If $|\overline{a}+\overline{b}|^2+|\overline{b}+\overline{c}|^2+|\overline{c}+\overline{a}|^2=9$, find angle between $\overline{b}$ and $\overline{c}$.

3. Find vector which is equally inclined to vectors $\hat{i}+\hat{j}+2\hat{k}$, $2\hat{i}-\hat{j}-\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$.

4. $\overline{a}=3\hat{i}-2\hat{j}+6\hat{k}$, $\overline{b}=2\hat{i}+2\hat{j}-\hat{k}$ and $\overline{c}=2\hat{i}-3\hat{j}+2\sqrt{3}\hat{k}$. Calculate $\overline{a}.|(\overline{a}\times\overline{b})\times\overline{c}|$.

5. If Δ denote the area of triangle ABC, then prove $4\Delta^2=\overline{a}^2\overline{b}^2-(\overline{a}\times\overline{b})^2$ where AB = $\overline{a}$ and AC = $\overline{b}$.

6. ABCD is a trapezium AD = λBC ; DB+CA = μAD and is collinear with AD, then prove $\lambda\mu+\lambda+1=0$.

7. A particle is acted upon by forces $4\hat{i}+\hat{j}-5\hat{k}$ and $5\hat{i}-3\hat{j}+4\hat{k}$ and is displaced from point (5, –3, 4) to point (3, 4, –1). Find work done.

8. The position vectors of points A, B, C and D are $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}+5\hat{j}, 3\hat{i}+2\hat{j}-3\hat{k}$ and $a\hat{i}-6\hat{j}-\hat{k}$. Find a if the points are coplanar.

9. Find the component of vector $\overline{b}$ in direction perpendicular to unit vector $\overline{a}$.

10. The diagonals of a parallelogram are $3\overline{a}-\overline{b}$ and $4\overline{a}+3\overline{b}$ and | a | = 3, | b | = 2. If angle between $\overline{a}$ and $\overline{b}$ be 45^o. calculate area of parallelogram.

11. $\overline{a}=\hat{i}-\hat{j}+\hat{k}, \overline{b}=\hat{j}-\hat{k}, \overline{c}=\hat{i}+\hat{j}+\hat{k}$. If $\overline{d}$ is a unit vector such that $\overline{a}\cdot\overline{d}=0$ and [b c d] = 0 Then calculate $\overline{d}$.

12. Calculate moment of a couple of forces $\overline{F}$ and $-\overline{F}$ acting through $2\hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}+\hat{j}-2\hat{k}$ when $\overline{F}=2\hat{i}+3\hat{j}+6\hat{k}$.

13. Find value of $[\overline{x}\times\overline{p}\ \overline{y}\times\overline{q}\ \overline{z}\times\overline{r}]+[\overline{x}\times\overline{q}\ \overline{y}\times\overline{r}\ \overline{z}\times\overline{p}]+[\overline{x}\times\overline{r}\ \overline{y}\times\overline{p}\ \overline{z}\times\overline{q}]$.

14. Vectors $\overline{a}$ and $\overline{b}$ are non-collinear. The value of λ for which vectors $\overline{p}=(\lambda-2)\overline{a}+\overline{b}$ and $q=(2\lambda+1)\overline{a}-\overline{b}$ are collinear is …

15. $\overline{p}$ is a unit vector such that $\overline{p}\times(i+2j+k)$ = $-3i+4j-6k$, find $\overline{p}$.

16. $\overline{a}, \overline{b}, \overline{c}$ are unit vectors such that $\overline{a}\cdot\overline{b}=6$ and $\overline{a}\cdot\overline{c}=0$. If angle between $\overline{b}$ and $\overline{c}$ is $\frac{\pi}{6}$ then $\overline{a}$ equals to

(a) $\pm 2(\overline{b}\times\overline{c})$ (b) $2(\overline{b}\times\overline{c})$ (c) $\pm\frac{1}{2}(\overline{b}\times\overline{c})$ (d) $-\frac{1}{2}(\overline{b}\times\overline{c})$

17. Prove $\overline{a}=2\hat{i}+3\hat{j}+\hat{k}, \overline{b}=3\hat{i}-2\hat{j}$, and $\overline{c}=2\hat{i}+3\hat{j}-13\hat{k}$ vectors form a right handed triple perpendicular vectors.

18. The sum of two vectors $\overline{a}$ and $\overline{b}=3\hat{i}-\hat{j}+2\hat{k}$ and v. product $\overline{a}\times\overline{b}=5\hat{i}-7\hat{j}-11\hat{k}$ find $\overline{a}$ and $\overline{b}$.

19. $\overline{a}, \overline{b}, \overline{c}$ are three mutually perpendicular vectors of equal magnitude. Find angle that $(\overline{a}+\overline{b}+\overline{c})$ makes with each of them.

20. $\overline{p}, \overline{q}, \overline{r}$ and $\overline{s}$ are four distinct vectors such that $\overline{p}\times\overline{q}=\overline{r}\times\overline{s}$ and $\overline{p}\times\overline{r}=\overline{q}\times\overline{s}$; prove $\overline{p}-\overline{s}$ is parallel to $q-r$

21. Find vectors moment of forces $\hat{i}+2\hat{j}-3\hat{k}$; $2\hat{i}+3\hat{j}+4\hat{k}$ and $-\hat{i}-\hat{j}+\hat{k}$ (all the three act at point (0, 1, 2)) about point A (1, –2, 0)

22. A unit vector in x-y plane makes angle 45^o with $\hat{i}-\hat{j}$ and an angle of 60^o with $4\hat{i}+3\hat{j}$. Prove that unit vector is $\frac{11}{14}\hat{i}-\frac{3}{14}\hat{j}$.

23. $\overline{p}, \overline{q}, \overline{r}$ are three non-parallel vectors of magnitude 2, 2 and 4 respectively. If $[\overline{p}\times(\overline{q}\times\overline{r})]+4\overline{q}=0$, then find angle between $\overline{p}$ and $\overline{r}$.

24. $\overline{p}=3i+4j-5k, q=i+j$. If vectors $\overline{r}$ be such that $\overline{p}\cdot\overline{r}=|\overline{r}|$ and $|\overline{r}-\overline{p}|=5\sqrt{2}, r\neq 0$ and angle between $\overline{p}\times\overline{q}$ and $\overline{r}$ be 30^o, then calculate $[(\overline{p}\times\overline{q})\times\overline{r}]$.

25. Vector $\overline{a}, \overline{b}, \overline{c}$ are non coplanar $\overline{r}\times\overline{a}=\overline{b}$ and $\overline{b}$ is perpendicular to $\overline{a}$ find $\overline{r}$, if $\overline{r}=x\overline{a}+y\overline{b}+z(\overline{a}\times\overline{b})$.

26. In a right angled triangle BC is hypotenuse and is $\overline{r}$, calculate $\overline{BC}\cdot\overline{BA}+\overline{CA}\cdot\overline{CB}+\overline{AC}\cdot\overline{AB}$.

27. Find vector which is equally inclined to vectors $2\hat{i}-\hat{j}-2\hat{k}; 3\hat{i}-2\hat{j}-6\hat{k}$ and $3\hat{i}-4\hat{k}$.

28. $\overline{a}, \overline{b}, \overline{c}$ are non-coplanar vector and $(\overline{a}\times\overline{b})\times\overline{c}-\overline{a}=\overline{b}$, $(\overline{b}\times\overline{c})\times\overline{a}-\overline{b}=\overline{c}, (\overline{c}\times\overline{a})\times\overline{b}-\overline{c}=\overline{a}$, prove that each is perpendicular to the sum of other two.

29. $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors of equal magnitude 2, and equally inclined to one-another at θ. If $\bar{b}\times\bar{a}-\bar{c}\times\bar{b}=p\bar{a}+q\bar{b}+r\bar{c}$ then express scalars p, q, r in terms of θ.

30. $\bar{a}, \bar{b}, \bar{c}$ are vectors of equal magnitude and inclined θ with each other. Simplify $\bar{a}\times\left[\bar{a}\times(\bar{b}\times\bar{c})\right]+\bar{b}\times\left[(\bar{b}\times\bar{c})\times\bar{a}\right]+\bar{c}\times\left[(\bar{c}\times\bar{a})\times\bar{b}\right]$

31. Vectors $\bar{x}, \bar{y}, \bar{z}$ each of magnitude $\sqrt{2}$ make angle of 30^o with each other. If $\bar{x}\times(\bar{y}\times\bar{z})=\bar{p}$, $\bar{y}\times(\bar{z}\times\bar{x})=\bar{q}$ and $x\times y=\bar{r}$ then express, $\bar{x}, \bar{y}, \bar{z}$ in terms of $\bar{p}, \bar{q}$ and $\bar{r}$.

32. $\bar{x}, \bar{y}, \bar{z}$ are unit vectors such that $\bar{x}+\bar{y}+\bar{z}=\bar{p}$, $\bar{x}\times(\bar{y}\times\bar{z})=\bar{q}$, $(\bar{x}\times\bar{y})\times\bar{z}=\bar{r}$ and $\bar{p}\cdot\bar{x}=\frac{5}{2}$, $\bar{p}\cdot\bar{y}=\frac{9}{4}$, $|\bar{p}|=\frac{5}{2}$, find $\bar{x}, \bar{y}, \bar{z}$.

33. In a trapazium ABCD, the vector $\overline{BC}=\lambda AD$ and vector $\bar{p}=\overline{AC}+\overline{BD}$, is collinear with $\overline{AD}$. If $\bar{p}=\mu\overline{AD}$ then

(a) $\lambda = \mu + 1$ (b) $\lambda + \mu = 1$ (c) $\mu = \lambda + 1$ (d) $\mu + 2 + \lambda$

34. Unit vectors $\bar{p}$ is coplanar with $\bar{a}=\hat{i}+2\hat{j}+\hat{k}$ and $\bar{b}=2\hat{i}-\hat{j}+\hat{k}$ is perpendicular to $\bar{a}$, then $\bar{p}$ is

(a) $\pm\frac{1}{\sqrt{210}}(11\hat{i}-8\hat{j}+5\hat{k})$ (b) $\pm\frac{1}{\sqrt{210}}(8\hat{i}-11\hat{j}+5\hat{k})$

(c) $\pm\frac{1}{\sqrt{220}}(-11\hat{i}+8\hat{j}+5\hat{k})$ (d) $\pm\frac{1}{\sqrt{230}}(8\hat{i}+11\hat{j}+5\hat{k})$

35. Unit vector $\bar{a}$ is perpendicular on $2\hat{i}-2\hat{j}-3\hat{k}$ and its projection on $3\hat{i}-4\hat{j}+5\hat{k}$ is $\frac{2\sqrt{2}}{3}$ then $\bar{a}$ is

(a) $\frac{1}{3}(2\hat{i}+\hat{j}-2\hat{k})$ (b) $\frac{1}{5}(3\hat{i}+5\hat{j})$

(c) $\frac{1}{3}(2\hat{i}-\hat{j}+2\hat{k})$ (d) $\frac{1}{5}(2\hat{i}-3\hat{j}+2\sqrt{3}\hat{k})$

36. Unit vector $\bar{p}$ is perpendicular to $\hat{i}-\hat{j}-\hat{k}$ and its projection on vectors $2\hat{i}+3\hat{j}-6\hat{k}$ and $-3\hat{i}+4\hat{j}+5\hat{k}$ are of magnitude 2 and $\sqrt{2}$. Then $\bar{p}$ is

(a) i + 2j – k (b) 2i – 2j + 3k (c) i – j + 2k (d) none of these

37. $\bar{a}, \bar{b}, \bar{c}$ are p. vectors of vertices A, B and C of triangle ABC. Show that a unit vector normal to plane of triangle is

(a) $\pm\dfrac{2\Delta}{[a\ b\ c]}$ (b) $\pm\dfrac{2\Delta}{[\overline{a}\times\overline{b}+\overline{b}\times\overline{c}+\overline{c}\times\overline{a}]}$

(c) $\pm\dfrac{1}{2}(\overline{b}-\overline{a})\times(\overline{c}-\overline{a})$ (d) $\pm\dfrac{1}{2\Delta}\left[\overline{a}\times\overline{b}+\overline{b}\times\overline{c}+\overline{c}\times\overline{a}\right]$

38. The vertices of a quadrilateral are A (4, 5, 1), B (0, –1, –1), C (3, 9, 4) and D (–4, 4, 4) area of quadrilateral is … sq. units.

(a) $\dfrac{1}{2}\sqrt{1755}$ (b) $\dfrac{5}{2}\sqrt{195}$ (c) $\dfrac{9}{2}\sqrt{195}$ (d) $\dfrac{5}{2}\sqrt{163}$

39. $\overline{a}, \overline{b}, \overline{c}$ are three vectors $(\overline{a}\neq 0)$ such that $\overline{a}\times\overline{b}=\overline{a}\times\overline{c}$ and $\overline{a}\cdot\overline{b}=\overline{a}\cdot\overline{c}$ then

(a) $\overline{b}=\overline{c}$ (b) $\overline{b}\cdot\overline{c}=0$ (c) $\overline{a}=\overline{b}=\overline{c}$ (d) none of these

40. Vector $\overline{c}$ perpendicular to vector $\overline{a}$, and $\overline{a}+\overline{b}=5\hat{i}+4\hat{j}$, and $\overline{a}-\overline{b}=-3\hat{i}-6\hat{k}$, If $\overline{c}=(\hat{i}-2\hat{j}+5\hat{k})=8$ then $\overline{c}$ is

(a) $4\hat{i}-5\hat{j}-2\hat{k}$ (b) $8\hat{i}-10\hat{j}-4\hat{k}$ (c) $5\hat{i}-2\hat{j}-4\hat{k}$ (d) $5\hat{i}-\hat{j}+4\hat{k}$

41. AB and CD two chords of a circle are perpendicular to each other and meet in P. If O is centre of P then $\overline{PA}+\overline{PB}+\overline{PC}+\overline{PD}$ =

(a) 2OB (b) $\overline{OC}\sim\overline{OB}$ (c) $2\overline{PO}$ (d) $\dfrac{2}{\overline{BC}}$

4. Vector Equations of Straight Lines & Planes

4.1 Equation of line passing through a point and parallel to a vector $\overline{b}$

In the figure A is the given point and $\overline{b}$ the given vector from A draw a line parallel to $\overline{b}$. On this line P is any point. Let position vector of A be $\overline{a}$ and of P be $\overline{r}$ with reference to origin O. then

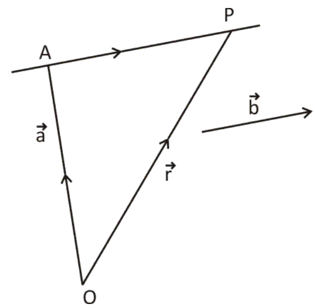


Fig 1

$$\overline{r} = \overline{OP} = \overline{OA} + \overline{AP}$$

$$= \overline{a} + t\overline{b} \qquad \ldots (1)$$

t is a scalar. AP parallel to $\overline{b} \Rightarrow AB = \pm\overline{b}$

equation of straight line is $\overline{r} = \overline{a} + t\overline{b}$

4.2 Equation of line through two points

Let A and B be two points where position vectors with reference to origin O are $\overline{a}$ and $\overline{b}$.

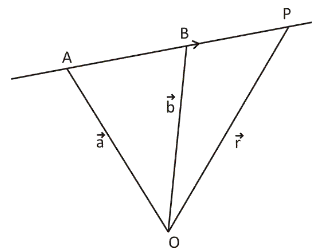


Fig 2

Clearly $AB = \overline{b} - \overline{a}$

t is the direction of straight line. Let P be the point on this straight line whose position vector is $\overline{r}$

$$\overline{OP} = \overline{OA} + \overline{AP} = \overline{OA} + t\overline{AB}$$

$$= \overline{a} + t(\overline{b} - \overline{a})$$

$$= (1-t)\overline{a} + t\overline{b}$$

4.3 Equation of the angle bisector of an angle

If $\hat{a}$ and $\hat{b}$ are unit vectors along the sides (arms) of an angle then $\overline{r} = \lambda(\hat{a} + \hat{b})$ is the equation of internal bisector of the angle and $\overline{r} = \lambda(\hat{a} - \hat{b})$ is the equation of the external bisector of the angle.

Let A be origin and position vectors of B and C be $\overline{b}$ and $\overline{c}$.

Equation of AD, bisector of angle A is $r = \lambda\left(\frac{\overline{b}}{|\overline{b}|} + \frac{\overline{c}}{|\overline{c}|}\right)$

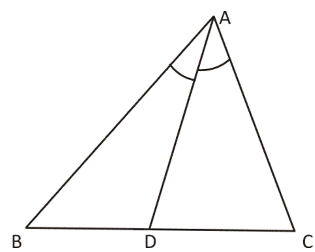


Fig 3

(b) If equation of arms of angle A are $\overline{r} = \overline{a} + \mu_1\overline{b}$ and $\overline{r} = \overline{a} + \mu_2\overline{c}$

then equation of angle bisector is $\overline{r} = \overline{a} + \lambda(\hat{b} + \hat{c})$

4.4 To prove that in ΔABC, the internal bisector of angle A divides BC in the ratio of AB : AC

let A be origin and p. vectors of B and C be $\bar{b}$ and $\bar{c}$ respectively and let $AB=\alpha, AC=\beta$. The bisector of angle A meets BC in D.

Eqn. BC is $\bar{r}=(1-t)\bar{b}+t\bar{c}$ ... (1)

Eqn. AD is $\bar{r}=\lambda\left(\frac{\bar{b}}{\alpha}+\frac{\bar{c}}{\beta}\right)$ ... (2)

Point D lies to both (1) and (2)

$$\therefore\ (1-t)\bar{b}+t\bar{c}=\lambda\left(\frac{\bar{b}}{\alpha}+\frac{\bar{c}}{\beta}\right)$$

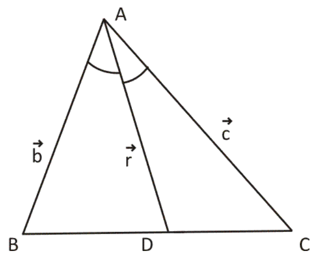


Fig 4

equating co-efficients of vectors

$$1-t=\frac{\lambda}{\alpha}; \qquad t=\frac{\lambda}{\beta}$$

$$\Rightarrow 1-\frac{\lambda}{\beta}=\frac{\lambda}{\alpha}\Rightarrow\lambda\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=1$$

$$\therefore\ \lambda=\frac{\alpha\beta}{\alpha+\beta}$$

Putting this value of λ in Eqn. (2) the P.V. of D is $\bar{r}=\frac{\alpha\beta}{\alpha+\beta}\left(\frac{\bar{b}}{\alpha}+\frac{\bar{c}}{\beta}\right)\Rightarrow\frac{\bar{b}\beta+\alpha\bar{c}}{\alpha+\beta}\Rightarrow\frac{AC\cdot\bar{b}+AB\cdot\bar{c}}{AC+AB}=\bar{r}$

$\therefore$ D divides BC, internal in the ratio of AB : BC.

Solved Example

Example 1: To prove that angle bisectors of a triangle meet at one point.

Sol: In Δ ABC, let p. v. of A, B and C be $\bar{a}, \bar{b}, \bar{c}$ and sides BC = α, CA = β and AB = γ.

$\overline{AB} = \bar{b} - \bar{a}$, $\overline{AC} = \bar{c} - \bar{a}$

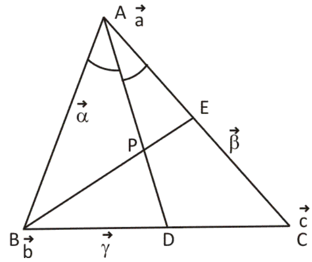


Fig 5

Eqn. angle bisector AD, $\bar{r} = \bar{a} + \lambda\left(\dfrac{\bar{b}-\bar{a}}{\gamma} + \dfrac{\bar{c}-\bar{a}}{\beta}\right)$

$$\Rightarrow \bar{r} = \left(1 - \frac{\lambda}{\gamma} - \frac{\lambda}{\beta}\right)\bar{a} + \frac{\lambda}{\gamma}\bar{b} + \frac{\lambda}{\beta}\bar{c} \qquad \text{... (1)}$$

Eqn. angle bisector of angle β, BE is

$$\bar{r} = \bar{b} + \mu\left(\frac{\bar{a}-\bar{b}}{\gamma} + \frac{\bar{c}-\bar{b}}{\alpha}\right)$$

$$= \frac{\mu\bar{a}}{\gamma} + \left(1 - \frac{\mu}{\gamma} - \frac{\mu}{\alpha}\right)\bar{b} + \frac{\mu}{\alpha}\bar{c} \qquad \text{... (2)}$$

the two angle bisectors meet at P, for which P.V. from the two eqns. must be same

$$\therefore \left(1 - \frac{\lambda}{\beta} - \frac{\lambda}{\gamma}\right)\bar{a} + \frac{\lambda}{\gamma}\bar{b} + \frac{\lambda}{\beta}, \bar{c} = \frac{\mu\bar{a}}{r} + \left(1 - \frac{\mu}{\gamma} - \frac{\mu}{\alpha}\right)\bar{b} + \frac{\mu}{\alpha}\bar{c}$$

$$\therefore 1 - \frac{\lambda}{\beta} - \frac{\lambda}{\gamma} = \frac{\mu}{\gamma} \qquad \text{... (3)}$$

$$\frac{\lambda}{b} = 1 - \frac{\mu}{\gamma} - \frac{\mu}{\alpha} \qquad \text{... (4)}$$

$$\frac{\lambda}{\beta} = \frac{\mu}{\alpha} \qquad \text{... (5)}$$

Putting value of μ from (5) in (3)

$$1 - \frac{\lambda}{\beta} - \frac{\lambda}{\gamma} = \frac{\alpha\lambda}{\beta\gamma} \Rightarrow \lambda\left(\frac{\alpha}{\beta\gamma} + \frac{1}{\beta} + \frac{1}{\gamma}\right) = 1$$

$$\therefore \gamma = \frac{\beta\gamma}{\alpha+\beta+\gamma}, \text{and } \mu = \frac{\alpha\gamma}{\alpha+\beta+\gamma} \text{ from 5}$$

Putting value λ in (A), p.v. of p is

$$\bar{r} = \bar{a} + \frac{\beta\gamma}{\alpha+\beta+\gamma}\left(\frac{\bar{b}-\bar{a}}{\gamma} + \frac{\bar{c}-\bar{a}}{\beta}\right)$$

$$= \bar{a} + \frac{\beta(\bar{b}-\bar{a})+\gamma(\bar{c}-\bar{a})}{\alpha+\beta+\gamma} = \frac{\bar{a}\alpha+\beta\bar{b}+\gamma\bar{c}}{\alpha+\beta+\gamma}$$

From symmetry it is clear that angle bisector of c shall pass through P.

$\therefore$ angle bisectors of the Δ are concurrent.

Example 2: In parallelogram ABCD, bisectors two consecutive angles meet at P. Use vector method to find $\angle BPC$.

Sol: See Fig. 6. Let $\overrightarrow{BC}$ be $a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\overrightarrow{BA}$ be $a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$.

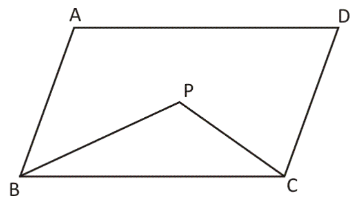


Fig 6

Eqn. BP of (angle bisector of B) is

$$\frac{a_1\hat{i}+b_1\hat{j}+c_1\hat{k}}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{a_2\hat{i}+b_2\hat{j}+c_2\hat{k}}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$\overrightarrow{CB} = -(a_1\hat{i}+b_1\hat{j}+c_1\hat{k})$;

$\overrightarrow{CD} = a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$ (as CD || BA)

Eqn. Of angle bisector of $\angle BCD$

$$-\frac{a_1\hat{i}+b_1\hat{j}+c_1\hat{k}}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{a_2\hat{i}+b_2\hat{j}+c_2\hat{k}}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Scalar product of BP and CP i.e. BP.CP

$$= \frac{a_1^2+b_1^2+c_1^2}{a_1^2+b_1^2+c_1^2} - \frac{a_2^2+b_2^2+c_2^2}{a_2^2+b_2^2+c_2^2} = 0 \Rightarrow \angle BPC = 90^\circ$$

Example 3 : D and E are points on sides BC and AC of traingle ABC such that BD=2DC and AE = 3EC. If AD and BE meet in P, then calculate BP/PE by vector method.

Sol : Let B be origin and P.V. of A and C be $\bar{a}$ and $\bar{c}$.

BD = 2DC=$\frac{2}{3}$ BC=$\frac{2}{3}$ $\bar{c}$

AE=3EC

i.e. E divided AC in the ration 3:1.

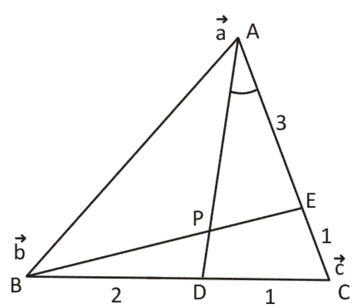


Fig 7

$\therefore$ P.V. of E is $\frac{3\bar{c}+\bar{a}}{4}$

Equ. of BE is $\bar{r} = t_1\frac{3\bar{c}+\bar{a}}{4}$

Equ. of AD is $r=(1-t_2)\bar{a}+t_2\frac{2}{3}\bar{c}$

$\therefore\ t_1\frac{3\bar{c}+\bar{a}}{4}=(1-t_2)\bar{a}+\frac{2}{3}\bar{c}.t_2$

Equating co-efficients $\frac{3}{4}t_1=\frac{2}{3}t_2 \Rightarrow t_2=\frac{3}{4}t_1$

And $\frac{1}{4}t_1=1-t_2=1-\frac{9}{8}t_1$ $\therefore\ (\frac{1}{4}+\frac{9}{8})t_1=1 \Rightarrow t_1=\frac{8}{11}$

$\therefore$ BP = $\frac{8}{11}$ BE $\therefore$ BP:PE =8:3

Example 4: Prove by vector method that st. lines joining the vertices of a tetrahedran to the centroid of opposite forces are concurant.

Sol: Let the P.V. of A, B, C and D be $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$. The P.V. of G_1, centroid of face BCD is $\frac{\bar{b}+\bar{c}+\bar{d}}{3}$.

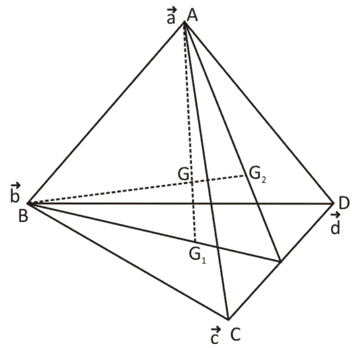


Fig 8

P.V. of centeroid G_2 of face ACD is $\frac{\bar{a}+\bar{c}+\bar{d}}{2}$

Eqn. of AG_1 $r=(1-t_1)a+\frac{\bar{b}+\bar{c}+\bar{d}}{3}t_1$

Eqn. of BG_2 $r=(1-t_2)b+\frac{\bar{a}+\bar{c}+\bar{d}}{3}t_2$

AG_1 and BG_2 meet in G.

for $G(1-t_1)\bar{a}+\frac{\bar{b}+\bar{c}+\bar{d}}{3}t_1=(1-t_2)\bar{b}+\frac{\bar{a}+\bar{c}+\bar{d}}{3}t_2$

comparing co-efficients of $\bar{a}$ and $\bar{b}$

$1-t_1=\frac{1}{3}t_2$ and $\frac{1}{3}t_1=1-t_2$

$\Rightarrow t_2 = 3(1-t_1)$ and $\therefore\ \frac{1}{3}t_1 = 1-3+3t_1$ $\therefore\ \left(3-\frac{1}{3}\right)t_1=2$ $\Rightarrow t_1=\frac{3}{4}$

$\therefore$ P.V. of G = $\frac{1}{4}a+\frac{\bar{b}+\bar{c}+\bar{d}}{3}\cdot\frac{3}{4}=\frac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$

and from symmetry it is clear that straight lines joining vertices to the centroids of opposite faces are concurrent.

4.5 Equation of a plane, when it goes through a point and parallel to two non-parallel vectors:

See figure 9, O is origin A is a point on plane, P.V. of A is $\overline{a}$. Plane is parallel to vectors $\overline{b}$ and $\overline{c}$ which are non-parallel.

From point A draw straight lines, AB and AC which are parallel to $\overline{b}$ and $\overline{c}$. Let P be a point in the plane whose P.V. is r. Draw from P, PM parallel to CA which meets AB in M.

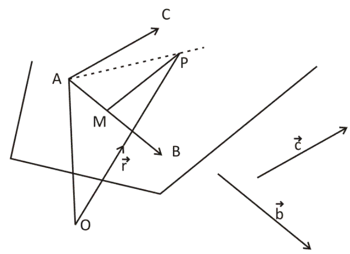


Fig 9

Now AM = t$\overline{b}$ and MP = s$\overline{c}$

where t and s are scalars.

$$\overrightarrow{OP} = r = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + (\overrightarrow{AM} + \overrightarrow{MP})$$

$$r = \overline{a} + t\overline{b} + s\overline{c}$$

This is the equation of plane.

4.6 Equation of plane passing through three points

Let the P.V. of three points A, B, C in plane be $\overline{a}$, $\overline{b}$ and $\overline{c}$

$\therefore$ The plane goes through $\overline{a}$ and is parallel to $\overline{b} - \overline{a}$ and $\overline{c} - \overline{a}$.

$\therefore$ $r = \overline{a} + t(\overline{b} - \overline{a}) + s(\overline{c} - \overline{a})$

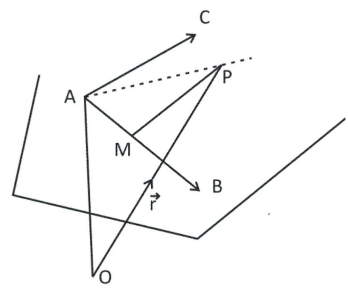


Fig 10

Solved Example

Example 5: Find the equation of plane which passes through $-2\hat{i}+6\hat{j}-6\hat{k}, -3\hat{i}+10\hat{j}-9\hat{k}$ and $-5\hat{i}-6\hat{k}$.

Sol. Let A be $(-2\hat{i}+6\hat{j}-6\hat{k})$; B $(-3\hat{i}+10\hat{j}-9\hat{k})$ and C, $(-5\hat{i}-6\hat{k})$

$\therefore\ \overline{AB} = 3\hat{i}+10\hat{j}-9\hat{k}+2\hat{i}-6\hat{j}+6\hat{k} = -\hat{i}+4\hat{j}-3\hat{k}$

$\overline{AC} = -5\hat{i}-6\hat{k}+2\hat{i}-6\hat{j}+6\hat{k} = -3\hat{i}-6\hat{j}$

Equation of plane is $\bar{r} = \bar{a} + t\overline{AB} + s\overline{AC}$

$\therefore\ r = (-2\hat{j}+6\hat{j}-6\hat{k}) + t(-\hat{i}+4\hat{j}-3\hat{k}) + s(-3\hat{i}-6\hat{j})$

Example 6: Find equation of plane which goes through origin, and points $-2\hat{i}+4\hat{j}+\hat{k}$ and $4\hat{i}+2\hat{k}$. Find point at which straight line joining $\hat{i}-2\hat{j}+\hat{k}$ and $-2\hat{j}+3\hat{k}$ meets the plane.

Sol.. O is origin A, $(2\hat{i}+4\hat{j}+\hat{k})$, B $(4\hat{i}+2\hat{k})$

$\overline{OA} = -2\hat{i}+4\hat{j}+\hat{k}, \overline{OB} = 4\hat{i}+2\hat{k}$

$\therefore$ Plane is $r = t(-2\hat{i}+4\hat{j}+\hat{k}) + s(4\hat{i}+2\hat{k})$ ... (1)

Equation of line through $P(\hat{i}-2\hat{j}+\hat{k})$ and $Q(-2\hat{j}+3\hat{k})$ is

$r = (1-\lambda)(\hat{i}-2\hat{j}+\hat{k}) + \lambda(-2\hat{j}+3\hat{k}) = \hat{i}-2\hat{j}+\hat{k}+\lambda(-\hat{i}+2\hat{k})$

for point of intersection

$t(-2\hat{i}+4\hat{j}+\hat{k}) + s(4\hat{i}+2\hat{k}) = \hat{i}-2\hat{j}+\hat{k}+\lambda(-\hat{i}+2\hat{k})$

equating co-efficients of $\hat{i}, \hat{j}$ and $\hat{k}$

(i) $-2t + 4s = 1 - \lambda$

(ii) $4t = -2 \Rightarrow t = -\frac{1}{2}$

(iii) $t + 2s = 1 + 2\lambda \Rightarrow 2t + 4s = 2 + 4\lambda$

(iii) – (i) $4t = 1 + 5\lambda \Rightarrow -2 = 1 + 5\lambda$

$\therefore\ \lambda = -\frac{3}{5}$, and t is $-\frac{1}{2}$

$\therefore\ \bar{r} = \hat{i}-2\hat{j}+\hat{k}-\frac{3}{5}(-\hat{i}+2\hat{k}) = \frac{1}{5}(8\hat{i}-10\hat{j}-\hat{k})$

$\therefore$ P.V. of point of intersection $\frac{1}{5}(8\hat{i}-10\hat{j}-\hat{k})$

Example 7: Prove by vector method or otherwise that the point of intersection of the diagonals of a trapazium lies on the line joining mid points of parallel sides.

Sol. ABCD is the trapazium, DC || AB. Let A be the origin and P.V. of B, C and D be $\overline{b}$, $\overline{c}$ and $\overline{d}$ respectively.

Equation of AC $r=t_1\overline{c}$ ... (1)

Equation of BD $r=(1-t_2)\overline{b}+t_2\overline{d}$ ... (2)

DC is parallel to AB $\therefore$ $\overline{c}-\overline{d}=\lambda\overline{b}$ ... (2)

$\therefore$ Eqn. (2) is $r=(1-t_2)\overline{b}+t_2(\overline{c}-\lambda\overline{b}) \Rightarrow r=(1-t_2-t_2\lambda)\overline{b}+t_2\overline{c}$ ... (3)

For, Point of intersection of (1) and (3)

$t_1\overline{c}=(1-t_2-\lambda t_2)\overline{b}+t_2\overline{c}$

$\therefore$ $t_1 = t_2$ and $(1+\lambda)t_2 - 1 = 0$

$\therefore$ $t_1=t_2=\dfrac{1}{1+\lambda}$

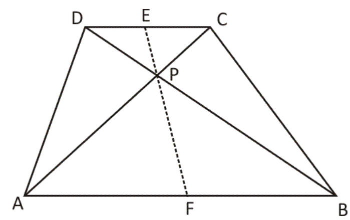


Fig 11

$\therefore$ from (1) P.V. of P point of intersection of diagonal AC and BD is $\dfrac{\overline{c}}{1+\lambda}$

Mid point of AB is E, P.V. $\dfrac{b}{2}$

Mid point of DC is F, P.V. $\frac{1}{2}(\overline{c}+\overline{d})$

from (1) P.V. of F = $\frac{1}{2}(\overline{c}+\overline{c}-\lambda\overline{b})=\frac{1}{2}(2\overline{c}-\lambda\overline{b})$

$\therefore$ $\overrightarrow{EF}=\frac{1}{2}(2\overline{c}-\lambda\overline{b})-\frac{\overline{b}}{2}=\frac{1}{2}\left[2\overline{c}-(\lambda+1)\overline{b}\right]$ = $\frac{1}{2}\left[2\overline{c}-(\lambda+1)\overline{b}\right]$ ... (3)

$EP=\dfrac{\overline{c}}{1+\lambda}-\dfrac{\overline{b}}{2}=\dfrac{1}{2(1+\lambda)}\left[2\overline{c}-(1+\lambda)\overline{b}\right]$ ... (4)

from (3) and (4) it is evident that P lies on EF

Example 8: Find point of intersection of the plane passing through $2\hat{i}+\hat{j}-3\hat{k}, 4\hat{i}-\hat{j}+2\hat{k}$ and $3\hat{i}+\hat{k}$ and the line joining $\hat{i}-2\hat{j}-\hat{k}$ and $2\hat{i}+3\hat{j}+\hat{k}$.

Sol. The equation of plane passing through 2i + j – 3k, 4i – j + 2k and 3i + k

$\Rightarrow r=(1-t-s)(2\hat{i}+\hat{j}-3\hat{k})+t(4\hat{i}-\hat{j}+2\hat{k})+s(3\hat{i}+\hat{k})$

$=(2+2t+s)\hat{i}+(1-s-2t)\hat{j}+(5t+4s-3)\hat{k}$ ... (A)

Equation of line joining the two points is

$r=(\hat{i}-2\hat{j}-\hat{k})+\lambda(2\hat{i}+3\hat{j}+\hat{k}-\hat{i}+2\hat{j}+\hat{k})=(\hat{i}-2\hat{j}-\hat{k})+\lambda(\hat{i}+5\hat{j}+2\hat{k})$

$=(1+\lambda)\hat{i}+(5\lambda-2)\hat{j}+(2\lambda-1)\hat{k}$ ... (B)

For point of intersection its P. Vector in (A) & (B) should be same comparing co-efficient of i, j, k

(i) $\therefore\ 2 + 2t + s = 1 + \lambda$ $\qquad 6\lambda = 4 \Rightarrow \lambda = \frac{2}{3}$

(ii) $1 - s - 2t = 5\lambda - 2$

(iii) $-3 + 5t + 4s = 2\lambda - 1$

Putting value of λ in (i) and (iii)

$s + 2t = -\frac{1}{3}$, $\qquad 5t+4s=\frac{10}{3}$

$\therefore\ t=-\frac{14}{9}$ $\qquad s=\frac{25}{9}$

Putting value of λ in (B), position vectors of point of intersection is $r=\frac{5}{3}i+\frac{4}{3}j+\frac{1}{3}k$

Three Dimension: Points on plane are A, (2, 1, –3), (4, –1, 2) B and (3, 0, 1) C.

Plane through C is a (x – 3) + b (y – 0) + c (z – 1) = 0

$$\left.\begin{array}{ll} pt.(2,1,-3) & -a+b-4c=0 \\ pt.(4,-1,2) & a-b+c=0 \end{array}\right\} \frac{a}{-3}=\frac{b}{4+1}=\frac{c}{0}$$

$$\therefore\ \frac{a}{1}=\frac{b}{1}=\frac{c}{0}$$

Plane is x – 3 + y = 0 $\Rightarrow$ x + y = 3

Eqn. of line through (1, –2, –1) and (2, 3, 1) is

$$\frac{x-1}{1}=\frac{y+2}{5}=\frac{z+1}{2}=r$$

any point on straight line is (r + 1, 5r – 2, 2r – 1)

If this line meets the plane then r + 1 + 5r – 2 = 3 $\Rightarrow r = \frac{2}{3}$

$\therefore$ Point is $\left(\frac{5}{3}, \frac{10}{3} - 2, \frac{4}{3} - 1\right) \Rightarrow \left(\frac{5}{3}, \frac{4}{3}, \frac{1}{3}\right)$

Practice Worksheet (Foundation Level) – 4(a)

1. Find equation of straight line through $(\hat{i}-2\hat{j}+3\hat{k})$ and parallel to vector $3\hat{i}+4\hat{j}+5\hat{k}$.

2. Find equation of straight line through $2\hat{i}-\hat{j}+2\hat{k}$ and $3\hat{i}-2\hat{j}+6\hat{k}$.

3. Show that $14\hat{j}-25\hat{k}$ lies on the straight line joining points $4\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}+2\hat{j}-4\hat{k}$.

4. Find point of intersection of straight lines, one through $\hat{i}+\hat{j}+\hat{k}$ and parallel to $3\hat{i}+\hat{j}-\hat{k}$ and the other passing through $2\hat{i}+\hat{j}+2\hat{k}$ and $3\hat{i}+\hat{j}+3\hat{k}$

5. ABC is triangle, point D, in BC divides it in the ratio of 3 : 1 and point E in AC divides it in the ratio of 2 : 1, AD and BE intersect in P. Find the ratio of BP : PE.

6. Show that the straight line joining mid points of non-parallel sides of a trapezium is parallel to parallel sides.

7. Prove by vector method that median of a triangle are concurrent and point of concurrency divides each median in the ratio of 2 : 1.

8. Prove by vector method that diagonals of a parallelogram bisect each other.

9. Prove that in triangle ABC, angle bisector of angle A divides BC in the ratio of AB : AC.

10. Find equation of plane through $\hat{i}+2\hat{j}+3\hat{k}$ and parallel to $3\hat{i}-4\hat{j}+5\hat{k}$ and $2\hat{i}-2\hat{j}-\hat{k}$

11. Find equation of plane which passes through $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}-2\hat{j}-\hat{k}$.

12. A plane goes through origin and through points (2, 1, –1) and (3, 1, –1). Find point of intersection of this plane and straight line which passes through points (2, 1, –1) and (1, –2, 1).

13. Find point of intersection of straight lines one of which goes through origin and point (2, 1, 2) and the other goes through points (1, 2, 2) and (4, 2, 4).

14. A, B, C and D are four points (2, 1, 3), (4, –1, 2), (4, 2, 1) and (4, 3, 1) respectively. Find point of intersection of AC and BD.

15. A straight line goes through (2, 2, 3) and (3, 3, 1); another straight line goes through (1, 1, 3) and (2, 2, 2). Find their point of intersection.

16. Show that straight lines through (i) (1, –1, 1) and (2, 2, 4) (ii) (1, 2, 5) and (3, 2, 3) and (iii) (4, 2, –1) and (3, 2, $\frac{3}{2}$) are concurrent. Find this point.

17. A plane goes through three points (1, –1, 3), (2, 4, 5) and (3, –2, 1). Straihgt line goes through origin and point (3, –2, 1) lies on it. If it meets the plane in point P, then find point P.

18. A plane goes through point (1, 1, 1) and is parallel to vectors $\bar{a} = 3\hat{i} - 2\hat{j} + 4\hat{k}$ and $\bar{b} = 4\hat{i} - \hat{j} - \hat{k}$. Find equation of plane which is parallel to this plane and goes through point (5, –2, 4).

19. If point (1, –1, 5) lies on plane $\bar{r} = 3\hat{i} + \hat{j} + \hat{k} + t(2\hat{i} - \hat{j} + \hat{k}) + s(4\hat{i} + \hat{j} - 3\hat{k})$, then calculate values of t and s.

20. Find equation of plane which goes through points (2, –1, 1) , (3, 2, 4) and (–1, –2, 4). Does point (4, 2, –3) lie on it.

4.7 Vector Equation of straight lines and plane in Product form

(a) Straight Lines: A straight line goes through point $\overline{a}$ and is parallel to vector $\overline{b}$.

If $\overline{r}$ is the P. vector of a point on it then $\overline{r}-\overline{a}$ is parallel to $\overline{b}$.

$\therefore\ (\overline{r}-\overline{a})\times\overline{b}=0 \Rightarrow \overline{r}\times\overline{b}=\overline{a}\times\overline{b}$

This is the equation of straight line in product form

(b) Equation of straight lines through two points

Let the position vector of points be $\overline{a}$ and $\overline{b}$ and of any point P on the straight line be $\overline{r}$.

Then $\overline{r}-\overline{a}$ is parallel to $\overline{b}-\overline{a}$

$\therefore$ Equation of straight line is $(\overline{r}-\overline{a})\times(\overline{b}-\overline{a})=0$

(c) Equation of straight line through a point and perpendicular to two non-parallel vectors.

Let the position vector of a point be $\overline{a}$ and two non-parallel vectors be $\overline{b}$ and $\overline{c}$. If r is any point on the straight line.

then $\overline{r}-\overline{a}$ shall be perpendicular to $\overline{b}$ and $\overline{c}$

i.e. $\overline{r}-\overline{a}$ shall be parallel to $\overline{b}\times\overline{c}$

$$(\overline{r}-\overline{a})\times(\overline{b}\times\overline{c})=0$$

4.8. Plane

(a) Equation of plane which goes through a point and parallel to two non parallel vectors

Let the position vector of point bc $\overline{a}$ and $\overline{b}$ and $\overline{c}$ be two non-parallel vectors. Let $\overline{r}$ be the position vector of any point on plane.

$\therefore\ \overline{r}-\overline{a}$ shall be parallel to $\overline{b}$ & $\overline{c}$

i.e. it shall be perpendicular to $\overline{b}\times\overline{c}$

$\therefore\ (\overline{r}-\overline{a})\cdot(\overline{b}\times\overline{c})=0$ or $\overline{r}\cdot(\overline{b}\times\overline{c})=\overline{a}\cdot(\overline{b}\times\overline{c})$

or $(\overline{r}\ \overline{b}\ \overline{c})=(\overline{a}\ \overline{b}\ \overline{c})$

is the equation of the plane.

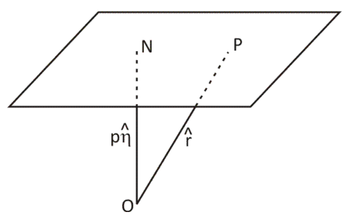


Fig 12

(b) The equation of a plane which goes through three points

Let the position vectors of three points be $\overline{a}$, $\overline{b}$ and $\overline{c}$ and $\overline{r}$ be the position vector of any point on it, then vectors, $\overline{r}-\overline{a}$, $\overline{a}-\overline{b}$, $\overline{b}-\overline{c}$ shall be coplanar.

i.e. $\left[\bar{r}-\bar{a}\ \ \bar{a}-\bar{b}\ \bar{b}-\bar{c}\right]=0$

$\Rightarrow\ (\bar{r}-\bar{a})\cdot\left[(\bar{a}-\bar{b})\times(\bar{b}-\bar{c})\right]=0$

or $\quad (\bar{r}-\bar{a})\cdot\left[\bar{a}\times\bar{b}-\bar{a}\times\bar{c}+\bar{b}\times\bar{c}\right]=0$

or $r\cdot\left[\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}\right]=\left[a\ b\ c\right]$

This is required equation of the plane.

(c) To find the equation of plane, whose distance from origin is p and the unit normal vector of plane be η

In the figure ON is perpendicular from origin O. |ON| = p, but unit vector normal vector is η

$$\overrightarrow{ON}=p\cdot\hat{n}$$

If P be any point on plane (position vector r) then projection of OP on ON is = p

$\therefore$ p = OP cos θ = $\bar{r}\cdot\hat{n}\Rightarrow\bar{r}\cdot\hat{n}=p$

This is the equation of plane in normal term.

(d) The equation of plane which goes through point A (p. v. $\bar{a}$) and which is perpendicular to η is $(\bar{r}\cdot\bar{a})\cdot\bar{n}=0\Rightarrow\bar{r}\cdot\bar{\eta}=\bar{a}\cdot\bar{\eta}$

where $\bar{r}$ is the position vector of any point on the plane.

Note: The distance of plane from origin is $p=\dfrac{\bar{r}\cdot\bar{\eta}}{|\bar{\eta}|}=\dfrac{\bar{a}\cdot\bar{\eta}}{|\bar{\eta}|}$

(e) Two planes are $r\cdot\eta_1=q_1$ and $r\cdot\eta_2=q_2$ then the equation of plane which goes through the line of intersection of these planes is

$$(r\cdot\eta_1-q_1)+\lambda(r\cdot\eta_2-q_2)=0$$

Solved Example

Example 9: Find the equation of straight line which goes through (1, 2, 3) and is perpendicular to the straight line passing through $3\hat{i}+2\hat{j}-4\hat{k}$ and $-2\hat{i}+7\hat{j}+5\hat{k}$

Sol. Let A be (1, 2, 3), B (3i + 2j – 4k), C (–2i + 7j + 5k), $\bar{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\bar{b}=3\hat{i}+2\hat{j}-4\hat{k}$, $\bar{c}=2\hat{i}+7\hat{j}+5\hat{k}$

Eqn. of line is $(\bar{r}-\hat{i}-2\hat{j}-3\hat{k})\times\left[(3\hat{i}+2\hat{j}-4\hat{k})\times(-2\hat{i}+7\hat{j}+5\hat{k})\right]=0$

or $\left[r-(\hat{i}+2\hat{j}+3\hat{k})\right]\times\left[38\hat{i}-7\hat{j}+25\hat{k}\right]=0$

or $r\times(38\hat{i}-7\hat{j}+25\hat{k})=71\hat{i}+89\hat{j}-83\hat{k}$

Example 10: The length of perpendicular from origin on a plane is 15 and its normal vector is 2i – j + 2k. Find equation of plane.

Sol. We know $\bar{r}\cdot\hat{\eta}=p\Rightarrow\bar{r}\cdot\frac{\bar{\eta}}{|\bar{\eta}|}=p$

$\frac{\bar{n}}{|\bar{n}|}$ = unit normal vector = $\frac{2\hat{i}-\hat{j}+2\hat{k}}{3}$

$\therefore\ r\cdot\frac{(2\hat{i}-\hat{j}+2\hat{k})}{3}$ = 15

$\therefore$ Plane is $r\cdot(2\hat{i}-\hat{j}+2\hat{k})=45$

Example 11: Find the equation of plane whose distance from origin is 10 and which is perpendicular is $2\hat{i}-3\hat{j}+6\hat{k}$.

Sol. Plane is perpendicular to $2\hat{i}-3\hat{j}+6\hat{k}$

$\therefore$ Direction of normal of plane is $2\hat{i}-3\hat{j}-6\hat{k}$

$\therefore$ Eqn. $\frac{r\cdot(2\hat{i}-3\hat{j}-6\hat{k})}{7}=10$

i.e. $r\cdot(2i-3j-6k)=70$

Note: The equation of this plane in three dimension is $(xi+yj+zk)\cdot(2i-3j-6k)=70$

i.e. 2x –3y – 6z = 70 $\perp$ from (0, 0) on it is $\frac{70}{\sqrt{2^2+3^2+6^2}}=10$

Example 12: Find the equation of plane which passes through $2\hat{i}+3\hat{j}+\hat{k}$ and is perpendicular to $3\hat{i}-2\hat{j}+6\hat{k}$. Find its distance from origin as well.

Sol. The normal of the plane is along $3i-2j+6k$.

$\therefore$ equation of plane is $(\bar{r}\cdot\bar{a})\cdot\eta=0$

$r\cdot(3\hat{i}-2\hat{j}+6\hat{k})=(2\hat{i}+3\hat{j}+\hat{k})\cdot(3i-2j+6k)$

r · (3i – 2j + 6k) = 6 – 6 + 6 = 6

Length of perpendicular from origin is $\dfrac{6}{|\bar{\eta}|}=\dfrac{6}{7}$

Example 13: A unit vector perpendicular to plane determined by P(1, –1, 2), Q (2, 0, –1), R (0, 2, 1) is …

I.I.T.

Sol. The normal of plane is along $\overrightarrow{PQ}\times\overline{PR}$

$\Rightarrow (\hat{i}+\hat{j}-3\hat{k})\times(-\hat{i}+3\hat{j}-\hat{k})=8\hat{i}-4\hat{j}+4\hat{k}$

Unit normal vector = $\dfrac{1}{4\sqrt{6}}(8\hat{i}-4\hat{j}+4\hat{k})$

$=\dfrac{1}{\sqrt{6}}(2\hat{i}-\hat{j}+\hat{k})$

Example 14: Find distance of point P, (i + j + k) from plane L, which passes through A, $(2\hat{i}+\hat{j}+\hat{k})$, B $(\hat{i}+\hat{j}+2\hat{k})$ and C $(\hat{i}+\hat{j}+2\hat{k})$.

I.I.T.

Sol. Normal of plane is parallel to $\overline{BA}\times\overline{BC}$

$=(\hat{i}-\hat{j})\times(-\hat{j}+\hat{k})=-\hat{i}-\hat{j}-\hat{k}$

$=\hat{i}+\hat{j}+\hat{k}$

Eqn. of plane is $\left[r-(2\hat{i}+\hat{j}+\hat{k})\right]\cdot(\hat{i}+\hat{j}+\hat{k})=0$

Plane ABC $\Rightarrow r\cdot(\hat{i}+\hat{j}+\hat{k})=2+1+1=4$

Plane parallel to plane ABC is

$r-(\hat{i}+\hat{j}+\hat{k})=\lambda$

It passes through $(\hat{i}+\hat{j}+\hat{k})$

$\therefore (\hat{i}+\hat{j}+\hat{k})(\hat{i}+\hat{j}+\hat{k})=\lambda=3$

$\therefore$ Plane parallel to ABC and passing through P is $r\cdot(\hat{i}+\hat{j}+\hat{k})=3$

Distance of plane from origin is $\frac{3}{\sqrt{3}} = \sqrt{3}$

Distance of plane ABC from origin = $\frac{4}{\sqrt{3}}$

$\therefore$ Distance of P from ABC = $\frac{4}{\sqrt{3}} - \frac{3}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

Example 15: A tetrahedron has vertices A(1, 2, 1) B (2, 1, 3), C (0, 0, 4) and origin O. Find angle between faces ABC and OAB

I.I.T.

Sol. Angle between two plane = angle between their normals.

Normal of face OAB

$= (\overline{OA} \times \overline{OB}) = (\hat{i} + 2\hat{j} + \hat{k}) \times (2\hat{i} + \hat{j} + 3\hat{k})$

$= 5\hat{i} - \hat{j} - 3\hat{k}$

Normal of face ABC = $\overline{AB} \times \overline{AC}$

$= (\hat{i} - \hat{j} + 2\hat{k}) \times (-\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{i} - 5\hat{j} - 3\hat{k}$

$\therefore$ cos θ = $\left(\frac{5i - j - 3k}{\sqrt{35}}\right) \cdot \left(\frac{\hat{i} - 5\hat{j} - 3\hat{k}}{\sqrt{35}}\right)$

$= \frac{5+5+9}{35} = \frac{19}{35} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{35}\right)$

Example 16: Show that the equation of plane through $(\hat{i} + 2\hat{j} - \hat{k})$ and perpendicular to the line of intersection of planes $r \cdot (3\hat{i} - \hat{j} + \hat{k}) = 1$ and $\bar{r} \cdot (\hat{i} + 4\hat{j} - 2\hat{k})$ = 0 is $\bar{r} \cdot (2\hat{i} - 7\hat{j} - 13\hat{k}) = 1$

Sol. Line of intersection of planes is perpendicular to normals of the plane. Therefore normal of the required plane shall be perpendicular to normals of these plane i.e. parallel to their cross product.

Cross product of normals = $(3\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 4\hat{j} - 2\hat{k})$

$= -2\hat{i} + 7\hat{j} + 13\hat{k}$

$\therefore$ Plane is $\left[\bar{r} - (\hat{i} + 2\hat{j} - \hat{k})\right] \cdot \left[-2\hat{i} + 7\hat{k} + 13\hat{k}\right] = 0$

$\Rightarrow \quad \bar{r} \cdot (-2\hat{i} + 7\hat{j} + 13\hat{k}) = -1$

Three Dimension: Given point (1, 2, –1) and planes are 3x – y + z = 1, x + 4y – 2z = 0

any plane through (1, 2, –1) is

a(x – 1) + b (y – 2) + c (z + 1) = 0

It is perpendicular to given planes

$$\therefore \left.\begin{aligned} 3a - b + c = 0 \\ a + 4b - 2c = 0 \end{aligned}\right\} \Rightarrow \frac{a}{2-4} = \frac{b}{1+6} = \frac{c}{12+1}$$

Plane is –2(x – 1) + 7(y – 2) + 13(z + 1) = 0

$\Rightarrow$ 2x – 7y – 13z = 1

In vector form $\bar{r}\cdot(2\hat{i} - 7\hat{j} - 13\hat{k}) = 1$

Example 17: A plane through (3, –1, 1) passes through the line of intersection of planes $\bar{r}\cdot(2\hat{i} - 3\hat{j} + \hat{k}) = 5$ and $\bar{r}\cdot(\hat{i} + 5\hat{j} - 2\hat{k}) = 1$. Find its equation.

Sol. Equation of plane through line of intersection of gives plane is

$$\left[\bar{r}\cdot(2\hat{i} - 3\hat{j} + \hat{k}) - 5\right] + \lambda\left[\bar{r}\cdot(\hat{i} + 5\hat{j} - 2\hat{k})\right] - 1 = 0$$

$$\Rightarrow r\cdot\left[(2+\lambda)\hat{i} + (5\lambda - 3)\hat{j} + (1 - 2\lambda)\hat{k}\right] = 5 + \lambda$$

It passes through $3\hat{i} - \hat{j} + \hat{k}$

$$\therefore \left(3\hat{i} - \hat{j} + \hat{k}\right)\cdot\left[(2+\lambda)\hat{i} + (5\lambda - 3)\hat{j} + (1 - 2\lambda)\hat{k}\right] = \lambda + 5$$

$\Rightarrow 6 + 3\lambda - 5\lambda + 3 + 1 - 2\lambda = \lambda + 5 \Rightarrow \lambda = 1$

Plane is $\bar{r}\cdot(3\hat{i} + 2\hat{j} - \hat{k}) = 6$

Three Dimensions: Planes are 2x – 3y + z = 5 and x + 5y – 2z = 1

Any plane through the line of intersection of these planes is

(2x – 3y + z – 5) + λ(x + 5y – 2z – 1) = 0

It passes through (3, –1, 1)

$\therefore$ (6 + 3 + 1 – 5) + λ(3 – 5 – 2 –1) = 0 $\Rightarrow \lambda = 1$

$\therefore$ Plane is x + 2y – z = 6

Example 18: Points P and Q are $(3\hat{i} + \hat{j} + 2\hat{k})$ and $(\hat{i} - 2\hat{j} - 4\hat{k})$. Find equation of plane through P and perpendicular to PQ. Find its distance from (–1, 1, 1)

Sol. Normal of plane is along PQ, as plane is perpendicular to PQ, $\overline{PQ} = -2\hat{i} - 3\hat{j} - 6\hat{k}$

$\therefore$ Plane through P is $\left[\bar{r} - (3\hat{i} + \hat{j} + 2\hat{k})\right]\cdot\left[-2\hat{i} - 3\hat{j} - 6\hat{k}\right] = 0$

$\Rightarrow r\cdot(-2i - 3j - 6k) = -21$

$\Rightarrow r\cdot(2\hat{i} + 3\hat{j} + 6\hat{k}) = 21$ ... (1)

Plane through (–1, 1, 1) parallel to plane (1) is

$r \cdot (2i + 3j + 6\hat{k}) = \lambda = (-\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} + 3\hat{j} + 6\hat{k})$

$\Rightarrow \quad r \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) = 7$ ... (2)

$\perp$ from origin plane (1) = $\frac{21}{\hat{\eta}} = \frac{21}{7} = 3$

$\perp$ from origin on plane (2) = $\frac{7}{\hat{\eta}} = \frac{7}{7} = 1$

$\therefore$ Distance of point (–1, 1, 1) from plane is 3 – 1 = 2

Example 19: Find the equation of line of intersection of planes $r \cdot (3\hat{i} - \hat{j} + \hat{k}) = 13$ and $r \cdot (\hat{i} + 4\hat{j} - 2\hat{k}) = 57$

Sol. The line of intersection of the two planes is parallel to the cross product of their normals. i.e. parallel to $(3\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 4\hat{j} - 2\hat{k}) = -2\hat{i} + 7\hat{j} + 13\hat{k}$

Suppose foot of perpendicular from origin on the line of intersection is $\bar{a}$. This is also on the plane formed by the normals of planes.

$\therefore \ \bar{a} = \lambda(3\hat{i} - \hat{j} + \hat{k}) + \mu(\hat{i} + 4\hat{j} - 2\hat{k})$ ... (α)

and $\therefore$ $\left[(3\lambda + \mu)\hat{i} + (4\mu - \lambda)\hat{j} + (\lambda - 2\mu)\hat{k}\right] \cdot \left[3\hat{i} - \hat{j} + \hat{k}\right] = 13$

$\Rightarrow \quad 9\lambda + 3\mu - 4\mu + \lambda + \lambda - 2\mu = 13$

$11\lambda - 3\mu = 13$ ... (1)

and $\left[\lambda(3\hat{i} - \hat{j} + \hat{k}) + \mu(\hat{i} + 4\hat{j} - 2\hat{k})\right] \cdot \left[\hat{i} + 4\hat{j} - 2\hat{k}\right] = 57$

$\Rightarrow \quad 3\lambda + \mu + 16\mu - 4\lambda - 2\lambda + 4\mu = 57$

$\Rightarrow \quad -3\lambda + 21\mu = 57 \Rightarrow -\lambda + 7\mu = 19$ ... (2)

from 1 and 2; $\lambda = 2, \mu =- 3$

Putting these values of λ and μ in eqn. (α)

$\bar{a} = 2(3\hat{i} - \hat{j} + \hat{k}) + 3(\hat{i} + 4\hat{j} - 2\hat{k}) = 9\hat{i} + 10\hat{j} - 4\hat{k}$

$\therefore$ Line of intersection is

$\bar{r} = 9\hat{i} + 10\hat{j} - 4\hat{k} + t(-2\hat{i} + 7\hat{j} + 13\hat{k})$

Example 20: Show that planes $\bar{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 0$, $\bar{r} \cdot (\hat{i} - \hat{j} + 4\hat{k}) =$ 12 and $r \cdot (7\hat{i} - 5\hat{k}) + 4 = 0$ have common line of intersection.

Sol. Equation of plane passing through the line of intersection of planes $\bar{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})=0$ and $r\cdot(\hat{i}-\hat{j}+4\hat{k})=2$ is

$$\bar{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})+\lambda\left[\bar{r}\cdot(\hat{i}-\hat{j}+4\hat{k})-2\right]=0$$

$$\bar{r}\cdot\left[(2+\lambda)\hat{i}+(5-\lambda)\hat{j}+(3+4\lambda)\hat{k}\right]-2\lambda=0 \qquad \text{... }(\alpha)$$

If the given planes have a common line of intersection then this plane (α) and the given third plane $\bar{r}\cdot(7\hat{j}-5\hat{k})+4=0$ should be identical.

$\therefore -2\lambda = 4$ $\quad \lambda = -2$ putting this value of λ in Eqn. (α)

$r\cdot\left[0\hat{i}+7\hat{j}-5\hat{k}\right]+4=0$ which is the third given plane.

Example 21: Position vector of points P and Q are $5\hat{i}+7\hat{j}-2\hat{k}$ and $3\hat{i}+3\hat{j}+6\hat{k}$ respectively. The vector A $(3\hat{i}-\hat{j}+\hat{k})$, passes through P and the vector B, $(-3\hat{i}+2\hat{j}+4\hat{k})$ passes through Q. A third vector $2\hat{i}+7\hat{j}-5\hat{k}$ intersects vector A and B. Find P.V. of point of intersection.

Sol. Let vector 3i – j + k act along L_1A and vector –3i+2j+4k act along L_2B. P is on L_1A and Q is on L_2B and vector.

2i + 7j + 4k meets L_1A in C and L_2B in D. Its direction is $\overrightarrow{CD}$.

From figure 14 P. vector of C $r_1=\overline{OP}+\lambda$ vector A

$\Rightarrow r_1=5\hat{i}+7\hat{j}-2\hat{k}+\lambda(3\hat{i}-\hat{j}+\hat{k})$ and P. vector of D ... (1)

$r_2=\overrightarrow{OQ}+\mu$

$= -3\hat{i}+3\hat{j}+6\hat{k}+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$ ... (2)

$\overrightarrow{CD}=r_2-r_1=\alpha(2\hat{i}+7\hat{j}-5\hat{k})$

rom (1) and (2), $\alpha(2\hat{i}+7\hat{j}-5\hat{k})$

$= (-8-3\mu-3\lambda)\,\hat{i} + (-4+2\mu+\lambda)\,\hat{j} + (8+4\mu-\lambda)\,\hat{k}$

$\therefore\ 2\alpha = -8-3\mu-3\lambda,$

$\Rightarrow \alpha = -4-\frac{3}{2}\mu-\frac{3}{2}\lambda$ ...(i)

and $7\alpha = -4+2\mu+\lambda,$ ... (ii)

$5\alpha = -8-4\mu+\lambda$ ... (iii)

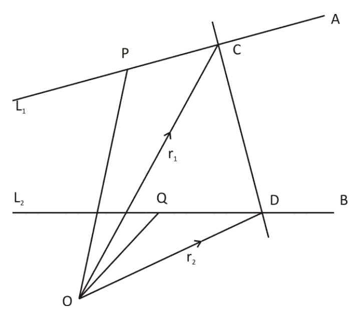


Fig 13

from (i) and (ii) $\quad 25\mu+23\lambda=-48$ ... (3)

(i) and (iii) $\quad 7\mu+17\lambda=-24$ ... (4)

From (4) and (3) $\mu = -1, \lambda = -1$

$\therefore\ \overline{OC} = \bar{r}_1 = 5\hat{i} + 7\hat{j} - 2\hat{k} - 3\hat{i} + \hat{j} - \hat{k} = 2\hat{i} + 8\hat{j} - 3\hat{k}$

$\overline{OD} = \bar{r}_2 = -3\hat{i} + 3\hat{j} + 6\hat{k} + 3\hat{i} - 2\hat{j} - 4\hat{k} = \hat{j} + 2\hat{k}$

Example 22: Vectors $\overline{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $CD = -3i + 2j + 4k$ are not coplanar and p. vector of A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{i} + 2\hat{k}$ respectively. Find p. vector of point P on AB and a point Q on CD such that PQ is perpendicular on both AB and CD.

Sol. Study the figure 15. Let $\bar{r}_1$ be P.V. of P.

$r_1 = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k}) = (6 + 3\lambda)\hat{i} + (7 - \lambda)\hat{j} + (4 + \lambda)\hat{k}$

Let P.V. of Q be r_2

$r_2 = (-9\hat{j} + 2\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) = -3\mu\hat{i} + (-9 + 2\mu)\hat{j} + (2 + 4\mu)\hat{k}$

$PQ = r_2 - r_1 = -3\mu\hat{i} + (-9 + 2\mu)\hat{j} + (2 + 4\mu)\hat{k} - (6 + 3\lambda)\hat{i} - (7 - \lambda)\hat{j} - (4 + \lambda)\hat{k}$

$= \hat{i}(-6 - 3\mu - 3\lambda) + \hat{j}(2\mu + \lambda - 16) + \hat{k}(-2 + 4\mu - \lambda)$

$PQ \perp AB \Rightarrow 3(-6 - 3\mu - 3\lambda) - (2\mu + \lambda - 16) + (-2 + 4\mu - \lambda) = 0$

$\therefore\ 7\mu + 4\lambda = -4$ ... (1)

$PQ \cdot CD = 0;$

$\therefore\ -3\ (-6 - 3\mu - 3\lambda) + 2(2\mu + \lambda - 16) + (-2 + 4\mu - \lambda) = 0$

$\Rightarrow 29\mu + 7\lambda = 22$ ... (2)

Solving (1) and (2) $\mu = 1, \lambda = -1$

$\therefore$ P. vector of P is $3\hat{i} + 8\hat{j} + 3\hat{k}$, of Q, $-3\hat{i} - 7\hat{j} - 6\hat{k}$

Example 23: P vector of two points A and C are $9\hat{i} - \hat{j} + 7\hat{k}$ and $7\hat{i} - 2\hat{j} + 7\hat{k}$. The point of intersection of vectors AB and CD is P. $\overline{AB}$ = $4\hat{i} - \hat{j} + 3\hat{k}$ and $\overline{CD}$ is $2\hat{i} - \hat{j} + 2\hat{k}$. If PQ is perpendicular to AB and CD and = 15, then find position vector of Q.

Sol. Let O be the origin and p.vector of P be r from ΔOAP

$\bar{r} = \overrightarrow{OA} + \overrightarrow{AP} = 9\hat{i} - \hat{j} + 7\hat{k} + \lambda(4\hat{i} - \hat{j} + 3\hat{k})$ ... (1)

From Δ OCP, $\bar{r} = \overrightarrow{OC} + \overrightarrow{CP}$

$r = 7\hat{i} - 2\hat{j} + 7\hat{k} + \mu(2\hat{i} - \hat{j} + 2\hat{k})$

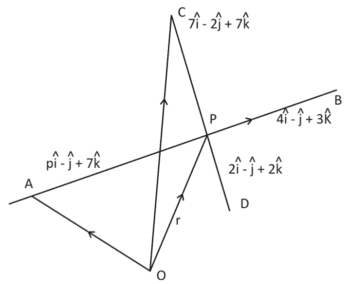


Fig 14

From (1) and (2)

$$\left.\begin{array}{l}\text{(i) } 9+4\lambda=7+2\mu \\ \text{(ii) } 1+\lambda=2+\mu \\ \text{(iii) } 7+3\lambda=7+2\mu\end{array}\right\}$$

From (iii) $\lambda=\frac{2}{3}\mu$

From (i) $9+\frac{8}{3}\mu=7+2\mu$

or $\frac{2}{3}\mu=-2 \Rightarrow \mu=-3$ and these satisfy (ii)

$\therefore$ $\overrightarrow{OP}=\bar{r}=\hat{i}+\hat{j}+\hat{k}$ $\Rightarrow$ P is (1, 1, 1)

Let PQ = $a\hat{i}+b\hat{j}+c\hat{k}$

$$\left.\begin{array}{l}PQ \perp AB \Rightarrow 4a-b+3c=0 \\ PQ \perp CD \Rightarrow 2a-b+2c=0\end{array}\right\} \therefore \frac{a}{1}=\frac{b}{-2}=\frac{c}{-2}$$

$\therefore$ D. Ratios are 1, –2, –2 $\Rightarrow$ D. cosine are $\frac{1}{3},-\frac{2}{3},-\frac{2}{3}$

P lies on $\frac{x-1}{\frac{1}{3}}=\frac{y-1}{-\frac{2}{3}}=\frac{z-1}{-\frac{2}{3}}=r=15$

$\therefore$ Q is (5 + 1, –10 + 1, –10 + 1) $\Rightarrow$ (6, –9, –9)

$\therefore$ P.V. of Q is $6\hat{i}-9\hat{j}-9\hat{k}$

Example 24: Find the equation of the plane which goes through origin and (3, –1, 2) and is parallel to straight line $r=4\hat{i}+3\hat{j}-\hat{k}+t(\hat{i}-4\hat{j}+7\hat{k})$

Sol. Let normal of plane be $a\hat{i}+b\hat{j}+c\hat{k}$

Plane is $\bar{r}\cdot(a\hat{i}+b\hat{j}+c\hat{k})=0$

It is parallel to straight line i.e. parallel to vector $\hat{i}-4\hat{j}+7\hat{k}$. It means normal is perpendicular to it.

$\therefore$ a · 1 – 4 · b + 7 · c = 0 $\Rightarrow$ a – 4b + 7c = 0 … (1)

Point $(3\hat{i}-\hat{j}+2\hat{k})$ lies on plane

$\therefore$ $(3\hat{i}-\hat{j}+2\hat{k})\cdot(a\hat{i}+b\hat{j}+c\hat{k})=3a-b+2c=0$

From (1) and (2) $\frac{a}{-8+7}=\frac{b}{21-2}=\frac{c}{-1+12}$

$\Rightarrow \frac{a}{-1}=\frac{b}{19}=\frac{c}{11} \Rightarrow \frac{a}{1}=\frac{b}{-19}=\frac{c}{-11}$

$\therefore$ Plane is $\bar{r}\cdot(\hat{i}-19\hat{j}-11\hat{k})=0$

Example 25: Find equation of plane through points A (2, 3, –4) and B (1, –1, –3) and parallel to x – axis.

Sol. Points are A $(2\hat{i}+3\hat{j}-4\hat{k})$,B$(\hat{i}-\hat{j}+3\hat{k})$

$\therefore$ Vector $\overline{AB}=-\hat{i}-4\hat{j}+7\hat{k}$

Plane is parallel to AB and x-axis $(\hat{i}+0\cdot\hat{j}+0\cdot\hat{k})$

Plane is parallel to AB and x-axis $(\hat{i}+0\cdot\hat{j}+0\cdot\hat{k})$

Normal shall be perpendicular to AB and x-axis

$\therefore$ Plane is $\left[\bar{r}-(2\hat{i}+3\hat{j}-4\hat{k})\right]\cdot\left[(-\hat{i}-4\hat{j}+7\hat{k})\times\hat{i}\right]=0$

$\Rightarrow \left[\bar{r}-(2\hat{i}+3\hat{j}-4\hat{k})\right]\cdot\left[4\hat{k}+7\hat{j}\right]=0$

$\Rightarrow r\cdot(4\hat{k}+7\hat{j})=5$

Example 26: Find the point where the line joining $3\hat{i}+4\hat{j}+\hat{k}$ and $5\hat{i}+\hat{j}+6\hat{k}$ crosses x-y plane

Sol. Equation of line is

r = $3\hat{i}+4\hat{j}+\hat{k}+\lambda(2\hat{i}-3\hat{j}+5\hat{k})$

Point Q in x-y plane should be $a\hat{i}+b\hat{j}$. It lies on line.

$\therefore$ $a\hat{i}+b\hat{j}$ = $3\hat{i}+4\hat{j}+\hat{k}+\lambda(2\hat{i}-3\hat{j}+5\hat{k})$

$\therefore$ a = 3 + 2λ, b = 4 – 3λ, O = 1 + 5λ $\Rightarrow \lambda = -\frac{1}{5}$

$\therefore$ $a=3-\frac{2}{5}=\frac{13}{5}$, $\qquad b=4+\frac{3}{5}=\frac{23}{5}$

$\therefore$ Point of intersection is $\left(\frac{13}{5},\frac{23}{5},0\right)$

i.e. $\frac{13}{5}\hat{i}+\frac{23}{5}\hat{j}$

Example 27: From a point P (a, b, c) perpendicular PA and PB are drawn to y-z and z-x planes. Find vector equation of plane OAB

Sol. The plane passes through (0, 0, 0), (0, b, c) and (a, 0, c). The equation of plane is

$$\bar{r}\cdot\left[(b\hat{j}+c\hat{k})\times(a\hat{i}+c\hat{k})\right]=0 \Rightarrow \bar{r}\cdot\left[bc\hat{i}+ac\hat{j}-ab\hat{k}\right]=0$$

$$\Rightarrow r\cdot\left[\frac{\hat{i}}{a}+\frac{\hat{j}}{b}-\frac{\hat{k}}{c}\right]=0$$

Example 28: Find eqn. of plane through (5, –2, 3) and $\perp$ to planes $\bar{r}\cdot(2\hat{i}+\hat{j}+\hat{k})=9$ and $\bar{r}\cdot(\hat{i}+3\hat{j}-2\hat{k})=4$.

Sol. Plane is $\perp$ to two given planes

$\therefore$ Its normal η = cross product of two normals

$= (2\hat{i}+\hat{j}+\hat{k})\times(\hat{i}+3\hat{j}-2\hat{k})$

$= 5(-\hat{i}+\hat{j}+\hat{k})$

$\therefore$ Plane is $\left[\bar{r}-(5\hat{i}-2\hat{j}+3\hat{k})\right]\cdot\left[-\hat{i}+\hat{j}+\hat{k}\right]=0$

$\Rightarrow \quad \bar{r}\cdot(\hat{i}-\hat{j}-\hat{k})=4$

Example 29: Intercepts by a plane on axes are 2, 3 and 1. Find distance of plane from origin.

Sol: Plane goes through (2, 0, 0), (0, 3, 0), (0, 0, 1)

$\therefore$ Points A is $2\hat{i}$, B is $3\hat{j}$ and C is $\hat{k}$

$\overline{AB}=-2\hat{i}+3\hat{k}$, $\overline{BC}=-3\hat{j}+\hat{k}$

$\therefore$ normal $\quad \eta=(-2\hat{i}+3\hat{k})\times(-3\hat{j}+\hat{k})$

$=3\hat{i}+2\hat{j}+6\hat{k}$

Plane is $(\bar{r}-2\hat{i})\cdot(3\hat{i}+2\hat{j}+6\hat{k})=0$

$r\cdot(3\hat{i}+2\hat{j}+6\hat{k})=6$

$$\Rightarrow r\cdot\frac{(3\hat{i}+2\hat{j}+6\hat{k})}{7}=\frac{6}{7}$$

$\therefore$ Distance of plane from origin is $\frac{6}{7}$

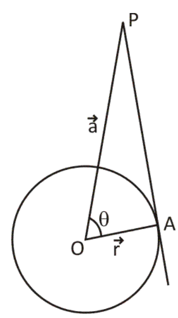


Fig 15

Example 30: Origin O is centre of a circle of radius, r. A is a fixed point on circle and P is any point on tangent at A. OP = a. Show that

$\bar{r}\cdot\bar{a}=r^2$ and taking $\bar{a}=(x,y)$ and $\bar{r}=\begin{pmatrix}x_1\\y_1\end{pmatrix}$ deduce equation of tangent at (x, y) to circle $x^2+y^2=r^2$

Sol. In the figure, $\overrightarrow{OA}=\bar{r},\overrightarrow{OP}=\bar{a}$ and $\angle POA=\theta$

$\therefore\quad \bar{r}\cdot\bar{a}=\overrightarrow{OA}\cdot\overrightarrow{OP}$

$=|\overrightarrow{OA}||\overrightarrow{OP}|\cos\theta$ = r.OPcosθ = r.r = r^2

and $\bar{a}\cdot\bar{r}=(x,y)\begin{pmatrix}x_1\\y_1\end{pmatrix}$

$\Rightarrow r^2=xx_1+yy_1$

which is eqn. of tangent at (x_1, y_1) to circle $x^2 + y^2 = r^2$

Practice Worksheet (Foundation Level) – 4(b)

1. The equation of line through (2, –1, 3) and parallel to $3\hat{i}+7\hat{j}-2\hat{k}$ is …

2. Equation of straight line passing through (1, –2, 1) and (2, 3, 4) is

3. Angle between the straight line $r=2\hat{i}-\hat{k}+t_1(3\hat{i}+4\hat{j})$ and $r=\hat{i}+\hat{j}+\hat{k}+t_2(2\hat{i}+\hat{j}+2\hat{k})$ is

4. For what value of a, are the points $2\hat{i}+3\hat{j}+4\hat{k}$, $a\hat{i}-3\hat{j}+5\hat{k}$ and $-9\hat{j}+6\hat{k}$ collinear.

5. For what value of λ are the vectors $\bar{a}+\lambda\bar{b}+3\bar{c}$, $\bar{a}-2\bar{b}+3\bar{c}$ and $6\bar{a}+14\bar{b}+4\bar{c}$ are coplanar.

6. The point of intersection of lines $\bar{r}=(\hat{i}+\hat{j}+\hat{k})+t_1(2\hat{i}-\hat{j}+\hat{k})$ and $\bar{r}=(2\hat{i}+\hat{j}-\hat{k})+t_2(3\hat{i}-2\hat{j}+4\hat{k})$ is …

7. (a) Equation of the plane through (2, 3, –1) and parallel to plane $r\cdot(\hat{i}+\hat{j}+\hat{k})=3$ is …

   (b) and its distance from the given plane is …

8. Find angle between the line $\bar{r}=2\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ and plane $\bar{r}\cdot(3\hat{i}-2\hat{j}+6\hat{k})=4$.

9. The distance of the plane $r\cdot(2i-2j+k)=3$ from (1, –1, 1) is …

10. Equation of a plane through origin and through the line of intersection of the planes $\bar{r}\cdot(\hat{i}+3\hat{j}-\hat{k})=5$ and $r\cdot(\hat{j}+2\hat{k})=5$ is …

11. Does the plane $\bar{r}\cdot(\hat{i}+2\hat{j}-\hat{k})=3$ contain the line $\bar{r}=(2\hat{i}+\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+4\hat{k})$.

12. The equation of plane containing the lines

    $r=2\hat{i}+\hat{j}-3\hat{k}+\lambda(\hat{i}+2\hat{j}+5\hat{k})$ and $r=3\hat{i}+7\hat{j}+7\hat{k}+\mu(3\hat{i}-2\hat{j}+5\hat{k})$ is…

13. Find equation of plane which bisects the line joining (1, 2, 4) and (3, 4, 6) at right angles.

14. The equation of plane through (2, 3, –4) and (1, –1, 3) parallel to x-axis is …

15. Find the equation of plane which goes through origin and point (4, –2, 3) and is parallel to the line $\bar{r}=4\hat{i}-3\hat{j}+\hat{k}+t(\hat{i}+4\hat{j}+3\hat{k})$

16. The equation of plane through (1, 2, 3) and perpendicular to planes $r\cdot(2\hat{i}-\hat{j}+\hat{k})=3$ and $r\cdot(\hat{i}-2\hat{j}-2\hat{k})=4$ is

17. Find the equation of plane which cuts off intercepts of 4, –3 and 6 from axes. Find its distance from origin as well.

18. P is (–2, 3, –1) and PM, PN, PT are perpendicular from P on x – y, y – z and z – x plane. Find the equation of the plane **M.N.T.**

19. The equation of plane through (3, –1, 2) perpendicular to the line of intersection of planes $\bar{r}\cdot(\hat{i}-\hat{j}+\hat{k})=5$ and $\bar{r}\cdot(2\hat{i}+3\hat{j}-4\hat{k})=10$ is …

20. The equation of a plane which is parallel to vectors $3\hat{i}+2\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}-2\hat{k}$ which are non parallel and which passes through (1, –2, 4) is

21. Find the equation of plane which passes through (1, 2, 3), (1, –3, 4) and (2, 1, –3).

22. The straight line $\bar{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(3\hat{i}+\hat{j}-2\hat{k})$ meets the plane $\bar{r}\cdot(4\hat{i}-\hat{j}+\hat{k})=10$ in P. P.V. of P is …

23. The straight line $\bar{r}=3\hat{i}-2\hat{j}+4\hat{k}+\lambda(2\hat{i}-\hat{j}+3\hat{k})$, $\bar{r}=\hat{i}+3\hat{j}+5\hat{k}+\mu(\hat{j}+\hat{k})$ meets in P perpendicular to the line segment (…) and (2, 1, 3) and is at a … from origin. Equation of plane is …

24. Show that planes $\bar{r}\cdot(3\hat{i}+4\hat{j}+2\hat{k})=0$ $\bar{r}\cdot(3\hat{i}-\hat{j}+3\hat{k})=2$ and $\bar{r}\cdot(5\hat{j}-\hat{k})+2=0$ have common lines of intersection.

25. Find equation of line of intersection of planes $r\cdot(\hat{i}+4\hat{j}-2\hat{k})=39$ and $\bar{r}\cdot(3\hat{i}-\hat{j}+\hat{k})=5$.

26. Find equation of plane which passes through the line of intersection of planes $\bar{r}\cdot(3\hat{i}-7\hat{j}+2\hat{k})$ = 15 and $\bar{r}\cdot(4\hat{i}+5\hat{j}-3\hat{k})=12$ and pt. (4, –2, 5)

27. Find equation of plane which bisects angle between planes $\bar{r}\cdot(2\hat{i}-2\hat{j}+\hat{k})=0$ and $\bar{r}\cdot(\hat{i}+2\hat{j}-2\hat{k})=0$ and passes through (3, –2, –1)

28. The image of point (3, –2, 1) in a plane is (–1, 3, –4). Find equation of this plane.

29. Point A is (2, 1, 3) and $\overline{AB}=4\hat{i}-\hat{j}+5\hat{k}$ straight line CD goes through (1, 3, 4) and has d. ratios 5, –3, 4. If AB and CD intersect at P, find P.

30. Planes $\bar{r}\cdot(2\hat{i}-2\hat{j}+\hat{k})=0$ and $\bar{r}\cdot(\hat{i}+2\hat{j}-2\hat{k})$ = λ lies opposite sides of point (3, –2, –1). If this point is equidistant from these planes then find value of λ.

31. α and $\beta=4\hat{i}+3\hat{j}$ are two perpendicular vectors in x – y plane. All vectors of x – y plane have projection 1 and 2 along α and β respectively. Find these vectors.

    $\left[\text{Hint}:\alpha=3\hat{i}-4\hat{j}\right]$

32. Find vector moment of vectors $\hat{i}+2\hat{j}-3\hat{k}$, $2\hat{i}+3\hat{j}+4\hat{k}$ and $-\hat{i}-\hat{j}+\hat{k}$ which act on a particle at P (0, 1, 2); about the point (1, –2, 0).

33. $\bar{r}\times\bar{b}=\bar{c}\times\bar{b}, \bar{r}\cdot\bar{a}=0$ and $\bar{a}\cdot\bar{b}\neq 0$. If these three statements hold good simultaneously, then express $\bar{r}$ in terms of $\bar{a}$, $\bar{b}$ and $\bar{c}$.

34. The vertices of a tetrahedron are O, (0, 0, 0), A, (1, 2, 0); B, (2, 0, 1) and C (0, 4, 5). Calculate angle between faces OAB and ABC.

35. $\bar{a} = 3\hat{i} + \hat{j} + 4\hat{k}$, $\bar{b} = \hat{i} - \hat{j}$ and $\bar{c} = \hat{i} + 5\hat{k}$. Find angle between $\bar{a} + \bar{b}$ and $\bar{a} + \bar{c}$. Find also unit vector perpendicular to $\bar{a} + \bar{b}$ and $\bar{a} + \bar{c}$.

36. A force $\bar{F} = 2\hat{i} + \hat{j} - \hat{k}$ acts at point A, whose p. vectors is $2\hat{i} - \hat{j}$. Calculate moment of force about origin. If force moves from A to B $(2\hat{i} + \hat{j})$, then calculate work done by force.

37. $\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{a} = \bar{a}p + \bar{b}q + \bar{c}r$ where $\bar{a}, \bar{b}, \bar{c}$ are vectors and p, q, r are scalars. If $\bar{a}, \bar{b}, \bar{c}$ be unit vector and be inclined at θ to one – another, then express p, q, r in terms of θ.

38. Position vectors of vertices of a triangle are A $(\hat{i} + \hat{j} + \hat{k})$, B $(3\hat{i} - \hat{j} + 2\hat{k})$ and C $(4\hat{i} - 3\hat{j} + \hat{k})$. The angle bisector of angle A meets BC in P. Find position vector of P and equation of AP.

39. Position vectors of points P and Q are $3\hat{i} + 4\hat{j} + 5\hat{k}$ and $-3\hat{i} + 4\hat{j} + 7\hat{k}$ respectively. Vectors $\bar{a} = 4\hat{i} - 2\hat{j} + \hat{k}$ passes through P and vector $\bar{b} = -4\hat{i} + 3\hat{j} + 2\hat{k}$ passes through Q. A third vector $-2\hat{i} - \hat{j} + 6\hat{k}$ meets vector $\bar{a}$ and $\bar{b}$ in C and D. Find C and D.

40. Position vectors of points A and C are $8\hat{i} - \hat{j} + 7\hat{k}$ and $\hat{i} + 9\hat{j} + \hat{k}$ respectively. Vectors $\overline{AB}$ an $\overline{CD}$ are $4\hat{i} - 5\hat{j} + 4\hat{k}$ and $3\hat{i} - 5\hat{j} + 2\hat{k}$; AB and CD intersect in P. $\overline{PQ}$ is perpendicular to $\overline{AB}$ and $\overline{CD}$ and its magnitude is $\sqrt{141}$. Find P.V. of Q.

5. Formulae and Concepts at a Glance

Linear Combination of Vectors

The linear combination of a finite set of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ is defined as a vector $\vec{r}$ such that $\vec{r} = k_1\vec{a}_1 + k_2\vec{a}_2 + ...+ k_n\vec{a}_n$, Where $k_1, k_2, ...k_n$ are any scalars.

Linearly Department and Independent Vectors

A system of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ is said to be linearly dependent if there exists scalars $k_1, k_2, ...k_n$ (not all zero) such that $k_1\vec{a}_1 + k_2\vec{a}_2 + ...+ k_n\vec{a}_n = \overline{0}$.

They are said to be linearly independent if $k_1\vec{a}_1 + k_2\vec{a}_2 + ...+ k_n\vec{a}_n = \overline{0}$ implies that $k_1 = k_2 = ... = k_n = 0$

- Two collinear vectors are always linearly dependent
- Three coplanar vectors are always linearly dependent
- Three non-coplanar non-zero vectors are always linearly independent.
- Three points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar if
- Four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar if

$$\lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3\vec{c} + \lambda_4\overleftarrow{d} = 0 \text{ with } \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$$

C (or Dot) product of Two Vectors

The Scalar product of $\vec{a}$ and $\vec{b}$ written as $\vec{a} . \vec{b}$ is defined to be the number $|\vec{a}|\,|\vec{b}| \cos\theta$ where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ i.e. $\vec{a} . \vec{b} = ab\cos\theta$.

Properties

$\vec{a} . \vec{b} = (a\cos\theta)b =$ (projection of $\vec{a}$ on $\vec{b}$) b = (projection of $\vec{b}$ on $\vec{a}$)a

- $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a}, \vec{b}$ are perpendicular to each other $\Rightarrow \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

Vector (or Cross) Product of Two Vectors

The vector product of two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a} \times \vec{b}$ is defined as the vector $|\vec{a}|\,|\vec{b}| \sin\theta\hat{n}$, where θ is the angle between the vectors $\vec{a} . \vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ (i.e., perpendicular to plane of $\vec{a}$ and $\vec{b}$). The sense of $\hat{n}$ is obtained by the right hand thumb rule i.e., $\vec{a}, \vec{b}$ and $\hat{n}$ form a right –handed screw.

It id evident that $|\vec{a} \times \vec{b}| = ab\sin\theta$.

Properties

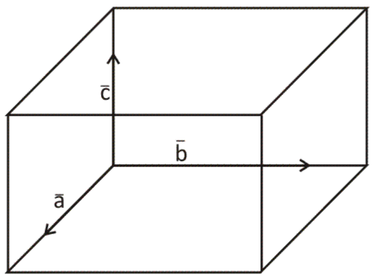
- $\vec{a}\times\vec{a}=0 \Rightarrow \hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\bar{0}$
- $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are collinear (if none of $\vec{a}$ or $\vec{b}$ is zero vector)
- $\hat{i}\times\hat{j}=\hat{k}, \hat{j}\times\hat{k}=\hat{i}, \hat{k}\times\hat{i}=\hat{j}$ and $\hat{j}\times\hat{i}=-\hat{k}, \hat{k}\times\hat{j}=-\hat{i}, \hat{i}\times\hat{k}=-\hat{j}$
- Any vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\lambda(\vec{a}\times\vec{b})$ where l is a real number. Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

$|\vec{a}\times\vec{b}|$ denotes the area of the parallelogram OACB, where $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$

Scalar Triple Product

It is defined for three vectors $\vec{a}, \vec{b}, \vec{c}$ in that order as the scalar $(\vec{a}\times\vec{b}).\vec{c}=[\vec{a}\vec{b}\vec{c}]$, which can also be written simply as $\vec{a}\times\vec{b}.\vec{c}$. It denotes the volume of the parallelepiped formed by taking a, b, c as the co-terminus edges.

i.e. V = magnitude of $\vec{a}\times\vec{b}.\vec{c}$ = $[\vec{a}\times\vec{b}.\vec{c}]$



Properties

- $\vec{a}\times\vec{b}.\vec{c}$ = $\vec{a}.\vec{b}\times\vec{c}$ i.e. position of dot and cross can be interchanged without altering the product. Hence it is also represented by $[\vec{a}\vec{b}\vec{c}]$.
- $[\vec{a}\vec{b}\vec{c}]$ = $[\vec{b}\vec{c}\vec{a}]$ = $[\vec{c}\vec{a}\vec{b}]$
- $[k\vec{a}\vec{b}\vec{c}]$ = k $[\vec{a}\vec{b}\vec{c}]$

Vector Triple Product

It is defined for three vectors $\vec{a}, \vec{b}, \vec{c}$ as the vector $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

In general $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$

$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$ if some or all of $\vec{a}, \vec{b}, \vec{c}$ are zero vectors or $\vec{a}$ and $\vec{c}$ are collinear.

APPLICATIONS OF VECTORS

(1) Section Formula

Let the position vector of a point P be $\vec{p}$ and that of another point Q be $\vec{q}$. If the line joining P and Q is divided by a point R in the ratio of m:n (internally or externally), then $\vec{r}=\frac{m\vec{q}+n\vec{p}}{m+n}$.For internal division take m:n as positive and for external division take m:n as negative (i.e., either of m or n as negative).

(2) Equation of Line in Vector Form

(i) Equation of the line passing vector through a given point $A(\vec{a})$ are parallel to a vector $(\vec{b})$ is given by $r=\vec{a}+\lambda\vec{b}$, where $\vec{r}$ is the position vector of any general point P on the line and λ is a real number.

(ii) Equation of the line passing through two given points $A(\vec{a})$ and $B(\vec{a})$ is given by $r=\vec{a}+\lambda(\vec{b}-\vec{a})$.

(3) Equation of Plane in Vector Form

Following are the four useful ways of specifying a plane.

(i) A plane at a perpendicular distance d from the origin and normal to a given direction $(\hat{n})$ has the equation $(\vec{r}-d\hat{n}).d\hat{n}=0$ or $\vec{r}.\hat{n}=d(\hat{n}$ is a unit vector).

(ii) A plane passing through the point $A(\vec{a})$ and normal $\hat{n}$ has the equation $(\vec{r}-\vec{a}).\hat{n}=0$.

(iii) Parametric equation of the plane passing through $A(\vec{a})$ and parallel to the plane of vectors $(\vec{b})$ and $(\vec{c})$ is given by $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}\Rightarrow\vec{r}.(\vec{b}\times\vec{c})=[\vec{a}\vec{b}\vec{c}]$

Some Miscellaneous Results

(i) Volume of the tetrahedron ABCD $=\frac{1}{6}[\overrightarrow{AB}\overrightarrow{AC}\overrightarrow{AD}]$

(ii) Area of the quadrilateral with diagonals $\vec{d}_1$ and $\vec{d}_2=\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$.

(iii) The internal dissector of angle between unit vectors $\hat{a}$ and $\hat{b}$ is along the vector $\hat{a}+\hat{b}$. The external bisector is along $\hat{a}$-$\hat{b}$.Equation of internal and external bisectors of the lines

$\vec{r}=\vec{a}+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}+\mu\vec{b}_2$ (intersecting at $A(\vec{a})$) are given by

$\vec{r}=\vec{a}+t\left(\frac{\vec{b}_1}{|\vec{b}_1|}\pm\frac{\vec{b}_2}{|\vec{b}_2|}\right)$.

Solved Example

Example 1: Find shortest distance between lines

$$\bar{r} = (3\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k}) \text{ and } \bar{r} = (\hat{i} + \hat{j} - 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$$

Sol: Here $\bar{a}_1 = 3\hat{i} - \hat{j}$, $\bar{a}_2 = \hat{i} + \hat{j} - 2\hat{k}$

$\bar{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$, $\bar{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$

$\bar{b}_1 \times \bar{b}_2 = (\hat{i} + 2\hat{j} - 3\hat{k}) \times (2\hat{i} + 4\hat{j} - 5\hat{k})$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i} - \hat{j} + 0.\hat{k}$$

and $|\bar{b}_1 \times \bar{b}_2| = 1$

$\bar{a}_2 - \bar{a}_1 = 2\hat{i} + 2\hat{j} - 2\hat{k}$

$$\therefore \text{ S.D.} = \frac{(-2\hat{i} + 2\hat{j} - 2\hat{k}).(2\hat{i} - \hat{j} + 0.\hat{k})}{\sqrt{4+1}}$$

$$= \frac{-4-2}{\sqrt{5}} \quad \Rightarrow \quad \text{S.D.} = \frac{6}{\sqrt{5}}$$

Note: See these lines do not intersect.

Suppose these intersect

$$\therefore (3\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k}) = (\hat{i} + \hat{j} - 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$$

$$\Rightarrow (3+\lambda)\hat{i} + (-1+2\lambda)\hat{j} - 3\lambda\hat{k} = (1+2\mu)\hat{i} + (1+4\mu)\hat{j} - (2+5\mu)\hat{k}$$

Equating co-efficient of $\hat{i}, \hat{j}$ and $\hat{k}$

$$\left.\begin{array}{ll} \text{(i)} & 3+\lambda = 1+2\mu \\ \text{(ii)} & -1+2\lambda = 1+4\mu \\ \text{(iii)} & -3\lambda = -2-5\mu \end{array}\right\}$$

from (iii) $\lambda = \frac{2+5\mu}{3}$

from (i) $3 + \frac{2+5\mu}{3} = 1 + 2\mu$

Solving μ = + 8, λ = 14

Putting these value in (ii), –1 + 28 = 1 + 32 does not satisfy i.e. lines do not intersect.

Example 2: Find shortest distance between straight lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{and} \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

Sol: The given lines are

$\hat{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}+4\hat{k})$ and $\hat{r}=(2\hat{i}+4\hat{j}+5\hat{k})+\mu(3\hat{i}+4\hat{j}+5\hat{k})$

Now, $\bar{a}_2-\bar{a}_1=(2\hat{i}+4\hat{j}+5\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=\hat{i}+2\hat{j}+2\hat{k}$

$\bar{b}_1\times\bar{b}_2=(2\hat{i}+3\hat{j}+4\hat{k})\times(3\hat{i}+4\hat{j}+5\hat{k})$

$$=\begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=-\hat{i}+2\hat{j}-\hat{k} \text{ magnitude = } \sqrt{6}$$

$$\therefore \text{ S.D.}=\frac{(\bar{a}_2-\bar{a}_1).(\bar{b}_1\times\bar{b}_2)}{|\bar{b}_1\times\bar{b}_2|}$$

$$=\frac{(\hat{i}+2\hat{j}+2\hat{k}).(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{6}}$$

$$=\frac{-1+4-2}{\sqrt{6}}=\frac{1}{\sqrt{6}}$$

Example 3: Vector $\bar{a}$ is perpendicular to vector $\bar{b}$ and vector $\bar{r}$ is a linear combination of vectors $\bar{a},\bar{b}$ an d $\bar{a}\times\bar{b}$. Find $\bar{r}$ if $\bar{r}\times\bar{a}=\bar{b}$

Sol: $\bar{a},\bar{b}$ and $\bar{a}\times\bar{b}$ are non coplanar

$\therefore\ r=x\bar{a}+y\bar{b}+z(\bar{a}\times\bar{b})$

where x, y, z are scalers and $\bar{a}\perp\hat{b}\Rightarrow\bar{a}.\bar{b}=\bar{c}$

Now $\bar{b}=\bar{r}\times\bar{a}=(x\bar{a}+y\bar{b}+z(\bar{a}\times\bar{b})\times\bar{a}$

$\Rightarrow\ \bar{r}\times\bar{a}=x(\bar{a}\times\bar{a})+y(\bar{b}\times\bar{a})+z[\bar{a}\times\bar{b}]\times\bar{a}$

$x.0+y(b\times a)+z[(\bar{a}.\bar{a})b-(b.a)a]$

$\Rightarrow\ \bar{r}\times\bar{a}=y(\bar{b}\times\bar{a})+z[(\bar{a}.\bar{a})\bar{b}]=\bar{b}$

Comparing co-efficients, y = 0, $z|\bar{a}.\bar{a}|=1$

$\Rightarrow\ z=\frac{1}{|\bar{a}|^2} \qquad \Rightarrow r=x\bar{a}+\frac{1}{[\bar{a}.\bar{a}]}(\bar{a}\times\bar{b})$

$$= x\bar{a} + \frac{1}{|\bar{a}|^2}(\bar{a}\times\bar{b})$$

Example 4: $\bar{a}, \bar{b}$ and $\bar{c}$ are non-zero vectors and $[(\bar{a}\times\bar{b}).\bar{c}] = |\bar{a}||\bar{b}||\bar{c}|$. Then prove

$\bar{a}.\bar{b} = \bar{b}.\bar{c} = \bar{c}.\bar{a} = 0$

Sol: $[(\bar{a}\times\bar{b}).\bar{c}]$ = volume of a parallelelopiped whose coterminous edges are OA, OB, OC (and $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}$)

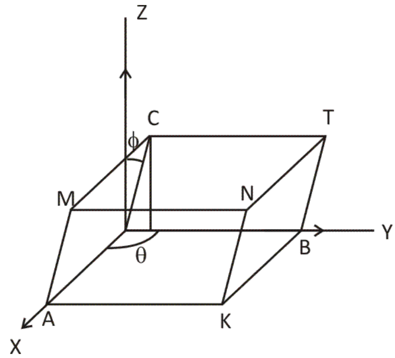


Fig 2

In the figure (2) volume of parallelopiped is = area of parallelogram OAKBX height CL

$= |(\bar{a}\times\bar{b})|.\bar{c} = \{|\bar{a}|.|\bar{b}|\sin\theta\}\{|\bar{c}|\cos\theta\}$

where ϕ is the angle OC makes with vertical

V = $|\bar{a}||\bar{b}||\bar{c}|\sin\theta.\cos\phi$

but given $V = |\bar{a}||\bar{b}||\bar{c}|$

$\therefore$ sin θ = 1, cos θ = 1 $\Rightarrow$ $\bar{a}\perp\bar{b}$ and $\bar{c}\perp\bar{b}$

$\therefore$ $\bar{a}.\bar{b} = \bar{a}.\bar{c} = \bar{c}.\bar{a} = 0$

Example 5: ABC is a triangle and $\bar{a}, \bar{b}$ and $\bar{c}$ are P.V. of A, B and C respectively. Point E divides AB in the ratio 2 : 3. CE meets median AD in P. Find P. Vector of P in terms of $\bar{a}, \bar{b}$ and $\bar{c}$.

Sol. Point E on AB divides AB in the ratio of 2 : 3

$\Rightarrow$ P.V. of $E = \dfrac{2\bar{b}+3\bar{a}}{5}$

D is mid-point of BC.

P.V. of D is $\dfrac{\bar{b}+\bar{c}}{2}$

P lies on CE and AD

Equation of CE is

$$r = c + \lambda\left(\frac{2\bar{b}+3\bar{a}-5\bar{c}}{5}\right)$$

$$= \bar{c} + \lambda\left(\frac{3}{5}\bar{a} + \frac{2}{5}\bar{b} - \bar{c}\right) = \frac{3}{5}\lambda\bar{a} + \frac{2}{5}\lambda\bar{b} + (1-\lambda)\bar{c}$$

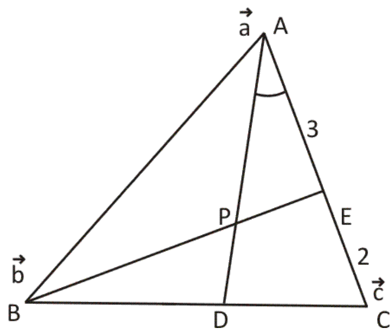


Fig 3

Equation of AD is

$$\bar{r} = \bar{a} + \mu\left(\frac{\bar{b} + \bar{c} - 2\bar{a}}{2}\right) = (1-\mu)\bar{a} + \frac{\mu\bar{b}}{2} + \frac{\mu\bar{c}}{2}$$

P lies on both.

$$\therefore \; \frac{3}{5}\lambda\bar{a} + \frac{2}{5}\lambda\bar{b} + (1-\lambda)\bar{c} = (1-\mu)\bar{a} + \frac{\mu\bar{b}}{2} + \frac{\mu\bar{c}}{2}$$

Comparing co-efficients of $\bar{b}$ $\frac{2}{5}\lambda = \frac{\mu}{2} = \mu = \frac{4}{5}\lambda$

Co-efficients of $\bar{a}$ $\frac{3\lambda}{5} = 1 - \mu = 1 - \frac{4}{5}\lambda \Rightarrow \lambda = \frac{5}{7}$

$\therefore$ P.V. of P is $\bar{c} + \frac{1}{7}(2b + 3a - 5c) = \frac{1}{7}(3\bar{a} + 2\bar{b} + 2\bar{c})$

Example 6: P. Vectors of points A, B, and C are $\hat{i} + \hat{j} + \hat{k}$, $3\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} + 3\hat{j} - 5\hat{k}$, respectively. Find vector BK perpendicular to AC

Sol. A, $\hat{i} + \hat{j} + \hat{k}$; B, $3\hat{i} - \hat{j} + 2\hat{k}$; C, $4\hat{i} + 3\hat{j} - 5\hat{k}$

$\therefore$ $\overline{AB} = 2\hat{i} - 2\hat{j} + \hat{k}, \overline{AC} = 3\hat{i} + 2\hat{j} - 6\hat{k}$

Let BK be perpendicular on AC, let it be $a\hat{i} + b\hat{j} + c\hat{k}$

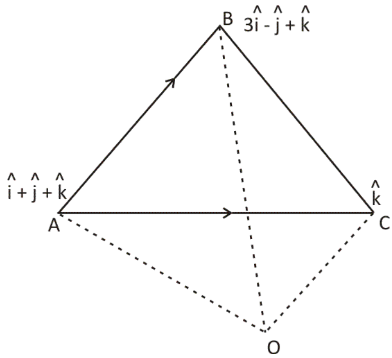


Fig 4

It is $\perp$ AC

$\therefore$ $(a\hat{i} + b\hat{j} + c\hat{k}).(3\hat{i} + 2\hat{j} - 6\hat{k}) = 0$

$\Rightarrow$ 3a + 2b – 6c = 0 ... (1)

K lies on AC and BK.

$\therefore$ $\bar{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} - 6\hat{k}) = 3\hat{i} - \hat{j} + 2\hat{k} + \mu(a\hat{i} + b\hat{j} + c\hat{k})$

Comparing co-efficients of $\hat{i}, \hat{j}$ and $\hat{k}$

1 + 3λ = 3 + μa $\Rightarrow$ 3aμ = 9λ – 6

1 + 2λ = –1 + μb $\Rightarrow$ 2bμ = 4λ + 4

1 – 6λ = 2 + μc $\Rightarrow$ –6cμ = 36λ + 6

adding (3a + 2b – 6c) μ = 49λ + 4 = 0 μ $\therefore \lambda = -\frac{4}{49}$

vector BK is $\hat{i} + \hat{j} + \hat{k} - \frac{4}{49}(3\hat{i} + 2\hat{j} - 6\hat{k}) - (3\hat{i} - \hat{j} + \hat{k})$

$$=\frac{1}{49}(37\hat{i}+41\hat{j}+73\hat{k})-(3\hat{i}-\hat{j}+\hat{k})$$

$$=-\frac{1}{49}(110\hat{i}-90\hat{j}+24\hat{k})$$

Example 7: vectors $\overrightarrow{OA}=2\hat{i}+3\hat{j}+6\hat{k}$ and $OB=\hat{i}-2\hat{j}+2\hat{k}$. C is a point in AB such that OC bisects angle ∠AOB. Find P.V. of C.

Sol. OC bisects ∠AOB in ΔAOB.

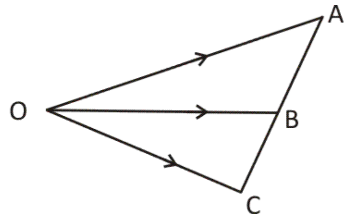


Fig 5

$\overrightarrow{OA}=2\hat{i}+3\hat{j}+6\hat{k}\Rightarrow|\overrightarrow{OA}|=7$

$\overrightarrow{OB}=\hat{i}-2\hat{j}+2\hat{k}\Rightarrow|\overrightarrow{OB}|=3$

∴ BC : CA = 3 : 7

P.V. B, $\hat{i}-2\hat{j}+2\hat{k}$ A, $2\hat{i}+3\hat{j}+6\hat{k}$

$$\text{P.V. of } c=\frac{6i+9j+18k+7i-14j+14k}{10}$$

$$=\frac{13\hat{i}-5\hat{j}+32\hat{k}}{10}$$

P.V. of c = $\frac{1}{10}(13\hat{i}-5\hat{j}+32\hat{k})$

Example 8: $\bar{a}=2\hat{i}+\hat{j}-2\hat{k},\bar{b}=\hat{i}+\hat{j}$ and $\bar{c}$ are vectors between $\hat{a}.\hat{c}=|\hat{c}|$, $\hat{c}-\hat{a}=2\sqrt{2}$, and angle $\hat{a}\times\hat{b}$ and $\hat{c}$ is 30°. Find $\left|(\bar{a}\times\bar{b})\times\bar{c}\right|$.

Sol. $|\hat{c}-\hat{a}|=2\sqrt{2}\Rightarrow|\bar{c}|^2+|\bar{a}|^2-2\bar{a}.\bar{c}=8$

But $\hat{a}.\hat{c}=|\hat{c}|$

$\therefore|\bar{c}|^2+|\bar{a}|^2-2|\bar{c}|=8 \qquad \because|\bar{a}|=3$

$\therefore\ |\bar{c}|^2-2|\bar{c}|+9=8\Rightarrow|\bar{c}|^2-2|\bar{c}|+1=0$

$\therefore\ (|\bar{c}|-1)^2=0\Rightarrow|\bar{c}|=1$

Now $(\bar{a}\times\bar{b})\times\bar{c}=|\bar{a}\times\bar{b}|.|\bar{c}|\sin 30^\circ=\frac{1}{2}.1.|\bar{a}\times\bar{b}|$

$$\bar{a}\times\bar{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -2\\ 1 & 1 & 0\end{vmatrix}=2\hat{i}-2\hat{j}+\hat{k}$$

$\therefore\ |\bar{a}\times\bar{b}|=3$

$\therefore\ [(a\times b)\times c]=\frac{1}{2}.1.3=\frac{3}{2}$

Example 9: $a=2\hat{i}+\hat{j}+\hat{k},\bar{b}=\hat{i}+2\hat{j}-\hat{k}$ and a unit vector $\bar{c}$ are coplanar. If $\bar{c}$ is perpendicular to $\bar{a}$, find $\bar{c}$.

Sol. $|\bar{a}|=\sqrt{6},|\bar{b}|=\sqrt{6}$, $\bar{c}$ is $\perp$ to $\bar{a}$ and lies in the plane of $\bar{a}$ and $\bar{b}$ $\Rightarrow$ $\bar{c}$ is $\perp$ $\bar{a}\times\bar{b}$

$\therefore$ Required vector = $\frac{\bar{a}\times(\bar{a}\times\bar{b})}{|\bar{a}\times(\bar{a}\times\bar{b})|}$

$=\frac{(\bar{a}.\bar{b})\bar{a}-(\bar{a}.\bar{a})\bar{b}}{|\bar{a}\times(\bar{a}\times\bar{b})|}$

$\{\bar{a}.\bar{b}=3 \quad \bar{a}.\bar{a}=6$

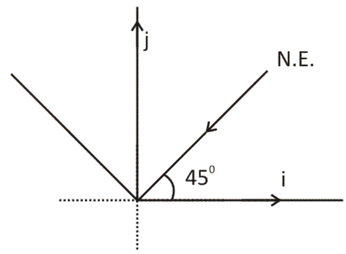


Fig 6

$\bar{c}=\frac{3\bar{a}-6\bar{b}}{|3\bar{a}-6\bar{b}|}=3\hat{a}-6\hat{b}=6\hat{i}+3\hat{j}+3\hat{k}-6\hat{i}-12\hat{j}+6\hat{k}$

$=\frac{-9\hat{i}+9\hat{k}}{9\sqrt{2}}=\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$

Example 10: A man cycling eastwards at 8 km/hr. Find the wind blown from north. On doubling his speed, the wind seems to blow from N.E. Find the velocity of wind

Sol. Let i and j represent velocities of man and wind along East and North.

i = 8 km/hr velocity of cycle.

–j = apparent velocity of wind.

If xi + yj is true velocity of wind, then xi + yj + (–i) = –j $\Rightarrow$ i(x – 1) + yj = –j ... (1)

Speed is doubled i.e. it is now 2i

Now apparent velocity is from N.E. – (i + j)

$\therefore$ xi + yj – 2i = –(i + j) $\Rightarrow$ i(x – 2) + yj = –i –j ... (2)

Solving (1) and (2) xi + yj = i – j

i.e. velocity is $8\sqrt{2}$ km/h from N.W.

Example 11: If $\bar{p}=x(\bar{\ell}\times\bar{m})+y(\bar{m}\times\bar{n})+z(\bar{n}\times\bar{\ell})$ and $[\bar{\ell}\ \bar{m}\ \bar{n}]=\sqrt{3}$ then evaluate x + y + z in terms of $\bar{l},\bar{m},\bar{n},\bar{p}$.

Sol. $\bar{p}=x(\bar{\ell}\times\bar{m})+y(\bar{m}\times\bar{n})+z(\bar{n}\times\bar{\ell})$

$\therefore\ \bar{\ell}.\bar{p} = x[\bar{\ell}\ \bar{\ell}\ \bar{m}] + y[\bar{\ell}\ \bar{m}\ \bar{n}] + z[\bar{\ell}\ \bar{n}\ \bar{\ell}]$

$= y[\bar{\ell}\ \bar{m}\ \bar{n}]$

Similarly $\bar{m}.\bar{p} = z[\bar{m}\ \bar{n}\ \bar{\ell}] = z[\bar{\ell}\ \bar{m}\ \bar{n}]$

$\bar{n}.\bar{p} = x[\bar{n}\ \bar{\ell}\ \bar{m}] = x[\bar{\ell}\ \bar{m}\ \bar{n}]$

$\therefore\ \bar{\ell}.\bar{p} + \bar{m}.\bar{p} + \bar{n}.\bar{p} = (x+y+z)[\bar{\ell}\ \bar{m}\ \bar{n}]$

$\therefore\ (x+y+z) = \frac{1}{\sqrt{3}}(\bar{\ell} + \bar{m} + \bar{n}).\bar{p}$

Example 12: The value of p for which angle between $\bar{a} = 3p^2 i + 5pj + k$ and $\bar{b} = 6i - j - pk$ is obtuse and angle between b and z axis is acute and less than $\pi/6$ arc

(a) $p > \frac{1}{2}$ or $p < 0$ (b) $0 < p < \frac{1}{2}$ (c) $\frac{1}{2} < p < 1$ (d) None of these

Sol. $\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\cos\theta < 0 \Rightarrow \bar{a}.\bar{b} = 18p^2 - 6p < 0 \Rightarrow 0 < p < \frac{1}{3}$

angle between $\bar{b}$ and z axis, $\frac{|\bar{b}|.|\hat{k}|}{|\bar{b}|.1} > 0$ but $\phi < \frac{\pi}{6}$

$\therefore\ \frac{p.1}{\sqrt{p^2+37}} > \frac{\sqrt{3}}{2} \Rightarrow p^2 - 111 > 0, p > \sqrt{111}$

$\therefore$ (d) correct

Example 13: $\bar{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\bar{b} = \hat{i} + 2\hat{j} - 2\hat{k}$; and vector $\bar{c}$ is perpendicular to $\bar{a}$. It is coplanar with $\bar{a}$ and $\bar{b}$ also find a unit vector $\bar{d}$ perpendicular to a and c

Sol. Let $\bar{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{d} = p\hat{i} + q\hat{j} + r\hat{k}$

$$\begin{vmatrix} x & y & z \\ 2 & -1 & 2 \\ 1 & 2 & -2 \end{vmatrix} = 0$$

(i) $\bar{a}, \bar{b}, \bar{c}$ are coplanar

$\therefore\ -2x + 6y + 5z = 0$ … (1)

$\bar{c} \perp \bar{a}\ \therefore\ 2x - y + 2z = 0$ … (2)

$\therefore\ \frac{x}{17}=\frac{y}{14}=\frac{z}{-10}$ $\qquad \therefore\ \bar{c}=17\bar{i}+14\bar{j}-10\bar{z}$

(ii) $\left.\begin{array}{ll}\bar{d}\perp \text{to } \bar{a} & \therefore 2p-q+2r=0 \\ \bar{d}\perp \text{to } \bar{c} & 17p+14q-10r=0\end{array}\right\}, d=pi+qj+rz$

$\therefore\ \frac{p}{-18}=\frac{q}{54}=\frac{r}{45}\Rightarrow\frac{p}{2}=\frac{q}{-6}=\frac{r}{-5}$

$\therefore\ \hat{d}=\frac{1}{\sqrt{4+36+25}}(2i-6j-5k)=\frac{1}{\sqrt{65}}(2i-6j-5k)$

Example 14: $\bar{r}\times\bar{p}=\bar{q}\times\bar{p}$ and $\bar{r}\times\bar{q}=\bar{p}\times\bar{q}, \bar{p}\neq 0$, $\bar{q}\neq 0$ and $\bar{p},\bar{q}$ are non coplanar, $\bar{p}$ is not perpendicular to $\bar{q}$ then $\bar{r}$ =

(a) $\bar{p}+\bar{q}$ (b) $p-\bar{q}$ (c) $\bar{p}\times\bar{q}+\bar{p}$ (d) $\bar{p}\times\bar{q}+\bar{q}$

Sol. $\bar{r}\times\bar{p}=\bar{q}\times\bar{p}\Rightarrow(\bar{r}-\bar{q})\times\bar{p}=0$

$\bar{r}\times\bar{q}=\bar{p}\times\bar{q}\Rightarrow(\bar{r}-\bar{p})\times\bar{q}=0$

$\therefore\ \bar{r}-\bar{p}$ parallel to $\bar{q}$, $\bar{r}-\bar{q}$ parallel to $\bar{p}$

$\therefore\ \bar{r}-\bar{p}=\lambda\bar{q}$ and $(\bar{r}-\bar{q})=\mu\bar{p}$

For pt. of intersection of these lines

$r=\bar{p}+\lambda\bar{q}=\bar{q}+\mu\bar{p}\Rightarrow\lambda=1,\mu=1.$

$\therefore\ \bar{r}=\bar{p}+\bar{q}$

Example 15: If $\bar{p}\times\bar{q}=\bar{r}$ and $\bar{q}\times\bar{r}=\bar{p}$, then (a) $\bar{p},\bar{q},\bar{r}$ are orthogonal in pairs (b) $|\bar{q}|=1, |\bar{p}|=|\bar{r}|$ (c) $\bar{p},\bar{q},\bar{r}$ orthogonal in pairs, but $|\bar{q}|\neq 1$ (d) $|\bar{p}|=1, |\bar{q}|=|\bar{r}|$

Sol. $\left.\begin{array}{l}\bar{p}\times\bar{q}=\bar{r}\perp\bar{p} \text{ and } \bar{r}\perp\bar{q} \\ \bar{q}\times\bar{r}=\bar{p}\Rightarrow\bar{q}+\bar{p} \text{ and } \bar{r}\perp\bar{p}\end{array}\right\}\text{orthogonalinpairs}$

and $\bar{p}\times\bar{q}=\bar{r}\Rightarrow(\bar{q}\times\bar{r})\times\bar{q}=\bar{r}\Rightarrow(\bar{q}\,.\,\bar{q})\bar{r}-(\bar{r}\,.\,\bar{q})\bar{q}=\bar{r}$

But $\bar{r}.\bar{q}=0\Rightarrow|\bar{q}|^2\,\bar{r}=\bar{r}$ $\qquad \Rightarrow(\bar{q}.\bar{q}).\bar{r}-(\bar{r}.\bar{q}).\bar{r}=r$

again $\bar{r}=\bar{p}\times\bar{q}\Rightarrow|\bar{r}|=|\bar{p}\times\bar{q}|=|\bar{q}||\bar{p}|\sin\frac{\pi}{2}$

$\Rightarrow|\bar{r}|=|\bar{p}|.1.1=|\bar{p}|$

$\therefore\ |\bar{q}|=1 \text{ and} |\bar{r}|=|\bar{p}|$

Example 16: Find all values of λ such that (x, y, z) $\neq$ (0, 0, 0) and

$$(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z=\lambda(x\hat{i}+y\hat{j}+z\hat{k})$$

Sol. Given $(\hat{i}+\hat{j}+\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z=-\lambda(x\hat{i}+y\hat{j}+z\hat{k})=0$

$$\Rightarrow (x+3y-4z-\lambda x)i+(x-3y+5z-\lambda y)j+(3x+y-\lambda z)=0$$

i, j, k are non-coplanar vectors.

$$\therefore \left.\begin{aligned}(1-\lambda)x+3y-4z=0\\ x+(-3-\lambda)y+5z=0\\ 3x+y-\lambda z=0\end{aligned}\right\}$$

These homogeneous equation in x, y, z shall have infinite solution. (not x = 0, y = 0, z = 0 solution) if

$$\begin{vmatrix}1-\lambda & 3 & -4\\ 1 & -(3+\lambda) & 5\\ 3 & 1 & -\lambda\end{vmatrix}=0$$

$\Rightarrow (1-\lambda)(-3-\lambda)(-\lambda)-5(1-\lambda)+3[15+\lambda]-4[1+9+3\lambda]=0$

$\Rightarrow (1-\lambda)(3\lambda+\lambda^2)-5+5\lambda+45+3\lambda-40-12\lambda=0$

$\Rightarrow \lambda^3+2\lambda^2+\lambda=0 \Rightarrow \lambda(\lambda+1)^2=0$

$\therefore$ λ can have two values, 0 and –1

Example 17: $\bar{a},\bar{b},\bar{c}$ are mutually perpendicular vectors of equal magnitude. Show that $\bar{a}+\bar{b}+\bar{c}$ is equally inclined to $\bar{a},b,\bar{c}$.

Sol. Given $\bar{a}.\bar{b}=\bar{b}.\bar{c}=\bar{c}.\bar{a}=0$

$\therefore\ (\bar{a}+\bar{b}+\bar{c})^2=\bar{a}^2+\bar{b}^2+\bar{c}^2$

and if $|\bar{a}|=|\bar{b}|=|\bar{c}|=m$ than $\bar{a}^2+\bar{b}^2+\bar{c}^2=3m^2$

$\therefore\ (\bar{a}+\bar{b}+\bar{c})^2=3m^2$ … (1)

Now $\bar{a}.(\bar{a}+\bar{b}+\bar{c})=m\sqrt{3}m.\cos\theta=\bar{a}.\bar{a}+0+0$

$\Rightarrow \sqrt{3}\ m^2\cos\theta=m^2 \Rightarrow \cos\theta=\dfrac{1}{\sqrt{3}}$

$\bar{b}.(\bar{a}+\bar{b}+\bar{c})=|\bar{b}||\bar{a}+\bar{b}+\bar{c}|\cos\phi=\bar{b}.\bar{b}$

$\Rightarrow m\ .\ \sqrt{3}\ m\cos\theta=m^2$

$\therefore \cos\theta=\dfrac{1}{\sqrt{3}} \Rightarrow 0$

$\therefore\ (\overline{a}+\overline{b}+\overline{c})$ is equally inclined to $\overline{a},\overline{b}$ and $\overline{c}$

Example 18: Given $\overline{r}\times\overline{b}=\overline{c}\times\overline{b},\overline{r}.\overline{a}=0$ and $\overline{a}.\overline{b}\neq 0$. What are the geometric meaning of these equation. If the above three holds good simultaneously, then determine vector $\overline{r}$ in terms of $\overline{a},\overline{b}$ and $\overline{c}$

Sol. Let A, B, C and P be points whose vectors are $\overline{a},\overline{b},\overline{c}$ and $\overline{r}$.

(i) $\overline{a}.\overline{b}\neq 0 \Rightarrow \overline{a}\neq 0,\overline{b}\neq 0$ and a not $\perp\overline{b}$

(ii) $r\times b=\overline{c}\times\overline{b}\Rightarrow(\overline{r}-\overline{c})\times\overline{b}=0$

$\overline{r}.\overline{c}$ may be zero or parallel to $\overline{b}$. But $\overline{r}-\overline{c}\neq 0$ as P and C are distinct points (not coincident) $\therefore\ (\overline{r}-\overline{c})$ is $||\overline{b}$

(iii) $\overline{r}.\overline{a}=0\Rightarrow$ either $\overline{r}=0$ or $\overline{a}=0$ or $\overline{r}\perp\overline{a}$

(iv) $\overline{a}\times[(\overline{r}-\overline{c})\times\overline{b}]=\overline{a}\times 0=0\Rightarrow\overline{a}\times[\overline{r}\times\overline{b}-\overline{c}\times\overline{b}]=\overline{a}\times(\overline{r}\times\overline{b})-\overline{a}\times(\overline{c}\times\overline{b})=0$

$\Rightarrow(\overline{a}.\overline{b})\overline{r}-(\overline{a}.\overline{r})\overline{b}-(\overline{a}.\overline{b})\overline{c}+(\overline{a}.\overline{c}).\overline{b}=0$

But $\overline{a}.\overline{r}=0$

$\therefore\ (\overline{a}.\overline{r})\overline{r}=(\overline{a}.\overline{b})\overline{c}-(\overline{a}.\overline{c})\overline{b}$

$$\therefore\ r=\frac{(\overline{a}.\overline{b})\overline{c}-(\overline{a}.\overline{c})\overline{b}}{\overline{a}.\overline{b}}$$

Practice Worksheet (Foundation Level) – Miscellaneous Exercise

1. Vectors $\overline{a}, \overline{b}$ and $\overline{a}+\overline{b}$ are unit vectors, then angle between $\overline{a}$ and $\overline{b}$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) $\frac{-\pi}{3}$

2. In question no. 1 $|\overline{a}-\overline{b}|$ is equal to

(a) 0 (b) $\sqrt{3}$ (c) 2 (d) $\frac{1}{2}$

3. The angle between any two diagonals of a cuboid is

(a) $\cos^{-1}\left(\frac{1}{3}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (c) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (d) none of these

4. The vertices of a quadrilateral are A, (2, 3); B, (5, 1); C, (6, 4) and D (4, 5) angle between diagonals is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

5. The vector area of parallelogram whose adjacent sides are $\overline{a}=\hat{i}+\hat{j}+3\hat{k}, b=-2\hat{i}-3\hat{j}+\hat{k}$ is

(a) $5\sqrt{3}$ (b) $5\sqrt{6}$ (c) $6\sqrt{5}$ (d) $4\sqrt{5}$

6. ABCD is a parallelogram. E in AB divides AB in the ratio of 1 : 2; DE and AC intersect in F. Use vector method and find AF/FC.

(a) 1 : 4 (b) 1 : 3 (c) 2 : 7 (d) 2 : 5

7. The three conterminous edges of a right triangular prism are $\hat{i}+2\hat{j}-3\hat{k}; 2\hat{i}-3\hat{j}+4\hat{k}$ and $3\hat{i}+5\hat{j}-\hat{k}$; Its volume is

(a) 16 sq. u. (b) 28 sq. u. (c) 23 sq. u. (d) 42 sq. u.

8. two adjacent sides of a parallelogram are $\hat{i}+5\hat{j}+4\hat{k}$ and $2\hat{i}-\hat{j}+\hat{k}$. Unit vectors along diagonals are:

(a) $\frac{1}{5\sqrt{2}}(3\hat{i}+4\hat{j}+5\hat{k}), \frac{-1}{\sqrt{46}}(\hat{i}-6\hat{j}-3\hat{k})$ (b) $\frac{1}{5\sqrt{2}}(3\hat{i}+4\hat{j}+5\hat{k}), \frac{1}{\sqrt{46}}(6\hat{i}+\hat{j}-3\hat{k})$

(c) $\frac{1}{5\sqrt{2}}(3\hat{i}+4\hat{j}+5\hat{k}), \frac{1}{\sqrt{46}}(\hat{i}+3\overline{k}-6\hat{j})$ (d) none of these

9. Vectors $\overline{a}-2\overline{b}+3\overline{c}, -2\overline{a}+3\overline{b}-4\overline{c}$ and $\overline{a}-\lambda\overline{b}+5\overline{c}$ are coplanar then λ is

(a) –3 (b) 5 (c) 3 (d) –5

10. Volume of parallelepiped whose conterminous edges are $2i-3j+k, i+j-k$ and $3i+2j+3k$ is:

(a) 30 sq. u. (b) 27 sq. u. (c) 29 sq. u. (d) 26 sq. u.

11. The vertices of a tetrahedron are (1, 1, 1), (5, –1, 2) (–2, 3, 1) and (–3, –2, 4). Its volume is

(a) 4 sq. u. (b) $\frac{25}{6}$sq.u. (c) $\frac{23}{6}$sq.u. (d) $\frac{9}{2}$sq.u.

12. The unit vector perpendicular to vectors $-\bar{i}+2\bar{j}+2\bar{k}$ and $\bar{i}-\bar{j}+3\bar{k}$ forming a right handed system is

(a) $\frac{1}{3\sqrt{10}}(8\hat{i}+5\hat{j}+\hat{k})$ (b) $\frac{-1}{3\sqrt{10}}(8\bar{i}+5\bar{j}-\bar{k})$

(c) $\frac{1}{3\sqrt{10}}(8\bar{i}+7\bar{j}-\bar{k})$ (d) $\frac{1}{3\sqrt{10}}(8\bar{i}+5\bar{j}-\bar{k})$

13. C is mid point of AB. P is any point outside AB then

(a) $\overline{PA}+\overline{PB}=\overline{PC}$ (b) $\overline{PA}+\overline{PB}=2\overline{PC}$ (c) $\overline{PA}+\overline{PB}+\overline{PC}=0$ (d) none of these

14. An unit vector in x – y plane which makes an angle of 45^o with i + j and an angle of 60^o with 3i – 4j is

(a) i (b) $\frac{1}{\sqrt{2}}(i+j)$ (c) $\frac{1}{\sqrt{2}}(i-j)$ (d) None of these

[Hint: Let xi + yj be the unit vector, then $x^2 + y^2 = 1$, x + y = 1, $3x - 4y = \frac{5}{2}$. No real value of x, y satisfy all the three –d]

15. $\bar{a}.i=3$ then $[\bar{a}\times k]$. $[2j-3k]$ =

(a) 9 (b) –6 (c) –9 (d) 6

16. $(\bar{a}\times\bar{b})^2+(\bar{a}.\bar{b})^2=144$ and | a | = 4, then | b | =

(a) 16 (b) 8 (c) 3 (d) 2

17. The unit vector perpendicular to $-\hat{i}+2\hat{j}+2\hat{k}$ and making equal angles with x and y axis is

(a) $\frac{1}{3}(2\hat{i}+2\hat{j}-\hat{k})$ (b) $\frac{1}{3}(-2\hat{i}-2\hat{j}+\hat{k})$

(c) $\frac{1}{\sqrt{17}}(2\hat{i}+3\hat{j}-2\hat{k})$ (d) $\frac{1}{\sqrt{17}}(-2\bar{i}-3\bar{j}+2\bar{k})$

18. A unit vector makes equal angles with vectors, $3i+4j$; $i+2j+2k$ and $4i+3k$. The vector is

(a) $\pm\frac{1}{\sqrt{491}}(17\hat{i}+11\hat{j}+9\hat{k})$ (b) $\pm\frac{1}{\sqrt{491}}(17\hat{i}-11\hat{j}+9\hat{k})$

(c) $\pm\frac{1}{\sqrt{491}}(17\hat{i}+11\hat{j}-9\hat{k})$ (d) $\pm\frac{1}{\sqrt{491}}(17\hat{i}-11\hat{j}-9\hat{k})$

19. $\overline{a},\overline{b}$ and $\overline{c}$ are vectors and $\overline{a}+\overline{b}+\overline{c}=0$ $|\overline{a}|=3,|\overline{b}|=4,|\overline{c}|=5$ then $\overline{a}.\overline{b}+\overline{b}.\overline{c}+\overline{c}.\overline{a}=$

(a) 47 (b) –25 (c) 25 (d) 28

20. $\overline{a},\overline{b},\overline{c}$ are three non-coplanar vectors than $(\overline{a}+\overline{b}+\overline{c}).[(\overline{a}+\overline{b})\times(\overline{a}+\overline{c})]$ is equal to **I.I.T.**

(a) 0 (b) $[\overline{a}\,\overline{b}\,\overline{c}]$ (c) $2[\overline{a}\,\overline{b}\,\overline{c}]$ (d) $-[abc]$

21. D, E, F are mid points of sides BC, CA and AB of triangle ABC. O is any point in the plane of the triangle, then $\overline{OD}+\overline{OE}+\overline{FO}=$

(a) $\overline{OC}$ (b) $\overline{CO}$ (c) $\overline{DF}$ (d) $\overline{OA}$

22. $\begin{vmatrix} \overline{a}.\overline{c} & \overline{a}.\overline{d} \\ \overline{b}.\overline{c} & \overline{b}.\overline{d} \end{vmatrix}=$

(a) $(\overline{a}\times\overline{c}).(\overline{b}\times\overline{d})$ (b) $(\overline{a}\times\overline{d}).(\overline{b}\times\overline{c})$

(c) $(\overline{a}\times\overline{b}).(\overline{c}\times\overline{d})$ (d) none of these

23. In a triangle ABC, $\overline{a},\overline{b}$ and $\overline{c}$ represent $\overline{BC},\overline{CA},\overline{AB}$, then

(a) $\overline{a}+\overline{b}=\overline{c}$ (b) $\overline{b}+\overline{c}=\overline{a}$

(c) $\overline{a}\times\overline{b}=\overline{b}\times\overline{c}$ (d) $[abc]=[bca]=[cab]$

24. $\overline{a}.\overline{b}=0$ and $\overline{a}+\overline{b}$ makes an angle of 60^o with $\overline{a}$ then

(a) $|\overline{b}|=2|\overline{a}|$ (b) $|\overline{a}|=2|\overline{b}|$ (c) $|\overline{b}|=\sqrt{3}|\overline{a}|$ (d) $|\overline{a}|=\sqrt{3}|\overline{b}|$

25. Vectors $\overline{a}=\hat{i}+2\hat{j}-2\hat{k},\overline{b}=2\hat{i}+2\hat{j}+4\hat{k}$ and $\hat{c}=3\hat{i}+2\hat{j}+5\hat{k}$ then magnitude of projection of $\overline{a}\times\overline{c}$ on $\overline{b}$ is

(a) $\frac{5}{\sqrt{6}}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{5}{2\sqrt{3}}$ (d) $\sqrt{3}$

26. $\overline{a},\overline{b},\overline{c}$ are unit vectors such that $\overline{a}.\overline{b}=0=\overline{a}.\overline{c}$ and angle between $\overline{b}$ and $\overline{c}$ is 45^o, $\overline{a}$ =

(a) $2(\overline{b}\times\overline{c})$ (b) $\frac{1}{2}(\overline{b}\times\overline{c})$ (c) $\frac{1}{\sqrt{2}}(\overline{b}\times\overline{c})$ (d) $\sqrt{2}(\overline{b}\times\overline{0})$

27. $\overline{a}\times(\overline{b}\times\overline{c})=(\overline{a}\times\overline{b})\times\overline{c}$ then

(a) $\overline{b}\times(\overline{c}\times\overline{a})=0$ (b) $(\overline{c}\times\overline{a})\times\overline{b}=0$

(c) $\overline{c}\times(\overline{a}\times\overline{b})=0$ (d) none of these

28. Position vector of A and B are $\overline{a}$ and $\overline{b}$. Points C and D divide AB internally & externally in the ratio of 3 : 2, respectively. Vector $\overline{CD}$ is

(a) $\frac{12}{5}(\overline{b}-\overline{a})$ (b) $\frac{12}{5}(\overline{a}-\overline{b})$

(c) $\frac{8}{5}(2\overline{a}+\overline{b})$ (d) $\frac{8}{5}(2\overline{b}+\overline{a})$

29. $\overline{a},\overline{b},\overline{c}$ are non-coplanar unit vectors such that $\overline{a}\times(\overline{b}\times\overline{c})=\frac{1}{\sqrt{2}}(\overline{b}+\overline{c})$, angle between $\overline{a}$ and $\overline{b}$ is

I.I.T.

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

30. Let α, β, γ be real numbers, the points with position vector $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$, $\beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}$ and $\gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$ on

(a) collinear (b) term equilateral triangle

(c) term isosceles triangle (d) form right angled triangle

31. The vector $\frac{1}{3}(2\hat{i}-2\hat{j}+\hat{k})$ is

(a) unit vector (b) is inclined at $\frac{\pi}{3}$ with $2\hat{i}-4\hat{j}+3\hat{k}$

(c) parallel to $\overline{i}+\overline{j}-\frac{1}{2}\overline{k}$ (d) perpendicular to $3\overline{i}+2\overline{j}-2\overline{k}$

32. A unit vector perpendicular to ai + bj + ck and i + j + k is

(a) $\frac{1}{\lambda}\left[(b-c)i+j(c-a)+k(a-b)\right]$ (b) $\frac{1}{\lambda}\left[(b-c)i-(a+c)j+k(a-b)\right]$

(c) $\frac{1}{\lambda}[(c-b)i+j(a-c)+k(b-a)]$ (d) $\frac{1}{\lambda}[bci+caj+abk]$

where $\lambda^2 = 2(a^2 + b^2 + c^2 - bc - ca - ab)$

33. $\overline{\alpha}$ and $\overline{\beta}$ are components of vector $5\hat{i}-2\hat{j}+6\hat{k}$ and along and perpendicular to $2\hat{i}-\hat{j}+2\hat{k}$, then β is

(a) $\frac{1}{3}(16\hat{i}-8\hat{j}+16\hat{k})$ (b) $\frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$

(c) $\frac{1}{3}(-\hat{i}-2\hat{j}+2\hat{k})$ (d) $\frac{1}{3}(-\hat{i}+2\hat{j}-2\hat{k})$

34. $\overline{a}=\hat{i}+\hat{j}+\hat{k}$, $\overline{b}=4\hat{i}+3\hat{j}+\hat{k}$, $\overline{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ and $|c|=\sqrt{3}$ and if these are linearly dependent vectors then

(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 1, \beta -1$ (c) $\alpha = -1, \beta = 1$ (d) $\alpha = \pm 1, \beta = 1$

35. $\overline{a}$ and $\overline{b}$ are two non-zero vectors. $\overline{a}$ is perpendicular to $\overline{b}$ of

(a) $|\overline{a}+\overline{b}|=|\overline{a}-\overline{b}|$ (b) $(\overline{a}+\overline{b}).(\overline{a}-\overline{b})=0$

(c) (a + b) | | (a – b) (d) none of these

36. $\overline{a}=\hat{i}+2\hat{j}-3\hat{k}$, $\overline{b}=3\hat{i}-\hat{j}+2\hat{k}$, $\overline{c}=2\hat{i}-3\hat{j}+\hat{k}$, then angle between $(\overline{a}\times\overline{b})$ and $(\overline{b}\times\overline{c})$ is

(a) $\cos^{-1}\left(\frac{43}{12\sqrt{57}}\right)$ (b) $\cos^{-1}\frac{\sqrt{3}}{\sqrt{57}}$

(c) $\cos^{-1}\frac{43}{15\sqrt{57}}$ (d) $\cos^{-1}\frac{43}{13\sqrt{57}}$

37. $(\overline{a}+2\overline{b}-\overline{c}).[(\overline{a}-\overline{b})\times(\overline{a}-\overline{b}-\overline{c})]=$

(a) $[\overline{a}\,\overline{b}\,\overline{c}]$ (b) $2[\overline{a}\,\overline{b}\,\overline{c}]$ (c) $3[\overline{a}\,\overline{b}\,\overline{c}]$ (d) $4[\overline{a}\,\overline{b}\,\overline{c}]$

38. Vector $\overline{c}$ is perpendicular to $2\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}-2\hat{j}+3\hat{k}$ and $c.[\hat{i}+2\hat{j}-7\hat{k}]$ = 10 then $\overline{c}$ is

(a) $7\hat{i}+5\hat{j}+\hat{k}$ (b) $7\hat{i}-5\hat{j}+\hat{k}$ (c) $7\hat{i}+5\hat{j}-\hat{k}$ (d) $7\hat{i}-5\hat{j}-\hat{k}$

39. The points with position vector $60\hat{i}+3\hat{j}$, $40\hat{i}+8\hat{j}$ and $a\hat{i}+5\hat{j}$ are collinear then a is

(a) –50 (b) 30 (c) 52 (d) 40

40. If $\overline{p}=3\hat{i}+2\hat{j}+\hat{k}$ and $\overline{q}=4\hat{j}-8\hat{k}$ and $\overline{p}\times\overline{a}=\overline{q}$ and $\overline{p}.\overline{a}=-2$ then $\hat{a}$ is

(a) $\frac{1}{\sqrt{6}}(\hat{i}-2\hat{j}+\hat{k})$ (b) $\frac{1}{\sqrt{6}}(\hat{i}-2\hat{j}-\hat{k})$

(c) $\frac{1}{\sqrt{30}}(5\hat{i}+2\hat{j}+\hat{k})$ (d) $\frac{1}{\sqrt{33}}(5\hat{i}-2\hat{j}+2\hat{k})$

41. The force $F=3\hat{i}+4\hat{j}-2\hat{k}$ acts through $\hat{i}+\hat{j}+\hat{k}$ and –F acts through $2\hat{i}-\hat{j}-\hat{k}$. The moment of couple is

(a) $-12\hat{i}+4\hat{j}-10\hat{k}$ (b) $12\hat{i}-4\hat{j}+10\hat{k}$

(c) $12\hat{i}+4\hat{j}+10\hat{k}$ (d) $12\hat{i}-4\hat{j}-10\hat{k}$

42. If $\bar{a}.\bar{b}=\bar{a}.\bar{c}$ and $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$ then

(a) $\bar{a}$ parallel $\bar{b}-\bar{c}$ (b) $\bar{a}$ perpendicular $(\bar{b}-\bar{c})$

(c) either $\bar{a}$ = 0 or $\bar{b}=\bar{c}$ (d) none of these

43. The unit vector perpendicular to plane through (1, 3, 2), (0, 1, 4) and (–1, –2, 0) is

(a) $\frac{1}{\sqrt{233}}(14\hat{i}-6\hat{j}+\hat{k})$ (b) $\frac{1}{\sqrt{233}}(14\hat{i}+6\hat{j}-\hat{k})$

(c) $\frac{1}{\sqrt{233}}(14\hat{i}+6\hat{j}+\hat{k})$ (d) $\frac{1}{7}(6\hat{i}-2\hat{j}+3\hat{k})$

44. $\bar{a}\times\bar{b}=\bar{c}\times\bar{d}$ and $\bar{a}\times\bar{c}=\bar{b}\times d$, then angle between $(\bar{a}-\bar{d})$ and $(\bar{b}-\bar{c})$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (b) 0 (d) not known

45. Equation of a plane 5 units away from origin and perpendicular to vector $2\hat{i}+6\hat{j}-3\hat{k}$ is

(a) $\bar{r}.(2\hat{i}+6\hat{j}-3\hat{k})$ =5 (b) $\bar{r}.(2\hat{i}+6\hat{j}-3\hat{k})=35$

(c) $\bar{r}.(-2\hat{i}-6\hat{j}+3\hat{k})=35$ (d) $\bar{r}.(-2\hat{i}-6\hat{j}+3\hat{k})=5$

46. The vertices of triangle ABC are A(2, 3, 4) B (4, 5, 5) and C (4, 1, 5). The equation of internal bisector of angle A is

(a) $r=\frac{2\lambda}{3}(2\hat{i}+\hat{k})$ (b) $r=\frac{2\lambda}{3}(2\hat{j}+\hat{k})$

(c) $\bar{r}=\frac{4}{3}\lambda\hat{j}$ (d) none of these

47. The equation of plane through (5, 2, –3) and perpendicular to planes $\bar{r}.(2\hat{i}-\hat{j}+\hat{k})=9$ and $\bar{r}.(\hat{i}+3\hat{j}-5\hat{k})+3=0$ is

(a) $\bar{r}.(3\hat{i}-11\hat{j}+9\hat{k})=10$ (b) $\bar{r}.(2\hat{i}+11\hat{j}+9\hat{k})=9$

(c) $\bar{r}.(2\hat{i}+11\hat{j}+7\hat{k})=11$ (d) $\bar{r}.(-2\hat{i}-11\hat{j}-7\hat{k})=19$

48. The vertices of triangle are (0, 2, 3), (4, 0, 3) and (4, 2, 0). The length of perpendicular from origin on the plane of triangle is

(a) $\frac{24}{\sqrt{61}}$ (b) $\frac{12}{\sqrt{61}}$ (c) 3 (d) $\frac{48}{\sqrt{61}}$

49. A plane is $\sqrt{2}$ distance away from origin. It is perpendicular to planes $\bar{r}.(i-2j+k)+7=0$. Its equation is

(a) $\bar{r}.(3\hat{i}+4\hat{j}+5\hat{k})=5$ (b) $\bar{r}.(3\hat{i}+4\hat{j}+5\hat{k})=\sqrt{2}$

(c) $\bar{r}.(3\hat{i}+4\hat{j}+5\hat{k})=10$ (d) none of these

50. The equation of plane which passes through (–1, –1, –1) and through the line of intersection of the planes $\bar{r}.(\hat{i}+3\hat{j}-\hat{k})=0$ and $\bar{r}.(\hat{j}+2\hat{k})=0$

(a) $\bar{r}.(\hat{i}+2\hat{j}+\hat{k})+4=0$ (b) $\bar{r}.(\hat{i}+2\hat{j}-\hat{k})+2=0$

(c) $\bar{r}..(\hat{i}+2\hat{j}-3\hat{k})=0$ (d) $\bar{r}.(\hat{i}+2\hat{j}+3\hat{k})+6=0$

51. The equation of a plane which goes through (5, –1, 3) and is parallel to vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ is

(a) $\bar{r}.(11\hat{i}+5\hat{j}-7\hat{k})=29$ (b) $\bar{r}.(11\hat{i}+5\hat{j}+7\hat{k})=71$

(c) $\bar{r}.(5\hat{i}-13\hat{j}+7\hat{k})=59$ (d) $\bar{r}.(5\hat{i}+5\hat{j}-7\hat{k})=-1$

52. The vector equation of a plane which contains st. lines $\bar{r}=20+j-3k+\lambda(i+2j+5k)$ is

(a) $\bar{r}.(10i+5j-4k)=13$ (b) $\bar{r}.(20i-10j-4k)=28$

(c) $\bar{r}.(10i+5j-4k)=37$ (d) none of these

53. The equation of a plane through (2, –4, 5) and perpendicular to the line of intersection of planes $\bar{r}.(3i+j-2k)=2$ and $\bar{r}.(2i+3j-4k)=1$

(a) $\bar{r}.(2i+8j-7k)+63=0$ (b) $\bar{r}.(2i-8j+7k)=7$

(c) $\bar{r}.(-2i-8j+7k)=-7$ (d) $\bar{r}.(2i-8j+7k)=7$

54. The equation of a plane through A (2, 3, –1), B (4, 5, 2) and C (3, 6, 5) is

(a) $\bar{r}.(3\hat{i}-9\hat{j}+4\hat{k})+25=0$ (b) $\bar{r}.(3\hat{i}+9\hat{j}-4\hat{k})=37$

(c) $\bar{r}.(3\hat{i}+9\hat{j}+4\hat{k})=27$ (d) $\bar{r}.(3\hat{i}-9\hat{j}-4\hat{k})=-17$

55. Equation of the plane perpendicular to planes $\bar{r}.(\hat{i}+\hat{j}+\hat{k})=3$ and $\bar{r}.(2\hat{i}+3\hat{j}+4\hat{k})=5$ and whose distance from origin is $\sqrt{6}$ is

(a) $\bar{r}.(\hat{i}-2\hat{j}+\hat{k})=\sqrt{6}$ (b) $\bar{r}.(\hat{i}-2\hat{j}+\hat{k})=6$

(c) $\bar{r}.(\hat{i}+2\hat{j}+\hat{k})=\sqrt{6}$ (d) $\bar{r}.(\hat{i}+2\hat{j}-\hat{k})=6$

56. A unit vector perpendicular to plane which passes through P(2, 1, –2), Q (3, 2, 1) and P (0, –1, –3) is

(a) $\frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$ (b) $\frac{1}{\sqrt{2}}(-\hat{i}+\hat{j})$ (c) $\frac{1}{\sqrt{2}}(\hat{i}-\hat{j})$ (d) $\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$

57. The equation on of plane through (1, –1, 1)and passing through the line of intersection of planes $\bar{r}.(3\hat{i}+\hat{j}-\hat{k})=7$ and $\bar{r}.(\hat{i}-2\hat{j}+4\hat{k})=9$ is

(a) $\bar{r}.(7\hat{j}-13\hat{k})+20=05$ (b) $\bar{r}.(5\hat{i}+7\hat{j}-13\hat{k})+15=0$

(c) $\bar{r}.(7\hat{j}+13\hat{k})=6$ (d) $\bar{r}.(5\hat{i}-7\hat{j}+13\hat{k})=25$

58. Straight lines $\bar{r}=\hat{i}+2\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}-\hat{k})$ and $\bar{r}=3\hat{i}+\hat{j}+2\hat{k}+\mu(a\hat{i}-3\hat{j}+2\hat{k})$ are coplanar then a =

(a) 1 (b) –1 (c) 2 (d) –3

59. The shortest distance between straight lines $\bar{r}=4\hat{i}-3\hat{j}+2\hat{k}+\lambda(-\hat{i}+2\hat{j}-3\hat{k})$ and $\bar{r}=\hat{i}+\hat{j}-3\hat{k}+\mu(5\hat{i}-3\hat{j}+4\hat{k})$ is

(a) $\frac{6}{\sqrt{171}}$ (b) $\frac{3}{5\sqrt{7}}$ (c) $\frac{2}{3\sqrt{19}}$ (d) $\frac{12}{\sqrt{171}}$

60. A is (1, –1, 2) and B is (3, 1, 4). The equation of plane which bisects AB is perpendicular to AB is

(a) $\bar{r}.(\hat{i}-\hat{j}+\hat{k})=5$ (b) $\bar{r}.(\hat{i}+\hat{j}-\hat{k})=5$

(c) $\bar{r}.(\hat{i}+\hat{j}+\hat{k})=5$ (d) $\bar{r}.(\hat{i}+\hat{j}+\hat{k})=10$

61. The straight line $\bar{r}=(i-j-2k)+\lambda(2i-j-2k)$ meets the plane $\bar{r}.(2\hat{i}+2\hat{j}-\hat{k})=10$ in pt.

(a) (–5, 3, –6) (b) (5, –3, –6) (c) (5, 0, 0) (d) (2, 3, 0)

62. The image of the point (1, –1, –2) in the plane $\bar{r}.(2\hat{i}+2\hat{j}-\hat{k})=10$ is

(a) (–1, 1, 2) (b) $\frac{41}{9}, \frac{23}{9}, -\frac{34}{9}$

(c) $\frac{25}{9}, \frac{7}{9}, \frac{-26}{9}$ (d) none of these

63. The image of the straight line $\bar{r} = (\hat{i} - \hat{j} - 2\hat{k}) + \lambda(2\hat{i} - \hat{j} - 2\hat{k})$ in the plane $\bar{r}.(2\hat{i} + 2\hat{j} - \hat{k}) = 10$ is

(a) $\bar{r} = (5\hat{i} - 3\hat{j} - 6\hat{k}) + \lambda(-2\hat{i} + 25\hat{j} + 10\hat{k})$ (b) $\bar{r} = (5\hat{i} - 3\hat{j} - 6\hat{k}) + \lambda(2\hat{i} + 25\hat{j} + 10\hat{k})$

(c) $\bar{r} = (5\hat{i} - 3\hat{j} - 6\hat{k}) + \lambda(2\hat{i} - 25\hat{j} + 10\hat{k})$ (d) none of these

64. The equation of a plane through (3, 2, –2) and (1, –3, 4) and parallel to y axis is

(a) $\bar{r}.(3\hat{i} + \hat{k}) + 7 = 0$ (b) $\bar{r}.(3\hat{i} + \hat{k}) = 7$

(c) $\bar{r}.(3\hat{i} - \hat{k}) = 11$ (d) $\bar{r}.(2\hat{i} - 3\hat{k}) = 12$

65. Point P is (2, 3, 4) PM, PN and PT are perpendiculars from P on x-y, y-z and z-x planes. The equation on the plane through M, N, T is

(a) $\bar{r}.(6\hat{i} + 4\hat{j} - 3\hat{k}) = 0$ (b) $\bar{r}.(6\hat{i} - 4\hat{j} + 3\hat{k}) = 12$

(c) $\bar{r}.(6\hat{i} - 4\hat{j} - 3\hat{k}) + 12 = 0$ (d) $\bar{r}.(6\hat{i} + 4\hat{j} + 3\hat{k}) = 24$

66. A plane cuts intercepts of 5, 3 and –2 an axes. Its distance from origin is …

(a) $\frac{30}{17}$ (b) $\frac{15}{17}$ (c) $\frac{30}{19}$ (d) $\frac{15}{19}$

67. The vector equation of a plane passing through $\bar{a}, \bar{b}$ and $\bar{c}$ is $\bar{r} = \alpha\bar{a} + \beta\bar{b} + \gamma\bar{c}$, then

(a) $\alpha + \beta + \gamma = 0$ (b) $\alpha + \beta + \gamma = 1$ (c) $\alpha + \beta = \gamma$ (d) $\alpha^2 + \beta^2 + \gamma^2 = 1$

68. The equation of a plane containing straight lines $\bar{r} - \bar{a} = t\bar{b}$ and $\bar{r} - \bar{b} = s\bar{a}$ is

(a) $[\bar{r}\,\bar{a}\,\bar{b}] = 0$ (b) $\bar{r}.\bar{a} = \bar{a}.\bar{b}$ (c) $\bar{r}.\bar{a} = \bar{r}.\bar{b}$ (d) $\bar{r}.\bar{b} = \bar{a}.\bar{b}$

69. Vector $a\hat{i} + b\hat{j} + c\hat{k}$ shall bisect the angle between $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$ if

(a) a = b (b) a = c (c) c = a + b (d) b = a + c

70. The point of intersection of planes through (3, 1, –2), (4, 0, 3) and (4, –1, –2) and the straight line through $\hat{i} + 2\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} - 2\hat{k}$ is

(a) $\left(10, -\frac{23}{2}, \frac{11}{2}\right)$ (b) (3, –1, –12)

(c) (5, –4, –7) (d) $\left(\frac{17}{3}, -\frac{16}{3}, -7\right)$

71. $\bar{a}, \bar{b}$ and $\bar{c}$ are non co-planar unit vectors and $\bar{a}\times(\bar{b}\times\bar{c}) = \frac{1}{\sqrt{2}}\bar{b} + \frac{\sqrt{3}}{2}\bar{c}$ angle between $\bar{a}$ and $\bar{b}$ is

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

72. $\bar{p}, \bar{q}, \bar{r}$ and $\bar{s}$ are outward drawn normals of faces of tetrahedron and equal to area of respective forces in magnitude, then $\bar{p}+\bar{q}+\bar{r}+\bar{s} =$

(a) 1 (b) Sum of areas of all four faces

(c) 0 (d) 4 AG, when A is vertex G centroid

73. $\bar{p} = i + j + k$ and $\bar{q} = i + j - k$ and $\bar{a}, \bar{b}$ are two vectors such that $\bar{p} = 2\bar{a} + \bar{b}$ and $\bar{q} = \bar{a} + 2\bar{b}$; then an angle between $\bar{a}$ and $\bar{b}$ is

(a) $\cos^{-1}\left(\frac{7}{9}\right)$ (b) $\cos^{-1}\left(\frac{7}{11}\right)$ (c) $\cos^{-1}\left(-\frac{7}{10}\right)$ (d) $\cos^{-1}\left(\frac{6\sqrt{2}}{10}\right)$

74. Vectors $\bar{a} = x\hat{i} + (x+1)\hat{j} + (x+2)\hat{k}$, $\bar{b} = (x+3)\hat{i} + (x+4)\hat{j} + (x+5)\hat{k}$ and $\bar{c} = (x+6)\hat{i} + (x+7)\hat{j} + (x+8)\hat{k}$ are co-planar for

(a) x , 0 (b) x > 0 (c) for all x (d) x = 1 only

75. The vector $3\hat{i} + 7\hat{j} - 4\hat{k}$ bisects the angle between $\bar{c}$ and $3\hat{j} - 4\hat{k}$ then unit vector $\bar{c}$ is

(a) $\frac{1}{5}(4i+3j)$ (b) $\frac{1}{5}(3i+4j)$ (c) $\frac{1}{5}(3i-4j)$ (d) none of these

76. Vector $\bar{a} = \hat{i} + \hat{j} + \hat{k}, \bar{b} = 2\hat{i} - \hat{j} + 2\hat{k}$, find vector $\bar{c}$ which is (i) coplanar with $\bar{a}$ and $\bar{b}$, (ii) perpendicular to $\bar{a}$ and (iii) $\bar{c}.\bar{b} = -6$

(a) $\hat{i} - 2\hat{j} + \hat{k}$ (b) $-\hat{i} + 2\hat{j} - \hat{k}$ (c) $2\hat{i} - \hat{j} - \hat{k}$ (d) $2\hat{i} + \hat{j} + 2\hat{k}$

77. A vector of magnitude $5\sqrt{6}$ is directed along the bisector of angle between $\bar{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\bar{b} = -2\hat{i} - \hat{j} + 2\hat{k}$ is

(a) $\pm\frac{5}{3}(\hat{i} - 7\hat{j} + 2\hat{k})$ (b) $\pm\frac{5}{3}(\hat{i} - 2\hat{j} + 7\hat{k})$

(c) $\pm\frac{5}{3}(2\hat{i} - \hat{j} + 7\hat{k})$ (d) $\pm\frac{5}{3}(2\hat{i} - 7\hat{j} - \hat{k})$

78. Let vectors $\bar{p}, \bar{q}$ and $\bar{a}, \bar{b}$, each determine a plane. These two planes shall be parallel if

(a) $(\bar{p}\times\bar{a})\times(\bar{q}\times\bar{b})=0$ (b) $(\bar{p}\times\bar{b})\times(\bar{q}\times\bar{a})=0$

(c) $(\bar{p}\times\bar{q})\times(\bar{a}\times\bar{b})=0$ (d) $(\bar{p}\times\bar{q})\cdot(\bar{a}\times\bar{b})=0$

79. Vectors $(9\hat{i}+3\hat{j}),(5\hat{i}+6\hat{j}),(8\hat{i}+\lambda\hat{j})$ have their initial point at (1, –1). If vectors terminate on the same line then λ =

(a) 7 (b) 8 (c) 9 (d) 10

80. The straight line joining $3\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}-2\hat{j}+3\hat{k}$ meets the plane y – z in point

(a) $5\hat{j}+7\hat{k}$ (b) $-5\hat{j}+7\hat{k}$ (c) $5\hat{j}-7\hat{k}$ (d) $-5\hat{j}-7\hat{k}$

81. The straight line $r=a\hat{i}+3\hat{j}-4\hat{k}+\lambda(\hat{i}-b\hat{j}+\hat{k})$ shall lie in plane $r.(4\hat{i}-2\hat{j}+\hat{k})=2$, if

(a) a = 3, b = $\frac{5}{2}$ (b) a = –3, b = $\frac{5}{2}$ (c) a = –3, b = $-\frac{5}{2}$ (d) a = 3, b = $-\frac{5}{2}$

82. The locus of a point equidistant from (3, 1, 4) and (5, –2, 3) is

(a) $\hat{r}.(2\hat{i}+3\hat{j}-\hat{k})+1=0$ (b) $\bar{r}.(2\hat{i}-3\hat{j}-\hat{k})=6$

(c) $r.(2\hat{i}+3\hat{j}+\hat{k})=10$ (d) $\hat{r}.(2\hat{i}-3\hat{j}-\hat{k})+6=0$

83. $\bar{a}=2\hat{i}+\hat{j}-\hat{k},\ \bar{b}=\hat{i}+2\hat{j}+\hat{k}$. What is that vector which lies in the plane of $\bar{a}$ and $\bar{b}$ and whose projection on $\bar{a}$ is equal to $3\sqrt{3/2}$

(a) $3(\hat{i}+\hat{j})$ (b) $3(\hat{i}+\hat{j}+\hat{k})$ (c) $(\hat{i}-\hat{j}+\hat{k})$ (d) $3(\hat{i}+\hat{k})$

84. Vertices of a tetrahedron are A (–1, 2, –1) . B (1, 1, 1), C (2, –1, 2) and D (–2, 4, 1) angle between faces ABC and ACD is

(a) $\cos^{-1}\left(\frac{-5}{2\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)$ (c) $\cos^{-1}\left(\frac{5}{\sqrt{13}}\right)$

85. Vertices of a tetrahedron are A (–1, 2, –1) B (2, 1, 2), C (2, –1, 2) and D (–2, 4, 3) angle between faces ABC and ACD is

(a) $\cos^{-1}\left(\frac{7}{2\sqrt{31}}\right)$ (b) $\cos^{-1}\left(\frac{5}{2\sqrt{31}}\right)$ (c) $\cos^{-1}\left(\frac{9}{2\sqrt{3}}\right)$ (d) none of these

86. Points P, Q, R, S are (1, 1, 1), (–1, 1, 1) and (1, 1, –1) and (1,-1,1)

(A) PQRS is

(a) quadrilateral (b) pyramid (c) plane (d) none of these

(B) The volume of PQRS is

(a) 2.5 cu. v. (b) $\frac{4}{3}$ cu. v. (c) $\frac{8}{3}$cu.v. (d) none of these

The triangle QRS is

(a) Isosceles (b) right angled (c) equilateral (d) scalene

87. a body is rotating about a vertical axis through $2\hat{i}+3\hat{j}-2\hat{k}$ with an angular velocity of $3\hat{i}-\hat{j}+2\hat{k}$ the velocity of a point at (4, 3, 4) is

(a) $6\hat{i}+14\hat{j}-2\hat{k}$ (b) $6\hat{i}-14\hat{j}-2\hat{k}$ (c) $3\hat{i}-7\hat{j}+\hat{k}$ (d) $6\hat{i}+22\hat{j}+2\hat{k}$

88. A right body is rotating about a vertical axis through the point (2, −1, 3). Particles at points (5, 2, 0) and (3, 3, 2) have velocities $6\hat{i}-9\hat{j}-3\hat{k}$ and $5\hat{i}-3\hat{j}-7\hat{k}$ angular velocity is

(a) $\hat{i}+2\hat{j}+\hat{k}$ (b) $\hat{i}+\hat{j}+2\hat{k}$ (c) $2\hat{i}+\hat{j}+\hat{k}$ (d) none of these

89. $\bar{a}=\hat{i}+\hat{j}+2\hat{k}$, $\bar{b}=4\hat{i}+3\hat{j}+8\hat{k}$ and $\bar{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linear dependent vectors, if | c | = 3, then

(a) $\alpha = 2, \beta = -2$ (b) $\alpha = 2, \beta = 2$ (c) $\alpha = 2, \beta = \pm 2$ (d) $\alpha = \pm 2, \beta = 2$

90. $\bar{a}=2\hat{i}+3\hat{j}-4\hat{k}, \bar{b}=3\hat{i}-2\hat{j}+\hat{k}$ unit vector perpendicular to $\bar{a}+\bar{b}$ and $\bar{a}-\bar{b}$ is

(a) $\frac{(5\hat{i}+14\hat{j}+13\hat{k})}{\sqrt{390}}$ (b) $\frac{(5\hat{i}-14\hat{j}+13\hat{k})}{\sqrt{390}}$

(c) $\frac{(15\hat{i}+4\hat{j}+3\hat{k})}{5\sqrt{10}}$ (d) $\frac{(5\hat{i}+4\hat{j}-3\hat{k})}{5\sqrt{2}}$

91. Vector $\bar{a}$ whose projection on vector $2\hat{i}-2\hat{j}+\hat{k}$, $-8\hat{i}+\hat{j}+4\hat{k}$ and $\hat{i}-2\hat{j}+2\hat{k}$ are $\frac{2}{3},\frac{3}{3}$ and $\frac{4}{3}$ respectively is

(a) $3\hat{i}-2\hat{j}+4\hat{k}$ (b) $2\hat{i}-3\hat{j}+4\hat{k}$ (c) $3\hat{i}+2\hat{j}+4\hat{k}$ (d) $2\hat{i}+3\hat{j}+4\hat{k}$

92. Straight lines $\bar{r}=(3\hat{i}-\hat{j}+2\hat{k})+\lambda(\hat{i}+2\hat{j}+\hat{k})$ and $\bar{r}=(\hat{i}+2\hat{j}-3\hat{k})+\mu(3\hat{i}-\hat{j}+6\hat{k})$ intersect in

(a) $4\hat{i}+\hat{j}+3\hat{k}$ (b) $4\hat{i}-\hat{j}-3\hat{k}$ (c) $-4\hat{i}+\hat{j}+3\hat{k}$ (d) $3\hat{i}+4\hat{j}+\hat{k}$

93. If scalar product of $3i-j+k$ with the unit vector parallel to the sum of $i+bj-k$ and $i-j-k$ is unity, then

(a) b = 1 (b) b = 2 (c) b = –2 (d) b = –1

94. Vector $\bar{a}$ makes equal angles with vectors $\frac{1}{5\sqrt{2}}(3\hat{i}+4\hat{j}-5\hat{k})$, $(4\hat{i}-3\hat{j}-5\hat{k})\frac{1}{5\sqrt{2}}$ and $\frac{1}{\sqrt{134}}(2\hat{i}+7\hat{j}-9\hat{k})$ and its magnitude is $\sqrt{51}$, vector a is

(a) $5\hat{i}-\hat{j}+5\hat{k}$ (b) $\hat{i}-5\hat{j}+5\hat{k}$ (c) $7\hat{i}+\hat{j}-\hat{k}$ (d) $\hat{i}+\hat{j}-7\hat{k}$

95. Straight lines $\bar{r}=2\hat{i}+3\hat{j}+\hat{k}+\lambda(a\hat{i}+2\hat{j}+3\hat{k})$ and $\bar{r}=3\hat{i}+\hat{j}+7\hat{k}+\mu(2\hat{i}+\hat{j}+2\hat{k})$ are coplanar then a is

(a) $\frac{3}{2}$ (b) $\frac{-7}{2}$ (c) $\frac{5}{2}$ (d) $\frac{7}{2}$

96. $\bar{b},\bar{c}$ and $\bar{d}$ are non-coplanar vectors then, $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})+(\bar{a}\times\bar{c})\times(\bar{d}\times\bar{b})+(\bar{a}\times\bar{d})\times(\bar{b}\times\bar{c})$ is

(a) parallel to $\bar{b}$ (b) parallel to $\bar{c}$

(c) parallel to $\bar{a}$ (d) none of these

97. Straight lines $r=(i+j+k)+\lambda(2i-3j+4k) \text{and} r=(a^2i+aj+k)+\mu(3i+2j-k)$ are coplanar, then

(a) a = 1, $\frac{9}{5}$ (b) a = –1, $-\frac{9}{5}$ (c) a = 1, $\frac{9}{5}$ (d) a = 1, $-\frac{9}{5}$

98. $\bar{a}=\hat{i}+\hat{j}+2\hat{k}, b=3\hat{i}-2\hat{j}+\hat{k}, c=2\hat{i}-\hat{j}+4\hat{k}$. Find vector $\bar{d}$ such that it is perpendicular to $\bar{a}$ and $\bar{b}$ and $\bar{c}.\bar{d}=-1$

(a) $\frac{1}{3}(\hat{i}-\hat{j}+\hat{k})$ (b) $\frac{1}{3}(i+\hat{j}-\hat{k})$ (c) $\frac{1}{6}(\hat{i}+\hat{j}-\hat{k})$ (d) $\frac{1}{6}(\hat{i}-\hat{j}-\hat{k})$

99. A unit vector in the plane of vectors $\bar{a}=\hat{i}+2\hat{j}$ and $\bar{b}=\hat{j}+2\hat{k}$ and perpendicular to $\bar{c}=2\hat{i}+\hat{j}+2\hat{k}$ is

(a) $\frac{1}{5\sqrt{5}}(-5\hat{i}-6\hat{j}+8\hat{k})$ (b) $\frac{1}{5\sqrt{5}}(5\hat{i}+6\hat{j}-8\hat{k})$

(c) $\frac{1}{5\sqrt{5}}(5\hat{i}-6\hat{j}+8\hat{k})$ (d) $\frac{1}{5\sqrt{5}}(5\hat{i}+6\hat{j}+8\hat{k})$

100. A unit vector $\bar{r}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and θ with $\hat{k}$, then $\bar{r}$ and θ are

(a) $\frac{1}{2}(\sqrt{2}i + j + \sqrt{2}k), \frac{\pi}{3}$ (b) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j} + 2\hat{k}), \frac{\pi}{6}$

(c) $\frac{1}{2}(\hat{i} + \sqrt{2}\hat{j} + \hat{k}), \frac{\pi}{3}$ (d) $\frac{1}{2}(\sqrt{2}\hat{i} - \hat{j} + \sqrt{2}\hat{k}), \frac{\pi}{6}$

101. The three coterminous edges of a parallelepiped are $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 4\hat{j} - \hat{k}$ and $\hat{i} + \hat{j} + 3\hat{k}$. The first two form the base, then height of parallelepiped is

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{8}{\sqrt{38}}$ (d) $\frac{4}{\sqrt{38}}$

102. The distance of point P (–1, –5, –10) from the point of intersection of straight line $\bar{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\ (3\hat{i} + 4\hat{j} + 2\hat{k})$ and plane $\bar{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$ is

(a) $\sqrt{89}$ (b) $\sqrt{131}$ (c) 15 (d) 13

103. Equation of the plane containing straight lines $\bar{x} - \bar{p} = \pm\bar{q}$ and $\bar{r} - \bar{q} = \lambda\bar{p}$ is

(a) $[\bar{r}\,\bar{p}\,\bar{q}] = 0$ (b) $\bar{r}.\bar{p} = \bar{p}.\bar{q}$ (c) $\bar{r} \times \bar{p} = (\bar{p} + \bar{q}) \times \bar{a}$ (d) $\bar{r} \times \bar{q} = \bar{p} \times \bar{q}$

104. $\bar{r}$ is a vector such that $\bar{r} \times (2\hat{i} - \hat{j} + 3\hat{k}) = 2\hat{i} - 11\hat{j} - 5\hat{k}$. If $|r| = \sqrt{11}$ then $\bar{r} =$

(a) $3\hat{i} - \hat{j} + \hat{k}$ (b) $\hat{i} - 3\hat{j} + \hat{k}$ (c) $3\hat{i} + \hat{j} - \hat{k}$ (d) $\hat{i} + \hat{j} - 3\hat{k}$

105. A (0, 0), B (4, $4\sqrt{3}$) and C ($-4\sqrt{3}, 4$) are vertices of a triangle, whose circumcentre is

(a) $2(1 - \sqrt{3})\hat{i} + 2(\sqrt{3} + 1)\hat{j}$ (b) $2(\sqrt{3} - 1)\hat{i} - 2(\sqrt{3} + 1)\hat{j}$

(c) $2(\sqrt{3} + 1)\hat{i} + 2(\sqrt{3} - 1)\hat{j}$ (d) none of these

106. $\bar{a} + 3\bar{b} + 4\bar{c} = 0$ and $\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a} = \lambda(\bar{b} \times \bar{c})$ then λ =

(a) 6 (b) 8 (c) 10 (d) 4

107. $\bar{a} = 2\hat{i} + \hat{j} + 2\hat{k}, \bar{b} = 3\hat{i} - \hat{j} + \hat{k}$, vector $\bar{r}$ is such that $\bar{a} \times \bar{r} = \hat{i} + 4\hat{j} - 3\hat{k}, \bar{r}.\bar{b} = 3$, then $\bar{r}$ is

(a) $\hat{i} - \hat{j} + \hat{k}$ (b) $\hat{i} + \hat{j} + \hat{k}$ (c) $\hat{i} + \hat{j} - \hat{k}$ (d) $\hat{i} - \hat{j} - \hat{k}$

108. $\bar{p}$ and $\bar{q}$ are unit vectors and angle between them is 45^o. The diagonals of a parallelogram are $3\bar{p} - 2\bar{q}$ and $4\bar{p} + \bar{q}$. Area of parallelogram is

(a) $\frac{11}{2\sqrt{2}}$sq.u. (b) $\frac{15}{2\sqrt{2}}$sq.u. (c) $\frac{9}{2\sqrt{2}}$sq.u. (d) none of these

109. O is circumcentre, G centroid, O_1 orthocentre of triangle ABC. Then $\overline{OA} + \overline{OB} + \overline{OC} =$

(a) $2\overline{OC}$ (b) $\overline{O_1O}$ (c) $\overline{OO_1}$ (d) $2\overline{OG}$

110. $\bar{a}=2\hat{i}-\hat{j}+\hat{k}$ and $\bar{b}=3\hat{i}+2\hat{j}-\hat{k}$, then angle between $2\bar{a}+\bar{b}$ and $2\bar{b}-\bar{a}$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

111. Let $\bar{a}=\hat{i}+\hat{j}, \bar{b}=2\hat{i}-\hat{k}$, then point of intersection of straight lines $\bar{r}\times\bar{a}=\bar{b}\times\bar{a}$ and $\bar{r}\times\bar{b}=\bar{a}\times\bar{b}$ is

(a) $-i-j-k$ (b) $3i+j-k$ (c) $2i+j-k$ (d) $i-j-k$

112. Centroid of a Δ is $\frac{1}{3}(\hat{i}+\hat{j}-\hat{k})$, its circumcentre is $\frac{1}{2}(2\hat{i}-\hat{j}+\hat{k})$, then orthocentre is

(a) $\hat{i}-2\hat{j}+2\hat{k}$ (b) $\hat{i}+2\hat{j}-2\hat{k}$

(c) $-\hat{i}+2\hat{j}-2\hat{k}$ (d) $-\bar{i}-2\bar{j}-2\bar{k}$

113. The two diagonals of a parallelogram are $2\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}+6\hat{j}-3\hat{k}$, then angle between diagonals and acute angle of parallelogram are

(a) $\cos^{-1}\left(\frac{13}{21}\right);\cos^{-1}\left(\frac{7}{\sqrt{42}}\right)$ (b) $\cos^{-1}\left(\frac{13}{21}\right);\cos^{-1}\left(\frac{9}{\sqrt{42}}\right)$

(c) $\cos^{-1}\left(\frac{13}{21}\right),\cos^{-1}\left(\frac{5}{\sqrt{42}}\right)$ (d) none of these

114. $\bar{p},\bar{q},\bar{r}$ are non-coplanar unit vectors such that $\bar{p}\times(\bar{q}\times\bar{r})=\frac{\sqrt{3}}{2}(\bar{q}-\bar{r})$ then angle between p and q is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

115. $\bar{a}=[5i+3j-3k], \bar{b}=(i+5j-7k)$, then magnitude of projection of $\bar{a}+\bar{b}$ on $\bar{a}-\bar{b}$ is

(a) $\frac{20}{3}$ (b) $\frac{8}{5\sqrt{2}}$ (c) $\frac{16}{3}$ (d) $\frac{12}{5\sqrt{2}}$

116. If $\bar{p}\times\bar{q}=c$ and $\bar{q}\times\bar{c}=\bar{p}$, then

(a) $\bar{p}=\bar{q}=\bar{c}$ (b) $\bar{p},\bar{q},\bar{c}$ are orthogonal but $|\bar{b}|\neq 1$

(c) $\bar{p},\bar{q},\bar{c}$ are orthogonal in pairs and $|\bar{p}|=|\bar{c}|=1$

(d) $\bar{p},\bar{q},\bar{c}$ are not orthogonal to each other

117. The position vectors of A, B, C vertices of a triangle ABC are $\hat{i}+\hat{j}-\hat{k}$, $3\hat{i}+2\hat{j}+\hat{k}$ and $5\hat{i}+4\hat{j}-\hat{k}$ respectively. The bisector of angle A meets BC in D; P.V. of D is

(a) $\frac{15\hat{i}-11\hat{j}+\hat{k}}{4}$ (b) $\frac{15\hat{i}+11\hat{j}-\hat{k}}{4}$

(c) $\frac{15\hat{i}+11\hat{j}+\hat{k}}{4}$ (d) $\frac{15\hat{i}-11\hat{j}-\hat{k}}{4}$

118. $\bar{a}+2\bar{b}+3\bar{c}=0$ and $|\bar{a}|=3, |\bar{b}|=2$ and $|\bar{c}|=1$ then angle between $\bar{b}$ and $\bar{c}$ is

(a) $\cos^{-1}\left(\frac{-2}{3}\right)$ (b) $\cos^{-1}\left(\frac{-3}{4}\right)$ (c) $\cos^{-1}\left(\frac{3}{4}\right)$ (d) $\cos^{-1}\left(\frac{-1}{2}\right)$

119. $\bar{a}=\hat{i}+2\hat{j}-\hat{k}, \bar{b}=2\hat{i}-\hat{j}-2\hat{k}$ and $\bar{c}=3\hat{i}+2\hat{j}-\hat{k}$ then projection of $\bar{a}\times\bar{c}$ on vector $\bar{b}$ has magnitude

(a) 4 (b) 0 (c) 10/3 (d) 8/3

120. $\bar{a}.\bar{b}=0$ and $\bar{a}+\bar{b}$ makes an angle of 60^{o} with $\bar{a}$, then $\bar{b}$ =

(a) $\sqrt{3}a$ (b) $2\bar{a}$ (c) $\sqrt{2}\,\bar{a}$ (d) none of these

121. | M_1 | is the moment of force $5\hat{i}+2\hat{j}+3\hat{k}$ acting through $2\hat{i}+3\hat{j}+4\hat{k}$ about the point $\hat{i}+\hat{j}+\hat{k}$ and $|M_2|$ is moment of force $4\hat{i}-3\hat{j}+\hat{k}$ acting through $5\hat{i}-7\hat{j}+3\hat{k}$, about the point $2\hat{i}-\hat{j}+\hat{k}$, then $|M_1|/|M_2|$ =

(a) < 1 (b) > 1 (c) $\frac{2\sqrt{130}}{25}$ (d) $2\sqrt{6}$

122. Perpendicular from P, $(\hat{i}+2\hat{j}-\hat{k})$ meets the plane in M. $(3\hat{i}-\hat{j}+4\hat{k})$. The equation of plane is

(a) $\hat{r}.(2\hat{i}-3\hat{j}+5\hat{k})=29$ (b) $\bar{r}.(2\hat{i}+3\hat{j}+5\hat{k})=23$

(c) $\bar{r}.(2\hat{i}+3\hat{j}-6\hat{k})-17=0$ (d) $\bar{r}.(2\hat{i}-3\hat{j}-5\hat{k})+11=0$

123. P is any point, $\bar{r}$, then $[\bar{r}-\bar{a}\ \ \bar{a}-\bar{b}\ \ \bar{b}-\bar{c}]$ represent

(a) a straight line perpendicular to $\bar{a}-\bar{b}$ and $\bar{b}-\bar{c}$

(b) a plane through $\bar{a},\bar{b},\bar{c}$

(c) that $\bar{r},\bar{a},\bar{b},\bar{c}$ are coplanar

(d) none of these

124. For all values of λ such that (x, y, z) $\neq$ (0, 0, 0) and (5x + 3j + 2k)x + (–2i + 4j + k)y + (10i + 4j + k)z = λ (xi – yj + z)

(a) 0, 1, 9 (b) –1, 9, –5 (c) ϕ (d) 1, 9, –5

125. $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors, then

$$\begin{vmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \end{vmatrix} =$$

(a) 0 (b) 1 (c) –1 (d) $\bar{a}\bar{b}\bar{c}$

126. A vector makes equal angles with $\bar{a} = \frac{1}{3\sqrt{3}}(\hat{i} + 5\hat{j} + \hat{k})$, $\bar{b} = \frac{1}{5}(-4\hat{i} - 3\hat{k})$ and $\bar{c} = \hat{j}$. Its magnitude is $\sqrt{51}$. The vector is

(a) $\pm(5\hat{i} - \hat{j} - 5\hat{k})$ (b) $\pm(5\hat{i} + \hat{j} - 5\hat{k})$

(c) $\pm(5\hat{i} + \hat{j} + 5\hat{k})$ (d) $7\hat{i} + \hat{j} - \hat{k}$

127. $\bar{a}$ is a unit vector and $\bar{b}$ is a non zero vector, not parallel to $\bar{a}$. If two sides of a triangle are $\sqrt{3}(\bar{a} \times \bar{b})$ and $\bar{b} - (\hat{a}.\bar{b})\hat{a}$, then the triangle is

(a) equilateral (b) isosceles (c) right angle (d) none of these

128. The component of vector $8\hat{i} + \hat{k}$ perpendicular to $\hat{i} + 2\hat{j} - 2\hat{k}$ is

(a) $\frac{1}{3}(22\hat{i} - 7\hat{j} + \hat{k})$ (b) $\frac{1}{3}(22\hat{i} - 4\hat{j} + 7\hat{k})$

(c) $\frac{1}{3}(26\hat{i} - 7\hat{j} - 7\hat{k})$ (d) none of these

129. Vectors $x\hat{i} + \hat{j} + \hat{k}, \hat{i} + y\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - z\hat{k}$ are coplanar $(x \neq 1, y \neq 1, z \neq 1)$ then $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1+z} =$

(a) 0 (b) 1 (c) –1 (d) 3

130. Plane $r.(i - 2j + 3k) = 2$ is rotated about its line of intersection with plane $r.(i + 2j - k) = 6$ through 90^o. In new position its equation is

(a) 5x + 4y – z = 21 (b) 5x + 4y + z = 24 (c) 5x – 4y + z = 24 (d)

131. The equation of plane $\bar{r} = (i - j) + \lambda(i + j + k) + \mu(i - 2j + 3k)$ in scalar product form is

(a) $r.(5\hat{i} + 3\hat{j} - 2\hat{k}) = 7$ (b) $r.(3\hat{i} + 5\hat{j} + 2\hat{k}) = 10$

(c) $r.(5\hat{i}-2\hat{j}+3\hat{k})=7$ (d) none of these

132. Vectors $\overline{p}$ and $\overline{q}$ are non-collinear. the value of x for which vectors $\overline{r}.(x-2)\overline{p}+\overline{q}$ and $s=(2x+1)\overline{p}-\overline{q}$ are collinear is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{2}$

133. The vector $\overline{OP}=5i+4j+2k$ turns through an angle of $\frac{\pi}{2}$ about O, the origin, passing through the positive side of j axis on its way. The vector in its new position is

(a) $\frac{1}{\sqrt{29}}(20\hat{i}-29\hat{j}+8\hat{k})$ (b) $\sqrt{\frac{10}{281}}(20\hat{i}+29\hat{j}+8\hat{k})$

(c) $-\frac{1}{\sqrt{281}}(10\hat{i}-17\hat{j}+8\hat{k})$ (d) $-\sqrt{\frac{2}{17}}(10\hat{i}-17\hat{j}+6\hat{k})$

134. If $\overline{\alpha}=x(\overline{a}\times\overline{b})+y(\overline{b}\times\overline{c})+z(\overline{c}\times\overline{a})$ and $[\overline{a}\,\overline{b}\,\overline{c}]=\frac{1}{8}$ then x + y + z =

(a) $8\overline{\alpha}\cdot(\overline{a}+\overline{b}+\overline{c})$ (b) $\overline{\alpha}\cdot(\overline{a}+\overline{b}+\overline{c})$

(b) $8\overline{\alpha}\cdot(\overline{a}-\overline{b}-\overline{c})$ (d) none of these

135. The value of p for which angle between $\overline{\alpha}=2p^2\hat{i}+4p\hat{j}+\hat{k}$ and $\overline{\beta}=7\hat{i}-2\hat{j}+p\hat{k}$ is obtuse and angle between β and z axis is acute and $<\frac{\pi}{6}$ are

(a) $0<p<\frac{1}{2}$ (b) $\frac{1}{2}<p<1$

(c) $p>\frac{1}{2}\text{ or}<0$ (d) none of these

136. Position vectors of vertices of equilateral triangle ABC are A ($\overline{a}$), B($\overline{b}$) and C($\overline{c}$). If orthocentre of triangle is at origin then

(a) $\overline{a}+\overline{b}+\overline{c}=0$ (b) $\overline{a}^2+\overline{b}^2+\overline{c}^2=0$

(c) a + b = c (d) none of these

137. $a=-i+j+k, b=2i+k$, then vector $\hat{c}$, satisfying conditions (i) that it is coplanar with $\overline{b}$ and $\overline{a}$ (ii) it is perpendicular to $\overline{b}$ (iii) that $\overline{a}.\overline{c}=7$, is

(a) $-\frac{3}{2}\hat{i}+\frac{5}{2}\hat{j}+3\hat{k}$ (b) $-3\hat{i}+3\hat{j}+6\hat{k}$

(c) $-6\hat{i}+5\hat{j}+\hat{k}$ (d) $-\hat{i}+2\hat{j}+2\hat{k}$

138. $\bar{a}+\bar{b}+\bar{c}=0$, $|\bar{a}|=7, |\bar{b}|=3, |\bar{c}|=4$ then angle between $\bar{b}$ and $\bar{c}$ is

(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

139. If $\bar{b}.\bar{c}=0$ and $\bar{b}+\bar{c}$ makes angle 30^o with $\bar{b}$, then

(a) $|\bar{c}|=2|\bar{b}|$ (b) $|\bar{b}|=2|\bar{c}|$

(c) $|\bar{b}|=\sqrt{3}|\bar{c}|$ (d) none of these

140. A unit vector orthogonal to $-2\hat{i}+3\hat{j}+5\hat{k}$ makes equal angles with x and y axis. Vector is

(a) $\frac{1}{3\sqrt{5}}(2\hat{i}+4\hat{j}+5\hat{k})$ (b) $\frac{1}{\sqrt{19}}(3\hat{i}+3\hat{j}-\hat{k})$

(c) $\frac{1}{\sqrt{51}}(5\hat{i}+5\hat{j}-\hat{k})$ (d) $\frac{1}{\sqrt{66}}(5\hat{i}+5\hat{j}-4\hat{k})$

141. $\bar{r}\times\bar{a}=\bar{b}\times\bar{a}; \bar{r}\times\bar{b}=\bar{a}\times\bar{b}$; $\bar{a}\neq 0$, $\bar{a}=\lambda\bar{b}$ and $\bar{a}$ is not perpendicular to $\bar{b}$, then r is

(a) $\bar{a}-\bar{b}$ (b) $\bar{a}+\bar{b}$ (c) $\bar{a}\times\bar{b}+\bar{a}$ (d) $\bar{a}\times\bar{b}+\bar{b}$

142. $\bar{a}=\hat{i}+\hat{j}-\hat{k}, \bar{b}=\hat{i}-\hat{j}+\hat{k}$ and $\bar{c}$ is a unit vector perpendicular to $\bar{a}$ and coplanar with $\bar{a}$ and $\bar{b}$ then a unit vector $\bar{p}$ perpendicular to both $\bar{a}$ and $\bar{c}$ is

(a) $\frac{1}{\sqrt{6}}(2\hat{i}-\hat{j}+\hat{k})$ (b) $\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$ (c) $\frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$ (d) $\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$

143. $\bar{p},\bar{q}$ and $\bar{r}$ are three distinct vectors of magnitude 2, 2, 4 respectively. $\bar{p}\times(\bar{q}\times\bar{r})+\bar{q}=0$ then acute angle between $\bar{p}$ and $\bar{r}$ is

(a) $\cos^{-1}\left(\frac{3}{16}\right)$ (b) $\cos^{-1}\left(\frac{1}{4}\right)$ (c) $\cos^{-1}\left(\frac{1}{8}\right)$ (d) $\frac{\pi}{10}$

144. The equation of line of intersection of planes $\bar{r}.(2\hat{i}-3\hat{j}+4\hat{k})$ + 27 and $r.(2\hat{i}+2\hat{j}+\hat{k})-7=0$ is

(a) $\bar{r}=(3\hat{j}-5\hat{k})+t(-11\hat{i}+6\hat{j}+10\hat{k})$ (b) $\bar{r}=(\hat{i}+3\hat{j})+t(-11\hat{i}+6\hat{j}+10\hat{k})$

(c) $\bar{r}=(5\hat{j}-3\hat{k})+t(-11\hat{i}+6\hat{j}+10\hat{k})$ (d) none of these

145. Shortest distance between lines $\bar{r}=(-\hat{i}+3\hat{j}+2\hat{k})+t(3\hat{i}+4\hat{j}+\hat{k})$ and $\bar{r}=(-\hat{i}-2\hat{j}+\hat{k})+t_2(-\hat{i}+4\hat{j}-2\hat{k})$ is

(a) $\frac{\sqrt{17}}{5}$ (b) $\frac{\sqrt{21}}{5}$ (b) $\frac{3\sqrt{3}}{5}$ (d) $\frac{-3}{\sqrt{17}}$

146. $\bar{a}=\hat{i}+\hat{j}+\hat{k}, \bar{b}=2\hat{i}-\hat{j}+2\hat{k}, \bar{c}=\hat{i}+3\hat{j}-\hat{k}$ then $\left[(\bar{a}\times\bar{b})\times(\bar{b}\times\bar{c})\right]\cdot(\bar{a}+\bar{b}+\bar{c})$ is =

(a) 90 (b) 6 (c) 54 (d) 24

147. The image of straight line $\bar{r}=(2\hat{i}+3\hat{j}-2\hat{k})+t(-\hat{i}-5\hat{j}+3\hat{k})$ in plane $\bar{r}\cdot(3\hat{i}+2\hat{j}-\hat{k})+2=0$ is

(a) $\bar{r}=(\hat{i}-2\hat{j}+\hat{k})+\frac{t}{7}(41\hat{i}-3\hat{j}+2\hat{k})$ (b) $\bar{r}=(\hat{i}-2\hat{j}+\hat{k})+\frac{1}{7}t(41\hat{i}-3\hat{j}+5\hat{k})$

(c) $\bar{r}=(\hat{i}+2\hat{j}+\hat{k})+\frac{1}{7}t(41\hat{i}+3\hat{j}+5\hat{k})$ (d) none of these

148. P. Vector of points A and C are $9\hat{i}+5\hat{j}+6\hat{k}$ and $6\hat{i}-\hat{j}-4\hat{k}$ and $\overline{AB}$ is $4\hat{i}-3\hat{j}+\hat{k}$ and $\overline{CD}=-4\hat{i}+7\hat{j}+3\hat{k}$. Point P is on AB and point Q is on CD, if PQ is perpendicular to AB and CD then $|\overrightarrow{PQ}|$ is

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{5}}$

149. A, B, C and D are points $3\hat{i}+4\hat{j}-\hat{k}, -\hat{i}+2\hat{j}-2\hat{k}, -\hat{i}+2\hat{j}+2\hat{k}$ and $2\hat{i}-\hat{j}+\hat{k}$ respectively, then lines AB and CD

(a) intersect (b) Do not intersect (c) are skew (d) intersect at infinitely

150. Shortest distance between lines AB and CD of Q. 149 is

(a) $\frac{72}{\sqrt{374}}$ (b) $\frac{30}{\sqrt{374}}$ (c) $\frac{76}{\sqrt{394}}$ (d) $\frac{74}{\sqrt{394}}$

151. $\bar{a}=2\hat{i}-3\hat{j}+\hat{k}, \bar{b}=2\hat{i}+\hat{j}-\hat{k}, \bar{c}=3\hat{i}-2\hat{j}+4\hat{k}$ then, $\left[(\bar{a}\times\bar{b})\times\bar{c}+(\bar{c}\times\bar{b})\times\bar{a}\right]\cdot\left[\bar{a}+\bar{b}+\bar{c}\right]$ is =

(a) 48 (b) 96 (c) 192 (d) 288

152. Show that the perpendicular distance of point $\bar{c}$ from line joining $\bar{a}$ and $\bar{b}$ is

$$\frac{\left|\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}\right|}{\left|\bar{b}-\bar{a}\right|}$$

153. $\bar{a}=2\hat{i}+2\hat{j}-\hat{k},\ \bar{b}=\hat{i}-2\hat{j}+2\hat{k},\ \bar{c}=2\hat{i}+\hat{j}-2\hat{k}$ a unit vector parallel to $(\bar{a}\times\bar{b})\times\bar{c}$ is

(a) $\frac{1}{\sqrt{29}}(-4\hat{i}+2\hat{j}-3\hat{k})$ (b) $\frac{1}{\sqrt{29}}(4\hat{i}-2\hat{j}+3\hat{k})$

(c) $\frac{1}{\sqrt{29}}(2\hat{i}-4\hat{j}+3\hat{k})$ (d) $\frac{1}{\sqrt{29}}(4\hat{i}-3\hat{j}+2\hat{k})$

154. Vectors $\ell\hat{i}+\ell\hat{j}+n\hat{k}, \hat{i}+\hat{k}$ and $n\hat{i}+n\hat{j}+n\hat{k}$ are co-planar then ℓ, m, n are

(a) in A.P. (b) in G. P. (c) in H.P. (d) ℓ + m + n = 0

155. The vector $-\hat{i}+\hat{j}-\hat{k}$ bisects the angle between vectir $\bar{c}$ and $3i+4j-$ then c is

(a) $-4\hat{i}-3\hat{j}-\hat{k}$ (b) $\frac{1}{3}(-2\hat{i}-5\hat{j}+\hat{k})$ (c) $\frac{1}{15}(2\hat{i}+5\hat{j}-5\hat{k})$ (d) none of these

156. The vector which is equally inclined to vectors $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}-\hat{k})$, $\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}+\hat{k})$ and $\frac{1}{\sqrt{3}}(-\hat{i}+\hat{j}+\hat{k})$ is

(a) $2i-2j+k$ (b) $2i+j-k$ (c) $i+j+k$ (d) none of these

157. $(\bar{a}+2\bar{b}-3\bar{c})\cdot(\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a})$ =

(a) 2 [a b c] (b) 0 (d) – [a b c] (d) –1

158. $\bar{a},\bar{b},\bar{c}$ are unit vectors and each in inclined with other at an angle of 60^o, then

$(\bar{a}+\bar{b}+\bar{c})\times(\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b})$ =

(a) 0 (b) –1 (c) 1 (d) none of these

159. $\bar{a},\bar{b},\bar{c}$ are non co-planar vectors and if vectors $5\bar{a}+\bar{b}+\bar{c}$, $6\bar{a}+2\hat{b}+k\hat{c}$ and $3\bar{a}-\bar{b}+\bar{c}$ are co-planar then k =

(a) 1 (b) –1 (c) 2 (d) –3

160. Vectors $\bar{a},\bar{b}$ and $\bar{c}$ are non-coplanar then vectors $\bar{b}-\bar{c}-\bar{a}, \bar{a}+\bar{b}+3\bar{c}$ and $\bar{b}-2\bar{a}-3\bar{c}$ are

(a) collinear (b) coplanar (c) independent (d) none of these

161. Equation of line of intersection of the planes $\bar{r}\cdot(3\hat{i}-\hat{j}+\hat{k})=8$ and $\bar{r}\cdot(\hat{i}+\hat{j}+2\hat{k})=3$ is

(a) $3i+j-2k+\lambda(3i+5j-4k)$ (b) $4i+2j-2k+\lambda(3i+5j-4\kappa)$

(c) $2i-j+k+\lambda(3i+5j-4k)$ (d) none of these

162. P. Vectors point A is i + j – k, of B is 3i + 2j + k & C is 4i – j + k. P divides AB in the rate of 2 : 3 and Q divides AC in the rate of 3 : 2 internally. Vector $\overline{PQ}$ is

(a) $\frac{1}{5}(5\hat{i}-8\hat{j}-2\hat{k})$ (b) $\frac{1}{5}(5\hat{i}-8\hat{j}+2\hat{k})$

(c) $\frac{1}{5}(5\hat{i}+8\hat{j}+2\hat{k})$ (d) none of these

163. Let $\overline{a}=\hat{j}-\hat{k}$ and $\overline{c}=\hat{i}+\hat{j}+\hat{k}$, then vector $\overline{b}$ satisfying $\overline{a}\times\overline{b}+\overline{c}=0$ and $\overline{a}.\overline{b}=3$ is

(a) $\hat{i}-\hat{j}-2\hat{k}$ (b) $\hat{i}-\hat{j}+2\hat{k}$

(c) $-\hat{i}+\hat{j}-2\hat{k}$ (d) $2\hat{i}-\hat{j}+2\hat{k}$

164. Vectors $\overline{a}=\hat{i}-\hat{j}+2\hat{k}$, $b=2\hat{i}+4\hat{j}+\hat{k}$ and $\overline{c}=\lambda\hat{i}+\hat{j}+\mu\hat{k}$ are mutually orthogonal, then (λ, μ) =

(a) (–2, 3) (b) (3, –2) (c) (–3, 2) (d) (2, –3)

EXERCISE- ON VECTORS

PART-A

1. The sides of a parallelogram are $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$. Find the unit vectors parallel to the diagonals.

2. The vector $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between the vector $\vec{c}$ and $3\hat{i} + 4\hat{j}$. Determine the unit vector along $\vec{\mathbf{c}}$.

3. Find the value of *a* such that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$, and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar.

4. Find the value of the constant *S* such that the scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with the unit vector parallel to the sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $S\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one.

5. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors of length 3, 4 and 5 respectively. Let $\vec{a}$ be perpendicular to $(\vec{b} + \vec{c})$, $\vec{b}$ to $(\vec{c} + \vec{a})$ and $\vec{c}$ to $(\vec{a} + \vec{b})$. Find the length of the vector $\vec{a} + \vec{b} + \vec{c}$.

6. Find a vector of magnitude $\sqrt{51}$ which makes equal angles with the three vectors
$\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$, $\vec{b} = \frac{1}{5}(-4\hat{i} - 3\hat{k})$ and $\vec{c} = \hat{j}$.

7. Given $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$, $\vec{c} = 3\hat{i} + \hat{j}$, find a unit vector in the direction of the resultant of these vectors. Also find a vector $\vec{r}$ which is normal to both $\vec{a}$ and $\vec{b}$. What is the inclination of $\vec{r}$ and $\vec{c}$.

8. If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{k}$, $\vec{c} = 2\hat{j} - \hat{k}$, find the area of the parallelogram having diagonals $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$

9. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar vectors, show that

$$[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{a}.\vec{c} & \vec{b}.\vec{c} & \vec{c}.\vec{c} \end{vmatrix} = [\vec{a} \quad \vec{b} \quad \vec{c}]^2$$

10. Using vector method, find the ratio in which the bisector of an angle in a triangle divides the opposite side.

11. For any two vector $\vec{a}$ and $\vec{b}$ prove that $\left(\frac{\vec{a}}{|\vec{a}|^2}-\frac{\vec{b}}{|\vec{b}|^2}\right)^2=\left(\frac{\vec{a}-\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)^2$

12. Prove that the median to the base of an isosceles triangle is perpendicular to the base.

13. Prove, by vector methods that the point of the intersection of the diagonals of a trapezium lies on the line passing through the mid-points of the parallel sides (you may assume that the trapezium is not a parallelogram.) **[I.I.T-98]**

14. Prove that internal disectors of the angles of a triangle are concurrent. Also find the position vector of the point of concurrently. **[I.I.T-2001]**

15. In a triangle ABC, D divide BC in the ratio 3:2 and E divide CA in the ratio 1:3 The lines AD and BE meet at H and CH meets AB in F. find the ratio in which F divide AB.

16. The median AD of a triangle ABC us bisected at E and BE is Produced to meet AC in F. Prove by vector method that EF = $\frac{1}{4}$BF.

17. In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods.

18. If $\vec{a}+\vec{b}+\vec{c}=0$, show that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$. Hence deduce that if $\vec{a}$, $\vec{b}$ and $\vec{c}$ represent the sides of a triangle *ABC*, then $\frac{\sin A}{|\vec{a}|}=\frac{\sin B}{|\vec{b}|}=\frac{\sin C}{|\vec{c}|}$

19. Prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.

20. Prove that $(\vec{a}\times\vec{b})^2=a^2b^2-(\vec{a}.\vec{b})^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b}\\ \vec{a}.\vec{b} & \vec{b}.\vec{b}\end{vmatrix}$

21. Given that vectors A, B, C from a triangle such that A = B + C. Find a, b, c, d such that the area of the triangle is $5\sqrt{6}$ where A = ai + bj + ck , B = di + 3j + 4k , C = 3i + j - 2k

22. Let $\hat{a}$ be a unit vector and $\vec{b}$ be a non-zero vector not parallel to $\hat{a}$. Find the angles of the triangle, two sides of which are represented by the vectors $\sqrt{3}\hat{a}\times\vec{b}$ and $\vec{b}-(\hat{a}.\vec{b})\hat{a}$.

23. Let $\vec{A}=2i+k$. $\vec{B}$ = i + j +k and $\vec{C}$ = 4i - 3j + 7k. find the vector R which satisfies $R\times B=C\times B$ and R. A = 0.

24. ABC is a triangle and D, E, F are three points on sides BC, CA and AB respectively such that BD: DC = 2: 3, CE : EA = 1 : 2, AF : FB = 3 : 1. Using vector method prove that AD, BE and CF are concurrent.

25. If $\hat{a}, \hat{b}$ and $\hat{c}$ be three unit vectors such that $\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b}$, find the angles which $\hat{a}$ makes with $\hat{b}$ and $\hat{c}$, $\hat{b}$ and $\hat{c}$ being non parallel.

PART- B

1. a) Vector $\vec{x}$, $\vec{y}$ and $\vec{z}$ each of magnitude $\sqrt{2}$ make in angles of 60^o with each other. If $\vec{x}\times(\vec{y}\times\vec{z}) = \vec{a}$, $\vec{y}\times(\vec{z}\times\vec{x}) = \vec{b}$ and $\vec{x}\times\vec{y} = \vec{c}$. Then find $\vec{x}$, $\vec{y}$ and $\vec{z}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$

   b) If $\vec{x}\times\vec{y} = \vec{a}$, and $\vec{y}\times\vec{z} = \vec{b}$, $\vec{x}.\vec{b} = \gamma$, $\vec{x}.\vec{y} = 1$ and $\vec{y}.\vec{z} = 1$ then find $\vec{x}$, $\vec{y}$ and $\vec{z}$ in terms of $\vec{a}$, $\vec{b}$ and γ.

2. Let V be the volume of the parallelopiped formed by the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If a_r, b_r, c_r, where r = 1, 2, 3, are nonnegative real numbers and $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$, show that $V \le L^3$.

   [I.I.T-2002]

3. a) Solve the following system of equations for vectors $\vec{x}$ and $\vec{y}$: $\vec{x} + \vec{y} = \vec{a}$, $\vec{x}\times\vec{y} = \vec{b}$, $\vec{x}.\vec{a} = 1$.

   b) Let $\hat{X}, \hat{Y}$ and $\hat{Z}$ be unit Vectors such that $\hat{X} + \hat{Y} + \hat{Z} = \vec{a}$, $\hat{X}\times(\hat{Y}\times\hat{Z}) = \vec{b}$, $(\hat{X}\times\hat{Y})\times\hat{Z} = \vec{c}$, $\vec{a}.\hat{X} = \frac{3}{2}$, $\vec{a}.\hat{Y} = \frac{7}{4}$ and $|\vec{a}| = 2$. Find $\hat{X}, \hat{Y}$ and $\hat{Z}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$

4. Vectors $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are not coplaner .the position vector of point A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{i} + 2\hat{k}$ respectively. Find the position vector of point P on the line AB and point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

5. The position vectors of the vertices A, B and C of a tetrahedron ABCD are $\hat{i} + \hat{j} + \hat{k}$, $\hat{i}$ and $3\hat{i}$ respectively the altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at point E If the length of the side AD is 4 and volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, Find the position vector of the point E for all its possible positions. **[I.I.T-96]**

6. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non coplanar unit vectors equally inclined to one another at an angle of θ. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$ find scalars p, q and r in terms of θ. **[I.I.T-97]**

7. The position vector of the points P and Q are $5\hat{i} + 7\hat{j} - 2\hat{k}$ and $-3\hat{i} + 3\hat{j} + 6\hat{k}$ respectively. the vector A = $3\hat{i} - \hat{j} + \hat{k}$ passes through the point P and vector B = $-3\hat{i} + 2\hat{j} + 4\hat{k}$ passes through the point Q. A third vector $2\hat{i} + 7\hat{j} - 5\hat{k}$ intersect vector A and B. find the position vectors of points of intersection.

8. The position vectors of two points A and C are $9\hat{i} - \hat{j} + 7\hat{k}$ and $7\hat{i} - 2\hat{j} + 7\hat{k}$ respectively. The point of intersection of Vectors $\overrightarrow{AB} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\overrightarrow{CD} = 2\hat{i} - \hat{j} + 2\hat{k}$ is P If the Vector $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and PQ =15 units Find the position vector of Q.

9. Let ABC and PQR be any two triangle in the same plane Assume that the perpendiculars from the points A, B, C to the sides OR, RP, PQ respectively are concurrent. Using vector methods or otherwise, prove that perpendicular from P, Q, R to BC , CA , AB respectively are also concurrent . **[I.I.T-2000]**

10. Let points P, Q and R have position vectors r_1 = 3i - 2j - k , r_2 = i + 3j +4k and r_3 = 2i + j - 2k relative to an origin O .find the distance of P from the plane OQR.

11. For any two vectors u and v prove that $(1 + |u|^2)(1 + |v|^2) = (1 - u \cdot v)^2 + |u + v + (u \times v)^2|$ **[I.I.T-1998]**

12. In the parallelogram ABCD the internal bisectors of the consecutive angles B and C intersect at P. Use vector method to find ∠BPC.

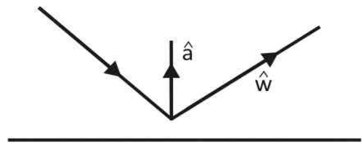
13. If $\vec{u}, \vec{v}, \vec{w}$ be three non coplanar unit vectors with angles between $\vec{u}$ and $\vec{v}$ is α between $\vec{v}$ and $\vec{w}$ is β and between $\vec{w}$ and $\vec{u}$ is γ. If $\vec{a}, \vec{b}, \vec{c}$ are the unit vectors along the bisectors of α, β, γ respectively , then prove that

$$\left[\vec{a}\times\vec{b}\;\; \vec{b}\times\vec{c}\;\; \vec{c}\times\vec{a}\right] = \frac{1}{16}\left[\vec{u}\,\vec{v}\,\vec{w}\right]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$ **[I.I.T-2003]**

14. $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four distinct vectors satisfying the condition $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then prove that $\vec{a}.\vec{b} + \vec{c}.\vec{d} \neq \vec{a}.\vec{c} + \vec{b}.\vec{d}$. **[I.I.T-2004]**

15. Given three non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ such that $(\vec{a} \times \vec{b}) \times \vec{c} + (\vec{b} \times \vec{c}) \times \vec{a} + (\vec{a} \times \vec{c}) \times \vec{b} = 0$. Show that either the vectors $\vec{a}$ and $\vec{c}$ are parallel or $\vec{b}$ is normal to the plane containing $\vec{a}$ and $\vec{c}$.

16. Incident ray is along the unit vector $\hat{v}$ and the reflected ray is along the unit vector $\vec{w}$. The normal is along unit vector $\hat{a}$ outwards. Express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$.



17. Let u and v be unit vectors. If w is a vector such that w + (w $\times$ u) = v, then prove that $|(u \times v).w| \leq \frac{1}{2}$ and that the equality holds if and only if u is perpendicular to v.

18. Given that $\vec{a}$ is perpendicular to $\vec{b}$ and p is a non-zero scalar, solve for $\vec{r}$: $p\vec{r} + (\vec{r}.\vec{b})\vec{a} = \vec{c}$.

19. $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of points A, B, C respectively prove that $(\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b})$ is perpendicular to the plane ABC.

20. Find the equation of the plane passing through the point A(3,-2,1) and perpendicular to the vector 4i+7j-4k. If PM be the perpendicular from P(1,2,-1) to this plane, find its length.

21. Find the length of the perpendicular from the point $-\hat{i}+2\hat{j}+5\hat{k}$ on the line passing through the point $3\hat{i}+4\hat{j}+5\hat{k}$ and parallel to the vector $\vec{p}=2\hat{i}-3\hat{j}+6\hat{k}$ Find also the position vector of the foot of the perpendicular.

22. A straight line 'L' cuts the line AB, AC and AD of a parallelogram ABCD at point B_1, C_1 and D_1 respectively. If $\vec{AB_1} = \lambda_1 \vec{AB}$, $\vec{AD_1}, = \lambda_2 \vec{AD}$ and $\vec{AC_1} = \lambda_3 \vec{AC}$ then Prove that $\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$.

23. P is any point on the circum-circle of Δ ABC, other than the vertices. H is the orthocentre of ΔABC, M is the mid-point of BC. Prove by vector methods that the line MD is perpendicular to AP.

24. Consider a triangle ABC, having AD as its median through vertex A. Let P be any point on the this median using vector methods or otherwise prove that for every position of point P on AD, area of Δ APB is equal to the area ΔAPC.

25. Let the area of a given triangle ABC be Δ. Points A_1, B_1 and C_1 are the mid-points of the sides BC, CA and AB respectively. Points A_2 is the midpoint of CA_1. Lines C_1A_1 and AA_2 meet the median BB_1 at points E and D respectively If

Δ_1 be the area of the quadrilateral A_1A_2DE, using vectors of otherwise prove that $\frac{\Delta_1}{\Delta}=\frac{11}{56}$.

OBJECTIVE PROBLEMS

Level-1

1. The position vectors of the point P and Q are $\vec{p}$ and $\vec{q}$ respectively. If O is the origin and R is a point in the interior of $\angle POQ$ such that OR bisects the angle $\angle POQ$ then unit vector along OR is:

   a) $\frac{\vec{p}+\vec{q}}{|\vec{p}||\vec{q}|}$ b) $\frac{\vec{p}}{|\vec{p}|}-\frac{q}{|\vec{q}|}$ c) $\frac{\vec{p}}{|\vec{p}|}+\frac{q}{|\vec{q}|}\Big/\left|\frac{\vec{p}}{|\vec{p}|}+\frac{q}{|\vec{q}|}\right|$ d) none of these

2. The vectors $\hat{i}+\hat{j}$, $\hat{j}+\hat{k}$ and $\hat{k}+\hat{i}$ are :

   a) Linearly independent b) linearly dependent

   c) Can not be discussed d) none of these

3. If $\vec{r}=3\hat{i}+2\hat{j}-5\hat{k}$, $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+3\hat{j}-2\hat{k}$ $\vec{c}=-2\hat{i}+\hat{j}-3\hat{k}$ such that $\vec{r}=\lambda\vec{a}+\mu\vec{b}+\gamma\vec{c}$, then:

   a) μ, $\lambda/2$, γ are in A.P. b) λ, μ, γ are G.P.

   c) λ, μ, γ are H.P. d) μ, λ, γ are in G.P.

4. A vector $\vec{a}=(x,y,z)$ makes an obtuse angle with y-axis , equal angles with $\vec{b}=(y,-2z,3x)$ and $\vec{c}=(2z,3x,-y)$ and *a* is perpendicular to $\vec{d}=(1,-1,2)$. If $|\vec{a}|=2\sqrt{2}$, then the vector $\vec{a}$ is

   a) $\left(\frac{2\sqrt{2}}{\sqrt{3}},-\frac{2\sqrt{2}}{\sqrt{3}},-\frac{2\sqrt{2}}{\sqrt{3}}\right)$ b) $\left(\frac{-2\sqrt{2}}{\sqrt{3}},\frac{2\sqrt{2}}{\sqrt{3}},\frac{2\sqrt{2}}{\sqrt{3}}\right)$

   c) $\left(\frac{2\sqrt{2}}{\sqrt{3}},-\frac{2\sqrt{2}}{\sqrt{3}},\frac{2\sqrt{2}}{\sqrt{3}}\right)$ d) none of these

5. If $|\vec{a}|=|\vec{b}|=|\vec{a}+\vec{b}|=1$, then $|\vec{a}-\vec{b}|$ is equal to :

   a) 1 b) $\sqrt{3}$

   c) 0 d) none of these

6. If $\vec{u}=\vec{a}-\vec{b}$ and $\vec{v}=\vec{a}+\vec{b}$ and $|\vec{a}|=|\vec{b}|$ = 2 then $|\vec{u}\times\vec{v}|$ is equal to :

(a) $2\sqrt{16-(\vec{a}.\vec{b})^2}$ (b) $\sqrt{16-(\vec{a}.\vec{b})^2}$

(c) $2\sqrt{4-(\vec{a}.\vec{b})^2}$ (d) $\sqrt{4-(\vec{a}.\vec{b})^2}$

7. If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is the angle between them, then $|\vec{a}+\vec{b}| < 1$ if :

(a) θ = π/2 (b) θ < π/3

(c) θ > 2π/3 (d) π/2 < θ < 2π/3

8. The number of distinct real values of ? for which the vectors $-\lambda^2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\lambda^2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\lambda^2\hat{k}$ are coplanar, is

a) Zero b) one

c) Two d) three **[IITJEE-07]**

9. Let $\vec{a},\vec{b},\vec{c}$ be the unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$. Which one of the following is correct? **[IITJEE-07]**

a) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=0$ b) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq 0$

c) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{a}\times\vec{c}\neq 0$ d) $\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}$ are mutually perpendicular

10. The value of a for which the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $a\hat{i}-3\hat{j}+\hat{k}$ respectively are the vertices of the right angled triangle with C = p/2 are **[AIEEE-06]**

a) -2 and-1 b) -2 and 1

c) 2 and -1 d) 2 and 1

11. If $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors such that $\vec{a}.\vec{b}\neq 0, \vec{a}.\vec{c}\neq 0$ then $\vec{a}$ and $\vec{c}$ are

a) Inclined at an angle of p/6 between them

b) Perpendicular

c) Parallel

d) Inclined at an angle of p/3 between them **[AIEEE-06]**

12. Let $\vec{a}=\hat{i}-\hat{k}$, $\vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$ and $\vec{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$. Then $[\vec{a}\ \vec{b}\ \vec{c}]$ depends on **[AIEEE-05]**

a) both *x* and *y* b) Neither *x* nor *y* c) only *y* d) only *x*

13. If $\vec{a},\vec{b},\vec{c}$ be the unit vectors then $|\hat{a}-\hat{b}|^2+|\hat{b}-\hat{c}|^2+|\hat{c}-\hat{a}|^2$ does not exceed

a) 4 b) 9 c) 8 d) 6 **[IITSc.-01]**

14. The unit vector which is orthogonal $\vec{a}=3\hat{i}+2\hat{j}+6\hat{k}$ and coplanar with $\vec{b}=2\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}+\hat{k}$ is

a) $\frac{6\hat{i}-5\hat{k}}{\sqrt{61}}$ b) $\frac{3j-\hat{k}}{\sqrt{10}}$

c) $\frac{2\hat{i}-5\hat{j}}{\sqrt{29}}$ d) $\frac{2\hat{i}+j-2\hat{k}}{3}$ **[IITSc.-04]**

15. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{a}.\vec{b}=1$ and $\vec{a}\times\vec{b}=\vec{j}-\vec{k}$, then b is

a) $\hat{i}-\hat{j}+\hat{k}$ b) $2\hat{j}-\hat{k}$ c) $\hat{i}$ d) $2\hat{i}$ **[IITSc.-04]**

16. The value of *a* so that the volume of the paralellopiped formed by the vectors $\hat{i}+a\hat{j}+\hat{k}$, $\hat{j}+a\hat{k}$, $a\hat{i}+\hat{k}$

a) √3 b) 2 c) 1/√3 d)3

17. If the projection of $\vec{x}$ along a unit vector $\vec{a}$ be 2 and $\vec{a}\times\vec{x}+\vec{b}=\vec{x}$, then vector component of $\vec{x}$ perpendicular to $\vec{a}$ is

a) $\frac{2\vec{a}+\vec{b}+\vec{a}\times\vec{b}}{2}$ b) $\frac{-2\vec{a}+\vec{b}+\vec{a}\times\vec{b}}{2}$

c) $\frac{\vec{a}-\vec{b}+\vec{a}\times\vec{b}}{2}$ d) $\frac{\vec{a}-2\vec{b}+\vec{a}\times\vec{b}}{2}$

18. Vectors $\vec{a},\vec{b},\vec{p},\vec{q}$ are related as $\vec{p}=\vec{a}+\vec{b}$, $\vec{q}=\vec{a}\times\vec{b}$, $\vec{p}.\vec{a}=1$, then $\vec{a}$ is

a) $\frac{\vec{p}-\vec{p}\times\vec{q}}{\vec{p}.\vec{p}}$ b) $\frac{\vec{p}+\vec{p}\times\vec{q}}{\vec{p}.\vec{p}}$

c) $\frac{\vec{p}+\vec{p}\times\vec{q}}{\vec{q}.\vec{q}}$ d) $\frac{\vec{q}-\vec{p}\times\vec{q}}{\vec{q}.\vec{q}}$

19. If $\vec{a},\vec{b},\vec{c}$ are the non-coplanar vectors and $\vec{d}$ is a unit vector then $\left|(\vec{a}.\vec{d})\vec{b}\times\vec{c}+(\vec{b}.\vec{d})\vec{c}\times\vec{a}+(\vec{c}.\vec{d})\vec{a}\times\vec{b}\right|$ equals

a) 1 b) $\left|[\vec{a}\,\vec{b}\,\vec{c}]\right|$ c) $[\vec{a}\,\vec{b}\,\vec{c}]^2$ d) none of these

20. $\vec{a}.\vec{b}=0$ and $\vec{a}+\vec{b}$ makes an angle of 60^o with $\vec{b}$, then

a) $|\vec{a}|=\sqrt{3}|\vec{b}|$ b) $|\vec{b}|=2|\vec{a}|$ c) $|\vec{a}|=2|\vec{b}|$ d) none of these

21. If $|\vec{a}|=2$ and $|\vec{b}|=3$ $\vec{a}.\vec{b}=0$ then $(\vec{a}\times(\vec{a}\times(a\times(\vec{a}\times\vec{b}))))$

a) $16\vec{b}$ b) $-16\vec{b}$ c) $16\vec{a}$ d) $-16\vec{a}$

22. If $2\hat{a}+3\hat{b}+5\hat{c}=0$ then the area of the triangle whose sides are represented by the vectors $\hat{a},\hat{b},\hat{c}$ is

a) 0 b) 3 c) 5 d) 8

23. $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{r}$ is the variable vector such that $\vec{r}.\hat{i}, \vec{r}\hat{j}$ and $\vec{r}\hat{k}$ are positive integers if $\vec{r}.\vec{a}\le 12$ then the number of values of $\vec{r}$ is

a) ${}^{12}C_9 - 4$ b) ${}^{12}C_9$ c) ${}^{12}C_4$ d) none of these

24. Let $\vec{V}$ = 2i + j - k and $\vec{W}$ = i + 3k. If $\vec{U}$ is a unit vector then the maximum value of the scalar triple product $[\vec{U}\vec{V}\vec{W}]$ is:

a) -1 b) $\sqrt{10}+\sqrt{6}$ c) $\sqrt{59}$ d) $\sqrt{60}$

25. If a and b are two unit vectors such that a + 2b and 5a - 4b are perpendicular to each other then the anglebetween a and b is :

a) 45^0 b) 60^0 c) $\cos^{-1}\left(\frac{1}{3}\right)$ d) cos-1$\left(\frac{2}{7}\right)$

Level-2

1. $\left[a\times\left(3\vec{b}+2\vec{c}\right)\ \vec{b}\times\left(\vec{c}-2\vec{a}\right)\ 2c\times\left(\vec{a}-3\vec{b}\right)\right]$ Equal to

a) $-18\left[\vec{a}\vec{b}\vec{c}\right]^2$ b) $18\left[\vec{a}\vec{b}\vec{c}\right]^2$

c) $-2\left[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\right]$ d) $6\left[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\right]$

2. Let the unit vectors $\hat{a}$ and $\hat{b}$ be perpendicular and the unit vector $\hat{c}$ be inclined at an angle of ? to both $\hat{a}$ and $\hat{b}$. If $\vec{c}=\alpha\hat{a}+\beta\hat{b}+\gamma\left(\hat{a}\times\hat{b}\right)$ then

a) $\alpha=\beta$ b) $\gamma^2=1-2\alpha^2$

c) γ^2= -cos2? d) $\beta^2=\frac{1+\cos 2\theta}{2}$

3. The vector $\vec{a}=x\hat{i}-2\hat{j}+5\hat{k}$ and $\vec{b}=\hat{i}+y\hat{j}-z\hat{k}$ are collinear if

a) x =1, y =-2, z =-5 b) x =1/2, y =-4, z = -10

c) x = -1/2, y =4, z =10 d) x =-1, y=2, z =5

4. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}|=\sqrt{3}$, then

a) α =1, β = -1 b) α =1, $\beta=\pm 1$

c) α =-1, $\beta=\pm 1$ d) $\alpha=\pm 1$, β = 1

5. Let $\hat{a}$ and $\hat{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-\left(\vec{a}.\vec{b}\right)\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$ then

a) $|\vec{u}|$ b) $|\vec{u}|+|\vec{u}.\vec{a}|$

c) $|\vec{u}|+|\vec{u}.\vec{b}|$ d) $|\vec{u}|+\vec{u}.\left(\vec{a}+\vec{b}\right)$

6. Let $\vec{p},\vec{q},\vec{r}$ be the three mutually perpendicular vectors of the same magnitude. If the vector x satisfies the equation $\vec{p}\times[(\vec{x}-\vec{q})\times\vec{p}]+\vec{q}\times[(\vec{x}-\vec{r})\times\vec{q}]+\vec{r}\times[(\vec{x}-\vec{p})\times\vec{r}]=0$ then x is given by

a) $\frac{1}{2}(\vec{p}+\vec{q}-2\vec{r})$ b) $\frac{1}{2}(\vec{p}+\vec{q}+\vec{r})$

c) $\frac{1}{3}(\vec{p}+\vec{q}+\vec{r})$ d) $\frac{1}{3}(2\vec{p}+\vec{q}-\vec{r})$

7. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. If $\vec{c}$ is the vector such that $\vec{a}.\vec{c}=|\vec{c}|, |\vec{c}-\vec{a}|=2\sqrt{2}$ and the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$ is 30^o, then $|(\vec{a}\times\vec{b})\times\vec{c}|$ =

a) 2/3 b) 3/2 c) 2 d) 3

8. The magnitude of vectors $\vec{a},\vec{b},\vec{c}$ are respectively 1, 1 and 2. If $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=0$ then the acute angle between a and c is

a) p/6 b) p/4 c)p/3 d) p/2

9. Let the vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ be such that $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=0$ Let P_1& P_2 be plane determined by the pairs of vectors a, b and c, d respectively. Then angle between P_1& P_2 is

a) 0 b) p/4 c) p/3 d) p/2

10. For three vectors $\vec{u},\vec{v},\vec{w}$ which of the following expressions is not equal to any of the remaining three?

a) $\vec{u}.(\vec{v}\times\vec{w})$ b) $(\vec{v}\times\vec{w}).\vec{u}$ c) $\vec{v}.(\vec{u}\times\vec{w})$ d) $(\vec{u}\times\vec{v}).\vec{w}$

11. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ $\vec{a}.\vec{b}=2$ and $\vec{a}\times\vec{b}=2\hat{i}+\hat{j}-3\hat{k}$, then

a) $\vec{a}+\vec{b}=5\hat{i}-4\hat{j}+2\hat{k}$ b) $\vec{a}+\vec{b}=3\hat{i}+2\hat{k}$

c) $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ d) $\vec{b}=\hat{i}-2\hat{j}-3\hat{k}$

12. If $\hat{a}, \vec{b}, \vec{c}$ are the unit vectors such that $\hat{a}.\hat{b}=0=\hat{a}.\hat{c}$ and angle between $\hat{b}$ and $\hat{c}$ is p/3. then the value of $|\vec{a}\times\vec{b}-\vec{a}\times\vec{c}|$ is

a) ½ b) 1 c) 2√2 d) none of these

13. Let $\hat{r}$ be the unit vectors satisfying $\vec{r}\times\vec{a}=\vec{b}$, where $|\vec{a}|=\sqrt{3}$ and $|\vec{b}|=\sqrt{2}$, then

a) $\vec{r}=\frac{2}{3}(\vec{a}+\vec{a}\times\vec{b})$ b) $\vec{r}=\frac{1}{3}(\vec{a}+\vec{a}\times\vec{b})$

c) $\vec{r} = \frac{2}{3}\left(\vec{a} - \vec{a}\times\vec{b}\right)$ d) $\vec{r} = \frac{1}{3}\left(-\vec{a} + \vec{a}\times\vec{b}\right)$

14. A vector $\vec{a}$ has a component 2*p* and 1 with respect to the rectangular Cartesian system. This system is rotated through a certain angle about the origin in the counter-clock wise sense. If with respect to new system $\vec{a}$ has components *p*+1 and 1, then

a) p =0 b) p=1 or p= –1/3

c) p= –1 or p=1/3 d) p=1 or p= –1

15. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$ If the vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, then *x* is equals

a) –2 b) 0 c) 1 d) 4

16. The vector $\hat{i} + x\hat{j} + 3\hat{k}$ is rotated through an angle ? and is doubled in magnitude, then it becomes $4\hat{i} + (4x-2)\hat{j} + 2\hat{k}$. The value of *x* is

a) -2/3, 2 b) 1/3, 2 c) 2/3, 0 d) 2, 7

17. $\hat{a}, \vec{b}, \vec{c}$ are the edges of the cube of unit length and $\vec{r}$ be the unit vector in the cube, then $|\vec{r}\times\vec{a}|^2 + |\vec{r}\times\vec{b}|^2 + |\vec{r}\times\vec{c}|^2$ is

a) 0 b) 1 c) 2 d) 3

18. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplaner vectors such that $\vec{b}\times\vec{c} = \vec{a}$, $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{c}\times\vec{a} = \vec{b}$ then

a) $[\vec{a}\vec{b}\vec{c}] = 1$ b) $[\vec{a}\vec{b}\vec{c}] = 1$

c) $|\vec{a}| + |\vec{b}| + |\vec{c}| = 1$ d) none of these

19. If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are unit vectors such that $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right) = 1$ and $\vec{a}.\vec{c} = \frac{1}{2}$ then

[IIT-2009, 3]

a) $\vec{a},\vec{b},\vec{c}$ are non-coplaner b) $\vec{b},\vec{c},\vec{d}$ are non-coplaner

c) $\vec{b},\vec{d}$ are non parallel d) $\vec{a},\vec{d}$ are parallel and $\vec{b},\vec{c}$ are parallel

20. Let two –collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector OP (where O is the origin) is given by $\hat{a}\cos t + \hat{b}\sin t$. When P is farthest from the origin O, let M be the length of OP and $\hat{u}$ be the unit vector along OP. Then **[IIT-2008, 3]**

a) $\hat{u}=\frac{\hat{a}+\hat{b}}{\left|\hat{a}+\hat{b}\right|}$ and $M=\left(1+\hat{a}.\hat{b}\right)^{\frac{1}{2}}$ b) $\hat{u}=\frac{\hat{a}-\hat{b}}{\left|\hat{a}-\hat{b}\right|}$ and $M=\left(1+\hat{a}.\hat{b}\right)^{\frac{1}{2}}$

c) $\hat{u}=\frac{\hat{a}+\hat{b}}{\left|\hat{a}+\hat{b}\right|}$ and $M=\left(1+2\hat{a}.\hat{b}\right)^{\frac{1}{2}}$ d) $\hat{u}=\frac{\hat{a}-\hat{b}}{\left|\hat{a}-\hat{b}\right|}$ and $M=\left(1+2\hat{a}.\hat{b}\right)^{\frac{1}{2}}$

21. Let the vectors PQ, QR, RS, ST, TU and UP represent the sides of the regular hexagon.

STATEMENT-1: PQ$\times$(RS + ST) $\neq$0

because

STATEMENT-2: PQ$\times$RS = 0 and PQ$\times$ST$\neq$0

a) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is the correct explanation STATEMENT-1

b) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is not the correct explanation STATEMENT-1

c) STATEMENT-1 is true, STATEMENT-2 is False

d) STATEMENT-1 is False, STATEMENT-2 is true **[IIT-2007, 3]**

22. The edges of a parallopiped are of unit lengths and are parallel to non-coplaner unit vectors $\hat{a}$, $\hat{b}$, $\hat{c}$ such that $\hat{a}.\hat{b}=\hat{b}.\hat{c}=\hat{c}.\hat{a}=1/2$ the the volume of the parallelopiped is **[IIT-2008, 3]**

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2\sqrt{2}}$ c) $\frac{\sqrt{3}}{2}$ d) $\frac{1}{\sqrt{3}}$

6. Straight Line

6.1 Introduction:

A straight line in a plane divides the plane in two halves. Plane is two dimensional. In co-ordinate geometry we have seen that perpendicular axes X'OX and Y'OY divide the plane in four parts called quadrants. But space is three dimensional, length breadth and height is associated with every object in space.

6.2 Co-ordiantes of a point in space :

The space is divided in 8 octants by three mutually perpendicular line X'OX, Y'OY, ZOZ' called x axis, y axis and z axis respectively. These intersect in O. O is called origin.

Let P be a point in space. PM is perpendicular from P on x – y plane. PM is distance of point P from x – y plane. |PM| is z co-ordinate of point from M draw MN parallel to OY. It shall be perpendicular to OX. MN is distance of the point from x – z plane. It is called y-co-ordinate of the on it. It is OT also. ON is perpendicular to z-y plane.

ON = distance of the point from y – z plane

ON = x co-ordinate of the point P.

$\therefore$ If P is (x, y, z) then

ON = x, NM = y and PM = z, triangle ONM in the plane XOY is right angled at N. OM is hypotenuse

$\therefore$ $OM^2 = ON^2 + NM^2 = ON^2 + OT^2 = x^2 + y^2$

Triangle POM is right angled at M (PM perpendicular on x – y plane)

$\therefore$ $OP^2 = OM^2 + PM^2 = x^2 + y^2 + z^2$

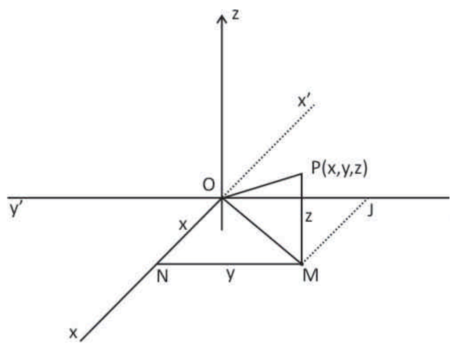


Fig 1

$\therefore$ Distance of a point P (x, y, z) from origin $OP = \sqrt{x^2 + y^2 + z^2}$

Note : In co-ordinate geometry of two dimension every point in the plane was represented by an ordered pair (a,b), similarly in space a point is represented by an ordered triand (a, b,c)

6.3 Distance between two points :

A $(x_1y_1z_1)$ and $B(x_2,y_2,z_2)$ are two points in space. Join AB. AM and BM are perpendicular on x – y plane from A and B, and AL is perpendicular on BN.

$BN = z_2$, $AM = z_1$

Point M is $(x_1\ y_1)$ and point N is $(x_2\ y_2)$ in x – y plane.

$\therefore\ MN^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

from right angled triangle ALB'

$$AB^2 = (AL)^2 + (BL)^2 = (MN)^2 + (BL)^2$$

$$= \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]$$

$$\therefore AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

6.4 Section Formula :

To find the co-ordinate of the point dividing the line joining A(x_1, y_1, z_1) and B (x_2, y_2, z_2) in the ratio of m : n. Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be the two points and let ($\bar{x}, \bar{y}, \bar{z}$) be the point P which divide AB in the ratio of m : n.

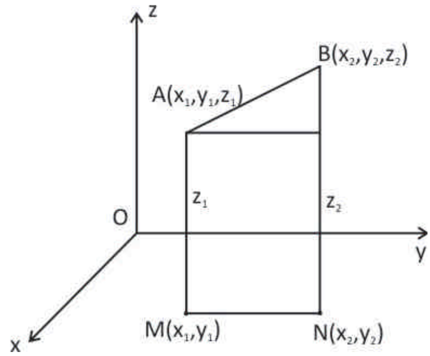


Fig 2

From A,P,B perpendicular AL, PM and BN are drawn on x – y plane. (LMN is one line, projection of AB on x-y plane) Through P drawn kPT perpendicular to AL and BN. It is parallel to LMN.

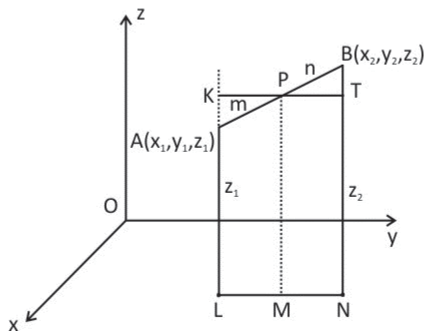


Fig 3

From similar triangles KAP and BPT

$$\frac{m}{n} = \frac{AP}{BP} = \frac{KA}{BT} = \frac{KP}{PT}$$

$$\frac{m}{n} = \frac{Ak}{BT} = \frac{\bar{z} - z_1}{z_2 - \bar{z}} \Rightarrow mz_2 - m\bar{z} = n\bar{z} - nz_1$$

$$\Rightarrow mz_2 + nz_1 = (m+n)\bar{z}$$

$$\Rightarrow \bar{z} = \frac{mz_2 + nz_1}{m+n}$$

Similarly $\bar{x} = \frac{mx_2 + nx_1}{m+n}, \bar{y} = \frac{my_2 + ny_1}{m+n}$

$\therefore$ P is $\bar{x} = \frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{my_2 + ny_1}{m+n}$

Corollary 1 : co-ordinate of mid point of A (x_1, y_1, z_1) and B (x_2, y_2, z_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

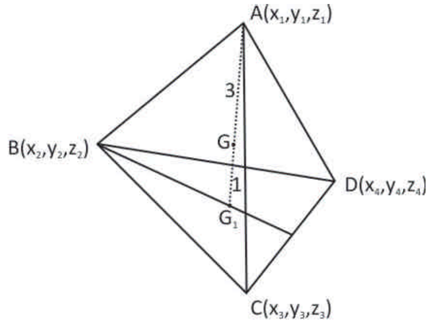


Fig 4a

Corollary 2 : Point dividing A(x_1, y_1, z_1) and B (x_2, y_2, z_2) externally in the ratio of m : n is $\left[\frac{mx_2+nx_1}{m-n},\frac{my_2+ny_1}{m-n},\frac{mz_2+nz_1}{m-n}\right]$

Corollary 3 : Centroid of the triangle ABC. When A is (x_1, y_1, z_1), B is (x_2, y_2, z_2) and C is (x_3, y_3, z_3), is

$$\left[\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right]$$

Example 1 : The vertices of a tetrahedron are (x_r, y_r, z_r), r = 1,2,3,4. Find its centroid.

Sol. In the fig. G is centroid of triangular face BCD of the tetrahedron.

$\therefore$ G_1 is

$$\left(\frac{x_2+x_3+x_4}{3},\frac{y_2+y_3+y_4}{3},\frac{z_2+z_3+z_4}{3}\right)$$

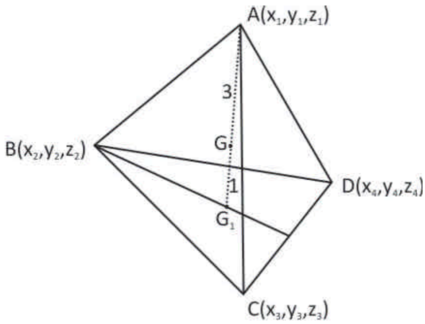


Fig 4b

$\therefore$ The centre of gravity of tetrahedron is at G which divides AG1 in the ratio of 3 : 1.

$\therefore$ x co-ordinates of G is

$$\frac{3.\left(\frac{x_2+x_3+x_4}{3}\right)+1.x_1}{3+1}$$

i.e. $\frac{x_1+x_2+x_3+x_4}{4}$

$\therefore$ Centroid G is $\left[\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right]$

Example 2 : Find the ratio in which the surface (sphere) $x^2 + y^2 + z^2 = 504$ divides the line joining the points (12, –4, 8) and (27, –9, 18)

Sol. A (12, –4, 8), B(27, – 9, 18). Let the surface divide AB in the ratio of l : 1 at P. Then co-ordinate of P are.

$\left[\frac{27\lambda+12}{\lambda+1},\frac{-9\lambda-4}{\lambda+1},\frac{18\lambda+8}{\lambda+1}\right]$ This point must lie on the surface.

$$\therefore (27\lambda+12)^2+(9\lambda+4)^2+(18\lambda+8)^2=504(\lambda+1)^2$$

$$\Rightarrow (729+81+324-504)\lambda^2+(648+72+288-1008)\lambda+(144+16+64-504)=0$$

$$\Rightarrow 630\lambda^2-280=0 \quad \Rightarrow 9\lambda^2-4=0$$

$$\therefore \lambda=\pm 2/3$$

∴ The surface divided by AB in the ratio of 2 : 3 and 2 : – 3.

∴ Point A is inside sphere, B is outside sphere.

6.5 Direction cosines of a line :

AB is the give line. OP is drawn parallel to AB from origin and equal to AB with OP as diagonal cuboid has been constructed whose three co terminous edges are OA, OB, OC are along x, y and z axis.

Let OP make angle α with x-axes. Then OP is the diagonal of rectangle APNO

$\therefore$ OA = OP cos α

If P (x_1, y_1, z_1) then OA = x_1 = OP cos α .

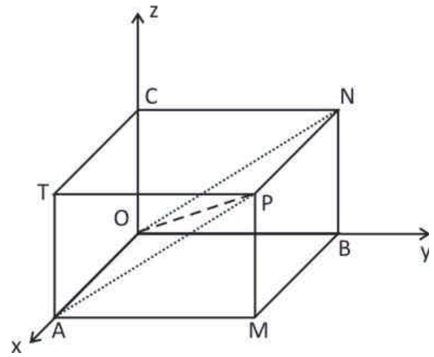


Fig 5

(ii) If OP makes angle β with y-axis then from rectangle TOBP ($\angle PBO = 90^0$) OB = OP cosβ = y_1 cosβ.

(iii) If OP makes angle γ with OZ then from rectangle COMP, OC = z_1 = OP cos r.

If a straight line makes angle α, β and γ with x, y and z axis respectively, then cos α, cosβ and cos γ are called direction cosines of the line. Generally these are denoted by l, m and n.

$l = \cos\alpha, m = \cos\beta,\ n = \cos\gamma$.

We have seen above that if straight line OP where P is (x_1, y_1, z_1) makes angle a , b and γ with axes. Then x_1 = OP cos α , y_1 = OP cosβ, z_1 = OPcos γ

And OP^2 = $x_1^2 + y_1^2 + z_1^2$

$\therefore (OP\cos\alpha)^2 + (OP\cos\beta)^2 + (OP\cos\gamma)^2 = OP^2$

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

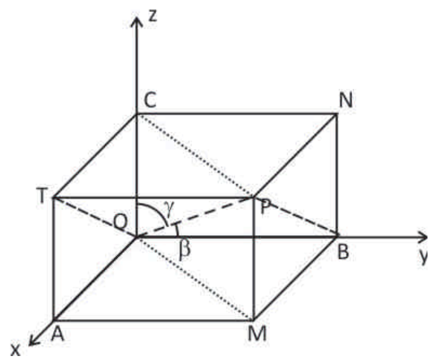


Fig 6

i.e. sum of the squares of direction cosines is equal to one.

(a) If a straight line makes equal angle with axes then its d.c. are $\cos\alpha, \cos\alpha, \cos\alpha$

$\therefore \cos^2\alpha, + \cos^2\alpha, + \cos^2\alpha, = 3\cos^2\alpha = 1$

$\therefore \cos\alpha = \frac{1}{\sqrt{3}}$

d.c. of the straightly line are $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

(b) If $\cos\alpha, \cos\beta, \cos\gamma$ are d.c. of a straight lien AB the d. cosines of straight line BA are

$\cos(180-\alpha).\cos(180-\beta), \cos(180-\gamma)$

i.e. $-\cos\alpha, -\cos\beta, -\cos\gamma$.

(c) If P is any point on the line OAB whose d.c. are l, m, n and OP = r. Then

P is (x, y, z) then x = l, y = mr, z= nr

(d) If OP = r and P is (x, y, z) then d.c. of OP are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$.

(e) D.c. of x-axis, 1, 0, 0; of y-axis 0,1,0; of z –axis 0,0,1

6.6 Direction cosines of the straight line joining two points :

Let the two points. Let the two points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. From P and Q drop perpendicular PM and QN on y axis. Then M and N are (0,y,0) and (0,y,0) respectively. And MN = $y_2 - y_1$ Now drawn QL perpendicular from Q on PM, $\angle PQL = \beta$, so that $\cos\beta$ = m.

Now m = cos β = $\frac{y_2 - y_1}{r}$ where r = PQ

similarly it can be shown that

$r = \cos\alpha = \frac{x_2 - x_1}{r}, n = \cos\gamma = \frac{z_2 - z_1}{r}$

Hence Direction cosines of PQ are

$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$

Alternate Method – If origin is shifted to P the new co-ordinates of Q with respect to new origin shall be $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ Hence direction cosines shall be

$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$

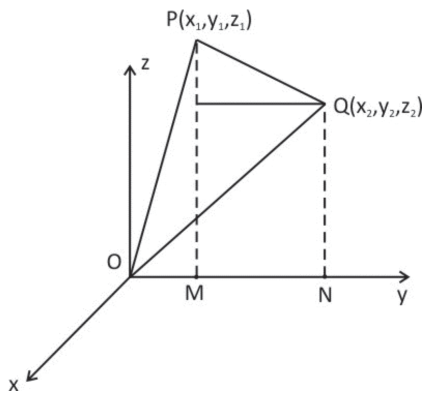


Fig 7

6.7 Direction Ratios :

Quantities proportional to direction cosines called direction ratios. Thus if a, b, c are the direction ratios of a straight line whose d. cosines are l, m, n then

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}=\pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow l=\frac{a}{\sqrt{a^2+b^2+c^2}},m=\frac{b}{\sqrt{a^2+b^2+c^2}},n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

$\therefore$ If 1,2,2 be the direction ratio of a straight line then by dividing these quantities.

By $\sqrt{1+4+4}=3$, we get d. cosines of straight lien d.c. are $\frac{1}{3},\frac{2}{3},\frac{2}{3}$

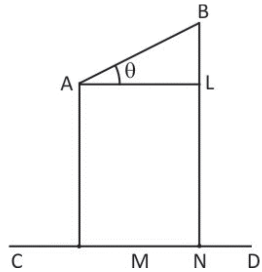


Fig 8

6.8 Projection of a line segment:

In fig. 8 straight line AB and CD lie in the same plane. AM and BN are perpendicular from A and B on CD, meet CD in M and N

MN = projection of AB on CD. If angle between these two lines is θ. Then MN=AB cosθ.

6.9 Projection of a straight line OP (o, origin) on a straight line whose direction cosines are l, m, n.

Let P be (x_1, y_1, z_1) Let us find the projection of straight line OP on straight line AB. Through O, draw OQ parallel to AB, Projection of OP and AB shall be equal to projection of OP on OQ. In fig 9 PM is perpendicular on x – y plane and MN is perpendicular on OY .

Projection of OP on AB=projection of OP on OQ=Sum of projection of ON, MN and MP on OQ

($\because$ vector $O\overline{P}=O\overline{N}+M\overline{N}+\overline{M}P$)

NK is perpendicular from N on OQ and OQ is inclined at β with OY.

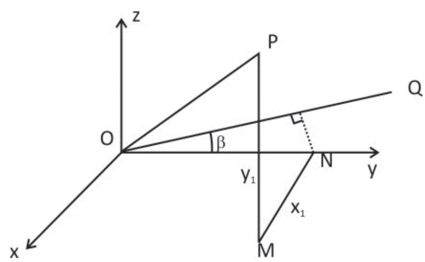


Fig 8

$\therefore$ projection of ON on OQ = OK = $y_1 \cos\beta$

Projection of NM on OQ = $x_1 \cos\alpha$ ($l = \cos\alpha$)

Projection of MN on OQ = $z_1 \cos\gamma$ ($n = \cos\gamma$)

$\therefore$ Projection of OP on AB = $x_1 \cos\alpha + y_1 \cos\beta + z_1 \cos\gamma = lx_1 + my_1 + nz_1$

Corollary – From above it is clear that if A is (x_1, y_1, z_1), $B(x_2, y_2, z_2)$ then projection of line segment AB on straight line whose d.c. are l, m, and n is

$$(x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$$

i.e. $$(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$$

6.10 Angle between two straight line :

Let OP and OQ be straight line parallel to the two given lines. $\angle POQ$ is the angle between them then; Let l_1, m_1, n_1 be d.c. of OP and l_2, m_2, n_2 d.c. OQ and $\angle POQ = \theta$

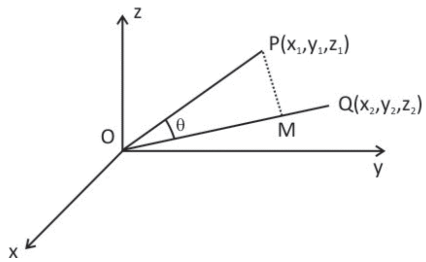


Fig 10

Perpendicular from P on OQ is OM

$\therefore$ OM is projection of OP on OQ.

$\therefore$ OM = OP $\cos q$ = $l_2x_1 + m_2y_1 + n_2 z_1$

$$\Rightarrow \cos\theta = \frac{l_2x_1}{OP} + \frac{m_2y_1}{OP} + \frac{n_2z_1}{OP}$$

But $\frac{x_1}{OP} = l_1, \frac{y_1}{OP} = m_1, \frac{z_1}{OP} = n_1$

$$\therefore \cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

Corollary 1 : If a_1, b_1, c_1 and a_2, b_2, c_2 are d. ratios of OP and OQ, then d.c. of OP are

$\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$ and d.c. of OQ are $\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$

$$\therefore \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Corollary 2 : (a) If $a_1a_2 + b_1b_2 + c_1c_2 = 0$ the straight lines are perpendicular to each other.

(b) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then straight line are parallel.

(c) Condition (a) and (b) are equivalent to $l_1 l_2 + m_1m_2 + n_1n_2 = 0$ and $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Example 3 : The vertices of a triangle are A(3,1, –2), B(5, 3, –1) and C(7, 4, –2) find the d.c. of the bisector of angle A.

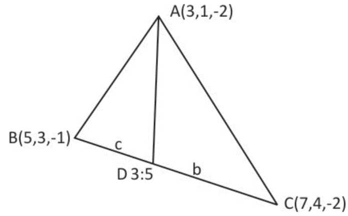


Fig 11

Sol. : $AB=\sqrt{(5-3)^2+(3-1)^2+(-1+2)^2}$

$= =\sqrt{4+4+1}=3$

$AC=\sqrt{(7-3)^2+(4-1)^2+0^2}=5$

And bisector of $\angle$A meets BC in D.

$\therefore$ BD : DC = AB : AC = 3 : 5

$\therefore$ D is $\left[\frac{21+25}{8},\frac{12+15}{8},\frac{-6-5}{8}\right]$ i.e. $\left[\frac{46}{8},\frac{27}{8},\frac{-11}{8}\right]$

Direction ratio of AD are $\left(\frac{46}{8}-3\right),\left(\frac{27}{8}-1\right),\left(-\frac{11}{8}+2\right)$ i.e. (22,19,5) d.c. are $\frac{22}{\sqrt{870}},\frac{19}{\sqrt{870}},\frac{5}{\sqrt{870}}$

Example 4 : Find the co-ordinate of the foot of perpendicular from P(2,1,3) on the line joining A(3,2,1) and B(1,3,4).

Fig 12

Sol. : Let perpendicular from P meets AB in M, and M divides AB in the ratio of k : 1

$\therefore$ Co-ordiante of M are $\left(\frac{k+3}{k+1},\frac{3k+2}{k+1},\frac{4k+1}{k+1}\right)$

$\therefore$ Direction ratios of PM $\left(\frac{k+3}{k+1}-2\right),\left(\frac{3k+2}{k+1}-1\right),\left(\frac{4k+1}{k+1}-3\right)\Rightarrow\left(\frac{1-k}{k+1},\frac{2k+1}{k+1},\frac{k-2}{k+1}\right)$

Direction ratio of AB, –2, 1, 3. PM is $\perp$ on AB.

$\therefore\ -2(1-k)+1.(2k+1)+3(k-2)=0$

$\Rightarrow 7k-7=0$ $\therefore$ k = 1

$\therefore$ M is $\left(2,\frac{5}{2},\frac{5}{2}\right)$

Example 5 : The projection of a line on axes are, 6,2,3, what is the length of line?

Sol. : Let AB the length of the line

$\therefore$ Projection on x axis = AB . *l* + AB. O + AB. P = AB . *l* = 6

and AB . m = 2 , AB . n = 3

Squaring and adding $AB^2 (l^2 + m^2 + n^2) = 6^2 + 2^2 + 3^2 = 49$

$\therefore$ AB . l = 7 $\Rightarrow AB = 7$

Example 6 : A, B, C, D are points (–2, 3, 4); (–4,4,6); (4,3,5) and (0,1,2) Prove by projection that $AB \perp CD$.

Sol.: A(–2,3,4), B(–4,4,6), C(4,3,5), D(0,1,2). D. Ratios of AB are –2,1–2

$\therefore$ d.c. are $-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$

Projection of CD on AB is

$$-\frac{2}{3}(0-4)+\frac{1}{3}(1-3)+\frac{2}{3}(2-5)$$

$$=\frac{8}{5}-\frac{2}{3}-\frac{6}{3}=0=CD\cos 90^0$$

$\therefore AB \perp CD.$

Example 7 : If line mates angle $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube, prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$.

Sol. In the fig the four diagonals are OG, EB, AD and CF. Let the side of cube be a, coordinates of G(a,a,a), E(0,0,a), B(a,a,0), A(a,0,0) D(0,a,a), C(0,a,0) and F(a,0,a)

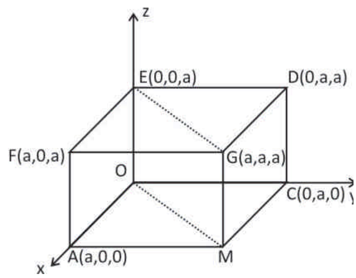


Fig 13

D.C. of OG $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ of EB $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ of AD $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ of CF $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Let the D.C of the line be l, m, n.

$$\therefore \cos\alpha=\frac{l+m+n}{\sqrt{3}}, \cos\beta=\frac{l+m-n}{\sqrt{3}}$$

$$\cos\gamma=\frac{-l+m+n}{\sqrt{3}}, \cos\delta=\frac{l-m+n}{\sqrt{3}}$$

$$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{1}{3}\left[(l+m+n)^2+(l+m-n)^2+(l-m+n)^2+(-l+m+n)^2\right]$$

$$=\frac{1}{3}\left[4\left(l^2+m^2+n^2\right)\right]=\frac{4}{3}$$

Example 8 : If the edges of a rectangular parallel piped (cuboid) are OA, OB and OC on axes of reference find, the angles between the diagonals.

Sol. : Diagonals in the fig, are ON, BT, AM and CL ON and CL are diagonal of rectangle OCNL, AM and BT are diagonal of rectangle TABM co-ordinate of N (a,b,c) B(0,b,0), T(a,0,c), C(0,0,c) A(a,0,0), M(0,b,c), L(a,b,0) D.R of ON (a,b,c) of CL (a, b, –c)

$$\cos\theta = \left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$$

D.R of AM (–a, b, c); of BT (a, –b, c)

$$\cos\theta = \frac{c^2-a^2-b^2}{\left(a^2+b^2+c^2\right)}$$

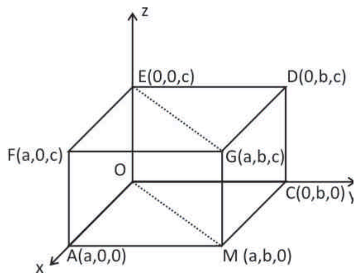


Fig 14

Example 9 : Show that the straight line whose d.c. are given by equations l + m + n = 0 and 2mn + 3nl –5 ln = 0 are perpendicular to each other.

Sol. : $l + m + n = 0 \Rightarrow n = -(l + m)$

Given $2mn + 3nl - 5lm = 0$

$$\Rightarrow -2m(l+m) - 3l(l+m) - 5lm = 0$$

$$\Rightarrow 2m^2 + 3l^2 + 10lm = 0$$

$$\Rightarrow 3\left(\frac{l}{m}\right)^2 + 10\frac{l}{m} + 2 = 0$$

$$\therefore \frac{l_1}{m_1} \times \frac{l_2}{m_2} = \frac{2}{3} \text{ (i) and } \therefore \frac{l_1}{m_1} + \frac{l_2}{m_2} = -\frac{10}{3} \text{ ....(ii)}$$

$$\Rightarrow \frac{l_1 l_2}{2} = \frac{m_1 m_2}{3} = k \quad \therefore l_1 l_2 = 2k, m_1 m_2 = 3k \text{ and}$$

$$\text{from (ii) } l_1 m_2 + m_1 l_2 = -\frac{10}{3} \times m_1 m_2 = -\frac{10}{3}(3k)$$

$$= -10k$$

and $n_1 = -(l_1 + m_1)$ and $n_1 = -(l_2 + m_2)$

$$\therefore n_1 n_2 = (l_1 + m_1)(l_2 + m_2) = l_1 l_2 + m_1 m_2 + l_1 m_2 + l_2 m_1$$

$$= 2k + 3k - 10k = -5k$$

$$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 2k + 3k - 5k = 0$$

$\therefore$ Lines are perpendicular.

Exercise 1(a)

1. The vertices of a triangle are A(5,4, –2), B(3,2, –3) and C(4,2,0) (a) What type of triangle ABC is? (b) Find C. G. of this triangle
2. The vertices of triangle are A(4,2,5), B(2,1,3) and C(5,7,1). Find angles of the triangle.
3. Find locus of point equidistant from points (2 –1, –3) and (–1,2,1). The locus shall be plane/straight line / or else?
4. The projections of a line segment on axes of a straight line going through origin are 4,3 and 5 respectively. Find length of the segment. (b) Find its direction cosines.
5. Points A(2,4,5), B(0,6,3), C(–1, –2,5) and D(3,2,1) are in space. Find angle between AB and CD.
6. The distance of a point from axes are $2\sqrt{3}$, $2\sqrt{5}$ and $3\sqrt{2}$ respectively. Find distance of this point from origin.
7. A(3,2,0), B(5,3,2) and C(–9,6, –3) are vertices of vertices of triangle ABC. The internal bisector of angle A meets BC in D. Find co-ordinates of D.
8. Projection of straight line OP on x – y plane is 5. If z co-ordinate of P be 5. Find OP.
9. Point P is (3,4,5). Find angles that the projections on axes of OP make with OP.
10. A straight line is perpendicular to two straight line whose d.r. are 1,3,5 and –2, 4, 3 find d.c. of this line.
11. If α,β,γ be angles that half-ray through origin makes with positive directions of axes, then find value of $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$.
12. Find angle between straight line whose d.c. are given by equations l+m+n = 0, and 2nl+2lm–mn =0.
13. Find angle between straight lines whose d.c. satisfy equations. 3*l*+m+5n=0 and 6mn–2n*l*+5 lm = 0
14. The d.c. of two straight lines satisfy equations, l+m=n and 2lm – 3mn + 2ln = 0. Find angle between them.
15. Point P is (1,3,5), then calculate d.c. of straight line OP.
16. A straight line goes through (–1,2,4) and (1,0,5) and points A and B are (3,4,5) and (4,5,3). Find projection of AB on the straight line.
17. The direction cosines of a straight line are 2, –1, 5, P is a point (3,6,2). Find projection of OP on the straight line.
18. The d.r of two straight line are 3,4,5 and 2,6,3. Find the d.c. of the straight line which is perpendicular to these lines.

19. OA, OB, OC are three conterminous edges of a cuboid along axes. Show that angle between the two diagonals is $\cos^{-1}\left(-\frac{1}{3}\right)$

20. a, b, c are the three conterminous edges of a cuboid, along axes, and a vertex coincides with origin, a>b>c angle between diagonals of greatest face is $\cos^{-1}\left(\frac{a^2-b^2}{2\sqrt{a^2+b^2}}\right)$, prove.

21. The vertex C of a tetrahedron is (0,0,c) and its two edges OA, OB of length a and b lie along x and y axis. O is common to the three. G is centroid of Δ ABC and G_1 of Δ OAB. Find angle between OG and CG_1.

6.11 (A) Equation of a straight line :

Let the straight line pass through A (x_1, y_1, z_1) and P(x,y,z) be any point on it. If Direction cosine of this line AP be l, m and n, then.

Projection of AP on x-axis $= (x - x_1) .1 + 0 + 0$

(AP = r) $= lr \Rightarrow (x - x_1) = lr$

Projection of AP on y-axis $= 0 + (y - y_1) . 1 + 0 = mr$

Projection of AP on z-axis $= 0 + 0 + (z - z_1) . 1 = n r$

$$\therefore \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$

Here as mentioned, r is distance of point (x, y, z) from point (x_1, y_1, z_1) on this line.

$\therefore$ Equation of straight line is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$.

(B) Equation of straight line through two points:

Let the points be (x_1, y_1, z_1) and (x_2, y_2, z_2). Any lines through (x_1, y, z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \text{ where l, m, n are its d.cs.}$$

But the d.r. joining two points are $x_2 - x_1, y_2 - y_1, z_2 - z_1 \propto$ to D. cosines.

$\therefore$ Equation of this lines is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

Note : Any point on straight line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ distance r from (x_1, y_1, z_1) is $lr + x_1, mr + y_1, nr + z_1$.

6.12 Equation of angle bisectors of the angle between two straight lines :

l_1, m_1, n_1 are d.c. of straight line AOB and l_2, m_2, n_2 are d.c. of straight line COD. Both pass through origin. Let p be a point on OB and point Q on OD such that OP = OQ = 1 unit. Mid point of PQ is M and it shall lie on the angle bisector EF. Similarly OR = – 1, then mid point of RP shall lie on the other angle bisector GH.

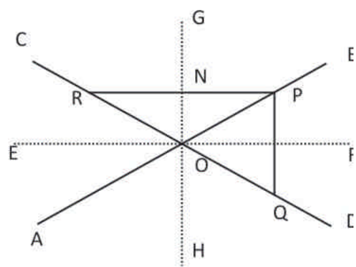


Fig 15

l_1, m_1, n_1 and l_2, m_2, n_2 are d.c. of two lines.

$\therefore$ M is $\left(\frac{l_1 + l_2}{2}, \frac{m_1 + m_2}{2}, \frac{n_1 + n_2}{2}\right)$, N is

$\left(\frac{l_1-l_2}{2},\frac{m_1-m_2}{2},\frac{n_1-n_2}{2}\right)$,

$\therefore$ D.C. of angle bisector EMF are $\left(\frac{l_1+l_2}{2},\frac{m_1+m_2}{2},\frac{n_1+n_2}{2}\right)$, and of other angle bisector $\left(\frac{l_1-l_2}{2},\frac{m_1-m_2}{2},\frac{n_1-n_2}{2}\right)$,

$\therefore$ Equations of these angle bisectors are $\frac{x-0}{(l_1+l_2)/2}=\frac{y-0}{(m_1+m_2)/2}=\frac{z-0}{(n_1+n_2)/2}$

$\Rightarrow \frac{2x}{l_1+l_2}=\frac{2y}{m_1+m_2}=\frac{2z}{n_1+n_2}$ and $\frac{2x}{l_1-l_2}=\frac{2y}{m_1-m_2}=\frac{2z}{n_1-n_2}$

Note : If point of intersection of the given straight line is not origin but (x_1, y_1, z_1) then equation of angle bisectors are.

$$\frac{2(x-x_1)}{l_1+l_2}=\frac{2(y-y_1)}{m_1+m_2}=\frac{2(z-z_1)}{n_1+n_2};\frac{2(x-x_1)}{l_1-l_2}=\frac{2(y-y_1)}{m_1-m_2}=\frac{2(z-z_1)}{n_1-n_2}$$

Example 10 : Find d.c. of straight line $\frac{x-2}{2}=\frac{2y-5}{-3}$ and z = 1

Sol. Given line is $\frac{x-2}{2}=\frac{y-5/2}{-3/2}=\frac{z-1}{0}$

D. ratio are 2, –3/2, 0 and $\sqrt{4+\frac{9}{4}}=\frac{5}{2}$

$\therefore$ D.C of straight line are $\frac{2}{5/2},\frac{-3/2}{5/2},0\Rightarrow\frac{4}{5},-\frac{3}{5},0$

Example 11 : Find length of perpendicular from P. (1,2,3) on straight line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$.

Sol. : $\sqrt{3^2+2^2+(-2)^2}=\sqrt{17}$

Given line is $\frac{x-6}{3/\sqrt{17}}=\frac{y-7}{2/\sqrt{17}}=\frac{z-7}{-2/\sqrt{17}}=r$ any point on its is $\left(\frac{3}{\sqrt{17}}r+6,\frac{2}{\sqrt{17}}r+7,\frac{-2}{\sqrt{17}}r+7\right)$

If it is a foot of perpendicular PQ from P on the line, then d.c. of PQ are proportional to

$$\left(\frac{3}{\sqrt{17}}r+6-1,\frac{2}{\sqrt{17}}r+7-2,\frac{-2}{\sqrt{17}}r+7-3\right)$$

P.Q. is perpendicular to given line

$$\therefore\frac{3}{\sqrt{17}}\left(\frac{3}{\sqrt{17}}r+5\right)+\frac{2}{\sqrt{17}}\left(\frac{2r}{\sqrt{17}}+5\right)-\frac{2}{\sqrt{17}}\left(\frac{-2r}{\sqrt{17}}+4\right)=0$$

$$\Rightarrow\frac{1}{\sqrt{17}}(9r+4r+4r)+15+10-8=0$$

$$\Rightarrow\sqrt{17}r+17=0 \qquad \therefore r=-\sqrt{17}$$

$$\therefore Q \text{ is }\left[\frac{-3}{\sqrt{17}}.\sqrt{17}+6,\frac{-2}{\sqrt{17}}\sqrt{17}+7,\frac{2}{\sqrt{17}}\sqrt{17}+7\right]$$

i.e. Q is (3,5,9) $\Rightarrow PQ=\sqrt{4+9+36}=7$

Example 12. Find equation of perpendicular from (1,6,3) on the straight line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and also foot of perpendicular.

Sol. : Let the straight line through (1,6,3) perpendicular on gives line $\frac{x}{1}=\frac{y-1}{2}=\frac{x-2}{3}$ be

$$\frac{x-1}{l}=\frac{y-6}{m}=\frac{z-3}{n}=(\lambda)$$

$\therefore$ $l.1+m.2+n.3=0 \Rightarrow l+2m+3n=0$ ....(1)

Any point on perpendicular line is $(l\lambda+1, m\lambda+6, n\lambda+3)$ any point on given line is (k, 2k+1, 3k +2).

If these are the same point (foot of perpendicular) then $\underset{(i)}{l\lambda+1=k}; \underset{(ii)}{m\lambda+6=2k+1}; \underset{(iii)}{n\lambda+3=3k+2}$.

From (i) $l=\frac{k-1}{\lambda}$

From (ii) $m=\frac{2k-5}{\lambda}$ and from (iii) $n=\frac{3k-1}{\lambda}$

$\therefore$ from (i) $\frac{k-1}{\lambda}+\frac{4k-10}{\lambda}+\frac{9k-3}{\lambda}=0 \Rightarrow k=1$

$\therefore$ foot of perpendicular is (k, 2k+1, 3k + 2) $\Rightarrow$ (1, 3, 5) and $l=\frac{0}{\lambda}, m=-\frac{3}{\lambda}, n=\frac{2}{\lambda}$

$\therefore$ Equation of perpendicular line is $\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}$

Example 13 : Find distance of point (2,4,6) from straight line through (–3, 4, 1) and whose direction ratios are 2 –2, 1.

Sol. : In the fig. point P is (2,4,6) and AB is straight line through A(–3,4,1) AN is projection of AP on straight line AB

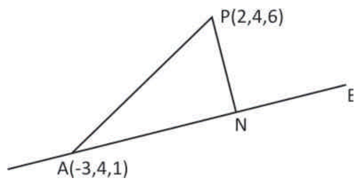


Fig 16

$$AP=\sqrt{(-5)^2+0^2+(-5)^2}=5\sqrt{2}$$

D.C. of straight line AB are $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$

AN = projection of AP on AB

$$=(2+3).\frac{2}{3}+0+(6-1)\frac{1}{3}=5$$

$$\therefore PN=\sqrt{(AP)^2-(AN)^2}=\sqrt{50-25}=5$$

Note : Question of example 11 can also be attempted in this way.

Example 14 : Find image of point (1,7,6) in straight line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.

Sol. : In the fig. LL' is the straight line P is point (1,7,6) PP' is perpendicular on LL' and PM = P'M. Then P' is image of P in straight line given line is $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.

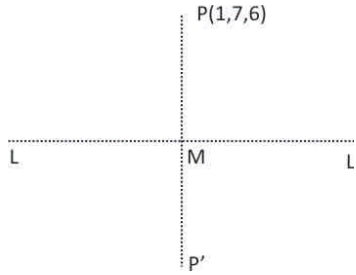


Fig 17

Any point on it is 2λ, $3\lambda+2$, $4\lambda+3$. Let it be M. D.Rs of PM $(2\lambda-1)$, $(3\lambda-5)$, $(4\lambda-3)$

PM $\perp$ PP' $\Rightarrow 2(2\lambda-1)+3(3\lambda-5)+4(4\lambda-3)=0$

$$\Rightarrow 29\lambda-2-15-12=0 \Rightarrow \lambda=1$$

$\therefore$ M is (2.1, 3.1 + 2, 4.1 + 3) i.e. (2,5,7)

If P' is (x_1, y_1, z_1) then $\frac{x_1+1}{2}=2\Rightarrow x_1=3$

$\frac{y_1+7}{2}=5\Rightarrow y_1=3;\frac{z_1+6}{2}=7\Rightarrow z_1=8,$ image (3, 3, 8)

Alternate : Let P' be (x_1, y_1, z_1).

D. Ratio of PP' are (x_1-1), (y_1-7), (z_1-6).

PP' $\perp$ LL' $\therefore$ $2(x_1-1)+3(y_1-7)+4(z_1-6)=0$

$\Rightarrow 2x_1+3y_1+4z_1=47$ ....(1)

Point $\frac{x_1+1}{2},\frac{y_1+7}{2};\frac{z_1+6}{2}$ is on LL'

$\therefore \frac{x_1+1}{4}=\frac{y_1+3}{6}=\frac{z_1}{8}\Rightarrow y_1=\frac{3x_1-3}{2},z_1=2x_1+2$

$\therefore$ from (i) $2x_1+3.\left(\frac{3x_1-3}{2}\right)+4.(2x_1+2)=47$

$\Rightarrow 4x_1+9x_1-9+16x_1+16=94\Rightarrow 29x_1=97$

$\therefore x_1=3, y_1=\frac{3.3-3}{2}=3, z_1=2.3+2=8$

$\therefore$ image is (3,3,8)

Example 15 : Find the distance of point (1, –2, 3) from plane x – y + = 5 measured parallel to the straight line whose direction ratios are 2, 3, – 6.

Sol. : Distance is measured parallel to the line

$\therefore$ equation of straight line through (1, –2, 3) parallel to it is $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}$

$\Rightarrow \frac{x-1}{2/7}=\frac{y+2}{3/7}=\frac{z-3}{-6/7}=\lambda$, point $\left(\frac{2}{7}\lambda+1, \frac{3}{7}\lambda-2, -\frac{6}{7}\lambda+3\right)$.

The distance of this point shall be l from plane. If the point $\left(\frac{2\lambda}{7}+1\right), \left(\frac{3\lambda}{7}-2\right), \left(-\frac{6}{7}\lambda+3\right)$ lies on plane.

$\therefore \left(\frac{2\lambda}{7}+1\right)-\left(\frac{3\lambda}{7}-2\right)+\left(-\frac{6}{7}\lambda+3\right)-5=0$

Solving λ =1 $\therefore$ distance of point measured parallel to the straight line from plane is1

Example 16 : Show that straight line $\frac{x-1}{3}=\frac{y+1}{4}=\frac{z+1}{5}$ and $\frac{x+2}{4}=\frac{y-2}{3}=\frac{z-1}{-2}$ do not intersect.

Sol.: D.C. of 1st line are $\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}$ any point on this straight line is $\left(\frac{3\lambda}{5\sqrt{2}}+1, \frac{4\lambda}{5\sqrt{2}}-1, \frac{\lambda}{\sqrt{2}}-1\right)$

D.C. of 2nd line are $\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, -\frac{2}{\sqrt{29}}$. Any point on this line is $\left(\frac{4\mu}{\sqrt{29}}-2, \frac{3\mu}{\sqrt{29}}+2, \frac{-2}{\sqrt{29}}+1\right)$, If we put $\frac{\lambda}{5\sqrt{2}}=k$ and $\frac{1}{\sqrt{29}}=t$, then points on lines are 3k + 1, 4k – 1, 5k – 1 and 4t – 2, 3t + 2, –2t + 1. If the line intersect then point should be common. $\Rightarrow 3k+1=4t-2; 4k-1=3t+2$ and $5k-1=-2t+1$.

Solving 3k +1 = 4t – 2, 4k – 1 = 3t + 2, k = 3, t = 3 these values of k and t do not satisfy.

5k – 1 = – 2 t + 1

$\therefore$ Lines do not intersect. These are skew lines.

Example 17 : Find point of intersection of lines $\frac{x-1}{2}=\frac{y+1}{4}=\frac{z}{-3}$ and $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z-2}{-4}$.

Sol. : Let $\frac{x-1}{2}=\frac{y+1}{4}=\frac{z}{-3}=\lambda$

$\therefore$ points $[(2\lambda+1),(4\lambda-1),-3\lambda]$ and if $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z-2}{-4}=\mu$ then point on it is $[3\mu-1, 5\mu-3, -4\mu+2]$ for point of intersection the two points must coincide i.e. $\left.\begin{array}{l}2\lambda+1=3\mu-1\\4\lambda-1=5\mu-3\end{array}\right\}\lambda=2,\mu=2$

and it satisfies $-3\lambda=-4\mu+2$.

$\therefore$ point of intersection is [5,7, –6]

Example 18: Prove that the straight line whose d.c. are given by relation al + bm + cn = 0 and fmn + gnl + hlm = 0 are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ and parallel if $\sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0$.

Sol. : $al+bm+cn=0 \Rightarrow n=-\left[\frac{al+bm}{c}\right]$ ....(1)

$fmn+gnl+hlm=0$ ....(2)

$\Rightarrow n[fm+gl]+hlm=0$

$\Rightarrow -\left(\frac{al+bm}{c}\right)(fm+gl)+hlm=0$

$\Rightarrow -agl-bm^2f-lmaf-bgml+chlm=0$

$\Rightarrow ag\left(\frac{l}{m}\right)^2+\frac{l}{m}(af+bg-ch)+bf=0$ ......(3)

If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c. of lines then from (3) $\frac{l_1.l_2}{m_1.m_2}=\frac{bf}{ag}\Rightarrow\frac{l_1l_2}{bf}=\frac{m_1m_2}{ag}$

$\Rightarrow\frac{l_1l_2}{cbf}=\frac{m_1m_2}{cag}$ and by symmetry $\frac{n_1n_2}{abh}$ for line to be perpendicular $l_1l_2+m_1m_2+n_1n_2=0 \Rightarrow cbf+cag+abh=0$

$\Rightarrow\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$

(ii) for lines to be parallel, roots of equation

(3) should be equal

i.e. $(af+bg-ch)^2-4abgf=0$

$\Rightarrow a^2f^2+b^2g^2+c^2h^2-2abgf-2bcgh-2cafh=0$

i.e. $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$

Exercise I (b)

1. Find d.c. of straight line 5x = 3y = 4z.

2. Point B is unit away from –1, 2, 3 on straight line $\frac{x+1}{2}=\frac{y-2}{6}=\frac{z-3}{3}$, find its co-ordiantes.

3. Point P lies on straight line joining (1,3,1) and (2,1,5). If z co-ordinates of P be 4, find P.

4. The vertices of a triangle are A,(2,3,0), B(3,5,2) and C (8,6,2). Find equation of internal bisector of angle A.

5. Find equation of straight line which passes through (1,3,5) and (5,3,1).

6. A is (5,3, –1) and B(7, –3, 2). Find projection of AB on straight line $\frac{x}{2}=\frac{y}{3}=\frac{z-2}{-6}$.

7. Find point of intersection of straight lines $\frac{x-3}{-4}=\frac{y-4}{-2}=\frac{z+5}{8}$ and $\frac{x-2}{-3}=\frac{y-5}{-3}=\frac{z-3}{0}$

8. Find equation of angle bisectors of lines $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z+1}{-2}$ and $\frac{x-4}{1}=\frac{y-3}{-2}=\frac{z+7}{2}$

9. Find distance of point (3,4,5) from straight line which passes through (–1, 2, 3) and whose direction ratios are 2, –3, 6.

10. The direction cosines of three lines are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 and these are mutually perpendicular. Find d.c. of a straight line which is equally inclined to these lines.

11. The equation a straight line is $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-4}{6}$; Find co-ordinate of a point P which is at a distance of 1 unit from point (1, –1, 4).

12. The straight line $\frac{x-3}{-2}=\frac{y+4}{2}=\frac{z-5}{1}$ meets the plane 2x + y + z = 1 in point……….

13. Find equation of a line through (2, –3, –4) perpendicular to straight lines $\frac{x}{a_1}=\frac{y}{b_1}=\frac{z}{c_1}$ and $\frac{x}{a_2}=\frac{y}{b_2}=\frac{z}{c_2}$.

14. The equation of two lines are $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{1}$ and $\frac{x+1}{-2}=\frac{y-5}{3}=\frac{z+3}{-6}$. Find equations of angle bisector of the angle between them.

15. Find image of point (1,6,3) in the straight line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

16. Point P is (–2, 3, –4) and PQ is perpendicular to straight line $\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z-3}{1}$ at Q. Find Q.

17. Find the distance of point (2, –3, 1) from the plane x – y + z = 8, measured parallel to the straight line whose d.c. are proportional to 2, 2, –1.

[Hint – Line through A(2, –3, 1) parallel to line is $\frac{x-2}{2}=\frac{y+3}{2}=\frac{z-1}{-1}$. If it meets plane x - y + z = 8 in B then AB is the required distance].

18. The d.c. of two straight lines are given by equation l – 2m + n = 0 and nl – 2mn –lm = 0. Find angle between the lines.

19. Show that if straight 1st line $\frac{x-1}{2}=\frac{y}{3}=\frac{z+4}{-2}$ and $\frac{x}{3}=\frac{y-1}{4}=\frac{z-1}{n}$ intersect then n = ….

20. Show that straight line $\frac{x-2}{2}=\frac{y-3}{2}=\frac{z}{1}$ and $\frac{x+5}{-3}=\frac{y-5}{-2}=\frac{z-1}{6}$ do not intersect.

6.13 Shortest Distance between two straight line :

If two straight lines are not parallel and also do not intersect then these lines are skew lines. They lie in different planes. Thus skew are those lines which lie in different planes. The problem of finding the shortest distance between two lines arises in skew lines. The shortest distance between these two lines is the line segment which is perpendicular to both these lines.

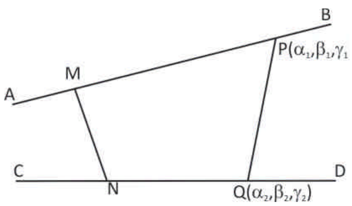


Fig 18

Let the equation of lines be

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \text{ and}$$

$$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}.$$

In the fig P is ($\alpha_1,\beta_1,\gamma_1$) and Q is ($\alpha_2,\beta_2,\gamma_2$). MN is perpendicular to both. Let p, q and r be the direction cosines of shortest distance MN.

$$\therefore \quad pl_1+qm_1+rn_1=0$$

$$pl_2+qm_2+rn_2=0$$

$$\therefore \quad \frac{p}{m_1n_2-m_2n_1}=\frac{q}{l_2n_1-l_1n_2}=\frac{r}{l_1m_2-l_2m_1}$$

$$=\frac{1}{\sqrt{(m_1n_2-m_2n_1)^2+(l_2n_1-l_1n_2)^2+(l_1m_2-l_2m_1)^2}}$$

$$=\frac{1}{\lambda}$$

and this shall given the value of p, q, r; clearly MN is projection of PQ on it.

$\therefore$ MN = $p(\alpha_2-\alpha_1)+q(\beta_2-\beta_1)+r(\gamma_2-\gamma_1)$

6.14 Equation of line intersecting two straight lines :

Let the two lines be $\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}$ and $\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$ any point on line (1) is $l_1\lambda+\alpha_1, m_1k+\beta_1, n_1k+\gamma_1$, any point on $l_2\lambda+\alpha_2, m_2k+\beta_2, n_2k+\gamma_2$. The line through these points is line intersecting both lines.

6.15 Shortest Distance between two skew straight lines

Skew straight lines do not intersect, neither they are parallel skew straight lines lie in different planes. But one can find shortest distance between them.

In the figure straight line

ℓ_1 is $\bar{r} = \bar{a}_1 + \lambda \bar{b}_1$

ℓ_2 is $\bar{r} = \bar{a}_2 + \lambda \bar{b}_2$

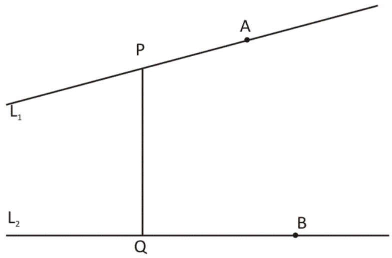


Fig 1

The p.v. of point A on straight line ℓ_1 is $\bar{a}_1$ and straight line is parallel to $\bar{b}_1$. Similarly p.v. of point B on nℓ_2 is $\bar{a}_2$ and it is parallel to vector $\bar{b}_2$. Let PQ be straight line, which is perpendicular to both ℓ_1 and ℓ_2, then line segment PQ is length of shortest distance between ℓ_1 and ℓ_2.

$\therefore$ PQ is parallel to $\bar{b}_1 \times \bar{b}_2$

If $\hat{\eta}$ is unit vector along $\bar{b}_1 \times \bar{b}_2$ then

$$\hat{\eta} = \frac{\hat{b}_1 \times \hat{b}_2}{|\bar{b}_1 \times \bar{b}_2|}$$

Clearly PQ is projection of AB on PQ

$$\therefore \text{PQ} = \overline{AB}.\hat{n} = \frac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}$$

Note: If two lines intersect then S.D. = 0

Example 19 : Find shortest distance between lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$.

Sol. : Let l, m, n be the d.c. of the shortest distance. S. D. segment is perpendicular to both lines.

$$\therefore \left.\begin{array}{l} 2l+3m+4n=0 \\ 3l+4m+5n=0 \end{array}\right\} \Rightarrow \frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

$$\Rightarrow \frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}=\frac{1}{\sqrt{6}}=\frac{l}{1}=\frac{m}{-2}=\frac{n}{1}$$

$$\therefore \text{ d.c. are } \left(\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$$

Point on line (1), A is (1,2,3) and point on line (2), B is (2,4,5).

S.D. = projection of AB on it $(2-1).\frac{1}{\sqrt{6}}+(4-2)\left(\frac{-2}{\sqrt{6}}\right)+(5-3).\frac{1}{\sqrt{6}}=\frac{1}{\sqrt{6}}$

Example 20. Find the equation of straight line which intersects the lines $\frac{x-3}{1}=\frac{y}{1}=\frac{z-3}{1}$ and $\frac{x+3}{1}=\frac{y}{1}=\frac{z+3}{2}$ and is parallel to $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z-6}{3}$.

Sol. : Any point A on straight line $\frac{x-3}{1}=\frac{y}{1}=\frac{z-3}{1}$ is $(\lambda_1+3,\lambda_1,\lambda_1+3)$ any point B of $\frac{x+3}{1}=\frac{y}{1}=\frac{z+3}{2}$ is $\lambda_2-3,\lambda_2,2\lambda_2-3$

$\therefore$ D.R. of line joining these points are i.e. AB is $\lambda_1-\lambda_2+6,\lambda_1-\lambda_2,\lambda_1-2\lambda_2+6$. The line is parallel to $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z-6}{3}$

$$\therefore \frac{\lambda_1-\lambda_1+6}{2}=\frac{\lambda_1-\lambda_2}{1}=\frac{\lambda_1-2\lambda_2+6}{3}$$

$$\therefore \lambda_1-\lambda_2+6=2\lambda_1-2\lambda_2 \Rightarrow \lambda_1-\lambda_2=6$$

and $3\lambda_1-3\lambda_2+18=2\lambda_1-4\lambda_2+12 \Rightarrow \lambda_1+\lambda_2=-6$

$$\therefore \lambda_1=0,\lambda_2=-6$$

Hence points A and B are (3,0,3), (–9, –6, –15)

$\therefore$ Equation of line intersecting the given two lines and parallel to the given line is $\frac{x-3}{-12}=\frac{y}{-6}=\frac{z-3}{-18}$ i.e. $\frac{x-3}{2}=\frac{y}{1}=\frac{z-3}{3}$

Example 21 : Find S.D. between lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ also find its equations and co-ordinates of points where it meets the lines.

Sol. : Let shortest distance segment meets the lines in P and Q, then

P is $(3\lambda+3,-\lambda+8,\lambda+3)$ and Q is $(-3k-3,2k-7,4k+6)$

$\therefore$D.C. of PQ are proportional to $3\lambda+3k+6,-\lambda-2k+15,\lambda-4k-3$. PQ is perpendicular to both the lines.

$\therefore (3\lambda+3k+6).3+(-\lambda-2k+15)(-1)+(\lambda-4k-3).1=0$

$\Rightarrow 11\lambda+7k=0$ ....(1)

and $-3(3\lambda+3k+6)+2(-\lambda-2k+15)+4(\lambda-4k-3)=0$

$\Rightarrow 7k+29k=0$ ....(2)

Solving (1) and (2) $\lambda=0,k=0$.

$\therefore$ P is (3,8,3) and Q is (–3, –7,6)

$\therefore$S.D. = $\sqrt{6^2+15^2+(-3)^2}=-3\sqrt{30}$

(ii) D. R. of PQ are $\propto$ (6, 15, –3) i.e. 2, 5, –1

Equation PQ; $\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{1}$

Example 22 : Find equation of line which goes through (2,4,6) and intersects lines $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$ and $\frac{x-2}{3}=\frac{y+1}{5}=\frac{z-1}{4}$.

Sol. Let the straight line through (2,4,6) be $\frac{x-2}{l}=\frac{y-4}{m}=\frac{z-6}{n}$ ....(1)

Any point on this line is $(l\lambda+2,m\lambda+4,n\lambda+6)$ If this point is on $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$,

then $\frac{l\lambda+2-1}{1}=\frac{m\lambda+4-1}{2}=\frac{n\lambda+6-1}{3}$

$\Rightarrow 2(l\lambda+1)=(m\lambda+3)$ and $3m\lambda+9=2n\lambda+10$

$\Rightarrow \lambda=\frac{1}{2l-m},\lambda-\frac{1}{3m-2n}$

$\therefore 2l-m=3m-2n \Rightarrow 2l-4m+2n=0$ ....(2)

If point $(l\lambda+2, m\lambda+4, n\lambda+6)$ is on line (2) then

$$\frac{l\lambda+2-2}{3}=\frac{m\lambda+4+1}{5}=\frac{n\lambda+6-1}{4}$$

$$\Rightarrow 5l\lambda=3m\lambda+15, 4m\lambda+20=5n\lambda+25$$

$$\Rightarrow \lambda=\frac{15}{5l-3m}, \lambda=\frac{5}{4m-5n}=\frac{15}{12m-15n}$$

$$\Rightarrow 5l-3m=12m-15n \Rightarrow 5l-15m+15n=0$$

$$\Rightarrow l-3m+3n=0 \qquad \text{......(3)}$$

Solving $\left.\begin{array}{l} l-2m+n=0 \\ l-3m+3n=0 \end{array}\right\} \frac{l}{-6+6}=\frac{m}{1-3}=\frac{n}{-3+2}$

$\therefore l : m : n = 0 : -2 : -1$

D.C. 0, $-2/\sqrt{5}, -1/\sqrt{5}$

Straight line is $\frac{x-2}{0}=\frac{y-4}{2}=\frac{z-6}{-1}$

Example 23: Find equation of straight line passing through the point of intersection of lines $\frac{x+4}{1}=\frac{y}{1}=\frac{z+1}{-2}$ is $\frac{x+4}{1}=\frac{y}{1}=\frac{z+5}{2}$ and which is parallel to $\frac{x-3}{4}=\frac{y-1}{3}=\frac{z-2}{-2}$.

Sol. : Any point on line $\frac{x+4}{1}=\frac{y}{1}=\frac{z+1}{-2}$ is $(\lambda-4, \lambda, -2\lambda-1)$ and any point on straight line $\frac{x+4}{1}=\frac{y}{1}=\frac{z+5}{2}$ is $(\mu-4, \mu, 2\mu-5)$ If the two points are same then this point is point of intersection of two lines. $\lambda-4=\mu-4, \lambda=\mu, -2\lambda-1=2\mu-5$ clearly $\lambda=\mu$; and $-2\lambda-1=2\lambda-5$ gives $\lambda=1$

$\therefore$ point of intersection of lines is $(-3, 1, -3)$. Line is parallel to $\frac{x-3}{4}=\frac{y-1}{3}=\frac{z}{-2}$

$\therefore$ Required line is $\frac{x-3}{4}=\frac{y-1}{3}=\frac{z+3}{-2}$

Example 24. Find the ratio in which surface $x^2 + y^2 + z^2 = 350$ divides the line joining points (3, –1, 2) and (9, –3, 6).

Sol. : The given points are (3, –1, 2) and (9, –3, 6). Let the surface divide the line joining these points in the ratio of $l : 1$.

This point is $\frac{9\lambda+3}{\lambda+1}, \frac{-3\lambda-1}{\lambda+1}, \frac{6\lambda+2}{\lambda+1}$. This should satisfy $x^2 + y^2 + z^2 = 350$.

$\therefore (9\lambda+3)^2+(-3\lambda-1)^2+(6\lambda+2)^2=(\lambda+1)^2.350$

$\Rightarrow \lambda^2(350-86-9-36)+\lambda(700-54-6-24)+(350-1-9-4)=0$

$\Rightarrow 224\lambda^2+616\lambda+336=0$

$\Rightarrow 4\lambda^2+11\lambda+6=0 \Rightarrow (\lambda+2)(4\lambda+3)=0$

$\therefore \lambda=-2, \lambda=-3/4$

$\therefore$ Surface divides AB in ratio –2 : 1 and –3 : 4

Example 25 : Show that the straight line whose d.c. are given by $al+bm+cn=0$ and $ul^2+vm^2+wn^2=0$ are perpendicular if $u(b^2+c^2)+v(c^2+a^2)+w(a^2+b^2)=0$ and parallel if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$.

Sol. : al + bm + cn = 0 $\Rightarrow n=-\frac{al+bm}{c}$

$\therefore ul^2+vm^2+wn^2=ul^2+vm^2+w\left(\frac{al+bm}{c}\right)^2=0$

$\Rightarrow c^2ul^2+c^2vm^2+w(a^2l^2+b^2m^2+2ablm)w=0$ ....(1)

$\Rightarrow (c^2u+wa^2)\frac{l^2}{m^2}+2abw\frac{l}{m}+c^2v+wb^2=0$

$\therefore \frac{l_1l_2}{m_1m_2}=\frac{c^2v+wb^2}{c^2u+wa^2}$

$\Rightarrow \frac{l_1l_2}{b^2w+c^2u}=\frac{m_1m_2}{c^2u+wa^2}$ = by sym. $\frac{n_1n_2}{a^2v+b^2u}$

Straight line shall be perpendicular if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow (b^2w+c^2v)k+(c^2u+wa^2)k+(a^2v+b^2u)k=0$

$\Rightarrow u(b^2+c^2)+v(c^2+a^2)+w(a^2+b^2)=0$

Straight lines shall be parallel if root of equation $(c^2u+wa^2)l^2+2abwlm+m^2(c^2v+wb^2)=0$ are equal i.e. $4a^2b^2w^2-4(c^2u+wa^2)(c^2v+wb^2)=0$ Solving we get $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$

Exercise 1(c)

1. Find length and equation of S.D. of lines $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ find also points whose S.D. meets the lines.

2. Find S.D. between lines x + a = 2y = – 12z and x = y + 2a = 6z – 6a.

3. Find S.D. between lines $\frac{x-3}{2}=\frac{y+1}{-3}=\frac{z-2}{4}$ and $\frac{x-1}{4}=\frac{y+3}{1}=\frac{z+1}{2}$.

4. Find S.D. between lines $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{1}$ and $\frac{x-4}{-1}=\frac{y+3}{4}=\frac{z-2}{3}$

5. Find S.D between straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and $\frac{x-2}{-1}=\frac{y+3}{2}=\frac{z-4}{3}$

6. Find S.D. between straight line 3x – 3 = 2y + 2 = 4z and $\frac{x+1}{2}=\frac{y+4}{5}=\frac{z-1}{4}$

7. Find foot of perpendicular from P(2,1,3) on straight line joining A(1, 2, 4) and B(3,0,5).

8. Find image of point (–2, –10,8) in the straight line $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z}{1}$

9. Find value of p if image of point (p, 2, 5) in the straight line $\frac{x+2}{3}=\frac{2y-3}{8}=\frac{z-4}{-1}$ is (–3,5,p)

10. Prove that the four points (5, –1,1); (7, –4,7); (4, –6,10) and (–1, –3, –4) are vertices of a rhombus.

11. Show that points (3, –2, 4), (–1,4, –2) and (1,1,1) are collinear.

12. If l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c. of two straight lines which include angle q, then show that actual d.c. of one of the angle bisector between them is $\frac{l_1+l_2}{2\cos\theta/2},\frac{m_1+m_2}{2\cos\theta/2},\ \frac{n_1+n_2}{2\cos\theta/2}$.

13. l_1, m_1, n_1 and l_2, m_2, n_2 are d.c. of two perpendicular straight line. Show that d.c. of straight line perpendicular to these line is $m_1n_2 - m_2n_1$, $n_1l_2 - n_2l_1$, $l_1m_2 - m_1l_2$

14. Find distance between parallel lines (a) $\frac{x}{2}=\frac{y}{3}=\frac{z}{4},\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{4}$

    (b) $x=\frac{y}{2}=\frac{z}{3}$ and $\frac{x-2}{1}=\frac{y-3}{2}=\frac{z+5}{3}$

15. Find equation of straight line through (–1, 4, –3) and parallel to line which goes through (–2, –3,4) and (–5,4, –2), also find distance between these lines

16. A straight line through point (–2, –6,7) with d.r. 2, 7, –5 is drawn to intersect straight lines $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$ and $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z-6}{4}$. Find length of intercept and points of intersection.

17. Prove that the equation of straight line through (a, b,c) and at right angle to straight lines $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$ and $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$ is $\frac{x-a}{m_1n_2-m_2n_1}=\frac{y-b}{l_1n_2-n_1l_2}=\frac{y-c}{l_1m_2-m_1l_2}$

18. In tetrahedron OABC, $OA^2 + BC^2 = OB^2 + CA^2$, $= OC^2 + AB^2$, then show that opposite edges are at right angles.

19. Find angle between lines whose direction cosine satisfy equations, 3m + n – l = 0, 3lm + 4ln – 3mn = 0.

20. The direction cosines of two straight lines satisfy equation l + 2m –3n = 0 and $l^2 - 2m^2 + n^2 = 0$; prove angle between them is $\cos^{-1}\left(\frac{1}{5\sqrt{3}}\right)$.

21. P' and Q' are images of P(2,3,6) and Q(5,3,6) in x-axis. Equation of PQ is

(a) 2x + 3y = 22 (b) y + 3 = 0 (c) 2x + 3z = 2 (d) none of these

22. PQ is perpendicular on straight line $\frac{x-3}{2}=\frac{y-5}{2}=\frac{z-7}{-1}$, If P is (2,1,6) then Q is

(a) (1,3,8) (b) (5,7,6) (c) (3,5,7) (d) (–1, 1, 9)

23. PQ is perpendicular from P(1,2,0) on straight line $\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-9}{-6}$. Equation of PQ is

(a) $\frac{x-2}{3}=\frac{y-4}{2}=\frac{z-1}{-6}2$ (b) $\frac{x-1}{3}=\frac{y-2}{4}=\frac{z}{3}$

(c) $\frac{x-1}{6}=\frac{y-2}{4}=\frac{z}{4}$ (d) $\frac{x-2}{3}=\frac{y-4}{2}=\frac{z}{2}$

24. Image of P(3, – 1,1) in

(a) y axis is …… (b) z-axis is …… (c) in y –z plane is …… (d) in z-x plane is ……

7. Plane

7.1 Definition :

A plane is a straight surface such that if two points are taken on it then the straight line joining them lies wholely in it.

7.2 General equation of plane :

A linear equation in x, y , z is

$ax + by + cz + d = 0$ ....(i)

is the general equation of plane.

Plane $ax + by + cz = 0$ passes through origin. Equation (i) can be written as

$$\frac{a}{d}x+\frac{b}{d}y+\frac{c}{d}z+1=0$$

which shows that the equation contains 3 co-efficient $\frac{a}{d},\frac{b}{d}$ and $\frac{c}{d}$.

Hence a plane can be determined if 3 independent conditions are given.

7.3 To prove that general equation of first degree in x, y, z always represent a plane.

First degree general equation in x, y, z is $ax + by + cz + d = 0$ ....(1)

Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) lie on this locus

$\therefore ax_1 + by_1 + cz_1 + d = 0$ ....(2)

$ax_2 + by_2 + cz_2 = 0$ ....(3)

Multiplying (3) by α and adding to (2)

$$a(x_1 + \alpha x_2) + b(y_1 + \alpha y_2) + c(z_1 + \alpha z_2) + (1+\alpha) d=0$$

or $$\frac{a(x_1+\alpha x_2)}{1+\alpha}+\frac{b(y_1+\alpha y_2)}{1+\alpha}+\frac{c(z_1+\alpha z_2)}{1+\alpha}+d=0$$

and it shows that point

$\left(\frac{x_1+\alpha x_2}{1+\alpha},\frac{y_1+\alpha y_2}{1+\alpha},\frac{z_1+\alpha z_2}{1+\alpha}\right)$ is also on locus (1) Different values of α shall give different points lying on straight line PQ.

Every point on straight line joining two arbitary points (here P and Q) lies on the locus, therefore by definition of plane the locus

$\therefore ax + by + cz + d = 0$ is a plane

7.4 One point form of equation of plane.

Let (x_1, y_1, z_1) be a point on plane

$ax + by + cz + d = 0$ ....(1)

$\therefore$ $ax_1 + by_1 + cz_1 + d = 0$ ....(2)

and (1) – (2) $\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

This is the equation of plane which passes through the given point (x_1, y_1, z_1)

7.5 Normal form of equation of a plane :

Let ABC be plane which meets axes, OX, OY and OZ in A, B, and C respectively and let OP be perpendicular from origin on the plane.

Let direction cosines of OP be $\cos\alpha, \cos\beta, \cos\gamma$; and let OP = p.

Then coo-ordinate of P are $(p\cos\alpha, p\cos\beta, p\cos\gamma)$ or (lp, mp, np) if d.c. of OP are l, m, n If Q is any other point (x, y, z) lying in plane. Then direction cosines of straight line P Q are x –lp, y –mp, z –np.

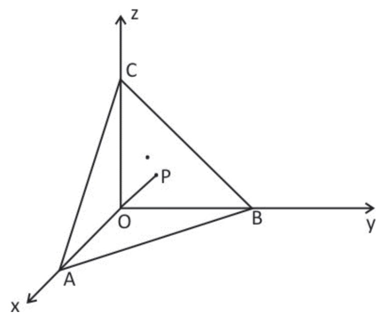


Fig 1

OP in $\perp$ to plane, therefore, OP $\perp$ PQ.

$$\therefore l(x - lp) + m(y - mp) + n(z - np) = 0$$

$$lx + my + nz = p(l^2 + m^2 + n^2) = p.1 = p$$

$$\Rightarrow lx + my + nz = p$$

This is the equation of plane in normal form, p is perpendicular from origin the plane, and l, m, n are its cosines.

(a) the point where perpendicular from origin meets the plane is (lp, mp, np)

(b) This also proves that equation of any plane is first degree equation in x, y, z.

7.6 Equation of plane – intercept form:

Let a plane cut, intercepts on axes OX, OY, OZ of length a, b and c respectively. If plane meets axes in A, B and C then A is (a, 0,0), B (0,b,0), C(0,0,c)

Let equation of plane be $\lambda_1 x + \lambda_2 y + \lambda_3 z + d = 0$ ....(1)

Then from point A $\lambda_1 a + d = 0, \lambda_1 = -\frac{d}{a}$

From point B $\lambda_2 a + d = 0, \lambda_2 = -\frac{d}{b}$

From point C $\lambda_3 a + d = 0, \lambda_3 = -\frac{d}{c}$

Putting these values of λ_1, λ_2 and λ_3 in equation (1)

$$-\frac{d}{a}x-\frac{d}{b}y-\frac{d}{c}z+d=0$$

Putting these values of λ_1, λ_2 and λ_3 in equation (1)

$$-\frac{d}{a}x-\frac{d}{b}y-\frac{d}{c}z+d=0$$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

This is the equation of plane in intercept form.

7.7 The general equation of plane can be reduced.

(a) In normal form (b) In intercept form as and when required

(a) General equation of plane is ax + by + cz + d = 0 ....(1)

Normal form is lx + my + nz – p = 0 ....(2)

(1) and (2) are identical

$$\therefore \frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{-p}{d}=\frac{\sqrt{l^2+m^2+n^2}}{a^2+b^2+c^2}$$

$$\Rightarrow \frac{l}{a}=\frac{m}{b}=\frac{n}{c}=-\frac{p}{d}=\frac{\pm 1}{\sqrt{a^2+b^2+c^2}}$$

p is always positive and so we shall take + sign or minus sign of the radical according as d is negative or positive. If d is positive then

$$l=\frac{-a}{\sqrt{a^2+b^2+c^2}}, m=\frac{-b}{\sqrt{a^2+b^2+c^2}}, n=\frac{-c}{\sqrt{a^2+b^2+c^2}}$$

and $p=\dfrac{d}{\sqrt{a^2+b^2+c^2}}$

If d is negative then, $l=\dfrac{a}{\sqrt{a^2+b^2+c^2}}$

$$m=\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\frac{c}{\sqrt{a^2+b^2+c^2}}, p=\frac{-d}{\sqrt{a^2+b^2+c^2}}$$

$\therefore$ Normal form of equation ax + by + cz = –d

is $$\frac{ax}{\sqrt{a^2+b^2+c^2}} \pm \frac{by}{\sqrt{a^2+b^2+c^2}} \pm \frac{cz}{\sqrt{a^2+b^2+c^2}} = \pm \frac{d}{\sqrt{a^2+b^2+c^2}}$$

signs being so taken that $= \frac{\pm d}{\sqrt{a^2+b^2+c^2}}$ is always positive.

If equation of a plane is 3x – 4y – 7z + 5 = 0 then $\sqrt{a^2+b^2+c^2} = \sqrt{3^2+4^2+7^2} = \sqrt{74}$

Then $\frac{-3x+4y+7z}{\sqrt{74}} = \frac{5}{\sqrt{74}}$

Normal form is $-\frac{3}{\sqrt{74}}x + \frac{4}{\sqrt{74}}y + \frac{7}{\sqrt{74}}z = \frac{5}{\sqrt{74}}$

$\frac{5}{\sqrt{74}}$ is length of perpendicular from origin it.

(b) Intercept form

ax + by + cz + d = 0 $\Rightarrow$ ax + by + cz = –d

$\Rightarrow \frac{x}{-d/a} + \frac{y}{-d/b} + \frac{z}{-d/c} = 1$

This is intercept form of equation, $-\frac{d}{a}, -\frac{d}{b}$ and $-\frac{d}{c}$ are length of intercepts on axes.

7.8 Angle between two plane:

We have seen in 2.5 that in the equation of plane ax +by + cz + d = 0 a, b, c are direction Rations of its normal.

$\therefore$ angle between two planes is equal to the angle between their normal.

If planes are $a_1x + b_1y + c_1z + d_1 = 0$ ....(1)

$a_2x + b_2y + c_2z + d_2 = 0$ ....(2)

then d.c. of their normals are $\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$ and

$\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$

$\therefore$ Angle q between them is given $\cos\theta = \pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Note : (a) Planes shall be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(b) Planes shall be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(c) Equation of a plane parallel to the plane ax + by + cz + d = 0 is ax + by +c z = $\lambda, \lambda \in R$

7.9 Some important types of planes:

(a) Equation of y –z plane is x = 0

Equation of z – x plane is y = 0

Equation of x –y plane is z = 0

(b) x = a or x = λ is a plane parallel to y – z plane y = λ is plane parallel to x – z plane.

y = λ is a plane parallel to x – z plane.

Z = λ is a plane parallel to y –x plane.

(c) Equation of a plane perpendicular to co-ordiante plates or parallel to axes.

Equation of y – z plane is x = 0 $\Rightarrow x.1 + 0.y + 0.z = 0$ ....(1)

Let the equation of the plane which is perpendicular to it (i.e.) perpendicular to y-z plane (i.e.) parallel to x-axis

ax + by + cz + d = 0 ....(2)

Plane (1) $\perp$(2) $\Rightarrow$ a . 1 + b . 0 + c . 0 = 0 $\Rightarrow$a = 0 and this reduced equation (2) as $by + cz + d = 0$

It follows that equation of plane $\perp$ to x – y plane is ax+by+d=0 and $\perp$ to x–z plane is ax+by+d=0

7.10 Equation of plane through the intersection of two planes :

Let the two planes be

$P = a_1x + b_1y + c_1z + d_1 = 0$ ....(1)

$Q = a_2x + b_2y + c_2z + d_2 = 0$ ....(2)

The equation P + l Q = 0 represents a plane which passes through the intersection of planes P = 0, Q = 0 i.e. plane $a_1x + b_1y + c_1z + d_1 + l (a_2x + b_2y + c_2z + d_2) = 0$ ....(3) is the equation of planes which pass through (1) or (2).]

Equation (3) is a linear equation in x, y, z and a linear equation x,y,z represents a plane.

$\therefore$ (3) is the equation of any plane passing through the intersection of two planes one and (2).

Example 1 : Find the equation of a plane through points (1,1,0), (1,2,1) and (–2,2, –1).

Sol. : Let the equation of plane through (1,2,1)

a(x – 1) + b (y – 2) + c(z –1) = 0 ....(1)

Points (1,1,0) and (–2,2,–1) lie on it

$$\left.\begin{aligned}\therefore a.0-b-c=0\\ -3a+0.-2c=0\end{aligned}\right\}\Rightarrow \frac{a}{2-0}=\frac{b}{3-0}=\frac{c}{-3}$$

$$\Rightarrow \frac{a}{2}=\frac{b}{3}=\frac{c}{-3}$$

$\therefore$ Equation of plane is $2(x-1)+3(y-2)-3(z-1)=0$

i.e. 2x + 3y – 3z = 4

Example 2 : Find equation of plane through (1,2,3) and parallel 3x + 4y – 5z + 15 = 0.

Sol. : Any plane parallel to 3x + 4y – 5z + 15 = 0

is 3x + 4y – 5z = l

point (1,2,3) lies on it $\therefore$ 3 + 8 – 16 = l = –4

$\therefore$ plane is 3x + 4y – 5z + 4 = 0

Example 3 : Find the equation of plane through (2,2,1) and (9,3,6) and perpendicular to 2x + 6y + 6z = 9.

Sol. : Any plane through (2,2,1) is

$$a(x-2)+b(y-2)+c(z-1)=0 \quad ....(1)$$

point (9,3,6) lies on it $\therefore$ 7a + b + 5c = 0 ....(2)

It is $\perp$ to 2x + 6y + 6z = 9; 2a + 6b + 6c = 0 ....(3)

$$\therefore \frac{a}{6-30}=\frac{b}{10-42}=\frac{c}{42-2} \qquad \Rightarrow \frac{a}{-24}=\frac{b}{-32}=\frac{c}{40}$$

$$\Rightarrow \frac{a}{3}=\frac{b}{4}=\frac{c}{-5}$$

$\therefore$ Equation of plane is $3(x-2)+4(y-2)-5(z-1)=0$

3x + 4y – 5z = 9

Example 4 : Find the equation of plane passing through the intersection of planes x + 2y + 3z – 4 = 0 and 2x + y – z + 5 = 0 and perpendicular to plane 5x + 3y + 3z + 18 = 0

Sol. : Any plane through the intersection of two given planes is

(x + 2y + 3z – 4) + l (2x + y – z + 5) = 0

$\Rightarrow (1+2\lambda)x+(2+\lambda)y+(3-\lambda)z-4+5\lambda=6$ ....(1)

It is perpendicular to 5x + 3y + 3z + 18 = 0

$\therefore 5(1+2\lambda)+3(2\lambda)+3(3-\lambda)=0$

$\Rightarrow 10\lambda+3\lambda-3\lambda+5+6+9=0 \quad \Rightarrow \lambda=-2$

$\therefore$ Plane is $(1-4)x+(2-2)y+(3+2)z-4-10=0$

i.e. 3x – 5y + 14 = 0

Example 5 : O is origin. Find equation of plane through P(2,3, –1) and perpendicular to OP.

Sol. : OP is perpendicular to plane

$\therefore$ OP is normal its D.R are 2,3, –1

$\therefore$ D.C. are $\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}}$

Equation of plane is $l(x-2)+m(y-3)+n(z+1)=0$

Where l, m, n are d.c. of normal

$\therefore$ planes is $\frac{2}{\sqrt{14}}(x-2)+\frac{3}{\sqrt{14}}(y-3)-\frac{1}{\sqrt{14}}(z+1)=0$

$\Rightarrow 2x+3y-z=\sqrt{14}$

Example 6 : A plane meets axes in (4,0,0), (0,5,0) and (0,0,3). Find length of perpendicular from origin on it.

Sol. : Intercept by plane on axes are 4, 5 and 3

$\therefore$ plane is $\frac{x}{4}+\frac{y}{5}+\frac{z}{3}=1 \quad \Rightarrow 15x+12y+20z=60$

To find the length of perpendicular from origin on it we shall transform the equation in normal form.

$\sqrt{15^2+12^2+20^2}=\sqrt{769}$

$\therefore$ Normal form of plane is $\frac{15x}{\sqrt{769}}+\frac{12y}{\sqrt{769}}+\frac{20z}{\sqrt{769}}=\frac{60}{\sqrt{769}}$

$\therefore$ Length of perpendicular from origin on the plane is $\frac{60}{\sqrt{769}}$.

Example 7 : A variable plane is at constant distance p from origin and meets the axes in A, B and C. O is origin. Show that the locus of centroid of tetrahedron OABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$

Sol. : Let plane be $lx+my+nz=p$ where p is perpendicular from origin on plane. It meets axes in A, B, and C, then A is $\left(\frac{p}{l},0,0\right)$,B$\left(0,\frac{p}{m},0\right)$,c$\left(0,0,\frac{p}{n}\right)$ Centroid of tetrahedron OABC is $\left(\frac{p}{l}+0+0+0\right)/4,\left(0+\frac{p}{m}+0+0\right)/4,\left(0+0+\frac{p}{n}+0\right)/4$

i.e. $\left(\frac{p}{4l},\frac{p}{4m},\frac{p}{4n}\right)$. If it is ($x_1$, y_1, z_1) then

$$x_1=\frac{p}{4l},y_1=\frac{p}{4m},z_1=\frac{p}{4n}\Rightarrow l=\frac{p}{4x_1},m=\frac{p}{4y_1},n=\frac{p}{4z_1}$$

But $l^2+m^2+n^2=1$ $\therefore\frac{p^2}{16x_1^2}+\frac{p^2}{4y_1^2}+\frac{p^2}{16z_1^2}=1$

$\therefore$ Locus of (x_1, y_1, z_1) is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$

Example 8 : Find equation of plane through (3,4,1) and (5, –6, 3) parallel to x-axis.

Sol. : Any plane parallel to x-axis is perpendicular to y –z plane. Let it be

by + cz + d = 0 ....(1)

It passes through (3,4,1) and (5, –6, 3)

$\therefore$ 4b + c + d = 0 and –6b + 3c + d = 0 $\Rightarrow\frac{b}{-2}=\frac{c}{-10}=\frac{d}{10}$

$\Rightarrow\frac{b}{1}=\frac{c}{5}=\frac{d}{-9}=\lambda(\text{let})$ $\therefore b=\lambda,c=5\lambda,d=-9\lambda$

$\therefore$ from equation (1) $\lambda y+5\lambda z-9\lambda=0$

Plane is y + 5z = 0

Example 9 : A plane meets co-ordinate axes in A, B and C. Centroid of triangle ABC is (3, –4, 1) find equation of plane.

Sol. Let equation of plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ where a, b, c are intercepts cut by plane on axes

$\therefore$A is (a, 0, 0) B(0, b, 0), C(0,0,C) But it is (3, –4, 1) $\therefore$ a = 9, b = – 12, c = 3.

Equation of plane is $\frac{x}{9}-\frac{y}{12}+\frac{z}{3}=1$

Example 10 : A variable plane is at a constant distance p, from origin and it meets axes in A, B and C. Through A, B and C planes are drawn parallel to co-ordinate planes. The locus of their point of intersection is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$. Prove:

Sol. : Let the plane be lx + my + nz = p ……….(1)

L, m, n are d.c. of normal of plane.

It meets axes in A, B and C

$\therefore$ A is $\left(\frac{p}{l},0,0\right)$, B is $\left(0,\frac{p}{m},0\right)$ C is $\left(0,0,\frac{p}{n}\right)$

Equation of plane through A parallel to y – z plane is $x=\frac{p}{l}$ and planes through B and C parallel to co-ordinate planes are $y=\frac{p}{m},z=\frac{p}{n}$. To find the locus of point of intersection, we have to find a relation in x, y, z and p. from this equations,. I.e. l, m and n has to be eliminated. From equation of planes.

$l=\frac{p}{x},m=\frac{p}{y}$ and $n=\frac{p}{z}$ and $l^2 + m^2 + n^2 = 1 \Rightarrow \frac{p^2}{x^2}+\frac{p^2}{y^2}+\frac{p^2}{z^2}=1$

$\therefore$ Locus is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$

Example 11 : A variable plane through a fixed point (a, b, c) meets the axes in A, B, C show that the locus of point of intersection of planes, through A, B and C parallel to co-ordinate planes is $ax^{-1} + by^{-1} + cz^{-1}=1$

Sol. : Equation of planes through (a,b,c) is $l(x-a)+m(y-b)+n(z-c)=0$

Where $l^2 + m^2 + n^2 = 1$, It meets axes in A, B and C

$\therefore$ A is $\left[\frac{la+mb+nc}{l},0,0\right]$,B$\left[0,\frac{la+mb+nc}{m},0\right]$ and $\left[0,0\frac{la+mb+nc}{n}\right]$

Equation of planes parallel to co-ordinate planes are $x=\frac{la+mb+nc}{l},y=\frac{la+mb+nc}{m}$,

$z=\frac{la+mb+nc}{n}$

$\Rightarrow \frac{a}{x}=\frac{al}{al+bm+nc},\frac{b}{y}=\frac{bm}{la+mb+nc},\frac{c}{z}=\frac{cn}{la+mb+nc}$

adding $\frac{a}{x}+\frac{b}{y}+\frac{c}{a}=1 \qquad \Rightarrow \frac{ax^{-1}+by^{-1}+cz^{-1}}{1}=1$

Exercise 2(a)

1. The equation of plane parallel to $5x - 3y + 4z + 8 = 0$ and passing through $(1, -1, 2)$ is ………
2. The equation of plane passing through the point of intersection of planes $3x + 4y + 5z + 6 = 0$ and $5x - 3y + 4z + 1 = 0$ and the point $(2, 4, -3)$ is ……….
3. A plane passes through $P(-2,2,1)$ and is perpendicular to OP, O is origin. Then the equation of plane is ……..
4. The equation of plane through $(4,5,1)$, $(3,9,4)$ and $(-4,4,4)$ is ….
5. Find the equation of a plane through $(2, -3,1)$ perpendicular to $4x - 3y + 2z-8=0$ and $x+y-2z+7= 0$.
6. A variable plane moves so that the sum of reciprocals of its intercepts on three axes is constant. Show that it passes through a fixed point.
7. The equation of the plane which passes through $(2, -3,1)$ and is perpendicular to the line joining the points $(3,4, -1)$ and $(2, -1, 5)$ is ….
8. The equation of a plane bisecting the line joining $(4, -2,5)$ and $(6, 4, -3)$ perpendicular is ……. (Hint – see AB is normal of plane)
9. The angle between the planes $2x - y + z = 0$ and $x + y + 2z = 3$ is …..
10. The equation of plane through $(2,3,6)$ and $(-2, 2,1)$ and perpendicular to plane. $3x-4y+2z=11$.
11. Find the equation of plane which passes through $(-2,3, -3)$ and $(1, -1,3)$ and parallel to y-axis.
12. Find the equation of plane through $(2, -3,4)$ and $(-1,2,3)$ and parallel to z – axis
13. Find angle between planes $3x + 4y - 5z = 7$ and $2x + 2y + z + 3 = 0$
14. Find length of perpendicular from origin on the plane $3x + 4y - 12z - 39 = 0$
15. The plane $4x - 6y + 9z = 18$ meets the axes in A, B and C.

    (a) Find centroid of triangle ABC (b) find centroid of tetrahedron OABC.
16. The two system of rectangular axes have the same origin. A plane cuts axes at distances p_1, q_1 and r_2; at distances p_2, q_2 and r_2 from origin prove:

    $$\frac{1}{p_1^2}+\frac{1}{q_1^2}+\frac{1}{r_1^2}=\frac{1}{p_2^2}+\frac{1}{q_2^2}+\frac{1}{r_2^2}$$

    [Hint : Perpendiculars dropped from origin on the two are equal, same]
17. The plane goes through $(4,3,2)$ and $(1,5,4)$ and is parallel to the line joining $(5,3,2)$ and $(-1,1,5)$ find it.

18. Find equation of plane which goes through of planes 2x + 3y – 5z = 2 and x – 4y + z + 6 = 0 and perpendicular from origin on it is $\sqrt{2}$.

19. Find equation of a plane through points (2,3,2) and (2,2,3) when it is at a distance of 4 from origin.

Sol. : Let plane be lx + my + nz = 4 ....(1)

And $l^2 + m^2 + n^2 = 1$, points (2,3,2), (2,2,3) are on plane

$$\therefore \left.\begin{array}{l} 2l + 3m + 2n = 4 \\ 2l + 2m + 3n = -4 \end{array}\right\} \begin{array}{l} \Rightarrow m - n = 0 \Rightarrow m = n \\ \text{and } 2l + 5n = 4 \Rightarrow l = \dfrac{4-5n}{2} \end{array}$$

$$l^2 + m^2 + n^2 = \left(\frac{4-5n}{2}\right)^2 + n^2 + n^2 = 1$$

$$\Rightarrow 33n^2 - 40n + 12 = 0$$

$$\Rightarrow (11n-6)(3n-2) = 0$$

$$\therefore n = \frac{2}{3}, m = \frac{2}{2}, l = \frac{1}{3} \text{ or } n = \frac{6}{11}, m = \frac{6}{11}, n = \frac{7}{11}$$

$\therefore$ plane is x + 2y + z = 12 or 6x + 6y + 7z = 44

7.11 Position of a point with respect to a plane :

Let the plane be $ax + by + cz + d = 0$ ....(1)

Let A (x_1, y_1, z_1) and B(x_2, y_2, z_2) be two points. The points can be on the same side of plane or these can be on opposite sides of plane.

Let AB meet plane in C and C divided AB in the ratio of k : 1

Then C is $\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$

If C divides AB internally then k shall be positive, and if c divides AB externally then k shall be negative.

Since C lies on plane

$$\therefore a\left(\frac{x_1+kx_2}{k+1}\right)+b\left(\frac{y_1+yx_2}{k+1}\right)+c\left(\frac{z_1+kz_2}{k+1}\right)+d=0$$

$$\Rightarrow a(x_1+kx_2)+b(y_1+ky_2)+c(z_1+kz_2)+d(1+k)=0$$

$$\Rightarrow k(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0$$

$$\therefore k=-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d} \qquad \text{..............(2)}$$

(a) If A and B are on the same side of plane then k is negative and from (2)

$ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ should have the same sign.

(b) If A and B are on opposite side of plane k should be positive.

$ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ should have positive signs.

7.12 Perpendicular distance of a point from a plane:

Let $ax + by + cz + d = 0$ be the plane, P (x_1, y_1, z_1) be the point. Equation of plane in normal form is $lx + my + nz = p$. Then in fig. ON = p. Plane parallel to this plane and passing through (x_1, y_1, z_1) is $lx + my + nz = p' = lx_1 + my_1 + nz_1$

Distance of (x_1, y_1, z_1) from plane is PQ = p'– p and PQ = OM – ON = $(lx_1+my_1+nz_1)-p$.

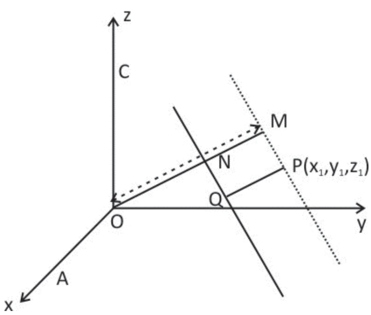


Fig 2

(b) the plane $ax + by + cz + d = 0$ in normal form

is $\frac{a}{\sqrt{a^2+b^2+c^2}}x+\frac{b}{\sqrt{a^2+b^2+c^2}}y+$

$$\frac{c}{\sqrt{a^2+b^2+c^2}}z+\frac{d}{\sqrt{a^2+b^2+c^2}}=0$$

$\therefore$ length of perpendicular is $=\frac{ax_1+by_1+cz_1}{\sqrt{a^2+b^2+c^2}}-\frac{-d}{\sqrt{a^2+b^2+c^2}}=\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$

Note : If (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points on the opposite sides of the plane then $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ will be opposite sign. Hence their perpendicular distance from the plane for shall be of opposite signs.

7.13 Distance between two parallel planes.

It is equal to the distance of a point on one plane from the other. OT is equal to difference of lengths of perpendicular from origin on the two planes.

(a) If planes are on the same sides of origin then p_1 and p_2 shall have the same signs distance $p_1 \sim p_2$.

(b) If planes are on opposite sides of origin then p_1 and p_2 shall have different signs. Distance between them = $p_1 \sim (-p_2)$.

7.14 Equation of bisector of the angle between two planes.

Let the two planes be $a_1x+b_1y+c_1z+d_1=0$, and $a_2x+b_2y+c_2z+d_2=0$. Bisector planes are locus of a point, which is equidistance from the two planes. Let (α,β,γ) be the point, then

$$\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2\alpha+b_2\beta+c_2\gamma}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$\therefore$ Locus is $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$

These are equations of two bisector planes. One bisects the acute angle between the two planes, while other bisects the obtuse angle between them.

(a) If two equation planes are so written that constant terms in both are positive or in both negative (i.e. of the same sign), then + sign in the equation (above) gives that equation of the bisector of that angle in which origin lies.

7.15 Bisector of acute angle or obtuse angle.

(a) Bisector of acute angle shall make angle less than of 45^0 with either plane. If it is more than 45^0 then it is bisector of obtuse angle.

(b) It origins lies in the acute angle between the planes, then than the angle between the normals of planes shall be obtuse, hence value of $\cos q$ shall be negative

$$\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = -ve \qquad ....(1)$$

(c) If origin lies in obtuse angle between the planes then angle between their normals shall be acute and

$$\frac{\sum a_1a_2}{\sum\sqrt{a_1^2}\sum\sqrt{a_2^2}} = +ve \qquad .... (2)$$

Conditions (1) and (2) shall decide. Whether origin lies in acute angle or obtuse angle between the planes.

7.16 Equation of a straight line :

Two planes if not parallel then they intersect in a line.

$\therefore$ a straight line can be represented by two equations of the two planes together.

$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ represents a straight line.

This equation of a line can be transformed into the symmetrical form equation of a straight line.

Let l, m, n be the d.c. of the straight line then

$$a_1l + b_1m + c_1n = 0$$
$$a_2l + b_2m + c_2n = 0$$

and $\therefore \dfrac{l}{b_1c_2 - b_2c_1} = \dfrac{m}{c_1a_2 - c_2a_1} = \dfrac{n}{a_1b_2 - a_2b_1}$ ....(α)

This (α) gives the direction cosines of the straight line. Now to find the point on line, we generally put z = 0 in both the equations and then solve them for x and y.

$$x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}, z = 0$$

an example shall clearify the process.

Example 12 :

Put in symmetrical form the equation of straight line.

x + y + z + 1 = 0, 4x + y – 2z + 2 = 0

Sol. : Give straight line is

x + y + z + 1 = 0, 4x + y – 2z + 2 = 0

Let l, m, n be the direction ratio of straight line

$$\left.\begin{array}{l}\therefore l+m+n=0\\ 4l+m-2n=0\end{array}\right\}\therefore \frac{l}{-2-1}=\frac{m}{4+2}=\frac{n}{1-4}$$

$$\Rightarrow \frac{l}{-3}=\frac{m}{6}=\frac{n}{-3}\quad \Rightarrow \frac{l}{1}=\frac{m}{-2}=\frac{n}{1}$$

Putting z = 0 in equations, then

$$\left.\begin{array}{l}x+y=-1\\ 4x+y=-2\end{array}\right\}\begin{array}{l}3x=-1, x=-1/3\\ \therefore y=-1+1/3=-2/3\end{array}$$

$\therefore$ Equation of straight line in symmetrical form is

$$\frac{x+1/3}{1}=\frac{y+2/3}{-2}=\frac{z}{1}$$

Its direction cosines are $\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}}$

Example 13: Find the angle between the lines x – 2y + z = 0 = x + y – z and x + 2y + z = 8x + 12y + 5z=0

Sol. : Line x – 2y + z = 0 = x + y – z

Let l, m, n be the direction rations of straight line

$$\left.\begin{array}{l}\therefore l-2m+n=0\\ l+m-n=0\end{array}\right\}\begin{array}{l}\Rightarrow 2l-m=0\Rightarrow m=2l\\ \text{and } n=2m-l=4l-l=3l\end{array}$$

$\therefore$ D. R. l, 2 l, 3 l i.e. 1,2,3

Line No. 2 $\left.\begin{array}{l}x+2y+z=0\\ 8x+12y+5z=0\end{array}\right\}\Rightarrow \begin{array}{l}l_1+2m_1+n_1=0\\ 8l_1+12m_1+5n_1=0\end{array}$

$$\therefore \quad \frac{l_1}{10-12}=\frac{m_1}{8-5}=\frac{n_1}{12-16}\Rightarrow \frac{l_1}{-2}=\frac{m_1}{3}=\frac{n_1}{-4}$$

$$\Rightarrow \frac{l_1}{2}=\frac{m_1}{-3}=\frac{n_1}{4}$$

If angle between lines is q then

$$\cos\theta = \frac{ll_1 + mm_1 + nn_1}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l^2 + m^2 + n^2}} = \frac{2-6+12}{\sqrt{14}\sqrt{29}} = \frac{8}{\sqrt{14}\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right)$$

Example 14 : Find distances of points (1,2,3) and (3,0, –1) from planes 3x+ 4y – 5z + 1 = 0 are the points on the same side of plane?

Sol. : Equation of plane 3x + 4y – 5z + 1 = 0

Constant term positive.

Distance of (1,2,3) from it $\alpha_1 = \frac{3+8-15+1}{5\sqrt{2}} = \frac{-3}{5\sqrt{2}}$

Distance of (3,0,1) from it $d_2 = \frac{9-5+1}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$. The points are on opposite sides of plane. Distance are $-\frac{3}{5\sqrt{2}};\frac{1}{\sqrt{2}}$.

Example 15 : Find the locus of a point the sum of squares of whose distances from planes x + y + z = 0; x 2y + z = 0 is equal to the square of its distance from the plane x = z

Sol. : Let the point be (x_1, y_1, z_1), then

$$\therefore \left(\frac{x_1 + y_1 + z_1}{\sqrt{3}}\right)^2 + \left(\frac{x_1 - 2y_1 + z_1}{\sqrt{6}}\right)^2 = \left(\frac{x_1 - z_1}{\sqrt{2}}\right)^2$$

$\therefore$ Locus is $2(x + y + z)^2 + (x - 2y + z)^2 = 3(x - z)^2$;

$\therefore 2[x^2 + y^2 + z^2 + 2xy + 2yz + 2zx] + [x^2 + 4y^2 + z^2 - 4xy - 4yz + 2xz] = 3x^2 + 3z^2 - 6xz$

$\Rightarrow \quad 6y^2 + 12xz = 0 \quad \Rightarrow \quad y^2 + 2xz = 0$

Example 16 : Find the equation of planes bisecting the angles between planes x + 2y – 2z = 3 and 3x – 4y + 12 z +1 = 0 and specify the one that bisects acute angle.

Sol. Given planes x + 2y – 2z – 3 = 0

3x – 4y + 12z + 1 = 0

The equation of two bisecting planes are $\frac{x+2y-2z-3}{3} = \pm\frac{3x-4y+12z+1}{13}$

$\Rightarrow 13x + 26y - 26z - 39 = \pm(9x - 12y + 36z + 3)$ planes are 2x+ 19y–31z = 21 and 11x+7y+5z =18. Now if angle between planes x + 2y – 2z – 3 = 0 and bisector plane 11x + 7y + 5z – 18 = 0 is q then

$$\cos\theta = \frac{11+14-10}{3\sqrt{195}} = \frac{15}{3\sqrt{195}}$$

$$= \frac{5\sqrt{195}}{195} = \frac{\sqrt{195}}{395} = \frac{10\sqrt{2}}{40} \text{ approx } = \frac{1}{6\sqrt{2}} \text{ approximately } < \frac{1}{\sqrt{2}}$$

$\frac{1}{2\sqrt{2}} < \frac{1}{\sqrt{2}}$ $\therefore q > 45^0 \Rightarrow$ this bisector plane bisects obtuse angle between planes.

$\therefore$ Bisector of acute angle is 2x + 19y – 31z = 21

Example 17 : Show that the equation $\frac{a}{y-z} + \frac{b}{z-x} + \frac{c}{x-y} = 0$ represents a pair of planes.

Sol.: A second degree equation in x, y, z containing terms x^2, y^2, z^2, xy, yz, zx represents a pair of planes if it can be expressed as product of two linear functions of x, y, z condition for it is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Where a, b, c are co-efficient of x^2, y^2 and z^2 respectively and 2f, 2g, 2h are co-efficient of yz; zx and xy respectively.

Now given equation is $\frac{a}{y-z} + \frac{b}{y-z} + \frac{c}{x-y} = 0$

$$\Rightarrow a(z-x)(x-y) + b(y-z)(x-y) + c(y-z)(z-x) = 0$$

$$\Rightarrow a(zx - zy - x^2 + xy) + b(yx - y^2 - zx + zy) + c(yz - xy - z^2 + zx) = 0$$

$$\Rightarrow ax^2 + by^2 + cz^2 - (b+c-a)yz - (c+a-b)zx - (a+b-c)xy = 0$$

Here A = a, B = b, C = c F = $-\frac{b+c-a}{2}, G = \frac{c+a-b}{2} = 0$ and $H = -\frac{a+b-c}{2}$ applying condition.

$$abc - \frac{1}{4}(b+c-a)(c+a-b)(a+b-c) - a\left(\frac{b+c-a}{2}\right)^2 - b\left(\frac{c+a-b}{2}\right)^2 - c\left(\frac{a+b-c}{2}\right)^2$$

$$= 4abc - (2ab - b^2 - a^2 + c^2)(a+b-c) - a(b^2 + c^2 + a^2 - 2ab - 2ac + 2bc)$$
$$- b(c^2 + a^2 + b^2 - 2bc - 2ab + 2ac) - c(a^2 + b^2 + c^2 + 2ab - 2ac - 2bc)$$

$$= 4abc - (a^2b + ab^2 + c^2b + cb^2 + a^2c + ac^2 - a^3 - b^3 - c^3 - 2abc)$$
$$- (ab^2 + ac^2 + a^3 - 2a^2b - 2a^2c + 2abc) - (bc^2 + ba^2 + b^3 - 2b^2c - 2ab^2 + 2abc)$$

$$-\left(a^2c+b^2c+c^3+2abc-2ac^2-bc^2\right)$$
$$=(6abc-6abc)+\left(2a^2b-2a^2b\right)+\left(2ab^2-2ab^2\right)+\left(2c^2b-2c^2b\right)+\left(2cb^2-2cb^2\right)+\left(2a^2c-2a^2c\right)$$
$$+\left(2a^2c-2a^2c\right)+\left(a^3+b^3+c^3-a^3-b^3-c^3\right)=0$$

$\therefore$ The given equation represents a pair of planes.

Example 8 : Find equation of plane through points (1,3,5) and (3, –1,4) parallel to straight lint 3x – 2 = 4y – 2 and 2y – 1 = 3z – 1

Sol. Given straight line is $\frac{x-2/3}{1/3}=\frac{y-1/2}{1/4};\frac{y-1/2}{1/2}=\frac{z-1/3}{1/3}$

$\Rightarrow\frac{x-2/3}{1/3}=\frac{y-1/2}{1/4}=\frac{z-1/3}{1/6}$ i.e. d.R are 4, 3, 2.

Let equation of plane be $a(x-1)+b(y-3)+c(z-5)=0$

Point (3, –1,4) lies on it $\therefore$ 2a – 4b – c = 0 ………..(1)

Plane normal $\perp$ to straight line $\therefore$ 4a + 3b + 2c = 0 ………..(2)

From (1) and (2) $\frac{a}{-5}=\frac{b}{-8}=\frac{c}{22}$ i.e. a : b : c $=5:8:-22$

$\therefore$ Plane is $5(x-1)+8(y-3)-22(z-5)=0$

i.e. 5x + 8y – 22z + 81 = 0

Example 19 : A variable plane is at a constant distance p from origin and meets axes in A, B and C. Show that locus of centroid of Δ ABC is $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$

Sol. : Let lx + my + nz = p be plane, $l^2+m^2+n^2=1$

$\therefore$ A is $\left[\frac{p}{l},0,0\right]$;B,$\left[0,\frac{p}{m},0\right]$;C$\left[0,0,\frac{n}{p}\right]$ centroid is $\left[\frac{p}{3l},\frac{p}{3m},\frac{p}{3n}\right]$

$\therefore$ if centroid is $(\bar{x},\bar{y},\bar{z})$ then similarly

$\bar{x}=\frac{p}{3l}\Rightarrow l=\frac{p}{3\bar{x}}\Rightarrow\frac{1}{\bar{x}}=\frac{3l}{p}$; and $\frac{1}{\bar{y}}=\frac{3m}{p};\frac{1}{\bar{z}}=\frac{3n}{p}$

$$\therefore\left(\frac{1}{\bar{x}}\right)^2+\left(\frac{1}{\bar{y}}\right)^2+\left(\frac{1}{\bar{z}}\right)^2=\frac{9\left(l^2+m^2+n^2\right)}{p^2}=\frac{9}{p^2}$$

Locus, $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$.

Example 20 : Find the distance of point (5, –3, 3) from straight line $\frac{2x-1}{2}=\frac{3y-4}{4}=\frac{z+4}{2}$ measured parallel to plane x + 2y + z = 10.

Sol. : Equation of plane parallel to the given plane x + 2y + z = 10 is x + 2y + z = λ. It goes through (5, –3, 3) $\Rightarrow$ 5 – 6 + 3 = 2 = λ

$\therefore$ plane is x + 2y + z = 2 ....(1)

If the straight line $\frac{2x-1}{2}=\frac{3y-4}{4}=\frac{z+4}{2}$ meets this plane is Q then PQ is the required distance.

Straight line is $\frac{x-1/2}{1}=\frac{y-4/3}{4/3}=\frac{z+4}{2}$

$\Rightarrow\frac{x-1/2}{3}=\frac{y-4/3}{4}=\frac{z+4}{6}$

$\Rightarrow\frac{x-1/2}{3/\sqrt{61}}=\frac{y-4/3}{4/\sqrt{61}}=\frac{y-4/3}{6/\sqrt{61}}$, and point on it is $\frac{3\lambda}{61}+\frac{1}{2},\frac{4\lambda}{\sqrt{61}}+4/3,\frac{6\lambda}{\sqrt{61}}-4$. It should be on plane (1) $\left(\frac{3\lambda}{\sqrt{61}}+\frac{1}{2}\right)+\left(\frac{8\lambda}{\sqrt{61}}+\frac{8}{3}\right)+\left(\frac{6\lambda}{\sqrt{61}}-4\right)=2$

$\Rightarrow\frac{17\lambda}{\sqrt{61}}+\frac{19}{6}-6=0\ \Rightarrow\lambda=\frac{\sqrt{16}}{6}$

$\therefore$ Point Q is $\frac{3}{6}+\frac{1}{2},\frac{4}{6}+\frac{4}{3},\frac{6}{6}-4\Rightarrow 1,2,-3$

$\therefore$ Distance $=\sqrt{(5-1)^2+(-3-2)^2+(3+3)^2}=\sqrt{77}$

Example 21 : A plane cuts intercepts OA = a, OB = b, OC = c on axes. Find area of triangle ABC.

Sol. Let Ax. Ay. Az be the projection of the area of triangle on plane x = 0, y = 0 and z = 0. Ax = area of Δ COB = $\frac{1}{2}$bc and $Ay=\frac{1}{2}ac, Az=\frac{1}{2}ab$.

$\therefore$ area of Δ ABC = $\sqrt{(Ax)^2+(Ay)^2+(Az)^2}=\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$

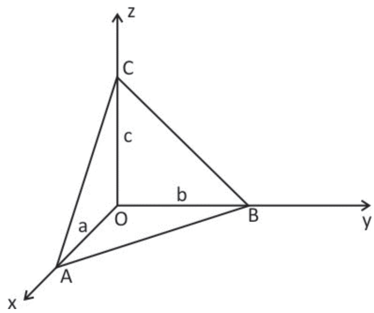


Fig 3

Example 22 : Find shortest distance between straight line 3x – 9y + 5z = 0 = x + y – z and 6x + 8y + 3z – 13 = x + 2y + z – 3

Sol. : First straight line is 3x – 9y + 5z = 0 and x + y – z = a let its d.c. be a_1, b_1 and c_1

$$\left.\begin{aligned}\therefore 3a_1-9b_1+5c_1=0\\ a_1+b_1-c_1=0\end{aligned}\right\}\Rightarrow\frac{a_1}{4}=\frac{b_1}{8}=\frac{c_1}{12}$$

$\Rightarrow \frac{a_1}{1} = \frac{b_1}{2} = \frac{c_1}{3}$

If z = 0 then 3x – 9y = 0, x + y = 0 $\Rightarrow$ x = 0, y = 0,

$\therefore$ equation of this straight line in symmetrical form is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ ....(1)

2nd straight line is 6x + 8y + 3z = 13, x + 2y + z = 3

Let its d.c. be a_2, b_2, c_2

$$\therefore \left.\begin{array}{l} 6a_2 + 8b_2 + 3c_2 = 0 \\ a_2 + 2b_2 + c_2 = 0 \end{array}\right\} \Rightarrow \frac{a_2}{2} = \frac{b_2}{-3} = \frac{c_2}{4}$$

and if z = 1 then 6x + 8y = 10, x + 2y = 2 solving x = 1, y = ½

$\therefore$Straight line is $\frac{x-1}{2} = \frac{y-1/2}{-3} = \frac{z-1}{4}$ ....(2)

Let l, m, n be d.R of the shortest distance lines

$$\therefore \left.\begin{array}{l} 2l - 3m + 4n = 0 \\ l + 2m + 3n = 0 \end{array}\right\} \Rightarrow \frac{l}{-17} = \frac{m}{-2} = \frac{n}{7}$$

$\Rightarrow \frac{l}{17} = \frac{m}{2} = \frac{n}{-7}$ $\therefore$ d.c. are $\frac{17}{\sqrt{342}}, \frac{2}{\sqrt{342}}, \frac{-7}{\sqrt{342}}$

$$\text{S.D.} = (1-0)l' + \left(\frac{1}{2} - 0\right)m' + (1-0)n' = (1-0)\frac{17}{\sqrt{342}} + \left(\frac{1}{2} - 0\right)\frac{2}{\sqrt{342}} + (1-0)\left(\frac{-7}{\sqrt{342}}\right)$$

$$= \left(\frac{11}{\sqrt{342}}\right)$$

Exercise 2(b)

1. Prove that points (5, –2, 3) and (3,3,3) are equidistant from plane $5x + 2y - 7z + 9 = 0$.
2. (a) Find distance between parallel planes, $2x - 2y + z + 1 = 0$ and $4x - 4y + 2z + 5 = 0$.

   (b) Find distance between planes $2x + 3y - 6z = 15$ and $4x + 6y - 12z + 21 = 0$
3. Show that point (1, 1, 1) lies between planes $2x + 3y - 6z + 12 = 0$ and $4x + 6y - 12z = 49$.
4. Find direction cosines of the straight lien $3x + 2y - z - 5 = 0 = 4x + y - 2z + 5$ and put it in symmetric form.
5. Find the equation of line of intersection of the planes $2x - y + 2z - 10 = 0$ and $6x + 3y - 2z + 15 = 0$ in symmetrical form.
6. Find the incentre of the tetrahedron formed by planes x = 0, y = 0, z = 0 and plane $x + 2y + 2z = 1$.

   [Hint – The three face of tetrahedron are y – z, z – x and x – y planes so incentre shall be (a, a, a) such that distance of (a,a,a) from fourth face $x + 2y + 2z - 1 = 0$ is equal to radius a]
7. Find the equation of plane bisecting the angle between planes $3x - 4y + 12z = 26$ and $x + 2y - 2z = 9$ not containing origin.
8. Find the equation of plane passing through the intersecting planes $x + 3y + 6 = 0$ and $3x - y - 4z = 0$ when its perpendicular distance from origin is unity.
9. (a) of the two planes $2x + 2y - z + 6 = 0$ and $\frac{1}{2} + \frac{1}{3}y + z + 1 = 0$, which is nearer to origin.

   (b) Find equation of line of intersection of these plane symmetrical form.
10. Find the equation of plane through the intersection of planes $2x - 2y + z + 5 = 0$ and $x + 2y + 2z - 7 = 0$ perpendicular to x-y plane.
11. Find the equation of plane which bisects the acute angle between planes $2x - y + 2z = 3$ and $2x - 2y + z = 6$.
12. Show that $6x^2 + 4y^2 - 10z^2 + 3yz + 4zx - 11xy = 0$ represents a pair of planes.
13. Find the points of intersection of this $x + y - 1 = 0 = y + z - 5$ and $2x + y + z - 3 = 0 = x - 3y + z + 4$.
14. Find point of intersection of straight lines $x + y + z - 6 = 0 = 2x - 3y + 4$ and $2x - y + z - 3 = 3x - y - z + 2$.
15. Find the shortest distance between the line $5x - y - z = 0 = x - 2y + z + 3$ and $7x - 4y - 2z = 0 = x - y + z + 3$.

16. Find shortest distance between the straight line $x+y-4=0=y+z-5$ and $2x+y+z-5=0=x-3y+z+1$.

17. Find the equation of straight line through (3, 4, –1) parallel to planes $2x + 3y + 4z = 5$ and $3x + 5y + 4z = 6$

18. Find the equation of a plane through points (2, 0, 4), (–1, 2, 1) and parallel to line $2x = \frac{y-3}{2} = \frac{3z+5}{3}$.

19. Find the distance of point (–2,3, –4) from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to plane $4x + 12y - 3z + 1 = 0$.

20. The plane $2x + 3y + 6z - 4 = 0$ meets the axes in A, B and C. Find the incentre of tetrahedron OABC.

21. Find equation of straight line through (0,–1,3) parallel to planes $2x - y + 3z = 8$ and $3x + y - 2z = 7$.

22. Show that the origin lies in the acute angle between planes $x + 2y + 2z = 9$ and $4x-3y+12z+13=0$. Find equation of that angle bisecting plane.

7.17 Angle between the line and a plane:

Let the plane P = 0 be ax + by + cz + d = 0 and straight line be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

Let it make angle θ with the plane. In fig. AB is the line, AC is any line in the plane. AN is normal of plane.

The line shall make angle $(90^0-\theta)$ with the normal of plane.

$$\therefore \cos(90^0-\theta)=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}=\sin\theta$$

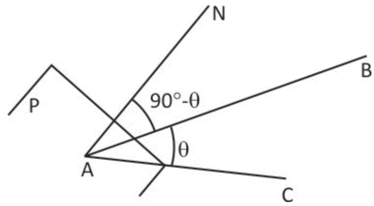


Fig 4

(a) If line is parallel to the plane, then $\sin\theta=0$ i.e. al + bm + cn = 0

(b) Line shall be perpendicular to the plane if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$

7.18 Condition for a line to lie in the plane:

Let plane be ax + by + cz + d = 0 and straight line be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$. The line shall be in the plane if every point of line lies in the plane.

$lr+\alpha, mr+\beta, nr+\gamma$ is any point on the line and for different values of r we can get all the points of line. The points lie in the plane then $a(lr+\alpha)+b(mr+\beta)+c(nr+\gamma)+d=0$

$$\Rightarrow (a\alpha+b\beta+c\gamma+d)+r(al+bm+cn)=0$$

This relation proves that every point of line shall lie in the plane if $a\alpha+b\beta+c\gamma+d=0$ and $al+bm+cn=0$

These are the required two conditions for lines.

7.19 Line of greatest Slope :

Plane ABCD intersects the horizontal plane of ground in AD. PQ is a line in the plane ABCD. If it is perpendicular to AD then only it is the line of greatest slope. Therefore in an inclined plane the line of greatest slope is that line which is perpendicular to the line of intersection of the inclined plane and the ground plane.

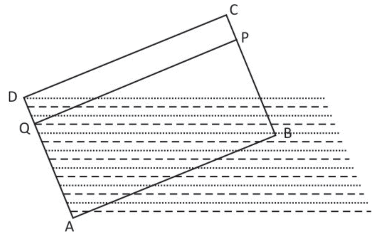


Fig 5

7.20 Coplanar lines

To find the condition that the two given lines are coplanar.

Let the lines be $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ....(1)

And $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ ....(2)

The lines shall be coplanar if they lie in the same plane.

Equation of plane containing lines (1) is a

$(x - x_1) + b (y - y_1) + c(z - z_1) =$ ....(3)

Where $al_1 + bm_1 + cn_1 = 0$ .....(4)

The plane (3) shall contain line (2) if

$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$ .....(5)

And $al_2 + bm_2 + cn_2 = 0$ ....(6)

If we eleminate a, b, c from equation (4), (5) and (6) we shall get the required condition.

$\therefore$ Condition is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$ ....(7)

This is the condition for the two straight line to be co-planer.

Note : If we elemiante a, b, c from (3), (4) and (6) we shall get the equation of the plane containing the lines

Plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$ ....(8)

Example 23 : Prove that straight line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ lies in the plane 4x + 4y – 5z = 3

Sol. : (i) Line should be perpendicular to normal of plane (ii) point (3,4,5) must lie in the plane. If these two conditions are satisfied, the straight line then shall lie in plane. (i) D.R of straight line are 2, 3, 4 and of normal of plane are 4,4, –5 = 0

(ii) point is (3,4,5) and 4.3 + 4.4 –5.5 – 3 =0 Both conditions satisfied Straight line lie in plane.

Example 24 : For what value of a and b, the straight line $\frac{x-a}{2}=\frac{y+2}{5}=\frac{z-3}{-b}$ shall in plane 4x + 2y + 3z + 1 = 0

Sol. : D. R. of straight line 2, 5, –b and of normal of plane 4,2,3 for straight line to lie in plane.

2.4 + 5.2 – 3.b = 0 $\Rightarrow$ b = 6

point (a, –2, 3) of straight line should lie in plane

$\therefore$ 4a + 2(–2) + 3.3 + 1 = 0 $\Rightarrow$ a = – 6/4

$\therefore$ For straight line to lie in plane $a=-\frac{3}{2}, b=6$.

Example 25 : Find angle between plane 4x – 3y + 5z = 8 and straight line $\frac{x-3}{2}=\frac{y+1}{2}=\frac{z}{-1}$

Sol. : Plane 4x – 3y + 5z = 8, D. R of normal, 4, –3, 5 straight line $\frac{x-3}{2}=\frac{y+1}{2}=\frac{x}{-1}$ d.R of line, 2, 2, –1

If q is the angle, then $\sin\theta=\frac{4.2+(-3).2+5(-1)}{\sqrt{9}\sqrt{50}}=\frac{-3}{3.5\sqrt{2}}=\frac{-1}{5\sqrt{2}}$ for acute angle $\theta=\sin^{-1}\left(\frac{1}{5\sqrt{2}}\right)$

Example 26 : Does straight line $\frac{x-1}{5}=\frac{y+2}{6}=\frac{z-3}{4}$ intersect the plane 2x + 3y – 6z + 20 = 0.

Sol. : For straight line intersect the plane, there should be a common point. On straight line $\frac{x-1}{5}=\frac{y+2}{6}=\frac{z-3}{4}, (5\lambda+1, 6\lambda-2, 4\lambda+3)$ is any point. For it to be on plane too $2(5\lambda+1)+3(6\lambda-2)\ -6(4\lambda+3)=-20$

Note : Line is not parallel to plane as $2\times5+3\times6-6\times4\neq0$. Line intersect the plane.

Example 27 : A plane is parallel to x – 1 = 2y – 5 =\ 2z and 3x = 4y – 11 = 3z – 4 and passes through (2,3,3). Find its equation.

Sol. : Given lines are $\frac{x-1}{1}=\frac{y-5/2}{1/2}=\frac{z}{1/2}$ ....(1)

And $\frac{x}{1/3}=\frac{y-11/6}{1/4}=\frac{z-4/3}{1/3}$ ....(2)

Equation of plane through (2,3,3) is a(x – 2) + b (y – 3) +c (z – 3) = 0 .... (3)

It is parallel to (1) $\therefore a+\frac{1}{2}b+\frac{1}{2}c=0$

It is parallel to (2) $\frac{1}{3}a+\frac{1}{4}b+\frac{1}{3}c=0$. The equation are 2a+b+c=0

And 4a + 3b + 4c = 0

$\therefore \frac{a}{4-3}=\frac{b}{4-8}=\frac{c}{6-4} \Rightarrow \frac{a}{1}=\frac{b}{-4}=\frac{c}{2}$

$\therefore$ equation of plane is $1.(x-2)-4(y-3)+2(z-3)=0$

$\Rightarrow x-4y+2z+4=0$

Example 28: Find equation of a plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z-2}{1}$ and the point (0, –7,3).

Sol. : Let the required plane be ax + by +cz + d = 0 ....(1)

It contains line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z-2}{1}$

$\therefore\ -3a+2b+c=0$ ....(2)

and $-a+3b+2c+d=0$ ....(3)

Plane (1) shall pass through (0, –7,3)

If $-7b+3c+d=0$ ....(4)

From (3) –(4) $-a+10b-c=0$ ....(5)

$-3a+2b+c=0$ ....(2)

$\therefore \frac{a}{10+2}=\frac{b}{3+1}=\frac{c}{-2+30} \Rightarrow \frac{a}{3}=\frac{b}{1}=\frac{c}{7}$

$\therefore$ from (3) – 3 + 3 + 14 + d = 0 $\Rightarrow d=-14$

$\therefore$ plane is 3x + y + 7z – 14 = 0

Example 29 : With the given axes the straight line $\frac{x}{3}=\frac{y}{-2}=\frac{z}{1}$ is vertical. Find d.c. of the line of greatest slope in the plane 2x + 3y – 3z = 18.

Sol. : Since $\frac{x}{3}=\frac{y}{-2}=\frac{z}{1}$ is vertical, the horizontal at plane is 3x – 2y + z = 0 through origin. Given plane is 2x + 3y – 3z = 18. Let l, m, n be the d . rations of the line of intersection of horizontal plane and given plane.

$$\therefore \left.\begin{matrix} 3l-2m+n=0 \\ 2l+3m-3n=0 \end{matrix}\right\} \Rightarrow \frac{l}{-3}=\frac{m}{11}=\frac{n}{13}$$

Let l_1, m_1, n_1 be d.R. of line of greatest slope. $\therefore -3l_1+11m_1+13n_1=0$

Line lies in plane $\Rightarrow 2l_1+3m_1-3n_1=0$

$$\therefore \frac{l_1}{-33-39}=\frac{m_1}{26-97}=\frac{n_1}{-9-22}$$

$$\Rightarrow \frac{l_1}{-72}=\frac{m_1}{17}=\frac{n_1}{-31}$$

$\therefore$ D.C. are $\frac{-72}{\sqrt{6434}},\frac{17}{\sqrt{6434}},\frac{-31}{\sqrt{6434}}$

Example 30 : Find the equation of plane through $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-9}{5}$ parallel to x-axis.

Sol. : Plane parallel to x axes is ay + cz + d = 0 ....(1)

Line $\frac{x-3}{2}=\frac{7-4}{3}=\frac{z-9}{5}$ lies in it

$$\therefore \quad 3b+5c=0 \Rightarrow c=-\left(\frac{3}{5}\right)b$$

and point (3,4, –9); 4b + 9c + d = 0

$$\Rightarrow d=-4b-9c=-4b+\frac{27}{5}b=\frac{7}{5}b$$

$$\therefore b:c:d=b:-\frac{3}{5}b:\frac{7}{5}b \Rightarrow 5:-3:7$$

$\therefore$ Equation of plane is 5y – 3z + 7 = 0

Example 31 : Find the equation of a plane through (1, 0, –1) and (3,2,2) and parallel to line x – 1 = $\frac{1-y}{2}=\frac{z-2}{3}$

Sol. : Let the plane through (1, 0, –1) be $a(x-1)+by+c(z+1)=0$ ....(1)

It is parallel to $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$

$\therefore \quad a-2b+3c=0$ ....(2)

Point (3,2,2) lies in it $\therefore 2a+2b+3c=0$ ....(3)

$\therefore \frac{a}{-12}=\frac{b}{3}=\frac{c}{6} \Rightarrow \frac{a}{4}=\frac{b}{-1}=\frac{c}{-2}$

$\therefore$ plane is 4(x – 1) –y – 2 (z +1) = 0

$\Rightarrow 4x-y-2z-6=0$

Example 32 : Prove straight line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar, Find point of intersection too.

Sol. : The 1st line passes through (1,2,3) and its d.r. are 2,3,4. The 2nd line passes through (2, 3, 4) and its d.r. are 3,4,5. These shall be coplanar if

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0, D=(15-16)-(10-12)+(8-9)=-1+2-1=0$$

$\therefore$ lines are coplanar

$$\text{Equation of planes is } \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$\Rightarrow -(x-1)+2(y-2)-(z-3)=0$

$\Rightarrow x-2y+z=0$

To find point of intersection any point of line (1); $2\lambda+1, 3\lambda+2, 4\lambda+3$

and any point of line (2) is ; $3\lambda_1+2,\ 4\lambda_1+3,\ 5\lambda_1+4$

For point of intersection –

$2\lambda+1=3\lambda_1+2 \qquad \Rightarrow \lambda+1=\lambda_1+1 \Rightarrow \lambda=\lambda_1$

$3\lambda+2=4\lambda_1+3$

$4\lambda+3=5\lambda_1+4 \Rightarrow 4\lambda+3=5\lambda+4 \Rightarrow 5\lambda+4 \Rightarrow -1=\lambda$

$\therefore$ point of intersection is (–1, –1, –1)

Example 33 : Prove that the lines $\frac{x-1}{2}=\frac{y-1}{3}=\frac{z-3}{4}$ and 4x – 3y + 1 = 0 = 5x – 3z + 2 are coplanar. Find point of intersection also.

Sol. : Let l, m, n be d.r. of 2nd line then $4l-3m+0n=0, 5l+m.0-3n=0$

$$\therefore \frac{l}{9}=\frac{m}{12}=\frac{n}{15}\Rightarrow\frac{l}{3}=\frac{m}{4}=\frac{n}{5}$$

planes are, 4x – 3y + 1 = 0, 5x – 3z + 2 = 0

$$\Rightarrow x=0, y=\frac{1}{3}, z=\frac{2}{3}$$

point is $\left(0,\frac{1}{3},\frac{2}{3}\right)$ straight line $\frac{x}{3}=\frac{y-1/3}{4}=\frac{z-2/3}{5}$ and x_1-x_2

$$\Rightarrow 1-0=1,\ 2-\frac{1}{3}=\frac{5}{3},\ 3-\frac{2}{3}=\frac{7}{3}$$

$$\therefore D=\frac{1}{3}\begin{vmatrix}3 & 5 & 7\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}=\frac{1}{3}[(3x(-1)-5(-2)+7(8-9)]=0$$

$\therefore$ Lines are coplanar any point of these lines are $2\lambda+1, 3\lambda+2, 4\lambda+3$ and $3\mu, 4\mu+1/3, 5\mu+2/3$ point of intersection.

$$\left.\begin{matrix}2\lambda+1=3\mu\\ 3\lambda+2=4\mu+1/3\end{matrix}\right\}\text{ Solving } \lambda=-1, \mu=-\frac{1}{3} \text{ and this satisfies } 4\lambda+3=5\mu+\frac{2}{3}$$

$\therefore$ point of intersection is (–1, –1, –1)

Example 34 : Ax, Ay and Az are projections of area A on three co-ordinate axes, then prove $A^2=(Ax)^2+(Ay)^2+(Az)^2$

Sol. : Let $\cos\alpha, \cos\beta, \cos\gamma$ be the d.c. of the normal to the plane area A.

$\therefore \alpha$ = angle that plane makes with y –z plane

$\therefore$ Ax = A cos α

Similarly Ay = Acosβ, Az = A cos γ

$$\therefore (Ax)^2+(Ay)^2+(Az)^2=A^2\left(\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)=A^2$$

Note : Equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ shall represent two planes if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

2. And angle between these planes is

$$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-bc-ca-ab}}{a+b+c}$$

3. The two planes gives by this equation shall be perpendicular a + b + c = 0

Example 35 : Find image of point (2, – 4, 5) in the plane x – 2y + z = 3.

Sol. : Equation of plane is x – 2 y + z = 0

D. R. of normal are 1, –2, 1

$\therefore$ Equation of line perpendicular to plane and passing through given point P (2, –4,5) is

$\frac{x-2}{1} = \frac{y+4}{-2} = \frac{z-5}{1} = \lambda$ (say) any point on it is ($\lambda+2, -2\lambda-4, \lambda+5$)

If this perpendicular line meets plane in M. then M is $(\lambda+2, -2\lambda-4, \lambda+5)$

And $\therefore$ $\lambda+2-2(-2\lambda-4)+(\lambda+5)=3$.

$\Rightarrow 6\lambda = -12 \qquad \Rightarrow \lambda = -2$

$\therefore$ M is (0, 0, 3). If P' is image of P in plane then P' lies on PM produced and M is mid point of PP'. If P' is (x_1, y_1, z_1) then

$$\frac{x_1+2}{2} = 0 \Rightarrow x_1 = -2, \frac{y_1-4}{2} = 0 \Rightarrow y_1 = 4, \frac{z_1+5}{2} = 3 \Rightarrow z_1 = 1$$

$\therefore$ image is (–2, 4, 1)

Example 36 : Find image of line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z-3}{5}$ in the plane x – y + z + 2 = 0

Sol.: Straight line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z-3}{5}$ co-ordinates of any point on line $(3\lambda+1, 4\lambda+2, 5\lambda+3)$. If this point lies in given plane x – y + z + 2 = 0. Then 3λ + 1 – 4λ –2 + 5λ + 3 +2 = 0 $\Rightarrow \lambda = -1$. This point P on plane is (–2, –2, –2) In fig. AP is line, B is image of A in the plane. PB is image of line in the plane. B is image of A in the plane. Therefore AB is perpendicular to the plane and M is mid point of AB.

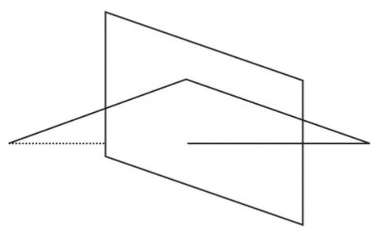


Fig 6

D. R. of normal of plane are 1, –1,1

$\therefore$ Equation AB is $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-3}{1} = (\mu)$

$\therefore$ B is ($\mu+1, -\mu+2, \mu+3$) and M is $\frac{\mu+1+1}{2}, \frac{-\mu+2+2}{2}, \frac{\mu+3+3}{2}$ on plane

$$\therefore \frac{\mu}{2}+1-\left(-\frac{\mu}{2}+2\right)+\frac{\mu}{2}+3+2=0 \Rightarrow \mu=-\frac{8}{3}$$

$$\therefore \text{B is } \left(-\frac{5}{3},\frac{14}{3},\frac{1}{3}\right)$$

Equation PB is $\frac{x+2}{-5/3+2}=\frac{y+2}{14/3+2}=\frac{x+2}{1/3+2}$

$$\Rightarrow \frac{x+2}{1/2}=\frac{y+2}{20/3}=\frac{z+2}{7/3}$$

$$\Rightarrow \frac{x+2}{3}=\frac{y+2}{40}=\frac{z+2}{14}$$

Example 37 : Find projection of line $\frac{x+1}{2}=\frac{y+2}{3}=\frac{z+3}{4}$ on plane x – 2y + 3z – 4 = 0.

Sol. : In question a straight line (fig AB) and a plane P_1 is given. Projection of straight line L_1 (AB) on plane is straight line L_2. L_2 lies in the plane P_1 and also in the plane perpendicular to it, and which contains straight line L_1.

Thus L_2 is line of intersection of plane P1 and the plane perpendicular to it and contains straight line L_1. Equation of plane is x – 2y + 3z – 4 = 0 ....(1) and let the plane containing straight line

$$\frac{x+1}{2}=\frac{y+2}{3}=\frac{z+3}{4}$$

be $ax+by+cz+d=0$ ....(2)

Point (–1, –2, –3) of straight line shall lie on it

$\therefore -a-2b-3c+d=0$ ....(3)

D.R. of normal of plane are a, b, c and it is perpendicular to given plane.

$\therefore a-2b+3c=0$ ....(4)

D.R. of straight line L_1, are 2, 3 and 4

$\therefore 2a+3b+4c=0$ ....(5)

from (4) and (5) $\frac{a}{17}=\frac{b}{-2}=\frac{c}{-7}=k(\text{say})$

$\therefore$ from (3) $-17k+4k+21k+d=0 \Rightarrow d=-8k$

Hence the plane is $17x-2y-7z-8=0$

Projection L_2 is line of intersection of planes x – 2y + 3z = 4 and 17 x – 2y – 7z – 8 = 0. Let its D. R. be l, m, n then

$$\left.\begin{array}{l} l-2m+3n=0 \\ 17l-2m-7n=0 \end{array}\right\} \Rightarrow \frac{l}{10}=\frac{m}{29}=\frac{n}{16}$$

Now if z = 0, then $\left.\begin{array}{l} x-2y=4 \\ 17x-2y=8 \end{array}\right\} \begin{array}{l} x=\frac{1}{4} \\ y=-\frac{15}{4} \end{array}$

Projection of L_1, straight line L_2 is $\frac{x-1/4}{10}=\frac{y+15/8}{20}=\frac{z}{16}$

Example 38 : A variable plane makes intercept on axes, the sum of whose squares in constant equal to k^2. Show that locus of foot of perpendicular from origin on it is $\left(x^{-2}+y^{-2}+z^{-2}\right)\left(x^2+y^2+z^2\right)=k^2$

Sol. : Let the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ......(1)

a, b, c variables and $a^2 + b^2 + c^2 = k^2$ given ......(2)

The d.c. of normal of plane are $\frac{1}{a},\frac{1}{b},\frac{1}{c}$

Hence from origin equation of normal to the planes is $\frac{x}{1/a}=\frac{y}{1/b}=\frac{z}{1/c}=r(\text{say})$ any point on normal is $\frac{r}{a},\frac{r}{b},\frac{r}{c}$, Let it be the foot of perpendicular (x_1, y_1, z_1)

$\therefore x_1=\frac{r}{a}\Rightarrow a=\frac{r}{x_1}$, similarly $b=\frac{r}{y},c=\frac{r}{z_1}$ (x_1, y_1, z_1) lies on plane (1)

$$\therefore \frac{x_1}{a}+\frac{y_1}{b}+\frac{z_1}{c}=1\Rightarrow \frac{x_1^2}{r}+\frac{y_1^2}{r}+\frac{z_1^2}{r}=1$$

$\therefore r=x_1^2+y_1^2+z_1^2$` ....(3)

and $a^2+b^2+c^2=k^2\Rightarrow \frac{r^2}{x_1^2}+\frac{r^2}{y_1^2}+\frac{r^2}{z_1^2}=k^2$

$$\therefore r^2\left(\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}\right)=k^2$$

Putting value of r in it from (3)

$\therefore$ Locus is $(x^2 + y^2 +z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = k^2$

Example 39 : A plane parallel to x – 1 = 2y – 5 = 2z and 3x = 4y – 11 = 3y – 4 passes through. (2, 3, 3). Find its equation.

Sol. : Straight lines are $\frac{x-1}{1}=\frac{y-5/2}{1/2}=\frac{z}{1/2};\frac{x}{1/3}=\frac{y-11/4}{1/4}=\frac{z-4}{1/3}$. If a, b, c are d.r. of normal of plane then, $a+\frac{1}{2}b+\frac{1}{c}c=0 \Rightarrow 2a+b+c=0$

$\frac{1}{3}a+\frac{1}{4}b+\frac{1}{3}c=0 \Rightarrow 4a+3b+4c=0$ $\therefore \frac{a}{1}=\frac{b}{-4}=\frac{c}{2}$

Plane is $a(x-2)+b(y-3)+c(z-3)=0$

$$x-2-4y+12+2z-6=0$$
$$x-4y+2z+4=0$$

Exercise 2 (c)

1. Find the equation of the plane through P(2,3,6) and perpendicular to OP. Where O is origin.
2. Find the length of perpendicular from origin on the plane. Which cuts off intercept of 5, – 4 and 3 from axes.
3. Find the equation of the plane which goes through (3, 2, –1) and is perpendicular to the line joining (–2, 4, 1) and (–4, 2, –3).
4. Find the equation of plane which bisects the join AB and is perpendicular to AB. A is (3,4,1) and B(– 5, 2, –3).
5. Find the equation of the plane which passes through (3, 1, –1), (2, –2, –2), (0, 3, 3).
6. Find the equation of the plane through (1,3,5) and parallel to the plane $x - 3y + 2z + 5 = 0$.
7. From a point P (a,b,c) perpendicular PM and PN are drawn to y – z and z – x planes respectively. Find the equation of OMN plane.
8. Find the equation of plane which passes through the line of intersection of planes $3x - 2y = 0$ and $5z - 3y = 0$ and is perpendicular to plane $4x + 5y - 3z = 8$.
9. Show that the four points (1, –1,0), (2, 1, –1), (1,1,1) and (3, 3, –2) are coplanar.
10. Find the equation on of plane through (–1,0,1) (3,5,7) and perpendicular to plane $3x + 4y - 5z = 11$.
11. Find the equation of plane which passes through (3,1, –2) and is perpendicular to planes $2x + 2y - z + 5 = 0$, $3x + 6y - 2z + 12 = 0$.
12. Find the distance of point (–2, –3, –4) from the plane $x + 2y - 2z + 12 = 0$.
13. Find the equation of plane which passes through the straight line $x + 2y - z + 5 = 0 = 3x + 4y + 2z$ and the point (2,–1,3).
14. A variable plane at a constant distance 3p from origin and which meets the axes in A, B and C. Find the locus of the centroid of the plane.
15. Find the equation of plane which passes through (–2,1,3) and (4,3,5) and is parallel to z-axis.
16. Find equation of plane through (2,2,1) and (1, –2, 3) and parallel to x-axis.
17. Find the distance between parallel planes $4x + 4y - 2z - 3 = 0$ and $2x + 2y - z + 11 = 0$
18. The distance of the point (1, –2,3) from the plane $x - y + z = 5$ measured parallel to the line whose direction ratios are 2, 3, –6 is ....
19. Find the point where the line passing through points (2, –3,1) and (3, –4, –5) intersects the plane $2x + y + z = 7$

20. Find the equation of the perpendicular from point (1, 6, 3) on the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$, find also the foot of perpendicular.

21. The equation of a line through (3, 2, –1) parallel to planes 2x + 3y + 4z = 5 and 3x + 5y + 4z = 10.

22. A plane parallel 2x – 1 = 3y – 2 = 2z and 3x = 4y – 12 = 3z – 4 and passing through (1, –1,1) is ….

23. P point is (1,3,6), A(2, –1,3), B (3,1,2) perpendicular from P meets AB on PN co-ordinate of N are.

24. Find angle between planes x + y – z = 4 and 3x + 4y – 5z = 7

25. Find angle between straight line $\frac{x-2}{2}=\frac{y-3}{1}=\frac{z-5}{4}$ and plane 2x – 3y + z = 5.

26. Prove that straight line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ lies in the plane 4x + 4y – 5z – 3 = 0.

27. Straight line $\frac{x-1}{3}=\frac{y-4}{2}=\frac{z-3}{4}$ lies in the plane ax + 2y – 2z – 5 = 0, find value of a and n.

28. Prove that straight line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and 4x + 3y + 1 = 0 = 5x – 3z + 2 are coplaner. Find their point of intersection.

29. Find distance of point (3,8,2) from straight line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}$, measure parallel to the plane 3x + 2y – 2z + 15 = 0

30. Find image of straight line $\frac{x+4}{3}=\frac{y+2}{5}=\frac{z+3}{4}$ in the plane x + 2y – 3z = 0.

31. Find distance of point (–1, –5, –10) from the point of intersection of straight line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane x – y + z = 5.

32. Find point of intersection of straight line $\frac{x-1}{3}=\frac{y-2}{-4}=\frac{z+1}{2}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z-3}{-2}$.

33. Put equation 2x + 3y – 4z = 0 = 2x – 2y + z + 5 of straight line in symmetric form.

34. The equation of a straight line is $\frac{x+1}{3}=\frac{y-2}{-3}=\frac{z-3}{4}$. Find its points of intersection with the planes 3x + 4y – 2z = 10 and x + 2y – z + 7 = 0.

35. Find shortest distance between straight lines $\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{-5}$ and $\frac{x+1}{1}=\frac{y-3}{-2}=\frac{z-2}{3}$.

36. Find image of (3, –2, 1) in the plane 4x + 2y +z = 0.

37. Find image of (1,3,5) in the straight line $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-1}{1}$

38. Prove that $\frac{x-3}{2}=\frac{y-1}{-3}=\frac{z-5}{1}$ and $\frac{x-2}{5/2}=\frac{y+3}{-1}=\frac{z-1}{3}$ are co-planer. Find equation of plane too.

39. According to position plane 4x – 3y + 7z = 0 is horizontal. Find equation of line of greatest slope through the point (2,1,1) in the plane zx + y – 5z = 0.

40. Find angle between plane and the straight line. Plane is 2x – 2y + z = 6, straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{2}$.

41. Find equation of projection of the straight line 3x – y + 2z – 1 = x + 2y – z – 2 = 0 on the plane 3x + 2y + z = 0 in symmetric form.

42. Find equation of projection of the straight line $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-1}{1}$ of plane 4x – 3 y + 2z = 0.

43. Find equation of planes bisecting the angle between planes 3x – 4y + 2z = 17 and 4x + 2y – 3z + 9 = 0. Which plane bisects the acute angle.

44. Find equation of plane bisecting the angle between planes 6x + 2y – 3z + 7 = 0 and 3x + 6y + 6z = 10 when origin lies in the angle.

45. Show that equation 2x2 – 3y – 4z2 + 7yz – 2zx + xy = 0 represents a pair of planes.

46. A plane meets axes, A, B and C and OA, OB, OC are 3, 4 and 5 respectively. Find area of triangle ABC.

47. Plane 3x + 2y – 4z = 0 is horizontal and plane x – 2y + 3z = 0 is inclined at it. Find equation of line of greatest slope through (4, –1, –2) of inclined plane.

48. With the given axes, the straight line $\frac{x}{4}=\frac{y}{-3}=\frac{z}{2}$ is vertical. Find d. rs of the line of greatest slope in the plane 3x – 2y + 4z = 16.

49. Find projection of straight line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ on the plane x + 2y + 2z = 17.

50. If straight line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{4}$ and $\frac{x+2}{3}=\frac{y+4}{-2}=\frac{z+3}{a}$ are coplanar, then find a, and the plane.

51. The equation of plane through points (1,1,1), (–1,3,5) and (2, –3, 1) is

(a) $4x + 3y - 2z = 5$ (b) $8x + 2y + 3z = 13$

(c) $3x + 5y - 2z = -10$ (d) None of these

52. Angle between planes $3x + 4y + 5z = p$ and $2x + 3y + 6z = 15$ is :

(a) $\cos^{-1}\frac{p}{15}$ (b) $\cos^{-1}\frac{48}{35\sqrt{6}}$ (c) $\cos^{-1}\frac{48}{5\sqrt{58}}$ (d) $\cos^{-1}\frac{48}{35\sqrt{2}}$

53. Equation of plane parallel to $3x - 2y + 6z = 14$ and at a distance of 2 from it in positive direction is:

(a) $3x - 2y + 6z = 28$ (b) $3x - 2y + 6z = 0$

(c) $3x - 2y + 6z = 42$ (d) None of these

54. Equation of plane parallel to $2x + 3y - z + 9 = 0$ and through (4, 6, –3) is

(a) $2x + 2y - z = 17$ (b) $2x + 2y - z + 17 = 0$

(c) $2x + 2y - z \pm 17 = 0$ (d) None of these

55. Equation of plane through (3,0,1) and (4,8, 12) and perpendicular to plane $x + y - z = 8$ is

(a) $19x + 12y + 7z = 64$ (b) $19x - 12y - 7z + 104 = 0$

(c) $19x - 12y + 7z = 64$ (d) none of these

56. The plane through points of intersection of planes $2x + 3y + 4z = 7$ and $3x - 2y + 4z = 9$ and perpendicular to plane $2x + 2y - z = 3$ is:

(a) $11x - 3y + 16z = 34$ (b) $19x - 15y + 8z = 13$

(c) $11x - 3y + 16z - 34 = 0$ (d)none of these

57. A plane meets axes in (5,0,0), (0,2,0) and (0,0,3). Length of perpendicular from origin on plane is:

(a) $\frac{30}{19}$ (b) $\frac{10}{7}$ (c) $\frac{5}{6}$ (d) $\frac{3}{4}$

58. Image of point (4,6,8) in straight line $\frac{x-3}{10}=\frac{y-2}{2}=\frac{z-2}{-3}$ is:

(a) (2, –2, –4) (b) (–2,2–4) (c) (–2, 2, –4) (d) (2, –2,4)

59. Image of point (3, –2,4) in the plane $2x + 3y + z = 18$ is :

(a) (11, 10, 8) (b) (4, 7, 6) (c) (7, 4, 6) (d) (–1, –8,2)

60. Equation of the plane through (3,1,2) and (–1, 3, 5) and parallel to x-axis is:

(a) $5y + z - 7 = 0$ (b) $3y - 2z + 1 = 0$

(c) $3y + 2z - 19 = 0$ (d) $5y - z - 3 = 0$

61. A plane meets axes in A, B and C. Centroid of triangle is (2, –3,5). Equation of plane is

(a) $\frac{x}{6}-\frac{y}{9}+\frac{z}{15}=1$ (b) $\frac{x}{6}+\frac{y}{9}-\frac{z}{15}=1$

(c) $3x - 2y + z = 18$ (d) $5x - 3y + 2z = 0$

62. A plane passes through P (3,5,7) and is perpendicular to OP (O, is origin). Find equation of plane.

(a) $3x + 5y + 7z + 83 = 0$ (b) $3x + 5y + 7z = 83$

(c) $3x + 5y + 7z = 17$ (d) $3x - 5y - 7z = 25$

63. Image of point P (3,4,5) in plane 1, is (1, 0, –1). If d.c. of normal of plane be proportional to λ, $\lambda+1$ $\lambda+2$, then equation of plane is :

(a) $x - 2y - 3z + 8 = 0$ (b) $x + 2y + 3z = 12$

(c) $x + 2y + 3z = 15$ (d) $x + 2y - 3z = 8$

64. Equation of plane through (3, –2 , 4) and through the intersection of plane $x + y - z = 8$ and $2x + 3y - 5z = 2$ is :

(a) $2y - 5z + 22 = 0$ (b) $x - y + z = 9$

(c) $x - 3z + 14 = 0$ (d) none of these

65. Shortest distance between planes $2x + 3y + 6z = 15$ and $4x + 6y + 12z = 10$ is :

(a) $\frac{5}{7}$ (b) $\frac{10}{7}$ (c) $\frac{15}{7}$ (d) none of these

66. A and B are points (2, –3, 4) and (–2,3,5) plane is $2x + y - 2z + 15 = 0$.

(a) points are on different sides of plane.

(b) Points are the same side of plane.

(c) Point A is nearer to plane as compared with B

(d) Segment AB is perpendicular to plane.

67. Plane $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$ meets the axes in A, B and C area of Δ ABC is:

(a) 6 (b) 12 (c) 3 (d) 9

68. Equation of plane through (1, 0, 2) and (3, 1, –3) and parallel to straight line $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ is

(a) 11x + 12 y + 2z = 15 (b) 11x + 12y + 2z – 15 = 0

(c) 11x – 12y – 2z = 27 (d) 11x + 12y + 2z = 15

69. The straight line x – y + z + 1 = 0 = 4x + 4y + 2z – 2 in symmetric form is

(a) $\frac{x+1/5}{3}=\frac{y+6/5}{2}=\frac{z}{5}$ (b) $\frac{x-1/5}{-3}=\frac{y-6/5}{-2}=\frac{z}{5}$

(c) $\frac{x-1/5}{-3}=\frac{y-6/5}{2}=\frac{z}{5}$ (d) None of these

70. The straight line $\frac{x-3}{2}=\frac{y+1}{1}=\frac{z-2}{-2}$ meets the plane 2x – y + z = 4 in :

(a) $(-7,-6,12)$ (b) $(2,1,-1)$ (c) $(3,1,-1)$ (d) none of these

71. Image of straight line $\frac{x-2}{-1}=\frac{y-1}{-1}=\frac{z+1}{3}$ in the plane 8x + y + 3z = 14 is :

(a) $\frac{x-4}{3}=\frac{y+3}{3}=\frac{z+5}{7}$ (b) $\frac{x-4}{3}=\frac{y+3}{-3}=\frac{z+5}{7}$

(c) $\frac{x-4}{3}=\frac{y+3}{-3}=\frac{z-5}{7}$ (d) $\frac{x-4}{-3}=\frac{y-3}{3}=\frac{z+5}{7}$

72. Statement – I : The point A (3,1,6) is the image of point B(–1,3,4) in the plane x – y +z = 5.

Statement II : The plane x – y + z = 5 bisects segment joining A(3,1,6) and B(1,3,4)

(a) Statement I is true, Statement II is false

(b) Statement I is false, Statement II is True

(c) Statement I is true, statement II is a correct explanation of statement I

(d) Statement I is true, statement II is not a correct explanation of statement I

8. Sphere

8.1 Definition :

Sphere is the locus of a point in space whose distance from a fixed point is always the same constant. The fixed point is called the centre of sphere and the fixed constant distance, is radius of the sphere.

Equation of Sphere :

Let C, (a,b,c) be the centre of the sphere and r the radius. If P is any point (x, y, z) on the surface of sphere then

$\therefore PC = r \Rightarrow PC^2 = r^2$

$\therefore (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ ....(1)

This is the required equation of sphere. If the centre of sphere is origin then its equation is $x^2 + y^2 + z^2 = r^2$

Expanding (1) $x^2 + y^2 + z^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2 = r^2$ ....(2)

In this equation –

(1) co-efficients of x^2, y^2, z^2 are equal (unity)

(2) There is no term of xy, yz or zx.

(3) The equation is second degree equation in x, y, z.

$\therefore$ General equation of sphere can be written as

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ .... (3)

If we compare it with equation 2, then

u = -a, v = -b, c = -w, d = $a^2+b^2+c^2-r^2$

$\therefore$ Centre of this sphere is $(-u,-v,-w)$ and $d = a^2 + b^2 + c^2 - r^2 \Rightarrow r = \sqrt{a^2 + b^2 + c^2 - d}$

radius = $\sqrt{u^2 + v^2 + w^2 - d}$

8.2 Equation of sphere on the line joining two points as diameter.

Let point A be (x_1, y_1, z_1) and B be (x_2, y_2, z_2) and let P be any point (x, y, z) on the sphere whose diameter is AB.

$\therefore$ PA is perpendicular to PB.

The direction ratio of PA are $x - x_1$, $y - y_1$, $z - z_1$; and of PB are $x - x_2$, $y - y_2$, $z - z_2$,

$PA \perp PB \Rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$. This is the required equation of the sphere.

Example 1 : Find the equation of the sphere which passes through (4,1,2), (0, – 2, 3), (2,0,1) and (3,4,0).

Sol. : Let sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Point (4,1,2) $\therefore$ $21 + 8u + 2v + 4w + d = 0$ ....(1)

Point (0, –2, 3) $\therefore$ $13 + 0 - 4v + 6w + d = 0$ ....(2)

Point (2, 0, 1) $\therefore$ $5 + 4u + 2w + d = 0$ ....(3)

Point (3, 4, 0) $25 + 6u + 8v + d = 0$ ....(4)

(1) – (2) $8 + 8u + 6v - 2w = 0$ ....(5)

(2) – (3) $8 - 4u - 4v + 4w = 0$ ....(6)

(4) – (3) $20 + 2u + 8v - 2w = 0$ ....(7)

(7) – (5) $12 - 6u + 2v = 0$ ....(8)

(6) – 2 x (7) $48 + 12v = 0 \Rightarrow v = -4$

$\therefore$ from (8) $6u = 12 + 2v = 12 - 8 = 4$

$\therefore$ $u = 2/3$

from (7) $2w = 20 + 4/3 - 32 = -32/3 \Rightarrow w = -16/3$

$\therefore$ from (1) $21 + \frac{16}{3} - 8 - \frac{64}{3} + d = 0 \Rightarrow d = 3$

$\therefore$ Equation of sphere is

$$x^2 + y^2 + z^2 + \frac{4}{3}x - 8y - \frac{32}{3}z + 3 = 0$$

$$\Rightarrow 3\left(x^2 + y^2 + z^2\right) + 4x - 24y - 32z + 9 = 0$$

Example 2 : Find the equation of the sphere which passes through (1, –3, 4), (1, –4, 2) and (1, –3, 0) and whose centre lies on x + y + z = 0

Sol. : Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (1)

Point (1, –3, 4) : $26 + 2u - 6v + 8w + d = 0$ (2)

Point (1, –4, 2) : $21 + 2u - 8v + 4w + d = 0$ (3)

Point (1, –2, 0) : $5 + 2u - 4v + d = 0$ (4)

$$\left.\begin{array}{ll} (2)-(3) & 5 + 2v + 4w = 0 \\ (3)-(4) & 16 - 4v + 4w = 0 \end{array}\right\} \Rightarrow -11 + 6v = 0$$

$\therefore$ $v = \frac{11}{6}$

and $\therefore\ 5+\frac{11}{3}+4w=0 \Rightarrow w=-\frac{13}{6}$

Centre lies on $x + y + z = 1$

$\therefore -u-v-w=1 \Rightarrow -u=1+\frac{11}{6}-\frac{13}{6}=\frac{2}{3}$

$\therefore u=-\frac{2}{3}$

from (4) $5-\frac{4}{3}-\frac{22}{3}+d=0 \Rightarrow d=\frac{11}{3}$

$\therefore$ Equation of sphere is

$$x^2+y^2+z^2-\frac{4}{3}x+\frac{11}{3}y-\frac{13}{3}z+\frac{11}{3}=0$$

$$\Rightarrow 3\left(x^2+y^2+z^2\right)-4x+11y-13z+11=0$$

Example 3 : A sphere goes through (1, 0, 0); (0, 1, 0) and (0,0,1) and its radius is as small as possible. Find its equation.

Sol. : Let equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad .....(1)$$

point (1,0,0) $\quad 1 + 2u + d = 0 \Rightarrow u=-\left(\frac{1+d}{2}\right)$

point (0,1,0) $\quad 1 + 2v + d = 0 \Rightarrow v=-\left(\frac{1+d}{2}\right)$

point (0,0,1) similarly $\quad 1 + 2w + d = 0 \Rightarrow w=-\left(\frac{1+d}{2}\right)$

radius R of sphere $R^2=u^2+v^2+w^2-d$

$$R^2=\frac{3}{4}(d+1)^2 \Rightarrow R=\frac{\sqrt{3}}{2}\sqrt{d^2+2d+1-\frac{4}{3}d}$$

$$R=\frac{\sqrt{3}}{2}\sqrt{d^2+\frac{2}{3}d+1}=\frac{\sqrt{3}}{2}\sqrt{\left(d+\frac{1}{3}\right)^2+\frac{8}{9}}$$

R shall be minimum when $d=-\frac{1}{3}$

$$\therefore u = v = w = -\left(\frac{1-\frac{1}{3}}{2}\right) = -\frac{1}{3}$$

$\therefore$ Equation of sphere $3\left(x^2+y^2+z^2\right)-2x-2y-2z-1=0$

Example 4 : Find the equation of sphere whose center is (3, 6, –4) and which touches the plane $2x-y-2z+1=0$

Sol. : Sphere touches the plane $2x-y-2z+1=0$

$\therefore$ perpendicular from center on it = radius of sphere

$$\therefore\ r = \frac{6-6+8+1}{3} = 3$$

Equation of sphere $(x-3)^2+(y-6)^2+(z+4)^2=3^2$

$\Rightarrow x^2+y^2+z^2-6x-12y+8z+52=0$

Example 5 : A plane passes through a fixed point (a, b, c). Show that the lines of the foot of perpendicular from origin on the plane is sphere.

Sol. : Equation of plane through (a, b, c) is

$l(x-a)+m(y-b)+n(z-c)=0$ ......(1)

l, m, n are direction cosines of normal of plane

$\therefore$ Equation of perpendicular from origin on the plane is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r$$

any point on this perpendicular is

$lr, mr, nr \Rightarrow (x_1, y_1, z_1)$

If this is the foot of perpendicular on plane then it must lie on plane.

$$l=\frac{x_1}{r}, m=\frac{y_1}{r}, n=\frac{z_1}{r}$$

$$\therefore \frac{x_1}{r}(x-a)+\frac{y_1}{r}(y-b)+\frac{z_1}{r}(z-c)=0$$

$\therefore$ Locus x (x –a) + y (y – b) + z (z –c) = 0

$x^2 + y^2 + z^2 - ax - by - cz = 0$

Second Method : In the fig, A is (a,b,c) P (α, β, γ) is foot of perpendicular $OP \perp AP$

D. R. of OP are α, β, γ

D.R. of AP are $(\alpha - a, \beta - b, \gamma - c)$

$\therefore \alpha(\alpha - a) + \beta(\beta - b) + \gamma(\gamma - c) = 0$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Locus of P $x^2 + y^2 + z^2 - ax - by - cz = 0$

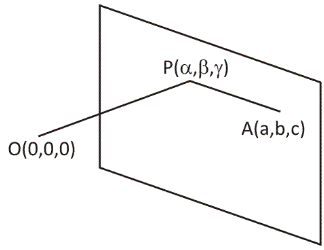


Fig 1

Example 6 : The center of a sphere c is (–4, 2, 3) and it touches y axis. Find the equation of sphere.

Sol.: Centre of sphere, c (–4, 2, 3)

It touches y axis at A, then A is (0, 2, 0)

$\therefore$ radius AC = $\sqrt{16 + 0 + 9} = 5$

Equation of sphere is

$(x+4)^2 + (y-2)^2 + (z-3)^2 = 25$

$x^2 + y^2 + z^2 + 8x - 4y - 6z + 4 = 0$

Example 7: Find the equation of the sphere which passes through (4,1,1) and touches x – y, y –z and z –x planes.

Sol. : Sphere touches all the three co-ordiante planes $\Rightarrow$ its center is at equal distance form planes. If this distance is α then center of sphere is (α, α, α). Its equation is :

$x^2 + y^2 + z^2 - 2\alpha x - 2\alpha y - 2\alpha z + d = 0$ ......(1)

It passes through (4,1,1)

$\therefore$ $18 - 12\alpha + d = 0$ ....(2)

And distance of (4,1,1) from center = α

$\therefore (\alpha - 4)^2 + (\alpha - 1)^2 + (\alpha - 1)^2 = \alpha^2$

$\Rightarrow 2\alpha^2 - 12\alpha + 18 = 0$ $\Rightarrow (\alpha - 3)^2 = 0$

$\therefore \alpha = 3$; from (2) d = –18 +36 = 18

$\therefore$ sphere is $x^2 + y^2 + z^2 - 6x - 6y - 6z + 18 = 0$

Example 8 : A point moves such that the sum of squares of its distances from 6 faces of a cube is constant. Show that its locus is a sphere.

Sol. Let center of cube be origin and its face parallel to co-ordinate axes. Let side of cube = 2a.

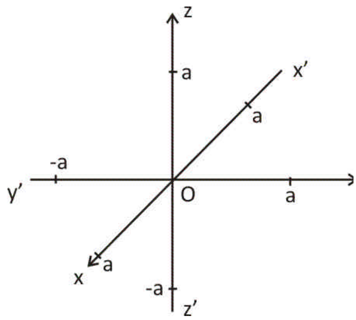


Fig 2

$\therefore$ Equation of opposite faces of the cubes are x = a, x = – a, y = a, y = – a, z = a, z =- – a.

Let (x, y, z) be the point $\therefore$ Locus is

$$(x-a)^2+(x+a)^2+(y-a)^2+(y+a)^2+(z-a)^2+(z+a)^2=k^2$$

$$\Rightarrow 2\left(x^2+y^2+z^2\right)+6a^2=k^2$$

$$\Rightarrow x^2+y^2+z^2+3a^2=\frac{k^2}{2}$$

which is equation of sphere.

Example 9 : Find the equation of sphere circums cribeing the tetrahedron whose faces are x = 0, y = 0, z=0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Sol. : The forces are

$x = 0$ .... (1)

$y = 0$ .....(2)

$z = 0$ .....(3)

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ....(4)

The point of intersection of any three faces shall give a vertex of tetrahedron.

Point of intersection of (1), (2) and (3) (0,0,0)

Point of intersection of (1), (2) and (4) (0,0,c)

Point of intersection of (1), (3) and (4) (0,b,0)

Point of intersection of (2), (3) and (4) (a,0,0)

$\therefore$ equation of sphere through these points is

$$x^2+y^2+z^2-ax-by-cz=0$$

Example 10 : A plane passes through a fixed point (a, b, c) and meets the axes in A,B and C show that the locus of the center of the sphere is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

Sol. : The sphere passes through O,A,B,C, i.e. it goes through origin, so sphere is

$$\therefore x^2+y^2+z^2+2ux+2vy+2wz=0$$

It meets axes in $x^2 + 2ux = 0$ $x = -2u$

$$\Rightarrow (-2u,0,0),(0,-2v,0)(0,0,-2w)$$

Equation of the plane through these points is

$$\frac{x}{-2u}+\frac{y}{-2v}+\frac{z}{-2w}=1$$

But it goes through (a, b, c)

$$\therefore \frac{a}{-u}+\frac{b}{-v}+\frac{c}{-w}=2$$

$(-u,-v,-w)$ is center of sphere

$\therefore$ Locus of the center of sphere is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

Example 11 : A sphere of constant radius k passes through origin and meets the axes in A, B and C. Prove that the foot of perpendicular from origin to the plane ABC lies on the surface

$$\left(x^2+y^2+z^2\right)^2\left(x^{-2}+y^{-2}+z^{-2}\right)=4k^2$$

Sol. : Let the sphere be $x^2 + y^2 + z^2 - ax - by - cz = 0$ .... (1)

It meets axes in (a,0,0), (0,b,0) and (0,0,c) and its radius $r=\frac{1}{2}\sqrt{a^2+b^2+c^2}$

The equation of plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ....(2)

Given $6=k=\frac{1}{2}\sqrt{a^2+b^2+c^2} \Rightarrow a^2+b^2+c^2=4k^2$ ....(3)

D.C. of normal of plane are proportional to $\frac{1}{a},\frac{1}{b},\frac{1}{c}$, hence equation of normal (perpendicular) from origin is

$$\frac{x}{1/a}+\frac{y}{1/b}=\frac{z}{1/c}=\lambda \qquad ....(4)$$

$\therefore \left(\frac{\lambda}{a},\frac{\lambda}{b},\frac{\lambda}{c}\right)$ is any point on this line distance l from origin. Let this point be (x_1, y_1, z_1) where the normal meets the plane.

$$\therefore x_1=\frac{\lambda}{a},\ y_1=\frac{\lambda}{b},\ z_1=\frac{\lambda}{c} \qquad ....(5)$$

and $\frac{x_1}{a}+\frac{y_1}{b}+\frac{z_1}{c}=1$ from (2)

$$\Rightarrow \frac{x_1^2}{\lambda}+\frac{y_1^2}{\lambda}+\frac{z_1^2}{\lambda}=1$$

$$= x_1^2 + y_1^2 + z_1^2 = \lambda \qquad ....(6)$$

and from (3) and (5)

$$\frac{\lambda^2}{x_1^2} + \frac{\lambda^2}{y_1^2} + \frac{\lambda^2}{z_1^2} = 4k^2$$

or $$\lambda^2\left(x_1^{-2} + y_1^{-2} + z_1^{-2}\right) = 4k^2 \qquad ....(7)$$

and from (6) $\left(x_1^2 + y_1^2 + z_1^2\right)^2\left(x_1^{-2} + y_1^{-2} + z_1^{-2}\right) = 4k^2$

$\therefore$ Locus of foot of perpendicular from origin to plane is

$$\left(x^2 + y^2 + z^2\right)^2\left(x^{-2} + y^{-2} + z^{-2}\right) = 4k^2$$

Example 12 : Show that the locus of center of sphere which passes through (1,2,3), (–2,1,4) and (3,–1,2) is a straight line. Find this equation of straight line in systematic form.

Sol. : Let (x_1, y_1, z_1) be the center of sphere

$\therefore$ Equation sphere $x^2 + y^2 + z^2 - 2x_1x - 2y_1y - 2z_1z + d = 0$

Point (1,2,3) lies on it

$\therefore$ $14 - 2x_1 - 4y_1 - 6z_1 + d = 0$ .... (1)

point (–2,1,4) $-4x + 2y_1 + 8z_1 = d + 14$ ....(2)

point (3, –1,2) $6x_1 - 2y_1 + 4z_1 = d + 21$ ....(3)

(1) is $2x_1 + 4y_1 + 6z_1 = d + 14$ ....(1)

(1) – (2) $6x_1 + 2y_1 - 2z_1 = -7$ ....(4)

(3) – (2) $10x_1 - 4y_1 - 4z_1 = -7$ ....(5)

$\therefore$ Centre lies on straight line

$$6x + 2y - 2z + 7 = 0 = 10x - 4y - 4z + 7 \qquad ....(6)$$

If l, m, n are d, R of this line then

$$6l + 2m - 2n = 10 \quad \Rightarrow \quad 10l - 4m - 4n = 0$$

$$\therefore \quad \frac{l}{-16} = \frac{m}{4} = \frac{n}{-44} \quad \Rightarrow \quad \frac{l}{4} = \frac{m}{-1} = \frac{n}{11}$$

For point, put z = 0 in equation (6)

$$6x + 2y = -7$$
$$10x - 4y = -7$$

Solving $x = -\frac{21}{22}, y = -\frac{7}{11}$

Equation of straight line in symmetric form is

$$\frac{x+21/22}{4}=\frac{y+7/11}{-1}=\frac{z}{11}$$

Exercise 3(a)

1. Find equation of sphere whose center is (a – b, a+ b, a) and radius a.
2. Centre of a sphere is ($\sqrt{3},-\sqrt{2},\sqrt{5}$) and radius $\sqrt{10}$. Find its equation.
3. The ends of a diameter of a sphere are (–1,2,3) and (2, –1,2).Find its equation.
4. The ends of a diameter of sphere are $(\alpha+\beta, 2\alpha+\beta, \alpha+2\beta)$ and $[(\alpha-\beta),(2\alpha-\beta),(\alpha-2\beta)]$. Find its equation.
5. A sphere passes through (1, –3,4), (1, –5,2) and (1, –3,0) and its center lies on x + y + z = 0 find its equations.
6. Find equation of sphere which goes through origin and cuts off intersects of length a, – b and c from axes.
7. A sphere goes through (3,0,0), (0,3,0) and (0,0,3) and its radius is as small as possible. Find its equation.
8. A sphere touches plane 3x + 2y – 6z – 4 = 0. If its center be (6, –2,4), then find its equation.
9. Find equation of sphere whose center is (2a, – a, a) and which passes through (6a, – 4a, a).
10. A sphere touches x –axis, If its center be (6, –2, 3) find its equation.
11. A sphere touches y-z plane. Find its equation if its center be (4, –2,3).
12. Find equation of sphere which passes through (7,9,1), (–2, –3,2), (1,5,5) and (–6,2,5).
13. The straight line joining origin to center of sphere is inclined at 120^0, 45^0, and 120^0 with the axes and is of length 8. Find equation of sphere if is radius is5.
14. Find equation of sphere which passes though (2,0,1); (–2,–4, 3) and (0,1,4) and whose center lies on the straight lien 5y + 2z = 0 = 2x – 3y.
15. The plane 3x – 4y – 2z = 12 meets axes in A, B and C. Find equation of sphere through O,A,B,C.
16. One end of diameter of sphere is $x^2+y^2+z^2-2x+4y-6z-7=0$ is (–1, 2, 4. Find the other end.
17. Find equation of sphere which circumscribed the tetrahedron whose faces are $\frac{y}{3}+\frac{z}{4}=0, \frac{z}{4}+\frac{x}{2}=0, \frac{x}{2}+\frac{y}{3}=0, \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$
18. Find radius of sphere which touches planes 2x+ 2 y – z + 6 = 0 and 4x + 4y- 2z + 15 = 0.
19. The center of sphere is point of intersection of straight lien $\frac{x}{2}=\frac{y+1}{3}=\frac{z-1}{1}$ with the plane x + 2y – z – 4 = 0 If radius is 5. Find equation of sphere.

20. Find equation of sphere which passes through origin and the points of intersection on axes by a plane whose normal form origin is of length 3 and its inclination with axes are angles $\cos^{-1}\left(\frac{1}{3}\right), \cos^{-1}\left(\frac{2}{3}\right), \cos^{-1}\left(\frac{2}{3}\right)$.

21. A plane passes through a fixed point (3,4,5) and is variable. It meets axes in A, B and C. Find locus of center of sphere through O,A,B and C.

22. Centre of sphere lies on straight lien $\frac{x-3}{2}=\frac{y+1}{-2}=\frac{z-2}{1}$ and $\frac{x-4}{3}=\frac{y+5}{-6}=\frac{z-3}{2}$ and its radius is 3 units. Find points of intersection of sphere on these lines.

23. The center of a sphere in point of intersection of straight line $\frac{x}{3}=\frac{y+1}{2}=\frac{z-1}{4}$ and $\frac{x+1}{-2}=\frac{y+1}{-1}=\frac{z-7}{1}$ and it touches the plane $2x + 2y - z + 9 = 0$. Find its equation.

24. Show that sphere $x^2 + y^2 + z^2 - 4x - 2z + 4 = 0$ and $x^2 + y^2 + z^2 - 8x + 4y - 4z + 20 = 0$ touches each other.

8.3 The section of a sphere cut by a plane.

Let a plane P cut a sphere S. Then set of points common to both sphere and the plane is called plane section. This plane section is always a circle.

Let the plane cut the sphere and P be any point on plane section. CP shall always be equal to r (radius of sphere, for all positions of P on sphere and plane. CN is perpendicular on the plane from center of sphere. CN = p.

The plane section on cantains the line QNP

$\angle CNP = 90^0$ and $\therefore (NP)^2 = r^2 - p^2$

$\Rightarrow NP = \sqrt{r^2 - p^2}$ = a constant as r and p are constants

$\therefore$ locus of P is circle whose center is N and radius NP = $\sqrt{r^2 - p^2}$. Therefore plane section is always a circle.

The section of the sphere by a plane which passes through the center of sphere is called great circle. Its radius is equal to the radius of sphere.

8.4 Any sphere through a given circle:

From the above discussion it is clear that

if $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ is a sphere

and ax + by + c z + d_2 = 0 ....(1) is a plane

then (α) is equation of circle and

$x^2 + y^2 + z^2 + 2u_1 + 2v_1y + 2w_1 z + d_1 + l\,(ax + by + cz + d_2) = 0$ ....(1)

is the equation of the sphere which passes through the circle (α).

It follows that if S = 0 is sphere and P = 0 is plane that cuts the sphere the equation of other spheres that pass through this circle section are S + $\lambda P = 0, \lambda \in R$

8.5 Sphere through the point of intersection of two sphere.

Let the two sphere be

$$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

$$\text{Now } S_1 + \lambda S_2 = x^2(1+\lambda) + y^2(1+\lambda) + z^2(1+\lambda) + 2x(u_1 + \lambda u_2) + 2y(v_1 + \lambda v_2)$$
$$+ 2z(w_1 + \lambda w_2) + (d_1 + \lambda d_2) = 0$$

and it represents a sphere.

$\therefore$ Equation of the sphere which passes through the points of intersection of two sphere $S_1 = 0$ and $S_2 =$ is $S_1 + \lambda S_2 = 0$.

8.6 Two sphere always intersect in a circle

The points of intersection of two sphere $S_1=0$ and $S_2 =0$ satisfy $S_1 - S_2 = 0$

i.e. $2(u_1 - u_2) + 2(v_1 + v_2) + 2(w_1 + w_2) + d_1 - d_2 = 0$ and this is the equation of a plane and the section of sphere by this plane is a circle. Therefore the points of intersection of two spheres lie on a circle.

Example 13 : Obtain the radius and centre of the circle $x^2 + y^2 + z^2 - 2x - 4y + 4z - 9 = 0$, $x - 2y + z + 8 = 0$.

Sol. : Centre of given sphere is (1, 2, –2) and radius $R = \sqrt{1+4+4+9} = 3\sqrt{2}$ p = length of perpendicular from center C on the plane = $\frac{1-4-2+8}{\sqrt{6}} = \frac{3}{\sqrt{6}}$

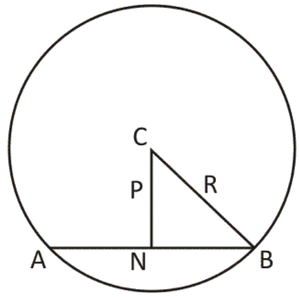


Fig 3

$\therefore$ radius of circle $r = \sqrt{R^2 - p^2}$

$$= \sqrt{18 - \frac{9}{6}} = 3\frac{\sqrt{11}}{6}$$

The equation of line through (1, 2, –2) perpendicular on plane is

$\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+2}{1}$. Any point on it is $\lambda+1, -2\lambda+2, \lambda-2$

It must satisfy equation of plane $\therefore \lambda+1+4\lambda-4+\lambda-2+8=0$

$$\Rightarrow 6\lambda = -3 \quad \Rightarrow \lambda = -\frac{1}{2}$$

center $(-1/2+1, +2, -1/2, -2)$

$\therefore$ Centre is $\left(\frac{1}{2}, 3, -\frac{5}{2}\right)$

Example 14 : Find the plane in which the circle of spheres $x^2 + y^2 + z^2 - 2x + 4y - 6z + 12 = 0$ and $x^2 + y^2 + z^2 + 6x - 7y - z - 12 = 0$ lies. Find also the sphere through this circle and passing through (1,1,1).

Sol. : Equation of plane is $S_1 - S_2 = 0$

i.e. $-8x + 11y - 5z + 24 = 0$ ....(1)

Equation of sphere through circle

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 12 = 0,\ 8x - 11y + 5z - 24 = 0$ is

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 12 + \lambda(8x - 11y + 5z - 24) = 0$$

It goes through (1,1,1)

$\therefore$ $3 - 2 + 4 - 6 + 12 + l(8 - 11 + 5 - 24) = 0$

$\Rightarrow$ $11 - 22\lambda = 0$ $\Rightarrow \lambda = 1/2$

$\therefore$ Sphere required is

$$2(x^2 + y^2 + z^2 - 2x + 4y - 6z + 12) + 8x - 11y + 5z - 24 = 0$$

$\Rightarrow 2(x^2+y^2+z^2)+4x-3y-7z=0$

Example 15 : Show that the equation of sphere though $\Theta\ x^2+y^2+z^2-2x-3y+4z+8=0$ $=x^2+y^2+z^2+4x+5y-6z+2$ and having its center on plane 4x – 5y – z = 3 is $x^2 + y^2 + z^2 + 7x + 9y - 11z - 1 = 0$

Sol. : The plane in which circle lies is $S_1 - S_2 = 0$

$\Rightarrow\ -6x-8y+10z+6=0 \Rightarrow 3x+4y-5z-3=0$

Sphere through given circle is

$x^2+y^2+z^2-2x-3y+4z+8+\lambda(3x+4y-5z-3)=0$ Centre of this sphere is

$$\left(\frac{2-3\lambda}{2},\frac{3-4\lambda}{2},\frac{-4+5\lambda}{2}\right)$$

Centre lies on plane 4x – 5y – z = 0

$\therefore 4(2-3\lambda)-5(3-4\lambda)+(4-5\lambda)=6$

$\Rightarrow 3\lambda=9$ $\therefore \lambda=3$

Equation sphere is $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 + 9x + 12y - 15z - 3 = 0$

i.e. $x^2 + y^2 + z^2 - 7x + 9y - 11z = 1$

Example 16 : Show that circle $x^2 + y^2 + z^2 - y + 2z = 0$, $x - y + z + 2 = 0$

$x^2 + y^2 + z^2 + x - 3y + z - 3 = 0$ $2x - y + 4z = 1$

lies on the ....

Sol : Equation of the two spheres through the given circles are $x^2 + y^2 + z^2 - y + 2z + l\ (x-y+z-2)=0$ and $x^2 + y^2 + z^2 + x - 3y + z - 3 + m\ (2x - y + 4z - 1) = 0$.

If the two sphere are the same then co-efficient of x, y and z in the two must be equal

i.e. $\lambda=1+2\mu$; (i) $1+\lambda=3+\mu$; (ii) $2+\lambda=1+4\mu$ (iii)

from (i), (ii) $1+2\mu=2+\mu$ $\Rightarrow \mu=1$, and $\lambda=3$, and this satisfy the third $2+\lambda=1+4\mu$

$\therefore$ the circles lie on the same sphere is $x^2 + y^2 + z^2 - y + 2z + 3x - 3y + 3z - 6 = 0$

$\Rightarrow x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0$

Example 17 : Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 = 5$, $x + 2y + 3z = 3$, and which touches the plane $4x + 3y = 15$.

Sol. : Equation of sphere through circle is $x^2 + y^2 + z^2 - 5 + \lambda x + 2\lambda y + 3\lambda z - 3\lambda = 0$ .....(1)

Centre of sphere $-\frac{\lambda}{2}, -\lambda, -\frac{3\lambda}{2}$

Radius = $\sqrt{\lambda^2/4+\lambda^2+9\lambda^2/4+3\lambda+5}$

$$=\frac{1}{2}\sqrt{14\lambda^2+12\lambda+20}$$

plane 4x + 3y – 15 = 0 shall touch sphere if perpendicular on it from center = Radius

$$\Rightarrow \frac{-2\lambda-3\lambda-15}{4}=\frac{1}{2}\sqrt{14\lambda^2+12\lambda+20}=-\lambda-3$$

$$\therefore 4(\lambda+3)^2=14\lambda^2+12\lambda+20 \Rightarrow 5\lambda^2-6\lambda-8=0$$

$$\Rightarrow (\lambda-2)(5\lambda+4)=0 \Rightarrow \lambda=2, \lambda=-\frac{4}{5}$$

$\therefore$ sphere is $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$.

And $5(x^2+y^2+z^2)-4x-8y-12z-13=0$

Example 18 : Find the equation of the sphere whose great circle is $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$.

Sol. : The equation of sphere through the circle is $x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda\ (2x + 3y + 4z - 8) = 0$

$x^2 + y^2 + z^2 + 2\lambda x + y(7 + 3\lambda) + z(-2 + 4\lambda) + 2 - 8\lambda = 0$

Centre of sphere is $\left(-\lambda, -\frac{7+3\lambda}{2}, -\frac{4\lambda-2}{2}\right)$.

This is also center of greatest circle. It must lie on plane $\Rightarrow -2\lambda-\frac{21+9\lambda}{2}+\frac{-16\lambda+8}{2}-8=0$

$$\Rightarrow -29\lambda-29=0 \Rightarrow \lambda=-1$$

$\therefore$ Sphere is $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$

8.7 Equation of tangent to sphere at a point on it:

Let sphere be, $S_1 = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ …… (1) and Let (x, y, z) be the point P on it.

$\therefore\ x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$ …….(2)

any straight line through (x_1, y_1, z_1) i.e. through P

$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n};\ \left(l^2 + m^2 + n^2 = 1\right)$ ……(3)

If (3) meets sphere (1) in points then the distances of point of intersection from P are given by

$(l\lambda + x_1)^2 + (m\lambda + y_1)^2 + (n\lambda + z_1)^2 + 2u(l\lambda + x_1) + 2v(m\lambda + y_1) + 2w(n\lambda + z_1) + \lambda = 0$

$\left(l^2 + m_2 + n^2\right)\lambda^2 + 2\lambda l(x_1 + u) + 2\lambda m(y_1 + v) + 2\lambda n(z_1 + w) + x_1^2 + y_1^2 + z_1^2 +$
$2ux_1 + 2vy_1 + 2wz_1 + \lambda = 0$

$\Rightarrow \lambda^2 + 2\lambda[l(x_1 + u) + m(y_1 + v) + n(z_1 + w)] = 0$ ………(4)

Now (λ = 0) gives P,

For the straight line to be tangent the other value of l should also be zero

$\therefore l(x_1 + u) + m(y_1 + v) + n(z_1 + w) = 0$ ………(5)

from (3) $l = \frac{x - x_1}{k}, m = \frac{y - y_1}{k}, n = \frac{z - z_1}{k}$

$\therefore$ from (5) $(x - x_1)(x_1 + u) + (y - y_1)(y_1 + v) + (z - z_1)(z_1 + w) = 0$

$\Rightarrow xx_1 + yy_1 + zz_1 + ux + vy + wz = x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1$ adding $ux_1 + vy_1 + wz_1 + d$ on both sides.

$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$

This is the equation of tangent at (x_1, y_1, z_1).

Note : The tangent plane at any point of the sphere is perpendicular to radius through that point.

The tangent plane to a sphere is perpendicular to the radius through the point of contact.

Example 19 (a) : Find equation of tangent plane of sphere $x^2 + y^2 + z^2 + 2x + 3y + 4z + 2=0$ at point (1, –2, –3).

(b) Find equation of tangent plane to sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$, parallel to plane $2x + 2y - z = 0$.

Sol. : (a) Given plane $x^2 + y^2 + z^2 + 2x + 3y + 4z + 2 = 0$

Tangent at (1, –2, –3) $(x.1)+y(-2) + z.(-3) + (x + 1) + \frac{3}{2}(y - 2) + 2(z - 3) + 2 = 0$

$\Rightarrow 2x - \frac{1}{2}y - z - 6 = 0 \quad \Rightarrow 4x - y - 2z - 12 = 0$

(b) Equation sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ centre (2, –1, 3) radius $R = \sqrt{4+1+9-5} = 3$ Plane parallel to the given plane is $2x + 2y - z + l = 0$

It shall be tangent palne if perpendicular from center (2, –1,3) on it = radius 3

i.e. $\frac{4-2-3+\lambda}{3} = 3 \Rightarrow \lambda = 10$ $\therefore$ tangent plane is $2x + 2y - z + 10 = 0$.

Example 20 : Show that the plane $3x + 2y - 6z + 8 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$. Find point of contact.

Sol. : Equation of sphere is $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ centre (1, 2, –1) $r = \sqrt{1+4+1+3} = 3$

Length of perpendicular from centre (1, 2, –1) on the plane $3x + 2y - 6z + 8 = 0$, if p then $p = \frac{3+4+6+8}{\sqrt{9+4+36}} = 3$ = radius of sphere

$\therefore$ given plane is tangent plane.

Point of contact line $\perp$ to plane through center is $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z+1}{-6} = \lambda$

$\therefore$ any point on it $\frac{3}{7}\lambda + 1, \frac{2}{7}\lambda + 2 - \frac{6}{7}\lambda - 1$

Radius = 3 $\therefore$ for point of contact λ =3 = radius

Point of contact is $\frac{3.3}{7} + 1, \frac{6}{7} + 2, -\frac{18}{7} - 1$.

Example 23.: Find equation of sphere inscribed in a tetrahedron whose faces are x = 0, y=0, z=0, and $x + 2y + 2z = 1$

Sol. : The faces of tetrahedron will be largest tangent places of sphere inscribed.

Centre of sphere shall be (α, α, α) and radius $= \alpha$ (if plane meets axes in positive direction). Perpendicular from center on plane shall be equal to radius.

$\therefore \frac{a+2a+2a-1}{3} = a \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$

$\therefore$ Equation of sphere is $\left(x-\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 + \left(z-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

i.e. $x^2 + y^2 + z^2 - x - y - z + \frac{1}{2} = 0$.

Note : If plane was x + 2y + 2z + 1 = 0, then center would have been (–a, –a, –a) and then a = 1/8.

Example 24. : Find equation of sphere through the center $x^2 + y^2 + z^2 = 1$; 2x + 4y+ 5z = 6 and touching the plane z = 0.

Sol. : Equation of sphere the rough the circle is $x^2 + y^2 + z^2 - 1 + \lambda(2x + 4y + 5z - 6) = 0$

centre is $\left(-\lambda, -2\lambda, -\frac{5}{2}\lambda\right)$, $r = \sqrt{5\lambda^2 + \frac{25\lambda^2}{4} + 6\lambda + 1}$; $z = 0$ is tangent to plane $\Rightarrow z$ co-ordinate of center is $-\frac{5}{2}\lambda = r$

$\therefore \frac{25}{4}\lambda^2 = 5\lambda^2 + \frac{25\lambda^2}{4} + 6\lambda + 1 \Rightarrow 5\lambda^2 + 6\lambda + 1 = 0 \Rightarrow \lambda = -1, -1/5$

Sphere are $x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0$ and $5(x^2 + y^2 + z^2) - 2x - 4y - 5z + 1 = 0$

Example 25 : A circle of radius unity is drawn in the plane x = 0 and its center is (0,2,3). Find equation of sphere whose section by x = 0 is this circle and which passes through (2,2,2).

Sol. : Equation of circle is $x^2 + (y-2)^2 + (z-3)^2 = 1$, x = 0

i.e. $x^2 + y^2 + z^2 - 4y - 6z + 12 = 0$, x = 0

Equation of sphere through this circle is

$x^2 + y^2 + z^2 - 4y - 6z + 12 + \lambda x = 0$

It passes through (2, 2, 2)

$\therefore 12 - 8 - 12 + 12 + 2\lambda = 0 \quad \Rightarrow \lambda = -2$

$\therefore$ Sphere is $x^2 + y^2 + z^2 - 2x - 4y - 6z + 12 = 0$

Exercise 3 (b)

1. Obtain the radius and center of the circle $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ $x + 2y + 2z - 15 = 0$.

2. The plane $4x + 4y - 2z = 2$ cuts the sphere $x^2 + y^2 + z^2 - 4x - 6y - 12 = 0$. Find radius and center of the section.

3. Prove that circle. $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$, $x^2 + y^2 + z^2 - 3x + 5z - 4y - 6 = 0$, $x + 2y - 7z = 0$ lie on the same sphere. Find equation of sphere.

4. Find equation of sphere through circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and passing through (1,2,3).

5. Find equation of sphere through the circle $x^2 + y^2 + z^2 - 4x + 6y - 12 = 0$, $3x - 4y + 6z = 4$ and passing through origin.

6. A circle center (2,3,0) and radius 1 is drawn in plane z = 0, find the equation of sphere passing through the circle and point (1,1,1).

7. Find equation of sphere which goes through (4,1,0), (2, –3,4), (1, 0, 0) and touches the plane $2x + 2y - z = 11$

8. If tangent plane to the sphere $x^2 + y^2 + z^2 = a2$ make intercepts p,q,r on axes, then prove $\frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2}=\frac{1}{a^2}$.

9. Show that sphere $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + z^2 - 6x - 4y - 12z + 45 = 0$ touches externally, find point of contact.

10. Find equation of sphere whose great circle is $x^2 + y^2 + z^2 - 2x + 4y + 6z = 2$, $3x + 2y - 6z = 3$.

11. Find equation of sphere whose great circle is $x^2 + y^2 + z^2 - 8x - 6y + 2z + 1 = 0$, $2x + 2y - z = 24$.

12. Show that sphere $x^2 + y^2 + z^2 = 64$ and $x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$, touché each other. Find point of contact.

13. Show that plane $2x + 2y + 2z = 21$ touches the sphere $x^2 + y^2 + z^2 - 4x - 6y + 8z + 4 = 0$. Find point of contact.

14. Find equation of sphere through $x^2 + y^2 + z^2 = 1$, $2x + 5y + 4z = 6$ and which touches the plane z=0.

15. Find equation of sphere which goes through (1,1,0), (0,1,1), (1,0,1) and touches the plane $x+y+z=0$

16. Find equation of sphere which touches co-ordinate planes and lies in first octant and passes through (1,9,4)

17. Find equation of sphere which lies in seventh octant and touches axes and the plane $2x+3y+6z+24=0$

18. The plane $-\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=1$ meets axes in A, B and C respectively. Find equation of sphere which passes through origin and A, B and C.

19. Find equation of sphere passing through circle $x^2 + y^2 + z^2 = 5$, $x - 2y + 2z + 5 = 0$ and touches plane $3x + 4y + 5 = 0$

20. A sphere cuts axes in A, B and C. Centroid of triangle ABC is (2, –4, –1); find equation of sphere through (O,A,B,C) , O is origin.

21. Find equation of tangent plane of sphere $x^2+y^2+z^2+2x-4y + 6z + 5 = 0$, parallel to $3x\ 2y+6z=15$.

Plane $3x + 2y + 6z + 7 = 0$ is tangent to a sphere whose centre lies on $\frac{x-2}{1}=\frac{y+1}{1}=\frac{z-1}{3}$ and $\frac{x+1}{3}=\frac{y+2}{1}=\frac{z-3}{2}$. Find equation of sphere.

8.8 Angle of intersection of two sphere :

Angle of intersection of two sphere is the angle between their tangent planes at the point of intersection.

Since radii of spheres to the common point of contact are perpendicular to the respective planes. Therefore angle between the radii is equal to the angle between these planes, therefore if r_1 and r_2 be radii, d distance between centers then angle between radii is given by

$$\cos\theta = \left(r_1^2 + r_2^2 - d^2\right)/2r_1r_2$$

8.9 Condition for Orthogenal cutting:

Let sphere be

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1 + 2w_1z + d_1 = 0$$

$$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

radius $r_1 = \sqrt{u_1^2 + v_1^2 + w_1^2 - d_1}$ center $\left(-u_1 - v_1, -w_1\right)$ radius $r_2 = \sqrt{u_2^2 + v_2^2 + w_2^2 - d_2}$, center $\left(-u_2, -v_2, -w_2\right)$ angle between radii is 90^0

$$\therefore d^2 = r_1^2 + r_2^2$$

$$\therefore \left(u_1 - u_2\right)^2 + \left(v_1 - v_2\right)^2 + \left(w_1 - w_2\right)^2$$

$$= u_1^2 + v_1^2 + w_1^2 - d_1 + u_2^2 + v_2^2 + w_2^2 - d_2$$

$$\Rightarrow 2\left(u_1u_2 + v_1v_2 + w_1w_2\right) = d_1 + d_2$$

Example 26 : Show that two spheres.

$x^2+y^2+z^2+6y+2z+8=0,\ x^2+y^2+z^2+6x+8y+4z+20=0$ cut orthogoanlly. Find their plane of intersection also.

Sol.: $u_1 = 0, v_1 = 3, w_1 = 1; u_2 = 3, v_2 = 4, w_2 = 2\ d_1 = 8, d_2 = 20$

$2(u_1u_2 + v_1v_2 + w_1w_2) = 2(0+12+2) = 8+20$

$\therefore$ sphere intersection orthogonally.

Equation of plane is $S_2 - S_1 = 0$

$6x + 2y + 2z + 12 = 0 \Rightarrow 3x + y + z + 6 = 0$

Example 27 : The spheres of radii r_1 and r_2 cut orthogonally, prove that the radius of common circle is $\frac{r_1r_2}{\sqrt{r_1^2+r_1^2}}$.

Sol.: Let the two sphere be $x^2 + y^2 + z^2 = r_1^2$ ....(1)

And $x^2 + y^2 + z^2 + 24x + 2vy + 2wz + d = 0$ .... (2)

Radius of second sphere = $r_2 = \sqrt{u^2+v^2+w^2-d}$ ....(3)

And center is (–u, –v, –w) the two cut orthogoanlly

$2(u_1 0 + v_1 0 + w_1 0) = d + (r_1^2) = 0$

$\therefore d - r_1^2 = 0$ ....(4)

The common circle shall lie on the plane.

$S_2 - S_1 = 0; 2(ux + vy + wz) + d + v_1^2 = 0$ ($\because d = r_1^2$)

$\Rightarrow$ $ux + vy + wz + r_1^2 = 0$ ....(5) ($\because d = r_1^2$)

If p = ON = perpendicular distance of (5) from center of sphere $S_1 = 0$ i.e. from (0,0,0) then

$$p^2 = \frac{r_1 4}{u^2+v^2+w^2} = \frac{r_1 4}{v_2^2+d} = \frac{r_1 4}{v_1^2+v_2^2}$$

with the help of (3) and (4). Radius of common circle, $r^2 = r_1^2 - p^2 = r_1^2 - \frac{r_1^4}{r_1^2+r_2^2} = \frac{r_1^2v_2^2}{r_1^2+r_2^2}$

$$= r = \frac{r_1 r}{\sqrt{r_1^2+r_2^2}}$$

Example 28 : Prove that a sphere which cuts the two given sphere S_1 =0, S_2 = 0 at right angles will cut $\lambda_1 S_1 + \lambda_2 S_2 = 0$ at right –angle for all values λ_1 and λ_2.

Sol. : Let $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

S cuts S_1 and S_2 orthogonally

$$\therefore \quad 2(uu_1 + vv_1 + ww_1) = d + d_1 \qquad \text{..........(1)}$$

$$2(uu_2 + vv_2 + ww_2) = d + d_2 \qquad \text{..........(2)}$$

Sphere $\lambda_1 S_1 + \lambda_2 S_2$ is

$$(\lambda_1 + \lambda_2)(x^2 + y^2 + z^2) + 2x(\lambda_1 u_1 + \lambda_2 u_2) + 2y(\lambda_1 v_1 + \lambda_2 v_2) + 2z(\lambda_1 w_1 + \lambda_2 w_2) + \lambda_1 d_1 + \lambda_2 d_2 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + 2x\frac{(\lambda_1 u_1 + \lambda_2 u_2)}{\lambda_1 + \lambda_2} + 2y\frac{(\lambda_1 v_1 + \lambda_2 v_2)}{\lambda_1 + \lambda_2} + 2z\frac{(\lambda_1 w_1 + \lambda_2 w_2)}{\lambda_1 + \lambda_2} + \frac{(\lambda_1 d_1 + \lambda_2 d_2)}{\lambda_1 + \lambda_2} = 0$$

(3)

Sphere S will cut sphere (3) orthogonally if

$$2\left[\frac{u(\lambda_1 u_1 + \lambda_2 u_2)}{\lambda_1 + \lambda_2} + \frac{v(\lambda_1 v_1 + \lambda_2 v_2)}{\lambda_1 + \lambda_2} + \frac{w(\lambda_1 w_1 + \lambda_2 w_2)}{\lambda_1 + \lambda_2}\right] = \quad d + \frac{(\lambda_1 d_1 + \lambda_2 d_2)}{\lambda_1 + \lambda_2}$$

......(4)

Now L.H.S. of (4)

$$= \frac{2\lambda_1}{\lambda_1 + \lambda_2}[uu_1 + vv_1 + ww_1] + \frac{2\lambda_2}{\lambda_1 + \lambda_2}[uu_2 + vv_2 + ww_2]$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}[d + d_1] + \frac{\lambda_2}{\lambda_1 + \lambda_2}(d + d_2) \text{ from (1) and (2)}$$

$$= \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2}.d + \frac{\lambda_1 d_1 + \lambda_2 d_2}{\lambda_1 + \lambda_2}$$

$$= d + \frac{\lambda_1 d_1 + \lambda_2 d_2}{\lambda_1 + \lambda_2} = \text{R.H.S. of (4)}$$

$\therefore$ relation satisfied for all values of λ_1 and λ_2

$\therefore$ The sphere S = 0 which cuts sphere S_1 = 0, S_2 = 0 orthogonally shall cut sphere $\lambda_1 S_1 + \lambda_2 S_2 = 0$ orthogonal.

Example 29 : Find equation of sphere which passes through the points (0,3,0) and (–2, –1, –4) and cuts spheres $x^2 + y^2 + z^2 + x - 3z - 2 = 0$ and $2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$ orthogonally.

Sol.: Given sphere $x^2 + y^2 + z^2 + x - 3z - 2 = 0$ ....(1)

$$u_1 = \frac{1}{2}, v_1 = 0, w_1 = -\frac{3}{2}, d_1 = -2$$

$$x^2 + y^2 + z^2 + \frac{1}{2}x + \frac{3}{2}y + \frac{4}{2} = 0 \quad \text{..........(2)}$$

$$u_2 = \frac{1}{4}, v_2 = \frac{3}{4}, w_2 = 0, d_2 = 2$$

Let S = 0, $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ....(3)

be the equation of required sphere.

It passes through (0,3,0) $9 + 6v + d = 0$ .... (4)

And through (–2, –1, –4) $21 - 4u - 2v - 8w + d = 0$ .... (5)

$\therefore$ from (5) – (4) $12 - 4u - 8v - 8w = 0$

$$\Rightarrow u + 2v + 2w = 3 \quad \text{........... (6)}$$

Sphere S = 0 cuts orthogonally the spheres (1) and (2)

$$\therefore \quad 2\left(\frac{1}{2}u - \frac{3}{2}w\right) = d + (-2)$$

$$\Rightarrow \quad u - 3w + 2 = d \quad \text{...........(7)}$$

and $2\left(\frac{1}{4}u + \frac{3}{4}v\right) = d + 2$

$$\Rightarrow \quad \frac{1}{2}u + \frac{3}{2}v - 2 = d \quad \text{..........(8)}$$

from (7) and (8) $\frac{1}{2}u + \frac{3}{2}v - 2 = u - 3 - w + 2$

$$\Rightarrow u - 3v - 6w + 8 = 0 \quad \text{............(9)}$$

from (4) and (7) $u + 6v - 3w + 11 = 0$ ....(10)

from (6) and (9) $5v + 8w - 11 = 0$

from (10) and (9) $9v + 3w + 3 = 0$

Solving $v = -1, w = 2$ and from (6) $u = 3 + 2 - 4 = 1$

And from (4) $d = -9 - 6v = -9 + 6 = -3$

$\therefore$ sphere is $x^2 + y^2 + z^2 + 2x - 2y + 4z - 3 = 0$

Example 30 : Find angle of intersection between the sphere $x^2 + y^2 + z^2 - 4x - 2y - 6z - z = 0$ and $x^2 + y^2 + z^2 + 8x - 8y - 10z - 7 = 0$

Sol. : Given sphere are $x^2 + y^2 + z^2 - 4x - 2y - 6z - 2 = 0$

Center (2,1,3), $r_1 = 4$

Other sphere $x^2 + y^2 + z^2 + 8x - 8y - 10z - 7 = 0$

Center (–4, 4, 5), $r_2 = 8$

Distance d between centers is $d = \sqrt{(-6)^2 + (3)^2 + (2)^2} = 7$

$$\therefore \cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} = \frac{16+64-49}{2.4.8}$$

$$\frac{-31}{64} \Rightarrow \theta = \cos^{-1}\left(\frac{31}{64}\right)$$

Example 31. : Find a if sphere $x^2 + y^2 + z^2 - 6x - 4y - 4z + 1 = 0$ and $x^2 + y^2 + z^2 - ax + 6y + 2z + 3 = 0$ cut each other othgonally.

Sol. : Sphere $x^2 + y^2 + z^2 - 6x - 4y - 4z + 2 = 0$

$u_1 = -3,\ v = -2,\ w_1 = -2,\ d_1 = 2$

and sphere $x^2 + y^2 + z^2 - ax + 6y + 2z + 3 = 0$

$$u_2 = -\frac{a}{2}, v_2 = 3, w_2 = 1, d_2 = 3$$

Orthogonally cutting $2\left[\frac{3a}{2} + (-6) - 2\right] = 3 + 2 \Rightarrow a = 7$

Exercise 3 (c)

1. Find the angle of intersection of sphere :

$$x^2+y^2+z^2-6x-2y+2z+2=0$$
$$x^2+y^2+z^2-2x-4y-6z+10=0$$

2. Find angle of intersection of the sphere $x^2+y^2+z^2-4x-2y+4z=0$ and $x^2+y^2+z^2-4x-8y-4z=12$

3. Find the condition that spheres $a(x^2+y^2+z^2)+2px+2qy+2rz+\lambda=0$ and $b(x^2+y^2+z^2)=\mu^2$ may cut orthogonally.

4. Find the equation of the sphere through the circle $x^2+y^2+z^2-2x+3y-4z+6=0$, $3x\ 4y+5z\ 15=0$ which cuts sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally.

5. Find the angle of intersection of the spheres

$$x^2+y^2+z^2-4x+2y-4=0$$
$$x^2+y^2+z^2-10x+2y-8z+26=0$$

6. Prove that sphere $x^2+y^2+z^2-2x+4y+6z+5=0$ and $x^2+y^2+z^2-4x+6y-4z-1=0$ cut orthogonally.

7. If sphere $x^2+y^2+z^2-6x+8y-2z+5=0$ and $x^2+y^2+z^2-2ax+4y+6z+3=0$ cut orthogopnally, find value of a.

8. If sphere $x^2 + y^2 + z^2 - 6ax + 8ax + 8ay - 4z - 6 = 0$, $x^2 + y^2 + z^2 - 4ax - 6y + 20z + 2 = 0$ cut orthogonally, then find the value of a.

9. If a sphere passes through (3,0,2), (–1,1,1), (2, –5,4) and its center lies on $2x + 3y + 4z - 6 = 0$ find its radius.

10. The plane $2x - 3y + 6z - 5 = 0$ touches the sphere. $x^2 + y^2 + z^2 - 2x - 4y + 4z = 0$, find point of contact.

Miscellaneous Exercise

1. P is a point on straight line $2x = \sqrt{2}y = 2z$ such that OP = $8\sqrt{2}$,

(a) Find projections of OP on axes.

(b) Find projection of OP on y – z, z – x, x – y planes.

(c) Find angle OP makes with x – y plane.

(d) OQ is another straight line $x = \frac{1}{2}y = \frac{1}{2}z$ and OQ = 9. Find angle POQ.

(e) Find equation of angle bisector of acute angle POQ.

(f) Find equation of plane POQ.

(g) Find equation of plane perpendicular to OP and passing through (3, $2\sqrt{2}$, –5)

(h) if this plane (g) meets axes in M, N, R, Find centroid of ΔMNR.

(i) Find equation of sphere OMNR.

(j) Find equation of tangent plane of this sphere at origin.

(k) Find section of sphere OMNR by plane MNR center and radius of circle.

(l) Find projection of OP on OQ.

(m) Find image of P in OQ

(n) Find equation of planes parallel to plane OPQ and at 5 unit distance from it.

(o) Find equation of PQ.

(p) Find centroid tetrahedron OMNR.

2. The three coterminous edge of a cube are along OA, OB, OC. Find equation of diagonal through B.

3. Find the angle between straight lines $\frac{2x-1}{3} = \frac{3-2y}{2} = \frac{z-4}{3}$ and $\frac{3x-2}{2} = \frac{y}{2} = \frac{z-3}{1}$.

4. The vertices of a triangle are A (1,2,3); (–1,0,1) and C(1,1, –1). Find angle B.

5. The vertices of a triangle are A (1,2,3), B(4,0, –3) and C (3,4,5). The bisector of angle A meets BC in D. Find co-ordiantes of D.

6. Find the ratio in which line joining (–2,4, –5) and (3, –5,4) is divided by (y–z) plane internally.

7. Find the projection of straight line joining (3,4,5) and (4,5,3) on the straight line joining (–1,2,4) and (1,0,5).

8. O is origin, P is (3,4,5) and Q(2, a, –2) find a if OP is perpendicular to OQ.

9. The straight line through (a,b,c) parallel to x axis is ……….

10. The equation of straight line through (–1,2,3) and perpendicular to plane 2x + 3y + z + 7 = 0 is …..

11. Find equation of plane through line $\frac{x-2}{4}=\frac{y-3}{-3}=\frac{z-4}{5}$ which is parallel to z axis.

12. O, is origin, P is (a,b,c). Find equation of plane perpendicular to OP.

13. P is (2,5,4) and straight line AB is $\frac{x}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ PN is perpendicular on AB. Find N.

14. Find equation of a line which is parallel to x + y + z = 6 and 2x – y + 3z = 10 and passes through (1,2,3).

15. Find straight line $\frac{x-2}{a}=\frac{y+1}{2}=\frac{z-1}{3}$ lies in the plane 4x + 2y – bz = 3. Find a and b.

16. Find point of intersection of straight line $\frac{x-3}{1}=\frac{y+1}{3}=\frac{z-2}{-5}$ and $\frac{x-5}{1}=\frac{y-3}{1}=\frac{z-1}{4}$.

17. Find image of point (3,4,5) in the plane x + y + z = 0.

18. Find distance of point (4, 1, 1) from the straight line x + y + z – 4 = 0 = x – 2y – z – 1.

19. Find angle between straight line $\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-1}{2}$ and z = 0

20. Find points in which straight line $\frac{x+1}{1}=\frac{y-12}{-5}=\frac{z-7}{-2}$ meets the curve $11x^2 - 5y^2 + z^2 = 0$

21. Shortest distance between lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-2}=\frac{y+7}{2}=\frac{z-6}{4}$ is…..

22. Distance of point (1,2,5) from plane 2x – y + z = 1 measured parallel to straight line $\frac{x}{1}=\frac{y}{-1}=\frac{z}{-1}$ is ……

23. Find equation of plane parallel to y axis and which passes thorugh point (2,3,4) and (3, 5, –6).

24. Find equation of a plane perpendicular to plane 2x – 3y + 6z + 8 = 0 and which passes through the line of intersection of planes.

    x + 2y + 3z – 4 =0 and 2x – y – z + 5 = 0.

25. A plane meets x = 0 where x = 0, 2y – 3z = 5 and meets z = 0, where z = 0, 7x + 4y = 10, find equation of plane.

26. The perpendicular from origin to a plane is of length 2 and makes angles α,β,γ with axes. Find equation of plane if $\frac{\cos\alpha}{-1}=\frac{\cos\beta}{4}=\frac{\cos\gamma}{8}$.

27. The straight line joining origin to center of a sphere is inclined at 60^0, 60^0 with x and y axes respectively and its length is 4. If the radius of sphere be unity, then the equation of sphere is....

28. Find equation of sphere through circle $x^2 + y^2 + z^2 + 7 = 0$, $3x - 2y + 6z + 14 = 0$ and passing through origin.

29. Find equation of sphere whose great circle is $x^2 + y^2 + z^2 + 10y - 4z = 8$, $x + y + z = 3$.

30. $2x - 2y + z + 12 = 0$ is tangent plane of sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$. Find point of contact.

31. One end of a diameter of sphere $x^2 + y^2 + z^2 - 6x - 12y - 2z + 37 = 0$ is (4, 8, –1). Find other end of diameter.

32. Two spheres with radius a and b respectively cut orthogonally. Find radius of common circle.

33. Find angle of intersection of spheres $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + z^2 - 6x - 8y - 24z + 25 = 0$

34. The straight line $\frac{x-4}{3}=\frac{y-4}{2}=\frac{z-1}{-2}$ touches sphere $x^2 + y^2 + z^2 - 6x - 2y - 10z + 26 = 0$. Find point of contact.

35. Plane $2x + 2y - z - 5 = 0$ meets the sphere $x^2 + y^2 + z^2 - 2x + 4y - 4z - 7 = 0$ Prove that center of section circle is (3,0,1).

36. Point A is (–1, –2,3) and B is (4,2,5) projection of AB on straight line $\frac{x}{3}=\frac{y}{2}=\frac{z}{6}$ is :

(a) 35 (b) 28 (c) 15 (d) 5

37. A straight line is inclined at 60^0 with z – axis and at 45^0 with x-axis. Its inclination with y axis is:

(a) $\tan^{-1}\sqrt{15}$ (b) $\tan^{-1}\sqrt{3}$ (c) $\tan^{-1}(-\sqrt{3})$ (d) $\frac{\pi}{6}$

38. Straight lines $2x - 1 = 3y + 1 = lx - 1$ and $3x + 1 = 2 - 5y = 3 - 2z$ are perpendicular to each other, then λ is :

(a) 3 (b) –5 (d) 5 (d) 7

39. A plane meets co-ordinate axes in A, B and C so that the centroid of tetrahedron OABC is :

(1, –3/2. –2). The equation of plane is :

(a) $\frac{x}{3}+\frac{y}{2}-\frac{z}{6}=1$ (b) $\frac{x}{4}+\frac{y}{6}-\frac{z}{8}=1$

(c) 6x –4y –3z = 24 (d) none of these

40. Point P is $\left(\frac{1}{3},-\frac{1}{3},\frac{-7}{3}\right)$, A is (2,1,3) B(4,2,1) perpendicular from P on AB meets AB in M, co-ordinates of M are:

(a) (3,0,3) (b) $\left(\frac{4}{3},-\frac{2}{3},1\right)$

(c) $\left(\frac{10}{3},\frac{5}{3},\frac{5}{3}\right)$ (d) none of these

41. Angle between planes $\frac{x}{2}+\frac{y}{3}-z=1$ and 2x + 2y + z = 5 is:

(a) $\cos^{-1}\left(\frac{4}{21}\right)$ (b) $\cos^{-1}\left(\frac{8}{21}\right)$

(c) $\cos^{-1}\left(\frac{16}{21}\right)$ (d) none of these

42. Angle between straight line $\frac{x-3}{2}=\frac{y-5}{-1}=\frac{z+3}{-2}$ and plane 2x – y – 2z = 0 is :

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

43. The straight line $\frac{x-3}{1}=\frac{y+1}{-1}=\frac{z-4}{-3}$ meets the plane 3x + 4y – 2z = 2 in point :

(a) $(4,-2,1)$ (b) (2,2,6) (c) (5, –3. –2) (d) none of these

44. The image of point (3, –5, 4) in the straight line $\frac{x-4}{3}=\frac{y}{-2}=\frac{z+1}{3}$ is:

(a) (11, 1, 0) (b) (15, 3, 2) (c) (1, 11, 0) (d) (7, 5, 3)

45. Equation of a straight line through (2,3, –5) equally inclined to axes are :

(a) x – 2 = y – 3 = z + 5 (b) $\frac{x-1}{2}=\frac{y-3}{3}=\frac{z-5}{5}$

(c) $\frac{x-2}{2}=\frac{y-3}{-2}=\frac{z-5}{2}$ (d) none of these

46. Lines $\frac{x}{-1}=\frac{y}{-2}=\frac{z}{3}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ are :

(a) parallel (b) perpendicular

(c) skew (d) intersect in a point

47. A plane is parallel to x – z plane and perpendicular from origin on it is 6; equation of plane is :

(a) $x + z = 6$ (b) $y + z = 6$ (c) $y \pm 6 = 0$ (d) none of these

48. A plane is parallel to $3x - 6y + 2z = 8$ and its distance from point (2, 1, –4) is 4. Its equation is

(a) $3x - 6y + 2z + 34 = 0$ (b) $3x - 6y + 3z + 4 = 0$ (c) $3x - 6y + 2z + 22 = 0$

(d) $3x - 6y + 2z - 22 = 0$

49. Equation of a plane perpendicular to planes $3x - y + z = 0$, $x + y + z = 0$ and $\sqrt{6}$ units away from origin is :

(a) $x + y + 2z \pm 6 = 0$ (b) $2x + y + z \pm 6 = 0$ (c) $x + y - 2z \pm 6 = 0$ (d) $x - y - 2z \pm 6 = 0$

50. Plane through line of intersection of planes $x - y + 2z - 3 = 0$, $2x - y - 3z = 0$ and through point (4, – 3, 2) is :

(a) $11x - 3y - 34z + 15 = 0$ (b) $11x - 3y + 34z - 15 = 0$

(c) $11x + 3y - 14z + 15 = 0$ (d) $5x - 7y - z - 39 = 0$

51. The center of a sphere lies on a straight line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$ at a distance of 6 from (1,2,3). If its radius is 4, then its equation is :

(a) $x^2 + y^2 + z^2 - 10x - 8y - 14z + 74 = 0$ (b) $x^2 + y^2 + z^2 + 6x + 2z = 6$

(c) $x^2 + y^2 + z^2 - 10x + 14z + 58 = 0$ (d) none of these

52. The co-ordinate of the foot of perpendicular from origin on plane are (2,3,1) equation of plane is :

(a) $2x + 3y + z - 14 = 0$ (b) $2x - 3y + z = 14$

(c) $2x + 3y + z + 14 = 0$ (d) none of these

53. A sphere lies in the 1st octant and touches all axes and its distance of center from origin is $3\sqrt{3}$. The equation of sphere is :

(a) $x^2 + y^2 + z^2 + 6(x + y + z) + 18 = 0$ (b) $x^2 + y^2 + z^2 = 9$

(c) $x^2 + y^2 + z^2 - 6(x + y + z) + 18 = 0$ (d) $x^2 + y^2 + z^2 - 6\sqrt{3}(x + y + z) + 54 = 0$

54. The plane $2x + 2y - z - 1 = 0$ cuts sphere $x^2 + y^2 + z^2 - 4x - 6y - 12 = 0$. Radius and center of section circle is :

(a) 4; (0,1,1) (b) 3; (0,1,1)

(c) 4 (1, – 1, –1) (d) 4(1, –1, 1)

55. The equation of sphere which passes through origin and (1,1,1), (–1, 1, 0) and (0,1, –1) is :

(a) $3(x^2 + y^2 + z^2) + x + 7y - z = 0$ (b) $x^2 + y^2 + z^2 - \frac{1}{3}(x + 7y + z) = 0$

(c) $3(x^2 + y^2 + z^2) + x - 7y + z = 0$ (d) none of these

56. Spheres $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$ and $x^2 + y^2 + z^2 - 4x - 2y + 6z - 2 = 0$ intersect at an angle of

(a) $\cos^{-1}\left(\frac{5}{12}\right)$ (b) $\cos^{-1}\left(\frac{5}{8}\right)$ (c) $\cos^{-1}\left(\frac{19}{24}\right)$ (d) $\frac{\pi}{4}$

57. The equation of a sphere which goes through origin and touches sphere $x^2 + y^2 + z^2 - x + 3y + 2x - 3 = 0$ at (1, 1, –1) is:

(a) $x^2 + y^2 + z^2 - 4x + 6y + 52 = 0$ (b) $x^2 + y^2 + z^2 + 2x - 4y + z = 0$

(c) $2(x^2 + y^2 + z^2) - 3x + y + 4z = 0$ (d) $2(x^2 + y^2 + z^2) - 3x - y + 4z = 0$

58. Spheres $x^2 + y^2 + z^2 - 16x - 6z + 48 = 0$ and $x^2 + y^2 + z^2 - 8x - 8y - 12z + 52 = 0$ cut each-other at an angle of

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

59. Show that straight line $\frac{x-3}{23} = \frac{y-5}{2} = \frac{z-6}{2}$ touches sphere $x^2 + y^2 + z^2 + 2x - 8y + 8z - 16 = 0$ the point of contact is :

(a) (6,7,8) (b) (9,9,10) (c) (3,5,6) (d) (–3, 1, 2)

60. Plane $2x + 2y - z - 5 = 0$ cuts sphere $x^2 + y^2 + z^2 - 2x + 4y - 4z - 7 = 0$. Centre of section circle is :

(a) (3,0,1) (b) (0,3,1) (c) (–3,2, –7) (d) (1,2,1)

61. Points A, B and C are (3,0,1), (5,2,2) and (0,2, –5) respectively. Projection of AB on AC is :

(a) $\frac{8}{7}$ (b) $\frac{8}{3}$ (c) $\frac{4}{7}$ (d) $\frac{6}{7}$

62. Points A, B and C are (–1, 2, –2), (3, 1, –1) and (3,4,2) respectively/ CM is perpendicular on AB co-ordinate of M are :

(a) (–3,1,1) (b) (–3, –1, 0) (c) (3, 1, –1) (d) (3,1,1)

63. The image of (–1,2,4) in the straight line $\frac{x-2}{3} = \frac{y+2}{-1} = \frac{z-1}{2}$ is :

(a) (2,5,4) (b) (2, – 5, –4) (c) (–2, 5, –4) (d) (–2,5,4)

64. The image of (2, –3,4) in plane 3x + 2y – 4z = 13, is

(a) (8, 1, –4) (b) (1,8,4) (c) (4, –8,1) (d) (8, –1, –4)

65. The point P lies on the straight line joining (2,4,3) and (4,3,1). If z co-ordinate of P be $\frac{5}{3}$ and P is :

(a) $\left(\frac{8}{3},\frac{8}{3},\frac{5}{3}\right)$ (b) $\left(3,3,\frac{5}{3}\right)$

(c) $\left(\frac{10}{3},\frac{-10}{3},\frac{5}{3}\right)$ (d) none of these

66. The angle bisector between lines $\frac{x-1}{3}=\frac{y+2}{-5}=\frac{z-3}{4}$ and $\frac{x-1}{1}=\frac{y+2}{-4}=\frac{z-3}{-1}$ is :

(a) $\frac{x-1}{14}=\frac{y+2}{-35}=\frac{z-3}{7}$ (b) $\frac{x-1}{12}=\frac{y-2}{8}=\frac{y-3}{10}$

(c) $\frac{x-1}{4}=\frac{y+2}{5}=\frac{z-3}{17}$ (d) $\frac{x-1}{3}=\frac{y+2}{-3}=\frac{z-3}{-5}$

67. Points A (–2,3,4), B(2, –3, –4) are such that $\angle APB = 90^0$ always, locus of P is :

(a) $x^2 + y^2 + z^2 = 29$ (b) $x^2 + y^2 + z^2 - 2x + 4y + 29 = 0$

(c) $x^2 + y^2 + z^2 - 2x - 6y - 8z + 4 = 0$ (d) none of these

68. Distance of the point (3, –2, 1) from straight line through (1, 1, –1) and having d.rt. 2, –2, 1 is :

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) $\frac{5}{3}$

69. M, mid point of AB is (–1,2,3) and straight line AB is $\frac{x+1}{2}=\frac{y-2}{-2}=\frac{z-3}{1}$. If AB = 10, then point A is :

(a) $\left(-\frac{7}{3},\frac{4}{3},-\frac{14}{3}\right)$ (b) $\left(\frac{7}{3},-\frac{4}{3},\frac{14}{3}\right)$

(c) $\left(-\frac{13}{3},\frac{16}{3},\frac{4}{3}\right)$ (d) $\left(\frac{13}{3},-\frac{16}{3},\frac{4}{3}\right)$

70. A corner of a cuboid is at origin and its edge are 5, 4 and 3 unit long along OX, OY and OZ. One end of its one diagonal is at origin and one end of other diagonal is on z axis angle between the two diagonolas is :

(a) $\cos^{-1}\left(\frac{8\sqrt{2}}{25}\right)$ (b) $\cos^{-1}\left(\frac{16\sqrt{2}}{25}\right)$

(c) $\cos^{-1}\left(\frac{12\sqrt{2}}{25}\right)$ (d) $\cos^{-1}\left(\frac{1}{5}\right)$

71. Shortest distance between straight lines $\frac{x-1}{3}=\frac{y-2}{4}=\frac{z+3}{1}$ and $\frac{x+2}{-2}=\frac{y+2}{2}=\frac{z-1}{-1}$ is

(a) $\frac{14}{3}$ (b) $\frac{60}{\sqrt{233}}$ (c) $\frac{16}{3}$ (d) $\frac{70}{\sqrt{233}}$

72. Centre of a sphere lies on straight line 2x = 4y = – 2z and $\frac{z-2}{1}=\frac{y-5/2}{2}=\frac{x-1}{2}$ and its radius is 3 units. Its equation is :

(a) $x^2 + y^2 + z^2 + 2x + y - 2z = \frac{27}{4}$ (b) $x^2 + y^2 + z^2 - 2x - y - 2z = \frac{27}{4}$

(c) $x^2 + y^2 + z^2 - 2x + 2y - z = \frac{27}{4}$ (d) none of these

73. Straight lines $\frac{x+1}{2}=\frac{y-3}{-3}=\frac{z+6}{4}$ and $\frac{x-2}{1}=\frac{y+1}{-2}=\frac{z-4}{-2}$ are :

(a) skew (b) coplanar (c) parallel (d) perpendicular

74. The co-ordinates of center of a sphere are roots of equation $x^3 + 5x^2 + 2x - 8 = 0$ taken in increasing order and radius is geometric mean of these roots. Sphere is :

(a) $x^2 + y^2 + z^2 + 8x + 4y - 2z + 17 = 0$ (b) $x^2 + y^2 + z^2 - 2x + 4y + 8z + 17 = 0$

(c) $x^2 + y^2 + z^2 + 8x - 4y + 2z + 17 = 0$ (d) none of these

75. The equation of plane through the line of intersection of planes 3x + 2y + 4z = 3 and 2x–3y–6z = 7, parallel to straight line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{-1}$ is :

(a) 6x – 5y + 10z – 27 = 0 (b) 13y + 26 z + 15 = 0

(c) 13 x + 51 = 0 (d) x + 2y + 1 = 0

76. Straight lines x = ay + b, z = cy + d and x = my + n, z = py +q shall be perpendicular if :

(a) am – cp – 1 = 0 (b) an + cq + 1 = 0

(c) am + cp + 1 = 0 (d) ap+cm+1=0

77. A sphere passes through origin. Plane 2x – 2y – z = 0 and 2x – 2y – z + 18 = 0 are its tangent planes. Equation of sphere is :

(a) $x^2 + y^2 + z^2 - 4x + 4y + 2z = 0$ (b) $x^2 + y^2 + z^2 - 6x + 2y - 2z = 0$

(c) $x^2 + y^2 + z^2 + 4x - 2y - 2z = 0$ (d) $x^2 + y^2 + z^2 - 3x + 5y - z = 0$

78. If straight line 4 = 0 is a diameter of sphere $S_1 = 0$ then number of diameter perpendicular to this diameter of sphere $S_1 = 0$ are :

(a) one (b) 4 (c) many but not unlimited (d) infinite

79. The distance of point (1,3,5) from plane $2x - 3y + 6y = 3$ measured in direction parallel to straight line $\frac{x-1}{1} = \frac{y-4}{-2} = \frac{x+1}{2}$ is :

(a) $\frac{20}{7}$ (b) 3 (c) 5 (d) $\frac{10}{3}$

80. Find l : m : n if straight lines $\frac{x-2}{l} = \frac{y-1}{m} = \frac{z+1}{n}, \frac{x+1}{2} = \frac{y-1}{-1} = \frac{z-7}{-2}, \frac{x-3}{1} = \frac{y+1}{-2} = \frac{z-3}{2}$ are concurrent.

(a) 1 : –2 : 4 (b) 4 : (–2) : 1 (c) 2 : –3 : 4 (d) 11 : –10 : 20

81. Points are A (3,4,5), B (4,7,9). Sum of the projections of AB on axes is :

(a) 15 (b) 24 (c) 8 (d) 0

82. Points are A (4,5,2), B(9,1, $\frac{11}{2}$), C(2,3,5), D(–3,7, $\frac{3}{2}$) then A B C D is :

(a) parallelogram (b) Rhombus (c) square (d) ordinary quadrilateral

83. The plane which contains straight line $\frac{x+1}{4} = \frac{y+1}{-3} = \frac{z-3}{-2}$ and $\frac{x-1}{-2} = \frac{y+5}{1} = \frac{z-2}{1}$ is :

(a) $2x + 3y + 2z + 9 = 0$ (b) $3x + 2y - 5z = 1$ (c) $x + 2z - 5 = 0$ (d) $4x - 3y + 11 = 0$

84. The sphere whose great circles is $x^2 + y^2 + z^2 - 2x + 4y - 6z - 36 = 0$, $3x + 4y - 20 = 0$ is :

(a) $x^2 + y^2 + z^2 - 8x - 4y - 6z + 4 = 0$ (b) $x^2 + y^2 + z^2 + 8x - 4y + 6z + 8 = 0$

(c) $x^2 + y^2 + z^2 - 8x + 4y + 6z - 4 = 0$ (d) none of these

85. The angle between spheres $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3d = 0$ and $x^2 + y^2 + z^2 - 4x - y + 6z - 2d = 0$ is $\cos\left(\frac{19}{24}\right)$, then d is :

(a) 3 (b) 2 (c) 1 (d) –1

86. Find condition that spheres $x^2 + y^2 + z^2 - 2ax + 8y - 2z - 2b = 0$ and $x^2 + y^2 + z^2 + 4x - 6y + 2bz + 4a = 0$ cut orthogonally

(a) $a + 2b = 0$ (b) $3a - 4b = 0$ (c) $a + 2b = 0$ (d) $a = -3$

87. A circle with center (2,0,4) and radius is drawn in y = 0, plane. The equation of sphere which passes through this circle and point (2, 1, 2) is:

(a) $x^2 + y^2 + z^2 - 4x + 4y - 8z + 11 = 0$ (b) $x^2 + y^2 + z^2 - 4x - 4y - 8z + 19 = 0$

(c) $x^2 + y^2 + z^2 + 4x + 4y - 8z - 5 = 0$ (d) none of these

88. z = 0 is horizontal plane. Plane 3x + 4y – z = 0 goes through point P (2, –3, –6). Find equation of line of greatest slope through P :

(a) $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z+6}{25}$ (b) $\frac{x-2}{3} = \frac{y+3}{-4} = \frac{z+6}{-25}$

(c) $\frac{x-2}{4} = \frac{y+3}{-3} = \frac{z+6}{0}$

89. The equation of sphere through circle $x^2 + y^2 + z^2 = 4x + 2y + 2z + 6 = 0$ and touching x= 0 is:

(a) $x^2 + y^2 + z^2 + x + 2y + 2z + 2 = 0$ (b) $x^2 + y^2 + z^2 - x - 2y - 2z + 2 = 0$

(c) $x^2 + y^2 + z^2 + 2x + 4y + 4z + 8 = 0$ (d) none of these

90. Plane x – 3y + 4z = 0 is horizontal equation of line of greatest slope through point (1, –2, –4) its plane 2x + 3y – z = 0 is :

(a) $\frac{x-1}{1} = \frac{y+2}{1} = \frac{z+4}{5}$ (b) $\frac{x-1}{-3} = \frac{y+2}{1} = \frac{z+4}{-3}$

(c) $\frac{x-1}{-4} = \frac{y+2}{1} = \frac{z+4}{-5}$ (d) none of these

91. The d.c. of two lines are given by equations $3l - m + n = 0$ and $3lm + 5ln + nmn = 0$ angle between lines is :

(a) $\cos^{-1}\left(\frac{-1}{\sqrt{705}}\right)$ (b) $\cos^{-1}\left(\frac{-1}{\sqrt{15}}\right)$

(c) $\cos^{-1}\left(\frac{-1}{\sqrt{47}}\right)$ (d) none of these

92. Locus of center of sphere passing through (1,0,0), (0,2,0) and (2, 3, 4) is a straight line whose equation in symmetric form is :

(a) $\frac{x-\frac{41}{10}}{8} = \frac{y-\frac{31}{10}}{4} = \frac{z}{-5}$ (b) $\frac{x+\frac{47}{10}}{8} = \frac{y+\frac{31}{10}}{4} = \frac{z}{-5}$

(c) $\frac{x+\frac{41}{10}}{-8} = \frac{y+\frac{31}{10}}{-4} = \frac{z}{5}$ (d) none of these

93. The straight line $\frac{x}{2}=\frac{y}{-2}=\frac{z}{1}$ is parallel to plane L_1, which contains the straight line $\frac{x-2}{4}=\frac{y-3}{3}=\frac{z+4}{-6}$ find equation of L_1 :

(a) 9x + 16 y + 14z – 10 = 0 (b) 9x + 16y + 14 z + 86 = 0

(c) 9x + 16y + 14z + 122 = 0 (d) none of these

94. The straight line $\frac{x-1}{3}=\frac{y+1}{-2}=\frac{z-4}{6}$ makes with plane z = 0 an angle :

(a) $\cos^{-1}\left(\frac{6}{7}\right)$ (b) $\cos^{-1}\left(\frac{3}{7}\right)$ (c) $\cos^{-1}\frac{\sqrt{13}}{7}$ (d) none of these

95. Equation of a sphere inscribed in a tetrahedron whose faces are x = 0, y = 0, z = 0 are 3x + 2y + 6z + 9 = 0 is :

(a) $x^2 + y^2 + z^2 + x + y + z + \frac{1}{2} = 0$ (b) $x^2 + y^2 + z^2 + 4x + 4y + 4z + 8 = 0$

(c) $x^2 + y^2 + z^2 - x - y - z + \frac{1}{2} = 0$ (d) $x^2 + y^2 + z^2 - 4x - 4y - 4z + 8 = 0$

96. 3x+ 4y – 5z +4 =0 and 3x + 4y – 5z – 36 = 0 are tangent plane of a sphere whose centre lies on straight line $\frac{x-2}{1}=\frac{y+1}{4}=\frac{z+2}{3}$. Equation of sphere is :

(a) $x^2 + y^2 + z^2 + 6x + 2y + 11 = 0$ (b) $x^2 + y^2 + z^2 - 6x - 6y - 2z + 11 = 0$

(c) $x^2 + y^2 + z^2 - 6x + 6y - 2z + 15 = 0$ (d) none of these

97. 3x + 4y + 2 = 0, 2x + 2y + z – 2 = 0 and 3x + 6y + 2z – 4 =- 0 are three tangent planes of a sphere, and center lies on x + y – z –1 = 0, equation of sphere is :

(a) $x^2 + y^2 + z^2 - 4x - 6x - 8z + 13 = 0$ (b) $x^2 + y^2 + z^2 + 4x - 6x - 8z + 13 = 0$

(c) $x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0$ (d) none of these

98. A (4,6,8), B(2,4,7) and C(6,3,2) are vertices of triangle ABC. Equation of angle bisector of $\angle$BAC is :

(a) $\frac{x-4}{8}=\frac{y-6}{23}=\frac{z-8}{25}$ (b) $\frac{x-4}{20}=\frac{y-6}{5}=\frac{z-8}{11}$

(c) $\frac{x-4}{4}=\frac{y-6}{11}=\frac{z-8}{13}$ (d) none of these

99. The angle bisector of obtuse angle between the straight line $\frac{x-2}{5}=\frac{y+3}{-4}=\frac{z-1}{3}$ and $\frac{x-2}{4}=\frac{y+3}{-1}=\frac{z-1}{1}$ is :

(a) $\frac{z-2}{25}=\frac{y+3}{17}=\frac{z-1}{14}$ (b) $\frac{x-2}{-5}=\frac{y+3}{-7}=\frac{z-1}{4}$

(c) $\frac{x-2}{9}=\frac{y+3}{-5}=\frac{z-1}{14}$ (d) $\frac{x-2}{1}=\frac{y+3}{-3}=\frac{z-1}{2}$

100. The straight line $\frac{x+1}{2}=\frac{y-3}{2}=\frac{z+4}{1}$:

(a) meets x – y plane in point ……….

(b) Projection of this straight line on x- y plane is ……

(c) Angle that line makes with x – y plane is ……….

(d) Intersects the straight line $\frac{x-1}{3}=\frac{y-1}{2}=\frac{z-3}{3}$ in point ……..

(e) Angle between these two lines is ………….

(f) Equation of plane containing these two lines is ………

(g) If this plane intersects axes in A, B and C then equation of sphere through O,A,B,C is

(h) Perpendicular from origin on plane ABC is ……….

Practice Worksheet (Competition Level)

1. Let a plane passes through origin and is parallel to the line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z+1}{-2}$ such that distances between plane and line is $\frac{5}{3}$.Then the equation of the plane is:

a) 2x +2y +z=0 b) x-2y-2z=0

c) x+2y+2z=0 d) 2x-2y+3z=0

2. A line is drawn from the point P(1,1,1) to intersect the plane x+2y+3z=4 at Q. The locus of the point Q is:

a) $\frac{x}{1}=\frac{y-5}{-2}=\frac{z+2}{1}$ b) $\frac{x}{-2}=\frac{y-5}{1}=\frac{z+2}{1}$

c) x = y = z d) None of these

3. The plane $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=1$ meets the axes in points A, B and C. The co-ordinate of the orthocentre of triangle ABC are:

a) $\left(\frac{a^2}{a+b+c},\frac{b^2}{a+b+c},\frac{c^2}{a+b+c}\right)$

b) $\left(\frac{a^{-1}}{a^{-2}+b^{-2}+c^{-2}},\frac{b^{-1}}{a^{-1}+b^{-2}+c^{-3}},\frac{c^{-1}}{a^{-2}+b^{-2}+c^{-2}}\right)$

c) $\left(\frac{a^3bc}{b^2c^2+c^2a^2+a^2b^2},\frac{b^3ac}{b^2c^2+c^2a^2+a^2b^2},\frac{c^3ab}{b^2c^2+c^2a^2+a^2b^2}\right)$

d) $\left(\frac{a\ b^2c^2}{b^2c^2+c^2a^2+a^2b^2},\frac{b\ a^2c^2}{b^2c^2+c^2a^2+a^2b^2},\frac{c\ a^2b^2}{b^2c^2+c^2a^2+a^2b^2}\right)$

4. The number of arbitrary constants that the general equations to the straight line in space contain is:

a) 2 b) 4 c) 5 d) 6

5. The planes x = c y + b z, y = a z + c x, z = b x + a y pass through one line if :

(a, b, c are unequal)

a) bc +ca + cb = abc b) $a^2 + b^2 + c^2 + ab + bc + ca = 0$

c) $a^3 + b^3 + c^3 - 3abc = 0$ d) $a^2 + b^2 + c^2 + 2abc = 0$

6. A plane makes intercepts OA , OB , OC whose measurements are a, b, c on the axis OX, OY, OZ. The area of ΔABC is

a) $\frac{1}{2}$ (ab + bc + ca) b) $\frac{1}{2}\left(a^2b^2+b^2c^2+c^2a^2\right)^{\frac{1}{2}}$

c) $\frac{1}{2}$ a b c (a + b + c) d) $\frac{1}{2}$ (a + b + c)(ab + bc + ca)

7. The distance of the point (1, -2, 3) from the plane x – y + z = 5 measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$

a) 1 b) 2 c) 3 d) $\frac{1}{7}$

8. The straight lines whose direction cosines are given by a l + b m + c n = 0 , f l^2 + g m^2 + h n^2 =0 are parallel to:

a) $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ b) $\frac{a}{f}+\frac{b}{g}+\frac{c}{h}=0$

c) $\frac{f}{a^2}+\frac{g}{b^2}+\frac{h}{c^2}=0$ d) $\frac{a^2}{f}+\frac{b^2}{g}+\frac{c^2}{h}=0$

9. A variable plane at a distance of 1 unit from the origin cuts the co-ordinate axis at A, B, C. If the centroid D (x, y, z) of triangle ABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=K$. Then the value of K is:

a) 3 b) 1 c) $\frac{1}{3}$ d) 9

10. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=\lambda_1$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}=\lambda_2$ intersect. Then the value of k is:

a) $\frac{3}{2}$ b) $\frac{9}{2}$ c) $\frac{-2}{9}$ d) $\frac{-3}{2}$

11. A plane mirror is placed at origin whose direction ratios of the normal are 1, -1, 1. If a ray of light coming along the positive direction of x-axis strikes the mirror and gets reflected.Then the direction cosines of the reflected ray are:

a) $\frac{1}{3},\frac{2}{3},\frac{2}{3}$ b) $\frac{-1}{3},\frac{2}{3},\frac{2}{3}$ c) $\frac{-1}{3},\frac{-2}{3},\frac{-2}{3}$ d) $\frac{-1}{3},\frac{-2}{3},\frac{2}{3}$

12. If $(4k_1, k_1^2, 1)$ and $(4k_2, k_2^2, 1)$ are the two points lying on the plane in which (2, 3, 2) and (1, 2, 3) are the

mirror image to each other.Than k_1k_2 is equal to:

a) $\frac{-3}{2}$ b) $\frac{-5}{2}$ c) $\frac{-7}{2}$ d) $\frac{-9}{2}$

13. if the plane $x - y + 2z + 1 = 0$ intersects the plane parallel to x-axis and passing through (1,1,1) , $\left(1,\frac{3}{2},\frac{1}{2}\right)$ at an angle θ.Then $\tan\theta$

a) $\sqrt{2}$ b) $\sqrt{3}$ c) $\sqrt{7}$ d) $\sqrt{11}$

14. Maximum distance if any point on $4x^2 + 9y^2 = 36$ from the plane $x + 2y + z + 1 = 0$ is:

a) $\sqrt{3}$ b) 2 c) $\sqrt{7}$ d) $\sqrt{6}$

15. Sum of reciprocals of intercept cut off by a plane containing $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and (-4, 4, 1), on the axis is:

a) 1 b) $\frac{1}{49}$ c) 2 d) $\frac{3}{49}$

16. If the plane passing through the line of intersection of ax + by = 0 with xy plane is rotated about the same line through 60^o and equation of the new plane is $ax + by \pm k\sqrt{a^2+b^2}.Z = 0$.Then k is:

a) 2 b) $\sqrt{3}$ c) $\sqrt{5}$ d) $\sqrt{6}$

17. If (p, q, v) is any point on the plane x+y+z+3 = 0. Then least value of $\left((p-2)^2+(q-3)^2+(r-4)^2\right)^{\frac{1}{2}}$ is:

a) $3\sqrt{3}$ b) $4\sqrt{3}$ c) $3\sqrt{2}$ d) $\sqrt{2}$

18. If lines $\frac{x-a}{b}=\frac{y-b}{c}=\frac{z-c}{a}$ and $\frac{x}{c}=\frac{y}{a}=\frac{z}{b}$ (a+b+c?0) are coplanar,then lines are always

a) Perpendicular b) Coincident

c) intersecting at angle θ $(0‹\theta‹\pi/2)$ d)can't be predicted

19. If the line x=y=z intersect the line sin A.x + sin B .y+sinC.z+2d^2, sin 2A.x +sin2B.y+sin2C.z=d^2. Then $\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$ is equal to: (where A+B+C=A)

a) $\frac{1}{16}$ b) $\frac{1}{8}$ c) $\frac{1}{32}$ d) $\frac{1}{12}$

20. In a three dimensional co-ordinate system P, Q and R are images of a point A(a, b, c) in xy-plane, the y-z plane and the z-x plane respectively. If G is the centroid of the triangle PQR then area of the triangle AOG is:(O is the origin)

a) 0 b) $a^2+b^2+c^2$ c) $\frac{2}{3}(a^2+b^2+c^2)$ d)None of these

SECTION – I (Total Marks : 21)

(Single Correct Answer Type)

This section contains **7 multiple choice questions.** Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

47. Let (x_0, y_0) be solution of the following equations

$(2x)^{\ln 2} = (3y)^{\ln 3}$

$3^{\ln x} = 2^{\ln y}$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6

Sol. (C)

$(2x)^{\ln 2} = (3y)^{\ln 3}$ ………….(i)

$3^{\ln x} = 2^{\ln y}$ ………… (ii)

$\Rightarrow (\log x)(\log 3) = (\log y)\log 2$

$\Rightarrow \log y = \frac{(\log x)(\log 3)}{\log 2}$ …………..(iii)

In (i) taking log both sides

(log2) {log2 + log x } = log3 {log3 + log y}

$(\log 2)^2 + (\log 2)(\log x) = (\log 3)^2 + \frac{(\log 3)^2(\log x)}{\log 2}$ from (iii)

$\Rightarrow (\log 2)^2 - (\log 3)^2 = \frac{(\log 3)^2 - (\log 2)^2}{\log 2}(\log x) \Rightarrow -\log 2 = \log x$

$\Rightarrow x = \frac{1}{2} \Rightarrow x_0 = \frac{1}{2}$

48. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P \subset Q$ and $Q - P \neq \varnothing$ (B) $Q \not\subset P$

(C) $P \not\subset Q$ (d) P = Q

Sol. (D)

In set P, $\sin\theta = \left(\sqrt{2}+1\right)\cos\theta \Rightarrow \tan\theta = \sqrt{2}+1$

In set Q, $\left(\sqrt{2}-1\right)\sin\theta = \cos\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \Rightarrow = Q$

49. Let $\vec{a} = \hat{i}+\hat{j}+\hat{k}, \vec{b} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

(a) $\hat{i}-3\hat{j}+3\hat{k}$ (b) $-3\hat{i}-3\hat{j}+\hat{k}$

(c) $3\hat{i}-\hat{j}+3\hat{k}$ (d) $\hat{i}+3\hat{j}-3\hat{k}$

Sol. (C)

$\vec{v} = \lambda\bar{a}+\mu\bar{b}$

$= \lambda\left(\hat{i}+\hat{j}+\hat{k}\right)+\mu\left(\hat{i}-\hat{j}++\right)$

Projection of $\bar{v}$ on $\bar{c}$

$\frac{\bar{v}.\bar{c}}{|\bar{c}|} = \frac{1}{\sqrt{3}} \Rightarrow \frac{\left[(\lambda+\mu)\hat{i}+(\lambda-\mu)\hat{j}+(\lambda+\mu)\hat{k}\right]\left(\hat{i}-\hat{j}-\hat{k}\right)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow \lambda+\mu-\lambda+\mu-\lambda-\mu = 1 \Rightarrow \mu-\lambda = 1 \Rightarrow \lambda = \mu-1$

$\bar{v} = (\mu-1)\left(\hat{i}+\hat{j}+\hat{k}\right)+\mu\left(\hat{i}-\hat{j}+\hat{k}\right) = \mu\left(2\hat{i}+2\hat{k}\right)-\hat{i}-\hat{j}-\hat{k}$

$\bar{v} = (2\mu-1)\hat{i}-\hat{j}+(2\mu-1)\hat{k}$

At $\mu = 2, \bar{v} = 3\hat{i}-\hat{j}+3\hat{k}$

50. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x\sin x^2}{\sin x^2+\sin\left(\ln 6-x^2\right)}dx$ is

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$ (C) $\ln\frac{3}{2}$ (D) $\frac{1}{6}\ln\frac{3}{2}$

Sol. (A)

$x^2 = t \Rightarrow 2xdx = dt$

$I = \frac{1}{2}\int_{\ln 2}^{\ln 3}\frac{\sin t}{\sin t+\sin\left(\ln 6-t\right)}dt$ and $I = \frac{1}{2}\int_{\ln 2}^{\ln 3}\frac{\sin\left(\ln 6-t\right)}{\sin\left(\ln 6-t\right)+\sin t}dt$

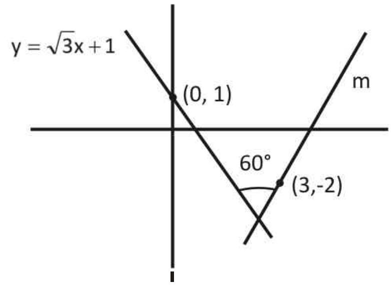
$2I = \frac{1}{2}\int_{\ln 2}^{\ln 3} 1dt \Rightarrow I = \frac{1}{4}\ln\frac{3}{2}$.

51. A straight line L through the point (3, – 2) is inclined at angle 60^0 to the line $\sqrt{3}x+y=1$. If L also intersects the x-axis, then the equation of L is

(A) $y+\sqrt{3}x+2-3\sqrt{3}=0$ (B) $y-\sqrt{3}x+2+3\sqrt{3}=0$

(C) $\sqrt{3}y-x+3+2\sqrt{3}=0$ (D) $\sqrt{3}y+x-3+2\sqrt{3}=0$

Sol. (B)



$$\left|\frac{m+\sqrt{3}}{1-\sqrt{3}m}\right|=\sqrt{3}$$

$$\Rightarrow m+\sqrt{3}=\pm\left(\sqrt{3}-3m\right)$$

$$\Rightarrow 4m=0\Rightarrow m=0$$

or $2m=2\sqrt{3}\Rightarrow m=\sqrt{3}$

$\therefore$ Equation is $y+2=\sqrt{3}(x-3)$

$$\Rightarrow \sqrt{3}x-y-\left(2+3\sqrt{3}=0\right)$$

52. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha$ and $\alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10}-2a_8}{2a_9}$ is

(A) 1 (B) 2 (C) 3 (D) 4

Sol. (C)

$$a_n=\alpha^n-\beta^n$$

$$\alpha^2-6\alpha-2=0$$

Multiply with α^8 on both sides

$\Rightarrow \alpha^{10}-6\alpha^9-2\alpha^8=0$ ....(i)

Similarly $\beta^{10}-6\beta^9-2\beta^8=0$ ....(ii)

(i) and (ii)

$$\Rightarrow \alpha^{10}-\beta^{10}-6\left(\alpha^9-\beta^9\right)=2\left(\alpha^8-\beta^8\right)$$

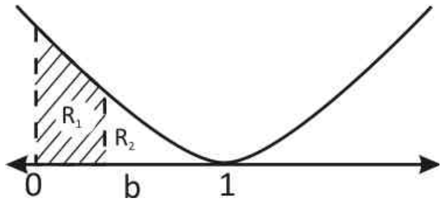
$$\Rightarrow a_{10}-6a_9=2a_8\Rightarrow \frac{a_{10}-2a_8}{2a_9}=3$$

53. Let the straight line x = b divides the area enclosed by y $(1 - x^2)$, y = 0 and x = 0 into two parts R_1 ($0 \leq x \leq b$) and R_2 ($b \leq x \leq 1$) such that $R_1-R_2=\frac{1}{4}$. Then be equals

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

Sol. (B)

$$\because \int_0^b (1-x)^2 dx - \int_b^1 (1-x)^2 dx = \frac{1}{4}$$



$$\Rightarrow \left.\frac{(x-1)^3}{3}\right|_0^b - \left.\frac{(x-1)^3}{3}\right|_b^1 = \frac{1}{4}$$

$$\Rightarrow \frac{(b-1)^3}{3} + \frac{1}{3} - \left(0 - \frac{(b-1)^3}{3}\right) = \frac{1}{4}$$

$$\Rightarrow \frac{2(b-1)^3}{3} = -\frac{1}{12} \Rightarrow (b-1)^3 = -\frac{1}{8} \Rightarrow b = \frac{1}{2}$$

SECTION – II (Total Marks : 16)

(Multiple Correct Answers Type)

This section contains **4 multiple choice questions.** Each question has four choices (A), (B), (C) and (D) out of

which **ONE or MORE** may be correct.

54. Let $f: R \to R$ be a function such that f(x + y) = f(x) + f(y), $\forall x, y \in R$. If f(x) is differentiable at x = 0, then

(A) f(x) is differentiable only in a finite interval containing zero

(B) f(x) is continuous $\forall x \in R$

(C) f'(x) is constant $\forall x \in R$

(D) f(x) is differentiable except at finitely many points

Sol. (B,C)

$\because f(0)=0$ and $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow \lim\limits_{h\to 0}\frac{f(h)}{h}=f'(0)=k(say)$

$\Rightarrow f(x)=kx+c \Rightarrow f(x)=kx(\because f(0)=0)$

55. The vector(s) which is/are coplanar with vectors $i+\hat{j}+2\hat{k}$ and $i+2\hat{j}+\hat{k}$, and perpendicular to the vector $i+\hat{j}+\hat{k}$ is/are

(A) $\hat{j}-\hat{k}$ (B) $-\hat{i}+\hat{j}$ (C) $\hat{i}-\hat{j}$ (D) $-\hat{j}+\hat{k}$

Sol. (A,D)

Any vector in the plane of $i+\hat{j}+2\hat{k}$ and $i+2\hat{j}+\hat{k}$ is $\vec{r}=(\lambda+2\mu)\hat{i}+(\lambda+2\mu)\hat{j}+(2\lambda+\mu)\hat{k}$

Also, $\vec{r}.\vec{c}=0 \Rightarrow \lambda+\mu=0$ $\Rightarrow \left[\vec{r}\vec{a}\vec{b}\right]=0$

56. Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2+4y^2=4$. If hyperbola passes through a focus of the ellipse, then

(A) the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$ (B) a focus of the hyperbola is (2, 0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$ (D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

Sol. (B, D)

Ellipse is $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$

$1^2 = 2^2(1 - e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$

$\therefore$ eccentricity of the hyperbola is $\frac{2}{\sqrt{3}} \Rightarrow b^2 = a^2\left(\frac{4}{3} - 1\right) \Rightarrow 3b^2 = a^2$

Foci of the ellipse are $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$

Hyperbola passes through $(\sqrt{3}, 0)$

$\frac{3}{a^2} = 1 \Rightarrow a^2 = 3$ and $b^2 = 1$

$\therefore$ Equation of hyperbola is $x^2 - 3y^2 = 3$

Focus of hyperbola is (ae, 0) $\equiv \left(\sqrt{3} \times \frac{2}{\sqrt{3}}, 0\right) \equiv (2, 0)$

57. Let M and N be two 3 x 3 non – singular skew symmetric matrices such that MN = NM. If P^T denotes the transpose of P, the $M^2N^2\ (MTN)^{-1}\ (MN^{-1})^T$ is equal to

(A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

Sol. (C)

MN = NM

$M^2N^2(M^TN)^{-1}\ (MN^{-1})^T$

$M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^T.M^T$

$= M^2N.\ (M^T)^{-1}\ (N^{-1})^TM^T = -\ M^2.N(M)^{-1}(N^T)^{-1}\ M^T$

$= +\ M^2NM^{-1}\ N^{-1}M^T = -M.NMM^{-1}\ N^{-1}M = -MNN^{-1}\ M = -M^2$

Note : A skew symmetric matrix of order 3 cannot be non-singular hence the equation is wrong.

SECTION-III (Total Marls : 15)

(Paragraph Type)

This section contains **2 paragraphs.** Based upon one of paragraphs **2 multiple choice questions** and based on the

other paragraph **3 multiple choice questions** have to be answered. Each of these questions has four choices (A), (B),

(C) and (D) out of **which ONLY ONE** is correct.

Paragraph for Question Nos. 58 to 59

Let U1 and U2 be two urns such that U1 contains 3 white and 2 red balls, and U2 contains only 1 white ball. A fair

coin is tossed. If head appears then 1 ball is drawn at random from U1 and put into U2. However, if tail appears then

2 balls are drawn at random from U1 and put into U2. Now 1 ball is drawn at random from U2.

58. The probability of the drawn ball from U2 being white is

(A) $\frac{13}{30}$ (B) $\frac{23}{30}$ (C) $\frac{19}{30}$ (D) $\frac{11}{30}$

Sol. (B)

$H \rightarrow$ 1 ball from U_1 to U_2

$T \rightarrow$ 2 ball from U_1 to U_2

E : 1 ball drawn from U_2

P/W

from

$$U_2 = \frac{1}{2}\times\left(\frac{3}{5}\times 1\right)+\frac{1}{2}\times\left(\frac{2}{5}\times\frac{1}{2}\right)+\frac{1}{2}\times\left(\frac{{}^3C_2}{{}^5C_2}\times 1\right)+\frac{1}{2}\times\left(\frac{{}^2C_2}{{}^5C_2}\times\frac{1}{3}\right)+\frac{1}{2}\times\left(\frac{{}^3C_1.{}^2C_1}{{}^5C_2}\times\frac{2}{3}\right)=\frac{23}{30}$$

59. Given that the drawn ball from U2 is white, the probability that head appeared on the coin is

(A) $\frac{17}{23}$ (B) $\frac{11}{23}$ (C) $\frac{15}{23}$ (D) $\frac{12}{23}$

Sol. (D)

$$P\left(\frac{H}{W}\right)=\frac{P(W/H)\times P(H)}{P(W/T).P(T)+(W/H).P(H)}=\frac{\frac{1}{2}\left(\frac{3}{5}\times 1+\frac{2}{5}\times\frac{1}{2}\right)}{23/30}=\frac{12}{23}$$

Paragraph for Question Nos. 60 to 62

Let a, b and c be three real numbers satisfying $\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

……….(E)

60. If the point P(a,b,c) with reference to (E), lies on the plane 2x + y + z = 1, then the value of 7a + b + c is

(A) 0 (B) 12 (C) 7 (D) 6

Sol. (D)

a + 8b + 7c = 0 & 9a + 2b + 3c = 0 & a + b + c = 0

Solving these we get

B = 6a $\Rightarrow c=-7a$

Now 2x + y + z = 0 $\Rightarrow 2a+6a+(-7a)=1\Rightarrow a=1, b=6, c=-7$

61. Let ω be a solution of $x^3 - 1 = 0$ with Im(ω) > 0. If a = 2 with b and c satisfying (E). then the value of $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is equal to

(A) – 2 (B) 2 (C) 3 (D) – 3

Sol. (A)

A = 2, b and c satisfies (E) $\Rightarrow$ B = 12, c = – 14

$$\frac{3}{\omega^2}+\frac{1}{\omega^b}+\frac{3}{\omega^c}=\frac{3}{\omega^2}+\frac{1}{\omega^{12}}+\frac{3}{\omega^{-14}}=-2$$

62. Let b = 6, with a and c satisfying (E). If a and b are the roots of the quadratic equation $ax^2 + bx + c = 0$ then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is

(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

Sol. (B)

$ax^2 + bx + c = 0 \Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow \alpha = 1, \beta = -7$

$$\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{1}-\frac{1}{7}\right)^n = 7$$

SECTION-IV (Total Marks : 28)

(Integer Answer Type)

This section contains **7** questions. The answer to each of the questions is a **single digit integer,** ranging from 0 to 9.

The bubble corresponding to the correct is to be darkened in the ORS.

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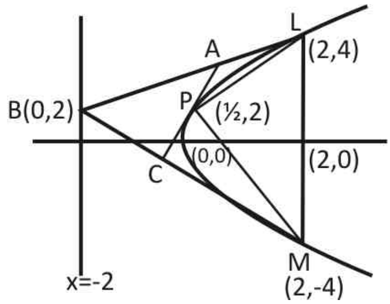
63. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is:

Sol. (2)



y2 = 8x = 4.2 x

$$\frac{\Delta LPM}{\Delta ABC} = 2$$

$$\frac{\Delta_1}{\Delta_2} = 2$$

64. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\cos 2\theta}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is:

Sol. (1)

$$\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right) \quad \text{, where } \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{2\cos^2\theta - 1}}\right)\right)$$

$$= \sin\left(\sin^{-1}(\tan\theta)\right) = \tan\theta$$

$$\frac{d(\tan\theta)}{d(\tan\theta)} = 1$$

65. Let $f : [1, \infty) \to [2, \infty)$ be a differentiable function such that f(1)=2. If $6\int_1^x f(t)dt = 3xf(x) - x^3$ for all x ≥ 1, then the value of f(2) is

Sol. (6)

$$6\int_1^x f(t)dt = 3xf(x) - x^3 \Rightarrow 6f(x) = 3f(x) + 3xf'(x) - 3x^2$$

$$\Rightarrow 3f(x) = 3xf'(x)3x^2 \Rightarrow xf'(x) - f(x) = x^2$$

$$\Rightarrow x\frac{dy}{dx} - y = x^2 \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = x \qquad \text{.............(i)}$$

I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\log_e x}$

Multiplying (i) both sides by $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 1 \Rightarrow \frac{d}{dx}\left(y, \frac{1}{x}\right) = 1$$

Integrating

$$\frac{y}{x} = x + c$$

Put x = 1, y = 2

$$\Rightarrow 2 = 1 + c \Rightarrow c = 1 \Rightarrow y = x^2 + x$$

$$\Rightarrow f(x) = x^2 + x \Rightarrow f(2) = 6$$

Note : If we put x = 1 in the given equation we get f(1) =1/3

66. The positive integer value of n > 3 satisfying the equation

$$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)} \text{ is:}$$

Sol. (7)

$$\frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} \Rightarrow \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} \frac{\left(2\sin\frac{\pi}{n}\cos\frac{2\pi}{n}\right)\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = 1$$

$$\Rightarrow \sin\frac{4\pi}{n} = \sin\frac{3\pi}{n} \Rightarrow \frac{4\pi}{n} + \frac{3\pi}{n} = \pi \Rightarrow n = 7.$$

67. Let $a_1, a_2, a_3, \ldots\ldots a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{P} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let m = 5n. If $\frac{S_m}{S_n}$ does not depends on n, then a_2 is.

Sol. (9)

$a_1, a_2, a_3, \ldots\ldots a_{100}$ is an A.P.

$$a_1 = 3, S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$$

$$\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}\left(6+\left(5n-1\right)d\right)}{\frac{n}{2}\left(6-d+nd\right)}$$

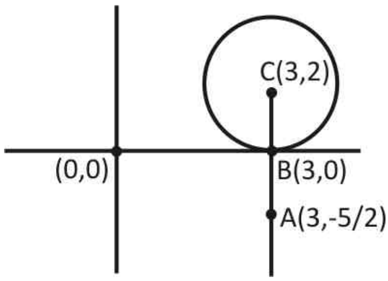
$\frac{S_m}{S_n}$ is independent of n of 6 – d = 0 $\Rightarrow d = 6$.

68. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is

Sol. (5)

Length AB= $\frac{5}{2}$

$\Rightarrow$ minimum value = 5



69. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, a^{8} and a^{10} and a >0 is

Sol. (8)

$$\frac{a^{-5}+a^{-4}+a^{-3}+a^{-3}+a^{-3}+a^{8}+a^{10}+1}{8} \ge 1$$

Minimum value = 8.

Answers

Practice Worksheet (Foundation Level) – 2 (a)

1) c	2) b	3) d	4) a	5) a	6) c
7) b	8) c	9) d	10) a	11) c	12) c
13) a	14) c	15) d	16) d		

Practice Worksheet (Foundation Level) – 2 (b)

1) $2\overline{a}-\overline{b}+4\overline{c}$ 2) $6\overline{a}+\frac{5}{2}\overline{b}+\frac{3}{2}\overline{c}$ 3) $\overline{a}+2\overline{b}+\overline{c}$ 8) $-\frac{1}{3}\ a=\frac{5}{2}, b=\frac{7}{4}$

15) 7 17) p = 1, q = 3 18) $\beta = 2, \alpha = \pm 2$

Practice Worksheet (Foundation Level) – 3 (a)

1) $\cos^{-1}\left(\frac{2+2\sqrt{3}}{15}\right)$ 2) $\cos^{-1}\left(\frac{-4}{21}\right)$ 4) –6, 2, 5 . $\frac{13}{\sqrt{17}}$

6) $\cos^{-1}\left(\frac{-1}{7\sqrt{3}}\right)$, $\cos^{-1}\left(\frac{1-2\sqrt{3}}{5\sqrt{3}}\right)$ 7) $\cos^{-1}\left(\frac{3}{4}\right)$ 9) $\frac{2}{7}(3\hat{i}-2\hat{j}+6\hat{k})$

10) $\frac{\pi}{2}$ 11) $\frac{1}{9}(62\hat{i}-11\hat{j}+20\hat{k})$ 12) $\frac{148}{7}$J

13) $\frac{9}{2}\hat{i}+2\hat{j}+\frac{3\sqrt{3}}{2}\hat{k}$ 16) $\frac{\pi}{2}$ 21) b, d 22) d

23) c	24) d	25) c	26) a
27) c	28) c	29) c	30) a

Practice Worksheet (Foundation Level) – 3(b)

1) 0 2) $\frac{1}{5\sqrt{2}}(3\hat{i}+5\hat{j}+4\hat{k})$ 3) $\pm\frac{1}{\sqrt{35}}(\hat{i}+5\hat{j}+3\hat{k})$

7) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ 8) $-7\hat{i}+\hat{j}+3\hat{k}$ 9) $\sqrt{621}$ sq. units

10) $5\sqrt{3}$ sq. units 11) $\sqrt{6}$ sq. units 12) $12\sqrt{2}$, direction $4\hat{i}+5\hat{j}-3\hat{k}$

13) $4\sqrt{5}$, direction $2\hat{i}-\hat{k}$ 14) $60\sqrt{2}$ direction $-4\hat{i}+5\hat{j}-3\hat{k}$ 15) $7\hat{i}-19\hat{j}-11\hat{k}$

17) (i) –I – 3j + 2k, –6i + 4j – 3k. (ii) $\sqrt{14}$, 7, $\sqrt{51}$. (iii) $\frac{1}{2}\sqrt{710}$

19) (i) $\overline{a}+\lambda\overline{b}$, (ii) $\lambda(\overline{a}\times\overline{b})$ or $\lambda(b\times\overline{a})$ (iii) $(\overline{a}-\overline{b})or\lambda(\overline{a}-\overline{b})\times(\overline{a}+\overline{b})$

21) d	22) d	23) b	24) c
25) b, c	26) b	27) d	28) c
29) a	30) d	31) b	32) d
33) a	34) c	35) a	36) a
37) b	38) a	39) b	

Practice Worksheet (Foundation Level) – 3(c)

1) 5 cu. u. 2) $\frac{5}{3}$ cu. u. 3) 6 4) 0

5) x – 13y + 11z = 0 7) 2 ℓ mn + ℓ m – mn + ℓ n + 1 = 0 8) p = –1 or 4 or –3

9) $c^2 = 4a^2 + 10ab + 25b^2$ 10) –36 12) $\pm\frac{1}{2\sqrt{5}}(2\hat{i}+4\hat{j}-5\hat{k})$

13) 3 14) $\pm(i+2j+3k)$ 17) $\frac{2\pi}{3}$ 18) 0

19) k or –k 20) 21) $\frac{8}{13}$ 22) –2 [a b c]

23) d 24) b 25) b 26) a

27) 0 28) 0

Practice Worksheet (Foundation Level) – 3(d)

1) $16\hat{i}-7\hat{j}-15\hat{k};28$ 2) $\frac{1}{2}\sqrt{\frac{1}{153}}$ 3) $90(\hat{i}+\hat{j})$ 4) 366

5) $24(\hat{i}+\hat{j}+\hat{k})$, $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$ 7) $\frac{6}{7}(8\hat{i}+3\hat{j}+5\hat{k})$ 11) $\frac{\pi}{3}$

12) $\pm(2\hat{i}+2\hat{j}+\hat{k})$ 13) $\frac{9}{2}$ 14) $\lambda=\frac{15}{4}$

16) zj – yk 17) $\lambda(-23\hat{i}+\hat{j}+4\hat{k})$ 18) p^2

19) $\gamma=\pm(5\hat{i}-\hat{j}-5\hat{k})$ 20) $\hat{a}=3\hat{i}+2\hat{j}+8\hat{k},\overline{b}=2\hat{i}+2\hat{j}-\hat{k}$

Practice Worksheet (Foundation Level) – 3 (e)

1) $\frac{7}{3}(\hat{i}-2\hat{j}+2\hat{k})$ 2) (i) 0, (ii) $\frac{1}{2}\overline{b}$, (iii) 3 (iv) zero 3) $\lambda(7\hat{i}-\hat{j}+3\hat{k})$

4) $50(\sqrt{3}+1)$ 7) 27 8) a = 1 9) $\bar{b}-\frac{(\bar{b}\cdot\bar{a})}{|\bar{a}|}\bar{a}$

10) $39\sqrt{2}$ 11) $\pm\frac{1}{\sqrt{14}}(-2\hat{i}+\hat{j}+3\hat{k})$ 12) $-9\hat{i}-8\hat{j}+7\hat{k}$

13) 1 14) $\frac{1}{3}$ 15) $-\frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k})$ 16) $\bar{a}$ 18) $\bar{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\bar{b}=\hat{i}-4\hat{j}+3\hat{k}$

19) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ 21) $-2\hat{i}+6\hat{j}-10\hat{k}$ 23) $\frac{2\pi}{3}$

25) $\bar{r}=x\bar{a}+\frac{\bar{a}\times\bar{b}}{|\bar{a}|^2}$ 24) $\sqrt{51}$ 26) $|\bar{x}|^2$

27) $\lambda(\hat{i}-3\hat{j}-2\hat{k})$ 28) $\bar{p}=\bar{x}=\frac{[a\ b\ c]}{(1-\cos\theta)(1+2\cos\theta)}$, $\bar{q}=-\frac{2\cos\theta[a\ b\ c]}{(1-\cos\theta)(1+2\cos\theta)}$

30) zero 31) $\bar{x}=\frac{1}{\sqrt{3}}(\bar{p}+\bar{q}+\bar{r})$, $\bar{y}=\frac{\bar{p}}{\sqrt{3}}$, $z=\frac{\bar{q}}{\sqrt{3}}$

33) $\bar{x}=\frac{1}{39}[27\bar{p}-96\bar{r}+72\bar{q}]$, $\bar{y}=\frac{1}{39}[3\bar{p}+24\bar{r}+8\bar{q}]$ $\bar{z}=\frac{1}{39}[9\bar{p}+72\bar{r}-2\bar{q}]$

34) a 35) a 36) a 37) b 38) b 39) a

40) b 41) c

Practice Worksheet (Foundation Level) – 4(a)

1) $\bar{r}=\hat{i}-2\hat{j}+3\hat{k}+t(3\hat{i}+4\hat{j}+5\hat{k})$ 2) $\bar{r}=2\hat{i}-\hat{j}+2\hat{k}+t(\hat{i}-\hat{j}+4\hat{k})$

4) $\hat{i}+\hat{j}+\hat{k}$,(1,1,1)

10) $r=\hat{i}+2\hat{j}+3\hat{k}+t(3\hat{i}-4\hat{j}+5\hat{k})+s(2\hat{i}-2\hat{j}+\hat{k})$ 11) $\bar{r}=\hat{i}-\hat{j}+\hat{k}+t(\hat{i}-3\hat{j})+s(-3\hat{j}-2\hat{k})$

12) $\frac{1}{9}(10\hat{i}-15\hat{j}+7\hat{k})$ 13) $4\hat{i}+2\hat{j}+4\hat{k}$,(4,2,4) 14) $4\hat{i}+2\hat{j}+\hat{k}$,(4,2,1)

15) $3\hat{i}+3\hat{j}+\hat{k}$;(3,3,1) 16) $2\hat{i}+2\hat{j}+4\hat{k}$,(2,2,4) 17) (3, –2, 1)

18) $\bar{r}=5\hat{i}-2\hat{j}+4\hat{k}-t(3\hat{i}-2\hat{j}+4\hat{k})+s(4\hat{i}-\hat{j}-\hat{k})$ 19) t = 1, s = 1

20) $\bar{r}=2\hat{i}-\hat{j}-\hat{k}+t(\hat{i}+3\hat{j}+5\hat{k})+s(3\hat{i}-\hat{j}+5\hat{k})$ point is not on plane

Practice Worksheet (Foundation Level) – 4(b)

1) $\bar{r}=2\hat{i}-\hat{j}+3\hat{k}+t(3\hat{i}+7\hat{j}-2\hat{k})$, $\bar{r}\times(3\hat{i}+7\hat{j}-2\hat{k})=-19\hat{i}+13\hat{j}+17\hat{k}$

2) $\bar{r} = \hat{i} - 2\hat{j} + \hat{k} + t(\hat{i} + 5\hat{j} + 3\hat{k})$, $\bar{r} \times (\hat{i} + 5\hat{j} + 3\hat{k}) = -11\hat{i} - 2\hat{j} + 7\hat{k}$

3) $\cos^{-1}\left(\frac{2}{3}\right)$ 4) a = 1 5) $\lambda = -2$ 6) $5\hat{i} - \hat{j} + 3\hat{k}$

7) $\bar{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 4, \frac{1}{\sqrt{3}}$ 8) $\sin^{-1}\left(\frac{\sqrt{185}}{21}\right)$ 9) $\frac{2}{3}$

10) $\bar{r} \cdot (10\hat{i} + 5\hat{j} - 4\hat{k}) = 37$ 11) yes

12) $\bar{r} \cdot (10\hat{i} + 5\hat{j} - 4\hat{k}) = 37$ 13) $\bar{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 10$ 14) $\bar{r} \cdot (7\hat{j} + 4\hat{k}) = 5$

15) $\bar{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) = 0$ 16) $\bar{r} \cdot (4\hat{i} + 5\hat{j} - 3\hat{k}) = 5$

17) $\bar{r} \cdot (3\hat{i} - 4\hat{j} + 2\hat{k}) = 12$ and $\frac{12}{\sqrt{29}}$

18) $\bar{r} \cdot (-3\bar{i} + 2\bar{j} - 6\bar{k}) = 12, \bar{r} \cdot (3\hat{i} - 2\hat{j} + 6\hat{k}) + 12 = 0$

19) $\bar{r} \cdot (\hat{i} + 6\hat{j} + 5\hat{k}) = 7$ 20) $\bar{r} \cdot (\hat{i} - \hat{j} - \hat{k}) + 1 = 0$

21) $\bar{r} \cdot (31\hat{i} + \hat{j} + 5\hat{k}) = 48$ 22) $4\hat{i} + 3\hat{j} - 3\hat{k}$ 23) $\hat{i} - \hat{j} + \hat{k}$

24) $\bar{r} \cdot (\hat{i} - 2\hat{j} + 2\hat{k}) = 30$ 26) $\bar{r} = 5\hat{i} + 7\hat{j} - 3\hat{k} + \lambda(-2\hat{i} - 7\hat{j} + 13\hat{k})$

27) $\bar{r} \cdot (7\hat{i} - 2\hat{j} - \hat{k}) = 27$ 28) $\bar{r} \cdot (3\hat{i} - \hat{k}) = 10$

29) $\bar{r} \cdot (4\hat{i} - 5\hat{j} + 5\hat{k}) + 6 = 0$ 30) (6, 0, 8) 31) $\lambda = 10$ 32) $(2\hat{i} - \hat{j})$

33) $-2\hat{i} + 6\hat{j} - 10\hat{k}$ 34) $\bar{r} = \frac{\bar{a} \times (\bar{c} + \bar{b})}{(\bar{a} - \bar{b})}$ 35) $\cos^{-1}\left(\frac{3}{\sqrt{105}}\right)$

36) $\cos^{-1}\left(\frac{13}{\quad}\right), \pm\frac{1}{3\sqrt{3}}(-\hat{i} - 5\hat{j} + \hat{k})$ 37) (a) $\hat{i} + 2\hat{j} + 4\hat{k}$, (b) 2 units

38) p = q = r = $[\bar{a}\ \bar{b}\ \bar{c}]$

39) (a) $\frac{1}{8}(27\hat{i} - 14\hat{j} + 13\hat{k})$ (b) $\bar{r} = (\hat{i} + \hat{j} + \hat{k}) + t(19\hat{i} - 22\hat{j} + 5\hat{k})$

40) C; $(-5\hat{i} + 8\hat{j} - 7\hat{k}; D, -7\hat{i} + 7\hat{j} + 9\hat{k})$ 41) P. V. Q, $14\hat{i} + 8\hat{j} - 2\hat{k}$

Practice Worksheet (Foundation Level) – 5 - Miscellaneous

1) c	2) b	3) a	4) b	5) b	6) b
7) c	8) a	9) c	10) b	11) c	12) d
13) b	14) d	15) b	16) c	17) a, b	18) a

19) b	20) d	21) a	22) c	23) c, d	24) c
25) a	26) d	27) b	28) a	29) d	30) b, a
31) a, c, d	32) a, c	33) b	34) d	35) a	36) c
37) c	38) a	39) c	40) b	41) a	42) c
43) a	44) c	45) b	46) a	47) c	48) a
49) c	50) c	51) a	52) a	53) d	54) a
55) b	56) c, b	57) a	58) b	59) a	60) c
61) b	62) b	63) a	64) b	65) d	66) c
67) b	68) a	69) d	70) a	71) a	72) c
73) c	74) c	75) b	76) b	77) a	78) c
79) c	80) b	81) d	82) b	83) a	84) a
85) (i) b (ii) b (iii) c		86) a	87) c	88) d	89) a
90) d	91) a	92) b	93) c	94) d	95) c
96) a	97) b	98) a, b	99) c	100) d	101 d
102 a	103 c	104 a	105) b	106) d	107) a
108) c	109) d	110) b	111) c	112) c	113) b
111) c	114) c	115) c	116) c	117) a	118) c
119) c	120) a, c	121) a	122) c, b	123) d	124) a
125) a	126) c	127) b	128) b	129) b	130) c
131) b	132) a	133) a	134) d	135) a	136) a
137) b	138) c	139) c	140) b	141) d	142) c
143) c	144) a	145) c	146) a	147) c	148) b, c
149) a	150) c	151)	152) d	153) b	154) a
155) c	156) b	157) a	158) a	159) a,b	160) c
161) b	162) c	163) c			

Practice Worksheet (Foundation Level) – 6(a)

1) $5x - 3y + 4z = 16$ 2) $8x + y + 9z + 7 = 0$ 3) $2x - 2y - z + 9 = 0$

4) $5x - 7y + 11z + 4 = 0$ 5) $4x + 10y + 7z + 15 = 0$ 7) $x + 5y - 6z + 19 = 0$

8) $x + 3y - 4z = 4$ 9) 60^0 10) $22x + 7y - 19z + 49 = 0$

11) $2x - 2z + 1 = 0$ 12) $5x + 3y - 1 = 0$ 13) $\cos^{-1}\left(\frac{3}{5\sqrt{2}}\right)$

14) 3 15) (a) $\left(\frac{3}{2}, -1, \frac{2}{3}\right)$ (b) $\left(\frac{9}{8} l - \frac{3}{4}, \frac{1}{2}\right)$ 16) $x - 3y + 2z = 1$

17) $10x - 3y + 18z - 67 = 0$ 18) $4x - 5y - 3z + 10 = 0$

Practice Worksheet (Foundation Level) – 6(b)

2) $\frac{1}{2}, \frac{51}{14}$ 4) $\frac{x+3}{3} = \frac{y-7}{-2} = \frac{z}{5}; \frac{3}{\sqrt{38}}, \frac{-2}{\sqrt{38}}, \frac{5}{\sqrt{38}}$ 5) $\frac{x-5/4}{1} = \frac{y+15/2}{-4} = \frac{z}{-3}$

6) $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right]$ 7) $22x + 14y + 10z = 195$ 8) $2x + y - 2z + 3 = 0$

9) (a) $\frac{1}{2}x + \frac{1}{3}y + z + 1 = 0$ (b) $\frac{x}{14} = \frac{y+3}{15} = \frac{z}{-2}$ 10) $3x - 6y + 17 = 0$

11) $4x - 3y + 3 - 9 = 0$ 13) (–1, 2, 3) 14 (1,2,3) 15) $(1/\sqrt{75})$

16) $\frac{20}{\sqrt{194}}$ 17) $\frac{x-3}{-8} = \frac{y-4}{4} = \frac{z+1}{1}$ 18) $16x + 3y - 14z + 24 = 0$

19) $\frac{17}{2}$ 20) (1,1,1) 21) $\frac{x}{1} = \frac{y+1}{-13} = \frac{z-5}{-5}$ 22) $25x + 17y + 62z = 78$

Practice Worksheet (Foundation Level) – 6(c)

1) $2x + 3y + 6 = 49$ 2) $\frac{60}{\sqrt{769}}$ 3) $x + y + 2z = 3$

4) $4x + y + 2z + 3 = 0$ 5) $10x - 7y + 11z - 12 = 0$ 6) $x - 3y + 2z = 0$

7) $\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 0$ 8) $45x - 33y + 5z = 0$ 9) $4x - y + 2z = 0$

10) $49x - 38y - z + 50 = 0$ 11) $2x + y + 6z + 5 = 0$ 12) 4

13) $x + 4y - 6z + 20 = 0$ 14) $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$ 15) $x - 3y + 5 = 0$

16) $y + 2z - 4 = 0$ 17) $\frac{25}{6}$ 18) One

19) (1, –2,7) 20) $\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}; (1,3,5)$ 21) $\frac{x-3}{-8} = \frac{y-2}{4} = \frac{z+1}{1}$

22) $x - z = 0$ 23) $\left(\frac{8}{3}, \frac{1}{3}, \frac{7}{3}\right)$ 24) $\cos^{-1}\left(\frac{12}{5\sqrt{6}}\right)$

25) $\sin^{-1}(2/7)$ 27) a = 3, $n = \frac{13}{2}$ 28) (–1, –1, –1)

29) 7 30) $\frac{x+7}{20} = \frac{y+7}{33} = \frac{z+7}{31}$ 31) 13

32) (4, –2,1) 33) $\frac{x+3/2}{1} = \frac{y-1}{2} = \frac{z}{2}$ 34) $(-4,5,-1)$;(2,–1,7)

35) $\frac{7}{\sqrt{189}}$ 36) $\left(\frac{9}{7};\frac{-20}{7}|\frac{4}{7}\right)$ 37) (0, –3, –4)

38) 16x + 7y – 11 z = 0 39) $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$ 40) Line is parallel to plane.

41) $\frac{x+4/15}{11} = \frac{y-2/5}{-9} = \frac{z}{-15}$ 42) $\frac{x-3}{7} = \frac{y+1}{2} = \frac{z}{-11}$

43) x + 6y – 5z + 26 = 0, 7x – 2y – z = 8, first bisect acute angle

44) 75x + 60y + 15z – 7 = 0

46) $\frac{1}{2}\sqrt{769}$ 47) $\frac{x-4}{55} = \frac{y+1}{2} = \frac{z+2}{-17}$ 48) 38,–35,–46

49) $\frac{x-1}{-2} = \frac{y-5}{5} = \frac{z-3}{-4}$ 50) $a = \frac{5}{3}$, 3x + 2y – 3z + 5 = 0 51) b

52) b 53) b 54) a

55) c 56) a 57) a

58) a 59) c 60) b

61) a 62) b 63) b

64) c 65) b 66) b

67) c 68) a 69) c

70) a 71) a 72)d

Practice Worksheet (Foundation Level) – 7(a)

1. $x^2 + y^2 + z^2 - 2x(a-b) - 2y(a+b) - 2az + a^2 + = 0$

2. $x^2 + y^2 + z^2 - 2\sqrt{3}x + 2\sqrt{2}y - 2\sqrt{5}z = 0$

3. $x^2 + y^2 + z^2 - x - y - 5z + 2 = 0$

4. $x^2 + y^2 + z^2 - 2\alpha x - 4\alpha y - 2\alpha z + 6(\alpha^2 - \beta^2) = 0$

5. $x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0$

6. $x^2+y^2+z^2-ax+by-cz=0$

7. $x^2+y^2+z^2-2x-2y-2z-3=0$

8. $(x-6)^2+(y+2)^2+(z-4)^2=\frac{7}{2}$

9. $x^2+y^2+z^2-4ax+2ay-2ax-19a^2=0$

10. $x^2+y^2+z^2-12x-6y+8z+36=0$

11. $x^2+y^2+z^2-8x+4y-6z+13=0$

12. $x^2+y^2+z^2+8x-14y+18z-79=0$

13. $x^2+y^2+z^2+8x-8\sqrt{2}y+8z+39=0$

14. $3(x^2+y^2+z^2)+4x+8y-12z-11=0$

15. $x^2+y^2+z^2-4x+3y+6z=0$

16. $(3,-6,2)$

17. $x^2+y^2+z^2-\frac{29}{2}x-\frac{29}{3}y-\frac{29}{4}z=0$

18. $\frac{1}{4}$

19. $x^2+y^2+z^2-4x-4y-4z=13$

20. $2(x^2+y^2+z^2)-18x-9y-9z=0$

21. $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=2$

22. $(3,-1,2),(-1,3,0)$ and $\left(\frac{16}{7},-\frac{11}{7},\frac{13}{7}\right),\left(-\frac{2}{7},\frac{25}{7},\frac{1}{7}\right)$

23. $x^2+y^2+z^2-6x-2y-10z+19=0$

Practice Worksheet (Foundation Level) – 7 (b)

1) (1,3,4), r = $\sqrt{7}$ 2) r = 4, center (0,1,1) 3) $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$

4) $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$ 5) $x^2 + y^2 + z^2 - 13x + 18y - 18z = 0$

6) $x^2 + y^2 + z^2 - 4x - 6y - 5z + 12 = 0$ 7) $x^2 + y^2 + z^2 + 6x - 34y - 34z - 7 = 0$

9) $\left(\frac{15}{7}, \frac{10}{7}, \frac{30}{7}\right)$ 10) $7(x^2 + y^2 + z^2) - 2x + 36y + 18z - 26 = 0$

11) $x^2 + y^2 + z^2 - 12x - 10y + 4z + 49 = 0$ 12) $\left(\frac{48}{7}, \frac{-16}{7}, \frac{24}{7}\right)$ 13) $\left(\frac{16}{3}, \frac{19}{3}, -\frac{7}{3}\right)$

14) $x^2 + y^2 + z^2 - \frac{2}{5}x - y - \frac{4}{5}z + \frac{1}{5} = 0$

$x^2 + y^2 + z^2 - 2x - 5y - 4z + 5 = 0$

15) $x^2 + y^2 + z^2 - x - y - z = 0$ 16) $x^2 + y^2 + z^2 - 14(x + y + z) + 98 = 0$

17) $(3x + 4)^2 + (3y + 4)^2 + (3z + 4)^2 = 16$, $(x + 6)^2 + (y + 6)^2 + (z + 6)^2 = 36$

18) $x^2 + y^2 + z^2 + 3x - 4y - 5z = 0$ 19) $x^2 + y^2 + z^2 + 2x - 4y + 4z + 5 = 0$
$x^2 + y^2 + z^2 + x - 2y + 2z = 0$

20) $x^2 + y^2 + z^2 - 6x + 12y + 3z = 0$21) $x^2 + y^2 + z^2 - 4x + 2y - 2z - 3 = 0$

Practice Worksheet (Foundation Level) – 7 (c)

1) $\cos^{-1}\left(-\frac{2}{3}\right)$ 2) $\cos^{-1}\left(\frac{5}{9}\right)$ 30 $a\mu^2 = b\lambda$

4) $x^2 + y^2 + z^2 - 13x + 19y - 25z + 100 = 0$

5) $\frac{\pi}{2}$ 7 $a = -\frac{1}{3}$ 8 a = 3, a = –1 9) $\sqrt{14}$ 10) $\left(\frac{13}{7}, \frac{5}{7}, \frac{4}{7}\right)$

Practice Worksheet (Foundation Level) - 7 - Miscellaneous

1) (a) $4\sqrt{2}, 8, 4\sqrt{2}$ (b) $4\sqrt{6}, 8, 4\sqrt{6}$ (c) $\frac{\pi}{6}$ (d) $\cos^{-1}\left(\frac{3+2\sqrt{2}}{6}\right)$ (e)

$\frac{x}{5} = \frac{y}{3\sqrt{2}+4} = \frac{z}{7}$

(f) $(2 - 2\sqrt{2})(x - z) + y = 0$ (g) $x + \sqrt{2}y + z = 0$ (h) $\left(\frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{2}{3}\right)$

(i) $x^2 + y^2 + z^2 - 2x - \sqrt{2}\,y - 2z = 0$ (j) $\sqrt{2}(x + z) + y = 0$

(k) circle center $\left(\frac{3}{4},\frac{\sqrt{2}}{4},\frac{3}{4}\right)$ radius $\frac{3}{2}$ (l) $\frac{4}{3}\left(3+2\sqrt{2}\right)$

(m) $\left(\frac{32-12\sqrt{2}}{9},\frac{48\sqrt{2}-8}{9};\frac{64+12\sqrt{2}}{9}\right)$

(n) $\left(2-2\sqrt{2}\right)(x-2)+y\pm5\sqrt{25-16\sqrt{2}}=0$ (o) $\frac{x-3}{4\sqrt{2}-3}=\frac{y-6}{2}=\frac{z-6}{4\sqrt{2}-6}$

(p) $\left(\frac{1}{2},\frac{1}{2\sqrt{2}}.\frac{1}{2}\right)$

2) $\frac{x-a}{1}=\frac{y}{-1}=\frac{z-a}{1}$ 3) $\cos^{-1}\left(\frac{12}{49}\right)$ 4) $\cos^{-1}\left(\frac{1}{3\sqrt{3}}\right)$

5) $\left(\frac{33}{10},\frac{21}{10},\frac{26}{10}\right)$ 6) 2 : 3 7) $\frac{2}{3}$ 8) 1

9) y – b = z – c = 0 x – a = λ, $\lambda \in R$ 10) $\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-3}{1}$

11) 3x + 4y – 18 = 0 12) ax + by + cz = $a^2 + b^2 + c^2$ 13) $\left(\frac{11}{14},\frac{25}{7},\frac{75}{14}\right)$

14) $\frac{x-1}{4}=\frac{y-2}{-1}=\frac{z-3}{-3}$ 15) $a=\frac{5}{4}, b=3$ 16) (4, 2, –3) 17) (–5, –4, –3)

18) $2\sqrt{\frac{3}{7}}$ 19) $\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$ 20) (1,2,3), (2, –3, 1) 21) $3\sqrt{30}$

22) $2\sqrt{3}$ 23) 10x + z = 24 24) 27 x- 16y – 17z + 74 = 0

25) 7x + 4y – 6z = 10, 26) x – 4y – 8z $\pm$ 18 = 0

27) $x^2 + y^2 + z^2 - 4x - 4y - 4\sqrt{3}z + 19 = 0$28) $2(x^2 + y^2 + z^2) - 3x + 2y - 6z = 0$

29) $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ 30) (–1, 4, –2) 31) (2, 4, 3)

32) $\frac{ab}{\sqrt{a^2+b^2}}$ 33) $\frac{\pi}{2}$ 34) (1,2,3) 36) (a) 37) (b,c) 38) (c)

39) (c) 40) (c) 41) (a) 42) (b)

43) a 44) (a) 45) (a) 46) (c)

47) (c) 48) (a,d) 49) (c) 50) (a)

51) (a) 53 (c) 54) (a) 55) (b)

56 (c)	57) (c)	58) (a)	59) (d)
60) (a)	61) (a)	62) (c)	63) (b)
64) (a)	65) (c)	66) (a)	67 (a)
68) (a)	69) (c)	70 (b)	71 (d)
72 (b)	73 (b,d)	74 (a)	75(b)
76 (c)	77 (a)	78 (c)	79 (b)
80 (a)	81 (c)	82 (b)	83 (c)
84 (a)	85 (c)	86 (d)	87 (b)
88 (a)	89 (a,c)	90 (c)	91 (a)
92 (b)	93 (a)	94 (c)	95 (a)
96 (b)	97 (a)	98 (c)	

99 (b) 100 (a) (7,11,0) (b) $8\sqrt{2}$ (c) $\sin^{-1}\left(\frac{1}{3}\right)$ (d) $[-11,-7,-9]$ (e) $\cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$

(f) $4x - 3y - 2z + 5 = 0$ (g) $x^2 + y^2 + z^2 + 5x - \frac{5}{3}y - \frac{5}{2}z = 0$ (h) $\frac{5}{\sqrt{29}}$

About the Book

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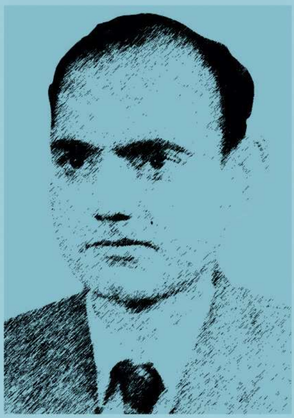
The book has been designed keeping in mind the present trend of IITJEE and other competitive examinations. For smart students who are raring to ride on the tides of this adventurous IITJEE, here is a unique book to satisfy their appetite! The book presented henceforth has been crafted whilst keeping in mind the present trend and style of the examination. The examination style has changed over the years and as such students need to prepare themselves accordingly. The book tries to explain the concepts in such an easy and comprehendible manner that a student will not have to go any further than this. Every lesson has been elaborated with related illustrations, core objectives and solved examples (subjective). Apart from objective questions, subjective answers have been included in order to lend an in-depth understanding. In order to help the student grasp the fundamentals, we have made it a point to illustrate and elucidate things starting from scratch. This will enable the student to get a better understanding of the topic.

About the Authors



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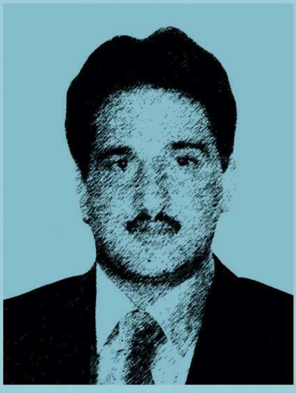
During this period he wrote three Mathematics books for Intermediate classes - Algebra (1968), Co-ordinate (1976) and Dynamics (1971). He also worked as Deputy-Secretary, Intermediate Board for 2 years and dealt services for one year at Head Quarters, Allahabad. Also had the privilege for setting board paper continuously for 5 years. After retirement in June 1982, he remained in constant touch with the subject and coached students for competitive examinations. After through study he has written these books for the benefits of students preparing for competitions.



Lalit Yadav, a graduate in engineering from world famous University of Roorkee (Now IIT-Roorkee), has been into shaping the future of students for past 15+ years and has gained expertise in mathematics. Having the reputation for excellence in coaching, motivating, and instructing students through challenging math curricula for competitive examinations, he has designed lesson plans and provided data driven instruction in mathematics as Head at many Institutes in North India providing coaching to the students for IIT-JEE examinations.

He has also monitored academic performance and provided additional attention to students in need. Also served as an advisor to students and liaison to the parents and directed team discipline efforts on a daily basis.

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