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Math Book

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3D Geometry

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JEE Syllabus :

Direction cosines and direction ratios, equation of a straight line in space, equation of a plane, distance of a point from a plane.

A. DISTANCE BETWEEN TWO POINTS

Let P and Q be two given points in space. Let the co-ordinates of the points P and Q be $(x_1, y_1\ z_1)$ and (x_2, y_2, z_2) with respect to a set OX, OY, OZ of rectangular axes. The position vectors of the points P and Q are given by $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Now we have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. $= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$$= (x_2 - x_1)\ \hat{i} - (y_2 - y_1)\ \hat{j} - (z_2 - z_1)\ \hat{k}.$$

$\therefore \quad PQ = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

B. SECTION FORMULA

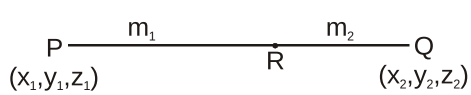
$$x = \frac{m_2x_1 + m_1x_2}{m_1 + m_2} \quad ; \quad y = \frac{m_2y_1 + m_1y_2}{m_1 + m_2} \quad ; \quad z = \frac{m_2z_1 + m_1z_2}{m_1 + m_2}$$

(for external division take –ve sign)

To determine the co-ordinates of a point R which divides the joining of two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m_1 : m_2$. Let OX, OY, OZ be a set of rectangular axes.

The position vectors of the two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ ....(1) and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ ....(2)



Also if the co-ordinates of the point R are (x, y, z), then $\overrightarrow{OR} = x\hat{i} + y\hat{j} + z\hat{k}$. .....(3)

Now the point R divides the join of P and Q in the ratio $m_1 : m_2$, so that

Hence $m_2\overrightarrow{PR} = m_1\overrightarrow{RQ}$ or $m_2(\overrightarrow{OR} - \overrightarrow{OP}) = m_1\ (\overrightarrow{OQ} - \overrightarrow{OR})$ or $\overrightarrow{OR} = \dfrac{m_1\overrightarrow{OQ} + m_2\overrightarrow{OP}}{m_1 + m_2}$

or $x\hat{i} + y\hat{j} + z\hat{k} = \dfrac{(m_1x_2 + m_2x_1)\hat{i} + (m_1y_2 + m_2y_1)\hat{j} + (m_1z_2 + m_2z_1)\hat{k}}{(m_1 + m_2)}$ [Using (1), (2) and (3)]

Comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ we get $x = \dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}$, $y = \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}$, $z = \dfrac{m_1z_2 + m_2z_1}{m_1 + m_2}$

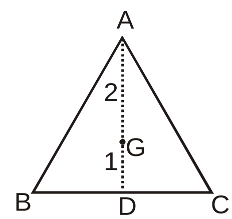
Remark : The middle point of the segment PQ is obtained by putting $m_1 = m_2$. Hence the co-ordinates of the middle point of PQ are $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2)\right)$

CENTROID OF A TRIANGLE :

Let ABC be a triangle. Let the co-ordinates of the vertices A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Let AD be a median of the ΔABC. Thus D is the mid point of BC.

$\therefore$ The co-ordinates of D are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$

Now if G is the centroid of ΔABC, then G divides AD in the ratio 2 : 1. Let the co-ordinates of G be (x, y, z). Then $x = \frac{2.\left(\frac{x_2+x_3}{2}\right)+1.x_1}{2+1}$, or $x = \frac{x_1+x_2+x_3}{3}$.



Similarly y = $\frac{1}{2}(y_1 + y_2 + y_3)$, z = $\frac{1}{2}(z_1 + z_2 + z_3)$.

CENTROID OF A TETRAHEDRON :

Let ABCD be a tetrahedron, the co-ordinates of whose vertices are (x_r, y_r, z_r), r = 1, 2, 3, 4.
Let G_1 be the centroid of the face ABC of the tetrahedron. Then the co-ordinates of G_1 are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

The fourth vertex D of the tetrahedron does not lie in the plane of ΔABC. We know from statics that the centroid of the tetrahedron divides the line DG_1 in the ratio 3 : 1. Let G be the centroid of the tetrahedron and if (x, y, z) are its co-ordinates, then

$$x = \frac{3.\frac{x_1+x_2+x_3}{3}+1.x_4}{3+1} \text{ or } x = \frac{x_1+x_2+x_3+x_4}{4}. \text{ Similarly } y = \frac{1}{4}(y_1+y_2+y_3+y_4),\ z = \frac{1}{4}(z_1+z_2+z_3+z_4).$$

Ex.1 P is a variable point and the co-ordinates of two points A and B are (–2, 2, 3) and (13, –3, 13) respectively. Find the locus of P if 3PA = 2PB.

Sol. Let the co-ordinates of P be (x, y, z).

$\therefore$ PA = $\sqrt{(x+2)^2+(y-2)^2+(z-3)^2}$ ....(1) and PB = $\sqrt{(x-13)^2+(y+3)^2+(z-13)^2}$ ....(2)

Now it is given that 3PA = 2PB i.e., $9PA^2 = 4PB^2$. ....(3)

Putting the values of PA and PB from (1) and (2) in (3), we get

$9\{(x + 2)^2 + (y - 2)^2 + (z - 3)^2\} = 4\{(x - 13)^2 + (y + 3)^2 + (z - 13)^2\}$

or $9\{x^2 + y^2 + z^2 + 4x - 4y - 6z + 17\} = 4\{x^2 + y^2 + z^2 - 26x + 6y - 26z + 347\}$

or $5x^2 + 5y^2 + 5z^2 + 140x - 60y + 50z - 1235 = 0$ or $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$

This is the required locus of P.

Ex.2 Find the ratio in which the xy-plane divides the join of (–3, 4, –8) and (5, –6, 4). Also find the point of intersection of the line with the plane.

Sol. Let the xy-plane (i.e., z = 0 plane) divide the line joining the points (–3,4, –8) and (5, –6, 4) in the ratio $\mu : 1$, in the point R. Therefore, the co-ordinates of the point R are

$$\left(\frac{5\mu-3}{\mu+1}, \frac{-6\mu+4}{\mu+1}, \frac{4\mu-8}{\mu+1}\right) \quad ....(1)$$

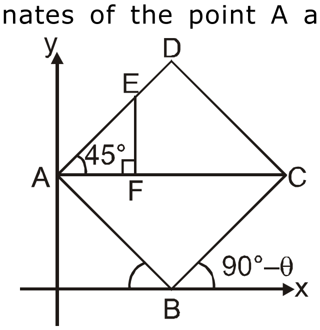
But on xy-plane, the z co-ordinate of R is zero

$\therefore$ $(4\mu - 8) / (\mu + 1) = 0$, or $\mu = 2$. Hence $\mu : 1 = 2 : 1$. Thus the required ratio is 2 : 1.

Again putting $\mu = 2$ in (1), the co-ordinates of the point R become (7/3, –8/3, 0).

Ex.3 ABCD is a square of side length 'a'. Its side AB slides between x and y-axes in first quadrant. Find the locus of the foot of perpendicular dropped from the point E on the diagonal AC, where E is the midpoint of the side AD.

Sol. Let vertex A slides on y-axis and vertex B slides on x-axis coordinates of the point A are $(0, a\sin\theta)$ and that of C are $(a\cos\theta + a\sin\theta, a\cos\theta)$



In ΔAEF, $AF = \frac{a}{2}\cos 45° = \frac{a}{2\sqrt{2}}$ and $FC = AC - AF = \sqrt{2}a - \frac{a}{2\sqrt{2}} = \frac{3a}{2\sqrt{2}}$

$\Rightarrow$ $AF : FC = \frac{a}{2\sqrt{2}} : \frac{3a}{2\sqrt{2}} = 1 : 3$

$\Rightarrow$ Let the coordinates of the point F are (x, y)

$$\Rightarrow x = \frac{3\times 0 + 1(a\cos\theta + a\sin\theta)}{4} = \frac{a(\sin\theta + \cos\theta)}{4}$$

$\Rightarrow$ $\frac{4x}{a} = \sin\theta + \cos\theta$ ....(1) and $y = \frac{3a\sin\theta + a\cos\theta}{4}$ $\Rightarrow$ $\frac{4y}{a} = 3\sin\theta + \cos\theta$...(2)

Form (1) and (2), $\sin\theta = \frac{2(y-x)}{a}$ and $\cos\theta = \frac{6x-2y}{a}$

$\Rightarrow$ $(y - x)^2 + (3x - y)^2 = \frac{a^2}{4}$ is the locus of the point F.

C. DIRECTION COSINES OF A LINE

If α, β, γ are the angles which a given directed line makes with the positive directions of the axes. of x, y and z respectively, then $\cos\alpha$, $\cos\beta$ $\cos\gamma$ are called the direction cosines (briefly written as d.c.'s) of the line. These d.c.'s are usually denote by ℓ, m, n.

Let AB be a given line. Draw a line OP parallel to the line AB and passing through the origin O. Measure angles α, β, γ, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the d.c.'s of the line AB. It can be easily seen that ℓ, m, n, are the direction cosines of a line if and only if $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector in the direction of that line.

Clearly OP′(i.e. the line through O and parallel to BA) makes angle $180° - \alpha$, $180° - \beta$, $180° - \gamma$ with OX, OY and OZ respectively. Hence d.c.'s of the line BA are $\cos(180° - \alpha)$, $\cos(180° - \beta)$, $\cos(180° - \gamma)$ i.e., are $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$.

If the length of a line OP through the origin O be r, then the co-ordinates of P are (ℓr, mr, nr) where ℓ, m, n are the d c.'s of OP.

If ℓ, m, n are direction cosines of any line AB, then they will satisfy $\ell^2 + m^2 + n^2 = 1$.

DIRECTION RATIOS :

If the direction cosines ℓ, m, n of a given line be proportional to any three numbers a, b, c respectively, then the numbers a, b, c are called direction ratios (briefly written as d.r.'s of the given line.

RELATION BETWEEN DIRECTION COSINES AND DIRECTION RATIOS :

Let a, b, c be the direction ratios of a line whose d.c.'s are ℓ, m, n. From the definition of d.r.'s. we have $\ell/a = m/b = n/c = k$ (say). Then $\ell = ka$, $m = kb$, $n = kc$. But $\ell^2 + m^2 + n^2 = 1$.

$\therefore$ $k^2 (a^2 + b^2 + c^2) = 1$, or $k^2 = 1/(a^2 + b^2 + c^2)$ or $k = \pm \dfrac{1}{\sqrt{(a^2+b^2+c^2)}}$.

Taking the positive value of k, we get $\ell = \dfrac{a}{\sqrt{(a^2+b^2+c^2)}}$, $m = \dfrac{b}{\sqrt{(a^2+b^2+c^2)}}$, $n = \dfrac{c}{\sqrt{(a^2+b^2+c^2)}}$

Again taking the negative value of k, we get $\ell = \dfrac{-a}{\sqrt{(a^2+b^2+c^2)}}$, $m = \dfrac{-b}{\sqrt{(a^2+b^2+c^2)}}$, $n = \dfrac{-c}{\sqrt{(a^2+b^2+c^2)}}$.

Remark. Direction cosines of a line are unique. But the direction ratios of a line are by no means unique. If a, b, c are direction ratios of a line, then ka, kb, kc are also direction ratios of that line where k is any non-zero real number. Moreover if a, b, c are direction ratios of a line, then $a\hat{i} + b\hat{j} + c\hat{k}$ is a vector parallel to that line.

Ex.4 Find the direction cosines $\ell + m + n$ of the two lines which are connected by the relation $\ell + m + n = 0$ and $mn - 2n\ell - 2\ell m = 0$.

Sol. The given relations are $\ell + m + n = 0$ or $\ell = -m - n$ ....(1) and $mn - 2n\ell - 2\ell m = 0$ ...(2)

Putting the value of ℓ from (1) in the relation (2), we get

$mn - 2n(-m - n) - 2(-m - n)m = 0$ or $2m^2 + 5mn + 2n^2 = 0$ or $(2m + n)(m + 2n) = 0$.

$\therefore$ $\dfrac{m}{n} = -\dfrac{1}{2}$ and -2. From (1), we have $\dfrac{\ell}{n} = \dfrac{-m-n}{n} = -\dfrac{m}{n} - 1$ ...(3)

Now when $\dfrac{m}{n} = -\dfrac{1}{2}$, (3) given $\dfrac{\ell}{n} = \dfrac{1}{2} - 1 = -\dfrac{1}{2}$. $\therefore$ $\dfrac{m}{1} = \dfrac{n}{-2}$ and $\dfrac{\ell}{1} = \dfrac{n}{-2}$

i.e. $\dfrac{\ell}{1} = \dfrac{m}{1} = \dfrac{n}{-2} = \dfrac{\sqrt{(\ell^2+m^2+n^2)}}{\sqrt{\{1^2+1^2+(-2)^2\}}} = \dfrac{1}{\sqrt{6}}$ $\therefore$ The d.c.'s of one line are $\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}$.

Again when $\dfrac{m}{2} = -2$, (3) given $\dfrac{\ell}{n} = 2 - 1 = 1$.

i.e. $\dfrac{\ell}{1} = \dfrac{m}{-2} = \dfrac{n}{1} = \dfrac{\sqrt{(\ell^2+m^2+n^2)}}{\sqrt{\{1^2+(-2)^2+1^2\}}} = \dfrac{1}{\sqrt{6}}$ $\therefore$ The d.c.'s of the other line are $\dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}$.

To find the projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the another line whose d.c.'s are ℓ, m, n.

Let O be the origin. Then $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

Now the unit vector along the line whose d.c.'s are ℓ,m,n $= \ell\hat{i} + m\hat{j} + n\hat{k}$.

$\therefore$ projection of PQ on the line whose d.c.'s are ℓ, m, n

$= [(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}] \cdot (\ell\hat{i} + m\hat{j} + n\hat{k}) = \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.

The angle θ between these two lines is given by cos θ = $\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$

If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are two sets of real numbers, then

$(\ell_1^2 + m_1^2 + n_1^2)(\ell_2^2 + m_2^2 + n_2^2) - (\ell_1\ell_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1\ell_2 - n_2\ell_1)^2 + (\ell_1m_2 - \ell_2m_1)^2$

Now, we have

$\sin^2\theta = 1 - \cos^2\theta = 1 - (\ell_1\ell_2 + m_1m_2 + n_1n_2)^2 = (\ell_1^2 + m_1^2 + n_1^2)(\ell_2^2 + m_2^2 + n_2^2) - (\ell_1\ell_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1\ell_2 - n_2\ell_1)^2 + (\ell_1m_2 - \ell_2m_1)^2 = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & n_1 \\ \ell_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix}^2$

Condition for perpendicularity $\Rightarrow \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

Condition for parallelism $\Rightarrow \ell_1 = \ell_2, m_1 = m_2, n_1 = n_2$. $\Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Ex.5 Show that the lines whose d.c.'s are given by $\ell + m + n = 0$ and $2mn + 3\ell n - 5\ell m = 0$ are at right angles.

Sol. From the first relation, we have $\ell = -m - n$. ....(1)

Putting this value of ℓ in the second relation, we have

$2mn + 3(-m - n)n - 5(-m - n)m = 0$ or $5m^2 + 4mn - 3n^2 = 0$ or $5(m/n)^2 + 4(m/n) - 3 = 0$ ....(2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the d,c,'s of the two lines. Then the roots of (2) are m_1/n_1 and m_2/n_2.

$\therefore$ product of the roots $= \dfrac{m_1}{n_1}\cdot\dfrac{m_2}{n_2} = -\dfrac{3}{5}$ or $\dfrac{m_1m_2}{3} = \dfrac{n_1n_2}{-5}$. ....(3)

Again from (1), $n = -\ell - m$ and putting this value of n in the second given relation, we have

$2m(-\ell - m) + 3\ell(-\ell - m) - 5\ell m = 0$ or $3(\ell/m)^2 + 10(\ell/m) + 2 = 0$.

$\therefore \quad \dfrac{\ell_1}{m_1}\cdot\dfrac{\ell_2}{m_2} = \dfrac{2}{3}$ or $\dfrac{\ell_1\ell_2}{2} = \dfrac{m_1m_2}{3}$ From (3) and (4) we have $\dfrac{\ell_1\ell_2}{2} = \dfrac{m_1m_2}{3}\ \dfrac{n_1n_2}{-5}$ = k (say)

$\therefore \quad \ell_1\ell_2 + m_1m_2 + n_1n_2 = (2 + 3 - 5)k = 0 \cdot k = 0$. $\Rightarrow$ The lines are at right angles.

Remarks :

(a) Any three numbers a, b, c proportional to the direction cosines are called the direction ratios

i.e. $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$ same sign either +ve or –ve should be taken throughout.

Note that d.r's of a line joining x_1, y_1, z_1 and x_2, y_2, z_2 are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$

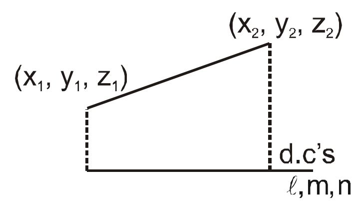
(b) If θ is the angle between the two lines whose d.c's are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2

$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

Hence if lines are perpendicular then $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

if lines are parallel then $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note that if three lines are coplanar then $\begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = 0$



(c) Projection of the join of two points on a line with d.c's ℓ, m, n are

$\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

(d) If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are the d.c.'s of two concurrent lines, show that the d.c.'s of two lines bisecting the angles between them are proportional to $\ell_1 \pm \ell_2$, $m_1 \pm m_2$, $n_1 \pm n_2$.

D. AREA OF A TRIANGLE

Show that the area of a triangle whose vertices are the origin and the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $\frac{1}{2}\sqrt{(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2}$.

The direction ratios of OA are x_1, y_1, z_1 and those of OB are x_2, y_2, z_2.

Also OA = $\sqrt{(x_1-0)^2 + (y_1-0)^2 + (z_1-0)^2} = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$

and OB = $\sqrt{(x_2-0)^2 + (y_2-0)^2 + (z_2-0)^2} = \sqrt{(x_2^2 + y_2^2 + z_2^2)}$.

$\therefore$ the d.c.' s of OA are $\frac{x_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}, \frac{y_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}, \frac{z_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}$

and the d.c.'s of OB are $\frac{x_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}, \frac{y_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}, \frac{z_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}$

Hence if θ is the angle between the line OA and OB, then

$$\sin\theta = \frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}}}{\sqrt{(x_1^2+y_1^2+z_1^2)}\sqrt{(x_2^2+y_2^2+z_2^2)}} = \frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_1)^2\}}}{OA.OB}$$

Hence the area of $\Delta OAB \;=\; \frac{1}{2} . OA . OB \sin\theta$ $\quad [\because \angle AOB = \theta]$

$$= \frac{1}{2} . OA. OB. \frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}}}{OA.OB} \quad = \frac{1}{2}\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}} .$$

Ex.6 Find the area of the triangle whose vertices are A(1, 2, 3), B(2, –1, 1)and C(1, 2, –4).

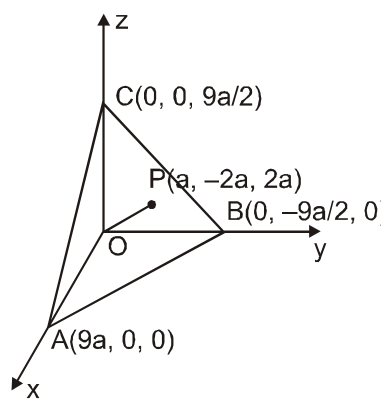
Sol. Let Δ_x, Δ_y, Δ_z be the areas of the projections of the area Δ of triangle ABC on the yz, zx and xy-planes respectively. We have

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 2 & -4 & 1 \end{vmatrix} = \frac{21}{2} \quad ; \quad \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & -4 & 1 \end{vmatrix} = \frac{7}{2}$$

$$\Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0 \quad \therefore \text{ the required area } \Delta = \sqrt{[\Delta_x^2 + \Delta_y^2 + \Delta_z^2]} = \frac{7\sqrt{10}}{2} \text{ sq. units.}$$

Ex.7 A plane is passing through a point P(a, –2a, 2a), a ≠ 0, at right angle to OP, where O is the origin to meet the axes in A, B and C. Find the area of the triangle ABC.

Sol. $OP = \sqrt{a^2 + 4a^2 + 4a^2} = |3a|$.



Equation of plane passing through P(a, –2a, 2a) is

A(x – a) + B(y + 2a) + C(z – 2a) = 0.

$\because$ the direction cosines of the normal OP to the plane ABC are proportional to

a – 0, –2a – 0, 2a – 0 i.e. a, –2a, 2a.

$\Rightarrow$ equation of plane ABC is

a(x – a) – 2a(y + 2a) + 2a(z – 2a) = 0

or $ax - 2ay + 2az = 9a^2$ ....(1)

Now projection of area of triangle ABC on ZX, XY and YZ planes are the triangles AOC, AOB and BOC respectively.

$\therefore (\text{Area } \Delta ABC)^2 = (\text{Area } \Delta AOC)^2 + (\text{Area } \Delta AOB)^2 + (\text{Area } \Delta BOC)^2$

$$= \left(\frac{1}{2}. AO . OC\right)^2 + \left(\frac{1}{2}. AO . BO\right)^2 + \left(\frac{1}{2}. BO . OC\right)^2$$

$$= \frac{1}{4}\left[\left(9a.\frac{9}{2}a\right)^2 + \left(9a . \frac{-9}{2}a\right)^2 + \left(\frac{-9}{2}a.\frac{9}{2}a\right)^2\right] = \frac{1}{4}, \frac{81^2 a^4}{4}\left(1 + 1 + \frac{1}{4}\right)$$

$$\Rightarrow (\text{Area } \Delta ABC)^2 = \frac{9^5}{4^3}a^4 \Rightarrow \text{Area of } \Delta ABC = \frac{3^5}{2^3}a^2 = \frac{243}{8}a^2 .$$

E. PLANE

(i) General equation of degree one in x, y, z i.e. $ax + by + cz + d = 0$ represents a plane.

(ii) Equation of a plane passing through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

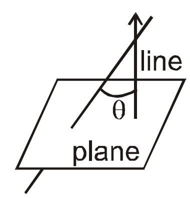
(iii) Equation of a plane if its intercepts on the co-ordinate axes are x_1, y_1, z_1 is $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$.

(iv) Equation of a plane if the length of the perpendicular from the origin on the plane is 'p' and d.c's of the perpendiculars as ℓ, m, n is $\ell x + my + nz = p$

(v) **Parallel and perpendicular planes :**
Two planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are

Perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, **parallel** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ and **Coincident** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$

(vi) Angle between a plane and a line is the complement of the angle between the normal to the plane and the line. If $\left.\begin{array}{ll}\text{Line} : & \vec{r} = \vec{a} + \lambda\vec{b} \\ \text{Plane} : & \vec{r}.\vec{n} = d\end{array}\right]$ then $\cos(90 - \theta) = \sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}$.



where θ is the angle between the line and normal to the plane.

(vii) Length of the $\perp^{ar}$ from a point (x_1, y_1, z_1) to a plane $ax + by + cz + d = 0$ is $p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$

(viii) Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

(ix) Planes bisecting the angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2 x + b_2y + c_2 z + d_2 = 0$ is given by $\left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \pm\left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$ of these two bisecting planes, one bisects the acute and the other obtuse angle between the given planes.

(x) Equation of a plane through the intersection of two planes P_1 and P_2 is given by $P_1 + \lambda P_2 = 0$

Ex.8 Reduce the equation of the plane x + 2y − 2z − 9 = 0 to the normal form and hence find the length of the perpendicular drawn form the origin to the given plane.

Sol. The equation of the given plane is x + 2y − 2z − 9 = 0

Bringing the constant term to the R.H.S., the equation becomes x + 2y − 2z = 9 ...(1)

[Note that in the equation (1) the constant term 9 is positive. If it were negative, we would have changed the sign throughout to make it positive.]

Now the square root of the sum of the squares of the coefficients of x, y, z in (1)

$= \sqrt{(1)^2 + (2)^2 + (-2)^2} = \sqrt{9} = 3$.

Dividing both sides of (1) by 3, we have $\frac{1}{2}x + \frac{2}{3}y - \frac{2}{3}z = 3$. ....(2)

The equation (2) of the plane is in the normal form $\ell x + my + nz = p$.

Hence the d.c.'s ℓ, m, n of the normal to the plane are $\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}$ and the length p of the perpendicular from the origin to the plane is 3.

Ex.9 Find the equation to the plane through the three points (0, −1, −1), (4, 5, 1) and (3, 9, 4).

Sol. The equation of any plane passing through the point (0, −1, −1) is given by

a(x − 0) + b{y − (−1)} + c{z − (−1)} = 0 or ax + b(y + 1) + c (z + 1) = 0 ....(1)

If the plane (1) passes through the point (4, 5, 1), we have 4a + 6b + 2c = 0 ....(2)

If the plane (1) passes through the point (3, 9, 4), we have 3a + 10b + 5c = 0 ....(3)

Now solving the equations (2) and (3), we have $\frac{a}{30-20} = \frac{b}{6-20} = \frac{c}{40-18} = \lambda$ (say).

$\therefore$ a = 10λ, b = −14λ, c = 22λ.

Putting these value of a, b, c in (1), the equation of the required plane is given by

λ[10x − 14(y + 1) + 22(z + 1)] = 0 or 10x − 14(y + 1) + 22(z + 1) = 0 or 5x − 7y + 11z + 4 = 0.

Ex.10 Find the equation of the plane through (1, 0, −2) and perpendicular to each of the planes 2x + y − z − 2 = 0 and x − y − z − 3 = 0.

Sol. The equation of any plane through the point (1, 0, −2) is

a (x − 1) + b (y − 0) + c(z + 2) = 0. ...(1)

If the plane (1) is perpendicular to the planes 2x + y − z − 2 = 0 and x − y − z − 3 = 0, we have

a (2) + b(1) + c(−1) = 0 i.e., 2a + b − c = 0, ...(2)

and a(1) + b(−1) + c(−1) = 0 i.e., a − b − c = 0. ...(3)

Adding the equation (2) and (3), we have $c = \frac{3}{2}a$. Subtracting (3) from (2), we have $b = -\frac{1}{2}a$.

Putting the values of b and c in (1), the equation of the required plane is given by

$a(x-1) - \frac{1}{2}ay + \frac{3}{2}a(z+2) = 0$ or 2x − 2 − y + 3z + 6 = 0 or 2x − y + 3z + 4 = 0.

Ex.11 Find the equation of the plane passing through the line of intersection of the planes $2x - 7y + 4z = 3$, $3x - 5y + 4z + 11 = 0$, and the point $(-2, 1, 3)$

Sol. The equation of any plane through the line of intersection of the given plane is

$$(2x - 7y + 4z - 3) + \lambda (3x - 5y + 4z + 11) = 0. \quad ....(1)$$

If the plane (1) passes through the point $(-2, 1, 3)$, then substituting the co-ordinates of this point in the equation (1), we have

$$\{2(-2) - 7(1) + 4(3) - 3\} + \lambda\{3(-2) - 5(1) + 4(3) + 1\} = 0 \text{ or } (-2) + \lambda(12) = 0 \text{ or } \lambda = 1/6.$$

Putting this value of λ in (1), the equation of the required plane is

$(2x - 7y + 4z - 3) + (1/6)(3x - 5y + 4z + 11) = 0$ or $15x - 47y + 28z = 7$.

Ex.12 A variable plane is at a constant distance 3p from the origin and meets the axes in A, B and C. Prove that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Sol. Let the equation of the variable plane be $x/a + y/b + z/c = 1$. .....(1)

It is given that the length of the perpendicular from the origin to the plane (1) is 3p.

$$\therefore \quad 3p = \frac{1}{\sqrt{(1/a^2 + 1/b^2 + 1/c^2)}} \text{ or } \frac{1}{9p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}, \quad .....(2)$$

The plane (1) meets the coordinate axes in the points A, B and C whose co-ordinates are respectively given by $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Let (x, y, z) be the co-ordinates of the centroid of the triangle ABC. Then $x = (a + 0 + 0)/3$, $y = (0 + b + 0)/3$, $z = (0 + 0 + c)/3$

i.e., $x = \frac{1}{3}a$, $y = \frac{1}{3}b$, $z = \frac{1}{3}c$. $\therefore$ $a = 3x$, $b = 3y$, $c = 3z$. .....(3)

The locus of the centroid of the triangle ABC is obtained by eliminating a, b, c between the equation (2) and (3). Putting the value of a, b, c from (3) in (2), the required locus is given by

$$\frac{1}{9p^2} = \frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} \text{ or } x^{-2} + y^{-2} + z^{-2} = p^{-2}.$$

Ex.13 Show that the origin lies in the acute angles between the planes $x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$. Find the planes bisecting the angles between them and point out the one which bisects the acute angle.

Sol. In order that the constant terms are positive, the equations of the given planes may be written as

$$-x - 2y - 2z + 9 = 0 \quad ...(1) \qquad \text{and} \quad 4x - 3y + 12z + 13 = 0.$$

We have $a_1a_2 + b_1b_2 + c_1c_2 = (-1).4 + (-2).(-3) + (-2).(12) = -4 + 6 - 24 = -22$ = negative.

Hence the origin lies in the acute angle between the planes (1) and (2)

The equation of the plane bisecting the angle between the given planes (1) and (2) when contains the origin is $\frac{x - 2y - 2z + 9}{\sqrt{(1 + 4 + 4)}} = \frac{4x - 3y + 12z + 13}{\sqrt{(16 + 9 + 144)}}$

or $13(-x - 2y - 2z + 9) = 3(4x - 3y + 12z + 13)$ or $25x + 17y + 62z - 78 = 0$ ...(3)

We have proved above that origin lies in the acute angle between the planes and so the equation (3) is the equation of the bisector plane which bisects the acute angle between the given planes.

The equation of the other bisector plane (i.e., the plane bisecting the obtuse angle) is

$$-\frac{x-2y-2z+9}{\sqrt{(1+4+4)}} = -\frac{4x-3y+12z+13}{\sqrt{(16+9+144)}} \quad \text{or} \quad x + 35y - 10z - 156 = 0 \qquad ....(4)$$

the equation (3) and (4) given the planes bisecting the angle between the given planes and the equation (3) is the bisector of the acute angle.

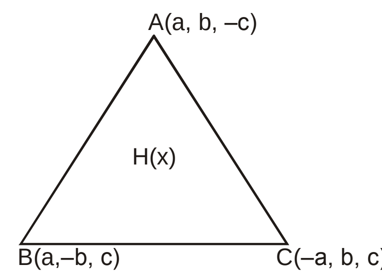
Ex.14 The mirror image of the point (a, b, c) about coordinate planes xy, xz and yz are A, B and C. Find the orthocentre of the triangle ABC.

Sol. Let the point P be (a, b, c) $\Rightarrow$ $A \equiv (a, b, -c)$, $B \equiv (a, -b, c)$ and $C \equiv (-a, b, c)$

Let the orthocentre of ΔABC be $H \equiv (x, y, z)$

$\Rightarrow$ $(x - a)(2a) + (y - b)(-2b) + (z + c)\,0 = 0$ $\Rightarrow$ $ax - by = a^2 - b^2$ ...(1)

Similarly, $by - cz = b^2 - c^2$ ...(2)



Also $\begin{vmatrix} x-a & y-b & z+c \\ 0 & 2b & -2c \\ -2a & 0 & 2c \end{vmatrix} = 0$ (As A, B, C and H are coplanar)

$\Rightarrow$ $bcx + acy + abz = abc$ ...(3)

for solving (1), (2) and (3),

$$D = \begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ bc & ac & ab \end{vmatrix} = a^2b^2 + b^2c^2 + a^2c^2 , \ D_1 = \begin{vmatrix} a^2-b^2 & -b & 0 \\ b^2-c^2 & b & -c \\ abc & ac & ab \end{vmatrix} = a^2(b^2 + c^2) - b^2c^2$$

$\Rightarrow$ Similarly $D_2 = b^2(c^2 + a^2) - a^2c^2$ and $D_3 = c^2(a^2 + b^2) - a^2b^2$

$\Rightarrow$ Orthocentre is $H \equiv \left(\frac{a^2(b^2+c^2)-b^2c^2}{a^2b^2+b^2c^2+c^2a^2}, \frac{b^2(c^2+a^2)-a^2c^2}{a^2b^2+b^2c^2+c^2a^2}, \frac{c^2(a^2+b^2)-a^2b^2}{a^2b^2+b^2c^2+c^2a^2}\right)$.

F. STRAIGHT LINE

(i) Equation of a line through $A(x_1, y_1, z_1)$ and having direction cosines ℓ , m , n are

$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the lines through (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

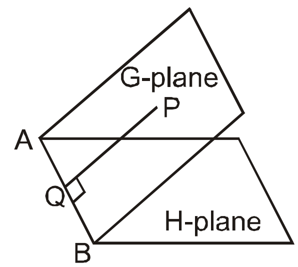
(ii) Intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represent the unsymmetrical form of the straight line.

(iii) General equation of the plane containing the line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is

$A(x - x_1) + B(y - y_1) + c(z - z_1) = 0$ where $A\ell + bm + cn = 0$.

(iv) Line of Greatest Slope

AB is the line of intersection of G-plane and H is the h orizontal plane. Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point 'P' perpendicular to the line of intersection of the given plane with any horizontal plane.



Ex.15 Show that the distance of the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$ is 13.

Sol. The equation of the given line are $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ = r (say). ....(1)

The co-ordinates of any point on the line (1) are $(3r + 2, 4r - 1, 12r + 2)$. If this point lies on the plane $x - y + z = 5$, we have $3r + 2 - (4r - 1) + 12r + 2 = 5$, or $11r = 0$, or $r = 0$.

Putting this value of r, the co-ordinates of the point of intersection of the line (1) and the given plane are $(2, -1, 2)$.

$\therefore$ The required distance = distance between the points $(2, -1, 2)$ and $(-1, -5, -10)$

$= \sqrt{(2+1)^2+(-1+5)^2+(2+10)^2} = \sqrt{(9+16+144)} = \sqrt{(169)} = 13$

Ex.16 Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $3x + 4y - 6z + 1 = 0$. Find also the co-ordinates of the point on the line which is at the same distance from the foot of the perpendicular as the origin is.

Sol. The equation of the plane is $3x + 4y - 6z + 1 = 0$. ....(1)

The direction ratios of the normal to the plane (1) are 3, 4, −6. Hence the line normal to the plane (1) has d.r.'s 3, 4, −6, so that the equations of the line through (0, 0, 0) and perpendicular to the plane (1) are $x/3 = y/4 = z/-6 = r$ (say) ....(2)

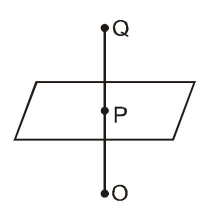
The co-ordinates of any point P on (2) are $(3r, 4r, -6r)$ ....(3)

If this point lies on the plane (1), then $3(3r) + r(4r) - 6(-6r) + 1 = 0$, or $r = -1/61$.

Putting the value of r in (3), the co-ordinates of the foot of the perpendicular P are $(-3/61, -4/61, 6/61)$.

Now let Q be the point on the line which is at the same distance from the foot of the perpendicular as the origin. Let (x_1, y_1, z_1) be the co-ordinates of the point Q. Clearly P is the middle point of OQ.

Hence we have $\frac{x_1+0}{2}=-\frac{3}{61}, \frac{y_1+0}{2}=\frac{4}{61}, \frac{z_1+0}{2}=\frac{6}{61}$



or $x_1 = 6/61, y_1 = -8/61, z_1 = 12/61$.

$\therefore$ The co-ordinates of Q are $(-6/61, -8/61, 12/61)$.

Ex.17 Find in symmetrical form the equations of the line 3x + 2y – z – 4 = 0 & 4x + y – 2z + 3 = 0 and find its direction cosines.

Sol. The equations of the given line in general form are 3x + 2y – z – 4 = 0 & 4x + y – 2z + 3 = 0 ..(1)

Let ℓ, m, n be the d.c.'s of the line. Since the line is common to both the planes, it is perpendicular to the normals to both the planes. Hence we have $3\ell + 2m - n = 0$, $4\ell + m - 2n = 0$.

Solving these, we get $\frac{\ell}{-4+1} = \frac{m}{-4+6} = \frac{n}{3-8}$ or $\frac{\ell}{-3} = \frac{m}{2} = \frac{n}{-5} = \frac{\sqrt{(\ell^2+m^2+n^2)}}{\sqrt{(9+4+25)}} = \frac{1}{\sqrt{(38)}}$

$\therefore$ the d.c.'s of the line are $-\frac{3}{\sqrt{(38)}}, \frac{2}{\sqrt{(38)}}, -\frac{5}{\sqrt{(38)}}$.

Now to find the co-ordinates of a point on the line given by (1), let us find the point where it meets the plane z = 0. Putting z = 0 i the equations given by (1), we have 3x + 2y – 4 = 0, 4x + y + 3 = 0.

Solving these, we get $\frac{x}{6+4} = \frac{y}{-16-9} = \frac{1}{3-8}$, or x = –2, y = 5.

Therefore the equation of the given line in symmetrical form is $\frac{x+2}{-3} = \frac{y-5}{2} = \frac{z-0}{-5}$.

Ex.18 Find the equation of the plane through the line 3x – 4y + 5z = 10, 2x + 2y – 3z = 4 and parallel to the line x = 2y = 3z.

Sol. The equation of the given line are 3x – 4y + 5z = 10, 2x + 2y – 3z = 4 ...(1)

The equation of any plane through the line (1) is (3x – 4y + 5z – 10) + λ (2x + 2y – 3z – 4) = 0

or $(3 + 2\lambda)x + (-4 + 2\lambda)y + (5 - 3\lambda)z - 10 - 4\lambda = 0$. ...(2)

The plane (1) will be parallel to the line x = 2y = 3z i.e. $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$ if

$(3 + 2\lambda) . 6 + (-4 + 2\lambda). 3 + (5 - 3\lambda).2 = 0$ or $\lambda(12 + 6 - 6) + 18 - 12 + 10 = 0$ or $\lambda = -\frac{4}{3}$.

Putting this value of λ in (2), the required equation of the plane is given by

$$\left(3-\frac{8}{3}\right)x + \left(-4-\frac{8}{3}\right)y + (5+4)z - 10 + \frac{16}{3} = 0 \quad \text{or} \quad x - 20y + 27z = 14.$$

Ex.19 Find the equation of a plane passing through the line $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-2}{-2}$ and making an angle of 30° with the plane x + y + z = 5.

Sol. The equation of the required plane is $(x - y + 1) + \lambda(2y + z - 6) = 0 \Rightarrow x + (2\lambda - 1)y + \lambda z + 1 - 6\lambda = 0$

Since it makes an angle of 30° with x + y + z = 5

$\Rightarrow \frac{|1+(2\lambda-1)+\lambda|}{\sqrt{3}.\sqrt{1+\lambda^2+(2\lambda-1)^2}} = \frac{\sqrt{3}}{2} \Rightarrow |6\lambda| = 3\sqrt{5\lambda^2 - 4\lambda + 2} \Rightarrow 4\lambda^2 = 5\lambda^2 - 4\lambda + 2$

$\Rightarrow \lambda^{2-} - 4\lambda + 2 = 0 \Rightarrow \lambda = (2 \pm \sqrt{2}) \Rightarrow (x - y + 1) + (2 \pm \sqrt{2})(2y + x - 6) = 0$ are two required planes.

Ex.20 Prove that the lines 3x + 2y + z – 5 = 0 = x + y – 2z – 3 and 2x – y – z = 0 = 7x + 10y – 8z – 15 are perpendicular.

Sol. Let ℓ_1, m_1, n_1 be the d.c.'s of the first line. Then $3\ell_1 + 2m_1 + n_1 = 0$, $\ell_1 + m_1 - 2n_1 = 0$. Solving, we get

$$\frac{\ell_1}{-4-1} = \frac{m_1}{1+6} = \frac{n_1}{3-2} \text{ or } \frac{\ell_1}{-5} = \frac{m_1}{7} = \frac{n_1}{1}.$$

Again let ℓ_2, m_2,n_2 be the d.c.'s of the second line, then $2\ell_2 - m_2 - n_2 = 0$, $7\ell_2 + 10m_2 - 8n_2 = 0$.

Solving, $\frac{\ell_2}{8+10} = \frac{m_2}{-7+16} = \frac{n_2}{20+7}$ or $\frac{\ell_2}{2} = \frac{m_2}{1} = \frac{n_2}{3}$.

Hence the d.c.'s of the two given lines are proportional to –5, 7, 1 and 2, 1, 3. We have

–5.2 + 7.1 + 1.3 = 0 $\therefore$ the given lines are perpendicular.

Ex.21 Find the equation of the plane which contains the two parallel lines

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1} \text{ and } \frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}.$$

Sol. The equation of the two parallel lines are

(x + 1)/3 = (y – 2)/2 = (z – 0)/1 ....(1) and (x – 3)/3 = (y + 4)/2 = (z – 1)/1. ....(2)

The equation of any plane through the line (1) is

a(x + 1) + b (y – 2) + cz = 0, ...(3) where 3a + 2b + c = 0. ....(4)

The line (2) will also lie on the plane (3) if the point (3, –4, 1) lying on the line (2) also lies on the plane (3), and for this we have a (3 + 1) + b (–4 – 2) + c. 1 = 0 or 4a – 6b + c = 0. ....(5)

Solving (4) and (5), we get $\frac{a}{8} = \frac{b}{1} = \frac{c}{-26}$.

Putting these proportionate values of a, b, c in (3), the required equation of the plane is

8(x + 1) + 1.(y – 2) – 26z = 0, or 8x + y – 26 + 6 = 0.

Ex.22 Find the distance of the point P(3, 8, 2) from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane 3x + 2y – 2z + 17 = 0.

Sol. The equation of the given line are (x – 1)/2 = (y – 3)/4 = (z – 2)/3 = r, (say). ...(1)

Any point Q on the line (1) is (2r + 1, 4r + 3, 3r + 2).

Now P is the point (3, 8, 2) and hence d.r.'s of PQ are

2r + 1 – 3, 4r + 3 – 8, 3r + 2 – 2 i.e. 2r – 2, 4r – 5, 3r.

It is required to find the distance PQ measured parallel to the plane 3x + 2y – 2z + 17 = 0 ...(2)

Now PQ is parallel to the plane (2) and hence PQ will be perpendicular to the normal to the plane (2).

Hence we have (2r – 2) (3) + (4r – 5) (2) + (2r) (–2) = 0 or 8r – 16 = 0, or r = 2.

Putting the value of r, the point Q is (5, 11, 8) = $\sqrt{[(3-5)^2 + (8-11)^2 + (2-8)^2]} = \sqrt{(4+9+36)} = 7$.

Ex.23 Find the projection of the line 3x − y + 2z = 1, x + 2y − z = 2 on the plane 3x + 2y + z = 0.

Sol. The equations of the given line are 3x − y + 2z = 1, x + 2y − z = 2. ....(1)

The equation of the given plane is 3x + 2y + z = 0. ....(2)

The equation of any plane through the line (1) is $(3x - y + 2z - 1) + \lambda(x + 2y - z - 2) = 0$

or $(3 + \lambda) x + (-1 + 2\lambda) y + (2 - \lambda) z - 1 - 2\lambda = 0$ ....(3)

The plane (3) will be perpendicular to the plane (2), if $3(3 + \lambda) + 2(-1 + 2\lambda) + 1 (2 - \lambda) = 0$ or $\lambda = -\frac{3}{3}$.

Putting this value of λ in (3), the equation of the plane through the line (1) and perpendicular to the plane (2) is given by $\left(3-\frac{3}{2}\right)x + (-1 -3)\, y + \left(2+\frac{3}{2}\right) z - 1 + 3 = 0$ or $3x - 8y + 7z + 4 = 0$. ....(4)

∴ The projection of the given line (1) on the given plane (2), is given by the equations (2) and (4) together.

Note : The symmetrical form of the projection given above by equations (2) and (4) is $\frac{x+\frac{4}{5}}{-11} = \frac{y-\frac{2}{5}}{9} = \frac{z}{5}$.

Ex.24 Find the image of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ in the plane x + 2y + z = 12

Sol. Any point on the given line is 2r + 1, −r − 1, 4r + 3. If this point lies on the planes,

then $2r + 1 - 2r - 2 + 4r + 3 = 12 \Rightarrow r = \frac{5}{2}$.

Hence the point of intersection of the given line and that of the plane is $\left(6, -\frac{7}{2}, 13\right)$.

Also a point on the line is (1, −1, 3).

Let (α, β, γ) be its image in the given plane. In such a case $\frac{\alpha-1}{1} = \frac{\beta+1}{2} = \frac{\gamma-3}{1} = \lambda$

$\Rightarrow$ $\alpha = \lambda + 1, \beta = 2\lambda - 1, \gamma = \lambda + 3$. Now the midpoint of the image and the point (1, −1, 3) lies on the plane i.e. $\left(1+\frac{\lambda}{2}, \lambda - 1, 3 + \frac{\lambda}{2}\right)$ lies in the plane $\Rightarrow \lambda = \frac{10}{3}$. Hence the image of (1, −1, 3) is $\left(\frac{8}{3}, \frac{7}{3}, \frac{14}{3}\right)$.

Hence the equation of the required line is $\frac{x-6}{\frac{10}{3}} = \frac{y+\frac{7}{2}}{\frac{-35}{6}} = \frac{z-13}{\frac{25}{3}}$ or $\frac{x-6}{4} = \frac{y+\frac{7}{2}}{-7} = \frac{z-13}{10}$.

Ex.25 Find the foot and hence the length of the perpendicular from the point (5, 7, 3) to the line $(x - 15)/3 = (y - 29)/8 = (z - 5)/(-5)$. Find the equations of the perpendicular. Also find the equation of the plane in which the perpendicular and the given straight line lie.

Sol. Let the given point (5, 7, 3) be P.

The equations of the given line are $(x - 15)/3 = (y - 29)/8 = (z - 5)/(-5) = r$ (say). ...(1)

Let N be the foot of the perpendicular from the point P to the line (1). The co-ordinates of N may be taken as $(3r + 15, 8r + 29, - 5r + 5)$. ...(2)

$\therefore$ the direction ratios of the perpendicular PN are

$3r + 15 - 5, 8r + 29 - 7, - 5r + 5 - 3$, i.e. are $3r + 10, 8r + 22, -5r + 2$. ...(3)

Since the line (1) and the line PN are perpendicular to each other, therefore

$3(3r + 10) + 8(8r + 22) - 5(-5r + 2) = 0$ or $98r + 196 = 0$ or $r = -2$

Putting this value of r in (2) and (3), the foot of the perpendicular N is (9, 13, 15) and the direction ratios of the perpendicular PN are 4, 6, 12 or 2, 3, 6.

$\therefore$ the equations of the perpendicular PN are $(x - 5)/2 = (y - 7)/3 = (z - 3)/6$. ...(4)

Length of the perpendicular PN

= the distance between P(5, 7, 3) and N(9, 13, 15) = $\sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2} = 14$.

Lastly the equation of the plane containing the given line (1) and the perpendicular (4) is given by

$$\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

or $(x - 15)(48 + 15) - (y - 29)(18 + 10) + (z - 5)(9 - 16) = 0$ or $9x - 4y - z =- 14 = 0$.

Ex.26 Show that the planes $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$, $8x - 31y - 33z = 0$ pass through the one line find its equations.

Sol. The rectangular array of coefficient is $\begin{vmatrix} 2 & -3 & -7 & 0 \\ 3 & -14 & -13 & 0 \\ 8 & -31 & -33 & -0 \end{vmatrix}$.

We have, $\Delta_4 = \begin{vmatrix} 2 & -3 & -7 \\ 3 & -14 & -13 \\ 8 & -31 & -33 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -1 \\ 3 & -11 & -4 \\ 8 & -23 & -9 \end{vmatrix}$ (by $C_2 + C_1$, $C_2 + 3C_1$)

$= \begin{vmatrix} 0 & 0 & -1 \\ -5 & -7 & -4 \\ -10 & -14 & -9 \end{vmatrix} = -1(70 - 70) = 0$, (by $C_1 + 2C_2$, $C_2 - C_2$)

since $\Delta_4 = 0$, therefore, the three planes either intersect in a line or form a triangular prism.

Now $\Delta_3 = \begin{vmatrix} 2 & -3 & 0 \\ 3 & -14 & 0 \\ 8 & -31 & 0 \end{vmatrix} = 0$ Similarly $\Delta_2 = 0$ and $\Delta_1 = 0$,

Hence the three planes intersect in a common line.

Clearly the three planes pass through (0, 0, 0) and hence the common line of intersection will pass through (0, 0, 0). The equations of the common line are given by any of the two given planes. Therefore the equations of the common line are given by $2x - 3y - 7z = 0$ and $3x - 14y - 13z = 0$.

$\therefore$ the symmetric form of the line is given by $\dfrac{x}{39-98} = \dfrac{y}{-21+26} = \dfrac{z}{-28+9}$ or $\dfrac{x}{-59} = \dfrac{y}{5} = \dfrac{z}{-19}$.

Ex.27 For what values of k do the planes $x - y + z + 1 = 0$, $kx + 3y + 2z - 3 = 0$, $3x + ky + z - 2 = 0$ **(i)** intersect in a point ; **(ii)** intersect in a line ; **(iii)** form a triangular prism ?

Sol. The rectangular array of coefficients is $\begin{vmatrix} 1 & -1 & 1 & 1 \\ k & 3 & 2 & 3 \\ 3 & k & 1 & -2 \end{vmatrix}$

Now we calculate the following determinants

$$\Delta_4 = \begin{vmatrix} 1 & -1 & 1 \\ k & 3 & 2 \\ 3 & k & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ k+3 & 3 & 5 \\ 3+k & k & k+1 \end{vmatrix} \quad \text{(adding 2nd column to 1st and 3rd)}$$

$$= (k+3)\begin{vmatrix} 0 & -1 & 0 \\ 1 & 3 & 5 \\ 1 & k & k+1 \end{vmatrix} = (k+3)(k+1-5) = (k+3)(k-4).$$

$$\Delta_2 = \begin{vmatrix} 1 & -1 & 1 \\ k & 3 & -3 \\ 3 & k & -2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ k+3 & 3 & 0 \\ 3+k & k & k-2 \end{vmatrix} = (k+3)(k-2), \quad \text{(adding 2nd column to 1st and 3rd)}$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ k & 2 & -3 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ k-2 & 2 & -5 \\ 2 & 1 & -3 \end{vmatrix} \quad \text{(adding (-1) times 2nd column to 1st and 3rd)}$$

$$= -\{(k-2)(-3) + 10\} = 3k - 16,$$

$$\text{and } \Delta_1 = \begin{vmatrix} -1 & 1 & 1 \\ 3 & 2 & -3 \\ k & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & -3 \\ k-2 & 1 & -2 \end{vmatrix} = -5(k-2) \quad \text{(adding 3rd column to 1st)}$$

(i) The given planes will intersect in a point if $\Delta_4 \neq 0$ and so we must have $k \neq -3$ and $k \neq 4$. Thus the given planes will intersect in a point for all real values of k other than –3 and 4.

(ii) If $k = -3$, we have $\Delta_4 = 0$, $\Delta_3 = 0$ but $\Delta_2 \neq 0$. Hence the given planes will form a triangular prism if $k = -3$.

(iii) If $k = 4$, we have $\Delta_4 = 0$ but $\Delta_3 \neq 0$. Hence the given planes will form a triangular prism if $k = 4$. We observe that for no value of k the given planes will have a common line of intersection.

Ex.28 Find the equation of the line passing through (1, 1, 1) and perpendicular to the line of intersection of the planes $x + 2y - 4z = 0$ and $2x - y + 2z = 0$.

Sol. Equation of the plane through the lines $x + 2y - 4z = 0$ and $2x - y + 2z = 0$ is

$x + 2y - 4z + \lambda(2x - y + 2z) = 0$ ...(1)

If (1, 1, 1) lies on this plane, then $-1 + 3\lambda = 0$

$\Rightarrow \lambda = \frac{1}{3}$, so that the plane becomes $3x + 6y - 12z + 2x - y + 2z = 0 \Rightarrow x + y - 2z = 0$ ....(2)

Also (1) will be perpendicular to (2) if $1 + 2\lambda + 2 - \lambda - 2(-4 + 2\lambda) = \Rightarrow \lambda = \frac{11}{3}$.

$\Rightarrow$ Equation of plane perpendicular to (2) is $5x - y + 2z = 0$. ...(3)

Therefore the equation of line through (1, 1, 1) and perpendicular to the given line is parallel to the normal to the plane (3). Hence the required line is $\frac{x-1}{5} = \frac{y-1}{-1} = \frac{z-1}{2}$

Alternate :

Solving the equation of planes $x + 2y - 4z = 0$ and $2x - y + 2z = 0$, we get $\frac{x}{0} = \frac{y}{-10} = \frac{z}{-5}$ ...(1)

Any point P on the line (1) can be written as $(0, -10\lambda, -5\lambda)$.

Direction ratios of the line joining P and Q(1, 1, 1) is $(1, 1, + 10\lambda, 1 + 5\lambda)$.

Line PQ is perpendicular to line (1) $\Rightarrow$ $0(1) - 10(1 + 10\lambda) - 5(1 + 5\lambda) = 0$

$\Rightarrow 0 - 10 - 100\lambda - 5 - 25x = 0$ or $125\lambda + 15 = 0 \Rightarrow = \frac{-15}{125} = \frac{-3}{25} \Rightarrow$ $P = \left(0, \frac{6}{5}, \frac{3}{5}\right)$

Direction ratios of PQ = $\left(-1, \frac{1}{5}, \frac{-2}{5}\right)$. Hence equations of lien are $\frac{x-1}{5} = \frac{y-1}{-1} = \frac{z-1}{2}$.

Ex.29 Find the shortest distance (S.D.) between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}, \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

Find also its equations and the points in which it meets the given lines.

Sol. The equations of the given lines are $(x - 3)/3 = (y - 8)/-1 = (z - 3)/1 = r_1$ (say) ...(1)

and $(x + 3)/(-3) = (y + 7)/2 = (z - 6)/4 = r_2$ (say) ...(2)

Any point on line (1) is $(3r_1 + 3, -r_1 + 8, r_1 + 3)$, say P. ...(3)

any point on line (2) is $(-3r_2 - 3, 2r_2 - 7, 4r_2 + 6)$, say Q. ...(4)

The d.r.'s of the line PQ are $(-3r_2 - 3) - (3r_1 + 3), (2r_2 - 7) - (-r_1 + 8), (4r_2 + 6) - (r_1 + 3)$

or $-3r_2 - 3r_1 - 6, 2r_2 + r_1 - 15, 4r_2 - r_1 + 3$. ...(5)

Let the line PQ be the lines of S.D., so that PQ is perpendicular to both the given lines (1) and (2), and so we have $3(-3r_2 - 3r_1 - 6) - 1(2r_2 + r_1 - 15) + 1.(4r_2 - r_1 + 3) = 0$

and $-3(-3r_2 - 3r_1 - 6) + 2.(2R_2 + r_1 - 15) + 4(4r_2 - r_1 + 3) = 0$

or $-7r_2 - 11r_1 = 0$ and $11r_2 + 7r_1 = 0$. Solving these equations, we get $r_1 = r_2 = 0$.

Substituting the values of r_1 and r_2 in (3), (4) and (5), we have P(3, 8, 3), Q(–3, –7, 6)

And the d.r.'s of PQ (the line of S.D.) are –6, –15, 3 or –2, –5, 1.

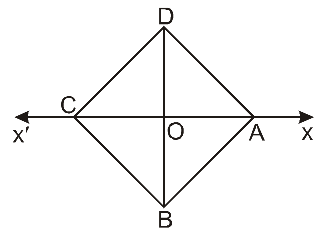
The length of S.D. = the distance between the points P and Q = $\sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2} = 3\sqrt{30}$.

Now the line PQ of shortest distance is the line passing through P(3, 8, 3) and having d.r.'s –2, –5, 1

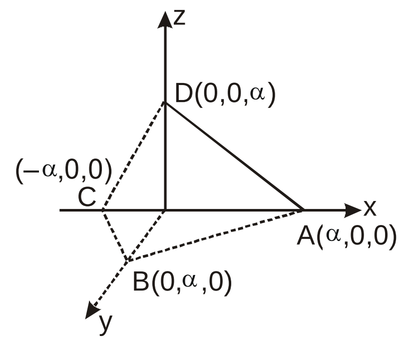
and hence its equations are given by $\frac{x-3}{-2} = \frac{y-8}{-5} = \frac{z-3}{1}$ or $\frac{x-3}{2} - \frac{y-8}{5} = \frac{z-3}{1}$.

Ex.30 A square ABCD of diagonal 2a is folded along the diagonal AC so that the planes DAC, BAC are at right angles. Find the shortest distance between DC and AB.

Sol. ABCD is a square of diagonal 2a, so that AC = BD = 2a. Let O, the centre of the square, be chosen as origin of co-ordinates and the diagonal CA be taken along x-axis. Hence the co-ordinates of the vertices A and C are
(a, 0, 0) and (–a, 0, 0) respectively.



Now as given in the problem, the square is folded over along the diagonal AC so that the planes DAC and BAC are at right angles. This implies that the lines OB and OD become at right angles. Also OA is perpendicular to the plane DOB. Hence the lines OA, OB, OD are mutually orthogonal. Let us now take OB and OD as y
and z axes respectively.



∴ The co-ordinates of B and D are (0, a, 0) and (0, 0, a) respectively.

The equations to AB are $\frac{x-a}{a}=\frac{y-0}{-a}=\frac{z-0}{0}$ .....(1)

The equation to DC are $\frac{x-0}{a}=\frac{y-0}{0}=\frac{z-a}{a}$ .....(2)

The equation of any plane through DC and parallel to AB [i.e. through the line (2) and parallel to the line (1)] is $\begin{vmatrix} x-0 & y-0 & z-0 \\ a & 0 & a \\ a & -a & 0 \end{vmatrix}$ = 0 or $x(a^2) - y(-a^2) + (z - a)(-a^2) = 0$ or $x + y - z + a = 0$ ...(3)

∴ The S.D. between DC and AB
= the length of perpendicular from a point (a, 0, 0) on AB [i.e. (1)] to the plane (3)

$$= \frac{a+0-0+a}{\sqrt{\{(1)^2+(1)^2+(-1)^2\}}}\frac{2a}{\sqrt{3}}.$$

Ex.31 Find the condition that the equation $\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ may represent a pair of planes, passing through the origin

Sol. Since it passes through the origin, let it represent the planes
$\ell_1 x + m_1 y + n_1 z = 0$ ...(1) and $\ell_2 x + m_2 y + n_2 z = 0$ ...(2)
⇒ $a x^2 + b y^2 + c z^2\ 2 f y z + 2 g z x + 2 h x y \equiv (\ell_1 x + m_1 y + n_1 z)(\ell_2 x + m_2 y + n_2 z) = 0$
comparing the coefficients of x^2, y^2, z^2, yz, zx and xy of both sides, we get,
$\ell_1 \ell_2 = a$; $m_1 m_2 = b$; $n_1 n_2 = c$;
$m_1 n_2 + m_2 n_1 = 2f$; $n_2 \ell_2 + n_2 \ell_1 = 2g$ and $\ell_1 m_2 + \ell_2 m_1 = 2h$ ...(3)

consider the product of two zero determinants $\begin{vmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} = 0$ and $\begin{vmatrix} \ell_2 & \ell_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$

i.e. $\begin{vmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} \ell_2 & \ell_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$ or $\begin{vmatrix} 2\ell_1\ell_2 & \ell_1 m_2 + \ell_2 m_1 & \ell_1 n_2 + \ell_2 n_1 \\ \ell_1 m_2 + \ell_2 m_1 & 2m_1m_2 & m_1n_2 + m_2n_1 \\ \ell_1 n_2 + \ell_2 n_1 & m_1 n_2 + m_2 n_1 & 2n_1n_2 \end{vmatrix} = 0$

putting the values of $\ell_1\ \ell_2$, $m_1\ m_2$ ...... etc. from (4), we get

$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ i.e. $a\,b\,c + 2\,f\,g\,h - a\,f^2 - b\,g^2 - c\,h^2 = 0$

which is the required condition for ϕ (x, y, z) = 0 to represent pair of planes passing through origin.

Ex.32 Prove that the product of distances of the planes represented by

$\phi(x, y, z) = a\,x^2 + b\,y^2 + c\,z^2 + 2\,f\,y\,z + 2\,g\,z\,x + 2\,h\,x\,y = 0$ from (a, b, c) is $\left| \dfrac{\phi(a,b,c)}{\sqrt{\sum a^2 + 4\sum h^2 - 2\sum ab}} \right|$.

Sol. Let the equation of two planes be $\alpha_1 x + \beta_1 y + \gamma_1 z = 0$ and $\alpha_2 x + \beta_2 y + \gamma_2 z = 0$

So, that $\phi(x, y, z) \equiv (\alpha_1 x + \beta_1 y + \gamma_1 z)(\alpha_2 x + \beta_2 y + \gamma_2 z) = 0$ ....(1)

Comparing the coefficients, we get $\alpha_1\alpha_2 = a$, $\beta_1\beta_2 = b$, $\gamma_1\gamma_2 = c$

$\beta_1\gamma_2 + \beta_2\gamma_1 = 2f$; $\gamma_1\alpha_2 + \gamma_2\alpha_1 = 2g$; $\alpha_1\beta_2 + \beta_2\alpha_1 = 2h$

Let p_1 and p_2 be the perpendiculars distances of the point (a, b, c) from the two planes then

$$p_1\,p_2 = \left| \frac{\alpha_1 a + \beta_1 b + \gamma_1 c}{\sqrt{\alpha_1^2 + \beta_1^2 + \gamma_1^2}} \right| \left| \frac{\alpha_2 a + \beta_2 b + \gamma_2 c}{\sqrt{\alpha_2^2 + \beta_2^2 + \gamma_2^2}} \right|$$

$$= \left| \frac{(\alpha_1\alpha_2 a^2 + \beta_1\beta_2 b^2 + \gamma_1\gamma_2 c^2)(\alpha_1\beta_2 + \beta_1\alpha_2)ab + (\beta_1\gamma_2 + \beta_2\gamma_1)bc + (\alpha_1\gamma_2 + \alpha_2\gamma_1)ac}{\sqrt{\alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + \gamma_1^2\gamma_2^2 + (\alpha_1^2\beta_2^2 + \alpha_1^2\beta_2^2) + (\beta_1^2\gamma_2^2 + \gamma_1^2\beta_2^2) + (\gamma_1^2\alpha_2^2 + \alpha_1^2\gamma_2^2)}} \right|$$

$$= \left| \frac{a\,.\,a^2 + b\,.\,b^2 + c\,.\,c^2 + 2hab + 2fbc + 2gac}{\sqrt{a^2 + b^2 + c^2 + \sum[(\alpha_1\beta_2 + \beta_2\alpha_1)^2 - 2\alpha_1\alpha_2.\beta_1\beta_2]}} \right| = \left| \frac{\phi(a,b,c)}{\sqrt{\sum a^2 + \sum[4h^2 - 2ab]}} \right|$$

$$\Rightarrow\quad p_1 p_2 = \left| \frac{\phi(a,b,c)}{\sqrt{\sum a^2 + a\sum h^2 - 2\sum ab}} \right|.$$

Ex.33 From a point (1, 1, 21), a ball is dropped onto the plane x + y + z = 3, where x, y-plane is horizontal and z-axis is along the vertical. Find the co-ordinates of the point where the ball hits the plane the second time. (use $s = ut - 1/2gt^2$ and $g = 10\ m/s^2$)

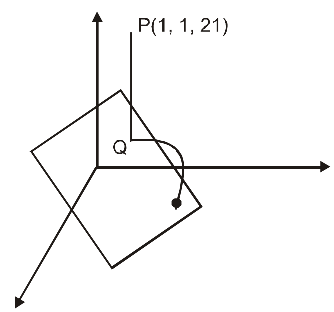
Sol. Since it falls along the vertical, the x-y coordinates of the ball will not change before it strikes the plane $\Rightarrow$ If Q be the point where the ball meets the plane 1st time, then Q ≡ (1, 1, 1)

Speed of the balls just before striking the plane is $\sqrt{2 \times 10 \times 20}$ = 20 m/s.

Now let θ be the angle between PQ and normal to the plane $\Rightarrow \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \cos 2\theta = -\frac{1}{3}, \sin 2\theta = \frac{2\sqrt{2}}{3}$

Now component of velocity in the direction of z-axis after it strikes the plane

$$= -20 \sin\left(2\theta - \frac{\pi}{2}\right) = -\frac{20}{3} \text{ m/s}$$



Hence in 't' time the z-coordinate of ball becomes

$$1 - \frac{20}{3}t - \frac{1}{2} \times 10t^2 = 1 - \frac{20}{3}t - 5t^2$$

The component of velocity in x-y plane is

$$20 \cos\left(2\theta - \frac{\pi}{2}\right) = 20 \sin 2\theta = \frac{20 \times 2\sqrt{2}}{3} = \frac{40\sqrt{2}}{3}$$

Using symmetry, the component along the x-axis = $\frac{40}{3}$ & the component along the y-axis = $\frac{40}{3}$

Hence x and y coordinates of the ball after t time = $1 + \frac{40}{3}t$

$\Rightarrow$ after t time the coordinate of the ball will become $\left(1 + \frac{40}{3}t, 1 + \frac{40}{3}t, 1 - \frac{20}{3}t - 5t^2\right)$

Its lies on the plane $\frac{80}{3}t - \frac{20}{3}t - 5t^2 = 0 \Rightarrow 20t - 5t^2 = 0 \Rightarrow t = 4$

$\Rightarrow$ coordinate of the point where the ball strikes the plane the second time = $\left[\frac{163}{3}, \frac{163}{3}, \frac{-317}{3}\right]$.

EXERCISE – I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. If the sum of the squares of the distances of a point from the three coordinate axes be 36, then its distance from the origin is

(A) 6 (B) $3\sqrt{2}$ (C) $2\sqrt{3}$ (D) $6\sqrt{2}$

2. The locus of a point P which moves such that $PA^2 - PB^2 = 2k^2$ where A and B are (3, 4, 5) and (–1, 3, –7) respectively is

(A) $8x + 2y + 24z - 9 + 2k^2 = 0$
(B) $8x + 2y + 24z - 2k^2 = 0$
(C) $8x + 2y + 24z + 9 + 2k^2 = 0$ (D) None of these

3. A line makes angles α, β, γ with the coordinates axes. If $\alpha + \beta = 90°$, then γ equal to

(A) 0 (B) 90° (C) 180° (D) None of these

4. The coordinates of the point A, B, C, D are $(4, \alpha, 2)$, $(5, -3, 2)$, $(\beta, 1, 1)$ & $(3, 3, -1)$. Line AB would be perpendicular to line CD when

(A) $\alpha = -1, \beta = -1$ (B) $\alpha = 1, \beta = 2$
(C) $\alpha = 2, \beta = 1$ (D) $\alpha = 2, \beta = 2$

5. The locus represented by $xy + yz = 0$ is

(A) A pair of perpendicular lines
(B) A pair of parallel lines
(C) A pair of parallel planes
(D) A pair of perpendicular planes

6. The equation of plane which passes through (2, –3, 1) & is normal to the line joining the points (3, 4, –1) & (2, – 1, 5) is given by

(A) $x + 5y - 6z + 19 = 0$ (B) $x - 5y + 6z - 19 = 0$
(C) $x + 5y + 3z + 19 = 0$ (D) $x - 5y - 6z - 19 = 0$

7. The equation of the plane passing through the point (1, – 3, –2) and perpendicular to planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$, is

(A) $2x - 4y + 3z - 8 = 0$ (B) $2x - 4y - 3z + 8 = 0$
(C) $2x - 4y + 3z + 8 = 0$ (D) None of these

8. A variable plane passes through a fixed point (1, 2, 3). The locus of the foot of the perpendicular drawn from origin to this plane is

(A) $x^2 + y^2 + z^2 - x - 2y - 3z = 0$
(B) $x^2 + 2y^2 + 3z^2 - x - 2y - 3z = 0$
(C) $x^2 + 4y^2 + 9z^2 + x + 2y + 3 = 0$
(D) $x^2 + y^2 + z^2 + x + 2y + 3z = 0$

9. The reflection of the point (2, –1, 3) in the plane $3x - 2y - z = 9$ is

(A) $\left(\frac{26}{7}, \frac{15}{7}, \frac{17}{7}\right)$ (B) $\left(\frac{26}{7}, \frac{-15}{7}, \frac{17}{7}\right)$
(C) $\left(\frac{15}{7}, \frac{26}{7}, \frac{-17}{7}\right)$ (D) $\left(\frac{26}{7}, \frac{15}{7}, \frac{-15}{7}\right)$

10. The distance of the point (–1, –5, –10) from the point of intersection of the line, $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane, $x - y + z = 5$, is

(A) 10 (B) 11 (C) 12 (D) 13

11. The distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line, $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ is

(A) 1 (B) 6/7 (C) 7/6 (D) None of these

12. The straight lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-3}{-2}$ are

(A) Parallel lines (B) intersecting at 60°
(C) Skew lines (D) Intersecting at right angle

13. If plane cuts off intercepts OA = a, OB = b, OC = c from the coordinate axes, then the area of the triangle ABC equal to

(A) $\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$ (B) $\frac{1}{2}(bc + ca + ab)$
(C) $\frac{1}{2}abc$ (D) $\frac{1}{2}\sqrt{(b+c)^2(c-a)^2 + (a-b)^2}$

14. A point moves so that the sum of the squares of its distances from the six faces of a cube given by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$ is 10 units. The locus of the point is

(A) $x^2 + y^2 + z^2 = 1$ (B) $x^2 + y^2 + z^2 = 2$
(C) $x + y + z = 1$ (D) $x + y + z = 2$

15. A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes in A, B, C. Locus of the point common to the planes through A, B, C and parallel to coordinate plane, is

(A) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$ = 1 (B) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ = 1

(C) ax + by + cz = 1 (D) None of these

16. Two systems of rectangular axes have same origin. If a plane cuts them at distances a, b, c and a_1, b_1, c_1 from the origin, then

(A) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}$

(B) $\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a_1^2}-\frac{1}{b_1^2}+\frac{1}{c_1^2}$

(C) $a^2 + b^2 + c^2 = a_1^2+b_1^2+c_1^2$

(D) $a^2 - b^2 + c^2 = a_1^2+b_1^2+c_1^2$

17. Equation of plane which passes through the point of intersection of lines $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-3}{2}$ and $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and at greatest distance from the point (0, 0, 0) is

(A) 4x + 3y + 5z = 25 (B) 4x + 3y + 5z = 50
(C) 3x + 4y + 5z = 49 (D) x + 7y − 5z = 2

18. The angle between the plane 2x − y + z = 6 and a plane perpendicular to the planes x + y + 2z = 7 and x − y = 3 is

(A) $\pi/4$ (B) $\pi/3$ (C) $\pi/6$ (D) $\pi/2$

19. The non zero value of 'a' for which the lines 2x − y + 3z + 4 = 0 = ax + y − z + 2 and x − 3y + z = 0 = x + 2y + z + 1 are co-planar is

(A) −2 (B) 4 (C) 6 (D) 0

20. If the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}, \frac{x-1}{3}=\frac{y-2}{-1}=\frac{z-3}{4}$ and $\frac{x+k}{3}=\frac{y-1}{2}=\frac{z-2}{h}$ are concurrent then

(A) h = −2, k = − 6 (B) h = $\frac{1}{2}$, k = 2

(C) h = 6, k = 2 (D) h = 2, k = $\frac{1}{2}$

21. The coplanar points A, B, C, D are (2 − x, 2, 2), (2, 2 − y, 2), (2, 2, 2 − z) and (1, 1, 1) respectively. Then

(A) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ = 1 (B) x + y + z = 1

(C) $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}$ = 1 (D) None of these

22. The direction ratios of a normal to the plane through (1, 0, 0), (0, 1, 0), which makes an angle of $\pi/4$ with the plane x + y = 3 are

(A) (1, $\sqrt{2}$, 1) (B) (1, 1, $\sqrt{2}$)
(C) (1, 1, 2) (D) ($\sqrt{2}$, 1, 1)

23. Let the points A(a, b, c) and B(a′, b′, c′) be at distances r and r′ from origin. The line AB passes through origin when

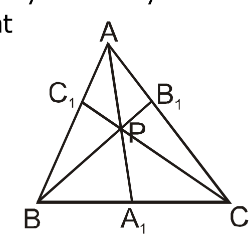
(A) $\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}$ (B) aa′ + bb′ + cc′ = rr′

(C) aa′ + bb′ + cc′ = $r^2 + r'^2$ (D) None of these

24. The base of the pyramid AOBC is an equilateral triangle OBA with each side equal to $4\sqrt{2}$, 'O' is the origin of reference, AC is perpendicular to the plane of Δ OBC and $|\overrightarrow{AC}|$ = 2. Then the cosine of the angle between the skew straight lines one passing through A and the mid point of OB and the other passing through O and the mid point of BC is

(A) $-\frac{1}{\sqrt{2}}$ (B) 0 (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{\sqrt{2}}$

25. In the adjacent figure 'P' is any arbitrary interior point of the triangle ABC such that the lines AA_1, BB_1, CC_1 are concurrent at P.



Value of $\frac{PA_1}{AA_1}+\frac{PB_1}{BB_1}+\frac{PC_1}{CC_1}$ is always equal to

(A) 1 (B) 2 (C) 3 (D) None of these

26. Let L be the line of intersection of the planes 2x + 3y + z = 1 and x + 3y + 2z = 2. If L makes an angle α with the positive x-axis, the cos α equals

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{1}{\sqrt{2}}$

27. If a line makes an angle of $\frac{\pi}{4}$ with the positive directions of each of x-axis and y-axis, then the angle that the line makes with the positive direction of the z-axis is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

28. If the angle θ between the line $\frac{x+1}{1}=\frac{y-1}{2}=\frac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda}\, z + 4 = 0$ is such that $\sin\theta = \frac{1}{3}$. The value of λ is

(A) $-\frac{4}{3}$ (B) $\frac{3}{4}$ (C) $-\frac{3}{5}$ (D) $\frac{5}{3}$

29. A line makes the same angle θ with each of the x and z-axis. If the angle β, which it makes with y-axis is such that $\sin^2 \beta = 3 \sin^2 \theta$, then $\cos^2\theta$ equals

(A) 2/3 (B) 1/5 (C) 3/5 (D) 2/5

30. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is

(A) 3/2 (B) 5/2 (C) 7/2 (D) 9/2

31. A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The co-ordinates of each of the points of intersection are given by

(A) (3a, 3a, 3a), (a, a, a) (B) (3a, 2a, 3a), (a, a, a)
(C) (3a, 2a, 3a), (a, a, 2a) (D) (2a, 3a, 3a,), (2a, a, a)

32. A tetrahedron has vertices at O(0, 0, 0), A(1, 2, 1), B(2, 1, 3) and C(–1, 1, 2). Then the angle between the face OAB and ABC will be

(A) $\cos^{-1}\left(\frac{19}{35}\right)$ (B) $\cos^{-1}\left(\frac{17}{31}\right)$
(C) 30° (D) 90°

33. The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar if

(A) k = 0 or –1 (B) k = 1 or –1
(C) k = 0 or –3 (D) k = 3 or –3

34. The two lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ will be perpendicular, iff

(A) $aa' + bb' + cc' + 1 = 0$
(B) $aa' + bb' + cc' = 0$
(C) $(a + a')(b + b') + (c + c') = 0$
(D) $aa' + cc' + 1 = 0$

35. The equation of plane which meet the co-ordinate axes whose centroid is (a, b, c)

(A) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (B) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$
(C) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$ (D) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{1}{3}$

36. Let O be the origin and P be the point at a distance 3 units from origin. If D.r.'s of OP are (1, –2, –2), then co-ordinates of P is given by

(A) 1, –2, –2 (B) 3, –6, –6
(C) 1/3, –2/3, –2/3 (D) 1/9, –2/9, –2/9

37. Angle between the pair of lines

$\frac{x-2}{1}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+1}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$

(A) $\cos^{-1}\left(\frac{13}{9\sqrt{38}}\right)$ (B) $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$
(C) $\cos^{-1}\left(\frac{4}{\sqrt{38}}\right)$ (D) $\cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{19}}\right)$

38. A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. The locus of the centroid of the tetrahedron OABC is

(A) $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$
(B) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p}$
(C) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=16$ (D) None of these

39. ABC is a triangle where A = (2, 3, 5), B = (–1, 2, 2) and C(λ, 5, μ). If the median through A is equally inclined to the axes then

(A) $\lambda = \mu = 5$ (B) $\lambda = 5, \mu = 7$
(C) $\lambda = 6, \mu = 9$ (D) $\lambda = 0, \mu = 0$

40. A mirror and a source of light are situated at the origin O and at a point on OX, respectively. A ray of light from the source strikes the mirror and is reflected. If the D.r.'s of the normal to the plane are 1, −1, 1, then D.C.'s of the reflected ray are

(A) $\frac{1}{3},\frac{2}{3},\frac{2}{3}$ (B) $-\frac{1}{3},\frac{2}{3},\frac{2}{3}$

(C) $-\frac{1}{3},-\frac{2}{3},-\frac{2}{3}$ (D) $-\frac{1}{3},-\frac{2}{3},\frac{2}{3}$

41. The shortest distance between the z-axis and the line, x + y + 2z − 3 = 0, 2x + 3y + 4z − 4 = 0 is

(A) 1 (B) 2 (C) 3 (D) None of these

42. The line, $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{-1}$ intersects the curve $xy = c^2$, z = 0 then c is equal to

(A) ±1 (B) $\pm\frac{1}{3}$ (C) $\pm\sqrt{5}$ (D) None of these

43. The equation of motion of a point in space is x = 2t, y = −4t, z = 4t where t measured in hours and the co-ordinates of moving point in kilometers. The distance of the point from the starting point O(0, 0, 0) in 10 hours is

(A) 20 km (B) 40 km (C) 60 km (D) 55 km

44. Minimum value of $x^2 + y^2 + z^2$ when ax+by+cz=p is

(A) $\frac{p}{\Sigma a}$ (B) $\frac{p^2}{\Sigma a^2}$ (C) $\frac{\Sigma a^2}{p}$ (D) 0

45. The direction cosines of a line equally inclined to three mutually perpendicular lines having D.C.'s as ℓ_1, m_1, n_1 ; ℓ_2, m_2, n_2 ; ℓ_3, m_3, n_3 are

(A) $\ell_1+\ell_2+\ell_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$

(B) $\frac{\ell_1+\ell_2+\ell_3}{\sqrt{3}},\frac{m_1+m_2+m_3}{\sqrt{3}},\frac{n_1+n_2+n_3}{\sqrt{3}}$

(C) $\frac{\ell_1+\ell_2+\ell_3}{3},\frac{m_1+m_2+m_3}{3},\frac{n_1+n_2+n_3}{3}$

(D) None of these

46. The co-ordinates of the point where the line joining the points (2, −3, 1), (3, −4, −5) cuts the plane 2x + y + z = 7 are

(A) (2,1,0) (B) (3,2,5) (C) (1,−2,7) (D) None of these

47. If the line joining the origin and the point (−2, 1, 2) makes angle θ_1, θ_2 and θ_3 with the positive direction of the coordinate axes, then the value of $\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3$ is

(A) −1 (B) 1 (C) 2 (D) −2

48. The square of the perpendicular distance of point P(p, q, r) from a line through A(a, b, c) and whose direction cosine are ℓ, m, n is

(A) $\Sigma\{(q-b)\,n-(r-c)\,m\}^2$ (B) $\Sigma\{(q+b)\,n-(r+c)\,m\}^2$

(C) $\Sigma\{(q-b)\,n+(r-c)\,m\}^2$ (D) None of these

EXERCISE – II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. Equation of the plane passing through $A(x_1, y_1, z_1)$ and containing the line $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$ is

(A) $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

(B) $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

(C) $\begin{vmatrix} x-d_1 & y-d_2 & z-d_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$

(D) $\begin{vmatrix} x & y & z \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

2. The equation of the line
$x + y + z - 1 = 0$, $4x + y - 2z + 2 = 0$ written in the symmetrical form is

(A) $\frac{x+1}{1}=\frac{y-2}{-2}=\frac{z-0}{1}$ (B) $\frac{x}{1}=\frac{y}{-2}=\frac{z}{1}$

(C) $\frac{x+1/2}{1}=\frac{y-1}{-2}=\frac{z-1/2}{1}$ (D) $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{z-2}{2}$

3. The acute angle that the vector $2\hat{i}-2\hat{j}+\hat{k}$ makes with the plane contained by the two vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $\hat{i}-\hat{j}+2\hat{k}$ is given by

(A) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (B) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(C) $\tan^{-1}\left(\sqrt{2}\right)$ (D) $\cot^{-1}\left(\sqrt{2}\right)$

4. The ratio in which the sphere $x^2 + y^2 + z^2 = 504$ divides the line joining the points (12, –4, 8) and (27, –9, 18) is

(A) 2 : 3 internally (B) 3 : 4 internally
(C) 2 : 3 externally (D) 3 : 4 externally

5. The equations of the planes through the origin which are parallel to the line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z+1}{-2}$ and distance $\frac{5}{3}$ from it are

(A) $2x + 2y + z = 0$ (B) $x + 2y + 2z = 0$
(C) $2x - 2y + z = 0$ (D) $x - 2y + 2z = 0$

6. If the edges of a rectangular parallelopiped are 3, 2, 1 then the angle between a pair of diagonals is given by

(A) $\cos^{-1}\frac{6}{7}$ (B) $\cos^{-1}\frac{3}{7}$ (C) $\cos^{-1}\frac{2}{7}$ (D) None of these

7. Consider the lines $\frac{x}{2}=\frac{y}{3}=\frac{z}{5}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ the equation of the line which

(A) bisects the angle between the lines is $\frac{x}{3}=\frac{y}{3}=\frac{z}{8}$

(B) bisects the angle between the lines is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

(C) passes through origin and is perpendicular to the given lines is $x = y = -z$

(D) None of these

8. The direction cosines of the lines bisecting the angle between the lines whose direction cosines are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 and the angle between these lines is θ, are

(A) $\frac{\ell_1+\ell_2}{\cos\frac{\theta}{2}},\frac{m_1+m_2}{\cos\frac{\theta}{2}},\frac{n_1+n_2}{\cos\frac{\theta}{2}}$

(B) $\frac{\ell_1+\ell_2}{2\cos\frac{\theta}{2}},\frac{m_1+m_2}{2\cos\frac{\theta}{2}},\frac{n_1+n_2}{2\cos\frac{\theta}{2}}$

(C) $\frac{\ell_1+\ell_2}{\sin\frac{\theta}{2}},\frac{m_1+m_2}{\sin\frac{\theta}{2}},\frac{n_1+n_2}{\sin\frac{\theta}{2}}$

(D) $\frac{\ell_1+\ell_2}{2\sin\frac{\theta}{2}},\frac{m_1+m_2}{2\sin\frac{\theta}{2}},\frac{n_1+n_2}{2\sin\frac{\theta}{2}}$

9. The equation of line AB is $\frac{x}{2}=\frac{y}{-3}=\frac{z}{6}$. Through a point P(1, 2, 5), line PN is drawn perpendicular to AB and line PQ is drawn parallel to the plane 3x + 4y + 5z = 0 to meet AB is Q. Then

(A) co-ordinate of N is $\left(\frac{52}{49},-\frac{78}{49},\frac{156}{49}\right)$

(B) the equation of PN is $\frac{x-1}{3}=\frac{y-2}{-176}=\frac{z-5}{-89}$

(C) the co-ordinates of Q is $\left(3,-\frac{9}{2},9\right)$

(D) the equation of PQ is $\frac{x-1}{4}=\frac{y-2}{-13}=\frac{z-5}{8}$

10. The planes 2x − 3y − 7z = 0, 3x − 14y − 13z = 0 and 8x − 31y − 33z = 0
(A) pass through origin
(B) intersect in a common line
(C) form a triangular prism
(D) None of these

11. If the length of perpendicular drawn from origin on a plane is 7 units and its direction ratios are −3, 2, 6, then that plane is
(A) −3x + 2y + 6z − 7=0
(B) −3x + 2y + 6z − 49=0
(C) 3x − 2y − 6z − 49=0
(D) −3x + 2y − 6z − 49=0

12. Let a perpendicular PQ be drawn from P(5, 7, 3) to the line $\frac{x-15}{3}=\frac{y-2}{8}=\frac{z-6}{-5}$ when Q is the foot. Then
(A) Q is (9, 13, −15)
(B) PQ = 14
(C) the equation of plane containing PQ and the given line is 9x − 4y − z − 14 = 0
(D) None of these

EXERCISE – III SUBJECTIVE QUESTIONS

1. Show that points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form an isosceles right angled triangle.

2. Prove that the tetrahedron with vertices at the points (0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0) is a regular tetrahedron. Find also the co-ordinates of its centroid.

3. Find the coordinates of the point equidistant from the point (a, 0, 0), (0, b, 0), (0, 0, c) and (0, 0, 0,).

4. Find the ratio in which the line joining the points (3, 5, –7) and (–2, 1, 8) is divided by the y-z plane. Find also the point of intersection on the plane and the line.

5. What are the direction cosines of a line that passes through the points P(6, –7, –1) and Q(2, –3, 1) and is so directed that it makes an acute angle α with the positive direction of x-axis.

6. Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 = n^2$.

7. Show that the foot of the perpendicular from the origin to the join of A(–9, 4, 5) and B(11, 0, –1) is the mid point of AB.

8. P and Q are the points (–1, 2, 1) and (4, 3, 5). Find the projection of PQ on a line which makes angles of 120° and 135° with y and z axes respectively and an acute angle with x-axis.

9. Find the equation of the planes passing through points (1, 0, 0) and (0, 1, 0) and making an angle of 0.25 π radians with plane x + y – 3 = 0.

10. Find the angle between the plane passing through point (1, 1, 1), (1, –1, 1), (–7, –3, –5) & x-z plane.

11. Find the equation of the plane containing parallel lines $(x - 4) = \frac{3-y}{4} = \frac{z-2}{5}$ and $(x - 3) = \lambda (y + 2) = \mu z$.

12. Find the equation of image of the line $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3}$ in the plane 3x – 3y + 10z = 26.

13. Find the distance between points of intersection of

(i) Lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ & $\frac{x-4}{5} = \frac{y-1}{2} = z$

(ii) Lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ & $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

14. Find the equation of the sphere described on the line (2, –1, 4) and (–2, 2, –2) as diameter. Also find the area of the circle in which the sphere is intersected by the plane 2x + y – z = 3.

15. Find the plane π passing through the points of intersection of the planes 2x + 3y – z + 1 = 0 and x + y – 2z + 3 = 0 and is perpendicular to the plane 3x – y – 2z = 4. Find the image of point (1, 1, 1) in plane π.

16. Find the equation of the straight line which passes through the point (2, –1, –1); is parallel to the plane 4x + y + z + 2 = 0 and is perpendicular to the line of intersection of the planes 2x + y = 0, x – y + z.

17. If the distance between point (α, 5α, 10α) from the point of intersection of the lines

$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} + 4\hat{j} + 12\hat{k})$ and

plane $\vec{r} = (\hat{i} - \hat{j} + \hat{k})$ = 5 is 13 units.

Find the possible values of α.

18. The edges of a rectangular parallelepiped are a, b, c; show that the angles between the four diagonals are given by $\cos^{-1} \frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}$.

19. Find the equation of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at an angle of $\pi/3$.

20. Find the equation of the projection of line $3x - y + 2z - 1 = 0$, $x + 2y - z - 2 = 0$ on the plane $3x + 2y + z = 0$.

21. Find the acute angle between the lines $\frac{x-1}{\ell} = \frac{y+1}{m} = \frac{z}{n}$ & $\frac{x+1}{m} = \frac{y-3}{n} = \frac{z-1}{\ell}$ where $\ell > m > n$ and ℓ, m, n are the roots of the cubic equation $x^3 + x^2 - 4x = 4$.

22. Let P(1, 3, 5) and Q(−2, 1, 4) be two points from which perpendiculars PM and QN are drawn to the x-z plane. Find the angle that the line MN makes with the plane $x + y + z = 5$.

23. If 2d be the shortest distance between the lines $\frac{y}{b} + \frac{z}{c} = 1$; $x = 0$ $\frac{x}{a} - \frac{z}{c} = 1$; $y = 0$ then prove that $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

24. Prove that the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ lies in the plane $3x + 4y + 6z + 7 = 0$. If the plane is rotated about the line till the plane passes through the origin then find the equation of the plane in the new position.

EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. A line $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z-k}{3}$ cuts the y-z plane and the x-y plane at A and B respectively. If $\angle AOB = \frac{\pi}{2}$, then find k, where O is the origin.

2. Find the volume of the tetrahedron with vertices P(2, 3, 2), Q(1, 1, 1), R(3, –2, 1) and S(7, 1, 4).

3. A sphere has an equation $|\vec{r}-\vec{a}|^2+|\vec{r}-\vec{b}|^2 = 72$ where $\vec{a}=\hat{i}+3\hat{j}-6\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}+2\hat{k}$. Find

(i) the centre of the sphere

(ii) the radius of the sphere

(iii) perpendicular distance from the centre of the sphere to the plane $\vec{r}=(2\hat{i}+2\hat{j}-\hat{k}) = -3$.

4. Find the equation of the sphere which is tangential to the plane x – 2y – 2z = 7 at (3, –1, –1) and passes through the point (1, 1, –3).

5. Let PM be the perpendicular from the point P(1, 2, 3) to the x-y plane. If OP makes an angle θ with the positive direction of the z-axis and OM makes an angle ϕ with the positive direction of the x-axis, where O is the origin, then find θ and ϕ.

6. Prove that the line $\frac{x}{1}=\frac{y}{1}=\frac{z-1}{-2}$ lies in the plane x + y + z = 1. Find the lines in the plane through the point (0, 0, 1) which are inclined at an angle $\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$ with the line.

7. Find the equations of the straight line passing through the point (1, 2, 3) to intersect the straight line x + 1 = 2 (y – 2) = z + 4 and parallel to the plane x + 5y + 4z = 0.

8. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at an angle of $\frac{\pi}{3}$.

9. Find the distance of the point P(–2, 3, –4) from the line $\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$ measured parallel to the plane 4x + 12y – 3z + 1 = 0.

10. Find the equation to the line passing through the point (1, –2, –3) parallel to the line 2x + 3y – 3z + 2 = 0 = 3x – 4y + 2z – 4.

11. Find the equation of the line passing through the point (4, –14, 4) and intersecting the line of intersection of the planes 3x + 2y – z = 5 and x – 2y – 2z = –1 at right angles.

12. Let P = (1, 0, –1); Q = (1, 1, 1) and R = (2, 1, 3) are three points.

(a) Find the area of the triangle having P, Q and R as its vertices.

(b) Given the equation of the plane through P, Q and R in the form ax + by + cz = 1.

(c) Where does the plane in part (b) intersect the y–axis.

(d) Give parametric equations for the line through R that is perpendicular to the plane in part (b).

13. Find the point where the line of intersection of the planes x – 2y + z = 1 and x + 2y – 2z = 5, intersect the plane 2x + 2y + z + 6 = 0.

14. Feet of the perpendicular drawn from the point P(2, 3, –5) on the axes of coordinates are A, B and C. Find the equation of the plane passing through their feet and the area of ΔABC.

15. Find the equation to the line which can be drawn from the point (2, −1, 3) perpendicular to the lines

$\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-3}{2}$ and $\frac{x-4}{3} = \frac{y}{2} = \frac{z+3}{1}$

16. Find the equation of the plane containing the straight line $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z}{5}$ and perpendicular to the plane x − y + z + 2 = 0.

17. Find the value of p so that the lines $\frac{x-1}{-3} = \frac{y-p}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ are in the same plane. For this value of p, find the coordinates of their point of intersection and the equation of the plane containing them.

18. Find the equations to the line of greatest slope through the point (7, 2, −1) in the plane x − 2y + 3z = 0 assuming that the axes are so placed that the plane 2x + 3y − 4z = 0 is horizontal.

19. The line $\frac{x+6}{5} = \frac{y+10}{3} = \frac{z+14}{8}$ is the hypotenuse of an isosceles right angled triangle whose opposite vertex is (7, 2, 4). Find the equation of the remaining sides.

20. Find the equation of the line which is reflection of the line $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3}$ in the plane 3x − 3y + 10z = 26.

21. Find the equation of the plane containing the line $\frac{x-1}{2} = \frac{y}{3} = \frac{z}{2}$ and parallel to the line $\frac{x-3}{2} = \frac{y}{5} = \frac{z-2}{4}$.

Find the also the S.D. between the two lines.

EXERCISE – V — JEE PROBLEMS

1. (i) Find the equation of the plane passing through the points (2, 1, 0), (5, 0, 1) and (4, 1, 1).

(ii) If P is the point (2, 1, 6) then find the point Q such that PQ is perpendicular to the plane in (i) and the mid point of PQ lies on it. **[JEE 2003, 4]**

2. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k equals **[JEE 2004(Scr.)]**
(A) 2/9 (B) 9/2 (C) 0 (D) –1

3. Let P be the plane passing through (1, 1, 1) and parallel to the lines L_1 and L_2 having direction ratios (1, 0, –1) and (–1, 1, 0) respectively. If A, B and C are the points at which P intersects the coordinate axes, find the volume of the tetrahedron whose vertices are A, B, C and the origin. **[JEE 2004, 2]**

4. (a) A variable plane at a distance of 1 unit from the origin cuts the co-ordiante axes at A, B and C. If the centroid D (x, y, z) of triangle ABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, then the values of k is
(A) 3 (B) 1 (C) 1/3 (D) 9
[JEE 2005 (Scr.), 3]

(b) Find the equation of the plane containing the line $2x - y + z - 3 = 0, 3x + y + z = 5$ and at a distance of $1/\sqrt{6}$ from the point (2, 1, –1). **[JEE 2005 (Mains), 4]**

5.(a) A plane passes through (1, –2, 1) and is perpendicular to two planes 2x – 2y + z = 0 and x – y + 2z = 4. The distance of the plane from the point (1, 2, 2) is **[JEE 2006, 3]**
(A) 0 (B) 1 (C) $\sqrt{2}$ (D) $2\sqrt{2}$

(b) Match the following **[JEE 2006, 6]**

Column–I	**Column–II**
(A) Two rays in the first quadrant x + y = | a | and ax – y = 1 intersects each other in the interval a ∈ (a_0, ∞), the value of a_0 is	(P) 2
(B) Point (α, β, γ) lies on the plane x + y + z = 2. Let $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$. $\hat{k} \times (\hat{k} \times \vec{a}) = 0$, then γ equal	(Q) 4/3
(C) $\left\|\int_0^1 (1-y^2)\,dy\right\| + \left\|\int_1^0 (y^2-1)\,dy\right\|$	(R) $\left\|\int_0^1 \sqrt{1-x}\,dx\right\| + \left\|\int_{-1}^0 \sqrt{1+x}\,dx\right\|$
(D) In a ΔABC, if sin A sin B sin C + cos A cos B = 1, then the value of sin C equal	(S) 1

(c) Match the following **[JEE 2006, 6]**

Column–I	**Column–II**
(A) $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then tan t equal	(P) 0
(B) Sides a, b, c of a triangle ABC are in A.P. and $\cos\theta_1 = \frac{a}{b+c}$,	(Q) 1
$\cos\theta_2 = \frac{b}{a+c}$, $\cos\theta_3 = \frac{c}{a+b}$	(R) $\frac{\sqrt{5}}{3}$
then $\tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2}$ equal	
(C) A line is perpendicular to x + 2y + 2z = 0 and passes through (0, 1, 0). The perpendicular distance of this line from the origin is	(S) 2/3

6.(a) Consider the planes 3x – 6y – 2z = 15 and 2x + y – 2z = 5. **[JEE 2007, 3+6]**
Statement-I : The parametric equations of the line of intersection of the given planes are x = 3 + 14t, y = 1 + 2t, z = 15t.
because
Statement-II : The vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of given planes.
(A) Statement-I is true, Statement-II is true; Statement-II is correct explanation for Statement-I
(B) Statement-I is true, Statement-II is true; Statement-II is **NOT** correct explanation for Statement-I
(C) Statement-I is true, Statement-II is False
(D) Statement-I is False, Statement-II is True

MATCH THE COLUMN

(b) Consider the following linear equations
ax + by + cz = 0
bx + cy + az = 0
cx + ay + bz = 0
Match the conditions/expressions in **Column–I** with statements in **Column–II.**

Column–I	Column–II
(A) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	(P) the equation represent planes meeting only at a single point.
(B) $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	(Q) the equation represent the line $x = y = z$
(C) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	(R) the equation represent identical planes
(D) $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	(S) the equation represent the whole of the three dimensional space.

7.(a) Consider three planes **[JEE 2008, 3+4+4+4]**

$P_1 : x - y + z = 1$

$P_2 : x + y - z = -1$

$P_3 : x - 3y + 3z = 2$

Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1 & P_1 and P_2 respectively.

Statement-I : At least two of the lines L_1, L_2 and L_3 are non-parallel.

because

Statement-II : The three planes do not have a common point.

(A) Statement-I is true, Statement-II is true; Statement-II is correct explanation for Statement-I

(B) Statement-I is true, Statement-II is true; Statement-II is **NOT** correct explanation for Statement-I

(C) Statement-I is true, Statement-II is False

(D) Statement-I is False, Statement-II is True

Paragraph for Question Nos. (i) to (iii)

(b) Consider the lines $L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$;

$L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$

(i) The unit vector perpendicular to both L_1 and L_2 is

(A) $\frac{-\hat{i}+7\hat{j}+7\hat{k}}{\sqrt{99}}$ (B) $\frac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$

(C) $\frac{-\hat{i}+7\hat{j}+5\hat{k}}{5\sqrt{3}}$ (D) $\frac{7\hat{i}-7\hat{j}-\hat{k}}{\sqrt{99}}$

(ii) The shortest distance between L_1 and L_2 is

(A) 0 (B) $\frac{17}{\sqrt{3}}$ (C) $\frac{41}{5\sqrt{3}}$ (D) $\frac{17}{5\sqrt{3}}$

(iii) The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the lines L_1 and L_2 is

(A) $\frac{2}{\sqrt{75}}$ (B) $\frac{7}{\sqrt{75}}$ (C) $\frac{13}{\sqrt{75}}$ (D) $\frac{23}{\sqrt{75}}$

8. (a) Let P(3, 2, 6) be a point in space and Q be a point on the line $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is **[JEE 2009, 3+3+4]**

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $-\frac{1}{8}$

(b) A line with positive direction cosines passes through the point P(2, –1, 2) and makes equal angles with the coordiantes axes. The line meets the plane $2x + y + z = 9$ at point Q. The length of the line segment PQ equals

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2

(c) Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations $3x - y - z = 0$ & $-3x + z = 0$, $-3x + 2y + z = 0$ Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is

9. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is **[JEE 2010]**

(A) $x + 2y - 2z = 0$ (B) $3x + 2y - 2z = 0$

(C) $x - 2y + z = 0$ (D) $5x + 2y - 4z = 0$

10. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then |d| is **[JEE 2010]**

11. If the distance of the point P(1, –2, 1) from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$ is 5, then the foot of the perpendicular from P to the plane is **[JEE 2010]**

(A) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ (B) $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$

(C) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (D) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

12. Match the statements in Column I with the values in Column II **[JEE 2010]**

Column–I	**Column–II**
(A) A line from the origin meets the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ & $\frac{x-\frac{8}{3}}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$ at P and Q respectively. If length PQ = d, then d^2 is	(P) – 4
(B) The values of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\frac{3}{5}\right)$ are	(Q) 0
(C) Non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}\cdot\vec{b} = 0$, $(\vec{b} - \vec{a})\cdot(\vec{b} + \vec{c}) = 0$ and $2\|\vec{b} + \vec{c}\| = \|\vec{b} - \vec{a}\|$, If $\vec{a} = \mu\vec{b} + 4\vec{c}$, then the possible values of μ are	(R) 4
(D) Let f be the function on $[-\pi, \pi]$ given by $f(0) = 9$ and $f(x) = \sin\left(\frac{9x}{2}\right)/\sin\left(\frac{x}{2}\right)$ for $x \neq 0$. Then the value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ is	(S) 5
	(T) 6

13. The point P is the intersection of the straight line joining the points Q(2, 3, 5) and R(1, –1, 4) with the plane 5x – 4y – z = 1. If S is the foot of the perpendicular drawn from the point T(2, 1, 4) to QR, then the length of the line segment PS is **[JEE 2012]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

14. The equation of a plane passing through the line of intersection of the planes x + 2y + 3z = 2 and x – y + z = 3 and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is **[JEE 2012]**

(A) 5x – 11y + z = 17 (B) $\sqrt{2}x + y = 3\sqrt{2} - 1$

(C) $x + y + z = \sqrt{3}$ (D) $x - \sqrt{2}y = 1 - \sqrt{2}$

15. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is (are) **[JEE 2012]**

(A) y + 2z = –1 (B) y + z = –1

(C) y – z = –1 (D) y – 2z = –1

Answer Ex–I — SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. B	**2.** C	**3.** B	**4.** A	**5.** D	**6.** A	**7.** A	**8.** A
9. B	**10.** D	**11.** A	**12.** D	**13.** A	**14.** B	**15.** A	**16.** A
17. B	**18.** D	**19.** A	**20.** D	**21.** A	**22.** B	**23.** A	**24.** D
25. A	**26.** A	**27.** D	**28.** D	**29.** C	**30.** C	**31.** B	**32.** A
33. C	**34.** D	**35.** C	**36.** A	**37.** B	**38.** A	**39.** C	**40.** D
41. B	**42.** C	**43.** C	**44.** B	**45.** B	**46.** C	**47.** A	**48.** A

Answer Ex–II — MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. AB	**2.** AB	**3.** BD	**4.** AC	**5.** AD	**6.** ABC	**7.** C	**8.** BD
9. ABCD	**10.** AB	**11.** BC	**12.** BC				

Answer Ex–III — SUBJECTIVE QUESTIONS

2. (1/2, 1/2, 1/2)

3. (a/2, b/2, c/2)

4. 3 : 2 ; (0, 13/5, 2)

5. (2/3, −2/3, −1/3)

6. 60°

8. $2 - 2\sqrt{2}$

9. $x + y \pm \sqrt{2}\, z = 1$

10. $\pi/2$

11. $11x - y - 3z = 35$

12. $\frac{x-4}{9} = \frac{y+1}{-1} = \frac{z-7}{-3}$

13. $\sqrt{26}$

14. $x^2 + y^2 + z^2 - y - 2z - 14 = 0,\ \frac{317\pi}{24}$

15. $7x + 13y + 4z - 9 = 0\ ;\ \left(\frac{12}{117} - \frac{-78}{117}, \frac{57}{117}\right)$

16. $\frac{x-2}{1} = \frac{y+1}{13} = \frac{z+1}{9}$

17. $\alpha = -1,\ \frac{80}{63}$

19. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}\ ;\ \frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

20. $\frac{x+1}{11} = \frac{y-1}{9} = \frac{z-1}{-15}$

21. $\cos^{-1}\frac{4}{9}$

22. $\sin^{-1}\frac{4}{\sqrt{30}}$

24. $x + y + z = 0$

Answer Ex–IV ADVANCED SUBJECTIVE QUESTIONS

1. $\frac{9}{2}$

2. 1/2 units

3. (i) $\left(\frac{3}{2},\frac{7}{2},-2\right)$ **(ii)** $\sqrt{\frac{39}{2}}$ **(iii)** 5 unit

4. $x^2 + (y - 5)^2 + (z - 5)^2 = 81$

5. $\theta = \cos^{-1}\frac{3}{\sqrt{14}}$ and $\phi = \cos^{-1}\frac{1}{\sqrt{5}}$

7. $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-3}$

8. $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ or $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$

9. $\frac{17}{2}$

10. $\frac{x-1}{6}=\frac{y+2}{13}=\frac{z+3}{17}$

11. x = 2t + 2; y = 2t + 1 and z = –t + 3

12. (a) $\frac{3}{2}$, **(b)** $\frac{2x}{3}+\frac{2y}{3}+\frac{z}{3} = 1$, **(c)** $\left(0,\frac{3}{2},0\right)$, **(d)** $\frac{x-2}{11}=\frac{y+1}{-10}=\frac{z-3}{2}$

13. (1, –2, –4)

14. $\frac{x}{2}+\frac{y}{3}+\frac{z}{-5} = 1$, Area = $\frac{19}{2}$ sq. units

15. $\frac{x-2}{11}=\frac{y+1}{-10}=\frac{z-3}{2}$

16. 2x + 3y +z + 4 = 0

17. p = 3, (2, 1, –1); x + y + z = 0

18. $\frac{x-7}{22}=\frac{y-2}{5}=\frac{z+1}{-4}$

19. $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$; $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$

20. $\frac{x-4}{9}=\frac{y+1}{-1}=\frac{z-7}{-3}$

21. x – 2y + 2z – 1 = 0; 2 units

Answer Ex–V JEE PROBLEMS

1. (i) x + y – 2z = 3 ; **(ii)** (6, 5, –2)

2. B

3. 9/2 cubic units

4. **(a)** D ; **(b)** 2x – y + z – 3 = 0 and 62x + 29y + 19z – 105 = 0

5. **(a)** D ; **(b)** (A)–S, (B)–P, (C)–Q, R, (D)–S ; **(c)** (A)–Q, (B)–S, (C)–R

6. **(a)** D ; **(b)** (A)–R ; (B)–Q ; (C)–P ; (D)–S

7. **(a)** D ; **(b)** **(i)** B ; **(ii)** D ; **(iii)** C

8. (a) A ; **(b)** C ; **(c)** 7

9. C

10. 6

11. A

12. (A)–T ; (B)–P,R ; (C)–Q ; (D)–R

13. A

14. A

15. B,C

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THREE DIMENSIONAL GEOMETRY (3-D)

EXERCISE – I — HINTS & SOLUTIONS

Sol.1 B

Given $2x^2 + 2y^2 + 2z^2 = 36$

$\Rightarrow x^2 + y^2 + z^2 = 18$

Distance from origin

$= \sqrt{x^2 + y^2 + z^2} = \sqrt{18} = 3\sqrt{2}$

Sol.2 C

$PA^2 - PB^2 = 2k^2$

$(x - 3)^2 + (y - 4)^2 + (z - 5)^2 - (x + 1)^2$

$- (y - 3)^2 - (z + 7)^2 - 2k^2$

$\Rightarrow 8x + 2y + 24z + 9 + 2k^2 = 0$

Sol.3 B

$\alpha + \beta = 90^o$

$\alpha = 90 - \beta$

$\cos\alpha = \sin\beta$

$\cos^2\alpha = 1 - \cos^2\beta$

$\cos^2\alpha + \cos^2\beta = 1$ ....(1)

& $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\cos^2\gamma = 0 \Rightarrow \gamma = 90^o$

Sol.4 A

$AB = (1, -3 - \alpha, 0)$

$CD = (3 - \beta, 2, -2)$

$AB \perp CD$

$(3 - \beta) + 2(-3 - \alpha) + 0 = 0$

$\beta + 2\alpha + 3 = 0$

Sol.5 D

$(xy + yz) = 0$

$x + z = 0$ and $y = 0$

Two perpendicular plane.

Sol.6 A

Normal vector of plane

$= (2 - 3, -1 -4, 5 + 1) = (-1, -5, 6)$

Equation of plane

$-x - 5y + 6z = k$

passes through $(2, -3, 1)$

$-2 + 15 + 6 = k \Rightarrow k = 19$

$-x - 5y + 6z = 19$

$x + 5y - 6z +19 = 0$

Sol.7 A

$x + 2y + 2z = 5$ $\quad \vec{n}_1 = (1, 2, 2)$

$3x + 3y + 2z = 8$ $\quad \vec{n}_2 = (3, 3, 2)$

Normal vector of plane $= \vec{n}_1 \times \vec{n}_1$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 3 & 3 & 2 \end{vmatrix} = -2\hat{i} + 4\hat{j} + 3\hat{k}$$

Equation of plane

$-2x + 4y - 3z = k$

passing through $(1, -3, -2)$

$k = -8$

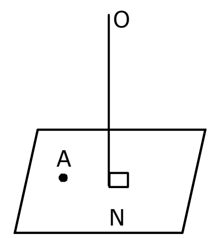
$-2x + 4y - 3z = -8$

$2x - 4y + 8z - 8 = 0$

Sol.8 A

Let N be foot of poerpendicular $= (\alpha, \beta, \gamma)$

$N(\alpha, \beta, \gamma)$



A (1, 2, 3)

Equation of plane willk be

$\alpha x + \beta y + \gamma z = k$

passing through (1, 2, 3)

$\Rightarrow k = \alpha + 2\beta + 3\gamma$

$\alpha x + \beta y + \gamma z = \alpha + 2\beta + 3\gamma$

this plane passes through (α, β, γ) also

$\alpha^2 + \beta^2 + \gamma^2 = \alpha + 2\beta + 3\gamma$

$x^2 + y^2 + z^3 - x - 2y - 3z = 0$

Sol.9 B

$N(\alpha, \beta, \gamma)$

$3x - 2y - z = 9$

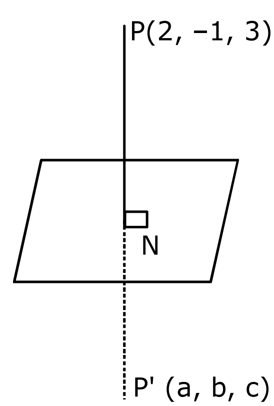
$$\frac{\alpha - 2}{3} = \frac{\beta + 1}{-2} = \frac{\gamma - 3}{-1} = \lambda$$

$\alpha = 3\lambda + 2, \beta = -2\lambda - 1, \gamma = -\lambda + 3$

N point lies on the plane

$3(3\lambda + 2) - 2(-2\lambda + 1) - (-\lambda + 3) = 9$

$\Rightarrow \lambda = \frac{2}{7}$



$N\left(\frac{20}{7}, \frac{-11}{7}, \frac{19}{7}\right)$

$N = \frac{P + P'}{2} \Rightarrow P^1 = 2N - P$

$\Rightarrow P^1 \left(\frac{26}{7}, \frac{-15}{7}, \frac{17}{7}\right)$

Sol.10 D

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$$

Use pases through P(2, −1, 2)
point P
So P_0I of line and plane is P (2, −1, 2)
(−1, −5, −10) so PQ = 13

Sol.11 A

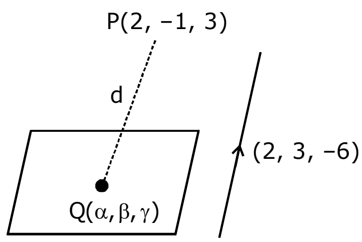
$$\frac{\alpha-1}{2} = \frac{\beta+2}{3} = \frac{\gamma-3}{-6} = \lambda$$

$\alpha = 2\lambda + 1$, b = 3l −2, $\gamma = -6\lambda + 3$
(α, β, γ) lie on the plane x + y + z= 5

$\Rightarrow \lambda = \frac{1}{7}$

$Q\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$

d = PQ = 1



Sol.12 D

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

& $\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-3}{3}$

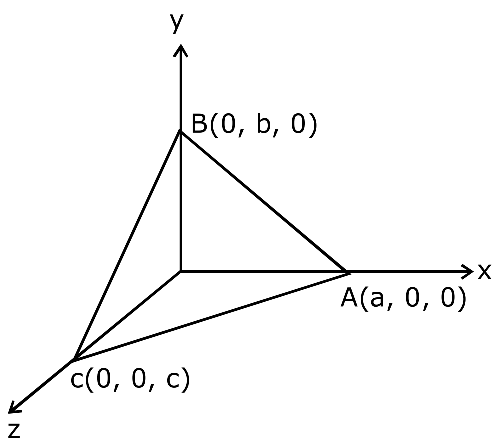
Both lines poasing through same point (1, 2, 3) that they intersect each other at point P.

Angle $\cos\theta = \frac{(1.2)+(2.2)+(3.(-2))}{\sqrt{1+4+9}\sqrt{4+4+4}} = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

Sol.13 A

Area = $\frac{1}{2}\,|\vec{AB} \times \vec{AC}|$



$= \frac{1}{2}\,|(-a, b, 0) \times (-a, 0, c)|$

$= \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$

Sol.14 B

Let Point P (α, β, γ)
Given that
$(\alpha - 1^2) + (\alpha + 1)^2 + (\beta - 1) + (\beta + 1)^2 + (\gamma - 1)^2 + (\gamma + 1)^2 = 10$
$2\alpha^2 + 2\beta^2 + 2\gamma^2 + 6 = 0$
$\alpha^2 + \beta^2 + \gamma^2 = 2 \Rightarrow x^2 + y^2 + z^2 = 2$

Sol.15 A

Let the Eq^n of plane

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

passes through (a, b, c)

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1$$

common point will be (α, β, γ)
so locus

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$$

Sol.16 A

Let the equation of planes

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ & $\frac{x}{a_1} + \frac{y}{b_1} + \frac{z}{c_1} = 1$

perpendicular distance from orign will be same
$p_1 = p_2$

$$\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = \left|\frac{-1}{\sqrt{\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}}}\right|$$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a_1^2} + \frac{1}{b_1^2} + \frac{1}{c_1^2}$$

Sol.17 B

$$\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2} = \lambda \quad ....(1)$$

$$\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \mu \; ....(2)$$

Variable point on line (1) & (2)

$(3\lambda+1, \lambda+2, 2\lambda+3)$ & $(\mu+3, 2\mu+1, 3\mu-2)$

$3\lambda + 1 = \mu + 3$

$\lambda + 2 = 2\mu + 1$

$2\lambda + 3 = 3\mu + 2$

$2\lambda + 3 = 3\mu + 2$

$2\lambda + 3 = 3\mu + 2$

By solving $\lambda = 1, \mu = 1$

Intersection point (4, 3, 5)

Equation of plane

$4x + 3y + 5z = k$

passes through (4, 3, 5) $\Rightarrow$ k = 50

$4x + 3y + 5z = 50$

Sol.18 D

$2x - y + z = 6 \; \vec{n}_1 = (2, -1, 1)$

normal vector of other plane

$$\vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = 2\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\text{angle } \cos\theta = \left|\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right| = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Sol.19 A

$$\vec{n}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -5\hat{i} + 5\hat{k}$$

$$\vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ a & 1 & -1 \end{vmatrix} = -2\hat{i} + (2+3a)j + (2+a)\hat{k}$$

p (0, –5, –3) ; R(0, –1/5, –3/5)

For compaire lines

$[\overrightarrow{PQ} \quad \vec{n}_1 \quad \vec{n}_2] = 0 \Rightarrow a = -2$

Sol.20 D

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = \lambda \Rightarrow \text{point } (\lambda, 2\lambda, 3\lambda)$$

$$\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{4} = M$$

$\Rightarrow$ Point (3M + 1, –M + 2, 4M + 3)

$$\frac{x+k}{3} = \frac{y-1}{2} = \frac{z-2}{h} = t$$

$\Rightarrow$ Point (3t – k, 2t + 1, ht + 2)

If all three lines are concurrent

$\lambda = 3\mu + 1; \; 2\lambda = -\mu + 2; \; 3\lambda = 4\mu + 3$

$\lambda = 1 \Rightarrow \mu = 1$

$3t - k = 1 ; \quad 2t + 1 = 2 \Rightarrow k = \frac{1}{2} \quad \Rightarrow t = \frac{1}{2}$

$ht + 2 = 3$

$ht = 1 \Rightarrow h = 2$

Sol.21 A

A (2 –x, 2, 2) B (2, 2 – y, 2) C (2, 2, 2 – z) D(1, 1, 1)

$\overrightarrow{AB} = (x, -y, 0), \; \overrightarrow{AC} = (x, 0, -2),$

$\overrightarrow{AD} = (x-1, -1, -1)$

If A, B, C, D are coplanar points then

$[\overrightarrow{AB} \; \overrightarrow{AC} \; \overrightarrow{AD}] = 0$

$$\begin{vmatrix} x & -y & 0 \\ x & 0 & -2 \\ x-1 & -1 & -1 \end{vmatrix} = 0 \quad \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

Sol.22 B

$|\overrightarrow{AC}| = 2$

$|\vec{a}| = |\vec{b}| = |\vec{a} - \vec{b}| = 4\sqrt{2}$

$|\vec{a} - \vec{c}| = 2$

$$\cos\theta = \frac{\left(\frac{\vec{b}}{2} - \vec{a}\right)\left(\frac{\vec{b}+\vec{c}}{2}\right)}{\left|\frac{\vec{b}}{2} - \vec{a}\right|\left|\frac{\vec{b}+\vec{c}}{2}\right|} = \frac{(\vec{b}-2\vec{a}).(\vec{b}+\vec{c})}{|\vec{b}-2\vec{a}|.|\vec{b}+\vec{c}|} = \frac{1}{\sqrt{2}}$$

Sol.23 A

A (a, b, c) B(a′, b′, c′)

Line $\overrightarrow{AB}$ = (a, b, c) + λ (a′ –a, b′ – b, c′ – c)
= (a + λa′, b + λb′, c + λc′) – λ(a, b, c)
It will passes through origin when
a + λa′ = b + λb′ = c + λc′ =0

$\Rightarrow \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

Sol.24 D

$|\overrightarrow{AC}| = 2$; $|\vec{a}| = |\vec{b}| = |\vec{a} - \vec{b}| = 4\sqrt{2}$

$|\vec{a} - \vec{b}| = 2$

$$\cos\theta = \frac{\left(\frac{\vec{b}}{2} - \vec{a}\right).\left(\frac{\vec{b}+\vec{c}}{2}\right)}{\left|\frac{\vec{b}}{2} - \vec{a}\right|\left|\frac{\vec{b}+\vec{c}}{2}\right|}$$

$$= \frac{(\vec{b} - 2\vec{a}).(\vec{b}+\vec{c})}{|\vec{b} - 2\vec{a}||\vec{b} - \vec{c}|}$$

put all the values $\cos\theta = \frac{1}{\sqrt{2}}$

Sol.25 A

Assume P is centroid

Sol.26 A

$$\text{Direction of line} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3\hat{i} - 3\hat{j} + 3\hat{k}$$

$$\text{O.D. (x-axis)} = \frac{3}{\sqrt{a+a+a}} = \frac{1}{\sqrt{3}}$$

Sol.27 D

$\ell = \cos\alpha = \frac{1}{\sqrt{2}}$

$\mu = \cos\beta = \frac{1}{\sqrt{2}}$

$\ell^2 + m^2 + n^2 = 1$

$n = 0 \Rightarrow \cos\gamma = 0 \quad \Rightarrow \gamma = \frac{\pi}{2}$

Sol.28 D

Direction of line = (1, 2, 2)

normal vector of plane = (2, –1, $\sqrt{\lambda}$)

$$\sin\theta = \frac{2 - 2 + 2\sqrt{\lambda}}{\sqrt{1+4+4}\sqrt{4+1+\lambda}} = \frac{1}{3}$$

$4\lambda = 5 + \lambda$

$\lambda = \frac{5}{3}$

Sol.29 C

$\cos^2\theta + \cos^2\beta + \cos^2\theta = 1$
$2\cos^2\theta = 1 - \cos^2\beta = \sin^2\beta$
$2\cos^2\theta = 3\sin^2\theta = 3 - 3\cos^2\theta$
$\cos^2\theta = 3/5$

Sol.30 C

$2x + y + 2z = 8$ ....(1)

$2x + y + 2z = -\frac{5}{2}$ ....(2)

$$\text{Distance} = \frac{8 + \frac{5}{2}}{\sqrt{4+1+4}} = \frac{21}{2\times 3} = \frac{7}{2}$$

Sol.31 B

x = y + a = z ....(1)
x + a = 2y = 2z ....(2)
we have option (B) & (C)
but ifwe look at option B
it will satisfy the given equation

Sol.32 A

Angle between two faces is equal to the angle between the normals $\vec{n}_1$ and $\vec{n}_2$.

$\vec{n}_1 \rightarrow$ normal of OAB

$\vec{n}_2$ = normal of ABC

$$\vec{n}_1 = \overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$= 5\hat{i} - \hat{j} - 3\hat{k}$ ...(1)

$$\vec{n}_2 = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix}$$

$= \hat{i} - 5\hat{j} - 3\hat{k}$ ...(2)

$\cos\theta = \dfrac{\vec{n_1}.\vec{n_2}}{|\vec{n_2}||\vec{n_2}|} = \dfrac{19}{35} \Rightarrow \theta = \cos^{-1}\left(\dfrac{19}{35}\right)$

Sol.33 C

$\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z-4}{-k}$;

$\dfrac{x-1}{k} = \dfrac{y-4}{2} = \dfrac{z-5}{1}$

A(2, 3, 4) B (1, 4, 5)

D.R. (1, 1, −k) D.R. (k, 2, 1)

Coplanar then = $\begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow$ k = 0 or k = −3

Sol.34 D

x = ay + b, z = cy+ d

and x = a′y + b′, z = c′h + d′

$\dfrac{x-b}{a} = y = \dfrac{z-d}{c}$

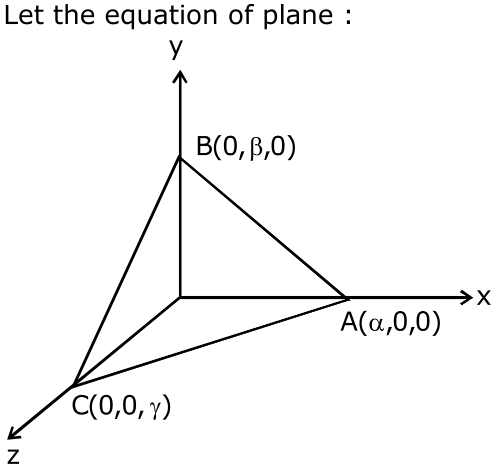
and $\dfrac{x-b'}{a'} = y = \dfrac{z-d'}{c'}$

poerpendicular then

aa′ + 1 + cc′ = 0

Sol.35 C

Let the equation of plane :



$\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 1$ ....(1)

$\dfrac{\alpha}{3} = a \qquad \Rightarrow \alpha = 3a$

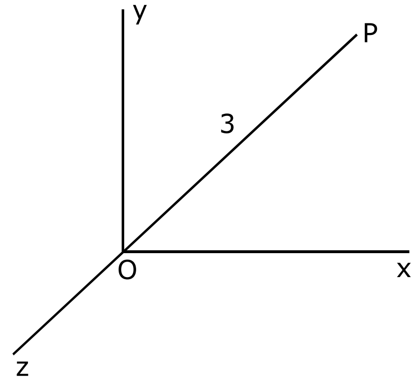
$\dfrac{\beta}{3} = b \qquad \Rightarrow \alpha = 3a$

$\dfrac{\gamma}{3} = c \qquad \Rightarrow \gamma = 3c$

$\dfrac{x}{3a} + \dfrac{y}{3b} + \dfrac{z}{3c} = 1 \Rightarrow \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 3$

Sol.36 A

D.R. of OP = (1, −2, −2)



D.C. of OP = $\left(\dfrac{1}{3}, \dfrac{-2}{3}, \dfrac{-2}{3}\right)$

Vector $\vec{OP} = |\vec{OP}| \left(\dfrac{1}{3}, \dfrac{-2}{3}, \dfrac{-2}{3}\right)$

= (1, −2, −2)

Sol.37 B

$\vec{a}$ = (1, 5, −3)

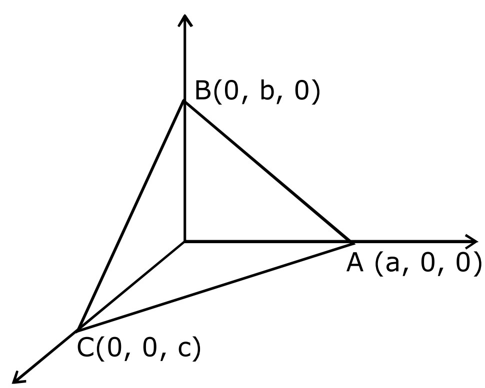
$\vec{b}$ = (−1, 8, 4)

$\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Sol.38 A

Ley the equation of plane

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$



given that p = $\dfrac{1}{\sqrt{\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}}}$

or $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$ ....(1)

Let centroid (u, v, w)

$u = \frac{1}{4} \quad \Rightarrow a = 4u$

$v = \frac{b}{4} \quad \Rightarrow b = 4v$

$w = \frac{c}{4} \quad \Rightarrow c = 4w$

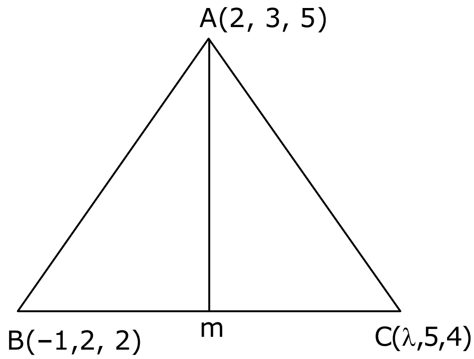
$$\frac{1}{16u^2} + \frac{1}{16v^2} + \frac{1}{16w^2} = \frac{1}{p^2}$$

$$\frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2} = \frac{16}{p^2}$$

$u^{-2} + v^{-2} + z^{-2} = 16p^{-2}$

Sol.39 C

A (2, 3, 5) B(−1, 2, 2) C(λ, 5, 4)



m $\left(\frac{\lambda-1}{2}, \frac{7}{2}, \frac{\mu+2}{2}\right)$

D.R> of median through A :

$$\left(\frac{\lambda-1}{2} - 2\frac{7}{2} - 3, \frac{\mu+2}{2} - 5\right)$$

$$\left(\frac{\lambda-5}{2}, \frac{1}{2}, \frac{\mu-8}{2}\right)$$

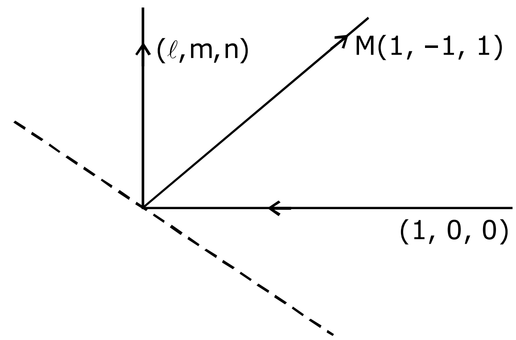
As thje median through A is equally inclined to He axis

∴ D.R.'s will be and equal to k.

$$\frac{\frac{\lambda-5}{2}}{k} = \frac{1}{2k} = \frac{\frac{\mu-8}{2}}{k} \quad \Rightarrow \lambda = 6 \text{ and } \mu = 9$$

Sol.40 D

The DC's of incident RAy arew (1, 0, 0).
Let the D.C's of reflectd ray be (λ, m, n)



⇒ The D.R.'s of the normal to polane of mirror is (l – 1, m, n)

$\frac{\ell-1}{1} = \frac{m}{-1} = \frac{n}{1}$

$\ell = \lambda + 1$, $m = -\lambda$, $n = \lambda$

$\ell^2 + m^2 + n^2 = 1$

$(\lambda + 1)^2 + \lambda^2 + \lambda^2 = 1$

$3\lambda^2 + 2\lambda = 0$

$\lambda = -2/3$

D.C's of reflected Ray $\left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$

or $\left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$

Sol.41 B

dir^n of line = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{vmatrix} = -2\hat{i} + \hat{k}$

DR' & = (−2, 0, 1)

$(\vec{n}_1 \times \vec{n}_2) \times \hat{k} = (-2\hat{i} + \hat{k}) \times \hat{k} = 2\hat{j}$

⇒ distance = 2

Sol.42 C

$\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1} = \lambda$

$(3\lambda + 2, 2\lambda - 1, 1 - \lambda)$

$z = 0 \quad \Rightarrow \quad \lambda = 1$

$xy = c^2$

$(3\lambda + 2)(2\lambda - 1) = c^2$

put $\lambda = 1 \Rightarrow c^2 = 5 \Rightarrow c = \pm\sqrt{5}$

Sol.43 C

Distance = $\sqrt{x^2 + y^2 + z^2}$

= $\sqrt{(2t)^2 + (4t)^2 + (4t)^2}$

= 6t t = 10

Distance = 60 km

Sol.44 B

Let the point P(x, y, z)

Asking minimum value of OP^2

$\Rightarrow \perp^r$ distance of origin from plane

$$d = \left|\frac{P}{\sqrt{a^2 + b^2 + c^2}}\right| \Rightarrow d^2 = \frac{P^2}{\Sigma a^2}$$

Sol.45 B

Since three lines are mutually perpendicular

$\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$; $\ell_2\ell_3 + m_2m_3 + n_2n_3 = 0$

$\ell_3\ell_1 + m_3m_1 + n_3n_1 = 0$

Also $\ell_1^2 + m_1^2 + n_1^2 = 1$; $\ell_2^2 + m_2^2 + n_2^2 = 1$;

$(\ell_1 + \ell_2 + \ell_3)^2 + (m_1 + n_2 + m_3)^2 + (n_1 + n_2 + n_3)^2$

$= (\Sigma\ell_1^2 + \Sigma\ell_2^2 + \Sigma\ell_1\ell_3^2 + 2\Sigma\ell_1\ell_2 + 2\Sigma\ell_2\ell_3 + 2\Sigma\ell_3\ell_1) = 3$

$\Rightarrow (\ell_1 + \ell_2 + \ell_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + m_3)^2 = 3$

Hence direction cosines of OP are

$$\left(\frac{\ell_1 + \ell_2 + \ell_3}{\sqrt{3}}, \frac{m_1 + m_2 + m_3}{\sqrt{3}}, \frac{n_1 + n_2 + n_3}{\sqrt{3}}\right)$$

Sol.46 C

Equation of lines :

$$\frac{x-2}{3-2} = \frac{y+3}{-4+3} = \frac{z-1}{-5-1}$$

$$\frac{x-2}{1} = \frac{y+3}{-1} = \frac{z-1}{-6} = \frac{z-1}{-6} = \lambda$$

Points $(\lambda + 2, -\lambda - 3, -6\lambda + 1)$

Point will be on given plane

$2(\lambda + 2) + (-\lambda - 3) + (-6\lambda + 1) = 7$

$\Rightarrow \lambda = -1$

Intersection point (1, –2, 7)

Sol.47 A

Direction ratio's of line = (–2, 1, 2)

Direction cosine's = $\left(\frac{-2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

$\cos\theta = \frac{-2}{3}$, $\cos\theta_2 = \frac{1}{3}$; $\cos\theta_3 = \frac{2}{3}$

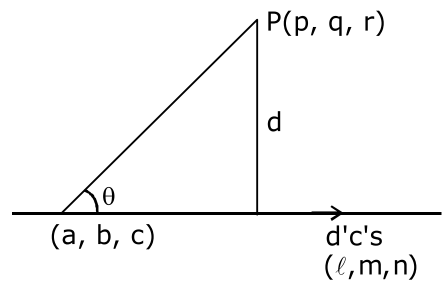
$\cos2\theta_1 + \cos2\theta_2 + \cos2\theta_3$

$= 2\,[\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3] - 3$

$= 2\left[\frac{4}{3} + \frac{1}{3} + \frac{4}{3}\right] - 3 = -1$

Sol.48 A

$$\frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n} \quad \text{Point } (p, q, r)$$



Let $\vec{r_1} = (p - a)\,\hat{i} + (q - b)\,\hat{j} + (r - c)\,\hat{j}$

$\vec{r_2} = \ell\hat{i} + m\hat{j} + n\hat{k}$

$$\cos\theta = \frac{\vec{r_1}\,.\,\vec{r_2}}{|\vec{r_1}|\,|\vec{r_2}|}$$

also $d = |\vec{r_1}|\sin\theta$

$d^2 = |\vec{r_1}|^2\sin^2\theta$

$= |\vec{r_1}|^2(1 - \cos^2\theta)$

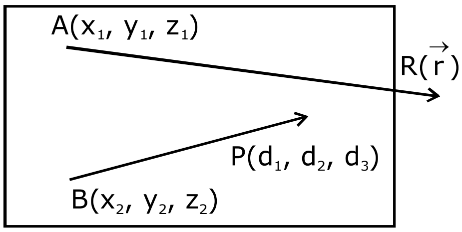
$= |\vec{r_1}|^2\left[\frac{(\vec{r_1}\,.\,\vec{r_2})^2}{|\vec{r_1}|^2|\vec{r_2}|^2}\right]$

$d^2 = |\vec{r_1}|^2 - (\vec{r_1}\cdot\vec{r_2})^2$

$= [(P - a)^2 + (q - b)^2 + (r - c)^2] - [\ell(p - a) + m(q - b) + n(r - c)]^2$

EXERCISE – II HINTS & SOLUTIONS

Sol.1 A,B



$[\vec{AR}\ \vec{AB}\ \vec{P}] = 0$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Sol.2 A,C

$x + y + z - 1 = 0$ & $4x + y - 2z + 2 = 0$
put $z = 0$
$x + y = 1$
$4x + y = -2 > x = -1, y = 2$
Point $(-1, 2, 0)$

$$\text{Direction} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{vmatrix}$$

$= \hat{i}(-2-1) - \hat{j}(-2-4) + \hat{k}(1-4)$

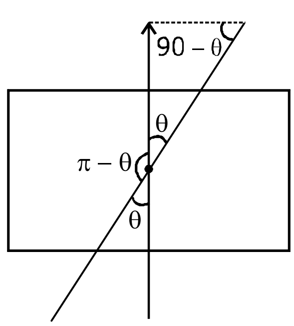
$= -3\hat{i} + 6\hat{j} - 3\hat{k} \quad = -3(1, -2, 1)$

Equation of line in symmetrical form

$$\frac{x+1}{1} = \frac{y-2}{-2} = \frac{z-0}{1}$$

(C) will also satisfy

Sol.3 C,D

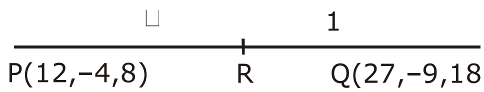


$$\text{Normal vector} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 5(1, -1, -1)$$

$$\cos(90-\theta) = \frac{2+2-1}{\sqrt{9}\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\sin\theta = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \cot\theta = \sqrt{2}$$

Sol.4 A,C



$$R\left[\frac{27\alpha+12}{\alpha+1}, \frac{-9\alpha+4}{\alpha+1}, \frac{18\alpha+8}{\alpha+1}\right]$$

Put R in given sphere

$$\left(\frac{27\alpha+12}{\alpha+1}\right)^2 + \left(\frac{-9\alpha+4}{\alpha+1}\right)^2 + \left(\frac{18\alpha+8}{\alpha+1}\right)^2 = 504$$

$\Rightarrow \alpha = 2/3$ internally
$\alpha = -2/3$ externally

Sol.5 A,D

$$\frac{x-1}{2} = \frac{y+3}{1} = \frac{z+1}{-2}$$

Direction of line $\vec{b} = (2, -1, -2)$

(A) Normal of plane $\vec{n} = (2, 2, 1)$

$\vec{b} \cdot \vec{n} = 4 - 2 - 2 = 0$

(B) $\vec{b} \cdot \vec{n} = 2 - 2 + 4 = 4$

(C) $\vec{b} \cdot \vec{n} = 4 + 2 - 2 = 4$

(D) $\vec{b} \cdot \vec{n} = 2 + 2 - 4 = 0$

Sol.6 A,B,C

$G(3, 2, 1)$

$\overrightarrow{OG} = (3, 2, 1)$

$\overrightarrow{BF} = (3, -2, 1)$

$$\cos\alpha = \frac{(3,2,1).(3,-2,1)}{\sqrt{14}\ \sqrt{14}}$$

$$\cos\alpha = \frac{3}{7} \Rightarrow \alpha = \cos^{-1}\frac{3}{7}$$

Similarly ratate the length 2 get all angle.

Sol.7 C

x = y = −z

DR'S (1,1,−1) = 0

$\Rightarrow$ (2, 3, 5). (1, 1, −1) = 0

2 (1, 2, 3). (1, 1, −1) = 0

Sol.8 B,D

$$DC'S = \frac{(\ell_1+\ell_2)\hat{i}+(m_1+m_2)\hat{j}+(n_1+n_2)\hat{k}}{\sqrt{(\ell_1+\ell_2)^2+(m_1+m_2)^2+(n_1+n_2)^2}}$$

$$|\vec{n}| = \sqrt{(\ell_1+\ell_2)^2+(m_1+m_2)^2+(n_1+n_2)^2}$$

$$= \sqrt{2+2\ell_1\ell_2+m_1m_2+n_1n_2}$$

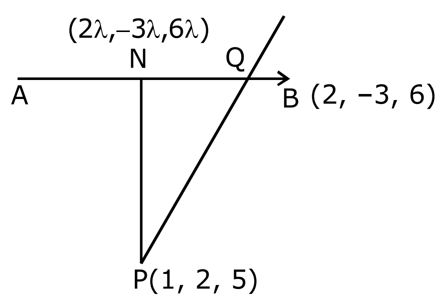
$\cos\theta = \ell_1\ell_2+m_1m_2+n_1n_2$

$|\vec{n}| = \sqrt{2+2\cos\theta} = 2\cos\frac{\theta}{2}$

angle is $\pi - \theta$

$|\vec{n}| = \sqrt{2+2\cos\theta} = 2\sin\frac{\theta}{2}$

Sol.9 A,B,C,D



PN = $(2\lambda - 1, -3\lambda + 2, 6\lambda - 5)$

$\overrightarrow{PN}$. (2, −3, 6) = 0

$2(2\lambda - 1) + 3(3\lambda + 2) + 6(6\lambda - 5) = 0$

$\lambda = \frac{26}{49} \quad \Rightarrow \quad N\left(\frac{52}{49}, \frac{-79}{49}, \frac{156}{49}\right)$

Equation of PN

$$\frac{x-1}{2\lambda-1} = \frac{y-2}{-3\lambda-2} = \frac{z-5}{6\lambda-5}$$

Put $\lambda = \frac{26}{49}$

$$\frac{x-1}{3} = \frac{y-2}{-176} = \frac{z-5}{-89}$$

Let a point $Q(2\mu, -3\mu, 6\mu)$

PQ will be $\perp^n$ to normal vector of given plane.

$\{(2\mu-1), (-3\mu-2), (6\mu-s), (3, 4, 5)\} = 0$

$3(2\mu - 1) + 4(-3\mu - 2) + 5(6\mu - 5) = 0$

$\Rightarrow \mu = \frac{3}{2}$

$Q\left(3, \frac{-9}{2}, 9\right)$

Equaton of PR

$$\frac{x-1}{2\mu-1} = \frac{y-2}{-3\mu-2} = \frac{z-5}{6\mu-5}$$

Put $\mu = \frac{3}{2}$

$$\frac{x-1}{4} = \frac{y-2}{-13} = \frac{z-5}{8}$$

Sol.10 A,B

2x − 3y − 7z = 0

3x − 14y − 13z = 0

8x − 31y − 33z = 0

Above three planes are passing thorugh origin.

and passes through common line.

Sol.11 B,C

$\hat{n} = \pm\left(\frac{-3,2,6}{7}\right) = \pm\left(\frac{-3}{7}, \frac{2}{7}, \frac{6}{7}\right)$

$-\frac{3x}{7} + \frac{2y}{7} + \frac{6z}{7} = 7$

− 3x + 2y + 6z − 49 = 0

and $\frac{3x}{7} - \frac{2y}{7} - \frac{6z}{7} = 7$

3x − 2y − 6z − 49 = 0

Sol.12 B,C

Let a point Q $(3\lambda + 15, 8\lambda + 2, -5\lambda + 6)$

PQ = $(2\lambda + 10, 8\lambda - 5, -5\lambda + 3)$

$3(3\lambda + 10) + 8(8\lambda - 5) - 5(-5\lambda + 3) = 0$

$9\lambda + 30 + 64\lambda - 40 + 25\lambda - 15 = 0$

$98\lambda = 35$

$\lambda = \frac{35}{98} \quad \Rightarrow$ PQ = 14 (B)

and plane equation 9x − 4y − 14 = 0

EXERCISE – III HINTS & SOLUTIONS

Sol.1 A (0, 7, 10) ; B(−1, 6, 6) ; C (−4, 9, 6)

$$\left.\begin{aligned} AB &= \sqrt{1+1+16} = \sqrt{18} \\ AC &= \sqrt{16+4+16} = 9 \\ BC &= \sqrt{9+9+10} = \sqrt{18} \end{aligned}\right\} AB = BC \text{ so isoceles } \Delta.$$

Sol.2

$$G = \left(\frac{0+0+1+1}{4}, \frac{0+1+0+1}{4}, \frac{0+1+1+0}{4}\right)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

Sol.3 Point equidistant from the points is center of tetrahedron.

Sol.4

α 1

(3,5,−7)   P   (−2,1,8)

$$P\left[\frac{-2\alpha+3}{\alpha+1}, \frac{\alpha+5}{\alpha+1}, \frac{8\alpha-7}{\alpha+1}\right]$$

$$\frac{-2\alpha+3}{\alpha+1} = 0 \Rightarrow \alpha = 3/2$$

$$P\left(0, \frac{13}{5}, 2\right)$$

Sol.5 QP = (4, −4, −2) = 2 (2, −2, −1)

So direction Ratio of line = (2, −2, −1)

direction cosine = $\left(\frac{2}{3}, \frac{-2}{3}, \frac{-1}{3}\right)$

Sol.6 $\ell + m + n = 0$ & $\ell^2 + m^2 = n^2$ (given)

$\ell = (m + n)$ put in IInd relatiin

(i) $m = 0$

$\Rightarrow \ell + n = 0$

$\Rightarrow \frac{m}{0} = \frac{\ell}{1} = -\frac{n}{1}$

$= \frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{O^2+1^2+\ell^2}} = \frac{1}{\sqrt{2}}$

(ii) $m + n = 0$

$\Rightarrow \frac{m}{1} = \frac{n}{-1} = \frac{\ell}{0}$

$\Rightarrow \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{\ell^2+1^2+0}} = \frac{1}{\sqrt{2}}$

So $(\ell_1, m_1, n_1) = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$;

$(\ell_1, m_2, n_2) = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$; $\theta = 60^o$

Sol.7 Equation of line joining A & B

$$\frac{x+9}{-20} = \frac{y-4}{4} = \frac{z-5}{6} = \ell \text{ (Let)}$$

Let a point C oin the line is

$(-20\lambda - 9, 4\lambda + 4, 6\lambda + 5)$

Now Co (where O is origin)

$\overrightarrow{CO} = (-20\lambda - 9, 4\lambda + 4, 6\lambda + 5)$

& $\overrightarrow{CO} \cdot \overrightarrow{AB} = 0$ (Q $\overrightarrow{CO}$ is $\perp$r to line)

$\Rightarrow 400\lambda + 180 + 16\lambda + 16 + 36\lambda + 30 = 0$

$$\lambda = -\frac{1}{2}$$

So point C = (1, 2, 2)

which is also the mid point of A & B.

Sol.8 $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$;

where $\beta = 120^o$, $\gamma = 135^o$

So $(\ell, m, n) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$, $= \vec{d}$ (say)

Projected $\overrightarrow{PQ}$ on $\vec{d} = \frac{\overrightarrow{PQ} \cdot \vec{d}}{|\vec{d}|} = 2 - 2\sqrt{3}$

Sol.9 Let the equation of plane is

Ax + By + Cz + 1 = 0

using points (1, 0, 0) & (0, 1, 0)

A = −1 & B = −1

& angle ; $\frac{1}{\sqrt{2}} = \frac{A.(1)+B.(1)}{\sqrt{1^2+1^2+C^2}.\sqrt{2}}$

$\Rightarrow C = \pm\sqrt{2}$

Sol.10 Let $\vec{a}$ (1,1,1); $\vec{b}$ (1,–1,1)& $\vec{c}$ (–7, –3, –5)

normal of the plane

$\vec{n}_1 = (\vec{b} - \vec{a}) \times (\vec{b} - \vec{c})$

& $\vec{n}_2 = (0, 1, 0)$

angle = $\frac{\pi}{2}$

Sol.11 Equation of L_1 : $\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}$

& of L_2 : $\frac{x-3}{1} = \frac{y+2}{-4} = \frac{z}{5}$ ($\because$ || lines)

Equation of any plane through L_1

a(x – 4) + b (y – 3) + C (z – 2) = 0 ...(i)

where a – 4b + 5c = 0 ...(ii)

also (3, –2, 0) lie oin plane (i)

using (ii) & (iii)

a + 5b + 2c = 0 ...(iii)

$\frac{a}{11} = \frac{b}{-1} = \frac{c}{-3}$

So Equation of plane 11x – y – 3z = 35

Sol.12 Line and plane are parallel.

So image of (1, 2, –3) about the plane

3x – 3y + 10z = 26

is (4, –1/7)

So equation of line is

$\frac{x-4}{9} = \frac{y+1}{-1} = \frac{z-7}{-3}$

Sol.13 $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \alpha$

p (2α + 1, 3α + 2, 4α + 3)

$2\ \frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1} = \mu$

(5μ + 4, 2μ + 1, μ)

2α + 1 = 5μ = 4

3α + 2 = 2μ = 1

4α + 3 = μ

a = –1

P(–1, –1, –1)

Similarly

PoI of other two lines

Q(4, 0, –1)

PQ = $\sqrt{26}$

Sol.14 Certre $\left(0, \frac{1}{2}, 1\right)$

Diameter= $\sqrt{(2+2)^2 + (1-1-2)^2 + (4+2)^2} = \sqrt{61}$

$\Rightarrow \alpha = \frac{\sqrt{61}}{2}$

Eq^n of sphere

$(x - 0)^2 + \left(y - \frac{1}{2}\right)^2 + (z - 1)^2 = \frac{61}{4}$

$\Rightarrow x^2 + y^2 + z^2 - y - 2z - 14 = 0$

Sol.15 π_1 : 2x + 3y – z + 1 = 0 ; $\vec{n}_1$ = (2, 3, –1)

π_2 : x + y – 2z + 3 = 0 ; $\vec{n}_2$ = (1, 1, –2)

Let the equation of the required polane :

$\pi = \pi_1 + \lambda\pi_2$ ...(i)

& normal of 5 plae is $(2+\lambda, 3+\lambda, -1-2\lambda) = \vec{n}$

also fnπ_3 : 3x – y – 2z = 4 ; $\vec{n}_3$ = (3, –1, –2)

& $\vec{n} \cdot \vec{n}_3 = 0 \Rightarrow \lambda = -\frac{5}{6}$

Put in (i) plane is 7x + 13y + 4z – 9 = 0

Sol.16 Line of intersection of planes

2x + y = 0 & x – y + z = 0

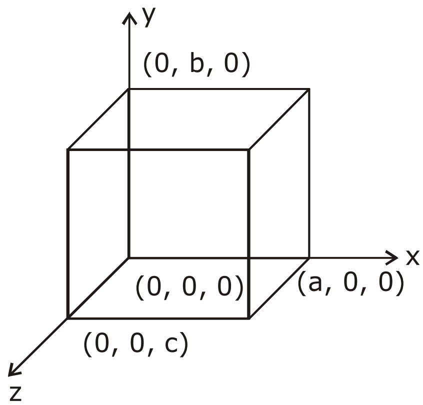
Sol.17 Point of intersection of line & plane

= (2, –1, 2)

using distance formula

$\alpha = -1, \frac{80}{63}$

Sol.18 Angle between diagonals



$$= \cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$$

Sol.19 $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

lines will be coplanar so

$$\begin{vmatrix} a & b & c \\ 2 & 1 & 1 \\ 3 & 3 & 0 \end{vmatrix} = 0 \Rightarrow a + b + c$$

$$\cos 60° = \left|\frac{2a + b + c}{a^2 + b^2 + c^2\sqrt{6}}\right|$$

$2b^2 + 2c^2 + 5bc = 0$

$(b + 2c)(2b + c) = 0$

$b = -2c$ or $b = -c/2$

$a = -c$ or $a = c/2$

$$\frac{x}{-c} = \frac{y}{-2c} = \frac{z}{c} \text{ or } \frac{x}{c/2} = \frac{y}{-c/2} = \frac{z}{c}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} \quad \text{or} \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$$

Sol.20 Equation of plane through given line's

$(3x - y + 2z - 1) + \lambda(x + 2y - z - 2) = 0$ ...(i)

This is perpoendicular to

$3x + 2y + z = 0$ ...(ii)

So $\lambda = -\frac{2}{3}$

Putting this in (i) ;

$3x - 8y + 7z + 4 = 0$ ...(iii)

using (ii) & (iii) equation of line.

Sol.21 $\cos\theta = \left|\frac{\ell m + mn + n\ell}{\ell^2 + m^2 + n^2}\right|$

using the given equation

$$\theta = \cos^{-1}\left(\frac{4}{9}\right)$$

Sol.22 M(1, 0, 5) & N (–2, 0, 4)

Equation of MN

$$\frac{x-1}{3} = \frac{y-0}{0} = \frac{z-5}{1}$$

angle between line & plane is

$$\sin\theta = \frac{(3, 0, 1).(1, 1, 1)}{\sqrt{3^2 + 1^2}.\sqrt{1^2 + 1^2 + 1^2}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{4}{\sqrt{30}}\right)$$

Sol.23 $x = 0;\ \frac{y}{b} + \frac{z}{c} = 1$

is a line in (y – z) plane with y intercept 'b' & zintercept 'c'.

$y = 0;\ \frac{x}{a} - \frac{z}{c} = 1$

is a line in (x – z) plane with x intercept 'a' & z intercept '–c'.

So using distancer between two skew lines

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Sol.24 Now plane passing through origin

Normal of plane = $\vec{a} \times \vec{b}$

$= (1, 2, -3) \times (2, -3, 1)$

$= -7\hat{i} - 7\hat{j} - 7\hat{k}$

$= -7(\hat{i} + \hat{j} + \hat{k})$

so Eq^n of plane is

$x + y + z = 0$

EXERCISE – IV HINTS & SOLUTIONS

Sol.1 Let a Point $(\lambda - 2, 2\lambda + 3, 3\lambda + k)$

In y – z plane $x = 0 \Rightarrow \lambda = 2$

$A(0, 7, 6 + k)$

In x – y lane $z = 0 \Rightarrow \lambda = -k/3$

$$B\left(\frac{-k}{3} - 2, \frac{-2k}{3} + 3, 0\right)$$

$$\overline{A}.\overline{B} = 0 \Rightarrow \left(\frac{-2k}{3} + 3\right) = 0$$

$$k = \frac{9}{2}$$

Sol.2 $\overrightarrow{PQ} = (-1, -2, -1)$

$\overrightarrow{PR} = (1, -5, -1)$

$\overrightarrow{PS} = (5, -2, 2)$

$$\text{volume} = \frac{1}{6}[\overrightarrow{PQ} \quad \overrightarrow{PR} \quad \overrightarrow{PS}] = \frac{1}{2}$$

Sol.3 **(i)**

$(x - 1)^2 + (y - 3)^2 + (z + 6)^2 + (x - 2)^2 + (y - 4)^2 + (z - 2)^2 = 72$

$\Rightarrow x^2 + y^2 + z^2 - 3x - 7y + 4z - 1 = 0$

$$\text{Center}\left(\frac{3}{2}, \frac{7}{2}, -2\right)$$

(ii) $$r = \sqrt{\left(\frac{3}{2} - 1\right)^2 + \left(\frac{7}{2} - 3\right)^2 + (2 - 6)^2}$$

$$= \sqrt{\frac{33}{2}}$$

(iii) plane: $2x + 2y - z + 3 = 0$

$$d = \left|\frac{2(3/2) + 2(7/2) + 2 + 3}{3}\right| = 5$$

Sol.5 $M(1, 2, 0)$

$\overrightarrow{OP} = (1, 2, 3)$

$\overrightarrow{OM} = (1, 2, 0)$

$$\cos\theta = \frac{3}{\sqrt{14}} \Rightarrow \theta = \cos^{-1}\frac{3}{\sqrt{14}}$$

$$\cos\phi = \frac{1}{\sqrt{5}} \Rightarrow \phi = \cos^{-1}\frac{1}{\sqrt{5}}$$

Sol.6 $$\frac{x}{1} = \frac{y}{1} = \frac{z - 1}{-2}$$

$\vec{a}$ = DR'S = (1, 1, –2) Fixed point P(0, 0, 1)

$x + y + z = 1$; $\vec{n} = (1, 1, 1)$

$\vec{n} \cdot \vec{a} = 1 + 1 - 2 = 0$

& point P satisfy the plane

$\Rightarrow$ line lies is the plane.

Let the line $\frac{x}{a} = \frac{y}{b} = \frac{z - 1}{c}$

$$\cos\theta = \frac{1}{\sqrt{6}} = \left|\frac{a + b - 2c}{\sqrt{a^2 + b^2 + c^2}\ \sqrt{6}}\right|$$

squareing

$3c^2 + 2ab = 4c(a + b)$ ....(1)

let the point is plane (1, 0, 0)

$\Rightarrow$ condition of copalanarity

$$\begin{vmatrix} 1 & 0 & -1 \\ a & b & c \\ 1 & 1 & -2 \end{vmatrix} = 0 \quad \text{........(2)}$$

Solve (1) & (2) and get (a, b, c)

Sol.7 $$\frac{x - 1}{a} = \frac{y - 2}{b} = \frac{z - 3}{c}$$

$$\frac{x + 1}{2} = \frac{y - 2}{1} = \frac{z + 4}{2}$$

Lines are coplaner.

$$\begin{vmatrix} a & b & c \\ 2 & 1 & 2 \\ 2 & 0 & 7 \end{vmatrix} = 0 \Rightarrow 7a - 10b - 2c = 0 \;......(1)$$

and a + 5b + 4c = 0 ....(2)

from (1) & (2)

$a = k,\ b = k,\ c = -\frac{3}{2}k$

$$\frac{x-1}{k} = \frac{y-1}{k} = \frac{z-3}{-\frac{3}{2}k}$$

$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-3}{-3}$$

Sol.8 $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

Lines will be coplaner so

$$\begin{vmatrix} a & b & c \\ 2 & 1 & 1 \\ 3 & 3 & 0 \end{vmatrix} = 0 \Rightarrow a = b + c$$

$$\cos 60° = \left|\frac{2a+b+c}{\sqrt{a^2+b^2+c^2}\sqrt{6}}\right|$$

$\Rightarrow 2b^2 + 2c^2 + 5bc = 0$

$\Rightarrow (b + 2c)(2b + c) = 0$

b = –2c or b = –c/2

a = –c a = c/2

$$\frac{x}{-c} = \frac{y}{-2c} = \frac{z}{c} \quad \text{or} \quad \frac{x}{c/2} = \frac{y}{-c/2} = \frac{z}{c}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} \quad \text{or} \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$$

Sol.9 $\frac{x+2}{3} = \frac{y+3}{2} = \frac{z+4/3}{5/3} = \lambda$

$$Q\left(3\lambda - 2,\ 2\lambda - 3,\ \frac{5\lambda - 4}{3}\right)$$

$$\overrightarrow{PQ} = \left(3\lambda,\ 2\lambda - \frac{9}{2},\ \frac{5\lambda+8}{3}\right)$$

$\vec{n} = (4, 12, -3)$

$\overrightarrow{PQ} \cdot \vec{n} = 0 \Rightarrow \lambda = 2$

$$\overrightarrow{PQ} = \left(6, -\frac{1}{2}, 6\right)$$

$$\text{distance} = |\overrightarrow{PQ}| = \frac{17}{2}$$

Sol.10 Direction of line

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -3 \\ 3 & -4 & 2 \end{vmatrix} = -6\hat{i} - 13\hat{j} - 17\hat{k}$$

$$\text{line : } \frac{x-1}{-6} = \frac{y+2}{-13} = \frac{z+3}{-17}$$

$$\text{or } \frac{x-1}{6} = \frac{y+2}{13} = \frac{z+3}{17}$$

Sol.11 $\frac{x-4}{a} = \frac{y+14}{b} = \frac{z-4}{c}$

direction of intersecting line = $\vec{n}_1 \times \vec{n}_2$ = (–6, 5, –8)

Put z = 0 in both the planes

3x + 2y = 5

x – 2y = –1 x = 1, y = 1

P (1, 1, 0)

Another line $\frac{x-1}{-6} = \frac{y-1}{5} = \frac{z-0}{-8}$

–6a + 5b – 8c = 0

both lines will be coplanar

$$\Rightarrow \begin{vmatrix} a & b & c \\ -6 & 5 & -8 \\ 3 & -15 & 4 \end{vmatrix} = 0 \qquad \Rightarrow 4a = 3c$$

If $a = k,\ c = \frac{4}{3}k,\ b = \frac{10}{3}k$

$$\frac{x-4}{3} = \frac{y+14}{10} = \frac{z-4}{4}$$

Sol.12 **(a)**

$\overrightarrow{PQ} = (0, 1, 2)$ $\overrightarrow{PR} = (1, 1, 4)$

$\overrightarrow{PQ} \times \overrightarrow{PR} = 2\hat{i} + 2\hat{j} - \hat{k}$

Area = $\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{3}{2}$

(b)

$2(x - 1) + 2(y - 0) - (z + 1) = 0$

$2x + 2y - z - 3 = 0$

$\frac{2}{3}x + \frac{2}{3}y - \frac{1}{3}z = 1$

(c)

$x = 0, z = 0$

$y = 3/2$

point $\left(0, \frac{3}{2}, 0\right)$

(d)

dir of line will be along the normal of plane

$\frac{x-2}{2} = \frac{y-1}{2} = \frac{z-3}{-1}$

Sol.13 Direction of intersection line

$= \vec{n}_1 \times \vec{n}_2$

put z = 0 in both planes

$x - 2y = 1$ $\quad$ $x = 3, y = 1$

$x + 2y = 5$

point (3, 1, 0)

line : $\frac{x-3}{2} = \frac{y-1}{3} = \frac{z-0}{4}$

variable point $(2\lambda + 3, 3\lambda + 1, 4\lambda)$

$2(2\lambda + 3) + 2(3\lambda + 1) + 4\lambda + 6 = 0$

$\Rightarrow \lambda = -1$

point (1, –2, –4)

Sol.14 A(2, 0, 0) ; B(0, 3, 0) ; C(0, 0, –5)

normal of plane = $\overrightarrow{AB} \times \overrightarrow{AC}$

$= (-15, -10, 6)$

Equation of plane

$-15(x - 2) - 10(y - 0) + 6(z - 0) = 0$

$\frac{x}{2} + \frac{y}{3} + \frac{z}{(-5)} = 1$

Area = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$= \frac{19}{2}$

Sol.15 direction of line = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{vmatrix}$

$= -2\hat{i} + 4\hat{j} - 2\hat{k}$

equation of line

$\frac{x-2}{-2} = \frac{y+1}{4} = \frac{z-3}{-2}$

$\frac{z-2}{1} = \frac{y+1}{-2} = \frac{z-3}{1}$

Sol.16 Normal of plane = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 5 \\ 1 & -1 & 1 \end{vmatrix}$

$= 2\hat{i} + 3\hat{j} + \hat{k}$

equation of plane

$2(x - 1) + 3(y - 2) + 1(z - 0) = 0$

$2x + 3y + z + 4 = 0$

Sol.17 coplanar $\Rightarrow \begin{vmatrix} -3 & 2 & 1 \\ 1 & -3 & 2 \\ 1 & p-7 & 5 \end{vmatrix} = 0 \Rightarrow p = 1$

$\frac{x-1}{-3} = \frac{y-1}{2} = \frac{z+2}{1} = \lambda$

& $\dfrac{x}{1} = \dfrac{y-7}{-3} = \dfrac{z+7}{2} = \mu$

$-3\lambda + 1 = \mu$ ....(1)

$2\lambda + 1 = -3\mu + 7$ ....(2)

$\lambda - 2 = 2\mu - 7$ ....(3)

$\lambda = -3/7$ & $\mu = 16/7$

Point of intersection $\left(\dfrac{16}{7}, \dfrac{1}{7}, \dfrac{-17}{7}\right)$

Normal of plane = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}$ $= (7, 7, 7)$ $= (1, 1, 1)$

Equation of plane $(x-1) + (y-1) + (z+2) = 0$

$x + y + z = 0$

Sol.18 $\vec{n}_1 = (1, -2, 3)$; $\vec{n}_2 = (2, 3, -4)$

Directionof line = $\vec{n}_1 \times (\vec{n}_1 \times \vec{n}_2)$

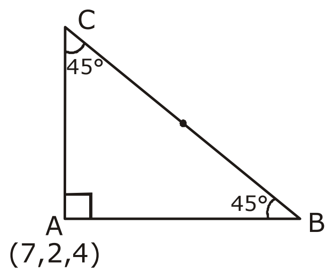
$= (-44, -10, 8)$

$\dfrac{x-7}{-44} = \dfrac{y-2}{-10} = \dfrac{z+1}{8}$

or $\dfrac{x-7}{22} = \dfrac{y-2}{5} = \dfrac{z+1}{-4}$

Sol.19 Let the DR's of AB = (a, b, c)

$\cos 45° = \left| \dfrac{5a + 3b + 8c}{\sqrt{a^2 + b^2 + c^2}\,\sqrt{96}} \right|$



$48(a^2 + b^2 + c^2) = (5a + 3b + 8c)^2$ .....(1)

condition of coplanarity

$\begin{vmatrix} 7+6 & 2+10 & 4+14 \\ a & b & c \\ 5 & 3 & 8 \end{vmatrix} = 0$

$\begin{vmatrix} 13 & 12 & 16 \\ a & b & c \\ 5 & 3 & 8 \end{vmatrix} = 0$ ....(2)

Solve (1) & (2) & get a, b, c

Sol.20 Line & plane are $\perp^r$ to each other

image of (1, 2, –3) in the plane is foot of $\perp^r$

(α, β, γ)

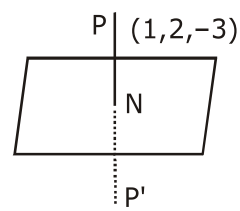
$\dfrac{\alpha-1}{3} = \dfrac{\beta-2}{-3} = \dfrac{\gamma+3}{10} = \lambda$

$N(3\lambda + 1, -3\lambda + 2, 10\lambda - 3)$

$\Rightarrow 3(3\lambda + 1) - 3(-3\lambda + 2) + 10(10\lambda - 3) = 26$

$\Rightarrow \lambda = 1/2$

$N\left(\dfrac{5}{2}, \dfrac{1}{2}, 2\right)$



$\dfrac{P + P'}{2} = N \Rightarrow P' = 2N - P$

$\Rightarrow P' \equiv (4, -1, 7)$

equation of line

$\dfrac{x-4}{9} = \dfrac{y+1}{-1} = \dfrac{z-7}{-3}$

Sol.21 Normal of plane = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ 2 & 5 & 4 \end{vmatrix} = 2\hat{i} - 4\hat{j} + 4\hat{k}$

plare will passes through (1, 0, 0)

$\Rightarrow 1(x-1) - 2y + 2z = 0$

$x - 2y + 2z = 1$

EXERCISE – V HINTS & SOLUTIONS

Sol.1 **(i)** Let the equation of plane be

$ax + by + cz + d = 0$ ....(1)

(1) passes through (2,1,0),(5,0,1)&(4,2,1)

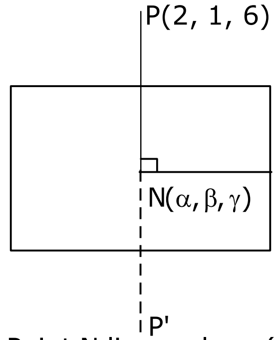
$\Rightarrow a = \frac{-d}{3}$; $b = -\frac{d}{3}$; $c = \frac{2}{3}d$

$\Rightarrow x + y - 2z - 3 = 0$ ....(2)

(ii) P (2, 1, 6)

$$\frac{\alpha-2}{1} = \frac{\beta-1}{1} = \frac{\gamma-6}{-2} = \lambda$$

$\alpha = \lambda + 2$; $\beta = \lambda + 1$, $\gamma = -2\lambda + 6$



Point N lie on plane (2)

$(\lambda + 2) + (\lambda + 1) - 2(-2\lambda + 6) - 3 = 0$

$\Rightarrow \lambda = 2$

N (4, 3, 2)

$2N = P + P^1 \Rightarrow P^1 = 2N - P$

$= (8, 6, 4) - (2, 1, 6)$

$= (6, 5, -2)$

Sol.2 **B**

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = \lambda \quad ....(1)$$

$$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = H \quad ....(2)$$

General point on (1) is $(2\lambda + 1, 3\lambda - 1, 4\lambda + 1)$

and on (2) is $(\mu + 3, 2\mu + k, \mu)$

so $2\lambda + 1 = \mu + 3$

$3\lambda - 1 = 2\mu + k$

$4\lambda + 1 = \mu$

So after solving we get $k = \frac{9}{2}$

Sol.3 Direction of plane $= \vec{L_1} \times \vec{L_2}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$\vec{n} = (1, 1, 1)$

Equation of plane

$x + y + z = d$ passes through (1, 1, 1)

$d = 3$

$x + y + z = 3$

A (3, 0, 0) ; B(0, 3, 0), C (0, 0, 3)

$$\text{Volumeof OABC} = \frac{1}{6}\begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \frac{27}{6} = \frac{9}{2}$$

cubic units.

Sol.4 **D**

(a) Let $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ be the variable plane

so

$$\left|\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right| = 1$$

a (a, 0, 0) B(a, b, 0) C(0, 0, c)

Centroid G of ΔABC is G $\left(\frac{a}{c}, \frac{b}{3}, \frac{c}{3}\right)$

$x = \frac{a}{3}$; $y = \frac{b}{3}$, $z = \frac{c}{3}$

& $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1$

$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9$

So k = 9

(b) Reqd. plane $\pi, + \lambda\pi_2 = 0$

$2x - y + z - 3 + \lambda(3x + y + z - 5) = 0$

$(3\lambda + 2)x + (\lambda - 1)y + (\lambda + 1)z$

$- 5\lambda - 3 = 0$ ...(1)

Distance of plane (1) from point

(2, 1, −1) is $\frac{1}{\sqrt{6}}$

$$\Rightarrow \left|\frac{6\lambda + 2 + \lambda - 1 - \lambda - 1 - 5\lambda - 3}{\sqrt{(3\lambda+2)^2 + (\lambda-1)^2 + (\lambda+1)^2}}\right| = \frac{1}{\sqrt{6}}$$

$\Rightarrow\ 6(\lambda-1)^2 = 11\lambda^2 + 12\lambda + 6$

$\Rightarrow \lambda = 0, -\dfrac{2y}{5}$

The planes are

$2x - y + z - 3 = 0$

and $62x + 29y + 19z - 109 = 0$

Sol.5 **(a)** $\vec{n}_1 = (2, -2, 1)\ \vec{n}_2 = (1, -1, 2)$

Normal vector of $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix}$

$= -3\hat{i} - 3\hat{j} - 0\hat{k}$

$\vec{n} = (-3, -3, 0)$

So plane will be

$-3x - 3y = k$

passes through $(1, -2, 1) \Rightarrow k = 3$

$-3x - 3y = 3$

$x + y + 1 = 0$

$d = \left|\dfrac{1+2+1}{\sqrt{2}}\right| = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$

(b) **(A)** Solving the two equations

$x = \dfrac{|a|+1}{a+1} > 0$ and $y = \dfrac{|a|-1}{a+1} > 0$

when $a + 1 > 0$ we get $a > 1$

$\Rightarrow a_0 = 1$ (S)

(B) $\vec{a} = (\alpha, \beta, \gamma) \Rightarrow \vec{a}.\hat{k} = \gamma$

$\hat{k} \times (\hat{k} \times \vec{a}) = (\hat{k}.\vec{a})\hat{k} - (\hat{k} - \hat{k})\vec{a}$

$= \gamma\hat{k} - (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$

$\Rightarrow \alpha\hat{i} + \beta\hat{j} = \overline{O} \Rightarrow \alpha = 0, \beta = 0$

$\alpha + \beta + \gamma = 2 \Rightarrow \gamma = 2$ (P)

(C) $\left|\int_0^1 (1-y^2)dy\right| + \left|\int_0^1 (y^2-1)dy\right|$

$= 2\int_0^1 (1-y^2)dy = \left|\dfrac{4}{3}\right|$

$\left|\int_0^1 \sqrt{1-x}\,dx\right| + \left|\int_{-1}^0 \sqrt{1-x}\,dx\right| = 2\int_0^1 \sqrt{1-x}\,dx$

$= 2\int_0^1 \sqrt{x}\,dx = \dfrac{4}{3}$ (Q)

(D) $\sin A \sin B \sin C + \cos A \cos B$

$\le \sin A \sin B + \cos A \cos B$

$\le \cos(A - B)$

$\cos(A - B) \ge 1$

$\Rightarrow \cos(A - B) = 1 \Rightarrow \sin C = 1$

(C) **(A)** $t = \sum_{i=1}^{\infty} \tan^{-1}\left(\dfrac{1}{2i^2}\right)$

$= \sum_{i=1}^{\infty} \tan^{-1}\left(\dfrac{2}{4i^2 - 1 + 1}\right)$

$= \sum_{i=1}^{\infty} [\tan^{-1}(2i+1) - \tan^{-1}(2i-1)]$

$= [(\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) + \ldots$

$\ldots + \tan^{-1}(2n+1) - \tan^{-1}(2n-1) \ldots \infty]$

$t = \tan^{-1}(2n+1) - \tan^{-1}1$

$t = \lim_{n\to\infty} \tan^{-1}\dfrac{2n}{1+(2n+1)}$

$\tan t = \lim_{n\to\infty} \dfrac{n}{n+1} = 1$ (Q)

(B) We have

$\cos\theta_1 = \dfrac{1 - \tan^2\frac{\theta_1}{2}}{1 + \tan^2\frac{\theta_1}{2}} = \dfrac{a}{b+c}$

$\Rightarrow \tan^2\dfrac{\theta_1}{2} = \dfrac{b+c-a}{b+c+a}$

Also $\cos\theta_3 = \dfrac{1 - \tan^2\frac{\theta_3}{2}}{1 + \tan^2\frac{\theta_3}{2}} = \dfrac{c}{a+b}$

$\Rightarrow \tan^2\dfrac{\theta_3}{2} = \dfrac{a+b-c}{a+b+c}$

$\tan^2\dfrac{\theta_1}{2} + \tan^2\dfrac{\theta_3}{2} = \dfrac{2b}{3b} = \dfrac{2}{3}$ (S)

(C) Line through (0, 1, 0) and $\perp^n$ to plane $x + 2y + 2z = 0$

is $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-0}{2} = \lambda$

Let $P(\lambda, 2\lambda + 1, 2\lambda)$ be the foot of $\perp^n$ on the straight line then

$\lambda.1 + (2\lambda + 1)\,2 + 2(2\lambda) = 0$

$\Rightarrow\ k = -\frac{2}{9}$

$P\left(-\frac{2}{9}, \frac{5}{9}, \frac{-4}{9}\right)$

$\perp^n$ distance $= \sqrt{\frac{4+25+16}{81}} = \frac{\sqrt{5}}{3}$ unit.

(R)

Sol.6 **(a)** $3x - 6y - 2z = 15$ & $2x + y - 2z = 5$

for z= 0 we get x = 3, y = −1

Direction vector of planes are

(3, −6, −2) & (2, 1, −2)

then the D.R.'s of line of intersection of plane is (14, 2, 15)

$\frac{x-3}{14} = \frac{y+1}{2} = \frac{z-0}{15} = \lambda$

statement-2 is correct.

(b) $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -\frac{1}{2}(a + b + c)$

$[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(A) If $a + b + c \neq 0$ and $\Sigma a^2 = \Sigma ab$

$\Rightarrow$ D = 0 and a = b = c

$\Rightarrow$ Equation represents identical planes

(B) D = 0 $\Rightarrow$ Equation will have infinite many solution

$ax + by = (a + b)z$

$bx + cy = (b + c)z$

$(b^2 - ac)y = (b^2 - ac)z$

$y = z$

$\Rightarrow\ ax + by + cy = 0$

$\Rightarrow\ ax = ay \Rightarrow x = y \Rightarrow x = y = z$

(C) $D \neq 0$

$\Rightarrow$ Planes meeting at only one point

(D) $a + b + c = 0$

$\Sigma a^2 = \Sigma ab$

$\Rightarrow\ a = b = c = 0$

Sol.7 **(a)** **D**

Given equations are

$x - y + z = 1$

$x + y - z = -1$

$x - 3y + 3z = 2$

The system of equations can be put in matrix from as

$Ax = B$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} R_3 \to R_3 + R_2$$

Which is inconsistent as $\rho(A : B) \neq \rho(A)$

$\Rightarrow$ The three planes do not have a common point.

$\Rightarrow$ Statement-2 is true.

Since, planes P_1, P_2, P_3 are pairmise intersection, then their lines of intersection are parallel.

Statement-1 is false.

(b) (i) $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$

Hence unit vector will be $= \frac{-\hat{i} - 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$

(ii) Shortest distance

$= \frac{(1+2)(-1) + (2-2)(-7) + C(1+3)5}{5\sqrt{3}}$

$= \frac{17}{5\sqrt{3}}$

(iii) Plane is given by

$-(x+1) - 7(y + 2) + 5(z + 1) = 0$

$\Rightarrow x + 7y - 5z + 10 = 0$

distance $= \left|\frac{1+7-5+10}{\sqrt{75}}\right| = \frac{13}{\sqrt{75}}$

Sol.8 **(a)** **A**

Any point Q on the line

$Q \equiv \{(1 - 34), (\mu - 1), (5\mu + 2)\}$

$\vec{PQ} = \{-3\mu - 2, \mu - 3, 5\mu - 4\}$

Now $1(-3\mu - 2) - 4(\mu - 3) + 3(5\mu - 4) = 0$

$\Rightarrow \mu = \frac{1}{4}$

(b) **C**

D.C. of the line are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Equation of line

$\vec{r} = (2, -1, 2) + \lambda\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

where λ is the distance.

variable point on lie is

$\left(2 + \frac{\lambda}{\sqrt{3}}, \frac{-1+\lambda}{\sqrt{3}}, \frac{2+\lambda}{\sqrt{3}}\right)$

Which lies on the plane $2x + y + z = 9$

$\Rightarrow \lambda = \sqrt{3}$ **[C]**

(C)

$3x - y - z = 0$

$-3x + z = 0$

$\Rightarrow y = 0$ and $z = 3x$

$-3x + 2y + z = 0$

$\Rightarrow x^2 + y^2 + z^2 = x^2 + z^2$

$= 9x^2 + x^2 \; 10x^2 \le 100$

$\Rightarrow x^2 \le 10 \quad \Rightarrow x = 0, \pm 1, \pm 2, \pm 3$ **[7]**

Sol.9 **C**

Plane 1 : $ax + by + cz = 0$

containing line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$

$2a + 3b + 4c = 0$ ....(i)

Plane 2 : $a^1x + b^1y + c^1z = 0$ is $\perp^n$ to plane containg lines

$\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$

$3a' + 4b' + 2c' = 0$

and $4a' + 2b' + 3c' = 0$

$\frac{a'}{12-4} = \frac{b'}{8-9} = \frac{c'}{6-16}$

$\Rightarrow 8a - b - 10c = 0$ ....(ii)

Equation of plane 1 : $x - 2y + z = 0$ **[C]**

Sol.10 $2\ell + 3m + 4n = 0$

$3\ell + 4m + 5n = 0$

$\frac{\ell}{-1} = \frac{m}{2} = \frac{n}{-1}$

Equation of plane will be

$a(x - 1) + b(y - 2) + c(z - 3) = 0$

$-1(x - 1) + 2(y - 2) - 1(z - 3) = 0$

$-x + 2y - z = 0$

$x - 2y + z = 0$

$\frac{|d|}{\sqrt{6}} = \sqrt{6} \quad \Rightarrow |d| = 6$

Sol.11 **A**

Distance of point P(1, −2, 1) from plane $x + 2y - 2z = d$ is $5 \Rightarrow \alpha = 10$

Equation of PQ $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2} = t$

$Q \equiv (t + 1, 2t - 2, -2t + 1)$

$PQ = 5 \Rightarrow t = \frac{5+\alpha}{9} = \frac{5}{3}$

$\Rightarrow Q \equiv \left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$

Sol.12 **(A)** Let the line

$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ intersect the lines

$\Rightarrow a + 3b + 5c = 0$

and $3a + b - 5c = 0$

$\Rightarrow a : b : c :: 5t : -5t : 2t$

on solving with given lines we get points of intersection $P \equiv (5, -5, 2)$ and

$Q \equiv \left(\frac{10}{3}, \frac{-10}{3}, \frac{8}{3}\right)$

$PQ^2 = d^2 = 6$

(B) $\tan^{-1}(x + 3) - \tan^{-1}(x - 3) = \sin^{-1}\left(\frac{3}{5}\right)$

$\tan^{-1}\left[\frac{(x+3)-(x-3)}{1+(x^2-9)}\right] = \tan^{-1}\frac{3}{4}$

$\Rightarrow \frac{6}{x^2-8} = \frac{3}{4} \quad \Rightarrow x = \pm 4$

(C) $\vec{a} = \mu\vec{b} + 4\vec{c} \Rightarrow m(|\vec{b}|)^2 = -4\vec{b}.\vec{c}$

and $|\vec{b}|^2 + \vec{b}.\vec{c} - \vec{a}.\vec{c} = 0$

Again as $2|\vec{b} + \vec{c}| = |\vec{b} - \vec{a}|$

Solving and eliminating $\vec{b}.\vec{c}$ and

eliminating $|\vec{a}|^2$

We get $(2\mu^2 - 10\mu)\ |b|^2 = 0 \Rightarrow \mu = 0, 5$

(D) $I = \frac{2}{\pi^5}\int_{-\pi}^{\pi} f(x)\,dx = \frac{\pi}{2}\int_{-\pi}^{\pi}\frac{\sin 9(x/2)}{\sin(x/2)}\,dx$

$= \frac{2}{\pi} \times 2\int_0^{\pi}\frac{\sin 9(x/2)}{\sin(x/2)}\,dx$

Let $\frac{x}{2} = \theta \qquad \Rightarrow dx = 2d\theta$

$I = \frac{8}{\pi}\int_0^{\pi/2}\frac{\sin 9\theta}{\sin\theta}\,d\theta$

$I = \frac{8}{\pi}\int_0^{\pi/2}\frac{(\sin 9\theta - \sin 7\theta)}{\sin\theta}$

$+ \frac{(\sin 7\theta - \sin 5\theta)}{\sin\theta} + \left(\frac{\sin 5\theta - \sin\theta}{\sin\theta}\right)$

$+ \frac{\sin\theta d\theta}{\sin\theta}$

$= \frac{16}{\pi}\int_{\pi}^{\pi/2}(\cos 9\theta + \cos 6\theta + \cos 4\theta + \cos 2\theta)\,d\theta$

$+ \frac{8}{\pi}\int_0^{\pi/2} d\theta$

$= \frac{16}{\pi}\left[\frac{\sin 8\theta}{8} + \frac{\sin 6\theta}{6} + \frac{\sin 4\theta}{4} + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + \frac{8}{\pi}[\theta]_0^{\pi/2}$

$= 0 + \frac{8}{\pi} \times \frac{\pi}{2} = 4$

Sol.13 A

Line $\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-5}{1} = \lambda$ (Let)

dso $(\lambda + 2, 4\lambda + 3, \lambda + 5)$

Line on plane $5x - 4y - z = 1$

$5\lambda + 10 - 16\lambda - 12 - \lambda - 5 = 1$

$-12\lambda = 8$

$\lambda = -2/3$ so $P\left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$

for foot of perpendiuclar of T(2, 1, 4)

$(\lambda\ 4\lambda + 2, \lambda + 1) . (1, 4, 1) = 0$

$\lambda + 16\lambda + 8 + \lambda + 1 = 0$

$\lambda = -9/18 \qquad \Rightarrow \lambda = -1/2$

So R(3/2, 1, 9/2), distance $a = 1/\sqrt{2}$

Sol.14 A

$(x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$

$(1 + \lambda)x + (2 - \lambda)y + (\lambda + 3)z - (2 + 3\lambda) = 0$

$\Rightarrow \frac{|(1+\lambda).3 + (2-\lambda)1 - (\lambda+3) - (2+3\lambda)|}{\sqrt{(1+\lambda)^2 + (2-\lambda)^2 + (\lambda+3)^2}} = \frac{2}{\sqrt{3}}$

$\Rightarrow \sqrt{3\lambda^2 + 4\lambda + 14} = \frac{2}{\sqrt{3}}$

$\Rightarrow \frac{|-2\lambda|}{\sqrt{3\lambda^2 + 4\lambda + 14}} = \frac{2}{\sqrt{3}}$

$= 3\lambda^2 + 4\lambda + 14$

$\lambda = -7/2$

$(x + 2y + 3z - z) - 7/2\ (x - y + z - 3) = 0$

$-5x + 11y - z + 17 = 0$

$5x - 11y + z = 17$

Sol.15 B, C

$\begin{vmatrix} 2 & 0 & 0 \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0 \Rightarrow k = \pm 2$

use the value of k for finding the equation of planes