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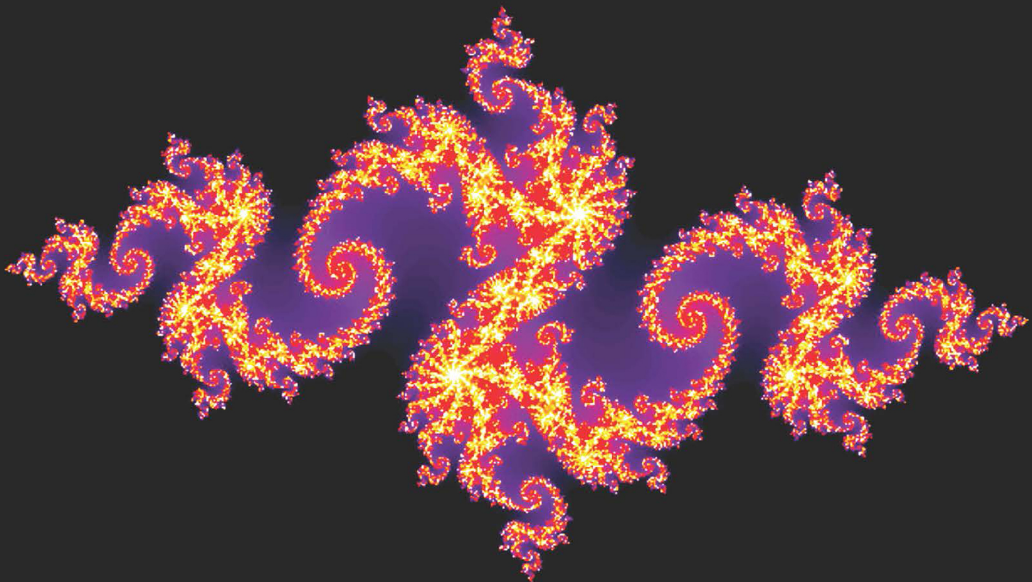
Magnifying Spacetime

How Physics Changes with Scale



Daniel Coumbe

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HOW PHYSICS CHANGES WITH SCALE

DANIEL COUMBE

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Preface

You quickly stride down the corridor, eager to reach the door at the end. The polished sign placed neatly above the handle comes into sharp focus, ‘*Interview Room - Google, Inc.*’ You take a deep breath and enter.

> The interviewer begins, "We will start with a little physics question if that’s OK?" Fresh from taking a course on physics you confidently nod. Then comes the bombshell, "Imagine you were shrunk to the size of a matchstick and put in a blender, how would you get out?"
>
> A long pause. Yet more silence... *OK, now you’ve been quiet for too long. Say something... Anything!* "Ermmm..." You frantically try to recall any relevant physics. You remember how things change with time, with speed, with acceleration. *But shrinking!? Changing with scale...?* A crack of panic swells within. "Ermmm..., did you say a blender?", desperately trying to stall for time.
>
> The interviewer nods coldly from across the room.
>
> "Well.. ummm..." you stutter, followed by another painfully long silence. The crack of panic opens to a chasm; you drop to the floor, assume the fetal position and gently sob.

But what is the answer to this reportedly common Google interview question? Well, shrinking yourself by a factor of say r will decrease your mass by a factor of r^3, because your body is 3-dimensional. However, muscular power depends on the cross-sectional area of muscle fibers, which will only decrease by a factor of r^2, because area is 2-dimensional. So proportionately you actually get stronger as you shrink — allowing you to simply jump out of the blender. Voila, a job at Google awaits you!

The effect of stretching or shrinking distances, that is performing a scale transformation, is not commonly emphasised at an introductory level. The scientific literature available on the subject of scale transformations is typically either too technical for non-professional physicists or too lacking in mathematical and conceptual detail. A major aim of this book is to fill this gap, providing an accessible but sufficiently detailed account of this fascinating subject. Primarily aimed at the undergraduate and graduate level, this text may also serve as a helpful resource for professional physicists. That being said, the scientifically literate layman should still find this work accessible due to the emphasis on conceptual understanding over gratuitous mathematical detail.

This book is particularly well-timed due to a recent surge of striking results, all pointing towards the crucial role of scale in quantum gravity. There are only a few things we know about quantum gravity with any confidence. One robust feature of quantum gravity is the thermodynamic behaviour of black holes, which shows up so consistently in such a diverse number of approaches it is highly likely that it will appear in some form in the final theory of quantum gravity. A second, much lesser-known, feature that is common to all leading approaches is that of dynamical dimensional reduction. This book is in part motivated by the desire to raise the profile of this exciting, and until now largely overlooked, phenomena. This will be achieved by reporting recent results from the cutting edge of modern research, in addition to highlighting a number of proposed explanations for this puzzling and unexpected observation.

In physics, experiment is the arbiter of truth. Ultimately these exciting new results must face the acid-test of experiment. Fortunately, recent experiments have begun to reach a sensitivity comparable to the scale at which quantum gravity is expected to become significant. These experiments have already helped constrain, and in some cases even exclude, models of quantum gravity. The delicate dance between experiment and theory, that has driven scientific progress for centuries, is now once again taking centre stage. However, before experiment spills its secrets in the next few years, we have a small window of opportunity in which to understand this phenomenon so that we can meet these results head-on with phenomenological models and experimental predictions. The hope is that this book will provide a timely and detailed account of this phenomena, perhaps even inspiring the next great breakthrough in our understanding of quantum gravity.

I have personally been involved in the search for a theory of quantum gravity for the last decade or so, and in particular, I have thought about many of the conceptual difficulties involved. I have authored numerous publications on quantum gravity in internationally leading research journals, most of which closely relate to the role of scale in fundamental physics. Therefore, I feel that I am able to divulge these new and exciting research findings, coming from myself and colleagues. It is my personal view that scale transformations and the associated symmetries will prove to be the key that will help unlock the enigma of quantum gravity, a view I hope to share with you over the coming pages.

The layout of this book is as follows. The Introduction will provide the reader with a basic understanding of general relativity and quantum mechanics, followed by a discussion of attempts to unify these two theories into a single theory of quantum gravity, and why this has so far proved such a difficult task. Chapters 1 and 2 build the necessary physical concepts and mathematical tools that will be used in the rest of the book, introducing such areas as global and local scale transformations as well as the fascinating subject of fractals. A comprehensive discussion on the evidence and implications of a minimum length scale is given in Chapter 3 and the renormalisation group is outlined in Chapter 4. Two modern approaches to quantum gravity, the asymptotic safety scenario and lattice quantum gravity, are discussed in detail in Chapters 5 and 6. The evidence for dynamical dimensional reduction is presented in Chapter 7, along with a discussion of possible implications and experimental tests. Finally, Chapter 8 will specify Weyl's attempt to modify Einstein's general theory of relativity via a local scale transformation of the metric tensor and explore some related ideas and implications for quantum gravity.

Ultimately, you should read this book for one reason, and only one reason; because you want to. Nature is beautiful, filled with diverse and interesting phenomena. But the source of all phenomena, the laws of physics themselves, are even more beautiful. This book will hopefully give you a flavour of the sublime symmetry underlying the laws of physics which operate over an enormous range of scales; from the size of the observable universe ($\sim 10^{26} m$) down to the unimaginably small Planck scale ($\sim 10^{-35} m$), and possibly beyond. Curiosity and intuition have their own reasons for existing — follow them — always.

Units, Conventions and Common Abbreviations

Max Planck, at the close of the 19th Century, proposed a natural set of units for length, time and mass, now aptly called Planck units. Planck units, by definition, normalise certain universal constants to unity, namely $\hbar = G_N = c = k_B = k_e = 1$, where $\hbar$ is the reduced Planck's constant, G_N is Newton's gravitational constant, c is the speed of light in a vacuum, k_B is Boltzmann's constant and k_e is Coulomb's constant. The Planck length l_P can be derived from purely dimensional arguments, being a natural length scale associated with the gravitational constant, the speed of light and Planck's constant, and is given by

$$l_P = \sqrt{\frac{\hbar G_N}{c^3}} \approx 1.6 \times 10^{-35} m. \tag{1}$$

The Planck time is simply the time it takes to traverse the Planck length at the speed c, hence

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G_N}{c^5}} \approx 5.4 \times 10^{-44} s. \tag{2}$$

The Planck mass is given by

$$m_P = \sqrt{\frac{\hbar c}{G_N}} \approx 2.2 \times 10^{-8} kg, \tag{3}$$

which unlike the other Planck units is much closer to an everyday scale, with m_P roughly corresponding to the mass of a flea's egg. Finally, the Planck temperature corresponds to the temperature at which an object emits radiation of wavelength l_P, hence

$$T_P = \frac{m_P c^2}{k_B} = \sqrt{\frac{\hbar c^5}{G_N k_B^2}}. \tag{4}$$

l_P, t_P, m_P and T_P define the most important base Planck units used in this work, from which a number of other derived units can be obtained. The derived Planck units most relevant to this book are the Planck area $A_P = l_P^2$, the Planck energy $E_P = m_P c^2$ and the Planck momentum $p_P = \hbar / l_P$.

In this book we shall work with a metric signature $(-,+,+,+)$, unless otherwise stated, so that the Minkowski metric, for example, is written as

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

We adopt the repeated index summation convention put forward by Einstein as a notational shorthand. This convention states that repeated indices within a single term imply a summation of that term over each value of the index. So for example, when the invariant spacetime interval is written in the shorthand convention as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{6}$$

this actually means

$$ds^2 = \sum_\mu \sum_\nu g_{\mu\nu} dx^\mu dx^\nu, \tag{7}$$

where the summation is taken over indices μ and ν because they are the ones repeated in Eq. (6).

Below we list some common abbreviations used throughout and the section in which they are first discussed or defined.

Table 1. A table of common abbreviations, their meanings, and the section in which they are first discussed

Abbreviation	Meaning	Section
CDT	Causal dynamical triangulations	6.4.
DSR	Doubly special relativity	3.3.
EDT	Euclidean dynamical triangulations	6.3.
ERGE	Exact renormalisation group equation	5.1.
LI	Lorentz invariance	3.3.
LQCD	Lattice quantum chromodynamics	6.1.
NTFP	Non-trivial fixed point	5.1.
QCD	Quantum chromodynamics	4.3.
QED	Quantum electrodynamics	Intro
RG	Renormalisation group	3

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First and foremost I wish to thank my beautiful wife Ivana and my wonderful daughter Ava for putting up with me while I wrote this book. Parts of this book are inspired by my Ph.D. thesis *Exploring a lattice formulation of lattice quantum gravity* obtained at the University of Glasgow in 2013. I wish to express my gratitude to everybody involved in the completion of my thesis, in particular, my advisor Jack Laiho and the external examiner Daniel Litim. I would like to thank Jan Ambjorn, Jerzy Jurkiewicz, Holger Bech Nielsen, Roberto Percacci, Gordon Donald, Antonio Morais, K. Grosvenor, V. Gueorguiev and C. MacLaurin for discussions on various parts of this work.

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Introduction

An almighty clash of iron and steel sent sparks high into the evening air. "Ping. . . ping. . . ping," an alarm sounded somewhere to his left. After grinding to an abrupt halt, he released the seat belt and opened the door. Looking up through a thin veil of smoke the driver saw his truck wedged under an Iron bridge.

Within minutes there were traffic jams, blasting horns, and screeching sirens. The police arrived first, followed by a myriad of ambulances and fire engines. After treating the driver for minor injuries, thoughts quickly turned to the problem of freeing the truck. They tried pulling it out using a winch; it didn't move. They debated cutting the truck into pieces, but the fuel tank would ignite. They considered removing a segment of the bridge, but an electrified power line lay overhead. After an hour of scrutinising blue-prints and debating they had no idea how to remove the truck. An abundance of structural engineers, architects, policemen, and firemen gawked at the problem, utterly dumbfounded. No one knew how to solve it.

> "Mum, what's going on over there?" said a little girl hanging her head out of the car window.
>
> "Looks like that silly man got his truck stuck under the bridge," she replied over the clash of horns.
>
> The girl got out of the car to stretch her legs. The very instant her feet touched the floor an idea struck. A pulse of electricity sparked her prefrontal cortex, shocking to life dead memories like Frankenstein's monster. Sepia photos of bicycle tyres and birthday balloons danced in her head.

> She hurried to the first Policeman and blurted, “why don’t you let some air out of the tyres and drive the truck out—?”

Just like the truck, our understanding of the universe gets stuck sometimes, it’s then that we need a new idea to free us again. As I shall explain, our current understanding of the universe is more deeply stuck than ever before. In 1915, Einstein beautifully and simply explained gravity as the curvature of spacetime, which defines our best description of the universe on large distance scales. A second, slightly later, revolution which came to be known as quantum mechanics, defines our best description of the universe on small distance scales. For about a century now physicists have been trying in vain to put these two puzzle pieces together to form a single unified picture of the universe, but with no success. The consensus is that we are missing something big, a new idea or way of looking at things before we become unstuck. Physics desperately needs another Einstein, another child-like thinker to look up and say “—let the air out of the tyres”.

Quantum mechanics describes three of the four fundamental interactions of nature, with general relativity describing the fourth, gravity. At short distances, quantum mechanical effects typically dominate, whereas general relativistic effects dominate over large distances and for strong gravitational fields. The vast majority of physical phenomena can be described independently by either general relativity or quantum mechanics, without the need for both. However, when strong gravitational fields interact over short distances, such as in the vicinity of the big-bang singularity or near black holes, the description of such phenomena demand a unification of quantum mechanics and general relativity. Such a unification would be a theory of quantum gravity.

Newtonian gravity attempted to explain the gravitational force via the inverse square law of gravitation, claiming that gravity was a universal property that acted instantaneously and on all massive bodies. This was an important mathematical step, but it did not fully explain the mechanism behind gravity. After publishing his special theory of relativity in 1905, Einstein wanted to generalise his theory such that all motion is relative. Einstein’s realisation of the equivalence of inertial and gravitational mass is called the equivalence principle and ultimately led to our current understanding of gravity known as the general theory of relativity. General relativity was not only able to reproduce the results of Newtonian gravity in appropriate limits but it also gave an intuitive explanation for the mechanism behind gravitation.

In general relativity [12] spacetime is represented by a four-dimensional manifold M on which there exists a metric $g_{\mu\nu}$ [13]. The metric $g_{\mu\nu}$ is a set of numbers describing the distance to neighbouring points on the manifold. The field equations of general relativity constrain the possible values that the curvature of spacetime can take. Introducing no additional geometrical structure into spacetime apart from the metric itself and requiring that the field equations contain no derivatives higher than second-order [13], one is uniquely led to the equations,

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{8}$$

$T_{\mu\nu}$ is the stress-energy-momentum tensor describing the amount and distribution of energy in spacetime, and $G_{\mu\nu}$ is the Einstein tensor describing how spacetime is curved by the presence of matter and is a function of the metric $g_{\mu\nu}$,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{9}$$

$R_{\mu\nu}$ is the Ricci tensor acting on the metric $g_{\mu\nu}$, and R is the Ricci scalar.[1] The Ricci scalar quantifies the curvature at each point on a Riemannian manifold and represents the difference in the volume of a ball embedded in curved spacetime with that in flat Euclidean spacetime. Mathematically, the Ricci scalar is defined by the product of the metric with the Ricci tensor,

$$R = R_{\mu\nu} g^{\mu\nu}. \tag{10}$$

In order to obtain exact solutions of the Einstein equations, one must impose certain symmetry constraints. For example, imposing spherical symmetry on a non-rotating massive body leads to the Schwarzschild solution. In this way, the number of coupled partial non-linear differential equations one obtains when writing Eq. (9) in full are reduced, and exact analytical solutions can be found.

The explanation of the gravitational force as a result of matter following the path of shortest distance in curved spacetimes is an elegant theoretical explanation that has been experimentally verified to high precision. The perihelion of

[1] Eq. (9) is for pure gravity (vanishing cosmological constant), however for a non-zero cosmological constant one would obtain $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$.

Mercury's orbit, gravitational lensing, and high-precision measurements of the decay of the orbital period of binary pulsars all agree well with general relativistic predictions [14].

Just as general relativity must replace Newtonian gravity in the large mass limit, quantum mechanics must also replace Newtonian mechanics in the small-distance limit. For example, until the development of quantum mechanics, physicists did not even have a satisfactory explanation for the stability of the atom. In classical mechanics, atoms are composed of electrons orbiting along well-defined paths about a central nucleus. Classical electromagnetism says that a charged particle with acceleration a and charge e will radiate energy with a power P given by Larmor's equation,

$$P = \frac{e^2 a^2}{6\pi\varepsilon_0 c^3}. \tag{11}$$

According to classical physics an electron, therefore, must radiate away all its energy and spiral in towards the nucleus. However, this is not observed in nature.

Max Planck's earlier radical solution to the so-called ultraviolet catastrophe of blackbody radiation – in which he posited that electromagnetic energy is not a continuous variable, but can only take discrete values, or quanta – also helped to resolve the problem of the apparent instability of the atom. Niels Bohr realised that if an electron's energy was also quantised then the atom could not collapse because electrons must have a non-zero minimum energy.

Our understanding of the atom was further developed by de Broglie who proposed that particles also have a wave-like nature. In this picture only integer wavelengths will be admitted by the electron's orbit, thereby explaining the particular values of the allowed energy levels of the electron within the atom. Heisenberg and Schrödinger developed these concepts further to give a complete description of a particle's quantum state at any instant of time in terms of the wavefunction Ψ. The evolution of the non-relativistic wavefunction is governed by the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right]\Psi(\mathbf{r},t), \tag{12}$$

where $|\Psi(\mathbf{r},t)|^2$ is interpreted as the probability density of finding the particle of mass m at a given location $\mathbf{r}$ at a time t. A wave-like description automatically introduces an uncertainty relation, for example between the position x and the momentum p_x, known as Heisenberg's uncertainty principle,

$$\triangle x \triangle p_x \geq \frac{\hbar}{2}. \tag{13}$$

A reformulation of quantum mechanics based on a path integral approach was developed by Wiener, Dirac and Feynman [15, 16, 17]. In this interpretation, the probability amplitude A for a particle initially located at x_i at time t_i, to be found at some later time t_f at position x_f is calculated by taking a weighted sum over all possible ways in which this can happen,

$$A\left(x_i, t_i \rightarrow x_f, t_f\right) = \int_{path} e^{\frac{i}{\hbar} S_{path}} = \langle x_f | e^{-\frac{iS_{path}}{\hbar}} | x_i \rangle. \tag{14}$$

The path integral approach to quantum mechanics has a number of advantages over the traditional Hamiltonian approach. Firstly, the path integral approach is intrinsically symmetric with regard to space and time and so provides a manifestly covariant version of quantum mechanics, therefore admitting the possibility of coordinate change in a straightforward manner [17]. Secondly, the path integral approach can be used to study nonperturbative quantum amplitudes using stochastic importance sampling which is important in the development of many physical theories (e.g., Ref. [18]).

Quantum mechanics has been subjected to high precision measurements for several decades, and no significant discrepancy between theory and experiment has ever been found. For example, the anomalous magnetic moment of the electron a given in terms of the $g-$factor is experimentally determined to be [19]

$$a = \frac{g-2}{2} = 0.00115965218073(28), \tag{15}$$

which is in agreement with the value calculated using quantum electrodynamics (QED) [20] to 10 significant figures.

General relativity and quantum mechanics, in their respective domains of applicability, are extremely accurate theories. However, even at a quick glance there exist points of tension that resist a straightforward unification of the two theories. Firstly, the energy-time uncertainty relation coupled with mass-energy equivalence implies that the smaller the region of spacetime under consideration the greater the allowed mass of particle-antiparticle pairs in the quantum vacuum. Hence, a prediction of general relativity in combination with the uncertainty principle is that as one probes spacetime on ever decreasing distance scales the geometry becomes increasingly turbulent; eventually creating infinite energy fluctuations as the distance scale is taken to zero. Secondly, the notion of time has fundamentally different meanings in general relativity and ordinary quantum mechanics. In ordinary quantum mechanics, time is an absolute quantity, the passage of which is completely independent of the state of the physical system. In general relativity, however, time is a purely relational concept whose passage is dependent on the particular configuration of the system.

Nature does not appear to encounter any such problems; after all, atoms do fall down in a gravitational field. Physicists tend to think, therefore, that the seeming incompatibility of general relativity and quantum mechanics necessitates the alteration of one or both theories, rather than that nature does not conform to a singular description.

Attempts at unifying general relativity and quantum mechanics have a long history (see Ref. [21] for an excellent overview). As early as 1916, Einstein recognised that his theory of general relativity must be modified to accommodate quantum effects [22] after realising that general relativity predicts that an electron in an atom will radiate away all of its energy in the form of gravitational waves, and eventually collapse on to the nucleus. In 1927, Oscar Klein realised a theory of quantum gravity must lead to a modification of space and time [23]. Technical publications on quantum gravity began appearing in the 1930s, most notably by Fierz and Pauli [24, 25], and by Blokhinstev and Gal'perin [26], whose work first recognised the spin-2 quantum of the gravitational field. In 1938, Heisenberg had the insight that the dimensionality of the gravitational coupling constant is likely to cause problems with a quantum theory of gravity [27].

The 1950s and '60s saw the application of tools recently developed in quantum field theory to the problem of quantum gravity, mainly because of the success in applying these techniques to the quantisation of the other fundamental interactions. Charles Misner introduced the concepts developed in Feynman's path integral approach to quantum mechanics to develop a weighted sum over geometries [28], which was further developed by Feynman [17]. The Wheeler-

DeWitt equation, first published in 1967 [29], describes the wavefunction of the universe by expressing the quantum mechanical Hamiltonian constraint using variables that are dependent on the metric. By the end of the '60s a complete set of Feynman rules for general relativity were known, thus paving the way for 't Hooft and Veltman to apply perturbative quantisation techniques to the gravitational field, eventually showing that the theory is perturbatively nonrenormalisable.

Although gravity was conclusively shown to be perturbatively nonrenormalisable by power counting in the early 70's, several advances were still made by using an approximation to the full theory of quantum gravity, known as semiclassical gravity. Semiclassical gravity is the quantum mechanical treatment of matter content on curved, but still classical, background geometries. In semiclassical gravity the stress-energy-momentum tensor takes on a quantum mechanical expectation value $\langle T_{\mu\nu}\rangle$ giving field equations for semiclassical general relativity,

$$8\pi G_N \langle T_{\mu\nu}\rangle = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{16}$$

Hawking used semiclassical gravity to study the effect of particle-antiparticle pair production near a black hole horizon. Because particles can be viewed as positive energy solutions to Dirac's equation, and antiparticles as negative energy solutions, Hawking was able to calculate the effect of antiparticles quantum mechanically tunnelling inwards through the horizon, or conversely particles tunnelling outwards through the horizon and escaping to infinity [30]. The result is that a black hole of mass M emits a thermal shower of particles with Hawking temperature T_{bh}, given by

$$T_{bh} = \frac{\hbar c^3}{8\pi k_B M G_N}. \tag{17}$$

This equation implies that the entropy of a black hole S_{BH} is proportional to its surface area A and not its volume,

$$S_{BH} = \frac{A}{4 l_P^2}. \tag{18}$$

This so-called Hawking-Bekenstein entropy formula will prove to be important in later discussions.

A similar phenomenon that can be derived using semiclassical gravity is the Unruh effect, first described in its full mathematical form by Bill Unruh in 1976 [31, 32, 33], whereby an observer with non-zero acceleration a in a Minkowski vacuum will measure a non-zero temperature T_U to be

$$T_U = \frac{\hbar a}{2\pi k_B c}. \tag{19}$$

This result, which has not yet been experimentally confirmed, implies that the definition of what constitutes a vacuum is dependent on the state of motion of the observer. Semiclassical results such as Hawking radiation, the black hole entropy formula and the Unruh effect show the range of new phenomena that arise when combining quantum mechanics with classical general relativity, and may act as a guiding principle in the formulation of a full theory of quantum gravity.

General relativity has been successfully formulated as an effective quantum field theory that is valid up to some low-energy cut-off scale, usually taken to be the Planck scale. For example, the work of Donoghue [34] uses an effective field theory formulation of gravity to calculate quantum corrections to the gravitational potential between two heavy masses, finding that gravity actually forms the best perturbative theory in nature [34]. However, as one increases the energy scale beyond the cut-off in a perturbative expansion new divergences appear that require an infinite number of counterterm coefficients to define the theory [35]. Since an infinite number of counterterm coefficients cannot be measured in a finite number of experiments the theory loses most of its predictive power at high energy scales. When considering small perturbations about flat Minkowski space one observes that the divergences cancel for one-loop diagrams. However, at the two-loop level and higher, such cancellations do not occur and divergences are once again present. The problem is compounded when one includes matter content, with nonrenormalisability occurring at the one-loop level [36].

A more intuitive way of understanding why gravity is perturbatively nonrenormalisable comes from simple dimensional arguments. Gravity is distinguished from the other fundamental interactions of nature by the fact that its coupling constant G_N is dimensionful. In d-dimensional spacetime Newton's gravitational coupling G_N has a mass dimension of $[G_N] = 2 - d$. This means that higher order loop corrections will generate a divergent number of counterterms of ever-increasing dimension. One can clearly see this from the perturbative quantum field theoretic treatment of gravity in d-dimensional space, where ultraviolet divergences at loop order L scale with momentum p as

$$\int p^{A-[G_N]L} dp, \tag{20}$$

where A is a process dependent quantity that is independent of L [37]. Equation (20) is clearly divergent for $[G_N] < 0$ because the integral will grow without bound as the loop-order L increases in the perturbative expansion [37]. It is interesting to note that $[G_N] < 0$ is only true when the dimension of spacetime $d > 2$. For $d = 2$ Newton's gravitational coupling, in fact, becomes dimensionless. Therefore, gravity as a perturbative quantum field theory becomes renormalisable by power counting in 2-dimensions. However, in our 4-dimensional universe, this is clearly not the case.

If one includes higher order derivative terms in the gravitational action and performs a resummation, such that the higher order derivative terms are incorporated into the graviton propagator, then gravity is renormalisable by power counting. However, this theory appears to be non-unitary at high energy scales due to the presence of ghost terms containing the wrong sign in the propagator [38]. The ghost poles, however, are of the order of the Planck mass m_P and so the unitarity violations only become significant in the high-energy regime where perturbation theory is already known to break down. This further highlights the need for a nonperturbative theory of quantum gravity.

One can still proceed with the standard perturbative approach by finding some mechanism for reducing the infinite number of couplings to a finite set. One attempt is to directly incorporate supersymmetry into general relativity, forming what has become known as supergravity. Supersymmetry postulates the existence of a fundamental symmetry of nature that exists between fermions and bosons, namely that each boson of integer-spin is associated with a fermion whose spin differs by a half-integer [39]. Applying supersymmetry as a local symmetry constraint in combination with the postulates of general relativity leads to the theory of supergravity. In supergravity every bosonic field is associated with a fermionic field with opposite statistics [40], thus the bosonic spin-2 graviton would have a fermionic supersymmetric partner of spin 3/2, the gravitino. It is hoped that supersymmetry can alleviate some of the ultraviolet divergences via the cancellation of divergent quantities in one field with those in the partner field, resulting in milder ultraviolet divergences, or perhaps even their complete elimination [41].

Viewing supergravity as an effective theory of a much larger superstring theory may resolve the issue of nonrenormalisability in quantum gravity altogether, at least order by order in perturbation theory [41]. Superstring theory posits that 0-dimensional point-like particles are actually 1-dimensional extended strings whose fundamental degrees of freedom are the vibrational modes of the string. In superstring theory discontinuities in standard Feynman diagrams are smoothed out into two-dimensional sheets, and so the infinite energy limit no longer corresponds to the zero distance limit. Divergences previously associated with the zero distance limit simply do not exist in superstring theory because there is no zero distance. Superstring theory is then perturbatively renormalisable, at least order by order in perturbation theory [41]. Supersymmetric string theory has enjoyed a number of successes, such as the emergence of the spin-2 gravitons as a fundamental mode of string oscillation [42, 43], calculations of the entropy of a special class of black holes that agree with the holographic principle and black hole thermodynamics [44, 45, 46], a calculation of the radiation spectrum emitted from black holes that agrees with Hawking radiation [47], as well as passing several nontrivial self-consistency checks. See Refs. [48, 49] for a comprehensive overview of string theory and its connection to other approaches to quantum gravity.

Another approach to quantum gravity adopts the background independent lessons of general relativity as a fundamental starting point and attempts to construct a quantum theory from it. This approach has resulted in Loop Quantum Gravity, which is background independent and therefore nonperturbative from the outset. The major successes of Loop Quantum Gravity include the agreement with Bekenstein's prediction of black hole entropy up to a constant factor [50], and at least in the low energy limit the emergence of spin-2 gravitons [51].

An alternative and more conservative candidate for a theory of quantum gravity comes from the asymptotic safety scenario, as first proposed by Weinberg [37]. If the asymptotic safety scenario is correct, gravity is effectively renormalisable when formulated nonperturbatively because the renormalisation group flow of couplings end on a nontrivial fixed point in the high-energy limit, and therefore remain finite over the entire range of energy scales. Furthermore, the ultraviolet critical surface of the nontrivial fixed point is required to be finite dimensional, and so in principle, gravity would be completely determined by a finite number of couplings up to arbitrarily high energies. Under this scenario, a perturbative expansion around a fixed background metric corresponds to perturbations about the low energy infrared fixed point, and taking the high-energy limit corresponds to following the renormalisation group flow towards the ultraviolet fixed point.

Chapter 1

Scale Transformations

We begin our exploration into the potentially important role of scale in the development of a successful theory of quantum gravity by first looking at the basics of global and local scale transformations.

1.1. Global Scale Transformations

Global scale transformations stretch or shrink distances by the same factor at every point throughout space. A simple example are those little model kits, of say a sailboat, which on the box state something like "1:500 scale kit". What this means, of course, is that if you scale every piece of the model by a factor of 500 you will reproduce the exact size and proportions of the real boat. You can't scale the mast by a factor of 500 and the hull by a factor of 400, as this will distort the shape of the object, and will not define a global scale transformation.

Now, let us try to be a bit more precise and describe a global scale transformation mathematically. To perform a global scale transformation of an object by some factor, for example, our model sailboat by a factor of 2, one needs to scale each point $p = (p_x, p_y, p_z)$ of the object by the same factor $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ in each direction. Such scaling can be represented by a product of matrices via

$$\Omega \cdot p = \begin{bmatrix} \Omega_x & 0 & 0 \\ 0 & \Omega_y & 0 \\ 0 & 0 & \Omega_z \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \Omega_x p_x \\ \Omega_y p_y \\ \Omega_z p_z \end{bmatrix}, \tag{21}$$

where the scaling factors are equal in each direction under a isotropic global scale transformation, i.e., $\Omega_x = \Omega_y = \Omega_z$. The application of a global scale transformation matrix to a 3-dimensional model boat is illustrated in Fig. 1., where in this case, the object is enlarged by a factor of $\Omega = 2$.

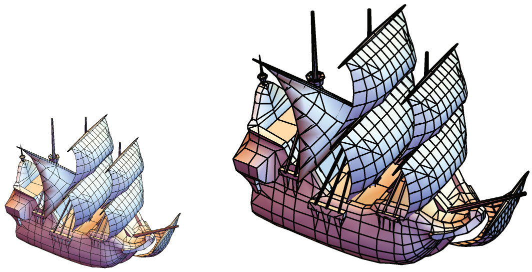

Figure 1. An example of a global scale transformation. The original boat (left) is scaled by a factor of 2, creating the uniformly transformed object (right).

Global scale transformations can be more concisely written as a transformation of spacetime coordinates

$$x^{\mu} \to \tilde{x}^{\mu} = x^{\mu}\Omega, \tag{22}$$

where Ω has the same value at each spacetime point and is thus a constant. Under this scale transformation of spacetime coordinates the metric tensor $g_{\mu\nu}$ transforms via [52]

$$g_{\mu}(x^{\mu}) \to g_{\mu}(x^{\mu})\Omega^2. \tag{23}$$

Are global scale transformations a symmetry of nature? Galileo first discovered a new symmetry of nature in 1632 by considering a thought experiment aboard a ship. Galileo told us to imagine going below decks of the ship and covering up all the windows, isolating the room from the outside. Next, we ask the captain to move at a constant speed over a perfectly calm sea. In this specific case, Galileo argued that we could not possibly know whether the ship was moving. You could perform any number of experiments in this isolated room—dropping balls, measuring distances, times and masses—and the results would be indistinguishable from those obtained at any other constant speed, including when the boat is stationary. Galileo found that changing between different constant velocities does not change the laws of physics. Since *symmetry is change without change*, this observation describes an underlying symmetry of nature known as Galilean invariance.

Might we play a similar game, but with scale? Imagine being below deck on our hypothetical ship once again. But this time we magnify everything in the room, including ourselves, by the same scale factor. Is there any experiment we can perform that will tell us if anything has changed? To investigate this question, imagine we have an air-tight box filled with a gas of iodine atoms. We shine a laser into the gas and measure the wavelength λ of the emitted light. We now perform a global scale transformation, magnifying everything within the room by a factor of say two, including the box of gas. Has anything changed?

Well, the wavelength of light emitted by a gas of iodine atoms with twice the volume is not two times longer, it is in fact exactly the same as the original. Although the box containing the atoms is enlarged, the atoms themselves and their associated spectra are fixed due to a fundamental atomic scale. However, the ratio of the size of the emitter (the box of gas) to the wavelength of emitted light λ *does* change with scale. It seems our experiment is capable of distinguishing between before and after the scale transformation. Therefore, the laws of physics are not symmetrical under a change of scale (a fact also discovered by Galileo) [53]. We now know that the essential reason for this asymmetry is the atomic nature of matter. An atom has a definite scale, which breaks scale invariance, at least globally [53].

In fact, it is a commonly held view in modern physics that no global symmetry can be realised in nature, mainly due to the fact that such a symmetry would imply action at a distance [54]. All known global symmetries are either broken (for example parity, time reversal invariance, and charge symmetry), approximate (such as isotropic spin invariance) or the product of a spontaneously broken local symmetry [54]. Given the apparently unnatural status of global scale transformations let's move on to local scale transformations.

1.2. Local Scale Transformations

The power of local symmetries is exemplified by the successes of gauge theories and general relativity. A founding principle of general relativity is a statement of local symmetry, the principle of equivalence, in which the laws of physics must be invariant under local spacetime coordinate transformations. Perhaps, then, the fundamental laws of physics might also be invariant with respect to local scale transformations? This question forms an important research direction that we shall return to later, but first, we must define what exactly we mean by a local scale transformation.

Local scale transformations are point-dependent, that is the factor Ω with which a particular spacetime point is rescaled depends on the location of that point. Returning to our ship illustration above, an example of a local scale transformation might be the rescaling of points depending on some function of their distance from an arbitrarily defined origin, say the centre of mass of the ship. Thus, in this case, points near the centre of mass could be rescaled by a smaller factor than those further away. Clearly, a global scale transformation is a specific type of local scale transformation, for which $\Omega(x) = \Omega$.

Local scale transformations act on spacetime coordinates via

$$x^{\mu} \rightarrow x^{\mu}\Omega\left(x^{\mu}\right), \tag{24}$$

and on the spacetime metric via

$$g_{\mu\nu}(x^{\mu}) \rightarrow g'_{\mu\nu}(x'^{\mu}) = g_{\mu\nu}(x^{\mu})\Omega^{2}\left(x^{\mu}\right), \tag{25}$$

where $\Omega\left(x^{\mu}\right)$ is a smooth non-vanishing point-dependent function that must lie in the range $0 < \Omega\left(x^{\mu}\right) < \infty$ [55].

Conformal transformations are nothing but localised scale transformations that preserve the angle between vectors as a function of scale. That is, given two vectors $\mathbf{v}$ and $\mathbf{w}$ the angle θ between them

$$\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\left(\mathbf{v}^{2}\mathbf{w}^{2}\right)^{\frac{1}{2}}}, \tag{26}$$

is a scale invariant quantity, where $\mathbf{v}\cdot\mathbf{w} = g_{\mu\nu}v^{\mu}w^{\nu}$. It is interesting to note that the Poincare group, the group of spacetime isometries, is in fact a subgroup of the conformal group, since the metric tensor is left invariant $g'_{\mu\nu} = g_{\mu\nu}$ ($\Omega = 1$) [56].

Under a d-dimensional infinitesimal coordinate transformation $x^{\mu} \rightarrow x^{\mu} + \varepsilon^{\mu}$ the infinitesimal spacetime interval transforms like

$$ds^2 \to ds^2 + (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)\, dx^\mu dx^\nu . \tag{27}$$

The constraint of Eq. (5) means that $\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$ must be proportional to the metric $\eta_{\mu\nu}$, more specifically

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}\,(\partial \cdot \varepsilon)\, \eta_{\mu\nu}, \tag{28}$$

where the proportionality constant $\frac{2}{d}\,(\partial \cdot \varepsilon)$ is set by performing a trace over both sides with $\eta_{\mu\nu}$ [56]. Comparing Eq. (8) with Eq. (5) tells us that

$$\Omega(x) = 1 + \frac{2}{d}\,(\partial \cdot \varepsilon)\,. \tag{29}$$

From Eq. (8) we also obtain

$$(\eta_{\mu\nu} \Box + (d-2)\partial_\mu \partial_\nu)\, \partial \cdot \varepsilon = 0. \tag{30}$$

If the number of dimensions d is greater than two then Eqs. (9) and (10) imply that the third derivative of ε equals zero, and therefore that ε is at most quadratic in x [56].

Finite conformal transformations can be obtained by integrating the infinitesimal expressions. This gives the Poincare group for which $\Omega = 1$, as

$$\begin{aligned} &x \to x' = x + a \\ &x \to x' = \Lambda x, \qquad \left(\Lambda^\mu_\nu \in SO(p,q)\right), \end{aligned} \tag{31}$$

the dilatations as

$$x \to x' = \lambda x, \qquad (\Omega = \lambda^{-2}), \tag{32}$$

and the special conformal transformations as

$$x \to x' = \frac{x + bx^2}{1 + 2b \cdot x + b^2 x^2}, \qquad \left(\Omega(x) = \left(1 + 2b \cdot x + b^2 x^2\right)^2\right). \tag{33}$$

Special conformal transformations can be seen to be made up of an inversion $x^\mu \to x^\mu / x^2$ followed by a translation $x^\mu \to x^\mu - b^\mu$ followed by a second inversion, namely

$$\frac{x'^{\mu}}{x'^{2}} = \frac{x^{\mu}}{x^{2}} - b^{\mu}. \tag{34}$$

In 1909, Bateman and Cunningham demonstrated that Maxwell's equations of electromagnetism are not only Lorentz invariant but also scale and conformal invariant. Consider two inertial coordinates systems $S(x,y,z,t)$ and $S'(x',y',z',t')$ consistent with an invariant speed of light. If a point moves with speed c in system S and is displaced an amount (dx,dy,dz) in a time dt, then

$$dx^2 + dy^2 + dz^2 - c^2 dt^2 = 0. \tag{35}$$

Similarly, if (dx',dy',dz') is the displacement of the same point observed in frame S' in a time dt', then

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = 0. \tag{36}$$

However, by differentiating the functional relations connecting $S(x,y,z,t)$ and $S'(x',y',z',t')$ we find that $dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2$ is a homogeneous quadratic function of (dx,dy,dz,dt) [57]. This means that we must have

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 \equiv \Omega^2(x^{\mu})(dx^2 + dy^2 + dz^2 - c^2 dt^2), \tag{37}$$

where $\Omega(x^{\mu})$ is the linear magnification and is a function of the spacetime coordinates x^{μ} alone [57].

Assuming that the application of such a transformation does not alter the shape of any elemental volume then this is a conformal transformation [57]. It has been shown that all conformal transformations exist in three possible classes, depending on the particular definition of $\Omega(x^{\mu})$ [57]. The first class is defined by $\Omega = 1$, which is the Lorentz group. The second class is defined by a constant Ω other than unity, which is the group of global scale transformations. The third class is defined by $\Omega(x^{\mu}) = k^2/\left(r^2 - c^2 t^2\right)$, where k is the radius of inversion of a 4-dimensional hypersphere [57], giving the group of conformal transformations consistent with inversion symmetry. Poincare and Einstein argued that only the $\Omega = 1$ (Lorentz group) case is symmetric with respect to all laws of nature, while the laws of optics and electrodynamics alone are invariant under the group of conformal transformations. We shall touch upon this point again in chapter 7.

Chapter 2

Fractals

2.1. The Coastline Paradox

Fractals are everywhere. Once you study and understand this beautiful mathematical structure, you can't help but see it everywhere in nature. Broccoli, rivers, snowflakes, lightning and sea shells are all examples of commonly occurring fractals. Put simply, an object is said to be fractal if it exhibits self-similarity at different distance scales, that is zooming in on any particular part of a fractal object reproduces the same geometric structure as the whole. A river viewed from space has the same tendril-like structure as one of its tributaries viewed from an aeroplane. Snap off a piece of Broccoli and it will have the same tree-like structure as the whole. The cover image of this book is a beautiful example of a particular fractal known as a quadratic Julia set, which is defined by points in the complex plane under the mapping $z \rightarrow z^2 + c$, with the complex number c given by $c = -0.79 + 0.15I$. Zooming in on any particular region of this image yields the same geometric structure at larger scales, the hallmark of a fractal. More precisely, a fractal is defined as a set for which the measured dimension strictly departs from the topological dimension.

As an example, consider trying to measure the length of the coastline of Great Britain. Imagine that the only equipment you are given is a 1km measuring stick. Painstakingly, you make your way around the entire coastline carefully placing the stick end-to-end, until finally obtaining a result. Now you are given a much shorter 1m measuring stick. Small indentations and coastal features that could not previously be measured now become accessible; the length you now measure is bigger. After a few such iterations you notice that the length of the coastline depends on the length of the stick you use to measure it; the smaller the stick, the smaller the features you can resolve and the longer the measured coastline. It is just as meaningless to ask what is the speed of a body without specifying a reference object, as it is to ask what is the length of a coastline without specifying a reference scale.

The mathematician Benoit Mandelbrot detailed this so-called coastline paradox in a 1967 *Science* publication [58]. Mandelbrot, building on the earlier work of Lewis Richardson [59], came up with an empirical law

$$L(\Delta x) = C\Delta x^{1-D_H}, \tag{38}$$

describing how the measured length $L(\Delta x)$ depends on the measurement scale Δx, where C is a positive constant and D_H is known as the Hausdorff dimension, which is equal to or greater than unity. A perfectly smooth and regular coastline would have a Hausdorff dimension of $D_H = 1$. The more irregular and fractal a coastline is, the greater the departure of the dimensionality from $D_H = 1$. Thus, one can quantify the fractal nature of a given geometry by its Hausdorff dimension. We now proceed to define and discuss various measures of the fractal dimension in more detail, including the Hausdorff dimension.

2.2. Fractal Dimensions

2.2.1. The Topological Dimension

The standard non-fractal definition of dimension is given by the familiar topological dimension D_T, which can be determined by simply counting the minimum number of coordinates to uniquely specify an event. For example, the topological dimension of a line is one and the topological dimension of a plane is two, with the value always being an integer. However, the validity of the topological dimension comes into question when considering a fractal geometry, in which dimensionality may deviate from the number of topological dimensions. In this section, we review a number of ways of defining the fractal dimension of a geometry that are relevant to this work.

2.2.2. The Hausdorff Dimension

The Hausdorff dimension, first proposed by Felix Hausdorff in 1918 [60], generalises the concept of dimension to non-integer values, thereby enabling one to quantify the dimension of a fractal geometry. To motivate a mathematical definition of the Hausdorff dimension we consider self-similarity and dimension under scale transformations. Take a line of length r and stretch it by a factor of three via a global scale transformation. The resulting line has a length of $3r$. Now take a circle of area A and scale its radius r by a factor of three. The new circle has an area of $9A$. Next, take a sphere of volume V and again scale the radius r by a factor of three, obtaining a sphere with a volume of $27V$. This simple set of scale transformations is depicted in Fig. 2., and the results are tabulated in Table 2..

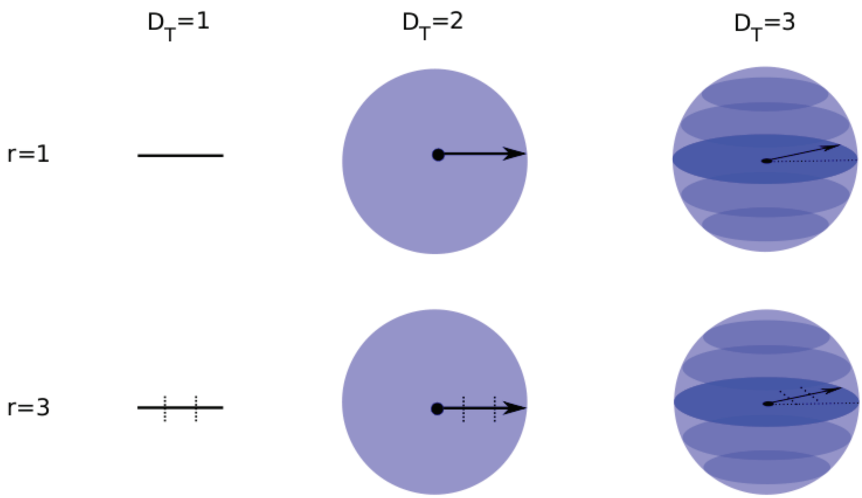

Figure 2. A schematic describing how the Hausdorff dimension is determined. The left-hand column shows a line segment with topological dimension 1. The line segment undergoes a scale transformation from $r = 1$ to $r = 3$. The central column shows a circle of topological dimension 2 whose area increases by a factor of 9 under such a scale transformation. The right-hand column shows a sphere with topological dimension 3, whose volume increases by a factor of 27 under such a scale transformation [1].

Table 2.: A table showing how the volume of a sphere with topological dimension D_T scales with radius r

D_T	r	V
1	3	3
2	3	9
3	3	27

As can be seen from Table 2. the relationship between the D_T-dimensional volume and the radius r is given by

$$V \propto r^{D_T}. \tag{39}$$

Rearranging so that D_T is on the left-hand side, and taking the limit $r \to 0$, gives us the mathematical definition of the Hausdorff dimension as,

$$D_H = \lim_{r \to 0} \frac{\ln(\mathrm{V}(\mathrm{r}))}{\ln(\mathrm{r})}. \tag{40}$$

2.2.3. The Spectral Dimension

Another measure of the fractal dimension of a space is the spectral dimension. The spectral dimension D_S defines the effective dimension of a fractal geometry and is related to the probability $P_r(\sigma)$ that a random walk will return to its origin after σ steps.

We can obtain an intuitive understanding of why the probability of return must be related to dimensionality by considering a random walk on a two and three-dimensional square lattice, as shown in Fig. 3.. In two dimensions, at every point the random walker has four options to pick from; he can either go forward, backward, left or right (assuming he cannot remain at the same point). In three dimensions, the random walker has six options to choose from; the same four as before plus one up and one down. So, in a higher number of dimensions, the random walker is less likely to return to their origin after a given number of steps, simply because there are more options.

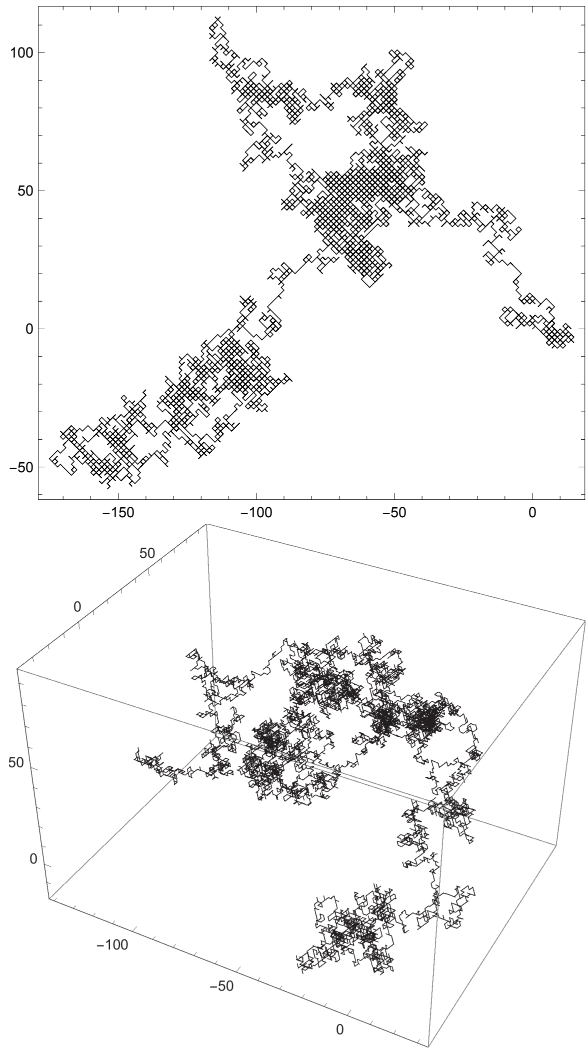

Figure 3. A random walk consisting of 10^4 diffusion steps in 2-dimensions (upper) and 3-dimensions (lower).

One can derive the spectral dimension (following Refs. [8, 61]) starting from the d-dimensional diffusion equation

$$\frac{\partial}{\partial\sigma}K_g(\zeta_0,\zeta,\sigma) - g^{\mu\nu}\bigtriangledown_\mu\bigtriangledown_\nu K_g(\zeta_0,\zeta,\sigma) = 0, \tag{41}$$

where K_g is known as the heat kernel and describes the probability density of diffusion from ζ_0 to ζ in a fictitious diffusion time σ. $\bigtriangledown$ is the covariant derivative of the metric $g_{\mu\nu}$. The diffusion process is taken over a d-dimensional closed Riemannian manifold M with a smooth metric $g_{\mu\nu}(\zeta)$. In the case of an infinite flat Euclidean space, Eq. (4) has the simple solution,

$$K_g(\zeta_0,\zeta,\sigma) = \frac{\exp\left(-\mathrm{d}_\mathrm{g}^2(\zeta,\zeta_0)/4\sigma\right)}{(4\pi\sigma)^{d/2}}, \tag{42}$$

where $d_g^2(\zeta,\zeta_0)$ is the geodesic distance between ζ and ζ_0.

The quantity that is measured in numerical simulations of the spectral dimension is the probability $P_r(\sigma)$ that the diffusion process will return to a randomly chosen origin after σ diffusion steps over the spacetime volume $V = \int d^d\zeta\sqrt{\det(\mathrm{g}(\zeta))}$,

$$P_r(\sigma) = \frac{1}{V}\int d^d\zeta\sqrt{\det(\mathrm{g}(\zeta))}K_g(\zeta_0,\zeta,\sigma). \tag{43}$$

The probability of returning to the origin in infinitely flat space is then just,

$$P_r(\sigma) = \frac{1}{\sigma^{d/2}}, \tag{44}$$

and so one can extract the spectral dimension D_S by taking the logarithmic derivative with respect to the diffusion time, giving

$$D_S = -2\frac{d\log\mathrm{P_r}(\sigma)}{d\log\sigma}. \tag{45}$$

Note that Eq. (8) is only strictly valid for an infinitely flat Euclidean space. However, one can still use this definition of the spectral dimension to compute the fractal dimension of a curved, or finite volume, by factoring in the appropriate corrections for large diffusion times σ.

The need for such a correction can be intuitively explained by realising that the volume of the lattice is proportional to the number of lattice sites, and so as the number of diffusion steps increases relative to the volume, the probability that the random walk will sample all available lattice sites at least once approaches unity, sending D_S to zero. More mathematically, the zero mode of the Laplacian $-\triangle_g$, which determines the behaviour of $P_r(\sigma)$ via its eigenvalues λ_n, will dominate the diffusion in this region, causing $P_r(\sigma) \to 1$ for very large σ [8]. One can, therefore, factor in the appropriate finite volume corrections by omitting values of $D_S(\sigma)$ for which σ is greater than some cut-off value that is proportional to the lattice volume. The curvature of the space on which the diffusion process occurs should also be corrected for due to the fact that it will change the probability that the diffusion process will return to the origin [8]. These, and other corrections to the spectral dimension will be discussed in more detail in section 7.2.2..

2.2.4. The Walk Dimension

The average square displacement $\langle r^2 \rangle$ for Brownian motion in flat space increases linearly with time T. This can be demonstrated by determining the expectation value of the probability density of the heat kernel of Eq. (5), which gives

$$\langle r^2 \rangle \equiv \langle x^2 \rangle = \int d^d x K_g(x, 0; T) x^2 \propto T, \tag{46}$$

where we have expressed K_g in terms of T and x rather than the parameters σ and ζ used in the definition of the spectral dimension above (Eq. (8)). Diffusion processes on fractals are in general anomalous, and so the linear relationship $\langle r^2 \rangle \propto T$ is generalised to the power law $\langle r^2 \rangle \propto T^{\frac{2}{D_w}}$ for $D_w \neq 2$, where the exponent D_w is the walk dimension [62].

A more intuitive definition of the walk dimension D_w is the fractal dimension of the path traced out by a random walker, as depicted in Fig. 3..

2.2.5. Myrheim-Meyer Dimension

The Myrheim-Meyer dimension is particularly useful as it is valid in Lorentzian spacetime [63, 64], which will prove helpful in later discussions of causal set theory in particular. Consider two causally related points a and b, with a in the causal past of b, in d-dimensional Minkowski spacetime. The intersection of the future of a and the past of b defines an interval $I[a, b]$, known as the Alexandrov interval. A point r in the interval $I[a,b]$, for which a is in the past of r and b is in the future of r, forms two new smaller intervals, $I[a,r]$ and $I[r,b]$. The volume of these intervals obey a scaling relation [65]

$$\frac{\langle \mathrm{Vol}\,(\mathrm{I[r,b]})\rangle_{\mathrm{r}}}{\mathrm{Vol}\,(\mathrm{I[a,b]})} = \frac{\Gamma\,(d+1)\,\Gamma\left(\frac{d}{2}\right)}{4\Gamma\left(\frac{3d}{2}\right)}. \tag{47}$$

One can, therefore, extract the dimension d from Eq. (10). This dimension can be generalised to arbitrary spacetimes to give a Lorentzian scaling dimension D_{MM}, known as the Myrheim-Meyer dimension. If the spacetime is not flat then curvature corrections can be added [66].

2.2.6. Correlation Dimension

The correlation dimension D_c quantifies the dimensionality associated with a random distribution of points in a n-dimensional space. Given a set of P points in a n-dimensional space

$$\overrightarrow{x}(i) = \{x_1(i), x_2(i), x_3(i), ..., x_n(i)\}, \qquad i = 1,2,...,P, \tag{48}$$

then the correlation integral $C(r)$ is given by

$$C(r) = \lim_{P\to\infty} \frac{N(<r)}{P^2}, \tag{49}$$

where $N(<r)$ is the number of pairs of points that are separated by a distance less than r. In the limit $P \to \infty$, the correlation integral for small values of r can be approximated by

$$C(r) \sim r^{D_c}, \tag{50}$$

where D_c is the correlation dimension.

The correlation dimension has a number of advantages over other definitions of the fractal dimension, such as being simple and easy to compute in a wide range of cases and being significantly less noisy when restricted to small data sets (small values of P).

2.3. Fractals Above Us and Below Us

We now take a look at two examples of fractal behaviour that have been observed at cosmological and quantum mechanical scales, utilising some of the measures of dimensionality discussed above.

2.3.1. Fractals in Cosmology

A founding principle of modern cosmology is the assumption that the spatial distribution of matter is homogeneous and isotropic over sufficiently large distance scales, known as the cosmological principle. At scales larger than $400h^{-1}Mpc$ observations confirm the statistical homogeneity and isotropy of galaxy distributions [67]. However, over smaller distance scales the universe does not obey the cosmological principle, with galaxies being unevenly distributed between dense clusters and vast voids, as evidenced by the red-shift survey *Las Campanas* [68] and others.[1]

It has been noted by a number of authors that the clustering of galaxies is ideally suited to a fractal description, since clumpiness in the spatial distribution of matter persists over a large range of distance scales [69, 70]. Modelling the universe using fractal geometry is useful because it allows the quantification of galaxy clustering via a fractal correlation dimension D_c, which is defined as

$$D_c = \frac{d\ln(\mathrm{N}(<\mathrm{R}))}{d\ln\mathrm{R}}, \tag{51}$$

where $N(<R)$ is the number of galaxies within a distance R from a given reference galaxy.

A perfectly homogeneous distribution of matter should have a dimensionality of $D_c = 3$. However, various observations have measured the fractal dimension associated with the distribution of galaxies to be between $D_c = 1.2-2.93$ at distances smaller than $400h^{-1}Mpc$ [67], well below the homogeneous value of $D_c = 3$. Over the largest measured distances of $300-400h^{-1}Mpc$ the fractal dimension was found to be $D_c = 2.93$ [71], very close to the homogeneous value. Between $30-60h^{-1}Mpc$ the value is within the range $D_c = 2.7-2.9$ [72], for $1-10h^{-1}Mpc$ it is $D_c = 2.25$ [73] and for the smallest measured distances of

[1] Note that we are talking exclusively about the distribution of visible matter, not dark matter.

$1-3.5h^{-1}Mpc$ it is $D_c = 1.2$ [74]. Clearly, the distribution of luminous matter is highly fractal on small scales but slowly approaches a smooth homogeneous and isotropic geometry over sufficiently large distances.

2.3.2. Fractals in Quantum Mechanics

Consider an experiment in which we measure the path length of a free quantum mechanical particle by measuring the particle's position with a spatial resolution Δx at sequential time intervals Δt [75]. Classically, the path will be a straight line of Hausdorff dimension $D_H = 1$ connecting the start and end points. Quantum mechanically, however, we must define the path by joining straight lines between the centres of regions in which the particle is known to exist. In the quantum mechanical paradigm, the particle's path is expected to become increasingly erratic as a consequence of confining the particle within a decreasing radius Δx. Furthermore, in quantum mechanics, we can only define a particle's path in a statistical sense, and so we introduce a notation for the expectation value of a particle's path $\langle l \rangle$.

We measure the position of a particle at times $t_N = t_0 + N\Delta t$, with $T = t_N - t_0 = N\Delta t$. The length of the particle's path will then be

$$\langle l \rangle = N \langle \Delta l \rangle, \tag{52}$$

where $\langle \Delta l \rangle$ is the average distance a particle travels in a time Δt. Here we consider a particle whose total average momentum is zero, so as to make the example similar to a typical diffusion process, which involve random walks with no preferred direction. The wavefunction of a particle measured to be initially located at the origin is given by

$$\psi_{\Delta x}(\mathbf{x}) = \frac{(\Delta x)^{\frac{3}{2}}}{\hbar^3} \int_{\Re^3} \frac{d^3 p}{2\pi^{3/2}} f\left(\frac{|\mathbf{p}|\Delta x}{\hbar}\right) \exp\left(i\mathbf{p}x/\hbar\right). \tag{53}$$

By defining a dimensionless vector $\mathbf{k} = \mathbf{p}(\Delta x)/\hbar$, we can write the normalisation condition for the wavefunction as

$$\int_{\Re^3} d^3 k |f(|\mathbf{k}|)|^2 = 1. \tag{54}$$

We keep Δt fixed and allow Δx to vary. In this case $f(|\mathbf{k}|)$ in Eq. (17) can be approximated by a Gaussian with the result

$$\langle \Delta l \rangle = \frac{\zeta \hbar \Delta t}{m \Delta x} \sqrt{1 + \left(\frac{2m(\Delta x)^2}{\hbar \Delta t} \right)^2}. \tag{55}$$

Where ζ is a constant of proportionality. Since $\langle l \rangle = N \langle \Delta l \rangle$ and $T = N \Delta t$, we can write

$$\langle l \rangle = \frac{\zeta \hbar T}{m \Delta x} \sqrt{1 + \left(\frac{2m(\Delta x)^2}{\hbar \Delta t} \right)^2}. \tag{56}$$

We now compare the path lengths in the classical limit L (i.e., when the particle's position is only measured at the start and end points of the diffusion, which are sufficiently separated in space), and in the quantum mechanical limit $\langle l \rangle$, finding

$$\frac{\langle l \rangle}{L} = \frac{\langle l \rangle}{vT} = \frac{\zeta \hbar}{m v \Delta x} \sqrt{1 + \left(\frac{2m(\Delta x)^2}{\hbar \Delta t} \right)^2}. \tag{57}$$

Using the relations $\Delta t = T/N$, $T = L/v$ and the definition of the de Broglie wavelength $\lambda_d = \hbar/(mv)$ we obtain

$$\frac{\langle l \rangle}{L} = \frac{\zeta \lambda_d}{\Delta x} \sqrt{1 + \left(\frac{4(\Delta x)^4 N^2}{\lambda_d^2 L^2} \right)}. \tag{58}$$

Defining $X \equiv \Delta x/\lambda_d$ and using the Hausdorff definition of resolution $N\Delta x = L$ gives

$$\frac{\langle l \rangle}{L} = \frac{\zeta}{X} \sqrt{1 + 4X^2} = \zeta \sqrt{4 + \frac{1}{X^2}}. \tag{59}$$

Since, in the classical limit we must have $\langle l \rangle / L = 1$ we can determine that $\zeta = 1/2$, yielding

$$\frac{\langle l \rangle}{L} = \sqrt{1 + \frac{1}{4X^2}}. \tag{60}$$

Using Hausdorff's definition of a scale invariant length L defined via

$$\langle L \rangle = \langle l \rangle (\Delta x)^{D_H - 1}, \tag{61}$$

we can transfer any scale dependence to the Hausdorff dimension D_H [75]. Equation (24) can be rearranged to give

$$D_H = \frac{\ln(\langle L \rangle / \langle l \rangle)}{\ln(\Delta x)} + 1. \tag{62}$$

Substituting Eq. (22) into Eq. (53) therefore gives an expression for how the Hausdorff dimension D_H changes as a function of distance scale Δx as

$$D_H = \frac{\ln\left(\left(1 + \frac{\lambda_d^2}{4\Delta x^2}\right)^{-\frac{1}{2}}\right)}{\ln(\Delta x)} + 1. \tag{63}$$

In the classical limit $\Delta x \to \infty$, Eq. (26) gives $D_H \to 1$, as one would expect from a classical path. However, in the limit $\Delta x \to 0$ we find that $D_H \to 2$. Therefore, quantum mechanical paths are fractal with a scale dependent Hausdorff dimension [75]. This conclusion was also suggested by Feynman and Hibbs who showed that quantum mechanical trajectories follow discontinuous nowhere differentiable paths, and thus have similarities with fractal curves [17].

Chapter 3

A Minimum Scale?

3.1. Atoms of Spacetime

Take a piece of string and cut it in half. Then half this half. If one keeps cutting the string with a hypothetical pair of scissors will we ever reach a minimum length? Well, eventually one would be left with just one atom. The ancient Greek philosopher Democritus is well-known for hypothesising the existence of atoms or *indivisibles*, quipping that "Nothing exists except atoms and empty space; everything else is opinion." However, it took until the 1800s for evidence of the existence of atoms to appear, and not until the early 1900s did Einstein convince the scientific community as a whole.

In fact, we do not need to perform elaborate experiments involving the hypothetical cutting of string. The existence of atoms can be deduced even at everyday scales. This insight is due to the Austrian physicist Ludwig Boltzmann who pointed out that a glass of water, for example, must be composed of microscopic degrees of freedom (atoms) due to the fact that it can be heated. The very fact that a glass of water can store heat energy and therefore have a temperature implies that it must have an associated entropy. As pointed out by Boltzmann, entropy S is proportional to the number of microscopic degrees of freedom W, in particular

$$S = k_B \ln\left(\mathrm{W}\right), \tag{64}$$

where $k_B \approx 1.38 \times 10^{-23} J/K$ is the Boltzmann constant. Thus, the simple fact that water can be heated implies that matter must be comprised of microscopic degrees of freedom (atoms) [76]. As we shall soon see, a similar argument can be made for atoms of spacetime.

Returning to our string cutting experiment, we know that we will eventually reach an individual atom. But what then? What if we continue to divide? Is there some minimum possible distance, an atom of spacetime? It is known that the atom consists of proton's, neutrons and electrons. But the proton and neutron are themselves made of more elementary constituents, called quarks. Quarks and electrons, according to modern physics are elementary particles of zero size. However, quantum field theory tells us that elementary particles are more accurately pictured as a cloud of virtual particles filling the space around a point-like object. So, what if we keep dividing space itself into ever smaller pieces? Contemporary research suggests that we will eventually encounter a minimal length, an atom of spacetime, known as the Planck length. We review the numerous arguments in favour of an atomic spacetime in section 3.2., but before that, we briefly examine what the thermodynamics of spacetime might tell us about a minimal length.

In 1972, the late Stephen Hawking proved that when any two black holes merge the resulting black hole must have a surface area that is greater than or equal to the sum of the individual black holes, leading to the conclusion that the surface area of the event horizon of a black hole A can never decrease $dA/dt \geq 0$. This conclusion is strikingly reminiscent of the second law of thermodynamics which states that entropy of an isolated system can never decrease $dS/dt \geq 0$.

Inspired by Hawking's ideas, Jakob Bekenstein made the audacious hypothesis that the entropy of a black hole was proportional to the area of its event horizon. Bekenstein considered firing a single photon of energy $E = \omega$ into a black hole with a gravitational radius of $r_S = 2G_N m$ [77, 78, 79]. Making the wavelength of the photon comparable to the size of the black hole ensures that we add only one bit of information since in this case, we have no information on where exactly the photon entered the horizon, only whether or not it did. Thus, the incident photon of wavelength $\lambda \approx r_S$ and energy $E \approx 1/(r_S)$ increases the mass of the black hole by $\delta m = 1/(r_S)$, which changes the area of the horizon by

$$\delta A = r_S \delta r_S = C l_P^2, \tag{65}$$

where C is a constant of proportionality and $l_P^2 = G_N$.

The entropy of a black hole is a measure of the amount of inaccessible information. Thus, sending one bit of information into the black hole must increase its entropy since that bit of information has become inaccessible. Bekenstein argued that we should then equate the increase in horizon area A with an increase in entropy S due to the hidden bit of information, leading to the equality

$$S = \frac{k_B A}{C A_P}, \tag{66}$$

where $A_P = l_P^2$ is the Planck area and C is an undetermined constant of proportionality.

According to the laws of thermodynamics, anything that has an entropy must also have a temperature. Hawking pounced on this simple fact. Using semiclassical gravity he showed that black holes must emit thermal radiation at a specific temperature

$$T = \frac{\hbar c^3}{8\pi k_B M G_N}, \tag{67}$$

and in the process determined Bekenstein's constant of proportionality as $C = 1/4$, giving

$$S_{BH} = \frac{k_B A}{4 A_P}. \tag{68}$$

Heat is nothing more than the random jiggling of microscopic components. As we have seen black hole horizons are thermodynamical objects; they have a temperature, entropy, and an associated heat content. Therefore, event horizons should be made up of microscopic constituents that can store this heat content, just as a glass of water stores heat in the motion of atoms. Furthermore, since a black hole's horizon is merely empty spacetime, the microscopic constituents of the horizon must be parts of spacetime itself. The very fact that black holes can be heated up or cooled down suggests that spacetime is comprised of microscopic degrees of freedom—atoms of spacetime [76].

3.2. Evidence for a Minimal Length

Presumably, an atom of spacetime requires an atom of length. Here, we review the evidence for a minimum length scale using arguments that are mainly independent of any particular approach to quantum gravity.

3.2.1. A Lower Bound on Distance Measurements

Under a plausible set of assumptions, Wigner and Salecker [80] demonstrated in a straightforward way why one should expect a lower bound on any distance measurement. Wigner and Salecker start by assuming a universal speed of light and set about detailing the practical limitations of any length measurement. The authors ask us to consider a clock of mass M that periodically emits photons, which reflect off a mirror positioned at a distance D from the clock and return to their source. Given the photon's time-of-flight as recorded by the clock, coupled with the universal speed of light, it is then possible to determine the distance D. But can D be measured with arbitrary precision?

Let's say that we know the position of the clock to within a distance Δx when it initially emits the photon. Therefore, due to the uncertainty principle, this means that we cannot possibly know the velocity of the clock to a precision greater than

$$\Delta v = \frac{1}{2M\Delta x}. \tag{69}$$

Since the photon takes a time $T = 2D/c$ to complete its return trip to the mirror, the clock has then moved a distance $T\Delta v$, giving a total position uncertainty Δx_{total} for the clock of

$$\Delta x_{total} = \Delta x + \frac{T}{2M\Delta x}, \tag{70}$$

which of course sets a limit on the precision with which we can possibly measure D. Variation with respect to Δx gives the minimum uncertainty Δx_{min} with which D can be known as [81]

$$\Delta x_{min} = \sqrt{\frac{T}{2M}}. \tag{71}$$

Now, if our measurement of D is smaller than the Schwarzschild radius $r_S = 2G_NM$ of a black hole of mass M then the measurement will be causally disconnected from the rest of the universe. Thus, substituting the condition $D > 2G_NM$ into Eq. (8) using $T = 2D/c$ sets a lower bound on distance measurements of

$$\Delta x_{min} \geq l_P, \tag{72}$$

where we have used $G_N = l_P^2$.

3.2.2. Black Hole Limitations

Scardigli has put forward a gedanken experiment involving micro-black holes that seems to demonstrate the inherent limitations of measuring infinitesimally small distance scales [82].

At the remote Planck scale spacetime permits large fluctuations of the metric tensor which is typically modelled as a spacetime foam of virtual microscopic black holes. This model of quantum foam was originally proposed by John Archibald Wheeler in 1955 [83] and has since been used to compute many different properties of quantum gravity [84, 85], most notably providing an explanation for the microscopic origin of black hole entropy [86, 87]. Following Ref. [82], we shall now show how the formation of such microscopic black holes affects the process of measuring small distance scales.

Our starting point is the standard Heisenberg uncertainty relation $\Delta p \Delta x \geq \hbar/2$. Since at very high energies we have $\Delta E \sim c\Delta p$ we can then rewrite the standard inequality as $\Delta E \Delta x \geq \hbar c/2$. This modified inequality tells us that when one observes a region of spatial extent Δx the spacetime metric within that region is expected to be subject to quantum fluctuations of energy $\Delta E \sim \hbar c/(2\Delta x)$. The gravitational radius R_g associated with the energy ΔE is of course

$$R_g = \frac{2G\Delta E}{c^4}, \tag{73}$$

which is usually much smaller than the spatial extent Δx.

However, we now imagine compressing the observed region, which increases the amplitude of energy fluctuations ΔE and hence enlarges the associated gravitational radius R_g via Eq. (10), until eventually it attains the same width Δx. This critical length occurs at the Planck length l_p, or conversely at the Planck energy E_p, at which point a microscopic black hole forms [88] preventing the measurement of any distance smaller than its Schwarzschild radius. Attempting to probe even smaller distance scales by increasing the energy beyond $\Delta E = E_p$ simply results in a larger gravitational radius and thereby a larger minimum observable distance. In this scenario, the minimum observable distance is therefore l_p.

As pointed out in Ref. [82] it is important to note that this does not necessarily mean that spacetime is discrete, in fact, the continuity of spacetime holds up to any length in this gedanken experiment, it merely indicates the inability to measure an infinitesimal length. Whether or not something exists if it cannot be measured is an interesting question that has appeared many times in the history of physics, most notably in arguments surrounding the existence of a spacetime aether.

In the example above we found that distance measurements have a lower bound due to the formation of black holes. Might there also exist a limit to the precision with which the black hole itself can be measured? A simple argument suggests that the answer to this question is yes [89]. For a black hole with a mass M approaching the Planck mass, the associated gravitational radius $R_g \sim G_N M$ approaches its Compton wavelength $\lambda \sim 1/M$. At this point, the definition of the Schwarzschild horizon becomes blurred due to the large uncertainty in the position of the black hole.

This simple argument can be made a little more precise by following the work of Maggiore [90]. Suppose we wish to measure the horizon area of a non-rotating black hole with charge Q and mass M. In Boyer–Lindquist coordinates, the horizon is located at the radius

$$R = G_N M \left(1 + \left(1 - \frac{Q^2}{G_N M^2} \right)^{\frac{1}{2}} \right). \tag{74}$$

Maggiore considers attempting to determine the horizon area of the black hole by measuring its Hawking radiation, tracking it back to its localised point of origin with the greatest precision possible. Here we are considering an extremal black hole $Q^2 = G_N M^2$ for which the temperature is zero. Individual photons are fired at the black hole from asymptotic infinity and are subsequently re-emitted as Hawking radiation at some later time.

In addition to the usual uncertainty in the location of the horizon due to the finite wavelength $\lambda = c/\omega$ of the emitted photon, there is an additional uncertainty due to the change in the black holes mass from $M + \omega$ to M during emission, causing a change in the horizon radius. Since the energy of the photon is only known to a precision Δp the error propagates into the uncertainty in horizon radius [81]

$$\Delta R \sim \left| \frac{\partial R}{\partial M} \right| \Delta p. \tag{75}$$

Assuming naked singularities are forbidden in nature $M^2 G_N \leq Q^2$ and using Eq. (11) we find

$$\Delta R \gtrsim 2 G_N \Delta p. \tag{76}$$

Maggiore then argues that the simplest way to combine the standard uncertainty ($\propto 1/\Delta p$) with this additional uncertainty ($\propto G_N \Delta p$) is to linearly add them, giving

$$\Delta R \gtrsim \frac{1}{\Delta p} + \beta G_N \Delta p, \tag{77}$$

where β is a constant of proportionality that must be fixed by a particular theory of quantum gravity. Minimising the position uncertainty ΔR in Eq. (14) with respect to the momentum uncertainty Δp yields a minimum value

$$\Delta R \approx \beta l_p. \tag{78}$$

3.2.3. Heisenberg's Microscope

Heisenberg first derived a version of the position-momentum uncertainty principle by considering how to measure the position of an electron using a microscope. The precision with which one can determine the position of an electron Δx is limited by the wavelength λ of the electromagnetic wave one uses to make the measurement, $\Delta x \approx \lambda$. Due to the quantisation of the electromagnetic field into photons with discrete momenta $p = \hbar/\lambda$, a photon scattering from an electron must impart a non-zero component of its momentum to the electron, thereby introducing a non-zero uncertainty Δp. Thus, the position-momentum uncertainty principle is an inescapable consequence of making measurements using discrete momentum carrying particles. The standard position-momentum uncertainty principle is given by

$$\Delta p \geq \frac{\hbar}{2\Delta x}. \tag{79}$$

However, Eq. (16) is almost certainly incomplete, as it does not account for gravitational interactions induced by the act of observation [91, 90, 92, 93]. Since any scattering particle must have non-zero momentum it must also have non-zero energy. The gravitational field of the scattering particle must then cause a non-zero acceleration of the electron, thus perturbing its original position. This gravitational contribution to uncertainty seems an inevitable result of combining general relativity and quantum mechanics and is consequently considered a likely feature of quantum gravity.

Consider trying to measure the distance between two point-like particles via the scattering of electromagnetic radiation. We can estimate the magnitude of perturbations about flat Minkowksi spacetime using linearised general relativity which defines the metric tensor via

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{80}$$

where $h_{\mu\nu}$ describes a small perturbation $(|h_{\mu\nu}| \ll 1)$ about the Minkowksi metric $\eta_{\mu\nu}$. The field equations of linearised general relativity are

$$\Box \left[h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \right] = -\frac{8\pi G}{c^4} T_{\mu\nu}, \tag{81}$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu} = h^{\alpha}_{\alpha}$ and $\Box = \frac{\partial^2}{\partial (ct)^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$, with the Lorentz gauge condition

$$\left[h^{\nu}_{\mu} - \frac{1}{2}\eta^{\nu}_{\mu} h \right]_{\nu} = 0. \tag{82}$$

For electromagnetic radiation, such as that used in our measurement of distance, moving in the x-direction the energy-momentum tensor is

$$T_{mu\nu} = F_{\mu\alpha}F^{\alpha}_{\nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \rho \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{83}$$

where $\rho = (E^2 + B^2)/2$ is the energy density of the incident radiation field. Using this energy-momentum tensor we obtain an inhomogeneous solution involving only one unknown function f

$$h^{\nu}_{\mu} = f \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Box f = -\left(\frac{8\pi G}{c^4}\right)\rho. \tag{84}$$

Choosing this density to be a product of components $\rho = \rho_{\parallel} + \rho_{\perp}$ implies $f = f_{\parallel} + f_{\perp}$, with

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_{\perp}(y,z) = \left(\frac{8\pi G}{c^4}\right)\rho_{\perp}, \qquad f_{\parallel} = \rho_{\parallel}. \tag{85}$$

By characterising the incident envelope of electromagnetic radiation as a cylinder of length L and radius R this gives (in cylindrical coordinates)

$$f = \frac{4G(E/c^2)}{L} g(r)\theta_L = \frac{4Gp}{c^3 L} g(r)\theta_L. \tag{86}$$

Using Eqs. (21) and (23) the spacetime metric (with a signature $(+,-,-,-)$) can then be shown to take the simple form

$$ds^2 = c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right) + f\left(cdt - dx\right)^2. \tag{87}$$

Since near the cylinder $g(r) \approx 1$ the deviation of the metric from Lorentzian is $\sim 4Gp/c^3 L$, which again results in a minimum distance [91, 81, 93]

$$\Delta x_{min} \approx l_P. \tag{88}$$

3.2.4. High-Energy Convergence

It is well known that a minimal distance is capable of curing one of the biggest sicknesses in quantum gravity, namely its perturbative nonrenormalisablility. To see why a minimal length has this useful property consider a massless scalar field in a curved spacetime background [92]. In a classical spacetime, the Green's function takes the form

$$D(x,y) \sim \frac{1}{l_0^2(x,y) + i\varepsilon}. \tag{89}$$

In the limit of zero proper distance, namely as $x \to y$, the Green's function blows up to an infinite value.

However, in a non-classical spacetime the divergent behaviour at small distances is improved. Quantum fluctuations of spacetime are expected to replace the unique propagator $D(x,y)$ by its quantum expectation value [92]

$$D(x,y) \to \langle D(x,y) \rangle \sim \frac{1}{l_0^2(x,y) + l_{min}^2 + i\varepsilon}. \tag{90}$$

It is now evident from Eq. (27) that this modified propagator remains finite even in the zero-distance limit $x \to y$, because of the presence of a minimum proper length l_{min}, which acts as an ultraviolet cutoff. As is well known, a Green's function that remains finite even in the zero distance limit removes the infinities that normally plague quantum field theories in the ultraviolet [92].

Fourier transforming and Wick rotating Eq. (27) to Euclidean space yields the Euclidean propagator in momentum space as

$$\langle D(k) \rangle \sim \frac{l_{min}}{|k|} K_1 \left(l_{min} |k| \right), \tag{91}$$

where K_1 is the modified Bessel function. Expanding the modified Bessel function $K_1 \left(|k| \right)$ shows that the existence of a minimal length induces a damping effect on the propagator in the high-energy limit, leading to an ultraviolet convergence.

$$\langle D(k) \rangle = \begin{cases} \frac{1}{k^2}, & \text{for } k \to 0 \\ \left(\frac{l_{min}}{|k|^3} \right)^{\frac{1}{2}} e^{-l_{min}|k|}, & \text{for } k \to \infty. \end{cases} \tag{92}$$

3.2.5. Fluctuations of the Conformal Factor

In this section, we follow the work of Refs. [94, 95] and derive how fluctuations of the conformal factor affect the measurement of the classical proper length l between two spacetime events x_1 and x_2 at a given time t in flat spacetime. We begin by considering the path integral of quantum gravity

$$Z = \int \mathcal{D}g e^{iS_{EH}}, \tag{93}$$

where S_{EH} is the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_N} \int R(-g)^{\frac{1}{2}} d^4x \equiv \frac{1}{12 l_P^2} \int R(-g)^{\frac{1}{2}} d^4x. \tag{94}$$

The dominant contributions to the path integral should come from the classical solution $g_{\mu\nu} = \bar{g}_{\mu\nu}$. Since we are considering only conformal fluctuations the path integral of Eq. (30) should contain only metrics which are conformal to $\bar{g}_{\mu\nu}$, therefore

$$g_{\mu\nu} = (1 + \phi(x))^2 \, \bar{g}_{\mu\nu}. \tag{95}$$

We can now express the path integral in terms of ϕ via

$$Z = \int \mathcal{D}\phi \exp\left(-\frac{i}{2l_P^2} \int \left[\phi^\mu \phi_\mu - \frac{1}{6}\bar{R}\,(1+\phi)^2\right] (-\bar{g})^{\frac{1}{2}} d^4x\right). \tag{96}$$

Using this formalism it is possible to determine the probability $P[\phi(x)]$ that the conformal fluctuations have a value $\phi(x)$ in flat spacetime (see Refs. [96] and [97] for details), with the result

$$P[\phi(x)] = N \exp\left(-\frac{1}{4\pi^2 l_P^2} \int d^3x d^3y \frac{\nabla\phi(x) \cdot \nabla\phi(y)}{|x-y|^2}\right), \tag{97}$$

where

$$\phi(x) = \int q_k \exp i\boldsymbol{k} \cdot \boldsymbol{x} \frac{d^3\boldsymbol{k}}{(2\pi)^3}. \tag{98}$$

Consider an experiment capable of measuring the classical length l with a spatial resolution λ, hence if $|x_2 - x_1| < \lambda$ the points x_1 and x_2 cannot be distinguished. The experiment, therefore, measures the *coarse-grained* value of the field $\phi(x)$ averaged over a distance $\sim \lambda$. Defining $f(r)$ as the sensitivity profile of the measurement apparatus then the coarse-grained value of the field is given by

$$\phi_f(x) \equiv \int \phi(x+r)\, f(r)\, d^3 r. \tag{99}$$

We wish to determine the probability amplitude that the field $\phi_f(x)$ has a specific value η. Using the probability distribution given by Eq. (34) we find

$$P[\phi=\eta] = \int \mathcal{D}\phi(x)\, \delta(\phi-\eta)\, P[\phi(x)] =$$
$$N\int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \int \mathcal{D}\phi(x) \exp\left(i\lambda[\phi_f-\eta]\right) \exp\left(-\frac{1}{4\pi^2 l_P^2}\int d^3x d^3y \frac{\nabla\phi\cdot\nabla\phi}{|x-y|^2}\right). \tag{100}$$

Performing the integrations gives (see [95] for details)

$$P[\eta] = \left(\frac{1}{2\pi\Delta^2}\right)^{\frac{1}{2}} \exp\left(-\frac{\eta^2}{2\Delta^2}\right), \tag{101}$$

with

$$\Delta^2 = l_P^2 \int \frac{d^3k}{(2\pi^3)} \frac{|f(k)|^2}{2|k|}. \tag{102}$$

Assuming a Gaussian sensitivity profile

$$f(r) = \left(\frac{1}{2\pi\lambda^2}\right)^{\frac{3}{2}} \exp\left(-\frac{|\mathrm{r}|^2}{2\lambda^2}\right) \tag{103}$$

gives

$$\Delta^2 = l_P^2 \int \frac{d^3k}{(2\pi^3)} \frac{|f(k)|^2}{|k|} = \frac{l_P^2}{4\pi^2\lambda^2}. \tag{104}$$

The resolution λ is equal to σ, and so λ measures the width of $f(r)$ for any given distribution. For length measurements using a large resolution scale

$\lambda \gg l_P$, the factor Δ in Eq. (39) is vanishingly small and the probability distribution given by Eq. (38) is sharply peaked around the classical value $\eta = 0$, meaning that spacetime fluctuations have a negligible effect on length measurements. Nevertheless, as the measurement accuracy increases $\lambda \to l_P$ spacetime fluctuations begin to significantly affect the measurement of length. Via Eqs. (38) and (32) one can show that the probability two events (t,x) and (t,y) have a spatial separation l is

$$P(l) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(1-l_0)^2}{2\sigma^2}\right), \tag{105}$$

where

$$l_0 = |x-y|, \qquad \sigma^2 = l_0^2 \frac{l_P^2}{4\pi^2\lambda^2}. \tag{106}$$

To have a well-defined notion of the distance between two points requires $\sigma^2 \ll l_0^2$ which implies $\lambda \gg l_P$. What does this mean? Well, as the measurement resolution becomes greater and greater the concept of a definite length becomes more and more meaningless, the best we can do is ascribe a probability that a given measurement will result in a particular length. Thus, the notion of a definite proper length becomes nonsensical at scales below $\lambda \approx l_P$.

In Refs. [94, 95] Padmanabhan also determines a lower bound to proper length by considering the expectation value of the line interval

$$\langle 0|ds^2|0\rangle \equiv \langle g_{\mu\nu}(x)\rangle dx^\mu dx^\nu = \left(1 + \langle \phi^2(x)\rangle\right) dx^\mu dx^\nu \bar{g}_{\mu\nu}. \tag{107}$$

Since $\langle \phi^2(x)\rangle$ diverges, a covariant point-splitting into two separate points x^ν and $y^\nu = x^\nu + dx^\nu$ is employed [81, 94]. Taking the coincidence limit $x^\nu \to y^\nu$ then gives

$$\langle 0|ds^2|0\rangle = \lim_{dx\to 0} (1 + \langle 0|\phi(x)\phi(x+dx)|0\rangle)\, \eta_{\mu\nu} dx^\mu dx^\nu, \tag{108}$$

in a flat Minkowski background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Using the fact that

$$\langle 0|\phi(x)\phi(y)|0\rangle = \frac{l_P^2}{4\pi^2} \frac{1}{(x-y)^2}, \tag{109}$$

allows us to write Eq. (45) as [81]

$$\langle 0|ds^2|0\rangle = \frac{l_P^2}{4\pi^2}\lim_{dx\to 0}\frac{1}{(x-y)^2}\eta_{\mu\nu}dx^\mu dx^\nu = \frac{l_P^2}{4\pi^2}. \tag{110}$$

Therefore, a non-zero minimum length $\sim l_P$ arises due to the divergent two-point function of the scalar fluctuation counteracting any attempt to shrink the spacetime distance to zero.

3.2.6. Modified Feynman Propagator

In this section we follow the work of Ref. [98], and examine the modification of the path integral due to the duality transformation $l \to l_P^2/l$.

The Feynman propagator in standard quantum field theory for a spin-zero free particle of mass m in D-dimensional Euclidean space is[1]

$$G(\boldsymbol{x},\boldsymbol{y}) = \int \frac{d^D\boldsymbol{p}}{(2\pi)^D}\frac{\exp\left(-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})\right)}{p^2+m^2}. \tag{111}$$

Since the discrete path integral for the Feynman propagator is given by

$$G(\boldsymbol{x},\boldsymbol{y}) = \sum_{paths}\exp\left(-ml(x,y)\right), \tag{112}$$

the action receives a contribution of $-ml$ from a path of length l. A minimal length scale l_P should then modify the path integral by suppressing the contribution from paths of length $l \ll l_P$. In Ref. [98] this suppression is implemented by assuming the contribution from a given path is invariant under the transformation $l \to l_P^2/l$. This so-called principle of duality is intimately linked with the introduction of a minimal length and seems to be related to the notion of T-duality in string theory [99], where it was shown that string theory can be incorporated into quantum field theory via such a transformation. It has also been noted that the generalised uncertainty principle (GUP) is invariant under this transformation [93].

[1] A spin-zero scalar field is considered for the sake of simplicity, however it is easy to extend the described formalism to more complicated scenarios.

To make the path integral of Eq. (49) meaningful Ref. [98] introduces a cubic lattice in D-dimensional Euclidean space with a lattice spacing ε. The propagator is then evaluated in the limit $\varepsilon \to 0$, yielding

$$G(\boldsymbol{p}) = \frac{2l_P}{\sqrt{p^2+m^2}} K_1\left(2l_P\sqrt{p^2+m^2}\right), \tag{113}$$

where K_1 is the first order modified Bessel function. Equation. (50) has the limiting behaviour

$$G(\boldsymbol{p}) = \begin{cases} \frac{1}{p^2+m^2}, & \text{for } m_P \gg \sqrt{p^2+m^2} \\ \frac{\exp\left(-2l_P\sqrt{p^2+m^2}\right)}{\sqrt{2l_P}(p^2+m^2)^{\frac{3}{4}}}, & \text{for } m_P \ll \sqrt{p^2+m^2}. \end{cases} \tag{114}$$

Thus, this duality transformation of the path integral suppresses contributions to amplitudes below the Planck scale. In position space, the Feynman propagator differs from the ordinary one by the shift $l^2 \to l^2 + l_P^2$, which in the limit of zero classical distance yields a minimal length $\sim l_P$.

3.2.7. Lattice Quantum Gravity

We now follow the work of Ref. [100] and study a lattice formulation of linearised gravity.

Consider a lattice regularisation of linearised gravity, where spacetime is approximated by a hypercubic lattice of points, with ordinary derivatives replaced by discrete lattice derivatives. Let us assume that there exists a lattice formulation of quantum gravity that admits a perturbative expansion of Newton's constant, such that the leading order term is equivalent to a lattice regularisation of the continuum theory.

The average physical distance between two points on such a hypercubic lattice along a given axis, say the x-axis, in $D = 4$ dimensions can be approximated by

$$\begin{aligned} \langle l^2 \rangle &= \langle g_{11} \rangle (\delta x)^2 \\ &= \left[1 + 16\pi G_N \delta_{ab} \langle b_1^a(x) b_1^b(x) \rangle\right] \varepsilon^2 \\ &= \varepsilon^2 + 16\pi G_N \gamma^2, \end{aligned} \tag{115}$$

where b^a_μ is related to the definition of the metric in tetrad gravity [100], ε is the lattice spacing, γ is a constant of order one, and we have used the approximation

$$\delta_{ab}\langle b^a_i(x) b^b_i(x)\rangle \approx \frac{\gamma^2}{\varepsilon^2}. \tag{116}$$

Utilising the fact that $G_N = l_P^2$, Eq. (52) then gives

$$\langle l^2\rangle = \varepsilon^2 + 16\pi\gamma^2 l_P^2, \tag{117}$$

so that as the lattice spacing is taken to zero ($\varepsilon \to 0$) we retain a non-zero average distance $\langle l^2\rangle \sim l_P^2$. Therefore, any formal reduction of the lattice spacing is counteracted by an increase in metric fluctuations such that the average physical separation between neighbouring points has a non-zero minimum.

Reference [100] also considers a more sophisticated lattice regularisation of quantum gravity first proposed by Regge [101] where the lattice links have a dynamical edge length. Using a specific gauge-fixing (see Ref. [102] for details) the mean-squared link length is also found to have the same functional form as Eq. (54), again suggesting a minimal length.

3.3. Special Relativity and a Minimal Length

Minimal length scenarios seem to directly contradict the most fundamental symmetry in modern physics, Lorentz invariance (LI). Since there is no lower bound to Lorentz contraction in special relativity, for any fundamental length in one observer's rest frame one can always define a Lorentz boosted frame that will measure a yet shorter length. The existence of a minimal length, therefore, seems to require a modification of special relativity.

Motivated by such considerations a few alternative formulations of special relativity have now been explored. Perhaps the most notable attempt is that of doubly special relativity (DSR), which was first proposed by Pavlopoulos as far back as 1967 [103]. DSR attempts to define an observer-independent maximal velocity c and minimal length l_{DSR}. However, the speed of light in DSR is in general wavelength dependent $c(\lambda)$ and only maximal in the infrared limit, i.e., for $\lambda/l_{DSR} \to \infty$. As such, DSR is able to maintain an observer-independent minimal length but typically at the cost of deforming or violating

LI. The first exploration of such a DSR-type model by Pavlopoulos estimated this minimal length to be approximately $l_{DSR} \sim 10^{-15}m$ [103]. However, in 2000 Amelino-Camelia sought a definition of DSR in which the Planck length defined an observer-independent minimum length scale, thus making DSR relevant to quantum gravity [104, 105]. Since then there have been a number of reformulations and closely related approaches. In particular, Kowalski-Glikman reformulated Amelino-Camelia's DSR in terms of an observer-independent Planck mass [106], rather than Planck length, and Magueijo and Smolin developed a related model based on an observer-independent Planck energy [107].

DSR theories typically consider an energy dependent speed of light with an associated dispersion relation that takes the form

$$E^2 = p^2c^2 + f\left(E, p; l_P\right), \tag{118}$$

where f is a function that has a leading order l_P dependence of $f\left(E, p; l_P\right) \approx l_P cEp^2$ [104]. This dispersion relation then leads to a deformed momentum-dependent speed of light

$$v_\gamma(p) = c\left(1 + \frac{1}{2}l_P|p|\right), \tag{119}$$

which could, at least in principle, be experimentally tested via time-of-flight observations of distant gamma-ray bursts [108].

Although well motivated, DSR suffers from a number of significant problems. For example, it is not clear how to describe everyday macroscopic objects within the DSR framework [109]. It is possible that a Planck scale deformation may be consistent with observations of low energy systems $E \ll E_P$, as the predicted deviations are beyond current experimental sensitivities [109]. However, for multiparticle systems with aggregate energy $E \gg E_P$, the predicted deviation directly contradicts experiment and even the everyday observation of macroscopic objects.

This so-called soccer-ball problem stems from the non-linearity of the theory, namely the transformation of the sum of momenta does not equal the sum of the transformation of momenta [110]

$$\tilde{\Lambda}\left(p_1 + p_2\right) \neq \tilde{\Lambda}\left(p_1\right) + \tilde{\Lambda}\left(p_2\right). \tag{120}$$

Also, if $\tilde{\Lambda}$ is the unit element of the group, then this equation is fulfilled, and thus the rest-frame singles out a preferred frame of reference in DSR [110]. A further problem is that DSR is also *a priori* constructed in momentum space, with a description in terms of physical position space presenting a number of difficulties [111, 112]. These problems constitute potentially serious obstacles for the development of DSR that must be satisfactorily addressed.

3.4. Phenomenological Quantum Gravity

Quantising gravity means quantising spacetime itself, which presumably requires the quantisation of length. Yet, as we have seen, special relativity precludes a minimum length scale. A second generic prediction of quantum gravity is the vacuum dispersion of light. Coupling the equivalence of energy and mass with Heisenberg's energy-time uncertainty principle dictates that the greater the resolving power with which we view spacetime the greater the observed energy fluctuations in the quantum vacuum. Thus, quantum gravity generically predicts that when viewed on sufficiently small distance scales one will observe large metric fluctuations of spacetime. Since higher energy photons probe spacetime on shorter distance scales than lower energy ones they are expected to encounter proportionately larger metric fluctuations, which hinder their passage through spacetime to a greater extent. Hence, if the spacetime fluctuations predicted by quantum gravity are real then the speed of light should depend on its energy and LI may be deformed or violated.

Since we do not have a complete theory of quantum gravity, the predicted energy dependent speed of light due to vacuum fluctuations is usually estimated by assuming the deformation admits a series expansion at small energies E compared with the energy scale E_{QG} at which quantum gravity is expected to dominate (presumably $E_{QG} \approx E_P$) [108, 113], yielding a deformed dispersion relation

$$E^2 \simeq p^2 \left(1 - \sum_{n=1}^{\infty} \pm \left(\frac{E}{E_{QG}} \right)^n \right), \tag{121}$$

and hence an energy dependent speed of light

$$c(E) \simeq \sqrt{1 - \sum_{n=1}^{\infty} \pm \left(\frac{E}{E_{QG}} \right)^n}. \tag{122}$$

In Eq. (59) the conventional speed of light in the zero-energy limit c is set equal to unity. The $\pm$ ambiguity in Eq. (59) denotes either subluminal $(+1)$ or superluminal (-1) motion [108]. In an effective field theoretic description $n = d - 4$, where d is the dimension of the leading order operator. Within effective field theory, it has been shown that odd values of d directly violate CPT invariance, and so a common expectation is that $n = 2$ will be the leading order term [114, 115].[2]

After more than a century adrift in a sea of speculation, quantum gravity is now becoming an experimental science. Recent observations of distant gamma-ray bursts by the Fermi space telescope find the speed of light to be independent of energy up to 7.62 E_p (with a 95%CL) and 4.8 E_p (with a 99%CL) for linear dispersion relations, where E_p is the Planck energy. The same data has even been used to tightly constrain quadratic dispersion relations [108]. Moreover, variations in the speed of light have been experimentally excluded to the incredible precision of $\Delta c/c < 6.94 \times 10^{-21}$, conclusively showing that space-time appears smooth beyond the Planck energy [116]. The collective data of the *Chandra* and *Fermi* space telescopes and the ground-based *Cherenkov* telescopes have already been used to rule out some approaches to quantum gravity based on their predictions of vacuum dispersion [117, 118, 119]. An increasingly large number of experiments, using a variety of different techniques, also find results consistent with an energy independent speed of light, including particle decay processes [120], time-of-flight comparisons [121], neutrino oscillation experiments [122] and ultra-high-energy cosmic ray observations [123].

There also exist compelling theoretical arguments preventing an energy dependent speed of light. For example, Polchinski has argued that almost all theories of quantum gravity predicting vacuum dispersion at high energies have already been ruled out by precision low-energy tests, due to the way such effects scale with energy, with the only possible exception being supersymmetric theories [124]. Vacuum dispersion also opens the door to causality violations [120],

[2] However, it is important to note that it is currently unknown how reliable effective field theory is at or near the Planck scale.

closed timelike curves [120] and other implausible scenarios. The accumulative weight of experimental and theoretical evidence now strongly suggests that spacetime remains a smooth manifold up to a significantly higher resolution than typically predicted by quantum gravity.

However, the more fundamental a symmetry is the more thoroughly it should be tested. This is especially true of LI. Over the coming decade, a brand new generation of telescopes and astronomical surveys will yield a thousand-fold gain in cosmological and astronomical data. Current detectors such as the Fermi Gamma-ray Space Telescope, Chandra X-ray Observatory, and Pierre Auger Observatory, in addition to future telescopes and surveys, will provide unprecedented amounts of experimental data that will further constrain violations of LI.

Chapter 4

The Renormalisation Group

4.1. Overview

The renormalisation group (RG) is a mathematical formalism defining how a physical system changes as a function of the scale at which it is viewed. When describing phenomena as a function of scale the value of parameters measured at large distances typically differs from those made at small distances. For example, in QED the charge of an electron appears to be scale dependent [125, 126]. Since virtual particle pairs permeate the vacuum they become polarised by the presence of the electron's charge. This polarisation tends to screen the strength of the electric charge as measured at large distances. However, at small distances there is less screening, therefore the strength of the electric charge is inversely proportional to distance, or conversely, is proportional to energy. The observation that coupling constants are not, in fact, constant, but are in general dependent on the energy scale at which they are measured, led to the idea of the renormalisation group (RG) [127].

A key concept in the definition of the renormalisation group is the introduction of a high-energy cut-off scale Λ, or conversely, a minimum length scale $l_{min} \sim \Lambda^{-1}$, beyond which physical contributions are neglected. Physical theories typically remain valid only over a limited range of length scales. A water wave is accurately described as the disturbance of a continuous fluid via the laws of hydrodynamics, without any knowledge of the underlying discrete atomic structure of water molecules. That is, hydrodynamics does not care about physical dynamics below some minimum length scale Λ^{-1}, where Λ^{-1} is much greater than the average inter-molecular distance between water molecules [127].

Likewise, when considering the motion of discrete water molecules atomic distances become relevant, however, one does not need to know about physics at or below nuclear scales. Thus, physical interactions below a yet smaller nuclear length scale Λ'^{-1} can be safely neglected in such a description. The question now arises as to whether we can change the cut-off scale Λ^{-1} continuously and determine how the resulting laws of physics vary? In the water wave example, a completely new set of equations and parameters are needed every time a new cut-off scale is defined making it difficult to study the effects of a continuously varying Λ^{-1}. However, there are a number of examples in which the observations made above can be generalised so that the cut-off scale Λ^{-1} can be varied continuously in a well-defined manner. Perhaps the most pedagogical example is given by the block-spin model.

4.2. Kadanoff's Block-Spin Model

Following Kadanoff [128], let's consider a 2-dimensional square array of atoms forming a perfect crystal lattice, as depicted in Fig. 4.. Each point on the lattice has either a spin up or a spin down value. In Fig. 4., filled lattice points represent atoms with spin down and unfilled lattice points denote atoms with spin up. Let the probability that neighbouring spins are aligned in the same direction at a given temperature T be characterised by a coupling parameter J.

Let's assume the physical state at each of the 64 lattice points in Fig. 4. can be completely characterised by its Hamiltonian $H(T,J)$ for given values of T and J. Now we divide the lattice into 2×2 blocks. The physical state of each block is then described by a new Hamiltonian $H(T',J')$, where the variables T' and J' quantify the average behaviour of each 2×2 block, so that we are now effectively analysing only $1/4$ of the atoms. Performing another iteration, by averaging over 4×4 square blocks, amounts to analysing $1/16$ of the atoms, where each block is now described by the Hamiltonian $H(T'',J'')$. Thus, with each successive iteration, or RG step, we are effectively decreasing the resolving power of a hypothetical microscope. Finding the RG transformation for a given system is then effectively equivalent to finding the transformation law that takes

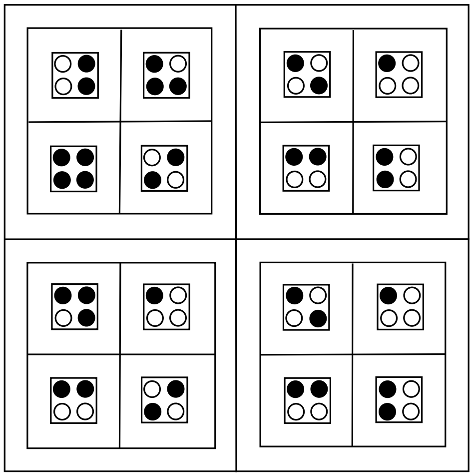

Figure 4. A 2-dimensional square lattice of atoms each with an associated spin. Filled lattice points represent atoms with spin down and unfilled lattice points denote atoms with spin up. The system may be described by block variables that average over the behaviour of a given block size. Each renormalisation step is then equivalent to averaging over successively smaller square block sizes.

$H(T,J) \to H(T',J')$ and $H(T',J') \to H(T'',J'')$, etc.

4.3. The Beta Function

How a particular coupling g changes with respect to the energy scale μ of a given physical process is governed by its so-called β-function

$$\beta(\tilde{g}(\mu)) = \mu\frac{\partial g}{\partial \mu} = \frac{\partial g}{\partial \ln \mu}. \tag{123}$$

Clearly, if the β-function of a particular theory goes to zero for some value of the coupling g then the theory has become scale invariant, since its coupling remains constant as a function of the scale μ.

Beta functions are typically estimated using an approximation method such as perturbation theory. Performing a perturbative expansion in powers of the coupling allows one to neglect higher-order contributions, as long as the coupling is assumed to be small from the outset. As an example, the beta function of the fine structure constant α in QED computed in perturbation theory is

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi}. \tag{124}$$

The positive sign of the QED beta function tells us that the magnitude of the coupling α increases with the energy scale μ, becoming infinite at finite energy, resulting in the infamous Landau pole [129]. However, this Landau pole is typically thought to be the result of applying perturbation theory beyond its range of validity. Namely, at very high energies (in the strong coupling regime) perturbation theory is no longer expected to remain valid and it may thus be possible to evade the troublesome Landau pole.

In quantum chromodynamics (QCD), the quantum field theory of the strong interaction, the one-loop beta function for the strong coupling α_s is given by

$$\beta(\alpha_s) = -\frac{\alpha_s^2}{2\pi}\left(11 - \frac{n_S}{3} - \frac{2n_f}{3}\right), \tag{125}$$

where n_S is the number of scalar Higgs Bosons and n_f the number of flavors of fundamental multiplets of quarks [130]. The negative sign of the QCD beta function means that α_s actually decreases with an increasing energy scale μ, a property known as asymptotic freedom [130]. Therefore, the strong coupling regime for QCD corresponds to low energy scales, where perturbation theory is no longer valid. Similarly, the coupling associated with the weak interaction decreases in magnitude with increasing energy.

Extrapolating the beta functions for the strong, weak and electromagnetic interactions to extremely high energies hints at a possible unification at the so-called grand unification energy scale of $E_{GU} \sim 10^{16} GeV$. However, it has been noted that this unification is not exact, with the three couplings missing each other by a non-negligible amount. As pointed out by a number of authors, this unification does appear to become exact if one includes supersymmetry [131].

4.4. Renormalisation Group Operators

As one increases the energy scale μ the magnitude of a coupling or observable under the renormalisation group transformation can either increase, decrease, or stay the same. Renormalisation group trajectories exhibiting such behaviour are said to be relevant, irrelevant or marginal trajectories, respectively. Relevant operators are essential to the description of macroscopic systems, whereas irrelevant ones are not. For a given marginal operator it is unclear whether it contributes to macroscopic physics because it can be ill-defined. Macroscopic physics is typically described by only a few observables, whereas microscopic physics is typically characterised by a very large number of variables. This implies that most observables are irrelevant, namely that the microscopic degrees of freedom are not relevant on the macroscopic level.

Chapter 5

The Asymptotic Safety Scenario

5.1. Weinberg's Great Idea

The problems raised by the perturbative nonrenormalisability of gravity have spawned a multitude of competing solutions. A common approach is to assume that the reason quantum mechanics and general relativity cannot be unified in a straightforward manner is that one of the two theories needs to be modified. The most popular notion is to regard quantum mechanics as the more fundamental theory, and that general relativity is just an effective field theory that is only valid up to some high-energy cut-off. Typically, this is the route taken by perturbative approaches to quantum gravity such as superstring theory. However, there does exist a possibility of defining a quantum field theory of gravity that is valid over all energy scales despite being perturbatively nonrenormalisable. This approach, known as the asymptotic safety scenario, was developed by Steven Weinberg in the 1970s [37].

In general relativity, the fundamental degrees of freedom are given by the metric field. Fluctuations of space-time would then modify the strength of the gravitational coupling G_N as a function of energy, in an analogous way to the scale dependence of electric charge in QED. We now follow the discussion of asymptotic safety given by Weinberg in [37] in which he considers a generalised set of couplings $g_i(\mu)$ as a function of a renormalisation scale μ. If $g_i(\mu)$ has mass dimensions of $[M]^{D_i}$ then we replace $g_i(\mu)$ with a dimensionless coupling $\tilde{g}_i(\mu)$,

$$\tilde{g}_i(\mu) = \mu^{-D_i} g_i(\mu). \tag{126}$$

The behaviour of the RG flow of the coupling g_i is governed by its β-function, which describes the rate of change of g_i with respect to the scale μ,

$$\beta_i(\tilde{g}(\mu)) = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \tilde{g}_i(\mu). \tag{127}$$

To prevent the couplings $\tilde{g}_i(\mu)$ from diverging as $\mu \to \infty$ we can force them to end on a fixed point, g^*, in the limit $\mu \to \infty$. This requires that the beta-function of $\tilde{g}$ vanish at this point,

$$\beta_i(g^*) = 0. \tag{128}$$

Additionally, the couplings must lie on a trajectory $\tilde{g}_i(\mu)$ which actually hits the fixed point. The surface formed from such trajectories is called the ultraviolet critical surface. If the couplings of a particular theory lie on the ultraviolet critical surface of a fixed point then the theory is said to be asymptotically safe.

There exist two types of fixed points, trivial fixed points where $g^* = 0$ and non-trivial fixed points (NTFP) where $g^* \neq 0$. The trivial fixed points are said to be non-interacting because the coupling constant vanishes; the immediate vicinity of this trivial fixed point is described by perturbation theory. This is the construction underlying asymptotic freedom. Conversely, a non-trivial fixed point lies at non-zero values of the coupling constants, and so is said to be an interacting fixed point.

QCD is a special type of asymptotically safe theory because it is also perturbatively renormalisable and asymptotically free. QCD contains a fixed point with only a finite number of attractive renormalisation group trajectories in the ultraviolet, namely the Yang-Mills coupling and the quark masses [132]. All other couplings are repelled from the fixed point and must be set to zero in order to give a well-defined ultraviolet limit. QCD is also asymptotically free because colour charge is anti-screened by quark-antiquark pairs, and so at asymptotically high energies quarks become free.

An important consideration when dealing with a NTFP is the dimensionality of its ultraviolet critical surface. The dimensionality of an ultraviolet critical surface is given by the number of renormalisation group trajectories that are attracted to the fixed point in the ultraviolet, i.e., the number of relevant trajectories. If an infinite number of such trajectories were attracted to the fixed point then we would be in the same situation as for perturbative renormalisation, namely a complete loss of predictive power [37]. The ideal situation is one in which the number of relevant couplings is minimised. For example, if there was just a single trajectory that ended at the fixed point in the ultraviolet then the theory would be maximally predictive.

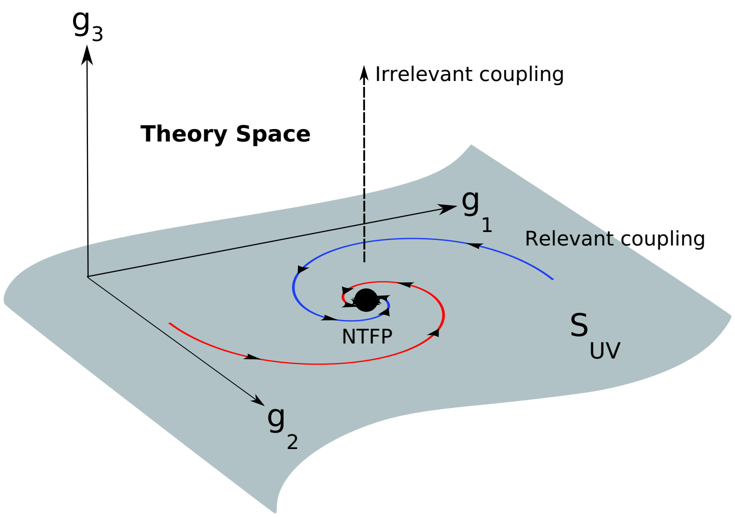

Figure 5. A schematic of the ultraviolet critical surface S_{UV}. Trajectories that are repelled from the NTFP are irrelevant couplings. Trajectories that are attracted to the fixed point are relevant couplings. The theory space is defined by the couplings g_i. In the case of the Einstein-Hilbert truncation the couplings are $G_N(k)$ and $\Lambda(k)$. Schematic derived from the work of Refs. [2, 3, 4].

General relativity without matter content is described by the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}(-2\Lambda + R), \tag{129}$$

where G_N is Newton's gravitational constant, Λ is the cosmological constant, g is the determinant of the metric $g_{\mu\nu}$, and R is the Ricci scalar curvature, a function of the metric. If we replace the couplings G_N and Λ by the scale dependent running couplings $G_N(k)$ and $\Lambda(k)$, and recognising that the Einstein-Hilbert action is now a scale dependent functional $\Gamma(k)$ we can write,

$$\Gamma(k) = \frac{1}{16\pi G_N(k)} \int d^4x\sqrt{-}g(-2\Lambda(k) + R + ...). \tag{130}$$

The functional of Eq. (5) in principle contains infinitely many couplings. The renormalisation group flow of the functional $\Gamma(k)$ is defined by the exact renormalisation group equation (ERGE), and so the ERGE contains the beta-functions of all of the couplings present in the functional. Although exact solutions have been found, e.g., for supersymmetric $O(N)$ theories in the infinite N limit [133, 134], the general form of the ERGE is not exactly solvable, and so one is typically forced to perform truncations of the functional $\Gamma(k)$ such that it contains only a finite number of couplings. The beta-functions can then be extracted from the ERGE without any further approximation [135]. The simplest truncation is the Einstein-Hilbert truncation, in which only the cosmological constant and Newton's constant are retained. The RG flow will typically induce terms quadratic in curvature. However, under the Einstein-Hilbert truncation, these contributions are neglected. A more accurate treatment takes the curvature squared terms into account, which is known as the curvature squared truncation. One can then extend this series of truncations up to terms polynomial in the scalar curvature; this has been done up to a polynomial of order eight and higher, demonstrating the existence of a fixed point with a three-dimensional critical surface [136, 137, 138]. Recent RG results have even reported calculations with truncations to a polynomial of order thirty-four in the Ricci scalar [139]. See Ref. [140] for an overview of asymptotic safety.

An example of how the ERGE is applied comes from performing a numerical integration of the β-function extracted from the ERGE under the Einstein-Hilbert truncation, i.e., for only $G_N(k)$ and $\Lambda(k)$. This gives the following picture [5]:

At the origin one observes the trivial fixed point corresponding to the non-interacting theory. For a non-zero value of Newton's constant and the cosmological constant one may observe a NTFP that corresponds to an interacting theory. The fixed point for gravity, if it exists, must be interacting; since gravity is perturbatively nonrenormalisable.

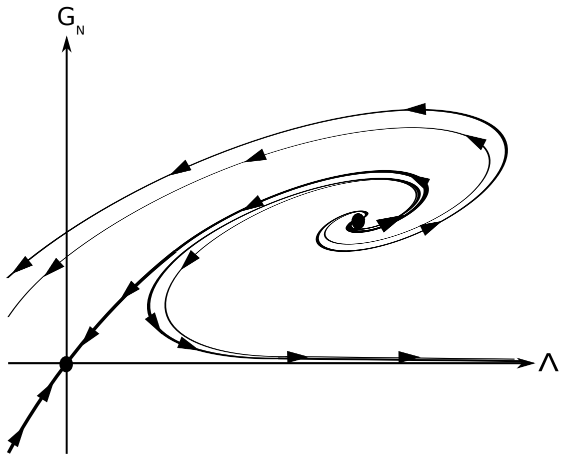

Figure 6. Renormalisation group flow of gravity in the Einstein-Hilbert truncation. Arrows indicate RG flow in the direction of increased coarse-graining, or conversely decreasing momentum scale k. Plot adapted from Ref. [5].

Although the renormalisation group studies are suggestive, the truncation of the effective action makes it difficult to systematically assess the reliability of the results obtained using this method[1]. A lattice formulation of gravity is thus desirable, given the possibility of performing calculations with controlled systematic errors. Before turning to a lattice formulation of quantum gravity we briefly review a potential problem with the asymptotic safety scenario.

5.2. A Potential Problem

There exists a long-standing argument against the asymptotic safety scenario. This challenge to asymptotic safety was first put forward by Banks [144] and later detailed by Shomer [145] and suggests gravity cannot possibly be a renormalisable quantum field theory. Their argument is that since a renormalisable quantum field theory is a perturbation of a conformal field theory (CFT) by relevant operators, the high-energy spectrum of any renormalisable quantum field

[1] Although systematic error control in ERGE studies is possible and has been exemplified in the work of Refs. [141, 142], for example. Error estimates in quantum gravity in dimensions greater than four have also been studied [143].

theory should be equivalent to that of a CFT. However, this is not true for gravity [145]. We now review this problem in more detail.

Using dimensional analysis, the extensivity of the quantities involved and the fact that a finite-temperature conformal field theory has no dimensionful scales other than the temperature T, the entropy S of any 4-dimensional CFT must scale according to

$$S \sim a(rT)^3, \tag{131}$$

and the energy E according to

$$E \sim br^3T^4, \tag{132}$$

where a and b are dimensionless constants and r is the radius of spacetime that we are considering. From Eqs. (6) and (7) we have

$$S \sim \frac{aE}{bT}. \tag{133}$$

From Eq. (7) we obtain

$$T \sim \frac{E^{\frac{1}{4}}}{b^{\frac{1}{4}} r^{\frac{3}{4}}}. \tag{134}$$

Simply substituting Eq. (9) into Eq. (8) tells us that the entropy of a CFT is[2]

$$S_{CFT} \sim \frac{a}{b^{\frac{3}{4}}} E^{\frac{3}{4}} r^{\frac{3}{4}}. \tag{135}$$

For gravity, on the other hand, it is expected that the high-energy spectrum will be dominated by black holes [144, 145, 146]. The entropy of a semi-classical black hole is given by the celebrated area law

$$S_{Grav} = \frac{A}{4G_N} = \frac{\pi r^2}{l_P^2}, \tag{136}$$

[2]The scaling of Eq. (10) differs from that proposed by Shomer [145], according to which $S \sim E^{\frac{3}{4}}$ in 4 dimensions, but agrees with that found by Falls and Litim [146]. As pointed out by Falls and Litim the scaling relation $S \sim E^{\frac{3}{4}}$ can only be correct if the radius r is assumed to be constant, an assumption that is unjustified in this context.

where r is the Schwarzschild radius of the black hole and $G_N = l_P^2$. It is now evident that Eqs. (10) and (11) do not scale in the same way. Thus, Banks and Shomer conclude that gravity cannot be a renormalisable quantum field theory [144, 145]. This is a potentially serious challenge to quantum gravity that must be satisfactorily addressed. We will revisit this problem in sections 7.3. and 8.2.4.

Chapter 6

Quantum Gravity on the Lattice

6.1. Lattice Regularisation

The quantisation of a continuous field theory involves introducing an infinite number of degrees of freedom, which can lead to divergent results. To prevent this, one typically regulates the theory via the introduction of an ultraviolet cut-off. A theory can be regularised by taking a high-energy limit, above which the field theory is no longer applicable, or conversely one can impose a minimum length scale. The latter is the approach taken by lattice field theories, which originated from the seminal work of K. Wilson [127]. Lattice field theories replace continuous spacetime with a discrete lattice of points with non-zero spacing a.

The method of lattice regularisation allows one to achieve a stable ultraviolet regularisation, yielding finite observables. Lattice methods also allow one to recover continuum limit physics by calculating observables at different lattice spacings and extrapolating to the continuum limit, i.e., sending $a \to 0$. In addition to taking $a \to 0$ one must also take the infinite volume limit, so as to eliminate finite-size effects, whilst simultaneously making appropriate adjustments to the bare coupling constants. The existence of a continuum limit requires the presence of a second-order phase transition because the divergent correlation length characteristic of a second-order phase transition allows one to take the lattice spacing to zero whilst simultaneously keeping observable quantities fixed in physical units.

For strongly interacting field theories the applicability of perturbation theory is limited, and nonperturbative lattice methods become essential. Lattice QCD (LQCD), for example, has become the central tool in developing our understanding of the strong interaction, allowing one to calculate the hadronic spectrum and weak matrix elements to high precision [147]. LQCD also contributes to our understanding of confinement, chiral symmetry breaking, and finite temperature QCD. See Refs. [148, 149] for more detailed accounts of the methods and successes of LQCD. Lattice regularisation methods have also been applied in Beyond the Standard Model (BSM) physics, most notably in supersymmetric lattice field theories and technicolour theories. See Ref. [150] for an overview of supersymmetric lattice field theory, and Ref. [151] for a more general overview of the application of lattice methods to BSM physics.

Motivated by the successful application of lattice methods to the strong interaction, and their use in studying BSM physics, one is led to ask whether applying similar methods to the gravitational interaction may aid our understanding of nonperturbative quantum gravity.

6.2. Geometric Observables

General relativity describes gravity as the curvature of spacetime, and quantum mechanics describes spacetime on the smallest distance scales. So if we are to unify general relativity and quantum mechanics we must understand what spacetime looks like on the smallest distances. One idea is that spacetime is discrete at the Planck scale, meaning that there exists a smallest possible unit of both space and time, namely the Planck length and the Planck time. A further complication comes from the fact that quantum mechanics describes observables as being in a superposition of all possible states until observation has occurred, and therefore that the geometry of spacetime at the Planck scale could consist of a superposition of all possible geometries. These complications make it very difficult to study the universe on such small scales. One simplifying approach is to approximate the smooth continuous geometry of spacetime by a lattice of locally flat n-dimensional triangles called simplices. In this way one can numerically generate a large ensemble of configurations that samples all the possibilities of curvature and geometry, allowing one to calculate specific observables by taking the expectation value over the entire set of configurations.

The simplicial approach to quantum gravity, as first proposed by Regge [101], allows one to make use of the considerable number of techniques used in lattice gauge theories. In a similar way to lattice gauge theories, one is not restricted to discrete spacetime, since one can allow the lattice size to approach infinity and take the simplicial edge length to zero. Under these limits, one would effectively remove the ultraviolet lattice regulator and recover continuum physics. Regge calculus is not dependent on simplices either; any shape that can be extended into a n-dimensional polytope can be used in the same way. Simplices are chosen because they are essentially the simplest possible polytope that one can use to approximate n-dimensional manifolds.

A n-dimensional simplex is a polytope, where the number of k-dimensional faces N_k is given by

$$N_k = \left(\begin{array}{c} n+1 \\ k+1 \end{array} \right) = \frac{(n+1)!}{(k+1)!\,((n+1)-(k+1))!}. \tag{137}$$

A 4-simplex, therefore, has 5 vertices, 10 edges, 10 triangular faces, and 5 tetrahedral faces. Simplices are fitted together along their $(n-1)$-dimensional faces creating a simplicial piecewise linear manifold. Simplicial quantum gravity can be subdivided into two main approaches, Regge calculus, and dynamical triangulations. The approach of Regge calculus is to fix the connectivity of the simplices but allow the edge lengths to vary; in this way the dynamics of the geometry is captured. Conversely, dynamical triangulations fixes the simplicial edge lengths but allows the connectivity of the simplices to define the dynamics of the geometry.

The interior geometry of a n-simplex is assumed to be flat, and so in the case of dynamical triangulations, the dynamics of a given simplex is entirely contained within the connectivity of the $(n+1)/2$ fixed edge lengths. The curvature in simplicial quantum gravity is defined by the deficit angle around a $(n-2)$-dimensional face at which arrangements of n-simplices converge. If we take the example of a lattice of 2-simplices, or triangles, then the idea becomes clear. Imagine joining 6 such triangles together so that they meet at a single shared vertex, as shown in Fig. 7..

One can define a complete angle of 2π radians about this vertex. However, upon removing one of these triangles one now has a deficit angle δ. The remaining simplices can now be joined together along their piecewise linear edges to form

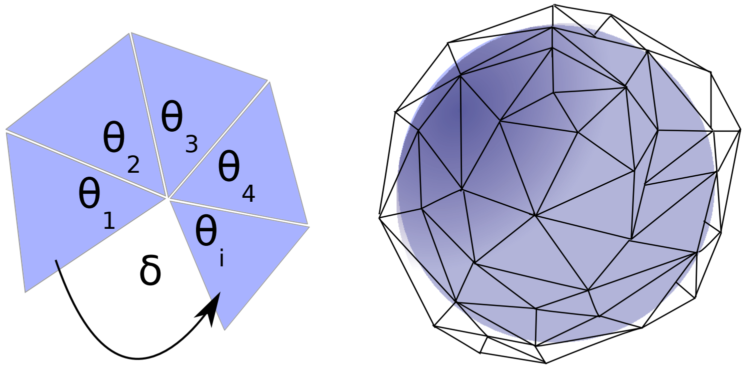

Figure 7. The deficit angle δ for a 2-dimensional lattice (left) and a simplicial approximation to a sphere (right) [1].

a curved geometry, as shown in Fig. 7. (right). The angle δ is then a measure of the extent to which the geometry is curved. The deficit angle δ is given by

$$\delta = 2\pi - \sum_i \theta_i. \tag{138}$$

The value of the deficit angle defines three different types of curvature. If $\sum_i \theta_i = 2\pi$ the geometry is flat. If $\sum_i \theta_i < 2\pi$ the geometry has positive Gaussian curvature. If $\sum_i \theta_i > 2\pi$ the geometry has negative Gaussian curvature. This idea can be extrapolated to an arbitrary dimension n, where flat n-dimensional simplices are joined together along their $(n-1)$-dimensional faces, defining the curvature at a hinge of dimension $(n-2)$.

Another geometric observable that can be computed in simplicial quantum gravity is the volume profile. One can calculate the volume profile of a simplicial ensemble in the following way. Firstly, one defines a randomly chosen vertex from the set of triangulations to be the origin o, one then shells radially outwards from this point by hopping to an adjacent vertex. We can thus define a geodesic distance τ from the origin o. By counting the number of simplices within a shell of radius τ one builds up a volume profile for the ensemble of triangulations as a function of geodesic distance from o. Then, just as one sums over all possible paths in the Feynman path integral approach to quantum mechanics to obtain the macroscopic classical path, one should average over all possible geometries to create an expectation value for the macroscopic geometry of the universe.[1]

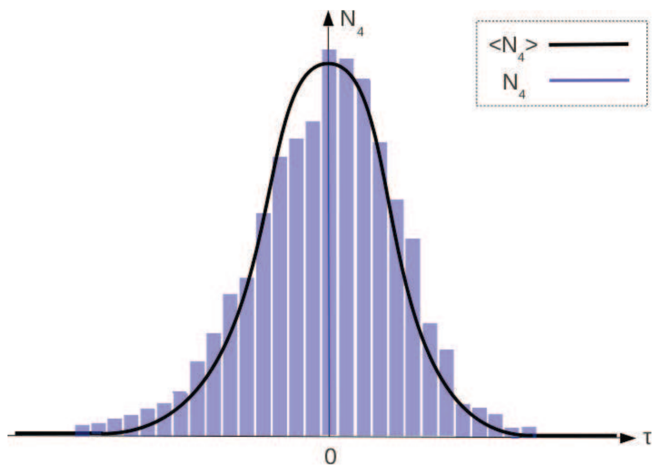

Figure 8. A schematic representing the number of 4-simplices N_4 as a function of geodesic distance τ form an arbitrarily defined origin o [1]. Collectively the bars form a single measurement of the volume distribution of N_4. One forms an expectation value $\langle N_4 \rangle$ by making many such measurements and averaging. The centre of volume is located at 0.

For volume profiles of the type seen in Fig. 8. one can determine the Hausdorff dimension of the ensemble of simplices by simply counting the expectation value for the number of simplices within a given geodesic distance from a randomly chosen vertex. Observing how the number of simplices scales with geodesic distance allows one to measure the fractal Hausdorff dimension.

The large-scale geometry of the universe can also be studied via the volume-volume correlator. In a theory with 4-dimensional building blocks the volume-volume correlator gives us a measure of the distribution of the 4-volume as a function of geodesic distance by determining the correlation of adjacent slices of 4-volume. The finite volume scaling behaviour of the volume-volume correlator

[1]The above description is specific to a model of simplicial quantum gravity with no explicit time direction, although the same principle applies to a theory with an explicit time direction except that in this case one calculates the spatial volume at proper time intervals t.

introduced in Ref. [152], can be written as

$$C_{N_4}(\delta) = \sum_{\tau=1}^{t} \frac{\langle N_4^{\text{slice}}(\tau) N_4^{\text{slice}}(\tau+\delta) \rangle}{N_4^2}, \tag{139}$$

where $N_4^{\text{slice}}(\tau)$ is the total number of 4-simplices in a spherical shell a geodesic distance τ from a randomly chosen simplex, N_4 is the total number of 4-simplices, and the normalisation of the correlator is chosen such that $\sum_{\delta=0}^{t-1} C_{N_4}(\delta) = 1$. If we rescale δ and $C_{N_4}(\delta)$, defining $x = \delta / N_4^{1/D_H}$, then the universal distribution $c_{N_4}(x)$ should be independent of the lattice volume, where $c_{N_4}(x) = N_4^{1/D_H} C_{N_4}\left(\delta / N_4^{1/D_H}\right)$. One can determine the fractal Hausdorff dimension, D_H, as the value that leaves $c_{N_4}(x)$ invariant under a change in four-volume N_4.

Since in appropriate limits a theory of quantum gravity should reduce to general relativity, one would expect that the phase diagram for a theory of quantum gravity should contain a semi-classical phase with a Hausdorff dimension $D_H = 4$ whose geometry is a solution to general relativity. One such solution is de Sitter space. A de Sitter space is the maximally symmetric geometrically flat solution to Einstein's field equations with no matter content and a positive cosmological constant. The de Sitter metric can be written as,

$$ds^2 = dt^2 + a(t)\, d\Omega_{(3)}^2 = -dt^2 + H_L^2 \cosh^2\left(\frac{\mathrm{t}}{\mathrm{H_L}}\right) \mathrm{d}\Omega_{(3)}^2, \tag{140}$$

where H_l is the Hubble length given by $H_l = c/H_0 = \sqrt{\frac{3}{\Lambda}}$. $\Omega_{(3)}$ is the 3-dimensional volume element [153],

$$\Omega_{(3)} = d\chi^2 + \sin^2\chi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{141}$$

Such a de Sitter phase has been shown to exist for a specific model of simplicial quantum gravity, thus verifying that this model has a solution of general relativity as its low energy limit.

One may naively think that a n-dimensional theory of simplicial quantum gravity will always result in a n-dimensional geometry, however, this is not necessarily the case. For dynamical triangulations the dynamics is contained in the connectivity of the n-simplices, where the geometry is updated by a set of local update moves. These local update moves can result in the deletion or insertion of vertices within simplices and so it is possible to obtain a geometric structure that has self-similar properties at different scales, meaning the geometry can be a fractal. A fractal geometry admits non-integer dimensions, and so recovering n-dimensional space from n-dimensional fundamental building blocks is a non-trivial result. The approach of simplicial quantum gravity allows the fractal dimension of the ensemble of triangulations to be computed numerically, typically this is done by computing the Hausdorff dimension and the spectral dimension.

6.3. Euclidean Dynamical Triangulations

Euclidean dynamical triangulations (EDT) is a particular implementation of a lattice regularisation of quantum gravity. The approach of EDT was originally studied in two-dimensions for the purpose of defining a nonperturbative regularisation of bosonic string theory [154, 155]. This two-dimensional approach proved successful; with gravity coupled to conformal matter being shown to correspond to bosonic string theory [156]. Results from lattice calculations agree with continuum calculations in non-critical string theory wherever they are compared [156]. Motivated by the successes of the two-dimensional theory, EDT was generalised to three [157, 158, 159] and four dimensions [160, 161]. Here, we explore EDT in four-dimensions.

Euclidean dynamical triangulations defines a spacetime of locally flat n-dimensional simplices of fixed edge length. The four-dimensional EDT formalism translates the continuum path integral

$$Z = \int \mathcal{D}g e^{iS_{EH}}, \tag{142}$$

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}(-2\Lambda + R), \tag{143}$$

into the discrete partition function

$$Z_E = \sum_T \frac{1}{C_T} e^{-S_{EH}}. \tag{144}$$

The integration over the metric $g_{\mu\nu}$ is replaced by a sum over all possible triangulations T. C_T is a symmetry factor that divides out the number of equivalent ways of labelling the vertices in the triangulation T. The discretised Einstein-Regge action in the EDT formalism is

$$S_E = -\kappa_2 N_2 + \kappa_4 N_4, \tag{145}$$

where N_i is the number of simplices of dimension i, and κ_2 and κ_4 are related to the bare Newton's constant G_N and the bare cosmological constant Λ, respectively.

The particularly simple form of Eq. (9) for the discrete Einstein-Regge action S_E is determined via the simplicial geometry, being dependent on the deficit angle θ around a triangular face, where $\theta = \arccos\left(\frac{1}{n}\right)$, and on the n-dimensional simplicial volume V_n, where

$$V_n = \frac{\sqrt{n+1}}{n!\sqrt{2^n}}. \tag{146}$$

The discrete Euclidean-Regge action is then

$$S_E = \kappa \sum 2V_2 \left(2\pi - \sum \theta\right) - \lambda \sum V_4, \tag{147}$$

where $\kappa = (8\pi G_N)^{-1}$ and $\lambda = \kappa\Lambda$. One can rewrite Eq. (11) in terms of the bulk variables N_i, and use Eq. (10) to obtain the simplicial volumes, giving

$$S_E \equiv -\frac{\sqrt{3}}{2}\pi\kappa N_2 + N_4 \left(\kappa \frac{5\sqrt{3}}{2} \arccos\frac{1}{4} + \frac{\sqrt{5}}{96}\lambda\right). \tag{148}$$

Upon substituting $\kappa_2 = \frac{\sqrt{3}}{2}\pi\kappa$ and $\kappa_4 = \kappa\frac{5\sqrt{3}}{2}\arccos\left(\frac{1}{4}\right) + \frac{\sqrt{5}}{96}\lambda$ one recovers the simple form of the discrete Einstein-Regge action in terms of the bulk variables N_2 and N_4, as given by Eq. (9).

Early studies of four-dimensional EDT reported a one-dimensional parameter space containing two phases that were separated by a transition at some critical value of the coupling κ_2^C, but neither of the phases resembled four-dimensional semi-classical general relativity [160, 162, 163, 164, 165]. When $\kappa_2 < \kappa_2^C$, a phase exists in which the simplicial geometry effectively collapses, and has an infinite fractal dimension [163]. In the regime $\kappa_2 > \kappa_2^C$ the geometry resembles that of thin polymer strands [163]. Measurements of the Hausdorff dimension of these two phases confirm that neither resembles a physical 4-dimensional phase [163].

The work of Catterall et al. in Ref. [164] studied the order of the phase transition between the two phases via an order parameter related to the fluctuation in the number of simplicial nodes and the scaling of the auto-correlation time. Tentatively, Catterall et al. concluded that the transition separating the two phases is continuous, and therefore has the possibility of being second-order. In Ref. [163] Ambjorn and Jurkiewicz measured the critical scaling exponents using numerical simulations in the vicinity of the transition, also concluding that the transition is continuous. However, a more careful study with sufficiently large lattice volumes and with an increased number of lattice configurations showed that the transition was, in fact, a discontinuous first-order critical point. Specifically, de Bakker in Ref. [166] analysed 32,000 and 64,000 4-simplices and observed a histogram with a double peak in the number of vertices that grows with volume. This behaviour is characteristic of a first-order transition. An independent study by Bialas et al. [167] also reached the same conclusion, showing that for lattice volumes smaller than 16,000 simplices the data was consistent with a continuous transition, but for a lattice volume of 32,000 simplices or greater a bimodal structure in the histogram of the number of vertices emerged.

These studies not only explained why the transition was previously thought to be continuous (because of large finite-size effects) but they also conclusively showed that the transition is first-order. The fact that the transition separating the two phases is first-order makes it unlikely that the theory has a well-defined continuum limit, at least in the simplest implementation of the model [167, 166]. This is because a second-order transition, with its diverging correlation length, is needed to define a continuum limit that is independent of the underlying discrete lattice structure.

The sum over triangulations in the original formulation of EDT all used a trivial measure, that is, all triangulations in the sum defining the path integral were weighted equally. The trivial measure is the simplest implementation, but this does not necessarily mean that it is the correct one. Brugmann and Marinari [168], in the early '90s, investigated the effect of adding a non-trivial measure term to the Euclidean path integral by studying the resulting geometric observables. Brugmann and Marinari reported that the inclusion of a non-trivial measure term induced a strong effect, and even suggested that its inclusion may be responsible for a change in the universality class of the theory.

In the late '90s Bilke et al. [169] also investigated the result of introducing a non-trivial measure term in the path integral of 4-dimensional simplicial quantum gravity. Bilke et al. reported the emergence of a new phase for an appropriate choice of couplings. Both the Brugmann and Marinari and Bilke et al. studies were suggestive, but the resulting phase diagram had yet to be explored in any great detail. A remaining hope is that a more thorough study of the phase diagram of EDT with a non-trivial measure could lead to a physical semiclassical phase, and ultimately to a nonperturbative theory of quantum gravity.

EDT including a non-trivial measure has a partition function of

$$Z_E = \sum_T \frac{1}{C_T} \left[\prod_{j=1}^{N_2} O(t_j)^{\beta} \right] e^{-S_E}. \tag{149}$$

The term in square brackets is a non-trivial measure term, which in the continuum corresponds to a nonuniform weighting of the measure by $[\det(g)]^{\beta/2}$. The product in Eq. (13) is over all 2-simplices, and $O(t_j)$ is the order of the 2-simplex j, i.e., the number of four-simplices to which the triangle belongs. One can vary the free parameter β as an additional independent coupling constant in the bare lattice action (after exponentiating the measure term [168]), bringing the total number of couplings to three. Up until now, the vast majority of work on EDT has considered the partition function with $\beta = 0$ only, apart from the already mentioned work of Refs. [168] and [169].

The term associated with the non-trivial measure β can be combined into the discrete Einstein-Hilbert action by exponentiating the measure term, giving

$$S_E = -\kappa_2 N_2 + \kappa_4 N_4 - \beta \sum_j \log O\left(\mathrm{t_j}\right). \tag{150}$$

The partition function of Eq. (13) is implemented via a Monte Carlo integration over the ensemble of 4-dimensional triangulations with fixed topology S^4. The bare cosmological constant, or equivalently κ_4, controls the number of 4-simplices N_4 in the ensemble because they appear as conjugate variables in the action of Eq. (9). One must therefore tune κ_4 such that an infinite volume limit can be taken [170]. It is convenient to ensure that numerical simulations are performed for a nearly fixed four-volume, and so a term $\delta\lambda|N_4^f - N_4|$ is introduced into the action such that N_4 is kept close to the target value N_4^f, i.e., $N_4^f \approx N_4$. However, this does not change the action for values of $N_4^f = N_4$, about which the volume fluctuates. The purpose of the term $\delta\lambda|N_4^f - N_4|$ is to prevent volume fluctuations about N_4^f from becoming too large to be easily handled in numerical simulations. While the simulation stabilises around the target volume N_4^f the parameter $\delta\lambda$ is set to 0.08, after stabilisation $\delta\lambda$ is reduced to 0.04. This term permits fluctuations in N_4 of magnitude $\delta N_4 = \left(\left\langle N_4^2\right\rangle^2 - \langle N_4\rangle^2\right)^{1/2} = \left(\frac{1}{2\delta\lambda}\right)^{1/2}$. One is then left with a two-dimensional parameter space, which is explored by varying κ_2 and β.

EDT has recently undergone something of a revival, due to the inclusion of this non-trivial measure term β [171, 172, 173, 174]. There is now evidence that properly interpreted the EDT formulation may provide a viable theory of quantum gravity at high energies, for a specific fine-tuning of the non-trivial measure term [173]. We present results from this new EDT model in section 7.2.3..

6.3.1. Conformal Instability

One of the major obstacles to formulating a Euclidean theory of quantum gravity is that the Euclidean action for Einstein gravity is unbounded from below. This section will outline this problem and discuss some of the ideas that have been proposed to resolve this issue.

To motivate the desire to Euclideanise the Lorentzian Einstein action and to highlight the fact that it is unbound from below we take a simple interacting scalar field theory as an example. The expectation value of an observable O for a scalar field theory is given by the path integral

$$\langle O\rangle = \frac{1}{Z}\int O[\phi]e^{iS[\phi]}[d\phi], \tag{151}$$

where the action is the functional

$$S[\phi] = \int dt \int d^3x \left(\left(\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2 - V(\phi)\right). \tag{152}$$

When performing computations it is not clear how one would directly evaluate the observable O numerically using the path integral of Eq. (15) because it contains complex oscillatory factors e^{iS}. In order to make computations tractable, one typically performs a change of variables of the type $t \to -i\tau$ such that the Lorentzian action transforms into the Euclidean action,

$$S_{Eucl}[\phi] = \int d\tau \int d^3x \left(\left(\frac{\partial \phi}{\partial \tau} \right)^2 + (\nabla \phi)^2 + V(\phi) \right). \tag{153}$$

The oscillatory factor e^{iS} appearing in the path integral is then transformed into e^{-S_E}, giving a path integral that is damped as opposed to being oscillatory, in addition to being amenable to numerical simulations. The Euclidean action given in Eq. (17) is positive definite for positive definite interaction potentials $V(\phi)$ [175]. However, the problem arises due to the gravitational interaction potential being negative in general relativity. The reason is that although gravitational waves have positive energy, the gravitational potential is always negative because gravity is always attractive [13].

The fundamental degrees of freedom in general relativity are given by the metric $g_{\mu\nu}$. General relativity is diffeomorphism invariant and so one is free to transform the metric $g_{\mu\nu}$ via a conformal transformation. Any two metrics $g_{\mu\nu}$ and $h_{\mu\nu}$ are said to be conformally equivalent if $g_{\mu\nu} = \Omega h_{\mu\nu}$ where Ω is some positive conformal factor. Following Ref. [13] we make a conformal transformation $g_{\mu\nu} = \Omega^2 h_{\mu\nu}$, where once again Ω is some positive conformal factor. The Einstein action then becomes

$$S_E(h_{\mu\nu}) = -\frac{1}{16\pi G_N} \int d^4x \sqrt{\det(g_{\mu\nu})} \left(\Omega^2 R + 6 g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \right). \tag{154}$$

One can see from Eq. (18) that the Einstein action can be made arbitrarily negative if the conformal factor Ω is rapidly varying. This means that the Euclidean Einstein action is unbounded from below and that the path integral of Eq. (18) could be divergent, depending on the behaviour of the gravitational measure in the strong field limit and in the regime of rapidly varying conformal factors Ω [176].

The *positive energy conjecture* outlined in Refs. [177, 178] seems to suggest that the total energy of an asymptotically flat gravitational field as measured at infinity is always positive and that the formation of black holes in regions of arbitrarily negative potentials prevents such scenarios from being physically realised [13]. If Euclidean general relativity is to accurately describe reality one would need to mathematically prevent the conformal instability appearing in the Euclidean action. A number of possible solutions to the unboundedness problem of the gravitational action have been put forward, most notably those of Gibbons, Hawking and Perry [179, 180]. The conformal instability of the Euclidean gravitational action was shown in Refs. [181, 182] to cancel, at least to one-loop in perturbation theory [176], and possibly nonperturbatively [177].

Because the Euclidean path integral is unbounded from below one may ask whether Euclidean dynamical triangulations is a well-defined theory at all. This is a legitimate concern because the partition function can become dominated by lattice configurations with arbitrarily large curvature. However, the hope is that by introducing an appropriate measure term into the Euclidean action one can suppress the contributions coming from configurations with arbitrarily large curvature, thus making the gravitational path integral convergent [173].

6.4. Causal Dynamical Triangulations

The difficulties encountered in the original EDT model, namely the absence of a physical phase and a second-order critical point, led Ambjorn and Loll [183] to introduce a causality constraint on the set of triangulations over which the path integral is taken, in the hope that it would fix these problems. The method of causal dynamical triangulations (CDT) distinguishes between space-like and time-like links on the lattice so that an explicit foliation of the lattice into space-like hypersurfaces of fixed topology (usually chosen to be S^3) can be introduced. This prevents branching of geometries into "baby universes" along the time direction.

Figure 9. shows the two fundamental building blocks of causal dynamical triangulations. The four-simplex labelled $(4,1)$ has four vertices on the space-like hypersurface of constant time $t = 0$ that are connected by space-like links l_s, and one vertex on the hypersurface $t = 1$. The space-like hypersurfaces t and $t+1$ are connected by time-like edges l_t. The two types of simplices can be glued together, along with their time-reversed counterparts, to form a causal slice of spacetime. One can then stack t such slices on top of one another,

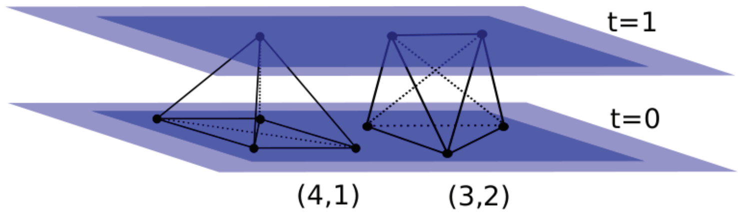

Figure 9. The building blocks of CDT. The $(4,1)$ and $(3,2)$ 4-simplices [1].

forming a causal spacetime of duration $\tau = t$. The causal structure of spacetime in CDT demands an extra parameter α that describes the ratio of the length of space-like links l_s to time-like links l_t (for Euclidean dynamical triangulations $\alpha = 1$) on the lattice and is given by,

$$\alpha = \frac{l_s^2}{l_t^2}. \tag{155}$$

The metric for CDT has the Lorentzian signature $(-1,1,1,1)$ resulting in a path integral, or partition function, of the type

$$Z = \sum_T \frac{1}{C_T} e^{-\imath S_E(T)}, \tag{156}$$

which contains complex probabilities and is not suitable for numerical integration. CDT must therefore be rotated to Euclidean signature $(1,1,1,1)$, producing a real action that is numerically tractable. It is important to note, however, that although the sum over geometries is performed with Euclidean metric signature the end result is strictly Lorentzian. This is because one starts by restricting the set of possible Euclidean geometries to only include those that meet the conditions of the causality constraint. One then Wick rotates the α parameter in the lower half of the complex plane and performs the Euclidean sum over geometries. At least in principle one can then Wick rotate back to Lorentzian signature. The inverse Wick rotation is not feasible using numerical methods alone, which is a familiar feature of numerical lattice field theory.

Numerical simulations in CDT generate a walk over the ensemble of triangulations based on Monte Carlo importance sampling with time extension t.

The local updating algorithm consists of a set of moves that change the geometry of the simplicial manifold locally, without altering its topological properties [152]. CDT appears to obtain the correct macroscopic properties of spacetime from a minimal set of assumptions. Numerical simulations using causally triangulated ensembles have demonstrated the existence of a phase whose ground state turns out to closely resemble the maximally symmetric vacuum solution of general relativity, namely de Sitter space [184] (phase C in Fig. 10.). Quantum fluctuations, that are well described by a semiclassical expansion about de Sitter space, have also been reported [185, 184]. The macroscopic dimension of this phase has been non-trivially determined and is found to be consistent with 4-dimensional spacetime [185, 7].

Figure 10. shows an up-to-date phase diagram of CDT containing four phases A, B, C, and the newly discovered bifurcation phase. As already mentioned, phase C appears to have physical geometric properties, exhibiting semiclassical de Sitter-like behaviour with an extended 4-dimensional geometry. However, phases A and B are likely unphysical in nature. Phase A consists of uncorrelated spatial slices and has a large-scale spectral dimension that is inconsistent with 4-dimensional spacetime ($D_S \sim 4/3$) [7]. Phase B consists of simplices with extremely high coordination number, such that the universe essentially collapses into a slice of minimal time extension [152]. The newly discovered bifurcation phase also appears to be unphysical due to some rather bizarre geometric properties [186, 6].

The transition dividing phases A and C has been found to be almost certainly first-order [187]. The A-B transition has not been studied in detail and its order is largely unknown, which is in part due to the fact that Monte Carlo simulations within this region of the parameter space become critically slow. In Ref. [187] Ambjorn et al. present strong evidence that the B-C transition is second-order. They determine the order of the transition in three main ways. Firstly, the histogram of the vertex number is studied, revealing a double peak structure that does not increase with simplicial volume in the vicinity of the B-C transition, behaviour indicative of a higher-order transition. Secondly, they measure the shift in the critical exponent $\tilde{\nu}$ of the asymmetry parameter Δ to be $\tilde{\nu} = 2.51$, which is in clear violation of the first-order value $\tilde{\nu} = 1$ [188]. Thirdly, measurements of geometric observables O using the so-called Binder cumulant [188] bolster the conclusion of a second-order transition. A recent study has shown that the boundary separating the bifurcation phase from phase C (see Fig. 10.) is also almost certainly second order [189]. Since in a lattice formulation of asymptotic safety the NTFP would appear as a second order critical point [171], this result raises the exciting possibility that CDT could make contact with Weinberg's asymptotic safety scenario [190].

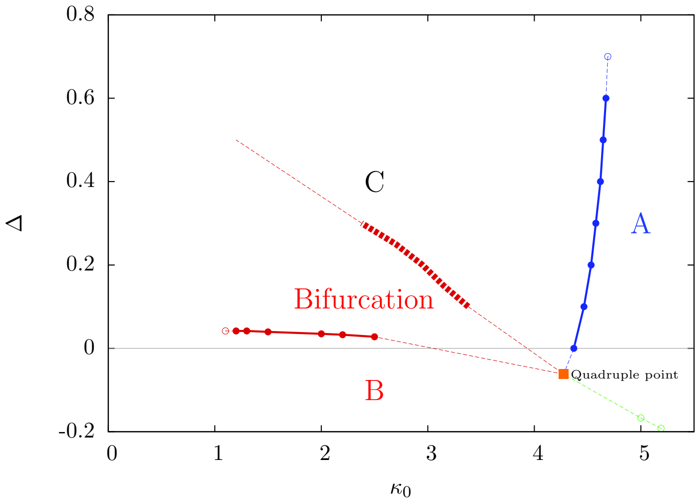

Figure 10. The 4-dimensional phase diagram of CDT spanned by the parameters κ_0 and Δ [6].

It has been shown that the phase diagram of CDT is similar to that of the generic phase diagram of Horava-Lifshitz gravity, assuming the average geometry in CDT can be correctly identified with the Lifshitz field [191]. Furthermore, in Ref. [192] Ambjorn et al. show that analytically solvable 2-dimensional CDT is equivalent to Horava-Lifshitz gravity in 2-dimensions. The global time foliation that is an inherent feature of CDT, and that explicitly breaks the isotropy between space and time, is also a defining feature of Horava-Lifshitz gravity [193]. Thus, it is possible that CDT might provide a unifying nonperturbative framework for anisotropic theories of quantum gravity [191].

Chapter 7

Is the Dimension of Spacetime Scale Dependent?

7.1. Why 4 Dimensions?

Why do we live in 4-dimensional spacetime? Why not 3, or 5, or 3.99? Other than our everyday experience there are a number of reasons to think that the universe is exactly 4-dimensional. Following the development of general relativity in 1915, with its assumption of a 4-dimensional Riemannian metric, Paul Ehrenfest began questioning the assumption of 4-dimensional spacetime as soon as 1917. Ehrenfest argued that the stability of orbits in Newtonian mechanics and Maxwell's equations uniquely required 4-dimensions. Furthermore, Maxwell's equations are conformally invariant only in 4-dimensional spacetime, and the Yang-Mills functional is dimensionless (scale invariant) if and only if the spacetime dimension is four. The smallest number of dimensions in which general relativity is non-trivial is $(3+1)$. In $(2+1)$-dimensions, for example, general relativity has no propagating gravitational degrees of freedom. However, given these interesting observations, we are yet to find a definitive explanation for why spacetime must be macroscopically 4-dimensional. This gap in our knowledge has inevitably led many physicists to propose theories with a dimension other than four.

Nordstrom was the first to hypothesise that spacetime might have more than four dimensions [194]. Theodore Kaluza brilliantly demonstrated the utility of higher-dimensional theories by extending general relativity to a five-dimensional spacetime. With no free parameters, Kaluza's theory was able to reproduce the four-dimensional Einstein field equations as well as Maxwell's electromagnetic equations, with an additional scalar field known as the dilaton [195]. Oscar Klein later added some important additions to Kaluza's work [196], which subsequently became known as Kaluza-Klein theory. In Kaluza-Klein theory the fifth spacetime dimension is compact and only becomes apparent at extremely high energies, thus demonstrating for the first time the notion of a scale dependent dimension. However promising, the Kaluza-Klein theory has a number of outstanding issues, including the stability of the fifth dimension, where the size of the extra dimension rapidly diverges to infinity or vanishes altogether. Nevertheless, the cat was out of the bag, and ever since physicists have invoked the idea of extra dimensions in theoretical physics models, most notably in string theory.

However, we might also ask whether the number of spacetime dimensions could be less than four at small distances? This possibility was first raised by a number of authors in the mid-1980s, such as Jourjine [197], Kaplunovsky and Weinstein [198], and Zeilinger and Svozil [199]. In 1985, Crane and Smolin [200] considered a model in which the spacetime vacuum is filled with a sea of virtual black holes with a scale invariant distribution. Using this model they found that the effective spacetime dimension turns out to be a non-integer value that is less than four, implying spacetime has a fractal geometry at high energies. Within the context of cosmology, Hu and O'Connor first noted an effective scale dependent dimension [201], an observation that has since been reported numerous times, as discussed in section 2.3.1..[1]

In four spacetime dimensions, gravity is perturbatively nonrenormalisable. However, in spacetime dimensions of two or less gravity is power-counting renormalisable. If the dimensionality of spacetime were to decrease to two or less one could in principle have a self-renormalising quantum field theory of gravity. Over the last decade, a number of independent field-theoretic approaches to quantum gravity have reported a scale dependent reduction in dimensionality. Individually, none of these independent approaches constitute substantial evidence in support of dimensional reduction; collectively, however, they form a compelling argument. The evidence for dimensional reduction in

[1] For an excellent overview of scale dependent dimensionality see Refs. [202] and [65].

quantum gravity will now be reviewed.

7.2. The Evidence for Dimensional Reduction

7.2.1. String Theory

Atick and Witten showed in Ref. [203] that high-temperature string theory behaves as if spacetime were two-dimensional, or more precisely $(1+1)$-dimensional. In string theory, there is thought to exist a limiting temperature known as the Hagedorn temperature, at which a heated gas of strings undergoes a phase transition. Below the Hagedorn temperature, or conversely at distances greater than the string scale $\sqrt{\alpha'}$, string theory is accurately characterised in terms of strings, and all is fine. However, at yet shorter distance scales this description begins to break down. Atick and Witten found that at distances r smaller than the string scale $\sqrt{\alpha'}$ physical systems have far fewer degrees of freedom per unit volume than relativistic field theories, with the free energy F scaling with the volume V and temperature T according to

$$F \sim VT^2, \qquad r < \sqrt{\alpha'}. \tag{157}$$

As pointed out in Ref. [203] the scaling behaviour of Eq. (1) is exactly what one would expect from a quantum field theory in $(1+1)$-dimensional spacetime. In addition, there are also hints from two-string scattering amplitudes [204] and high-energy open string scattering [205] that have been interpreted as suggesting $(1+1)$-dimensional behaviour at high energies.

There are two important caveats relating to the above discussion. First, this apparent reduction from $(3+1)$ to $(1+1)$-dimensional spacetime occurs for strings on a flat background, and so may not apply in the more realistic setting of non-Euclidean geometry. Secondly, it is inconsistent to use external measuring probes in string theory to examine spacetime at short distances, as stressed by Gross and Mende [206].

7.2.2. Causal Dynamical Triangulations

The CDT approach to quantum gravity has produced some of the strongest evidence for dynamical dimensional reduction on short distance scales. CDT approximates continuous spacetime via a lattice network of n-dimensional simplices. The spectral dimension is calculated in CDT by examining how a test particle diffuses throughout this lattice network of simplices, similar to the random walk shown in Fig. 3.. The test particle begins from a randomly chosen simplex and jumps between neighbouring simplices a total of σ times. Such random walks are repeated numerous times, with the number of diffusion processes that return to the origin after σ diffusion steps counted. From this one can determine an average probability of return, which is used to compute the spectral dimension of the simplicial manifold via Eq. (8).

In *Spectral Dimension of the Universe* Ambjorn et al. [8] present the first result for the running spectral dimension $D_S(\sigma)$ in CDT using a fit to the functional form,

$$D_S(\sigma) = a - \frac{b}{c+\sigma}, \tag{158}$$

where σ is the diffusion time, an effective measure of the distance scale probed by the diffusion process. The numerical coefficients of this fit were determined to be $a = 4.02$, $b = 119$, and $c = 54$. The physical significance of the particular fit function chosen by Ambjorn et al. (Eq. (2)) is not known from first principles, it is simply the functional form that best describes the dependence of the spectral dimension over the entire range of their data.[2] The work of Refs. [207, 208, 209, 210] have attempted to derive this function from first principles using analytical methods.

Their result for the large-scale spectral dimension of CDT is quoted as

$$D_S(\sigma \to \infty) = 4.02 \pm 0.10, \tag{159}$$

while the short-distance spectral dimension was found to be

$$D_S(\sigma \to 0) = 1.80 \pm 0.25. \tag{160}$$

The authors of [8] correctly claim that the value for $D_S(\sigma \to 0)$ is therefore consistent with the integer two. However, large discretisation errors are typically associated with the small-scale spectral dimension. One reason for this is

[2] The alternative fit functions considered in Ref. [8] were $a - b e^{-c\sigma}$ and $a - \frac{b}{\sigma^c}$.

because for a small number of diffusion steps the behaviour of $D_S(\sigma)$ can be significantly different when considering an even or odd number of diffusion steps. This effect can be understood by considering a simplified diffusion process on a one-dimensional piecewise straight line; in this scenario, it is impossible to begin and end at the origin after an odd number of diffusion steps, whereas this is not true for an even number of diffusion steps (see Fig. 11.). A similar effect is present in 4-dimensional triangulations, where an oscillatory behaviour is observed at small σ values, although this discrepancy between odd and even integer steps becomes negligible for sufficiently large σ values [8], as can be seen in Fig. 12.. The relatively large error associated with Eq. (4) leaves some ambiguity in the conclusions that can be drawn from this result.

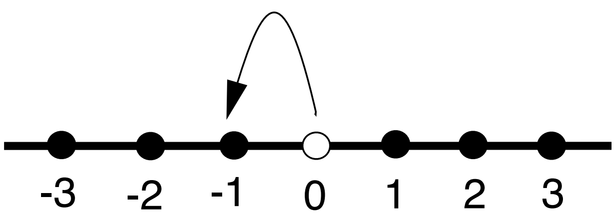

Figure 11. An illustration of how an odd or even number of diffusion steps can impact the probability of return close to the origin for a simple 1-dimensional random walk.

A more recent publication has revisited these calculations of the spectral in CDT [7], providing a more comprehensive study with reduced systematic and statistical errors. The most important results of this work are presented in Fig. 13. and Tab. 3.. In Ref. [7] it is found that the long-distance spectral dimension is consistent with the expected semiclassical dimensionality of 4, and that the spectral dimension monotonically decreases to a value of around $3/2$ on small distance scales when using sufficiently fine lattices. However, at the canonical point in the phase diagram of CDT ($\kappa_0 = 2.2, \Delta = 0.6$), which has been shown to closely approximate 4-dimensional de Sitter space in the infrared, the small-distance spectral dimension is more consistent with the integer 2, with the measured value being 1.970 ± 0.266.

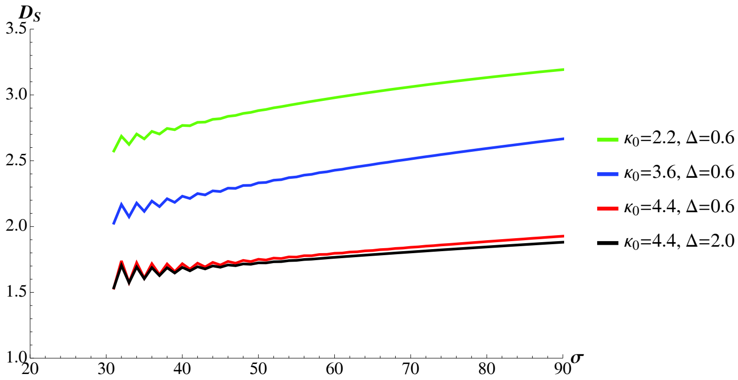

Figure 12. Fluctuations in the spectral dimension due to the difference between odd and even diffusion steps σ over short distance scales [7]. This effect introduces a systematic discretisation error that can be reduced by omitting data values below a specific cut-off in σ [7].

As demonstrated by the fits to the $(2.2, 0.6)\,160K$ data in Fig. 13., both the correlated and uncorrelated fits yield similar results. Utilising the full covariance matrix in the estimation of χ^2 yields a relatively large $\chi^2/d.o.f = 1.92$. Since there currently exists no consensus on the correct functional form that the spectral dimension should take we employ an uncorrelated version of the fit function of Eq. (2) as our ansatz, making a more accurate estimate of systematic errors by changing the fit function and varying the fit range. This procedure results in the central values of $D_S(\infty)$ and $D_S(0)$ of Tab. 3. by using the uncorrelated fit function of Eq. (2) over the data range $\sigma \in [50, 490]$ in steps of 4 for the point $(2.2, 0.6)$ and $\sigma \in [60, 490]$ in steps of 4 for the other three points. The errors quoted in Tab. 3. are also determined by adding statistical errors in quadrature.

A better estimate of the systematic error of spectral dimension measurements is obtained by changing the σ range over which the fit function of Eq. (2) is used. Moreover, due to the absence of any theoretical explanation for the particular form of Eq. (2) it is important to estimate the contribution to the systematic error associated with the application of the alternative fit functions $D_S(\sigma) = a - b\exp(-c\sigma)$ and $D_S(\sigma) = a - (b/(c+\sigma))^d$, where a, b, c, and d

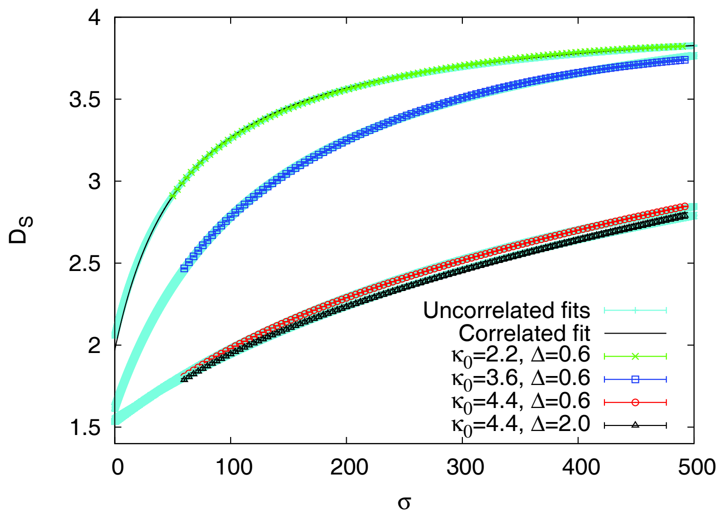

Figure 13. The spectral dimension D_S as a function of the diffusion time σ for 4 different points in phase C of CDT [7]. The $D_S(\sigma)$ curves for points with $\Delta = 0.6$ are calculated using a lattice volume of 160,000 $N_{4,1}$ simplices. $D_S(\sigma)$ at the point $\kappa_0 = 4.4, \Delta = 2.0$ is computed using a lattice volume of 300,000 $N_{4,1}$ simplices. The error band denotes an uncorrelated fit to the data using Eq. (2) over the fit range $\sigma \in [50, 494]$ for the point $\kappa_0 = 2.2, \Delta = 0.6$ and $\sigma \in [60, 492]$ for the other three points. An extrapolation to $\sigma = 0$ and $\sigma \to \infty$ is made using the function of Eq. (2). The uncorrelated fit shows only the central value for comparison. Errors presented in this figure are statistical only, however Tab. 3. includes estimates of the total statistical and systematic errors.

are free fit parameters, whose values can be found in Tables 4. and 5..

7.2.3. Euclidean Dynamical Triangulations

We now take a look at the spectral dimension in the Euclidean model of dynamical triangulations (EDT). In Ref. [171] the spectral dimension is calculated using an ensemble of degenerate triangulations with a volume of 4000 four-simplices. The simulation was performed for the bare parameters $\kappa_2 = 2.1$ and $\beta = -1.0$. Approximately 1000 configurations were used to determine the average return probability given by Eq. (5). The errors are determined from a single

Table 3. A table of the large-distance $D_S(\sigma \to \infty)$ and short-distance $D_S(\sigma \to 0)$ spectral dimension for 4 different points in phase C of CDT. $D_S(\sigma \to \infty)$ and $D_S(\sigma \to 0)$ are computed using the functional form $a - \frac{b}{c+\sigma}$ [8]

(κ_0, Δ)	$D_S(0)$	$D_S(\infty)$
$(2.2, 0.6)$	1.970 ± 0.266	4.05 ± 0.17
$(3.6, 0.6)$	1.576 ± 0.093	4.31 ± 0.32
$(4.4, 0.6)$	1.534 ± 0.058	4.12 ± 0.16
$(4.4, 2.0)$	1.540 ± 0.060	4.14 ± 0.12

Table 4. The fit parameters a, b, c and d for the two alternative fit functions used in estimating the systematic error for the bare parameters $(2.2, 0.6)$ and $(3.6, 0.6)$ with $N_{4,1} = 160,000$

Fit-function	(2.2,0.6)				(3.6,0.6)			
	a	b	c	d	a	b	c	d
$a - b\exp(-c\sigma)$	3.74	1.73	0.013	-	3.74	2.14	0.0078	-
$a - (b/(c+\sigma))^d$	4.20	108.17	21.69	0.62	4.01	479.57	339.14	2.43

elimination jackknife procedure. Fluctuations were reduced by histogramming the data into bin sizes of two from $\sigma = 2 - 80$ and in bin sizes of four from $\sigma = 80 - 288$. The fit is to the same functional form suggested in Ref. [8], namely

$$D_S(\sigma) = a - \frac{b}{c+\sigma}, \tag{161}$$

with a, b, and c determined by the fit. The fit uses the full covariance matrix in the estimate of χ^2 with σ ranging from 10 to 146 in increments of 4, giving a $\chi^2/dof = 35/32$ and a confidence level (corrected for finite sample size) of CL=0.37. Variations of the fit function were used to estimate a systematic error on the asymptotic value of D_S, based on the assumptions that $D_S(\sigma)$ is a

Table 5. The fit parameters a, b, c and d for the two alternative fit functions used in estimating the systematic error for the bare parameters $(4.4, 0.6)$ with $N_{4,1} = 160,000$ simplices, and for the bare parameters $(4.4, 2.0)$ with $N_{4,1} = 300,000$

Fit-function	(4.4,0.6)				(4.4,2.0)			
	a	b	c	d	a	b	c	d
$a - b\exp(-c\sigma)$	3.83	2.18	0.0016	-	4.12	2.46	0.0013	-
$a - (b/(c+\sigma))^d$	3.99	1213.56	586.11	1.24	4.00	1337.74	648.10	1.25

monotonic function.[3]

The preferred fit yields $D_S(\infty) = 4.04 \pm 0.26$, and $D_S(0) = 1.457 \pm 0.064$, where the errors include both the statistical error and a systematic error associated with varying the fit function and the fit range added in quadrature [171].[4] Calculations of the spectral dimension with combinatorial triangulations yield similar results, but require significantly larger lattice volumes.

A more recent study of the spectral dimension in EDT for a specific fine-tuning of the non-trivial measure term β adds to this result [173]. Applying the same fit function of Eq. (5) yields a spectral dimension in the range $D_S = 2.7 - 3.3$ at large distances [173], and a small-distance value in the range $D_S = 1.4 - 1.6$, as shown in Tab. 6.. Clearly, the large-distance result $D_S = 2.7 - 3.3$ is inconsistent with 4-dimensional semi-classical general relativity.

However, this inconsistency could be down to finite volume or discretisation effects. To determine whether or not this is the case we consider an additional extrapolation of $D_S(\infty)$ of the form

$$D_S(\infty) = c_0 + c_1 \frac{1}{V} + c_2 a^2, \tag{162}$$

where c_i is a fit parameter, V is the lattice volume and a is the lattice spacing (see Ref. [173] for the motivation behind this particular ansatz). Extrapolation to the continuum and infinite volume limit using the fit function of Eq. (6) gives the large-distance value

[3] Reuter and Saueressig in Ref. [62] use renormalisation group methods to show that the spectral dimension should exhibit a long plateau at $\sim 4/3$ for small values of the diffusion time σ, and that as $\sigma \to 0$ the spectral dimension will increase again to $D_S = 2$, therefore predicting that the behaviour of the spectral dimension as a function of diffusion time is not monotonic, although this prediction may not be correct due to the truncation that is used.

[4] The alternative fit functions considered are those suggested in Ref. [8], namely $a - b\mathrm{e}^{-c\sigma}$ and $a - \frac{b}{\sigma^c}$.

Table 6. The spectral dimension in the small and large-distance limits, including the $\chi^2/d.o.f.$ of each measurement, for different values of the non-trivial measure term β and for different lattice volumes N_4

β	N_4	$D_S(0)$	$D_S(\infty)$	$\chi^2/d.o.f.$
-0.8	8000	1.445(16)	2.75(11)	1.14
-0.6	4000	1.431(29)	2.756(89)	1.38
0	4000	1.464(49)	2.809(51)	1.12
0	8000	1.484(21)	3.090(41)	1.25
0	16000	1.484(37)	3.30(12)	0.92
0.8	4000	1.44(15)	2.797(64)	0.91
1.5	4000	1.64(26)	2.655(93)	1.60

$$D_S(\infty) = 3.94 \pm 0.16, \tag{163}$$

and the small-distance value

$$D_S(0) = 1.44 \pm 0.19. \tag{164}$$

7.2.4. Horava-Lifshitz Gravity

Horava-Lifshitz gravity is a proposed theory of quantum gravity that explicitly breaks the four-dimensional diffeomorphism symmetry of general relativity in the high-energy regime [193]. Although in Horava-Lifshitz gravity the symmetry between space and time is explicitly broken at the shortest scales, it is hoped that LI is recovered in the large-distance limit. Horava-Lifshitz gravity allows an anisotropic scaling of spacetime at short distances, or conversely at high energies. The degree of anisotropy is quantified via the dynamical critical exponent z. At low energies, or large distances, $z = 1$. However, at high energies, or small distances, the dynamical critical exponent runs to $z = 3$. The payoff that comes from treating space and time differently at high energies is a theory that is better behaved in the ultraviolet limit [193].

In $(D+1)$-dimensional spacetime with anisotropic scaling exponent z the spectral dimension of Horava-Lifshitz gravity is found to be [211]

$$D_S = 1 + \frac{D}{z}. \tag{165}$$

Thus, at small distances ($z = 3$) one obtains $D_S = 2$ and at large distances ($z = 1$) one recovers the semi-classically expected value $D_S = 4$. In addition to a similar running spectral dimension, Horava-Lifshitz gravity shares a number of other salient features with CDT, most importantly it also assumes a fixed foliated structure of spacetime [211, 191].

7.2.5. Asymptotic Safety

The requirement that general relativity be asymptotically safe demands that the renormalisation group flow of its couplings has a high-energy fixed point, a fixed point that is by definition scale invariant. General relativity contains at least one (the other possibility being the cosmological constant Λ) dimensionful coupling that breaks exact scale invariance, Newton's constant G_N. However, in d-dimensional spacetime Newton's constant has a canonical mass dimension of $[G_N] = 2 - d$, so that in $d = 2$ dimensions general relativity becomes precisely scale invariant. Intuitively, this explains why we should expect the dimension of spacetime to approach two at the high-energy fixed point of asymptotic safety; because scale invariance demands it. We now proceed to demonstrate in a more rigorous setting that this is indeed the observed behaviour.

The beta-function describing how the dimensionless gravitational couplings $g_k = k^{d-2} G_k$ and $\lambda_k = k^{-2} \Lambda_k$ run as a function of the RG scale k is given by

$$\beta(g, \lambda) = \mu \frac{\partial g}{\partial \mu} = (d - 2 + \eta(g, \lambda))\, g. \tag{166}$$

The high-energy fixed point required by asymptotic safety is parametrised by an anomalous dimension of

$$\eta(g, \lambda) = 2 - d = -2, \qquad (d = 4). \tag{167}$$

This simple fact can be used to calculate the effective behaviour of the dressed graviton propagator as a function of momentum within the vicinity of the fixed point. An expansion of the truncated gravitational action about flat space (without tensor structures) gives an inverse propagator at the threshold value $k = \sqrt{p^2}$ of

$$\mathcal{G}_k(p)^{-1} = Z(k)p^2 \propto \left(p^2\right)^{1-\eta/2}, \tag{168}$$

because $Z(k) \propto k^{-\eta}$ when $\eta \equiv -\partial_t \ln Z$ is approximately constant [3]. The Fourier transform of $\mathcal{G}_k(p)$ determines the Euclidean propagator in d-dimensional position space to be

$$\mathcal{G}(x;y) \propto \frac{1}{|x-y|^{d-2+\eta}}, \qquad \eta \neq 2-d. \tag{169}$$

For $\eta = -2$ the dressed propagator becomes $\tilde{\mathcal{G}} = p^{-4}$, which in position space is given by

$$\mathcal{G}(x;y) \propto -\frac{1}{8\pi^2}\ln\left(\mu|\mathrm{x}-\mathrm{y}|\right) \qquad (\mathrm{d}=4), \tag{170}$$

where μ is a constant with dimensions of mass [3]. For a field theory in flat space a propagator that scales like $\tilde{\mathcal{G}} = p^{-4}$ is usually associated with unitarity issues. However, for large values of k the spacetime considered has a large curvature that is proportional to k^2, and so this may not be an issue in this case.

The propagator of Eq. (14) has a particularly striking feature; it is exactly what one obtains from a $1/p^2$-propagator in 2-dimensional spacetime. Remarkably, this result seems to suggest that in asymptotic safety spacetime behaves as if it were two-dimensional when viewed using extremely high-energy gravitons.

Furthermore, in Ref. [62] Reuter and Saueressig compute the spectral dimension D_S and walk dimension D_W of the effective space-times associated with asymptotically safe gravity. In the large-distance limit they find $D_S = D_T$ and $D_W = 2$, where D_T is the topological dimension of the spacetime. At intermediate distances, referred to as the semiclassical regime, this dimensionality dynamically changes to $D_S = 2D_T/(D_T+2)$ and $D_W = 2+D_T$. Finally, in the large energy limit (the NTFP regime) $D_S = D_T/2$ and $D_W = 4$. Which, in the case of $D_T = 4$ topological dimensions yields a reduction in the number of spacetime dimensions from $D_S = 4$ in the large-distance limit to $D_S = 2$ in the small-distance limit, for the spectral dimension. Contrary to this, however, the walk dimension appears to non-monotonically increase from $D_W = 2$ at large distances to $D_W = 4$ at small distances [62].

7.2.6. Loop Quantum Gravity

Loop quantum gravity is a canonical quantisation scheme that is widely considered to be among the leading approaches to quantum gravity. The first piece of evidence for dimensional reduction in loop quantum gravity was found by Modesto [212], who noted that the average area operator $\langle A_l \rangle$ at a given (quantised) length scale l in loop quantum gravity can be written as

$$\langle A_l \rangle = \frac{\sqrt{l^2\left(l^2+l_P^2\right)}}{\sqrt{l_0^2\left(l_0^2+l_P^2\right)}}\langle A_{l_0} \rangle, \tag{171}$$

where $\langle A_{l_0} \rangle$ is the average area at the infrared length scale l_0 (with $l \leq l_0$) and l_P is the Planck length. Note that for sufficiently large length scales l the average area of Eq. (15) scales in the usual way, however, as l decreases the area operator begins to deviate from the usual expression, which already hints at an unusual scaling behaviour in the small-distance limit. We now proceed to analyse this behaviour in more detail in order to see what exactly it has to say about spacetime dimensionality at extremely small distances.

The scaling behaviour of the average area operator $\langle A_l \rangle$ can be converted into a scale dependent inverse metric tensor, which in the case of a four-dimensional spin foam model gives

$$\langle g^{\mu\nu} \rangle_k = \frac{k^2}{k_0^2}\langle g^{\mu\nu} \rangle_{k_0}. \tag{172}$$

Applying the scale dependence of the spacetime metric of Eq. (16) to the definition of the spectral dimension

$$D_S = -2\frac{\partial \mathrm{lnP_r(T)}}{\partial \mathrm{ln(T)}}, \tag{173}$$

where T is the diffusion time, allows us to write the spectral dimension of spacetime as

$$D_S = 2T\frac{\int d^4 k e^{-k^2F(k)T} k^2 F(k)}{\int d^4 k e^{-k^2F(k)T}}, \tag{174}$$

where $F(k)$ is defined to be $F(k) = 1 + k^2/k_0^2$. Using Eq. (16) we obtain the limiting behaviour

$$D_S = \begin{cases} 2, & \text{for } k \geq E_P \\ 4, & \text{for } k \ll E_P, \end{cases} \tag{175}$$

a result that is in perfect agreement with string theory [203], causal dynamical triangulations [8, 7] and asymptotic safety [3, 62].

The spectral dimension has also been calculated using the same methodology within the Barrett-Crane spinfoam model [213]. Within this model, the spectral dimension is once again found to run from approximately 4 at large distances to approximately 2 at extremely small distance scales.

7.2.7. The Wheeler-DeWitt Equation

Performing the canonical quantisation of general relativity by expressing the quantum mechanical Hamiltonian constraint using metric variables leads to the celebrated Wheeler-DeWitt equation [29], which makes the rather grand claim of describing the wavefunction of the entire universe, at least to some approximation. The Wheeler-DeWitt equation can be written as

$$\left[16\pi l_p^2 G_{abcd} \frac{\delta}{\delta q_{ab}} \frac{\delta}{\delta q_{cd}} - \frac{1}{16\pi l_p^2} \sqrt{q^{(3)}} R \right] \Psi[q] = 0, \tag{176}$$

where the so-called DeWitt super-metric is given by $G_{abcd} = 1/2 q^{-1/2}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})$ and q_{ab} is the spatial metric tensor at a given time.

The wave function of the universe $\Psi[q]$ describes the metric at all distance scales. In order to examine the behaviour at small distances we may take the strong coupling limit $l_P \to \infty$ [214]. In this limit the behaviour is essentially that of a Kasner space [215], which has a heat kernel given by [216, 217]

$$K(x, x, \sigma) \approx \frac{1}{4\pi\sigma^2} \left(1 + \frac{a\sigma}{t^2} + \ldots \right). \tag{177}$$

For a fixed probing scale (fixed value of σ) the second term will come to dominate for sufficiently small values of t, which once again yields a small-distance spectral dimension of $D_S = 2$ [65].

7.2.8. Causal Set Theory

Causal set theory is yet another approach to quantum gravity that exhibits dimensional reduction. In causal set theory spacetime is axiomatically discrete but still Lorentz invariant, being made from a set of points related by a partial order describing their causal relationship with one another [218]. One definition of the fractal dimension that respects the Lorentzian nature of causal set theory is the Myrheim-Meyer dimension D_{MM} discussed in section 2.2.5.. There are three main cases for which the Myrheim-Meyer dimension has been calculated in causal set theory. First, for a random causal set with a very small number of elements (between 4 and 6) the average Myrheim-Meyer dimension has been shown to be in the range $D_{MM} = 2.15 - 2.27$ [219]. Secondly, for larger 'generic' causal sets (~ 50 elements), which are dominated by structures known as Kleitman-Rothschild (KR) orders [220], the Myrheim-Meyer dimension is $D_{MM} = 2.38$ [219]. Thirdly, there are preliminary indications that causal sets defined via a random sprinkling of points in Minkowski spacetime result in the value $D_{MM} \approx 2$ for small sub-intervals [221].

The behaviour of the spectral dimension has also been studied using causal sets. Eichhorn and Mizera [222] have calculated the spectral dimension in causal set theory, finding that D_S actually increases at short distances, or conversely at high energies. However, the d'Alembertian used in this work is somewhat naive and does not reproduce the usual flat spacetime d'Alembertian that approximates Minkowski space [65]. Using a more appropriate d'Alembertian, however, yields the by now familiar reduction to $D_S = 2$ in the small-distance limit [223].

7.2.9. Non-Commutative Geometry

Noncommutative geometry attempts to replace the Riemannian geometry of general relativity with a more general formalism, seeking to unify gravity with the standard model at very high energy scales [224]. Benedetti has analysed the scale dependent spectral dimension in so-called κ-Minkowksi spacetime [225], a specific case of noncommutative geometry. For large diffusion times σ, which correspond to probing spacetime on large distance scales, the spectral dimension agrees with the semiclassical expectation of 4. However, for small σ the spectral dimension reduces to $D_S = 3$ [225]. Although the behaviour of the dimensional reduction in noncommutative geometry is qualitatively similar to that found in e.g., causal dynamical triangulations or the asymptotic safety scenario, the small-distance value of $D_S = 3$ is notably different. However, it is shown in Ref. [226] that one can also obtain a small-distance spectral dimension $D_S = 2$, depending on the particular choice of deformed Laplacian

Studies of Snyder space, a different example of noncommutative spacetime, found that at high temperatures various measures of the dimension approach the value of 2 [227]. Using a variation of the spectral action within a more formal approach to noncommutative geometry [228] also yields $D_S = 4$ in the large-distance limit and $D_S = 2$ in the small-distance limit [224].

7.3. A Possible Solution to an Old Problem

We now return to the problem discussed in section 5.2.. Banks [144] and Shomer [145] argue that any quantum field theory must scale in the same way as a conformal field theory in the high-energy limit if it is to be renormalisable. However, they find that this is not true for gravity, and so conclude that gravity cannot be a renormalisable quantum field theory. Crucially, however, this argument assumes spacetime is 4-dimensional.

In a more general d-dimensional setting the entropy S_{CFT} of a conformal field theory is given by [146]

$$S_{CFT} \sim \left(\frac{E}{R^{d-1}}\right)^{\frac{d-1}{d}} R^{d-1}, \tag{178}$$

where R is the radius of spacetime under consideration and E is the energy within this region. On the other hand, the entropy S_{Grav} for a high-energy theory of gravity in d-dimensional spacetime scales as [146]

$$S_{Grav} \sim \left(\frac{E}{R^{d-1}}\right)^{\frac{1}{2}} R^{d-1}. \tag{179}$$

Therefore, Eqs. (22) and (23) agree, if and only if, spacetime is 2-dimensional at high energies—which is exactly what the phenomenon of dimensional reduction is telling us.[5]

7.4. Dimensional Reduction in the Sky

In Ref. [229] it is shown that given any dispersion relation one can compute an associated spectral dimension, and that in principle this process can be inverted. That is, starting from a given analytical form of the spectral dimension on the complex plane one can derive an associated dispersion relation [229]. Practically, however, this process is often difficult due to the fact that the exact analytical form of the spectral dimension on the complex plane is typically unknown.

Nevertheless, it is fairly straightforward to show that a spectral dimension that varies in any way as a function of the diffusion time σ must lead to a deformed dispersion relation. To show this we begin by noting that the probability of return used in the definition of the spectral dimension in momentum space is given by

$$P_r(\sigma) = \int e^{\sigma \triangle_p} d\mu, \tag{180}$$

where $d\mu = dEd^3\vec{p}/(2\pi)^4$ and $\triangle_p$ are the invariant measure and Laplace operator in momentum space, respectively [230]. In 4-dimensional Euclidean space an undeformed dispersion relation yields $\triangle_p = -E^2 - p^2 = 0$, which preserves a dimensionality of $D_S = 4$ over all distance scales as determined via Eqs. (8) and (24). If, however, the spectral dimension varies with σ then we must have $\triangle_p \neq 0$, and the associated dispersion relation must be deformed to some extent [230].

It has been suggested that a diverse range of scenarios for a running spectral dimension can be derived from a deformed dispersion relation with the generic form [211, 231]

$$E^2 = p^2\left(1 + (\lambda p)^{2\gamma}\right), \tag{181}$$

[5] Note that if we instead assume that the radius is independent of mass and entropy, as Shomer did [145], then entropy scales as $S_{CFT} \sim E^{\frac{d-1}{d}}$ and $S_{Grav} \sim E^{(d-2)/(d-3)}$. In this case, gravity scales in the same way as a conformal field theory at high energies if $d = 3/2$, which agrees with the small-scale spectral dimension found in a few approaches to quantum gravity [7, 173].

where γ is a free exponent characterising the extent of vacuum dispersion. In a spacetime with $D_H + 1$ Hausdorff dimensions, where D_H is the number of spatial dimensions, the dispersion relation of Eq. (25) results in a small-distance spectral dimension of [211]

$$D_S = 1 + \frac{D_H}{1+\gamma}. \tag{182}$$

This running dimension may have important cosmological implications, depending on the particular role γ and λ take in cosmological scenarios, as we shall discuss later in this section. But first we must give a little theoretical background on cosmological perturbations.

Cosmological perturbations are density variations in the early universe that are generally considered to seed the formation of stars, galaxies and galaxy clusters. Assuming the dynamics in the early universe is still governed by general relativity, cosmological perturbations should be described by

$$f_k'' + \left[-\bigtriangledown_k^2 - \frac{a''}{a} \right] f_k = 0, \tag{183}$$

where $\bigtriangledown_k$ is the momentum component of the total Laplace operator, which in the classical $(3+1)$-dimensional case is $\bigtriangledown_k^2 = -c^2 k^2$ [230]. The scale factor is denoted by a whose precise definition depends on whether the perturbations are scalar or tensor modes, k is a co-moving constant labelling propagating modes, and f_k are the mode functions. We can now see how a deformed dispersion relation resulting from a running spectral dimension may impact cosmology, entering via a modification of the momentum space Laplace operator $-\bigtriangledown_k^2 \to a^2\Omega^2(k/a)$ in Eq. (27) [230]. Using the equation of state $w = p/\rho$ we may define the parameter ε via $\varepsilon = \frac{3}{2}(1+w)$, which will prove useful later.

The fractional energy density of fluctuations is given by

$$\delta(x) = \frac{\rho(x)}{\langle\rho\rangle} - 1 = \int dk \delta_k e^{ikx}, \tag{184}$$

where ρ is the energy density, $\langle\rho\rangle$ is the average energy density and k is the wave-number of fluctuations. Primordial fluctuations can be quantified by a power spectrum, giving power variations as a function of distance scale. The power spectrum $P(k)$ is defined by an ensemble average of Fourier components

$$\langle \delta_k \delta_{k'} \rangle = \frac{2\pi^2}{k^3} \delta\left(k - k'\right) P(k) . \tag{185}$$

The definition of the scalar power spectrum is

$$P_S(k) \equiv \frac{k^3}{2\pi^2} \frac{|f_k|^2}{a_S^2} \propto k^{n_S - 1}, \tag{186}$$

where n_S is the spectral index and a_S is the scale factor associated with scalar modes. The so-called Bunch-Davies vacuum normalisation of the mode functions f_k sets

$$|f_k|^2 = \frac{1}{2a\Omega(k/a)}. \tag{187}$$

Thus, for $n_S = 1$ the scalar power spectrum is precisely scale invariant. However, results from the Planck satellite conclusively demonstrate a small deviation from exact scale invariance, namely the measured spectral index is $n_S = 0.9616 \pm 0.0094$.

Now, as shown in Ref. [232] the modified dispersion of Eq. (25) dictates that the spectral index be given by

$$n_S - 1 = \frac{\varepsilon(\gamma - 2)}{\gamma - \varepsilon + 1}. \tag{188}$$

As is evident from Eq. (32) a value of $\gamma = 2$ leads to a precisely scale invariant power spectrum. Returning to Eq. (26) we find that a value of $\gamma = 2$ is exactly what we would expect from a spectral dimension of $D_S = 2$ in the small-distance limit. Hence, if the spectral dimension of the early universe was exactly two then this mechanism is capable of producing scale invariant primordial fluctuations without the need for inflation.

However, data from the Planck satellite tells us that the cosmological perturbations are close to, but not exactly, scale invariant. There are two obvious ways in which dimensional reduction can produce deviations from strict scale invariance, so as to be consistent with the observational data. First, the small-distance spectral dimension may be slightly larger than 2, which via Eqs. (25) and (32) would make n_S slightly smaller than 1, although this depends on the equation of state w. A spectral dimension slightly greater than two, say $D_S \approx 2.01$, is at least consistent with numerical CDT simulations at the canonical point, although the errors in this measurement are currently far too large.

Secondly, non-exact scale invariance of primordial fluctuations may result from a spectral dimension that does in fact asymptote to $D_S = 2$, but in a very slow and special way. In Ref. [232] the authors claim this requires a very special expression for the exponent γ of

$$\gamma(p) = 2 - \frac{2}{1 + C\log\left(1 + (\lambda \mathrm{p})^2\right)}, \tag{189}$$

with the dispersion relation now a function of $\gamma(p)$. For an equation of state $w = 1/3$ the authors of Ref. [232] show that this model leads to $n_S \approx 0.9633$, which agrees with the experimentally determined value within errors.

7.5. Experimental Tests

Dimensional reduction is typically predicted to only become significant at or near the remote Planck scale. Clearly, this creates a challenge when trying to experimentally test whether this phenomena is real. However, as we shall see in this section the situation is not entirely hopeless.

7.5.1. Cosmology

Due to the extremely high energy scales present in the early universe, coupled with the fact that inflation is thought to magnify Planck scale physics up to macroscopic scales, cosmology may provide one of the strongest experimental tests of dimensional reduction. In [230] Mielczarek has used observations of gamma-ray bursts to constrain the energy scale E_* at which dimensional reduction can occur to be $E_* > 0.7\left(4 - D_S^{UV}\right)^{1/2} \cdot 10^{10} GeV$ with a 95%CL. Furthermore, he determines a constraint on D_S^{UV} via observations of the scalar power spectrum made by the Planck and BICEP2 experiments. We will now review this latter constraint on D_S^{UV} in more detail.

Mielczarek estimates a modified dispersion relation at high energies associated with the CDT model to be [230]

$$\Omega_{CDT}(p) \approx \frac{2}{3}E_* \left(\frac{p}{E_*}\right)^{3-3\varepsilon} \propto p^{3-3\varepsilon}. \tag{190}$$

As discussed previously in section 7.4. the power spectrum of primordial fluctuations is given by

$$P_S(k) \equiv \frac{k^3}{2\pi^2}\frac{|f_k|^2}{a_S^2} \propto k^{n_S-1}, \tag{191}$$

with

$$|f_k|^2 = \frac{1}{2a\Omega_{CDT}(p)}, \tag{192}$$

in the case of CDT. The spectral index can then be written in terms of the tensor to scalar ratio r and the high-energy spectral dimension D_S^{UV} as

$$n_S - 1 = \frac{d\ln \mathrm{P_S(k)}}{d\ln \mathrm{k}} \approx \frac{3r(D_S^{UV}-2)}{(D_S^{UV}-1)r - 48(\varepsilon - r)}. \tag{193}$$

Since the Keck array and Planck satellite constrain $r < 0.12$ and $n_S = 0.9616 \pm 0.0094$ this means $\varepsilon > 0.835$ and therefore the ultraviolet spectral dimension is constrained to be

$$D_S^{UV} > 2.835, \tag{194}$$

which has some tension with the observed CDT value $D_S^{UV} = 2$. However, it must be clearly stated that there are a number of questionable assumptions and approximations that this constraint relies on, particularly in the calculation of the high-energy dispersion of CDT $\Omega_{CDT}(p)$.

Another experimental test on cosmological scales may be possible via measurements of the cosmic microwave background (CMB), relic electromagnetic radiation left over from the Big Bang. Some multifractional models, in which the geometry of spacetime changes as a function of the resolution scale, predict log-periodic oscillations in the CMB that are now tightly constrained by CMB measurements [233]. It is also possible to analyse the CMB spectrum directly, rather than indirectly via its density perturbations. Using such direct measurements, Caruso and Oguri found that the spacetime dimension differs from $d = 4$

by at most one part in 10^{-5}, at least in the low energy regime associated with direct measurements of the CMB spectrum [234].

Experimental searches for violations of LI have received a lot of attention in recent years [120]. In addition to the constraints on an energy dependent speed of light from observations of distant gamma-ray bursts (see section 3.4.), there are also experimental constraints on violations of LI coming from inter-galactic cosmic rays. Specifically, Greisen [235], Zatsepin and Kuzmin [236] (GZK) determined a theoretical upper limit of 5×10^{19}eV on the energy that inter-galactic cosmic ray protons can have. This upper bound is due to the fact that cosmic ray protons with an energy exceeding this limit will interact with photons in the CMB, decaying via well-defined channels, and thus not reaching our detectors. Essentially, this limit is set by the physics of special relativity; a violation of the GZK bound may signal a departure from special relativity. Observations made by the Pierre Auger Observatory, The Telescope Array Project and the Akeno Giant Air Shower Array, among others, have reported cosmic ray events above the GZK upper limit, although some have questioned whether these results actually violate LI [237]. However, many in the quantum gravity community have taken these observations at face value and used them to motivate modifications of special relativity [238]. If these results hold, quantifying the extent of LI violations via cosmic ray observations may help constrain vacuum dispersion, and by proxy the associated ultraviolet spectral dimension.

7.5.2. GeV Scales

Examining the electron anomalous magnetic moment in a fractal spacetime, Svozil and Zeilinger found that the Hausdorff dimension of spacetime is $D_H > 4 - 5 \times 10^{-7}$ [199]. A yet tighter constraint has been placed by Schafer and Muller, for which $D_H > 4 - 3.6 \times 10^{-11}$, via a calculation of the Lamb shift [239].

Multifractional models have thus far provided some of the strongest bounds in this energy regime. In particular, modifications of the precisely measured Lamb shift and electron anomalous magnetic moment constrain the characteristic length scale l_{DR} of dynamical dimensional reduction to be as low as $l_{DR} < 10^{-27}m$ for a particular set of parameters [240]. Clearly this constraint is still far from the Planck scale.

7.5.3. TeV **Scales**

The authors of Ref. [241] claim to already have experimental evidence for dimensional reduction. Cosmic rays detected in the Pamir mountains have an observed energy flux that is strongly aligned along a straight line in the target plane, a result that is hard to explain with conventional physics, and may be consistent with a $(1+1)$-dimensional spacetime at high energies [242].

Experimental searches for dimensional reduction using particle accelerators such as the LHC have thus far produced null results, with no significant deviations from 4-dimensionality observed at these scales [241, 243]. As indicated in Ref. [244], dimensional reduction present at TeV scales could also have a telltale signature in primordial gravitational waves, albeit a signature that strongly depends on the specific model [245].

7.6. What is Dimensional Reduction Really Telling Us?

7.6.1. Overview

The ubiquity and consistency of dimensional reduction in quantum gravity has motivated the search for an underlying theoretical explanation. There currently exist a few proposed explanations for the observation of dimensional reduction.

One proposal is that of scale invariance. There is growing evidence that gravity may be nonperturbatively renormalisable as described by Weinberg's asymptotic safety scenario, which requires a non-trivial fixed point at high energies towards which the couplings defining the theory flow [246]. At such a fixed point gravity must be scale invariant, and hence Newton's constant must be dimensionless. Only in 2-dimensional spacetime is Newton's constant dimensionless, and so in this scenario going to higher energies and hence flowing towards the fixed point should correspond to the dimensionality of spacetime reducing to the value 2. This explanation is not entirely satisfying as it does not explain why such a fixed point should exist in the first place [202].

A second proposed explanation for dimensional reduction is that of asymptotic silence. General relativity exhibits so-called asymptotic silence in the vicinity of a space-like singularity, which is the narrowing or focusing of light cones close to the Planck scale leading to a causal decoupling of nearby spacetime points. In this scenario, each point has a preferred spatial direction, and geodesics see a reduced $(1+1)$-dimensional spacetime [65]. See Ref. [65] for a more complete review of possible explanations for dimensional reduction.

We now proceed to examine another possible explanation for dimensional reduction in more detail, via a scale dependent rescaling of length. Devoting more space to this explanation does not necessarily mean it is any more credible than those discussed above, however it is more relevant to the theme of scale transformations.

7.6.2. Scale Dependent Length

If we take observations of dimensional reduction literally, assuming spacetime really does become lower-dimensional at extremely small distance scales, then logically we must accept its consequences. The implications of dimensional reduction are rather radical, including superluminal motion [229, 232] and deformed LI [247, 248, 106, 238]. Given these extreme consequences, in addition to the absence of any theoretical explanation underpinning dimensional reduction, a more conservative stance is to question the reality of this phenomena. In this section, we examine whether dimensional reduction is a mirage, whose appearance is merely a symptom of new, but less radical, physics. To investigate this line of research we focus specifically on the appearance of dimensional reduction in the CDT model of quantum gravity. However, due to the consistent appearance of dimensional reduction over a wide range of independent approaches the conclusions reached may be more widely applicable.

As detailed in section 2.2.3., the spectral dimension D_S is a measure of the effective dimension of a manifold over varying length scales (see e.g., Refs. [152, 7] for more details), and is related to the probability P_r that a diffusion process returns to the origin after a diffusion time σ. A spectral dimension that varies with distance scale implies either a systematic violation, a non-systematic violation or a deformation of LI [10]. Systematic LI violations are defined as position independent and imply the existence of a preferred global frame of reference [120]. Non-systematic violations of LI occur when the symmetry stochastically varies as a function of spacetime position [120]. If the exact low-energy symmetry is deformed, but not broken, we have the milder case of deformed LI [249]. In any case, whether LI is violated or merely deformed by a scale dependent spectral dimension, the associated dispersion relation $E(p)$ must nevertheless be modified [249].

The dispersion relation

$$E^2(p) = p^2\left(1 + (\lambda p)^{2\gamma}\right), \tag{195}$$

is an example of a systematic violation of LI, as it implies a global choice of frame in which energy is a distinguished component of momentum [120, 232]. Since CDT allows for the possibility of a preferred global frame of reference [250, 251], a reduction of the spectral dimension in CDT may be linked with a dispersion relation of the form of Eq. (39). In this case, Eq. (39) can be associated with a modified speed of light

$$c_m(\lambda p) = \frac{E}{p} = \sqrt{1 + (\lambda p)^{2\gamma}}, \tag{196}$$

where λp is a momentum scale and γ a positive integer.

In a spacetime with $(D_H + T_H)$ Hausdorff dimensions the spectral dimension can be written as [232, 211]

$$D_S = T_H + \frac{D_H}{1+\gamma}. \tag{197}$$

Based on a saddle-point approximation, Ref. [229] gives a model-independent expression for the spectral dimension in terms of the phase v_{phase} and group velocity v_{group} in $(D_T + 1)$ topological dimensions as

$$D_S = 1 + D_T \frac{v_{phase}}{v_{group}} + \ldots. \tag{198}$$

Electromagnetic waves in a vacuum should yield $v_{group}/v_{phase} = 1$, which we use to define a dimensionless speed of light $c_m = v_{group}/v_{phase}$. Equation (42) thus returns a $(3+1)$ dimensional spacetime for $c_m = 1$. However, if $D_S < 4$ then the dimensionless value for the speed of light c_m must exceed unity.

We can now see that dimensional reduction generically implies a scale dependent speed of light. Furthermore, any theory with dimensional reduction must at the very least deform LI [231, 247, 248, 106, 238], although in special circumstances it may preserve the relativity principle [247]. A scientific radical might then be tempted to interpret dimensional reduction as evidence for a variable speed of light [252, 253], or for theories that violate LI [193]. However, there may exist a more conservative explanation, which we will now explore.

The canonical point in the CDT phase diagram has a running spectral dimension that is most accurately described by a fit to the functional form

$$D_S = a - \frac{b}{c+\sigma}, \tag{199}$$

where a, b and c are free fit parameters [8, 7]. This specific form of the spectral dimension is further supported by analytical calculations [209, 254]. The return probability is then obtained by integration of Eq. (43), giving

$$P_r(\sigma) = \frac{1}{\sigma^{a/2}\left(1+\frac{c}{\sigma}\right)^{\frac{b}{2c}}}. \tag{200}$$

Extensive numerical simulations yield the fit parameters $a = 4.02$, $b = 119$ and $c = 54$ [8]. A more recent and exhaustive study at the same canonical point in the CDT phase diagram finds very similar results, namely $a = 4.06$, $b = 135$ and $c = 67$ [7]. Thus, two sets of independent calculations both suggest $a \simeq 4$ and $b/2c \simeq 1$, which from Eq. (44) implies a probability of return associated with dimensional reduction of

$$P_r(\sigma) \simeq \frac{1}{\sigma^2 + c\sigma}. \tag{201}$$

It is known that the probability of return in the absence of any dimensional reduction is $P(\sigma) = \sigma^{-2}$, in infinitely flat 4-dimensional Euclidean space. Since the path length traced out by a diffusing particle is proportional to the diffusion time σ, we ask what function $\Gamma(\sigma)$ is needed to rescale the path length such that we obtain the observed probability of return of Eq. (45)? To answer this question we form the equality

$$\frac{1}{\Gamma(\sigma)^2\sigma^2} = \frac{1}{\sigma^2 + c\sigma}, \tag{202}$$

which yields

$$\Gamma(\sigma) = \sqrt{1+\frac{c}{\sigma}}. \tag{203}$$

Therefore, the appearance of dimensional reduction may be explained by a specific scale dependent path length, a characteristic feature of fractal curves [75] and quantum mechanical paths (see section 2.3.2.). Since σ is proportional to the square of the resolution scale with which we probe spacetime r, we can therefore write

$$\frac{\langle l \rangle}{l} \equiv \Gamma(r) = \sqrt{1 + \frac{c}{r^2}}, \tag{204}$$

where the scale dependent path is denoted by $\langle l \rangle$ and the scale independent classical path by l. Thus, resolving spacetime with a resolution r increases the classical path length l by a factor $\Gamma(r)$.

A defining characteristic of CDT is the restriction to a fixed foliation. This foliation of the lattice introduces a fixed time coordinate via space-like hypersurfaces spaced at equal time intervals $t = 0$, $t = 1$, ..., $t = N$. Since time intervals are fixed in this way in CDT they cannot vary with scale, unlike spatial intervals. Given a scale dependent spatial interval coupled with a scale independent time interval one would expect the speed $c_m(\sigma)$ of diffusing particles to then scale like

$$c_m(\sigma) = \frac{\Gamma(\sigma)\, l}{t}, \tag{205}$$

where l and t are the scale independent path length and duration, respectively.

This claim can actually be tested by explicitly tracking the path a fictitious diffusing particle traces out in a CDT network. Diffusion begins with the test particle starting from a randomly chosen simplex, and proceeds by diffusing throughout the network by making σ hops between adjacent triangulations. The random walk can be explicitly tracked at each point so that one can obtain information about how exactly the path length varies as a function of geodesic distance. The path length simply equals the number of diffusion steps σ multiplied by the average distance between neighbouring simplices, which is encoded by the proportionality constant ζ. The time t_d it takes a particle to diffuse between two arbitrary points can, therefore, be determined by counting the number of times it intersects a space-like hypersurface. An effective velocity v_d for the diffusing particle is then given by[6]

[6] Since the spectral dimension is defined in Euclidean signature the diffusion rate v_d is interpreted as an effective velocity [10].

$$v_d = \frac{\zeta\sigma}{t_d}. \tag{206}$$

Figure 14. shows actual measurements of v_d averaged over 10^6 independent diffusion processes for the canonical point in the de Sitter phase (phase C of Fig. 10.) of CDT using two different lattice volumes, and for a constant of proportionality $\zeta = 0.18$ [9]. Figure 14. demonstrates that the measured velocity of diffusing particles v_d in a typical CDT network of simplices closely matches the predicted scale dependent diffusion speed of Eq. (49). Since time intervals are fixed in CDT, Fig. 14. provides numerical evidence for a scale dependent length, which exactly matches the functional form necessary to explain the appearance of dimensional reduction.[7]

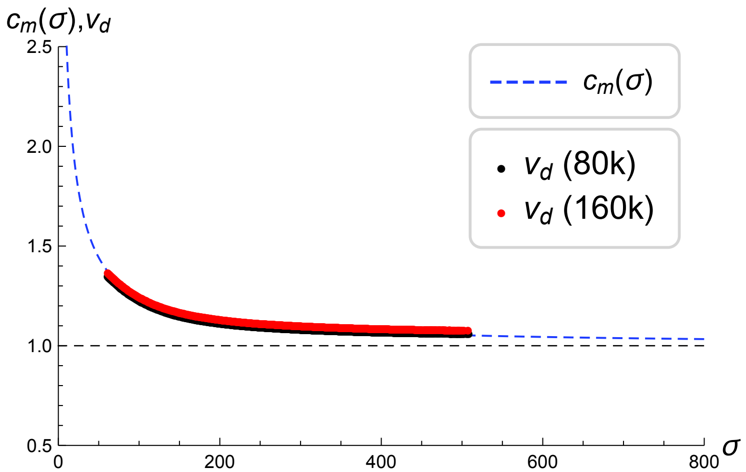

Figure 14. A comparison between the modified speed of light $c_m(\sigma) = \Gamma(\sigma)$ with $c = 54$ as determined in Ref. [8] (the dashed curve) and numerical measurements of the effective velocity v_d computed by averaging over 10^6 diffusion processes using 80,000 and 160,000 simplices with $\zeta = 0.18$ [9]. The horizontal dashed line denotes the classical scale invariant speed.

[7] Note that Horava-Lifshitz gravity also assumes a fixed foliation and observes the same reduction of the spectral dimension. It is possible that other approaches and methods that imply dimensional reduction may also be implicitly assuming scale invariant time intervals [3], although this is yet to be substantiated.

If we could remove the restriction of a fixed time coordinate in CDT, time intervals may vary with scale in precisely the same way as space intervals, as one would expect from a covariant theory in which space and time are treated equally. As shown in Refs. [10, 9, 255], by explicitly assuming that time intervals vary with scale in precisely the same way as spatial intervals, it is possible to maintain a scale invariant speed of light.

This suggests *it is the different treatment of space and time as a function of scale that is responsible for the appearance of dimensional reduction.* In CDT, time cannot vary as a function of scale due to the fixed foliation of hypersurfaces of constant time, but space can. Therefore, a diffusion process will not experience any variation of duration with scale, only a variation of distance. It is therefore possible that removing the restriction of fixed scale invariant time intervals may vanquish dimensional reduction altogether [10, 9, 255]. This idea will be further explored in section 7.

7.6.3. A Dual Description?

Following Hausdorff, a new definition of length $\langle L\rangle$ can be introduced that is independent of the measurement resolution r via a rescaling by the number of spatial Hausdorff dimensions D_H [75],

$$\langle L\rangle = \langle l\rangle\,(r)^{D_H-1}\,. \tag{207}$$

The ratio of the invariant Hausdorff length $\langle L\rangle$ and the variable length $\langle l\rangle$ is then equal to $1/\Gamma(r)$ so that

$$\frac{\langle L\rangle}{\langle l\rangle} = \frac{1}{\Gamma} = (r)^{D_H-1}\,. \tag{208}$$

Which gives,

$$D_H = \frac{\ln(1/\Gamma)}{\ln(r)} + 1, \tag{209}$$

quantifying how the spatial Hausdorff dimension D_H of diffusing particles varies with the scale at which they are resolved r.

Returning to Eq. (40), namely

$$c_m = \frac{E}{p} = \sqrt{1+(\lambda p)^{2\gamma}}, \tag{210}$$

we can normalise such that $c_m = 1$ when $\gamma = 0$, giving

$$c_m = \frac{E}{p} = \frac{1}{\sqrt{2}}\sqrt{1+(\lambda p)^{2\gamma}}. \tag{211}$$

Substituting $\gamma = 0$ into Eq. (41) and rearranging gives the condition

$$D_H = D_S - T_H. \tag{212}$$

Equations (53) and (56) for fixed $D_S = 2$ gives the number of temporal Hausdorff dimensions T_H as

$$T_H = 1 - \frac{\ln(1/\Gamma)}{\ln(r)}. \tag{213}$$

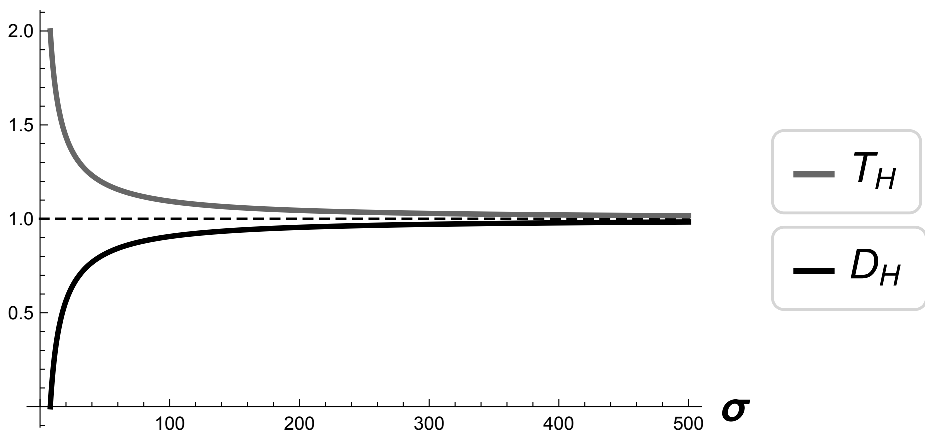

Figure 15. The spatial D_H and temporal T_H Hausdorff dimensions of CDT diffusion paths as a function of distance scale σ [10]. Note that $D_H + T_H = 2$ over all distance scales.

As shown in Fig. 15., in the large-distance limit we recover the expected result $D_H = 1$ and $T_H = 1$, whereas in the small-distance limit we find $D_H \to 0$

and $T_H \to 2$. CDT diffusion paths thus appear to indicate that a spatial dimension transforms into a temporal dimension at extremely small distances, which may be viewed as a kind of inverse Wick rotation [256]. Such a scale dependent signature change has also been reported elsewhere in CDT [6] and in loop quantum gravity [257].

Therefore, it seems there may exist a dual description of dimensional reduction in CDT: the initially linear ($D_H = 1$) diffusion paths on macroscopic scales dissolve into a series of points ($D_H = 0$) on microscopic scales, while simultaneously the number of temporal dimensions increases from $T_H = 1$ to $T_H = 2$. This situation is reminiscent of asymptotic silence and results reported in [258].

Chapter 8

Scale Dependent Spacetime

8.1. Einstein and Weyl

It was the potent combination of symmetry and simplicity that led to Einstein's discovery of general relativity. The symmetry that guided Einstein's thinking during this creative process was that of general covariance, the requirement that the laws of physics be invariant under any smooth local change of spacetime coordinates

$$x^\mu \to x^\mu + \varepsilon k^\mu(x^\mu), \tag{214}$$

where $k^\mu(x^\mu)$ is a smooth function over spacetime.

To illustrate how symmetry and simplicity almost inevitably lead to general relativity, we begin by considering a completely general gravitational action that is local in the sense that it can be defined via the integral of a Lagrangian. Therefore, we begin with the generic form

$$S = \int \mathcal{L} d^4V, \tag{215}$$

where $\mathcal{L}$ is a scalar Lagrangian density and d^4V is an element of four-volume. In order to respect general covariance both $\mathcal{L}$ and d^4V must be scalars, that is they must be invariant under any smooth local change of spacetime coordinates.

The region surrounding any point on a curved Riemannian manifold is locally flat. The volume element in a locally Minkowskian system of coordinates

x^{μ} is given by

$$d^4V = dx^0 dx^1 dx^2 dx^3. \tag{216}$$

By making an arbitrary change of coordinates $x \to x'$ the volume element d^4x transforms according to

$$d^4x = \left| \frac{\partial x}{\partial x'} \right| d^4x', \tag{217}$$

where $|\partial x/\partial x'|$ is the Jacobian determinant. Similarly, the metric tensor transforms as

$$g_{\mu\nu} = \frac{\partial x^a}{\partial x'^{\mu}} \frac{\partial x^b}{\partial x'^{\nu}} g_{ab}, \tag{218}$$

and the determinant of the metric tensor $|g_{\mu\nu}|$ as

$$|g'_{\mu\nu}| = \left| \frac{\partial x}{\partial x'} \right| \left| \frac{\partial x}{\partial x'} \right| |g_{\mu\nu}|. \tag{219}$$

Equation (6) then implies

$$\left| \frac{\partial x}{\partial x'} \right| = \frac{\sqrt{|g'_{\mu\nu}|}}{\sqrt{|g_{\mu\nu}|}}, \tag{220}$$

which upon substituting into Eq. (4) gives

$$\sqrt{|g_{\mu\nu}|} d^4x = \sqrt{|g'_{\mu\nu}|} d^4x'. \tag{221}$$

We therefore conclude that the four-volume element is

$$d^4V = \sqrt{|g_{\mu\nu}|} d^4x, \tag{222}$$

as dictated by local coordinate invariance.

Determining the scalar Lagrange density is a little more involved, but still requires only the application of symmetry and simplicity. The goal is to determine the simplest function of the metric and its derivatives that respects general covariance. The scalar Lagrangian should be some function of spacetime only. The simplest scalar is a constant, let's call it k_0. However, this cannot be the correct Lagrangian since $\mathcal{L} = k_0 \neq 0$ gives no solution to the Euler-Lagrange equations whatsoever. Next, we try a Lagrangian that depends only on the metric tensor $g_{\mu\nu}$ and not its derivatives. But this doesn't work either, because without any derivatives of the metric the theory would be void of any interesting dynamics. The next simplest Lagrangian is of the form

$$\mathcal{L} = k_1 R + k_0, \tag{223}$$

where k_0 and k_1 are constants and R is the Ricci scalar, the simplest curvature invariant of a Riemannian manifold. Thus the Lagrangian density of Eq. (10) defines the simplest function of the metric and its derivatives ($R = g^{\mu\nu} R_{\mu\nu}$) that respects local coordinate invariance. The full Einstein-Hilbert action is therefore found to be

$$S_{EH} = \int R \sqrt{|g_{\mu\nu}|} d^4x. \tag{224}$$

Variation of this action with respect to the metric tensor returns the celebrated Einstein field equations of general relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}, \tag{225}$$

where $T_{\mu\nu}$ is the stress-energy-momentum tensor describing the amount and distribution of energy in spacetime. $R_{\mu\nu}$ is the Ricci curvature tensor, defining the rate with which a small volume changes when parallel propagated along a geodesic in curved space. The field equations of general relativity have been confirmed by numerous experiments over the previous century, including the spectacular recent observation of gravitational waves. There is no doubt that the application of symmetry principles is remarkably successful, nowhere has this been more evident than in the application of general covariance as a guiding principle in the construction of general relativity.

But might we go further? Is there a yet greater symmetry of spacetime whose application might lead us to new and more successful physics? Shortly after the development of general relativity, Hermann Weyl put forward a generalisation of Riemannian geometry that at least mathematically seemed to unify electromagnetism and gravity [259]. Weyl's proposal was to keep spacetime coordinates fixed

$$x^{\mu} \rightarrow x^{\mu}, \tag{226}$$

but perform a position dependent rescaling of the metric tensor

$$g_{\mu\nu} \rightarrow g_{\mu\nu}\Omega^2(x^{\mu}), \tag{227}$$

where the Weyl factor $\Omega(x^{\mu})$ is an arbitrary function of spacetime coordinates. Note that this so-called Weyl transformation is very similar to a conformal transformation but differs in one crucial respect; under a conformal transformation coordinates transform, but under a Weyl transformation they do not. Hence, a Weyl transformation is not a simple diffeomorphism, as it can physically change measurable quantities.

Applying a Weyl transformation to the Ricci tensor R_{ab} in 4-dimensional spacetime we find

$$R_{ab} \rightarrow R_{ab}\frac{1}{\Omega^2}\left(4\Omega_{,a}\Omega_{,b} - \Omega_{,c}\Omega^{;c}g_{ab}\right) - \frac{1}{\Omega}\left(2\Omega_{;ab} + g_{ab}\Box\Omega\right). \tag{228}$$

Likewise, the Ricci scalar transforms according to

$$R \rightarrow \frac{1}{\Omega^2}\left(R - 6\frac{\Box\Omega}{\Omega}\right), \tag{229}$$

where $\Box\Omega$ is the d'Alembertian of the Weyl factor. Applying Eq. (16) to the Einstein-Hilbert action of Eq. (11) clearly demonstrates that general relativity is not invariant under a position dependent rescaling of the metric tensor.

However, Weyl found the simplest action that respects both local coordinate invariance and local rescalings of the metric tensor (Eq. (14)) to be

$$S_W = -\alpha_G \int C_{abcd}C^{abcd}d^4x\sqrt{-g}, \tag{230}$$

where C_{abcd} is known as the Weyl tensor and is given by [260]

$$C_{abcd} = R_{abcd} + \frac{1}{6}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) - \frac{1}{2}\left(g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac}\right). \tag{231}$$

The Weyl action can then more explicitly be written as

$$S_W = -\alpha_G \int \left(R_{abcd}R^{abcd} - 2R_{cd}R^{cd} + \frac{1}{3}R^2 \right) d^4x\sqrt{-g}. \tag{232}$$

Using the fact that

$$\int \left(R_{abcd}R^{abcd} - 4R_{cd}R^{cd} + R^2 \right) \tag{233}$$

is a topological invariant it can be equated with zero [261], which simplifies the action of Eq. (19) to

$$S_W = -2\alpha_G \int \left(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2 \right) d^4x\sqrt{-g}, \tag{234}$$

where $R_{\mu\nu}$ is the Ricci tensor and α_G is a dimensionless gravitational coupling [262]. This theory produces fourth-order equations for fluctuations about a fixed background, which may or may not lead to problems with unitarity [263]. Variations of this action with respect to the metric tensor give field equations, known as the Bach equations [264], of

$$-4\alpha_G W^{\mu\nu} + T^{\mu\nu} = 0, \tag{235}$$

with [260]

$$W^{\mu\nu} = \frac{1}{2}g^{\mu\nu}(R)^{;\beta}_{;\beta} + R^{\mu\nu;\beta}_{;\beta} - R^{\mu\beta;\nu}_{;\beta} - R^{\nu\beta;\mu}_{;\beta} - 2R^{\mu\beta}R^{\nu}_{\beta} + \frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}g^{\mu\nu}R^{;\beta}_{;\beta} + \frac{2}{3}(R)^{;\mu;\nu} + \frac{2}{3}RR^{\mu\nu} - \frac{1}{6}g^{\mu\nu}R^2, \tag{236}$$

where semicolons denote covariant derivatives.

Although Einstein initially admired Weyl's theory, calling it "*a stroke of genius of the first rate,*" he later raised a serious objection that has never been satisfactorily resolved. Put simply, Einstein's objection centred on the fact that a local rescaling of the metric tensor $g_{\mu\nu} \to g_{\mu\nu}\Omega^2(x)$ inevitably implies a non-invariant spacetime interval

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to ds^2\Omega^2(x). \tag{237}$$

Although light-like spacetime intervals ($ds^2 = 0$) are invariant under a Weyl transformation, space-like ($ds^2 < 0$) and time-like ($ds^2 > 0$) intervals are not. The fact that time-like intervals are not invariant under a Weyl transformation is a serious problem for the following reason. In Weyl's theory the metric tensor can vary arbitrarily as a function of spacetime coordinates so that the length of a vector is not constant under parallel transport. For example, since the spacetime interval can be physically measured by the ticking of a clock, proper time is a path-dependent quantity in this scenario. Einstein argued that a path-dependent proper time has a number of unphysical consequences, that contradict both experiment and everyday experience, for example, the spacing of atomic spectral lines would vary depending on the path history of the particular element, something that has never been observed [259]. In response to Einstein's criticism, Weyl made numerous revisions of his theory and even made several attempts to make the interval ds^2 invariant. Although Weyl was ultimately unsuccessful in refuting Einstein's argument, his ideas would later be applied in a different context, which would become the foundation of modern gauge theory.

8.2. Renormalising Spacetime

8.2.1. Motivation

Since Weyl's approach seems to have unphysical consequences, how are we to proceed? The central problem with Weyl gravity is that it predicts a position dependent spacetime interval, which contradicts experiment. However, what if we consider a metric that varies with scale, rather than position? Would this contradict experiment in an obvious way? Would it help in our search for a theory of quantum gravity? We investigate these questions in this section.

A scale dependent metric tensor has a number of motivations, including the renormalisation of gravity. Renormalisation is a procedure for dealing with divergences that arise in quantum field theories. A key insight into renormalisation was the realisation that the parameters appearing in the mathematics of the original theory were not measurable quantities, as they corresponded to values in the infinite-energy limit. These so-called bare parameters must be converted into renormalised parameters defined at a finite energy scale, in order to make contact with reality. The pieces of the bare parameters that remain following this conversion are then reinterpreted as counter-terms, which can cancel out the troublesome divergences. If all divergences can be removed using a finite number of such procedures, the theory is said to be renormalisable [11]. Renormalisation has been enormously successful when applied to three out of four fundamental interactions, with experiment confirming theoretical predictions with astonishing precision. However, general relativity is nonrenormalisable, as confirmed via explicit calculation [35].

But why does renormalisation not work for gravity, when it works so well for the other three interactions? To answer this question we must first understand how the other interactions are renormalised. Starting with QED, the relativistic quantum field theory of the electromagnetic interaction, it is important to note that the Lagrangian is a function of both bare parameters (charge and mass) and bare fields. Renormalising only the bare parameters will not yield a finite theory; it is crucial that the bare fields are also renormalised. In the case of perturbative QCD, the quantum field theory of the strong interaction, we find exactly the same thing. In order to make QCD finite, it is not enough to renormalise the coupling constant and quark mass, one must also renormalise the quark and gluon fields. In fact, in all quantum field theories, fluctuations modify bare fields such that they become a function of scale [11]. For example, a bare field ϕ is converted into a renormalised field $\tilde{\phi}$ via field renormalisation $\tilde{\phi} = \phi Z^{-1/2}(k)$, where $Z(k)$ encodes how ϕ depends on the so-called coarse-graining scale k, the resolving power of a hypothetical microscope [265]. However, the field renormalisation of gravity has been largely neglected [76]. Renormalising the gravitational field is equivalent to renormalising spacetime itself, which would require defining a scale dependent metric tensor $g_{\mu\nu}(k)$.

In addition to renormalisability, a scale dependent metric is expected for physical reasons. For instance, quantum field theory tells us that the allowed energy of particle-antiparticle pairs in the spacetime vacuum increases as we decrease the distance scale with which we resolve spacetime. General relativity therefore implies the metric should fluctuate with a magnitude that depends on scale. A second reason derives from the measurement process itself; any position measurement requires the introduction of a scattering particle of non-zero momenta. Thus, gravitational interactions within the region under observation are an inevitable feature of the measurement process. The larger the momentum of the scattering particle, the larger the gravitational perturbation within the observed region [93]. Taking measurements at two different coarse-graining scales k and k' will, therefore, result in two different spacetime metrics. Results in string theory [266] and loop quantum gravity [212] also support the idea of a scale dependent spacetime metric.

However, one must be cautious when dealing with a scale dependent metric tensor. For instance, scale dependent fluctuations of the metric tensor are often associated with the vacuum dispersion of light. This is due to the fact that photons with a higher energy can resolve spacetime on smaller scales, and so they will be hindered to a greater extent by vacuum fluctuations during their time of flight, resulting in an energy dependent speed of light. Recent astronomical data, however, from the *Fermi* space telescope has shown the speed of light to be independent of energy up to at least $7.62E_P$ (with a $95\%CL$) and $\simeq 4.8E_P$ (with a $99\%CL$) [108] for linear dispersion relations. These experimental results may provide an important hint as to how to correctly incorporate a scale dependent metric; it must be done in such a way that the speed of light remains scale independent.

Under a Weyl transformation, the angle between vectors is scale invariant, which includes the angle between null vectors that define the light-cone. Therefore, a Weyl transformation permits a scale dependent metric as well as a scale independent speed of light. However, Einstein's criticism essentially rules out a normal Weyl transformation, which defines a position-dependent rescaling of the metric.

Given these motivations, Ref. [11] postulates a *Weyl-like* transformation in which the bare metric tensor $g_{\mu\nu}(x^\mu)$ at each spacetime point x^μ transforms according to

$$g_{\mu\nu}(x^\mu) \to \Omega^2(k) g_{\mu\nu}(x^\mu) \equiv \tilde{g}_{\mu\nu}(x^\mu)_k, \tag{238}$$

where $\Omega(k)$ is a dimensionless function of the coarse-graining scale k, rather than a function of position.[1]

[1] In order to reproduce classical physics in the appropriate limit, Eq. (25) must be subject to the constraint $\Omega(k) \to 1$ as $k \to 0$.

8.2.2. Estimating $\Omega(k)$

An infinitesimal version of the Weyl-like transformation of Eq. (25) is given by $\delta g_{\mu\nu}(k) = \varepsilon(k) g_{\mu\nu}$, where $\varepsilon(k)$ is dimensionless and $|\varepsilon(k)| \ll 1$ [267, 268]. Thus,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}(k) = g_{\mu\nu}\left(1+\varepsilon(k)\right) \equiv \tilde{g}_{\mu\nu}(x^{\mu})_k, \tag{239}$$

hence $\Omega^2(k) = 1+\varepsilon(k)$. Under this infinitesimal transformation vector length scales as

$$\tilde{r}^2 = \left(1+\varepsilon(k)\right) r^2 = \left(r^2 + r^2\varepsilon(k)\right). \tag{240}$$

Assuming the perturbation $\delta g_{\mu\nu}$ is small, $\varepsilon(k)$ can be expanded as a perturbative series. Because the Weyl-like factor must be dimensionless a series expansion in the dimensionless combination $l_0 k$ is carried out, where l_0 is a constant having the dimensions of length. The product $r^2\varepsilon(k)$ in Eq. (27) therefore becomes

$$r^2\varepsilon(k) = r^2 \sum_{n=0}^{\infty} a_n \left(l_0 k\right)^n, \tag{241}$$

where a_n are expansion coefficients.

If the physical radius of curvature is much greater than $1/k$ for the renormalised metric $\tilde{g}_{\mu\nu}(x^{\mu})_k$ then the WKB approximation of mode functions can be used. Within the WKB approximation scheme, one can demonstrate that the coarse-graining scale k is related to the linear extension of the averaging region r via $k \simeq \pi/r$ [265]. Considering small perturbations $\delta g_{\mu\nu}$ about flat spacetime gives

$$\tilde{r}^2 \simeq r^2 + r^2 \sum_{n=0}^{\infty} a_n \left(\frac{l_0}{\pi r}\right)^n. \tag{242}$$

Since we must have $\tilde{r} = r$ as $r \rightarrow \infty$ we may exclude all terms with $n < 2$ in Eq. (29). Note that negative values of n are also excluded by the condition $\tilde{r} = r$ as $r \rightarrow \infty$. The functional form of $\varepsilon(k)$ can be further constrained by assuming the existence of a minimum resolvable distance scale $\tilde{r} = l_0$ (see section 3.2. for the evidence of a minimal length). The requirement of a minimum resolvable distance $\tilde{r} = l_0$ as $r \to 0$ excludes any terms for which $n > 2$ in Eq. (29). Therefore, $n = 2$ is the only term that reproduces classical physics in the appropriate limit and is consistent with a minimum length. This argument suggests that the only factor $\Omega(k)$ that gives $\tilde{r} = l_0$ as $r \to 0$ as well as $\tilde{r} = r$ as $r \to \infty$ is

$$\Omega(k) \simeq \sqrt{1 + (lk)^2}. \tag{243}$$

Equation (30) has a functional form that is in good agreement with the existing literature [94, 98, 93, 100, 10, 9].

8.2.3. Immediate Implications

In this section we compute some of the primary physical implications of this Weyl-like transformation of spacetime [11].

In tensor notation, the length r of any contravariant Riemannian vector ζ can be written as

$$r^2 = g_{\mu\nu}\zeta^\mu\zeta^\nu, \tag{244}$$

where ζ^μ and ζ^ν are arbitrary vectors and $g_{\mu\nu}$ is the symmetric metric tensor [11]. Under our postulated Weyl-like transformation

$$g_{\mu\nu}(x^\mu) \to \Omega^2(k) g_{\mu\nu}(x^\mu) \equiv \tilde{g}_{\mu\nu}(x^\mu)_k, \tag{245}$$

the length of any contravariant Riemannian vector $\tilde{r}$ in the renormalised metric $\tilde{g}_{\mu\nu}(x^\mu)_k$ now becomes [269]

$$\tilde{r}^2 = \tilde{g}_{\mu\nu}(x^\mu)_k \zeta^\mu\zeta^\nu = \Omega^2(k) g_{\mu\nu}\zeta^\mu\zeta^\nu = \Omega^2(k) r^2. \tag{246}$$

Such a scale dependent distance nicely links back to section 7.6.2., in which it is shown that the appearance of dimensional reduction in CDT may be explained by a scale dependent path length, especially due to the fact that the form of $\Omega(k)$ given by Eq. (30) under the WKB approximation $k \simeq \pi/r$ is strikingly similar to that required to account for the appearance of dimensional reduction in CDT.

The length of an infinitesimal vector in the new renormalised metric is similarly given by

$$d\tilde{r}^2 = \tilde{g}_{\mu\nu}(x^\mu)_k d\zeta^\mu d\zeta^\nu = \Omega^2(k) g_{\mu\nu} d\zeta^\mu d\zeta^\nu = \Omega^2(k) dr^2. \tag{247}$$

Since the null interval $ds^2 = 0$ and consequently the infinitesimal speed of light $c = dr/dt$ are invariant under this Weyl-like transformation, Eq. (34) tells us that infinitesimal time intervals dt must also scale according to

$$dt \rightarrow \Omega(k)dt \equiv d\tilde{t}. \tag{248}$$

Therefore, distance and duration are both dependent on scale in this scenario, analogous to how distance and duration are dependent on relative speed in special relativity.

The spacetime interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ transforms according to

$$ds^2 \rightarrow \Omega^2(k)ds^2 \equiv d\tilde{s}^2, \tag{249}$$

as can be verified using the transformation of Eq. (32) on $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. This Weyl-like transformation therefore changes space-like and time-like intervals, but leaves null intervals ($ds^2 = 0$) invariant. Moreover, unlike the normal Weyl transformation, the spacetime interval of Eq. (36) is not position dependent and so it evades Einstein's criticism.

A local scale transformation, such as our Weyl-like transformation, acts on momentum p in the opposite way to the position r [270], namely

$$p \rightarrow \frac{p}{\Omega(k)} \equiv \tilde{p}. \tag{250}$$

The mathematical reason for this is that in natural units the product of position and momentum must be dimensionless. A more physical reason is that resolving a smaller region of spacetime requires using higher frequency modes of momentum [270]. Since the speed of light $c = E/p$ should be invariant under this Weyl-like transformation, this implies the energy E should scale identically, namely

$$E \rightarrow \frac{E}{\Omega(k)} \equiv \tilde{E}. \tag{251}$$

8.2.4. Implications for Quantum Gravity

We now focus on what this Weyl-like transformation, if correct, might say about quantum gravity. In particular, we examine two unsolved and long-standing problems in quantum gravity; high-energy non-conformal scaling and nonrenormalisability.

Recalling our discussion in section 5.2., we found that for a conformal field theory in 4-dimensions entropy scales as

$$S_{CFT} \sim \frac{a}{b^{\frac{3}{4}}} E^{\frac{3}{4}} r^{\frac{3}{4}}, \tag{252}$$

but for a 4-dimensional theory of gravity at high energies entropy scales as

$$S_{Grav} = \frac{\pi r^2}{l_P^2}. \tag{253}$$

Since Eqs. (39) and (40) do not agree it is argued in Refs. [144, 145] that gravity cannot possibly be a renormalisable quantum field theory in 4-dimensional spacetime, a potentially serious problem for quantum gravity in general.

Applying the Weyl-like transformation of Eq. (25) we found that distance transforms according to $r \rightarrow \Omega(k) r$. Gravity, at high energies now has an entropy

$$S_{Grav} \rightarrow \frac{\pi \Omega^2(k) r^2}{l_P^2}. \tag{254}$$

What function $\Omega(k)$ is required to make the high-energy spectrum of gravity scale in the same way as a conformal field theory? In other words, what function is necessary to make $S_{Grav} = S_{CFT}$? To answer this question we write the equality

$$\frac{\pi \Omega^2(k) r^2}{l_P^2} = \frac{a}{b^{\frac{3}{4}}} E^{\frac{3}{4}} r^{\frac{3}{4}}, \tag{255}$$

a condition that fixes the functional form of $\Omega(k)$ to be

$$\Omega^2(k) \simeq C \frac{l_P^2}{r^2}, \tag{256}$$

where we have assumed $E \simeq \pi/r$ and defined the dimensionless constant $C \equiv a/(\pi^{\frac{1}{4}} b^{\frac{3}{4}})$.

Since the WKB approximation of mode functions implies $k \simeq \pi/r$ [265], Eq. (43) implies that $\Omega(k)$ must transform like $\Omega(k) \rightarrow \sqrt{C} l_P k$ at high energies.[2]

[2] In highly curved spacetimes r may deviate from π/k, however Ref. [271] demonstrates that $r \propto 1/k$ holds even as $k \rightarrow \infty$.

At low energies, however, we must recover the classical result $\Omega(k) = 1$. A Weyl-like factor of

$$\Omega(k) \simeq \sqrt{1 + C\,(l_P k)^2} \tag{257}$$

satisfies both these conditions, since $\Omega(k) \to \sqrt{C} l_P k$ in the high-energy limit ($1/k \ll l_P$) and $\Omega(k) \to 1$ in the low-energy limit $1/k \gg l_P$. Thus, if the Weyl-like factor is given by Eq. (44) then the Banks [144] and Shomer [145] argument may be resolved, since the high-energy spectrum of gravity now scales in the same way as a CFT. Note, comparing Eq. (30) and Eq. (44) suggests $l_0 \propto l_P$, where l_P is the Planck length. Therefore, $\Omega(k)$ will be practically indistinguishable from unity, except near the almost inaccessible Planck scale.

We now look at applying our Weyl-like transformation to the stubborn problem of renormalisation in quantum gravity. Of course, the proposed transformation of spacetime is not claimed to definitively solve this problem, but rather the hope is that it may provide some key insights that can be further developed.

A massless free scalar field $\Phi(x^\mu)$ in flat Euclidean space has a Green's function [94]

$$G(x_i^\mu, x_j^\mu) \equiv \int \mathcal{D}\Phi(x_j^\mu)\Phi(x_i^\mu)\exp^{\frac{i}{2}S[\Phi]} = \frac{(4\pi^2)^{-1}}{\left(x_j^\mu - x_i^\mu\right)^2}. \tag{258}$$

Given a proper distance r defined by $r^2 = \left(x_j^\mu - x_i^\mu\right)^2$ that transforms according to Eq. (30), with $k \simeq \pi/r$, then the Green's function becomes

$$G(x_i^\mu, x_j^\mu) \to \tilde{G}(x_i^\mu, x_j^\mu) \simeq \frac{(4\pi^2)^{-1}}{\left(x_i^\mu - x_j^\mu\right)^2 + \pi^2 l_0^2}. \tag{259}$$

In the coincidence limit ($x_i^\mu \to x_j^\mu$) this gives the finite result

$$\tilde{G}(x_i^\mu, x_j^\mu)_{x_i^\mu \to x_j^\mu} = \frac{(4\pi^2)^{-1}}{\pi^2 l_0^2}, \tag{260}$$

where $\tilde{r} = l_0$ is a minimum renormalised distance. A Green's function that remains finite in the limit of zero classical distance is known to remove the ultraviolet divergences present in quantum field theories [94, 81].

Let's now examine this idea within the context of a quantum field theoretic description of gravity. In d-dimensional spacetime, momentum p scales with loop-order $\mathcal{L}$ in perturbative quantum gravity via

$$\int p^{A-[G_N]\mathcal{L}} dp, \tag{261}$$

where $[G_N] = 2-d$ is the canonical mass dimension of Newton's constant and A is a process-dependent quantity that is independent of the loop-order $\mathcal{L}$ [37]. In 4-dimensional spacetime we then have $[G_N] = -2$, and so that the integral of Eq. (48) diverges as $\mathcal{L}$ increases in the perturbative expansion. This explicitly demonstrates the fact that gravity as a perturbative quantum field theory is power-counting nonrenormalisable. However, it may be possible to make the integral of Eq. (48) finite under a specific transformation of momentum, a possibility we will now explore.

Computing the differential $d\tilde{p}/dp$ using Eqs. (37) and (30) at a specific coarse-graining scale $k \simeq p$ yields

$$d\tilde{p} = (\Omega(p))^{-3} dp. \tag{262}$$

Substituting Eqs. (37) and (49) into Eq. (48) changes the perturbative expansion of gravity in 4-dimensional spacetime to

$$\int p^{A+2\mathcal{L}} dp \rightarrow \int \frac{p^{A+2\mathcal{L}}}{(\Omega(p))^{3+A+2\mathcal{L}}} dp. \tag{263}$$

Since for small p we have $\Omega(p) \rightarrow 1$ nothing changes. However, for large momenta $\Omega(p)$ scales like $\Omega(p) \sim p$, so the integrand of Eq. (50) becomes $\sim p^{-3}$.

Figure 16. plots the integrand of Eq. (50), labelled by $f(p)$, as a function of p (with $l = A = 1$) for $\mathcal{L} = 0-5$ loop-orders. For small p the integrand $f(p)$ looks like it may diverge, however as p increases the scaling behaviour p^{-3} comes to dominate and $f(p)$ reaches a finite maximum before eventually approaching zero as $p \rightarrow \infty$ at all loop-orders. Figure 16. provides evidence that the Weyl-like transformation of Eq. (25), if correct, may suppress divergences usually present in quantum field theoretic formulations of gravity, raising the exciting possibility that gravity could be renormalisable in this scenario [11].

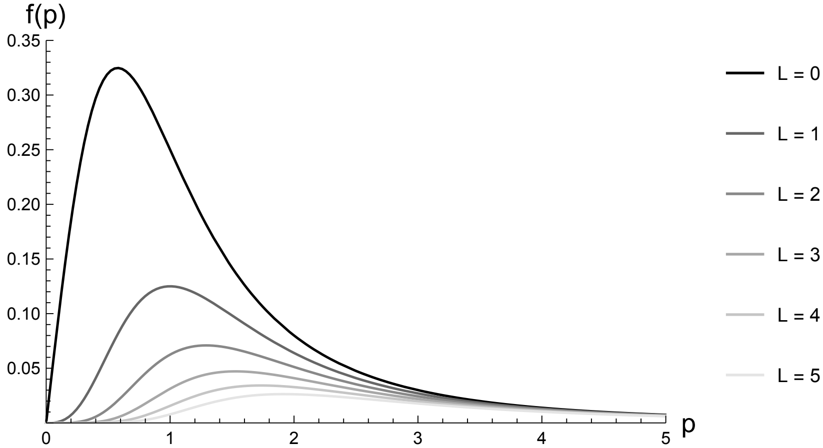

Figure 16. The integrand of Eq. (50) ($f(p)$) as a function of the momentum scale p, with $l = A = 1$ for $\mathcal{L} = 0 - 5$ loop orders [11].

Final Thoughts

There are many more orders of magnitude in scale between the Planck length $\sim 10^{-35}m$ and us $\sim 10^{0}m$, than between us and the observable universe $\sim 10^{26}m$. Arguably, we know more about the large-scale universe above than the small-scale universe below. It seems highly likely that there are new discoveries waiting to be made below the sensitivity of current experiment. A powerful mechanism for exploring this uncharted territory is provided by the renormalisation group, a mathematical formalism for *Magnifying Spacetime*.

Hopefully, this book has given you an insight into the importance of scale in fundamental physics research, and a brief overview of what we currently know about how physics changes with scale. However, there remain a lot of open questions, the answers to which may herald new physics.

Is spacetime fractal? As we have seen there is a small variation in the ultraviolet dimensionality of spacetime across different approaches and definitions of dimension. However, the most remarkable fact is that the dimension of spacetime departs from the topological dimension at all. Regardless of the exact details of dimensional reduction, this consistent observation implies one remarkable thing; the small-scale structure of spacetime itself may be fractal.

Is dimensional reduction real? A reduction to 2-dimensions at extremely small distances has the desirable feature that spacetime becomes self-renormalising, since only in 2-dimensions does the gravitational coupling become dimensionless. Yet, dimensional reduction is also associated with a number of radical and possibly unphysical features, such as an energy dependent speed of light and the deformation or violation of Lorentz invariance. As we have indicated in section 8.2., a simple rescaling of the metric tensor may account for the appearance of dimensional reduction and reproduce its desirable features, without the radical consequences associated with taking dimensional reduction literally. If, on the other hand, dimensional reduction is real, what are the consequences for cosmology, particle physics, and special relativity? Can dimensional reduction be experimentally tested in the near future?

Is there a minimum length? As reviewed in section 2, there is a growing body of evidence suggesting an answer in the affirmative. However, how can we consistently reconcile such a minimal length with special relativity? What is the fate of Lorentz invariance, the most fundamental symmetry we know of, at or near the Planck scale? If Lorentz invariance is violated at extremely small distances what is the symmetry breaking mechanism? And why is it so close to being an exact symmetry at low energies?

What is the correct gravitational action? A gravitational action that is scale invariant stands a good chance of being renormalisable. The Einstein-Hilbert action of general relativity is not scale invariant and so not renormalisable. The Weyl action discussed in section 8.2. is scale invariant. However, if Weyl's action is correct how does one evade Einstein's criticism? Could a metric tensor that depends on the coarse-graining scale k, as discussed in section 8.2., evade Einstein's criticism and lead to a scale invariant and renormalisable gravitational action?

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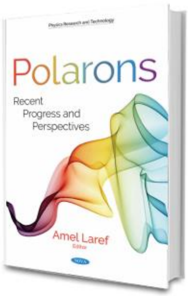

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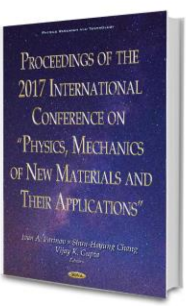

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