Engineering Mathematics-II

(M201)

WBUT—2015

Third Edition

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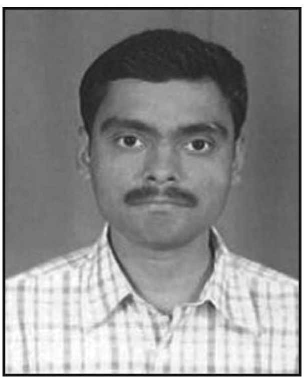

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Dr Kar and Prof. Karmakar have jointly published two other books *Engineering Mathematics I* and *Engineering Mathematics III* for WBUT with McGraw Hill Education (India).

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Dedicated

To my teacher

Dr Sanjib Kr Datta

and

My beloved family members

- Sourav Kar

To the Holy Mother

Maa Sarada

- Subrata Karmakar

Contents

Preface

One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts.

Albert Einstein (1879–1955)

Mathematics is an essential subject for any branch of engineering and technology, and to facilitate progressive learning West Bengal University of Technology (WBUT) has introduced a different syllabus for Mathematics for different semesters at the BTech level. This book has been written as per the latest WBUT syllabus for first-year second-semester BTech students. The syllabus for the second semester includes different kinds of topics but a student can hardly find a good book in the market which covers all these topics. The main objective of writing this book is to meet the demand of a good book which can build the fundamental concepts as well as help the students in their semester examination. Every topic in the book is explained in a lucid manner and is illustrated with different types of examples. Also step-wise clarification of different methods of solving problems is given. Though the book has been written according to the WBUT syllabus, other university students can also use this book for their curriculum.

Salient Features

- Written according to the WBUT syllabus
- Excellent coverage of the topics like Ordinary Differential Equations, Concepts of Graphs, Matrix Representation and Isomorphism of Graphs, Trees, Shortest Paths and Algorithm, Laplace Transforms, Improper Integrals, Beta and Gamma Functions
- Step-wise clarification of different methods of solving problems
- Solved 2001–2011 WBUT examination questions in each chapter
- Solved 2012 to 2014 WBUT examination papers
- Rich pedagogy:
    - *280 Solved Examples*
    - *95 Short and Long-Answer-Type Questions*
    - *95 Multiple Choice Questions*

Chapter Organization

The contents of the book are divided into 12 chapters according to the latest WBUT syllabus.

In **Chapter 1** we discuss the fundamental concepts of ordinary differential equations.

One of the most important kinds of differential equations is of first order and first degree. **Chapter 2** covers the concepts of various types of differential equations of first order and first degree along with the methods of solving them.

Chapter 3 deals with differential equations of first order and higher degree. Among these, Clairaut's form is one of the important topics.

Different applications of engineering and science require linear differential equations of higher order. In **Chapter 4** we present the methods of solution of linear differential equations of higher orders with constant coefficients. We also discuss the methods of solution of Cauchy–Euler equations and Cauchy–Legendre equations. At the end of the chapter, the methods of solution of simultaneous differential equations are described.

In the past few decades, graph theory has become one of the essential subjects of study in almost every field of science and technology. Graph theory can be applied to represent almost every problem which has discrete arrangements of objects. **Chapter 5** deals with the origin and fundamentals of graph theory, such as definitions, properties, different kinds of graphs, etc.

Though the pictorial representation is convenient for studying graphs, matrix representation of graphs is also essential. Representation of a graph in a matrix form means that it can be fitted to a computer; besides several results of matrix algebra can be applied on the structural properties of graphs. Among different types of matrix representation of graphs, adjacency matrix and incidence matrix are quite common. **Chapter 6** discusses adjacency matrix of graphs, incidence matrix of graphs and circuit matrix. One of the important applications of matrix representation of graphs is to see whether two graphs are isomorphic or not. In this chapter we have also described various techniques for checking isomorphism including the applications of adjacency matrix and incidence matrix.

Chapter 7 gives basic properties of trees along with the concept of spanning trees. We discuss how different kinds of searching algorithm are dependent on rooted trees and binary trees. Another topic included in this chapter is representation of different algorithms for finding minimal spanning trees, such as Krushkal's algorithm and Prim's algorithm. The concept of cut-set is very important in network theory. Here we discuss cut-set and fundamental cut-set with examples.

The most common problem is to find a path with the shortest length in different branches of science and technology. This is also used in operation research. There are several methods for finding shortest paths. **Chapter 8** discusses Dijkstra's algorithm as well as BFS algorithm to find the shortest distance.

Chapter 9 deals with the different types of improper integrals and their convergence followed by a discussion of special type of improper integrals called beta and gamma functions, along with their applications. Here we also give interrelations between beta and gamma functions.

In **Chapter 10** we first discuss the Laplace transform of some standard functions. Then we give details of the various properties of Laplace transform illustrated with different suitable examples. Here we also represent the Laplace transform of the unit step function (Heaviside's function) and Dirac's Delta function. The application of these functions makes the method particularly powerful for problems in engineering with inputs that have discontinuities or complicated periodic functions.

Chapter 11 deals with the inverse Laplace transform. Different properties of the inverse Laplace transform illustrated with various kinds of examples are provided. Here we include different techniques of finding inverse Laplace transform, such as partial fraction, convolution, etc.

In **Chapter 12** we discuss the method of solving linear ordinary differential equations with constant coefficients illustrated with different examples. To solve the equations we require an understanding of the concepts and properties of Laplace transform as well as inverse Laplace transform, which have been discussed in Chapters 10 and 11.

Various kinds of solved examples covering all the topics are given throughout the chapters including WBUT examination questions. Adequate questions are also given in the exercises of every chapter along with a section on multiple choice questions. Model question papers are provided at the end of the book.

Acknowledgements

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Last but not the least, we are very thankful to the entire team at McGraw Hill Education (India), for publishing this book in a short span of time.

Sourav Kar
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Publisher's Note

McGraw Hill Education (India) invites comments and suggestions from readers, all of which can be sent to *info.india@mheducation.com*. Please mention the title and authors' name in the subject line.

ROADMAP TO THE SYLLABUS

Engineering Mathematics-II

This text is suitable for the paper code: M 201.

Module I

Ordinary Differential Equations (ODE)–First order and first degree

Exact equations, Necessary and sufficient condition of exactness of a first order and first degree ODE (statement only), Rules for finding Integrating factors, Linear equation, Bernoulli's equation. General solution of ODE of first order and higher degree (different forms with special reference to Clairaut's equation).

GO TO

CHAPTER 1	FUNDAMENTAL CONCEPTS OF ORDINARY DIFFERENTIAL EQUATIONS (ODE)
CHAPTER 2	ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE
CHAPTER 3	ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

Module II

ODE-Higher order and first degree

General linear ODE of order two with constant coefficients, CF and PI, D–operator methods for finding PI, Method of variation of parameters, Cauchy–Euler equations, Solution of simultaneous linear differential equations.

GO TO

CHAPTER 4	ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER AND FIRST DEGREE

Module III

Basics of Graph Theory

Graphs, Digraphs, Weighted graph, Connected and disconnected graphs, Complement of a graph, Regular graph, Complete graph, Subgraph; Walks, Paths, Circuits, Euler Graph, Cut sets and cut vertices, Matrix representation of a graph, Adjacency and incidence matrices of a graph, Graph isomorphism, Bipartite graph.

GO TO

CHAPTER 5 BASIC CONCEPTS OF GRAPH THEORY
CHAPTER 6 MATRIX REPRESENTATION AND ISOMORPHISM OF GRAPHS

Module IV

Tree

Definition and properties, Binary tree, Spanning tree of a graph, Minimal spanning tree, Properties of trees, Algorithms: Dijkstra's algorithm for shortest path problem, Determination of minimal spanning tree using DFS, BFS, Kruskal's and Prim's algorithms.

GO TO

CHAPTER 7 TREE
CHAPTER 8 SHORTEST PATH AND ALGORITHM

Module V

Improper Integral

Basic ideas of improper integrals, Working knowledge of beta and gamma functions (convergence to be assumed) and their interrelations.

Laplace Transform (LT): Definition and existence of LT, LT of elementary functions, First and second shifting properties, Change of scale property; LT of $\frac{f(t)}{t}$, LT of $t^n f(t)$, LT of derivatives of $f(t)$, LT of $\int f(u)du$. Evaluation of improper integrals using LT, LT of periodic and step functions, Inverse LT: Definition and its properties; Convolution Theorem (statement only) and its application to the evaluation of inverse LT, Solution of linear ODE with constant coefficients (initial value problem) using LT.

Contd....

GO TO

CHAPTER 9	IMPROPER INTEGRALS
CHAPTER 10	LAPLACE TRANSFORM (LT)
CHAPTER 11	INVERSE LAPLACE TRANSFORM
CHAPTER 12	SOLUTION OF LINEAR ODE USING LAPLACE TRANSFORM

CHAPTER

1

Fundamental Concepts of Ordinary Differential Equations (ODE)

1.1 INTRODUCTION

Ordinary Differential Equations play an important role in different branches of science and technology. In the practical field of application problems are expressed as differential equations and the solution to these differential equations are of much importance. In this chapter, we will discuss the fundamental concepts of ordinary differential equations followed by chapters which deal with the various analytical methods to solve different forms of differential equations.

1.2 ORDINARY DIFFERENTIAL EQUATIONS (ODE)

An equation involving an independent variable and a dependent variable with its ordinary derivatives with respect to the independent variable or involving differentials is called an ordinary differential equation.

If x is the independent variable and y is a dependent variable, then the equation involving x, y and one or more of the following

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}, \ldots$$

is called an ordinary differential equation.

The general form of ordinary differential equations is

$$F\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y, x\right) = 0$$

where x is the independent variable and y is a dependent variable.

Let us give some examples of ordinary differential equations,

a) $\dfrac{dy}{dx} = a$

b) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3y = \sin x$

c) $\left(xy + x^2\right) dx + \left(y^2 + \sin x\right) dy = 0$

Observation: The word **ordinary** states the fact that there is only one independent variable in the equation. If there exists more than one independent variable along with the partial derivatives in a equation, it is called a partial differential equation.

The example of a partial differential equation is

$$\frac{\partial^2 z}{\partial^2 x} + 5\frac{\partial z}{\partial y} = x^2 + y^2$$

where z is the dependent variable and x, y are the independent variables.

1.3 ORDER AND DEGREE OF ORDINARY DIFFERENTIAL EQUATIONS

The **order** of an ordinary differential equation is the order of the highest ordered derivative involved in the equation and the **degree** of an ordinary differential equation is the power of the highest ordered derivative after making the equation rational and integrable as far as derivatives are concerned.

Example 1 **The order and degree of the differential equation**

$$\frac{d^4y}{dx^4} + 5\left(\frac{dy}{dx}\right)^3 + 3y = e^x$$

is 4 **and** 1 **respectively.**

Example 2 **Determine the order and degree of the differential equation**

$$\sqrt{y + \left(\frac{dy}{dx}\right)^2} = 1 + x$$

Sol. The differential equation can be written as

$$\sqrt{y+\left(\frac{dy}{dx}\right)^2} = 1+x$$

or,

$$y+\left(\frac{dy}{dx}\right)^2 = (1+x)^2$$

Therefore, the order of the differential equation is 1 and degree is 2.

Example 3 **Determine the order and degree of the differential equation**

$$\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} = \frac{dy}{dx}+y+2$$

Sol. The differential equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}+y+2\right)^2$$

Therefore, the order of the differential equation is 2 and degree is 3.

1.4 LINEAR AND NON—LINEAR ORDINARY DIFFERENTIAL EQUATIONS

An ordinary differential equation of first degree which contains the dependent variable as first degree term only and no other term which is a product of the dependent variable or its function and its derivatives or any trancendental function of the dependent variable, will be called a **linear differential equation**. Otherwise the differential equation is called a **non–linear differential equation**.

The general form of a linear differential equation is

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y = f(x)$$

where $a_0(x), a_1(x), \ldots, a_{n-1}(x), a_n(x)$ and $f(x)$ are functions of x or constants.

Here, we cite some examples of **linear** differential equations

a) $x\frac{dy}{dx}+6y=9$

b) $5x^2\frac{d^2y}{dx^2}+3e^x\frac{dy}{dx}+y=\cos x+\log x$

Following are some examples of **non–linear** differential equations

a) $(2x + y)^2 \dfrac{dy}{dx} + 9y = e^x$

Here, co-efficient of $\dfrac{dy}{dx}$ is a function of x and y. i.e. product of y and $\dfrac{dy}{dx}$ is present.

b) $x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 8y = 6x^2y$

Here, degree of $\dfrac{dy}{dx}$ is more than one.

c) $\dfrac{dy}{dx} + 8y^2 = 2x^2$

Here, degree of y is more than one.

Observations: **An ordinary differential equation is non–linear when**

a) There exist terms which is a product of the dependent variable or its function and its derivatives.

b) There exists any trancendental function of the dependent variable.

c) The coefficient of derivatives is a function of x and y.

d) The degree of the differential coefficient is not equal to one.

e) The degree of y is more than one.

In this context we state a very important fact that **every linear differential equations is of first degree but every first degree differential equation may not be linear.**

Example 4 **The differential equation**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 5y^2 = x^2$$

is of first degree but since y^2 is present, it is a non–linear differential equation.

1.5 FORMATION OF ORDINARY DIFFERENTIAL EQUATIONS

Case I: **Generally differential equations are formed by eliminating the constants from a relation consisting of independent variable and dependent variable.**

Let us consider the most simple relation (with one constant)

$$y = m, \text{ a constant}$$

Differentiating w.r.to. x, we have

$$\frac{dy}{dx} = 0$$

which is a first order first degree differential equation.

Now, consider the relation (with one constant)

$$y = mx, \text{where } m \text{ is a constant}$$

Differentiating w.r.to. x, we have

$$\frac{dy}{dx} = m$$

Eliminating m, we have

$$\frac{dy}{dx} = \frac{y}{x}$$

which is a first order first degree differential equation.

Now, suppose the relation is (with two constants)

$$y = mx + c, \text{where } m \text{ and } c \text{ are two constants}$$

Differentiating w.r.to. x, we have

$$\frac{dy}{dx} = m$$

Again differentiating w.r.to. x, we have

$$\frac{d^2y}{dx^2} = 0$$

which is a second order first degree differential equation.

It should be noted that when we are eliminating one constant, we are getting a first order differential equation where as elimination of two constants yields a second order differential equation.

So, in general when we have n constants in a relation, eliminating them we get a differential equation of order n. In other words, solution to a differential equation of order n contains n arbitrary constants.

Example 5 **Find a differential equation from the following relation**

$$y = e^x(A\cos x + B\sin x)$$

Sol. Here we observe that the number of constants are A and B and we have to eliminate the constants.

Now,

$$y = e^x(A\cos x + B\sin x) \tag{1}$$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$$or, \quad \frac{dy}{dx} - y = e^x(-A\sin x + B\cos x) \tag{2}$$

Differentiating (2) with respect to x, we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$$

$$or, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - y$$

$$or, \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Therefore, the required differential equation is of second order and given by

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Example 6 **Find a differential equation from the following relation**

$$y = \frac{A}{x} + B$$

Sol. Here we observe that the number of constants are A and B and we have to eliminate the constants.
Now,

$$y = \frac{A}{x} + B \tag{1}$$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = -\frac{A}{x^2} \tag{2}$$

Differentiating (2) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{2A}{x^3} \tag{3}$$

Eliminating A from (2) and (3), we have

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}\left(-x^2\frac{dy}{dx}\right)$$

i.e., $$\frac{d^2y}{dx^2} = -\frac{2}{x}\frac{dy}{dx}$$

i.e., $$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0.$$

Therefore, the required differential equation is of second order

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0.$$

Case II: **Differential equations can be formed from different kinds of physical and geometric problems.**

Example 7 **Suppose a curve is defined by the condition that the sum of x and y intercepts of its tangents is always equal to m. Express this by a differential equation.**

Sol. The equation of the tangent at any point (x, y) on the curve is given by

$$Y - y = \frac{dy}{dx}(X - x)$$

or, $$\frac{X}{\frac{y - x\frac{dy}{dx}}{-\frac{dy}{dx}}} = \frac{Y}{y - x\frac{dy}{dx}} = 1$$

By the given condition we have

$$\left(\frac{y - x\frac{dy}{dx}}{-\frac{dy}{dx}}\right) + \left(y - x\frac{dy}{dx}\right) = m$$

or, $$\left(y - x\frac{dy}{dx}\right) + \left(x - y\frac{dx}{dy}\right) = m$$

or, $$x\left(\frac{dy}{dx}\right)^2 - (x + y - m)\frac{dy}{dx} + y = 0$$

This is a first order second degree differential equation.

1.6 TYPES OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

1.6.1 General Solution or Complete Solution

A solution in which number of arbitary constants is equal to the order of the differential equation is called general or complete solution.

Example 8 **The general solution of the differential equation**

$$\frac{d^2y}{dx^2} + 4y = 0$$

is

$$y = C_1 \cos 2x + C_2 \sin 2x$$

where C_1 and C_2 are arbitary constants. Here, the order is 2 and the number of arbitary constants is also 2.

1.6.2 Particular Solution

A solution obtained from the general solution, by giving particular values to the arbitary constants is called particular solution.

Example 9 **The general solution of the differential equation**

$$\frac{d^2y}{dx^2} + 4y = 0$$

is

$$y = C_1 \cos 2x + C_2 \sin 2x$$

where C_1 and C_2 are arbitary constants. Here, one particular solution of the differential equation is

$$y = \cos 2x + 2 \sin 2x$$

Here,

$$C_1 = 1 \text{ and } C_2 = 2$$

1.6.3 Singular Solutions

The general solution of any ordinary differential equation sometimes does not include all the solutions of the differential equation. In other words, there may exist such a solution which cannot be obtained by giving any particular values to those arbitrary constants of the general solution. This kind of solution is known as singular solution.

Example 10 **The general solution of the differential equation**

$$y = px + \frac{a}{p}, \textbf{ where } p = \frac{dy}{dx}$$

is

$$y = cx + \frac{a}{c}$$

where c is an arbitary constant.

The singular solution is

$$y^2 = 4ax$$

This solution cannot be obtained by giving any particular value to c in the general solution.

1.7 GEOMETRICAL INTERPRETATION OF SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (ODE)

Once a solution to an ordinary differential equation has been found, it is possible to exhibit its properties geometrically by plotting its graph for various specific values of the parameters involved.

Geometrically,

1) The general solution to any ordinary differential equation is the equation of a family of curves.
2) The particular solution is the equation of a particular curve of the family.
3) The singular solution is the envelope (when it exists) of the family of curves representing the general solution.

WORKED OUT EXAMPLES

Example 1.1 **Find the order and degree of the following differential equations**

a) $x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^3 = 0$

b) $\left(1 + \frac{d^2y}{dx^2}\right)^{\frac{3}{4}} = \frac{d^2y}{dx^2}$

c) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = \log x$

Sol.

a) The order and degree of the differential equation

$$x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^3 = 0$$

is 2 and 3 respectively, since the highest derivative is of order 2 and the power of highest derivative is 3.

b) The order and degree of the differential equation

$$\left(1+\frac{d^2y}{dx^2}\right)^{\frac{3}{4}} = \frac{d^2y}{dx^2}$$

or,
$$\left(1+\frac{d^2y}{dx^2}\right)^{3} = \left(\frac{d^2y}{dx^2}\right)^{4}$$

is 2 and 4 respectively, since the highest derivative is of order 2 and the power of highest derivative is 4.

c) The order and degree of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = \log x$$

is 2 and 1 respectively, since the highest derivative is of order 2 and the power of highest derivative is 1.

Example 1.2 **Form an ordinary differential equation from the relation**

$$y = a(x-a)^2$$

where a is a constant.

Sol. Here,

$$y = a(x-a)^2$$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = 2a(x-a)$$

or,
$$\frac{y}{\frac{dy}{dx}} = \frac{(x-a)}{2}$$

or,
$$a = x - \frac{2y}{\frac{dy}{dx}}$$

Putting the value of a in the relation, we have

$$y = \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left\{x - x - \frac{2y}{\frac{dy}{dx}}\right\}^2$$

or,
$$y\left(\frac{dy}{dx}\right)^3 = \left(x\frac{dy}{dx} - 2y\right)4y^2$$

or, $$\left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right)$$

which is the required differential equation.

Example 1.3 **Find a differential equation from the relation.**

$$y = \log\cos(x+a) + b$$

where a and b are constants.

Sol. Here,

$$y = \log\cos(x+a) + b$$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = -\frac{\sin(x+a)}{\cos(x+a)} = -\tan(x+a)$$

Differentiating again with respect to x, we have

$$\frac{d^2y}{dx^2} = -\sec^2(x+a)$$

or, $$\frac{d^2y}{dx^2} = -\left\{1+\tan^2(x+a)\right\}$$

or, $$\frac{d^2y}{dx^2} = -\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$$

or, $$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$$

which is the required differential equation.

EXERCISES

Short and Long Answer Type Questions

1) Determine the order and degree of the following differential equations

a) $\frac{dy}{dx} + y = 0$

[**Ans:** order−1,degree−1]

b) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

[**Ans:** order−2, degree−1]

c) $\left(\frac{d^3y}{dx^3}\right)^2 = x^2\frac{dy}{dx}$

[**Ans:** order−3, degree−2]

d) $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^5 + \left(\frac{dy}{dx}\right)^4 + 5y = 8x$

[**Ans:** order−3, degree−1]

e) $\left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^{\frac{1}{3}} = \frac{d^2y}{dx^2}$

[**Ans:** order−2, degree−3]

2) Form a differential equation from the following relations by eliminating arbitary constants

a) $y^2 = 4ax$

[**Ans:** $2x\frac{dy}{dx} - y = 0$]

b) $xy = ae^x + be^{-x}$

[**Ans:** $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$]

c) $y = a + b\cos x$

[**Ans:** $\sin x\frac{d^2y}{dx^2} - \cos x\frac{dy}{dx} = 0$]

d) $y = ax^2 + bx$

[**Ans:** $\frac{d^2y}{dx^2} - \left(\frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{2}{x^2}\right)y = 0$]

e) $y = (a + bx)\,e^{mx}$

[**Ans:** $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y = 0$]

3) Show that the equation $ax + by + c = 0$ leads to $\frac{d^2y}{dx^2} = 0$ after the elimination of the constants a, b and c. How do you explain the equation is of second order?

4) Obtain the differential equation of all circles, each of which touches the axis of x at the origin.

[**Ans:** $(x^2 - y^2)\frac{dy}{dx} = 2xy$]

5) Obtain the differential equation of all parabolas, each of which has a latus rectum $4a$ whose axes are parallel to the x axis.

[**Ans:** $\left(\frac{dy}{dx}\right)^2 + 2a\frac{d^2y}{dx^2} = 0$]

Multiple Choice Questions

1) The differential equation representing the family of curves $y = a\cos(x + b)$ is

a) $\frac{d^2y}{dx^2} = k$

b) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$

c) $\frac{d^2y}{dx^2} - y = 0$

d) $\frac{d^2y}{dx^2} + y = 0$

2) The order and degree of the differential equation $\frac{d^2y}{dx^2} = \left\{y + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$ are

a) 4,2 b) 1,2 c) 2,4 d) 1,4

3) The solution of the differential equation $2x\frac{dy}{dx} - y = 3$ represent

a) family of straight lines b) a circle

c) a family of parabola d) a parabola

4) The family of curves $y = ax + a^2$ is represented by a differential equation of degree

a) 2 b) 3 c) 1 d) 4

5) The order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 - 3\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^4 = y^4$

a) 3,4   b) 3,2   c) 2,4   d) 1,4

Answers:

1. (d)   2(c)   3(d)   4(c)   5(b)

CHAPTER

2

Ordinary Differential Equations of First Order and First Degree

2.1 INTRODUCTION

Many practical problems related to different branches of science and technology, when mathematically expressed read as problems of ordinary differential equations. One of the most important kind of differential equations are of first order and first degree. In this chapter, we will discuss the concepts of various types of differential equations of first order and first degree along with the methods of solutions for them.

2.1.1 First Order and First Degree Differential Equations

Any differential equation of first order and first degree is of the form

$$\frac{dy}{dx} = f(x, y)$$

Let,

$$f(x, y) = -\frac{M(x, y)}{N(x, y)}$$

then the standard form of differential equation of first order is

$$\boxed{M(x, y)\,dx + N(x, y)\,dy = 0.}$$

Example 1 **The following equations represent the differential equations of first order and first degree.**

(i) $\dfrac{dy}{dx} = x^2 + y^2$

(ii) $(x^2 + y)\,dx + (y^2 + x)\,dy = 0$

(iii) $\dfrac{dy}{dx} + x^2 y = \sin x$

2.1.2 Existence and Uniqueness Theorem for the Solution

If in the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

$f(x, y)$ be continuous in a domain R containing the point (x_0, y_0), then there exist a function $y = g(x)$ which satisfies the equation (1) and takes the value y_0 for $x = x_0$.

In addition, if the partial derivative $\dfrac{\partial f}{\partial y}$ is continuous, then this solution of the equation is unique.

The condition that $y = g(x)$ has the value y_0 at $x = x_0$ is known as the initial condition of the equation.

2.1.3 Classifications

Assuming that a first order and first degree differential equation has a solution, the equations can be classified according to the methods of solutions. Some of these types are

(i) Exact differential equations
(ii) Homogeneous differential equations
(iii) Linear differential equations

2.2 EXACT DIFFERENTIAL EQUATIONS

2.2.1 Definition

The first order and first degree differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called an exact differential equation if there exist a function $u(x, y)$ **such that the term** $M(x, y)\,dx + N(x, y)\,dy$ **can be expressed as**

$$\boxed{du = M(x, y)\,dx + N(x, y)\,dy}$$

Then the equation becomes $du = 0$ and the solution is given by

$$\boxed{u(x, y) = c}$$

where c is an arbitrary constant.

Example 2 **The equation $x\,dx + y\,dy = 0$ is an exact differential equation, since we have**

$$x\,dx + y\,dy = d\left[\frac{1}{2}\left(x^2 + y^2\right)\right] = du \quad \text{(say)}$$

where $u = \frac{1}{2}\left(x^2 + y^2\right)$.

Therefore, the equation becomes $du = 0$ and its solution is $\dfrac{1}{2}\left(x^2 + y^2\right) = c$, where c is an arbitrary constant.

Example 3 **Prove that**

$$\frac{y}{x}\,dx + \log x\,dy = 0$$

is an exact differential equation and find its general solution.

Sol. Here,

$$\frac{y}{x}\,dx + \log x\,dy = d(y \log x)$$

Therefore,

$$\frac{y}{x}\,dx + \log x\,dy = 0$$

can be written as

$$d(y \log x) = 0$$

Hence it is an exact differential equation.
The general solution is

$$(y \log x) = c$$

where c is an arbitrary constant.

Observations:

1) The total differential of the function $u(x, y)$ is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

If we write in the form

$$du = M(x, y)\,dx + N(x, y)\,dy,$$

we see that

$$\frac{\partial u}{\partial x} = M(x, y) \text{ and } \frac{\partial u}{\partial y} = N(x, y).$$

2) **Every exact equation can be written as**

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$$

or, $$d[u(x, y)] = 0$$

or, $$u(x, y) = c$$

where c is an arbitrary constant.

3) **If $M(x, y)$ and $N(x, y)$ are simple enough, sometimes it is possible to say that whether or not a function $u(x, y)$ exists.**

2.2.2 Criterion for Exactness

Theorem 2.1: **If $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous in any region in space, then the necessary and sufficient condition that the differential equation**

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Proof: The condition is necessary

Let us assume that the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact.
Therefore, there exist $u(x, y)$ such that

$$M(x, y)\,dx + N(x, y)\,dy = du$$

Now,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

Therefore,

$$\frac{\partial u}{\partial x} = M(x, y) \text{ and } \frac{\partial u}{\partial y} = N(x, y)$$

Now,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

Assuming,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

we have,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The condition is sufficient

Let us assume that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Let there exist $V(x, y)$, such that

$$\frac{\partial V}{\partial x} = M$$

$$or, \qquad V = \int M\, dx \text{ treating } y \text{ as constant}$$

Now,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)$$

Again

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)$$

Therefore,

$$N = \frac{\partial V}{\partial y} + f(y) = \frac{\partial V}{\partial y} + \phi'(y)$$

where $f(y)$ is a function of y only and $f(y) = \phi'(y)$ (say).

Therefore,

$$
\begin{aligned}
M\,dx + N\,dy &= \frac{\partial V}{\partial x}\,dx + \left\{\frac{\partial V}{\partial y} + \phi'(y)\right\}\,dy \\
&= \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \phi'(y)\,dy \\
&= dV + \phi'(y)\,dy \\
&= d\{V + \phi(y)\} = du
\end{aligned}
$$

where $u = V + \phi(y)$.
Hence, $M\,dx + N\,dy$ is an exact differential and

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is an exact differential equation.

2.2.3 Methods of Solution

Working Procedure 1:

Step 1 Calculate $\int M(x, y)\,dx$ treating y as constant.
Step 2 Calculate $\int N(x, y)\,dy$ for those terms of N which do not contain x.
Step 3 Add the results of Step 1 and Step 2.

Therefore, the general solution of first order exact differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is given by

$$\boxed{\int M(x, y)\,dx + \int (\text{terms of } N(x, y) \text{ not containing } x)\,dy = c}$$

Example 4 **Show that the differential equation**

$$y\,dx + (x + \cos y)\,dy = 0$$

is an exact differential equation and find the general solution.

Sol. Here,

$$M(x, y) = y \text{ and } N(x, y) = (x + \cos y)$$

Now,

$$\frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = 1$$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

Hence, the differential equation is exact.
The general solution is,

$$\int M(x, y)\, dx + \int (\text{terms of } N(x, y) \text{ not containing } x)\, dy = c$$

i.e.

$$\int y\, dx + \int \cos y\, dy = c$$

or, $$xy + \sin y = c$$

where c is an arbitrary constant.

Working Procedure 2:

Step 1: Calculate $\int M(x, y)\, dx$ treating y as constant.
Step 2: Calculate $\int N(x, y)\, dy$ treating x as constant.
Step 3: Add the results of Step 1 and Step 2 (deleting those terms which have already occurred in Step 1) and equate to an arbitrary constant.

Example 5 **Show that the differential equation**

$$(e^x \sin y + e^{-y})\, dx + (e^x \cos y - xe^{-y})\, dy = 0$$

is an exact differential equation and find the general solution.

Sol. Here,

$$M(x, y) = (e^x \sin y + e^{-y}) \text{ and } N(x, y) = (e^x \cos y - xe^{-y})$$

Now,

$$\frac{\partial M}{\partial y} = e^x \cos y - e^{-y} \text{ and } \frac{\partial N}{\partial x} = e^x \cos y - e^{-y}$$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^x \cos y - e^{-y}$$

Hence, the differential equation is exact.
Now,

$$\int M(x, y)\, dx = \int (e^x \sin y + e^{-y})\, dx = e^x \sin y + xe^{-y}$$

and

$$\int N(x,y)\,dy = \int (e^x \cos y - xe^{-y})\,dy = e^x \sin y + xe^{-y}$$

Therefore, the general solution is

$$e^x \sin y + xe^{-y} = c$$

where c is an arbitary constant.

2.2.4 Integrating Factors for Non-exact Differential Equations to Make It Exact

A function $f(x, y)$ is said to be an integrating factor (I.F.) of the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

when the differential equation is not exact and we can find a function $u(x, y)$ such that

$$\boxed{f(x,y)\{M(x,y)\,dx + N(x,y)\,dy\} = du}$$

The following table represents some integrating factors which can be found by inspection.

S. No.	Expression	Integrating Factors (IF)	Exact Differential
1.	$x\,dy - y\,dx$	$\dfrac{1}{x^2}$	$\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
2.	$x\,dy - y\,dx$	$\dfrac{1}{x^2+y^2}$	$\dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$
3.	$x\,dx + y\,dy$	$\dfrac{1}{x^2+y^2}$	$\dfrac{x\,dx + y\,dy}{x^2+y^2} = \frac{1}{2}d\left[\log\left(x^2+y^2\right)\right]$
4.	$x\,dy - y\,dx$	$\dfrac{1}{xy}$	$\dfrac{x\,dy - y\,dx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$
5.	$y\,dx - x\,dy$	$\dfrac{1}{y^2}$	$\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
6.	$x\,dx + y\,dy$	$\dfrac{1}{x^2+y^2}$	$\dfrac{x\,dx + y\,dy}{x^2+y^2} = \frac{1}{2}d\left[\log\left(x^2+y^2\right)\right]$
7.	$x\,dx + y\,dy$	$\dfrac{1}{xy}$	$\dfrac{x\,dx + y\,dy}{xy} = d\{\log xy\}$

Observations:

1) **Integrating factor is the multiplying factor by which the non-exact differential equation can be made an exact differential equation.**
2) **A differential equation which is not exact may have an infinite number of integrating factors.**

Example 6 **Solve**

$$(x^4e^x - 2axy^2)\,dx + 2ax^2y\,dy = 0$$

by finding integrating factor by inspection.

Sol. Here,

$$M(x,y) = (x^4e^x - 2axy^2) \text{ and } N(x,y) = 2ax^2y$$

Now,

$$\frac{\partial M}{\partial y} = -4axy \text{ and } \frac{\partial N}{\partial x} = 4axy$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the differential equation is not exact.
The equation can be written as

$$x^4e^x\,dx + 2a(x\,dy - y\,dx)xy = 0$$

or,
$$x^4e^x\,dx + 2ax^3y\frac{(x\,dy - y\,dx)}{x^2} = 0$$

or,
$$x^4e^x\,dx + 2ax^3yd\left(\frac{y}{x}\right) = 0$$

or,
$$e^x\,dx + 2a\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) = 0$$

Integrating both sides, the general solution is

$$e^x + a(\frac{y}{x})^2 = c$$

where c is an arbitary constant.

Example 7 **Solve**

$$(x^4y^2 - y)\,dx + (x^2y^4 - x)\,dy = 0$$

by finding integrating factor by inspection.

Sol. Here,

$$M(x,y) = (x^4y^2 - y) \text{ and } N(x,y) = (x^2y^4 - x)$$

Now,

$$\frac{\partial M}{\partial y} = 2x^4y - 1 \text{ and } \frac{\partial N}{\partial x} = 2xy^4 - 1$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the differential equation is not exact.
The equation can be written as

$$x^4 y^2\, dx - y\, dx + x^2 y^4\, dy - x\, dy = 0$$

or, $$x^2 y^2 (x^2\, dx + y^2\, dy) - (x\, dy + y\, dx) = 0$$

or, $$x^2\, dx + y^2\, dy - \frac{(x\, dy + y\, dx)}{x^2 y^2} = 0$$

or, $$x^2\, dx + y^2\, dy - \frac{d(xy)}{x^2 y^2} = 0$$

Integrating both sides, the general solution is

$$\frac{x^3}{3} + \frac{y^3}{3} + \frac{1}{xy} = c$$

where c is an arbitary constant.

2.2.5 Rules for Finding Integrating Factors for Non-exact Differential Equations

Let us consider the differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

which is not an exact differential equation.

i.e., $$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation becomes an exact differential equation when we multiply the equation by a suitable integrating factors (IF). The following are the rules for finding integrating factors.

Rule 1: When,

$$\boxed{\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)}$$

$f(x)$ is a function of x only, then,

$$\boxed{\text{IF} = e^{\int f(x)\, dx}}$$

Example 8 **Show that the differential equation**

$$\left(xy^2 - e^{\frac{1}{x^3}}\right) dx - x^2 y\, dy = 0$$

is not exact. Find the integrating factor and the general solution.

Sol. Here,

$$M(x, y) = \left(xy^2 - e^{\frac{1}{x^3}}\right) \text{ and } N(x, y) = -x^2 y$$

Now,

$$\frac{\partial M}{\partial y} = 2xy \text{ and } \frac{\partial N}{\partial x} = -2xy$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore, the differential equation is not exact.
Here,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy + 2xy}{-x^2 y} = -\frac{4}{x} = f(x) \quad \text{(say)}$$

Therefore, the integrating factor is

$$\text{IF} = e^{\int f(x)\, dx} = e^{\int -\frac{4}{x}\, dx} = e^{-4 \log x} = \frac{1}{x^4}$$

Multiplying the differential equation by IF, we get

$$\frac{\left(xy^2 - e^{\frac{1}{x^3}}\right)}{x^4} dx - \frac{x^2 y}{x^4} dy = 0$$

or, $$\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}}\right) dx - \frac{y}{x^2} dy = 0$$

which is an exact differential equation.
Therefore, the general solution is (**using working procedure 1 of Art. 2.2.3**)

$$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}}\right) dx + \int 0\, dy = c$$

or, $$y^2 \int \frac{dx}{x^3} + \frac{1}{3} \int e^{\frac{1}{x^3}} d\left(\frac{1}{x^3}\right) = c$$

$$or, \qquad \frac{-y}{2x^2} + \frac{1}{3} e^{\frac{1}{x^3}} = c$$

$$or, \qquad 2x^2 e^{\frac{1}{x^3}} - 3y = 6cx^2$$

where c is an arbitrary constant.

Rule 2: When,

$$\boxed{\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = g(y)}$$

$g(y)$ is a function of y only, then,

$$\boxed{\text{IF} = e^{-\int g(y)\, dy}}$$

Example 9 **Show that the differential equation**

$$y \log y \, dx + (x - \log y)\, dy = 0$$

is not exact. Find the integrating factor and the general solution.

Sol. Here,

$$M(x, y) = y \log y \text{ and } N(x, y) = (x - \log y)$$

Now,

$$\frac{\partial M}{\partial y} = 1 + \log y \text{ and } \frac{\partial N}{\partial x} = 1$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore, the differential equation is not exact.
Here,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = \frac{1 + \log y - 1}{y \log y} = \frac{1}{y} = g(y) \quad \text{(say)}$$

Therefore, the integrating factor is

$$\text{IF} = e^{-\int g(y)\, dx} = e^{\int -\frac{1}{y}\, dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the differential equation by IF, we get

$$\frac{y \log y}{y}\, dx + \frac{(x - \log y)}{y}\, dy = 0$$

or, $$\log y\, dx + \left(\frac{x}{y} - \frac{\log y}{y}\right) dy = 0$$

which is an exact differential equation.
Therefore, the general solution is (**using working procedure 1 of Art. 2.2.3**)

$$\int \log y\, dx - \int \frac{\log y}{y}\, dy = c$$

or, $$x \log y - \frac{(\log y)^2}{2} = c$$

where c is an arbitrary constant.

Rule 3: When $M(x, y)$ and $N(x, y)$ are homogeneous functions of x and y of same degree and $Mx + Ny \neq 0$, then,

$$\boxed{\text{IF} = \frac{1}{Mx + Ny}}$$

Example 10 **Show that the differential equation**

$$(x^4 + y^4)\, dx - xy^3\, dy = 0$$

is not exact. Find the integrating factor and the general solution.

Sol. Here,

$$M(x, y) = (x^4 + y^4) \text{ and } N(x, y) = -xy^3$$

Now,

$$\frac{\partial M}{\partial y} = 4y^3 \text{ and } \frac{\partial N}{\partial x} = -y^3$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore, the differential equation is not exact.
Here, $M(x, y)$ and $N(x, y)$ are homogrneous functions of degree 4 and $Mx + Ny = x^5 \neq 0$.

Therefore,

$$\text{IF} = \frac{1}{Mx + Ny} = \frac{1}{x^5}$$

Multiplying the differential equation by IF, we get

$$\frac{(x^4 + y^4)}{x^5}\, dx - \frac{xy^3}{x^5}\, dy = 0$$

or,
$$\left(\frac{1}{x} + \frac{y^4}{x^5}\right) dx - \frac{y^3}{x^4}\, dy = 0$$

which is an exact differential equation.
Therefore, the general solution is (**using working procedure 1 of Art. 2.2.3**)

$$\int \left(\frac{1}{x} + \frac{y^4}{x^5}\right) dx = c$$

or,
$$\log x - \frac{y^4}{4x^4} = c$$

or,
$$4x^4 \log x - y^4 = 4cx^4$$

where c is an arbitrary constant.

Rule 4: When $M(x, y)\, dx + N(x, y)\, dy = 0$ can be expressed in the form $y\phi(xy)\, dx + x\psi(xy)\, dy = 0$ and $Mx - Ny \neq 0$, then,

$$\boxed{\text{IF} = \frac{1}{Mx - Ny}}$$

Example 11 **Show that the differential equation**

$$(x^2y^2 + xy + 1)y\, dx + (x^2y^2 - xy + 1)x\, dy = 0$$

is not exact. Find the integrating factor and the general solution.

Sol. Here,

$$M(x, y) = (x^2y^2 + xy + 1)y \text{ and } N(x, y) = (x^2y^2 - xy + 1)x$$

Now,

$$\frac{\partial M}{\partial y} = 3y^2x^2 + 2xy + 1 \text{ and } \frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore, the differential equation is not exact.

The differential equation is of the form $y\phi(xy)\,dx + x\psi(xy)\,dy = 0$ and

$$Mx - Ny = 2x^2y^2 \neq 0$$

Therefore, the integrating factor is

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

Multiplying the differential equation by IF, we get

$$\frac{(x^2y^2 + xy + 1)y}{2x^2y^2}\,dx + \frac{(x^2y^2 - xy + 1)x}{2x^2y^2}\,dy = 0$$

$$or, \qquad \left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \left(\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right)dy = 0$$

which is an exact differential equation.
Therefore, the general solution is (**using working procedure 1 of Art. 2.2.3**)

$$\int \left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx - \int \frac{1}{2y}\,dy = c$$

$$\frac{yx}{2} + \frac{1}{2}\log x - \frac{1}{xy} - \frac{1}{2}\log y = c$$

where c is an arbitrary constant.

Rule 5: When $M(x, y)\,dx + N(x, y)\,dy = 0$ can be expressed in the form

$$x^a y^b(my\,dx + nx\,dy) + x^c y^d(py\,dx + qx\,dy) = 0$$

where a, b, c, d, m, n, p, q are constants, then,

$$\text{IF} = x^h y^k$$

where h and k are constants determined by the simultaneous equations

$$\frac{a + h + 1}{m} = \frac{b + k + 1}{n} \text{ and } \frac{c + h + 1}{p} = \frac{d + k + 1}{q}$$

Example 12 **Show that the differential equation**

$$3y\,dx - 2x\,dy + x^2y^{-1}(10y\,dx - 6x\,dy) = 0$$

is not exact. Find the integrating factor and the general solution.

Sol. Here the differential equation

$$3y\,dx - 2x\,dy + x^2y^{-1}(10y\,dx - 6x\,dy) = 0$$

is of the form

$$x^a y^b(my\,dx + nx\,dy) + x^c y^d(py\,dx + qx\,dy) = 0$$

where $a = 0, b = 0, c = 2, d = -1, m = 3, n = -2, p = 10$ and $n = -6$.
Therefore, the integrating factor is

$$\text{IF} = x^h y^k$$

where h and k are constants determined by the simultaneous equations

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \text{ and } \frac{c+h+1}{p} = \frac{d+k+1}{q}$$

We have,

$$\frac{0+h+1}{3} = \frac{0+k+1}{-2} \text{ and } \frac{2+h+1}{10} = \frac{-1+k+1}{-6}$$

or, $$2h + 3k = -5 \text{ and } 3h + 5k = -9$$

or, $$h = 2 \text{ and } k = -3$$

Therefore,

$$\text{IF} = x^2 y^{-3}$$

Multiplying the differential equation by IF, we get

$$x^2y^{-3}(3y\,dx - 2x\,dy) + x^2y^{-3}\{x^2y^{-1}(10y\,dx - 6x\,dy)\} = 0$$

or, $$3x^2y^{-2}\,dx - 2x^3y^{-3}\,dy + 10x^4y^{-3}\,dx - 6x^5y^{-4}\,dy = 0$$

or, $$(3x^2y^{-2} + 10x^4y^{-3})\,dx + (-2x^3y^{-3} - 6x^5y^{-4})\,dy = 0$$

which is an exact differential equation.
Therefore, the general solution is (**using working procedure 1 of Art. 2.2.3**)

$$\int (3x^2y^{-2} + 10x^4y^{-3})\,dx = c$$

or, $$x^3y^{-2} + 2x^5y^{-3} = c$$

where c is an arbitrary constant

Rule 6: When

$$\boxed{\frac{1}{yN - xM}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \phi(xy)}$$

then

$$\boxed{\text{IF} = e^{\int \phi(xy)d(xy)}}$$

Example 13 **Show that the differential equation**

$$(x^4y^2 - y)\,dx + (x^2y^4 - x)\,dy = 0$$

is not exact. Find the integrating factor and the general solution.

Sol. Here,

$$M(x, y) = (x^4y^2 - y) \text{ and } N(x, y) = (x^2y^4 - x)$$

Now,

$$\frac{\partial M}{\partial y} = 2x^4y - 1 \text{ and } \frac{\partial N}{\partial x} = 2xy^4 - 1$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the differential equation is not exact.
Now,

$$\frac{1}{yN - xM}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(2x^4y - 1) - (2xy^4 - 1)}{y(x^2y^4 - x) - x(x^4y^2 - y)}$$

$$= \frac{-2}{xy} = \phi(xy) \quad \text{(say)}$$

Therefore,

$$\text{IF} = e^{\int \phi(xy)d(xy)} = e^{\int \frac{-2}{xy}d(xy)} = e^{-2\log(xy)} = \frac{1}{x^2y^2}$$

Multiplying the differential equation by the integrating factor we get,

$$\frac{(x^4y^2 - y)}{x^2y^2}\,dx + \frac{(x^2y^4 - x)}{x^2y^2}\,dy = 0$$

or,

$$\left(x^2 - \frac{1}{x^2y}\right)dx + \left(y^2 - \frac{1}{xy^2}\right)dy = 0$$

which is an exact differential equation.
The general solution is (**using working procedure 1 of Art. 2.2.3**)

$$\int \left(x^2 - \frac{1}{x^2y}\right)dx + \int y^2\,dy = c$$

or,

$$\frac{x^3}{3} + \frac{y^3}{3} + \frac{1}{xy} = c$$

where c is an arbitrary constant.

Rule 7: When

$$\frac{x^2}{xM + yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \psi\left(\frac{y}{x}\right)$$

then

$$\text{IF} = e^{\int \psi\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)}$$

Example 14 **Show that the differential equation**

$$\left(\frac{2x^2}{y} + \frac{x}{y}\right) dx + 2x\, dy = 0$$

is not exact. Find the integrating factor and the general solution.

Sol. Here,

$$M(x, y) = \left(\frac{2x^2}{y} + \frac{x}{y}\right) \text{ and } N(x, y) = 2x$$

Now,

$$\frac{\partial M}{\partial y} = \frac{-2x^2}{y^2} - \frac{x}{y^2} \text{ and } \frac{\partial N}{\partial x} = 2$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the differential equation is not exact.
Now,

$$\frac{x^2}{xM + yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{x^2\left(2 + \frac{x}{y^2} + \frac{2x^2}{y^2}\right)}{xy\left(2 + \frac{x}{y^2} + \frac{2x^2}{y^2}\right)}$$

$$= \frac{x}{y} = \left(\frac{y}{x}\right)^{-1} = \psi\left(\frac{y}{x}\right) \quad \text{(say)}$$

Therefore,

$$\text{IF} = e^{\int \psi\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)} = e^{\int \left(\frac{y}{x}\right)^{-1} d\left(\frac{y}{x}\right)} = e^{\log\left(\frac{y}{x}\right)} = \left(\frac{y}{x}\right)$$

Multiplying both sides of the equation by the integrating factor, we have

$$\frac{y}{x}\left(\frac{2x^2}{y} + \frac{x}{y}\right) dx + 2y\, dy = 0$$

or, $$(2x+1)\,dx + 2y\,dy = 0$$

which is an exact differential equation.
The general solution is (**using working procedure 1 of Art. 2.2.3**)

$$\int (2x+1)\,dx + \int 2y\,dy = c$$

or, $$x^2 + x + y^2 = c$$

where c is arbitrary constant.

Rule 8: When

$$\frac{y^2}{xM + yN}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \psi\left(\frac{x}{y}\right)$$

then

$$\text{IF} = e^{\int \psi\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right)}$$

2.3 HOMOGENEOUS EQUATIONS

2.3.1 Definition

When the functions $M(x, y)$ and $N(x, y)$ are homogeneous functions of x and y of same degree, then the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called a homogeneous differential equation.

2.3.2 Method of Solution

The differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

can be written as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Let us take the transformation

$$y = vx$$

then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting the value of $\frac{dy}{dx}$ and y, we have

$$v + x\frac{dv}{dx} = f(v)$$

or,
$$\frac{dv}{f(v) - v} = \frac{dx}{x}$$

Integrating and putting,

$$v = \frac{y}{x}$$

we get the required general solution.

Example 15 **Solve**

$$(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$$

Sol. The given differential can be written as

$$\frac{dy}{dx} = \frac{(x^2 + y^2)}{(xy - x^2)}$$

Let us take the transformation

$$y = vx$$

then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting the value of $\frac{dy}{dx}$ and y, we have

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{v - 1}$$

or,
$$x\frac{dv}{dx} = \frac{1 + v^2}{v - 1} - v$$

or,
$$\frac{v - 1}{v + 1}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$v - 2\log(v + 1) = \log cx$$

or,
$$\frac{y}{x} - 2\log\left(\frac{y}{x} + 1\right) = \log cx$$

where c is an arbitary constant.

2.3.3 Equations that are both Homogeneous and Exact

Let the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is both homogeneous and exact. Then its general solution is given by

$$Mx + Ny = c$$

provided degree of homogeneity is not equal to -1.

Example 16 **Solve the differential equation**

$$(x^3 + 2y^2x)\,dx + (y^3 + 2x^2y)\,dy = 0$$

Sol. Here,

$$M(x, y) = (x^3 + 2y^2x) \text{ and } N(x, y) = (y^3 + 2x^2y)$$

Now,

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}$$

So the equation is exact.
Also the equation is homogeneous, since $M(x, y)$ and $N(x, y)$ both are homogeneous functions of degree 3.
Hence, the solution is given by

$$Mx + Ny = c$$

$$or, \quad (x^3 + 2y^2x)x + (y^3 + 2x^2y)y = c$$

$$or, \quad x^4 + 4x^2y^2 + y^4 = c$$

where c is an arbitrary constant.

2.4 LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS

2.4.1 Definition

An ordinary differential equation which contains the dependent variable and its derivatives as first degree terms only and no such term which is a product of the dependent variable or its function and its derivatives, or any trancendental function of the dependent variable, will be called a linear differential equation. Otherwise, the differential equation is called a non-linear differential equation.

A linear differential equation of first order is of the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

where $P(x)$ and $Q(x)$ are functions of x or constants. This equation is known as linear in y.

Following are some examples of linear differential equations of first order. (linear in y)

a) $\frac{dy}{dx} + 5y = 8x$

b) $x\frac{dy}{dx} + 6x^2y = 9\sin x$

Following are some examples of non-linear differential equation of first order

a) $(x+y)^2\frac{dy}{dx} + 5y = 3x$

b) $\frac{dy}{dx} + 5xy^2 = 3x$

Observations:

1) **An ordinary differential equation is non–linear when,**
   a) **There exist terms which is a product of the dependent variable or its function and its derivatives.**
   b) **There exists any trancendental function of the dependent variable.**
   c) **The degree of the dependent variable or its derivatives is more than one.**
2) **The coefficients of differential equation is either a function of independent variable or constants.**
3) **Every linear differential equations is of first degree but every first degree differential equation may not be linear.**

For example the equation

$$\frac{dy}{dx} + 2xy^2 = 3e^x$$

is of first degree but not linear.

2.4.2 Method of Solution

Let us assume that by multiplying the left–hand side of the equation given in section 2.4.1 by $\mu = \mu(x)$, we get

$$\frac{d}{dx}(y\mu) = \mu\frac{dy}{dx} + P(x)\mu y$$

$$or, \quad \mu\frac{dy}{dx} + y\frac{d\mu}{dx} = \mu\frac{dy}{dx} + P(x)\mu y$$

$$or, \quad y\frac{d\mu}{dx} = P(x)\mu y$$

$$or, \quad \frac{d\mu}{dx} = P(x)\mu$$

$$or, \qquad \frac{d\mu}{\mu} = P(x)\,dx$$

$$or, \qquad \mu = e^{\int P(x)\,dx}$$

which is the integrating factor, i.e., IF is $e^{\int P(x)\,dx}$.

Multiplying both sides of the equation given in section 2.4.1 by the integrating factor, we have

$$e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}p(x)\,dx = e^{\int P(x)\,dx}Q(x)$$

$$or, \qquad \frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = Q(x)e^{\int P(x)\,dx}$$

Integrating both sides, the general solution is

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + c$$

$$or, \qquad y(\mathrm{IF}) = \int Q(x).(\mathrm{IF})\,dx + c$$

where c is an arbitrary constant.

Remarks:

1) The solution to the linear equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

can be put in the form

$$y = \frac{Q(x)}{P(x)} - e^{-\int P(x)\,dx}\left[e^{\int P(x)\,dx}d\left(\frac{Q}{P}\right) + C\right]$$

where C is an arbitary constant.

2) If u and v are two solutions of the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and if $v = uz$, then

$$z = 1 + Ce^{-\int \frac{Q}{u}\,dx}$$

Example 17 **Solve**

$$x\frac{dy}{dx} + 5y = 6$$

Sol. The equation can be wrtten in the form

$$\frac{dy}{dx} + \left(\frac{5}{x}\right) y = 6$$

and is a linear differential equation of first order.

Here, $P(x) = \frac{5}{x}$ and $Q(x) = 6$.

Therefore,

$$\text{IF} = e^{\int P(x)\, dx} = e^{\int \frac{5}{x}\, dx} = e^{5 \log x} = x^5.$$

Hence, the solution is given by

$$y(\text{IF}) = \int Q(x)(\text{IF})\, dx + c$$

or, $$yx^5 = \int 6x^5\, dx + c$$

or, $$yx^5 = x^6 + c.$$

where c is an arbitrary constant.

Example 18 **Solve**

$$\frac{dy}{dx} + \frac{4x}{x^2+1} y = \frac{1}{(x^2+1)^3}$$

Sol. The equation

$$\frac{dy}{dx} + \frac{4x}{x^2+1} y = \frac{1}{(x^2+1)^3}$$

is a linear differential equation of first order.

Therefore,

$$\text{IF} = e^{\int \frac{4x}{x^2+1} dx} = e^{2\log(x^2+1)} = (1+x^2)^2$$

Multiplying both sides of the differential equation by the integrating factor, we have

$$(1+x^2)^2 \frac{dy}{dx} + (1+x^2)^2 \frac{4x}{(x^2+1)} y = (1+x^2)^2 \frac{1}{(x^2+1)^3}$$

or, $$(1+x^2)^2 \frac{dy}{dx} + 4x(x^2+1) = \frac{1}{(x^2+1)}$$

or, $$(1+x^2)^2\, dy + 4x(x^2+1)\, dx = \frac{1}{(x^2+1)}\, dx$$

or, $$d\{(1+x^2)^2 y\} = \frac{1}{(x^2+1)}\, dx$$

Integrating both sides, the general solution is

$$(1 + x^2)^2 y = \tan^{-1} x + c$$

or, $$y = \frac{\tan^{-1} x}{(1 + x^2)^2} + \frac{c}{(1 + x^2)^2}$$

where c is an arbitrary constant.

2.4.3 Another Form of Linear Differential Equation of First Order (Linear in *x*)

A linear differential equation of first order is also of the form

$$\boxed{\frac{dx}{dy} + P(y)x = Q(y)}$$

where $P(y)$ and $Q(y)$ are functions of y or constants. This is known as linear in x.

Consider integrating factor as

$$\boxed{\text{IF} = e^{\int P(y)\,dy}.}$$

Multipling by IF, we have

$$\frac{d}{dy}\left\{xe^{\int P(y)\,dy}\right\} = Q(y)e^{\int P(y)\,dy}$$

On integration, the solution as

$$\boxed{\begin{aligned} xe^{\int P(y)\,dy} &= \int \left\{Q(y)e^{\int P(y)\,dy}\right\} dy + c \\ \text{or,} \quad x.(\text{IF}) &= \int Q(y).(\text{IF})\,dy + c \end{aligned}}$$

where c is an arbitrary constant.

Note: Any differential equation of first order and first degree, not linear in y, may be linear in x and vice versa.

Example 19 **Solve**

$$(1 + y^2)\,dx - (\tan^{-1} y - x)\,dy = 0$$

Sol. The differential equation can be written as

$$(1 + y^2)\frac{dx}{dy} + x = \tan^{-1} y$$

or, $$\frac{dx}{dy} + \frac{x}{(1 + y^2)} = \frac{\tan^{-1} y}{(1 + y^2)}$$

which is a linear equation in x.

The integrating factor is

$$\text{IF} = e^{\int \frac{1}{(1+y^2)} dy} = e^{\tan^{-1} y}$$

Multiplying both sides by the integrating factor, we get

$$\frac{d}{dy}(xe^{\tan^{-1} y}) = \frac{\tan^{-1} y}{(1+y^2)} e^{\tan^{-1} y}$$

Integrating, we get

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{(1+y^2)} e^{\tan^{-1} y}\, dy$$

or, $$xe^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

or, $$x = (\tan^{-1} y - 1) + ce^{-\tan^{-1} y}$$

where c is an arbitary constant.

2.5 BERNOULLI'S EQUATION

2.5.1 Definition

The differential equation of first order and first degree is of the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$$

where $P(x)$ and $Q(x)$ are functions of x or constants is called Bernoulli's Equation.

This equation is not linear but by change in dependent variable, it can be brought into the linear form.

2.5.2 Method of Solution

The equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

can be put in the form

$$y^{-n}\frac{dy}{dx} + y^{1-n}P(x) = Q(x)$$

Putting,

$$z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

or, $$y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dz}{dx}$$

Therefore, the equation reduces to

$$\frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x)$$

or, $$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

Putting,

$$P_1(x) = (1-n)P(x) \text{ and } Q_1(x) = (1-n)Q(x)$$

the differential equation becomes

$$\frac{dz}{dx} + P_1(x)z = Q_1(x)$$

which is a first order linear equation.

Example 20 **Solve**

$$x\frac{dy}{dx} + y = xy^2$$

Sol. The equation is a Bernoullis's equation.
The differential equation

$$x\frac{dy}{dx} + y = xy^2$$

can be written as

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = x$$

Let,

$$\frac{1}{y} = z$$

then,

$$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

or, $$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$$

Substituting these values in the original differential equation, we have

$$-\frac{dz}{dx} + z = x$$

or, $$\frac{dz}{dx} - z = -x$$

which is a linear first order differential equation.

The integrating factor is

$$\text{IF} = e^{\int -dx} = e^{-x}$$

Multiplying both sides by the integrating factor, we have

$$e^{-x}\frac{dz}{dx} - e^{-x}z = -xe^{-x}$$

or, $$d\{e^{-x}z\} = -xe^{-x}\,dx$$

Integrating both sides, we have

$$e^{-x}z = \int -xe^{-x}\,dx = -x\int e^{-x}\,dx - \int e^{-x}\,dx = xe^{-x} + e^{-x} + c$$

or, $z = x + 1 + ce^{x}$

where c is an arbitrary constant.
Therefore, the general solution is

$$\frac{1}{y} = x + 1 + ce^{x}$$

Example 21 **Solve**

$$\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$$

Sol. The differential equation can be written as

$$\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{(\log y)} = \frac{1}{x^2}$$

Putting,

$$\frac{1}{\log y} = v$$

we have

$$\frac{-1}{(\log y)^2}\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$$

Therefore, the differential equation reduces to

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x^2}$$

or, $$\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2}$$

which is a linear equation.
The integrating factor is

$$\text{IF} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

Multiplying both sides of the equation by the integrating factor and integrating, we get

$$\frac{v}{x} = \frac{1}{2x^2} + c$$

where c is an arbitary constant.
Therefore, the general solution is

$$\frac{1}{x \log y} = \frac{1}{2x^2} + c$$

or, $$x = \left(\frac{1}{2} + cx^2\right) \log y$$

WORKED OUT EXAMPLES

Example 2.1 **Solve**

$$(x^2 + y^2 + 2x)\,dx + xy\,dy = 0$$ [WBUT-2007]

Sol. Here,

$$M(x, y) = (x^2 + y^2 + 2x) \text{ and } N(x, y) = xy$$

Now,

$$\frac{\partial M}{\partial y} = 2y \text{ and } \frac{\partial N}{\partial x} = y$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the differential equation is not exact.
Now,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x}$$

Therefore,

$$\text{IF} = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$$

Multiplying both sides by the integrating factor, we have

$$x(x^2 + y^2 + 2x)\,dx + x^2 y\,dy = 0$$

or, $$(x^3 + xy^2 + 2x^2)\,dx + x^3 y\,dy = 0$$

which is an exact differential equation.

The general solution is

$$\int (x^3 + xy^2 + 2x^2)\,dx = c$$

or, $$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{2x^3}{3} = c$$

where c is an arbitrary constant.

Example 2.2 **Solve**

$$\frac{dy}{dx} - \frac{\tan y}{(1+x)} = (1+x)e^x \sec y$$

[WBUT-2007]

Sol. The given equation can be written as

$$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$$

Let,

$$\sin y = v$$

or, $$\cos y \frac{dy}{dx} = \frac{dv}{dx}$$

Therefore, the differential equation is written as

$$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)e^x$$

which is a linear equation.
The integrating factor is

$$\text{IF} = e^{\int \frac{-1}{(1+x)}\,dx} = e^{-\log(1+x)} = \frac{1}{1+x}$$

Multiplying both sides by the integrating factor and integrating, we have

$$\frac{v}{1+x} = \int e^x\,dx = e^x + c$$

or, $$v = (1+x)(e^x + c)$$

or, $$\sin y = (1+x)(e^x + c)$$

where c is an arbitary constant.

Example 2.3 **Solve**

$$x\frac{dy}{dx} + y = y^2 \log x$$

[WBUT-2008]

Sol. The given equation can be written as

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{\log x}{x}$$

Putting,

$$\frac{1}{y} = z$$

or, $$y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$$

Therefore, the differential reduces to

$$-\frac{dz}{dx} + \frac{z}{x} = \frac{\log x}{x}$$

or, $$\frac{dz}{dx} - \frac{z}{x} = -\frac{\log x}{x}$$

which is a linear equation.
The integrating factor is

$$\text{IF} = e^{\int \frac{-1}{x}\,dx} = e^{-\log x} = \frac{1}{x}$$

Multiplying by the integrating factor and integrating, we have

$$\frac{z}{x} = -\int \frac{\log x}{x^2}\,dx$$

or, $$\frac{z}{x} = \frac{(1+\log x)}{x} + c$$

or, $$z = cx + (1 + \log x)$$

or, $$\frac{1}{y} = cx + (1 + \log x)$$

where c is an arbitary constant.

Example 2.4 **Solve**

$$e^x \sin y\, dx + (e^x + 1)\cos y\, dy = 0$$

[WBUT-2005, 2006]

Sol. Here,

$$M(x, y) = e^x \sin y \text{ and } N(x, y) = (e^x + 1)\cos y$$

Now,

$$\frac{\partial M}{\partial y} = e^x \cos y \text{ and } \frac{\partial N}{\partial x} = e^x \cos y$$

Since,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

the differential equation is an exact differential equation.
The general solution is

$$\int e^x \sin y \, dx + \int \cos y \, dy = c$$

or,

$$e^x \sin y + \sin y = c$$

where c is an arbitary constant.

Example 2.5 **Solve**

$$2 \sin y^2 \, dx + xy \cos y^2 \, dy = 0$$

Sol. Here,

$$M(x, y) = 2 \sin y^2 \text{ and } N(x, y) = xy \cos y^2$$

Now,

$$\frac{\partial M}{\partial y} = 4y \cos y^2 \text{ and } \frac{\partial N}{\partial x} = y \cos y^2$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the differential equation is not exact.
We have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y \cos y^2}{xy \cos y^2} = \frac{3}{x} = f(x) \quad \text{say}$$

Therefore,

$$\text{IF} = e^{\int f(x) \, dx} = e^{\int \frac{3}{x} \, dx} = e^{3 \log x} = x^3$$

Multiplying the equation by IF

$$2x^3 \sin y^2 \, dx + x^4 y \cos y^2 \, dy = 0$$

which is an exact differential equation.
The general solution is

$$\int 2x^3 \sin y^2 \, dx = c$$

or, $$x^4 \sin y^2 = 2c$$

where c is an arbitrary constant

Example 2.6 **Solve**

$$(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$$

Sol. Here,

$$M(x, y) = (3x^2y^4 + 2xy) \text{ and } N(x, y) = (2x^3y^3 - x^2)$$

Now,

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x \text{ and } \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the differential equation is not exact.
We have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy} = \frac{2}{y} = g(y) \quad \text{say}$$

Therefore,

$$\text{IF} = e^{\int g(y)\,dy} = e^{-\int \frac{2}{y}\,dy} = e^{-2\log y} = \frac{1}{y^2}$$

Multiplying the equation by IF

$$\frac{1}{y^2}(3x^2y^4 + 2xy)\,dx + \frac{1}{y^2}(2x^3y^3 - x^2)\,dy = 0$$

or, $$(3x^2y^2 + \frac{2x}{y})\,dx + (2x^3y - \frac{x^2}{y^2})\,dy = 0$$

which is an exact differential equation.
The general solution is

$$\int (3x^2y^2 + \frac{2x}{y})\,dx = c$$

or, $$x^3y^2 + \frac{x^2}{y} = c$$

where c is an arbitrary constant.

Example 2.7 **Solve**

$$x\frac{dy}{dx} - y = x\sqrt{x^2 + y^2}$$

Sol. The given differential equation can be written as

$$x\,dy - y\,dx = x\sqrt{x^2+y^2}\,dx$$

$$or, \quad \frac{x\,dy - y\,dx}{x^2} = \frac{x\sqrt{x^2+y^2}\,dx}{x^2}$$

$$or, \quad \frac{x\,dy - y\,dx}{x^2} = \frac{\sqrt{x^2+y^2}\,dx}{x}$$

$$or, \quad d\left(\frac{y}{x}\right) = \sqrt{1+\left(\frac{y}{x}\right)^2}\,dx$$

$$or, \quad \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = dx$$

Integrating, we get the general solution

$$\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \int dx$$

$$or, \quad \sin^{-1}\left(\frac{y}{x}\right) = x + c$$

$$or, \quad y = x\sin(x+c)$$

where c is an arbitary constant.

Example 2.8 **Solve**

$$(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$$

Sol. The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

which is a linear equation in x.
Here,

$$\text{IF} = e^{\int \frac{1}{1+y^2}\,dy} = e^{\tan^{-1} y}$$

Multiplying both sides by the integrating factor, we have

$$e^{\tan^{-1} y}\frac{dx}{dy} + e^{\tan^{-1} y}\frac{x}{1+y^2} = e^{\tan^{-1} y}\frac{\tan^{-1} y}{1+y^2}$$

$$or, \quad \frac{d}{dy}(xe^{\tan^{-1} y}) = e^{\tan^{-1} y}\frac{\tan^{-1} y}{1+y^2}$$

$$or, \quad d(xe^{\tan^{-1} y}) = e^{\tan^{-1} y}\frac{\tan^{-1} y}{1+y^2}\,dy$$

Integrating both sides, we have

$$xe^{\tan^{-1} y} = \int e^{\tan^{-1} y} \frac{\tan^{-1} y}{1 + y^2} \, dy$$

Putting,

$$\tan^{-1} y = z$$

i.e.,
$$\frac{1}{1 + y^2} \, dy = dz$$

Therefore,

$$\int e^{\tan^{-1} y} \frac{\tan^{-1} y}{1 + y^2} \, dy = \int ze^z dz = ze^z - e^z + c = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

Therefore, the general solution is

$$xe^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

or,
$$x = (\tan^{-1} y - 1) + ce^{-\tan^{-1} y}$$

where c is an arbitrary constant.

Example 2.9 **Solve**

$$(x + y + 1)\frac{dy}{dx} = 1$$

Sol. The given differential equation can be written as

$$\frac{dx}{dy} = (x + y + 1)$$

or,
$$\frac{dx}{dy} - x = y + 1$$

which is a linear equation in x.
The integrating factor is

$$\text{IF} = e^{\int -dy} = e^{-y}$$

Multiplying both sides of the equation by the integrating factor, we have

$$e^{-y}\frac{dx}{dy} - e^{-y}x = e^{-y}(y + 1)$$

or,
$$\frac{d}{dy}(xe^{-y}) = e^{-y}(y + 1)$$

or,
$$d(xe^{-y}) = e^{-y}(y + 1) \, dy$$

Integrating both sides, we have

$$xe^{-y} = \int e^{-y}(y+1)\,dy$$

or, $$xe^{-y} = -ye^{-y} - 2e^{-y} + c$$

or, $$x + y + 2 = ce^{y}$$

where c is an arbitry constant.

Example 2.10 **Solve**

$$\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3 \cos^2 y$$

Sol. The differential equation can be written as

$$\sec^2 y \frac{dy}{dx} + \frac{2\tan y}{x} = x^3$$

Putting,

$$\tan y = z$$

or, $$\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$$

Therefore, the differential equation becomes

$$\frac{dz}{dx} + \frac{2z}{x} = x^3$$

which is a linear equation in z.
The integrating factor is

$$\text{IF} = e^{\int \frac{2}{x}\,dx} = e^{2\log x} = x^2$$

Multiplying both sides of the differential equation by the integrating factor, we have

$$x^2 \frac{dz}{dx} + 2zx = x^5$$

or, $$\frac{d}{dx}(zx^2) = x^5$$

or, $$d(zx^2) = x^5\,dx$$

Integrating both sides, we have

$$zx^2 = \int x^5\,dx = \frac{x^6}{6} + c$$

or, $$x^2 \tan y = \frac{x^6}{6} + c$$

where c is an arbitrary constant.

Example 2.11 **Solve**

$$\frac{dy}{dx} + y = y^3(\cos x - \sin x)$$

[WBUT-2009]

Sol. The given differential equation can be written as

$$y^{-3}\frac{dy}{dx} + y^{-2} = (\cos x - \sin x)$$

Let us take the transformation

$$y^{-2} = z \Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$$

Therefore, the differential equation reduces to

$$-\frac{1}{2}\frac{dz}{dx} + z = (\cos x - \sin x)$$

or, $$\frac{dz}{dx} - 2z = 2(\sin x - \cos x)$$

which is a linear equation.
The integrating factor is

$$\text{IF} = e^{\int -2\,dx} = e^{-2x}$$

Multiplying the differential equation by the integrating factor and integrating, we get

$$\int d(e^{-2x}z) = 2\int e^{-2x}(\sin x - \cos x)$$

or, $$e^{-2x}z = 2e^{-2x}\left\{\frac{1}{5}\cos x - \frac{3}{5}\sin x\right\} + C$$

or, $$z = 2\left\{\frac{1}{5}\cos x - \frac{3}{5}\sin x\right\} + Ce^{2x}$$

where C is arbitary constant.
Therefore, the general solution is

$$y^{-2} = 2\left\{\frac{1}{5}\cos x - \frac{3}{5}\sin x\right\} + Ce^{2x}$$

or, $$\frac{1}{y^2} = 2\left\{\frac{1}{5}\cos x - \frac{3}{5}\sin x\right\} + Ce^{2x}$$

Short and Long Answer Type Questions

1. Solve the following equations using method of inspection for exactness

a) $(x+y)(dx - dy) = dx + dy$

[**Ans:** $x - y = \log(x+y) + c$]

b) $(x + 2y - 2)\,dx + (2x - y + 3)\,dy = 0$

[**Ans:** $(x^2 - y^2) + 2(3x - 2y + 2xy) = c$]

c) $(x + y)\,dy + (y - x)\,dx = 0$

[**Ans:** $(x^2 - y^2) - 2xy = c$]

d) $x\dfrac{dy}{dx} + y = y^2 \log x$

[**Ans:** $1 + cxy = y(1 + \log x)$]

e) $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$

[**Ans:** $x = cye^{\frac{1}{xy}}$]

2. Show that the following equations are not exact. Find the integrating factors and the general solution.

a) $x^2\,dy - (xy + 2y^2)\,dx = 0$

[**Ans:** $x + y\log x^2 = cy$]

b) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$\left[\textbf{Ans: } \dfrac{x}{y} - \log\frac{x^2}{y^3} = c\right]$

c) $(x^3 + y^3)\,dx - xy^2\,dy = 0$

[**Ans:** $y^3 - 3x^3\log x = cx^2$]

d) $(y^3 - 2xy^2)\,dx + (2xy^2 - x^3)\,dy = 0$

[**Ans:** $x^2y^2(y^2 - x^2) = c$]

e) $y(1 + xy)\,dx + x(1 - xy)\,dy = 0$

$\left[\textbf{Ans: } \log\dfrac{x}{y} - \dfrac{1}{xy} = c\right]$

f) $y(x^2y^2 + 2)\,dx + x(2 - 2x^2y^2)\,dy = 0$

$\left[\textbf{Ans: } x = cy^2e^{\frac{1}{x^2y^2}}\right]$

g) $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$

$\left[\textbf{Ans: } \dfrac{x}{y}\sec xy = c\right]$

h) $(x^2 + y^2)\,dx - 2xy\,dy = 0$

[**Ans:** $x^2 - y^2 = cx$]

i) $(x^2 + y^2 + 1)\,dx + x(x - 2y)\,dy = 0$

[**Ans:** $x(x + y) - (y^2 + 1) = cx$]

j) $(x^2 + xy^4)\,dx + 2y^3\,dy = 0$

[**Ans:** $\frac{1}{2}y^4e^{x^2} + \int x^2e^{x^2}\,dx = c$]

k) $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

[**Ans:** $\mathrm{x}^3y^3 + x^2 = cy$]

l) $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

[**Ans:** $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^2} = c$]

m) $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

[**Ans:** $3x^2y^4 + 6xy^2 + 2y = c$]

n) $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^3)\,dy = 0$

[**Ans:** $-\dfrac{1}{xy} + 2\log x - y = c$]

o) $(x^4y^2 - y)\,dx + (x^2y^4 - x)\,dy = 0$

[**Ans:** $x^3 + y^3 + \frac{3}{xy} = c$]

p) $x\,dy - y\,dx = (x^2 + y^2)(x\,dx + y\,dy)$

[**Ans:** $y = x\tan\left(\dfrac{x^2 + y^2 + c}{2}\right)$]

q) $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

[**Ans:** $6x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{-3}{2}}y^{\frac{3}{2}} = c$]

r) $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

[**Ans:** $x^3y^3 + x^2 = cy$]

s) $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

[**Ans:** $\frac{-2}{3}x^{\frac{-3}{2}}y^{\frac{3}{2}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} = c$]

t) $x(3y\,dx + 2x\,dy) + 8y^4(y\,dx + 3x\,dy) = 0$

[**Ans:** $x^2y^3(x + 4y^4) = c$]

3. Solve the following linear equations.

a) $(x^2 + 1)\dfrac{dy}{dx} + 2xy = 4x^2$

[**Ans:** $3(x^2 + 1)y = 4x^3 + c$]

b) $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

[**Ans:** $6x^2\tan y = x^6 + c$]

c) $x \cos x \dfrac{dy}{dx} + (x \sin x + \cos x) y = 1$

[**Ans:** $xy \sec x = \tan x + c$]

d) $\dfrac{dy}{dx} + \dfrac{1-2x}{x^2} y = 1$

[**Ans:** $y = x^2(1 + ce^{\frac{1}{x}})$]

e) $x(x^2-1)\dfrac{dy}{dx} + (1-2x^2)y + ax^3 = 0$

[**Ans:** $y = ax + cx\sqrt{1-x^2}$]

f) $x \log x \dfrac{dy}{dx} + y = 2 \log x$

[**Ans:** $y \log x = (\log x)^2 + c$]

g) $y \log y \dfrac{dx}{dy} + x - \log y = 0$

[**Ans:** $x \log y = \frac{1}{2}(\log y)^2 + c$]

h) $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$

[**Ans:** $\dfrac{1}{xy} = \dfrac{1}{2x^2} + c$]

i) $y(2xy + e^x)\, dx - e^x\, dy = 0$

[**Ans:** $y^{-1}e^x + x^2 = c$]

j) $\dfrac{1}{1+y^2}\dfrac{dy}{dx} + 2x \tan^{-1} y = x^3$

[**Ans:** $\tan^{-1} y = \frac{1}{2}(x^2-1) + ce^{-x^2}$]

Multiple Choice Questions

1. The order of the differential equation $\left(\dfrac{d^3y}{dx^3}\right)^2 + y = \sin x$ is

   a) 4 b) 3 c) 2 d) 1

2. The integrating factor of the differential equation $\dfrac{dy}{dx} + 2y = \sin x$ is

   a) e^{2x} b) e^{3x} c) e^{-3x} d) e^x

3. If the integrating factor of $(2x^2y - y - ax^3)\, dx + (x - x^3)\, dy = 0$ is $e^{\int P\, dx}$ then P is

   a) $2x^3 - 1$ b) $\dfrac{2x^2-1}{x(1-x^2)}$ c) $\dfrac{2x^2-1}{ax^3}$ d) $\dfrac{2x^2-ax^3}{x(1-x^2)}$

4. The differential equation $\left(y + \frac{1}{x} + \frac{1}{x^2 y}\right) dx + (x - \frac{1}{y} + \frac{a}{xy^2}) \, dy = 0$ is exact then the value of a is

a) 2 b) 1 c) 0 d) -1

5. The integrating factor of $(2xy - 3y^3)\, dx + (4x^2 + 6xy^2)\, dy = 0$ is

a) $x^2 y$ b) $x^2 y^2$ c) xy^2 d) xy^3

6. The differential equation $(xe^{axy} + 2y)\, dx + ye^{xy}\, dy = 0$ is exact, then the value of a is

a) 3 b) 1 c) 2 d) 0

7. If $x^h y^k$ is the integrating factor of the differential equation$(2y\, dx + 3x\, dy) + 2xy(3y\, dx + 4x\, dy) = 0$ then the values of h and k are

a) 1, 3 b) 2, 1 c) 2, 2 d) 1, 2

8. If $x^h y^k$ is the integrating factor of the differential equation$(3x^{-1} + 2y^4)\, dx - (xy^3 - 3y^{-1})\, dy = 0$ then the values of h and k are

a) $-3, -3$ b) $-3, 3$ c) $3, -3$ d) 1, 2

9. The integrating factor of the differential equation $\frac{dy}{dx} + \frac{x}{2(1 - x^2)} y = \frac{x}{2}$ is

a) $(1 - x^2)^{-\frac{1}{4}}$ b) $\sqrt{1 - x^2}$ c) $\log(1 - x^2)$ d) none of these

10. The integrating factor of the differential equation $\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{e^{-\tan^{-1} y}}{1 + y^2}$ is

a) $\tan^{-1} y$ b) $\mathrm{e}^{\tan^{-1} y}$ c) $\mathrm{e}^{\cot^{-1} y}$ d) e^y

11. The differential equation $(y^2 e^{xy^2} + 4x^3)\, dx + (2xye^{xy^2} - 3y^2)\, dy = 0$ is

a) linear, homogeneous and exact

b) non-linear, homogeneous and exact

c) non-linear, non-homogeneous and exact

d) none of these

12. The differential equation $(x^3 + 3y^2 x)\, dx + (y^3 + 3x^2 y)\, dy = 0$ is

a) homogeneous and exact

b) non-homogeneous and exact

c) non-homogeneous and non-exact

d) none of these

Answers:

1(b)	2(a)	3(b)	4(b)	5(a)	6(b)	7(d)	8(a)
9(a)	10(b)	11(c)	12(a)				

CHAPTER 3

Ordinary Differential Equations of First Order and Higher Degree

3.1 INTRODUCTION

This chapter deals with differential equations of first order and higher degree. In this chapter, we will discuss the various techniques for solving them.

Clairaut's form is one of the important topic. Each and every topic is illustrated with different kinds of suitable examples. At the end of the chapter different solved problems of university examinations have been included.

3.2 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

3.2.1 Definition

A differential equation of the form

$$\boxed{f\left(x, y, \frac{dy}{dx}\right) = 0}$$

where the degree of the differential equation is greater than one it is known as a differential equation of first order and higher degree.

If we write $\frac{dy}{dx} = p$, then the general form of differential equation of first order and not of first degree is

$$\boxed{f(x, y, p) = 0}$$

We can write the equation in the form

$$\boxed{a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_{n-1} p + a_n = 0}$$

where

$$a_0 (\neq 0), a_1, a_2, \ldots, a_{n-1}, a_n$$

are functions of x and y or constants and n is a positive integer.

3.2.2 Method of Solution

There is no general method for solving differential equation of first order and not of first degree, but there are certain techniques for solving such equations when these equations are one of the following particular types :

(a) Equations solvable for p

(b) Equations solvable for y

(c) Equations solvable for x

(d) Clairaut's equation

(e) Equations not containing x

(f) Equations not containing y

(g) Equations homogeneous in x and y

3.3 EQUATIONS SOLVABLE FOR p

3.3.1 Definition

Let,

$$f(x, y, p) = 0$$

be any differential equation of first order and n^{th} degree, the equation is of the form

$$a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_{n-1} p + a_n = 0$$

which can be put in the form

$$[p - f_1(x, y)][p - f_2(x, y)] \cdots [p - f_n(x, y)] = 0$$

is called differential equations solvable for p.

3.3.2 Method of Solution

Let us express the differential equation in the form

$$\boxed{[p - f_1(x, y)][p - f_2(x, y)] \ldots [p - f_n(x, y)] = 0}$$

Equating each of the factors to zero, we get n differential equations of first order and first degree.
The equations are

$$\boxed{\frac{dy}{dx} = f_1(x, y), \frac{dy}{dx} = f_2(x, y), \ldots, \frac{dy}{dx} = f_n(x, y)}$$

Let the solutions be

$$\boxed{\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, \ldots, \phi_n(x, y, c_n) = 0}$$

where $c_1, c_2, \ldots, c_n$ are arbitary constants.
The general solution of the differential equation is

$$\boxed{\phi_1(x, y, c)\phi_2(x, y, c) \ldots \phi_n(x, y, c) = 0}$$

by making $c_1 = c_2 = \ldots = c_n = c$, where c is an arbitary constant since the given differential is of first order.

Example 1 **Solve**

$$p^2 - p(e^x + e^{-x}) + 1 = 0$$

Sol. The differential equation can be written as

$$p^2 e^x e^{-x} - p(e^x + e^{-x}) + 1 = 0$$

$or,$ $$(p - e^x)(p - e^{-x}) = 0$$

Therefore, either,

$$(p - e^x) = 0$$

$or,$ $$(p - e^{-x}) = 0$$

When,

$$(p - e^x) = 0$$

$or,$ $$\frac{dy}{dx} = e^x$$

$or,$ $$dy = e^x \, dx$$

Integrating, the solution is

$$(y - e^x + c_1) = 0$$

when,

$$(p - e^{-x}) = 0$$

or, $$\frac{dy}{dx} = e^{-x}$$

or, $$dy = e^{-x}\, dx$$

Integrating, the solution is

$$(y - e^{-x} + c_2) = 0$$

where c_1 and c_2 are arbitary constants.
Therefore, the general solution is

$$(y - e^x + c)(y - e^{-x} + c) = 0$$

where c is an arbitary constant.

Example 2 **Solve**

$$p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$$

Sol. The equation can be written as

$$p^3 - px^2 - pxy - py^2 + x^2y + xy^2 = 0$$

or, $$p(p^2 - x^2) - xy(p - x) - y^2(p - x) = 0$$

or, $$(p - x)\{p^2 + px - xy - y^2\} = 0$$

or, $$(p - x)(p - y)(p + x + y) = 0$$

When,

$$(p - x) = 0$$

or, $$dy = x\, dx$$

the solution is

$$(2y - x^2 + c_1) = 0$$

When,

$$(p - y) = 0$$

or, $$\frac{dy}{y} = dx$$

the solution is

$$(y - c_2e^{-x}) = 0$$

When,

$$(p + x + y) = 0$$

or,

$$\frac{dy}{dx} + y = -x$$

which is a linear equation of first order.

$$\text{I.F} = e^{\int dx} = e^x$$

Multiplying both sides by the integrating factor, we have

$$d(e^x y) = -xe^x\,dx$$

Integrating, the solution is

$$e^x y = -(x-1)e^x + c_3$$

or,

$$(y + x - 1 - c_3 e^{-x}) = 0$$

where c_1 and c_2 and c_3 are arbitary constants.
Therefore, the general solution is

$$(2y - x^2 + c)(y - ce^{-x})(y + x - 1 - ce^{-x}) = 0$$

3.4 EQUATIONS SOLVABLE FOR *y*

3.4.1 Definition

Let

$$f(x, y, p) = 0$$

be any differential equation of first order and n^{th} degree, the equation is of the form

$$a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_{n-1} p + a_n = 0$$

which can be put in the form

$$\boxed{y = F(x, p)}$$

and is called a differential equation solvable for *y*.

3.4.2 Method of Solution

Let us express the differential equation as

$$y = F(x, p)$$

Differentiating w.r.t x on both sides, we get

$$p = F_1\left(x, p, \frac{dp}{dx}\right)$$

which is a differential equation of first order and first degree.
Solving this equation, we have

$$x = \phi(p, c)$$

where c is an arbitrary constant.
Putting the value of x in the equation $y = F(x, p)$, we have

$$y = \psi(p, c)$$

Therefore, the general parametric form of solution is

$$\boxed{x = \phi(p, c) \text{ and } y = \psi(p, c)}$$

where p is the parameter.
The general solution is obtained by eliminating p from x and y.

Note: If it is not possible to eliminate p, we may represent the general solution in parametric form

Example 3 **Solve**

$$y = p^2 x + p$$

Sol. The differential equation is solvable for y
Differentiating both sides w.r.t x, we have

$$\frac{dy}{dx} = p^2 + 2px\frac{dp}{dx} + \frac{dp}{dx}$$

$$or, \quad p = \frac{dp}{dx}(2px + 1) + p^2$$

$$or, \quad \frac{dp}{dx} = \frac{p - p^2}{1 + 2px}$$

$$or, \quad \frac{dx}{dp} - \frac{2}{1-p}x = \frac{1}{p(1-p)}$$

which is a linear equation.
The integrating factor is,

$$I.F = e^{\int -\frac{2}{1-p}dp} = e^{2\log(p-1)} = (p-1)^2$$

Multiplying by the integrating factor and integrating, we get

$$x(p-1)^2 = \log p - p + c$$

$$or, \qquad x = \frac{\log p - p + c}{(p-1)^2}$$

Putting the value of x in the equation, we have

$$y = p^2 \left\{ \frac{\log p - p + c}{(p-1)^2} \right\} + p$$

Therefore, the general parametric solution is

$$x = \frac{\log p - p + c}{(p-1)^2}; \; y = p^2 \left\{ \frac{\log p - p + c}{(p-1)^2} \right\} + p$$

where p is the parameter and c is an arbitary constant.

Example 4 **Solve**

$$e^y - p^3 - p = 0$$

Sol. The given differential equation can be written as

$$y = \log(p^3 + p)$$

Differentiating both sides with respect to x, we have

$$p = \frac{1}{(p^3 + p)}(3p^2 + 1)\frac{dp}{dx}$$

$$or, \qquad \frac{dp}{dx} = \frac{p^2(p^2+1)}{(3p^2+1)}$$

$$or, \qquad dx = \frac{(3p^2+1)}{p^2(p^2+1)}dp$$

$$or, \qquad dx = \left(\frac{2}{1+p^2} + \frac{1}{p^2} \right) dp$$

Integrating both sides, we get

$$x = 2\tan^{-1} p - \frac{1}{p} + c$$

and

$$y = \log(p^3 + p)$$

Therefore, the solution is

$$x = 2\tan^{-1} p - \frac{1}{p} + c \text{ and } y = \log(p^3 + p)$$

where p is the parameter and c is an arbitary constant.

3.5 EQUATIONS SOLVABLE FOR x

3.5.1 Definition

Let,

$$f(x, y, p) = 0$$

be any differential equation of of first order and n^{th} degree, the equation is of the form

$$a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_{n-1} p + a_n = 0$$

which can be put in the form

$$\boxed{x = F(y, p)}$$

is called a differential equation solvable for x.

3.5.2 Method of Solution

Let us express the differential equation as

$$x = F(y, p)$$

Differentiating both sides w.r.t y, we get

$$\frac{1}{p} = F_1\left(y, p, \frac{dp}{dy}\right)$$

which is a differential equation of first order and first degree.
Solving this equation we have

$$y = \phi(p, c)$$

where c is an arbitrary constant.
Putting the value of x in the equation $x = F(y, p)$, we have

$$x = \psi(p, c)$$

Therefore, the general parametric form of solution is

$$\boxed{y = \phi(p, c) \text{ and } x = \psi(p, c)}$$

where p is the parameter.
The general solution is obtained by eliminating p from x and y.

Example 5 **Solve**

$$y = 2px + y^2 p^3$$

Sol. The differential equation can be written as

$$x = \frac{y}{2p} - \frac{y^2 p^3}{2}$$

which is solvable for x.
Differentiating both sides w.r.t y, we have

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - y^2 p\frac{dp}{dy}$$

$$or, \quad \frac{1}{2p} + yp^2 = \frac{-y}{p}\frac{dp}{dy}\left(\frac{1}{2p} + yp^2\right)$$

$$or, \quad \left(\frac{1}{2p} + yp^2\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Therefore,

$$\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0 \text{ if } \left(\frac{1}{2p} + yp^2\right) \neq 0$$

$$or, \quad p\,dy + y dp = 0$$

$$or, \quad d(py) = 0$$

Integrating, we get

$$py = c$$

$$or, \quad y = \frac{c}{p}$$

where c is an arbitary constant.
Putting the value of y in original equation, we have

$$x = \frac{c}{2p^2} + c^2 p$$

Therefore, the general parametric solution is

$$y = \frac{c}{p};\ x = \frac{c}{2p^2} + c^2 p$$

Eliminating p, the general solution is

$$27y^4 + 32x^3 = 0$$

Example 6 **Solve**

$$y^2 \log y = xyp + p^2$$

Sol. The differential equation can be written as

$$x = \frac{y \log y}{p} - \frac{p}{y}$$

Differentiating with respect to y, we have

$$\frac{dx}{dy} = \frac{p(1+\log y) - y\log y\dfrac{dp}{dy}}{p^2} - \frac{y\dfrac{dp}{dy} - p}{y^2}$$

$$or, \quad \frac{1}{p} = \frac{1}{p} + \frac{1}{p}\log y - \frac{y}{p^2}\log y\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$or, \quad \frac{1}{y}\frac{dp}{dy}\left(1 + \frac{y^2}{p^2}\log y\right) = \frac{p}{y^2}\left(1 + \frac{y^2}{p^2}\log y\right)$$

$$or, \quad \left(1 + \frac{y^2}{p^2}\log y\right)\left(\frac{1}{y}\frac{dp}{dy} - \frac{p}{y^2}\right) = 0$$

Therefore,

$$\left(\frac{1}{y}\frac{dp}{dy} - \frac{p}{y^2}\right) = 0$$

$$or, \quad \frac{dp}{dy} = \frac{p}{y}$$

$$or, \quad \frac{dp}{p} = \frac{dy}{y}$$

Integrating both sides, we get

$$\log p = \log y + \log c$$

$$or, \quad y = \frac{p}{c}$$

and

$$x = \frac{y\log y}{p} - \frac{p}{y}$$

$$or, \quad x = \frac{\log\left(\frac{p}{c}\right)}{c} - c$$

$$or, \quad x = \frac{\log p - \log c}{c} - c$$

Eliminating p from y and x, we have the general solution

$$\log y = cx + c^2$$

where c is an arbitary constant.

3.6 CLAIRAUT'S EQUATION

3.6.1 Definition:

A differential equation of the form

$$\boxed{y = px + f(p)}$$

where

$$p = \frac{dy}{dx}$$

is known as Clairaut's equation.

3.6.2 Method of Solution:

Differentiating the differential equation both sides w.r.t x, we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$or,$

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$or,$

$$\frac{dp}{dx}[x + f'(p)] = 0$$

Case 1: When,

$$\frac{dp}{dx} = 0$$

Therefore,

$$p = c$$

where c is an arbitary constant.
Therefore, the general solution is

$$\boxed{y = cx + f(c)}$$

Case 2: When,

$$[x + f'(p)] = 0$$

$or,$

$$x = -f'(p)$$

we have,

$$y = px + f(p)$$

Eliminating p from these two equations, we will get a solution. It is to be noted that the solution does not include any arbitray constant. So it is not a general solution, but a solution of the given equation and is called **singular solution.**

Example 7 **Find the general solution and the singular solution**

$$y = px + \frac{a}{p}$$

Sol. The differential equation is of Clairaut's form.
Differentiating both sides w.r.t x, we have

$$p = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

or, $$x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx} = 0$$

or, $$\left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$$

Either,

$$\frac{dp}{dx} = 0$$

or, $$\left(x - \frac{a}{p^2}\right) = 0$$

When,

$$\frac{dp}{dx} = 0$$

or, $$p = c$$

The general solution is

$$y = cx + \frac{a}{c}$$

Now again,

$$\left(x - \frac{a}{p^2}\right) = 0$$

or, $$x = \frac{a}{p^2}$$

Also, we have

$$y = px + \frac{a}{p}$$

Eliminating p, we have the singular solution

$$y^2 = 4ax.$$

Example 8 **Find the general solution and the singular solution**

$$\sin\left(x\frac{dy}{dx}\right)\cos y = \cos\left(x\frac{dy}{dx}\right)\sin y + \frac{dy}{dx}$$

Sol. The differential equation can be written as

$$\sin\left(x\frac{dy}{dx}\right)\cos y - \cos\left(x\frac{dy}{dx}\right)\sin y = \frac{dy}{dx}$$

or, $$\sin\left(x\frac{dy}{dx} - y\right) = \frac{dy}{dx}$$

or, $$\sin(xp - y) = p$$

or, $$xp - y = \sin^{-1} p$$

or, $$y = xp - \sin^{-1} p$$

which is a Clairaut's equation.
Differentiating both sides w.r.t x, we have

$$p = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

or, $$x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx} = 0$$

or, $$\left(x - \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$$

When,

$$\frac{dp}{dx} = 0$$

or, $$p = c$$

where c is an arbitary constant, the general solution is

$$y = xc - \sin^{-1} c$$

Now again,

$$\left(x - \frac{1}{\sqrt{1-p^2}}\right) = 0$$

or, $$p = \frac{\sqrt{x^2-1}}{x}$$

Putting this value of p in the equation, we have the singular solution as

$$y = x\frac{\sqrt{x^2-1}}{x} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}$$

or, $$y = \sqrt{x^2 - 1} - \sin^{-1}\frac{\sqrt{x^2 - 1}}{x}$$

Observations:

1) In many cases, the equation

$$f(x, y, p) = 0$$

is not in Clairaut's form but by some suitable transformations it can be reduced to Clairaut's form.

Example 9 **Use the transformations**

$$u = x^2 \textbf{ and } v = y^2$$

and transform the equation

$$xyp^2 - (x^2 + y^2 - 1)p + xy = 0$$

into Clairaut's equation and solve.

Sol. Here,

$$u = x^2 \text{ and } v = y^2$$

So,

$$du = 2x\,dx \text{ and } dv = 2y\,dy$$

Therefore,

$$\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du}$$

Let,

$$\frac{dy}{dx} = p \text{ and } \frac{dv}{du} = P$$

then,

$$\frac{y}{x}p = P$$

or, $$p = \frac{xP}{y}$$

Putting the value of p in the equation,

$$xy\left(\frac{xP}{y}\right)^2 - (x^2 + y^2 - 1)\frac{xP}{y} + xy = 0$$

or, $$x^2P^2 - (x^2 + y^2 - 1)P + y^2 = 0$$

or, $$uP^2 - (u + v - 1)P + v = 0$$

or, $$v = uP + \frac{P}{P-1}$$

which is of Clairaut's form.
Therefore, the general solution is

$$v = uc + \frac{c}{c-1}$$

or, $$y^2 = cx^2 + \frac{c}{c-1}$$

where c is an arbitary constant.

Example 10 **Reduce the differential equation**

$$xp^2 - 2yp + x + 2y = 0$$

to Clairaut's equation using the transformations

$$x^2 = u \textbf{ and } y - x = v$$

and solve.

Sol. Here,

$$x^2 = u \text{ and } y - x = v$$

So,

$$2x = \frac{du}{dx} \text{ and } \frac{dy}{dx} - 1 = \frac{dv}{dx}$$

Therefore,

$$\frac{dv}{du} = \frac{\frac{dy}{dx} - 1}{2x}$$

or, $$P = \frac{p-1}{2x}$$

or, $$p = 2xP + 1$$

where $p = \dfrac{dy}{dx}$ and $P = \dfrac{dv}{du}$.

Therefore, the differential reduces to

$$x\{2xP + 1\}^2 - 2y\{2xP + 1\} + x + 2y = 0$$

or, $$4x^2P^2 - 4(y - x)P + 2 = 0$$

or, $$4vP = 4uP^2 + 2$$

or, $$v = uP + \frac{1}{2P}$$

which is Clairaut's form.
The general solution is

$$v = cu + \frac{1}{2c}$$

where c is an arbitary constant.
Putting,

$$x^2 = u \text{ and } y - x = v$$

we have,

$$(y - x) = cx^2 + \frac{1}{2c}$$

or, $$2c^2x^2 - 2c(y - x) + 1 = 0$$

2) Equations of the type

$$y = xf(p) + g(p)$$

is known as Lagrange's equation, which is an extended form of Clairaut's equation.

3.7 EQUATIONS NOT CONTAINING *x*

3.7.1 Definition

A differential equation of the form

$$f(y, p) = 0$$

is a differential equation not containing x.

3.7.2 Method of Solution

Case 1: If the differential equation is solvable for p, then

$$p = g(y)$$

or, $$\frac{dy}{dx} = g(y)$$

and the solution is

$$\int \frac{dy}{g(y)} = x + c$$

where c is an arbitary constant.

Case 2: If the differential equation is solvable for y, then

$$y = \phi(p)$$

and the solution is obtained by solving it for y

Example 11 **Solve**

$$y = p \tan p + \log \cos p$$

Sol. Differentiating both sides w.r.t x, we have

$$p = \{p \sec^2 p + \tan p - \tan p\} \frac{dp}{dx}$$

$$or, \quad p = p \sec^2 p \frac{dp}{dx}$$

$$or, \quad p\left(1 - \sec^2 p \frac{dp}{dx}\right) = 0$$

Therefore,

$$\left(1 - \sec^2 p \frac{dp}{dx}\right) = 0$$

$$or, \quad dx = \sec^2 p dp$$

Integrating, we have

$$x = \tan p + c$$

where c is an arbitary constant.
Here,

$$y = p \tan p + \log \cos p$$

Therefore, the general parametric solution is

$$x = \tan p + c$$
$$y = p \tan p + \log \cos p$$

where p is the parameter.

3.8 EQUATIONS NOT CONTAINING y

3.8.1 Definition

A differential equation of the form

$$f(x, p) = 0$$

is a differential equation not containing y.

3.8.2 Method of Solution

Case 1: If the differential equation is solvable for p, then

$$p = g(x)$$

or,
$$\frac{dy}{dx} = g(x)$$

and the solution is

$$y = \int g(x)\, dx + c$$

where c is an arbitary constant.

Case 2: If the differential equation is solvable for x, then

$$x = \phi(p).$$

Example 12 **Solve**

$$\tan^{-1} p = x - \frac{p}{1+p^2}$$

Sol. The given equation is a differential equation not containing y. It can be written as

$$x = \tan^{-1} p + \frac{p}{1+p^2}$$

So, this is solvable for x.
Differentiating both sides w.r.t y , we have

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2) - 2p^2}{(1+p^2)^2}\frac{dp}{dy}$$

or,
$$\frac{1}{p} = \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

or,
$$dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating both sides, we have

$$y = c - \frac{1}{1+p^2}$$

where c is an arbitary constant.
Therefore, the required solution is

$$x = \tan^{-1} p + \frac{p}{1+p^2}$$

$$y = c - \frac{1}{1+p^2}$$

where p is the parameter.

Here, we present another special type of differential equation of first order and higher degree.

3.9 EQUATIONS HOMOGENEOUS IN *X* AND *Y*

3.9.1 Definition

Any differential equation of the form

$$f\left(p, \frac{y}{x}\right) = 0$$

is known as differential equations homogeneous in x and y.

3.9.2 Method of Solution

Case 1: When the differential equation is written as

$$p = \psi\left(\frac{y}{x}\right)$$

and the solution is obtained by methods previously discussed.

Case 2: When the differential equation is written as

$$\frac{y}{x} = \phi(p)$$

or,

$$y = x\phi(p)$$

and the solution is obtained by the method of solvable for y.

Example 13 **Solve**

$$y = px + p^2x$$

Sol. The differential equation can be written as

$$y = x(p + p^2) \tag{1}$$

or,

$$\frac{y}{x} = (p + p^2)$$

which is in the form

$$\frac{y}{x} = \phi(p)$$

Differentiating both sides of (1) w.r.t x, we have

$$p = (p + p^2) + x(1 + 2p)\frac{dp}{dx}$$

$$or, \qquad x(1+2p)\frac{dp}{dx} = -p^2$$

$$or, \qquad \frac{(1+2p)}{p^2}dp = -x\,dx$$

Integrating both sides, we have

$$-\frac{1}{p} + 2\log p = -\frac{x^2}{2} + c$$

$$or, \qquad x = 2\sqrt{c + \frac{1}{p} - 2\log p}$$

where c is an arbitary constant.
Therefore, the general solution is

$$x = 2\sqrt{c + \frac{1}{p} - 2\log p}$$

$$y = 2(p+p^2)\sqrt{c + \frac{1}{p} - 2\log p}$$

where p is the parameter.

WORKED OUT EXAMPLES

Example 3.1 **Solve**

$$y = (p+p^2)x + p^{-1} \text{ where } p = \frac{dy}{dx}$$ [WBUT-2003]

Sol. The differential equation is

$$y = (p+p^2)x + p^{-1}$$

Differentiating both sides with respect to x, we have

$$p = p + p^2 + x\left(\frac{dp}{dx} + 2p\frac{dp}{dx}\right) - \frac{1}{p^2}\frac{dp}{dx}$$

$$or, \qquad \frac{dx}{dp} + x\left(\frac{1+2p}{p^2}\right) = \frac{1}{p^4}$$

which is a linear equation.
The integrating factor is

$$\text{IF} = e^{\int\left(\frac{1+2p}{p^2}\right)dp} = p^2 e^{-\frac{1}{p}}$$

Multiplying both sides of the equation by the integrating factor and integrating, we get

$$p^2 x e^{-\frac{1}{p}} = \int \frac{1}{p^2} e^{-\frac{1}{p}} dp$$

or, $$p^2 x e^{-\frac{1}{p}} = e^{-\frac{1}{p}} + c$$

or, $$x = \frac{1 + ce^{\frac{1}{p}}}{p^2}$$

Therefore,

$$y = \left(1 + \frac{1}{p}\right)\left(1 + ce^{\frac{1}{p}}\right) + \frac{1}{p}$$

Therefore, the solution is

$$x = \frac{1 + ce^{\frac{1}{p}}}{p^2} \text{ and } \left(1 + \frac{1}{p}\right)\left(1 + ce^{\frac{1}{p}}\right) + \frac{1}{p}$$

where p is the parameter and c is arbitary constant.

Example 3.2 **Solve**

$$y = px + \sqrt{a^2 p^2 + b^2} \text{ where } p = \frac{dy}{dx}$$ [WBUT-2005]

Sol. The differential equation is a Clairaut's equation.
Differentiating both sides w.r.t x, we have

$$p = p + x\frac{dp}{dx} + \frac{2a^2 p}{2\sqrt{a^2 p^2 + b^2}} \frac{dp}{dx}$$

or, $$\left\{x + \frac{a^2 p}{\sqrt{a^2 p^2 + b^2}}\right\} \frac{dp}{dx} = 0$$

When,

$$\frac{dp}{dx} = 0$$

or, $$p = c$$

The general solution is

$$y = cx + \sqrt{a^2 c^2 + b^2}$$

where c is an arbitary constant.

When,

$$\left\{x + \frac{a^2 p}{\sqrt{a^2 p^2 + b^2}}\right\} = 0$$

or, $$x = -\frac{a^2 p}{\sqrt{a^2 p^2 + b^2}}$$

Putting the value of x in the diffeential equation, we have

$$y = -\frac{a^2 p^2}{\sqrt{a^2 p^2 + b^2}} + \sqrt{a^2 p^2 + b^2}$$

Eliminating p from x and y, the singular solution is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example 3.3 **Solve**

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$ [WBUT-2006]

Sol. The differential equation can be written as

$$y = xp + p^2$$

which is a Clairaut's equation.
Differentiating both sides w.r.t x, we have

$$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or, $$(x + 2p)\frac{dp}{dx} = 0$$

When,

$$\frac{dp}{dx} = 0$$

or, $$p = c$$

the general solution is

$$y = xc + c^2$$

where c is an arbitary constant.
When,

$$(x + 2p) = 0$$

or, $$x = -2p$$

we have,

$$y = -2p^2 + p^2 = -p^2$$

Eliminating p from x and y, we have the singular solution

$$y = -\left(\frac{-x}{2}\right)^2$$

$or,$ $$x^2 + 4y = 0$$

Example 3.4 **Find the general solution of**

$$p = \cos(y - px) \text{ where } p = \frac{dy}{dx}$$ [WBUT-2007]

Sol. The differential equation can be written as

$$y = px + \cos^{-1} p$$

which is a Clairaut's equation.
Differentiating both sides w.r.t x, we have

$$p = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

$or,$ $$\left\{x - \frac{1}{\sqrt{1-p^2}}\right\}\frac{dp}{dx} = 0$$

When,

$$\frac{dp}{dx} = 0$$

$or,$ $$p = c$$

The general solution is

$$y = cx + \cos^{-1} c$$

where c is an arbitary constant.
When,

$$x = \frac{1}{\sqrt{1-p^2}}$$

and

$$y = px + \cos^{-1} p$$

Eliminating p from x and y we get the general solution.

Example 3.5 **Solve**

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$ [WBUT-2006]

Sol. The differential equation can be written as

$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$$

where,

$$p = \frac{dy}{dx}$$

Therefore,

$$\frac{p^2 - 1}{p} = \frac{x^2 - y^2}{xy}$$

or, $$p^2xy - xy = px^2 - py^2$$

or, $$p^2xy - xy - px^2 + py^2 = 0$$

or, $$(px + y)(py - x) = 0$$

When,

$$(px + y) = 0$$

or, $$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating both sides, we have

$$\log y + \log x = \log c_1$$

or, $$(xy - c_1) = 0$$

When,

$$(py - x) = 0$$

or, $$y\,dy = x\,dx$$

Integrating both sides, we have

$$(y^2 - x^2 - 2c_2) = 0$$

Therefore, the general solution is

$$(xy - c)(y^2 - x^2 - 2c) = 0$$

where c is any arbitary constant.

Example 3.6 **Solve**

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$ [WBUT-2006]

Sol. The differential equation can be written as

$$y = px + p^2$$

where,

$$p = \frac{dy}{dx}$$

which is a Clairaut's equation.
Differentiating both sides with respect to x, we have

$$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$or, \quad \frac{dp}{dx}(x + 2p) = 0$$

When ,

$$\frac{dp}{dx} = 0$$

$$or, \quad p = c$$

The general solution is

$$y = cx + c^2$$

When,

$$(x + 2p) = 0$$

$$or, \quad x = -2p$$

and

$$y = px + p^2$$

Eliminating p from x and y, the singular solution is

$$y = x\left(\frac{-x}{2}\right) + \left(\frac{-x}{2}\right)^2$$

$$or, \quad y = -\frac{x^2}{4}$$

Example 3.7 **Solve**

$$p^2 + 2py\cot x = y^2$$

Sol. The differential equation can be written as

$$p^2 + 2py\cot x = y^2(\cos ec^2 x - \cot^2 x)$$

$$or, \quad p^2 + 2py\cot x + y^2\cot^2 x = y^2\cos ec^2 x$$

$$or, \quad (p + y\cot x)^2 - y^2\cos ec^2 x = 0$$

$$or, (p + y\cot x + y\cos ecx)(p + y\cot x - y\cos ecx) = 0$$

Solving the differential equation by solvable for p , we have either,

$$(p + y\cot x + y\cos ecx) = 0$$

or,

$$(p + y\cot x - y\cos ecx) = 0$$

When,

$$(p + y\cot x + y\cos ecx) = 0$$

$$or, \quad \frac{dy}{dx} = -(y\cot x + y\cos ecx)$$

$$or, \quad \frac{dy}{y} = -(\cot x + \cos ecx)\,dx$$

$$or, \quad -\frac{dy}{y} = \frac{\cos x + 1}{\sin x}\,dx = \frac{\sin x}{(1 - \cos x)}\,dx$$

Integrating, we have

$$-\log y - \log c_1 = \log(1 - \cos x)$$

$$or, \quad y(1 - \cos x) + c_1 = 0$$

When,

$$(p + y\cot x - y\cos ecx) = 0$$

$$or, \quad \frac{dy}{dx} = (-y\cot x + y\cos ecx)$$

$$or, \quad \frac{dy}{y} = \frac{(1 - \cos x)}{\sin x}\,dx = \frac{\sin x}{(1 + \cos x)}\,dx$$

Integrating, we have

$$\log y - \log c_2 = -\log(1 + \cos x)$$

$$or, \quad y(1 + \cos x) + c_2 = 0$$

Therefore, the general solution is

$$\{y(1 - \cos x) + c\}\{y(1 + \cos x) + c\} = 0$$

where c is any arbitary constant.

Example 3.8 **Solve**

$$xy\left\{\left(\frac{dy}{dx}\right)^2 - 1\right\} = (x^2 - y^2)\frac{dy}{dx}$$

Sol. The given equation can be written as

$$xy(p^2 - 1) = (x^2 - y^2)p$$

or, $$(xp + y)(yp - x) = 0$$

When,

$$(xp + y) = 0$$

or, $$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating, we have

$$\log y = -\log x + \log c_1$$

or, $$xy - c_1 = 0$$

When,

$$(yp - x) = 0$$

or, $$y\,dy = x\,dx$$

Integrating, we have

$$y^2 = x^2 + c_2$$

or, $$y^2 - x^2 - c_2 = 0$$

Therefore, the general solution is

$$(xy - c)(y^2 - x^2 - c) = 0$$

where c is any arbitary constant.

Example 3.9 **Reduce the equation**

$$xy(y - px) = x + py$$

to Clairaut's equation and solve.

Sol. Putting,

$$x^2 = u \text{ and } y^2 = v$$

We have,

$$2x\,dx = du \text{ and } 2y\,dy = dv$$

Therefore,

$$\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du}$$

or, $$\frac{y}{x}p = P$$

or, $$p = \frac{xP}{y}$$

Where,

$$p = \frac{dy}{dx} \text{ and } P = \frac{dv}{du}$$

Putting the value of p in the differential equation, we have

$$xy\left(y - \frac{x^2}{y}P\right) = x + y\frac{xP}{y}$$

or, $$(y^2 - x^2P) = 1 + P$$

or, $$v = uP + (1 + P)$$

which is a Clairaut's equation.
Hence, the general solution is

$$v = uc + (1 + c)$$

where c is an arbitary constant.
Therefore, the general solution of the differential equation is

$$y^2 = x^2c + (1 + c)$$

Example 3.10 **Reduce the equation**

$$xp^2 - 2yp + x + 2y = 0$$

to Clairaut's form by using the substitution

$$x^2 = u \textbf{ and } y - x = v$$

and then find the general solution.

Sol. Here,

$$x^2 = u \text{ and } y - x = v$$

Then,

$$2x = \frac{du}{dx} \text{ and } \frac{dy}{dx} - 1 = \frac{dv}{dx}$$

Therefore,

$$\frac{dv}{du} = \frac{\frac{dy}{dx} - 1}{2x}$$

or, $$P = \frac{p-1}{2x}$$

or, $$p = 2xP + 1$$

Where,

$$p = \frac{dy}{dx} \text{ and } P = \frac{dv}{du}$$

Therefore, the given differential equation becomes

$$x\{2xP + 1\}^2 - 2y\{2xP + 1\} + x + 2y = 0$$

or, $$4x^3P^2 - 4x(y - x)P + 2x = 0$$

or, $$4x^2P^2 - 4(y - x)P + 2 = 0$$

Putting,

$$x^2 = u \text{ and } y - x = v$$

the differential equation reduces to

$$4vP = 4uP^2 + 2$$

or, $$v = uP + \frac{1}{2P}$$

which is a Clairaut's equation.
The general solution is

$$v = uc + \frac{1}{2c}$$

where c is an arbitary constant.
Therefore, the general solution of the differential equation is

$$(y - x) = x^2c + \frac{1}{2c}$$

Short and Long Answer Type Questions

Solve the following differential equations.

1. $p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$

   [**Ans:** $(2y - x^2 - c_1)(y - c_2e^x)(y + x - 1 - ce^{-x}) = 0$]

2. $x^2p^2 + xyp - 6y^2 = 0$

   [**Ans:** $(x^3y - c_1)(y - c_2x^2) = 0$]

3. $p(p + y) = x(x + y)$

   [**Ans:** $(y - \frac{1}{2}x^2 + c_1)(y + x + c_2e^{-x} - 1) = 0$]

4. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

   [**Ans:** $(y + x^2 - c_1)(y - c_2)(xy + c_3y + 1) = 0$]

5. $(x^2 + x)p^2 + (x^2 + x - 2xy - y)p + y^2 - xy = 0$

   [**Ans:** $\{y - c_1(x + 1)\}\{y + x\log(c_2x)\} = 0$]

6. $y = 2px - p^2$

   [**Ans:** $x = \dfrac{c}{p^2} + \dfrac{2}{3}p;\ y = \dfrac{2c}{p} + \dfrac{1}{3}p^2$]

7. $y = p\sin p + \cos p$

   [**Ans:** $x = \sin p + c;\ y = p\sin p + \cos p$]

8. $p\tan p - y + \log\cos p = 0$

   [**Ans:** $x = \tan p + c;\ y = p\tan p + \log\cos p$]

9. $xp^2 - 2yp + ax = 0$

   [**Ans:** $2y = cx^2 + \dfrac{a}{x}$]

10. $p^3 - 4xyp + 8y^2 = 0$

    [**Ans:** $y = c(c - x)^2$]

11. $x = y + a\log p$

    [**Ans:** $x = c + a\log\dfrac{p}{p - 1};\ y = c - a\log(p - 1)$]

12. $6p^2y^2 - y + 3px = 0$

    [**Ans:** $y^3 = 3cx + 6c^2$]

Solve the following differential equations reducing to Clairaut's form.

13. $x^2(y - px) = p^2y$

    [**Ans:** $y = cx^2 + c^2$]

14. $xy(y - px) = x + py$

[**Ans:** $y^2 - 1 = c(x^2 + 1)$]

15. $xyp^2 - (x^2 + y^2 - 1)p - xy = 0$

$$\left[\textbf{Ans: } y^2 = cx^2 + \frac{c}{c-1}\right]$$

16. $x^2 + y^2 - (p + p^{-1})xy = c^2$

$$\left[\textbf{Ans: } y^2 = ax^2 + \frac{c^2 a}{c-1}\right]$$

17. Reduce the differential equation

$$y^2(y - px) = x^4 p^2$$

using the substitution

$$x = \frac{1}{u} \text{ and } y = \frac{1}{v}$$

into Clairaut's equation and solve.

[**Ans:** $c^2xy + cy - x = 0$]

18. Reduce the differential equation

$$(2x^2 + 1)p^2 + (x^2 + y^2 + 2xy + 2)p + 2y^2 + 1 = 0$$

using the substitution

$$x + y = u \text{ and } xy - 1 = v$$

into Clairaut's equation and solve.

[**Ans:** $xy - 1 = c(x + y) + c^2$]

19. Reduce the differential equation

$$(px^2 + y^2)(px + y) = (p + 1)^2$$

using the substitution

$$xy = u \text{ and } y + x = v$$

into Clairaut's equation and solve.

[**Ans:** $cxy - c^2(x + y) + 1 = 0$]

20. Reduce the differential equation

$$xp^2 - 2yp + x + 2y = 0$$

using the substitution

$$x^2 = u \text{ and } y - x = v$$

into Clairaut's equation and solve.

Multiple Choice Questions

1. The general solution of $(xp+3y)(xp-2y)=0$ is

   a) $(\log y+6\log x+c_1)(\log y+2\log x+c_2)=0$

   b) $(x^3y+c_1)\left(\frac{x^2}{y}+c\right)=0$

   c) $(x^3y^2+c_1)(x^2+c_2y^2)=0$

   d) $(x^3y-5)(x^2+cy)=0$

2. The singular solution of $y=px-\frac{1}{4}p^2$ is

   a) $y=x^2$ b) $y=x-\dfrac{1}{4}$ c) $y=0$ d) $y=2x-1$

3. The general solution of $py=p^2(x-b)+a$ is

   a) $y^2=4a(x-b)$ b) $cy=c^2(x-b)+a$

   c) $y=(x-b)+a$ d) none of these

4. The general solution of the differential equation $y=px+f(p)$ is

   a) $y=c^2x+f(c)$ b) $y=cx+f(c^2)$

   c) $y=cx+f(c)$ d) none of these

5. The general solution of $p=\log(px-y)$ is

   a) $y=cx-c$ b) $y=cx-e^c$

   c) $y=c^2x-e^{-c}$ d) none of these

Answers:

1 (b) 2 (a) 3 (b) 4 (c) 5 (b)

CHAPTER

4

Ordinary Differential Equations of Higher Order and First Degree

4.1 INTRODUCTION

The importance and methods of solution of first order differential equations have been discussed in the previous chapters.

Different applications of engineering and sciences encounter linear differential equations of higher order with constant coefficients and coefficients as functions of x. In Section 4.2 and Section 4.3 we discuss the methods of solution of linear differential equations with constant coefficients. In Section 4.4 we discuss the methods of solution of Cauchy–Euler equations and Cauchy–Legendres equations.

In Section 4.5 we discuss the methods of solution of simultaneous differential equations.

4.2 GENERAL LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

4.2.1 Definition

A differential equation of the form

$$\boxed{a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = X}$$

where $a_0(\neq 0), a_1, ..a_n$ are constants and X is a function of x which is a linear differential equation of n^{th} order with constant coefficients.

The differential equation can be written as

$$\frac{d^n y}{dx^n} + \frac{a_1}{a_0}\frac{d^{n-1}y}{dx^{n-1}} + \frac{a_2}{a_0}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + \frac{a_{n-1}}{a_0}\frac{dy}{dx} + \frac{a_n}{a_0}y = \frac{X}{a_0}$$

or,

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1}\frac{dy}{dx} + p_n y = F(x)$$

where

$$p_1 = \frac{a_1}{a_0}, p_2 = \frac{a_2}{a_0}, \ldots p_{n-1} = \frac{a_{n-1}}{a_0}, p_n = \frac{a_n}{a_0}, a_0(\neq 0)$$

4.2.2 Method of Solution

Let us consider the differential equation

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1}\frac{dy}{dx} + p_n y = F(x)$$

where $p_1, p_2, \ldots, p_n$ are constants and $F(x)$ is a function of x.

The general solution of the equations is composed of two parts- **Complementary Function(CF) and Particular Integral(PI).**

Therefore,

$$\boxed{y = \text{CF} + \text{PI} = y_c + y_p}$$

where, y_c is called **complementary function(CF)** and y_p is called **particular integral(PI).**

Method of Finding Complementary Functions The solution of the equation **(called reduced equation)**

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1}\frac{dy}{dx} + p_n y = 0$$

is called the **complementary function (CF)** of the differential equation.
Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation then

$$(m^n + p_1 m^{n-1} + p_2 m^{n-2} + \cdots + p_{n-1}m + p_n)e^{mx} = 0$$

Since, $e^{mx} \neq 0$, we have the **auxilliary equation(AE)**

$$\boxed{(m^n + p_1 m^{n-1} + p_2 m^{n-2} + \cdots + p_{n-1}m + p_n) = 0}$$

which is an algebraic equation of degree n and has n roots.

Case 1: **When all the roots of the auxilliary equation (AE) are real and distinct.**
Let the roots of the auxilliary equation be real and distinct, say, $m_1, m_2, \ldots m_n$, then the complementary function is

$$\boxed{\text{CF} = y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}}$$

where, $C_1, C_2, \ldots C_n$ are arbitary constants.

Case 2: **When all the roots of the auxilliary equation (AE) are real but all of them are not distinct.**
Let among the *n* real roots *p* roots are repeated, say, m (p-times), $m_{p+1}, \ldots m_n$ then the complementary function is

$$\boxed{\text{CF} = y_c = (C_1 + C_2 x + \cdots + C_p x^{p-1})e^{mx} + C_{p+1}e^{m_{p+1}x} + \cdots + C_n e^{m_n x}}$$

where $C_1, C_2, \ldots C_n$ are arbitary constants.

Case 3: **When one pair of roots of the auxilliary equation(AE) are complex.**
Let the two roots be complex of the form $\alpha \pm i\beta$ among the n roots. Then the complementary function is

$$\boxed{\text{CF} = y_c = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}}$$

where $C_1, C_2, \ldots C_n$ are arbitary constants.

Method of finding Particular Integrals Different methods are used to find particular integrals are,

1) ***D*-operator method**
2) **Variation of parameters method**
3) **Method of undetermined coefficients.**

We will discuss these methods in detail in the following sections for differential equations of second order.

4.3 GENERAL SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

4.3.1 Definition

An ordinary differential equation of the form

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)}$$

where P and Q are constants and $F(x)$ is a function of x, which is known as second order linear ordinary differential equations with constant coefficients.

4.3.2 Method of Solution

The general solution of the equations is composed of two parts - **complementary function (CF) and particular integral (PI).**
Therefore,

$$y = \text{CF} + \text{PI} = y_c + y_p$$

where, y_c is called **complementary function(CF) and y_p is called particular integral(PI).**

4.3.3 Method of Finding Complementary Functions

Let us consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)$$

where, P and Q are constants and $F(x)$ is a function of x.
The reduced equation is

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0}$$

and the complimentary function is the solution of the reduced equation.
Let,

$$y = e^{mx}$$

be a trial solution.
Then the **auxilliary equation** is

$$\boxed{m^2 + Pm + Q = 0}$$

Case 1: **When the two roots of auxilliary equation is real and distinct say m_1 and m_2, then**

$$\boxed{\text{CF} = y_c = C_1e^{m_1x} + C_2e^{m_2x}}$$

where C_1 and C_2 are arbitary constants.

Example 1 **Solve**

$$\frac{d^2y}{dx^2} - 4y = 0$$

Sol. Let,

$$y = e^{mx}$$

be a trial solution of the equation

$$\frac{d^2y}{dx^2} - 4y = 0$$

Then, the auxilliary equation is

$$m^2 - 4 = 0$$

or, $$(m+2)(m-2) = 0$$

or, $$m = 2, -2$$

Since, the roots of the auxilliary equation are real and dictinct, the solution is

$$y = C_1 e^{2x} + C_2 e^{-2x}$$

where C_1 and C_2 are arbitary constants.

Case 2: **When the two roots of auxilliary equation are real and equal, i.e.** $m_1 = m_2 = m$ **(say), then**

$$\boxed{\text{CF} = y_c = (C_1 + C_2 x)e^{mx}}$$

where C_1 **and** C_2 **are arbitary constants.**

Example 2 **Solve**

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Sol. Let,

$$y = e^{mx}$$

be a trial solution of the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Then, the auxilliary equation is

$$m^2 - 4m + 4 = 0$$

or, $$(m-2)^2 = 0$$

or, $$m = 2, 2$$

Since the roots of the auxilliary equation are real and equal, the solution is

$$y = (C_1 + C_2 x)e^{2x}$$

where C_1 and C_2 are arbitary constants.

Case 3: **When the two roots are complex, say $m = \alpha \pm i\beta$, then**

$$\boxed{\text{CF} = y_c = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)}$$

where C_1 and C_2 are arbitary constants.

Example 3 **Solve**

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = 0$$

Sol. The differential equation can be written as

$$(D^2 + 3D + 5)y = 0$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the equation

$$(D^2 + 3D + 5)y = 0$$

Then, the auxilliary equation is

$$m^2 + 3m + 5 = 0$$

or,

$$m = \frac{-3 \pm \sqrt{11}i}{2} = \frac{-3}{2} \pm i\frac{\sqrt{11}}{2}$$

Since, the roots of the auxilliary equation are complex, the solution is

$$y = e^{\frac{-3}{2}x}\left(C_1 \cos\left(\frac{\sqrt{11}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}x\right)\right)$$

where C_1 and C_2 are arbitary constants.

4.3.4 *D*-Operator Method of Finding Particular Integrals (PI)

Let us consider

$$\boxed{D = \frac{d}{dx}}$$

which is called **differential operator or D-operator.**

Properties of D-operator

1. $D(f(x) + g(x)) = Df(x) + Dg(x)$
2. $D(af(x)) = aD(f(x))$
3. $D^m D^n f(x) = D^{m+n}(f(x))$
4. $(D - \alpha)(D - \beta)f(x) = (D - \beta)(D - \alpha)f(x) = [D^2 - (\alpha + \beta)D + \alpha\beta]f(x)$

Let us consider the differential equation,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)$$

where, P and Q are constants and $F(x)$ is a function of x.
Then, using D-operator the differential equation can be written as

$$D^2y + PDy + Qy = F(x)$$

or,

$$\boxed{(D^2 + PD + Q)y = F(x)}$$

Let,

$$f(D) = (D^2 + PD + Q)$$

Then, the equation can be written as

$$f(D)y = F(x)$$

Now,

$$y = \frac{1}{f(D)}F(x)$$

is a function of x not involving any arbitary constants, but the value of y satisfies the differential equation.
Thus,

$$\boxed{\textbf{PI} = y_p = \frac{1}{f(D)}F(x)}$$

which is called the **Particular Integral (PI)** of the differential equation.
Here, we discuss different cases related to particular integral using D-operator.

Case 1: **When $f(D) = D$, then**

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{D}F(x) = \int F(x)dx}$$

Case 2: **When $f(D) = D - m$, where m is constant, then**

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{D-m}F(x) = e^{mx}\int e^{-mx}F(x)dx}$$

Case 3: **When $f(D) = (D - m_1)(D - m_2)$, where m_1, m_2 are constant, then $\frac{1}{f(D)}$ can be expressed as (using partial fraction)**

$$\frac{1}{f(D)} = \frac{a_1}{(D-m_1)} + \frac{a_2}{(D-m_2)}$$

and accordingly

$$\boxed{\begin{aligned}\text{PI} = y_p &= \frac{1}{f(D)}F(x) = \frac{a_1}{(D-m_1)}F(x) + \frac{a_2}{(D-m_2)}F(x)\\ &= a_1e^{m_1x}\int e^{-m_1x}F(x)dx + a_2e^{m_2x}\int e^{-m_2x}F(x)dx\end{aligned}}$$

Case 4: **When $f(D) = (D - m)^2$, where m is a constant, then $\frac{1}{f(D)}$ can be expressed as (using partial fraction)**

$$\frac{1}{f(D)} = \frac{a_1}{(D-m)} + \frac{a_2}{(D-m)^2}$$

and accordingly

$$\begin{aligned}\text{PI} = y_p &= \frac{1}{f(D)}F(x) = \frac{a_1}{(D-m)}F(x) + \frac{a_2}{(D-m)^2}F(x)\\ &= \frac{a_1}{(D-m)}F(x) + \frac{a_2}{(D-m)}\left\{\frac{1}{(D-m)}F(x)\right\}\\ &= a_1e^{mx}\int e^{-mx}F(x)dx + a_2\frac{1}{(D-m)}\left\{e^{mx}\int e^{-mx}F(x)dx\right\}\\ &= a_1e^{mx}\int e^{-mx}F(x)dx + a_2e^{mx}\left\{\int e^{-mx}\left\{e^{mx}\int e^{-mx}F(x)dx\right\}dx\right\}\\ &= a_1e^{mx}\int e^{-mx}F(x)dx + a_2e^{mx}\left\{\int\left\{\int e^{-mx}F(x)dx\right\}dx\right\}\end{aligned}$$

Alternative Method **Let,**

$$f(D) = (D - m_1)(D - m_2),$$

where m_1, m_2 are constants,

then

$$\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{(D-m_1)(D-m_2)}F(x)$$

Now, consider

$$u = \frac{1}{(D-m_2)}F(x)$$
$$\Rightarrow \frac{du}{dx} - m_2 u = F(x)$$

This is a first order linear differential equation, so by solving this we have u. Therefore, PI becomes

$$\text{PI} = y_p = \frac{1}{(D-m_1)}u$$

Again, choose

$$v = \frac{1}{(D-m_1)}u$$
$$\Rightarrow \frac{dv}{dx} - m_1 v = u$$

This is a first order linear differential equation, so by solving this we have v. Hence, PI is given by

$$\text{PI} = y_p = v.$$

Notes: (i) This method is applicable to all the cases stated above.

(ii) See Worked Out Examples 4.1, 4.2 and 4.3.

4.3.5 Shortcut Methods of Finding Particular Integrals in Some Special Cases Using *D*-Operator Method

Method 1 **When**

$$\boxed{F(x) = e^{mx}}$$

where m is constant.

a) **When**

$$f(m) \neq 0$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{f(D)}e^{mx} = \frac{1}{f(m)}e^{mx}; \quad f(m) \neq 0}$$

Example 4 **Solve**

$$\frac{d^2y}{dx^2} - 5y = e^x$$

Sol. The differential equation can be written as

$$(D^2 - 5)y = e^x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 5)y = 0$$

Then, the auxilliary equation is

$$m^2 - 5 = 0$$

or,

$$m = \pm\sqrt{5}$$

Therefore, the complementary function is

$$\text{CF} = y_c = C_1 e^{-\sqrt{5}x} + C_2 e^{\sqrt{5}x}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned} \text{PI} = y_p &= \frac{1}{(D^2 - 5)} e^x \\ &= \frac{1}{(1^2 - 5)} e^x \\ &= -\frac{e^x}{4} \end{aligned}$$

Therefore, the general solution is

$$y = Y_c + y_p = C_1 e^{-\sqrt{5}x} + C_2 e^{\sqrt{5}x} - \frac{e^x}{4}$$

b) **When**

$$f(m) = 0$$

$$\mathbf{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{f(D)}e^{mx} = x\frac{1}{f'(D)}e^{mx} = x\frac{1}{f'(m)}e^{mx}, f'(m) \neq 0$$

Example 5 **Solve**

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$$

Sol. The differential equation can be written as

$$(D^2 - 3D + 2)y = e^x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 3D + 2)y = 0$$

Then, the auxilliary equation is

$$m^2 - 3m + 2 = 0$$

or,

$$(m-1)(m-2) = 0$$

or,

$$m = 1, m = 2$$

Therefore, the complementary function is

$$\text{CF} = y_c = C_1e^x + C_2e^{2x}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned}\text{PI} = y_p &= \frac{1}{(D^2 - 3D + 2)}e^x \\ &= x\frac{1}{2D-3}e^x \\ &= -xe^x\end{aligned}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1e^x + C_2e^{2x} - xe^x$$

c) **When,**

$$f(m) = 0, f'(m) = 0$$

$$\boxed{\begin{aligned}\text{PI} = y_p &= \frac{1}{f(D)}F(x) = \frac{1}{f(D)}e^{mx} = x^2\frac{1}{f''(D)}e^{mx} \\ &= x^2\frac{1}{f''(m)}e^{mx};\ f''(m) \neq 0\end{aligned}}$$

Method 2 **Let**

$$\boxed{F(x) = \sin ax \textbf{ or } \cos ax}$$

a) **When**

$$\boxed{f(D) = \phi(D^2)}$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax;\ \phi(-a^2) \neq 0}$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{\phi(D^2)}\cos ax = \frac{1}{\phi(-a^2)}\cos ax;\ \phi(-a^2) \neq 0}$$

Example 6 **Solve**

$$\frac{d^2y}{dx^2} - 5y = 3\sin 2x + 5\cos 3x$$

Sol. The differential equation can be written as

$$(D^2 - 5)y = 3\sin 2x + 5\cos 3x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 5)y = 0$$

Then, the auxilliary equation is

$$m^2 - 5 = 0$$

or,

$$m = \pm\sqrt{5}$$

Therefore, the complementary function is

$$\text{CF} = y_c = C_1 e^{-\sqrt{5}x} + C_2 e^{\sqrt{5}x}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned}
\text{PI} = y_p &= \frac{1}{(D^2 - 5)}(3\sin 2x + 5\cos 3x) \\
&= 3\frac{1}{(D^2 - 5)}\sin 2x + 5\frac{1}{(D^2 - 5)}\cos 3x \\
&= 3\frac{1}{(-2^2 - 5)}\sin 2x + 5\frac{1}{(-3^2 - 5)}\cos 3x \\
&= \frac{\sin 2x}{-3} - \frac{5}{11}\cos 3x
\end{aligned}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1 e^{-\sqrt{5}x} + C_2 e^{\sqrt{5}x} - \frac{\sin 2x}{3} - \frac{5}{11}\cos 3x$$

b) **When**

$$\boxed{f(D) = \phi(D^2, D)}$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{\phi(D^2, D)}\sin ax = \frac{1}{\phi(-a^2, D)}\sin ax;\ \phi(-a^2, D) \neq 0}$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{\phi(D^2, D)}\cos ax = \frac{1}{\phi(-a^2, D)}\cos ax;\ \phi(-a^2, D) \neq 0}$$

Example 7 **Solve**

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = 2\cos 2x + 7\sin x$$

Sol. The differential equation can be written as

$$(D^2 + 3D + 5)y = 2\cos 2x + 7\sin x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + 3D + 5)y = 0$$

Then, the auxilliary equation is

$$m^2 + 3m + 5 = 0$$

or, $$m = \frac{-3 \pm \sqrt{11}i}{2}$$

Therefore, the complementary function is

$$\text{CF} = y_c = e^{-\frac{3}{2}x}\left(C_1 \cos\frac{\sqrt{11}}{2}x + C_2 \sin\frac{\sqrt{11}}{2}x\right)$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned}
\text{PI} = y_p &= \frac{1}{(D^2 + 3D + 5)}(2\cos 2x + 7\sin x)\\
&= 2\frac{1}{(D^2 + 3D + 5)}\cos 2x + 7\frac{1}{(D^2 + 3D + 5)}\sin x\\
&= 2\frac{1}{(-2^2 + 3D + 5)}\cos 2x + 7\frac{1}{(-1^2 + 3D + 5)}\sin x\\
&= \frac{2}{(3D + 1)}\cos 2x + \frac{7}{(3D + 4)}\sin x\\
&= \frac{2(3D - 1)}{(3D + 1)(3D - 1)}\cos 2x + \frac{7(3D - 4)}{(3D + 4)(3D - 4)}\sin x\\
&= \frac{2(3D - 1)}{(9D^2 - 1)}\cos 2x + \frac{7(3D - 4)}{(9D^2 - 16)}\sin x\\
&= \frac{2(3D - 1)}{(9(-2^2) - 1)}\cos 2x + \frac{7(3D - 4)}{(9(-1^2) - 16)}\sin x\\
&= \frac{2(3D - 1)}{-37}\cos 2x + \frac{7(3D - 4)}{-25}\sin x\\
&= \frac{12}{37}\sin 2x + \frac{2}{37}\cos 2x - \frac{21}{25}\cos x + \frac{28}{25}\sin x
\end{aligned}$$

Therefore, the general solution is

$$y = y_c + y_p = e^{-\frac{3}{2}x}\left(C_1 \cos\frac{\sqrt{11}}{2}x + C_2 \sin\frac{\sqrt{11}}{2}x\right) + \frac{12}{37}\sin 2x + \frac{2}{37}\cos 2x - \frac{21}{25}\cos x + \frac{28}{25}\sin x$$

c) **When**

$$\boxed{\frac{1}{f(D)} = \frac{\psi(D)}{\phi(D^2)}}$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{\psi(D)}{\phi(D^2)}\sin ax = \frac{\psi(D)}{\phi(-a^2)}\sin ax}$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{\psi(D)}{\phi(D^2)}\cos ax = \frac{\psi(D)}{\phi(-a^2)}\cos ax}$$

d) **When**

$$\boxed{f(D) = \phi(D^2) and \phi(-a^2) = 0}$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{f(D)}\sin ax = x\frac{1}{f'(D)}\sin ax}$$

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{f(D)}\cos ax = x\frac{1}{f'(D)}\cos ax}$$

Example 8 **Solve**

$$\frac{d^2y}{dx^2} + 4y = \sin 2x + 9\cos 2x$$

Sol. The differential equation can be written as

$$(D^2 + 4)y = \sin 2x + 9\cos 2x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2+4)y = 0$$

Then, the auxilliary equation is

$$m^2 + 4 = 0$$

or, $$m = \pm 2i$$

Therefore, the complementary function is

$$\text{CF} = y_c = C_1 \cos 2x + C_2 \sin 2x$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned}
\text{PI} = y_p &= \frac{1}{(D^2+4)}(\sin 2x + 9\cos 2x) \\
&= \frac{1}{(D^2+4)}\sin 2x + 9\frac{1}{(D^2+4)}\cos 2x \\
&= x\frac{1}{2D}\sin 2x + 9x\frac{1}{2D}\cos 2x \text{ since } (-2^2+4) = 0 \\
&= \frac{x}{2}\int \sin 2x dx + \frac{9x}{2}\int \cos 2x dx \\
&= \frac{-x\cos 2x}{4} + \frac{9x\sin 2x}{4}
\end{aligned}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1\cos 2x + C_2 \sin 2x - \frac{x\cos 2x}{4} + \frac{9x \sin 2x}{4}$$

Method 3 **Let,**

$$\boxed{F(x) = P_n(x)}$$

where $P_n(x)$ is a polynomial of degree n.

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{f(D)}P_n(x) = [f(D)]^{-1}P_n(x)}$$

where $[f(D)]^{-1}$ is expanded in a binomial expansion in ascending powers of D.

Example 9 **Solve**

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2 + x$$

Sol. The differential equation can be written as

$$(D^2 + D + 1)y = x^2 + x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + D + 1)y = 0$$

Then, the auxiliary equation is

$$m^2 + m + 1 = 0$$

or,

$$m = \frac{-1 \pm \sqrt{3}i}{2}$$

Therefore, the complementary function is

$$\text{CF} = y_c = e^{\frac{-1}{2}x}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right)$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned}
\text{PI} = y_p &= \frac{1}{(D^2 + D + 1)}(x^2 + x) \\
&= \{1 + (D^2 + D)\}^{-1}(x^2 + x) \\
&= \left\{1 - (D^2 + D) + \frac{(D^2 + D)^2}{2!} - \cdots\right\}(x^2 + x) \\
&= \left\{1 - D - \frac{D^2}{2}\right\}(x^2 + x) \\
&= x^2 + x - 2x - 1 - 1 \\
&= x^2 - x - 2
\end{aligned}$$

Therefore, the general solution is

$$y = y_c + y_p = e^{\frac{-1}{2}x}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right) + x^2 - x - 2$$

Method 4 **Let,**

$$\boxed{F(x) = e^{mx}V}$$

where V is any function of x.

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{f(D)}e^{mx}V = e^{mx}\frac{1}{f(D+m)}V}$$

Example 10 **Solve**

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2e^{3x}$$ [WBUT-2009, 2010]

Sol. The differential equation can be written as

$$(D^2 - 5D + 6)y = x^2e^{3x}$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 5D + 6)y = 0$$

Then, the auxilliary equation is

$$m^2 - 5m + 6 = 0$$

or, $$(m-2)(m-3) = 0$$

or, $$m = 2, m = 3$$

Therefore,
the complementary function is

$$\text{CF} = y_c = C_1e^{2x} + C_2e^{3x}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\text{PI} = y_p = \frac{1}{(D^2 - 5D + 6)}x^2e^{3x}$$

$$= e^{3x}\frac{1}{(D+3)^2 - 5(D+3) + 6}x^2$$

$$= e^{3x}\frac{1}{D^2+D}x^2$$

$$= e^{3x}\frac{1}{D}\frac{1}{(D+1)}x^2$$

$$= e^{3x}\frac{1}{D}(1+D)^{-1}x^2$$

$$= e^{3x}\frac{1}{D}\{1-D+D^2-\cdots\}x^2$$

$$= e^{3x}\frac{1}{D}\{x^2-2x+2\}$$

$$= e^{3x}\{\frac{x^3}{3}-x^2+2x\}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1e^{2x} + C_2e^{3x} + e^{3x}\left\{\frac{x^3}{3}-x^2+2x\right\}$$

Method 5 **Let,**

$$\boxed{F(x) = xV}$$

where V is any function of x.

$$\boxed{\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{f(D)}xV = \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V}$$

Example 11 **Solve**

$$(D^2+1)y = x\cos x$$

Sol. Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2+1)y = 0$$

Then,
the auxilliary equation is

$$m^2+1 = 0$$

or,

$$m = \pm i$$

Therefore, the complementary function is

$$\text{CF} = y_c = C_1 \cos x + C_2 \sin x$$

where $C_1$and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned}
\text{PI} &= \frac{1}{(D^2+1)} x \cos x \\
&= \left\{ x - \frac{1}{(D^2+1)} 2D \right\} \frac{1}{(D^2+1)} \cos x \\
&= \left\{ x - \frac{1}{(D^2+1)} 2D \right\} \frac{x \sin x}{2}, \\
&\quad \text{since } \frac{1}{(D^2+1)} \cos x = x \frac{1}{2D} \cos x = \frac{x \sin x}{2} \\
&= \left\{ \frac{x^2 \sin x}{2} - \frac{1}{(D^2+1)} D(x \sin x) \right\} \\
&= \frac{x^2 \sin x}{2} - \frac{1}{(D^2+1)} (\sin x + x \cos x) \\
&= \frac{x^2 \sin x}{2} - \frac{1}{(D^2+1)} \sin x - \frac{1}{(D^2+1)} x \cos x \\
&= \frac{x^2 \sin x}{2} - \frac{1}{(D^2+1)} \sin x - \text{PI} \\
&= \frac{x^2 \sin x}{2} + \frac{x \cos x}{2} - \text{PI}, \\
&\quad \text{since } \frac{1}{(D^2+1)} \sin x = x \frac{1}{2D} \sin x = \frac{-x \cos x}{2}
\end{aligned}$$

$$or, \quad 2\,(\text{PI}) = \frac{x^2 \sin x}{2} + \frac{x \cos x}{2}$$

$$or, \quad \text{PI} = \frac{x^2 \sin x}{4} + \frac{x \cos x}{4}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x + \frac{x^2 \sin x}{4} + \frac{x \cos x}{4}$$

Method 6 **Let,**

$$\boxed{F(x) = x^n V}$$

where V is any function of x.

$$\text{PI} = y_p = \frac{1}{f(D)}F(x) = \frac{1}{f(D)}xV = \left\{x - \frac{1}{f(D)}f'(D)\right\}^n \frac{1}{f(D)}V$$

Example 12 **Solve**

$$(D^2 - 1)y = x^2 \sin x$$

Sol. Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 1)y = 0$$

Then, the auxilliary equation is

$$m^2 - 1 = 0$$

or,

$$m = \pm 1$$

Therefore, the complementary function is

$$\text{CF} = y_c = C_1 e^x + C_2 e^{-x}$$

where $C_1$and C_2 are arbitary constants.
The particular integral is

$$\text{PI} = y_p = \frac{1}{(D^2 - 1)} x^2 \sin x$$

$$= \left\{x - \frac{2D}{(D^2-1)}\right\}^2 \frac{1}{(D^2-1)} \sin x$$

$$= \left\{x - \frac{2D}{(D^2-1)}\right\}\left\{x - \frac{2D}{(D^2-1)}\right\}\left(-\frac{1}{2}\sin x\right)$$

$$= \left\{x - \frac{2D}{(D^2-1)}\right\}\left\{\frac{-x}{2}\sin x + \frac{1}{(D^2-1)}\cos x\right\}$$

$$= \left\{x - \frac{2D}{(D^2-1)}\right\}\left\{\frac{-x}{2}\sin x - \frac{1}{2}\cos x\right\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{(D^2-1)}\{D(x\sin x + \cos x)\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{(D^2-1)}\{\sin x + x\cos x - \sin x\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{(D^2-1)}x\cos x$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \left\{x - \frac{2D}{(D^2-1)}\right\}\frac{1}{(D^2-1)}\cos x$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \left\{x - \frac{2D}{(D^2-1)}\right\}\left(\frac{-1}{2}\cos x\right)$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x - \frac{x\cos x}{2} - \frac{1}{(D^2-1)}\sin x$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x - \frac{x\cos x}{2} + \frac{\sin x}{2}$$

Therefore, the general solution is

$$y = y_c + y_p$$
$$= C_1e^x + C_2e^{-x} - \frac{x^2}{2}\sin x - \frac{x}{2}\cos x - \frac{x\cos x}{2} + \frac{\sin x}{2}$$

4.3.6 Method of Variation of Parameters

Let us consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)$$

where, P and Q are constants and $F(x)$ is a function of x.
The reduced equation is

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0}$$

and the complimentary function is the solution of the reduced equation.
Let the **Complementary function (CF)** be

$$\boxed{\text{CF} = y_c = C_1y_1 + C_2y_2}$$

where y_1 and y_2 are linearly independent solutions of the reduced equation and C_1 and C_2 are arbitary constants.
The solution of the differential equation is

$$y = y_c + y_p$$

Let us consider the Particular integral as

$$\text{PI} = y_p = C_1(x)y_1 + C_2(x)y_2 \tag{1}$$

where $C_1(x)$ and $C_2(x)$ are functions of x.

Note: Here C_1 and C_2 are arbitary constants and they are replaced by two functions $C_1(x)$ and $C_2(x)$ respectively. Here we can consider any two arbitrary functions, such as $u(x)$ and $v(x)$ instead of $C_1(x)$ and $C_2(x)$. So there should not be any confusion in the readers mind for the selection of the functions. We can choose any two functions arbitrarily.

Variation of Parameters is the method of finding the functions $C_1(x)$ and $C_2(x)$. Differentiating (1) w.r.t x, we get

$$y_p' = C_1'(x)y_1 + C_2'(x)y_2 + C_1(x)y_1' + C_2(x)y_2'$$

$C_1(x)$ and $C_2(x)$ are so chosen that

$$C_1'(x)y_1 + C_2'(x)y_2 = 0 \tag{2}$$

Therefore,

$$y_p' = C_1(x)y_1' + C_2(x)y_2' \tag{3}$$

Differentiating (3) w.r.t x, we have

$$y_p'' = C_1'(x)y_1' + C_2'(x)y_2' + C_1(x)y_1'' + C_2(x)y_2'' \tag{4}$$

Substituting the values of y, y' and y'' in (1), we have

$$C_1'(x)y_1' + C_2'(x)y_2' = F(x) \tag{5}$$

Solving equations (2) and (5), we have

$$C_1'(x) = -\frac{y_2F(x)}{W} \text{ and } C_2'(x) = \frac{y_1F(x)}{W}$$

where,

$$W\textbf{(called Wronskian)} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

The functions $C_1(x)$ and $C_2(x)$ are given by

$$C_1(x) = -\int \frac{y_2F(x)}{W}dx \text{ and } C_2(x) = \int \frac{y_1F(x)}{W}dx$$

Therefore, the **Particular Integral (PI)** is

$$\text{PI} = y_p = \left\{-\int \frac{y_2F(x)}{W}dx\right\} y_1 + \left\{\int \frac{y_1F(x)}{W}dx\right\} y_2$$

Therefore, the general solution is

$$y = y_c + y_p = C_1y_1 + C_2y_2 + \left\{-\int \frac{y_2F(x)}{W}dx\right\} y_1 + \left\{\int \frac{y_1F(x)}{W}dx\right\} y_2$$

Working Procedure Let us consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)$$

where, P and Q are constants and $F(x)$ is a function of x.

Step 1: Find the complementary function of the given differential equation. Let the **Complementary function (CF)** be

$$\boxed{\text{CF} = y_c = C_1y_1 + C_2y_2}$$

Step 2: Replace the constants of Complementary functions by functions of x and the **Particular Integral (PI)** becomes

$$\boxed{\text{PI} = y_p = C_1(x)y_1 + C_2(x)y_2}$$

Step 3: Calculate the determinant **(called Wronskian)**

$$\boxed{W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0}$$

Step 4: Calculate $C_1(x)$ and $C_2(x)$ by

$$\boxed{C_1(x) = -\int \frac{y_2F(x)}{W}dx \text{ and } C_2(x) = \int \frac{y_1F(x)}{W}dx}$$

Step 5: The Particular Integral (PI) is

$$\boxed{\text{PI} = y_p = \left\{-\int \frac{y_2F(x)}{W}dx\right\} y_1 + \left\{\int \frac{y_1F(x)}{W}dx\right\} y_2}$$

and the general solution is

$$\boxed{y = y_c + y_p = C_1y_1 + C_2y_2 + \left\{-\int \frac{y_2F(x)}{W}dx\right\} y_1 + \left\{\int \frac{y_1F(x)}{W}dx\right\} y_2}$$

Example 13 **Solve by the variation of parameters**

$$\frac{d^2y}{dx^2} + 9y = \sec 3x$$ [WBUT-2005, 2008, 2009]

Sol. The reduced equation is

$$\frac{d^2y}{dx^2} + 9y = 0$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation, then the auxilliary equation is

$$m^2 + 9 = 0$$

or, $$(m+3i)(m-3i) = 0$$

or, $$m = \pm 3i$$

The complementary function is

$$\text{CF} = y_c = C_1 y_1 + C_2 y_2 = (C_1 \cos 3x + C_2 \sin 3x)$$

where C_1 and C_2 are arbitary constants.
Let us consider the particular integral as

$$\text{PI} = y_p = C_1(x)\cos 3x + C_2(x)\sin 3x$$

where $C_1(x)$ and $C_2(x)$ are functions of x.

Variation of Parameters method is the method of finding the functions $C_1(x)$ and $C_2(x)$.

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3 \neq 0$$

Therefore,

$$C_1(x) = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{\sin 3x \sec 3x}{3} dx = \frac{1}{9}\log\cos 3x$$

and

$$C_2(x) = \int \frac{y_1 F(x)}{W} dx = \int \frac{\cos 3x \sec 3x}{3} dx = \frac{1}{3}x$$

Therefore, the particular integral is

$$\begin{aligned} \text{PI} = y_p &= C_1(x) y_1 + C_2(x) y_2 \\ &= \left(\frac{1}{9}\log\cos 3x\right)\sin 3x + \left(\frac{1}{3}x\right)\cos 3x \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y = y_c + y_p &= (C_1 \cos 3x + C_2 \sin 3x) \\ &\quad + \left(\frac{1}{9}\log\cos 3x\right)\sin 3x + \left(\frac{1}{3}x\right)\cos 3x \end{aligned}$$

The following method is for further reading and not included in the syllabus

4.3.7 Method of Undetermined Coefficients for Finding Particular Integrals (PI)

Let us consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)$$

where, P and Q are constants and $F(x)$ is a function of x.
The reduced equation is

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0}$$

and the complimentary function is the solution of the reduced equation.
Let the **complementary function (CF)** be

$$\boxed{\text{CF} = y_c = C_1y_1 + C_2y_2}$$

where y_1 andy_2 are linearly independent solutions of the reduced equation and C_1 and C_2 are arbitary constants.
The solution of the differential equation is

$$y = \text{CF} + \text{PI} = y_c + y_p$$

Method of undetermined coefficients give the value of particular integral using $F(x)$ of the differential equation.

Case 1: When $F(x) = p_n(x)$ a polynomial of degree n.

a) If $P \neq 0, Q \neq 0$, we assume

$$y_p = A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n$$

where $A_0, A_1, \ldots, A_n$, are constants to be determined.

b) If $P \neq 0, Q = 0$, we assume

$$y_p = x(A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)$$

where $A_0, A_1, \ldots, A_n$ are constants to be determined.

c) If $P = 0, Q = 0$, we assume

$$y_p = x^2(A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)$$

where $A_0, A_1, \ldots, A_n$ are constants to be determined.

Case 2: When $F(x) = e^{ax}$

a) If a is not a root of the auxilliary equation $m^2 + Pm + Q = 0$, we assume

$$y_p = Ae^{ax}$$

where A is a constant to be determined.

b) If a is a simple root of the auxilliary equation $m^2 + Pm + Q = 0$, we assume

$$y_p = Axe^{ax}$$

where A is a constant to be determined.

c) If a is a double root of the auxilliary equation $m^2 + Pm + Q = 0$, we assume

$$y_p = Ax^2e^{ax}$$

where A is a constant to be determined.

Case 3: When $F(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$ where C_1 and C_2 are constants

a) If $\cos \alpha x$ or $\sin \alpha x$ are not present in CF, we assume

$$y_p = A \cos \alpha x + B \sin \alpha x$$

where, A and B are constants to be determined.

b) If $\cos \alpha x$ or $\sin \alpha x$ are present in CF, we assume

$$y_p = x(A \cos \alpha x + B \sin \alpha x)$$

where, A and B are constants to be determined.

Example 14 **Solve by the method of undetermined coefficients**

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 24e^{-3x}$$

Sol. Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

then, the Auxilliary equation is

$$m^2 + 6m + 9 = 0$$

or, $$(m+3)^2 = 0$$

or, $$m = -3, -3$$

Therefore, the complementary function is

$$\text{CF} = y_c = (C_1 + C_2 x)e^{-3x}$$

Since, $m = -3$ is a double root of the auxilliary equation, we assume

$$y_p = Ax^2 e^{-3x}$$

where, the constant A has to be determined by the method of undetermined coefficients.
Therefore,

$$\begin{aligned}
(D^2 + 6D + 9)y_p &= 24e^{-3x} \\
\textit{or,} \quad (D^2 + 6D + 9)Ax^2 e^{-3x} &= 24e^{-3x} \\
\textit{or,} \quad D^2(Ax^2 e^{-3x}) + 6D(Ax^2 e^{-3x}) + 9(Ax^2 e^{-3x}) &= 24e^{-3x} \\
\textit{or,} \quad Ae^{-3x}(2 - 12x + 9x^2) + 6Ae^{-3x}(2x - 3x^2) + 9Ax^2 e^{-3x} &= 24e^{-3x} \\
\textit{or,} \quad 2Ae^{-3x} &= 24e^{-3x} \\
\textit{or,} \quad A &= 12
\end{aligned}$$

Therefore,

$$y_p = 12x^2 e^{-3x}$$

Then, the general solution is

$$y = y_c + y_p = (C_1 + C_2 x)e^{-3x} + 12x^2 e^{-3x}$$

4.4 HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

4.4.1 Cauchy–Euler Differential Equations

Definition The differential equation of the form

$$x^2 \frac{d^2 y}{dx^2} + p_1 x \frac{dy}{dx} + p_2 y = F(x)$$

where, p_1, p_2 are constants and $F(x)$ is a function of x, is a second order homogeneous linear equation.

This differential equation is called Cauchy–Euler differential equation.

Method of Solution The differential equation

$$x^2 \frac{d^2 y}{dx^2} + p_1 x \frac{dy}{dx} + p_2 y = F(x)$$

where, p_1, p_2 are constants and $F(x)$ is a function of x.

$$x^2D^2y + p_1xDy + p_2y = F(x)$$

or,

$$\left(x^2D^2 + p_1xD + p_2\right)y = F(x)$$

where,

$$D = \frac{d}{dx}$$

Let us consider the transformation

$$x = e^z \text{ or } \log x = z$$

Then,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

or,

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx} = Dy \text{ and } \frac{dy}{dz} = D'y$$

where,

$$D = \frac{d}{dx} \text{ and } D' = \frac{d}{dz}$$

Therefore,

$$xDy = D'y$$

Similarly,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)\\ &= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}\\ &= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\end{aligned}$$

or,

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

or,

$$x^2D^2y = D'(D'-1)y$$

Substituting the values of xDy, x^2D^2y we get,

$$D'(D'-1)y + p_1D'y + p_2y = \varphi(z)$$

which is a linear equation with constant coefficients.

The solution is found by the methods of solving linear equation with constant coefficients.

Example 15 **Solve**

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = x\sin(\log x) \qquad \text{[WBUT-2005, 2011]}$$

Sol. The differential equation can be written as

$$x^2D^2y - xDy + 4y = x\sin(\log x)$$

where,

$$D = \frac{d}{dx}$$

Let us consider the transformation

$$x = e^z \text{ or } \log x = z$$

Then,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

or,

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx} = Dy \text{ and } \frac{dy}{dz} = D'y$$

where,

$$D = \frac{d}{dx} \text{ and } D' = \frac{d}{dz}$$

Therefore,

$$xDy = D'y$$

Similarly,

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)\\ &= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}\\ &= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\end{aligned}$$

$$or, \qquad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$or, \qquad x^2D^2y = D'(D'-1)y$$

Substituting the values of xDy, x^2D^2y, we get

$$D'(D'-1)y - D'y + 4y = e^z \sin z$$

$$or, \qquad (D'^2 - 2D' + 4)y = e^z \sin z$$

which is a linear equation.
Let,

$$y = e^{mz}$$

be a trial solution of the reduced equation

$$(D'^2 - 2D' + 4)y = 0$$

then, the auxiliary equation is

$$m^2 - 2m + 4 = 0$$

$$or, \qquad m = 1 \pm \sqrt{3}i$$

Therefore, the complementary function is

$$\text{CF} = y_c = e^z(C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z)$$

where, C_1 and C_2 are arbitary constants.
The particular integral is

$$\text{PI} = y_p = \frac{1}{(D'^2 - 2D' + 4)}e^z \sin z$$

$$= e^z\frac{1}{\{(D'+1)^2 - 2(D'+1) + 4)}\sin z$$

$$= e^z\frac{1}{(D'^2 + 3)}\sin z$$

$$= e^z\frac{1}{(-1^2 + 3)}\sin z = \frac{e^z \sin z}{2}$$

Therefore, the general solution is

$$y = y_c + y_p = e^z(C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z) + \frac{e^z \sin z}{2}$$

Putting $z = \log x$, the general solution becomes

$$y = x\left[C_1 \cos\left(\sqrt{3}\log x\right) + C_2 \sin\left(\sqrt{3}\log x\right)\right] + \frac{x \sin(\log x)}{2}$$

4.4.2 Cauchy–Legendre Differential Equations

Definition **The differential equation of the form**

$$(ax+b)^2\frac{d^2y}{dx^2}+p_1(ax+b)\frac{dy}{dx}+p_2y=F(x)$$

where, p_1, p_2, a, b are constants and $F(x)$ is a function of x which is a homogeneous linear equation of order 2.

This differential equation is called Cauchy–Legendre differential equation.

Method of Solution The differential equation

$$(ax+b)^2\frac{d^2y}{dx^2}+p_1(ax+b)\frac{dy}{dx}+p_2y=F(x)$$

where, p_1, p_2, a, b are constants and $F(x)$ is a function of x can be written as,

$$(ax+b)^2D^2y+p_1(ax+b)Dy+p_2y=F(x)$$

where,

$$D=\frac{d}{dx}$$

Let us consider the transformation

$$(ax+b)=e^z \text{ or } \log(ax+b)=z$$

Then,

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{a}{(ax+b)}\frac{dy}{dz}$$

or,
$$(ax+b)\frac{dy}{dx}=a\frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx}=Dy \text{ and } \frac{dy}{dz}=D'y$$

where,

$$D=\frac{d}{dx} \text{ and } D'=\frac{d}{dz}$$

Therefore,

$$(ax+b)Dy=aD'y$$

Similarly,

$$(ax+b)^2D^2y=a^2D'(D'-1)y$$

Substituting the values of $(ax+b)Dy$, $(ax+b)^2D^2y$, we get

$$a^2D'(D'-1)y+p_1aD'y+p_2y=\varphi(z)$$

which is a linear equation with constant coefficients.

The solution is found by the methods of solving linear equations with constant coefficients.

Example 16 **Solve**

$$(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1$$

Sol. The differential equation can be written as

$$(3x+2)^2D^2y+5(3x+2)Dy-3y=x^2+x+1$$

where,

$$D=\frac{d}{dx}$$

Let us consider the transformation

$$(3x+2)=e^z \text{ or } \log(3x+2)=z$$

Then,

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{3}{(3x+2)}\frac{dy}{dz}$$

or,

$$(3x+2)\frac{dy}{dx}=3\frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx}=Dy \text{ and } \frac{dy}{dz}=D'y$$

where,

$$D=\frac{d}{dx} \text{ and } D'=\frac{d}{dz}$$

Therefore,

$$(3x+2)Dy=3D'y$$

Similarly,

$$(3x+2)^2D^2y=3^2D'(D'-1)y$$

Substituting the values of $(3x+2)Dy$, $(3x+2)^2D^2y$, we get

$$3^2D'(D'-1)y + 15D'y - 3y = \left(\frac{e^z-2}{3}\right)^2 + \left(\frac{e^z-2}{3}\right) + 1$$

or,
$$(9D'^2 + 6D' - 3)y = \frac{e^{2z} - e^z + 7}{9}$$

Let,

$$y = e^{mz}$$

be a trial solution of the reduced equation

$$(9D'^2 + 6D' - 3)y = 0$$

then, the auxiliary equation is

$$9m^2 + 6m - 3 = 0$$

or,
$$m = \frac{1}{3}, -1$$

Therefore, the Complementary function is

$$\text{CF} = y_c = C_1e^{\frac{1}{3}z} + C_2e^{-z}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned}\text{PI} = y_p &= \frac{1}{(9D'^2+6D'-3)}\left(\frac{e^{2z}-e^z+7}{9}\right)\\ &= \frac{1}{9}\left[\frac{e^{2z}}{45} - \frac{e^z}{12} + \left(\frac{-1}{3}\right)(1-2D'-3D'^2)^{-1}(7)\right]\\ &= \frac{1}{9}\left[\frac{e^{2z}}{45} - \frac{e^z}{12} - \frac{7}{3}\right]\end{aligned}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1e^{\frac{1}{3}z} + C_2e^{-z} + \frac{1}{9}\left[\frac{e^{2z}}{45} - \frac{e^z}{12} - \frac{7}{3}\right]$$

Putting

$$\log(3x+2) = z$$

the general solution is,

$$\begin{aligned}y = y_c + y_p &= C_1(3x+2)^{\frac{1}{3}} + C_2(3x+2)^{-1}\\ &\quad + \frac{1}{9}\left[\frac{(3x+2)^2}{45} - \frac{(3x+2)}{12} - \frac{7}{3}\right]\end{aligned}$$

4.5 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

We have discussed how to solve differential equations which consist of two variables only, of which one is independent and the other is a dependent variable. Now, we will discuss how to solve differential equations in which the number of independent variable is one, but the number of dependent variables is more than one.

4.5.1 Definition

If x and y are two dependent variables and t be any independent variable, then simultaneous linear differential equations are of the form

$$\frac{dx}{dt} = F(t, x, y) \text{ and } \frac{dy}{dt} = G(t, x, y)$$

4.5.2 Method of Solution

Let,

$$D = \frac{d}{dt}$$

then, the simultaneous linear differential equation is

$$\frac{dx}{dt} = F(t, x, y) \text{ and } \frac{dy}{dt} = G(t, x, y)$$

can be written as

$$Dx = F(t, x, y) \quad (1)$$

$$Dy = G(t, x, y) \quad (2)$$

Substituting the value of y from equation (1) in equation (2), we get a second order linear differential equation with constant coefficients where the dependent variable is x and the independent variable is t.

Solving the second order differential equation by the known methods we get the value of x.

Putting the value of x in one of the equations gives the value of y.

The method is illustrated in the following examples.

Example 17 **Solve**

$$\frac{dx}{dt} - 7x + y = 0; \frac{dy}{dt} - 2x - 5y = 0$$ [WBUT-2007]

Sol. The given equation can be written as

$$(D - 7)x + y = 0 \quad (1)$$

and

$$-2x + (D-5)y = 0 \tag{2}$$

where,

$$D = \frac{d}{dt}$$

Operating $(D-5)$ on (1) and subtracting by (2), we get

$$[(D-5)(D-7)+2]x = 0$$

or,

$$(D^2 - 12D + 37) = 0$$

Let,

$$y = e^{mt}$$

be a trial solution of the equation, then the auxilliary equation is

$$m^2 - 12m + 37 = 0$$

or,

$$m = 6 \pm i$$

Therefore,

$$x = e^{6t}(C_1 \cos t + C_2 \sin t) \tag{3}$$

where, C_1 and C_2 are arbitary constants.
Differentiating (3) with respect to t, we have

$$\frac{dx}{dt} = 6e^{6t}(C_1 \cos t + C_2 \sin t) + e^{6t}(-C_1 \sin t + C_2 \cos t)$$

From (1) we have,

$$y = 7x - Dx$$

or,

$$y = 7e^{6t}(C_1 \cos t + C_2 \sin t) - 6e^{6t}(C_1 \cos t + C_2 \sin t) - e^{6t}(-C_1 \sin t + C_2 \cos t)$$

or,

$$y = e^{6t}\{(C_1 - C_2)\cos t + (C_1 + C_2)\sin t\}$$

Therefore, the general solution is

$$x = e^{6t}(C_1 \cos t + C_2 \sin t)$$

and

$$y = e^{6t}\{(C_1 - C_2)\cos t + (C_1 + C_2)\sin t\}$$

where C_1 and C_2 are arbitary constants.

Example 18 **Solve**

$$\frac{dx}{dt} + y = e^t; \frac{dy}{dt} - x = e^{-t}$$ [WBUT-2003]

Sol. The simultaneous differential equations are

$$\frac{dx}{dt} + y = e^t \quad (1)$$

and

$$\frac{dy}{dt} - x = e^{-t} \quad (2)$$

From (1) we have,

$$y = e^t - \frac{dx}{dt} \quad (3)$$

Therefore, from (2) and (3) we have

$$\frac{d}{dt}\left(e^t - \frac{dx}{dt}\right) - x = e^{-t}$$

or, $$\frac{d^2x}{dt^2} + x = e^t - e^{-t}$$

Let,

$$x = e^{mt}$$

be a trial solution of the reduced equation

$$\frac{d^2x}{dt^2} + x = 0$$

then, the auxilliary equation is

$$m^2 + 1 = 0$$

or, $$m = \pm i$$

Therefore, the complementary function is

$$\text{CF} = x_c = C_1 \cos t + C_2 \sin t$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\text{PI} = x_p = \frac{1}{(D^2+1)}(e^t - e^{-t})$$

$$= \frac{(e^t - e^{-t})}{2}$$

Therefore,

$$x = x_c + x_p = C_1 \cos t + C_2 \sin t + \frac{(e^t - e^{-t})}{2}$$

Now,

$$\frac{dx}{dt} = -C_1 \sin t + C_2 \cos t + \frac{(e^t + e^{-t})}{2}$$

From (1),

$$y = e^t - \frac{dx}{dt}$$

or,

$$y = e^t - \left\{-C_1 \sin t + C_2 \cos t + \frac{(e^t + e^{-t})}{2}\right\}$$

or,

$$y = C_1 \sin t - C_2 \cos t + \frac{1}{2}(e^t - e^{-t})$$

Therefore, the solution is

$$x = C_1 \cos t + C_2 \sin t + \frac{(e^t - e^{-t})}{2}$$

and

$$y = C_1 \sin t - C_2 \cos t + \frac{1}{2}(e^t - e^{-t})$$

where C_1 and C_2 are arbitary constants.

WORKED OUT EXAMPLES

Example 4.1 Solve

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = x^2$$

Sol. The differential equation can be written as

$$(D^2 - 5D + 6)y = x^2$$

where $D = \frac{d}{dx}$. Let

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 5D + 6)y = 0$$

then the auxilliary equation is

$$(m^2 - 5m + 6) = 0$$

$$or, \qquad (m-3)(m-2) = 0$$

$$or, \qquad m = 3, 2$$

Therefore the complementary function is

$$C.F = y_c = C_1 e^{3x} + C_2 e^{2x}$$

where C_1 and C_2 are arbitary constants.

The particular integral is

$$P.I = y_p = \frac{1}{(D^2 - 5D + 6)} x^2$$

$$= \frac{1}{(D-3)(D-2)} x^2$$

$$= \frac{1}{(D-3)} \left[\frac{1}{(D-2)} x^2 \right]$$

$$= \frac{1}{(D-3)} \left[e^{2x} \int e^{-2x} x^2 dx \right], \text{ using } \textbf{Case 2} \text{ of } \textbf{Section 4.3.4}$$

$$= \frac{1}{(D-3)} \left[e^{2x} \left\{ \frac{x^2 e^{-2x}}{-2} - \int 2x \frac{e^{-2x}}{-2} dx \right\} \right]$$

$$= \frac{1}{(D-3)} \left[e^{2x} \left\{ -\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} \right\} \right]$$

$$= -\frac{1}{2} \frac{1}{(D-3)} \left[x^2 + x + \frac{1}{4} \right]$$

$$= -\frac{1}{2} e^{3x} \int e^{-3x} \left[x^2 + x + \frac{1}{4} \right] dx, \text{ using } \textbf{Case 2} \text{ of } \textbf{Section 4.3.4}$$

$$= \frac{1}{6} \left[x^2 + \frac{5x}{3} + \frac{19}{18} \right].$$

Therefore the general solution is

$$y = Y_c + y_p = C_1 e^{3x} + C_2 e^{2x} + \frac{1}{6} \left[x^2 + \frac{5x}{3} + \frac{19}{18} \right].$$

Alternatively (using **Case 3** of **Section 4.3.4)** we can also compute **P.I.** as the following: The particular integral is

$$P.I = y_p = \frac{1}{(D^2 - 5D + 6)} x^2$$

$$= \frac{1}{(D-3)(D-2)} x^2$$

Using partial fraction we have

$$\frac{1}{(D-3)(D-2)} = \left[\frac{1}{(D-3)} - \frac{1}{(D-2)}\right]$$

So we can write

$$P.I = y_p = \left[\frac{1}{(D-3)} - \frac{1}{(D-2)}\right] x^2$$

$$= \frac{1}{(D-3)} x^2 - \frac{1}{(D-2)} x^2$$

$$= \left[e^{3x} \int e^{-3x} x^2 dx\right] - \left[e^{2x} \int e^{-2x} x^2 dx\right]$$

$$= -\frac{1}{3}\left[x^2 + \frac{2x}{3} + \frac{2}{9}\right] + \frac{1}{2}\left[x^2 + x + \frac{1}{2}\right]$$

$$= \frac{1}{6}\left[x^2 + \frac{5x}{3} + \frac{19}{18}\right].$$

Example 4.2 Solve

$$\frac{d^2y}{dx^2} + 4y = \sec 2x$$

Sol. The differential equation can be written as

$$(D^2 + 4)y = \sec 2x$$

where $D = \frac{d}{dx}$. Let

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + 4)y = 0$$

then the auxilliary equation is

$$(m^2 + 4) = 0 \Rightarrow (m + 2i)(m - 2i) \Rightarrow m = \pm 2i$$

Therefore, the complementary function is

$$C.F = y_c = C_1 \cos 2x + C_2 \sin 2x$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned} P.I = y_p &= \frac{1}{(D^2+4)} \sec 2x \\ &= \frac{1}{(D+2i)(D-2i)} \sec 2x \end{aligned}$$

Using partial fraction, we have

$$\frac{1}{(D+2i)(D-2i)} = \frac{1}{4i}\left[\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right]$$

So we can write

$$\begin{aligned} P.I = y_p &= \frac{1}{4i}\left[\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right] \sec 2x \\ &= \frac{1}{4i}\left[\frac{1}{(D-2i)} \sec 2x - \frac{1}{(D+2i)} \sec 2x\right] \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{(D-2i)} \sec 2x &= e^{2ix} \int e^{-2ix} \sec 2x dx \\ &= e^{2ix} \int \frac{\cos 2x - i \sin 2x}{\cos 2x} dx \\ &= e^{2ix}\left[x + \frac{i}{2} \log(\cos 2x)\right] \\ &= [\cos 2x + i \sin 2x]\left[x + \frac{i}{2} \log(\cos 2x)\right] \\ &= \left[x \cos 2x - \frac{1}{2} \sin 2x \log(\cos 2x)\right] + i\left[x \sin 2x + \frac{1}{2} \cos 2x \log(\cos 2x)\right] \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{1}{(D+2i)} \sec 2x &= e^{-2ix} \int e^{2ix} \sec 2x dx \\ &= \left[x \cos 2x - \frac{1}{2} \sin 2x \log(\cos 2x)\right] - i\left[x \sin 2x + \frac{1}{2} \cos 2x \log(\cos 2x)\right] \end{aligned}$$

Putting the above values we have the particular integral as

$$P.I. = y_p = \frac{1}{4i}\left[\frac{1}{(D-2i)}\sec 2x - \frac{1}{(D+2i)}\sec 2x\right]$$

$$= \frac{x\sin 2x}{2} + \frac{\cos 2x \log(\cos 2x)}{4}$$

Therefore the general solution is

$$y = Y_c + y_p = C_1\cos 2x + C_2\sin 2x + \frac{x\sin 2x}{2} + \frac{\cos 2x\log(\cos 2x)}{4}.$$

Example 4.3 Solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = xe^{-2x}$$

Sol. The differential equation can be written as

$$(D^2 + D - 2)y = xe^{-2x}$$

where $D = \frac{d}{dx}$. Let

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + D - 2)y = 0$$

then the auxilliary equation is

$$(m^2 + m - 2) = 0 \Rightarrow (m-1)(m+2) \Rightarrow m = 1, -2$$

Therefore, the complementary function is

$$C.F = y_c = C_1e^x + C_2e^{-2x}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$P.I = y_p = \frac{1}{(D^2 + D - 2)}xe^{-2x}$$

$$= \frac{1}{(D-1)(D+2)}xe^{-2x}$$

$$= \frac{1}{(D-1)}\left[\frac{1}{(D+2)}xe^{-2x}\right] \quad (1)$$

Now consider

$$u = \frac{1}{(D+2)} x e^{-2x}$$

$$\Rightarrow \frac{du}{dx} + 2u = x e^{-2x}$$

This is a first order linear differential equation, so by solving this we have u as

$$u.\left(e^{2x}\right) = \int x e^{-2x}.e^{2x} dx, \text{ since I.F. is } e^{\int 2dx} = e^{2x}.$$

$$or, \qquad u.\left(e^{2x}\right) = \frac{x^2}{2}$$

$$or, \qquad u = \frac{1}{2} x^2 e^{-2x}$$

Now from Eq. (1), we have the P.I. as

$$P.I = y_p = \frac{1}{(D-1)} u$$

$$= \frac{1}{(D-1)} \left(\frac{1}{2} x^2 e^{-2x}\right) \tag{2}$$

Again consider

$$v = \frac{1}{(D-1)} \left(\frac{1}{2} x^2 e^{-2x}\right)$$

$$\Rightarrow \frac{dv}{dx} - v = \frac{1}{2} x^2 e^{-2x}$$

This is a first order linear differential equation, so by solving this, we have v as

$$v.\left(e^{-x}\right) = \int \frac{1}{2} x^2 e^{-2x}.e^{-x} dx, \text{ since I.F. is } e^{\int (-1)dx} = e^{-x}.$$

$$or, v.\left(e^{-x}\right) = \frac{1}{2} \int x^2 e^{-3x} dx = -\frac{1}{3}\left[x^2 + \frac{2x}{3} + \frac{2}{9}\right] e^{-3x}$$

$$or, \qquad v = -\frac{1}{3}\left[x^2 + \frac{2x}{3} + \frac{2}{9}\right] e^{-2x}$$

Hence from Eq. (2) we have the P.I. as

$$P.I = y_p = v = -\frac{1}{3}\left[x^2 + \frac{2x}{3} + \frac{2}{9}\right] e^{-2x}$$

Therefore, the general solution is

$$y = Y_c + y_p = C_1 e^x + C_2 e^{-2x} - \frac{1}{3}\left[x^2 + \frac{2x}{3} + \frac{2}{9}\right] e^{-2x}$$

$$= C_1 e^x + C_3 e^{-2x} - \frac{1}{3}\left[x^2 + \frac{2x}{3}\right] e^{-2x}, \text{ where } C_3 = C_2 - \frac{2}{27}, \text{ another constant.}$$

Example 4.4 **Solve**

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \cos x \qquad \text{[WBUT-2002, 2011]}$$

Sol. The differential equation can be written as

$$(D^2 - 5D + 6)y = e^x \cos x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 5D + 6)y = 0$$

then, the auxilliary equation is

$$m^2 - 5m + 6 = 0$$

or,
$$(m-2)(m-3) = 0$$

or,
$$m = 2, m = 3$$

Therefore, the complementary function is

$$y_c = C_1 e^{2x} + C_2 e^{3x}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$y_p = \frac{1}{(D^2 - 5D + 6)}(e^x \cos x)$$

$$= e^x \frac{1}{\{(D+1)^2 - 5(D+1) + 6\}} \cos x$$

$$= e^x \frac{1}{(D^2 - 3D + 2)} \cos x$$

$$= e^x \frac{1}{(1-3D)} \cos x$$

$$= e^x \frac{(1+3D)}{(1-9D^2)} \cos x$$

$$= e^x \frac{(1+3D)}{10} \cos x$$

$$= e^x \frac{(\cos x - 3\sin x)}{10}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1 e^{2x} + C_2 e^{3x} + e^x \frac{(\cos x - 3\sin x)}{10}$$

Example 4.5 **Solve**

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$$ [WBUT-2002]

Sol. The differential equation can be written as

$$x^2 D^2 y + xDy + y = \log x \sin(\log x)$$

where,

$$D = \frac{d}{dx}$$

Let us consider the transformation

$$x = e^z \text{ or } \log x = z$$

Then,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

or,

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx} = Dy \text{ and } \frac{dy}{dz} = D'y$$

where,

$$D = \frac{d}{dx} \text{ and } D' = \frac{d}{dz}$$

Therefore,

$$xDy = D'y$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

or, $$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

or, $$x^2D^2y = D'(D'-1)y$$

Substituting the values of xDy, x^2D^2y, we get

$$D'(D'-1)y + D'y + y = z\sin z$$

or, $$(D'^2+1)y = z\sin z$$

which is a linear differential equation with constant coefficients.
Let,

$$y = e^{mz}$$

be a trial solution of the reduced equation

$$(D'^2+1)y = 0$$

then, the auxilliary equation is

$$m^2 + 1 = 0$$

or, $$m = \pm i$$

Therefore, the complementary function is

$$y_c = C_1\cos z + C_2\sin z$$

The particular integral is

$$y_p = \frac{1}{(D'^2+1)}z\sin z$$

$$= \left\{z - \frac{1}{(D'^2+1)}2D'\right\}\frac{1}{(D'^2+1)}\sin z$$

$$= \left\{z - \frac{1}{(D'^2+1)}2D'\right\}z\frac{1}{2D'}\sin z$$

$$= \frac{-1}{2}\left\{z - \frac{1}{(D'^2+1)}2D'\right\}z\cos z$$

$$= \frac{-1}{2}\left\{z^2\cos z - \frac{1}{(D'^2+1)}2D'z\cos z\right\}$$

$$= \frac{-1}{2}\left\{z^2\cos z - \frac{1}{(D'^2+1)}2(\cos z - z\sin z)\right\}$$

$$= \frac{-1}{2}z^2\cos z + \frac{1}{(D'^2+1)}\cos z - \frac{1}{(D'^2+1)}z\sin z$$

$$= \frac{-1}{2}z^2\cos z + z\frac{1}{2D'}\cos z - y_p$$

$$= \frac{-1}{2}z^2\cos z + z\sin z - y_p$$

$$or, \quad 2y_p = \frac{-1}{2}z^2\cos z + z\sin z$$

$$or, \quad y_p = \frac{-z^2\cos z}{4} + \frac{z\sin z}{2}$$

Therefore, the complete solution is

$$y = y_c + y_p = C_1\cos z + C_2\sin z - \frac{z^2\cos z}{4} + \frac{z\sin z}{2}$$

$$= C_1\cos\log x + C_2\sin\log x - \frac{(\log x)^2\cos\log x}{4} + \frac{\log x\sin\log x}{2}$$

Example 4.6 **Solve by the method of variation of parameters**

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x$$ [WBUT-2002]

Sol. The differential equation can be written as

$$(D^2+4)y = 4\tan 2x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2+4)y = 0$$

then, the auxilliary equation is

$$m^2 + 4 = 0$$

$$or, \quad m = \pm 2i$$

Therefore, the complementary function is

$$y_c = C_1 y_1 + C_2 y_2 = C_1 \cos 2x + C_2 \sin 2x$$

where C_1 and C_2 are arbitary constants.
Let the particular integral be

$$y_p = C_1(x) \cos 2x + C_2(x) \sin 2x$$

where $C_1(x)$ and $C_2(x)$ are arbitary constants.
Variation of Parameters method is the method of finding the functions $C_1(x)$ and $C_2(x)$.
Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0$$

Therefore,

$$C_1(x) = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{\sin 2x.4\tan 2x}{2} dx$$
$$= -2\int \sin^2 2x \sec 2x \, dx = \sin 2x - \log(\sec 2x + \tan 2x)$$

and

$$C_2(x) = \int \frac{y_1 F(x)}{W} dx = \int \frac{\cos 2x.4\tan 2x}{2} dx = 2\int \sin 2x dx = -\cos 2x$$

Therefore, the particular integral is

$$y_p = \{\sin 2x - \log(\sec 2x + \tan 2x)\} \cos 2x - (\cos 2x) \sin 2x$$
$$= \{\log(\sec 2x + \tan 2x)\} \cos 2x$$

Thus, the complete solution is

$$y = y_c + y_p = C_1 \cos 2x + C_2 \sin 2x + \{\log(\sec 2x + \tan 2x)\} \cos 2x$$

Example 4.7 **Solve**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x^2 + 3e^{2x} + 4\sin x$$

Sol. The differential equation can be written as

$$(D^2 + 4D + 4)y = x^2 + 3e^{2x} + 4\sin x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + 4D + 4)y = 0$$

then, the auxilliary equation is

$$m^2 + 4m + 4 = 0$$

or, $$(m+2)^2 = 0$$

or, $$m = -2, m = -2$$

Therefore, the complementary function is

$$\text{CF} = y_c = (C_1 + C_2 x)e^{-2x}$$

The particular integral is

$$\text{PI} = y_p = \frac{1}{(D^2+4D+4)}(x^2 + 3e^{2x} + 4\sin x)$$

$$= \frac{1}{(D^2+4D+4)}x^2 + 3\frac{1}{(D^2+4D+4)}e^{2x} + 4\frac{1}{(D^2+4D+4)}\sin x$$

$$= \frac{1}{4}\left(1 + \frac{D^2+4D}{4}\right)^{-1} x^2 + 3\frac{1}{(2^2+8+4)}e^{2x} + 4\frac{1}{(-1^2+4D+4)}\sin x$$

$$= \frac{1}{4}\left\{1 - \frac{D^2+4D}{4} + \left(\frac{D^2+4D}{4}\right)^2 - \dots\right\} x^2 + \frac{3}{16}e^{2x} + 4\frac{1}{(4D+3)}\sin x$$

$$= \frac{1}{4}\left\{1 - D + \frac{15}{4}D^2\right\} x^2 + \frac{3}{16}e^{2x} + 4\frac{(4D-3)}{(4D+3)(4D-3)}\sin x$$

$$= \frac{1}{4}\left\{x^2 - 2x + \frac{15}{2}\right\} + \frac{3}{16}e^{2x} + 4\frac{(4D-3)}{(16D^2-9)}\sin x$$

$$= \frac{1}{4}\left\{x^2 - 2x + \frac{15}{2}\right\} + \frac{3}{16}e^{2x} + 4\frac{(4D-3)}{-25}\sin x$$

$$= \frac{1}{4}\left\{x^2 - 2x + \frac{15}{2}\right\} + \frac{3}{16}e^{2x} + \frac{4}{25}(3\sin x - 4\cos x)$$

Therefore, the general solution is

$$y = y_c + y_p = (C_1 + C_2 x)e^{-2x} + \frac{1}{4}\left\{x^2 - 2x + \frac{15}{2}\right\} + \frac{3}{16}e^{2x} + \frac{4}{25}(3\sin x - 4\cos x)$$

Example 4.8 **Solve by the method of variation of parameters**

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 9e^x$$ [WBUT-2004]

Sol. The differential equation can be written as

$$(D^2 - 3D + 2)y = 9e^x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 3D + 2)y = 0$$

then, the auxilliary equation is

$$m^2 - 3m + 2 = 0$$

or,

$$(m-1)(m-2) = 0$$

or,

$$m = 1, m = 2$$

Therefore, the complementary function is

$$y_c = C_1e^x + C_2e^{2x}$$

Let the particular integral be

$$y_p = C_1(x)e^x + C_2(x)e^{2x}$$

where $C_1(x)$ and $C_2(x)$ are arbitary constants.
Variation of Parameters method is the method of finding the functions $C_1(x)$ and $C_2(x)$.
Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$$

Therefore,

$$C_1(x) = -\int \frac{y_2F(x)}{W}dx = -\int \frac{e^{2x}9e^x}{e^{3x}}dx = -9x$$

and

$$C_2(x) = \int \frac{y_1F(x)}{W}dx = \int \frac{e^x9e^x}{e^{3x}}dx = \int 9e^{-x}dx = -9e^{-x}$$

Therefore, the particular integral is

$$y_p = (-9x)e^x + (-9e^{-x})e^{2x} = -9xe^x - 9e^x$$

Thus, the complete solution is

$$y = y_c + y_p = C_1(x)e^x + C_2(x)e^{2x} - 9xe^x - 9e^x$$

Example 4.9 **Solve**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \sin(\log x) + x\cos(\log x) \qquad \text{[WBUT-2004]}$$

Sol. The differential equation can be written as

$$x^2D^2y + xDy - y = \sin(\log x) + x\cos(\log x)$$

where,

$$D = \frac{d}{dx}$$

Let us consider the transformation

$$x = e^z \text{ or } \log x = z$$

Then,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

or,

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx} = Dy \text{ and } \frac{dy}{dz} = D'y$$

where,

$$D = \frac{d}{dx} \text{ and } D' = \frac{d}{dz}$$

Therefore,

$$xDy = D'y$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

or, $$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

or, $$x^2D^2y = D'(D'-1)y$$

Substituting the values of xDy, x^2D^2y we get

$$D'(D'-1)y + D'y - y = \sin z + e^z \cos z$$

or,
$$(D'^2 - 1)y = \sin z + e^z \cos z$$

which is a linear differential equation with constant coefficients.
Let,

$$y = e^{mz}$$

be a trial solution of the reduced equation

$$(D'^2 - 1)y = 0$$

then, the auxilliary equation is

$$m^2 - 1 = 0$$

or,
$$m = \pm 1$$

Therefore, the complementary function is

$$y_c = C_1 e^z + C_2 e^{-z}$$

The particular integral is

$$\begin{aligned}
y_p &= \frac{1}{(D'^2 - 1)}(\sin z + e^z \cos z) \\
&= \frac{\sin z}{-2} + e^z \frac{1}{(D'+1)^2 - 1} \cos z \\
&= \frac{\sin z}{-2} + e^z \frac{1}{D'^2 + D'} \cos z \\
&= \frac{\sin z}{-2} + e^z \frac{1}{D' - 1} \cos z \\
&= \frac{\sin z}{-2} + e^z \frac{(D'+1)}{(D'^2 - 1)} \cos z \\
&= \frac{\sin z}{-2} + e^z \frac{(D'+1)}{-2} \cos z \\
&= \frac{\sin z}{-2} + e^z \frac{(\cos z - \sin z)}{-2}
\end{aligned}$$

Therefore, the complete solution is

$$\begin{aligned}
y = y_c + y_p &= C_1 e^z + C_2 e^{-z} + \frac{\sin z}{-2} + e^z \frac{(\cos z - \sin z)}{-2} \\
&= C_1 x + \frac{C_2}{x} - \frac{\sin \log x}{2} - \frac{x(\cos \log x - \sin \log x)}{2}
\end{aligned}$$

Example 4.10 **Solve**$(D^2 - 4)y = \cos^2 x$

Sol. Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 4)y = 0$$

then, the auxilliary equation is

$$m^2 - 4 = 0$$

or, $$m = \pm 2$$

Therefore, the complementary function is

$$\text{CF} = y_c = C_1 e^{-2x} + C_2 e^{2x}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\begin{aligned}
\text{PI} = y_p &= \frac{1}{(D^2 - 4)} \cos^2 x \\
&= \frac{1}{(D^2 - 4)} \frac{(1 + \cos 2x)}{2} \\
&= \frac{1}{2} \frac{1}{(D^2 - 4)} 1 + \frac{1}{2} \frac{1}{(D^2 - 4)} \cos 2x \\
&= \frac{-1}{8} \left(1 - \frac{D^2}{4}\right)^{-1} 1 + \frac{1}{2} \frac{1}{(-2^2 - 4)} \cos 2x \\
&= \frac{-1}{8} \left\{1 + \frac{D^2}{4}\right\} 1 - \frac{1}{16} \cos 2x \\
&= \frac{-1}{8} - \frac{1}{16} \cos 2x
\end{aligned}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1 e^{-2x} + C_2 e^{2x} - \frac{1}{8} - \frac{1}{16} \cos 2x$$

Example 4.11 **Solve by the variation of parameters**

$$\frac{d^2 y}{dx^2} + 4y = 4 \sec^2 2x$$ [WBUT-2006]

Sol. The differential equation can be written as

$$(D^2 + 4)y = 4 \sec^2 2x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + 4)y = 0$$

then, the auxilliary equation is

$$m^2 + 4 = 0$$

or,

$$m = \pm 2i$$

Therefore, the complementary function is

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

where C_1 and C_2 are arbitary constants.
Let the particular integral be

$$y_p = C_1(x) \cos 2x + C_2(x) \sin 2x$$

where $C_1(x)$ and $C_2(x)$ are functions of x.
Variation of Parameters method is the method of finding the functions $C_1(x)$ and $C_2(x)$.

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0$$

Therefore,

$$\begin{aligned} C_1(x) &= -\int \frac{y_2 F(x)}{W} dx = -\int \frac{\sin 2x . 4\sec^2 2x}{2} dx \\ &= -2\int \sec 2x \tan 2x dx = -\sec 2x \end{aligned}$$

and

$$\begin{aligned} C_2(x) &= \int \frac{y_1 F(x)}{W} dx = \int \frac{\cos 2x . 4\sec^2 2x}{2} dx \\ &= 2\int \sec 2x dx = \log(\sec 2x + \tan 2x) \end{aligned}$$

Therefore, the particular integral is

$$\begin{aligned} y_p &= (-\sec 2x) \, . \cos 2x + \log(\sec 2x + \tan 2x) . \sin 2x \\ &= -1 + \sin 2x \log(\sec 2x + \tan 2x). \end{aligned}$$

The general solution is

$$y = C_1 \cos 2x + C_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$$

where C_1 and C_2 are arbitary constants.

Example 4.12 **Solve by the variation of parameters**

$$\frac{d^2y}{dx^2} + y = \sec^3 x \tan x \qquad \text{[WBUT-2007]}$$

Sol. The differential equation can be written as

$$(D^2 + 1)y = \sec^3 x \tan x$$

where,

$$D = \frac{d}{dx}$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + 1)y = 0$$

then, the auxilliary equation is

$$m^2 + 1 = 0$$

or, $$m = \pm i$$

The complementary function is

$$y_c = C_1 \cos x + C_2 \sin x$$

Let the particular integral be

$$y_p = C_1(x) \cos x + C_2(x) \sin x$$

where $C_1(x)$ and $C_2(x)$ are functions of x.

Variation of Parameters method is the method of finding the functions $C_1(x)$ and $C_2(x)$.

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0$$

Therefore,

$$C_1(x) = -\int \frac{y_2 F(x)}{W} dx = -\int \sin x \sec^3 x \tan x dx$$

$$= \int \tan^2 x \sec^2 x dx = \frac{\tan^3 x}{3}$$

and

$$C_2(x) = \int \frac{y_1 F(x)}{W} dx = \int \cos x \sec^3 x \tan x dx dx$$
$$= \int \tan x \sec^2 x dx = \frac{\tan^2 x}{2}$$

Therefore, the particular integral is

$$y_p = \left(\frac{\tan^3 x}{3}\right) \cos x + \left(\frac{\tan^2 x}{2}\right) \sin x$$

Thus, the complete solution is

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x + \left(\frac{\tan^3 x}{3}\right) \cos x + \left(\frac{\tan^2 x}{2}\right) \sin x$$

Example 4.13 **Solve**

$$(D^2 + 4)y = x \sin^2 x$$ [WBUT-2008, 2010]

Sol. Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + 4)y = 0$$

then, the auxilliary equation is

$$m^2 + 4 = 0$$

or,

$$m = \pm 2i$$

Therefore, the complementary function is

$$\text{CF} = y_c = C_1 \cos 2x + C_2 \sin 2x$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$\text{PI} = y_p = \frac{1}{(D^2+4)} x \sin^2 x$$
$$= \frac{1}{(D^2+4)} \frac{x(1-\cos 2x)}{2}$$
$$= \frac{1}{2} \frac{1}{(D^2+4)} x - \frac{1}{2} \frac{1}{(D^2+4)} x \cos 2x$$
$$= \frac{1}{8} (1 + \frac{D^2}{4})^{-1} x - \frac{1}{2} \left\{ x - \frac{2D}{(D^2+4)} \right\} \frac{1}{(D^2+4)} \cos 2x$$

Now,

$$\frac{1}{(D^2+4)}x\cos 2x = \left\{x - \frac{2D}{(D^2+4)}\right\}\frac{1}{(D^2+4)}\cos 2x$$

$$= \left\{x - \frac{2D}{(D^2+4)}\right\}x\frac{1}{2D}\cos 2x$$

$$= \left\{x - \frac{2D}{(D^2+4)}\right\}\frac{x\sin 2x}{4}$$

$$= \frac{x^2\sin 2x}{4} - \frac{2D}{(D^2+4)}\frac{x\sin 2x}{4}$$

$$= \frac{x^2\sin 2x}{4} - \frac{1}{2}\left\{\frac{1}{(D^2+4)}(\sin 2x + 2x\cos 2x)\right\}$$

$$= \frac{x^2\sin 2x}{4} - \frac{1}{2}\frac{1}{(D^2+4)}\sin 2x - \frac{1}{(D^2+4)}x\cos 2x$$

$$= \frac{x^2\sin 2x}{4} - \frac{1}{2}x\frac{1}{2D}\sin 2x - \frac{1}{(D^2+4)}x\cos 2x$$

$$= \frac{x^2\sin 2x}{4} + \frac{x\cos 2x}{4} - \frac{1}{(D^2+4)}x\cos 2x$$

or, $$\frac{1}{(D^2+4)}x\cos 2x = \frac{x^2\sin 2x}{8} + \frac{x\cos 2x}{8}$$

Therefore, the particular integral is

$$y_p = \frac{1}{8}\left(1+\frac{D^2}{4}\right)^{-1}x - \frac{1}{2}\left(\frac{x^2\sin 2x}{8} + \frac{x\cos 2x}{8}\right)$$

$$= \frac{x}{8} - \frac{1}{2}\left(\frac{x^2\sin 2x}{8} + \frac{x\cos 2x}{8}\right)$$

Thus, the general solution is

$$y = y_c + y_p = C_1\cos 2x + C_2\sin 2x + \frac{x}{8} - \frac{1}{2}\left(\frac{x^2\sin 2x}{8} + \frac{x\cos 2x}{8}\right)$$

where C_1 and C_2 are arbitary constants.

Example 4.14 **Solve by the variation of parameters**

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$ [WBUT-2010]

Sol. The reduced equation is

$$\frac{d^2y}{dx^2} + a^2y = 0$$

Let,

$$y = e^{mx}$$

be a trial solution of the reduced equation, then the auxilliary equation is

$$m^2 + a^2 = 0$$

or, $$(m + ai)(m - ai) = 0$$

or, $$m = \pm ai$$

The complementary function is

$$\text{CF} = y_c = C_1y_1 + C_2y_2 = (C_1 \cos ax + C_2 \sin ax)$$

where C_1 and C_2 are arbitary constants.
Let us consider the particular integral as

$$\text{PI} = y_p = C_1(x) \cos ax + C_2(x) \sin ax$$

where $C_1(x)$ and $C_2(x)$ are functions of x.

Variation of Parameters method is the method of finding the functions $C_1(x)$ and $C_2(x)$.

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a \neq 0$$

Therefore,

$$C_1(x) = -\int \frac{y_2F(x)}{W} dx = -\int \frac{\sin ax \sec ax}{3} dx = \frac{1}{a^2} \log \cos ax$$

and

$$C_2(x) = \int \frac{y_1F(x)}{W} dx = \int \frac{\cos ax \sec ax}{a} dx = \frac{1}{a} x$$

Therefore, the particular integral is

$$\text{PI} = y_p = C_1(x)y_1 + C_2(x)y_2 = \left(\frac{1}{a^2} \log \cos ax\right) \sin ax + \left(\frac{1}{a} x\right) \cos ax$$

Thus, the general solution is,

$$y = y_c + y_p = (C_1 \cos ax + C_2 \sin ax) + \left(\frac{1}{a^2} \log \cos ax\right) \sin ax + \left(\frac{1}{a} x\right) \cos ax$$

EXERCISES

Short and Long Answer Type Questions

1. Solve the following differential equations:

a) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = e^{2x} + x^3 + \cos 2x$

$$\left[\textbf{Ans: } y = (C_1 + C_2x)e^{2x} + \frac{1}{2}x^2e^{2x} + \frac{1}{8}(2x^3 + 6x^2 + 9x + 6) - \frac{1}{8}\sin 2x\right]$$

b) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^{-2x}\sin 2x$

$$\left[\textbf{Ans: } y = (C_1 + C_2x)e^{x} - \frac{1}{10}e^{-2x}(\cos 2x + 2\sin 2x)\right]$$

c) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^{x}\cos^2 x$

$$\left[\textbf{Ans: } y = C_1e^{x}\cos(\sqrt{3}x + C_2) + \frac{1}{6}e^{x} - \frac{1}{2}e^{x}\cos 2x\right]$$

d) $\dfrac{d^2y}{dx^2} + 4y = x^2\sin^2 x$

$$\left[\textbf{Ans: } y = C_1\cos 2x + C_2\sin 2x + \left[\frac{1}{8}(x^2 - \frac{1}{2}) - \frac{1}{2}\left\{\frac{x^2}{16}\cos 2x + \frac{1}{4}\left(\frac{x^3}{3} - \frac{x}{8}\right)\right\}\right]\right]$$

e) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = x^2 + \cos x$

$$\left[\textbf{Ans: } y = e^{x}(C_1\cos\sqrt{2}x + \sin\sqrt{2}x) + \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}\left(x^2 + \frac{4}{3}x + \frac{2}{9}\right)\right]$$

f) $\dfrac{d^2y}{dx^2} + y = \sin 3x\cos 2x$

$$\left[\textbf{Ans: } y = C_1\cos x + C_2\sin x + \frac{1}{48}(-\sin 5x - 12x\cos x)\right]$$

g) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = xe^{-x}\sin x$

$$\left[\textbf{Ans: } y = C_1e^{-x} + C_2e^{-2x} + e^{-x}\left[-\frac{x}{2}(\cos x + \sin x) + \sin x - \frac{1}{2}\cos x\right]\right]$$

h) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = 20x^2$

$$\left[\textbf{Ans: } y = C_1e^{4x} + C_2e^{5x} + x^2 + \frac{9x}{10} + \frac{61}{200}\right]$$

2. Solve the following differential equations by variation of parameter method:

a) $\frac{d^2y}{dx^2} + y = \cos ecx$

$$[\textbf{Ans: } y = C_1 \cos x + C_2 \sin x - x\cos x + \sin x \log \sin x]$$

b) $\frac{d^2y}{dx^2} + y = \sec x \tan x$

$$[\textbf{Ans: } y = C_1 \cos x + C_2 \sin x + x\cos x + \sin x \log \sec x - \sin x]$$

c) $\frac{d^2y}{dx^2} - 4y = e^{2x}$

$$\left[\textbf{Ans: } y = C_1e^{2x} + C_2e^{-2x} + \frac{x}{4}e^{2x} - \frac{e^{-2x}}{16}\right]$$

d) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$

$$[\textbf{Ans: } y = e^x(C_1 \cos x + C_2 \sin x) + e^x \cos x \log(\sec x + \tan x)]$$

e) $\frac{d^2y}{dx^2} + 4y = 4\sec^2 2x$

$$[\textbf{Ans: } y = C_1 \cos 2x + C_2 \sin 2x - 1 + \sin 2x \log \sec 2x + \tan 2x]$$

3. Solve the following homogeneous linear equations with variable coefficients:

a) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$

$$[\textbf{Ans: } y = (C_1 + C_2 \log x)x^2 + x^2(\log x)^2]$$

b) $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

$$\left[\textbf{Ans: } y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{e^x}{x^2}\right]$$

c) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

$$\left[\textbf{Ans: } y = \frac{C_1}{x} + C_2x^3 - \frac{1}{9}x^2(3\log x + 2)\right]$$

d) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=\log x\sin(\log x)$

$$\left[\textbf{Ans: } y=C_1\cos(\log x)+C_2\sin(\log x)-\frac{1}{4}(\log x)^2\cos(\log x)\right]$$

e) $(2x+5)^2\dfrac{d^2y}{dx^2}-6(2x+5)\dfrac{dy}{dx}+8y=8(2x+5)^2$

$$[\textbf{Ans: } y=C_1(2x+5)^{2+\sqrt{2}}+C_2(2x+5)^{2-\sqrt{2}}-(2x+5)^2]$$

f) $(x+1)^2\dfrac{d^2y}{dx^2}+(x+1)\dfrac{dy}{dx}+y=4\cos\log(1+x)$

$$[\textbf{Ans: } y=C_1\cos\log(1+x)+C_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x)]$$

g) $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-3y=x^2\log x$ [WBUT-2008]

4. Solve the following simultaneous differential equations:

a) $\dfrac{dx}{dt}-5x-4y=0;\ \dfrac{dy}{dt}-y+x=0$

$$[\textbf{Ans: } x=-2C_1e^{3t}+C_2(1+2t)e^{-3t};\ y=C_1e^{3t}-C_2e^{-3t}]$$

b) $\dfrac{dx}{dt}+2x-3y=t;\ \dfrac{dy}{dt}+2y-3x=e^{2t}$

$$\left[\textbf{Ans: } x=C_1e^{-5t}+C_2e^t+\frac{3}{7}e^{2t}-\frac{2}{5}t-\frac{13}{25};\ y=-C_1e^{5t}+C_2e^t+\frac{4}{7}e^{2t}-\frac{3}{5}t-\frac{12}{25}\right]$$

c) $\dfrac{dx}{dt}+4x+3y=t;\ \dfrac{dy}{dt}+2x+5y=e^t$

$$\left[\textbf{Ans: } x=C_1e^{-2t}+C_2e^{-7t}-\frac{31}{196}+\frac{5}{14}t-\frac{1}{8}e^t;\ y=\frac{2}{3}C_1e^{-2t}+C_2e^{-7t}+\frac{9}{98}-\frac{1}{7}t+\frac{5}{24}e^t\right]$$

Multiple Choice Questions

1. The differential equation $x^2\dfrac{d^2y}{dx^2}+6x\dfrac{dy}{dx}+6y=x\log x$ is solved by the transformation

   a) $\log x=z$ b) $x=z$ c) $e^x=z$ d) none of these

2. The particular integral of $(D^2+2)y=x^2$ is

a) $\frac{1}{2}(x^2-1)$ b) $\frac{1}{2}(1-x^2)$ c) $\frac{x^2}{2}$ d) $1+x^2$

3. The particular integral of $(D^2+4)y = \sin 3x$ is

a) $\frac{\sin 3x}{4}$ b) $\frac{\sin 3x}{3}$ c) $\frac{\sin 3x}{-5}$ d) $\frac{\sin 3x}{5}$

4. The particular integral of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^3e^{2x}$ is

a) $\frac{e^{2x}x^4}{20}$ b) $\frac{e^{x}x^5}{20}$ c) $\frac{e^{2x}x^5}{20}$ d) $\frac{e^{2x}x^4}{60}$

5. The complementary function of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x$ is

a) $C_1e^{-x} + C_2e^{-2x}$ b) $e^{-x} + e^{-2x}$

c) $e^{-x}(C_1 + C_2x)$ d) none of these

6. Using the transformation $x = e^z$ the differential equation $x^2\frac{d^2y}{dx^2} - 5y = \log x$ reduces to

a) $\frac{d^2y}{dz^2} + \frac{dy}{dz} - 5y = z$ b) $\frac{d^2y}{dz^2} - \frac{dy}{dz} - 5y = z$

c) $\frac{d^2y}{dz^2} - 2\frac{dy}{dz} - 5y = z$ d) none of these

7. The complementary function of the equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = 3x$ is

a) $C_1x + C_2e^{3x}$ b) $C_1e^{x} + C_2e^{3x}$

c) $C_1 + C_2e^{3x}$ d) none of these

8. The particular integral of the equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \cos\log x$ is

a) $-\cos x$ b) *cosx* c) $-\cos(\log x)$ d) $\cos(\log x)$

9. From the system of differential equations $\frac{dx}{dt} + 5x + y = e^t$; $\frac{dy}{dt} - x + 3y = e^{2t}$ the differential equation of x and t is

a) $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 4e^t - e^{2t}$ b) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 16x = 4e^t - e^{2t}$

c) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 16x = 0$ d) none of these

10. From the pair of equations $\dfrac{dx}{dt} = 4x - 2y; \dfrac{dy}{dt} = x + y$ we get

a) $x = C_1e^{2t} + C_2e^{4t}$ b) $x = C_1e^{2t} + C_2e^{3t}$

c) $x = C_1e^{t} + C_2e^{-t}$ d) $x = C_1e^{5t} + C_2e^{2t}$

Answers:

1 (c)	2 (a)	3 (c)	4 (c)	5 (a)	6 (a)	7 (c)	8 (c)
9 (a)	10 (b)						

CHAPTER

5

Basic Concepts of Graph Theory

5.1 INTRODUCTION

Graph Theory originated from finding the solution of a long standing problem known as **Königsberg Bridge Problem** solved by **Leonard Euler** in **1736**. Two islands ***C*** and ***D*** formed by river Pregel in **Königsberg** (capital of East Prussia), now renamed as Kaliningrad in West Soviet Russia are connected to each other and to the banks ***A*** and ***B*** with seven bridges. The problem was to start from any of the four lands ***A***, ***B***, ***C***, ***D***, one has to walk over each of the seven bridges exactly once and come back to the starting point. The problem is represented in the following figure.

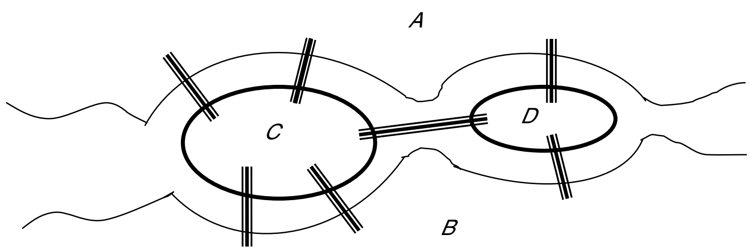


Figure 5.1 Königsberg Bridge Problem

The problem was solved by Leonard Euler by means of a graph. This was the first result written by Euler ever in the graph theory. Euler represented the problem in the form of following graph:

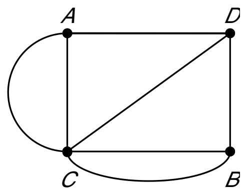


Figure 5.2

where each of the dots (vertices) in the graph represent the lands and the connecting lines or curves (edges) represent the seven bridges.

Euler proved that a solution for this problem does not exist, i.e., it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

After the solution of **Königsberg Bridge Problem** in 1736, **Kirchhoff** developed some results related to graph theory for their applications in the network theory. Then a lot of mathematicians (scientists, engineers) have contributed some remarkable theories in this field. Some of them are **Caley, Möbious, König** and so on.

For the last few decades, graph theory has become one of the essential subjects in almost every field of science and technology. Graph theory can be applied to represent almost every problem which has a discrete arrangement of objects.

In this chapter basically, we deal with the fundamentals of graph theory such as definitions, properties, different kinds of graphs, etc.

5.2 GRAPHS

A graph is a collection of some points (or dots) and some lines or curves joining some or all of the points. These points are known as vertices and the lines or curves are known as the edges in a graph. For example, in a city the electricity poles are the points (or vertices) and the joining wires are the edges. Here we give the formal definition of a graph.

Definition **A graph G is a pair of sets (V, E), where V is a set of vertices and E is the set of edges. Formally, a graph G consists a finite non empty set of vertices V and a set E of 2-element subsets of V called edges.**

The sets V and E are the vertex set and edge set of G. Sometimes we write G as $G(V, E)$.

Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, then each of $v_1, v_2, \ldots, v_n$ represent a vertex in the graph whereas each of $e_1, e_2, \ldots, e_m$ represent an edge in the graph. Here, each edge is associated with a pair of vertices, i.e., edges are formed by joining two vertices.

Example 1 In the following figure A, B, C, D are the vertices and joining A and B we have the edge AB, similarly we have the edges AD, BC, etc. Also note in the figure that there may be more than one joining (edge) between two dots (vertices) such as joining between A and D.

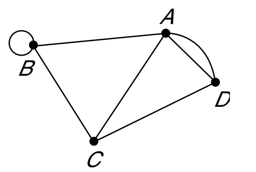


Figure 5.3

A directed graph or **digraph** G is a graph in which each edge $e = (v_i, v_j)$ has a direction from its initial vertex v_i to its terminal vertex v_j.

In a digraph $G(V, E)$, each edge e is associated with an ordered pair of vertices.

Example 2 Here in the following graph each edge is with some direction. In the edge AB, there is a direction from A to B, i.e., A is the initial or starting vertex and B is the terminal vertex or end vertex,.

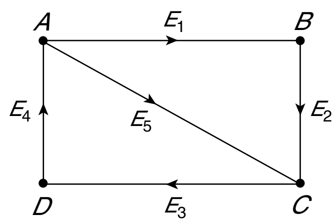


Figure 5.4

A graph having no directions is often called an undirected graph.

5.3 SOME IMPORTANT TERMS RELATED TO A GRAPH

Let us consider the following graph G:

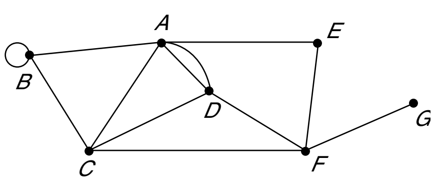


Figure 5.5

1) Self Loop

An edge which has the same vertex as the begining and end vertex is called a self loop.

From the above figure we can see that the graph G at vertex B has a self loop.

2) Parallel Edges

Two or more edges having the same pair of vertices are called parallel edges.

From the above figure we can see that the graph G has parallel edges joining the vertices A and D.

Observations

(a) For an undirected graph, a parallel edge is given by two distinct edges connecting the same two vertices.

(b) For a digraph, a parallel edge is given by two distinct directed edges from one vertex to another. The direction of parallel edges will be the same.

3) Simple Graph

A graph G with no self-loops and parallel edges is called a simple graph.

Example 3 The following graph represents a simple graph with 4 vertices and 4 edges.

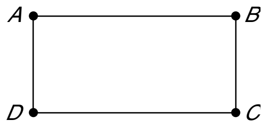


Figure 5.6

4) Multigraph

A graph G with parallel edges is called a multigraph.

Example 4 The following represents a multigraph.

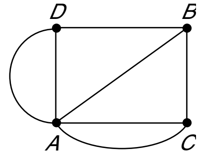


Figure 5.7

5) Pseudograph

A graph G with self-loops and parallel edges is called a pseudograph.

Example 5 The following represents a pseudograph.

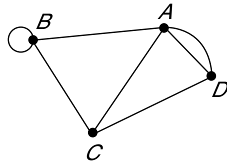


Figure 5.8

6) Adjacent vertices

Two vertices are said to be adjacent if they are the end vertices of the same edge. Two adjacent vertices are known as **neighbours** of each other. In the above figure A and C are adjacent, whereas B and D are not adjacent.

7) Incidence

If a vertex is an end vertex of an edge, then the vertex and the edge are called incident to each other. In the above figure the vertex A and the edge AC are incident to each other.

8) Degree of a vertex

The degree of a vertex in an undirected graph G is the number of edges incident to it, with the exception that a loop at a vertex contributes twice to the degree of that vertex. The degree of a vertex is denoted by $d(v)$.

Outdegree and Indegree of a Vertex of a Digraph

The **outdegree** of a vertex in a digraph is the number of edges leaving the vertex. The **indegree** of a vertex in a digraph is the number of edges entering the vertex.

Example 6 Find the degree of the vertices in the following graph.

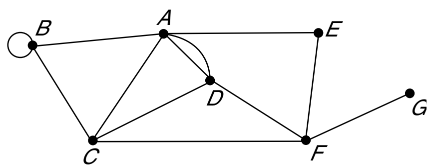


Figure 5.9

We see from the graph in the figure, the degrees of the vertices A, B, C, D, E, F and G are respectively $5, 4, 4, 4, 2, 4$ and 1 respectively.

Therefore, we write

$$d(A) = 5, d(B) = 4, d(C) = 4, d(D) = 4$$
$$d(E) = 2, d(F) = 4 \text{ and } d(G) = 1$$

9) Pendant Vertex

A vertex whose degree is one is called pendant vertex, i.e., only one edge is incident on pendant vertex.

We see from the graph in the above figure, $d(G) = 1$. Therefore, the vertex G is a pendant vertex.

10) Isolated Vertex

A vertex whose degree is zero is called isolated vertex, i.e, no edge is incident on isolated vertex.

11) Null Graph or Discrete Graph

A graph with no edges is called a null graph or discrete graph.

Example 7 The following graph is an example of a **null graph or discrete graph** since there are no edges between A, B and C.

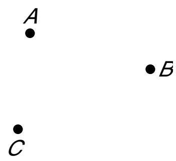


Figure 5.10

Here all the vertices A, B and C are **islolated vertices** since no edge is incident on them.

5.4 ORDER AND SIZE OF A GRAPH

In a graph $G(V, E)$, $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. Ordinarily, $V(G)$ is assumed to be a finite set, in which case $E(G)$ must also be finite, and we say that G is **finite**. If G is finite, $|V(G)|$ denotes the number of vertices in G and is called the **order of the graph** G. Similarly, if G is finite, $|E(G)|$ denotes the number of edges in G, and is called the **size of the graph** G.

Example 8 From the following graph we see that the order of the graph is 6 and the size of the graph is 7.

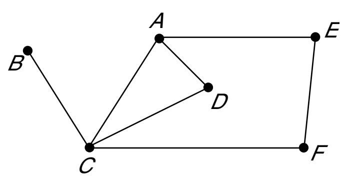


Figure 5.11

Definition The **minimum degree of the graph** G is the minimum degree among the vertices of G and is denoted by $\delta(G)$ and the **maximum degree of the graph** G is the maximum degree among the vertices of G and is denoted by $\Delta(G)$.

Result **In a graph G of order n, for a vertex v, we have**

$$0 \leq \delta(G) \leq d(v) \leq \Delta(G) \leq n - 1$$

Theorem 5.1 (Handshaking Theorem) **The sum of degrees of all vertices in a graph G is twice the number of edges in the graph, i.e, the sum of degrees of all vertices in a graph G is always even.**

Symbolically, in a graph $G(V, E)$ with $|E| = e$ number of edges, we have

$$\boxed{\sum_{v \in V} d(v) = 2e}$$

This theorem is also called the first theorem of graph theory.

Proof We prove the result by induction on e, the number of edges in G.

For $e = 1$, the result is obvious.

Now let $G(V, E)$ be a graph of size e (i.e., with the e number of edges) and the theorem is true for any graph of size $< e$.

Let uv be an edge in G and let $G'(V', E')$ be the graph obtained by deleting the edge uv from G. So, $G'(V', E')$ is a graph of size $< e$.

Therefore, by induction hypothesis

$$\sum_{v \in V'} d(v) = 2e', \quad \text{where } e' = |E'| = e - 1.$$

Now if we add the edge uv to G', then the sum of the degrees of the vertices is increased by 2, so that

$$\sum_{v \in V} d(v) = \sum_{v \in V'} d(v) + 2 = 2e' + 2 = 2(e' + 1) = 2e.$$

Hence, the result is proved.

Theorem 5.2 **The number of odd degree vertices in a graph is always even.**

[WBUT-2003,2008,2011]

Proof Let V and W be the set of vertices of odd degree and even degree respectively. Then by handshaking theorem the sum of degrees of odd degree vertices and even degree vertices of a graph G is equal to twice the number of edges of the graph, i.e.,

$$\sum_{v_i \in V} d(v_i) + \sum_{v_i \in W} d(v_i) = 2e$$

or,

$$\sum_{v_i \in V} d(v_i) = 2e - \sum_{v_i \in W} d(v_i)$$

Since, $\sum_{v_i \in W} d(v_i)$ is even , therefore, $\sum_{v_i \in V} d(v_i)$ is also even, i.e, the number of odd degree vertices in a graph is always even.

5.5 DEGREE SEQUENCE OF A GRAPH

If the degrees of the vertices of a graph G are listed in a sequence s , then s **is called a degree sequence of** G**.**

A degree sequence

$$s:\{d_1, d_2, d_3, \ldots, d_n\}(n \geq 2)$$

is called **graphical** if there is some simple nondirected graph with degree sequence s.

Theorem 5.3 (Havel-Hakimi Theorem) **There exists a simple graph with degree sequence**

$$s:\{d_1, d_2, d_3, \ldots, d_n\}(n \geq 2)$$

where

$$d_1 \leq d_2 \leq \ldots \leq d_n$$

if and only if there exists one with degree sequence

$$\{d_1^1, d_2^1, d_3^1, \ldots, d_{n-1}^1\}$$

where

$$d_k^1 = \left\{ \begin{array}{l} d_k \text{ for } k = 1, 2, \ldots n - d_n - 1 \\ \\ d_k - 1 \text{ for } k = n - d_n, \ldots, n-1 \end{array} \right\}$$

Proof is beyond the scope of the book.

Another form of Havel–Hakimi theorem

Theorem 5.4 **A non-increasing sequence**

$$s:\{d_1, d_2, d_3, \ldots, d_n\}(n \geq 2)$$

of nonnegative integers, where $d_1 \geq 1$ is graphical if and only if the sequence

$$s_1:\{d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots d_n\}$$

is graphical.

Proof is beyond the scope of the book.

Working Procedure Let

$$s:\{d_1, d_2, d_3, \ldots, d_n\}(n \geq 2)$$

be a degree sequence.

Step 1 Delete d_1 from s and subtract 1 from the next d_1 terms and we obtain a degree sequence

$$s_1:\{d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n\}$$

Step 2 Reorder the sequence s_1 so that a non-increasing sequence results.

Step 3 Apply Step1 and get another degree sequence and reorder it.

Step 4 Apply Step 1 repeatedly so that we get a degree sequence of a simple graph.

Example 9 **Examine whether the degree sequence $\{5, 4, 3, 3, 2, 2, 2, 1, 1, 1\}$ is graphical or not.**

Sol. Here, in the degree sequence $\{5, 4, 3, 3, 2, 2, 2, 1, 1, 1\}$ the number of vertices is 10 and the number of odd degree vertices are 6 which is even.

Since, a graph has even number of odd degree vertices, therefore, the degree sequence may be graphical.

Now, let us apply Havel-Hakimi theorem on the degree sequence

$$\{5, 4, 3, 3, 2, 2, 2, 1, 1, 1\}$$

Deleting 5 from the degree sequence and subtracting 1 from the next 5 terms the resulting degree sequence is

$$\{3, 2, 2, 1, 1, 2, 1, 1, 1\}$$

Reordering the degree sequence, we have the degree sequence

$$\{3, 2, 2, 2, 1, 1, 1, 1, 1\}$$

Deleting 3 from the degree sequence and subtracting 1 from the next 3 terms the resulting degree sequence is

$$\{1, 1, 1, 1, 1, 1, 1, 1\}$$

Since, the resulting degree sequence is a degree sequence of a simple graph, therefore, the degree sequence $\{5, 4, 3, 3, 2, 2, 2, 1, 1, 1\}$ is graphical.

5.6 SOME SPECIAL TYPE OF GRAPHS

1) **Complete Graph**

A simple graph in which there is exactly one edge between each pair of distinct vertices, is called a complete graph. A complete graph on n vertices is denoted by K_n.

Example 10 The following Graph G represents K_5 since there is exactly one edge between each pair of distinct vertices, where the number of vertices are 5.

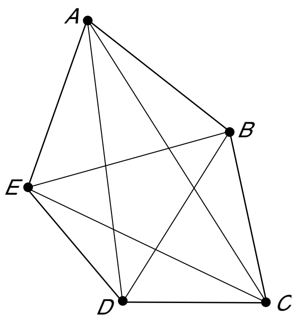


Figure 5.12

2) Regular Graph

A simple graph whose every vertex has same degree is called a regular graph. If every vertex in a regular graph has degree n then the graph is called *n*-regular.

Example 11 The following represents a 2-regular graph, since every vertex has the same degree 2.

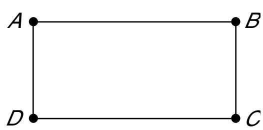


Figure 5.13

3) Bipartite Graph

If the vertex set V of a simple graph G is partioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2 so that no edge in G connects either two vertices in V_1 or two vertices in V_2, then G is called Bipartite graph.

Example 12 The following represents a Bipartite graph.

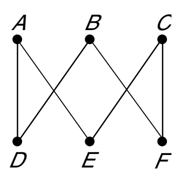


Figure 5.14

Here if we choose $V_1 = \{A, B, C\}$ and $V_2 = \{D, E, F\}$, then the graph satisfies the definition to be Bipartite graph.

4) Complete Bipartite Graph

In a bipartite graph if every vertex of V_1 is connected with every vertex of V_2 by an edge then G is called a completely bipartite graph.

A complete bipartite graph is denoted by $K_{m,n}$, where the set V_1 and V_2 contains m and n number of vertices respectively.

Example 13 The following represents a complete bipartite graph.

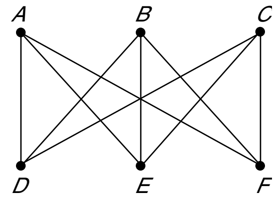


Figure 5.15

Here if we choose $V_1 = \{A, B, C\}$ and $V_2 = \{D, E, F\}$, then the graph satisfies the definition to be complete bipartite graph.

The graph is denoted by $K_{3,3}$ since V_1 and V_2 both have the three vertices.

5.7 SOME IMPORTANT THEOREMS ON GRAPHS

Theorem 5.5 **The maximum number of edges in a simple graph with n vertices is $\dfrac{n(n-1)}{2}$.** [WBUT-2004, 2005, 2006]

Proof Let G be a simple graph having n number of vertices and e number of edges.

A simple graph G has no self-loops and parallel edges.

Then, by handshaking lemma

$$\sum_{i=1}^{n} d(v_i) = 2e$$

or,

$$d(v_1) + d(v_2) + \cdots + d(v_n) = 2e$$

We know that the maximum degree of each vertex in the graph G is $(n-1)$.

Therefore,

$$d(v_1) = d(v_2) = \cdots = d(v_n) = (n-1)$$

and

$$(n-1) + (n-1) + \cdots + (n-1) \text{ (adding } n - \text{times)} = 2e$$

or,

$$n(n-1) = 2e$$

or,

$$e = \frac{n(n-1)}{2}$$

Therefore, maximum number of edges in a simple graph with n vertices is $\dfrac{n(n-1)}{2}$

Theorem 5.6 **The number of edges connected with a vertex of a simple graph of n vertices cannot exceed $n-1$.**

Proof Let G be a simple graph. A simple graph G has no self-loops and parallel edges.

Let v be any vertex of the graph G. Since, G is simple there is no self-loops and parallel edges.

The number of vertices of the graph G is n, so v can be adjacent to at most all the remaining $(n-1)$ vertices of G.

Therefore, maximum of edges connected with v is$(n-1)$, i.e, the number of edges of a simple graph of n vertices cannot exceed $n-1$.

Theorem 5.7 **A complete graph with *n* vertices consists of $\dfrac{n(n-1)}{2}$ number of edges.** [WBUT-2008]

Proof A simple graph in which there is exactly one edge between each pair of distinct vertices is called a complete graph.

Let G be a complete graph with n vertices and e edges. Since the complete graph has no loop and no parallel edges, the number of adjacent vertices of every vertex is $n-1$. Hence, the degree of each vertex is $(n-1)$.

Therefore, the total degree of the vertices is

$$(n-1)+(n-1)+\ldots+(n-1) \text{ (adding } n- \text{times)} = n(n-1)$$

By handshaking theorem, the sum of degrees of all vertices in a graph G is twice the number of edges in the graph, i.e,

$$2e = n(n-1)$$

$$or, \qquad e = \frac{n(n-1)}{2}$$

Therefore, a complete graph with n vertices consists of $\dfrac{n(n-1)}{2}$ number of edges.

Theorem 5.8 **A bipartite graph with n vertices has at most $\left(\dfrac{n^2}{4}\right)$ edges.**

Proof In a bipartite graph the vertex set V is partitioned into two vertex sets V_1 and V_2.

Let V_1 consists of m vertices then, V_2 consists of $(n-m)$ vertices. The largest number of edges of the graph can be obtained, when each of the m vertices in V_1 is connected to each of the $(n-m)$ vertices in V_2.

Therefore, the total no. of edges is

$$m(n-m) = f(m)(say)$$

Now,

$$f'(m) = n - 2m \text{ and } f''(m) = -2$$

when,

$$f'(m) = n - 2m = 0 \Rightarrow m = \frac{n}{2} \text{ and } f''\left(\frac{n}{2}\right) = -2 < 0$$

Therefore, $f(m)$ is maximum , when $m = \frac{n}{2}$.
Thus, the maximum number of edges of a bipartite graph is

$$f\left(\frac{n}{2}\right) = \frac{n}{2}\left(n - \frac{n}{2}\right) = \frac{n^2}{4}$$

5.8 SUBGRAPHS

5.8.1 Definition

Let us assume G and H are graphs. Then, H is called a subgraph of G if and only if the vertex set of H ,$V(H)$ is a subset of the vertex set of G, $V(G)$ and the edge set of H, $E(H)$ is a subset of the edge set of G, $E(G)$.

Therefore H is a subgraph of G, then

(a) All the vertices of H are in G.
(b) All the edges of H are in G.
(c) Every edge of H has the same end points in H as in G.

5.8.2 Different Types of Subgraphs

Let us consider the graph $G(V, E)$.

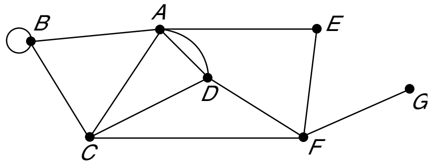


Figure 5.16

1) Vertex deleted subgraph
A subgraph H of a graph G is called a vertex deleted subgraph of G if H is obtained from G by the deletion of one or more vertices and the corresponding edges incident on them.

Example 14 Let us delete the vertex E from the graph G, then the edges AE and EF are deleted and we get the subgraph H as the following:

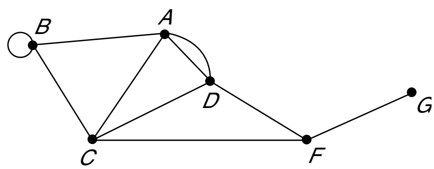


Figure 5.17

2) Edge deleted subgraph

A subgraph H of a graph G is called a edge deleted subgraph of G if H is obtained from G by the deletion of one or more edges not the corresponding vertices in which they are incident.

Example 15 Let us delete the edges CF and DF from the graph G then we get the edge deleted subgraph H as the following:

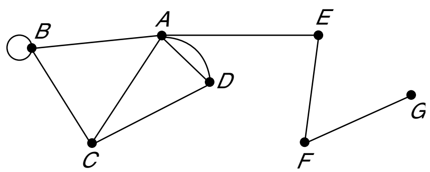


Figure 5.18

3) Underlying simple subgraph

A subgraph H of a graph G is called an underlying simple subgraph if H is obtained from G by deleting all loops and more than one parallel edges.

Example 16 Deleting all parallel edges and self-loops from G we have the simple subgraph H as the following:

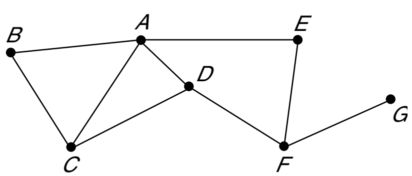


Figure 5.19

4) Spanning subgraph

A subgraph H of G is called a spanning subgraph of G if and only if $V(H) = V(G)$.

Example 17 The following represents a spanning subgraph H of G, since $V(H) = V(G)$. Here, it is to be noted that the number edges of H are fewer than that of G.

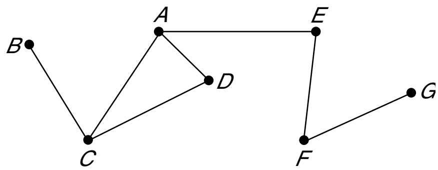


Figure 5.20

5) Induced subgraph of a graph

Let W is any subset of $V(G)$, then the subgraph induced by W is the subgraph H of G obtained by taking $V(H) + W$ and $E(H)$ to be those edges of G that join the pair of vertices in W.

5.9 COMPLEMENT OF A GRAPH

The complement $\overline{G}$ of a graph G is the graph whose vertex set is $V\ (G)$ and such that for each pair of vertices (u, v) of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G.

Example 18 **Draw the complement of the following graph.** [WBUT-2004]

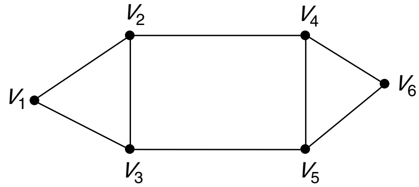


Figure 5.21

Sol. The complement of the given graph is

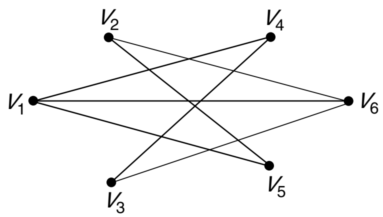


Figure 5.22

5.10 WALK, TRAIL, PATH AND CIRCUIT

5.10.1 Walk

A $u - v$ walk W in a graph G is a sequence of vertices in G, beginning with u and ending at v, such that consecutive vertices in the sequence are adjacent.

We express W as

$$W : u = v_0, v_1, v_2, \ldots, v_k = v$$

where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, \ldots, k-1$

Example 19 Let us consider the following graph:

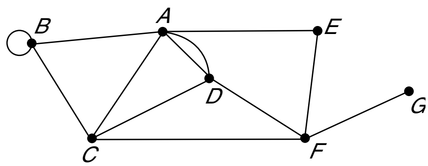


Figure 5.23

Different walks can be found from the graph, viz.

$$W_1 : u = A, B, C, F, G = v$$
$$W_2 : u = D, F, E, A, D, F = v$$

and so on.

Observations

(1) A finite sequence of vertices and edges beginning and ending with vertices, such that each edge is incident to its preceeding and following vertices is called a walk.

(2) The origin and terminal vertex of a walk may be same.

From the graph in the figure, we see that

$$u = D, F, E, A, D = v$$

is a $u - v$ walk with the origin and terminal vertex as D.

(3) A vertex may appear twice or more in a walk.

From the graph in the figure, we see that

$$u = D, F, E, A, D, F = v$$

is a $u - v$ walk where the vertex D appear twice.

(4) An edge may appear twice or more in a walk.

(5) A self-loop can be included in a walk.
From the graph in the figure, we see that

$$u = A, B, B, C, D = v$$

is a $u - v$ walk where a self loop is included at the vertex B.

(6) A walk W is said to be closed if $u = v$ while W is open if $u \neq v$.

(7) The number of edges encountered in a walk including multiple occurances of an edge is called length of the walk.

(8) A walk of length 0 is called a trivial walk.

(9) If a $u-v$ walk in a graph is followed by a $v-w$ walk, then a $u-w$ walk results.

5.10.2 Trail

A $u-v$ trail in a graph G is defined to be a $u-v$ walk in which no edge is traversed more than once.

5.10.3 Path

A $u - v$ path in a graph G is defined to be a $u - v$ walk in which no vertices are repeated.

Example 20 Let us consider the following graph:

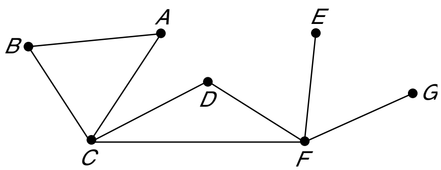


Figure 5.24

We see that

$$W : u = A, B, C, F, G = v$$

is a $u - v$ walk. Since no edge is traversed more than once, the walk W is also a $u - v$ trail.

Since no vertices are repeated, the walk W is also a $u - v$ path.

Observations

(1) Although the definition of a trial stipulates that no edge can be repeated, no such condition is placed on vertices.

(2) If no vertex in a walk is repeated then no edge is repeated either. Hence every path is a trial.

(3) A $u - v$ path followed by a $v - w$ path is a $u - w$ walk W, but not necessarily a $u - w$ path, as vertices in W may be repeated.

(4) While not every walk is a path, if a graph contains a $u - v$ walk then it must also contain $u - v$ path.

(5) A path does not intersect itself.

(6) A single edge having two adjacent vertices which is not a self-loop is a path of length 1.

(7) A self-loop is a walk but not a path. Even a self-loop cannot be included in a path.

(8) Considering a path as a subgraph, the terminal vertices of a path are of degree one and the intermediate vertices are of degree two.

5.10.4 Circuit

A circuit in a graph G is a closed trial of length 3 or more. Therefore, a circuit begins and ends at the same vertex but repeats no edges.

Example 21 Let us consider the following graph:

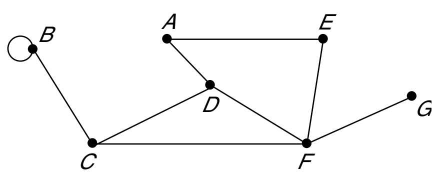


Figure 5.25

The closed $u - v$ trail

$$u = A, D, F, E, A = v$$

which is a trail of length 4 is a circuit.

Observations

(1) A circuit can be described by choosing any of its vertices as a beginning and ending vertex, provided the vertices are listed in the same cyclic order.

(2) A circuit that repeats no vertex, except for the first and last is called a cycle.

(3) A k-cycle is a cycle of length k.

(4) A cycle of odd length is called an odd cycle and a cycle of even length is called an even cycle.

(5) A graph that is a path of order n is denoted by P_n, while a graph that is a cycle of order $n \geq 3$ is denoted by C_n.

5.10.5 Theorems on Walks, Paths and Circuits

Theorem 5.9 **If a graph G contains a $u-v$ walk of length l, then G contains a $u-v$ path of length at most l.**

Proof Let G be a graph and among all $u-v$ walks in G, let

$$P : u = u_0, u_1, u_2, \dots, u_k = v$$

be a $u-v$ walk of smallest length k.

Therefore,

$$k \leq l$$

We claim that P is a $u-v$ path.

Let us assume that P is not a $u-v$ path, then some vertex of G must be repeated in P, say $u_i = u_j$ for some i and j with $0 \leq i < j \leq k$. If we

then delete the vertices $u_{i+1}, u_{i+2}, \dots, u_j$ from P we arrive at the $u-v$ walk whose length is less than k, which is impossible.

Therefore, as claimed, P is a $u-v$ path of length $k \leq l$.

Theorem 5.10 **If a graph has exactly two vertices of odd degree there must be a path joining these two vertices.**

Proof Let G be a graph with all even degree vertices except the vertices v_i and v_j which are odd degree vertices. If G is a connected graph then there exists a path joining v_i and v_j, and the theorem is proved.

If G is not connected, then suppose $G_1$and G_2 are two components such that $v_i \in G_1$ and $v_j \in G_2$. Then $G_1$is a graph having only one vertex of odd degree, but the number of odd degree vertices of a graph is even. So it is not possible. Therefore v_i and v_j must belong to G_1. Since,G_1 is connected there is a path joining v_i and v_j.

Theorem 5.11 **Every vertex, except the terminal vertices of a $u-v$ walk, whose all edges are distinct is an even degree vertex.**

Alternatively, every vertex, except the terminal vertices of a $u-v$ trail, is an even degree vertex.

Proof Let us consider a $u-v$ walk. Let v_k be a vertex of this walk such that $v_k \neq u$ and $v_k \neq v$. If v_k occurs only once in the walk then there must exist one edge preceeding v_k and one edge succeeding v_k. Therefore, v_k becomes a vertex of degree two.

Let us suppose v_k occurs more than once in the walk, say p-times. Since no edge repeats in a walk, at each time of occurance v_k gets degree two, and therefore, the degree of v_k is $2p$.

Therefore, every vertex is an even degree vertex.

Theorem 5.12 **Every $u-v$ trail contains a $u-v$ path.**

Proof In a $u-v$ trail, u is the origin and v is the terminus. Both u and v are of degree one. We know that every vertex, except u and v is of even degree in a $u-v$ trail. So, $u-v$ trail contains only two odd vertices.

Since, in a graph having exactly two vertices of odd degree, there is a path joining these two vertices, therefore, $u - v$ trail contains a $u - v$ path.

5.11 CONNECTED AND DISCONNECTED GRAPHS

A **path** in a graph G is a finite alternating sequence of vertices and edges beginning and ending with vertices, such that each edge is incident on the vertices preceding and following it.

A graph G is said to be a connected graph if there is some path from any vertex to any other vertex otherwise, the graph is called disconnected.

Note: By saying the vertices u and v is connected in the graph G only means that there is some $u - v$ path in G; it does not imply that u and v are joined by an edge. It is obvious that u is joined to v, then u is connected to v as well.

1. **Minimally connected graph**
   A graph G is said to be minimally connected graph if
   (i) G is connected.
   (ii) Deletion of any edge from G leaves the graph disconnected.

5.11.1 Components of a Graph

A connected subgraph of a graph G is called the component of G, if it is not contained in any bigger subgraph of G which is connected.

A disconnected graph consists of two or more connected subgraphs. Each of these connected subgraphs is known as **component.**

Sometimes components of a graph are also called connected components.

Example 22 Find the connected components of the graph:

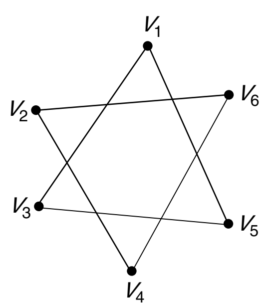


Figure 5.26

Sol. The number of connected components of the graph is 2. The connected components are

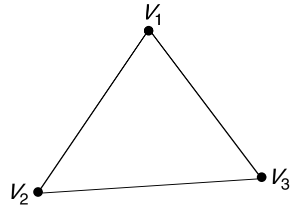


Figure 5.27

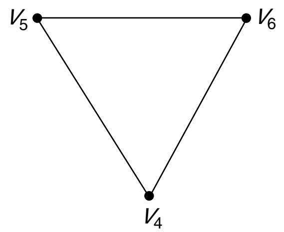


Figure 5.28

5.11.2 Distance and Diameter of a Connected Graph

Let G be a connected graph.

The distance between two vertices u and v is the shortest path between u and v and is denoted by $d(u, v)$.

The diameter of the connected graph G is the maximum distance between any two vertices of G and is denoted by $diam(G)$.

5.11.3 Theorems on Connected Graphs

Theorem 5.13 **The minimum number of edges in a connected graph with n vertices is $n - 1$.** [WBUT-2005,2006]

Proof Let us prove the theorem by mathematical induction.

Let $m =$ number of edges of the connected graph.

We have to show

$$m \geq n - 1$$

When $m = 0$, clearly $n = 1$, that is, the graph is an isolated vertex.

When $m = 1$, clearly $n = 2$, that is, an edge is formed by two connecting vertices.

Let the result be true for $m = k$. We shall show that the result is true for $m = k + 1$.

Let G be a graph with $k + 1$ number of edges and e be any edge of G. Deleting the edge e, let $G - e$ be a subgraph of G. The graph $G - e$ has k edges and n vertices.

Case 1: If $G-e$ is also connected, then by our hypothesis

$$k \geq n-1$$

or, $$k+1 \geq n-1+1=n>n-1$$

Case 2: If $G-e$ becomes disconnected then it has two connected components. Let the components have k_1 and k_2 number of edges and n_1, n_2 number of vertices respectively, where $n=(n_1+n_2)$.

So, by our hypothesis

$$k_1 \geq n_1-1 \text{ and } k_2 \geq n_2-1$$

Therefore, $$k_1+k_2 \geq n_1-1+n_2-1=(n_1+n_2)-2=n-2$$

or, $$k+1 \geq n-1$$

Therefore, the result is true for $m=k+1$.

Thus, by mathematical induction the minimum number of edges in a connected graph with n vertices is $n-1$.

Note: The minimum number of edges in a simple graph (not necessarily connected) with n vertices is $n-k$, where k is the number of components of the graph.

Theorem 5.14 **The maximum number of edges a simple graph with n number of vertices and k components can be $\dfrac{(n-k)(n-k+1)}{2}$.** [WBUT-2008, 2007, 2002]

Proof Let G be a simple graph, i.e, G does not have self-loops and parallel edges. Let the number of vertices in each of the k components of the graph G be $n_1, n_2, \ldots, n_k$. Therefore,

$$n_1+n_2+\cdots+n_k = n \text{ where } n>1$$

or, $$\sum_{i=1}^{k} n_i = n$$

Now,

$$(n_1-1)+(n_2-1)+\cdots+(n_k-1) = (n-k)$$

or, $$\sum_{i=1}^{k}(n_i-1) = (n-k)$$

or, $$\left\{\sum_{i=1}^{k}(n_i-1)\right\}^2 = (n-k)^2$$

or, $$\sum_{i=1}^{k}(n_i-1)^2+\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}(n_i-1)(n_j-1) = n^2-2nk+k^2$$

since, $(n_i - 1) > 0; (n_j - 1) > 0 \Rightarrow \sum_{i=1}^{k} \sum_{\substack{j=1 \\ i \neq j}}^{k} (n_i - 1)(n_j - 1) > 0$

we have,

$$\sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 - 2nk + k^2$$

$$\text{or,} \quad \sum_{i=1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$\text{or,} \quad \sum_{i=1}^{k} (n_i^2 - 2n_i) + k \leq n^2 - 2nk + k^2$$

$$\text{or,} \quad \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k \leq n^2 - 2nk + k^2$$

$$\text{or,} \quad \sum_{i=1}^{k} n_i^2 - 2n + k \leq n^2 - 2nk + k^2$$

$$\text{or,} \quad \sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k$$

$$\text{or,} \quad \sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$$

Since, the maximum number of edges in a simple graph with n vertices is $\dfrac{n(n-1)}{2}$.

Therefore, the maximum number of edges in the i^{th} component of the graph G is $\dfrac{n_i(n_i-1)}{2}$.

Thus, the maximum number of edges in G is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\left\{\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right\}$$

$$\text{or,} \quad \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$\text{or,} \quad \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}n$$

$$\text{or,} \quad \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \leq \frac{1}{2}\{n^2 - (k-1)(2n-k)\} - \frac{1}{2}n$$

$$\text{or,} \quad \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \le \frac{1}{2}\{n^2-(k-1)(2n-k)-n\}$$

$$\text{or,} \quad \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \le \frac{1}{2}\{n^2+k^2-2nk-k+2n-n\}$$

$$\text{or,} \quad \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \le \frac{1}{2}\{(n-k)^2+(n-k)\}$$

$$\text{or,} \quad \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \le \frac{1}{2}(n-k)(n-k+1)$$

Theorem 5.15 **If a graph has exactly two vertices of odd degree, then there is a path joining these two vertices.**

Proof

Case 1: Let us suppose the graph G is connected.

Let v_1 and v_2 be the only vertices of G which are of odd degree. Since the number of odd degree vertices in a graph is even, clearly there is a path connecting v_1 and v_2.

Case 2: Let us suppose the graph G is disconnected.

Since G is disconnected, the components of G are connected. Hence, v_1 and v_2 should belong to the same component of G and there is a path connecting v_1 and v_2.

Theorem 5.16 **A connected graph having at least two vertices has a pendant vertex if the number of edges is less than the number of vertices.**

Proof Let us assume, if possible, the graph G has no pendant vertex.

Since G is connected it has no vertex of degree 0. So degree of every vertex is greater and equal to 2.

Let us assume G has n vertices and e edges and $d(v_i)$ is the degree of the vertex v_i.

Therefore,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\text{or,} \quad 2e \ge \sum_{i=1}^{n} 2 \quad \text{since } d(v_i) \ge 2$$

$$\text{or,} \quad 2e \ge 2n$$

$$\text{or,} \quad e \ge n$$

This contradicts the hypothesis that the number of edges is less than n.

Theorem 5.17 **If G is a disconnected graph, then $\overline{G}$ is a connected graph.**

Proof Since G is a disconnected graph, G contains two or more components.

Let u and v be two vertices of $\overline{G}$. We have to show that u and v are connected in $\overline{G}$.

Case 1: If u and v belong to different components of G, then u and v are not adjacent in G and so u and v are adjacent in $\overline{G}$. Hence $\overline{G}$ contains a $u - v$ path of length 1.

Case 2: Let u and v belong to the same component of G. Let w be a vertex of G that belongs to a different component of G. Then the edge $v - w \notin E(G)$, implying that, the edges $u - w, v - w \in E(\overline{G})$ and so $u - w - v$ is a $u - v$ path of $\overline{G}$.

Theorem 5.18 **Let G be a graph of order n. If**

$$d(u) + d(v) \geq n - 1$$

for every two non-adjacent vertices u and v of G, then G is connected and $diam(G) \leq 2$.

Proof Let $u, v \in V(G)$. If $uv \in E(G)$, then $u - v$ is a path and u and v are certainly connected. Hence, we may assume that $uv \notin E(G)$.

Therefore,

$$d(u) + d(v) \geq n - 1$$

implies that there must be a vertex w that is adjacent to both u and v. Therefore, $u - w - v$ is a path in G and G is connected and $diam(G) \leq 2$.

Note: If G is a graph of order n, such that $d(v) \geq \frac{(n-1)}{2}$, then for every vertex v of G, G must be connected.

5.11.4 Strongly and Weakly Connected Digraphs

A digraph G is called a strongly connected digraph if there exists at least one directed path from every vertex to every other vertex.

Example 23 The following graph is a strongly connected graph:

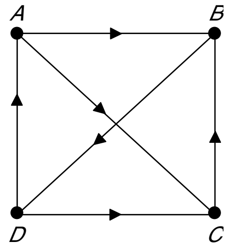


Figure 5.29

since, there is a path from each of the possible pair of vertices, (A, B), (A, C), (A, D), (B, C), (B, D) and (C, D).

A digraph G is called a weakly connected digraph if its corresponding undirected graph is connected, but G is not strongly connected.

5.11.5 Unilaterally Connected Digraph

A directed graph is called a unilaterally connected digraph if between every pair of adjacent vertices there exists only one directed path.

Example 24 The following graph is unilaterally connected c, since there is no path from C to the other vertices but C can be reached from them.

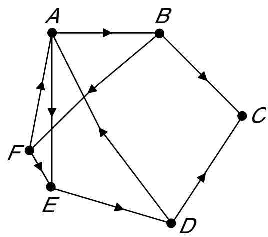


Figure 5.30

Observation: **In a directed graph,**

Sum of out degrees of all vertices = sum of in degrees of all vertices = number of edges in the directed graph.

5.12 EULER GRAPH

5.12.1 Definition

If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line and the graph is called Euler Graph.

Euler graphs do not have any isolated vertices and are therefore connected.

Example 25 The following two graphs represent Euler graphs:

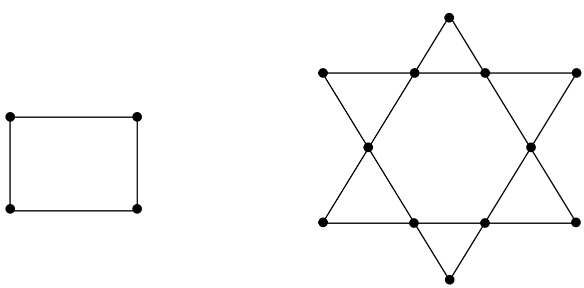


Figure 5.31

5.12.2 Theorems on Euler Graphs

Theorem 5.19 **A connected graph G is an Euler graph if and only if all the vertices of G are of even degree.** [WBUT-2002]

Proof Let G be a Euler graph. G therefore contains an Euler line which is a closed walk. In tracing this walk we observe that every time the walk meets a vertex v it goes through two new edges incident on v- with one we entered v and with the other exited. This is true not only of the intermediate vertices of the walk but also of the terminal vertex, because we entered the same vertex at the beginning and end of the walk, respectively. Therefore, if G is an Euler graph, the degree of every vertex is even.

Theorem 5.20 **A connected graph is Eulerian if and only if it can be decomposed into cycles.**

Proof Let G be a graph which can be decomposed into cycles, that is, G is a union of edge disjoint cycles. Since the degree of every vertex in a cycle is two, the degree of every vertex in G is even. Hence, G is an Euler graph.

Conversely, let G be an Euler graph. Consider a vertex v_1. There are at least two edges incident at v_1. Let one of these edges be between v_1 and v_2. Since the vertex v_2 is also of even degree, it must have at least another edge, say between v_2 and v_3. Proceeding in this way we arrive at a vertex that has previously been traversed, thus forming a cycle. Let us remove the cycle from G. All vertices in the remaining graph not necessarily connected must also be of even degree. From the remaining graph remove another cycle in exactly as we removed the previous cycle from G. Continuing this process we are left with no edges, hence the theorem.

WORKED OUT EXAMPLES

Example 5.1 **Suppose G is a non directed graph with 12 edges. If G has 6 vertices each of degree 3 and the rest have degree less than 3, find the minimum number of vertices G can have.** [WBUT-2008, 2007, 2006]

Sol. Let us assume G has n vertices of degree less than 3.
By handshaking theorem

$$2e = \sum d(v_i)$$

$$or, \quad 6 \times 3 + \sum_{i=1}^{n} d(v_i) = 2 \times 12$$

$$or, \quad \sum_{i=1}^{n} d(v_i) = 6$$

$$or, \quad d(v_1) + d(v_2) + \ldots + d(v_n) = 6$$

$$or, \qquad 6 \leq 3+3+\ldots+3 = 3n$$

$$or, \qquad n > 2$$

The minimum value of n is 3.
Therefore, the graph G has $6+3=9$ vertices.

Example 5.2 **Define complement of a graph. Find the complement of the graph.**

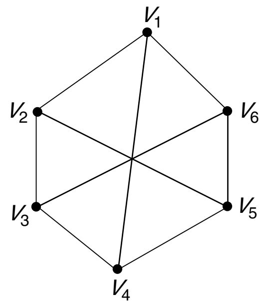


Figure 5.32

[WBUT-2007]

Sol. The complement of the graph is

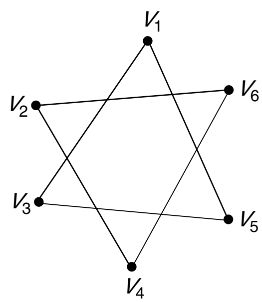


Figure 5.33

Example 5.3 **A graph G is given below**

(a) Find the distance between V_1 and V_4.

(b) Find the $diam(G)$.

(c) Find one circuit which includes V_1. [WBUT-2006]

Sol. (a) Distance between V_1 and V_4 is the length of the shortest path between V_1 and V_4. The shortest path is

$$V_1 - V_2 - V_3 - V_4$$

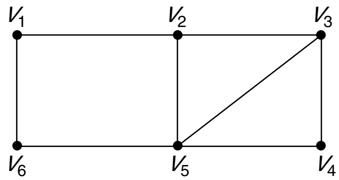


Figure 5.34

and the distance is 3.

(b) The diameter of G is the maximum distance any two vertices, the $diam(G) = 4$.

(c) One circuit which includes V_1 is

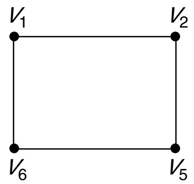


Figure 5.35

Example 5.4 **If a simple regular graph has n vertices and 24 edges, find all possible values of n.** [WBUT-2006]

Sol. Since the graph is a regular graph, let k be the degree of every vertex. Therefore, the sum of the degrees of all vertices is nk.
i.e,

$$nk = 2 \times 24$$

$$or, \qquad n = \frac{48}{k} \qquad (1)$$

We know that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.
Therefore,

$$\frac{n(n-1)}{2} \geq 24$$

$$or, \qquad n(n-1) \geq 48 \qquad (2)$$

Now k is a positive integer and from (1), we have

$$k = 1 \Rightarrow n = 48$$

which satisfies 2.

For,

$$k = 2 \Rightarrow n = 24$$

which satisfies 2.
For,

$$k = 3 \Rightarrow n = 16$$

which satisfies 2.
For,

$$k = 4 \Rightarrow n = 12$$

which satisfies 2.
For,

$$k = 5 \Rightarrow n \text{ is not a integer}$$

which does not satisfy 2.
For,

$$k = 6 \Rightarrow n = 8$$

which satisfies 2.
For,

$$k = 7 \Rightarrow n \text{ is not a integer}$$

which does not satisfy 2.
For,

$$k = 8 \Rightarrow n = 6$$

which satisfies 2.
Therefore, the possible values of n are $48, 24, 16, 12, 8$.

Example 5.5 **Draw a graph (if exists) having the following properties or explain why no such graph exists: A graph with 4 edges, 4 vertices with degree sequence 1, 2, 3, 4.** [WBUT-2004]

Sol. Here, in the degree sequence $\{4, 3, 2, 1\}$ the number of vertices is 4 and the number of odd degree vertices are 2 which is even.

Since, a graph has even number of odd degree vertices, therefore, the degree sequence may be graphical.

Now, let us apply Havel–Hakimi theorem on the degree sequence

$$\{4, 3, 2, 1\}$$

Deleting 4 from the degree sequence and subtracting 1 from the next 3 terms the resulting degree sequence is

$$\{2, 1, 0\}$$

Since, the resulting degree sequence is not a degree squence of a simple graph, therefore, the degree sequence $\{4, 3, 2, 1\}$ is not graphical.

Therefore, a graph with 4 edges, 4 vertices with degree sequence 1, 2, 3, 4 does not exist.

Example 5.6 **A graph G has order 14 and size 27. The degree of each vertex of G is 3, 4 and 5. There are 6 vertices of degree 4. How many vertices of G has degree 3 and how many have degree 5?**

Sol. Let x be the number of vertices of G having degree 3. Since the order of G is 14 and 6 vertices have degree 4, eight vertices have degree 3 or 5.

Therefore, there are $8 - x$ vertices of degree 5.

Summing the degrees of the vertices and applying handshaking theorem we have,

$$\begin{aligned} 3.x + 4.6 + 5.(8 - x) &= 2.27 \\ \textit{or}, \quad 3x + 24 + 40 - 5x &= 54 \\ \textit{or}, \quad -2x &= -10 \\ \textit{or}, \quad x &= 5 \end{aligned}$$

Therefore, the graph has 5 vertices of degree 3 and 3 vertices of degree 5.

Example 5.7 **Prove that a finite graph with at least one edge and without any circuit has at least two vertices of degree 1.**

Sol. Since the graph has no, circuit, we must have a vertex, say v_1 at which only one edge is incident, i.e, $d(v_1) = 1$.

Let e_1 be this edge which is incident at v_1. Since the graph has no circuit so other end of e_1 is not v_1. Let it be v_2. If there exists no other edge which is incident to v_2 then $d(v_2) = 1$ otherwise, let e_2 be the edge which is incident at v_2.

Arguing in the similar way and preceeding in this way we get a vertex v_k having degree one and $v_k \neq v_1$.

Example 5.8 **Let G be a graph with 15 vertices and 4 components. Show that G has at least one component having at least 4 vertices. Find the largest number of vertices that a component of G can have.**

Sol. Let us consider the 4 components contain n_1, n_2, n_3 and n_4 number of vertices.

Then,

$$n_1 + n_2 + n_3 + n_4 = 15 \tag{1}$$

If each of n_1, n_2, n_3 and $n_4 \leq 3$, then

$$n_1 + n_2 + n_3 + n_4 \leq 12$$

which contradicts (1).

If one component has 4 vertices then the other three components has 11 vertices. So, component among the remaining three must contain at least 4 vertices.

If one component of these three contains exactly 4 vertices then the remaining two components contain 7 vertices.

Now, these two components may have the following combination of vertices viz., $(1, 6), (2, 5), (3, 4), (4, 3), (5, 1)$ and $(6, 1)$.

Therefore, the maximum number of vertices in a component is 6.

Example 5.9 **Find the minimum and the maximum number of edges of a simple graph with 12 vertices and 5 components.**

Sol. The maximum number of edges a simple graph with n number of vertices and k components can be $\dfrac{(n-k)(n-k+1)}{2}$, and the minimum number of edges a simple graph with n number of vertices and k components can be $(n-k)$.

Therefore, the minimum and the maximum number of edges of a simple graph with 12 vertices and 5 components is $(12-5) = 7$ and $\dfrac{(12-5)(12-5+1)}{2} = 28$ respectively.

Example 5.10 **Let G be a graph with n vertices and e edges, then prove that G has a vertex of degree k such that $k \geq \dfrac{2e}{n}$.**

Sol. By handshaking theorem

$$2e = \sum_{i=1}^{n} d(v_i)$$

or, $$2e = d(v_1) + d(v_2) + \cdots + d(v_n)$$

or, $$2e \leq k + k + \cdots + k$$

or, $$2e \leq nk$$

or, $$k \geq \frac{n}{2e}$$

Example 5.11 **Let G be a graph with n vertices and $(n-1)$ edges, then prove that G either has a pendant vertex or an isolated vertex.**

Sol. If possible, let us assume G has no pendant vertex or isolated vertex. Then G does not contain any vertices of degree 1 or 0.

Hence, $d(v_i) \geq 2$ for any vertex v_i of $G, i = 1, 2, \ldots, n$.

By handshaking theorem

$$2e = \sum_{i=1}^{n} d(v_i)$$

$$or, \quad d(v_1) + d(v_2) + \cdots + d(v_n) = 2(n-1)$$

$$or, \quad 2(n-1) \geq 2 + 2 + \cdots + 2 = 2n$$

$$or, \quad 2n - 2 \geq 2n$$

$$or, \quad -2 \geq 0$$

which is a contradiction.
Hence, our assumption is wrong and G contains either a pendant vertex or isolated vertex.

Example 5.12 **Let G be a bipartite graph of order 22 with partite sets U and W, where $|U| = 12$. Suppose that every vertex in U has degree 3; while every vertex of W has degree 2 or 4. How many vertices of G have degree 2?**

Sol. The size of the graph G is

$$3\,|U| = 3 \times 12 = 36$$

The vertex set W has $22 - 12 = 10$ vertices.

Let x be the number of vertices of degree 2, then $(10 - x)$ is the number of vertices of degree 4. Therefore,

$$2 \times x + 4 \times (10 - x) = 36$$

$$or, \quad -2x = -4$$

$$or, \quad x = 2$$

Therefore, the number of vertices of degree 2 is 2.

Example 5.13 **The degree of every vertex of a graph G of order 25 and size 62 is 3, 4, 5 and 6. There are two vertices of degree 4 and 11 vertices of degree 6. How many vertices of G have degree 5?**

Sol. Let x be the number of vertices of degree 5. Therefore, by handshaking theorem, we have

$$2e = \sum_{i=1}^{n} d(v_i)$$

$$or, \quad (12 - x) \times 3 + 2 \times 4 + 5 \times x + 6 \times 11 = 2 \times 62$$

$$or, \quad 2x = 14$$

$$or, \quad x = 7$$

Therefore, the number of vertices of degree 5 is 7.

Example 5.14 **Decide whether the sequence $\{7, 7, 4, 3, 3, 3, 2, 1\}$ is graphical.**

Sol. Deleting the first term 7 from the non-increasing sequence $\{7, 7, 4, 3, 3, 3, 2, 1\}$ and subtracting 1 from the next seven terms, we have, the sequence

$$\{6, 3, 2, 2, 2, 1, 0\}$$

Deleting the first term 6 and subtracting 1 from the next six terms of the sequence $\{6, 3, 2, 2, 2, 1, 0\}$, we have the sequence

$$\{2, 1, 1, 1, 0, -1\}$$

Since, the degree sequence contains the negative number -1 and no vertex can have a negative degree, therefore the graph is not graphical.

EXERCISES

Short and Long Answer Type Questions

1) From the following graph:

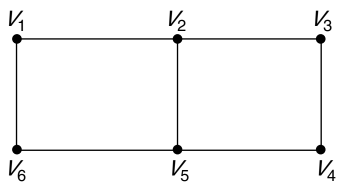


Figure 5.36

a) all the paths between v_1 and v_4.
b) all trails between v_1 and v_4.
c) distance between v_1 and v_4.
d) all circuits.

[**Ans:** a) $\{v_1, v_2, v_3, v_4\}, \{v_1, v_6, v_5, v_4\}, \{v_1, v_6, v_5, v_3, v_4\}$, $\{v_1, v_6, v_2, v_5, v_4\}, \{v_1, v_2, v_3, v_5, v_4\}, \{v_1, v_2, v_5, v_4\}$
b) $\{v_1, v_2, v_3, v_4\}, \{v_1, v_6, v_5, v_4\}, \{v_1, v_6, v_5, v_3, v_4\}$, $\{v_1, v_6, v_2, v_5, v_4\}, \{v_1, v_2, v_3, v_5, v_4\}, \{v_1, v_2, v_5, v_4\}$, $\{v_1, v_6, v_5, v_3, v_2, v_5, v_4\}, \{v_1, v_6, v_5, v_2, v_3, v_5, v_4\}$
c) distance between v_1 and v_4 is 3
d) $\{v_2, v_3, v_5, v_2\}, \{v_1, v_2, v_5, v_6, v_1\}, \{v_1, v_2, v_3, v_5, v_6, v_1\}$, $\{v_2, v_3, v_4, v_5, v_2\}, \{v_3, v_4, v_5, v_3\}, \{v_1, v_2, v_3, v_4, v_5, v_6, v_1\}$

2) Prove that there exists no simple graph with five vertices having degrees 4, 4, 4, 2, 2.

3) Find the number of edges of a graph with five vertices of degree 0, 2, 2, 3 and 9.

[**Ans:** 8]

4) Find the number of connected components of the graph:

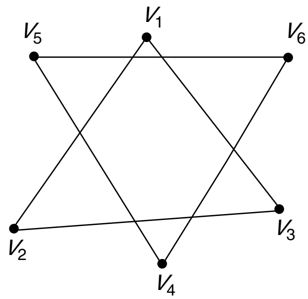


Figure 5.37

[**Ans:**

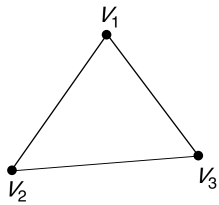


Figure 5.38

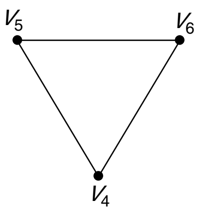


Figure 5.39

5) Find all the connected subgraphs from the following graph by deleting each vertex. List out the simple paths from A to F in each of the subgraph.

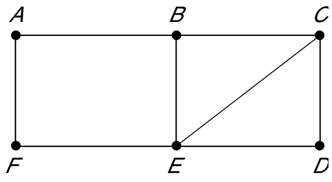


Figure 5.40

[**Ans:** The subgraphs are

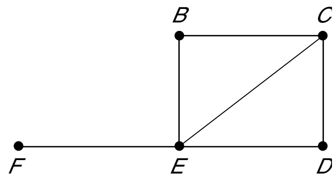


Figure 5.40 (a)

In figure (a) there is no path between A and F.

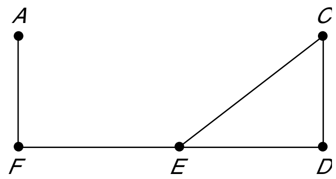


Figure 5.40 (b)

In figure (b) the paths are A-F

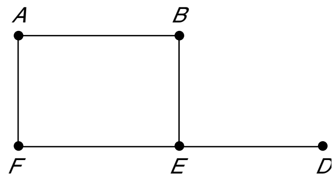


Figure 5.40 (c)

In figure (c) the paths are A-F, A-B-E-F

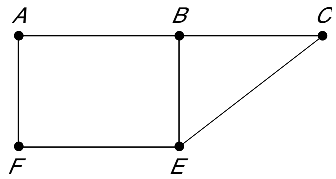


Figure 5.40 (d)

In figure (d) the paths are A-F, A-B-E-F, A-B-C-E-F

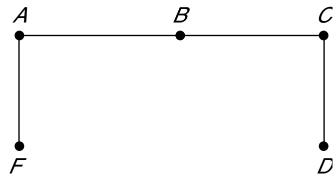


Figure 5.40 (e)

In figure (e) the paths are A-F

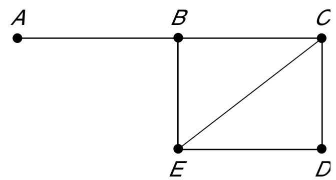


Figure 5.40 (f)

In figure (f) there is no path between A and F.

7) Draw a graph with 5 vertices A, B, C, D, E such that $d(A) = 3$, B is an odd vertex, $d(C) = 2$ and D and E are adjacent.

8) Draw the complete graph K_5 with vertices A, B, C, D, E. Draw all the subgraphs of K_5 with 4 vertices.

9) Using Havel–Hakimi theorem determine which of the following sequences are graphical. For each of those that are graphical, construct a graph.

   a) $\{5, 3, 3, 3, 3, 2, 2, 2, 1\}$

   b) $\{6, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1\}$

   c) $\{7, 5, 4, 4, 4, 3, 2, 1\}$

10) For which integer $x(0 \leq x \leq 7)$ the sequence $\{x, 7, 7, 5, 5, 4, 3, 2\}$ is graphical?

[**Ans:** $x = 5$ and $x = 3$]

11) Prove that a simple graph with n (≥ 2) vertices has at least two vertices of same degree. [WBUT-2004]

12) Find the maximum number of vertices in a connected graph having 17 edges. [WBUT-2008]

13) Find the minimum number of edges in a connected graph having 21 vertices. [WBUT-2008]

Multiple Choice Questions

1) An edge whose two end vertices coincide is called

   a) ring b) adjacent edge c) loop d) none of these

2) The number of edges between any two vertices may be

a)one b) two c) three d) any number of edges

3) A simple graph has

a) no parallel edges b) no self-loops

c) no parallel edges and self-loops d) no isolated vertex

4) The degree of an isolated vertex is

a) 0 b) 1 c) 2 d) 3

5) A pendant vertex is of degree

a) 0 b) 1 c) 2 d) 3

6) A complete graph must be a

a) circuit b) regular graph c) nonsimple graph d) null graph

7) The vertex set of a spanning subgraph of a graph G

a) is a proper subset of the vertex set of G

b) may not be a proper subset of the vertex set of G

c) identical with the vertex set of G

d) none of these

8) Which of the statement is correct?

a) Every walk is a path

b) Every circuit is a path.

c) Every loop is a circuit.

d) The origin and terminus of a walk are always distinct.

9) Which of the statement is correct?

a) Path is an open walk.

b) Every walk is a trail.

c) Every trail is a path.

d) Every vertex cannot appear twice in a walk.

10) A self-loop cannot be included in a

a) walk b) circuit c) trail d) path

Answers:

1 (c) 2 (d) 3 (c) 4 (a) 5 (b) 6 (b) 7 (c) 8 (c)
9 (a) 10 (d)

CHAPTER

6

Matrix Representation and Isomorphism of Graphs

6.1 INTRODUCTION

Matrix representation of graph is very essential although, the pictorial representation is more convenient for studying graphs. Representing a graph in a matrix form lies in the fact that it can be fitted to the computer, besides this many results of matrix algebra can be applied on the structural properties of graphs. There are different types of matrix representation of graphs among which adjacency matrix and incidence matrix are common. In this chapter we have discussed adjacency matrix of graphs, incidence matrix of graphs and circuit matrix. One of the important application of matrix representation of graphs is to see whether two graphs are isomorphic or not. In this chapter also, we have included various techniques for checking isomorphism, and the applications of the adjacency matrix and incidence matrix.

6.2 ADJACENCY MATRIX OF A GRAPH

6.2.1 Adjacency Matrix Representation of Simple Connected Graph

Let G be a simple graph (i.e., having no parallel edges and self-loops) with n vertices $v_1, v_2, \ldots, v_n$, then the adjacency matrix of G is given by a $n \times n$ matrix

$$A(G) = (a_{ij})_{n \times n}$$

where,

$$a_{ij} = \begin{cases} 1; & \text{when } v_i v_j \text{ is an edge of } G \\ 0; & \text{if there is no edge between } v_i \text{ and } v_j \end{cases}$$

Observations: **The following are the properties of adjacency matrix:**

1) **The order of the adjacency matrix is $n \times n$, where n is the number of vertices.**
2) **A simple graph has no self-loops, each diagonal entry of $A(G)$ i.e, $a_{ii} = 0$ for $i = 1, 2, 3, \ldots, n$**
3) **The adjacency matrix of a simple graph is symmetrical, i.e, $a_{ij} = a_{ji}$ for all i and j.**
4) **Given any symmetric matrix A with zero-one entries and which contains 0s on its diagonal, there exists a simple graph G whose adjacency matrix is A.**
5) **The number of 1s in a row or column of the adjacency matrix $A(G)$ is equal to the degree of the corresponding vertex.**

Example 1 **Find the adjacency matrix of the following simple graph.**

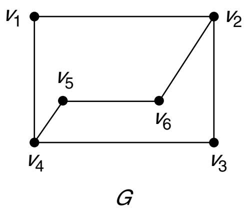


Figure 6.1

Sol. Here, the graph G is a simple graph. The adjacency matrix of G is

$$A(G) = (a_{ij})_{n \times n}$$

where,

$$a_{ij} = \begin{cases} 1; & \text{when } v_i v_j \text{ is an edge of } G \\ 0; & \text{if there is no edge between } v_i \text{ and } v_j \end{cases}$$

Here, the number of vertices $n = 6$. The adjacency matrix is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

6.2.2 Adjacency Matrix Representation of any Connected Graph

Let G be a **Connected Graph** with n-vertices $v_1, v_2, \ldots, v_n$ having no parallel edges (but may be with the self-loops), then the adjacency matrix of G is given by a $n \times n$ matrix

$$A(G) = (a_{ij})_{n\times n}$$

where,

$$a_{ij} = \begin{cases} 1; \text{ when } v_i v_j \text{ is an edge of } G \\ 0; \text{ if there is no edge between } v_i \text{ and } v_j \end{cases}$$

A self loop at the vertex v_i corresponds to $a_{ii} = 1$

Example 2 **Find the adjacency matrix of the following connected graph.**

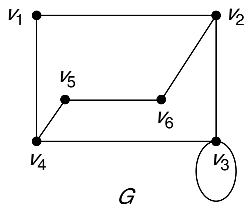


Figure 6.2

Sol. The Adjacency matrix of G is

$$A(G) = (a_{ij})_{n\times n}$$

where,

$$a_{ij} = \begin{cases} 1; \text{ when } v_i v_j \text{ is an edge of } G \\ 0; \text{ if there is no edge between } v_i \text{ and } v_j \end{cases}$$

A self loop at the vertex v_i corresponds to $a_{ii} = 1$
Here, $n = 6$ and the vertex v_3 has self loop so $a_{33} = 1$
Therefore, the adjacency matrix is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Theorem 6.1 **Let G be a graph with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and the adjacency matrix is $A(G) = (a_{ij})_{n\times n}$. Then the entry $a_{ij}^{(k)}$ in row i and coloumn j of $A^{k(G)}$ is the number of distinct $v_i - v_j$ walks of length k in G.**

6.2.3 Adjacency Matrix of a Digraph

Let G be a **digraph** with n-vertices $v_1, v_2, \ldots, v_n$ and having no parallel edges (but may be with the self loops), then the adjacency matrix of G is given by a $n \times n$ matrix

$$A(G) = (a_{ij})_{n\times n}$$

where,

$$a_{ij} = \left\{ \begin{array}{l} 1;\ \text{when there is an edge directed from } v_i \text{ to } v_j \\ 0;\ \text{if there is no edge between } v_i \text{ and } v_j \end{array} \right\}$$

A self-loop at the vertex v_i corresponds to $a_{ii} = 1$

Observations:

1) **If an adjacency matrix is not symmetric then it corresponds to a digraph.**
2) **The number of 1 in each row is the out-degree of the corresponding vertex and the number of 1 in each column is the in-degree of the corresponding vertex.**
3) **If A be the adjacency matrix for a digraph G, then A^T, transpose of A represents the adjacency matrix for digraph G', obtained by reversing the direction of every edge in G.**

Example 3 **Find the adjacency matrix of the following digraph.**

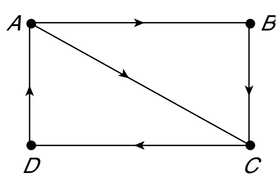


Figure 6.3

Sol. The adjacency matrix of a digraph is

$$A(G) = (a_{ij})_{n\times n}$$

where,

$$a_{ij} = \left\{ \begin{array}{l} 1;\ \text{when there is an edge directed from } v_i \text{ to } v_j \\ 0;\ \text{if there is no edge between } v_i \text{ and } v_j \end{array} \right\}$$

Here, $n = 4$. Therefore, the adjacency matrix is

$$A(G) = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

It is obvious that the matrix is not symmetric.

6.2.4 Adjacency Matrix of a Pseudograph

Let G be a **Pseudograph** with parallel edges and self loops with n-vertices $v_1, v_2, \ldots, v_n$, then the adjacency matrix of G is given by a $n \times n$ matrix

$$A(G) = (a_{ij})_{n \times n}$$

where,

$$a_{ij} = \left\{ \begin{array}{l} \text{number of edges that are incident on both } v_i \text{ and } v_j \\ 0; \text{ if there is no edge between } v_i \text{ and } v_j \end{array} \right\}$$

A self loop at the vertex v_i corresponds to $a_{ii} = 1$

The adjacency matrix of a pseudograph is a symmetric matrix.

Example 4 **Find the adjacency matrix of the following pseudograph.**

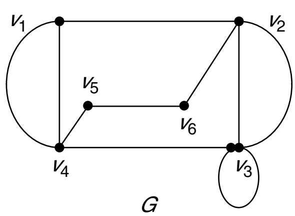


Figure 6.4

Sol. The Adjacency matrix of a pseudograph is

$$A(G) = (a_{ij})_{n \times n}$$

where,

$$a_{ij} = \left\{ \begin{array}{l} \text{number of edges that are incident on both } v_i \text{ and } v_j \\ 0; \text{ if there is no edge between } v_i \text{ and } v_j \end{array} \right\}$$

A self-loop at the vertex v_i corresponds to $a_{ii} = 1$

Here, $n = 6$ and there is a self-loop at the vertex v_3. So $a_{33} = 1$

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

6.2.5 Adjacency Matrix of a Disconnected Graph

Let G be a disconnected graph having two components G_1 and G_2. Then the adjacency matrix of G is given by a block diagonal form as

$$\boxed{A(G) = \begin{bmatrix} A(G_1) & O \\ O & A(G_2) \end{bmatrix}}$$

where $A(G_1)$ and $A(G_2)$ are adjacency matrices of G_1 and G_2 and O is the null matrix.

Example 5 **Find the adjacency matrix of the following disconnected graph.**

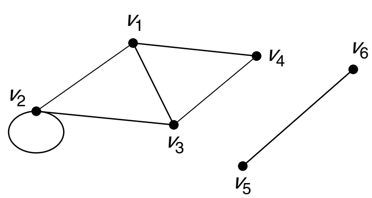


Figure 6.5

[WBUT-2006]

Sol. Here, the graph G has two components G_1 containing the vertices v_1, v_2, v_3, v_4 and G_2 containing the vertices v_5, v_6.

Therefore, the adjacency matrix of G is given by a block diagonal form as

$$A(G) = \begin{bmatrix} A(G_1) & O \\ O & A(G_2) \end{bmatrix}$$

where $A(G_1)$ and $A(G_2)$ are adjacency matrices of G_1 and G_2 and O is the null matrix.

Now,

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

and

$$A(G_2) = \begin{array}{c} \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cc} v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Therefore,

$$A(G) = \begin{bmatrix} A(G_1) & O \\ O & A(G_2) \end{bmatrix}$$

$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

6.2.6 Construction of Graph from a given Adjacency Matrix

Step 1 The order of the adjacency matrix is the number of vertices of the graph.

Step 2 If the adjacency matrix is in block diagonal form then the graph is a disconnected graph.

Step 3 If the matrix or the blocks are not symmetric then the graph is a digraph.

Step 4 For a connected graph if

$$a_{ij} = \left\{ \begin{array}{l} 1;\ \text{when } v_i v_j \text{ is an edge of } G \\ 0;\ \text{if there is no edge between } v_i \text{ and } v_j \end{array} \right\}$$

A self loop at the vertex v_i corresponds to $a_{ii} = 1$.

Step 5 For a digraph

$$a_{ij} = \left\{ \begin{array}{l} 1;\ \text{when there is an edge directed from } v_i \text{ to } v_j \\ 0;\ \text{if there is no edge between } v_i \text{ and } v_j \end{array} \right\}$$

A self-loop at the vertex v_i corresponds to $a_{ii} = 1$

Step 6 For a disconnected graph

$$A(G) = \begin{bmatrix} A(G_1) & O \\ O & A(G_2) \end{bmatrix}$$

where $A(G_1)$ and $A(G_2)$ are adjacency matrices of G_1 and G_2 and O is the null matrix.

Example 6 **Construct a graph whose adjacency matrix is given by**

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Sol. Here, the order of the matrix is 6. Therefore, the number of vertices is 6.

Let the vertices be A, B, C, D, E and F respectively. After labelling the vertices the matrix becomes

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Here, the graph is a disconnected graph since the adjacency matrix can be represented as

$$A(G) = \begin{bmatrix} A(G_1) & O \\ O & A(G_2) \end{bmatrix}$$

where,

$$A(G_1) = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \text{ and } A(G_2) = \begin{array}{c} \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{ccc} D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

We also see that each block diagonal is symmetric, so the graph is undirected.

Now we notice that in $A(G_1)$, $a_{22} = 1$ (i.e., at the position 2^{nd} row 2^{nd} column value is 1), so there is a self-loop at the vertex B.

Therefore, the graph is represented as

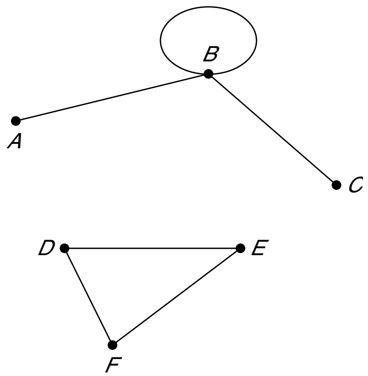


Figure 6.6

6.3 INCIDENCE MATRIX OF A GRAPH

6.3.1 Incidence Matrix Representation of Simple Connected Graph

Let G be a simple graph (i.e., having no parallel edges and self-loops) with n-vertices $v_1, v_2, \ldots, v_n$, and m edges $e_1, e_2, \ldots, e_m$ then the incidence matrix of G is given by a $n \times m$ matrix

$$I(G) = (a_{ij})_{n\times m}$$

where,

$$a_{ij} = \begin{cases} 1; \text{ when edge } e_j \text{ is incident on } v_i \\ 0; \text{ if there is no edge } e_j \text{ incident on } v_i \end{cases}$$

Observations: **The following are the properties of incidence matrix:**

1) **The order of the incidence matrix $I(G)$ is $n \times m$ where n is the number of vertices and m is the number of edges.**
2) **Each column of $I(G)$ has exactly two unit entries.**
3) **A row with all zeros corresponds to an isolated vertex.**
4) **A row with a single unit entry corresponds to a pendant vertex.**
5) **Degree of a vertex is equal to the number of 1s in the row of the vertex.**

6) The permutation of any two rows or columns on an incidence matrix of a graph G corresponds to re-labelling the vertices and edges of G.

Example 7 **Find the incidence matrix of the following graph**

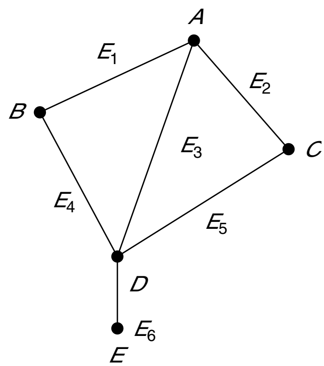


Figure 6.7

Sol. Here we have 5 vertices and 6 edges, so the incidence matrix is of order 5×6 and is given by

$$I(G) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

6.3.2 Incidence Matrix Representation of any Connected Graph

Let G be a connected graph having no self-loops (but may be with the parallel edges) and with n-vertices $v_1, v_2, \ldots, v_n$, and m edges $e_1, e_2, \ldots, e_m$ then the incidence matrix of G is given by a $n \times m$ matrix

$$I(G) = (a_{ij})_{n \times m}$$

where,

$$a_{ij} = \begin{cases} 1; & \text{when edge } e_j \text{ is incident on } v_i \\ 0; & \text{if there is no edge } e_j \text{ incident on } v_i \end{cases}$$

Example 8 **Find the incidence matrix of the following graph.**

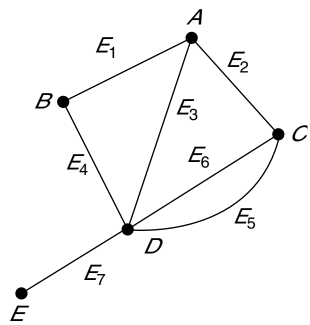


Figure 6.8

Sol. Here we have 5 vertices and 7 edges, so the incidence matrix is of order 5×7 and is given by

$$I(G) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

6.3.3 Incidence Matrix Representation of a Connected Digraph

Let G be a **Connected Digraph** with no self-loops (but may be with the parallel edges) with n-vertices $v_1, v_2, \ldots, v_n$, and m edges $e_1, e_2, \ldots, e_m$ then the incidence matrix of G is given by a $n \times m$ matrix

$$I(G) = (a_{ij})_{n \times m}$$

where,

$$a_{ij} = \begin{Bmatrix} 1; & \text{when edge } e_j \text{ is incident out of } v_i \\ -1; & \text{when edge } e_j \text{ is incident into } v_i \\ 0; & \text{if there is no edge } e_j \text{ incident out of or into } v_i \end{Bmatrix}$$

Example 9 **Find the incidence matrix of the following Digraph**

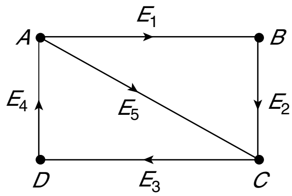


Figure 6.9

Sol. Here we have 4 vertices and 5 edges, so the incidence matrix is of order 4×5 and is given by

$$I(G) = \begin{matrix} \\ A \\ B \\ C \\ D \end{matrix} \begin{matrix} \begin{matrix} E_1 & E_2 & E_3 & E_4 & E_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{matrix}$$

6.3.4 Incidence Matrix of a Disconnected Graph

Let G be a disconnected graph having two components G_1 and G_2. Then the incidence matrix of G is given by a block diagonal form as

$$\boxed{I(G) = \begin{bmatrix} I(G_1) & O \\ O & I(G_2) \end{bmatrix}}$$

where $I(G_1)$ and $I(G_2)$ are incidence matrices of G_1 and G_2 and O is the null matrix.

Example 10 **Find the incidence matrix of the following Disconnected Graph.**

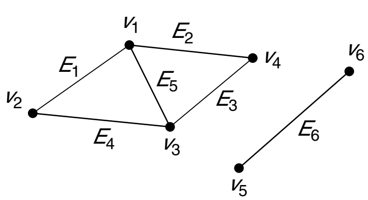


Figure 6.10

Sol. Here, the graph G has two components G_1 containing the vertices v_1, v_2, v_3, v_4 and G_2 containing the vertices v_5, v_6.

Incidence matrix for G_1 is

$$I(G_1) = \begin{matrix} & E_1 & E_2 & E_3 & E_4 & E_5 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Incidence matrix for G_2 is

$$I(G_2) = \begin{matrix} v_5 \\ v_6 \end{matrix} \overset{E_6}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Hence, incidence matrix for G is

$$I(G) = \begin{bmatrix} I(G_1) & O \\ O & I(G_2) \end{bmatrix} = \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{matrix} \overset{\begin{matrix} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{matrix}}{\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

6.3.5 Construction of Graph/Digraph from Incidence Matrix

The following steps are to be followed to construct a graph from a given incidence matrix

Step 1 The number of rows and columns correspond to number of vertices and edges of the graph/digraph.

Step 2 If the entries are zeros and one then it represents an undirected graph and if the entries are $0, 1$ and -1 then it represents digraph.

Step 3 Take the first column which corresponds to the edge e_1 and see from the entries in rows which are 1 are the vertices which form the edge e_1.

For an edge of a digraph take the first column which corresponds to the edge e_1 and see from the entries in rows which are 1 and -1 are the vertices which form the edge e_1

Step 4 Draw the edge e_1 and its connecting vertices.

Step 5 Repeat the steps 3 and 4 for rest of the columns and draw the corresponding edges.

Step 6 The resulting figure is the required undirected graph or digraph.

Example 11 **Draw the graph whose incidence matrix is given by**

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

[WBUT-2006]

Sol. The entries of the matrix are $0, 1, -1$. Therefore the graph corresponding to the matrix is a digraph.

Since the number of rows and columns of the matrix are equal to the number of vertices and edges of the digraph, therefore, the number of vertices are 6 and the number of edges are 5.

Let us suppose, the vertices are $v_1, v_2, v_3, v_4, v_5, v_6$ and the edges are e_1, e_2, e_3, e_4, e_5.

After labelling the vertices the matrix becomes

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

Here the third row, i.e., the row corresponding to the vertex v_3 contains only zeros, so no edge is incident on the vertex v_3. Therefore, v_3 is an isolated vertex.

Hence, the digraph is given by

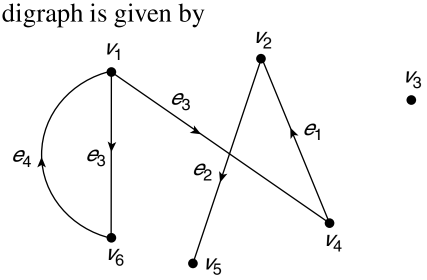


Figure 6.11

6.4 CIRCUIT MATRIX

6.4.1 Definition

Let G be a **Graph** containing n circuits and m edges then the circuit matrix of G is

$$C(G) = (a_{ij})_{n \times m}$$

where,

$$a_{ij} = \begin{cases} 1; & \text{when } i\text{-th circuit contains } j\text{-th edge} \\ 0; & \text{otherwise} \end{cases}$$

Observations: **The following are the properties of Circuit Matrix**

(1) Each row of a circuit matrix is a circuit.

(2) All vertex in a coloumn of a circuit matrix correspond to a non circuit edges.

(3) The number of edges in a circuit is equal to number of 1s in a row of circuit matrix.

(4) A circuit matrix is capable of representing a loop.

Example 12 **Find the circuit matrix of the following graph**

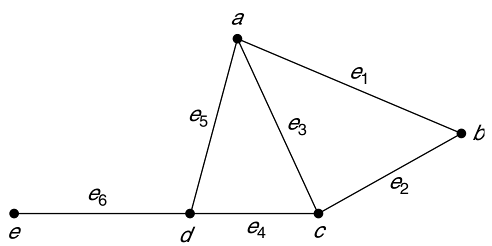


Figure 6.12

Sol. The graph has three circuits, viz., $A = (a, b, c)$, $B = (a, c, d)$ and $C = (a, b, c, d)$ and 6 edges.
The circuit matrix of the graph is

$$C(G) = (a_{ij})_{n \times m}$$

where,

$$a_{ij} = \begin{cases} 1; & \text{when ith circuit contains jth edge} \\ 0; & \text{otherwise} \end{cases}$$

Therefore, the circuit matrix of the graph is

$$C(G) = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

6.5 GRAPH ISOMORPHISM

6.5.1 Definition

Two graphs G_1 and G_2 are isomorphic if there is a function $f : V(G_1) \to V(G_2)$ from the vertices of G_1 to the vertices of G_2 such that

(a) **f is one to one**

(b) **f is onto, and**

(c) **for each pair of vertices u and v of G_1, $\{u, v\} \in E(G_1)$ if and only if $\{f(u), f(v)\} \in E(G_2)$.**

[WBUT-2009]

Observations:

1) **The condition that for each pair of vertices u and v of G_1, $\{u, v\} \in E(G_1)$ if and only if $\{f(u), f(v)\} \in E(G_2)$ says that vertices u and v are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. Therefore, the function f preserves adjacency.**
2) **If G_1 and G_2 are isomorphic and f is an isomorphism of G_1 to G_2 then the isomorphism f is by no means unique, there may be several isomorphisms from G_1 to G_2.**
3) **If an isomorphism f exists, then**

   (a) $|V(G_1)| = |V(G_2)|$

   (b) $|E(G_1)| = |E(G_2)|$

   (c) **If $v \in V(G_1)$, then $d_{G_1}(v) = d_{G_2}(f(v))$ and thus, the degree sequence of G_1 and G_2 are the same. (In other words, the number of same degree vertices are always same.)**

   (d) **If $\{v, v\}$ is a loop in G_1 then $\{f(v), f(v)\}$ is a loop in G_2.**

   (e) **If two graphs are isomorphic, they will contain circuits of same length k where $k > 2$.**

6.5.2 Isomorphism Problem

The problem of determining whether or not two graphs are isomorphic is known as an isomorphism problem.

Theoretically, it is always possible to determine whether or not two graphs G_1 and G_2 are isomorphic by keeping the order of vertices of G_1 fixed and reordering the vertices of G_2 to check the mapping $f : V(G_1) \to V(G_2)$ for isomorphism. But if G_1 and G_2 are two graphs with the same number of vertices, say n, then there are $n!$ such one-to-one, onto maps $f : V(G_1) \to V(G_2)$, so to check all such $n!$ mappings for isomorphism, specially for large n is too laborious and sometimes it is quite impossible. Therefore, it serves no practical purpose.

6.5.3 Working Procedure to find Whether Two Graphs G_1 and G_2 are Isomorphic or Not

Working Procedure 1

Step 1 Check whether the two graphs G_1 and G_2 have the same number of vertices.

Step 2 Check whether the two graphs G_1 and G_2 have the same number of edges.

Step 3 Find the incidence matrices for both the graphs $I(G_1)$ and $I(G_2)$.

Step 4 Check whether, the two graphs G_1 and G_2 have the same degree sequence. In other words, check whether the number of same degree vertices are same for the two graphs G_1 and G_2.

Step 5 Two graphs G_1 and G_2 are isomorphic, if and only if, the incidence matrix of G_1 is obtained from the permutation of rows and columns of the incidence matrix of G_2 and vice-versa.

Example 13 **Examine if the two graphs are isomorphic.**

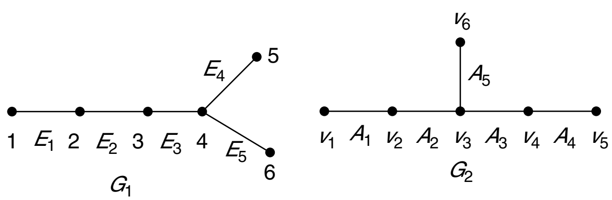


Figure 6.13

[WBUT-2006]

Sol. Both the graphs G_1 and G_2 have 6 vertices and 5 edges.

In both the graphs G_1 and G_2, there are three vertices of degree 1, two vertices of degree 2, one vertex is of degree 3.

So, the number of same degree vertices are same, for the two graphs G_1 and G_2.

The incidence matrix of the graph G_1 is

$$I(G_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and the incidence matrix of the graph G_2 is

$$I(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

We have, two graphs G_1 and G_2 are isomorphic if and only if the incidence matrix of G_1 is obtained by the permutation of rows and columns in the incidence matrix of G_2.

Here we see that first four columns are identical in $I(G_1)$ and $I(G_2)$, but the $5th$ column is still different. Therefore $I(G_1)$ cannot be obtained by the permutation of rows and columns of $I(G_2)$. Hence G_1 and G_2 are not isomorphic.

Working Procedure 2

Step 1 Check whether the two graphs G_1 and G_2 have the same number of vertices.

Step 2 Check whether the two graphs G_1 and G_2 have the same number of edges.

Step 3 Check whether the two graphs G_1 and G_2 have the same degree sequence. In other words, check whether the number of same degree vertices are same for the two graphs G_1 and G_2.

Step 4 Write the adjacency matrices $A(G_1)$ and $A(G_2)$ of G_1 and G_2 respectively. If the adjacency matrices are same, then the graphs G_1 and G_2 are isomorphic.

Step 5 If the adjacency matrices $A(G_1)$ and $A(G_2)$ of G_1 and G_2 are not same, then to establish isomorphism between G_1 and G_2, we have to find a permutation matrix P such that

$$PA(G_1)P^T = A(G_2)$$

Step 6 Since $A(G_1)$and $A(G_2)$ are n^{th} order matrices, P is a n^{th} order matrix obtained by permuting the rows of the unit matrix I_n.

Step 7 If there exists a permutation matrix P such that

$$PA(G_1)P^T = A(G_2)$$

then the two graphs are isomorphic, otherwise not.

Example 14 **Examine whether the following two graphs are isomorphic**

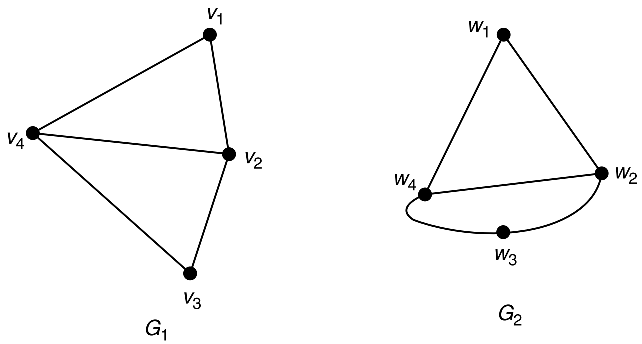


Figure 6.14

[WBUT-2007]

Sol. Both the graphs G_1 and G_2 have 4 vertices and 5 edges.
Also, in the graphs G_1 and G_2 there are 2 vertices of degree 3 and 2 vertices of degree 2.
Hence, the necessary condition of isomorphism are satisfied.
The adjacency matrices of the two graphs are

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \text{ and } A(G_2) = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Since,

$$A(G_1) = A(G_2)$$

Therefore, the two graphs G_1 and G_2 are isomorphic.

Example 15 **Show that the graphs G_1 and G_2 are isomorphic.**

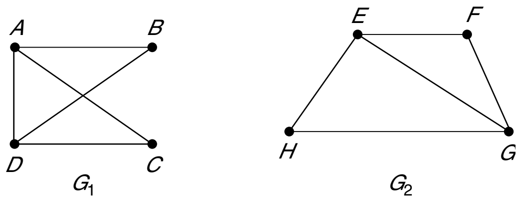


Figure 6.15

Sol. The two graphs G_1 and G_2 have 4 vertices each, 5 edges each and 2 vertices of degree 2 and 2 vertices of degree 3.

The adjacency matrices of G_1 and G_2 are

$$A(G_1) = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \text{ and } A(G_2) = \begin{array}{c} \\ E \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{cccc} E & F & G & H \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The matrices $A(G_1)$ and $A(G_2)$ of G_1 and G_2 are not same.

To establish isomorphism between G_1 and G_2, we have to find a permutation matrix P such that

$$PA(G_1)P^T = A(G_2)$$

Since $A(G_1)$ and $A(G_2)$ are fourth order matrices, P is also a fourth order matrix obtained by permuting the rows of the unit matrix I_4.

There are $4! = 24$ different forms for P. It is difficult to find the appropriate P from among the 24 matrices by trial that will satisfy

$$PA(G_1)P^T = A(G_2)$$

Using the degree of the vertices of G_1 and G_2 we find the permutation matrix P.

In the adjacency matrix $A(G_1)$, the degree of the vertex A, $d(A) = 3$ and in the adjacency matrix $A(G_2)$, the degree of the vertex E $d(E) = 3$, that is, the first vertex of G_1 corresponds to the first vertex of G_2.

Hence, the first row of I_4 can be taken as the first row of P.

Now, in the adjacency matrix $A(G_1)$, the degree of the vertex D, $d(D) = 3$ and in the adjacency matrix $A(G_2)$, the degree of the vertex G, $d(G) = 3$, that is, the fourth vertex of G_1 corresponds to the third vertex of G_2.

Hence, the fourth row of I_4 is taken as the third row of P.

Now, in the adjacency matrix $A(G_1)$, the degree of the vertex B, $d(B) = 2$ and in the adjacency matrix $A(G_2)$, the degree of the vertex F, $d(F) = 2$, that is, the second vertex of G_1 corresponds to the second vertex of G_2.

Hence, the second row of I_4 is taken as the second row of P.

Now, in the adjacency matrix $A(G_1)$, the degree of the vertex C, $d(C) = 2$ and in the adjacency matrix $A(G_2)$, the degree of the vertex H, $d(H) = 2$, that is, the third vertex of G_1 corresponds to the fourth vertex of G_2.

Hence, the third row of I_4 is taken as the fourth row of P.

Therefore,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now,

$$\begin{aligned} PA(G_1)P^T &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = A(G_2) \end{aligned}$$

Therefore, the two graphs G_1 and G_2 are isomorphic.

Working Procedure 3

Step 1 Check whether the two graphs G_1 and G_2 have the same number of vertices.

Step 2 Check whether the two graphs G_1 and G_2 have the same number of edges.

Step 3 Check whether the two graphs G_1 and G_2 have the same degree sequence. (In other words check whether the number of same degree vertices are same.for the two graphs G_1 and G_2)

Step 4 Check whether the two graphs G_1 and G_2 contain circuits of the same length k, where $k > 2$.

In both the circuits, if the degrees of the ordered vertices are same, then their adjacency matrices are also same and the two graphs G_1 and G_2 are isomorphic.

Step 5 In both the circuits, if the degrees of the ordered vertices are not same then their adjacency matrices also are not same and the two graphs G_1 and G_2 are not isomorphic.

Example 16 **Show that the graphs G and G' are isomorphic.**

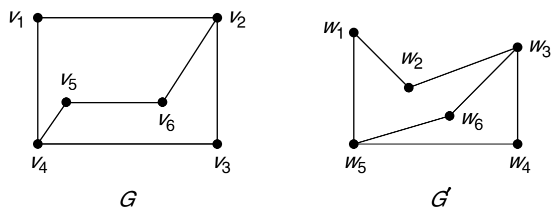


Figure 6.16

[WBUT-2006]

Sol. The two graphs G and G' have 6 vertices and 7 edges each.

They both have four vertices of degree 2 and two vertices of degree 3.

So, the number of same degree vertices are same for the two graphs G_1 and G_2.

The graphs will be isomorphic if they contain circuits of the same length k, where $k > 2$.

We see that G has a circuit of length 5 which pass through $v_4 - v_1 - v_2 - v_6 - v_5 - v_4$ and G' has a circuit of length 5 which pass through $w_5 - w_6 - w_3 - w_2 - w_1 - w_5$.

The degrees of the ordered vertices in both the circuits are $3, 2, 3, 2, 2$.

The adjacency matrix of the two graphs (according to the sequence of vertices in the circuits in the respective graphs) are given by

$$A(G) = \begin{array}{c} \\ v_4 \\ v_1 \\ v_2 \\ v_6 \\ v_5 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} v_4 & v_1 & v_2 & v_6 & v_5 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \text{ and } A(G') = \begin{array}{c} \\ w_5 \\ w_6 \\ w_3 \\ w_2 \\ w_1 \\ w_4 \end{array} \begin{array}{c} \begin{array}{cccccc} w_5 & w_6 & w_3 & w_2 & w_1 & w_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here, the adjacency matrix are same, therefore the two graphs G and G' are isomorphic.

6.5.4 Another Important Result on Isomorphism of Two Graphs

Two graphs G and H are isomorphic, if and only if, their complements $\overline{G}$ and $\overline{H}$ are isomorphic.

We can also apply this technique for checking isomorphism which will be illustrated in the next example.

Example 17 **Examine if the two graphs are isomorphic**

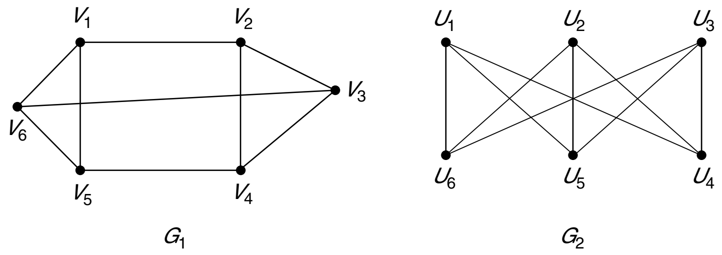


Figure 6.17

Sol. Let us consider the complements, $H_1 = \overline{G_1}$ and $H_2 = \overline{G_2}$. The graphs G_1 and G_2 are isomorphic, if and only if, the complements H_1 and H_2 are isomorphic.

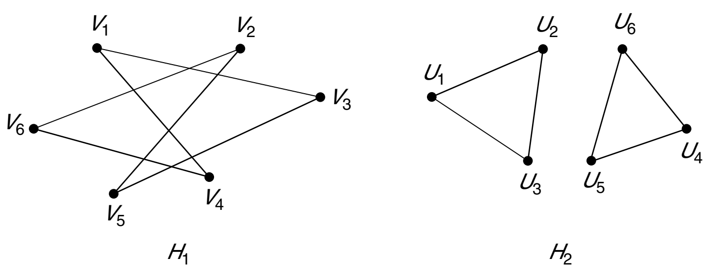


Figure 6.18

The graphs H_1 and H_2 both have 6 vertices and 6 edges.

Also, in both the graphs all the vertices is of degree 2. So, the number of same degree vertices are same for the graphs H_1 and H_2.

Since all the vertices are of same degree, let us consider three vertices V_1, V_2 and V_3 of H_1 that map into the three vertices U_1, U_2 and U_3 of H_2 and three vertices V_4, V_5 and V_6 of H_1 that map into the three vertices U_4, U_5 and U_6 of H_2.

U_1, U_2 and U_3 of H_2 are pairwise adjacent and form a triangle. Similarly, U_4, U_5 and U_6 of H_2 are pairwise adjacent and form a triangle, but H_1 does not contain a triangle.

Therefore, H_1 and H_2 are not isomorphic and consequently G_1 and G_2 are not isomorphic.

Note: Here in the problem for checking isomorphism, we can also apply directly any one of the above stated working procedures.

Theorem 6.2 **Let G and H are isomorphic graphs. Then**

a) G is bipartite, if and only if, H is bipartite, and

b) G is connected, if and only if, H is connected.

Proof Beyond the scope of the book.

WORKED OUT EXAMPLES

Example 6.1 **Draw the graph whose incidence matrix is given by**

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

[WBUT-2007]

Sol. Here in the matrix we have 5 rows and 6 columns. So the number of vertices in the graph is 5 and the number of edges in the graph is 6.

Since all the entries in the matrix are 0 or 1, the graph is an undirected graph.

Let the vertices be V_1, V_2, V_3, V_4, V_5 and the edges be $E_1, E_2, E_3, E_4, E_5, E_6$. So, with the labels the matrix becomes

$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore, the graph is given by

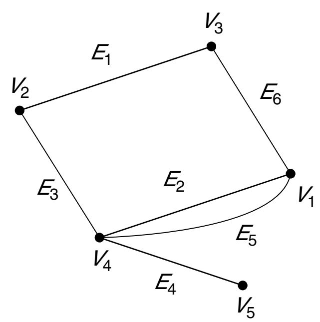


Figure 6.19

Example 6.2 **Determine the adjacency matrix and incidence matrix of the following graph.**

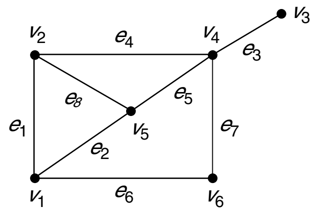


Figure 6.20

[WBUT-2002, 2011]

Sol. Here we have six vertices and eight edges.

Therefore, the order of the adjacency matrix is 6×6 and is given by

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Thus, the order of the incidence matrix is 6×8 and is given by

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example 6.3 **Examine whether the following two graphs G and G' are isomorphic.**

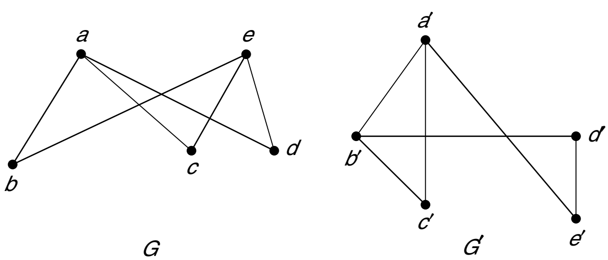


Figure 6.21

[WBUT-2008, 2005]

Sol. The two graphs G and G' have 5 vertices each, 5 edges each and they both have 3 vertices of degree 2 and 2 vertices of degree 3.

The adjacency matrices of G and G' are

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \text{ and } A(G') = \begin{array}{c} \\ a' \\ b' \\ c' \\ d' \\ e' \end{array} \begin{array}{c} \begin{array}{ccccc} a' & b' & c' & d' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The matrices $A(G)$ and $A(G')$ of G and G' are not same.

To establish isomorphism between G and G', we have to find a permutation matrix P such that

$$PA(G)P^T = A(G')$$

Since $A(G)$ and $A(G')$ are fifth order matrices, P is also a fifth order matrix got by permuting the rows of the unit matrix I_5.

There are $5! = 120$ different forms for P. It is difficult to find the appropriate P from among the 120 matrices by trial that will satisfy

$$PA(G)P^T = A(G')$$

Using the degree of the vertices of G and G' we find the permutation matrix P.

In the adjacency matrix $A(G)$, the degree of the vertex a, $d(a) = 3$ and in the adjacency matrix $A(G')$, the degree of the vertex a', $d(a') = 3$, that is, the first vertex of G corresponds to the first vertex of G'.

Hence, the first row of I_5 can be taken as the first row of P.

Now, in the adjacency matrix $A(G)$, the degree of the vertex b, $d(b) = 2$ and in the adjacency matrix $A(G')$, the degree of the vertex c', $d(c') = 2$, that is, the second vertex of G corresponds to the third vertex of G'.

Hence, the second row of I_5 is taken as the third row of P.

Now, in the adjacency matrix $A(G)$, the degree of the vertex c, $d(c) = 2$ and in the adjacency matrix $A(G')$, the degree of the vertex d', $d(d') = 2$, that is, the third vertex of G corresponds to the fourth vertex of G'.

Hence, the third row of I_5 is taken as the fourth row of P.

Now, in the adjacency matrix $A(G)$, the degree of the vertex d, $d(d) = 2$ and in the adjacency matrix $A(G')$, the degree of the vertex e', $d(e') = 2$, that is, the fourth vertex of G corresponds to the fifth vertex of G'.

Hence, the fourth row of I_5 is taken as the fifth row of P.

Now, in the adjacency matrix $A(G)$, the degree of the vertex d, $d(e) = 3$ and in the adjacency matrix $A(G')$, the degree of the vertex b', $d(b') = 3$, that is, the fifth vertex of G corresponds to the second vertex of G'.

Hence, the fifth row of I_5 is taken as the second row of P.

Therefore,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now,

$$PA(G)P^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \neq A(G')$$

Therefore, the graphs G and G' are not isomorphic.

Example 6.4 **Examine whether the following graphs G and G' are isomorphic. Give reasons.**

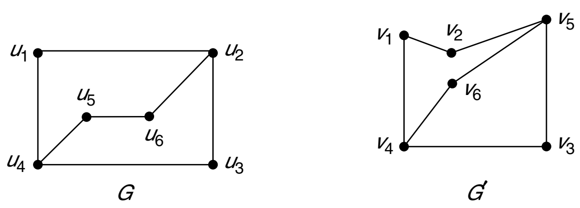


Figure 6.22

[WBUT-2009]

Sol. Both the graphs G and G' have 6 vertices and 7 edges.

Also, in the graphs $G$and G' there are 2 vertices of degree 3 and 4 vertices of degree 2.

Hence, the necessary condition of isomorphism are satisfied.

Now, there is only one circuit of length 5 from u_2 to u_2, viz, $u_2 - u_6 - u_5 - u_4 - u_1 - u_2$ in G and there is only one circuit of length 5 from v_5 to v_5 in G'viz, $v_5 - v_2 - v_1 - v_4 - v_6 - v_5$.

The adjacency matrices of the two graphs (according to the sequence of vertices in the circuits in the respective graphs) are given by

$$A(G) = \begin{array}{c} \\ u_2 \\ u_6 \\ u_5 \\ u_4 \\ u_1 \\ u_3 \end{array} \begin{array}{c} u_2\ u_6\ u_5\ u_4\ u_1\ u_3 \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad A(G') = \begin{array}{c} \\ v_5 \\ v_2 \\ v_1 \\ v_4 \\ v_6 \\ v_3 \end{array} \begin{array}{c} v_5\ v_2\ v_1\ v_4\ v_6\ v_3 \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore, the two graphs G and G' are isomorphic.

Example 6.5 **Using circuits show the the graphs are not isomorphic.**

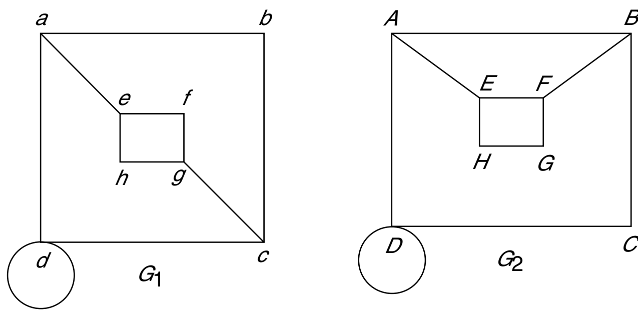


Figure 6.23

Sol. Both the graphs G_1 and G_2 have 8 vertices and 11 edges.

Also in both the graphs G_1 and G_2, there are 4 vertices of degree 3, 3 vertices of degree 2 and 1 vertex of degree 4.

Hence, the necessary condition of isomorphism are satisfied.

Now, there is only one circuit of length 4 from a to a ,viz, $a - b - c - d - a$ in the graph G_1 and there are two circuits of length 4 from A to A in the graph G_2 viz, $A - B - C - D - A$ and $A - B - F - E - A$.

Since for two isomorphic graphs, they must contain the same number of circuits of same length.

Therefore, the two graphs G_1 and G_2 are not isomorphic.

EXERCISES

Short and Long Answer Type Questions

1) Find the adjacency matrix of the following graphs:

a)

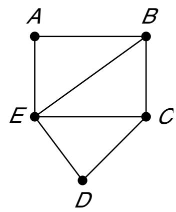


Figure 6.24

$$\left[\textbf{Ans}: \quad A(G) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \right]$$

b)

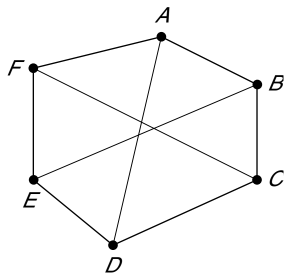


Figure 6.25

$$\left[\textbf{Ans}: \quad A(G)= \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \right]$$

c)

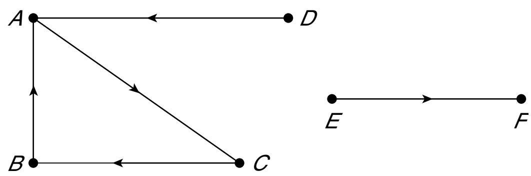


Figure 6.26

Ans: $\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} A\ B\ C\ D\ E\ F \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

2) Draw the graphs of the following adjacency matrices:

a) $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} A\ B\ C\ D \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$

Ans:

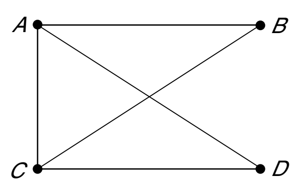


Figure 6.27

b) $\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} A\ B\ C\ D\ E \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$

Ans:

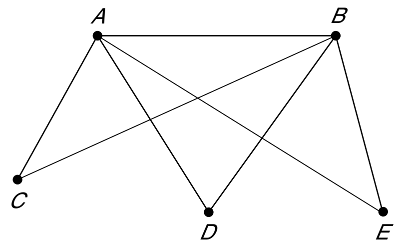


Figure 6.28

c)

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ans:

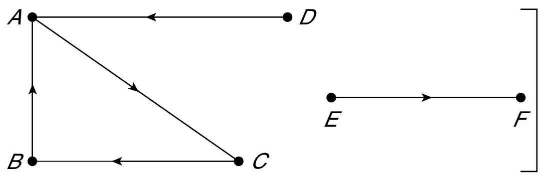


Figure 6.29

3) Find the incidence matrix of the following graphs:

a)

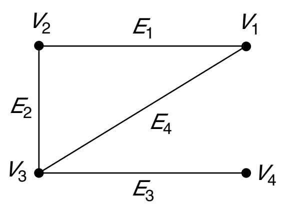


Figure 6.30

Ans:

$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b)

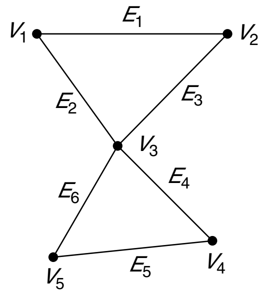


Figure 6.31

$$\left[\textbf{Ans}: \quad \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}\right]$$

c)

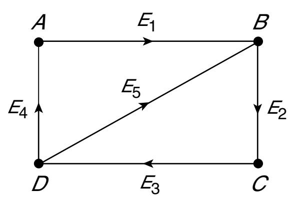


Figure 6.32

$$\left[\textbf{Ans}: \quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \end{array}\right]$$

4) Draw the graphs of the following incidence matrices:

a)
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Ans:

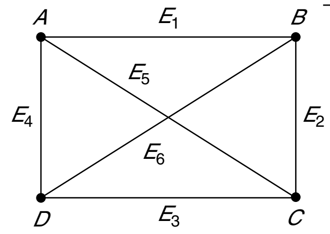


Figure 6.33

b)

	E_1	E_2	E_3	E_4	E_5	E_6	E_7
V_1	1	0	0	0	−1	0	−1
V_2	−1	1	0	0	1	0	0
V_3	0	−1	−1	0	0	0	1
V_4	0	0	1	1	0	−1	0
V_5	0	0	0	−1	1	0	0

Ans:

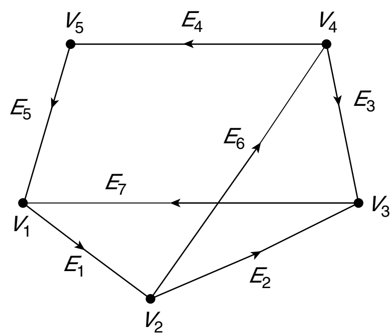


Figure 6.34

5. Examine the isomorphism of the following graphs:

a)

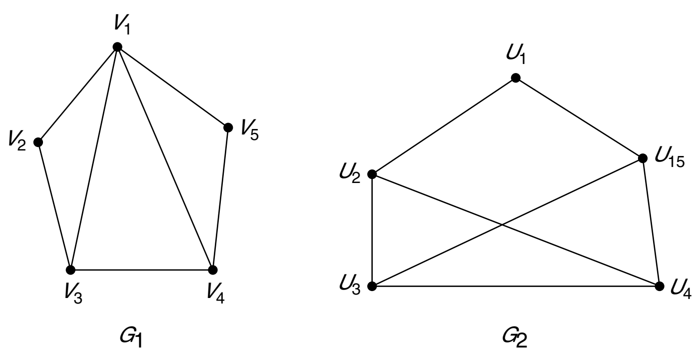


Figure 6.35

[**Ans:** not isomorphic]

b)

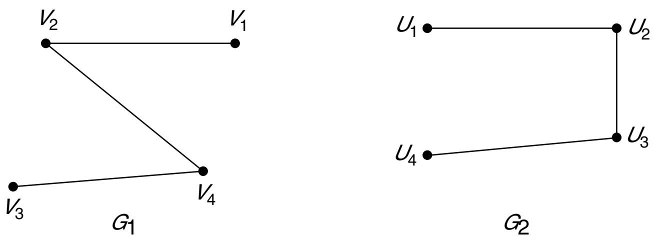


Figure 6.36

[**Ans:** isomorphic]

c)

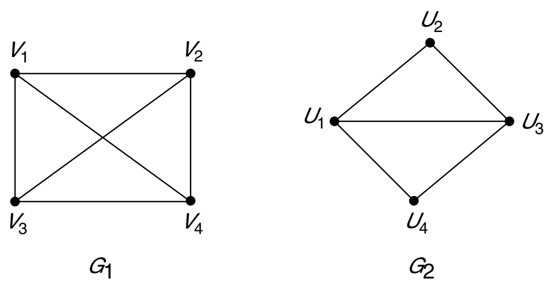


Figure 6.37

[**Ans:** isomorphic]

Multiple Choice Questions

1. The columns of an incidence matrix of a graph corresponds to

   a) vertices b) edges c) regions d) none of these

2. If the number of vertices and edges excluding self loops is same then the incidence matrix of the graph is

   a) symmetric b) identical c) square d) null

3. Incidence matrix of a graph is called a binary matrix since

   a) it corresponds two vertex and edges

   b) it contains only two elements 0 and 1

   c) it has only two rows

   d) none of these

4. If the inncidence matrix of a graph G_1 is obtained from that of G_2 by permutation of rows and columns, then G_1 and G_2 are

   a) isomorphic b) may be non-isomorphic

   c) dual of each other d) complement of each other

5. If the incidence matrix of a graph has four zero row vectors, then the graph has

   a) four parallel edges b) four loops

   c) four pendant vertices d) four isolated vertices

6. For an adjacency matrix which one of the following is false

   a) is a square matrix b) is a symmetric matrix

   c) it may be nonsingular d) it must be nonsingular

7. Adacency matrix of a graph is

   a) symmetric b) skew symmetric

   c) singular d) none of these

8. For a simple graph with 7 vertices and 8 edges if the 3rd row contains four 1 then

   a) degree of $v_3 = 4$ b) degree of $v_1 = 4$

   c) degree of $v_3 = 3$ d) degree of $v_3 = 1$

Answers:

1 (b) 2 (c) 3 (b) 4 (a) 5 (d) 6 (d) 7 (d) 8(a)

CHAPTER

7

Tree

7.1 INTRODUCTION

Tree is a very important topic in the graph theory, specially spanning trees. There is a huge application in the different branches of science and technology, specially in the field of computer science, Communication Engineering, circuit theory and network, etc. In this chapter we give basic properties of trees along with the concept of spanning trees. Different kinds of searching algorithm are dependent on rooted trees and binary trees. It is also included in the chapter. Here also we represent different algorithms for finding minimal spanning trees such as Krushkal Algorithm, Prim's Algorithm. The concept of Cut set is very much important in the network theory. Here we will discuss cut set, Fundamental cut set illustrated with examples.

7.2 DEFINITION AND PROPERTIES

7.2.1 Definition

A graph is called an acyclic graph if it has no cycle or circuit.

A connected acyclic graph is called a **tree**.

In a tree **pendent vertices** are also known as leaves. A nonpendent vertex in a tree is called an **internal vertex**.

A collection of some trees is known as **forest**.

Example 1 **Here the following figure represents a tree with 6 vertices.**

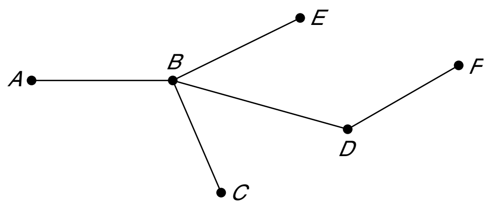


Figure 7.1

Here A, C, E, F are the pendent vertices (leaves) of the tree and B, D are the internal vertices of the tree.

Theorem 7.1 **A graph G is a tree if any two vertices of G are connected by a unique path.**

Alternative Statement Prove that if there is one and only one path between every pair of vertices in a graph G then G is a tree. [WBUT 2004]

Proof First let us consider that the graph G be a tree, then by definition G is connected and so any two vertices of G are connected by a path. The path must be unique for otherwise, if two vertices are connected by two different paths, then the union of these paths contains a cycle, which is a contradiction. So the path must be unique.

Conversely let G be a connected graph such that any two vertices are connected by a unique path. To prove that G is a tree, we have to show there is no cycle in G. Let, if possible, G contain a cycle and u and v are the two vertices of the cycle. Then there are two paths connecting u and v, which contradicts the hypothesis. Hence G is a tree.

Theorem 7.2 **Prove that a tree with n vertices has $(n-1)$ edges.**
[WBUT 2003, 2006, 2011]

Proof Let us consider a tree $T(V, E)$ with n number of vertices, i.e., $|V| = n$. Here we have to prove $|E| = n - 1$.

We prove the theorem by the method of induction on the number of vertices. It is obvious that the result is true when $n = 1, 2$. Let the result be true when the no. of vertices is $< n$.

Let $e = uv$ be an edge of T. Deleting the edge e from T, disconnects the graph and $T - \{e\}$ consists of 2 components, each of which is a tree. Let the components be $T_1(V_1, E_1)$ and $T_2(V_2, E_2)$ where $|V_1| = n_1$ and $|V_2| = n_2$.

Therefore,

$$T - \{e\} = T_1(V_1, E_1) \cup T_2(V_2, E_2),$$

where

$$V_1 \cup V_2 = V \text{ and } V_1 \cap V_2 = \phi \text{ together imply } n_1 + n_2 = n.$$

Since $n_1 < n$ and $n_2 < n$, by hypothesis the result is true for $T_1(V_1, E_1)$ and $T_2(V_2, E_2)$.

i.e., $$|E_1| = n_1 - 1 \text{ and } |E_2| = n_2 - 1.$$

Again, since $E = E_1 \cup E_2 \cup \{e\}$, we have

$$\begin{aligned} |E| &= |E_1| + |E_2| + 1 \\ &= n_1 - 1 + n_2 - 1 + 1 \\ &= n_1 + n_2 - 1 = n - 1. \end{aligned}$$

This proves the theorem.

Theorem 7.3 **Prove that a connected graph with n vertices and $(n - 1)$ edges is a tree.** [WBUT 2005]

Proof Let G be a connected graph with n vertices and $n - 1$ edges. Let us suppose, if possible, G is not a tree, then G must contain at least one cycle. Let e be any edge of the any one of the cycles. Now deleting the edge e from G yields a subgraph $G - \{e\}$, which is also connected.

Now $G - \{e\}$ has $n - 2$ edges, but it is not possible, because any connected graph having n vertices must have at least $n - 1$ number of edges. This leads to a contradiction. Hence, G is a tree.

Theorem 7.4 **A graph $G(V, E)$ is a tree if it is acyclic and $|E| = |V| - 1$.**

Alternative Statement A graph $G(V, E)$ with n vertices is a tree if it is acyclic and $|E| = n - 1$].

Proof Let $G(V, E)$ be an acyclic graph with n vertices and $(n-1)$ edges, i.e., $|V| = n$ and $|E| = |V| - 1 = n - 1$.

To show that G is a tree, we have to prove that G is connected.

Let us suppose, if possible, $G(V, E)$ is disconnected and consists of k (≥ 2) components. Suppose the components are $G_i(V_i, E_i)$ $(i = 1, 2, \ldots n.)$. Then,

$$|V| = \sum_{i=1}^{k} |V_i| \text{ and } |E| = \sum_{i=1}^{k} |E_i|.$$

Since G is acyclic, each of the components $G_i(V_i, E_i)$ must be acyclic and correspondingly they are all trees.

Therefore, $|E_i| = |V_i| - 1$.

Now,

$$|E| = \sum_{i=1}^{k} |E_i| = \sum_{i=1}^{k} (|V_i| - 1)$$

$$= \sum_{i=1}^{k} |V_i| - k = |V| - k$$

$$\Rightarrow |E| < |V| - 1.$$

But this leads to a contradiction to the hypothesis that $|E| = |V| - 1$.

Hence G is connected and correspondingly G is a tree.

Theorem 7.5 **A graph is a tree if and only if it is minimally connected.**

Proof First let us consider that T be a tree. Then by definition it is connected. Let $e = uv$ be an edge of T. Since there is an unique path between any pair of vertices, the edge $e = uv$ is the only path connecting u and v. So by deletion of the path leaves the graph disconnected. Hence T is minimally connected.

Conversely let G be a minimally connected graph. To show that G is a tree, we are to prove that G is acyclic.

If possible, let us assume, G contains a cycle and let $e = uv$ be an edge of the cycle. If we delete $e = uv$ from the graph G, then for every path containing the edge e, there must be a path in G containing the other path in the cycle connecting u and v. So, G fails to be minimally connected. This leads to a contradiction to the hypothesis. Hence, G must be acyclic and correspondingly G is a tree.

Theorem 7.6 **Any tree (with more than one vertex) must have atleast two pendent vertices.**

Proof Let $T(V, E)$ be a tree with $n(\geq 2)$ vertices. Then, $|E| = n - 1$.

Again we have $\sum_{v \in V} d(v) = 2.\,|E| = 2\,(n-1) = 2n - 2$.

Since each of the n vertices is of degree ≥ 1, then the problem becomes distributing $2n - 2$ degrees into n vertices. It is clear that there must be at least two vertices of degree 1. Therefore, there exists at least two pendent vertices.

7.3 ROOTED AND BINARY TREES

7.3.1 Rooted Trees

Let T be a tree. We choose a vertex of T arbitrarily and fix it, which is called root of the tree. Now the tree is drawn by assigning levels $0, 1, 2, \ldots, k$ at each of the vertices. The **root** is of level 0. All adjacent vertices differ by exactly one level and each vertex at level $i+1$ is adjacent to exactly one vertex at level i. Such a tree is called the **rooted tree**. The number k is called the **height (or depth)** of the rooted tree.

In a rooted tree all the vertices adjacent to any vertex u and of a level below u are called children of u.

The following figure shows the rooted trees of level 4, i.e., of height 4.

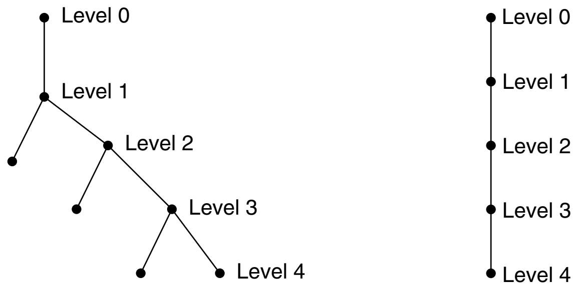


Figure 7.2 Rooted Trees

In a rooted tree always, root can be distinguished from all other vertices. For this reason sometimes the root is marked distinctly. Generally the term tree means tree without any root.

7.3.2 Binary Trees

Binary trees are special kind of rooted trees of which every vertex has either no child or exactly 2 children.

In other words, a binary tree is a such kind of tree in which there is exactly one vertex of degree two and each of the remaining vertices are either of degree one or degree three.

The vertex of degree two is distinguishable from the other vertices. Basically this is the root. Since binary tree is a rooted tree, here also root will be at the level 0.

The following figure shows a binary tree of height 3:

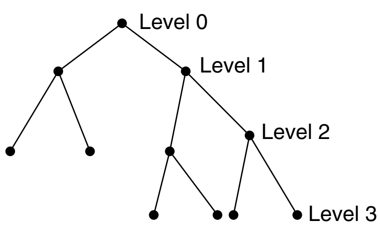


Figure 7.3 Binary Tree

It is to be noticed in the above figure that there exist leaves (pendent vertices) at the level 2 and level 3.

Balanced (or complete) binary tree Let T be a binary tree of height h. If the leaves are at level h only, then it is called balanced or complete binary tree.

In the figure below, a balanced binary tree of height 3 is presented and it is clear that leaves are only at level 3.

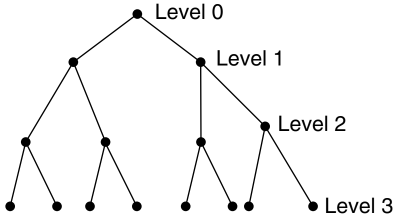


Figure 7.4 Balanced Binary Tree

Theorem 7.7 **The number of vertices in a binary tree is always odd.**

Proof Let us consider a binary tree with n number of vertices. By definition we know that in a binary tree one vertex is, of degree 2 and other vertices are either of degree 1 or degree 3. Therefore, number of odd degree vertices is $n-1$. Again we have a result that in a graph, the number of odd degree vertices is always even, So $n-1$ must be an even integer and correspondingly n is odd. Hence, the number of vertices in a binary tree is odd.

Theorem 7.8 **The number of leaves (pendent vertices) in a binary tree with n vertices is given by $\frac{1}{2}(n+1)$.**

Proof Suppose n_p be the number of pendent vertices in a binary tree T with n vertices. So, number of one degree vertices is n_p.

Here in the tree we have one vertex of degree 2 and rest of the vertices are of degree 3, i.e., $(n - n_p - 1)$ number of vertices are of degree 3.

Therefore, sum of the degrees of n vertices in T is

$$n_p \times 1 + 2 \times 1 + (n - n_p - 1) \times 3 = 3n - 2n_p - 1.$$

Again, we have that sum of the degrees of vertices in a graph is $2 \times$ (number of edges).

Since T is a tree with n vertices, it has $n-1$ number of edges.

Therefore,

$$3n - 2n_p - 1 = 2 \times (n-1) \Rightarrow n_p = \frac{1}{2}(n+1).$$

Hence, the theorem is proved.

Theorem 7.9 **The number of internal vertices (non-pendent vertices) in a binary tree is one less than the number of pendent vertices.**

Proof Let T be a binary tree with n vertices. Also, let n_i and n_p be the number of internal vertices and pendent vertices respectively.

Then, $n = n_i + n_p$. Now, by the last theorem we have $n_p = \frac{1}{2}(n+1)$.
Therefore,

$$n_i = n - n_p = n - \frac{1}{2}(n+1) = \frac{1}{2}(n-1)$$

Now,

$$n_p - n_i = \frac{1}{2}(n+1) - \frac{1}{2}(n-1) = 1$$
$$\Rightarrow n_i = n_p - 1.$$

Hence, the result is proved.

Theorem 7.10 **If T be a binary of height h with n vertices, then $n \leq 2^{h+1} - 1$.**

Proof Since the height of the binary tree is h, we have levels $0, 1, 2, \ldots, h$.
Now there is only one vertex at the level 0, which is root.
Number of vertices that can be at the level 1 is at most 2^1 $(= 2)$.
Number of vertices that can be at the level 2 is at most 2^2 $(= 4)$.
Similarly, number of vertices that can be at the level h is at most 2^h.
Since the number of vertices in the given binary tree is n, we have

$$n \leq 1 + 2^1 + 2^2 + \cdots + 2^h$$
$$= \frac{2^{h+1} - 1}{2 - 1} \text{ (Sum of the geometric progression.)}$$

i.e., $$n \leq 2^{h+1} - 1.$$

Hence, the result is proved.

Theorem 7.11 **The minimum height of a binary tree with n vertices is $\lceil log_2(n+1) \rceil - 1$, where $\lceil x \rceil$ denotes the least integer greater than or equal to x ($\lceil x \rceil$ is known as ceiling of x).**

In other words, $h \geq \lceil log_2(n+1) \rceil - 1$, where h is the height of the binary tree.

Proof From the last theorem, for a binary tree T of height h with n vertices, we have

$$n \leq 2^{h+1} - 1 \Rightarrow 2^{h+1} \geq n + 1$$

or, $$\log_2\left(2^{h+1}\right) \geq \log_2(n+1)$$

or, $$h + 1 \geq \log_2(n+1).$$

Since $h + 1$ is an integer, we get

$$h + 1 \geq \lceil \log_2(n+1) \rceil$$

or, $$h \geq \lceil \log_2(n+1) \rceil - 1.$$

Hence, the theorem is proved.

Remark For a balanced binary tree $h = \lceil \log_2(n+1) \rceil - 1$.

7.4 SPANNING TREE OF A GRAPH

A spanning subgraph of a graph $G(V, E)$ is a subgraph of G whose vertex set is the same as the vertex set V of G.

A **spanning tree** of a graph G is a spanning subgraph of G which is a tree.

[WBUT 2002]

Example 2 **Construct spanning trees from the following graph.**

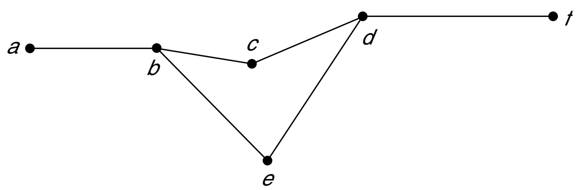


Figure 7.5

[WBUT 2006]

Sol. The spanning trees are given by the following:

(i)

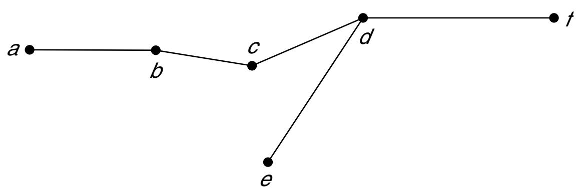


Figure 7.6

(ii)

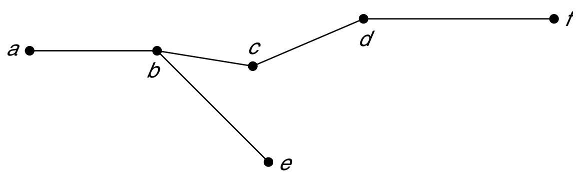


Figure 7.7

(iii)

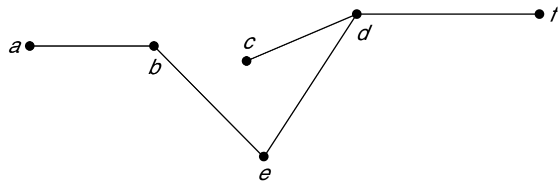


Figure 7.8

(iv)

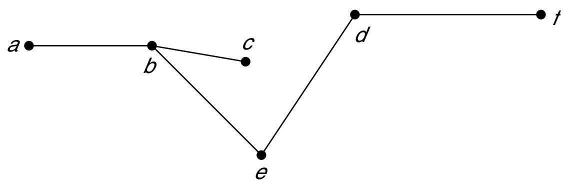


Figure 7.9

Theorem 7.12 **A graph G has a spanning tree if G is connected.** [WBUT 2005]

Proof At first we suppose that G has a spanning tree T. Since T is connected and has all the vertices of G, it follows that G is also connected.

Conversely let us suppose G is connected, we are to prove that it has a spanning tree. If G has no circuits, then G becomes a spanning tree of itself. Otherwise we consider a connected spanning subgraph H of G with the minimum number of edges. Now H cannot have any circuit. Let, if possible, C be a circuit in H. Then removal of the edges of C leaves another connected spanning subgraph of G with the fewer edges than H. But it leads to a contradiction that H has the minimum number of edges. Thus H has no circuits. Since H is also connected, H is a spanning tree.

Theorem 7.12 **Caley's Theorem**

The complete graph K_n, has n^{n-2} different spanning trees

Proof Beyond the scope of the book.

Example 3 The complete graph K_3 has $3^{3-2} = 3^1 = 3$ different spanning trees, as shown in the following figure:

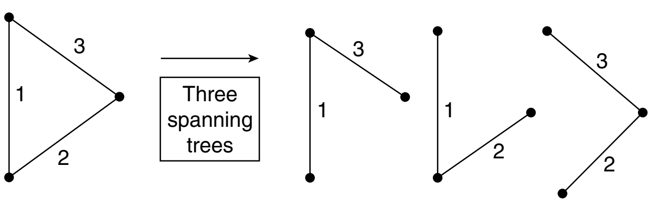


Figure 7.10

7.5 BREADTH FIRST SEARCH (BFS) ALGORITHM FOR FINDING SPANNING TREE OF A GRAPH

Input A connected graph.

At first discard the parallel edges and self-loops of the graph if they exist.

The algorithm is based on stage by stage labelling the vertices.

Select any vertex u of the graph and label it as 0.

Next, label the other vertices in every stage based on the following rule:

Traverse all the unlabelled vertices in G which are adjacent to the vertices of label k and label all of them as $k + 1$. Then join the vertices of label $k + 1$ with their corresponding adjacent k labelled vertices in such a manner that no circuit is formed.

Continue the process of stage to stage labelling the vertices and joining until all the vertices are labelled and joined.

Output A spanning tree of the given graph.

Example 4 Find a spanning tree of the following graph by BFS algorithm.

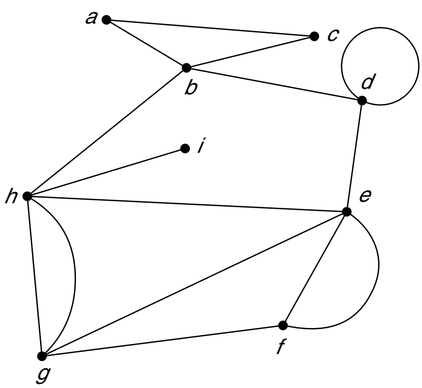


Figure 7.11

Sol. First we discard the loop and parallel edges.

Next, we select the vertex g and is labelled by 0.

Its adjacent unlabelled vertices are h, f and e. They are labelled by $0+1 = 1$. The labels are shown in the figure. Then we join each of them with g by the edges (g, h) , (g, e) and (g, f) , since joining of the edges does not result any cycle.

Next we see that the unlabelled adjacent vertices of h are b and i. We label each of them by $1 + 1 = 2$. Also, unlabelled adjacent vertex of e is only d. We label it by $1 + 1 = 2$. But there is no adjacent vertex of f which is unlabelled. Now we join the vertices b and i to h by the edges (h, b) and (h, i) respectively. Also, we join d with e by the edge (e, d). It is to be noted, that no cycle has been formed by the above joinings.

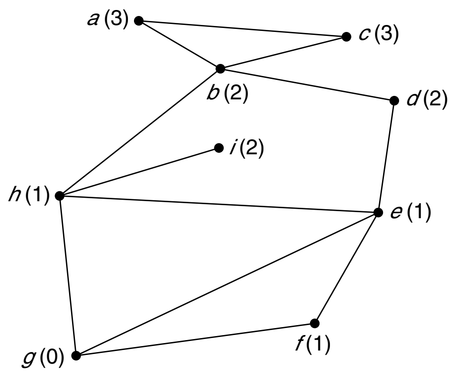


Figure 7.12

Now, the unlabelled adjacent vertices of b are a and c. We label each of them by $2 + 1 = 3$. We join the vertices a and c to b by the edges (b, a) and (b, c) respectively. Here also it is to be noted that no cycle has been formed.

We stop the process since no unlabelled vertices are left in the graph. Now the required spanning tree can be found by drawing the joining edges (g, h), (g, e), (g, f), (h, b), (h, i), (e, d), (b, a) and (b, c) successively which shown in the following figure:

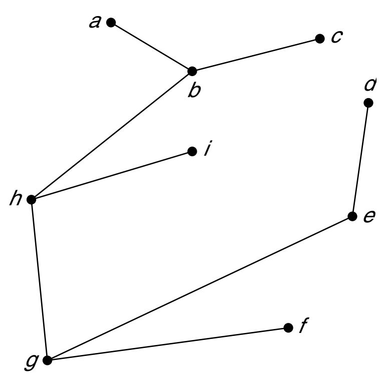


Figure 7.13

7.6 DEPTH FIRST SEARCH (DFS) ALGORITHM FOR FINDING SPANNING TREE OF A GRAPH

Input A connected graph.

At first discard the parallel edges and self loops of the graph if exists.

Select any vertex u of the graph. Find out a path in the graph as long as possible, starting from the vertex u by successively connecting the other vertices.

Let the path be $P_1 : u - v$, which is starting from u and ending at v. Now we backtrack starting from the vertex v along the path P_1 and suppose a be the first vertex (in the path P_1), from which another path P_2 (as long as possible) can be started containing no other vertices of P_1 except the starting vertex a.

Next suppose that b be the next vertex on the way of backtracking, from which another path P_3 (as long as possible) can be started containing no other vertices of P_1 and P_2 except the starting vertex b.

Continue this process of constructing above said paths until all the vertices are traversed by any one of the paths.

All the paths together represent a spanning tree.

Output A spanning tree of the given graph.

Example 5 Find a spanning tree by the DFS Algorithm.

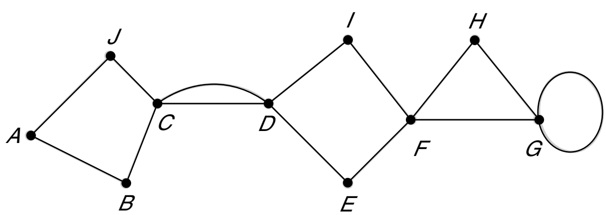


Figure 7.14

Sol. First we discard the parallel edge and self-loop.

Next we select the vertex B arbitrarily. Now we find a path starting from B as long as possible. The path is given by

$$P_1 : B - C - D - I - F - H - G$$

which is shown in the follwing figure by bold face lines:

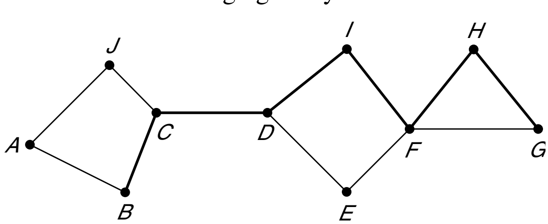


Figure 7.15

Next we start backtracking starting from G. First we backtrack from G to H. There is no path starting from H containing vertices other than vertices of path P_1. Now we backtrack from H to F. Here we get a path starting from F and the path is $P_2 : F - E$, not containing the vertices of P_1. The path P_2 is shown by the single dotted line in the following figure:

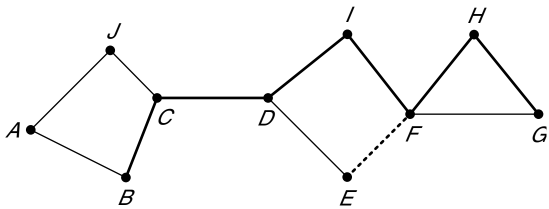


Figure 7.16

Again we start re-tracking from F through the vertices of P_1 and let C be the first vertex in that way from where a path starts not containing the vertices of P_1 and P_2. The path is $P_3 : C - J - A$, which is shown by double dotted lines in the following figure:

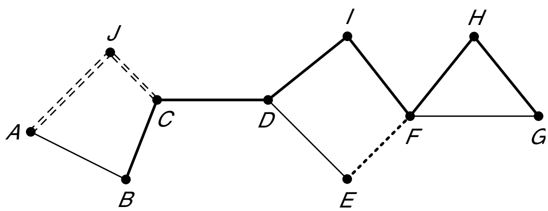


Figure 7.17

Now we stop the process because all the vertices are traversed by any one of the paths. The required spanning tree is formed by the combination of the three paths P_1, P_2 and P_3, which is shown in the following figure:

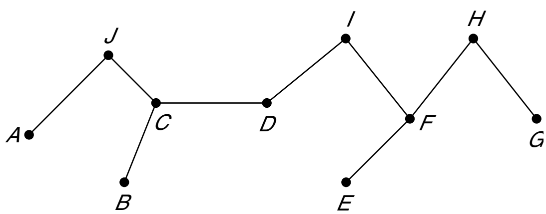


Figure 7.18

7.7 FUNDAMENTAL CIRCUITS

Let T be any spanning tree of a graph $G(V, E)$ with n vertices and e number of edges.

An edge in T is called the branch of the tree T. So the number of branches (w.r.t the spanning tree T) is $n - 1$.

An edge of G that is not an edge of T is known as **chord** (**or Tie**) w.r.t. T. So the number of chords (w.r.t. the spanning tree T) is $e - (n - 1)$, i.e., $e - n + 1$.

Co-tree (or chord set or tie set)

Co-tree of a spanning tree T in a connected graph G is the collection of all chords of G w.r.t T. It is denoted by $\overline{T}$. This is also known as complement of a spanning tree.

The number of branches of any spanning tree of a graph is called its **rank,** where as the number of chords is known as the **nullity** of the graph.

Example 6 Consider the folowing graph shown in the figure.

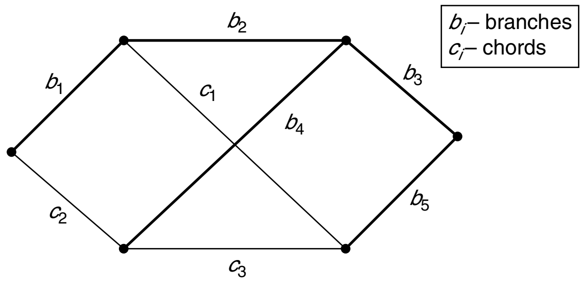


Figure 7.19

Here in the graph, the spanning tree T is shown by the bold face lines. The 5 branches are b_1, b_2, b_3, b_4, b_5. The 3 chords are c_1, c_2, c_3.

Here in the graph $n = 6$ and $e = 8$. So the number of chords are $e - n + 1 = 8 - 6 + 1 = 3$.

Here the co-tree of the spanning tree T (or complement of T) is given by

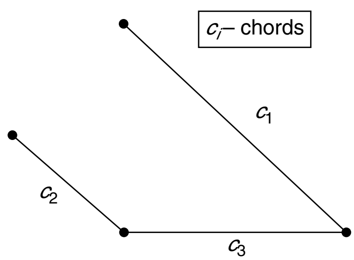


Figure 7.20

Here in this example rank is 5 and nullity is 3.

Theorem 7.13 If c be a chord of a connected graph G w.r.t one of its spanning tree T, then $T + \{c\}$ contains a unique cycle.

Proof Let $c = uv$ be a chord of G w.r.t the spanning tree T. Then c is not an edge of T. Again u and v both are vertices of T, so they are connected by an unique path. Now if we consider $T + \{c\}$, then u and v will be connected by an additional path and union of these two paths result a unique cycle.

It is obvious from the above theorem that addition of a chord to T in G creates a cycle. Such a cycle which contains only one chord is called fundamental cycle or circuit.

Also from above we know that there are $e - n + 1$ number of chords in G w.r.t T, so G has $e - n + 1$ number of fundamental circuits w.r.t the spanning tree T.

Example 7 Consider the last example. Here the number of chords are 3. Therefore the number of fundamental circuits are also 3.

The fundamental cycles are $\{b_1, b_2, b_4, c_2\}$, $\{b_2, b_3, b_5, c_1\}$ and $\{b_4, b_3, b_5, c_3\}$ which are shown in the following figure:

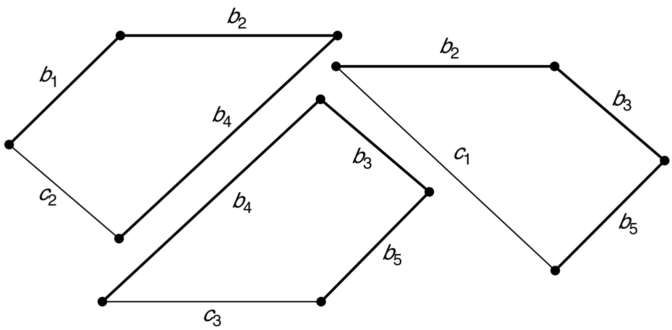


Figure 7.21

Though $\{b_1, b_2, b_3, b_5, c_3, c_2\}$ also forms a cycle, but it is not a fundamental cycle since it contains two chords which is shown in the following figure:

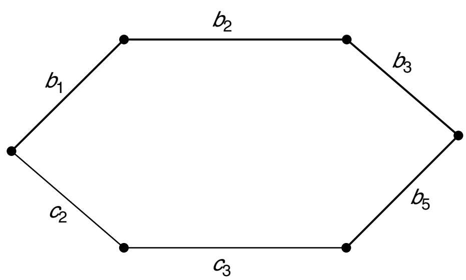


Figure 7.22

7.8 HOW TO GENERATE ALL SPANNING TREES (CYCLIC INTERCHANGE OR ELEMENTARY TREE TRANSFORMATION)

The method of generating a spanning tree from another spanning tree by addition of a chord and deletion of a branch is known as **Cyclic interchange** or **Elementary tree transformation**.

In this method first we select any arbitrary spanning tree from the given graph and then we have branches and chords with respect to the selected spanning tree. Now addition of a chord results a fundamental cycle, then delete a branch from the cycle which results another spanning tree. Continue this process until we get all the spanning trees.

An example of application of cyclic interchange is given below:

Example 8 Find all the spanning trees in the following graph.

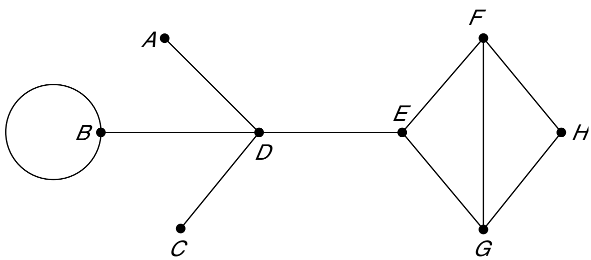


Figure 7.23

[WBUT 2007]

Sol. Let the given graph be G. Here we will apply the method of elimentary tree transformstion or cyclic interchange to get all the spanning trees.

Removing loops from G first we consider the following spanning tree ***Sp tree 1***.

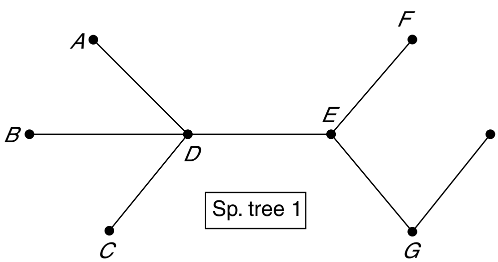


Figure 7.24

Now we will find other spanning trees starting from ***Sp tree 1***. Here we see that FH and FG are the chords of G w.r.t ***Sp tree 1***.

Adding the chord FH to ***Sp tree 1***, we have the cycle $EFGHE$. Now removing the branch FE from the cycle $EFGHE$ we have the following spanning tree.

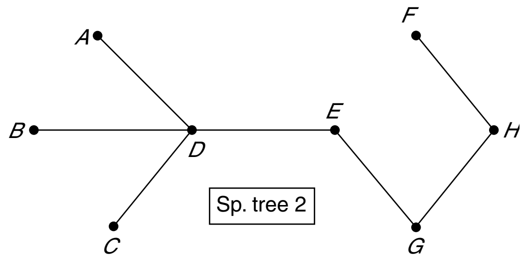


Figure 7.25

Next removing the branch EG from the cycle $EFGHE$ we have the following spanning tree:

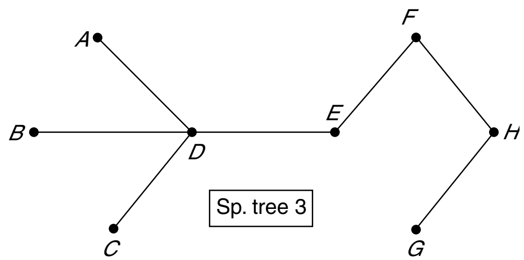


Figure 7.26

Also removing the branch GH from the cycle $EFGHE$ we have the following spanning tree:

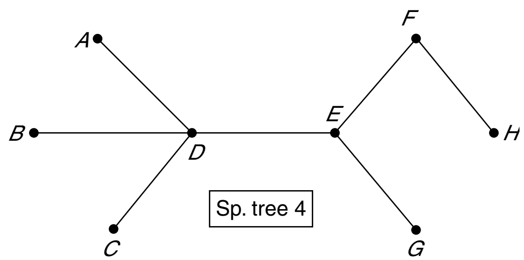


Figure 7.27

Adding the chord FG to ***Sp tree 1*** we have the cycle $EFGE$. Now removing the branch EF from the cycle $EFGE$ we have the following spanning tree:

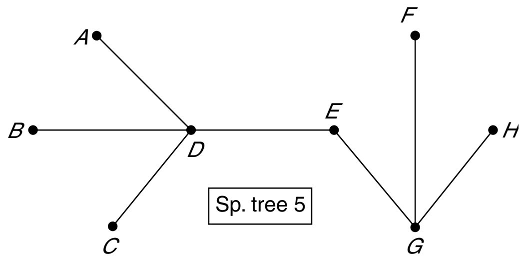


Figure 7.28

Next removing the branch EG from the cycle $EFGE$ we have the following spanning tree:

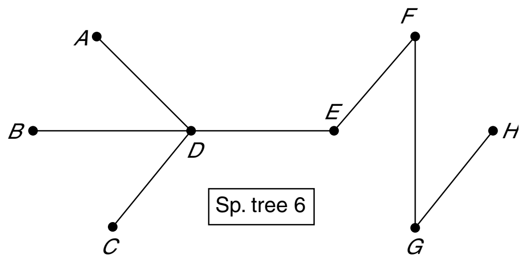


Figure 7.29

Adding the chord FG to ***Sp tree 2*** and removing the branch GH, we have the following spanning tree:

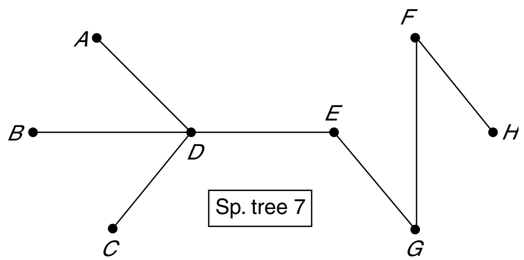


Figure 7.30

Adding the chord FG to ***Sp tree 3*** and removing the branch GH, we have the following spanning tree:

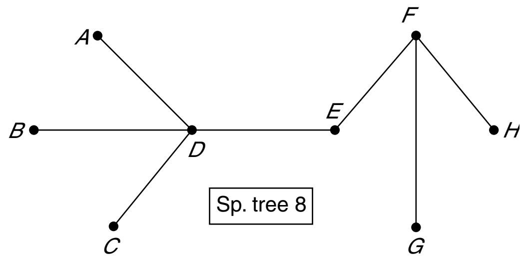


Figure 7.31

So the above 8 spanning trees are all the spanning trees.

7.9 MINIMAL (OR SHORTEST) SPANNING TREE

Let $G(V, E)$ be a connected graph. If a nonnegative real number is assigned to every edge of the graph, then the graph is called an edge weighted graph or weighted graph. The number that is assigned to the edge may be the distance between the two cities (or vertices) or it may be the cost of building the road between the cities etc.

Let T be a spanning tree of G. The sum of the weights of the edges of T is known as the weight of T. A spanning tree having the minimum weight (or shortest length) is called the shortest spanning tree or minimal spanning tree.

Note: A graph may have more than one minimal spanning tree which is known as alternative minimal spanning tree.

7.10 KRUSKAL'S ALGORITHM FOR FINDING MINIMAL (SHORTEST) SPANNING TREE

Input A connected weighted graph with n vertices.

Step I Arrange all the edges of G, except the loops in the order of non-decreasing weights.

Step II Select the first edge of the list.

Step III Add the next edge of smallest weight to the previous one which does result any cycle.

Step IV Repeat Step III until $n - 1$ edges have been selected.

Output A minimal (shortest) spanning tree with $n - 1$ edges.

Example 9 Using Kruskal's Algorithm find the minimal spanning tree of the following graph.

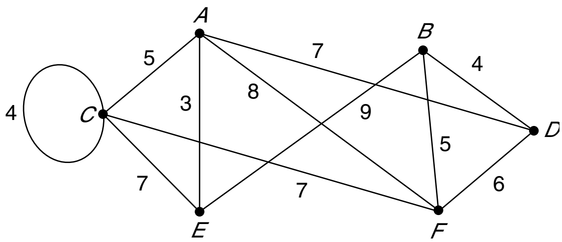


Figure 7.32

Sol. First we arrange all the edges of G, except the loops in the order of non-decreasing weights and write it in the following form:

Edges	**AE**	**BD**	**AC**	**BF**	**DF**	**CE**	**CF**	**AD**	**AF**	**BE**
Weights	3	4	5	5	6	7	7	7	8	9

Step 1 Select the first edge AE from the list, since it has the minimum weight.

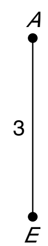


Figure 7.33

Step 2 The next edge of smallest weight is BD. We can add it to the previous one because it does not form any cycle.

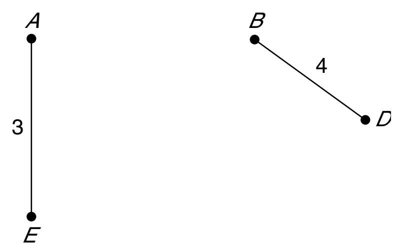


Figure 7.34

Step 3 The next edge of smallest weight is AC. We can add it to the previous one because it does not form any cycle.

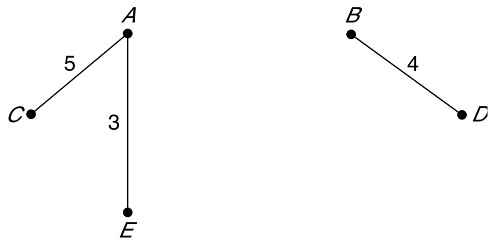


Figure 7.35

Step 4 The next edge of smallest weight is BF. We can add it to the previous one because it does not form any cycle.

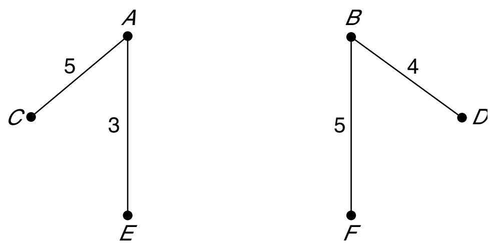


Figure 7.36

Step 5 The next edge of smallest weight is DF. We discard it because it results in a cycle. Also we discard CE due to the same reason.

Step 6 Next we add the edge CF to the previous one because it does not form any cycle.

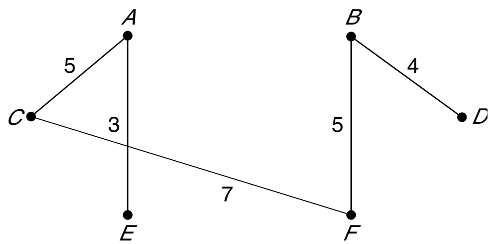


Figure 7.37

Since the number of vertices in the given graph is 6 and the tree in the last step contains 5 $(= 6 - 1)$ edges, the required minimal spanning tree is given by the Step 6.

Weight of the minimal spanning tree $= 3 + 4 + 5 + 5 + 7 = 24$.

7.11 PRIM'S ALGORITHM FOR FINDING MINIMAL (SHORTEST) SPANNING TREE

Input A connected weighted graph G with n number of vertices

First we remove all the self-loops if they exist. Also remove all the parallel edges except the edge with the minimum weight for any pair of vertices.

Next label the n vertices by $V_1, V_2, \ldots, V_n$. Weights of the edges are tabulated in an $n \times n$ table, of which the ij^{th} element w_{ij} is given by the following rule

$$w_{ij} = \text{ weight of the edge between } V_i \text{ and } V_j.$$
$$w_{ij} = \infty, \text{ if there is no direct edge between } V_i \text{ and } V_j.$$

In $n \times n$ table, we replace the diagonal values and put '$-$' and it is to be noted that the entries in the table are symmetric w.r.t its diagonal.

Now start from the vertex V_1. Connect it to the nearest adjacent vertex, i.e., the vertex for which there is the smallest entry in the row 1 of the table. Suppose the vertex is V_i. Now consider the edge connecting V_1 and V_i as one subgraph and connect this subgraph to its nearest neighbour, i.e., the vertex other than V_1 and V_i for which there is the smallest entry in the row 1 and row i of the table. Suppose the vertex is V_k. Next consider the tree with the vertices V_1 , V_i and V_k as one subgraph and continue the process of connecting until all the n vertices are connected by $n-1$ edges. If there is a tie for selecting the smallest entry in any row then we choose arbitrarily.

Output A minimal (shortest) spanning tree with $n-1$ edges.

Example 10 Using Prim's Algorithm find the minimal spanning tree of the following graph:

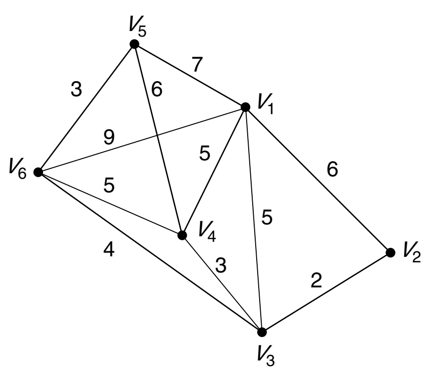


Figure 7.38

Here we have 6 vertices. So, the minimal spanning tree will be with the $6-1=5$ edges.

First we make the weight table in the following manner:

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	–	6	5	5	7	9
V_2	6	–	2	∞	∞	∞
V_3	5	2	–	3	∞	4
V_4	5	∞	3	–	6	5
V_5	7	∞	∞	6	–	3
V_6	9	∞	4	5	3	–

Step 1 We start from the vertex V_1 and smallest entry in the row 1 is 5 for both (V_1, V_3) and (V_1, V_4). We select (V_1, V_3) arbitrarily which results in the following subgraph:

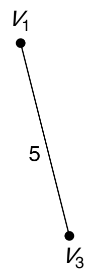


Figure 7.39

Step 2 Then the smallest entry in the row 1 and row 3 is 2 for (V_3, V_2). We connect it to the above subgraph which results in the following:

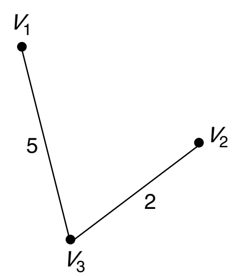


Figure 7.40

Step 3 Then the smallest entry in the row 1, row 3 and row 2 is 3 for (V_3, V_4). We connect it to the above subgraph which results in the following:

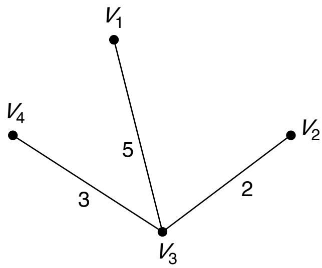


Figure 7.41

Step 4 Then the smallest entry in the row 1, row 3, row 2 and row 4 is 4 for (V_3, V_6). We connect it to the above subgraph which results in the following:

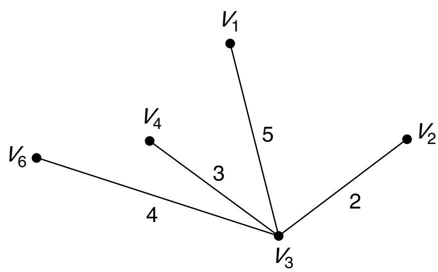


Figure 7.42

Step 5 Then the smallest entry in the row 1, row 3, row 2, row 4 and row 6 is 3 for (V_6, V_5). We connect it to the above subgraph which results in the following:

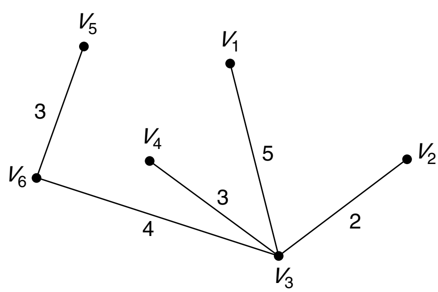


Figure 7.43

Since the number of vertices in the given graph is 6 and the subgraph in the last step contains 5 $(= 6 - 1)$ edges, the required minimal spanning tree is given by the Step 5.

Weight of the minimal spanning tree $= 5 + 2 + 3 + 4 + 3 = 17$.

7.12 CUT SET AND CUT VERTICES

7.12.1 Cut Set

Let $G(V, E)$ be a connected graph. A cut set for G is defined to be the smallest set of edges such that removal of the set disconnects the graph but removal of any proper subset of this set leaves a connected subgraph of G.

Example 11 **Let us find out some of the cut sets in the following graph:**

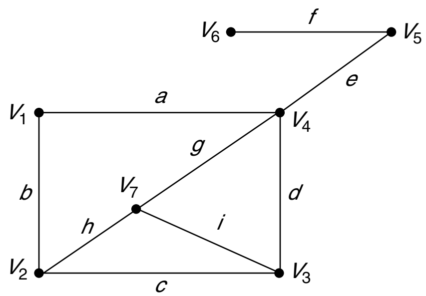


Figure 7.44

Here some of the cut sets are $\{a, b\}$, $\{f\}$, $\{e, f\}$, $\{a, g, d\}$, $\{a, g, i, c\}$, etc.

Here $\{a, g, i, d\}$ is not a cut set, though if we remove the edges the graph becomes disconnected. Because one of its proper subset $\{a, g, d\}$ is a cut set.

7.12.2 Cut point or Cut vertices

Let $G(V, E)$ be any connected graph. A cut vertex for G is a vertex v such that $G - \{v\}$ has more components than G or becomes disconnected.

The subgraph $G - \{v\}$ is obtained by deleting the vertex v along with the edges incident to it.

Example 12 **Find out all the cut vertices from the following graph:**

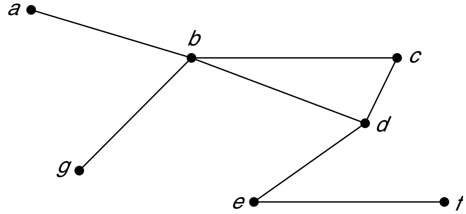


Figure 7.45

Sol. The vertex d is a cut vertex, since removal of d yields more than one components as shown in the following:

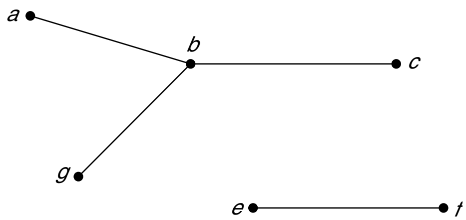


Figure 7.46

Also the vertex b is a cut vertex, since removal of b yields more than one components as shown in the following:

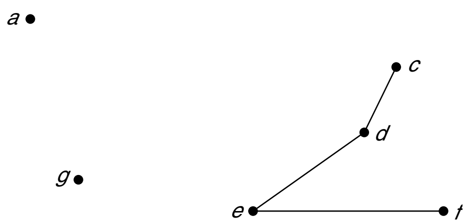


Figure 7.47

Apart from above the vertex e is also a cut vertex, since removal of e yields more than one component as shown in the following:

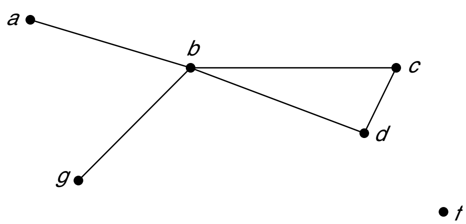


Figure 7.48

Theorem 7.14 In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut set.

Proof Let H be a minimal set of edges containing at least one branch of every spanning tree of G. Therefore $G - H$, the subgraph that remains after removing the edges of H from G, does not contain any spanning tree of G. For this reason $G - H$ is disconnected (one component of which may consist of an isolated vertex only). Again, since H is the minimal set of edges with the given property, if any edge e of H is returned to the subgraph $G - H$, then it will create at least one spanning tree. Thus,

$G - H + \{e\}$ will be a connected graph. Therefore, H is such a minimal set of edges whose removal from G disconnects the graph G whereas removal of any proper subset of H leaves the graph connected. Hence, by definition H is a cut set.

Theorem 7.15 A cut set and any spanning tree must have at least one edge in common.

Proof Let, if possible, H be a cut set that has no common edge with a spanning tree, then removal of the cut set H does not affect the spanning tree. Therefore removal of the cut set H does not separate the graph into more than one component. This contradicts the definition of the cut set. Hence, H must have at least one common edge with the spanning tree.

Theorem 7.16 Every circuit has an even number of edges common with every cut set.

Proof By definition corresponding to every cut set H, the vertex set V of the graph $G(V, E)$ is divided into two disjoint subsets V_1 and V_2 which are the vertex sets of the two components, when the edges of the cut set are removed from the graph. Consider a curcuit C.

Now, first we consider the case that all the vertices of C are entirely within the set V_1 or V_2, then the number of edges C common with the cut set H is zero.

Secondly, we consider the case when some of the vertices of C are in V_1 and rest of the vertices are in V_2. Then to traverse the circuit we traverse back and forth between the sets V_1 and V_2. Since the circuit is a closed path, the number of edges that we traverse between the vertex sets V_1 and V_2 is even. Again, the connecting edges between V_1 and V_2 are nothing but parts of the cut set H. Therefore, the number of edges common to C and H is even.

7.13 FUNDAMENTAL CUT SETS

Let us consider any spanning tree T in any connected graph G. Also let $\{b\}$ be any branch in T. Since every branch of any spanning tree is a cut set in that tree, $\{b\}$ is a cut set in T and so it creates two partitions of the vertices of T as the two disjoint sets (one at each end of $\{b\}$). Now if we consider any cut set H (in the graph G), removal of which creates the same partition of vertices of G, done by the branch $\{b\}$, then the cut set H will contain only one branch $\{b\}$ of T and the rest of the edges (if exists) of H are chords with respect to T. Such a kind of cut set which contains exactly one branch of a spanning tree T is called fundamental cut set or basic cut set with respect to T.

Example 13 Here we will show all the fundamental cut sets with respect to a spanning tree T. The spanning tree T is shown in the heavy lines in the figure.

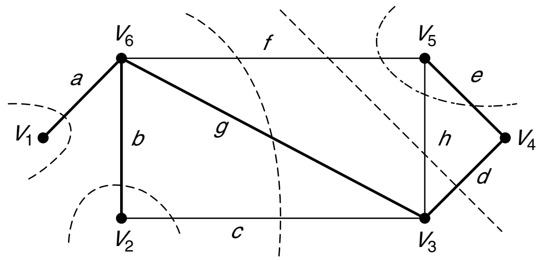


Figure 7.49

It is clear from the figure that $\{a\}$ is a cut set and it consists of only one branch $\{a\}$ of T. So, $\{a\}$ is a fundamental cut set with respect to spanning tree T.

Also $\{b, c\}$ is a fundamental cut set with respect to spanning tree T, since the cut set $\{b, c\}$ contains ony one branch $\{b\}$ of T.

Similarly $\{f, g, c\}$, $\{f, h, d\}$ and $\{f, h, e\}$ are all fundamental cut sets with respect to spanning tree T,

In the figure, the fundamental cut sets are shown by the dotted lines cutting through the cut sets.

7.14 EDGE CONNECTIVITY AND VERTEX CONNECTIVITY

7.14.1 Edge Connectivity

Let G be a connected graph. Then, the **edge connectivity** is defined as the number of edges in the smallest cut set (cut set having least number of edges) of the graph G.

In other words, the **edge connectivity** of a connected graph G is the minimum number of edges which disconnect the graph when removed.

Observations:

(1) If the graph G is disconnected, i.e., G has more than one component, then edge connectivity of G is the minimum number of edges removed, which increases the number of components of G.

(2) The edge connectivity for a tree is always one.

Example 14 In the following graph the edge connectivity is 3, since removal of minimum 3 edges disconnects the graph.

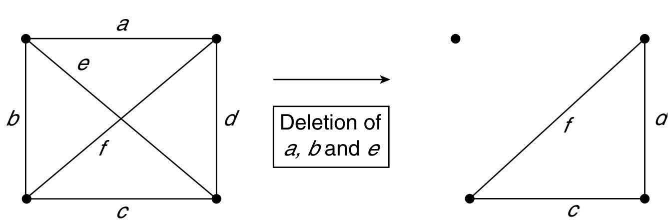


Figure 7.50

7.14.2 Vertex Connectivity

Let G be a connected graph. Then the **vertex connectivity** is defined as the minimum number of vertices, when removed (along with the edges incident to it) will disconnect the graph G.

Observations:

(1) If the graph G is disconnected, i.e., G has more than one component, then vertex connectivity of G is the minimum number of vertices removal of which (along with the edges incident to it) increases the number of components of G.

(2) The vertex connectivity for a tree is always one.

(3) According to the above definition vertex connectivity is only meaningful for those graphs having three or more vertices or for the graphs which are not complete.

Example 15 In the following graph the vertex connectivity is 2, since removal of minimum 2 vertices disconnects the graph.

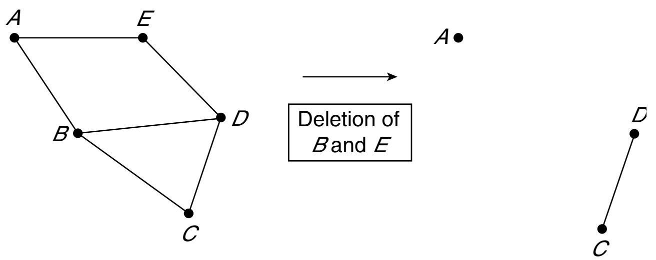


Figure 7.51

7.14.3 Some Results of Edge Connectivity and Vertex Connectivity

Result 1 The edge connectivity of a graph cannot exceed the degree of the smallest degree vertex.

Result 2 The vertex connectivity of any graph cannot exceed the edge connectivity of that graph.

Result 3 The maximum vertex connectivity of any graph with n vertices and e edges $(e \geq n-1)$ is $\left\lfloor \frac{2e}{n} \right\rfloor$, where $\left\lfloor \frac{2e}{n} \right\rfloor$ is defined as the integral part of the number $\frac{2e}{n}$ ($\lfloor a \rfloor$ is known as **floor** of a).

7.14.4 Separable Graph

A connected graph is said to be separable if the vertex connectivity of the graph is one. Otherwise it is known as non-separable graph.

Observations:

1) **A tree is always a separable graph.**
2) **Every cut set in a nonseparable graph with more than two vertices contains at least two edges.**

Example 16 Here in the following graph the vertex connectivity is 1, since removal of minimum number of 1 vertex disconnects the graph. Hence, this is a separable graph.

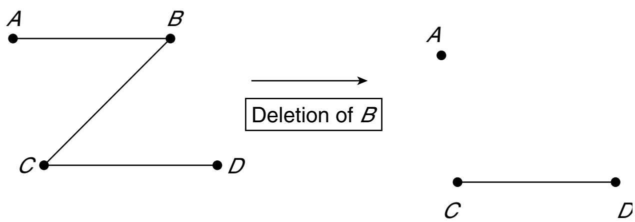


Figure 7.52

WORKED OUT EXAMPLES

Example 7.1 **Obtain a minimum spanning tree of the following graph using Krushkal's Algorithm.**

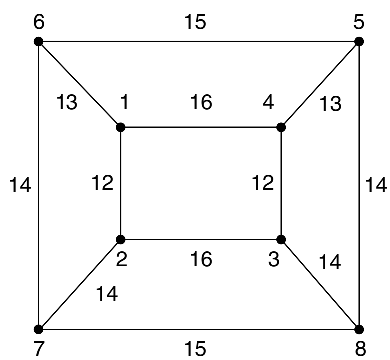


Figure 7.53

[WBUT 2002]

Sol. First we arrange all the edges of graph, in the order of non-decreasing weights and write in the following form:

Edges	$\{1,2\}$	$\{3,4\}$	$\{1,6\}$	$\{4,5\}$	$\{6,7\}$	$\{2,7\}$	$\{3,8\}$	$\{5,8\}$
Weights	12	12	13	13	14	14	14	14

Edges	$\{6,5\}$	$\{7,8\}$	$\{1,4\}$	$\{2,3\}$
Weights	15	15	16	16

Step 1 Select the first edge $\{1,2\}$ from the list, since it is of minimum weight.

Step 2 The next edge of smallest weight is $\{3,4\}$. We can add it to the previous one because it does not form any cycle.

Step 3 The next edge of smallest weight is $\{1,6\}$. We can add it to the previous one because it does not form any cycle.

Step 4 The next edge of smallest weight is $\{4,5\}$. We can add it to the previous one because it does not form any cycle.

Step 5 The next edge of smallest weight is $\{6,7\}$. We can add it to the previous one because it does not form any cycle.

Step 6 The next edge of smallest weight is $\{2,7\}$. We discard it because it results in a cycle.

Step 7 Next we add the edge $\{3,8\}$ to the previous one because it does not form any cycle.

Step 8 The next edge of smallest weight is $\{5,8\}$. We discard it because it results in a cycle.

Step 9 Next we add the edge $\{6,5\}$ to the previous one because it does not form any cycle.

In this step the tree becomes

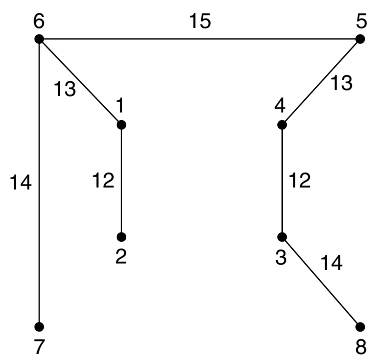


Figure 7.54

Since the number of vertices in the given graph is 8 and the tree in the last step contains 7 $(= 8 - 1)$ edges, the required minimal spanning tree is given by the Step 9.

Weight of the minimal spanning tree $= 12+12+13+13+14+14+15 = 93.$

Example 7.2 **Obtain a minimum spanning tree of the following graph using Krushkal's Algorithm.**

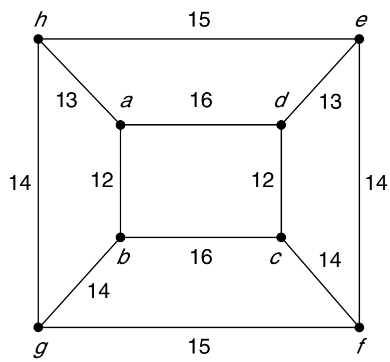


Figure 7.55

[WBUT 2003]

Sol. The given graph is similar to the graph in **Example 7.1** except for the names of the vertices. So, to get the minimal spanning tree, proceed in a manner similar to **Example 7.1.**

Example 7.3 **Obtain a minimum spanning tree of the following graph using Krushkal's Algorithm.**

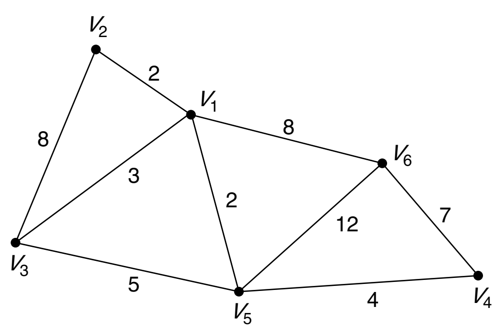


Figure 7.56

[WBUT 2003]

Sol. First we arrange all the edges of the graph, in the order of non-decreasing weights and write in the following form:

Edges	V_1V_2	V_1V_5	V_1V_3	V_5V_4	V_3V_5	V_4V_6	V_1V_6	V_3V_2	V_5V_6
Weights	2	2	3	4	5	7	8	8	12

Step 1 Select the first edge V_1V_2 from the list since it is of minimum weight.

Step 2 The next edge of smallest weight is V_1V_5. We can add it to the previous one because it does not form any cycle.

Step 3 The next edge of smallest weight is V_1V_3. We can add it to the previous one because it does not form any cycle.

Step 4 The next edge of smallest weight is V_5V_4. We can add it to the previous one because it does not form any cycle.

Step 5 The next edge of smallest weight is V_3V_5. We discard it because it results in a cycle.

Step 6 The next edge of smallest weight is V_4V_6. We can add it to the previous one because it does not form any cycle.

In this step the tree becomes

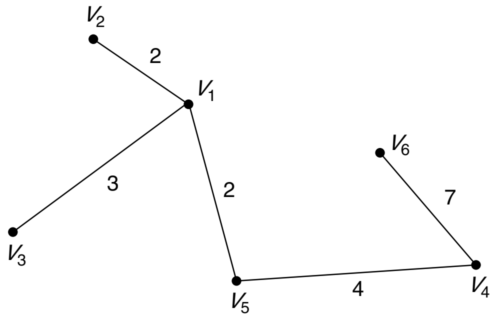


Figure 7.57

Since the number of vertices in the given graph is 6 and the tree in the last step contains 5 $(= 6 - 1)$ edges, the required minimal spanning tree is given by the Step 6.

Weight of the minimal spanning tree $= 2 + 2 + 3 + 4 + 7 = 18$.

Example 7.4 **Obtain a minimum spanning tree of the following graph using Krushkal's Algorithm.**

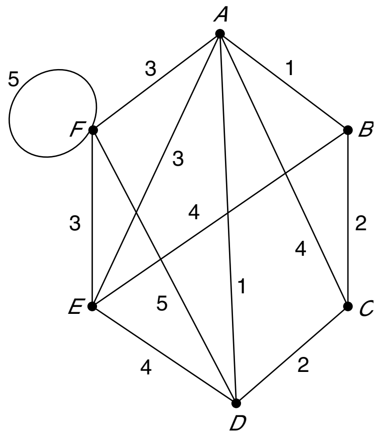


Figure 7.58

[WBUT 2005]

Sol. First we arrange all the edges of the graph except the loops in the order of non-decreasing weights and write it in the following form:

Edges	**AB**	**AD**	**BC**	**CD**	**AF**	**AE**	**FE**	**ED**	**BE**	**AC**	**FD**
Weights	1	1	2	2	3	3	3	4	4	4	5

Step 1 Select the first edge AB from the list since it is of minimum weight.

Step 2 The next edge of smallest weight is AD. We can add it to the previous one because it does not form any cycle.

Step 3 The next edge of smallest weight is BC. We can add it to the previous one because it does not form any cycle.

Step 4 The next edge of smallest weight is CD. We discard it because it results in a cycle.

Step 5 The next edge of smallest weight is AF. We can add it to the previous one because it does not form any cycle.

Step 6 The next edge of smallest weight is AE. We can add it to the previous one because it does not form any cycle.

In this step the tree becomes

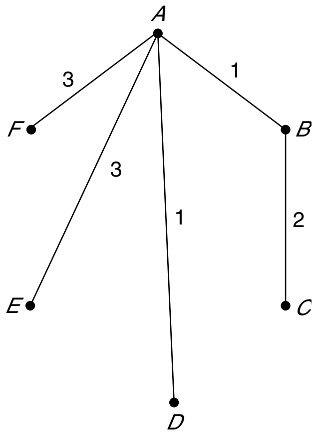


Figure 7.59

Since the number of vertices in the given graph is 6 and the tree in the last step contains 5 $(= 6 - 1)$ edges, the required minimal spanning tree is given by the Step 6.

Weight of the minimal spanning tree $= 1 + 1 + 2 + 3 + 3 = 10.$

Example 7.5 **Find by Prim's Algorithm a minimum spanning tree of the following graph.**

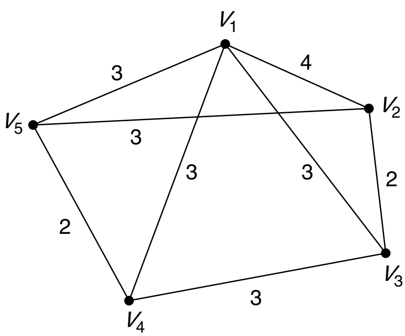


Figure 7.60

[WBUT 2004, 2007]

Sol. Here we have 5 vertices. So, the minimal spanning tree will be with the $5-1 = 4$ edges.

First we make the 5×5 weight table, of which ij^{th} element w_{ij} is given by the following rule:

$$w_{ij} = \text{ weight of the edge between } V_i \text{ and } V_j$$
$$w_{ij} = \infty, \text{ if there is no direct edge between } V_i \text{ and } V_j$$

We leave the diagonal empty by putting '–'. So, the table is

	V_1	V_2	V_3	V_4	V_5
V_1	–	4	3	3	3
V_2	4	–	2	∞	3
V_3	3	2	–	3	∞
V_4	3	∞	3	–	2
V_5	3	3	∞	2	–

Step 1 We start from the vertex V_1 and smallest entry in the row 1 is 3 for (V_1, V_3), (V_1, V_4) and (V_1, V_4). We select (V_1, V_3) arbitrarily which results the following subgraph

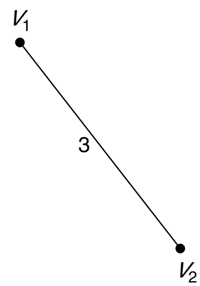


Figure 7.61

Step 2 Next the smallest entry in the row 1 and row 3 is 2 for (V_3, V_2). We connect it to the above subgraph which results the following:

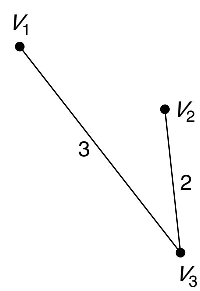


Figure 7.62

Step 3 Next the smallest entry in the row 1, row 3 and row 2 is 3 for (V_1, V_4) (V_1, V_5) (V_2, V_5) (V_3, V_4). We select (V_1, V_4) arbitrarily and connect it to the above subgraph which results the following:

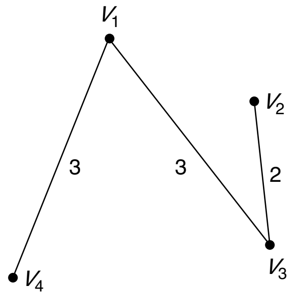


Figure 7.63

Step 4 Next the smallest entry in the row 1, row 3, row 2 and row 4 is 2 for (V_4, V_5). We connect it to the above subgraph which results the following:

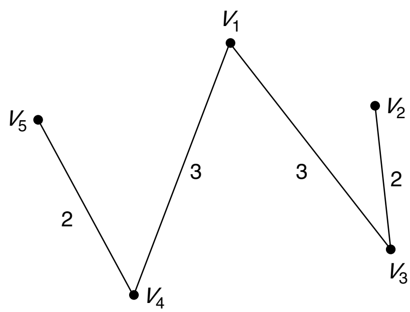


Figure 7.64

Since the number of vertices in the given graph is 5 and the subgraph in the last step contains 4 $(= 5 - 1)$ edges, the required minimal spanning tree is given by the Step 4.

Weight of the minimal spanning tree $= 3 + 2 + 3 + 2 = 10$.

Example 7.6 **Find by Prim's Algorithm the minimum spanning tree of the following graph.**

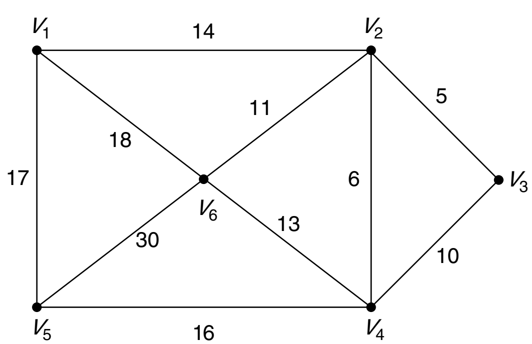


Figure 7.65 [WBUT 2005, 2006, 2011]

Sol. Here we have 6 vertices. So, the minimal spanning tree will be with the $6-1=5$ edges.

First, we make the 6×6 weight table, of which ij^{th} element w_{ij} is given by the following rule:

$$w_{ij} = \text{weight of the edge between } V_i \text{ and } V_j$$
$$w_{ij} = \infty, \text{ if there is no direct edge between } V_i \text{ and } V_j$$

We leave the diagonal empty (put '–'). So, the table is

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	–	14	∞	∞	17	18
V_2	14	–	5	6	∞	11
V_3	∞	5	–	10	∞	∞
V_4	∞	6	10	–	16	13
V_5	17	∞	∞	16	–	30
V_6	18	11	∞	13	30	–

Step 1 We start from the vertex V_1 and smallest entry in the row 1 is 14 for (V_1, V_2). We select (V_1, V_2) which results in the following subgraph

Figure 7.66

Step 2 Next the smallest entry in the row 1 and row 2 is 5 for (V_2, V_3). We connect it to the above subgraph which results in the following:

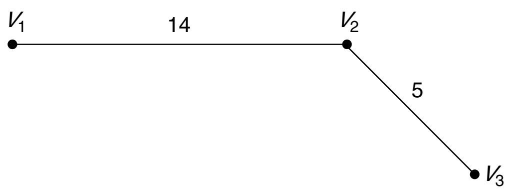


Figure 7.67

Step 3 Next the smallest entry in the row 1, row 2 and row 3 is 6 for (V_2, V_4). We connect it to the above subgraph which results in the following:

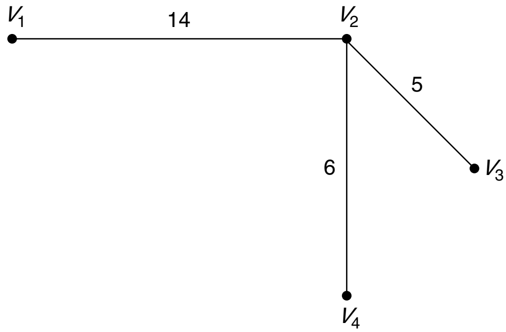


Figure 7.68

Step 4 Next the smallest entry in the row 1, row 2, row 3 and row 4 is 10 for (V_3, V_4). We cannot connect it to the above subgraph because it results in a cycle.

Step 5 Next the smallest entry in the row 1, row 2, row 3 and row 4 is 11 for (V_2, V_6). We connect it to the above subgraph which results in the following:

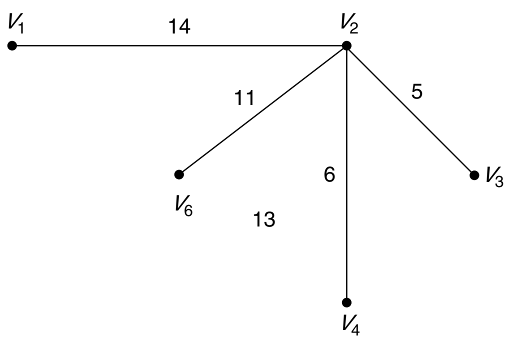


Figure 7.69

Step 6 Next the smallest entry in the row 1, row 2, row 3, row 4 and row 6 is 13 for (V_4, V_6). We cannot connect it to the above subgraph because it results in a cycle.

Step 7 Next the smallest entry in the row 1, row 2, row 3, row 4 and row 6 is 16 for (V_4, V_5). We connect it to the above subgraph which results in the following:

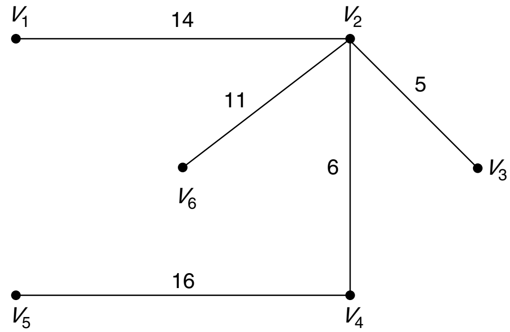


Figure 7.70

Since the number of vertices in the given graph is 6 and the subgraph in the last step contains 5 $(= 6 - 1)$ edges, the required minimal spanning tree is given by the Step 7.

Weight of the minimal spanning tree $= 14 + 5 + 6 + 11 + 16 = 52$.

Example 7.7 **Find by Prim's Algorithm the minimum spanning tree of the following graph.**

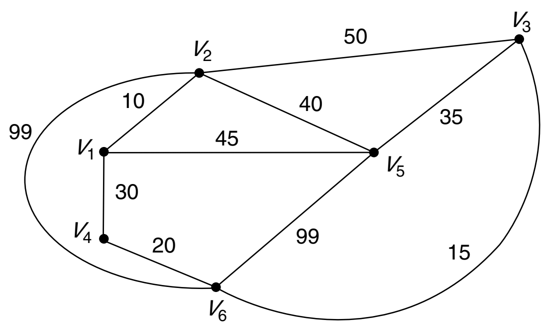


Figure 7.71

[WBUT 2005]

Sol. Here we have 6 vertices. So, the minimal spanning tree will be with the $6-1 = 5$ edges.

First we make the 6×6 weight table, of which ij^{th} element w_{ij} is given by the following rule

$$w_{ij} = \text{ weight of the edge between } V_i \text{ and } V_j$$
$$w_{ij} = \infty, \text{ if there is no direct edge between } V_i \text{ and } V_j$$

We leave the diagonal empty by putting '–'. So, the table is

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	–	10	∞	30	45	∞
V_2	10	–	50	∞	40	99
V_3	∞	50	–	∞	35	15
V_4	30	∞	∞	–	∞	20
V_5	45	40	35	∞	–	99
V_6	∞	99	15	20	99	–

Step 1 We start from the vertex V_1 and smallest entry in the row 1 is 10 for (V_1, V_2). We select (V_1, V_2) which results in the following subgraph:

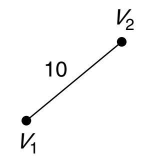


Figure 7.72

Step 2 Next the smallest entry in the row 1 and row 2 is 30 for (V_1, V_4). We connect it to the above subgraph which results in the following:

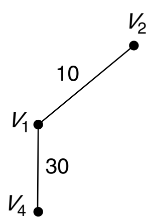


Figure 7.73

Step 3 Next the smallest entry in the row 1, row 2 and row 4 is 20 for (V_4, V_6). We connect it to the above subgraph which results in the following:

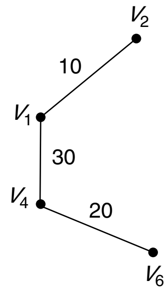


Figure 7.74

Step 4 Next the smallest entry in the row 1, row 2, row 4 and row 6 is 15 for (V_6, V_3). We connect it to the above subgraph which results in the following:

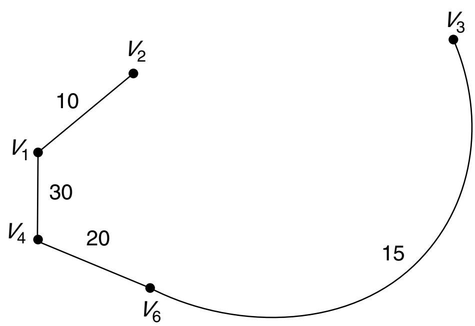


Figure 7.75

Step 5 Next the smallest entry in the row 1, row 2, row 3, row 4 and row 6 is 35 for (V_3, V_5). We connect it to the above subgraph which results in the following:

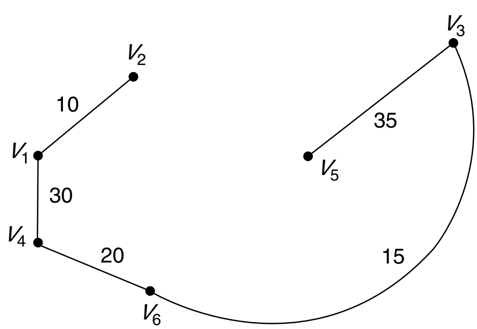


Figure 7.76

Since the number of vertices in the given graph is 6 and the subgraph in the last step contains 5 $(= 6 - 1)$ edges, the required minimal spanning tree is given by the Step 5.

Weight of the minimal spanning tree $= 10 + 30 + 20 + 15 + 35 = 110$.

Example 7.8 **Find by Prim's Algorithm the minimum spanning tree of the following graph.**

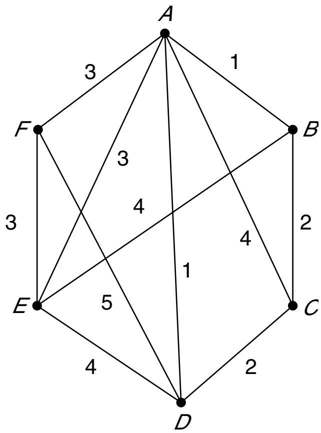


Figure 7.77

[WBUT 2004]

Sol. Here we have 6 vertices. So, the minimal spanning tree will be with the $6-1=5$ edges.

First we make the 6×6 weight table, of which ij^{th} element w_{ij} is given by the following rule:

$$w_{ij} = \text{ weight of the edge between } V_i \text{ and } V_j$$
$$w_{ij} = \infty, \text{ if there is no direct edge between } V_i \text{ and } V_j$$

We leave the diagonal empty by putting '$-$'. So, the table is

	A	B	C	D	E	F
A	$-$	1	4	1	3	3
B	1	$-$	2	∞	4	∞
C	4	2	$-$	2	∞	∞
D	1	∞	2	$-$	4	5
E	3	4	∞	4	$-$	3
F	3	∞	∞	5	3	$-$

Step 1 We start from the vertex A and smallest entry in the row 1 is 1 for (A, B) and (A, D). We select (A, B) arbitrarily which results the following subgraph:

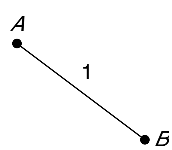


Figure 7.78

Step 2 Next the smallest entry in the row 1 and row 2 is 1 for (A, D). We connect it to the above subgraph which results the following:

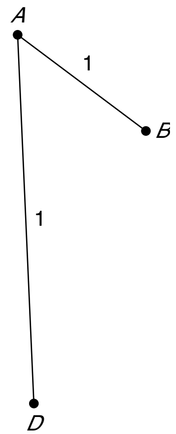


Figure 7.79

Step 3 Next the smallest entry in the row 1, row 2 and row 4 is 2 for (B, C).and (D, C). We select (B, C) arbitrarily and connect it to the above subgraph which results in the following:

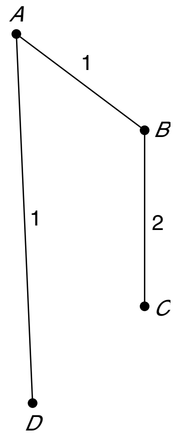


Figure 7.80

Step 4 Next the smallest entry in the row 1, row 2, row 3 and row 4 is 2 for (D, C). We cannot connect it to the above subgraph because it results in a cycle.

Step 5 Next the smallest entry in the row 1, row 2, row 3 and row 4 is 3 for (A, E) and (A, F). We select (A, E) arbitrarily and connect it to the above subgraph which results in the following:

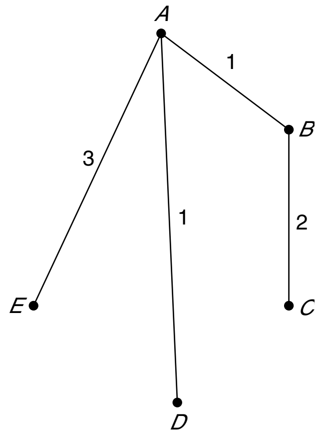


Figure 7.81

Step 6 Next the smallest entry in the row 1, row 2, row 3, row 4 and row 5 is 3 for (A, F) and (E, F). We select (A, F) arbitrarily and connect it to the above subgraph which results in the following:

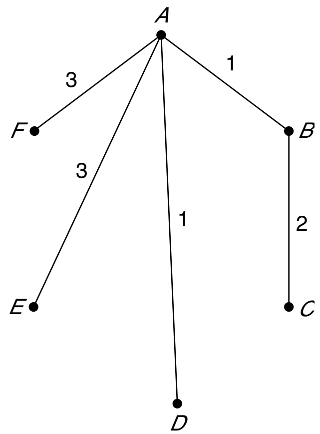


Figure 7.82

Since the number of vertices in the given graph is 6 and the subgraph in the last step contains 5 $(= 6 - 1)$ edges, the required minimal spanning tree is given by the Step 5.

Weight of the minimal spanning tree $= 1 + 1 + 2 + 3 + 3 = 10$.

Example 7.9 **If G be a connected graph with n vertices and e edges, then show that it has a unique circuit if and only if $n = e$.**

Sol. First we consider that the graph G with n vertices and e edges contains a unique circuit, say C. If we delete an edge e_1 from C, the graph remains connected and also becomes acyclic. This means $G - \{e_1\}$ becomes a tree with n vertices. Therefore, $G - \{e_1\}$ must have $n - 1$ edges. Since only one edge is deleted from G, the total number of edges of G is $(n - 1) + 1 = n$ which implies the fact that $e = n$.

Conversely let G be a connected graph with n vertices and e edges such that $e = n$. We are to prove that it contains a unique circuit.

Since G is connected and the number of edges are not equal to $n - 1$, G is not a tree. Therefore, it is not acyclic and must contain at least one circuit. If possible let it contain 2 circuits, say C_1 and C_2. Now if we delete two edges e_1 and e_2 respectively from C_1 and C_2, the the graph remains connected and also becomes acyclic. Therefore, $G - \{e_1 \cup e_2\}$ is a tree with n vertices and so it has $n - 1$ edges. Since two edges are deleted from G, total number of edges of G is $(n-1)+2 = n+1$, i.e., $e = n+1$. This leads to a contradiction to our hypothesis that $e = n$. So, G cannot have two circuits.

In a similar manner it can be proved that G cannot have more than two circuits. Therefore, G contains unique circuit.

Example 7.10 **If G be a graph having no cycles, n vertices and k components, then prove that G has $(n - k)$ edges.**

Sol. Let n_i be the number of vertices of the i^{th} component where $i = 1, 2, 3, \ldots, k$.

Therefore, total number of vertices in G is

$$n = n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i$$

Since G has no cycles, any one of the components cannot contain any cycle. Again since by definition each component is connected, all are trees.

So, the number of edges in the i^{th} component is $(n_i - 1)$.

So, the total number of edges in G is

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k.$$

EXERCISES

Short and Long Answer Type Questions

1) Find the spanning trees of the following graphs:

a)

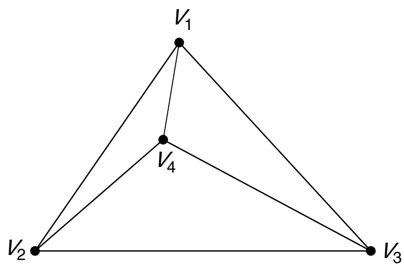


Figure 7.83

Ans:

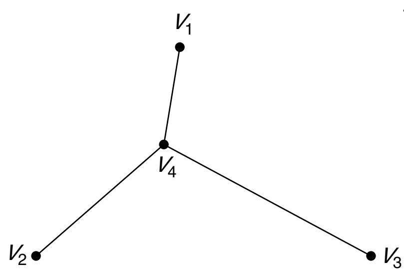


Figure 7.84

b)

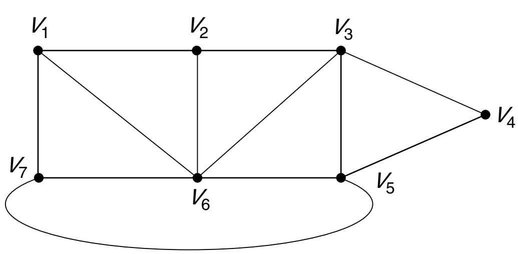


Figure 7.85

Ans:

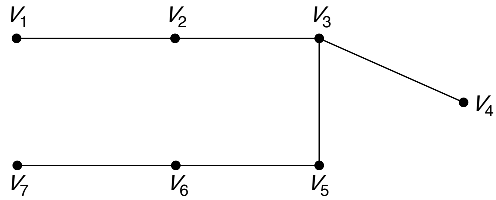


Figure 7.86

c)

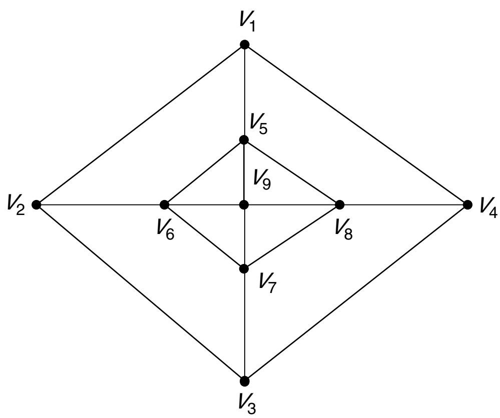


Figure 7.87

Ans:

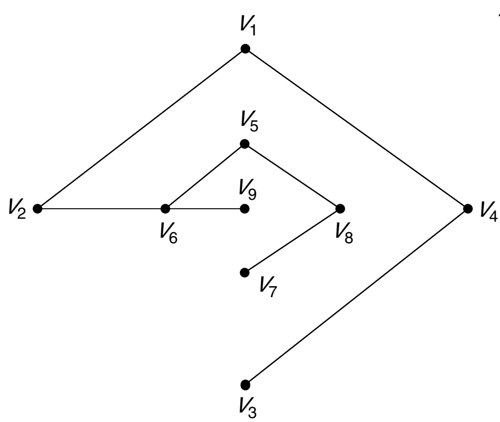


Figure 7.88

2) Find a minimal spanning tree by Kruskal's algorithm and find the corresponding minimum weight

a)

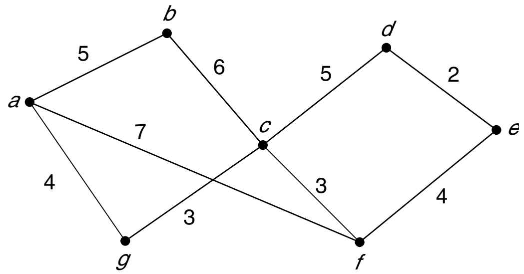


Figure 7.89

[**Ans:** Minimum weight = 21

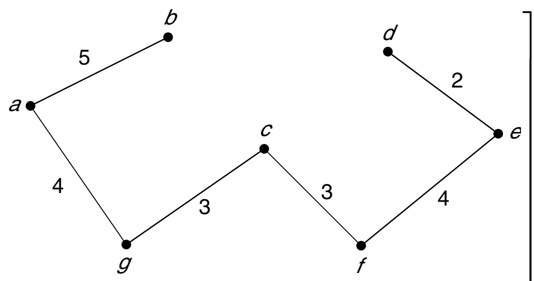


Figure 7.90]

b)

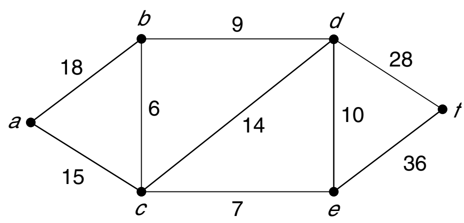


Figure 7.91

[**Ans:** Minimum weight = 65

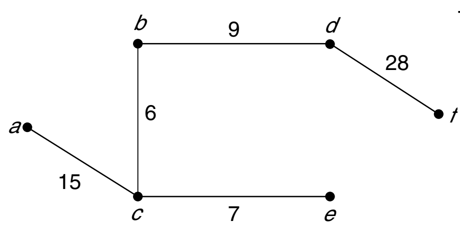


Figure 7.92]

c)

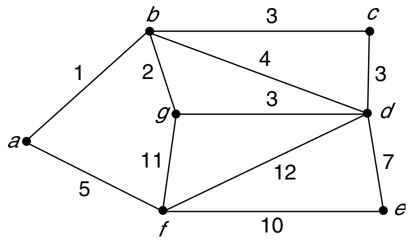


Figure 7.93

[**Ans:** Minimum weight = 21

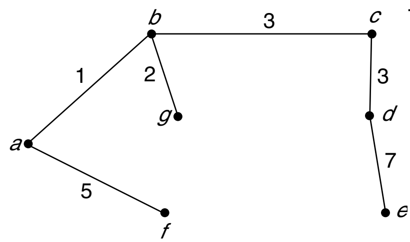


Figure 7.94]

3) Find a minimal spanning tree by Prim's algorithm and find the corresponding minimum weight.

a)

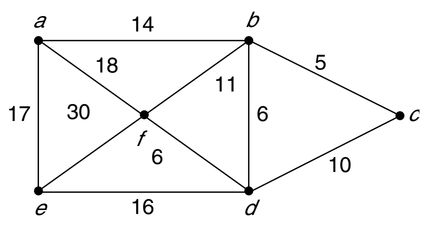


Figure 7.95

[**Ans:** Minimum weight = 52

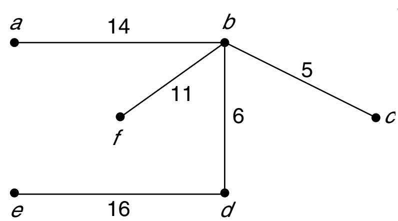


Figure 7.96]

b)

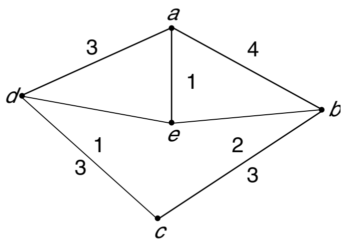


Figure 7.97

[**Ans:** Minimum weight = 7

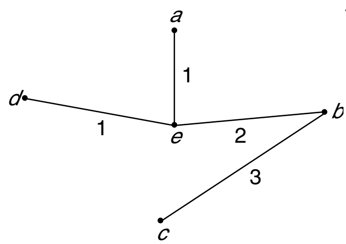


Figure 7.98]

c)

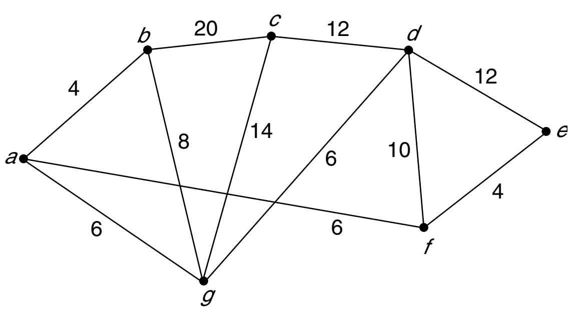


Figure 7.99

[**Ans:** Minimum weight = 38

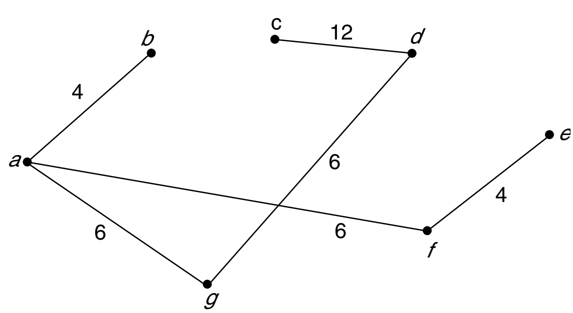


Figure 7.100]

4) Use BFS algorithm to find a spanning tree of the following graphs:

a)

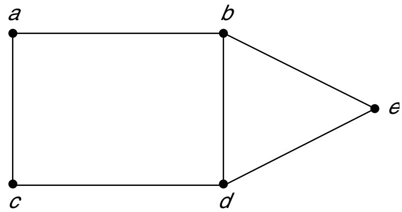


Figure 7.101

Ans:

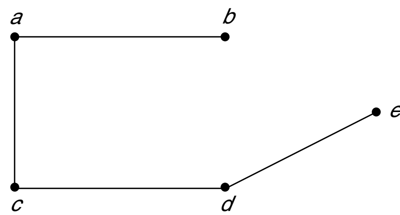


Figure 7.102

b)

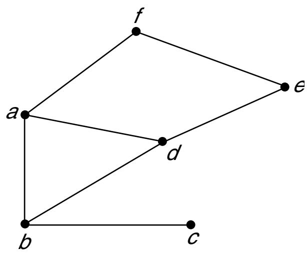


Figure 7.103

Ans:

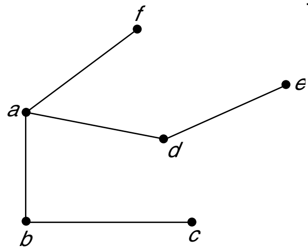


Figure 7.104

c)

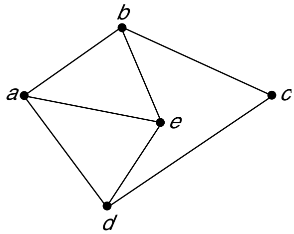


Figure 7.105

Ans:

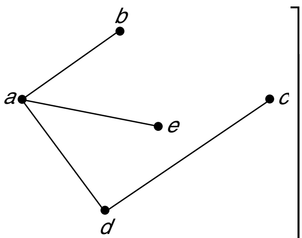


Figure 7.106

5) Use DFS algorithm to find a spanning tree of the following graphs:

a)

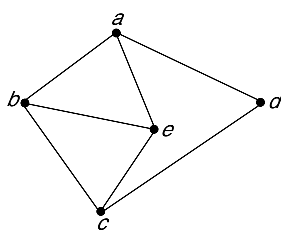


Figure 7.107

Ans: The path is a-e-b-c-d

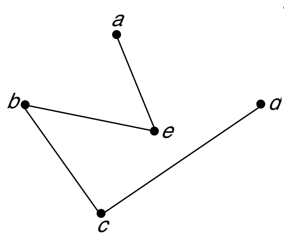


Figure 7.108

b)

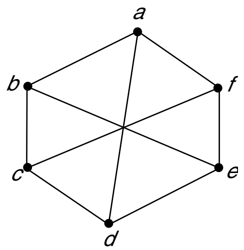


Figure 7.109

Ans: The path is a-b-c-d-e-f

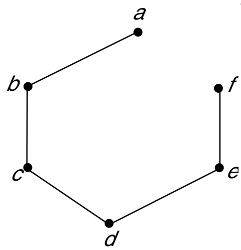


Figure 7.110

6) Find a spanning tree T of the following graph G. Find all the fundamental circuits of G with respect to T.

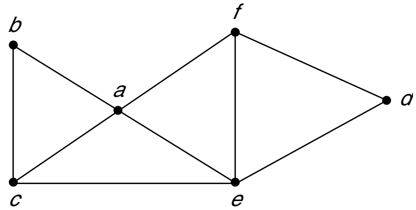


Figure 7.111

Ans: One of the spanning tree T of the graph G is

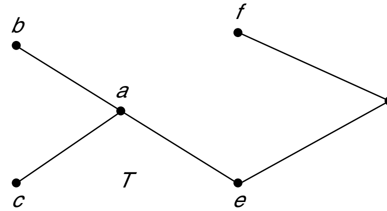


Figure 7.112

and the fundamental circuits are $b-c-a-b$, $c-e-a-c$, $f-e-d-f$.

Multiple Choice Questions

1) A tree is always a
a) self-complement graph b) Euler graph
c) simple graph d) Hamiltonian graph

2) Tree is a connected graph without any
a) pendant vertex b) circuit c) odd vertex d) even vertex

3) A minimally connected graph cannot have a
a) circuit b) component c) even vertex d) pendant vertex

4) A binary tree has exactly
a) two vertices of degree two b) one vertices of degree two
c) one vertex of degree one d) one vertices of degree three

5) Sum of the degrees of all vertices of a binary tree is even if the tree has
a) odd number of vertices b) even number of vertices
c) four vertices d) none of these

6) A binary tree has exactly
a) one root b) two root c) three root d) none of these

7) Addition of an edge between any two vertices of a tree creates
a) Euler line b) circuit c) longest path d) regular graph

8) The minimum number of pendant vertices in a tree with five vertices is
a) 1 b) 2 c) 3 d) 4

9) If T be a spanning tree in a graph G then the cotree of T contains
a) all the chords of T b) all the vertices of G which are not of T
c) all the edges of G d) none of these

10) A fundamental circuit is obtained from
a) any tree b) a spanning tree c) a binary tree d) longest path

11) A cut set always splits the graph into
a) three b) more than three c) two d) none of these

12) Cut set is defined for
a) digraph only b) connected graph only
c) arbitrary graph d) weighted graph

13) A tree having no cut vertex is a graph of

a) three vertices b) two vertices c) two edges d) none of these

14) The vertex connnectivity of a tree is

a) 0 b) 1 c) 2 d) more than 2

15) The edge connnectivity of a tree is

a) 0 b) 1 c) 2 d) more than 2

Answers:

1 (c)	2 (b)	3 (a)	4 (b)	5 (a)	6 (a)	7 (b)
8 (b)	9 (a)	10 (b)	11 (c)	12 (c)	13 (b)	14 (b)
15 (b)						

CHAPTER

8

Shortest Path and Algorithm

8.1 INTRODUCTION

Suppose a, b, c, . . . are some cities (or communication centers or electronic chips in a circuit board etc.) connected by highways (or telephone lines or electric wires etc). In the graph, cities and connecting highways are represented by vertices and edges respectively.

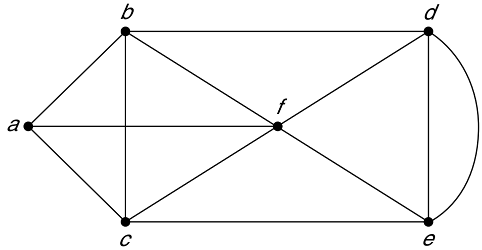


Figure 8.1

There is a path (union of edges) between every pair of vertices. But the most common problem is to find the path having shortest length in different branches of science and technology. It is also used in operation research.

There are several methods for finding shortest paths. Here, we represent Dijkstra's algorithm as well as BFS algorithm to find the shortest distance. University examination problems are solved at the end of the chapter.

8.2 SHORTEST PATHS IN UNWEIGHTED GRAPHS

Let $G(V, E)$ be a connected graph and u, v are two vertices of the graph. Now u and v can be connected by more than one path. Among all the paths possible between the vertices u and v, the path containing minimum number of edges is called the shortest path between u and v.

In the following graph, let us consider the two vertices A and Z.

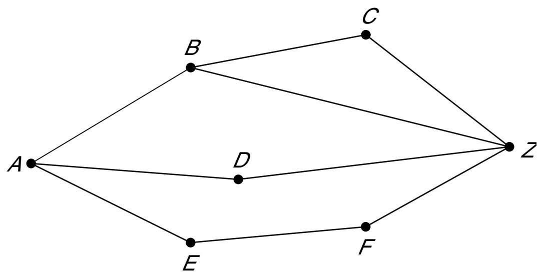


Figure 8.2

Now, the possible paths between A and Z are

(1) $A - B - C - Z$ containing three edges.
(2) $A - B - Z$ containing two edges.
(3) $A - D - Z$ containing two edges.
(4) $A - E - F - Z$ containing three edges.

Then the shortest path between these two vertices is given by the path containing two edges, since it is minimum. So, the paths are given by (2) and (3).

It should be noted that there may be more than one shortest path between any two vertices.

8.3 SHORTEST PATHS IN WEIGHTED GRAPHS

Let $G(V, E, w)$ be a connected weighted graph, where w is a function from E to the set of positive real numbers. Let us consider V as a set of cities and E as a set of highways connecting these cities. The weight of an edge $\{i, j\}$, denoted by w_{ij} is usually referred to as the length of the edge $\{i, j\}$. It has an obvious interpretation as the distance between the cities i and j, although other interpretations such as yearly cost to maintain the highways etc can exist.

The length of a path in G is defined as the sum of the weights of the edges in the path. Our interest is to determine a shortest path (path of minmum weight) from one vertex to another vertex in G.

In the following weighted graph, let us consider the two vertices A and Z.

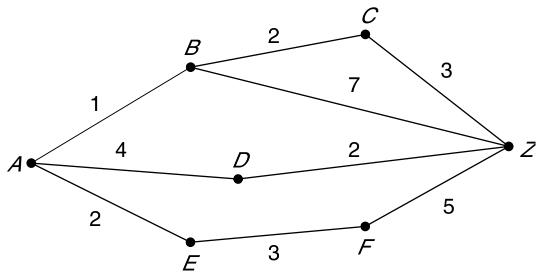


Figure 8.3

Now, the possible paths between A and Z are

(1) $A - B - C - Z$ containing three edges AB, BC, CZ and sum of the weights of the edges is $1 + 2 + 3 = 6$. So, the length of the path is 6.
(2) $A - B - Z$ containing two edges. AB, BZ and sum of the weights of the edges is $1 + 7 = 8$. So, the length of the path is 8.
(3) $A - D - Z$ containing two edges. AD, DZ and sum of the weights of the edges is $4 + 2 = 6$. So, the length of the path is 6.
(4) $A - E - F - Z$ containing three edges. AE, EF, FZ and sum of the weights of the edges is $2 + 3 + 5 = 10$. So, the length of the path is 10.

Then, the shortest path between these two vertices is given by the path of length 6, since it is minimum. So, the paths are given by (1) and (3).

It should be noted that there may be more than one shortest path between any two vertices.

There are different kinds of shortest path problems:

(1) Shortest path between two specified vertices.
(2) Shortest path from a specified vertex to all others.
(3) Shortest path between all pairs of vertices.

Sometimes type (1) becomes identical to type (2) because in the process of finding the shortest path from a specified vertex to another specified vertex, we may have to determine the shortest path to all other vertices.

There are several methods of finding the shortest paths. One effective method is described by **EW Dijkstra**. Here, we present **Dijkstra's Algoritham** for finding the shortest path between two specified vertices.

8.4 DIJKSTRA'S ALGORITHAM FOR FINDING THE SHORTEST PATH BETWEEN TWO SPECIFIED VERTICES

A simple weighted graph $G(V, E, w)$ with n vertices is described by an $n \times n$ matrix $W = (w_{ij})_{n\times n}$, where

$$w_{ij} = \text{weight (or distance or cost) of the edge from vertex } i \text{to } j$$
$$w_{ii} = 0$$
$$w_{ij} = \infty, \text{ if there is no edge from vertex } i \text{ to } j.$$

If the graph is not simple, i.e., if the graph G contains any self loop, then discard it. Also if G contains parallel edges between any two vertices, discard all except the edge having the least weight.

Let us consider that we are to find out the shortest path from a specified vertex s to another specified vertex t.

At each stage in the algorithm some vertices have permanent labels and others have temporary labels. Label of a vertex v is denoted by $L(v)$.

Assign first, the permanent label 0 to the starting vertex s, i.e., $L(s) = 0$ and a temporary label ∞ to the remaining vertices. Subsequently in each iteration another vertex gets a permanent label.

Step I Every vertex j that is not yet permanently labelled gets a new temporary label whose value is given by

$$L(j) = \min\{\text{old } L(j),\ (\text{old } L(i) + w_{ij})\}$$

where i is the latest vertex permanently labelled in the last iteration and w_{ij} is the direct distance between the vertices i and j. If i and j are not joined by an edge, then $w_{ij} = \infty$.

Step II The smallest value among all the temporary labels is marked and this is the permanent label of the corresponding vertex. In case of a tie, select any one for permanent labelling.

Step III Step I and Step II are repeated alternately until the destination vertex t gets a permanent label.

Every stage of lebelling will be displayed in the table. In the table permanent label of every vertices will be shown enclosed in a square (□).

Here, the shortest distance of the destination vertex t is found by its value of the permanent label. The shortest path will be found by the backtraking technique from the table.

Backtracking technique for finding the shortest path Starting from the permanent label of the destination vertex t, we proceed (towards the upward direction in the computation table) through the previously assigned temporary labels of t until we get a change in the label. Check the vertices that have got a permanent label at this stage. Move to that vertex. Now, apply the similar backtracking technique until we reach at the permanent label of the source vertex s.

Observations:

(1) The above algorithm can be applied for a directed graph as well as undirected graph. In case of a Di-graph $w_{ij} \neq w_{ji}$ whereas for a undirected graph $w_{ij} = w_{ji}$.

(2) The algorithm can be used for unweighted graphs. In this case, construct the table $W = (w_{ij})_{n \times n}$ for a graph with n vertices as follows, where

$$w_{ij} = 1, \text{ if there is an edge from vertex } i \text{ to } j$$
$$w_{ii} = 0$$
$$w_{ij} = \infty, \text{ if there is no edge from vertex } i \text{ to } j$$

Then proceed as stated above in the algorithm.

(3) There may exist more than one shortest path between two specified vertices. This kind of situation arises when we have a tie for selecting permanent label among the temporary labels at any stage in the table.

Example 1 **Find the shortest path and shortest distance from A to E using Dijkstra's Algorithm.**

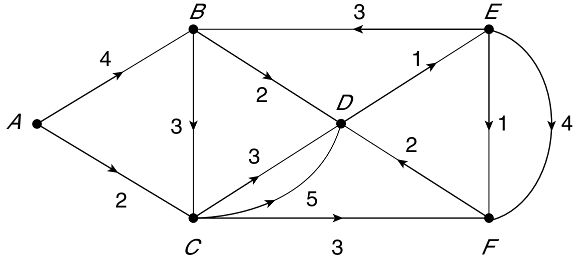


Figure 8.4

The given graph is not simple weighted. So, we make it simple.

First we delete the edge $\overrightarrow{CD}$ of weight 5, because 3 is the minimum weight. We also delete the edge $\overrightarrow{EF}$ of weight 4, because 1 is the minimum weight. Then the graph becomes simple and is given by

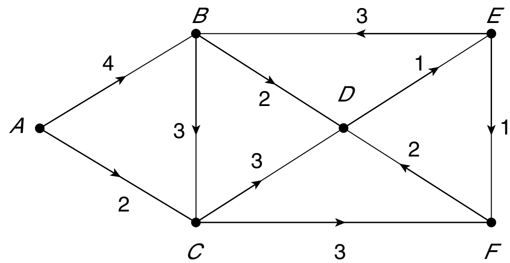


Figure 8.5

The weight table $W = (w_{ij})_{6\times 6}$ is formed on the basis

w_{ij} = weight (or distance or cost) of the edge from vertex $i$to j
$w_{ii} = 0$
$w_{ij} = \infty$, if there is no edge from vertex i to j

and is given by the following

	A	B	C	D	E	F
A	0	4	2	∞	∞	∞
B	∞	0	3	2	∞	∞
C	∞	∞	0	3	∞	3
D	∞	∞	∞	0	1	∞
E	∞	3	∞	∞	0	1
F	∞	∞	∞	2	∞	0

Here, we are to find the shortest path from the vertex A to the vertex E. So we start our computation by assigning permanent label 0 to the vertex A, i.e., $L(A) = 0$ and temporary label ∞ to all others. Permanent label is shown by enclosing in a square ($\square$) in the computation table. Now, at the every stage we compute temporary labels for all the vertices except those that already have permanent labels, and only some of them will get permanent labels. We continue this process until the destination vertex E gets a permanent label.

Temporary label of vertex j, which is not yet permanently labelled is given by

$$\boxed{L(j) = \min\{\text{old } L(j),\ (\text{old } L(i) + w_{ij})\}}$$

where i is the latest vertex permanently labelled in the last stage and w_{ij} is the direct distance between the vertices i and j.

The computation is shown in the following table:

	A	B	C	D	E	F	
St. 1	$\boxed{0}$	∞	∞	∞	∞	∞	: A has got the permanent label 0 and all others have temporary label ∞.
St. 2	$\boxed{0}$	4	2	∞	∞	∞	: Calculation of temporary labels and 2 is the minimum among all.
St. 3	$\boxed{0}$	4	$\boxed{2}$	∞	∞	∞	: C has got the permanent label.
St. 4	$\boxed{0}$	4	$\boxed{2}$	5	∞	5	: Calculation of temporary labels and 4 is the minimum among all.
St. 5	$\boxed{0}$	$\boxed{4}$	$\boxed{2}$	5	∞	5	: B has got the permanent label.
St. 6	$\boxed{0}$	$\boxed{4}$	$\boxed{2}$	5	∞	5	: Calculation of temporary labels and 5 is the minimum among all.
St.7	$\boxed{0}$	$\boxed{4}$	$\boxed{2}$	$\boxed{5}$	∞	5	: There is a tie, so we select D arbitrarily. D has got the permanent label.
St. 8	$\boxed{0}$	$\boxed{4}$	$\boxed{2}$	$\boxed{5}$	6	5	: Calculation of temporary labels and 5 is the minimum among all.
St. 9	$\boxed{0}$	$\boxed{4}$	$\boxed{2}$	$\boxed{5}$	6	$\boxed{5}$	: F has got the permanent label.
St. 10	$\boxed{0}$	$\boxed{4}$	$\boxed{2}$	$\boxed{5}$	6	$\boxed{5}$	: Calculation of temporary label and 6 is the only label.
St. 11	$\boxed{0}$	$\boxed{4}$	$\boxed{2}$	$\boxed{5}$	$\boxed{6}$	$\boxed{5}$	: Destination vertex E has got the permanent label.

For better clarification of the table the computations are shown as follows:

Calculation of temporary labels in stage 2: Here A is the latest permanently labelled vertex.

$$\begin{aligned} L(B) &= \min\{\text{old } L(B),\ (\text{old } L(A) + w_{AB})\} \\ &= \min\{\infty,\ (0+4)\} = 4. \end{aligned}$$

$$\begin{aligned} L(C) &= \min\{\text{old } L(C),\ (\text{old } L(A) + w_{AC})\} \\ &= \min\{\infty,\ (0+2)\} = 2. \end{aligned}$$

$$\begin{aligned} L(D) &= \min\{\text{old } L(D),\ (\text{old } L(A) + w_{AD})\} \\ &= \min\{\infty,\ (0+\infty)\} = \infty \end{aligned}$$

$$\begin{aligned} L(E) &= \min\{\text{old } L(E),\ (\text{old } L(A) + w_{AE})\} \\ &= \min\{\infty,\ (0+\infty)\} = \infty. \end{aligned}$$

$$\begin{aligned} L(F) &= \min\{\text{old } L(F),\ (\text{old } L(A) + w_{AF})\} \\ &= \min\{\infty,\ (0+\infty)\} = \infty. \end{aligned}$$

Calculation of temporary labels in stage 4: Here C is the latest permanently labelled vertex.

$$\begin{aligned} L(B) &= \min\{\text{old } L(B),\ (\text{old } L(C) + w_{CB})\} \\ &= \min\{4,\ (2+\infty)\} = 4. \\ L(D) &= \min\{\text{old } L(D),\ (\text{old } L(C) + w_{CD})\} \\ &= \min\{\infty,\ (2+3)\} = 5 \\ L(E) &= \min\{\text{old } L(E),\ (\text{old } L(C) + w_{CE})\} \\ &= \min\{\infty,\ (2+\infty)\} = \infty. \\ L(F) &= \min\{\text{old } L(F),\ (\text{old } L(C) + w_{CF})\} \\ &= \min\{\infty,\ (2+3)\} = 5. \end{aligned}$$

Calculation of temporary labels in stage 6: Here B is the latest permanently labelled vertex.

$$\begin{aligned} L(D) &= \min\{\text{old } L(D),\ (\text{old } L(B) + w_{BD})\} \\ &= \min\{5,\ (4+2)\} = 5 \\ L(E) &= \min\{\text{old } L(E),\ (\text{old } L(B) + w_{BE})\} \\ &= \min\{\infty,\ (4+\infty)\} = \infty. \\ L(F) &= \min\{\text{old } L(F),\ (\text{old } L(B) + w_{BF})\} \\ &= \min\{5,\ (4+\infty)\} = 5. \end{aligned}$$

Calculation of temporary labels in stage 8: Here D is the latest permanently labelled vertex.

$$\begin{aligned} L(E) &= \min\{\text{old } L(E),\ (\text{old } L(D) + w_{DE})\} \\ &= \min\{\infty,\ (5+1)\} = 6. \\ L(F) &= \min\{\text{old } L(F),\ (\text{old } L(D) + w_{DF})\} \\ &= \min\{5,\ (5+\infty)\} = 5. \end{aligned}$$

Calculation of temporary labels in stage 10: Here F is the latest permanently labelled vertex.

$$\begin{aligned} L(E) &= \min\{\text{old } L(E),\ (\text{old } L(F) + w_{FE})\} \\ &= \min\{6,\ (5+\infty)\} = 6. \end{aligned}$$

In the final stage 11 of the table, the destination vertex E has permanent label and its value is 6. So the required shortest distance is 6.

Now we apply backtrack technique for finding shortest path. Starting from the permanent label of E (from stage 11) we traverse back and see that in stage 7, it is changed and at that stage D has got the permanent label. So we move to D. Repeating the same thing we see that in stage 3, the label of D is changed and at that stage C has got the permanent label. So we move to C. Now if we apply the same technique, then in Stage 1 the label of C is changed and at that stage source vertex A has got the permanent label. So we reach at the source vertex and stop the process.

Hence, the shortest path is given by

$$A \to C \to D \to E.$$

8.5 BREADTH FIRST SEARCH (BFS) ALGORITHM TO FIND THE SHORTEST PATH FROM A SPECIFIED TO ANOTHER SPECIFIED VERTEX

This method is applicable to the unweighted graph.

Let G be any unweighted graph and suppose we are to find the shortest path between the vertices u and v.

The algorithm is based on stage by stage labelling the vertices.

Select the starting vertex u of the graph and label it as 0.

Then label the other vertices in every stage based on the rule given below.

Traverse all the unlabelled vertices in G which are adjacent to the vertices of label k and label all of them as $k + 1$. If no such vertex exists, then there exists no path starting from the vertex of label k.

Continue the process of stage to stage labelling of the vertices until the destination vertex v gets labelled.

The value of the label asigned to the vertex v is the shortest distance from u to v.

Apply the following back tracking method to find the shortest path from u to v:

> Suppose the destination vertex v is labelled as r. Then find an adjacent vertex whose label is $r - 1$. If there is a tie, select any one and let the selected adjacent vertex be w. Then move to the vertex w and apply the same technique.
>
> Continue the process of back tracking until the starting vertex u is reached. The path through the vertices which are traversed on the way of backtracking gives the shortest path.

Observation: **There may exist more than one shortest path between two specified vertices. This kind of situation arises when there is a tie for selecting vertices on the way of backtracking.**

Example 2 **Find by BFS algorithm the shortest path from the vertex v_2 to v_6 in the following graph.** [WBUT 2005]

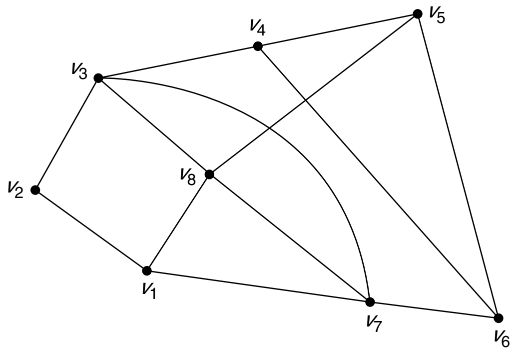


Figure 8.6

Sol. First we select the starting vertex v_2 and label it as 0.

Its adjacent unlabelled vertices are v_3 and v_1. They are labelled as $0+1 = 1$. The labels are shown in the figure.

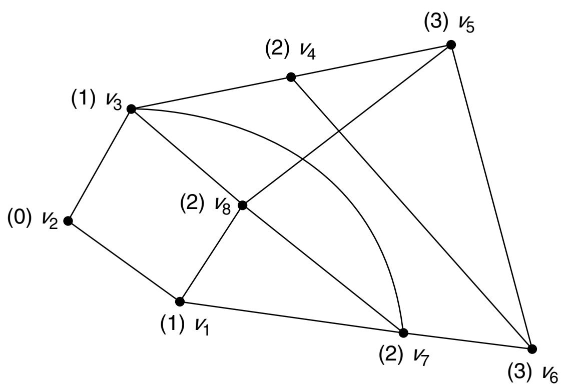


Figure 8.7

Then we see that the unlabelled adjacent vertices of v_3 are v_4 and v_8. We label each of them as $1 + 1 = 2$. The only unlabelled adjacent vertex of v_1 is v_7. We label it as $1 + 1 = 2$.

Now the unlabelled adjacent vertices of v_4 are v_5 and v_6. We label each of them as $2 + 1 = 3$.

We stop the process of labelling since the destination vertex v_6 gets labelled and the value of the label is 3. Therefore, the shortest distance between the vertices v_2 and v_6 is 3.

To get the shortest path between the vertices v_2 and v_6 we apply the backtracking method starting from destination vertex v_6, whose label is 3.

Now the vertices, v_4 and v_7 are adjacent to v_6 and labelled as $3 - 1 = 2$. There is a tie between the two vertices. We select v_4 arbitrarily and move to v_4.

Next the vertex v_3 is adjacent to v_4 and labelled as $2 - 1 = 1$. So we move to v_3.

Again the vertex adjacent to v_3 and labelled as $1 - 1 = 0$, is the starting vertex v_2. So we stop the process, since the starting vertex is reached.

Hence, the required shortest path is

$$v_2 \longrightarrow v_3 \longrightarrow v_4 \longrightarrow v_6.$$

WORKED OUT EXAMPLES

Example 8.1 **Using Dijkstra's Algorithm find the length of the shortest path from a to z in the following graph:**

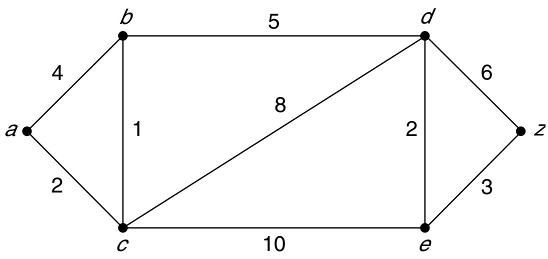


Figure 8.8

[WBUT 2003, 2007]

Sol. The given graph is a simple weighted connected graph. The weight table $W = (w_{ij})_{6\times 6}$ is formed on the basis

w_{ij} = weight (or distance or cost) of the edge from vertex $i$to j
$w_{ii} = 0$
$w_{ij} = \infty$, if there is no edge from vertex i to j

and is given by the following

	a	b	c	d	e	z
a	0	4	2	∞	∞	∞
b	4	0	1	5	∞	∞
c	2	1	0	8	10	∞
d	∞	5	8	0	2	6
e	∞	∞	10	2	0	3
z	∞	∞	∞	6	3	0

Here, we are to find the shortest path from the vertex a to the vertex z. So we start our computation by assigning the permanent label 0 to the vertex a, i.e., $L(a) = 0$ and temporary label ∞ to all other vertices. Permanent label is shown by enclosing in a square (□) in the computation table. Now at every stage we compute temporary labels for all the vertices except those that already have permanent labels and only some of them will get permanent labels. We continue this process until the destination vertex z gets the permanent label.

Temporary label of vertex j, which is not yet permanently labelled is given by

$$\boxed{L(j) = \min\{\text{old } L(j),\ (\text{old } L(i) + w_{ij})\}}$$

where i is the latest vertex permanently labelled in the last stage and w_{ij} is the direct distance between the vertices i and j.

The computation is shown in the following table:

	a	b	c	d	e	z	
St. 1	$\boxed{0}$	∞	∞	∞	∞	∞	: a has got the permanent label 0 and all others have temporary label ∞.
St. 2	$\boxed{0}$	4	2	∞	∞	∞	: Calculation of temporary labels and 2 is the minimum among all.
St. 3	$\boxed{0}$	4	$\boxed{2}$	∞	∞	∞	: c has got the permanent label.
St. 4	$\boxed{0}$	3	$\boxed{2}$	10	12	∞	: Calculation of temporary labels and 3 is the minimum among all.
St. 5	$\boxed{0}$	$\boxed{3}$	$\boxed{2}$	10	12	∞	: b has got the permanent label.
St. 6	$\boxed{0}$	$\boxed{3}$	$\boxed{2}$	8	12	∞	: Calculation of temporary labels and 8 is the minimum among all.
St.7	$\boxed{0}$	$\boxed{3}$	$\boxed{2}$	$\boxed{8}$	12	∞	: d has got the permanent label.
St. 8	$\boxed{0}$	$\boxed{3}$	$\boxed{2}$	$\boxed{8}$	10	14	: Calculation of temporary labels and 10 is the minimum among all.
St. 9	$\boxed{0}$	$\boxed{3}$	$\boxed{2}$	$\boxed{8}$	$\boxed{10}$	14	: e has got the permanent label.
St. 10	$\boxed{0}$	$\boxed{3}$	$\boxed{2}$	$\boxed{8}$	$\boxed{10}$	13	: Calculation of temporary labels and 13 is the only label.
St. 11	$\boxed{0}$	$\boxed{3}$	$\boxed{2}$	$\boxed{8}$	$\boxed{10}$	$\boxed{13}$	: Destination vertex z has got the permanent label.

In the final stage 11 of the table, the destination vertex z has permanent label and its value is 13. So the required shortest distance is 13.

Now we apply backtrack technique for finding shortest path. Starting from permanent label of z (from stage 11) we traverse back and see that in stage 9, it is changed and at that stage e has got the permanent label. So we move to e. Repeating the same thing we see that in stage 7, the label of e is changed and at that stage d has got the permanent label. So we move to d. Now if we apply similar technique, then in Stage 5 the label of d is changed and at that stage b has got the permanent label. So we move to b. Now if we apply similar technique, then in Stage 3 the label of b is changed and at that stage c has got the permanent label. Now if we apply similar technique, then in Stage 1 the label of c is changed and at that stage source vertex a has got the permanent label. So we reach at the source vertex and stop the process.

Hence, the shortest path is given by

$$a \to c \to b \to d \to e \to z.$$

Example 8.2 **Apply Dijkstra's Algorithm to determine a shortest path from a to z in the following graph:**

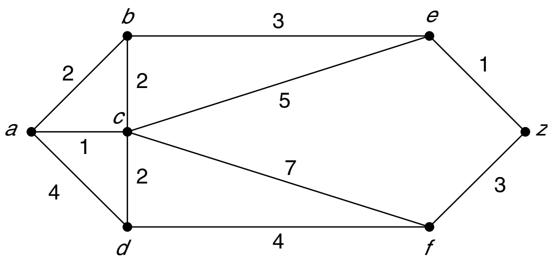


Figure 8.9

[WBUT 2005]

Sol. The given graph is a simple weighted connected graph. The weight table $W = (w_{ij})_{7\times 7}$ is formed on the basis

w_{ij} = weight (or distance or cost) of the edge from vertex $i$to j
$w_{ii} = 0$
$w_{ij} = \infty$, if there is no edge from vertex i to j

and is given by the following

	a	b	c	d	e	f	z
a	0	2	1	4	∞	∞	∞
b	2	0	2	∞	3	∞	∞
c	1	2	0	2	5	7	∞
d	4	∞	2	0	∞	4	∞
e	∞	3	5	∞	0	∞	1
f	∞	∞	7	4	∞	0	3
z	∞	∞	∞	∞	1	3	0

Here, we are to find the shortest path from the vertex a to the vertex z. So we start our computation by assigning the permanent label 0 to the vertex a, i.e., $L(a) = 0$ and temporary label ∞ to all others. Permanent label is shown by enclosing in a square (□) in the computation table. Now at every stage we compute temporary labels for all the vertices except those that already have permanent labels and only some of them will get the permanent label. We continue this process until the destination vertex z gets the permanent label.

Temporary label of vertex j, which is not yet permanently labelled is given by

$$\boxed{L(j) = \min\{\text{old } L(j),\ (\text{old } L(i) + w_{ij})\}}$$

where i is the latest vertex permanently labelled in the last stage and w_{ij} is the direct distance between the vertices i and j.

The computation is shown in the following table:

	a	b	c	d	e	f	z	
St. 1	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∴ a has got the permanent label 0 and all others have temporary label ∞.
St. 2	$\boxed{0}$	2	1	4	∞	∞	∞	∴ Calculation of temporary labels and 1 is the minimum among all.
St. 3	$\boxed{0}$	2	$\boxed{1}$	4	∞	∞	∞	∴ c has got the permanent label.
St. 4	$\boxed{0}$	2	$\boxed{1}$	3	6	8	∞	∴ Calculation of temporary labels and 2 is the minimum among all.
St. 5	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	3	6	8	∞	∴ b has got the permanent label.
St. 6	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	3	5	8	∞	∴ Calculation of temporary labels and 3 is the minimum among all.
St.7	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{3}$	5	8	∞	∴ d has got the permanent label.
St. 8	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{3}$	5	7	∞	∴ Calculation of temporary labels and 5 is the minimum among all.
St. 9	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{3}$	$\boxed{5}$	7	∞	∴ e has got the permanent label.
St. 10	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{3}$	$\boxed{5}$	7	6	∴ Calculation of temporary labels and 6 is the minimum among all.
St. 11	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{3}$	$\boxed{5}$	7	$\boxed{6}$	∴ Destination vertex z has got the permanent label.

In stage 11 of the table, the destination vertex z has permanent label and its value is 6. So the required shortest distance is 6.

Now we apply backtrack technique for finding the shortest path. Starting from permanent label of z (from stage 11) we traverse back and see that in stage 9, it is changed and at that stage e has got the permanent label. So we move to e. Repeating the same thing we see that in stage 5, the label of e is changed and at that stage b has got the permanent label. So we move to b.

Now if we apply similar technique, then in Stage 1 the label of b is changed and at that stage source vertex a has got the permanent label. So we reach at the source vertex and stop the process.

Hence, the shortest path is given by

$$a \to b \to e \to z.$$

Example 8.3 **Apply Dijkstra's Algorithm to determine a shortest path from a to f in the following graph:**

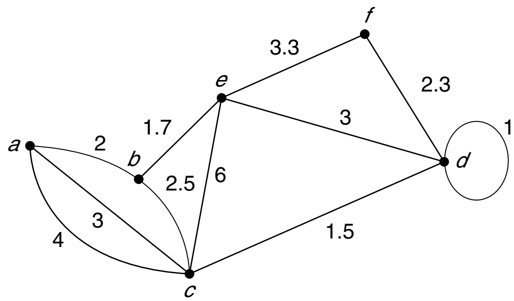


Figure 8.10

[WBUT 2004, 2008]

Sol. The given graph is not simple. We first discard the parallel edge ac with weight 4, because 3 is the minimum weight. We also delete the self loop at the vertex d. The we have the following simple weighted connected graph:

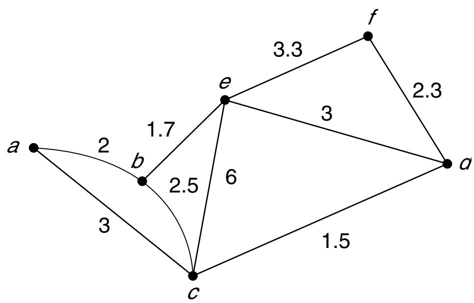


Figure 8.11

The weight table $W = (w_{ij})_{6\times 6}$ is formed on the basis

$w_{ij} =$ weight (or distance or cost) of the edge from vertex $i$to j
$w_{ii} = 0$
$w_{ij} = \infty,$ if there is no edge from vertex i to j

and is given by the following

	a	b	c	d	e	f
a	0	2	3	∞	∞	∞
b	2	0	2.5	∞	1.7	∞
c	3	2.5	0	1.5	6	∞
d	∞	∞	1.5	0	3	2.3
e	∞	1.7	6	3	0	3.3
f	∞	∞	∞	2.3	3.3	0

Here, we are to find the shortest path from the vertex a to the vertex f. So we start our computation by assigning the permanent lebel 0 to the vertex a, i.e., $L(a) = 0$ and temporary label ∞ to all others. Permanent label is shown by enclosing in a square (□) in the computation table. Now at every stage we compute temporary labels for all the vertices except those that already have permanent labels and only some of them will get the permanent label. We continue this process until the destination vertex f gets the permanent label.

Temporary label of vertex j, which is not yet permanently labelled is given by

$$\boxed{L(j) = \min\{\text{old } L(j),\ (\text{old } L(i) + w_{ij})\}}$$

where i is the latest vertex permanently labelled in the last stage and w_{ij} is the direct distance between the vertices i and j.

The computation is shown in the following table:

	a	b	c	d	e	f	
St. 1	$\boxed{0}$	∞	∞	∞	∞	∞	: a has got permanent label 0 and all others have temporary label ∞.
St. 2	$\boxed{0}$	2	3	∞	∞	∞	: Calculation of temporary labels and 2 is the minimum among all.
St. 3	$\boxed{0}$	$\boxed{2}$	3	∞	∞	∞	: b has got the permanent label.
St. 4	$\boxed{0}$	$\boxed{2}$	3	∞	3.7	∞	: Calculation of temporary labels and 3 is the minimum among all.
St. 5	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	∞	3.7	∞	: c has got the permanent label.
St. 6	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	4.5	3.7	∞	: Calculation of temporary labels and 3.7 is the minimum among all.
St.7	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	4.5	$\boxed{3.7}$	∞	: e has got the permanent label.
St. 8	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	4.5	$\boxed{3.7}$	7	: Calculation of temporary labels and 4.5 is the minimum among all.

St. 9	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4.5}$	$\boxed{3.7}$	7	: d has got the permanent label.
St. 10	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4.5}$	$\boxed{3.7}$	6.8	: Calculation of temporary labels and 6.8 is the only label.
St. 11	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4.5}$	$\boxed{3.7}$	$\boxed{6.8}$	: Destination vertex f has got the permanent label.

In the final stage 11 of the table, the destination vertex f has a permanent label and its value is 6.8. So the required shortest distance is 6.8.

Now we apply backtrack technique for finding shortest path. Starting from permanent label of f (from stage 11) we traverse back and see that in stage 9, it is changed and at that stage d has got the permanent label. So we move to d. Repeating the same thing we see that in stage 5, the label of d is changed and at that stage c has got the permanent label. So we move to c. Now if we apply similar technique, then in Stage 1 the label of c is changed and at that stage source vertex a has got the permanent label. So we reach at the source vertex and stop the process.

Hence, the shortest path is given by

$$a \to c \to d \to f.$$

Example 8.4 **Apply Dijkstra's Algorithm to determine a shortest path from s to z in the following graph:**

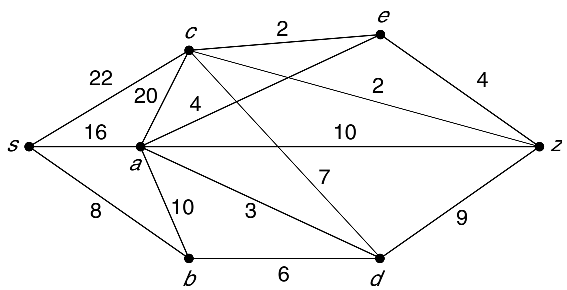


Figure 8.12

[WBUT 2009]

Sol. The given graph is simple weighted connected graph. The weight table $W = (w_{ij})_{7\times 7}$ is formed on the basis

$w_{ij} =$ weight (or distance or cost) of the edge from vertex $i$to j
$w_{ii} = 0$
$w_{ij} = \infty,$ if there is no edge from vertex i to j

and is given by the following

	s	a	b	c	d	e	z
s	0	16	8	22	∞	∞	∞
a	16	0	10	20	3	4	10
b	8	10	0	∞	6	∞	∞
c	22	20	∞	0	7	2	2
d	∞	3	6	7	0	∞	9
e	∞	4	∞	2	∞	0	4
z	∞	10	∞	2	9	4	0

Here, we are to find the shortest path from the vertex s to the vertex z. So we start our computation by assigning the permanent lebel 0 to the vertex s i.e., $L(s) = 0$ and temporary label ∞ to all others. Permanent label is shown by enclosing in a square (□) in the computation table. Now at every stage we compute temporary labels for all the vertices except those that already have permanent labels and only some of them will get the permanent label. We continue this process until the destination vertex z gets the permananent label.

Temporary label of vertex j, which is not yet permanently labelled is given by

$$\boxed{L(j) = \min\{\text{old } L(j),\ (\text{old } L(i) + w_{ij})\}}$$

where i is the latest vertex permanently labelled in the last stage and w_{ij} is the direct distance between the vertices i and j.

The computation is shown in the following table:

	s	a	b	c	d	e	z	
St. 1	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∵ s has got the permanent label 0 and all others have temporary label ∞.
St. 2	$\boxed{0}$	16	8	22	∞	∞	∞	∵ Calculation of temporary labels and 8 is the minimum among all.
St. 3	$\boxed{0}$	16	$\boxed{8}$	22	∞	∞	∞	∵ b has got the permanent label.
St. 4	$\boxed{0}$	16	$\boxed{8}$	22	14	∞	∞	∵ Calculation of temporary labels and 14 is the minimum among all.
St. 5	$\boxed{0}$	16	$\boxed{8}$	22	$\boxed{14}$	∞	∞	∵ d has got the permanent label.
St. 6	$\boxed{0}$	16	$\boxed{8}$	21	$\boxed{14}$	∞	23	∵ Calculation of temporary labels and 16 is the minimum among all.
St.7	$\boxed{0}$	$\boxed{16}$	$\boxed{8}$	21	$\boxed{14}$	∞	23	∵ a has got the permanent label.
St. 8	$\boxed{0}$	$\boxed{16}$	$\boxed{8}$	21	$\boxed{14}$	20	23	∵ Calculation of temporary labels and 20 is the minimum among all.

St. 9	[0]	[16]	[8]	21	[14]	[20]	23	: e has got the permanent label.
St. 10	[0]	[16]	[8]	21	[14]	[20]	23	: Calculation of temporary labels and 21 is the minimum among all.
St. 11	[0]	[16]	[8]	[21]	[14]	[20]	23	: c has got the permanent label.
St. 12	[0]	[16]	[8]	[21]	[14]	[20]	23	: Calculation of temporary labels and 23 is the only label.
St. 13	[0]	[16]	[8]	[21]	[14]	[20]	[23]	: Destination vertex z has got the permanent label.

In the stage 13 of the table, the destination vertex z has the permanent label and its value is 23. So the required shortest distance is 23.

Now we apply backtrack technique for finding shortest path. Starting from the permanent label of z (from stage 13) we traverse back and see that in stage 5, it is changed and at that stage d has got the permanent label. So we move to d. Repeating the same thing we see that in stage 3, the label of d is changed and at that stage b has got the permanent label. So we move to b. Now if we apply similar technique, then in Stage 1 the label of b is changed and at that stage source vertex s has got the permanent label. So we reach at the source vertex and stop the process.

Hence, the shortest path is given by

$$s \to b \to d \to z.$$

Example 8.5 **Apply Dijkstra's Algorithm to determine a shortest path from a to z in the following graph:**

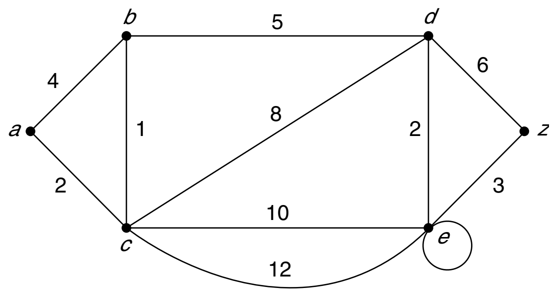


Figure 8.13

[WBUT 2006, 2011]

Sol. The given graph is not a simple one. To make it simple first we discard the self loop on the vertex e. Then delete the edge $\{c, e\}$ of weight 12, because 10 is the minimum weight. Then the graph becomes simple and is given by

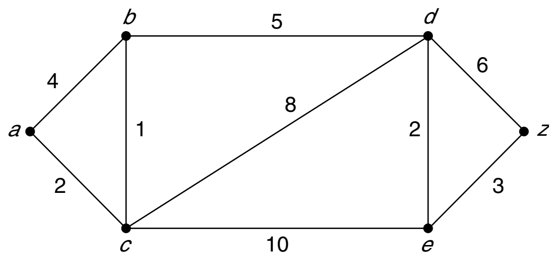


Figure 8.14

Now we apply Dijkstra's Algorithm to determine a shortest path from a to z, which is same as **Example 8.1.**

EXERCISES

Short and Long Answer Type Questions

1) Apply BFS Algorithm to determine a shortest path from v_2 to v_6 in the following graph:

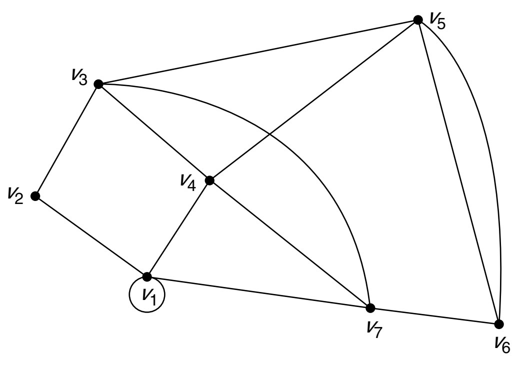


Figure 8.15

[**Ans:** $v_2 \to v_3 \to v_7 \to v_6$ or, $v_2 \to v_1 \to v_7 \to v_6$. Shortest distance 3.]

2) Apply Dijkstra's Algorithm to determine a shortest path from s to z in the following graph:

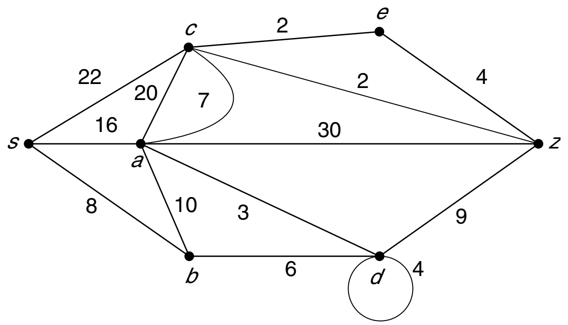


Figure 8.16

[**Ans:** $s \to b \to d \to z$. Shortest distance 23.]

3) Apply Dijkstra's Algorithm to determine a shortest path from a to b in the following graph:

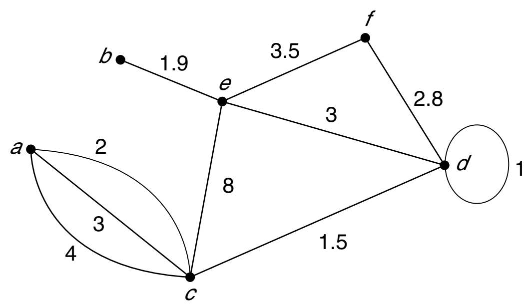


Figure 8.17

[**Ans:** $a \to c \to d \to b$. Shortest distance 8.4.]

4) Apply Dijkstra's Algorithm to determine a shortest path from A to E in the following graph:

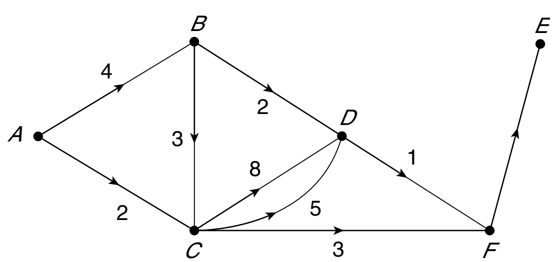


Figure 8.18

[**Ans:** $A \to B \to D \to F \to E$ or, $A \to C \to F \to E$. Shortest distance 10.]

5) Apply Dijkstra's Algorithm to determine a shortest path from A to F in the following graph:

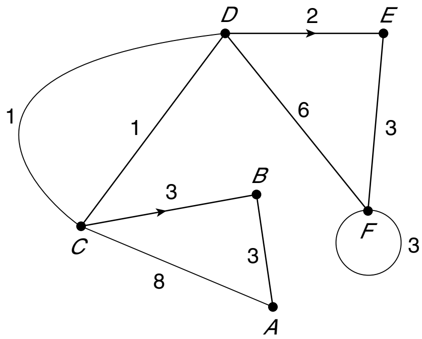


Figure 8.19

[**Ans:** $A \to C \to D \to E \to F$. Shortest distance 14.]

6) Apply Dijkstra's Algorithm to determine a shortest path from A to E in the following graph:

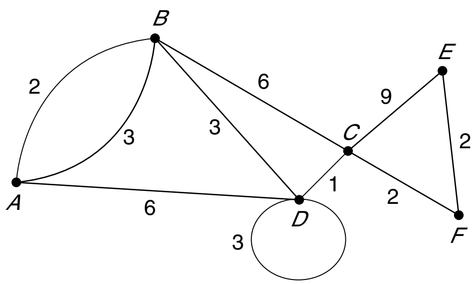


Figure 8.20

[**Ans:** $A \to B \to D \to C \to F \to E$. Shortest distance 10.]

7) Apply Dijkstra's Algorithm to determine a shortest path from A to G in the following graph:

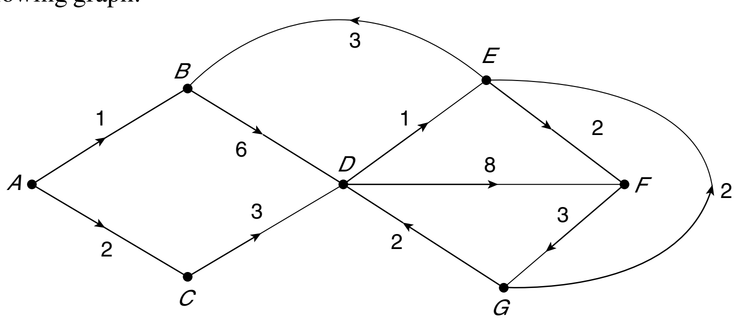


Figure 8.21

[**Ans:** $A \to C \to D \to E \to F \to G$. Shortest distance 11.]

8) Apply BFS Algorithm to determine a shortest path from E to H in the following graph:

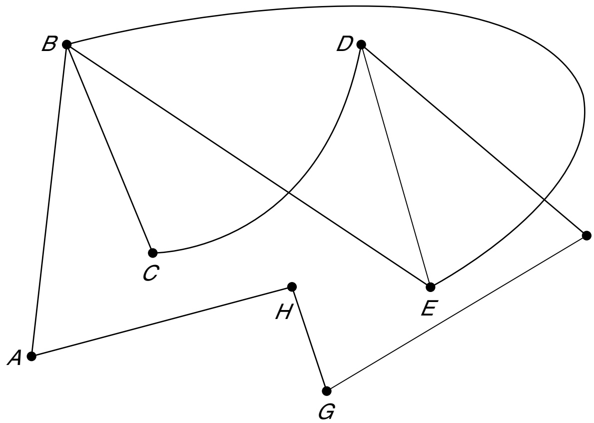


Figure 8.22

[**Ans:** $E \to B \to A \to H$. Shortest distance 3.]

9) Apply Dijkstra's Algorithm to determine a shortest path from A to G in the following graph:

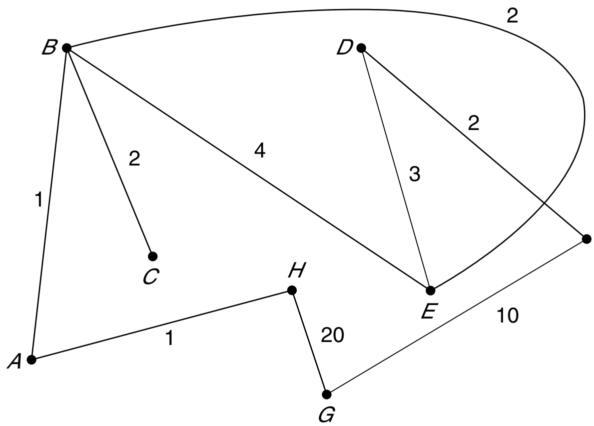


Figure 8.23

[**Ans:** $A \to B \to E \to D \to F \to G$. Shortest distance 18.]

10) Apply BFS Algorithm to determine a shortest path from G to L in the following graph:

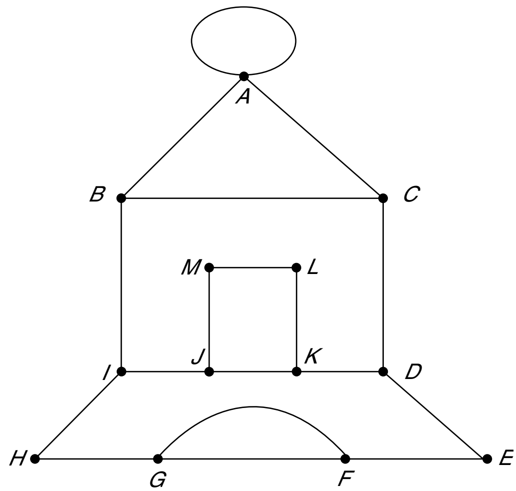


Figure 8.24

[**Ans:** $G \to H \to I \to J \to M \to L$. or, $G \to H \to I \to J \to K \to L$ or, $G \to F \to E \to D \to K \to L$. Shortest distance 5.]

Multiple Choice Questions

1) To find a shortest path between two vertices in a graph we apply
   a) Krushkal Algorithm b) Prims Algorithm
   c) Dijkstra's Algorithm d) None of these

2) The length of shortest path between two vertices in an unweighted graph is
   a) number of the edges in the path
   b) number of the vertices in the path
   c) total number of the vertices and edges in the path
   d) none of these

3) The length of shortest path between two vertices in a weighted graph is
   a) sum of weights of the edges in the path
   b) number of the edges in the path
   c) total number of the vertices and edges in the path
   d) none of these

4) The shortest distance between two vertices in a weighted graph in Dijkstra's Algorithm is given by

   a) value of permanent label of destination vertex

   b) value of permanent label of starting vertex

   c) addition of values of permanent labels of all vertices in the shortest path

   d) none of these

5) The number of shortest paths between two vertices in a graph

   a) is always only one   b) may be more than one

   c) is always at least one   d) none of these

Answers:

1) c   2) a   3) a   4) a   5) b

CHAPTER

9

Improper Integrals

9.1 INTRODUCTION

When dealing with different problems of science and technology we have to face different definite integrations where either the limits a and b are infinite or the integrand $f(x)$ is unbounded in $a \leq x \leq b$. These type of integrals are called improper integrals. In this chapter, we will discuss different types of improper integrals and their convergence followed by a discussion on special type of improper integrals called beta and gamma functions with their applications. Here also we give inter-relations between beta and gamma functions.

9.2 DEFINITION OF IMPROPER INTEGRALS

A definite integral $\int_a^b f(x)\,dx$ is called a **proper integral** when

i) the limits a and b are finite and

ii) the integrand $f(x)$ is bounded and integrable in $a \leq x \leq b$.

Improper Integral If either

i) a limit is infinite or both, i.e., ($a = -\infty$ or $b = \infty$ or both) or

ii) the integrand $f(x)$ becomes infinite in $a \leq x \leq b$. Then, the integral $\int_a^b f(x)\,dx$ is called an **improper integral.**

Some examples of improper integrals are

$$\int_0^\infty \frac{1}{x^3}\,dx, \int_1^3 \frac{1}{(x-1)(x-3)}\,dx, \int_{-\infty}^\infty \frac{1}{\sqrt{x}}\,dx$$

9.3 TYPES OF IMPROPER INTEGRALS

Improper integrals are of three types.

Type I The interval increases without limit (integrals with unbounded ranges)

Type II The integrand has a finite number of infinite discontinuities.

Type III **Combination of Type I and Type II**

Integrand has a finite number of infinite discontinuities and integrals with unbounded ranges.

9.3.1 Type I : Integrals with Unbounded Ranges

There are three kinds of unbounded ranges over which integrals may be taken.

Case 1: Let $f(x)$ be bounded and integrable in $a \le x \le B$ for every $B > a$. Then, the improper integral

$$\int_a^\infty f(x)\,dx$$

is said to exist or converge if

$$\lim_{B\to\infty}\int_a^B f(x)\,dx$$

exists finitely and we write

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{B\to\infty}\int_a^B f(x)\,dx}$$

Example 1 **Verify whether the improper integral**

$$\int_0^\infty \frac{dx}{1+x^2}$$

exists or not?

Sol. Here,

$$f(x) = \frac{1}{1+x^2}$$

which is bounded and integrable in $0 \le x \le B$ for every $B > 0$.

Now,

$$\lim_{B\to\infty}\int_0^B f(x)\,dx = \lim_{B\to\infty}\int_0^B \frac{dx}{1+x^2}$$

$$= \lim_{B\to\infty}[\tan^{-1}x]_0^B$$

$$= \lim_{B\to\infty}[\tan^{-1}B - \tan^{-1}0]$$

$$= \lim_{B\to\infty}\tan^{-1}B = \frac{\pi}{2}$$

Therefore,

$$\int_0^\infty \frac{dx}{1+x^2}$$

exists and is equal to $\frac{\pi}{2}$.

Case 2: Let $f(x)$ be bounded and integrable in $A \le x \le b$ for every $A < b$. Then, the improper integral

$$\int_{-\infty}^{b} f(x)\,dx$$

is said to exist or converge if

$$\lim_{A\to-\infty}\int_A^b f(x)\,dx$$

exists finitely and we write

$$\boxed{\int_{-\infty}^{b} f(x)\,dx = \lim_{A\to-\infty}\int_A^b f(x)\,dx}$$

Case 3: Let $f(x)$ be bounded and integrable in $A \le x \le a$ for every $A < a$ and in $a \le x \le B$ for every $B > a$ and

$$\lim_{A\to-\infty}\int_A^a f(x)\,dx$$

and

$$\lim_{B\to\infty}\int_a^B f(x)\,dx$$

exists finitely, and we write

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx = \lim_{A\to-\infty}\int_A^a f(x)\,dx + \lim_{B\to\infty}\int_a^B f(x)\,dx}$$

Example 2 **Evaluate**

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

if it converges. [WBUT-2011]

Sol. We know,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

$$= \lim_{A\to-\infty}\int_A^a f(x)\,dx + \lim_{B\to\infty}\int_a^B f(x)\,dx;\, A\le x\le a, a\le x\le B$$

Considering the integral is convergent, we have

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^{\infty} xe^{-x^2}\,dx$$

$$= \lim_{A\to-\infty}\int_A^0 xe^{-x^2}\,dx + \lim_{B\to\infty}\int_0^B xe^{-x^2}\,dx$$

$$= \lim_{A\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_A^0 + \lim_{B\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^B$$

$$= \lim_{A\to-\infty}\left[\frac{1}{2}e^{-A^2}-\frac{1}{2}\right] + \lim_{B\to\infty}\left[\frac{1}{2}-\frac{1}{2}e^{-B^2}\right]$$

$$= -\frac{1}{2}+\frac{1}{2} = 0$$

9.3.2 Type II : Integrand has a Finite Number of Infinite Discontinuities

Case 1: Let us suppose $f(x)$ has an infinite discontinuity only at the left-hand point a, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0+} \int_{a+\epsilon}^b f(x)\,dx, 0 < \epsilon < b - a$$

where the limit exists and is finite.

Example 3 **Verify whether the improper integral**

$$\int_0^1 \frac{1}{x}\,dx$$

exists or not and find the value of the integral.

Sol. Here,

$$f(x) = \frac{1}{x}$$

which has an infinite discontinuity at the left end point $x = 0$.
Therefore,

$$\begin{aligned}\int_0^1 \frac{1}{x}\,dx &= \lim_{\epsilon \to 0+} \int_{0+\epsilon}^1 \frac{1}{x}\,dx \\ &= \lim_{\epsilon \to 0+} [\log x]_\epsilon^1 \\ &= \lim_{\epsilon \to 0+} [\log 1 - \log \epsilon] \\ &= -\infty\end{aligned}$$

Therefore, the improper integral

$$\int_0^1 \frac{1}{x}\,dx$$

does not exist.

Case 2: Let us suppose $f(x)$ has an infinite discontinuity only at the right-hand point b, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0+} \int_a^{b-\epsilon} f(x)\,dx, 0 < \epsilon < b - a$$

where the limit exists and is finite.

Example 4 **Verify whether the improper integral**

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

exists or not and find the value of the integral.

Sol. Here,

$$f(x) = \frac{1}{\sqrt{1-x^2}}$$

which has an infinite discontinuity at $x = 1$.
Therefore,

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}} \\ &= \lim_{\epsilon \to 0+} [\sin^{-1} x]_0^{1-\epsilon} \\ &= \lim_{\epsilon \to 0+} \sin^{-1}(1-\epsilon) \\ &= \sin^{-1} 1 = \frac{\pi}{2}\end{aligned}$$

Therefore, the improper integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

exists and its value is $\frac{\pi}{2}$.

Case 3: Let us suppose $f(x)$ has an infinite discontinuity at the point $x=c$, $a<c<b$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+}\int_a^{c-\epsilon} f(x)\,dx + \lim_{\delta\to 0+}\int_{c+\delta}^{b} f(x)\,dx$$

where the limits exists and is finite.

Observations:

1) When, $f(x)$ has an infinite discontinuity at the point $x=c,\ a<c<b$, then taking $\epsilon=\delta$, we have

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{\epsilon\to 0+}\int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon\to 0+}\int_{c+\epsilon}^{b} f(x)\,dx\\ &= \lim_{\epsilon\to 0+}\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx\right]\end{aligned}$$

is called **Cauchy principal value of integral.**

2) In Case 3, sometimes the Cauchy principal value of the integral exists when according to the general definition, the integral does not exist.

Example 5 **Prove that**

$$\int_{-1}^{1}\frac{1}{x^3}\,dx$$

exists in cauchy principal value sense but not general sense.

Sol. Here, $x=0$ is a point of infinite discontinuity. So

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x^3}\,dx &= \lim_{\epsilon\to 0+}\int_{-1}^{-\epsilon}\frac{1}{x^3}\,dx + \lim_{\delta\to 0+}\int_{\delta}^{1}\frac{1}{x^3}\,dx\\ &= \lim_{\epsilon\to 0+}\left[-\frac{1}{2x^2}\right]_{-1}^{-\epsilon} + \lim_{\delta\to 0+}\left[-\frac{1}{2x^2}\right]_{\delta}^{1}\\ &= \lim_{\epsilon\to 0+}\left[\frac{1}{2}-\frac{1}{2\epsilon^2}\right] + \lim_{\delta\to 0+}\left[-\frac{1}{2}+\frac{1}{2\delta^2}\right]\end{aligned}$$

Since, $\lim\limits_{\epsilon\to 0+} \dfrac{1}{2\epsilon^2}$ and $\lim\limits_{\delta\to 0+} \dfrac{1}{2\delta^2}$ do not exist, the integral does not exist in general sense.

But if we put $\epsilon = \delta$, then

$$\int_{-1}^{1} \frac{1}{x^3}\, dx = \lim_{\epsilon\to 0+} \left[\int_{-1}^{-\epsilon} \frac{1}{x^3}\, dx + \int_{\epsilon}^{1} \frac{1}{x^3}\, dx \right]$$

$$= \lim_{\epsilon\to 0+} \left[\left(\frac{1}{2} - \frac{1}{2\epsilon^2} \right) + \left(-\frac{1}{2} + \frac{1}{2\epsilon^2} \right) \right] = 0.$$

Therefore, the integral exists as cauchy principal value sense.

9.3.3 Type III Combination of Type I and Type II

Integrand has a finite number of infinite discontinuities and integrals with unbounded ranges.

For example, the integral $\int_0^\infty \dfrac{dx}{(1-x)^2}$ is an improper integral of **Type III**. Since the integral has an unbounded range and $x = 1$ is a point of infinite discontinuity.

9.4 NECESSARY AND SUFFICIENT CONDITIONS FOR CONVERGENCE OF IMPROPER INTEGRALS

9.4.1 Convergence of Type I Improper Integral

Definition The improper integral $\int_a^\infty f(x)\, dx$ is said to **converge absolutely**, when $\int_a^\infty |f(x)|\, dx$ is convergent and $f(x)$ is bounded and integrable in the arbitrary interval $a \le x \le B$ for every $B > a$.

The improper integral $\int_a^\infty f(x)\, dx$ is said to be **conditionally convergent**, when $\int_a^\infty f(x)\, dx$ is convergent but $\int_a^\infty |f(x)|\, dx$ is not convergent.

Theorem 9.1 If $\int_a^\infty f(x)\, dx$ is an absolutely convergent integral, then $\int_a^\infty f(x)\, dx$ is convergent.

Proof Beyond the scope of the book.

Theorem 9.2 If $\int_{-\infty}^{b} f(x)\, dx$ is an absolutely convergent integral, then $\int_{-\infty}^{b} f(x)\, dx$ is convergent.

Proof Beyond the scope of the book.

Note: The converse of the above theorems are not true.

Theorem 9.3 **Test of Convergence (Limit Test)**

Let, $f(x)$ and $g(x)$ be integrable functions when $x \geq a$ and $g(x)$ is positive. If,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lambda \neq 0$$

then, the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge absolutely or both diverge.

Proof Beyond the scope of the book.

Corollary 1 If,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} \to 0$$

and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges absolutely.

Corollary 2 If,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} \to \pm\infty$$

and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Theorem 9.4 **(Comparison Integral)**

The improper integral $\int_a^\infty \frac{dx}{x^\mu} (a > 0)$ exists if $\mu > 1$ and does not exist if $\mu \leq 1$.

Proof Beyond the scope of the book.

Theorem 9.5 Let $f(x)$ be an integrable function, then $\int_a^\infty f(x)\,dx\ (a > 0)$

a) converges absolutely if,

$$\lim_{x \to \infty} x^\mu f(x) = \lambda(\text{finite}); \mu > 1$$

b) diverges if,

$$\lim_{x \to \infty} x^\mu f(x) = \lambda(\neq 0) \text{ or } \pm\infty; \mu \leq 1$$

Proof Beyond the scope of the book.

Example 6 **Examine the convergence of the improper integral**

$$\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$$

Sol. Here,

$$f(x) = \frac{1}{x\sqrt{1+x^2}}$$

Now, taking $\mu = 2 > 1$

$$\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} x^2 \frac{1}{x\sqrt{1+x^2}}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{1+x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^2}+1}} = 1$$

Therefore, the integral

$$\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$$

converges absolutely and hence is convergent.

9.4.2 Convergence of Type II improper integral

Definition The improper integral $\int_a^b f(x)\,dx$ is said to **converge absolutely** where $x = a$ is the point of infinite discontinuity, when $\int_a^b |f(x)|\,dx$ is convergent and $f(x)$ is bounded and integrable in the interval $a + \epsilon \le x \le b$ where $0 < \epsilon < b - a$

The improper integral $\int_a^b f(x)\,dx$ is said to be **conditionally convergent**, when $\int_a^b f(x)\,dx$ is convergent but $\int_a^b |f(x)|\,dx$ is not convergent where $x = a$ is the point of infinite discontinuity.

Theorem 9.6 If $\int_a^b f(x)\,dx$ is an absolutely convergent integal, then $\int_a^b f(x)\,dx$ is convergent where $x = a$ is the point of infinite discontinuity.

Proof Beyond the scope of the book.

Note: The converse of the above theorem is not true.

Theorem 9.7 **Test of convergence (Limit Test)**

Let $f(x)$ and $g(x)$ be integrable functions where $a < x \leq b$ and $g(x)$ are positive. If,

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lambda \neq 0$$

then the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge absolutely or both diverge.

Proof Beyond the scope of the book.

Corollary 1 If,

$$\lim_{x\to a+} \frac{f(x)}{g(x)} \to 0$$

and $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converge absolutely.

Corollary 2 If,

$$\lim_{x\to a+} \frac{f(x)}{g(x)} \to \pm\infty$$

and $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges.

Theorem 9.8 **(Comparison Integral)** The improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ exists if $\mu < 1$ and does not exists if $\mu \geq 1$, where $x = a$ is the point of infinite discontinuity.

Theorem 9.9 Let $f(x)$ be an integrable function in $a < x \leq b$, i.e., $(a+\epsilon, b)$ where $0 < \epsilon < b - a$, then $\int_a^b f(x)\,dx$ is

a) absolutely convergent if,

$$\lim_{x\to a+} (x-a)^\mu f(x) = \lambda\,;\, 0 < \mu < 1$$

b) divergent if,

$$\lim_{x\to a+} (x-a)^\mu f(x) = \lambda(\neq 0) \text{ or } \pm\infty; \mu \geq 1$$

where $x = a$ is the point of infinite discontinuity.

Example 7 Discuss the convergence of $\int_0^1 \frac{x^{m-1}}{1+x}\,dx$

Sol. The integral $\int_0^1 \frac{x^{m-1}}{1+x}\,dx$ is a proper integral when $m \geq 1$ but is an improper integral when $m < 1$.
Now,

$$\lim_{x \to 0+} (x-0)^{\mu} f(x) = \lim_{x \to 0+} x^{1-m} \frac{x^{m-1}}{1+x} = 1; \text{ where } \mu = 1-m$$

Therefore, the integral $\int_0^1 \frac{x^{m-1}}{1+x}\,dx$ is convergent when,

$$0 < \mu < 1$$

or, $$0 < 1-m < 1$$

or, $$0 < m < 1$$

9.4.3 Absolute Convergence and Convergence of Type III Improper Integral

Improper integrals of Type III can be expressed in terms of improper integrals of Type I and Type II, hence the convergence is dealt by using the results already established.

9.5 GAMMA FUNCTION

9.5.1 Definition

The improper integral

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx \text{ for } n > 0$$

is called **Gamma Function or Second Eulerian Integral.**

9.5.2 Properties of Gamma Functions

Property 1: **For any** $a > 0$

$$\int_0^{\infty} e^{-ax} x^{n-1}\,dx = \frac{\Gamma(n)}{a^n} \text{ for } n > 0$$

Proof Let,

$$ax = y \Rightarrow a\,dx = dy \Rightarrow\; dx = \frac{dy}{a}$$

Therefore,

$$\int_0^\infty e^{-ax}x^{n-1}\,dx = \int_0^\infty e^{-y}\left(\frac{y}{a}\right)^{n-1}\frac{dy}{a}$$

$$= \frac{1}{a^n}\int_0^\infty e^{-y}y^{n-1}\,dy = \frac{\Gamma(n)}{a^n} \quad \text{for, } n > 0$$

Property 2:

$$\boxed{\Gamma(n+1) = n\Gamma(n) \text{ for } n > 0}$$

Proof We have,

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$$

Integrating by parts taking e^{-x} as first function and x^{n-1}as second function, we have

$$\begin{aligned}
\Gamma(n) &= \int_0^\infty e^{-x}x^{n-1}\,dx \\
&= \left[e^{-x}\int x^{n-1}\,dx\right]_0^\infty - \int_0^\infty\left\{\frac{d}{dx}(e^{-x})\int x^{n-1}\,dx\right\}dx \\
&= \left[\frac{e^{-x}x^n}{n}\right]_0^\infty + \frac{1}{n}\int_0^\infty e^{-x}x^n\,dx \\
&= 0 + \frac{1}{n}\Gamma(n+1);\ \text{since } e^{-\infty} = 0 \\
&= \frac{1}{n}\Gamma(n+1)
\end{aligned}$$

or, $$\Gamma(n+1) = n\Gamma(n)$$

Property 3:

$$\boxed{\Gamma(1) = 1}$$

Proof By direct computation,

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\, dx = \int_0^\infty e^{-x}\, dx$$
$$= [-e^{-x}]_0^\infty$$
$$= [1 - e^{-\infty}]$$
$$= 1 \text{ since } e^{-\infty} = 0$$

Property 4: **When n is a positive integer**

$$\boxed{\Gamma(n+1) = n!}$$

Proof We know,

$$\Gamma(n+1) = n\Gamma(n) \text{ for } n > 0$$

When n is a positive integer,

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

Similarly,

$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$\Gamma(2) = 1\Gamma(1)$$

Therefore,

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)\ldots 2\Gamma(2)$$
$$= n(n-1)(n-2)\ldots 2.1\Gamma(1)$$
$$= n(n-1)(n-2)\ldots 2.1 \text{ since } \Gamma(1) = 1$$
$$= n!$$

9.6 BETA FUNCTION

9.6.1 Definition

The improper integral

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx;\ m,n>0}$$

is called **beta function or first Eulerian integral.**

9.6.2 Properties of Beta Function

Property 1:

$$\boxed{B(m,n) = B(n,m);\ m,n>0}$$

Proof Putting

$$x = 1-y \Rightarrow\ dx = -dy$$

we have,

$$\begin{aligned} B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \\ &= \lim_{\epsilon\to 0+}\lim_{\delta\to 0+}\int_{\epsilon}^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx \\ &= \lim_{\epsilon\to 0+}\lim_{\delta\to 0+}\int_{1-\epsilon}^{\delta}(1-y)^{m-1}y^{n-1}(-dy) \\ &= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = B(n,m) \end{aligned}$$

Property 2:

$$\boxed{B(m,n) = \int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\,dx = B(n,m); m,n>0}$$

Proof Putting

$$x = \frac{1}{1+y} \Rightarrow dx = -\frac{1}{(1+y)^2}\,dy$$

we have,

$$\begin{aligned}
B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \\
&= \lim_{\epsilon\to 0+}\lim_{\delta\to 0+}\int_{\epsilon}^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx \\
&= \lim_{\epsilon\to 0+}\lim_{\delta\to 0+}\int_{\frac{1}{\epsilon}-1}^{\frac{\delta}{1+\delta}} \frac{1}{(1+y)^{m-1}}\frac{y^{n-1}}{(1+y)^{n-1}}(-1)\frac{1}{(1+y)^2}\,dy \\
&= \int_0^{\infty}\frac{y^{n-1}}{(1+y)^{m+n}}\,dy \\
&= \int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\,dx
\end{aligned}$$

Similarly,

$$B(n,m) = \int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Since,

$$B(m,n) = B(n,m)$$

Therefore,

$$B(m,n) = \int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\,dx = B(n,m);\quad m,n>0$$

Property 3:

$$\boxed{B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta;\ m,n>0}$$

Proof Putting

$$x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta d\theta$$

Therefore,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \lim_{\epsilon\to 0+}\lim_{\delta\to 0+}\int_{\epsilon}^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx$$

$$= \lim_{\epsilon\to 0+}\lim_{\delta\to 0+}\int_{\sin^{-1}\sqrt{\epsilon}}^{\sin^{-1}\sqrt{1-\delta}} \sin^{2m-2}\theta\cos^{2n-2}\theta(2\sin\theta\cos\theta d\theta)$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

Property 4:

$$\boxed{B\left(\frac{1}{2},\frac{1}{2}\right) = \pi}$$

Proof We have,

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta; m,n > 0$$

Putting $m = n = \frac{1}{2}$, we have

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} d\theta = 2[\theta]_0^{\frac{\pi}{2}} = \pi$$

9.7 INTER-RELATION BETWEEN GAMMA FUNCTION AND BETA FUNCTION

$$\boxed{B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}; m,n > 0}$$ [WBUT-2011]

Proof We know,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \text{ for } n > 0$$

Putting

$$x = az \implies dx = adz$$

we have,

$$\Gamma(n) = \int_0^\infty e^{-az}(az)^{n-1} adz = a^n \int_0^\infty e^{-az} z^{n-1} dz$$

Writing x for z and z for a, we get

$$\Gamma(n) = z^n \int_0^\infty e^{-zx} x^{n-1}\, dx$$

$$= \int_0^\infty e^{-zx} x^{n-1} z^n\, dx$$

Therefore,

$$\Gamma(n) e^{-z} z^{m-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1}\, dx$$

Integrating both sides w.r.t z between the limits 0 to ∞, we have

$$\Gamma(n) \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz \right] dx$$

Putting $z(1+x) = y \Rightarrow (1+x)dz = dy$

$$= \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-y} \frac{y^{m+n-1}}{(1+x)^{m+n}}\, dy \right] dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \left[\int_0^\infty e^{-y} y^{m+n-1}\, dy \right] dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx \int_0^\infty e^{-y} y^{m+n-1}\, dy$$

$$\text{or,} \qquad \Gamma(n)\Gamma(m) = \Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\text{or,} \qquad \Gamma(n)\Gamma(m) = \Gamma(m+n)B(m,n)\text{ ; since } B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\text{or,} \qquad B(m,n) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$$

9.8 SOME STANDARD RESULTS USING INTER-RELATION OF BETA AND GAMMA FUNCTIONS

Result 1

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

Proof We have,

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}; m,n > 0$$

Putting $m = n = \frac{1}{2}$,

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2;\ \text{since } \Gamma(1) = 1$$

Now,

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$$

Therefore,

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \pi$$

$$\text{or,} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Result 2

$$\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$$

Proof Here,

$$\int_0^\infty e^{-x^2}\,dx$$

Taking $x^2 = z \Rightarrow 2x\,dx = dz \Rightarrow\ dx = \dfrac{dz}{2x} = \dfrac{dz}{2\sqrt{z}}$

$$= \int_0^\infty e^{-z}\frac{dz}{2\sqrt{z}}$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{-1}{2}}\,dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi};\ \text{since}\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Result 3 **Duplication Formula**

$$\boxed{2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2m); m > 0}$$

Proof We have,

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}; m,n > 0$$

Again,

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta; m,n > 0$$

Therefore,

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

Putting $m = n$, we have

$$B(m,m) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2m-1}\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1} \varphi d\phi$$

taking $2\theta = \varphi$

$$or, \quad B(m,m) = \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1}\varphi d\phi = \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta d\theta \tag{1}$$

Putting $n = \frac{1}{2}$

$$B\left(m, \frac{1}{2}\right) = \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta d\theta \tag{2}$$

Therefore from (1) and (2), we have

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta d\theta$$

$$= \frac{1}{2^{2m-1}}\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{2^{2m-1}}\frac{\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)} \text{ since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$or, \quad 2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2m); m > 0$$

Result 4

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}; p > -1, q > -1$$

Proof Here,

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta d\theta = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{p}{2}}(\cos^2\theta)^{\frac{q}{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{p}{2}}(1-\sin^2\theta)^{\frac{q}{2}} d\theta$$

Taking $\sin^2\theta = x \Rightarrow 2\sin\theta\cos\theta d\theta = dx \Rightarrow d\theta$

$$= \frac{dx}{2\sin\theta\cos\theta} = \frac{dx}{2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$$

When $\theta = 0, x = 0$ and $\theta = \frac{\pi}{2}, x = 1$

$$= \int_0^1 x^{\frac{p}{2}}(1-x)^{\frac{q}{2}} \frac{dx}{2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$$

$$= \frac{1}{2}\int_0^1 x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}} dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{p+1}{2}-1}(1-x)^{\frac{q+1}{2}-1} dx$$

$$= \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)};$$

since $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}; m, n > 0$

Result 5

$$\boxed{\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}; 0 < m < 1}$$

Proof We have,

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}; m,n > 0$$

and

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Therefore,

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx; m,n > 0$$

Putting

$$n - 1 - m; 0 < m < 1$$

we have,

$$B(m, 1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(1)} = \Gamma(m)\Gamma(1-m)$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)}\,dx \text{ since } \Gamma(1) = 1$$

or, $$B(m,1-m) = \Gamma(m)\Gamma(1-m) = \int_0^1 \frac{x^{m-1}}{(1+x)}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)}\,dx \qquad (1)$$

Putting

$$x = \frac{1}{y}$$

we have,

$$\int_1^\infty \frac{x^{m-1}}{(1+x)}\,dx = \lim_{B\to\infty} \int_1^B \frac{x^{m-1}}{(1+x)}\,dx$$

$$= \lim_{\frac{1}{B}\to 0} \int_{\frac{1}{B}}^1 \frac{y^{-m}}{1+y}\,dy = \int_0^1 \frac{y^{-m}}{1+y}\,dy = \int_0^1 \frac{x^{-m}}{1+x}\,dx \qquad (2)$$

Therefore from 1 and 2, for $0 < m < 1$

$$B(m, 1-m) = \Gamma(m)\Gamma(1-m) = \int_0^1 \frac{x^{m-1}}{(1+x)}\,dx + \int_0^1 \frac{x^{-m}}{1+x}\,dx$$

$$or, \qquad \Gamma(m)\Gamma(1-m) = \int_0^1 (x^{m-1} + x^{-m})\left(1 - \frac{x}{1+x}\right)dx$$

$$or, \qquad \Gamma(m)\Gamma(1-m) = \int_0^1 (x^{m-1} + x^{-m})\,dx - \int_0^1 (x^{m-1} + x^{-m})\frac{x}{1+x}\,dx \qquad (3)$$

Now,

$$\int_0^1 (x^{m-1} + x^{-m})\,dx = \lim_{\epsilon \to 0+} \int_\epsilon^1 (x^{m-1} + x^{-m})\,dx = \lim_{\epsilon \to 0+} \left[\frac{x^m}{m} + \frac{x^{1-m}}{1-m}\right]_\epsilon^1$$

$$or, \quad \int_0^1 (x^{m-1} + x^{-m})\,dx = \frac{1}{m} + \frac{1}{1-m} \qquad (4)$$

Again,

$$\int_0^1 (x^{m-1} + x^{-m})\frac{x}{1+x}\,dx = \int_0^1 \frac{(x^m + x^{1-m})}{1+x}\,dx$$

$$= \int_0^1 (x^m + x^{1-m})(1 - x + x^2 - x^3 + ...\infty)\,dx$$

$$or, \quad \int_0^1 (x^{m-1} + x^{-m})\frac{x}{1+x}\,dx = \frac{1}{m+1} + \frac{1}{2-m} - \frac{1}{m+2} - \frac{1}{3-m} + ...\infty \quad (5)$$

Therefore from 3, 4 and 5, we have

$$\Gamma(m)\Gamma(1-m) = \frac{1}{m} + \frac{1}{1-m} + \frac{1}{m+1} + \frac{1}{2-m} - \frac{1}{m+2} - \frac{1}{3-m} + \cdots\infty$$

$$= \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{k+m} + \frac{1}{k+1-m}\right)$$

$$or, \quad \Gamma(m)\Gamma(1-m) = \pi \operatorname{cosec} m\pi = \frac{\pi}{\sin m\pi}$$

WORKED OUT EXAMPLES

Example 9.1 **Evaluate**

$$\int_0^{\frac{\pi}{2}} \sin^6 x \cos^5 x \, dx$$

using beta and gamma functions.

Sol. We have,

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}; p > -1, q > -1$$

Here $p = 6, q = 5$
Therefore,

$$\int_0^{\frac{\pi}{2}} \sin^6 x \cos^5 x \, dx = \frac{1}{2} \frac{\Gamma\left(\frac{7}{2}\right)\Gamma(3)}{\Gamma\left(\frac{13}{2}\right)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{7}{2}\right)\Gamma(3)}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\Gamma\left(\frac{7}{2}\right)}, \text{ since } \Gamma(n+1) = n\Gamma(n)$$

$$= \frac{1}{2} \frac{\Gamma(3)}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}}$$

$$= \frac{1}{2} \frac{2!}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}}$$

$$= \frac{8}{11\cdot 9\cdot 7} = \frac{8}{693}$$

Example 9.2 **Prove that**

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\Gamma(n+1)}$$

Sol.

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\frac{2n+1}{2}\right)$$

$$= \Gamma\left(\frac{2n-1}{2}+1\right)$$

$$= \frac{2n-1}{2}\Gamma\left(\frac{2n-1}{2}\right)$$

$$= \frac{2n-1}{2}\Gamma\left(\frac{2n-3}{2}+1\right)$$

$$= \frac{2n-1}{2}\frac{2n-3}{2}\Gamma\left(\frac{2n-3}{2}\right)$$

$$= \frac{2n-1}{2}\frac{2n-3}{2}\frac{2n-5}{2}\cdots\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n-1)(2n-3)(2n-5)\cdots5\cdot3\cdot1}{2^n}\sqrt{\pi}$$

Multiplying the numerator and denominator by $2n(2n-2)(2n-4)\cdots4\cdot2$, we have

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{2n(2n-1)(2n-2)(2n-3)\cdots5\cdot4\cdot3\cdot2\cdot1}{2^n 2n(2n-2)(2n-4)\cdots4\cdot2}\sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)}{2^n 2^n n(n-1)(n-2)\cdots3\cdot2\cdot1}\sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}\sqrt{\pi}$$

Example 9.3 **Evaluate**

$$\int_0^t x^{\alpha+k-1}(t-x)^{\beta+k-1}\,dx$$

and find its value when $\alpha=\beta=\dfrac{1}{2}$

Sol. Putting

$$x = ty \Rightarrow dx = t\,dy$$

When $x=0, y=0$ and $x=t, y=1$

Therefore,

$$I = \int_0^t x^{\alpha+k-1}(t-x)^{\beta+k-1}\,dx$$
$$= \int_0^1 t^{\alpha+\beta+2k-1} y^{\alpha+k-1}(1-y)^{\beta+k-1}\,dy$$
$$= t^{\alpha+\beta+2k-1}\int_0^1 y^{\alpha+k-1}(1-y)^{\beta+k-1}\,dy$$
$$= t^{\alpha+\beta+2k-1}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha+\beta+2k)}$$

When $\alpha = \beta = \dfrac{1}{2}$

$$I = t^{2k}\frac{\Gamma\left(\frac{1}{2}+k\right)\Gamma\left(\frac{1}{2}+k\right)}{\Gamma(1+2k)}$$
$$= t^{2k}\frac{\dfrac{\Gamma(2k+1)}{2^{2k}\Gamma(k+1)}\sqrt{\pi}\dfrac{\Gamma(2k+1)}{2^{2k}\Gamma(k+1)}\sqrt{\pi}}{\Gamma(1+2k)}$$
$$= t^{2k}\frac{\Gamma(2k+1)}{2^{4k}\{\Gamma(k+1)\}^2}\pi$$
$$= t^{2k}\frac{\Gamma(2k+1)}{2^{4k}\{k!\}^2}\pi$$

Example 9.4 **Show that for** $l > 0, m > 0$

$$\int_a^b (x-a)^{l-1}(b-x)^{m-1}\,dx = (b-a)^{l+m-1}B(l,m)$$

Sol. Putting

$$x = (b-a)y + a$$

such that when $x = a, y = 0$ and when $x = b, y = 1$.

Therefore,

$$\begin{aligned}\int_a^b (x-a)^{l-1}(b-x)^{m-1}\,dx &= \int_0^1 [(b-a)y+a-a]^{l-1}[b-(b-a)y-a]^{m-1}(b-a)\,dy \\ &= \int_0^1 (b-a)^{l-1+1+m-1}y^{l-1}(1-y)^{m-1}\,dy \\ &= \int_0^1 (b-a)^{l+m-1}y^{l-1}(1-y)^{m-1}\,dy \\ &= (b-a)^{l+m-1}\int_0^1 y^{l-1}(1-y)^{m-1}\,dy \\ &= (b-a)^{l+m-1}B(l,m)\end{aligned}$$

Example 9.5 **Show that**

$$\int_0^1 x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}(1+2x)^{-1}\,dx = \frac{1}{9^{\frac{1}{3}}}B\left(\frac{2}{3},\frac{1}{3}\right)$$

Sol. Putting

$$\frac{x}{1-x} = \frac{at}{1-t}$$

where a is a constant.
Therefore,

$$x = \frac{at}{1-(1-a)t}$$

or,

$$dx = \frac{adt}{[1-(1-a)t]^2}$$

when $x=0, t=0$

Therefore,

$$\int_0^1 x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}(1+2x)^{-1}\,dx$$

$$= \int_0^1 \left[\frac{at}{1-(1-a)t}\right]^{-\frac{1}{3}} \left[\frac{1-t}{1-(1-a)t}\right]^{-\frac{2}{3}} \left[\frac{1-t+3at}{1-(1-a)t}\right]^{-1} \frac{adt}{[1-(1-a)t]^2}$$

$$= \int_0^1 \frac{a^{\frac{2}{3}}t^{-\frac{1}{3}}[1-t(1-3a)]^{-1}}{(1-t)^{\frac{2}{3}}}dt$$

Choosing, $a = \dfrac{1}{3}$, the integral becomes

$$-\int_0^1 \left(\frac{1}{3}\right)^{\frac{2}{3}} t^{\frac{2}{3}-1}(1-t)^{\frac{1}{3}-1}dt = \frac{1}{9^{\frac{1}{3}}}B\left(\frac{2}{3},\frac{1}{3}\right)$$

Example 9.6 **Evaluate the integrals**

$$\int_0^\infty e^{-ax}x^{m-1}\cos bx\,dx$$

$$\int_0^\infty e^{-ax}x^{m-1}\sin bx\,dx$$

Hence, or otherwise, show that

$$\int_0^\infty x^{m-1}\cos bx\,dx = \frac{\Gamma(m)}{b^m}\cos\left(\frac{m\pi}{2}\right)$$

and

$$\int_0^\infty x^{m-1}\sin bx\,dx = \frac{\Gamma(m)}{b^m}\sin\left(\frac{m\pi}{2}\right)$$

Sol. We know,

$$\int_0^\infty e^{-kx}x^{m-1}\,dx = \frac{\Gamma(m)}{k^m}$$

Taking,

$$k = a - ib; |k| > 0$$

we have,

$$\int_0^\infty e^{-(a-ib)x} x^{m-1}\, dx = \frac{\Gamma(m)}{(a-ib)^m}$$

or,
$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1}\, dx = \frac{\Gamma(m)(a+ib)^m}{(a-ib)^m (a+ib)^m}$$

or,
$$\int_0^\infty e^{-ax}(\cos bx + i\sin bx) x^{m-1}\, dx = \frac{\Gamma(m)(a+ib)^m}{(a^2+b^2)^m}$$

Writing

$$a + ib = r(\cos\theta + i\sin\theta)$$

and separating the real and imaginary parts, we have

$$\int_0^\infty e^{-ax} x^{m-1} \cos bx\, dx = \frac{\Gamma(m)\cos m\theta}{(a^2+b^2)^{\frac{m}{2}}}$$

$$\int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx = \frac{\Gamma(m)\sin m\theta}{(a^2+b^2)^{\frac{m}{2}}}$$

where

$$\theta = \tan^{-1} \frac{b}{a}$$

Taking $a = 0$ and $\theta = \dfrac{\pi}{2}$

$$\int_0^\infty x^{m-1} \cos bx\, dx = \frac{\Gamma(m)}{b^m} \cos\left(\frac{m\pi}{2}\right)$$

and

$$\int_0^\infty x^{m-1} \sin bx\, dx = \frac{\Gamma(m)}{b^m} \sin\left(\frac{m\pi}{2}\right)$$

Example 9.7 **Evaluate**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$$

Sol.

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}} x \cos^0 x\, dx \times \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x \cos^0 x\, dx$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \times \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{1}{4}\pi\frac{\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$$

$$= \pi$$

Example 9.8 **Evaluate**

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx$$

Sol.

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x \cos^{\frac{-1}{2}} x\, dx$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{2}\frac{\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{1}$$

$$= \frac{1}{2}\pi \text{ cosec } \frac{\pi}{4}$$

$$= \frac{\pi}{\sqrt{2}}$$

Example 9.9 **Prove that**

$$\int_0^1 \sqrt{1 - x^4\, dx} = \frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{6\sqrt{2\pi}}$$

Sol. Putting

$$x^4 = u$$

we have,

$$\int_0^1 \sqrt{1 - x^4\, dx} = \frac{1}{4}\int_0^1 u^{-\frac{3}{4}}(1-u)^{\frac{1}{2}}\, du$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{2}\right)$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{6}\frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{6}\frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{\pi \operatorname{cosec}\frac{\pi}{4}}$$

$$= \frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{6\sqrt{2\pi}}$$

Example 9.10 **Using**

$$\sin\frac{\pi}{a}\sin\frac{2\pi}{a}\sin\frac{3\pi}{a}\cdots\sin\frac{(a-1)\pi}{a}=\frac{a}{2^{n-1}}$$

where a is a positive integer >1, prove that

i) $\Gamma\left(\frac{1}{a}\right)\Gamma\left(\frac{2}{a}\right)\Gamma\left(\frac{3}{a}\right)\cdots\Gamma\left(\frac{a-1}{a}\right)=\left\{\frac{(2\pi)^{a-1}}{a}\right\}^{\frac{1}{2}}$

ii) $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\cdots\Gamma\left(\frac{8}{9}\right)=\frac{16}{3}\pi^4$

Sol.

i) We have,

$$\sin\frac{\pi}{a}\sin\frac{2\pi}{a}\sin\frac{3\pi}{a}\cdots\sin\frac{(a-1)\pi}{a}=\frac{a}{2^{n-1}}$$

Now,

$$\left\{\Gamma\left(\frac{1}{a}\right)\Gamma\left(\frac{2}{a}\right)\Gamma\left(\frac{3}{a}\right)\cdots\Gamma\left(\frac{a-1}{a}\right)\right\}^2$$

$$=\left\{\Gamma\left(\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)\right\}\left\{\Gamma\left(\frac{2}{a}\right)\right\}\Gamma\left(1-\frac{2}{a}\right)\cdots\left\{\Gamma\left(1-\frac{1}{a}\right)\Gamma\left(\frac{1}{a}\right)\right\}$$

$$=\pi\text{ cosec }\frac{\pi}{a}\times\pi\text{ cosec }\frac{2\pi}{a}\times\cdots\times\pi\text{ cosec }\frac{(a-1)\pi}{a}$$

$$=\pi^{a-1}\frac{2^{a-1}}{a}$$

Therefore,

$$\Gamma\left(\frac{1}{a}\right)\Gamma\left(\frac{2}{a}\right)\Gamma\left(\frac{3}{a}\right)\cdots\Gamma\left(\frac{a-1}{a}\right)=\left\{\frac{(2\pi)^{a-1}}{a}\right\}^{\frac{1}{2}}$$

ii) Putting $a=9$ in the expression

$$\Gamma\left(\frac{1}{a}\right)\Gamma\left(\frac{2}{a}\right)\Gamma\left(\frac{3}{a}\right)\cdots\Gamma\left(\frac{a-1}{a}\right)=\left\{\frac{(2\pi)^{a-1}}{a}\right\}^{\frac{1}{2}}$$

we have,

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\cdots\Gamma\left(\frac{8}{9}\right)=\left\{\frac{(2\pi)^{9-1}}{9}\right\}^{\frac{1}{2}}$$

$$=\frac{16}{3}\pi^4$$

EXERCISES

Short and Long Answer Type Questions

1) Prove that $B\left(\frac{1}{2},\frac{1}{2}\right)=\pi=\int\limits_0^\infty \frac{dt}{\sqrt{t}(1+t)}$

2) Prove that $\int\limits_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt = n^x B(x,n+1); n=1,2,3,\ldots$

3) Prove that $B(x+1,y)=\frac{x}{x+y}B(x,y)$

4) Prove that $\Gamma\left(\frac{1}{2}\right)\Gamma(2x)=2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$

5) Show that

a) $\int\limits_0^\infty e^{-x}x^{\frac{3}{2}}\,dx=\frac{3}{4}\sqrt{\pi}$

b) $\int\limits_0^\infty \sqrt{x}e^{-x^3}\,dx=\frac{\sqrt{\pi}}{3}$

c) $\int\limits_0^\infty 5^{-x^2}\,dx=\frac{1}{2\sqrt{\log 5}}\sqrt{\pi}$

6) Find the value of the following improper integrals:

a) $\int\limits_0^{\frac{\pi}{2}} \sin^9 x$

$\left[\textbf{Ans: } \frac{128}{315}\right]$

b) $\int\limits_0^{\frac{\pi}{2}} \cos^4 x\,dx$

$\left[\textbf{Ans: } \frac{3\pi}{16}\right]$

c) $\int\limits_0^{\frac{\pi}{2}} \sin^4 x\cos^5 x\,dx$

Ans: $\dfrac{8}{315}$

d) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x \cos^4 x \, dx$

Ans: $\dfrac{3\pi}{256}$

7) Prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$

8) Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^p x \, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x \, dx = \frac{\pi}{2(p+1)}$

9) Prove that $\displaystyle\int_0^\infty e^{-x^4} \, dx \times \int_0^\infty e^{-x^4} x^2 \, dx = \frac{\pi}{8\sqrt{2}}$

10) $\displaystyle\int_0^1 \frac{x^2 \, dx}{(1-x^4)^{\frac{1}{2}}} \times \int_0^\infty \frac{dx}{(1+x^4)^{\frac{1}{2}}} = \frac{\pi}{4\sqrt{2}}$

Multiple Choice Questions

1) The value of $\displaystyle\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$ is

a) $\dfrac{\sqrt{\pi}}{2}$ b) $\dfrac{\sqrt{\pi}}{4}$ c) $\dfrac{\pi}{2}$ d) 1

2) The value of $\Gamma(m)\Gamma(1-m)$ is

a) $\dfrac{2\pi}{\sin m\pi}$ b) $\dfrac{\pi}{\sin m\pi}$ c) $\dfrac{3\pi}{\sin m\pi}$ d) $\dfrac{m\pi}{\sin m\pi}$

3) The value of $\Gamma\left(\dfrac{1}{2}\right) =$ is

a) $\sqrt{\pi}$ b) $\dfrac{\sqrt{\pi}}{2}$ c) $\dfrac{\sqrt{\pi}}{4}$ d) π

4) $B\left(\dfrac{1}{2}, \dfrac{1}{2}\right) =$

a) $\frac{\pi}{2}$ b) $\frac{\pi}{4}$ c) π d) $\frac{\pi}{8}$

5) The value of $2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta; m, n > 0$ is

a) $B(m, m)$ b) $B(n\ , n)$ c) $B(m, n)$ d) none of these

6) The value of $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^4 x\, dx$ is

a) $\frac{3\pi}{256}$ b) $\frac{5\pi}{256}$ c) $\frac{7\pi}{256}$ d) $\frac{11\pi}{256}$

7) $\Gamma(1) =$

a) 0 b) 2 c) -1 d) 1

8) $\Gamma(n+1) =$ where n is a positive integer

a) $n!$ b) n c) $(n+1)!$ d) $n+1$

9) $\int_0^{\frac{\pi}{2}} \cos^4 x\, dx$

a) $\frac{5\pi}{16}$ b) $\frac{\sqrt{\pi}}{16}$ c) $\frac{\pi}{16}$ d) $\frac{3\pi}{16}$

10) $\int_0^{\infty} e^{-x} x^{\frac{3}{2}}\, dx =$

a) $\frac{3}{4}\sqrt{\pi}$ b) $\frac{3}{5}\sqrt{\pi}$ c) $\frac{1}{4}\sqrt{\pi}$ d) $\frac{5}{4}\sqrt{\pi}$

Answers:

1 (a) 2 (b) 3 (a) 4 (c) 5 (c) 6 (a) 7 (d) 8(a)
9 (d) 10 (a)

CHAPTER

10

Laplace Transform (LT)

10.1 INTRODUCTION

Pierre Simon Marquis de Laplace (1749–1827), great French mathematician, was a professor in Paris. He developed the foundation of potential theory and was also one of the important contributors to special functions, probability theory and astronomy. But his powerful techniques of practical Laplace transform were developed over a century later by the English electrical engineer **Oliver Heaviside** (1850–1925) and for that reason it is also often called *Heaviside calculus*.

In this chapter, first we discuss the Laplace transform of some standard functions. Next we give details of the various properties of Laplace transform illustrated with different suitable examples. Here we also represent the Laplace transform of the unit step function (Heaviside's function) and Dirac delta function because the use of these functions make the method particularly powerful for problems in engineering with inputs that have discontinuities or complicated periodic functions.

10.2 DEFINITION AND EXISTANCE OF LAPLACE TRANSFORM (LT)

10.2.1 Definition

Let $f(t)$ be a function defined for $t \geq 0$. Then Laplace transform of $f(t)$ is denoted by $L\{f(t)\}$ or $F(p)$ and is defined as

$$\boxed{L\{f(t)\} = F(p) = \int_0^\infty e^{-pt} f(t)dt}$$

where $p\ (> 0)$ is the transform parameter and e^{-pt} is called the kernel of the transform.

[WBUT 2003]

Note:

(1) The Laplace transform exists if the above improper integral exists, otherwise we say Laplace transform does not exist.

(2) In some of the books one may get the Laplace transform with the parameter s and also the transformed function may be denoted by $\overline{f}(p)$ or $\overline{f}(s)$ or by any other function. So there should not be any confusion in the readers mind. Basically we take two different functions to distinguish between the function under transformation and the function that we get after transform.

10.2.2 Piecewise Continuous Functions

A function $f(t)$ is called piecewise continuous in a closed interval if the interval can be divided into a finite number of sub-intervals such that in each of which the function is continuous and it has finite left and right-hand limits.

10.2.3 Functions of Exponential Order

A function $f(t)$ is of exponential order σ as $t \to \infty$ if σ, $k(> 0)$ and $t_0(\geq 0)$ can be found such that

$$\boxed{\left|e^{-\sigma t} f(t)\right| < k,\ i.e., |f(t)| < ke^{\sigma t} \text{ for } t \geq t_0}$$

For example, Let $f(t) = t^2$. Then,

$$|f(t)| = \left|t^2\right| < 1.e^{3t} \quad \text{for } t \geq 0$$

So, $f(t) = t^2$ is of exponential order 3.

Note: Bounded functions like $\sin t, \cos t$**, etc., are always of some exponential order.**

10.2.4 Sufficient Condition for Existance of Laplace transform

Theorem 10.1 **If a function $f(t)$ defined for $t \geq 0$ satisfies the following conditions**

(i) $f(t)$ is of exponential order σ as $t \to \infty$ and

(ii) $f(t)$ is piecewise continuous over every finite interval $t \geq 0$, then its Laplace transform $L\{f(t)\}$ exists for $p > \sigma$.

Proof It is given that $f(t)$ is of some exponential order σ as $t \to \infty$, then we have constants $k(> 0)$ and $t_0(\geq 0)$ such that

$$|f(t)| < ke^{\sigma t} \quad \text{for } t \geq t_0 \tag{1}$$

By definition, we have

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt = \int_0^{t_0} e^{-pt} f(t)\, dt + \int_{t_0}^\infty e^{-pt} f(t)\, dt$$

$$= I_1 + I_2 \ \text{(Say)} \tag{2}$$

Since $f(t)$ is piecewise continuous on every finite interval $0 \le t \le t_0$, I_1 exists. Again,

$$|I_2| = \left| \int_{t_0}^\infty e^{-pt} f(t)\, dt \right| \le \int_{t_0}^\infty \left| e^{-pt} f(t) \right| dt$$

$$= \int_{t_0}^\infty e^{-pt} |f(t)|\ dt \le \int_{t_0}^\infty e^{-pt} .k e^{\sigma t}\ dt \quad (\text{ by } (1)\ )$$

$$= k \int_{t_0}^\infty e^{-(p-\sigma)t}\ dt = k \left[-\frac{e^{-(p-\sigma)t}}{(p-\sigma)} \right]_{t_0}^\infty$$

$$= -\lim_{t\to\infty} \frac{e^{-(p-\sigma)t}}{(p-\sigma)} + k \frac{e^{-(p-\sigma)t_0}}{(p-\sigma)} = k \frac{e^{-(p-\sigma)t_0}}{(p-\sigma)}$$

since $e^{-(p-\sigma)t} \to 0$ as $t \to \infty$ for $p > \sigma$.

So, $|I_2|$ exists for $p > \sigma$.

Hence from (2) we conclude that $L\{f(t)\}$ exists for $p > \sigma$.

This proves the theorem.

Note:

(1) The conditions of the above theorem are only sufficient for the existance of Laplace transform, by no means they are necessary. In other words if the conditions are not satisfied by the function then Laplace transform may or may not exist.

(2) Uniqueness: If the Laplace transform of a given function exists, it is always unique.

10.3 LINEARITY PROPERTY OF LAPLACE TRANSFORM

Let $f_1(t)$ and $f_2(t)$ be functions of t, whose Laplace transform exists, then

$$\boxed{L\{c_1 f_1(t) \pm c_2 f_2(t)\} = c_1 L\{f_1(t)\} \pm c_2 L\{f_2(t)\}}$$

where c_1, c_2 are arbitrary constants.

10.4.1 $L\{f(t)\}$ or $F(p)$ for the function $f(t) = 1$

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)dt$$

i.e.,
$$L\{1\} = \int_0^\infty e^{-pt}.1dt = \int_0^\infty e^{-pt}dt$$
$$= -\frac{1}{p}\left[e^{-pt}\right]_0^\infty = \frac{1}{p}, \quad p > 0.$$

10.4.2 $L\{f(t)\}$ or $F(p)$ for the function $f(t) = t$

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

i.e.,
$$L\{t\} = \int_0^\infty e^{-pt} \cdot t\, dt = \left[-\frac{1}{p}e^{-pt} \cdot t\right]_0^\infty - \int_0^\infty \left(-\frac{1}{p}e^{-pt}\right) dt$$
$$= \left[\left(-\frac{1}{p}\right)\lim_{t\to\infty} e^{-pt} \cdot t + 0\right] - \frac{1}{p^2}\left[e^{-pt}\right]_0^\infty$$
$$= \left(-\frac{1}{p}\right)\lim_{t\to\infty}\frac{t}{e^{pt}} + \frac{1}{p^2}, \quad p > 0$$

Again $\lim_{t\to\infty}\frac{t}{e^{pt}}$ is of the form $\left[\frac{\infty}{\infty}\right]$. So using L'Hospital's rule we have

$$\lim_{t\to\infty}\frac{t}{e^{pt}} = \lim_{t\to\infty}\frac{\frac{d}{dt}(t)}{\frac{d}{dt}(e^{pt})} = \lim_{t\to\infty}\frac{1}{p.e^{pt}} = 0$$

Hence,

$$L\{f(t)\} = \frac{1}{p^2}, \quad p > 0$$

10.4.3 $L\{f(t)\}$ or $F(p)$ for the function $f(t) = t^n$ (n is a positive integer)

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt$$

i.e.,
$$L\{t^n\} = \int_0^\infty e^{-pt}.t^n\,dt = I_n \text{ (say)} \tag{1}$$

Therefore,

$$I_n = \left[-\frac{1}{p}e^{-pt}\cdot t^n\right]_0^\infty - \int_0^\infty n\cdot t^{n-1}\left(-\frac{1}{p}e^{-pt}\right)dt$$

$$= \left[\left(-\frac{1}{p}\right)\lim_{t\to\infty} e^{-pt}\cdot t^n + 0\right] + \frac{n}{p}\int_0^\infty e^{-pt}\cdot t^{n-1}\,dt$$

$$= \left(-\frac{1}{p}\right)\lim_{t\to\infty}\frac{t^n}{e^{pt}} + \frac{n}{p}\int_0^\infty e^{-pt}\cdot t^{n-1}\,dt,\quad p > 0 \tag{2}$$

Again $\lim_{t\to\infty}\frac{t^n}{e^{pt}}$ is of the form $\left[\frac{\infty}{\infty}\right]$. So using L'Hospital's rule we have

$$\lim_{t\to\infty}\frac{t^n}{e^{pt}} = \lim_{t\to\infty}\frac{\frac{d}{dt}(t^n)}{\frac{d}{dt}(e^{pt})} = \lim_{t\to\infty}\frac{n\cdot t^{n-1}}{p\cdot e^{pt}}\quad\left[\text{Form }\frac{\infty}{\infty}\right]$$

Proceeding similarly $(n-1)$ times more we get,

$$\lim_{t\to\infty}\frac{t^n}{e^{pt}} = \lim_{t\to\infty}\frac{n!}{p^n\cdot e^{pt}} = 0$$

So from **(2)**, we have

$$I_n = \frac{n}{p}\int_0^\infty e^{-pt}\cdot t^{n-1}\,dt \tag{3}$$

So from **(1)** and **(3)**

$$I_n = \frac{n}{p}\cdot I_{n-1}$$

Replacing n by $(n-1)$, we have

$$I_{n-1} = \frac{(n-1)}{p}\cdot I_{n-2} \tag{4}$$

Therefore,

$$I_n = \frac{n}{p} \cdot \frac{(n-1)}{p} \cdot I_{n-2} \tag{5}$$

Again, replacing n by $(n-1)$ in (4) and putting in (5) we have

$$I_n = \frac{n}{p} \cdot \frac{(n-1)}{p} \cdot \frac{(n-2)}{p} \cdot I_{n-3}$$

Proceeding in a similar manner, we get

$$I_n = \left[\left(\frac{n}{p}\right)\left(\frac{n-1}{p}\right)\left(\frac{n-2}{p}\right)\cdots\left(\frac{2}{p}\right)\left(\frac{1}{p}\right)\right] \cdot I_0 \tag{6}$$

By (1), we have

$$I_0 = \int_0^\infty e^{-pt}\, dt = \frac{1}{p},\ p > 0$$

So, putting the value of I_0 in (6), we get

$$I_n = \left[\left(\frac{n}{p}\right)\left(\frac{n-1}{p}\right)\left(\frac{n-2}{p}\right)\cdots\left(\frac{2}{p}\right)\left(\frac{1}{p}\right)\right] \cdot \frac{1}{p}$$

i.e., $$L\{t^n\} = \frac{n!}{p^n} \cdot \frac{1}{p} = \frac{n!}{p^{n+1}},\ p > 0.$$

10.4.4 *L*{*f*(*t*)} or *F*(*p*) for the function *f*(*t*) = t^n (*n* > -1)

[WBUT 2002]

First we recall the following terms from the last chapter:

$$\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx,\ \ n > 0 \ \text{ and } \ \Gamma(n+1) = n\Gamma(n) = n!$$

Now,

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

i.e., $$L\{t^n\} = \int_0^\infty e^{-pt} \cdot t^n\, dt \ \ (\text{consider } pt = x,\ \text{then } p \cdot dt = dx)$$

$$= \int_0^\infty e^{-x} \cdot \left(\frac{x}{p}\right)^n \frac{dx}{p} = \frac{1}{p^{n+1}} \int_0^\infty e^{-x} \cdot x^n \, dx$$

$$= \frac{1}{p^{n+1}} \int_0^\infty e^{-x} \cdot x^{(n+1)-1} \, dx = \frac{\Gamma(n+1)}{p^{n+1}} \text{ for } n+1 > 0 \text{ i.e., } n > -1$$

10.4.5 $L\{f(t)\}$ or $F(p)$ for the function $f(t)=e^{at}$

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

i.e.,

$$L\{e^{at}\} = \int_0^\infty e^{-pt} \cdot e^{at} \, dt$$

$$= \int_0^\infty e^{-(p-a)t} \, dt = \left[\frac{e^{-t(p-a)t}}{p-a}\right]_0^\infty, \quad p > a$$

$$= \frac{1}{p-a}, \quad p > a$$

10.4.6 $L\{f(t)\}$ or $F(p)$ for the function $f(t) = \sin(at)$

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

i.e.,

$$L\{\sin(at)\} = \int_0^\infty e^{-pt} \cdot \sin(at) \, dt$$

$$= \left[\frac{e^{-pt}(-p\sin(at) - a\cos(at))}{p^2+a^2}\right]_0^\infty, \quad p > 0$$

$$= \frac{a}{p^2+a^2}, \quad p > 0$$

Alternative Method We have from **section 10.4.5**

$$L\{e^{at}\} = \frac{1}{p-a} \quad p > a$$

So,

$$L\{e^{iat}\} = \frac{1}{p - ia} \text{ and } L\{e^{-iat}\} = \frac{1}{p + ia}$$

Again,

$$f(t) = \sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

So,

$$\begin{aligned} L\{f(t)\} &= L\{\sin at\} = L\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\} \\ &= \frac{1}{2i}\left[L\left\{e^{iat}\right\} - L\left\{e^{-iat}\right\}\right] \text{ (By the linearity property)} \\ &= \frac{1}{2i}\left[\frac{1}{p - ia} - \frac{1}{p + ia}\right] = \frac{1}{2i}\left[\frac{2ia}{(p - ia)(p + ia)}\right] \\ &= \frac{a}{p^2 + a^2}, \quad p > 0 \end{aligned}$$

10.4.7 *L*{*f*(*t*)} or *F*(*p*) for the function *f*(*t*) = cos(*at*)

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

$$\begin{aligned} L\{\cos(at)\} &= \int_0^\infty e^{-pt} \cdot \cos(at)\; dt \\ &= \left[\frac{e^{-pt}(-p\cos(at) + a\sin(at))}{p^2 + a^2}\right]_0^\infty, \quad p > 0 \\ &= \frac{p}{p^2 + a^2}, \quad p > 0 \end{aligned}$$

Alternative Method We have from **section 10.4.5**

$$L\{e^{at}\} = \frac{1}{p - a} \quad p > a$$

So,

$$L\{e^{iat}\} = \frac{1}{p - ia} \text{ and } L\{e^{-iat}\} = \frac{1}{p + ia}$$

Again,

$$f(t) = \cos at = \frac{e^{iat} + e^{-iat}}{2}$$

So,

$$\begin{aligned}
L\{f(t)\} &= L\{\cos at\} = L\left\{\frac{e^{iat} + e^{-iat}}{2}\right\} \\
&= \frac{1}{2}\left[L\left\{e^{iat}\right\} + L\left\{e^{-iat}\right\}\right] \quad \text{(By the linearity property)} \\
&= \frac{1}{2}\left[\frac{1}{p - ia} + \frac{1}{p + ia}\right] = \frac{1}{2}\left[\frac{2p}{(p - ia)(p + ia)}\right] \\
&= \frac{p}{p^2 + a^2}, \quad p > 0
\end{aligned}$$

10.4.8 *L*{*f*(*t*)} or *F*(*p*) for the function *f*(*t*) = sinh (*at*)

We have from **section 10.4.5**

$$L\{e^{at}\} = \frac{1}{p - a} \quad p > a$$

So,

$$L\{e^{-at}\} = \frac{1}{p + a} \quad p > a$$

Again,

$$f(t) = \sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

So,

$$\begin{aligned}
L\{f(t)\} &= L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\} \\
&= \frac{1}{2}\left[L\left\{e^{at}\right\} - L\left\{e^{-at}\right\}\right] \quad \text{(By the linearity property)} \\
&= \frac{1}{2}\left[\frac{1}{p - a} - \frac{1}{p + a}\right] = \frac{1}{2}\left[\frac{2a}{(p - a)(p + a)}\right] \\
&= \frac{a}{p^2 + a^2}, \quad p > |a|
\end{aligned}$$

10.4.9 $L\{f(t)\}$ or $F(p)$ for the function $f(t) = \cosh(at)$

We have from **section 10.4.5**

$$L\{e^{at}\} = \frac{1}{p-a} \quad p > a$$

So,

$$L\{e^{-at}\} = \frac{1}{p+a} \quad p > a$$

Again,

$$f(t) = \cosh(at) = \frac{e^{at} - e^{-at}}{2}$$

So,

$$\begin{aligned} L\{f(t)\} = L\{\cosh at\} &= L\left\{\frac{e^{at} + e^{-at}}{2}\right\} \\ &= \frac{1}{2}\left[L\left\{e^{at}\right\} + L\left\{e^{-at}\right\}\right] \quad \text{(By the linearity property)} \\ &= \frac{1}{2}\left[\frac{1}{p-a} + \frac{1}{p+a}\right] = \frac{1}{2}\left[\frac{2p}{(p-a)(p+a)}\right] \\ &= \frac{p}{p^2 + a^2}, \quad p > |a| \end{aligned}$$

All the results of the above are represented in the following tabular form for ready reference.

10.4.10 Table Showing Laplace Transform

$f(t)$	$L\{f(t)\}$ or $F(p)$
1	$\frac{1}{p}$, $p > 0$
t	$\frac{1}{p^2}$, $p > 0$
t^n (n is a positive integer)	$\frac{n!}{p^{n+1}}$, $p > 0$
t^n $(n > -1)$	$\frac{\Gamma(n+1)}{p^{n+1}}$, $p > 0$
e^{at}	$\frac{1}{p-a}$, $p > a$

$\sin(at)$	$\dfrac{a}{p^2+a^2},\ p>0$
$\cos(at)$	$\dfrac{p}{p^2+a^2},\ p>0$
$\sinh(at)$	$\dfrac{a}{p^2-a^2},\ p>\lvert a\rvert$
$\cosh(at)$	$\dfrac{p}{p^2-a^2},\ p>\lvert a\rvert$

Example 1 **Applying definition find $L\{k\}$, k being a nonzero constant.**

[WBUT 2004]

Sol. By definition, we have

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt$$

$$\text{So, } L\{k\} = \int_0^\infty e^{-pt}\cdot k\,dt = k\int_0^\infty e^{-pt}\,dt$$

$$= k\cdot\left[\frac{e^{-pt}}{-p}\right]_0^\infty = k\cdot\left[0+\frac{1}{p}\right] = \frac{k}{p},\ \text{since } p>0.$$

Example 2 **Find $L\{f(t)\}$, where**

$$f(t) = \left\{\begin{array}{ll} 1, & \text{if } 0<t<2 \\ 2, & \text{if } t>2 \end{array}\right\}.$$

[WBUT 2002]

Sol. By definition, we have

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt = \int_0^2 e^{-pt}\cdot 1\,dt + \int_2^\infty e^{-pt}\cdot 2\,dt$$

$$= \left[\frac{e^{-pt}}{-p}\right]_0^2 + 2\cdot\left[\frac{e^{-pt}}{-p}\right]_2^\infty$$

$$= \left[\frac{e^{-2p}}{-p}+\frac{1}{p}\right] + 2\cdot\left[0+\frac{e^{-2p}}{p}\right],\ \text{since } p>0$$

$$= \frac{e^{-2p}}{p}+\frac{1}{p}.$$

Example 3 **Find $L\{f(t)\}$, where**

$$f(t) = \begin{cases} e^t, & \text{if } 0 < t \le 1 \\ 0, & \text{if } t > 1 \end{cases}$$

[WBUT 2003]

Sol. By definition, we have

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt = \int_0^1 e^{-pt} \cdot e^t\, dt + \int_1^\infty e^{-pt} \cdot 0\, dt$$

$$= \int_0^1 e^{-(p-1)t}\, dt = \left[\frac{e^{-(p-1)t}}{-(p-1)}\right]_0^1$$

$$= \frac{e^{-(p-1)}}{-(p-1)} + \frac{1}{p-1} = \frac{1-e^{-(p-1)}}{p-1}.$$

Example 4 **Find $L\{at+b\}$** [WBUT 2005]

Sol. By linearity property of Laplace transform

$$L\{at+b\} = L\{at + b\cdot 1\} = aL\{t\} + bL\{1\}$$

$$= a \cdot \frac{1}{p^2} + b \cdot \frac{1}{p} \text{ (Using results of **Art. 10.4.1** and **Art. 10.4.2**)}$$

$$= \frac{a+bp}{p^2}.$$

10.5 FIRST AND SECOND SHIFTING PROPERTIES

10.5.1 First Shifting (or Translation) Theorem

If $f(t)$ be piecewise continuous for all $t \ge 0$ and of some exponential order σ as $t \to \infty$ such that the Laplace transform of $f(t)$ exists and is $F(p)$ then the Laplace transform of $e^{at} f(t)$ is given by $F(p-a)$.

In other words, if

$$\boxed{L\{f(t)\} = F(p), \textbf{then } L\left\{e^{at} f(t)\right\} = F(p-a)}$$

Proof From the definition, we have

$$L\{f(t)\} = F(p) = \int_0^\infty e^{-pt} f(t)\, dt$$

Now,

$$F(p-a)=\int_0^\infty e^{-(p-a)t}f(t)\,dt=\int_0^\infty e^{-pt}\left\{e^{at}f(t)\right\}dt$$
$$=L\left\{e^{at}f(t)\right\}$$

This proves the theorem.

Example 5 **Find** $L\{e^t \sin t \cos t\}$ [WBUT 2006].

Sol. Let $f(t)=\sin t\cos t=\frac{1}{2}\sin 2t.$

Then,

$$L\{f(t)\}=L\left\{\frac{1}{2}\sin 2t\right\}=\frac{1}{2}L\{\sin 2t\}$$
$$=\frac{1}{2}\cdot\frac{2}{p^2+4}=\frac{1}{p^2+4}=F(p)\text{ (By **Art. 10.4.6**)}$$

Now, by the first shifting theorem

$$L\left\{e^t f(t)\right\}=F(p-1)$$

or,

$$L\left\{e^t\sin t\cos t\right\}=\frac{1}{(p-1)^2+4}.$$

Example 6 **Find** $L\left\{(1+te^{-t})^2\right\}$**.**

Sol. Here,

$$L\left\{(1+te^{-t})^2\right\}=L\left\{1+2te^{-t}+t^2e^{-2t}\right\}$$
$$=L\{1\}+2L\left\{te^{-t}\right\}+L\left\{t^2e^{-2t}\right\}$$

Now,

$$L\{t\}=\frac{1}{p^2}\Rightarrow L\left\{te^{-t}\right\}=\frac{1}{(p+1)^2}\quad\text{(by first shifting theorem)}$$

and

$$L\left\{t^2\right\}=\frac{2}{p^3}\Rightarrow L\left\{t^2e^{-2t}\right\}=\frac{2}{(p+2)^3}\quad\text{(by first shifting theorem)}$$

Therefore,

$$L\left\{(1+te^{-t})^2\right\}=L\{1\}+2L\left\{te^{-t}\right\}+L\left\{t^2e^{-2t}\right\}$$
$$=\frac{1}{p}+\frac{4}{(p+1)^2}+\frac{2}{(p+2)^3}.$$

10.5.2 The Unit Step Function (or Heaviside's unit function)

The function $H(t-a)$ which is defined as

$$H(t-a)=\left\{\begin{array}{ll}0, & \text{if } t<a\\ 1, & \text{if } t\geq a\end{array}\right\}\quad where\ a>0$$

is called the unit step function (or Heaviside's unit function).

Note: Properties of the unit step function and Laplace transform of the function will be discussed in the later sections elaborately.

10.5.3 Second Shifting Theorem

If

$$L\{f(t)\}=F(p) \text{ and } g(t)=\left\{\begin{array}{ll}f(t-a), & t>a\\ 0, & t<a\end{array}\right\}$$

then,

$$L\{g(t)\}=e^{-ap}F(p).$$

Alternative Statement (using unit step function) **If** $L\{f(t)\}=F(p)$, **then**

$$L\{f(t-a)\cdot H(t-a)\}=e^{-ap}F(p).$$

Proof By definition, we have

$$\begin{aligned}
L\{g(t)\} &= \int_0^\infty e^{-pt}g(t)\,dt\\
&= \int_0^a e^{-pt}g(t)\,dt+\int_a^\infty e^{-pt}g(t)\,dt\\
&= \int_0^a e^{-pt}\cdot 0\,dt+\int_a^\infty e^{-pt}f(t-a)\,dt\\
&= \int_0^\infty e^{-p(a+u)}f(u)\,du \text{ (putting } t-a=u,\ dt=du)\\
&= e^{-ap}\int_0^\infty e^{-pu}f(u)\,du
\end{aligned}$$

So,

$$L\{g(t)\} = e^{-ap} \int_0^\infty e^{-pt} f(t)\, dt \quad \text{(Replacing } u \text{ by } t\text{)}$$

$$= e^{-ap} F(p).$$

Example 7 **Find $L\{g(t)\}$ where**

$$g(t) = \begin{Bmatrix} \cos\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{Bmatrix}$$

Sol. Let $f(t) = \cos t$.
So,

$$L\{f(t)\} = L\{\cos t\} = \frac{p}{p^2+1} = F(p)$$

Also

$$g(t) = \begin{Bmatrix} f\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{Bmatrix}$$

Then by second shifting theorem

$$L\{g(t)\} = e^{-\frac{\pi}{3}p} F(p) = e^{-\frac{\pi}{3}p} \cdot \frac{p}{p^2+1}.$$

Example 8 **Find $L\{g(t)\}$ where**

$$g(t) = \begin{Bmatrix} e^{t-3}, & t > 3 \\ 0, & t < 3 \end{Bmatrix}$$

Sol. Let $f(t) = e^t$.
So,

$$L\{f(t)\} = L\left\{e^t\right\} = \frac{1}{p-1} = F(p)$$

Also

$$g(t) = \begin{Bmatrix} f(t-3), & t > 3 \\ 0, & t < 3 \end{Bmatrix}$$

Then by second shifting theorem

$$L\{g(t)\} = e^{-3p} F(p) = e^{-3p} \cdot \frac{1}{p-1} = \frac{e^{-3p}}{p-1}.$$

10.6 CHANGE OF SCALE PROPERTY

If $L\{f(t)\} = F(p)$**, then**

$$\boxed{L\{f(at)\} = \frac{1}{a}\cdot F\left(\frac{p}{a}\right) \text{ for non zero constant } a.}$$

Proof By definition, we have

$$\begin{aligned} L\{f(at)\} &= \int_0^\infty e^{-pt} f(at)\, dt \\ &= \frac{1}{a}\int_0^\infty e^{-p\left(\frac{u}{a}\right)} f(u)\, du \quad (\text{putting } at = u,\ a \cdot dt = du) \\ &= \frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)t} f(t)\, dt \ (\text{replacing } u \text{ by } t) \\ &= \frac{1}{a}\cdot F\left(\frac{p}{a}\right). \end{aligned}$$

Example 9 **If** $L\{f(t)\} = \dfrac{p^2 - p + 1}{(2p+1)^2(p-1)}$**, apply change of scale property to show that**

$$L\{f(2t)\} = \frac{p^2 - 2p + 4}{4(p+1)^2(p-2)}.$$

[WBUT 2002]

Sol. Here $L\{f(t)\} = \dfrac{p^2 - p + 1}{(2p+1)^2(p-1)} = F(p)$

By change of scale property, we have

$$L\{f(2t)\} = \frac{1}{2}\cdot F\left(\frac{p}{2}\right) = \frac{1}{2}\cdot \frac{\left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right) + 1}{\left[2\left(\frac{p}{2}\right) + 1\right]^2\left[\left(\frac{p}{2}\right) - 1\right]}$$

$$= \frac{\dfrac{p^2 - 2p + 4}{4}}{2\cdot (p+1)^2\left(\dfrac{p-2}{2}\right)} = \frac{p^2 - 2p + 4}{4(p+1)^2(p-2)}.$$

Example 10 **Given that**

$$L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right)$$

Find $L\left\{\frac{\sin at}{t}\right\}$ [WBUT 2007]

Sol. Let $f(t) = \frac{\sin t}{t}$.
So,

$$L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right) \Rightarrow L\{f(t)\} = F(p) = \tan^{-1}\left(\frac{1}{p}\right)$$

Now by change of scale property, we have

$$L\{f(at)\} = \frac{1}{a}\cdot F\left(\frac{p}{a}\right)$$

i.e., $$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\cdot \tan^{-1}\left(\frac{1}{\left(\frac{p}{a}\right)}\right)$$

i.e., $$\frac{1}{a}\cdot L\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\cdot \tan^{-1}\left(\frac{a}{p}\right)$$

i.e., $$L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{p}\right)$$

10.7 LAPLACE TRANSFORM OF DERIVATIVES OF FUNCTIONS

10.7.1 Laplace transform of First Order Derivative of a Function

Theorem 10.2 **If**

(i) $f(t)$ **be continuous for all** $t \geq 0$ **and of exponential order** σ **as** $t \to \infty$ **and**
(ii) $f'(t)$ **be piecewise continuous for all** $t \geq 0$ **and of some exponential order as** $t \to \infty$, **then** $L\left\{f'(t)\right\}$ **exists for** $p > \sigma$ **and is given by**

$$\boxed{L\left\{f'(t)\right\} = pL\{f(t)\} - f(0) = p\cdot F(p) - f(0).}$$

where $L\{f(t)\} = F(p)$.

Proof First we prove the result when $f'(t)$ is continuous for all $t \geq 0$.

By the definition of Laplace transform, we have

$$L\{f'(t)\} = \int_0^\infty e^{-pt} f'(t)\, dt$$

$$= \left[e^{-pt} f(t)\right]_0^\infty dt - \int_0^\infty (-p) e^{-pt} f(t)\, dt$$

$$= \lim_{t\to\infty} e^{-pt} f(t) - f(0) + p \cdot L\{f(t)\}$$

$$= \lim_{t\to\infty} e^{-pt} f(t) - f(0) + p \cdot F(p) \tag{1}$$

Since $f(t)$ is of exponential order σ, we can find $k(> 0)$ such that

$$|f(t)| \leq k \cdot e^{\sigma t} \text{ for } t \geq 0$$

So,

$$\left|e^{-pt} f(t)\right| = e^{-pt} |f(t)| \leq e^{-pt} \cdot k \cdot e^{\sigma t}$$

$$= k \cdot e^{-(p-\sigma)t} \to 0 \text{ as } t \to \infty, \text{ for } p > \sigma$$

Therefore,

$$\lim_{t\to\infty} e^{-pt} f(t) = 0 \text{ for } p > \sigma.$$

Hence from (1), we see that $L\left\{f'(t)\right\}$ exists for $p > \sigma$ and is given by

$$L\left\{f'(t)\right\} = p \cdot F(p) - f(0)$$

Now, let us suppose $f'(t)$ is piecewise continuous for all $t \geq 0$.

Then, we have

$$L\{f'(t)\} = \int_0^\infty e^{-pt} f'(t)\, dt$$

Now we break up the RHS of the above as the sum of the integrals such that $f'(t)$ is continuous in each of these sub-intervals.

Then using the earlier concept, we have

$$L\left\{f'(t)\right\} = p \cdot F(p) - f(0).$$

Note:

(1) Suppose $f(t)$ is not continuous at $t = 0$, but right-hand limit $f(0+0)$ exists, then

$$\boxed{L\left\{f'(t)\right\} = p \cdot F(p) - f(0+0)}$$

(2) Suppose $f(t)$ is not continuous at $t = a\ 0 < a < \infty$, but right-hand limit $f(a+0)$ and left-hand limit $f(a-0)$ exist, then

$$\boxed{L\left\{f'(t)\right\} = p \cdot F(p) - f(0) - e^{-ap}\left[f(a+0) - f(a-0)\right]}$$

10.7.2 Laplace transform of Second Order Derivative of a Function

Theorem 10.3 **If**

(i) both $f(t)$ and $f'(t)$ be continuous for all $t \geq 0$ and of exponential order σ as $t \to \infty$ and

(ii) $f''(t)$ be piecewise continuous for all $t \geq 0$ and of some exponential order as $t \to \infty$,
then $L\left\{f''(t)\right\}$ exists for $p > \sigma$ and is given by

$$\boxed{L\left\{f''(t)\right\} = p^2 L\{f(t)\} - pf(0) - f'(0) = p^2 F(p) - pf(0) - f'(0).}$$

where $L\{f(t)\} = F(p)$.

Proof Let us consider $g(t) = f'(t)$. Then we have,

$$g'(t) = f''(t) \text{ and } g(0) = f'(0).$$

Now by the **Theorem 10.2** we get,

$$L\left\{g'(t)\right\} = pL\{g(t)\} - g(0).$$

Therefore,

$$\begin{aligned} L\left\{f''(t)\right\} &= pL\left\{f'(t)\right\} - f'(0) \\ &= p\left[p \cdot L\{f(t)\} - f(0)\right] - f'(0) \\ &= p^2 L\{f(t)\} - pf(0) - f'(0) \\ &= p^2 F(p) - pf(0) - f'(0) \end{aligned}$$

This proves the theorem.

10.7.3 Laplace transform of n^{th} Order Derivative of a Function (Generalised Form)

Theorem 10.4 **If**

(i) $f(t), f'(t); f''(t), \ldots, f^{(n-1)}(t)$ be continuous for all $t \geq 0$ and of exponential σ order as $t \to \infty$ and

(ii) $f^{(n)}(t)$ **be piecewise continuous for all** $t \geq 0$ **and of some exponential order as** $t \to \infty$**, then** $L\left\{f^{(n)}(t)\right\}$ **exists for** $p > \sigma$ **and is given by**

$$\boxed{\begin{aligned} L\{f^{(n)}(t)\} &= p^n L\{f(t)\} - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \\ &= p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \end{aligned}}$$

where $L\{f(t)\} = F(p)$.

Proof Beyond the scope of the book.

Example 11 **Using the Laplace transform of derivative of function show that**

$$L\{\cos t\} = \frac{p}{p^2 + 1}$$

Sol. We know $\cos t = \frac{d}{dt}(\sin t)$. Let, $f(t) = \sin t$.
Then, we get

$$L\{f(t)\} = L\{\sin t\} = \frac{1}{p^2 + 1} = F(p)$$

Now by **Theorem 10.2**, we have

$$L\left\{f'(t)\right\} = p \cdot F(p) - f(0)$$

i.e.,

$$L\{\cos t\} = p \cdot \frac{1}{p^2 + 1} - 0 = \frac{p}{p^2 + 1}.$$

10.8 LAPLACE TRANSFORM OF INTEGRALS

Theorem 10.5 **If** $L\{f(t)\} = F(p)$**, then**

$$\boxed{L\left\{\int_0^t f(x)\, dx\right\} = \frac{F(p)}{p}}$$

Proof Let

$$g(t) = \int_0^t f(x)\, dx$$

then $g'(t) = f(t)$ and $g(0) = 0$

Now,

$$L\left\{g'(t)\right\} = p \cdot L\{g(t)\} - g(0)$$

i.e.,
$$L\{f(t)\} = p \cdot L\{g(t)\}$$

i.e.,
$$L\{g(t)\} = \frac{L\{f(t)\}}{p} = \frac{F(p)}{p}.$$

Hence, we have

$$L\left\{\int_0^t f(x)\,dx\right\} = \frac{F(p)}{p}$$

This proves the theorem.

Example 12 **Find** $L\{\int_0^t \cos t dt\}$

Sol. Let $f(t) = \cos t$, then

$$L\{f(t)\} = L\{\cos t\} = \frac{p}{p^2+1} = F(p)$$

Now by **Theorem 10.5**, we have

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(p)}{p}$$

$$\Rightarrow L\left\{\int_0^t \cos t\,dt\right\} = \frac{1}{p} \cdot \frac{p}{p^2+1} = \frac{1}{p^2+1}.$$

Example 13 **Find** $L\{\int_0^t e^{2t} \cdot t^2 dt\}$

Sol. Let, $f(t) = e^{2t} \cdot t^2$. We know $L\left\{t^2\right\} = \frac{2}{p^3}$.

Then, by first shifting property

$$L\{f(t)\} = L\left\{e^{2t} \cdot t^2\right\} = \frac{2}{(p-2)^3} = F(p)$$

Now by **Theorem 10.5** we have

$$L\left\{\int_0^t e^{2t} \cdot t^2\,dt\right\} = \frac{F(p)}{p}$$

$$\Rightarrow L\left\{\int_0^t \cos t\, dt\right\} = \frac{1}{p}\cdot\frac{2}{(p-2)^3} = \frac{2}{p\,(p-2)^3}.$$

10.9 LAPLACE TRANSFORM OF FUNCTIONS ON MULTIPLICATION BY t^n (n is any positive integer)

Theorem 10.6 **If** $L\{f(t)\} = F(p)$, **then**

$$\boxed{L\{t\cdot f(t)\} = -\frac{d}{dp}\{F(p)\} = -F'(p)}$$

Proof It is given that $L\{f(t)\} = F(p)$, so we have

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{1}$$

Applying Leibnitz's rule for differentiation under the sign of integration, differentiating (1) w.r.t. p we obtain

$$\frac{d}{dp}\{F(p)\} = \frac{d}{dp}\left\{\int_0^\infty e^{-pt} f(t)\, dt\right\}$$

or, $$\frac{d}{dp}\{F(p)\} = \int_0^\infty \frac{\partial}{\partial p}\left\{e^{-pt} f(t)\right\}\, dt = \int_0^\infty (-t)e^{-pt} f(t)\, dt$$

or, $$\frac{d}{dp}\{F(p)\} = -\int_0^\infty e^{-pt}\{t\cdot f(t)\}\, dt = -L\{t\cdot f(t)\}$$

Therefore,

$$L\{t\cdot f(t)\} = -\frac{d}{dp}\{F(p)\} = -F'(p)$$

This proves the result.

Theorem 10.7 **(Generalised Form):**
If $L\{f(t)\} = F(p)$, **then**

$$\boxed{L\left\{t^n\cdot f(t)\right\} = (-1)^n\frac{d^n}{dp^n}\{F(p)\} = (-1)^n F^{(n)}(p)}$$

for positive integer n. [WBUT 2005]

Proof We apply the principle of mathematical induction to prove the following

$$L\left\{t^n \cdot f(t)\right\} = (-1)^n \frac{d^n}{dp^n}\{F(p)\} = (-1)^n F^{(n)}(p) \qquad (1)$$

For $n = 1$, (1) becomes

$$L\{t \cdot f(t)\} = -\frac{d}{dp}\{F(p)\} = -F'(p)$$

which is true by **Theorem 10.6.**

Let us assume that (1) is true for $n = k$, we are to prove for $n = k + 1$

From (1) we have,

$$L\left\{t^k \cdot f(t)\right\} = (-1)^k \frac{d^k}{dp^k}\{F(p)\}$$

$$or, \quad \int_0^\infty e^{-pt}\left\{t^k \cdot f(t)\right\} dt = (-1)^k \frac{d^k}{dp^k}\{F(p)\} \qquad (2)$$

Applying Leibnitz's rule for differentiation under the sign of integration, differentiating (2) w.r.t. p we obtain

$$\frac{d}{dp}\left\{\int_0^\infty e^{-pt}\left\{t^k \cdot f(t)\right\} dt\right\} = \frac{d}{dp}\left\{(-1)^k \frac{d^k}{dp^k}\{F(p)\}\right\}$$

$$or, \quad \int_0^\infty \frac{\partial}{\partial p}\left\{e^{-pt} \cdot t^k \cdot f(t)\right\} dt = (-1)^k \frac{d^{k+1}}{dp^{k+1}}\{F(p)\}$$

$$or, \quad \int_0^\infty (-t)e^{-pt} \cdot t^k \cdot f(t)\, dt = (-1)^k \frac{d^{k+1}}{dp^{k+1}}\{F(p)\}$$

$$or, \quad \int_0^\infty e^{-pt} \cdot \left\{t^{k+1} \cdot f(t)\right\} dt = -\left[(-1)^k \frac{d^{k+1}}{dp^{k+1}}\{F(p)\}\right]$$

Therefore,

$$L\left\{t^{k+1} \cdot f(t)\right\} = (-1)^{k+1} \frac{d^{k+1}}{dp^{k+1}}\{F(p)\}$$

So, (1) is true for $n = k + 1$ when it is true for $n = k$.

Hence the result (1) is true for $n = 1, 2, 3, \ldots,$ i.e., for all positive integers n.

Example 14 **Find** $L\{t\cos 2t\}$

Sol. Let $f(t) = \cos 2t$, then

$$L\{f(t)\} = L\{\cos 2t\} = \frac{p}{p^2+4} = F(p)$$

Now by **Theorem 10.6,** we have

$$L\{t \cdot f(t)\} = -\frac{d}{dp}\{F(p)\}$$

$$i.e., \quad L\{t\cos 2t\} = -\frac{d}{dp}\left\{\frac{p}{p^2+4}\right\} = -\frac{\left(p^2+4\right)\cdot 1 - p\cdot(2p)}{\left(p^2+4\right)^2}$$

$$= \frac{p^2-4}{\left(p^2+4\right)^2}.$$

Example 15 **Find** $L\left\{t^n e^{at}\right\}$

Sol. Let $f(t) = e^{at}$, then

$$L\{f(t)\} = L\left\{e^{at}\right\} = \frac{1}{p-a} = F(p)$$

Now by **Theorem 10.7,** we have

$$L\left\{t^n \cdot f(t)\right\} = (-1)^n \frac{d^n}{dp^n}\{F(p)\}$$

$$i.e., \; L\left\{t^n e^{at}\right\} = (-1)^n \frac{d^n}{dp^n}\left\{\frac{1}{p-a}\right\}$$

$$= (-1)^n \cdot \left\{(-1)^n \cdot \frac{n!}{(p-a)^{n+1}}\right\} \left(\begin{array}{c}\text{by successive}\\ \text{differentiation}\end{array}\right)$$

$$= \frac{n!}{(p-a)^{n+1}}.$$

10.10 LAPLACE TRANSFORM OF FUNCTIONS ON DIVISION BY t

Theorem 10.8 **If** $L\{f(t)\} = F(p)$, **then**

$$\boxed{L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(p)dp}$$

provided the integral exists.

Proof It is given that $L\{f(t)\} = F(p)$, so by definition we have

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{1}$$

Integrating both sides of (1) w.r.t. p, ranging from p to ∞ we obtain

$$\int_p^\infty F(p) = \int_p^\infty \left\{ \int_0^\infty e^{-pt} f(t)\, dt \right\} \cdot dp \tag{2}$$

Since p and t are independent variables, we can interchange the order of integration of RHS and performing that, we have

$$\int_p^\infty F(p) = \int_0^\infty \left\{ \int_p^\infty e^{-pt} \cdot dp \right\} f(t)\, dt$$

$$= \int_0^\infty \left[-\frac{e^{-pt}}{t} \right]_p^\infty f(t)\, dt = \int_0^\infty \frac{e^{-pt}}{t} f(t)\, dt$$

$$= \int_0^\infty e^{-pt} \left\{ \frac{f(t)}{t} \right\} dt = L\left\{ \frac{f(t)}{t} \right\}$$

Hence the theorem is proved.

Example 16 **Find** $L\left\{ \frac{\sin t}{t} \right\}$ [WBUT 2003]

Sol. Let $f(t) = \sin t$. Then by **Art. 10.4.6** we have,

$$L\{f(t)\} = L\{\sin t\} = \frac{1}{p^2+1} = F(p)$$

Now by above **Theorem 10.8**

$$L\left\{ \frac{f(t)}{t} \right\} = \int_p^\infty F(p)\, dp$$

so, $$L\left\{ \frac{\sin t}{t} \right\} = \int_p^\infty \frac{1}{p^2+1}\, dp = \left[\tan^{-1} p \right]_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} p = \cot^{-1} p = \tan^{-1}\left(\frac{1}{p} \right).$$

Example 17 **Find** $L\left\{\frac{1-e^t}{t}\right\}$ [WBUT 2004]

Sol. Let $f(t) = 1 - e^t$. Then,

$$L\{f(t)\} = L\left\{1-e^t\right\} = L\{1\} - L\left\{e^t\right\}$$
$$= \frac{1}{p} - \frac{1}{p-1} = F(p) \text{ (by**Art.10.4.1** and **Art.10.4.5**)}$$

Now by above **Theorem 10.8**

$$L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(p)\,dp$$

$$\text{so, } L\left\{\frac{1-e^t}{t}\right\} = \int_p^\infty \left(\frac{1}{p} - \frac{1}{p-1}\right) dp = \left[\log p - \log(p-1)\right]_p^\infty, \; p > 1$$

$$= \left[\log\left(\frac{p}{p-1}\right)\right]_p^\infty = \lim_{p\to\infty} \log\left(\frac{p}{p-1}\right) - \log\left(\frac{p}{p-1}\right)$$

$$= \lim_{p\to\infty} \log\left(\frac{1}{1-\frac{1}{p}}\right) - \log\left(\frac{p}{p-1}\right) = 0 - \log\left(\frac{p}{p-1}\right)$$

$$= \log\left(\frac{p-1}{p}\right), \; p > 1$$

10.11 LAPLACE TRANSFORM OF UNIT STEP FUNCTION (HEAVISIDE'S UNIT FUNCTION)

First we recall the definition of **unit step function (or Heaviside's unit function)** from the **section 10.5.2**

10.11.1 Definition

The function $H(t-a)$ which is defined as

$$H(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a \end{cases} \quad \text{where } a > 0$$

is called the unit step function (or Heaviside's unit function). The graph of the function is given by the following **Figure 10.1.**

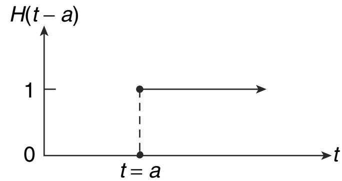


Figure 10.1

Theorem 10.9 **Laplace Transform of unit step function $H(t-a)$ is given by $\dfrac{e^{-ap}}{p}$.**

In other words,

$$\boxed{L\{H(t-a)\} = \frac{e^{-ap}}{p}}$$

Proof By definition, we have

$$\begin{aligned} L\{H(t-a)\} &= \int_0^\infty e^{-pt} \cdot H(t-a)\,dt \\ &= \int_0^a e^{-pt} \cdot 0\,dt + \int_a^\infty e^{-pt} \cdot 1\,dt \\ &= 0 + \left[-\frac{e^{-pt}}{p}\right]_a^\infty = \frac{e^{-ap}}{p} \end{aligned}$$

Hence the theorem is proved.

Theorem 10.10 **Let $f(t)$ be a function defined as**

$$f(t) = \begin{Bmatrix} f_1(t), & \text{if } t < a \\ f_2(t), & \text{if } t \geq a \end{Bmatrix}$$

Then $f(t)$ can be expressed involving unit step function $H(t-a)$ as the following form

$$\boxed{f(t) = f_1(t) + \{f_2(t) - f_1(t)\} \cdot H(t-a)}$$

Proof By the definition of the unit step function, we get

$$H(t-a) = \begin{Bmatrix} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a \end{Bmatrix} \tag{1}$$

Now, we consider the expression

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\} \cdot H(t-a)$$

Using (1) we have from above

$$f(t) = \begin{Bmatrix} f_1(t) + \{f_2(t) - f_1(t)\} \cdot 0, & \text{if } t < a \\ f_1(t) + \{f_2(t) - f_1(t)\} \cdot 1, & \text{if } t \geq a \end{Bmatrix}$$

$$= \begin{Bmatrix} f_1(t), & \text{if } t < a \\ f_2(t), & \text{if } t \geq a \end{Bmatrix}$$

Hence the result is proved.

Theorem 10.11 **Let $f(t)$ be a function defined as**

$$f(t) = \begin{Bmatrix} f_1(t), & \text{if } t < a_1 \\ f_2(t), & \text{if } a_1 \leq t < a_2 \\ f_3(t), & \text{if } t \geq a_2 \end{Bmatrix}$$

Then $f(t)$ can be expressed involving unit step function $H(t-a)$ as the following form

$$\boxed{f(t) = f_1(t) + \{f_2(t) - f_1(t)\} \cdot H(t-a_1) + \{f_3(t) - f_2(t)\} \cdot H(t-a_2).}$$

Proof The proof can be done in the line of **Theorem 10.10**.

In the next theorem we state the generalised form of the above two theorems.

Theorem 10.12 **Let $f(t)$ be a function defined as**

$$\boxed{f(t) = \begin{Bmatrix} f_1(t), & \text{if } t < a_1 \\ f_2(t), & \text{if } a_1 \leq t < a_2 \\ \cdots & \cdots \quad \cdots \quad \cdots \\ f_{n-1}(t), & \text{if } a_{n-2} \leq t < a_{n-1} \\ f_n(t), & \text{if } t \geq a_{n-1} \end{Bmatrix}.}$$

Then $f(t)$ can be expressed involving unit step function $H(t-a)$ as the following form:

$$\boxed{f(t) = f_1(t) + \{f_2(t) - f_1(t)\} \cdot H(t-a_1) + \ldots + \{f_n(t) - f_{n-1}(t)\} \cdot H(t-a_{n-1})}$$

Example 18 **Express the following function in terms of unit step function**

$$f(t) = \begin{Bmatrix} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{Bmatrix}$$

and find its Laplace transform. [WBUT 2002]

Sol. It is given that

$$f(t) = \begin{Bmatrix} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{Bmatrix}$$

Then $f(t)$ can be expressed in terms of unit step function as the following form (see **Theorem 10.10**)

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\} \cdot H(t-2)$$

where $f_1(t) = t-1$, $f_2(t) = 3-t$ and $H(t-2)$ is the unit step function. Therefore,

$$\begin{aligned} f(t) &= (t-1) + \{(3-t) - (t-1)\} \cdot H(t-2) \\ &= (t-1) + 2\,(2-t) \cdot H(t-2). \end{aligned}$$

Now,

$$\begin{aligned} L\{f(t)\} &= L\{(t-1) + 2 \cdot (2-t) \cdot H(t-2)\} \\ &= L\{t-1\} + 2 \cdot L\{(2-t) \cdot H(t-2)\} \\ &= L\{t\} - L\{1\} - 2 \cdot L\{(t-2) \cdot H(t-2)\} \qquad (1) \end{aligned}$$

Again by second shifting theorem (See Alternative statement of **Art. 10.5.3**), we have

$$L\{(t-2) \cdot H(t-2)\} = e^{-2p} L\{t\} = e^{-2p} \cdot \frac{1}{p^2} \qquad (2)$$

Using (2) in (1), we obtain

$$\begin{aligned} L\{f(t)\} &= \frac{1}{p^2} - \frac{1}{p} - 2 \cdot e^{-2p} \cdot \frac{1}{p^2} \\ &= \left(1 - 2 \cdot e^{-2p}\right) \frac{1}{p^2} - \frac{1}{p}. \end{aligned}$$

10.12 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

Definition of Periodic Function A function $f(t)$ is called periodic function if

$$\boxed{f(t+nk) = f(t) \text{ for } n = 1, 2, 3, \ldots}$$

where $k(>0)$ is called the period of the function

For example $\sin x$, $\cos x$ are the periodic functions.

Theorem 10.13 **Let $f(t)$ be a periodic function of period k and is piecewise continuous for $0 < t < k$. Then Laplace transform of $f(t)$ is given by**

$$L\{f(t)\} = \frac{1}{1-e^{-pk}} \int_0^k e^{-pt} \cdot f(t)\, dt$$

Proof By the definition,

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-pt} \cdot f(t)\, dt \\ &= \int_0^k e^{-pt} \cdot f(t)\, dt + \int_k^{2k} e^{-pt} \cdot f(t)\, dt + \int_{2k}^{3k} e^{-pt} \cdot f(t)\, dt + \ldots\infty \\ &= \sum_{n=0}^{\infty} \int_{nk}^{(n+1)k} e^{-pt} \cdot f(t)\, dt \end{aligned}$$

Put $t = x + nk$, so, $dt = dx$. Therefore, we have from above

$$L\{f(t)\} = \sum_{n=0}^{\infty} \int_0^k e^{-p(x+nk)} \cdot f(x+nk)\, dx$$

$$i.e., L\{f(t)\} = \sum_{n=0}^{\infty} \int_0^k e^{-p(t+nk)} \cdot f(t+nk)\, dt = \sum_{n=0}^{\infty} \int_0^k e^{-pnk} \cdot e^{-pt} \cdot f(t+nk)\, dt.$$

Since $f(t)$ is periodic with period k i.e., $f(t+nk) = f(t)$, we have

$$L\{f(t)\} = \sum_{n=0}^{\infty} \left[e^{-pnk} \cdot \int_0^k e^{-pt} \cdot f(t)\, dt \right]$$

Since the integration is independent of n, the order of summation and integration can be interchanged and we get

$$\begin{aligned} L\{f(t)\} &= \int_0^k e^{-pt} \cdot f(t)\, dt \cdot \sum_{n=0}^{\infty} e^{-pnk} \\ &= \int_0^k e^{-pt} \cdot f(t)\, dt \cdot \left(1 + e^{-pk} + e^{-2pk} + e^{-3pk} + \ldots\infty\right) \end{aligned}$$

$$= \int_0^k e^{-pt} \cdot f(t)\, dt \cdot \left[1 + e^{-pk} + \left(e^{-pk}\right)^2 + \left(e^{-pk}\right)^3 + \ldots \infty\right]$$

Since,

$$\left[1 + e^{-pk} + \left(e^{-pk}\right)^2 + \left(e^{-pk}\right)^3 + \ldots \infty\right] = (1 + e^{-pk})^{-1} = \frac{1}{1 + e^{-pk}}$$

we have from above

$$L\{f(t)\} = \frac{1}{1 - e^{-pk}} \int_0^k e^{-pt} \cdot f(t)\, dt$$

This proves the theorem.

Example 19 **Find the Laplace transform of a periodic function $f(t)$ with the period $2c$ given by**

$$f(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2c \end{cases}$$

[WBUT 2003]

Sol. Here, let $f(t)$ be a periodic function of period $2c$.
Then Laplace transform of $f(t)$ is given by

$$L\{f(t)\} = \frac{1}{1 - e^{-2pc}} \int_0^{2c} e^{-pt} \cdot f(t)\, dt \text{ (by **Th. 10.13**)}$$

$$= \frac{1}{1 - e^{-2pc}} \left\{ \int_0^c e^{-pt} \cdot t\, dt + \int_c^{2c} e^{-pt} \cdot (2c - t)\, dt \right\} \quad (1)$$

Now, by integration of parts

$$\int_0^c e^{-pt} \cdot t\, dt = \left[t \cdot \frac{e^{-pt}}{-p}\right]_0^c - \int_0^c 1 \cdot \frac{e^{-pt}}{-p}\, dt$$

$$= c \cdot \frac{e^{-pc}}{-p} - \left[\frac{e^{-pt}}{p^2}\right]_0^c$$

$$= -\frac{c \cdot e^{-pc}}{p} - \frac{e^{-pc}}{p^2} + \frac{1}{p^2} \quad (2)$$

and also

$$\int_{c}^{2c} e^{-pt} \cdot (2c - t)\, dt = \left[(2c - t) \cdot \frac{e^{-pt}}{-p}\right]_{c}^{2c} - \int_{c}^{2c} (-1) \cdot \frac{e^{-pt}}{-p}\, dt$$

$$= c \cdot \frac{e^{-pc}}{p} + \left[\frac{e^{-pt}}{p^2}\right]_{c}^{2c}$$

$$= \frac{c \cdot e^{-pc}}{p} + \frac{e^{-2pc}}{p^2} - \frac{e^{-pc}}{p^2} \qquad (3)$$

Using (2) and (3) in (1), we get

$$L\{f(t)\} = \frac{1}{1 - e^{-2pc}} \left\{ \left(-\frac{c \cdot e^{-pc}}{p} - \frac{e^{-pc}}{p^2} + \frac{1}{p^2} \right) + \left(\frac{c \cdot e^{-pc}}{p} + \frac{e^{-2pc}}{p^2} - \frac{e^{-pc}}{p^2} \right) \right\}$$

$$= \frac{1}{1 - \left(e^{-pc}\right)^2} \cdot \frac{1}{p^2} \cdot \left(1 - 2e^{-pc} + e^{-2pc}\right)$$

$$= \frac{1}{\left(1 + e^{-pc}\right)\left(1 - e^{-pc}\right)} \cdot \frac{1}{p^2} \cdot \left(1 - e^{-pc}\right)^2$$

$$= \frac{\left(1 - e^{-pc}\right)}{p^2 \cdot \left(1 + e^{-pc}\right)}.$$

10.13 INITIAL VALUE THEOREM

If

(i) $f(t)$ **is continuous for all** $t \geq 0$ **and of some exponential order as** $t \to \infty$**,**

(ii) $f'(t)$ **be piecewise continuous for all** $t \geq 0$ **and of some exponential order as** $t \to \infty$ **and**

(iii) $L\{f(t)\} = F(p)$**, then**

$$\boxed{\lim_{t \to 0} f(t) = \lim_{p \to \infty} p \cdot L\{f(t)\} = \lim_{p \to \infty} p \cdot F(p)}$$

Proof By **Theorem 10.2** we have,

$$L\{f'(t)\} = p \cdot F(p) - f(0)$$

or, $$\int_{0}^{\infty} e^{-pt} \cdot f'(t)\, dt = p \cdot F(p) - f(0)$$

or, $$\lim_{p \to \infty} \int_{0}^{\infty} \left[e^{-pt} \cdot f'(t)\right] dt = \lim_{p \to \infty} [p \cdot F(p) - f(0)] \qquad (1)$$

Since $f'(t)$ is piecewise continuous for all $t \geq 0$ and of some exponential order as $t \to \infty$, we get

$$\lim_{p\to\infty} \int_0^\infty \left[e^{-pt} \cdot f'(t)\right] dt = \int_0^\infty \lim_{p\to\infty} \left[e^{-pt} \cdot f'(t)\right] dt = 0 \qquad (2)$$

Using (2) in (1), we obtain

$$0 = \lim_{p\to\infty} [p \cdot F(p) - f(0)]$$

or,

$$f(0) = \lim_{p\to\infty} p \cdot F(p)$$

Since $f(t)$ is continuous, $\lim_{t\to 0} f(t) = f(0)$

Therefore,

$$\lim_{t\to 0} f(t) = \lim_{p\to\infty} p \cdot F(p)$$

This proves the theorem.

Note: Suppose $f(t)$ is not continuous at $t = 0$, but $\lim_{t\to 0} f(t)$ exists, then the theorem is still true using Note (1) of Theorem 10.2.

10.14 FINAL VALUE THEOREM

If

(i) $f(t)$ **be continuous for all** $t \geq 0$ **and of some exponential order as** $t \to \infty$,

(ii) $f'(t)$ **be piecewise continuous for all** $t \geq 0$ **and of some exponential order as** $t \to \infty$ **and**

(iii) $L\{f(t)\} = F(p)$, **then**

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{p\to 0} p \cdot L\{f(t)\} = \lim_{p\to 0} p \cdot F(p)}$$

Proof By **Theorem 10.2** we have,

$$L\{f'(t)\} = p \cdot F(p) - f(0)$$

or,

$$\int_0^\infty e^{-pt} \cdot f'(t)\, dt = p \cdot F(p) - f(0)$$

or,

$$\lim_{p\to 0} \int_0^\infty \left[e^{-pt} \cdot f'(t)\right] dt = \lim_{p\to 0} [p \cdot F(p) - f(0)]$$

$$or, \quad \int_0^\infty \lim_{p\to 0}\left[e^{-pt}\cdot f'(t)\right] dt = \lim_{p\to 0}\left[p\cdot F(p) - f(0)\right]$$

$$or, \quad \int_0^\infty f'(t)\, dt = \lim_{p\to 0} p\cdot F(p) - f(0)$$

$$or, \quad [f(t)]_0^\infty = \lim_{p\to 0} p\cdot F(p) - f(0)$$

$$or, \quad \lim_{t\to\infty} f(t) - f(0) = \lim_{p\to 0} p\cdot F(p) - f(0)$$

Therefore,

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} p\cdot F(p).$$

This proves the theorem.

Note: Suppose $f(t)$ is not continuous at $t = 0$ but $\lim_{t\to 0} f(t)$ exists, then the theorem is still true using Note (1) of Theorem 10.2.

10.15 LAPLACE TRANSFORM OF SOME SPECIAL INTEGRALS

10.15.1 The Sine Integral

The sine integral, denoted by $S_i(t)$, is defined as

$$\boxed{S_i(t) = \int_0^t \frac{\sin u}{u} du}$$

and its Laplace transform is given by

$$\boxed{L\{S_i(t)\} = \frac{1}{p}\tan^{-1}\left(\frac{1}{p}\right)}$$

Proof By definition, we have

$$S_i(t) = \int_0^t \frac{\sin u}{u} du = \int_0^t \frac{1}{u}\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \ldots\infty\right) du$$

$$= \int_0^t \left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \frac{u^6}{7!} + \ldots\infty\right) du$$

$$= \left[u - \frac{u^3}{3.3!} + \frac{u^5}{5.5!} - \frac{u^7}{7.7!} + \ldots \infty \right]_0^t$$

$$= t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \ldots \infty$$

Now,

$$L\{S_i(t)\} = L\left\{ t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \cdots \infty \right\}$$

$$= L\{t\} - L\left\{\frac{t^3}{3.3!}\right\} + L\left\{\frac{t^5}{5.5!}\right\} - L\left\{\frac{t^7}{7.7!}\right\} + \cdots \infty$$

$$= \frac{1}{p^2} - \frac{1}{3.3!} \cdot \frac{3!}{p^4} + \frac{1}{5.5!} \cdot \frac{5!}{p^6} - \frac{1}{7.7!} \cdot \frac{7!}{p^8} + \ldots \infty$$

$$= \frac{1}{p^2} - \frac{1}{3} \cdot \frac{1}{p^4} + \frac{1}{5} \cdot \frac{1}{p^6} - \frac{1}{7} \cdot \frac{1}{p^8} + \cdots \infty$$

$$= \frac{1}{p} \left(\frac{1}{p} - \frac{\left(\frac{1}{p}\right)^3}{3} + \frac{\left(\frac{1}{p}\right)^5}{5} - \frac{\left(\frac{1}{p}\right)^7}{7} + \cdots \infty \right)$$

$$= \frac{1}{p} \tan^{-1}\left(\frac{1}{p}\right)$$

This proves the result.

10.15.2 The Cosine Integral

The Cosine integral, denoted by $C_i(t)$, is defined as

$$\boxed{C_i(t) = \int_t^{\infty} \frac{\cos u}{u} du}$$

and its Laplace transform is given by

$$\boxed{L\{C_i(t)\} = \frac{1}{2p} \log(p^2 + 1)}$$

Proof Let us consider,

$$f(t) = C_i(t) = \int_t^{\infty} \frac{\cos u}{u}\, du = -\int_{\infty}^{t} \frac{\cos u}{u}\, du$$

$$or, \qquad f(t) = \int_{\infty}^{t} \left(-\frac{\cos u}{u}\right) du$$

such that

$$f'(t) = -\frac{\cos t}{t} \Rightarrow t \cdot f'(t) = -\cos t$$

Therefore,

$$L\left\{t \cdot f'(t)\right\} = -L\{\cos t\}$$

$$or, \qquad -\frac{d}{dp} L\left\{f'(t)\right\} = -\frac{p}{p^2+1}$$

$$or, \qquad \frac{d}{dp}[p \cdot F(p) - f(0)] = \frac{p}{p^2+1}, \text{where } L\{f(t)\} = F(p)$$

$$or, \qquad \frac{d}{dp}[p \cdot F(p)] = \frac{p}{p^2+1}$$

Therefore,

$$p \cdot F(p) = \int \frac{p}{p^2+1}\, dp = \frac{1}{2}\log(p^2+1) + c(\text{constant}) \tag{1}$$

Again by final value theorem,

$$\lim_{p \to 0} p \cdot F(p) = \lim_{t \to \infty} f(t)$$

$$or, \qquad \lim_{p \to 0}\left[\frac{1}{2}\log(p^2+1) + c\right] = \lim_{t \to \infty} \int_{t}^{\infty} \frac{\cos u}{u}\, du$$

$$or, \qquad 0 + c = 0 \Rightarrow c = 0.$$

So from (1) we obtain,

$$p \cdot F(p) = \frac{1}{2}\log(p^2+1)$$

$$or, \qquad F(p) = \frac{1}{2p}\log(p^2+1)$$

$$or, \qquad L\{f(t)\} = \frac{1}{2p}\log(p^2+1)$$

This proves the result.

10.15.3 The Exponential Integral

The exponential integral, denoted by $E_i(t)$, is defined as

$$\boxed{E_i(t) = \int_t^{\infty} \frac{e^{-u}}{u} du}$$

and its Laplace transform is given by

$$\boxed{L\{E_i(t)\} = \frac{1}{p}\log(p+1)}$$

Proof Let us consider

$$f(t) = E_i(t) = \int_t^{\infty} \frac{e^{-u}}{u}\, du = -\int_{\infty}^{t} \frac{e^{-u}}{u}\, du$$

or,
$$f(t) = \int_{\infty}^{t} \left(-\frac{e^{-u}}{u}\right) du$$

such that

$$f'(t) = -\frac{e^{-t}}{t} \Rightarrow t \cdot f'(t) = -e^{-t}.$$

Therefore,

$$L\left\{t \cdot f'(t)\right\} = -L\left\{e^{-t}\right\}$$

or,
$$-\frac{d}{dp}L\left\{f'(t)\right\} = -\frac{1}{p+1}$$

or,
$$\frac{d}{dp}\left[p \cdot F(p) - f(0)\right] = \frac{1}{p+1}, \text{ where } L\{f(t)\} = F(p)$$

or,
$$\frac{d}{dp}\left[p \cdot F(p)\right] = \frac{1}{p+1}$$

Therefore,

$$p \cdot F(p) = \int \frac{1}{p+1}\, dp = \log(p+1) + c \text{ (constant)} \qquad (1)$$

Again by final value theorem,

$$\lim_{p \to 0} p \cdot F(p) = \lim_{t \to \infty} f(t)$$

or,
$$\lim_{p \to 0}\left[\log(p+1) + c\right] = \lim_{t \to \infty} \int_t^{\infty} \frac{e^{-u}}{u}\, du$$

or,
$$0 + c = 0 \Rightarrow c = 0.$$

So from (1) we obtain,

$$p \cdot F(p) = \log(p+1)$$

or,

$$F(p) = \frac{1}{p}\log(p+1)$$

or,

$$L\{f(t)\} = \frac{1}{p}\log(p+1)$$

This proves the result.

WORKED OUT EXAMPLES

Example 10.1 **Find $L\{f(t)\}$ where**

$$f(t) = \begin{Bmatrix} 0, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{Bmatrix}$$

[WBUT 2005]

Sol. By definition, we have

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

$$= \int_0^1 e^{-pt} \cdot 0\, dt + \int_1^2 e^{-pt} \cdot t\, dt + \int_2^\infty e^{-pt} \cdot 0\, dt$$

$$= \int_1^2 e^{-pt} \cdot t\, dt = \left[t \cdot \frac{e^{-pt}}{-p}\right]_1^2 - \int_1^2 1 \cdot \frac{e^{-pt}}{-p}\, dt$$

$$= \left[2 \cdot \frac{e^{-2p}}{-p} - 1 \cdot \frac{e^{-p}}{-p}\right] - \left[\frac{e^{-pt}}{p^2}\right]_1^2$$

$$= \frac{e^{-p}}{p} - 2 \cdot \frac{e^{-2p}}{p} - \frac{e^{-2p}}{p^2} + \frac{e^{-p}}{p^2}$$

$$= \left(\frac{1}{p} + \frac{1}{p^2}\right) e^{-p} - \left(\frac{2}{p} + \frac{1}{p^2}\right) e^{-2p}$$

Example 10.2 **Find $L\{f(t)\}$ where**

$$f(t) = \begin{Bmatrix} 1, & \text{If } t > \alpha \\ 0, & \text{If } t < \alpha \end{Bmatrix}$$

[WBUT 2006]

Sol. By definition, we have

$$L\{f(t)\} = \int_0^{\infty} e^{-pt} f(t)\, dt$$

$$= \int_0^{\alpha} e^{-pt} \cdot 0\, dt + \int_{\alpha}^{\infty} e^{-pt} \cdot 1\, dt = 0 + \left[\frac{e^{-pt}}{-p}\right]_{\alpha}^{\infty}$$

$$= 0 - \frac{e^{-p\alpha}}{-p}, \text{ since } p > 0$$

$$= \frac{1}{p} e^{-p\alpha}.$$

Example 10.3 **Find Laplace transform of**

$$f(t) = \begin{Bmatrix} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{Bmatrix}$$

[WBUT 2008]

Sol. By definition, we have

$$L\{f(t)\} = \int_0^{\infty} e^{-pt} f(t)\, dt$$

$$= \int_0^{\pi} e^{-pt} \cdot \sin t\, dt + \int_{\pi}^{\infty} e^{-pt} \cdot 0\, dt = \int_0^{\pi} e^{-pt} \cdot \sin t\, dt \tag{1}$$

From integral calculas we know,

$$\int e^{at} \cdot \sin(bt)\; dt = \frac{e^{at}}{a^2 + b^2} [a \sin bt - b \cos bt] \tag{2}$$

Using (2) in (1) we obtain

$$L\{f(t)\} = \int_0^{\pi} e^{-pt} \cdot \sin t\, dt$$

$$= \left[\frac{e^{-pt}}{(-p)^2 + 1^2} (-p \sin t - \cos t)\right]_0^{\pi}$$

$$= \frac{e^{-\pi p}}{p^2 + 1} - \left(\frac{-1}{p^2 + 1}\right)$$

$$v = \frac{1}{p^2 + 1} \left(e^{-\pi p} + 1\right).$$

Example 10.4 **Find $L\{f(t)\}$, where**

$$f(t) = \begin{Bmatrix} \sin\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t \le \frac{\pi}{3} \end{Bmatrix}$$

[WBUT 2004]

Sol. Let $g(t) = \sin t$, then

$$f(t) = \begin{Bmatrix} g\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t \le \frac{\pi}{3} \end{Bmatrix}.$$

Also,

$$L\{g(t)\} = L\{\sin t\} = \frac{1}{p^2+1} = G(p), \text{ say.}$$

Now by second shifting theorem,

$$L\{f(t)\} = e^{\frac{-\pi}{3}p} G(p)$$
$$= e^{\frac{-\pi}{3}p} \cdot \frac{1}{p^2+1}.$$

Example 10.5 **Find the Laplace transform of $\frac{\sin at}{t}$. Hence show that**

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

[WBUT 2005, 2007]

Sol. Let $f(t) = \sin at$. Then by **Art. 10.4.6** we know

$$L\{f(t)\} = L\{\sin at\} = \frac{a}{p^2+a^2} = F(p)$$

Again by **Theorem 10.8**, we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(p)\, dp$$

so, $$L\left\{\frac{\sin at}{t}\right\} = \int_p^\infty \frac{a}{p^2+a^2}\, dp = a \cdot \frac{1}{a}\left[\tan^{-1}\frac{p}{a}\right]_p^\infty$$

i.e., $$L\left\{\frac{\sin at}{t}\right\} = \frac{\pi}{2} - \tan^{-1}\frac{p}{a}$$

Putting $a = 1$, we have

$$L\left\{\frac{\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1} p$$

Therefore, using definition we can write

$$\int_0^\infty e^{-pt} \cdot \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2} - \tan^{-1} p$$

Taking limit as $p \to 0$, we have

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

Example 10.6 (Dirac Delta function)
Let a function $\delta_\varepsilon(t)$ be defined as

$$\delta_\varepsilon(t) = \left\{\begin{matrix} 0, \ t < 0 \\ \frac{1}{\varepsilon}, \ 0 < t < \varepsilon \\ 0, \ t > \varepsilon \end{matrix}\right\}$$

and

$$\lim_{\varepsilon \to 0} \delta_\varepsilon(t) = \delta(t).$$

Prove that $L\{\delta(t)\} = 1$.

Sol. By definition, we have

$$\begin{aligned} L\{\delta_\varepsilon(t)\} &= \int_0^\infty e^{-pt} \cdot \delta_\varepsilon(t)\, dt \\ &= \int_0^\varepsilon e^{-pt} \cdot \frac{1}{\varepsilon}\, dt + \int_\varepsilon^\infty e^{-pt} \cdot 0\, dt \\ &= \frac{1}{\varepsilon}\left[\frac{e^{-pt}}{-p}\right]_0^\varepsilon + 0 = \frac{1 - e^{-p\varepsilon}}{p\varepsilon}. \end{aligned}$$

Therefore,

$$\int_0^\infty e^{-pt} \cdot \delta_\varepsilon(t)\, dt = \frac{1 - e^{-p\varepsilon}}{p\varepsilon}$$

Now taking $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0}\left(\int_0^\infty e^{-pt}\cdot\delta_\varepsilon(t)\,dt\right)=\lim_{\varepsilon\to 0}\left(\frac{1-e^{-p\varepsilon}}{p\varepsilon}\right)$$

i.e.,
$$\int_0^\infty e^{-pt}\cdot\left(\lim_{\varepsilon\to 0}\delta_\varepsilon(t)\right)\,dt=\frac{1}{p}\lim_{\varepsilon\to 0}\left(\frac{1-e^{-p\varepsilon}}{\varepsilon}\right)$$

$$\left[\text{Indeterminate form of } \frac{0}{0}\right]$$

So, applying L'Hospital's rule on RHS, we obtain

$$\int_0^\infty e^{-pt}\cdot\delta(t)\,dt=\frac{1}{p}\lim_{\varepsilon\to 0}\left(\frac{p\cdot e^{-p\varepsilon}}{1}\right)$$

i.e.,
$$L\{\delta(t)\}=\frac{1}{p}\cdot p=1.$$

Example 10.7 **Express the following function**

$$F(t)=\begin{Bmatrix} e^{-t}, & 0<t<2\\ 0, & t\geq 2\end{Bmatrix}$$

in terms of unit step function and hence find $L\{F(t)\}$. [WBUT 2004]

Sol. It is given that

$$F(t)=\begin{Bmatrix} e^{-t}, & 0<t<2\\ 0, & t\geq 2\end{Bmatrix}$$

Then $F(t)$ can be expressed in terms of unit step function as the following form (see **Theorem 10.10**).

$$F(t)=F_1(t)+\{F_2(t)-F_1(t)\}\cdot H(t-2)$$

where $F_1(t)=e^{-t}$, $F_2(t)=0$ and $H(t-2)$ is the unit step function. Therefore,

$$\begin{aligned}F(t)&=e^{-t}+\left\{0-e^{-t}\right\}\cdot H(t-2)\\&=e^{-t}-e^{-t}\cdot H(t-2).\end{aligned}$$

Now,

$$\begin{aligned}L\{F(t)\}&=L\left\{e^{-t}-e^{-t}\cdot H(t-2)\right\}\\&=L\left\{e^{-t}\right\}-L\left\{e^{-t}\cdot H(t-2)\right\}\end{aligned}\tag{1}$$

Again,

$$L\left\{e^{-t}\right\} = \frac{1}{p+1} \tag{2}$$

and by **Theorem 10.9** we have $L\{H(t-2)\} = \dfrac{e^{-2p}}{p} = f(p)$ (say).
Then by first shifting theorem,

$$L\left\{e^{-t} \cdot H(t-2)\right\} = f(p+1) = \frac{e^{-2(p+1)}}{p+1} \tag{3}$$

Using (2) and (3) in (1), we obtain

$$L\{F(t)\} = \frac{1}{p+1} - \frac{e^{-2(p+1)}}{p+1} = \frac{1-e^{-2(p+1)}}{p+1}.$$

Example 10.8 **Evaluate**

$$L\left\{\sin\sqrt{t}\right\}$$

Sol. By Taylor's series expansion, we know that

$$\sin\sqrt{t} = \sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3!} + \frac{\left(\sqrt{t}\right)^5}{5!} - \frac{\left(\sqrt{t}\right)^7}{7!} + \cdots\infty$$

Therefore,

$$\begin{aligned}
L\left\{\sin\sqrt{t}\right\} &= L\left\{\sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3!} + \frac{\left(\sqrt{t}\right)^5}{5!} - \frac{\left(\sqrt{t}\right)^7}{7!} + \cdots\infty\right\} \\
&= L\left\{\sqrt{t}\right\} - \frac{1}{3!}L\left\{\left(\sqrt{t}\right)^3\right\} + \frac{1}{5!}L\left\{\left(\sqrt{t}\right)^5\right\} - \frac{1}{7!}L\left\{\left(\sqrt{t}\right)^7\right\} + \cdots\infty \\
&= \frac{\Gamma\left(\frac{1}{2}+1\right)}{p^{\frac{1}{2}+1}} - \frac{1}{3!}\frac{\Gamma\left(\frac{3}{2}+1\right)}{p^{\frac{3}{2}+1}} + \frac{1}{5!}\frac{\Gamma\left(\frac{5}{2}+1\right)}{p^{\frac{5}{2}+1}} - \frac{1}{7!}\frac{\Gamma\left(\frac{7}{2}+1\right)}{p^{\frac{7}{2}+1}} + \cdots\infty \\
&= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{p^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{p^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}{p^{\frac{7}{2}}} - \frac{1}{7!}\frac{\frac{7}{2}\Gamma\left(\frac{7}{2}\right)}{p^{\frac{9}{2}}} + \cdots\infty \\
&= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{p^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{p^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{p^{\frac{7}{2}}} - \frac{1}{7!}\frac{\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{p^{\frac{9}{2}}} + \cdots\infty
\end{aligned}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{p^{\frac{3}{2}}}\left\{1 - \left(\frac{1}{4p}\right) + \frac{1}{2!}\left(\frac{1}{4p}\right)^2 - \frac{1}{3!}\left(\frac{1}{4p}\right)^3 + \cdots\infty\right\}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{p^{\frac{3}{2}}} e^{-\frac{1}{4p}}$$

$$= \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}} e^{-\frac{1}{4p}} \quad \text{since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 10.9 **Evaluate**

$$L\{\cosh^2 2t\}$$

Sol. Here,

$$L\{\cosh^2 2t\}$$

$$= \frac{1}{2} L\{\cosh^2 2t\}$$

$$= \frac{1}{2} L\{1 + \cosh 4t\}$$

$$= \frac{1}{2} L\{1\} + \frac{1}{2} L\{\cosh 4t\}$$

$$= \frac{1}{2}\frac{1}{p} + \frac{1}{2}\frac{p}{p^2 - 16}$$

$$= \frac{1}{2}\frac{p^2 - 16 + p^2}{p(p^2 - 16)}$$

$$= \frac{(p^2 - 8)}{p(p^2 - 16)}$$

Example 10.10 **Prove that if**

$$L\{F(t) = \frac{1}{p} e^{-\frac{1}{p}}$$

then

$$L\{e^{-t} F(3t)\} = \frac{e^{-\frac{3}{p}+1}}{p + 1}$$

Sol. Let,

$$L\{F(t) = \frac{1}{p} e^{-\frac{1}{p}} = f(p)$$

By change of scale property, we have

$$L\{F(3t)\} = \frac{1}{3} f(\frac{p}{3}) = \frac{1}{3}\frac{3}{p} e^{-\frac{3}{p}} = \frac{1}{p} e^{-\frac{3}{p}}$$

By first shifting theorem, we have

$$L\{e^{-t} F(3t)\} = \frac{1}{p+1} e^{-\frac{3}{p+1}}$$

Example 10.11 **Show that** $\int_0^\infty e^{-t} \frac{\sin t}{t} dt = \frac{\pi}{4}$

Sol. We know,

$$L\{\sin t\} = \frac{1}{p^2+1}$$

By division by t property, we have if $L\{f(t)\} = F(p)$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(p) dp$$

Therefore,

$$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty \frac{1}{p^2+1} dp = \left[\tan^{-1} p\right]_0^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} p = \frac{\pi}{2} - \tan^{-1} p \quad (1)$$

By definition of Laplace transform, we have

$$L\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-pt} \frac{\sin t}{t} dt \quad (2)$$

Therefore, from 1 and 2 we have

$$\int_0^\infty e^{-pt} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1} p$$

Putting $p = 1$, we have

$$\int_0^\infty e^{-t} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Example 10.12 **Using Laplace transform prove that**

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt = \log\left(\frac{4}{6}\right)$$

Sol. We know,

$$L\{\cos 6t - \cos 4t\} = L\{\cos 6t\} - L\{\cos 4t\}$$

$$= \frac{p}{p^2+36} - \frac{p}{p^2+16}$$

By division by t property, we have if $L\{f(t)\} = F(p)$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(p)dp$$

Therefore,

$$L\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \int_p^\infty \left\{\frac{p}{p^2+36} - \frac{p}{p^2+16}\right\} dp$$

$$= \int_p^\infty \frac{p}{p^2+36} dp - \int_p^\infty \frac{p}{p^2+16} dp$$

$$= \frac{1}{2}\log\left[\frac{p^2+36}{p^2+16}\right]_p^\infty$$

$$= \frac{1}{2}\lim_{p\to\infty}\log\left[\frac{p^2+36}{p^2+16}\right] - \frac{1}{2}\log\left[\frac{p^2+36}{p^2+16}\right]$$

$$= -\frac{1}{2}\log\left[\frac{p^2+36}{p^2+16}\right]$$

$$= \frac{1}{2}\log\left[\frac{p^2+16}{p^2+36}\right]$$

$$L\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \frac{1}{2}\log\left[\frac{p^2+16}{p^2+36}\right] \quad (1)$$

By definition of Laplace transform, we have

$$L\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \int_0^\infty e^{-pt}\frac{\sin t}{t} dt \quad (2)$$

Therefore, from 1 and 2 we have

$$\int_0^\infty e^{-pt}\frac{\cos 6t - \cos 4t}{t}dt = \frac{1}{2}\log\left[\frac{p^2+16}{p^2+36}\right]$$

Putting $p = 0$, we have

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}dt = \frac{1}{2}\log\left[\frac{16}{36}\right] = \log\frac{4}{6}$$

Example 10.13 **Express the following function in terms of unit step function**

$$\begin{aligned} f(t) &= 0, t < 0 \\ &= (t-1), 1 < t < 2 \\ &= (3-t), 2 < t < 3 \\ &= 0, t > 3 \end{aligned}$$

and find its Laplace transform.

Sol. Here

$$\begin{aligned} f(t) = f_1(t) &= 0, t < 0 \\ = f_2(t) &= (t-1), 1 < t < 2 \\ = f_3(t) &= (3-t), 2 < t < 3 \\ = f_4(t) &= 0, t > 3 \end{aligned}$$

In terms of unit step function,

$$\begin{aligned} f(t) &= f_1(t) + \{f_2(t) - f_1(t)\}H(t-1) + \{f_3(t) - f_2(t)\}H(t-2) \\ &\quad + \{f_4(t) - f_3(t)\}H(t-3) \\ &= 0 + \{t - 1 - 0\}H(t-1) + \{3 - t - t + 1\}H(t-2) \\ &\quad + \{0 - 3 + t\}H(t-3) \\ &= (t-1)H(t-1) - 2(t-2)H(t-2) + (t-3)H(t-3) \end{aligned}$$

Therefore,

$$\begin{aligned} L\{f(t) &= L\{(t-1)H(t-1) - 2(t-2)H(t-2) + (t-3)H(t-3)\} \\ &= L\{(t-1)H(t-1)\} - 2L\{(t-2)H(t-2)\} \\ &\quad + L\{(t-3)H(t-3)\} \end{aligned}$$

$$= e^{-p}L\{t\} - 2e^{-2p}L\{t\} + e^{-3p}L\{t\},$$

since, $L\{f(t-a)H(t-a) = e^{-ap}L\{f(t)\}$

$$= (e^{-p} - 2e^{-2p} + e^{-3p})L\{t\}$$

$$= \frac{(e^{-p} - 2e^{-2p} + e^{-3p})}{p^2}$$

Example 10.14 **Evaluate**

$$L\{\cosh t \int_0^t e^x \cosh x dx\}$$

Sol. We have,

$$\int_0^t e^x \cosh x dx = \int_0^t e^x \frac{(e^x + e^{-x})}{2} dx$$

$$= \frac{1}{2} \int_0^t (e^{2x} + 1) dx$$

$$= \frac{1}{2}\left(\frac{e^{2t}}{2} + t - \frac{1}{2}\right)$$

Therefore,

$$L\{\cosh t \int_0^t e^x \cosh x dx\} = L\left\{\frac{1}{2}\cosh t\left(\frac{e^{2t}}{2} + t - \frac{1}{2}\right)\right\}$$

$$= \frac{1}{4}L\left\{e^{2t}\cosh t\right\} + \frac{1}{2}L\{t\cosh t\} - \frac{1}{4}L\{\cosh t\}$$

$$= \frac{1}{4}\frac{(p-2)}{(p-2)^2 - 1} + \frac{1}{2}(-1)\frac{d}{dp}\left(\frac{p}{p^2-1}\right) - \frac{1}{4}\left(\frac{p}{p^2-1}\right)$$

$$= \frac{1}{4}\frac{(p-2)}{p^2 - 4p + 3} + \frac{p^2+1}{2(p^2-1)^2} - \frac{1}{4}\left(\frac{p}{p^2-1}\right)$$

Example 10.15 **If** $Lf(t) = F(p)$ **then prove that**

a) $L\{f(t)\sin at\} = \dfrac{1}{2i}\{F(p-ai) - F(p+ai)\}$

b) $L\{f(t)\cos at\} = \frac{1}{2}\{F(p-ai)+F(p+ai)\}$

Sol.

a) We know,

$$\sin at = \frac{1}{2i}(e^{ait} - e^{-ait})$$

Therefore,

$$\begin{aligned} L\{f(t)\sin at\} &= L\{f(t)\frac{1}{2i}(e^{ait} - e^{-ait})\} \\ &= \frac{1}{2i}[L\{e^{ait}f(t)\} - L\{e^{-ait}f(t)\}] \\ &= \frac{1}{2i}\{F(p-ai)+F(p+ai)\} \end{aligned}$$

b) We know

$$\cos at = \frac{1}{2}(e^{ait} + e^{-ait})$$

Therefore,

$$\begin{aligned} L\{f(t)\cos at\} &= L\{f(t)\frac{1}{2}(e^{ait} + e^{-ait})\} \\ &= \frac{1}{2}[L\{e^{ait}f(t)\} + L\{e^{-ait}f(t)\}] \\ &= \frac{1}{2}\{F(p-ai)+F(p+ai)\} \end{aligned}$$

EXERCISES

Short and Long Answer Type Questions

1) From definition find the Laplace transform of the following functions:

   a) $F(t) = \sin t, 0 < t < \pi$
      $= 0, t > \pi$

$$\left[\textbf{Ans: } \frac{e^{-p\pi}+1}{s^2+1}\right]$$

b) $F(t) = 0, 0 < t < 1$
$= t, 1 < t \le 2$
$= 0, t > 2$

$$\left[\textbf{Ans: } \left(\frac{1}{p} + \frac{1}{p^2}\right) e^{-p} - \left(\frac{1}{p^2} + \frac{2}{p}\right) e^{-2p}\right]$$

c) $F(t) = e^t, 0 < t < 5$
$= 3, t > 5$

$$\left[\textbf{Ans: } \frac{3}{5} e^{-5p} + \frac{1 - e^{-5(p-1)}}{p - 1}\right]$$

d) $F(t) = (t - 1)^2, t > 1$
$= 0, 0 < t < 1$

$$\left[\textbf{Ans: } \frac{2e^{-p}}{p^3}\right]$$

2) Evaluate

a) $L\{\sin(at + b)\}$

$$\left[\textbf{Ans: } \frac{a \cos b + p \sin b}{p^2 + a^2}\right]$$

b) $L\{e^{2t} - 1)^2\}$

$$\left[\textbf{Ans: } \frac{1}{p - 4} - \frac{2}{p - 2} + \frac{1}{p}\right]$$

c) $L\{t^3 e^{-3t}\}$

$$\left[\textbf{Ans: } \frac{6}{(p + 3)^4}\right]$$

d) $L\{e^{-t} \sin^2 t\}$

$$\left[\textbf{Ans: } \frac{2}{(p - 1)(p^2 - 2p + 5)}\right]$$

e) $L\{e^{-4t} \cosh 2t\}$

$$\left[\textbf{Ans: } \frac{p + 4}{p^2 + 8p + 12}\right]$$

f) $L\{t^2 e^{-2t} \cos t\}$

$$\left[\textbf{Ans: } \frac{2(p^3 + 10p^2 + 25p + 22)}{(p^2 + 4p + 5)^3}\right]$$

g) $L\{te^{-t}\cosh t\}$

$$\left[\textbf{Ans: } \frac{p^2+2p+2}{(p^2+2p)^2}\right]$$

h) $L\left\{\dfrac{\cos at - \cos bt}{t}\right\}$

$$\left[\textbf{Ans: } -\frac{1}{2}[\log(\frac{p^2+a^2}{p^2+b^2})\right]$$

i) $L\left\{\dfrac{e^{-t}\sin t}{t}\right\}$

$$\left[\textbf{Ans: } \cot^{-1}(p+1)\right]$$

j) $L\{(1+te^{-t})^3\}$

$$\left[\textbf{Ans: } \frac{1}{p}+\frac{3}{(p+1)^2}+\frac{6}{(p+2)^3}+\frac{6}{(p+4)^4}\right]$$

3) Express the following functions in terms of unit step function and obtain their Laplace transforms

a) $F(t) = \cos t, 0 < t < \pi$

$= \cos 2t, \pi < t < 2\pi$

$= \cos 3t, t > 2\pi$

$$\left[\textbf{Ans: } \frac{p}{p^2+1}+\frac{pe^{-\pi p}}{p^2+4}+\frac{pe^{-\pi p}}{p^2+1}+\frac{pe^{-2\pi p}}{p^2+9}-\frac{pe^{-2\pi p}}{p^2+4}\right]$$

b) $F(t) = 4, 0 < t < 1$

$= -2, 1 < t < 3$

$= 5, t > 3$

4) Using Laplace transforms evaluate the following improper integrals

a) $\int_0^\infty te^{-3t}\sin t\,dt$

$$\left[\textbf{Ans: } \frac{3}{50}\right]$$

b) $\int_0^\infty te^{t}\cot s\,dt$

[**Ans**: 0]

c) $\int_0^\infty e^{-t}\frac{\sin t}{t}dt$

$\left[\textbf{Ans: } \frac{\pi}{4}\right]$

Multiple Choice Questions

1) $L\{\sin 3t \sin 2t\} =$

a) $\frac{p}{(p^2+1)(p^2+25)}$ b) $\frac{12p}{(p^2+1)(p^2+25)}$

c) $\frac{12p}{(p^2+1)^2}$ d) none of these

2) $L\{\cos^3 t\} =$

a) $\frac{(p^2+5)}{(p^2+1)(p^2+9)}$ b) $\frac{(p^2+5)}{2(p^2+1)(p^2+9)}$

c) $\frac{p(p^2+5)}{2(p^2+1)(p^2+9)}$ d) none of these

3) $L\{f(t) =$

a) $\int_0^\infty e^{pt}f(t)dt$ b) $\int_0^\infty e^{-pt}f(t)dt$

c) $\int_{-\infty}^\infty e^{pt}f(t)dt$ d) $\int_{-\infty}^\infty e^{-pt}f(t)dt$

4) $L\{3t+5\} =$

a) $\frac{3}{p^2}+5$ b) $\frac{3}{p^2}+\frac{5}{p}$ c) $\frac{6}{p}$ d) $\frac{8}{p^2}$

5) If $L\{f(t)\} = F(p)$ then $L\{e^{at}f(t)\} =$

a) $e^{ap}F(p)$ b) $pF(p)$ c) $F(p+a)$ d) $F(p-a)$

6) $L\{e^{-3t}\sin 4t\} =$

a) $\frac{4}{p^2+6p-7}$ b) $\frac{p}{p^2+6p-7}$ c) $\frac{1}{p^2+6p-7}$ d) $\frac{p}{p^2+6p+25}$

7) If $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right)$, then $L\left\{\frac{\sin at}{t}\right\} =$

a) $\tan^{-1}\left(\frac{1}{p^2}\right)$ b) $\tan^{-1}\left(\frac{a}{p}\right)$ c) $\tan^{-1}\left(\frac{1}{ap}\right)$ d) $\frac{1}{a}\tan^{-1}\left(\frac{p}{a}\right)$

Answers:

1 (b) 2 (c) 3 (b) 4 (b) 5 (d) 6 (d) 7 (b)

CHAPTER

11

Inverse Laplace Transform

11.1 INTRODUCTION

In this chapter we deal with the Inverse Laplace Transform. We give different properties of the Inverse Laplace Transform illustrated with various kinds of examples. Here we include different techniques of finding Inverse Laplace Transform such as partial fraction, convolution, etc. At the end of the chapter, solutions of university examination problems have been included.

11.2 DEFINITION AND UNIQUENESS OF INVERSE LAPLACE TRANSFORM

11.2.1 Definition

If $L\{f(t)\} = F(p)$, then $f(t)$ is called the Inverse Laplace Transform of $F(p)$ and denoted by

$$\boxed{f(t) = L^{-1}\{F(p)\}}$$

11.2.2 Null Function

Null function $N(t)$ is such a function which satisfies the condition

$$\boxed{\int_0^t N(t)dt = 0 \text{ for all } t > 0}$$

11.2.3 Lerch's Theorem on Uniqueness

If $L\{f_1(t)\} = L\{f_2(t)\}$, then the functions

$$\boxed{f_1(t) - f_2(t) = N(t)}$$

where $N(t)$ is null function for all $t > 0$

Remark: **In other words the above theorem states, that Inverse Laplace Transform is always unique.**

11.3 LINEARITY PROPERTY OF INVERSE LAPLACE TRANSFORM

If $F_1(p)$ and $F_2(p)$ are the Laplace Transforms of $f_1(t)$ and $f_2(t)$ i.e., if $L\{f_1(t)\} = F_1(p)$ and $L\{f_2(t)\} = F_2(p)$ then

$$\boxed{L^{-1}\{c_1F_1(p) \pm c_2F_2(p)\} = c_1L^{-1}\{F_1(p)\} \pm c_2L^{-1}\{F_2(p)\} = c_1f_1(t) \pm c_2f_2(t)}$$

where c_1, c_2 are arbitrary constants.

11.3.1 Table of Inverse Laplace Transform

(1) $L\{1\} = \dfrac{1}{p} \Rightarrow L^{-1}\left\{\dfrac{1}{p}\right\} = 1$

(2) $L\{t^n\} = \dfrac{n!}{p^{n+1}} \Rightarrow L^{-1}\left\{\dfrac{n!}{p^{n+1}}\right\} = t^n \Rightarrow L^{-1}\left\{\dfrac{1}{p^{n+1}}\right\} = \dfrac{t^n}{n!}$,
where n is a positive integer.

(3) $L\{t^n\} = \dfrac{\Gamma(n+1)}{p^{n+1}} \Rightarrow L^{-1}\left\{\dfrac{\Gamma(n+1)}{p^{n+1}}\right\} = t^n \Rightarrow L^{-1}\left\{\dfrac{1}{p^{n+1}}\right\} = \dfrac{t^n}{\Gamma(n+1)}$,
$n > -1$

(4) $L\{e^{at}\} = \dfrac{1}{p-a} \Rightarrow L^{-1}\left\{\dfrac{1}{p-a}\right\} = e^{at}$

(5) $L\{\sin at\} = \dfrac{a}{p^2+a^2} \Rightarrow L^{-1}\left\{\dfrac{a}{p^2+a^2}\right\} = \sin at \Rightarrow L^{-1}\left\{\dfrac{1}{p^2+a^2}\right\} = \dfrac{\sin at}{a}$

(6) $L\{\cos at\} = \dfrac{p}{p^2+a^2} \Rightarrow L^{-1}\left\{\dfrac{p}{p^2+a^2}\right\} = \cos at$

(7) $L\{\sinh at\} = \dfrac{a}{p^2-a^2} \Rightarrow L^{-1}\left\{\dfrac{a}{p^2-a^2}\right\} = \sinh at \Rightarrow L^{-1}\left\{\dfrac{1}{p^2-a^2}\right\} = \dfrac{\sinh at}{a}$

(8) $L\{\cosh at\} = \dfrac{p}{p^2 - a^2} \Rightarrow L^{-1}\left\{\dfrac{p}{p^2 - a^2}\right\} = \cosh at$

Example 1 **Find** $L^{-1}\left\{\dfrac{3}{p^2} + \dfrac{4}{2p-1} + \dfrac{2+3p}{4p^2+1}\right\}$

Sol.

$$
\begin{aligned}
&L^{-1}\left\{\frac{3}{p^3} + \frac{4}{2p-1} + \frac{2+3p}{4p^2+1}\right\} \\
&= 3L^{-1}\left\{\frac{1}{p^3}\right\} + 4L^{-1}\left\{\frac{1}{2p-1}\right\} + L^{-1}\left\{\frac{2+3p}{4p^2+1}\right\} \\
&= 3L^{-1}\left\{\frac{1}{p^{2+1}}\right\} + 4\cdot\frac{1}{2}L^{-1}\left\{\frac{1}{\left(p-\frac{1}{2}\right)}\right\} + 2L^{-1}\left\{\frac{1}{4p^2+1}\right\} + 3L^{-1}\left\{\frac{p}{4p^2+1}\right\} \\
&= 3\cdot\frac{t^2}{2!} + 2\cdot e^{\frac{1}{2}t} + 2\cdot\frac{1}{4}L^{-1}\left\{\frac{1}{p^2+\left(\frac{1}{2}\right)^2}\right\} + 3\cdot\frac{1}{4}L^{-1}\left\{\frac{p}{p^2+\left(\frac{1}{2}\right)^2}\right\} \\
&= \frac{3t^2}{2} + 2\cdot e^{\frac{1}{2}t} + \frac{1}{2}\cdot\frac{\sin\left(\frac{1}{2}t\right)}{\frac{1}{2}} + \frac{3}{4}\cos\left(\frac{1}{2}t\right) \\
&= \frac{3t^2}{2} + 2\cdot e^{\frac{1}{2}t} + \sin\left(\frac{1}{2}t\right) + \frac{3}{4}\cos\left(\frac{1}{2}t\right).
\end{aligned}
$$

Example 2 **Find** $L^{-1}\left\{\dfrac{3+4p}{9p^2-4}\right\}$

Sol.

$$
\begin{aligned}
&L^{-1}\left\{\frac{3+4p}{9p^2-4}\right\} \\
&= L^{-1}\left\{\frac{3}{9p^2-4}\right\} + L^{-1}\left\{\frac{4p}{9p^2-4}\right\} \\
&= 3\cdot\frac{1}{9}L^{-1}\left\{\frac{1}{p^2-\left(\frac{2}{3}\right)^2}\right\} + 4\cdot\frac{1}{9}L^{-1}\left\{\frac{p}{p^2-\left(\frac{2}{3}\right)^2}\right\} \\
&= \frac{1}{3}\cdot\frac{\sinh\left(\frac{2}{3}t\right)}{\frac{2}{3}} + \frac{4}{9}\cos\left(\frac{2}{3}t\right) \\
&= \frac{1}{2}\sinh\left(\frac{2}{3}t\right) + \frac{4}{9}\cos\left(\frac{2}{3}t\right).
\end{aligned}
$$

Example 3 **Evaluate** $L^{-1}\left\{\dfrac{2p+3}{(p-1)\left(p^2+1\right)}\right\}$

Sol. Here, we apply the method of partial fraction to compute the given inverse Laplace transform.

Suppose

$$\frac{2p+3}{(p-1)\left(p^2+1\right)} = \frac{A}{(p-1)} + \frac{Bp+C}{\left(p^2+1\right)}$$

$$= \frac{A\left(p^2+1\right)+(Bp+C)(p-1)}{(p-1)\left(p^2+1\right)}$$

$$= \frac{(A+B)\,p^2+(-B+C)\,p+A-C}{(p-1)\left(p^2+1\right)}$$

Equating the coefficients of like powers of p in the numerator of both the sides, we get

$$A+B=0,\ \ -B+C=2,\ A-C=3$$

Now,

$$\begin{bmatrix} A+B=0 \\ -B+C=2 \end{bmatrix} \Rightarrow A+C=2$$

$$\begin{bmatrix} A+C=2 \\ A-C=3 \end{bmatrix} \Rightarrow A=\frac{5}{2},\ C=\frac{-1}{2}$$

Putting,

$$A=\frac{5}{2}\ \text{in}\ A+B=0,\ \text{we get}\ B=\frac{-5}{2}.$$

So,

$$\frac{2p+3}{(p-1)\left(p^2+1\right)} = \frac{\left(\frac{5}{2}\right)}{(p-1)} + \frac{\left(\frac{-5}{2}\right)p+\left(\frac{-1}{2}\right)}{\left(p^2+1\right)}$$

Therefore,

$$L^{-1}\left\{\frac{2p+3}{(p-1)\left(p^2+1\right)}\right\}$$

$$= L^{-1}\left\{\frac{\left(\frac{5}{2}\right)}{(p-1)} + \frac{\left(\frac{-5}{2}\right)p+\left(\frac{-1}{2}\right)}{\left(p^2+1\right)}\right\}$$

$$= \frac{5}{2} \cdot L^{-1}\left\{\frac{1}{(p-1)}\right\} + L^{-1}\left\{\frac{\left(\frac{-5}{2}\right)p}{(p^2+1)} + \frac{\left(\frac{-1}{2}\right)}{(p^2+1)}\right\}$$

$$= \frac{5}{2} \cdot e^t + \left(\frac{-5}{2}\right) L^{-1}\left\{\frac{p}{(p^2+1)}\right\} + \left(\frac{-1}{2}\right) \cdot L^{-1}\left\{\frac{1}{(p^2+1)}\right\}$$

$$= \frac{5}{2} \cdot e^t - \frac{5}{2}\cos t + \frac{1}{2}\sin t$$

11.4 FIRST AND SECOND SHIFTING PROPERTIES

11.4.1 First Shifting (or Translation) Theorem

Theorem 11.1 **If** $L^{-1}\{F(p)\} = f(t)$ **then**

$$\boxed{L^{-1}\{F(p-a)\} = e^{at} f(t) = e^{at} L^{-1}\{F(p)\}}$$

Proof From the definition we have,

$$F(p) = L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

Now,

$$F(p-a) = \int_0^\infty e^{-(p-a)t} f(t) dt = \int_0^\infty e^{-pt}\left\{e^{at} f(t)\right\} dt = L\left\{e^{at} f(t)\right\}$$

i.e., $$L^{-1}\{F(p-a)\} = e^{at} f(t) = e^{at} L^{-1}\{F(p)\}$$

This proves the theorem.

Example 4 **Evaluate** $L^{-1}\left\{\frac{p+1}{(p^2+4p+13)}\right\}$

Sol.

$$L^{-1}\left\{\frac{p+1}{(p^2+4p+13)}\right\}$$

$$= L^{-1}\left\{\frac{(p+2)-1}{(p+2)^2+3^2}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{p-1}{p^2+3^2}\right\}, \text{ since } L^{-1}\{F(p-a)\} = e^{at} L^{-1}\{F(p)\}$$

$$= e^{-2t}\left[L^{-1}\left\{\frac{p}{p^2+3^2}\right\} - L^{-1}\left\{\frac{1}{p^2+3^2}\right\}\right]$$

$$= e^{-2t}\left[\cos 3t - \frac{\sin 3t}{3}\right]$$

Example 5 **Evaluate** $L^{-1}\left\{\frac{p}{(p+1)^{\frac{3}{2}}}\right\}$

Sol.

$$L^{-1}\left\{\frac{p}{(p-1)^{\frac{5}{2}}}\right\}$$

$$= L^{-1}\left\{\frac{(p-1)+1}{(p+1)^{\frac{5}{2}}}\right\}$$

$$= e^t L^{-1}\left\{\frac{p+1}{p^{\frac{5}{2}}}\right\},\ \text{since } L^{-1}\{F(p-a)\} = e^{at}L^{-1}\{F(p)\}$$

$$= e^t\left[L^{-1}\left\{\frac{1}{p^{\frac{3}{2}}}\right\} + L^{-1}\left\{\frac{1}{p^{\frac{5}{2}}}\right\}\right]$$

$$= e^t\left[\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} + \frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\right]$$

$$= e^t\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\right]$$

$$= e^t\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}} + \frac{t^{\frac{3}{2}}}{\frac{3}{4}\sqrt{\pi}}\right]$$

$$= \frac{2e^t}{\sqrt{\pi}}\left[t^{\frac{1}{2}} + \frac{2\cdot t^{\frac{3}{2}}}{3}\right]$$

11.4.2 Second Shifting Theorem

Theorem 11.2 **If** $L^{-1}\{F(p)\} = f(t)$ **then,**

$$L^{-1}\left\{e^{-ap}F(p)\right\} = g(t),\ \textbf{where}\ g(t) = \begin{Bmatrix} f(t-a), & t > a \\ 0, & t < a \end{Bmatrix}$$

Alternative statement (using Unit Step Function)

If $L^{-1}\{F(p)\} = f(t)$ **then,**

$$L^{-1}\left\{e^{-ap}F(p)\right\} = f(t-a) \cdot H(t-a)$$

where $H(t-a)$ **is unit step function.**

Proof By definition, we have

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt$$

i.e.,

$$e^{-ap}F(p) = \int_0^\infty e^{-p(t+a)} f(t)\, dt$$

Putting, $t + a = x$, $dt = dx$ we obtain from above,

$$\begin{aligned} e^{-ap}F(p) &= \int_a^\infty e^{-px} f(x-a)\, dx \\ &= \int_0^a e^{-px} \cdot 0\, dx + \int_a^\infty e^{-px} f(x-a)\, dx \\ &= \int_0^a e^{-pt} \cdot 0\, dt + \int_a^\infty e^{-pt} f(t-a)\, dt \text{ (Replacing } x \text{ by } t) \\ &= \int_0^\infty e^{-pt} g(t)\, dt = L\{g(t)\} \end{aligned}$$

Therefore,

$$L^{-1}\left\{e^{-ap}F(p)\right\} = g(t),\ \text{ where } g(t) = \begin{Bmatrix} f(t-a), & t > a \\ 0, & t < a \end{Bmatrix}$$

Example 6 **Evaluate** $L^{-1}\left\{\dfrac{(p+1)\,e^{-2p}}{p^2+p+1}\right\}$

Sol. First we find,

$$L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\}$$

$$= L^{-1}\left\{\frac{\left(p+\frac{1}{2}\right)-\frac{1}{2}}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{\frac{-1}{2}t}L^{-1}\left\{\frac{p-\frac{1}{2}}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}, \text{ since } L^{-1}\{F(p-a)\} = e^{at}L^{-1}\{F(p)\}$$

$$= e^{\frac{-1}{2}t}\left[L^{-1}\left\{\frac{p}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= e^{\frac{-1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right]$$

By second shifting theorem, we have

$$L^{-1}\left\{e^{-ap}F(p)\right\} = f(t-a)\cdot H(t-a)$$

Therefore,

$$L^{-1}\left\{e^{-2p}\cdot\frac{(p+1)}{p^2+p+1}\right\}$$

$$= e^{\frac{-1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$= e^{\frac{-1}{2}(t-2)}\left[\cos\left\{\frac{\sqrt{3}}{2}(t-2)\right\} - \frac{1}{\sqrt{3}}\sin\left\{\frac{\sqrt{3}}{2}(t-2)\right\}\right]\cdot H(t-2)$$

where,

$$H(t-2) = \begin{Bmatrix}1, & t>2\\ 0, & t<2\end{Bmatrix}$$

11.5 CHANGE OF SCALE PROPERTY

Theorem 11.3 **If $L^{-1}\{F(p)\} = f(t)$ then,**

$$L^{-1}\{F(ap)\} = \frac{1}{a} \cdot f\left(\frac{t}{a}\right) \text{ for non zero constant } a.$$

[WBUT 2002]

Proof By the definition we have,

$$F(p) = L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

So,

$$F(ap) = \int_0^\infty e^{-apt} f(t)\, dt$$

$$= \int_0^\infty e^{-apt} f(t)\, dt \text{ (putting } at = u,\ a \cdot dt = du)$$

$$= \frac{1}{a} \int_0^\infty e^{-pu} f\left(\frac{u}{a}\right) du$$

$$= \frac{1}{a} \int_0^\infty e^{-pt} f\left(\frac{t}{a}\right) dt \text{ (Replacing } u \text{ by } t)$$

$$= \frac{1}{a} \cdot L\left\{f\left(\frac{t}{a}\right)\right\} = L\left\{\frac{1}{a} \cdot f\left(\frac{t}{a}\right)\right\}$$

Hence,

$$L^{-1}\{F(ap)\} = \frac{1}{a} \cdot f\left(\frac{t}{a}\right)$$

Alternative Proof Since, $L^{-1}\{F(p)\} = f(t)$, we have $F(p) = L\{f(t)\}$
Now,

$$L\left\{\frac{1}{a} \cdot f\left(\frac{t}{a}\right)\right\} = \frac{1}{a} \cdot L\left\{f\left(\frac{1}{a} \cdot t\right)\right\} \quad (1)$$

Using Change of scale property of LT **(see Art. 10.6 of Ch. 10),** we obtain

$$L\left\{f\left(\frac{1}{a} \cdot t\right)\right\} = \frac{1}{\frac{1}{a}} \cdot F\left(\frac{p}{\frac{1}{a}}\right) = a \cdot F(ap)$$

Therefore, from (1) we have

$$L\left\{\frac{1}{a}\cdot f\left(\frac{t}{a}\right)\right\}=\frac{1}{a}\cdot a\cdot F\left(ap\right)=F\left(ap\right)$$

i.e.,
$$L^{-1}\left\{F(ap)\right\}=\frac{1}{a}\cdot f\left(\frac{t}{a}\right)$$

Example 7 **Using change of scale find** $L^{-1}\left\{\dfrac{p}{\left(4p^2+1\right)}\right\}$

Sol. We know,

$$L^{-1}\left\{\frac{p}{\left(p^2+1\right)}\right\}=\cos t$$

Here, $F(p)=\dfrac{p}{\left(p^2+1\right)}$ and $f(t)=\cos t$

By change of scale property, we have

$$L^{-1}\left\{F(2p)\right\}=\frac{1}{2}\cdot f\left(\frac{t}{2}\right)$$

$$\Rightarrow\quad L^{-1}\left\{\frac{2p}{\left(4p^2+1\right)}\right\}=\frac{1}{2}\cdot\cos\frac{t}{2}$$

$$\Rightarrow\quad L^{-1}\left\{\frac{1p}{\left(4p^2+1\right)}\right\}=\frac{1}{4}\cdot\cos\frac{t}{2}$$

11.6 INVERSE LAPLACE TRANSFORM OF DERIVATIVES OF FUNCTIONS

11.6.1 Inverse Laplace Transform of first order derivative of a function

Theorem 11.4 **If** $L^{-1}\left\{F(p)\right\}=f(t)$ **then,**

$$L^{-1}\left\{\frac{d}{dp}F(p)\right\}=-t\cdot L^{-1}\left\{F(p)\right\}$$

$$i.e.,\ L^{-1}\left\{F'(p)\right\}=-t\cdot f(t)$$

Proof Since $L^{-1}\{F(p)\} = f(t)$, we have $F(p) = L\{f(t)\}$
Now by **Theorem. 10.6 of Ch. 10**, we have

$$L\{t \cdot f(t)\} = -\frac{d}{dp}\{F(p)\} = -F'(p)$$

$$or, \qquad F'(p) = -L\{t \cdot f(t)\} = L\{-t \cdot f(t)\}$$

$$\Rightarrow L^{-1}\left\{F'(p)\right\} = -t \cdot f(t).$$

Hence, the result is proved.

11.6.2 Inverse Laplace Transform of n^{th} Order Derivative of a Function (Generalised Form)

Theorem 11.5 **If** $L^{-1}\{F(p)\} = f(t)$ **then,**

$$L^{-1}\left\{\frac{d^n}{dp^n}F(p)\right\} = (-1)^n \cdot t^n \cdot L^{-1}\{F(p)\}$$

$$i.e., L^{-1}\left\{F^{(n)}(p)\right\} = (-1)^n \cdot t^n \cdot f(t)$$

Proof Since $L^{-1}\{F(p)\} = f(t)$, we have $F(p) = L\{f(t)\}$.
Now by **Theorem. 10.7 of Ch. 10,** we have

$$L\left\{t^n \cdot f(t)\right\} = (-1)^n \frac{d^n}{dp^n}\{F(p)\} = (-1)^n F^{(n)}(p)$$

$$or, \qquad F^{(n)}(p) = (-1)^n L\left\{t^n \cdot f(t)\right\} = L\left\{(-1)^n t^n \cdot f(t)\right\}$$

$$\Rightarrow L^{-1}\left\{F^{(n)}(p)\right\} = (-1)^n \cdot t^n \cdot f(t).$$

This proves the result.

Example 8 **Find** $L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\}$

Sol.

$$L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\} = L^{-1}\left\{-\frac{1}{2} \cdot \frac{d}{dp}\left(\frac{1}{(p^2+1)}\right)\right\}$$

$$= -\frac{1}{2} \cdot L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{(p^2+1)}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1 \cdot t \cdot L^{-1}\left\{\frac{1}{(p^2+1)}\right\}$$

$$= \frac{t}{2}\sin t.$$

Example 9 **Find** $L^{-1}\left\{\log\frac{p+5}{p+4}\right\}$

Sol. Let

$$F(p) = \log\left(\frac{p+5}{p+4}\right) = \log(p+5) - \log(p+3)$$

Therefore,

$$F'(p) = \frac{1}{p+5} - \frac{1}{p+3}$$

So,

$$L^{-1}\{F'(p)\} = L^{-1}\left\{\frac{1}{p+5}\right\} - L^{-1}\left\{\frac{1}{p+3}\right\} = e^{-5t} - e^{-3t}$$

Since,

$$L^{-1}\left\{\frac{d}{dp}F(p)\right\} = -t \cdot L^{-1}\{F(p)\}$$

We have,

$$-t \cdot L^{-1}\{F(p)\} = e^{-5t} - e^{-3t}$$

or,

$$L^{-1}\{F(p)\} = \frac{e^{-3t} - e^{-5t}}{t}$$

Hence,

$$L^{-1}\left\{\log\frac{p+5}{p+4}\right\} = \frac{e^{-3t} - e^{-5t}}{t}$$

11.7 INVERSE LAPLACE TRANSFORM OF INTEGRALS

Theorem 11.6 **If** $L^{-1}\{F(p)\} = f(t)$ **then,**

$$L^{-1}\left\{\int_p^\infty F(p)\,dp\right\} = \frac{f(t)}{t}.$$

Proof Since $L^{-1}\{F(p)\} = f(t)$, we have $F(p) = L\{f(t)\}$.
Now by **Theorem. 10.8 of Ch. 10,** we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_{p}^{\infty} F(p)dp$$

$$\Rightarrow L^{-1}\left\{\int_{p}^{\infty} F(p)\,dp\right\} = \frac{f(t)}{t}$$

Hence, the result is proved.

11.8 INVERSE LAPLACE TRANSFORM OF FUNCTIONS ON MULTIPLICATION BY p^n (*n* is any positive integer)

11.8.1 Multiplication by p

Theorem 11.7 **If** $L^{-1}\{F(p)\} = f(t)$ **and** $f(0) = 0$ **then**

$$\boxed{L^{-1}\{p \cdot F(p)\} = f'(t)}$$

Proof Since $L^{-1}\{F(p)\} = f(t)$, we have $F(p) = L\{f(t)\}$
Now by **Theorem 10.2 of Ch. 10,** we have

$$\begin{aligned} L\left\{f'(t)\right\} &= pL\{f(t)\} - f(0) \\ &= p \cdot F(p) - f(0) \\ &= p \cdot F(p),\ \text{since } f(0) = 0 \end{aligned}$$

Therefore,

$$L^{-1}\{p \cdot F(p)\} = f'(t)$$

11.8.2 Multiplication by p^n (Generalized Form)

Theorem 11.8 **If** $L^{-1}\{F(p)\} = f(t)$ **and** $f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = 0$ **then,**

$$\boxed{L^{-1}\left\{p^n \cdot F(p)\right\} = f^{(n)}(t)}$$

Proof Since, $L^{-1}\{F(p)\} = f(t)$, we have $F(p) = L\{f(t)\}$

Now by **Theorem 10.4 of Ch. 10**, we have

$$L\left\{f^{(n)}(t)\right\} = p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

$$= p^n \cdot F(p), \text{ since } f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = 0$$

Therefore,

$$L^{-1}\left\{p^n \cdot F(p)\right\} = f^{(n)}(t)$$

11.9 INVERSE LAPLACE TRANSFORM OF FUNCTIONS ON DIVISION BY p

Theorem 11.9 **If $L^{-1}\{F(p)\} = f(t)$ then,**

$$\boxed{L^{-1}\left\{\frac{F(p)}{p}\right\} = \int_0^t f(x)dx}$$

Proof Since $L^{-1}\{F(p)\} = f(t)$, we have $F(p) = L\{f(t)\}$
Now by **Theorem 10.5 of Ch. 10,** we have

$$L\left\{\int_0^t f(x)\,dx\right\} = \frac{F(p)}{p}$$

$$\Rightarrow L^{-1}\left\{\frac{F(p)}{p}\right\} = \int_0^t f(x)\,dx$$

Theorem 11.10 **(Generalized Form): If $L^{-1}\{F(p)\} = f(t)$ then,**

$$\boxed{L^{-1}\left\{\frac{F(p)}{p^n}\right\} = \int_0^t \int_0^t \cdots \int_0^t f(t)dt^n}$$

Proof Beyond the scope of the book.

Example 10 **Find** $L^{-1}\left\{\dfrac{1}{p^2\left(p^2+1\right)}\right\}$

Sol. We know that $L^{-1}\left\{\dfrac{1}{p^2+1}\right\} = \sin t$

Here, $F(p) = \dfrac{1}{p^2+1}$ and $f(t) = \sin t$

Now by division rule, we have

$$L^{-1}\left\{\frac{F(p)}{p}\right\} = \int_0^t f(x)dx$$

Therefore,

$$L^{-1}\left\{\frac{1}{p\left(p^2+1\right)}\right\} = \int_0^t \sin x \, dx = 1 - \cos t$$

Similarly, applying division rule to the above result we obtain

$$L^{-1}\left\{\frac{1}{p^2\left(p^2+1\right)}\right\} = \int_0^t (1 - \cos x) \; dx = t - \sin t$$

11.10 CONVOLUTION

11.10.1 Definition

Let $f_1(t)$ and $f_2(t)$ be two continuous functions for all $t \geq 0$ and of some exponential order as $t \to \infty$, then their convolution is denoted by $f_1 * f_2$ and is defined as

$$\boxed{f_1 * f_2 = \int_0^t f_1(x) \cdot f_2(t-x)dx}$$

Note: RHS is sometimes called convolution integral.

Properties:

(1) Commutative property

$$\boxed{f_1 * f_2 = f_2 * f_1.}$$

(2) Associative property

$$\boxed{f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3.}$$

(3) Distributive property w.r.t addition

$$\boxed{\begin{aligned} &f_1 * (f_2 + f_3) = (f_1 * f_2) + (f_1 * f_3) \\ &\text{and} \quad (f_1 + f_2) * f_3 = (f_1 * f_3) + (f_2 * f_3) \end{aligned}}$$

11.10.2 Convolution Theorem

Let $f_1(t)$ and $f_2(t)$ be two continuous functions for all $t \geq 0$ such that their Laplace transform exists and also let

$$L^{-1}\{F_1(p)\} = f_1(t) \text{ and } L^{-1}\{F_2(p)\} = f_2(t),$$

then, convolution theorem states that

$$\boxed{L^{-1}\{F_1(p) \cdot F_2(p)\} = \int_0^t f_1(x) \cdot f_2(t-x)dx = f_1 * f_2.}$$

In other words,

$$\boxed{L\{f_1(t) * f_2(t)\} = F_1(p) \cdot F_2(p) = L\{f_1(t)\} \cdot L\{f_2(t)\}}$$

Again, since $f_1 * f_2 = f_2 * f_1$, convolution theorem can also be stated as,

$$\boxed{L^{-1}\{F_1(p) \cdot F_2(p)\} = \int_0^t f_1(t-x) \cdot f_2(x)dx = f_1 * f_2.}$$

[WBUT 2006]

Proof Beyond the scope of the book.

Note: According to our requirement we choose any one of the above forms.

Example 11 **Use convolution theorem to find**

$$L^{-1}\left\{\frac{1}{(p-2)\left(p^2+1\right)}\right\}.$$

[WBUT 2004]

Sol. Here, we have to find

$$L^{-1}\left\{\frac{1}{(p-2)\left(p^2+1\right)}\right\} = L^{-1}\left\{\frac{1}{(p-2)} \cdot \frac{1}{\left(p^2+1\right)}\right\}$$
$$= L^{-1}\{F_1(p) \cdot F_2(p)\}$$

Where,

$$F_1(p) = \frac{1}{(p-2)} = L\{f_1(t)\} \text{ (say)}$$
$$\Rightarrow f_1(t) = L^{-1}\left\{\frac{1}{(p-2)}\right\} = e^{2t}$$

and

$$F_2(p) = \frac{1}{(p^2+1)} = L\{f_2(t)\} \text{ (say)}$$

$$\Rightarrow f_2(t) = L^{-1}\left\{\frac{1}{(p^2+1)}\right\} = \sin t.$$

By convolution theorem, we have

$$L^{-1}\{F_1(p) \cdot F_2(p)\} = \int_0^t f_1(t-x) \cdot f_2(x) dx$$

$$i.e., \; L^{-1}\left\{\frac{1}{(p-2)(p^2+1)}\right\} = \int_0^t e^{2(t-x)} \cdot \sin x \, dx = e^{2t} \int_0^t e^{-2x} \cdot \sin x \, dx$$

$$= e^{2t}\left[\frac{e^{-2x} \cdot (-2\sin x - \cos x)}{(-2)^2 + 1^2}\right]_0^t$$

$$= e^{2t}\left[\frac{e^{-2t} \cdot (-2\sin t - \cos t)}{5} - \frac{-1}{5}\right]$$

$$= \frac{1}{5}\left(e^{2t} - 2\sin t - \cos t\right).$$

WORKED OUT EXAMPLES

Example 11.1 **Evaluate**

$$L^{-1}\left\{\frac{4p+5}{(p-4)^2(p+3)}\right\}$$

[WBUT 2003]

Sol. Here, we apply the method of partial fraction to compute the given inverse Laplace transform. Suppose

$$\frac{4p+5}{(p-4)^2(p+3)} = \frac{A}{(p-4)} + \frac{B}{(p-4)^2} + \frac{C}{(p+3)}$$

$$= \frac{A(p-4)(p+3) + B(p+3) + C(p-4)^2}{(p-4)^2(p+3)}$$

$$= \frac{(A+C)p^2 + (-A+B-8C)p + (-12A+3B+16C)}{(p-4)^2(p+3)}$$

Equating the the coefficients of like powers of p in the numerator of both hand sides, we get

$$A + C = 0, \quad -A + B - 8C = 4, \quad -12A + 3B + 16C = 5$$

Now,

$$\begin{bmatrix} -A + B - 8C = 4 \\ \\ -12A + 3B + 16C = 5 \end{bmatrix} \Rightarrow 9A - 40C = 7$$

$$\begin{bmatrix} A + C = 0 \\ \\ 9A - 40C = 7 \end{bmatrix} \Rightarrow A = \frac{1}{7},\ C = \frac{-1}{7}$$

Putting, $A = \frac{1}{7}, C = \frac{-1}{7}$ in $-A + B - 8C = 4$, we get $B = 3$.
So,

$$\frac{4p+5}{(p-4)^2\,(p+3)} = \frac{1}{7} \cdot \frac{1}{(p-4)} + 3 \cdot \frac{1}{(p-4)^2} - \frac{1}{7} \cdot \frac{1}{(p+3)}$$

Therefore,

$$L^{-1}\left\{\frac{4p+5}{(p-4)^2\,(p+3)}\right\}$$

$$= L^{-1}\left\{\frac{1}{7} \cdot \frac{1}{(p-4)} + 3 \cdot \frac{1}{(p-4)^2} - \frac{1}{7} \cdot \frac{1}{(p+3)}\right\}$$

$$= \frac{1}{7} \cdot L^{-1}\left\{\frac{1}{(p-4)}\right\} + 3L^{-1}\left\{\frac{1}{(p-4)^2}\right\} - \frac{1}{7} \cdot L^{-1}\left\{\frac{1}{(p+3)}\right\}$$

$$= \frac{1}{7} \cdot e^{4t} + 3e^{4t} \cdot L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{1}{7} \cdot e^{-3t} \quad \left(\begin{array}{c}\text{Applying first shifting} \\ \text{Th. on the 2}^{\text{nd}} \text{ term.}\end{array}\right)$$

$$= \frac{1}{7} \cdot e^{4t} + 3e^{4t} \cdot t - \frac{1}{7} \cdot e^{-3t}$$

Example 11.2 **Find** $L^{-1}\left\{\frac{\left(3p^2+4\right)}{(p^2+1)\cdot\left(p^2+4\right)}\right\}$

Sol. Here, we apply the method of partial fraction to compute the inverse Laplace transform and let us consider,

$$\frac{\left(3p^2+4\right)}{(p^2+1)\cdot\left(p^2+4\right)} = \frac{A}{(p^2+1)} + \frac{B}{\left(p^2+4\right)} = \frac{A\left(p^2+4\right) + B(p^2+1)}{(p^2+1)\cdot\left(p^2+4\right)}$$

$$i.e., \quad \frac{\left(3p^2+4\right)}{(p^2+1)\cdot\left(p^2+4\right)} = \frac{(A+B)\,p^2 + (4A+B)}{(p^2+1)\cdot\left(p^2+4\right)}$$

Equating the coefficients of like powers of p in the numerator of both hand sides, we get

$$A + B = 3 \text{ and } 4A + B = 4$$

Solving, we obtain $A = \frac{1}{3}$ and $B = \frac{8}{3}$.
Therefore,

$$L^{-1}\left\{\frac{(p^2+6)}{(p^2+1)\cdot(p^2+4)}\right\} = L^{-1}\left\{\frac{\frac{1}{3}}{(p^2+1)} + \frac{\frac{8}{3}}{(p^2+4)}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{(p^2+1)}\right\} + \frac{4}{3}L^{-1}\left\{\frac{2}{(p^2+4)}\right\}$$

$$= \frac{1}{3}\sin t - \frac{4}{3}\sin 2t$$

$$= \frac{1}{3}(\sin t - 4\sin 2t)$$

Example 11.3 **Find** $L^{-1}\left\{\frac{(p+3)}{p^2(p-3)(p-1)}\right\}$

Sol. Now, we apply the method of partial fraction to compute the inverse laplace transform and let us consider,

$$\frac{(p+3)}{p^2(p-3)(p-1)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p-3} + \frac{D}{p-1}$$

$$i.e., \quad \frac{(p+3)}{p^2(p-3)(p-1)} = \frac{\left\{\begin{array}{l}Ap(p-3)(p-1) + B(p-3)(p-1)\\ +Cp^2(p-1) + Dp^2(p-3)\end{array}\right\}}{p^2(p-3)(p-1)}$$

Equating the numerators from both hand sides, we have

$$p + 3 = Ap(p-3)(p-1) + B(p-3)(p-1) + Cp^2(p-1) + Dp^2(p-3)$$

Putting, $p = 0, 1, 2, 3$ successively, we get

$$B = 1,\ D = -2,\ -2A - B + 4C - 4D = 5 \text{ and } C = \frac{1}{3}$$

$$\Rightarrow A = \frac{-19}{3},\ B = 1,\ C = \frac{1}{3} \text{ and } D = 2.$$

Therefore,

$$\frac{(p+6)}{p^2(p-3)(p-1)} = \frac{-19}{3}\frac{1}{p} + \frac{1}{p^2} + \frac{1}{3}\cdot\frac{1}{p-3} + 2\cdot\frac{1}{p-1}$$

Hence,

$$L^{-1}\left\{\frac{(p+6)}{p^2(p-3)(p-1)}\right\}$$
$$= L^{-1}\left\{\frac{-19}{3}\frac{1}{p} + \frac{1}{p^2} + \frac{1}{3}\cdot\frac{1}{p-3} + 2\cdot\frac{1}{p-1}\right\}$$
$$= \frac{-19}{3}L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\left\{\frac{1}{p^2}\right\} + \frac{1}{3}\cdot L^{-1}\left\{\frac{1}{p-3}\right\} + 2\cdot L^{-1}\left\{\frac{1}{p-1}\right\}$$
$$= \frac{-19}{3} + t + \frac{1}{3}\cdot e^{3t} + 2\cdot e^{t}$$

Example 11.4 **Find** $L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\}$ **by Convolution theorem.** [WBUT 2005]

Sol. Here, we have to find

$$L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\} = L^{-1}\left\{\frac{p}{\left(p^2+1\right)}\cdot\frac{1}{\left(p^2+1\right)}\right\}$$
$$= L^{-1}\left\{F_1(p)\cdot F_2(p)\right\}$$

where,

$$F_1(p) = \frac{p}{\left(p^2+1\right)} = L\left\{f_1(t)\right\} \text{(say)}$$

$$\Rightarrow f_1(t) = L^{-1}\left\{\frac{p}{\left(p^2+1\right)}\right\} = \cos t$$

and

$$F_2(p) = \frac{1}{\left(p^2+1\right)} = L\left\{f_2(t)\right\} \text{(say)}$$

$$\Rightarrow f_2(t) = L^{-1}\left\{\frac{1}{\left(p^2+1\right)}\right\} = \sin t$$

By convolution theorem, we have

$$L^{-1}\left\{F_1(p)\cdot F_2(p)\right\} = \int_0^t f_1(x)\cdot f_2(t-x)dx$$

$$i.e.,\ L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\} = \int_0^t \cos x \cdot \sin(t-x)\ dx$$

$$= \frac{1}{2}\int_0^t \{\sin t - \sin(2x-t)\}\ dx$$

$$= \frac{1}{2}\left[x \sin t + \frac{\cos(2x-t)}{2}\right]_0^t = \frac{1}{2} t \sin t$$

Example 11.5 **Apply convolution theorem to prove that**

$$\int_0^t \sin x \cdot \cos(t-x)\ dx = \frac{t}{2}\sin t$$

[WBUT 2007]

Sol. Let,

$$f(t) = \int_0^t \sin x \cdot \cos(t-x)\ dx \tag{1}$$

$$= \int_0^t f_1(x) \cdot f_2(t-x)\ dx = f_1(t) * f_2(t)$$

where,

$$f_1(t) = \sin t \ \text{ and } \ f_2(t) = \cos t$$

Then by convolution theorem,

$$L\{f(t)\} = L\{f_1(t) * f_2(t)\} = L\{f_1(t)\} \cdot L\{f_2(t)\}$$

$$= L\{\sin t\} \cdot L\{\cos t\} = \frac{1}{p^2+1} \cdot \frac{p}{p^2+1}$$

i.e., $$L\{f(t)\} = \frac{p}{\left(p^2+1\right)^2}$$

$$\Rightarrow f(t) = L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\}$$

Now,

$$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}$$
$$= L^{-1}\left\{-\frac{1}{2}\cdot\frac{d}{dp}\left(\frac{1}{(p^2+1)}\right)\right\} = -\frac{1}{2}\cdot L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{(p^2+1)}\right)\right\}$$
$$= -\frac{1}{2}(-1)^1\cdot t\cdot L^{-1}\left\{\frac{1}{(p^2+1)}\right\}, \text{(by **Th.11.5**)}$$
$$= \frac{t}{2}\cdot\sin t$$

Therefore,

$$f(t) = \frac{t}{2}\cdot\sin t \tag{2}$$

From (1) and (2) we have the required result as

$$\int_0^t \sin x\cdot\cos(t-x)\,dx = \frac{t}{2}\sin t$$

Example 11.6 **Apply convolution theorem to prove that**

$$B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)},\quad m,n>0$$

Sol. We know from definition

$$B(m,n) = \int_0^t x^{m-1}\cdot(1-x)^{n-1}\,dx,\quad m,n>0$$

Let,

$$f(t) = \int_0^t x^{m-1}\cdot(t-x)^{n-1}\,dx$$
$$= \int_0^t f_1(x)\cdot f_2(t-x)dx = f_1(t)*f_2(t)$$

where,

$$f_1(t) = t^{m-1} \text{ and } f_2(t) = t^{n-1}$$

Now, by convolution theorem

$$L\{f(t)\} = L\{f_1(t) * f_2(t)\} = L\{f_1(t)\} \cdot L\{f_2(t)\}$$

$$= L\{t^{m-1}\} \cdot L\{t^{n-1}\} = \frac{\Gamma(m)}{p^m} \cdot \frac{\Gamma(n)}{p^n} = \frac{\Gamma(m).\Gamma(n)}{p^{m+n}}$$

Hence,

$$f(t) = L^{-1}\left\{\frac{\Gamma(m)\cdot\Gamma(n)}{p^{m+n}}\right\} = \Gamma(m)\cdot\Gamma(n)\cdot L^{-1}\left\{\frac{1}{p^{m+n}}\right\}$$

$$= \Gamma(m)\cdot\Gamma(n)\cdot\frac{1}{\Gamma(m+n)}\cdot t^{m+n-1}$$

Therefore,

$$\int_0^t x^{m-1}\cdot(t-x)^{n-1}\,dx = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\cdot t^{m+n-1}$$

Putting, $t = 1$, we have

$$\int_0^t x^{m-1}\cdot(1-x)^{n-1}\,dx = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\cdot(1)^{m+n-1}$$

$$\Rightarrow B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

This proves the result.

Example 11.7 **Evaluate**

$$L^{-1}\{\tan^{-1}(p+2)\}$$

Sol. Let,

$$F(p) = \tan^{-1}(p+2)$$

Differentiating w.r.t p, we have

$$F'(p) = \frac{1}{(p+2)^2+1}$$

Now,

$$L^{-1}\{F'(p)\} = L^{-1}\left\{\frac{1}{(p+2)^2+1}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= e^{-2t}\sin t$$

$$L^{-1}\{F'(p)\} = e^{-2t}\sin t \tag{1}$$

By the derivative property, if $L^{-1}\{F(p)\} = f(t)$ then,

$$L^{-1}\left\{F'(p)\right\} = -t \cdot L^{-1}\{F(p)\}$$

Therefore,

$$L^{-1}\left\{\frac{1}{(p+2)^2+1}\right\} = -tL^{-1}\{\tan^{-1}(p+2)\}$$

or, $$L^{-1}\{\tan^{-1}(p+2)\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{(p+2)^2+1}\right\}$$

$$= -\frac{1}{t}e^{-2t}\sin t$$

Example 11.8 **Evaluate**

$$L^{-1}\left\{p\log\frac{p}{\sqrt{p^2+1}} + \cot^{-1} p\right\}$$

Sol. Let,

$$F(p) = p\log\frac{p}{\sqrt{p^2+1}} + \cot^{-1} p$$

$$= p\left\{\log p - \frac{1}{2}\log(p^2+1)\right\} + \cot^{-1} p$$

Differentiating w.r.t p, we have

$$F'(p) = \left\{\log p - \frac{1}{2}\log(p^2+1)\right\} + p\left\{\frac{1}{p} - \frac{p}{(p^2+1)}\right\} - \frac{1}{p^2+1}$$

$$= \log p - \frac{1}{2}\log(p^2+1)$$

Again,

$$F''(p) = \frac{1}{p} - \frac{p}{(p^2+1)}$$

Now,

$$L^{-1}(F''(p)\} = L^{-1}\left\{\frac{1}{p} - \frac{p}{(p^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{p}{(p^2+1)}\right\}$$

$$= 1 - \cos t$$

By the derivative property, if $L^{-1}\{F(p)\} = f(t)$ then,

$$L^{-1}\left\{F''(p)\right\} = (-1)^2 t^2 \cdot L^{-1}\{F(p)\}$$

Therefore,

$$L^{-1}\left\{\frac{1}{p} - \frac{p}{(p^2+1)}\right\} = (-1)^2 t^2 L^{-1}\left\{p\log\frac{p}{\sqrt{p^2+1}} + \cot^{-1} p\right\}$$

$$or, \quad L^{-1}\left\{p\log\frac{p}{\sqrt{p^2+1}} + \cot^{-1} p\right\} = \frac{1}{t^2} L^{-1}\left\{\frac{1}{p} - \frac{p}{(p^2+1)}\right\}$$

$$or, \quad L^{-1}\left\{p\log\frac{p}{\sqrt{p^2+1}} + \cot^{-1} p\right\} = \frac{1}{t^2}(1 - \cos t)$$

Example 11.9 **Prove that**

$$L^{-1}\left\{\frac{1}{p}\log(1 + \frac{1}{p^2})\right\} = 2\int_0^t \frac{1-\cos u}{u} du$$

Sol. Let,

$$F(p) = \log\left(1 + \frac{1}{p^2}\right)$$

Therefore,

$$F'(p) = -\frac{2}{p(p^2+1)}$$

and

$$\begin{aligned} L^{-1}\{F'(p)\} &= -2L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} \\ &= -2\left[L^{-1}\left(\frac{1}{p}\right) - L^{-1}\left(\frac{p}{p^2+1}\right)\right] \\ &= -2(1 - \cos t) \end{aligned}$$

By the derivative property, if $L^{-1}\{F(p)\} = f(t)$ then,

$$L^{-1}\left\{F'(p)\right\} = -t \cdot L^{-1}\{F(p)\}$$

Therefore,

$$-2(1 - \cos t) = -tL^{-1}\left\{\log\left(1 + \frac{1}{p^2}\right)\right\}$$

$$or, \quad L^{-1}\left\{\log\left(1 + \frac{1}{p^2}\right) = \frac{2(1 - \cos t)}{t}\right\}$$

By division by p property, we have the following,
If $L^{-1}\{F(p)\} = f(t)$ then,

$$L^{-1}\left\{\frac{F(p)}{p}\right\} = \int_0^t f(u)du$$

Therefore,

$$L^{-1}\left\{\frac{1}{p}\log\left(1+\frac{1}{p^2}\right)\right\} = 2\int_0^t \frac{1-\cos u}{u}du$$

Example 11.10 **Evaluate**

$$L^{-1}\left\{\frac{1}{p^3(p+1)}\right\}$$

Sol. We know,

$$L^{-1}\left\{\frac{1}{(p+1)}\right\} = e^{-t}$$

By division by p property, we have the following,
If $L^{-1}\{F(p)\} = f(t)$ then,

$$L^{-1}\left\{\frac{F(p)}{p}\right\} = \int_0^t f(u)du$$

Therefore,

$$L^{-1}\left\{\frac{1}{p(p+1)}\right\} = \int_0^t e^{-u}du = 1 - e^{-t}$$

Again, applying division by p property, we have

$$L^{-1}\left\{\frac{1}{p^2(p+1)}\right\} = \int_0^t (1-e^{-u})du$$

$$= t + e^{-t} - 1$$

Again, applying division by p property, we have

$$L^{-1}\{\frac{1}{p^3(p+1)}\} = \int_0^t (u + e^{-u} - 1)du$$

$$= \frac{t^2}{2} - e^{-t} - t + 1$$

IMPORTANT NOTE: The following problems are solved with the transform parameter s because some university examination problems involve parameter s. Though throughout the chapter we have discussed the topic using the parameter p, but to keep the originality of the examination paper problems, the following change has been made.

Example 11.11 **Find**

$$L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$$

[WBUT 2003]

Sol.

$$L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$$

$$= L^{-1}\left\{\frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{\frac{-1}{2}t} \cdot L^{-1}\left\{\frac{s+\frac{1}{2}}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} \text{ (Applying first shifting Th.)}$$

$$= e^{\frac{-1}{2}t} \cdot \left[L^{-1}\left\{\frac{s}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} + L^{-1}\left\{\frac{\frac{1}{2}}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= e^{\frac{-1}{2}t} \cdot \left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$= e^{\frac{-1}{2}t} \cdot \left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$

Example 11.12 **Find**

$$L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right\}$$

[WBUT 2006]

Sol.

$$\begin{aligned}
&L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right\}\\
&=L^{-1}\left\{\frac{1}{a^2-b^2}\cdot\left(\frac{\left(s^2+a^2\right)-\left(s^2+b^2\right)}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right)\right\}\\
&=\frac{1}{a^2-b^2}\cdot L^{-1}\left\{\frac{1}{\left(s^2+b^2\right)}-\frac{1}{\left(s^2+a^2\right)}\right\}\\
&=\frac{1}{a^2-b^2}\cdot\left[L^{-1}\left\{\frac{1}{\left(s^2+b^2\right)}\right\}-L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)}\right\}\right]\\
&=\frac{1}{a^2-b^2}\cdot\left[\frac{\sin bt}{b}-\frac{\sin at}{a}\right]
\end{aligned}$$

Example 11.13 **Find**

$$L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\}$$

[WBUT 2006]

Sol.

$$\begin{aligned}
L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\}&=L^{-1}\left\{\frac{(s+1-1)^2}{(s+1)^5}\right\}\\
&=e^{-t}L^{-1}\left\{\frac{(s-1)^2}{s^5}\right\}\text{ (by first shifting theorem)}\\
&=e^{-t}L^{-1}\left\{\frac{s^2-2s+1}{s^5}\right\}\\
&=e^{-t}\left[L^{-1}\left\{\frac{s^2}{s^5}\right\}-L^{-1}\left\{\frac{2s}{s^5}\right\}+L^{-1}\left\{\frac{1}{s^5}\right\}\right]\\
&=e^{-t}\left[L^{-1}\left\{\frac{1}{s^3}\right\}-2L^{-1}\left\{\frac{1}{s^4}\right\}+L^{-1}\left\{\frac{1}{s^5}\right\}\right]
\end{aligned}$$

$$= e^{-t}\left[\frac{t^2}{2!} - 2\cdot\frac{t^3}{3!} + \frac{t^4}{4!}\right]$$

$$= \frac{e^{-t}}{24}\left[12t^2 - 8t^3 + t^4\right]$$

Example 11.14 **Apply Convolution theorem to evaluate**

$$L^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\}$$

[WBUT 2006]

Sol. See solved Example 11.4.

Example 11.15 **Apply convolution theorem to find the inverse Laplace transform of**

$$\frac{1}{\left(s^2+1\right)\left(s^2+9\right)}$$

[WBUT 2002]

Sol. Here, we have to find

$$L^{-1}\left\{\frac{1}{\left(s^2+1\right)\left(s^2+9\right)}\right\} = L^{-1}\left\{\frac{1}{\left(s^2+1\right)}\cdot\frac{1}{\left(s^2+9\right)}\right\}$$

$$= L^{-1}\left\{F_1(s)\cdot F_2(s)\right\}$$

where,

$$F_1(s) = \frac{1}{\left(s^2+1\right)} = L\left\{f_1(t)\right\}\text{(say)}$$

$$\Rightarrow f_1(t) = L^{-1}\left\{\frac{1}{\left(s^2+1\right)}\right\} = \sin t$$

and

$$F_2(s) = \frac{1}{\left(s^2+9\right)} = L\left\{f_2(t)\right\}\text{(say)}$$

$$\Rightarrow f_2(t) = L^{-1}\left\{\frac{1}{\left(s^2+9\right)}\right\} = \frac{\sin 3t}{3}$$

By convolution theorem, we have

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(t-x)\cdot f_2(x)dx$$

i.e., $$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \int_0^t \sin(t-x)\cdot\frac{\sin 3x}{3}\,dx$$

$$= \frac{1}{3}\int_0^t (\sin t\cdot\cos x - \sin x\cdot\cos t)\cdot\sin 3x\,dx$$

$$= \frac{1}{3}\left\{\sin t\int_0^t \sin 3x\cdot\cos x dx - \cos t\int_0^t \sin 3x\cdot\sin x\,dx\right\}$$

$$= \frac{1}{3}\left\{\frac{\sin t}{2}\int_0^t (\sin 4x + \sin 2x)\,dx - \frac{\cos t}{2}\int_0^t (\cos 2x - \cos 4x)\,dx\right\}$$

$$= \frac{1}{6}\left\{\sin t\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right]_0^t - \cos t\left[\frac{\sin 2x}{2} - \frac{\sin 4x}{2}\right]_0^t\right\}$$

$$= \frac{1}{6}\left\{\sin t\left[-\frac{\cos 4t}{4} - \frac{\cos 2t}{2} + \frac{1}{4} + \frac{1}{2}\right] - \cos t\left[\frac{\sin 2t}{2} - \frac{\sin 4t}{2}\right]\right\}$$

$$= \frac{1}{24}(3\sin t - \sin 3t)$$

Example 11.16 **Apply Convolution theorem to find**

$$L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$$

[WBUT 2007]

Sol. Here, we have to find

$$L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+9)}\cdot\frac{1}{(s^2+9)}\right\}$$

$$= L^{-1}\{F_1(s)\cdot F_2(s)\}$$

where,

$$F_1(s) = \frac{s}{\left(s^2+9\right)} = L\left\{f_1(t)\right\}(\text{say})$$

$$\Rightarrow f_1(t) = L^{-1}\left\{\frac{s}{\left(s^2+9\right)}\right\} = \cos 3t$$

and

$$F_2(s) = \frac{1}{\left(s^2+9\right)} = L\left\{f_2(t)\right\}(\text{say})$$

$$\Rightarrow f_2(t) = L^{-1}\left\{\frac{1}{\left(s^2+9\right)}\right\} = \frac{\sin 3t}{3}.$$

By convolution theorem, we have

$$L^{-1}\left\{F_1(s)\cdot F_2(s)\right\} = \int_0^t f_1(x)\cdot f_2(t-x)dx$$

$$i.e.,\ L^{-1}\left\{\frac{s}{\left(s^2+9\right)^2}\right\} = \int_0^t \cos 3x \cdot \frac{\sin 3\,(t-x)}{3}\,dx$$

$$= \frac{1}{3}\cdot\frac{1}{2}\int_0^t \left\{\sin 3t - \sin\,(6x-3t)\right\}\,dx$$

$$= \frac{1}{6}\left[x\sin 3t + \frac{\cos\,(6x-3t)}{6}\right]_0^t = \frac{1}{6}t\sin 3t.$$

Example 11.17 **Apply Convolution theorem to evaluate**

$$L^{-1}\left\{\frac{1}{\left(s^2+2s+5\right)^2}\right\}$$

[WBUT 2008]

Sol. Here, we have to find

$$L^{-1}\left\{\frac{1}{\left(s^2+2s+5\right)^2}\right\} = L^{-1}\left\{\frac{1}{\left(s^2+2s+5\right)}\cdot\frac{1}{\left(s^2+2s+5\right)}\right\}$$

$$= L^{-1}\left\{F(s)\cdot F(s)\right\}$$

Where,

$$F(s) = \frac{1}{\left(s^2 + 2s + 5\right)} = L\{f(t)\} \text{ (say)}$$

$$\Rightarrow f(t) = L^{-1}\left\{\frac{1}{\left(s^2 + 2s + 5\right)}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2 + 2^2}\right\}$$

$$= e^{-t} \cdot L^{-1}\left\{\frac{1}{s^2 + 2^2}\right\}, \text{ by first shifting property}$$

$$= e^{-t} \cdot \frac{\sin 2t}{2} = \frac{1}{2}e^{-t} \cdot \sin 2t$$

By convolution theorem, we have

$$L^{-1}\{F(s) \cdot F(s)\} = \int_0^t f(x) \cdot f(t - x)dx$$

$$i.e., \; L^{-1}\left\{\frac{1}{\left(s^2 + 2s + 5\right)^2}\right\} = \int_0^t \left(\frac{1}{2}e^{-x} \cdot \sin 2x\right) \cdot \left(\frac{1}{2}e^{-(t-x)} \cdot \sin 2(t - x)\right) dx$$

$$= \frac{1}{4}e^{-t} \int_0^t \sin 2x \cdot \sin 2(t - x) \; dx$$

$$= \frac{1}{4}e^{-t} \cdot \frac{1}{2} \int_0^t [\cos(4x - 2t) - \cos 2t] \; dx$$

$$= \frac{1}{8}e^{-t} \cdot \left[\frac{\sin(4x - 2t)}{4} - x \cos 2t\right]_0^t$$

$$= \frac{1}{8}e^{-t} \cdot \left[\frac{\sin 2t}{4} - t \cos 2t - \frac{\sin(-2t)}{4}\right]$$

$$= \frac{1}{16}e^{-t} \cdot [\sin 2t - 2t \cos 2t]$$

Example 11.18 **Evaluate**

$$L^{-1}\left\{\frac{1}{(s-1)^2 (s-2)^3}\right\}$$

[WBUT 2009]

Sol. Here, we have to find

$$L^{-1}\left\{\frac{1}{(s-1)^2(s-2)^3}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2}\cdot\frac{1}{(s-2)^3}\right\}$$
$$= L^{-1}\{F_1(s)\cdot F_2(s)\}$$

where,

$$F_1(s) = \frac{1}{(s-1)^2} = L\{f_1(t)\}\,(\text{say})$$

$$\Rightarrow f_1(t) = L^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^t\cdot L^{-1}\left\{\frac{1}{s^2}\right\} = e^t\cdot t$$

and

$$F_2(s) = \frac{1}{(s-2)^3} = L\{f_2(t)\}\,(\text{say})$$

$$\Rightarrow f_2(t) = L^{-1}\left\{\frac{1}{(s-2)^3}\right\} = e^{2t}\cdot L^{-1}\left\{\frac{1}{s^3}\right\} = e^{2t}\cdot\frac{t^2}{2}.$$

By convolution theorem, we have

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(t-x)\cdot f_2(x)\,dx$$

$$i.e.,\ L^{-1}\left\{\frac{1}{(s-1)^2(s-2)^3}\right\} = \int_0^t e^{(t-x)}\cdot(t-x)\cdot e^{2x}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}e^t\int_0^t (t-x)x^2\cdot e^x\,dx$$

$$= \frac{1}{2}e^t\left\{t\int_0^t x^2\cdot e^x\,dx - \int_0^t x^3\cdot e^x\,dx\right\} \quad (1)$$

Now, by integration of parts, we have

$$\int_0^t x^3\cdot e^x\,dx = \left[x^3\cdot e^x\right]_0^t - \int_0^t 3x^2\cdot e^x\,dx$$

$$= t^3\cdot e^t - 3\int_0^t x^2\cdot e^x\,dx \quad (2)$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{1}{(s-1)^2(s-2)^3}\right\}$$

$$=\frac{1}{2}e^t\left\{t\int_0^t x^2\cdot e^x\,dx-\left(t^3\cdot e^t-3\int_0^t x^2\cdot e^x\,dx\right)\right\}$$

$$=\frac{1}{2}e^t\left\{(t+3)\int_0^t x^2\cdot e^x\,dx-t^3\cdot e^t\right\} \tag{3}$$

Again, by integration of parts, we have

$$\int_0^t x^2\cdot e^x\,dx=\left[x^2\cdot e^x\right]_0^t-\int_0^t 2x\cdot e^x\,dx$$

$$=t^2\cdot e^t-2\left\{\left[x\cdot e^x\right]_0^t-\int_0^t e^x\,dx\right\}$$

$$=t^2\cdot e^t-2t\cdot e^t+2\left[e^x\right]_0^t$$

$$=t^2\cdot e^t-2t\cdot e^t+2e^t-2 \tag{4}$$

Using (4) in (3), we obtain

$$L^{-1}\left\{\frac{1}{(s-1)^2(s-2)^3}\right\}$$

$$=\frac{1}{2}e^t\left\{(t+3)\left(t^2\cdot e^t-2t\cdot e^t+2e^t-2\right)-t^3\cdot e^t\right\}$$

$$=\frac{1}{2}e^t\left\{t\left(-2t\cdot e^t+2e^t-2\right)+3\left(t^2\cdot e^t-2t\cdot e^t+2e^t-2\right)\right\}$$

$$=\frac{1}{2}e^t(t^2\cdot e^t-4t\cdot e^t+6e^t)-\frac{1}{2}e^t(2t+6)$$

$$=\frac{1}{2}(t^2-4t+6)e^{2t}-(t+3)e^t$$

Short and Long Answer Type Questions

1) Find the inverse Laplace transform of the following functions:

a) $F(p) = \dfrac{6}{2p-3} - \dfrac{3+4p}{9p^2-16} - \dfrac{6p-8}{9+16p^2}$

$$\left[\textbf{Ans: } 3e^{\frac{3t}{2}} - \frac{1}{4}\sinh\frac{4t}{3} - \frac{4}{9}\cosh\frac{4t}{3} + \frac{2}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}\right]$$

b) $F(p) = \dfrac{3p-14}{p^2-4p+8}$

[**Ans:** $e^{2t}(3\cos 2t - 4\sin 2t)$]

c) $F(p) = \dfrac{3p-2}{p^2-4p+20}$

[**Ans:** $e^{2t}(3\cos 4t + \sin 4t)$]

d) $F(p) = \dfrac{p^2}{(p+2)^3}$

[**Ans:** $e^{-2t}(1 - 4t + 2t^2)$]

e) $F(p) = \dfrac{e^{-2p}}{p^2}$

[**Ans:** $t-2, t > 2; 0, t < 2$]

2) Evaluate $L^{-1}\left\{\log\dfrac{p+2}{p+1}\right\}$

$$\left[\textbf{Ans: } \frac{e^{-t} - e^{-2t}}{t}\right]$$

3) Evaluate $L^{-1}\left\{\tan^{-1}\left(\dfrac{p-2}{3}\right)\right\}$

$$\left[\textbf{Ans: } -\frac{1}{t}e^{2t}\sin 3t\right]$$

4) Evaluate $L^{-1}\left\{\dfrac{1}{p^3(p^2+1)}\right\}$

$$\left[\textbf{Ans: } \frac{t^2}{2} + \cos t - 1\right]$$

5) Evaluate $L^{-1}\left\{\dfrac{p^2}{(p^2+4)^2}\right\}$

$$\left[\textbf{Ans: } \frac{1}{4}\,(\sin 2t + 2t\cos t)\right]$$

6) Evaluate $L^{-1}\left\{\dfrac{1}{p}\log\dfrac{p+2}{p+1}\right\}$

$$\left[\textbf{Ans: } \frac{2t}{3} + \frac{1}{9} - \frac{1}{9}e^{-3t}\right]$$

7) Evaluate $L^{-1}\left\{\log\dfrac{p^2+1}{p^2+p}\right\}$

$$\left[\textbf{Ans: } \frac{1}{t}(1 + e^{-t} - 2\cos t)\right]$$

8) Find the inverse Laplace transform of the following functions using partial fraction:

a) $F(p) = \dfrac{p-1}{(p+3)(p^2+2p+2)}$

$$\left[\textbf{Ans: } \frac{4}{5}e^{-3t} + \frac{e^{-t}}{5}\,(4\cos t - 3\sin t)\right]$$

b) $F(p) = \dfrac{6p^2+22p+18}{(p+1)(p+2)(p+3)}$

$$[\textbf{Ans: } e^{-t} + 2e^{-2t} + 3e^{-3t}]$$

c) $F(p) = \dfrac{2p+1}{(p+2)^2(p-1)^2}$

$$\left[\textbf{Ans: } \frac{1}{3}t\left(e^{t} + e^{-2t}\right)\right]$$

9) Find the inverse Laplace transform of the following functions using convolution theorem:

a) $F(p) = \dfrac{1}{(p^2-1)(p^2+1)}$

$$\left[\textbf{Ans: } \frac{1}{5}(e^{2t} - 2\sin t - \cos t)\right]$$

b) $F(p) = \dfrac{p-1}{(p^2+2p+2)(3+p)}$

$$\left[\textbf{Ans: } \frac{1}{5}e^{-t}(4\cos t - 3\sin t)\right]$$

c) $F(p) = \dfrac{1}{p^3(p+1)}$

$\left[\textbf{Ans: } 1 - t + \dfrac{t^2}{2} - e^{-t}\right]$

d) $F(p) = \dfrac{3(p^2+2p+3)}{(p^2+2p+2)(p^2+2p+5)}$

[**Ans:** $e^{-t}\sin t + e^{-t}\sin 2t$]

e) $F(p) = \dfrac{1}{(p-2)^4(p+3)}$

$\left[\textbf{Ans: } \dfrac{e^{2t}}{30}\left(t^3 - \dfrac{3}{5}t^2 + \dfrac{6}{25}t - \dfrac{6}{125}\right)\right]$

Multiple Choice Questions

1) $L^{-1}\left\{\dfrac{4}{p^2-7} + \dfrac{2}{p^2+7}\right\} =$

a) $\dfrac{4\sinh 7t}{7} - \dfrac{2\sin 7t}{7}$ b) $\dfrac{4\cos 7t}{7} + \dfrac{2\sin 7t}{7}$

c) $\dfrac{4\cos\sqrt{7}t}{7} - \dfrac{2\sin\sqrt{7}t}{7}$ d) $\dfrac{4\sinh\sqrt{7}t}{\sqrt{7}} + \dfrac{2\sin\sqrt{7}t}{\sqrt{7}}$

2) $L^{-1}\left\{\dfrac{p}{p^2+5} + \dfrac{1}{p^2-4}\right\} =$

a) $\cos 5t + \dfrac{\sinh 2t}{2}$ b) $\cos\sqrt{5}t + \dfrac{\sinh 2t}{2}$

c) $\cos\sqrt{5}t + \dfrac{\sinh t}{2}$ d) none of these

3) $L^{-1}\left\{\dfrac{24}{(p-1)^5}\right\} =$

a) $\dfrac{24t^3}{e^t}$ b) $\dfrac{24t^4}{e^t}$ c) $\dfrac{2t^4}{e^t}$ d) $\dfrac{t^4}{e^t}$

4) $L^{-1}\left\{\dfrac{e^{-ap}}{p^2+1}\right\} =$

a) $\sin t\,H(t-a)$ b) $\sin t\,H(t+a)$ c) $H(t-a)$ d) $-\sin t\,H(t-a)$

5) $L^{-1}\left\{\dfrac{p}{(p+3)^2+4}\right\} =$

a) $\cos 2t - \frac{3}{2}\sin 2t$ b) $e^{-3t}(\cos 2t - \frac{3}{2}\sin 2t\ )$

c) e^{-3t} d) none of these

6) $\mathrm{L}^{-1}\{\log(s+3)\} =$

a) e^{-3t} b) $\dfrac{e^{-3t}}{t}$ c) $-\dfrac{e^{-3t}}{t}$ d) $\dfrac{e^{3t}}{t}$

7) If $L^{-1}\{F(p)\} = f(t)$, then $L^{-1}\left[\int\limits_p^\infty f(u)du\right] =$

a) $\dfrac{1}{p}F\left(\dfrac{p}{a}\right)$ b) $\dfrac{F(p)}{p}$ c) $\dfrac{f(t)}{t}$ d) $\dfrac{F(u)}{u}$

Answers:

1 (d) 2 (b) 3 (d) 4 (d) 5 (b) 6 (c) 7 (c)

CHAPTER

12

Solution of Linear ODE using Laplace Transform

12.1 INTRODUCTION

The Laplace transform is a very important tool for solving differential equations. Basically, solving an ODE is to reduce to an algebraic problem (plus transformations applied to it). This switching from calculus to algebra is known as operational calculus. The Laplace transform method is the most important operational method because it has two main advantages over the usual methods. Here, problems can be solved more directly, like, solving initial value problems without first determining a general solution, and for non-homogeneous ODE without first solving the corresponding homogeneous ODE. In this chapter, we discuss the method of solving linear ordinary differential equations with constant coefficients illustrated with various kinds of examples. To solve the equations, we require the concepts and properties of Laplace transform as well as inverse Laplace transform, which have been already discussed in Chapter 10 and 11.

12.1.1 Solution of Ordinary Differential Equations with Constant Coefficients

For the sake of convenience, let us consider a second order linear ordinary differential equation with constant coefficients

$$\boxed{\frac{d^2y}{d^2t} + c_1\frac{dy}{dt} + c_2 y = f(t)} \qquad (1)$$

where c_1 and c_2 are constants and $f(t)$ is a function of t. Also the given initial conditions are

$$\boxed{y(0) = A \text{ and } y'(0) = B} \tag{2}$$

To find the solution $y(t)$ the following steps are to be followed:

Step 1 Take the Laplace transform of both hand sides of (1).

Step 2 Put the conditions (2).

Step 3 Form the algebraic equation which involves $L\{y(t)\}$ and p, where p is the transform parameter.

Step 4 Express $L\{y(t)\}$ as a function of p, say $L\{y(t)\} = F(p)$.

Step 5 Take the inverse Laplace transform, such that, $y(t) = L^{-1}\{F(p)\}$.

Note: Similar steps should be followed for any order of linear ordinary differential equation with constant coefficients.

WORKED-OUT EXAMPLES

Example 12.1 **Solve**

$$y''(t) + y(t) = \sin(2t),$$

$y(0) = 0$ **and** $y'(0) = 1$, **with the help of Laplace transform.** [WBUT 2003]

Sol. The given equation is

$$y''(t) + y(t) = \sin(2t). \tag{1}$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{y''(t)\right\} + L\{y(t)\} = L\{\sin(2t)\}$$

$$\Rightarrow \left[p^2 \cdot L\{y(t)\} - p \cdot y(0) - y'(0)\right] + L\{y(t)\} = \frac{2}{p^2+4}$$

Putting the conditions $y(0) = 0$ and $y'(0) = 1$, we get

$$\left[p^2 \cdot L\{y(t)\} - p \cdot 0 - 1\right] + L\{y(t)\} = \frac{2}{p^2+4}$$

$$i.e., \; (p^2+1) \cdot L\{y(t)\} = \frac{2}{p^2+4} + 1 = \frac{(p^2+6)}{(p^2+4)}$$

$$i.e., \qquad L\{y(t)\} = \frac{(p^2+6)}{(p^2+1)\cdot(p^2+4)} = F(p), \text{ (say)}$$

Therefore,

$$y(t) = L^{-1}\{F(p)\} = L^{-1}\left\{\frac{(p^2+6)}{(p^2+1)\cdot(p^2+4)}\right\} \tag{2}$$

Now, we apply the method of partial fraction to compute the inverse and let us consider

$$\frac{(p^2+6)}{(p^2+1)\cdot(p^2+4)} = \frac{A}{(p^2+1)} + \frac{B}{(p^2+4)} = \frac{A(p^2+4)+B(p^2+1)}{(p^2+1)\cdot(p^2+4)}$$

$$i.e., \quad \frac{(p^2+6)}{(p^2+1)\cdot(p^2+4)} = \frac{(A+B)\,p^2+(4A+B)}{(p^2+1)\cdot(p^2+4)}$$

Equating the coefficients of like powers of p in the numerator of both hand sides, we get

$$A + B = 1 \text{ and } 4A + B = 6 \tag{3}$$

Solving (3) we obtain $A = \frac{5}{3}$ and $B = -\frac{2}{3}$.
Hence from (2),

$$y(t) = L^{-1}\left\{\frac{(p^2+6)}{(p^2+1)\cdot(p^2+4)}\right\} = L^{-1}\left\{\frac{\frac{5}{3}}{(p^2+1)} + \frac{-\frac{2}{3}}{(p^2+4)}\right\}$$

$$= \frac{5}{3}L^{-1}\left\{\frac{1}{(p^2+1)}\right\} - \frac{1}{3}L^{-1}\left\{\frac{2}{(p^2+4)}\right\}$$

$$= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t = \frac{1}{3}(5\sin t - \sin 2t).$$

So, the required solution is $y(t) = \frac{1}{3}(5\sin t - \sin 2t)$.

Example 12.2 **Solve by Laplace transform the equation**

$$y''(t) + y(t) = 8\cos t$$

given $y(0) = 1$ **and** $y'(0) = -1$ [WBUT 2005]

Sol. The given equation is

$$y''(t) + y(t) = 8\cos t \tag{1}$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{y''(t)\right\} + L\left\{y(t)\right\} = 8 \cdot L\left\{\cos t\right\}$$

$$\Rightarrow \left[p^2 \cdot L\left\{y(t)\right\} - p \cdot y(0) - y'(0)\right] + L\left\{y(t)\right\} = \frac{8p}{p^2+1}$$

Putting the conditions $y(0) = 1$ and $y'(0) = -1$, we get

$$\left[p^2 \cdot L\left\{y(t)\right\} - p \cdot 1 - (-1)\right] + L\left\{y(t)\right\} = \frac{8p}{p^2+1}$$

i.e.,
$$(p^2+1) \cdot L\left\{y(t)\right\} = \frac{8p}{p^2+1} + p - 1$$

i.e.,
$$L\left\{y(t)\right\} = \frac{8p}{\left(p^2+1\right)^2} + \frac{p-1}{p^2+1} = F(p)$$

Therefore,

$$\begin{aligned} y(t) = L^{-1}\left\{F(p)\right\} &= L^{-1}\left\{\frac{8p}{\left(p^2+1\right)^2} + \frac{p-1}{p^2+1}\right\} \\ &= L^{-1}\left\{\frac{8p}{\left(p^2+1\right)^2}\right\} + L^{-1}\left\{\frac{p}{p^2+1}\right\} - L^{-1}\left\{\frac{1}{p^2+1}\right\} \\ &= 8 \cdot L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\} + \cos t - \sin t \qquad (2) \end{aligned}$$

Again,

$$\begin{aligned} L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\} &= \left(-\frac{1}{2}\right) L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2+1}\right)\right\} \\ &= \left(-\frac{1}{2}\right)(-t) \cdot L^{-1}\left\{\frac{1}{p^2+1}\right\}, \text{ by } \textbf{Th. 11.4 of Ch. 11} \\ &= \frac{t}{2}\sin t. \qquad (3) \end{aligned}$$

Using (3) in (2), we get

$$y(t) = 8 \cdot \frac{t}{2}\sin t + \cos t - \sin t = 4t\sin t + \cos t - \sin t.$$

So, the required solution is $y(t) = 4t\sin t + \cos t - \sin t$.

Example 12.3 **Solve the differential equation using Laplace transform**

$$y'' - 3y' + 2y = 4t + e^{3t}$$

where $y(0) = 1$ **and** $y'(0) = -1$ [WBUT 2002]

Sol. The given equation is

$$y'' - 3y' + 2y = 4t + e^{3t} \tag{1}$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{y''(t)\right\} - 3L\left\{y'(t)\right\} + 2L\{y(t)\} = L\left\{4t + e^{3t}\right\}$$

$$\Rightarrow \left[p^2 \cdot L\{y(t)\} - p \cdot y(0) - y'(0)\right] - 3\left[p \cdot L\{y(t)\} - y(0)\right] + 2L\{y(t)\}$$

$$= 4L\{t\} + L\left\{e^{3t}\right\}$$

Putting the conditions $y(0) = 1$ and $y'(0) = -1$, we get

$$\left[p^2.L\{y(t)\} - p \cdot 1 - (-1)\right] - 3[p \cdot L\{y(t)\} - 1] + 2L\{y(t)\} = \frac{4}{p^2} + \frac{1}{p-3}$$

i.e., $$(p^2 - 3p + 2) \cdot L\{y(t)\} = \frac{4}{p^2} + \frac{1}{p-3} + p - 4$$

i.e., $$(p-2)(p-1) \cdot L\{y(t)\} = \frac{4p - 12 + p^2}{p^2(p-3)} + p - 4$$

$$= \frac{(p-2)(p+6)}{p^2(p-3)} + p - 4$$

i.e., $$L\{y(t)\} = \frac{(p+6)}{p^2(p-3)(p-1)} + \frac{p-4}{(p-2)(p-1)} = F(p)$$

Therefore,

$$y(t) = L^{-1}\{F(p)\} = L^{-1}\left\{\frac{(p+6)}{p^2(p-3)(p-1)} + \frac{p-4}{(p-2)(p-1)}\right\}$$

$$= L^{-1}\left\{\frac{(p+6)}{p^2(p-3)(p-1)}\right\} + L^{-1}\left\{\frac{p-4}{(p-2)(p-1)}\right\} \tag{2}$$

Now, we apply the method of partial fraction to compute the inverse and let us consider

$$\frac{(p+6)}{p^2(p-3)(p-1)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p-3} + \frac{D}{p-1}$$

i.e., $$\frac{(p+6)}{p^2(p-3)(p-1)} = \frac{\left\{\begin{array}{l} Ap(p-3)(p-1) + B(p-3)(p-1) \\ +Cp^2(p-1) + Dp^2(p-3) \end{array}\right\}}{p^2(p-3)(p-1)}$$

Equating the numerators from both hand sides, we have

$$p+6 = Ap(p-3)(p-1) + B(p-3)(p-1) + Cp^2(p-1) + Dp^2(p-3)$$

Putting $p = 0, 1, 2, 3$ successively, we get

$$B = 2,\ D = -\frac{7}{2},\ -2A - B + 4C - 4D = 8 \text{ and } C = \frac{1}{2}$$

$$\Rightarrow A = 3,\ B = 2,\ C = \frac{1}{2} \text{ and } D = -\frac{7}{2}$$

Therefore,

$$\frac{(p+6)}{p^2\,(p-3)\,(p-1)} = \frac{3}{p} + \frac{2}{p^2} + \frac{1}{2}\cdot\frac{1}{p-3} - \frac{7}{2}\cdot\frac{1}{p-1} \tag{3}$$

Again, suppose

$$\frac{p-4}{(p-2)(p-1)} = \frac{E}{(p-2)} + \frac{F}{(p-1)} = \frac{(p-1)\,E + (p-2)\,F}{(p-2)(p-1)}$$

Equating the numerators from both hand sides, we have

$$p-4 = (p-1)E + (p-2)F$$

Putting $p = 1, 2$ successively, we get

$$F = 3 \text{ and } E = -2$$

Therefore,

$$\frac{p-4}{(p-2)(p-1)} = \frac{-2}{(p-2)} + \frac{3}{(p-1)} \tag{4}$$

Using (3) and (4) in (2), we obtain

$$y(t) = L^{-1}\left\{\frac{3}{p} + \frac{2}{p^2} + \frac{1}{2}\cdot\frac{1}{p-3} - \frac{7}{2}\cdot\frac{1}{p-1}\right\} + L^{-1}\left\{\frac{-2}{(p-2)} + \frac{3}{(p-1)}\right\}$$

$$= 3L^{-1}\left\{\frac{1}{p}\right\} + 2L^{-1}\left\{\frac{1}{p^2}\right\} + \frac{1}{2}\cdot L^{-1}\left\{\frac{1}{p-3}\right\} - \frac{7}{2}\cdot L^{-1}\left\{\frac{1}{p-1}\right\}$$

$$- 2L^{-1}\left\{\frac{1}{p-2}\right\} + 3L^{-1}\left\{\frac{1}{p-1}\right\}$$

$$= 3 + 2t + \frac{1}{2}\cdot e^{3t} - \frac{7}{2}\cdot e^{t} - 2e^{2t} + 3e^{t}$$

$$= 3 + 2t - \frac{1}{2}\cdot e^{t} - 2e^{2t} + \frac{1}{2}\cdot e^{3t}$$

So, the required solution is

$$y(t) = 3 + 2t - \frac{1}{2} \cdot e^t - 2e^{2t} + \frac{1}{2} \cdot e^{3t}.$$

Example 12.4 **Solve the following differential equation by Laplace transform**

$$(D^2 - 1)y = \alpha \cos\ \text{h}(nt)$$

where $y(0) = 0$ **and** $y'(0) = 2$ [WBUT 2006]

Sol. The given equation is

$$(D^2 - 1)y = \alpha \cos\ \text{h}(nt)$$

$$\text{i.e.,} \quad y''(t) - y(t) = \alpha \cos\ \text{h}(nt) \tag{1}$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{y''(t)\right\} - L\{y(t)\} = \alpha \cdot L\{\cos\ \text{h}(nt)\}$$

$$\Rightarrow \left[p^2 \cdot L\{y(t)\} - p \cdot y(0) - y'(0)\right] - L\{y(t)\} = \alpha \cdot \frac{p}{p^2 - n^2}$$

Putting the conditions $y(0) = 0$ and $y'(0) = 2$, we get

$$\left[p^2 \cdot L\{y(t)\} - p \cdot 0 - 2\right] - L\{y(t)\} = \frac{\alpha p}{p^2 - n^2}$$

$$\text{i.e.,} \quad (p^2 - 1) \cdot L\{y(t)\} = \frac{\alpha p}{p^2 - n^2} + 2$$

$$\text{i.e.,} \quad L\{y(t)\} = \frac{\alpha p}{(p^2 - 1)\left(p^2 - n^2\right)} + \frac{2}{(p^2 - 1)} = F(p)$$

Therefore,

$$y(t) = L^{-1}\{F(p)\} = L^{-1}\left\{\frac{\alpha p}{(p^2 - 1)\left(p^2 - n^2\right)} + \frac{2}{(p^2 - 1)}\right\}$$

$$= \alpha \cdot L^{-1}\left\{\frac{p}{(p^2 - 1)\left(p^2 - n^2\right)}\right\} + 2 \cdot L^{-1}\left\{\frac{1}{(p^2 - 1)}\right\} \tag{2}$$

Suppose,

$$\frac{p}{(p^2 - 1)\left(p^2 - n^2\right)} = \frac{Ap + B}{(p^2 - 1)} + \frac{Cp + D}{\left(p^2 - n^2\right)}$$

$$\text{i.e.,} \quad \frac{p}{(p^2 - 1)\left(p^2 - n^2\right)} = \frac{(Ap + B)\left(p^2 - n^2\right) + (Cp + D)(p^2 - 1)}{(p^2 - 1)\left(p^2 - n^2\right)}$$

$$= \frac{(A+C)p^3 + (B+D)p^2 + (-n^2A - C)p + (-n^2B - D)}{(p^2-1)\left(p^2-n^2\right)}$$

Equating the coefficients of like powers of p in the numerator of both hand sides, we get

$$\left[\begin{array}{l} A+C=0 \Rightarrow C=-A \\ -n^2A-C=1 \Rightarrow -n^2A+A=1 \Rightarrow A=\dfrac{1}{1-n^2} \end{array}\right] \Rightarrow A=\frac{1}{1-n^2},\ C=-\frac{1}{1-n^2}$$

$$\left[\begin{array}{l} B+D=0 \Rightarrow D=-B \\ -n^2B-D=0 \Rightarrow -n^2B+B=0 \Rightarrow (1-n^2)B=0 \end{array}\right] \Rightarrow B=0,\ D=0$$

Therefore,

$$\frac{p}{(p^2-1)\left(p^2-n^2\right)} = \frac{1}{1-n^2}\cdot\frac{p}{(p^2-1)} - \frac{1}{1-n^2}\cdot\frac{p}{\left(p^2-n^2\right)} \tag{3}$$

Using (3) in (2), we get

$$\begin{aligned} y(t) &= \alpha \cdot L^{-1}\left\{\frac{1}{1-n^2}\cdot\frac{p}{(p^2-1)} - \frac{1}{1-n^2}\cdot\frac{p}{\left(p^2-n^2\right)}\right\} \\ &\quad + 2\cdot L^{-1}\left\{\frac{1}{(p^2-1)}\right\} \\ &= \frac{\alpha}{1-n^2}\cdot L^{-1}\left\{\frac{p}{(p^2-1)}\right\} - \frac{\alpha}{1-n^2}\cdot L^{-1}\left\{\frac{p}{\left(p^2-n^2\right)}\right\} \\ &\quad + 2\cdot L^{-1}\left\{\frac{1}{(p^2-1)}\right\} \\ &= \frac{\alpha}{1-n^2}\cdot\cos\ \mathrm{h}t - \frac{\alpha}{1-n^2}\cdot\cos\ \mathrm{h}(nt) + 2\sin \mathrm{h}t \end{aligned}$$

So, the required solution is

$$y(t) = \frac{\alpha}{1-n^2}\cdot\cos\ \mathrm{h}t - \frac{\alpha}{1-n^2}\cdot\cos\ \mathrm{h}(nt) + 2\sin\ \mathrm{h}t.$$

Example 12.5 **Solve the following differential equation by Laplace transform**

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$$

where $y(0) = 0$ **and** $y'(0) = 1$. [WBUT 2008]

Sol. The given equation is

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=e^{-t}\sin t$$

i.e., $$y''+2y'+5y=e^{-t}\sin t \tag{1}$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{y''(t)\right\}+2L\left\{y'(t)\right\}+5L\left\{y(t)\right\}=L\left\{e^{-t}\sin t\right\}$$

$$\Rightarrow\left[p^2\cdot L\left\{y(t)\right\}-p\cdot y(0)-y'(0)\right]+2\left[p\cdot L\left\{y(t)\right\}-y(0)\right]+5L\left\{y(t)\right\}$$
$$=L\left\{e^{-t}\sin t\right\}$$

Since $L\left\{\sin t\right\}=\dfrac{1}{p^2+1}$, by first shifting theorem $L\left\{e^{-t}\sin t\right\}=\dfrac{1}{(p+1)^2+1}$
Putting the conditions $y(0)=0$ and $y'(0)=1$, we get

$$\left[p^2\cdot L\left\{y(t)\right\}-p\cdot 0-1\right]+2\left[p\cdot L\left\{y(t)\right\}-0\right]+5L\left\{y(t)\right\}=\frac{1}{(p+1)^2+1}$$

i.e., $$(p^2+2p+5)\cdot L\left\{y(t)\right\}=\frac{1}{(p^2+2p+2)}+1$$

i.e., $$L\left\{y(t)\right\}=\frac{1}{(p^2+2p+2)(p^2+2p+5)}+\frac{1}{(p^2+2p+5)}=F(p)$$

Therefore,

$$\begin{aligned}y(t)&=L^{-1}\left\{F(p)\right\}\\&=L^{-1}\left\{\frac{1}{(p^2+2p+2)(p^2+2p+5)}+\frac{1}{(p^2+2p+5)}\right\}\end{aligned}\tag{2}$$

Again, it is obvious that

$$\begin{aligned}\frac{1}{(p^2+2p+2)(p^2+2p+5)}&=\frac{1}{3}\left\{\frac{(p^2+2p+5)-(p^2+2p+2)}{(p^2+2p+2)(p^2+2p+5)}\right\}\\&=\frac{1}{3}\frac{1}{(p^2+2p+2)}-\frac{1}{3}\frac{1}{(p^2+2p+5)}\end{aligned}$$

So, from (2) we obtain

$$
\begin{aligned}
y(t) &= L^{-1}\left\{\frac{1}{3}\frac{1}{(p^2+2p+2)} - \frac{1}{3}\frac{1}{(p^2+2p+5)} + \frac{1}{(p^2+2p+5)}\right\} \\
&= L^{-1}\left\{\frac{1}{3}\frac{1}{(p^2+2p+2)} + \frac{2}{3}\frac{1}{(p^2+2p+5)}\right\} \\
&= \frac{1}{3}L^{-1}\left\{\frac{1}{(p^2+2p+2)}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{(p^2+2p+5)}\right\} \\
&= \frac{1}{3}L^{-1}\left\{\frac{1}{(p+1)^2+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{(p+1)^2+4}\right\} \\
&= \frac{1}{3}\cdot e^{-t}\cdot L^{-1}\left\{\frac{1}{p^2+1}\right\} + \frac{1}{3}\cdot e^{-t}\cdot L^{-1}\left\{\frac{2}{p^2+2^2}\right\} \\
&\quad \text{(by first shifting theorem)} \\
&= \frac{1}{3}\cdot e^{-t}\cdot \sin t + \frac{1}{3}\cdot e^{-t}\cdot \sin 2t = \frac{1}{3}\cdot e^{-t}\,(\sin t + \sin 2t)
\end{aligned}
$$

So, the required solution is

$$
y(t) = \frac{1}{3}\cdot e^{-t}\,(\sin t + \sin 2t)\,.
$$

Example 12.6 **Solve the following differential equation by Laplace transform**

$$
\frac{d^2y}{dt^2} + y = t\cos t
$$

where $y(0) = 0$ **and** $y'(0) = 0$

Sol. The given equation is

$$
\frac{d^2y}{dt^2} + 4y = t\cos t
$$

i.e.,
$$
y'' + 4y = t\cos t \tag{1}
$$

Taking Laplace transform on both hand sides of (1), we have

$$
\begin{aligned}
& L\left\{y''(t)\right\} + 4L\{y(t)\} = L\{t\cos t\} \\
\Rightarrow & \left[p^2\cdot L\{y(t)\} - p\cdot y(0) - y'(0)\right] + 4L\{y(t)\} = L\{t\cos t\}
\end{aligned}
$$

Since $L\{\cos t\} = \dfrac{p}{p^2+1}$,

$$L\{t\cos t\} = (-1)\frac{d}{dp}\left\{\frac{p}{p^2+1}\right\} = -\frac{(p^2+1)\cdot 1 - p(2p)}{(p^2+1)^2}$$

$$= \frac{(p^2-1)}{(p^2+1)^2} = \frac{(p^2+1)-2}{(p^2+1)^2} = \frac{1}{p^2+1} - \frac{2}{(p^2+1)^2}$$

$$i.e., \quad L\{t\cos t\} = \frac{1}{p^2+1} - \frac{2}{(p^2+1)^2} \tag{2}$$

Putting the conditions $y(0) = 0$ and $y'(0) = 0$, we get

$$\left[p^2 \cdot L\{y(t)\} - p\cdot 0 - 0\right] + 4L\{y(t)\} = \frac{1}{p^2+1} - \frac{2}{(p^2+1)^2}$$

$$i.e., \qquad (p^2+4)\cdot L\{y(t)\} = \frac{1}{p^2+1} - \frac{2}{(p^2+1)^2}$$

$$i.e., L\{y(t)\} = \frac{1}{(p^2+4)(p^2+1)} - \frac{2}{(p^2+4)(p^2+1)^2} = F(p), \text{ (say)}$$

Therefore,

$$y(t) = L^{-1}\{F(p)\} = L^{-1}\left\{\frac{1}{(p^2+4)(p^2+1)} - \frac{2}{(p^2+4)(p^2+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(p^2+4)(p^2+1)}\right\} - L^{-1}\left\{\frac{2}{(p^2+4)(p^2+1)^2}\right\} \tag{3}$$

Now, we apply the method of partial fraction to compute the inverse and let us consider

$$\frac{2}{(p^2+4)(p^2+1)^2} = \frac{A}{(p^2+4)} + \frac{B}{(p^2+1)} + \frac{C}{(p^2+1)^2}$$

$$i.e., \quad \frac{2}{(p^2+4)(p^2+1)^2} = \frac{A(p^2+1)^2 + B(p^2+4)(p^2+1) + C(p^2+4)}{(p^2+4)(p^2+1)^2}$$

$$= \frac{(A+B)p^4 + (2A+5B+C)p^2 + (A+4B+4C)}{(p^2+4)(p^2+1)^2}$$

Equating the coefficients of like powers of p in the numerator of both hand sides, we get

$$A+B=0,\ 2A+5B+C=0,\ A+4B+4C=2$$

$$\left[\begin{array}{c} 2A + 5B + C = 0 \\ \\ A + 4B + 4C = 2 \end{array}\right] \Rightarrow 7A + 16B = -2$$

$$\left[\begin{array}{c} A + B = 0 \\ \\ 7A + 16B = -2 \end{array}\right] \Rightarrow A = \frac{2}{9},\ B = \frac{-2}{9}$$

Putting $A = \frac{2}{9}$, $B = \frac{-2}{9}$ in $2A + 5B + C = 0$, we have $C = \frac{2}{3}$.
Therefore,

$$\frac{2}{(p^2+4)\left(p^2+1\right)^2} = \frac{2}{9} \cdot \frac{1}{(p^2+4)} - \frac{2}{9} \cdot \frac{1}{\left(p^2+1\right)} + \frac{2}{3} \cdot \frac{1}{\left(p^2+1\right)^2} \tag{4}$$

Again,

$$\begin{aligned} \frac{1}{(p^2+4)\left(p^2+1\right)} &= \frac{1}{3}\left\{\frac{(p^2+4) - \left(p^2+1\right)}{(p^2+4)\left(p^2+1\right)}\right\} \\ &= \frac{1}{3}\left\{\frac{1}{\left(p^2+1\right)} - \frac{1}{(p^2+4)}\right\} \end{aligned} \tag{5}$$

Using (4) and (5) in (3), we have

$$\begin{aligned} y(t) = &\frac{1}{3} \cdot L^{-1}\left\{\frac{1}{\left(p^2+1\right)} - \frac{1}{(p^2+4)}\right\} \\ &- L^{-1}\left\{\frac{2}{9} \cdot \frac{1}{(p^2+4)} - \frac{2}{9} \cdot \frac{1}{\left(p^2+1\right)} + \frac{2}{3} \cdot \frac{1}{\left(p^2+1\right)^2}\right\} \\ = &\frac{1}{3} \cdot L^{-1}\left\{\frac{1}{\left(p^2+1\right)}\right\} - \frac{1}{3} \cdot L^{-1}\left\{\frac{1}{(p^2+4)}\right\} - \frac{2}{9} \cdot L^{-1}\left\{\frac{1}{(p^2+4)}\right\} \\ &+ \frac{2}{9} \cdot L^{-1}\left\{\frac{1}{\left(p^2+1\right)}\right\} - \frac{2}{3} \cdot L^{-1}\left\{\frac{1}{\left(p^2+1\right)^2}\right\} \\ = &\frac{5}{9} \cdot L^{-1}\left\{\frac{1}{\left(p^2+1\right)}\right\} - \frac{5}{9} \cdot L^{-1}\left\{\frac{1}{(p^2+4)}\right\} - \frac{2}{3} \cdot L^{-1}\left\{\frac{1}{\left(p^2+1\right)^2}\right\} \\ = &\frac{5}{9}\sin t - \frac{5}{9}\frac{\sin 2t}{2} - \frac{1}{3} \cdot L^{-1}\left\{\frac{2}{\left(p^2+1\right)^2}\right\} \end{aligned}$$

By (2), we have

$$t\cos t = L^{-1}\left\{\frac{1}{p^2+1}\right\} - L^{-1}\left\{\frac{2}{\left(p^2+1\right)^2}\right\}$$

$$i.e.,\ L^{-1}\left\{\frac{2}{\left(p^2+1\right)^2}\right\} = L^{-1}\left\{\frac{1}{p^2+1}\right\} - t\cos t = \sin t - t\cos t$$

Therefore, the required solution is

$$\begin{aligned} y(t) &= \frac{5}{9}\sin t - \frac{5}{9}\cdot\frac{\sin 2t}{2} - \frac{1}{3}\cdot(\sin t - t\cos t) \\ &= \frac{2}{9}\sin t - \frac{5}{18}\sin 2t + \frac{1}{3}t\cos t. \end{aligned}$$

Example 12.7 **Solve by method of Laplace transform**

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = \sin x$$

given $y(0) = 0$ **and** $y'(0) = 0$. [WBUT 2004]

Sol. The given equation is

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = \sin x$$

$$i.e., \quad y''(x) + 2y'(x) - 3y(x) = \sin x. \tag{1}$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{y''(x)\right\} + 2L\left\{y'(x)\right\} - 3L\left\{y(x)\right\} = L\left\{\sin x\right\}$$

$$\Rightarrow \left[p^2\cdot L\{y(x)\} - p\cdot y(0) - y'(0)\right] + 2\left[p\cdot L\{y(x)\} - y(0)\right] - 3L\{y(x)\} = \frac{1}{p^2+1}$$

Putting the conditions $y(0) = 0$ and $y'(0) = 0$, we get

$$\left[p^2\cdot L\{y(x)\} - p\cdot 0 - 0\right] + 2\left[p\cdot L\{y(x)\} - 0\right] - 3L\{y(x)\} = \frac{1}{p^2+1}$$

$$i.e., \quad (p^2 + 2p - 3)\cdot L\{y(x)\} = \frac{1}{\left(p^2+1\right)}$$

$$\begin{aligned} i.e., \quad L\{y(x)\} &= \frac{1}{(p^2+2p-3)\left(p^2+1\right)} \\ &= \frac{1}{(p-1)(p+3)\left(p^2+1\right)} = F(p) \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= L^{-1}\{F(p)\} \\ &= L^{-1}\left\{\frac{1}{(p-1)(p+3)\left(p^2+1\right)}\right\} \end{aligned} \tag{2}$$

Now, we apply the method of partial fraction to compute the inverse and let us consider

$$\begin{aligned} \frac{1}{(p-1)(p+3)\left(p^2+1\right)} &= \frac{A}{(p-1)} + \frac{B}{(p+3)} + \frac{Cp+D}{\left(p^2+1\right)} \\ &= \frac{\left\{\begin{array}{l} A\left(p^2+1\right)(p+3) + B(p-1)\left(p^2+1\right) \\ +(Cp+D)(p-1)(p+3) \end{array}\right\}}{(p-1)(p+3)\left(p^2+1\right)} \end{aligned}$$

Equating the numerators from both hand sides, we have

$$1 = A(p^2+1)(p+3) + B(p-1)(p^2+1) + (Cp+D)(p-1)(p+3)$$

Putting $p = 1, -3, 0, -1$ successively we get

$$A = \frac{1}{8},\ B = \frac{-1}{40},\ 3A - B - 3D = 1 \text{ and } 4A - 4B + 4C - 4D = 1$$

$$\Rightarrow A = \frac{1}{8},\ B = \frac{-1}{40},\ C = \frac{-1}{10},\ D = \frac{-1}{5}.$$

Therefore,

$$\begin{aligned} \frac{1}{(p-1)(p+3)\left(p^2+1\right)} &= \frac{1}{8}\cdot\frac{1}{(p-1)} - \frac{1}{40}\cdot\frac{1}{(p+3)} \\ &\quad - \frac{1}{10}\cdot\frac{p}{\left(p^2+1\right)} - \frac{1}{5}\cdot\frac{1}{\left(p^2+1\right)} \end{aligned} \tag{3}$$

Using (3) in (2), we obtain

$$\begin{aligned} y(x) &= \frac{1}{8}\cdot L^{-1}\left\{\frac{1}{(p-1)}\right\} - \frac{1}{40}\cdot L^{-1}\left\{\frac{1}{(p+3)}\right\} \\ &\quad - \frac{1}{10}\cdot L^{-1}\left\{\frac{p}{\left(p^2+1\right)}\right\} - \frac{1}{5}\cdot L^{-1}\left\{\frac{1}{\left(p^2+1\right)}\right\} \\ &= \frac{1}{8}e^{x} - \frac{1}{40}e^{-x} - \frac{1}{10}\cos x - \frac{1}{5}\sin x \end{aligned}$$

Hence, the required solution is

$$y(x) = \frac{1}{8}e^{x} - \frac{1}{40}e^{-x} - \frac{1}{10}\cos x - \frac{1}{5}\sin x.$$

Example 12.8 **Solve the following differential equation by Laplace transform**

$$\frac{d^2x}{dt^2} + 4x = \sin 3t$$

where $x(0) = 0$ **and** $x'(0) = 0$ [WBUT 2007]

Sol. The given equation is

$$\frac{d^2x}{dt^2} + 4x = \sin 3t$$

or, $$x''(t) + 4x(t) = \sin 3t \qquad (1)$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{x''(t)\right\} + 4L\{x(t)\} = L\{\sin 3t\}$$

$$\Rightarrow \left[p^2 \cdot L\{x(t)\} - p \cdot x(0) - x'(0)\right] + 4L\{x(t)\} = \frac{3}{p^2+9}.$$

Putting the conditions $x(0) = 0$ and $x'(0) = 0$, we get

$$\left[p^2 \cdot L\{x(t)\} - p \cdot 0 - 0\right] + 4L\{x(t)\} = \frac{3}{p^2+9}$$

i.e., $$(p^2+4) \cdot L\{x(t)\} = \frac{3}{p^2+9}$$

i.e., $$L\{x(t)\} = \frac{3}{\left(p^2+9\right)(p^2+4)} = F(p)$$

Therefore,

$$x(t) = L^{-1}\{F(p)\} = L^{-1}\left\{\frac{3}{\left(p^2+9\right)(p^2+4)}\right\} \qquad (2)$$

Again,

$$\frac{3}{\left(p^2+9\right)(p^2+4)} = \frac{3}{5}\left\{\frac{\left(p^2+9\right) - \left(p^2+4\right)}{\left(p^2+9\right)(p^2+4)}\right\}$$

$$= \frac{3}{5} \cdot \left\{\frac{1}{(p^2+4)} - \frac{1}{\left(p^2+9\right)}\right\} \qquad (3)$$

Using (3) in (2), we get

$$x(t) = \frac{3}{5} \cdot L^{-1} \left\{ \frac{1}{(p^2+4)} - \frac{1}{(sp^2+9)} \right\}$$

$$= \frac{3}{5} \cdot \left[L^{-1} \left\{ \frac{1}{(p^2+2^2)} \right\} - L^{-1} \left\{ \frac{1}{(p^2+3^2)} \right\} \right]$$

$$= \frac{3}{5} \cdot \left[\frac{\sin 2t}{2} - \frac{\sin 3t}{3} \right] = \frac{1}{10} [3 \sin 2t - 2 \sin 3t]$$

So, the required solution is

$$x(t) = \frac{1}{10} [3 \sin 2t - 2 \sin 3t] .$$

Example 12.9 **Solve the following differential equation by Laplace transform**

$$\frac{d^2y}{dt^2} + 9y = 1$$

where $y(0) = 0$ **and** $y\left(\frac{\pi}{2}\right) = -1$ [WBUT 2008]

Sol. The given equation is

$$\frac{d^2y}{dt^2} + 9y = 1$$

or, $$y''(t) + 9y(t) = 1 \qquad (1)$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{y''(t)\right\} + 9L\{y(t)\} = L\{1\}$$

$$\Rightarrow \left[p^2 \cdot L\{y(t)\} - p \cdot y(0) - y'(0)\right] + 9L\{y(t)\} = \frac{1}{p}$$

Suppose $y'(0) = k$ (constant).
Putting the conditions $y(0) = 1$ and $y'(0) = k$, we get

$$\left[p^2 \cdot L\{y(t)\} - p \cdot 1 - k\right] + 9L\{y(t)\} = \frac{1}{p}$$

i.e., $$(p^2+9) \cdot L\{y(t)\} = \frac{1}{p} + p + k$$

i.e., $$L\{y(t)\} = \frac{1}{p(p^2+9)} + \frac{p}{(p^2+9)} + \frac{k}{(p^2+9)} = F(p), \text{ (say)}$$

Therefore,

$$y(t) = L^{-1}\{F(p)\} = L^{-1}\left\{\frac{1}{p(p^2+9)} + \frac{p}{(p^2+9)} + \frac{k}{(p^2+9)}\right\}$$

$$= L^{-1}\left\{\frac{1}{p(p^2+9)}\right\} + L^{-1}\left\{\frac{p}{(p^2+9)}\right\} + L^{-1}\left\{\frac{k}{(p^2+9)}\right\} \tag{2}$$

Again,

$$L^{-1}\left\{\frac{1}{(p^2+9)}\right\} = \frac{\sin(3t)}{3}$$

So, $$L^{-1}\left\{\frac{1}{p(p^2+9)}\right\} = \int_0^t \frac{\sin(3x)}{3}\,dx,$$ by **Th. 11.9 of Ch. 11**

$$= \left[-\frac{\cos(3x)}{9}\right]_0^t = \frac{1}{9} - \frac{1}{9}\cos(3t) \tag{3}$$

Using (3) in (2), we get

$$y(t) = \frac{1}{9} - \frac{1}{9}\cos(3t) + \cos(3t) + k\cdot\frac{\sin(3t)}{3}$$

$$= \frac{1}{9} + \frac{8}{9}\cos(3t) + k\cdot\frac{\sin(3t)}{3} \tag{4}$$

Using $y\left(\frac{\pi}{2}\right) = -1$ in (4), we obtain

$$y\left(\frac{\pi}{2}\right) = \frac{1}{9} + \frac{8}{9}\cos\left(3\frac{\pi}{2}\right) + k\cdot\frac{\sin\left(3\frac{\pi}{2}\right)}{3}$$

i.e., $$-1 = \frac{1}{9} + \frac{8}{9}\cdot 0 + k\cdot\frac{(-1)}{3}$$

or, $$k = \frac{10}{3}.$$

Hence from (4), we get the required solution as

$$y(t) = \frac{1}{9} + \frac{8}{9}\cos(3t) + \frac{10}{3}\cdot\frac{\sin(3t)}{3}$$

$$= \frac{1}{9}\left(1 + 8\cos 3t + 10\sin 3t\right).$$

Example 12.10 **Solve the following differential equation by Laplace Transform**

$$\frac{d^2y}{dt^2} + k^2 y = f(t)$$

and express the general solution in terms of constant k and $f(t)$.

Sol. Here first we consider the initial conditions as $y(0) = A$ and $y'(0) = B$. The given equation is

$$\frac{d^2y}{dt^2} + k^2y = f(t)$$

$$or, \qquad y''(t) + k^2y(t) = f(t) \tag{1}$$

Taking Laplace transform on both hand sides of (1), we have

$$L\left\{y''(t)\right\} + k^2L\{y(t)\} = L\{f(t)\}$$

$$\Rightarrow \left[p^2 \cdot L\{y(t)\} - p \cdot y(0) - y'(0)\right] + k^2L\{y(t)\} = \overline{f}(p),$$

where $L\{f(t)\} = \overline{f}(p)$, say.
Putting the conditions $y(0) = A$ and $y'(0) = B$, we get

$$\left[p^2 \cdot L\{y(t)\} - p \cdot A - B\right] + k^2L\{y(t)\} = \overline{f}(p)$$

$$i.e., \qquad (p^2 + k^2) \cdot L\{y(t)\} = pA + B + \overline{f}(p) \cdot$$

$$i.e., \; L\{y(t)\} = \frac{Ap}{(p^2+k^2)} + \frac{B}{(p^2+k^2)} + \frac{\overline{f}(p)}{(p^2+k^2)} = F(p), \text{ (say)}$$

Therefore,

$$y(t) = L^{-1}\{F(p)\} = L^{-1}\left\{\frac{Ap}{(p^2+k^2)} + \frac{B}{(p^2+k^2)} + \frac{\overline{f}(p)}{(p^2+k^2)}\right\}$$

$$= AL^{-1}\left\{\frac{p}{(p^2+k^2)}\right\} + BL^{-1}\left\{\frac{1}{(p^2+k^2)}\right\} + L^{-1}\left\{\frac{\overline{f}(p)}{(p^2+k^2)}\right\}$$

$$= A\cos kt + B\frac{\sin kt}{k} + L^{-1}\left\{\frac{\overline{f}(p)}{(p^2+k^2)}\right\} \tag{2}$$

Again, by convolution theorem

$$L^{-1}\left\{\frac{\overline{f}(p)}{(p^2+k^2)}\right\} = L^{-1}\left\{\frac{1}{(p^2+k^2)} \cdot \overline{f}(p)\right\}$$

$$= \left(\frac{\sin kt}{k}\right) * f(t)$$

$$= \int_0^t \frac{\sin k(t-x)}{k} \cdot f(x)\,dx \tag{3}$$

Using (3) in (2), we get the solution as

$$y(t) = A\cos kt + B\frac{\sin kt}{k} + \int_0^t \frac{\sin k\,(t-x)}{k}\cdot f(x)\,dx.$$

EXERCISES

Short and Long Answer type Questions

Solve the following differential equations by Laplace transform:

1) $\dfrac{dy}{dt} - y = 1$ where $y(0) = 2$

[**Ans:** $y = 1 + e^{-t}$]

2) $\dfrac{dy}{dt} - y = 0$ where $y(0) = A$

[**Ans:** $y = Ae^{-t}$]

3) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 2y = 0$ where $y(0) = 1$ and $y'(0) = 1$

[**Ans:** $y = e^t \cos t$]

4) $\dfrac{d^2y}{dt^2} + y = t$ where $y(0) = 1$ and $y'(0) = -2$

[**Ans:** $y = t + \cos t - 3\sin t$]

5) $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = 4e^{2t}$ where $y(0) = -3$ and $y'(0) = 5$

[**Ans:** $y = -7e^t + 4e^{2t} + 4te^{2t}$]

6) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = e^{-t}\sin t$ where $y(0) = 0$ and $y'(0) = 1$ [WBUT 2008]

[**Ans:** $y = \dfrac{1}{3}\cdot e^{-t}(\sin t + \sin 2t)$]

7) $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 9y = \sin t$ where $y(0) = 1$ and $y'(0) = 0$

[**Ans:** $y = \dfrac{1}{50}(53 + 155t)e^{-3t} - 3\cos t + 4\sin t$]

8) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 3y = t\cos t$ where $y(0) = 0$ and $y'(0) = 0$ [WBUT 2010]

[**Ans:** $y = \dfrac{-3}{32}e^{3t} - \dfrac{1}{32}e^{-t} + \dfrac{1}{4}t\cos t + \dfrac{1}{8}\cos t$]

9) $\dfrac{d^2y}{dt^2} - (a+b)\dfrac{dy}{dt} - aby = 0, a \neq b$

[**Ans:** $y = c_1 e^{at} + c_2 e^{bt}$, where $c_1,\ c_2$ are arbitrary constants.]

10) $\dfrac{d^2y}{dt^2} + y = 6\cos 2t$ where $y(0) = 3$and $y'(0) = 1$

[**Ans:** $y = 5\cos t - 2\cos 2t + \sin t$]

11) $\dfrac{d^2y}{dt^2} + k^2 y = a\cos mt$ where $y(0) = 0$ and $y'(0) = 0$

$\left[\textbf{Ans: } y = \dfrac{a}{k^2 - m^2}\left(-\cos kt + \cos mt\right)\right]$

12) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + y = 0$ where $y(0) = 0$ and $y(1) = 2$

[**Ans:** $y = 2te^{t-1}$]

13) $y''(t) + y(t) = 8\cos t$ given $y(0) = 1$and $y'(0) = -1$ [WBUT 2005]

[**Ans:** $y = 4t\sin t + \cos t - \sin t$]

14) $\dfrac{d^3y}{dt^3} + \dfrac{dy}{dt} = e^{2t}$ where $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 0$

$\left[\textbf{Ans: } y = -\dfrac{1}{2} + \dfrac{1}{10}e^{2t} - \dfrac{1}{5}\sin t + \dfrac{2}{5}\cos t\right]$

15) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = \sin x$ given $y(0) = 0$ and $y'(0) = 0$ [WBUT 2004]

$\left[\textbf{Ans: } y = \dfrac{1}{8}e^{x} - \dfrac{1}{40}e^{-x} - \dfrac{1}{10}\cos x - \dfrac{1}{5}\sin x\right]$

16) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = t^2 + 2t$ given $y(0) = 4$ and $y'(0) = 2$

$\left[\textbf{Ans: } y = 2 + \dfrac{1}{3}t^3 + 2e^{-t}\right]$

17) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = 2(1 + t - t^2)$given $y(0) = 0$ and $y'(0) = 3$

[**Ans:** $y = t^2 - e^{-2t} + e^{t}$]

18) $\dfrac{d^2y}{dt^2} + 4y = H(t-2)$ where $H(t-2)$ is a unit step function and $y(0) = 0$ and $y'(0) = 1$

$\left[\textbf{Ans: } y = \dfrac{1}{2}\sin 2t\right]$

SOLUTIONS OF UNIVERSITY QUESTIONS (W.B.U.T)

B. TECH SEM-II (NEW) 2012
MATHEMATICS - II (M 201)

Time Alloted : 3 Hours *Full Marks : 70*

Group A (Multiple Choice Questions)

1. Choose the correct alternative from any ten of the following: (10 × 1 = 10)

(i) The integrating factor of

$$(2xy - 3y^3)dx + (4x^2 + 6xy^2)dy = 0$$

a) x^2y b) x^2y^2 c) xy^2 d) xy^3.

Solution: No alternative is correct.

(ii) The substitution $x = e^z$ transforms the differential equation

$$x^2\frac{d^2y}{dx^2} - 5y = \log_e x$$

to

a) $\frac{d^2y}{dz^2} + \frac{dy}{dz} - 5y = z$ b) $\frac{d^2y}{dz^2} - \frac{dy}{dz} + 5y = z$

c) $\frac{d^2y}{dz^2} - \frac{dy}{dz} + 3y = 0$ d) $\frac{d^2y}{dz^2} - \frac{dy}{dz} - 5y = z$

Solution: The correct alternative is $\boxed{\text{d}}$

(iii) If the differential equation

$$\left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \left(x - \frac{1}{y} + \frac{A}{xy^2}\right)dy = 0$$

is exact, then the value of A is

a) 2 b) 1 c) -1 d) 0

Solution: The correct alternative is $\boxed{\text{b}}$

(iv) The value of $\int_0^\infty e^{-x}x^{\frac{3}{2}}dx$ is

a) $\frac{3}{4}\sqrt{\pi}$ b) $\frac{5}{4}\sqrt{5}$ c) $\frac{3}{5}\sqrt{\pi}$ d) $\frac{1}{4}\sqrt{\pi}$.

Solution: The correct alternative is $\boxed{\text{a}}$

(v) The value of $\Gamma(6)$ is

a) 720 b) 5 c) 6 d) 120

Solution: The correct alternative is $\boxed{\text{d}}$

(vi) The Laplace transform of $e^{-3t}\sin 4t$ is

a) $\dfrac{4}{s^2+6s-7}$ b) $\dfrac{s}{s^2+6s-7}$ c) $\dfrac{1}{s^2+6s-7}$ d) $\dfrac{s}{s^2+6s+24}$

Solution: No alternative is correct.

(vii) The maximum number of edges in a connected graph of 7 vertices is

a) 6 b) 7 c) 21 d) 14

Solution: The correct alternative is $\boxed{\text{c}}$

(viii) The maximum degree of any vertex in a simple graph with 10 vertices is

a) 10 b) 5 c) 20 d) 9

Solution: The correct alternative is $\boxed{\text{d}}$

(ix) Tree is a connected graph without any

a) odd vertex b) even vertex

c) circuit d) pendant vertex

Solution: The correct alternative is $\boxed{\text{c}}$

(x) The improper integral $\displaystyle\int_0^1 \frac{dx}{(x-\alpha)^n}$ converges for

a) $n<1$ b) $n\geq 1$ c) $n=1$ d) none of these

Solution: The correct alternative is $\boxed{\text{a}}$

(xi) The particular integral of $(D^2-4D+4)y = x^3e^{2x}$ is

a) $\dfrac{e^{2x}x^4}{20}$ b) $\dfrac{e^{2x}x^5}{20}$ c) $\dfrac{e^{2x}x^4}{60}$ d) $\dfrac{e^{x}x^4}{20}$

Solution: The correct alternative is $\boxed{\text{b}}$

(xii) The inverse Laplace transform of $\left(\dfrac{4}{s^2-7}+\dfrac{2}{s^2+7}\right)$ is

a) $\dfrac{1}{7}\left\{4\cos(\sqrt{7}t)-2\sin(\sqrt{7}t)\right\}$

b) $\dfrac{1}{7}\left\{4\cos(7t)+2\sin(7t)\right\}$

c) $\dfrac{1}{\sqrt{7}}\left\{4\sin(\sqrt{7}t)+2\sinh(\sqrt{7}t)\right\}$

d) $\dfrac{1}{7}\left\{4\sinh(7t)-2\sin(7t)\right\}$

Solution: No alternative is correct.

(xiii) The general solution of $p = \log_e(px - y)$ is

a) $y = cx - c$ b) $y = cx - e^x$

c) $y = c^2x - e^x$ d) none of these

Solution: The correct alternative is $\boxed{\text{b}}$

Group B (Short Answer Type Questions)

Answer any three of the following. **3 × 5 = 15**

2. Solve

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log_e x \sin(\log_e x)$$

Solution: *See Example 4.2 on Page 4.39*

3. Evaluate

$$L^{-1}\left(\frac{s+4}{s(s-1)(s^2+4)}\right)$$

Solution: Let,

$$\left(\frac{s+4}{s(s-1)(s^2+4)}\right) = \frac{A}{s} + \frac{B}{(s-1)} + \frac{Cs+D}{(s^2+4)}$$

$$or, \left(\frac{s+4}{s(s-1)(s^2+4)}\right) = \frac{A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)s(s-1)}{s(s-1)(s^2+4)}$$

$$or, s+4 = A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)s(s-1)$$

Putting $s = 0$,

$$A = -1$$

Putting $s = 1$,

$$B = 1$$

Putting $s = -1$,

$$C - D = 1$$

Putting $s = 2$

$$2C + D = -2$$

Therefore,

$$C = -\frac{1}{3} \text{ and } D = -\frac{4}{3}$$

Therefore,

$$L^{-1}\left(\frac{s+4}{s(s-1)(s^2+4)}\right) = L^{-1}\left(\frac{-1}{s}\right) + L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-\frac{1}{3}s-\frac{4}{3}}{(s^2+4)}\right)$$

$$= -L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{3}L^{-1}\left(\frac{s}{(s^2+4)}\right) - \frac{2}{3}L^{-1}\left(\frac{2}{(s^2+4)}\right)$$

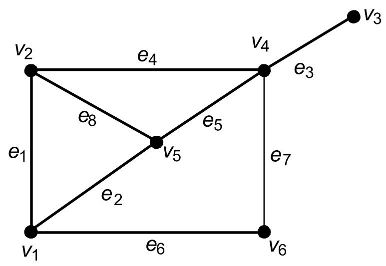
$$= -1 + e^t - \frac{1}{3}\cos 2t - \frac{2}{3}\sin 2t$$

4. Use Beta and Gamma function to evaluate

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx$$

Solution: *See Example 9.8 on Page 9.31*

5. Determine adjacency matrix of the following graph:



Solution: *See Example 6.2 on Page 6.24*

6. Solve

$$\frac{dx}{dt} + 3x + y = e^t, \frac{dy}{dt} - x + y = e^{2t}$$

Solution: The simultaneous differential equation are

$$\frac{dx}{dt} + 3x + y = e^t \tag{1}$$

and

$$\frac{dy}{dt} - x + y = e^{2t} \tag{2}$$

From Eq. (1) we have

$$y = e^t - 3x - \frac{dx}{dt} \tag{3}$$

Therefore from Eqs. (2) and (3), we have

$$\frac{d}{dt}(e^t - 3x - \frac{dx}{dt}) - x + e^t - 3x - \frac{dx}{dt} = e^{2t}$$

$$or, \quad e^t - 3\frac{dx}{dt} - \frac{d^2x}{dt^2} - x + e^t - 3x - \frac{dx}{dt} = e^{2t}$$

$$or, \quad \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 2e^t - e^{2t}$$

Let,

$$x = e^{mt}$$

be a trial solution of the reduced equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$$

then the auxilliary equation is

$$m^2 + 4m + 4 = 0$$

$$or, \quad m = -2, m = -2$$

Therefore the complementary function is

$$C.F = x_c = (C_1 t + C_2)e^{-2t}$$

where C_1 and C_2 are arbitary constants.

The particular integral is

$$P.I = x_p = \frac{1}{(D^2 + 4D + 4)}(2e^t - e^{2t})$$

$$= 2\frac{1}{(D^2 + 4D + 4)}e^t - \frac{1}{(D^2 + 4D + 4)}e^{2t}$$

$$= \frac{2}{9}e^t - \frac{1}{16}e^{2t}$$

Therefore,

$$x = x_c + x_p = (C_1 t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}$$

Now,

$$y = e^t - 3x - \frac{dx}{dt}$$

$$or, y = e^t - 3\left\{(C_1t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}\right\}$$
$$-\frac{d}{dt}\left\{(C_1t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}\right\}$$

$$or, y = e^t - 3\left\{(C_1t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}\right\}$$
$$-C_1(e^{-2t} - 2te^{-2t}) + 2C_2e^{-2t} - \frac{2}{9}e^t + \frac{1}{8}e^{2t}$$

$$or, y = -C_1(1+t)e^{-2t} - C_2e^{-2t} + \frac{1}{9}e^t + \frac{5}{16}e^{2t}$$

Therefore the solution is

$$x = (C_1t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}$$

and

$$y = -C_1(1+t)e^{-2t} - C_2e^{-2t} + \frac{1}{9}e^t + \frac{5}{16}e^{2t}$$

where C_1 and C_2 are arbitary constants.

GROUP C (Long Answer Type Questions)

Answer any three of the following. $3 \times 15 = 45$

7. a) Solve the following differential equation using Laplace transform

$$(D^2 + 2D + 5)y = e^{-t}\sin t, y(0) = 0, y'(0) = 1$$

Solution: *See Example 12.5 on Page 12.8*

b) Apply the variation of parameters to solve

$$\frac{d^2y}{dx^2} + y = \sec^3 x \tan x$$

Solution: *See Example 4.9 on Page 4.50*

c) Show that

$$\int_0^\infty e^{-4x}x^{\frac{3}{2}}dx = \frac{3}{128}\sqrt{\pi}$$

Solution: We know

$$\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}$$

therefore,

$$\int_0^\infty e^{-4x}x^{\frac{3}{2}}dx = \int_0^\infty e^{-4x}x^{\frac{5}{2}-1}dx$$

$$= \frac{\Gamma(\frac{5}{2})}{4^{\frac{5}{2}}} = \frac{\frac{3}{2}\Gamma(\frac{3}{2})}{2^5}$$

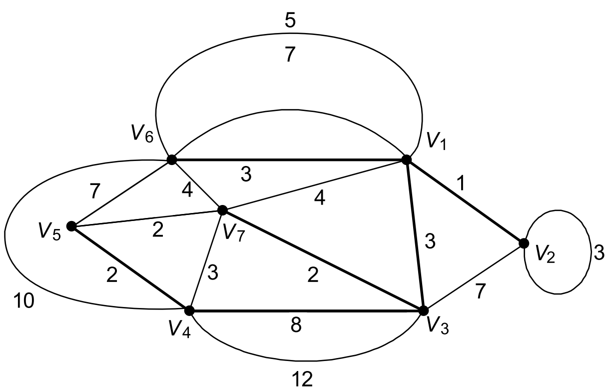
$$= \frac{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{2^5}$$

$$= \frac{3}{128}\sqrt{\pi} \text{ since } \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

8. a) Draw the graph whose incidence matrix is $\begin{pmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix}$

Solution: The graph is not possible

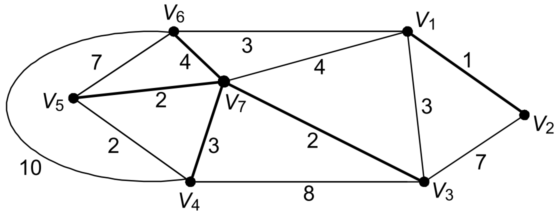
b) By Dijkstra's procedure, find the shortest path and the length of the shortest path from the vertex v_2 to v_5 in the following graph:



Solution: The given graph is not simple**.** We first discard the parallel edges having maximum weight and retaining the edge with minimum weight . We also delete the self-loop. Then we have the following.simple weighted connected graph:

The weight table $W = (w_{ij})_{7\times 7}$ formed on the basis of

w_{ij} = weight (or distance or cost) of the edge from vertex i to j,
$w_{ii} = 0$,
$w_{ij} = \infty$, if there is no edge from vertex i to j.



and is given by the following:

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	1	3	∞	∞	3	4
V_2	1	0	7	∞	∞	∞	∞
V_3	3	7	0	8	∞	∞	2
V_4	∞	∞	8	0	2	10	3
V_5	∞	∞	∞	2	0	7	2
V_6	3	∞	∞	10	7	0	4
V_7	4	∞	2	3	2	4	0

Here we are to find the shortest path from the vertex V_2 to the vertex V_5. So we start our computation by assigning permanent label 0 to the vertex V_2 i.e., $L(V_2) = 0$ and temporary label ∞ to all others. Permanent label is shown by enclosing in a square $(\square)$ in the computation table. Now at every stage, we compute temporary labels for all the vertices except those which have already permanent labels and minimum of them will get permanent label. We continue this process until the destination vertex V_5 gets the permanent label.

Temporary label of vertex j, which is not yet permanently labelled is given by

$$\boxed{L(j) = \min\{\text{old } L(j),\ (\text{old } L(i) + w_{ij})\}}$$

where i is the latest vertex permanently labelled in the last stage and w_{ij} is the direct distance between the vertices i and j.

The computation is shown in the following table:

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	
St. 1	∞	$\boxed{0}$	∞	∞	∞	∞	∞	: V_2 has got permanent label 0 and all others have temporary label ∞.
St. 2	1	$\boxed{0}$	7	∞	∞	∞	∞	: Calculation of temporary labels and 1 is the minimum among all.
St. 3	$\boxed{1}$	$\boxed{0}$	7	∞	∞	∞		: V_1 has got permanent label

St. 4	$\boxed{1}$	$\boxed{0}$	4	∞	∞	4	5	: Calculation of temporary labels and 4 is the minimum among all.
St. 5	$\boxed{1}$	$\boxed{0}$	$\boxed{4}$	∞	∞	4	5	$\because V_3$ has got permanent label
St. 6	$\boxed{1}$	$\boxed{0}$	$\boxed{4}$	12	∞	∞	6	: Calculation of temporary labels and 6 is the minimum among all.
St. 7	$\boxed{1}$	$\boxed{0}$	$\boxed{4}$	12	∞	∞	$\boxed{6}$	$\because V_7$ has got permanent label
St. 8	$\boxed{1}$	$\boxed{0}$	$\boxed{4}$	9	8	10	$\boxed{6}$	: Calculation of temporary labels and 8 is the minimum among all.
St. 9	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	9	$\boxed{8}$	10	$\boxed{6}$	: Destination vertex V_5 has got permanent label.

In the final stage 9 of the table the destination vertex V_5 has permanent label and its value is 8. So the required shortest distance is 8.

Now we apply backtrack technique for finding shortest path. Starting from permanent label of V_5 (from Stage 9) we traverse back and see that in Stage 7, it is changed and at that Stage V_7 has got permanent label. So we move to V_7. Again doing the similar thing we see that in Stage 5, the label of V_7 is changed and at that Stage V_3 has got a permanent label. So we move to V_3. Now if we apply similar technique, then in Stage 3 the label of V_3 is changed and at that stage V_1 has got permanent label. So we we move to V_1. Now if we apply similar technique, then in Stage 1 the label of V_1 is changed and at that stage V_2 has got permanent label which is the starting vertex. So we stop the process.

Hence, the shortest path is given by

$$V_2 \to V_1 \to V_3 \to V_7 \to V_5.$$

c) Solve

$$y = 2px - p^2.$$

Solution: Differentiating w.r.t x, we have

$$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$or, \quad 2(p-x)\frac{dp}{dx} = p$$

$$or, \quad \frac{dx}{dp} = 2 - \frac{2x}{p}$$

$$or, \quad \frac{dx}{dp} + \frac{2x}{p} = 2$$

which is a linear equation.

The integrating factor is

$$I.F = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$

Muliplying the I.F. with the differential equation it reduces to

$$\frac{d}{dp}(p^2x) = 2p^2$$

Integrating we have

$$p^2x = \frac{2}{3}p^3 + c$$

$$or, \qquad x = \frac{2}{3}p + cp^{-2}$$

where c is an arbitrary constant.

Therefore,

$$y = 2p(\frac{2}{3}p + cp^{-2}) - p^2 = \frac{1}{3}p^2 + 2cp^{-1}$$

9. a) Examine whether the differential equation

$$\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2ydy = 0$$

is exact or not and then solve it.

Solution: *See Example 8 on Page 2.11*

b) Prove that a complete graph of n vertices has $\frac{n(n-1)}{2}$ number of edges.

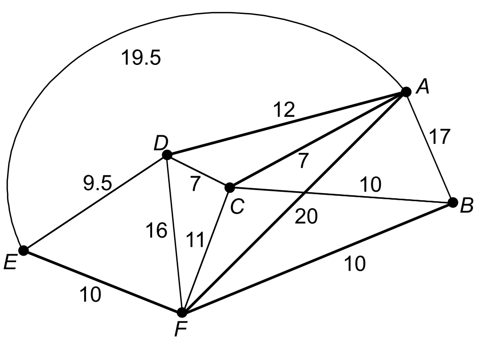
Solution: *See Theorem 5.7 on Page 5.12*

c) Apply convolution theorem to evaluate

$$L^{-1}\left(\frac{1}{(s^2+2s+5)^2}\right)$$

Solution: *See Example 11.17 on Page 11.31*

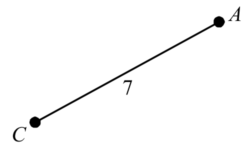
10. a) By Kruskal's algorithm, find a minimal (or shortest) spanning tree and the corresponding weight of the spanning tree of the following graph:



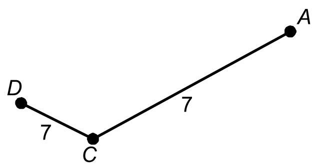
Solution: First we arrange all the edges of G, except the loops in the order of non-decreasing weights and write in the following form:

Edges	**AC**	**DC**	DE	**BC**	**BF**	**EF**	**DA**	**DF**	**AB**	**AE**	**AF**
Weights	7	7	9.5	10	10	10	12	16	17	19.5	20

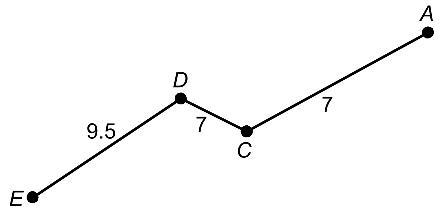
Step 1: Select the first edge AC from the list, since it has the minimum weight.



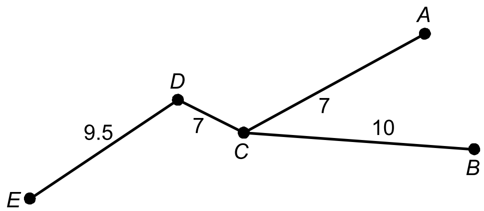
Step 2: The next edge of smallest weight is DC. We can add it to the previous one because it does not form any cycle.



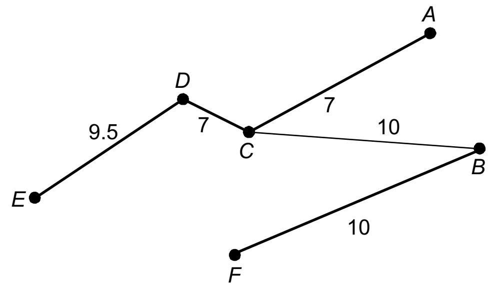
Step 3: The next edge of smallest weight is DE. We can add it to the previous one because it does not form any cycle.



Step 4: The next edge of smallest weight is BC. We can add it to the previous one because it does not form any cycle.



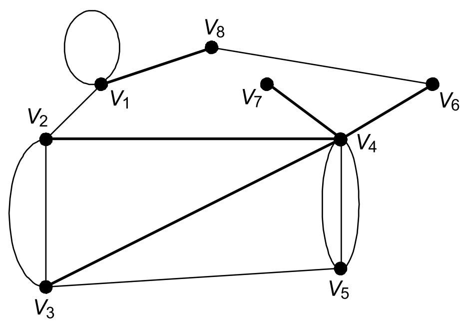
Step 5: The next edge of smallest weight is BF. We can add it to the previous one because it does not form any cycle.



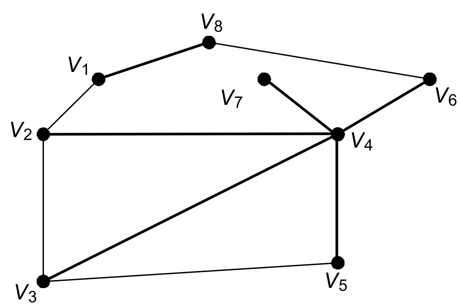
Since the number of vertices in the given graph is 6 and the tree in the last step contains $5\ (= 6 - 1)$ edges, the required minimal spanning tree is given by the Step 5.

Weight of the minimal spanning tree $= 7 + 7 + 9.5 + 10 + 10 = 43.5$.

b) Find using BFS algorithm a spanning tree in the following graph:

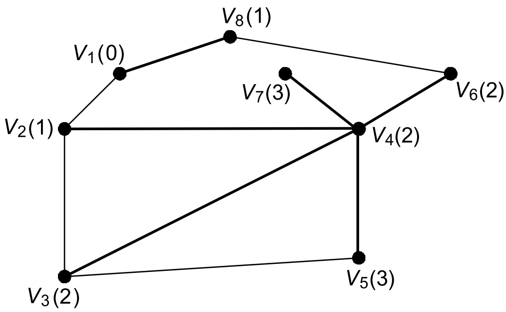


Solution: First we discard the loop and parallel edges.



Next we select the vertex V_1 and is labelled by 0.

Its adjacent unlabelled vertices are V_2 and V_8. They are labelled by $0+1=1$. The labels are shown in the figure. Then we join each of them with V_1 by the edges (V_1, V_2) and (V_1, V_8) since joining of the edges does not result any cycle.



Next we see that the unlabelled adjacent vertices of V_2 are V_3 and V_4. We label each of them by $1+1=2$. Also unlabelled adjacent vertex of V_8 is V_6 only. we label it by $1+1=2$. Now we join the vertices V_3 and V_4 to V_2 by the edges (V_2, V_3) and (V_2, V_4) respectively. Also we join V_8 with V_6 by the edge (V_8, V_6). It is to be noted that no cycle has been formed by the above joinnings.

Now the unlabelled adjacent vertices of V_4 are V_5 and V_7. We label each of them by $2+1=3$. Now we join the vertices V_5 and V_7 to V_4 by the edges (V_4, V_5) and (V_4, V_7) respectively. Here also it to be noted that no cycle has been formed.

We stop the process since no unlabelled vertices are left in the graph. Now the required spanning tree can be found by drawing the joining edges (V_1, V_2), (V_1, V_8), (V_2, V_3) , (V_2, V_4), (V_8, V_6), $(V_4, V_5)$and (V_4, V_7) successively which shown in the following figure:

c) Examine the convergence of the improper integral

$$\int_0^2 \frac{dx}{x(x-2)}$$

Solution: Here, $x = 0, 2$ are the points of infinite discontinuity.

Now,

$$\int_0^2 \frac{dx}{x(x-2)} = \int_0^1 \frac{dx}{x(x-2)} + \int_1^2 \frac{dx}{x(x-2)}$$

$$= \lim_{\epsilon \to 0+} \int_{\epsilon}^1 \frac{dx}{x(x-2)} + \lim_{\delta \to 0-} \int_1^{2-\delta} \frac{dx}{x(x-2)}$$

$$= \lim_{\epsilon \to 0+} \int_{\epsilon}^1 \frac{1}{2}\left(\frac{1}{x} + \frac{1}{2-x}\right) dx + \lim_{\delta \to 0-} \int_1^{2-\delta} \frac{1}{2}\left(\frac{1}{x} + \frac{1}{2-x}\right) dx$$

$$= \lim_{\epsilon \to 0+} \frac{1}{2} \left[\log|x| - \log|2-x|\right]_{\epsilon}^{1} + \lim_{\delta \to 0-} \frac{1}{2} \left[\log|x| - \log|2-x|\right]_{1}^{2-\delta}$$

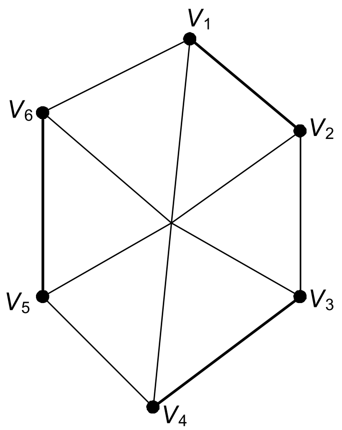
$$= \frac{1}{2} \lim_{\epsilon \to 0+} \left[-\log \frac{\epsilon}{2-\varepsilon}\right] + \frac{1}{2} \lim_{\delta \to 0-} \left[\frac{2-\delta}{\delta}\right]$$

Since,

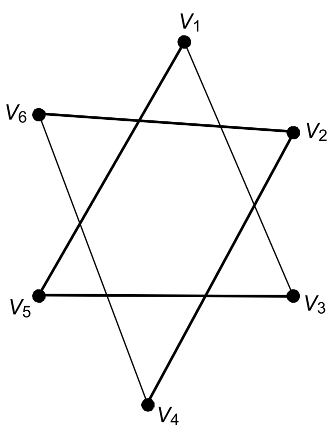
$$\lim_{\epsilon \to 0+} \left[-\log \frac{\epsilon}{2-\varepsilon}\right] \text{ and } \lim_{\delta \to 0-} \left[\frac{2-\delta}{\delta}\right]$$

does not exist,therefore the given improper integral is not convergent.

11. a) Define complement of a graph. Find the complement of the graph:



Solution: The complement $\overline{G}$ of a graph G is the graph whose vertex set is $V(G)$ and such that for each pair of vertices (u, v) of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G.



b) Solve

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$$

Solution: The differential equation can be written as

$$x^2D^2y+4xDy+2y=e^x$$

where,

$$D=\frac{d}{dx}$$

Let us consider the transformation

$$x=e^z \text{ or } \log x=z$$

Then,

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$$

$or,$ $$x\frac{dy}{dx}=\frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx}=Dy \text{ and } \frac{dy}{dz}=D'y$$

where

$$D=\frac{d}{dx} \text{ and } D'=\frac{d}{dz}$$

Therefore,

$$xDy\ =D'y$$

Similarly,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\frac{dy}{dx})=\frac{d}{dx}(\frac{1}{x}\frac{dy}{dz})$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$$

$or,$ $$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

$or,$ $$x^2D^2y=D'(D'-1)y$$

Substituting the values of xDy, x^2D^2y, we get

$$D'(D'-1)y + 4D'y + 2y = z$$

$$or, \qquad (D'^2 + 3D' + 2)y = z$$

which is a linear equation.

Let

$$y = e^{mz}$$

be a trial solution of the reduced equation

$$(D'^2 + 3D' + 2)y = 0$$

then the auxiliary equation is

$m^2 + 3m + 2 = 0$

$$or, m = -1, m = -2$$

Therefore, the complementary function is

$$C.F = y_c = C_1e^{-z} + C_2e^{-2z}$$

where C_1 and C_2 are arbitary constants.

The particular integral is

$$P.I = y_p = \frac{1}{(D'^2 + 3D' + 2)}z$$

$$= \frac{1}{2}\frac{1}{(1 + \frac{D'^2+3D'}{2})}z$$

$$= \frac{1}{2}(1 + \frac{D'^2 + 3D'}{2})^{-1}z$$

$$= \frac{1}{2}(z - \frac{3}{2})$$

$$= \frac{1}{2}z - \frac{3}{4}$$

Therefore, the general solution is

$$y = y_c + y_p = C_1e^{-z} + C_2e^{-2z} + \frac{1}{2}z - \frac{3}{4}$$

Putting $z = \log x$,the general solution becomes

$$y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{1}{2}\log x - \frac{3}{4}$$

(c) Prove that in a binary tree with n vertices, the number of internal vertices is one less than the number of pendant vertices.

Solution: *See Theorem 7.9 on Page 7.6*

SOLUTIONS OF UNIVERSITY QUESTIONS (W.B.U.T)

B. TECH SEM-II (NEW) 2013
MATHEMATICS - II (M 201)

Time Alloted : 3 Hours *Full Marks : 70*

Group A (Multiple Choice Questions)

1. Choose the correct alternatives for any ten of the following: (10 $\times$ 1 = 10)

i) The general solution of

$$y = px - \log p$$

a) $y = cx - \log c$ b) $y = 1 + \log x$

c) $y = 1 + \log x + c$ d) none of these..

Solution: The correct alternative is $\boxed{\text{a}}$

ii) The particular integral of

$$\frac{d^2y}{dx^2} + y = \cos x$$

a) $\frac{1}{2}\sin x$ b) $\frac{1}{2}\cos x$

c) $\frac{1}{2}x\sin x$ d) $\frac{1}{2}x\cos x$

Solution: The correct alternative is $\boxed{\text{c}}$

iii) $\frac{1}{D-1}x^2$ is equal to

a) $x^2 + 2x + 2$ b) $-\left(x^2 + 2x + 2\right)$

c) $2x - x^2$ d) $-\left(2x - x^2\right)$

Solution: The correct alternative is $\boxed{\text{b}}$

iv) The general solution of

$$\frac{d^2y}{dx^2} + y = 0$$

is

a) $Ae^x + Be^{-x}$ b) $(A + Bx)e^x$

c) $(A + Bx)\cos x$ d) $A\cos x + B\sin x$

Solution: The correct alternative is $\boxed{d}$

v) A simple graph can have

a) no pendent vertex
b) no isolated vertex
c) no circuit
d) none of these.

Solution: The correct alternative is $\boxed{d}$

vi) A simple graph with 20 vertices and 5 components has at least

a) 15 edges
b) 10 edges
c) 190 edges
d) 120 edges.

Solution: The correct alternative is $\boxed{a}$

vii) Which of the following is incorrect about a tree T with n vertices ?

a) There exist multiple paths between every pair of vertices in T
b) T is minimally connected
c) T is connected and curcuitless
d) T has $(n-1)$ edges.

Solution: The correct alternative is $\boxed{a}$

viii) If the incidence matrix of a graph has five identical columns, then G has

a) five loops
b) five isolated vertices
c) five parallel edges
d) five edges in series.

Solution: The correct alternative is $\boxed{c}$

ix)

$$L^{-1}\left(\frac{s}{s^2-a^2}\right)=$$

a) $\sin at$
b) $\sinh at$
c) $\cos at$
d) $\cosh at$

Solution: The correct alternative is $\boxed{d}$

x) $L\{H(t-a)\}$, H being Hevisides unit step function, is

a) e^{-as}
b) se^{-as}
c) $\frac{e^{-as}}{s}$
d) none of these.

Solution: The correct alternative is $\boxed{c}$

xi) Laplace transform of $\frac{\sin 2t}{t}$ is

a) $\cot^{-1}\frac{s}{2}$
b) $\cot^{-1}\frac{2}{s}$
c) $\frac{2}{s^2+4}$
d) $\frac{2}{s^2-4}$.

Solution: The correct alternative is $\boxed{a}$

xii) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$ equals to

a) $\frac{2\pi}{\sqrt{3}}$ b) $\frac{3\pi}{\sqrt{2}}$

c) $\frac{\pi}{\sqrt{3}}$ d) $\frac{\pi}{\sqrt{2}}$.

Solution: The correct alternative is $\boxed{a}$

xiii) $\int_{-\infty}^{\infty} xe^{-x^2}\,dx =$

a) -1 b) 0

c) 1 d) none of these

Solution: The correct alternative is $\boxed{b}$

GROUP B (Short Answer Type Questions)

Answer any three of the following: **($3 \times 5 = 15$)**

2. Solve:

$$\left(x^2y - 2xy^2\right)dx + \left(3x^2y - x^3\right)dy = 0$$

Solution: Here,

$$M(x,y) = \left(x^2y - 2xy^2\right) \text{ and } N(x,y) = \left(3x^2y - x^3\right)$$

Now,

$$\frac{\partial M}{\partial y} = x^2 - 4xy \text{ and } \frac{\partial N}{\partial x} = 6xy - 3x^2$$

Since,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore, the differential equation is not exact.

Here, $M(x,y)$ and $N(x,y)$ are both homogeneous function of degree 3 and $Mx + Ny = x^2y^2 \neq 0$.

When $M(x,y)$ and $N(x,y)$ are homogeneous functions of x and y of same degree and $Mx + Ny \neq 0$, then,

$$\boxed{I.F = \frac{1}{Mx + Ny}}$$

Therefore, here

$$\mathbf{I.F} = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$$

Multiplying the diffferential equation by I.F, we get

$$\frac{\left(x^2y-2xy^2\right)}{x^2y^2}dx+\frac{\left(3x^2y-x^3\right)}{x^2y^2}dy=0$$

$$or, \left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(\frac{3}{y}-\frac{x}{y^2}\right)dy=0$$

which is an exact differential equation.
Therefore, the general solution is **(Using Working Procedure 1 of Art. 2.2.3)**

$$\int M(x,y)dx+\int (\text{terms of } N(x,y) \text{ not containing } x)\,dy=c$$

$$or, \qquad \int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

$$or, \qquad \frac{x}{y}-2\log x+3\log y=c$$

$$or, \qquad \frac{x}{y}+\log\frac{y^3}{x^2}=c$$

where c is an arbitrary constant.

3. Solve the following simultaneous ODE:

$$\frac{dx}{dt}-7x+y=0,\ \frac{dy}{dt}-2x-5y=0.$$

Solution: *See Example 17 on Page 4.35*

4. Prove that the number of edges in a simple graph cannot exceed $\frac{n(n-1)}{2}$.

Solution: *See Theorem 5.5 on Page 5.11*

5. Prove that a graph is a tree if and only if it is minimally connected.

Solution: *See Theorem 7.5 on Page 7.4*

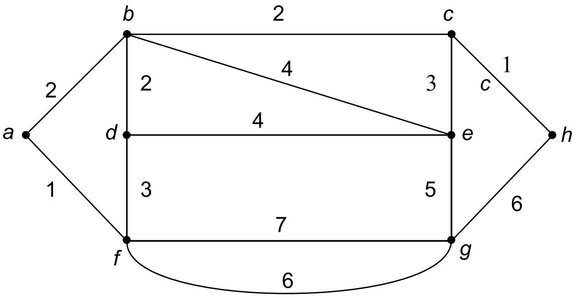
6. Define gamma function. Show that $\Gamma(n+1)=n\Gamma(n)$.

Solution: *See Property 2 of Section 9.5 on Page 9.13*

GROUP C (Long Answer Type Questions)

Answer any three of the following: **(3 × 5 = 15)**

7. a) Apply Dijkstra's algorithm to find shortest path between the vertices a and h in the following graph:



Solution: See Section 8.4 and Example 1 on page 8.5. The shortest path is given by

$$a \to b \to c \to h.$$

b) Solve:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$$

Solution: The differential equation can be written as

$$(D^2 - 2D + 1)y = xe^x$$

where

$$D = \frac{d}{dx}$$

Let

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 - 2D + 1)y = 0$$

then the Auxilliary equation is

$$m^2 - 2m + 1 = 0 \implies m = 1, 1$$

Therefore the complementary function is

$$y_c = (C_1 + C_2 x)\, e^x$$

where C_1 and C_2 are arbitary constants.
The Particular integral is

$$y_p = \frac{1}{(D^2 - 2D + 1)}(xe^x) = \frac{1}{(D-1)^2}(xe^x)$$

$$= e^x \frac{1}{(D+1-1)^2} x = e^x \frac{1}{D^2} x$$

$$= e^x \frac{1}{D} \int x dx = e^x \int \frac{x^2}{2} dx = e^x \frac{x^3}{6}$$

Therefore the general solution is

$$y = y_c + y_p = (C_1 + C_2 x)\, e^x + \frac{1}{6} x^3 e^x$$

c) Construct a digraph from the following incidence matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Solution: The entries of the matrix are $0, 1, -1$, therefore the graph corresponding to the matrix is a digraph.

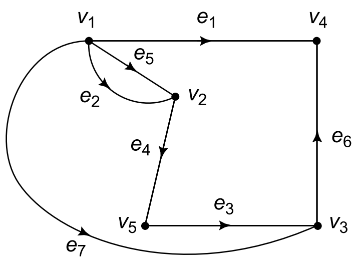
Since the number of rows and columns of the matrix are respectively equal to the number of vertices and edges of the digraph, therefore the number of vertices are 5 and the number of edges are 7.

Let the vertices be v_1, v_2, v_3, v_4, v_5 and the edges are $e_1, e_2, e_3, e_4, e_5, e_6, e_7$

After labelling the vertices and edges the matrix becomes

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Hence the required digraph is given by



8. a) Prove that a tree with n vertices has $(n-1)$ edges.

Solution: *See Theorem 7.2 on Page 7.2*

b) Solve the following by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + y = \tan x$$

Solution: The differential equation can be written as

$$(D^2 + 1)y = \tan x$$

where,

$$D = \frac{d}{dx}$$

Let

$$y = e^{mx}$$

be a trial solution of the reduced equation

$$(D^2 + 1)y = 0$$

then the auxilliary equation is

$$m^2 + 1 = 0 \implies m = \pm i$$

Therefore the complementary function is

$$y_c = C_1 y_1 + C_2 y_2 = C_1 \cos x + C_2 \sin x$$

where C_1 and C_2 are arbitary constants.
Let the particular integral is

$$y_p = u(x)\cos x + v(x)\sin x$$

where $u(x)$ and $v(x)$ are arbitary functions.
Variation of Parameters method is the method of finding the functions $u(x)$ and $v(x)$. Now, wronskians

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0$$

Here, $u(x)$ and $v(x)$ are given respectively by

$$u(x) = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{\sin x . \tan x}{1} dx = -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{(1 - \cos^2 x)}{\cos x} dx = -\int (\sec x - \cos x)\, dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

and

$$v(x) = \int \frac{y_1 F(x)}{W} dx = \int \frac{\cos x . \tan x}{1} dx = \int \sin x dx = -\cos x$$

Therefore the particular integral is

$$\begin{aligned} y_p &= u(x)\cos x + v(x)\sin x = -\cos x \log(\sec x + \tan x) \\ &\quad + \cos x \sin x - \cos x \sin x \\ &= -\cos x \log(\sec x + \tan x) \end{aligned}$$

Therefore the general solution is

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x - \cos x \log(\sec x + \tan x)$$

c) Solve the following differential equation by Laplace Transform:

$$(D^2 + 6D + 9)y = 0, y(0) = y'(0) = 1.$$

Solution: The given equation is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$$

$$i.e., \qquad y'' + 6y' + 9y = 0. \qquad (1)$$

Taking Laplace Transform on both hand sides of (1), we have

$$L\{y''(t)\} + 6L\{y'(t)\} + 9L\{y(t)\} = 0$$

$$\Rightarrow \left[p^2.L\{y(t)\} - p.y(0) - y'(0)\right] + 6\left[p.L\{y(t)\} - y(0)\right] + 9L\{y(t)\} = 0$$

Putting the conditions $y(0) = y'(0) = 1$, we get

$$\left[p^2.L\{y(t)\} - p.1 - 1\right] + 6\left[p.L\{y(t)\} - 1\right] + 9L\{y(t)\} = 0$$

$$i.e., \qquad (p^2 + 6p + 9).L\{y(t)\} - p - 7 = 0$$

$$i.e., \qquad L\{y(t)\} = \frac{p+7}{(p^2 + 6p + 9)} = F(p), \text{ (say)}$$

Therefore

$$y(t) = L^{-1}\{F(p)\} = L^{-1}\left\{\frac{p+7}{(p^2+6p+9)}\right\}$$

$$= L^{-1}\left\{\frac{(p+3)+4}{(p+3)^2}\right\} = L^{-1}\left\{\frac{1}{(p+3)} + \frac{4}{(p+3)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(p+3)}\right\} + L^{-1}\left\{\frac{4}{(p+3)^2}\right\}$$

$$= e^{-3t} + 4e^{-3t}L^{-1}\left\{\frac{1}{p^2}\right\} \text{ (by first shifting theorem)}$$

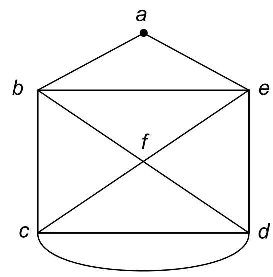
$$= e^{-3t} + 4e^{-3t}.t = (1+4t)e^{-3t}.$$

So, the required solution is

$$y(t) = (1+4t)e^{-3t}.$$

9. a) i) Define Euler circuit. Write the necessary and sufficient condition for a graph to contain an Euler curcuit.

ii) Find, if possible, an Euler curcuit in the following graph:



Solution: i) *See Section 5.12.1 on Page 5.26 and see Theorem 5.19 on page 5.27.*

ii) $\{a, b, e, f, b, c, f, d, c, d, e, a\}$ is a curcuit containing all the edges. Hence this is an Eulerian curcuit.

b) Using convolution theorem prove that

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{t \sin at}{2a}$$ [WBUT-2011]

Solution:

Here we are to find

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)}.\frac{1}{(s^2+a^2)}\right\}$$

$$= L^{-1}\{F_1(s).F_2(s)\}$$

where

$$F_1(s) = \frac{s}{(s^2+a^2)} = L\{f_1(t)\} \text{ (say)}$$

$$\Rightarrow f_1(t) = L^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} = \cos at$$

and

$$F_2(s) = \frac{1}{(s^2+a^2)} = L\{f_2(t)\} \text{ (say)}$$

$$\Rightarrow f_2(t) = L^{-1}\left\{\frac{1}{(s^2+a^2)}\right\} = \frac{\sin at}{a}.$$

By convolution theorem, we have

$$L^{-1}\{F_1(s).F_2(s)\} = \int_0^t f_1(x).f_2(t-x)dx$$

$$i.e.,\ L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos ax.\frac{\sin a(t-x)}{a}\,dx$$

$$= \frac{1}{a}.\frac{1}{2}\int_0^t \{\sin at - \sin(2ax-at)\}\ dx$$

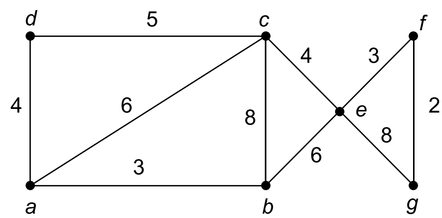
$$= \frac{1}{2a}\left[x\sin at + \frac{\cos(2ax-at)}{a}\right]_0^t = \frac{t\sin at}{2a}.$$

c) Prove that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Solution: *See Result 2 of Section 9.8 on Page 9.19.*

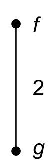
10. a) By Kruskal's algorithm, find a minimal spanning tree in the following graph:



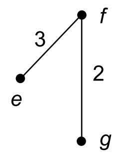
Solution: First we arrange all the edges of the graph in the order of non-decreasing weights and write them in the following form

Edges	fg	fe	ab	ec	ad	dc	be	ac	bc	eg
Weights	2	3	3	4	4	5	6	6	8	8

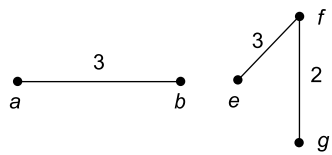
Step 1: Select the first edge **fg** from the list, since it has the minimum weight.



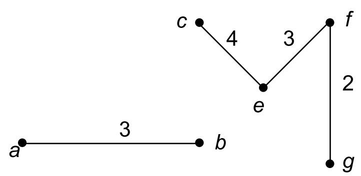
Step 2: The next edge of smallest weight is **fe**. We can add it to the previous one because it does not form any cycle.



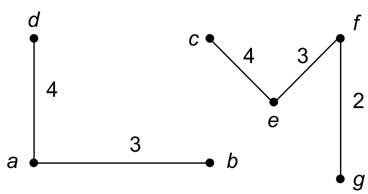
Step 3: The next edge of smallest weight is **ab**. We can add it to the previous one because it does not form any cycle.



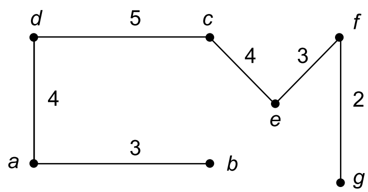
Step 4: The next edge of smallest weight is **ec**. We can add it to the previous one because it does not form any cycle.



Step 5: The next edge of smallest weight is **ad**. We can add it to the previous one because it does not form any cycle.



Step 6: The next edge of smallest weight is **dc**. We can add it to the previous one because it does not form any cycle.



Since the number of vertices in the given graph is 7 and the tree in the last step contains 6 $(= 7 - 1)$ edges, the required minimal spanning tree is given by the **Step 6**.

Weight of the minimal spanning tree $= 2 + 3 + 3 + 4 + 4 + 5 = 21$..

b) Find the Laplace transform of $f(t)$ defined as:

$$f(t) = \left\{ \begin{array}{l} \frac{t}{k}, \text{ when } 0 < t < k \\ 1, \text{ when } t > k \end{array} \right\}$$

Solution: By definition we have

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt = \int_0^k e^{-pt} \cdot \frac{t}{k}\, dt + \int_k^\infty e^{-pt} \cdot 1\, dt$$

$$= \frac{1}{k}\left\{ \left[t \cdot \frac{e^{-pt}}{-p} \right]_0^k - \int_0^k 1 \cdot \frac{e^{-pt}}{-p}\, dt \right\} + \left[\frac{e^{-pt}}{-p} \right]_k^\infty$$

$$= \frac{1}{k}\left\{ k \cdot \frac{e^{-pk}}{-p} - \cdot \left[\frac{e^{-pt}}{p^2} \right]_0^k \right\} + \frac{e^{-pk}}{p}, \text{ since } p > 0$$

$$= -\frac{e^{-pk}}{p} - \frac{1}{kp^2} \cdot \left(e^{-pk} - 1 \right) + \frac{e^{-pk}}{p}$$

$$= \frac{1}{kp^2} \cdot \left(1 - e^{-pk} \right)$$

c) Solve:

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$$

Solution: The differential equation can be written as

$$\left(x^2 D^2 - 3xD + 4 \right) y = 2x^2$$

where,

$$D = \frac{d}{dx}$$

Let us consider the transformation

$$x = e^z \text{ or } \log x = z$$

Then,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

or, $$x\frac{dy}{dx} = \frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx} = Dy \text{ and } \frac{dy}{dz} = D'y$$

where

$$D = \frac{d}{dx} \text{ and } D^{'} = \frac{d}{dz}$$

Therefore,

$$xDy \ = D'y$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$$
$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

or, $$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

or, $$x^2D^2y = D'(D^{'} - 1)y$$

Substituting the values of xDy, x^2D^2y we get

$$\left\{D'(D^{'} - 1) - 3D' + 4\right\}y = 2e^{2z}$$

or, $$(D'^2 - 4D' + 4)y = 2e^{2z}$$

or, $$(D' - 2)^2y = 2e^{2z}$$

which is a linear differential equations with constant coefficients. Let

$$y = e^{mz}$$

be a trial solution of the reduced equation

$$(D' - 2)^2y = 0$$

then the auxilliary equation is

$$(m - 2)^2 = 0 \implies m = 2, 2$$

Therefore the complementary function is

$$y_c = (C_1 + C_2 z)\, e^{2z}$$

The particular integral is

$$y_p = \frac{1}{(D'-2)^2} 2e^{2z} = e^{2z} \frac{1}{(D'+2-2)^2} 2$$

$$= e^{2z} \frac{1}{D'^2} 2 = e^{2z} \frac{1}{D'} \int 2dz = e^{2z} \int 2z dz = z^2 e^{2z}$$

Therefore the complete solution is

$$y = y_c + y_p = (C_1 + C_2 z)\, e^{2z} + z^2 e^{2z}$$

$$= (C_1 + C_2 \log x)\, x^2 + (\log x)^2\, x^2$$

11. a) Evaluate:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)\,(s^2+b^2)}\right]$$ [WBUT-2011]

Solution: Here we apply the method of partial fraction to compute the given inverse laplace transform. Consider

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{A}{(s^2+a^2)} + \frac{B}{(s^2+b^2)}$$

$$= \frac{A(s^2+b^2) + B(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}$$

$$= \frac{(A+B)\, s^2 + \left(A.b^2 + B.a^2\right)}{(s^2+a^2)(s^2+b^2)}$$

Equating the the coefficients of like powers of s in the numerator of both hand sides, we get

$$A + B = 1,\ A.b^2 + B.a^2 = 0$$

Solving we have

$$A = \frac{a^2}{a^2-b^2} \text{ and } B = \frac{-b^2}{a^2-b^2}$$

Therefore, using these values we get

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{a^2}{a^2-b^2}\frac{1}{(s^2+a^2)} - \frac{b^2}{a^2-b^2}\frac{1}{(s^2+b^2)}$$

$$= \frac{1}{a^2-b^2}\left\{\frac{a^2}{(s^2+a^2)} - \frac{b^2}{(s^2+b^2)}\right\}$$

Therefore,

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{a^2-b^2}L^{-1}\left\{\frac{a^2}{(s^2+a^2)} - \frac{b^2}{(s^2+b^2)}\right\}$$

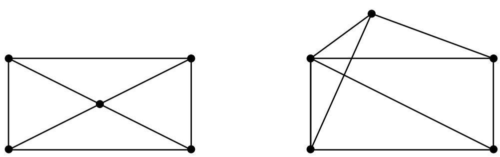
$$= \frac{1}{a^2-b^2}\left[a^2L^{-1}\left\{\frac{1}{(s^2+a^2)}\right\}\right.$$

$$\left.-b^2L^{-1}\left\{\frac{1}{(s^2+b^2)}\right\}\right]$$

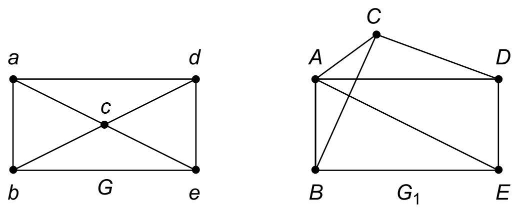
$$= \frac{1}{a^2-b^2}\left[a^2\frac{\sin at}{a} - b^2\frac{\sin bt}{b}\right]$$

$$= \frac{1}{a^2-b^2}\left[a\sin at - b\sin bt\right]$$

b) Examine whether the following graphs are isomorphic or not:



Solution: First we label the graphs as the following



Here it is clear that in the first graph G, there are four vertices of degree 3 which are a, b, d, e, but in the second graph G_1 there are all five vertices of degree 3 which are A, B, C, D, E. So the graphs are not isomorphic as they are violating the necessary condition that the number of same degree vertices must be same for the graphs.

c) Solve:

$$y = px + \sqrt{a^2p^2+b^2}$$

Solution: *See Example 3.2 on Page 3.21*

SOLUTIONS OF UNIVERSITY QUESTIONS (W.B.U.T)

B. TECH SEM-II (NEW) 2014
MATHEMATICS - II (M 201)

Time Alloted : 3 Hours *Full Marks : 70*

Group A (Multiple Choice Questions)

1. Choose the correct alternatives for any ten of the following: (10 ×1=10)

i) The general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

where A and B are arbitary constants is

a) $Ae^{2x} + Be^{-2x}$ b) $(A + B)e^{2x}$

c) $A\cos 2x + B\sin 2x$ d) $(A + Bx)\cos 2x$

Solution: The correct alternative is c

ii) If the differential equation

$$\left(y + \frac{1}{x} + \frac{1}{x^2y}\right) dx + \left(x + \frac{1}{y} + \frac{A}{xy^2}\right) dy = 0$$

is exact, then the value of A is

a) 2 b) 1

c) -1 d) 0

Solution: The correct alternative is b

iii) The number of edges in a tree with n vertices is

a) n b) $n - 1$

c) $n + 1$ d) none of these

Solution: The correct alternative is b

iv) A binary tree has exactly

a) two vertices of degree two

b) one vertex of degree two

c) one vertex of degree one

d) none of these

Solution: The correct alternative is $\boxed{\text{b}}$

v) $L^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$

a) $1+e^t$ b) $1-e^t$

c) $1+e^{-t}$ d) $1-e^{-t}$

Solution: The correct alternative is $\boxed{\text{d}}$

vi) The value of $\Gamma\left(\dfrac{1}{2}\right)$

a) π b) $\sqrt{\pi}$

c) $\dfrac{1}{\pi}$ d) $\dfrac{1}{\sqrt{\pi}}$.

Solution: The correct alternative is $\boxed{\text{b}}$

vii) The general solution of the differential equation $y = px + f(p)$ is

a) $y = c^2x + f(c)$

b) $y = cx + f(c^2)$

c) $y = cx + f(c)$

d) none of these

Solution: The correct alternative is $\boxed{\text{c}}$

viii) The improper integral $\displaystyle\int_0^1 \frac{dx}{(b-x)^n}$ converges for

a) $n > 1$ b) $n < 1$

c) $n \geq 1$ d) none of these

Solution: The correct alternative is $\boxed{\text{d}}$

ix) The sum of the degrees of all vertices of a graph is 40, the number of edges is

a) 20 b) 25

c) 40 d) none of these

Solution: The correct alternative is $\boxed{\text{a}}$

x) $\dfrac{1}{(D-3)}e^{3x}$

a) xe^{3x} b) $3e^{3x}$

c) x^2e^{3x} d) none of these.

Solution: The correct alternative is $\boxed{a}$

xi) The value of $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)$ is

a) $3\frac{\sqrt{\pi}}{4}$ b) $\frac{3}{2}\pi$

c) $\frac{3}{4}\pi$ d) none of these

Solution: The correct alternative is $\boxed{c}$

xii) $L(t\cos t) =$

a) $\frac{s}{s^2+1}$ b) $\frac{s+1}{s^2+1}$

c) $\frac{2s}{s^2+1}$ d) $\frac{s^2-1}{(s^2+1)^2}$

Solution: The correct alternative is $\boxed{d}$

xiii) The integrating factor of the differential equation

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1}y}}{1+y^2}$$

is

a) $\tan^{-1}y$ b) $e^{\tan^{-1}y}$

c) $e^{\cot^{-1}y}$ d) e^y

Solution: The correct alternative is $\boxed{b}$

GROUP B (Long Answer Type Questions)

Answer any three of the following: **(3 ×15 = 45)**

2. Solve

$$(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$$

Solution: See Example 2.6 on Page 2.33

3. Solve

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x$$

Solution: The differential equation can be written as

$$x^2D^2y - xDy - 3y = x^2\log x$$

where,

$$D = \frac{d}{dx}$$

Let us consider the transformation

$$x = e^z \text{ or } \log x = z$$

Then,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$or, x\frac{dy}{dx} = \frac{dy}{dz}$$

Let us consider

$$\frac{dy}{dx} = Dy \text{ and } \frac{dy}{dz} = D'y$$

where

$$D = \frac{d}{dx} \text{ and } D' = \frac{d}{dz}$$

Therefore,

$$xDy = D'y$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{1}{x}\frac{dy}{dz})$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$or, x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$or, x^2D^2y = D'(D'-1)y$$

Substituting the values of xDy, x^2D^2y we get

$$D'(D'-1)y - D'y - 3y = e^{2z}z$$

$$or, (D'^2 - 2D' - 3)y = e^{2z}z$$

which is a linear equation.
Let

$$y = e^{mz}$$

be a trial solution of the reduced equation

$$(D'^2 - 2D' - 3)y = 0$$

then the Auxxiliary equation is

$$m^2 - 2m - 3 = 0$$

$$or, m = 3, -1$$

Therefore, the complementary function is

$$C.F = y_c = (C_1 e^{3z} + C_2 e^{-z})$$

where C_1 and C_2 are arbitary constants.
The Particular integral is

$$P.I = y_p = \frac{1}{(D'^2 - 2D' - 3)} e^{2z} z$$

$$= e^{2z} \frac{1}{\{(D'+2)^2 - 2(D'+2) - 3)} z$$

$$= e^{2z} \frac{1}{(D'^2 + 2D' - 3)} z$$

$$= \frac{e^{2z}}{-3} \left[1 - \frac{D'^2 + 2D'}{3}\right]^{-1} z$$

$$= \frac{e^{2z}}{-3} \left[1 + \frac{D'^2 + 2D'}{3} + ...\right] z$$

$$= \frac{e^{2z}}{-3} \left[z + \frac{2}{3}\right]$$

Therefore the general solution is

$$y = y_c + y_p = (C_1 e^{3z} + C_2 e^{-z}) - \frac{e^{2z}}{3} \left[z + \frac{2}{3}\right]$$

Putting $z = \log x$, the general solution becomes

$$y = C_1 x^3 + C_2 x^{-1} - \frac{x^2}{3} \left[\log x + \frac{2}{3}\right]$$

4. Show that $\int\limits_{-1}^{1} \frac{1}{x^3}$ exists in the Cauchy principle value sense but not in the general sense.

Solution: See Example 5 on Page 9.7

5. Prove that the maximum number of edges in a graph with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$.

Solution: See Theorem 5.14 on Page 5.22

6. State the Convolution theorem for Laplace transform.Use this theorem to find

$$L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}$$

Solution: For the statement of Convolution theorem see Article 11.10.2 on Page 11.16.For the problem see Example11 on Page 11.16.

GROUP C (Long Answer Type Questions)

Answer any three of the following: **(3 × 15 = 45)**

7. a) Solve

$$\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2y\,dy = 0$$

Solution: See Example 8 on Page 2.11

b) Prove that

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Solution: See Article 9.7 on Pages 9.17 and 9.18

c) Show that

$$\int\limits_{0}^{\infty} \sqrt{\tan x}\,dx = \frac{\pi}{\sqrt{2}}$$

Solution: See Example 9.8 on Page 9.31

8. a) Solve the following simultaneous equation:

$$\frac{dx}{dt} + 3x + y = e^t, \; \frac{dy}{dt} - x + y = e^{2t}$$

Solution: The simultaneous differential equation are

$$\frac{dx}{dt} + 3x + y = e^t \tag{1}$$

and

$$\frac{dy}{dt} - x + y = e^{2t} \tag{2}$$

From (1) we have

$$y = e^t - 3x - \frac{dx}{dt} \tag{3}$$

Therefore from (2) and (3) we have

$$\frac{d}{dt}(e^t - 3x - \frac{dx}{dt}) - x + e^t - 3x - \frac{dx}{dt} = e^{2t}$$

$$or, e^t - 3\frac{dx}{dt} - \frac{d^2x}{dt^2} - x + e^t - 3x - \frac{dx}{dt} = e^{2t}$$

$$or, \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 2e^t - e^{2t}$$

Let

$$x = e^{mt}$$

be a trial solution of the reduced equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$$

then the auxilliary equation is

$$m^2 + 4m + 4 = 0$$

$$or, m = -2, m = -2$$

Therefore the complementary function is

$$C.F = x_c = (C_1 t + C_2)e^{-2t}$$

where C_1 and C_2 are arbitary constants.
The particular integral is

$$P.I = x_p = \frac{1}{(D^2 + 4D + 4)}(2e^t - e^{2t})$$

$$= 2\frac{1}{(D^2 + 4D + 4)}e^t - \frac{1}{(D^2 + 4D + 4)}e^{2t}$$

$$= \frac{2}{9}e^t - \frac{1}{16}e^{2t}$$

Therefore,

$$x = x_c + x_p = (C_1 t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}$$

Now,

$$y = e^t - 3x - \frac{dx}{dt}$$

$$or, y = e^t - 3\left\{(C_1 t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}\right\}$$
$$-\frac{d}{dt}\left\{(C_1 t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}\right\}$$

$$or, y = e^t - 3\left\{(C_1 t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}\right\}$$
$$-C_1(e^{-2t} - 2te^{-2t}) + 2C_2e^{-2t} - \frac{2}{9}e^t + \frac{1}{8}e^{2t}$$

$$or, y = -C_1(1+t)e^{-2t} - C_2e^{-2t} + \frac{1}{9}e^t + \frac{5}{16}e^{2t}$$

Therefore the solution is

$$x = (C_1 t + C_2)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}$$

and

$$y = -C_1(1+t)e^{-2t} - C_2e^{-2t} + \frac{1}{9}e^t + \frac{5}{16}e^{2t}$$

where C_1 and C_2 are arbitary constants.

b) Find the inverse Laplace transform of

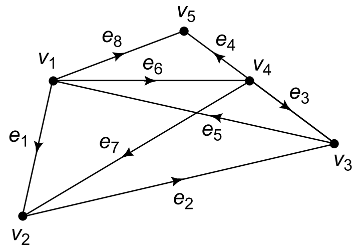
$$\frac{s^2 + s - 2}{s(s+3)(s-2)}$$

Solution: Let

$$\frac{s^2 + s - 2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{(s+3)} + \frac{C}{(s-2)}$$

$$or, \frac{s^2 + s - 2}{s(s+3)(s-2)} = \frac{A(s+3)(s-2) + Bs(s-2) + Cs(s+3)}{s(s+3)(s-2)}$$

$$or, s^2 + s - 2 = A(s+3)(s-2) + Bs(s-2) + Cs(s+3)$$



Putiing,$s = 0$

$$-2 = -6A$$

$$or, A = \frac{1}{3}$$

Putting, $s = 2$

$$4 = 10C$$

$$or, C = \frac{2}{5}$$

Putting, $s = -3$

$$4 = 15B$$

$$or, B = \frac{4}{15}$$

Therefore,

$$\frac{s^2 + s - 2}{s(s+3)(s-2)} = \frac{1}{3}\frac{1}{s} + \frac{4}{15}\frac{1}{(s+3)} + \frac{2}{5}\frac{1}{(s-2)}$$

and

$$L^{-1}\left\{\frac{s^2 + s - 2}{s(s+3)(s-2)}\right\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{15}L^{-1}\left\{\frac{1}{(s+3)}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{(s-2)}\right\}$$

$$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

c) Find the incidence matrix for the graph given below:

Solution: The Incidence matrix of G is given by a $n \times m$ matrix

$$I(G) = (a_{ij})_{n \times m}$$

where

$$a_{ij} = \begin{cases} 1; \text{ when edge } e_j \text{ is incident out of } v_i \\ -1; \text{ when edge } e_j \text{ is incident into } v_i \\ 0; \text{ if there is no edge } e_j \text{ incident out of or into } v_i \end{cases}$$

Here we have 5 vertices and 8 edges, so the incidence matrix is of order 5×8 and is given by

$$I(G) = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ v_1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ v_2 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ v_3 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ v_5 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{matrix}$$

9. a) Prove that the number of vertices in a binary tree is always odd.

Solution: See Theorem 7.7 on Page 7.6

b) Solve

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2e^{3x}$$

Solution: See Example 10 on Page 4.18

c) Use Laplace transform to find the integral $\int_0^\infty e^{-4t} \sin t dt$

Solution: We know

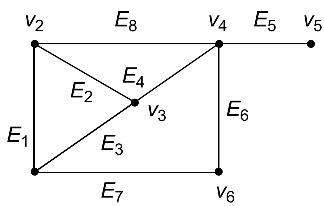
$$L\{\sin t\} = \frac{1}{1+s^2}$$

or,
$$\int_0^\infty e^{-st} \sin t dt = \frac{1}{1+s^2}$$

Putting, $s = 4$ both sides we have

$$\int_0^\infty e^{-4t} \sin t dt = \frac{1}{17}$$

10. a) Determine the adjacency matrix of the given graph



Solution: The graph is a simple graph. The adjacency matrix is

$$A(G) = (a_{ij})_{n\times n}$$

where

$$a_{ij} = \left\{ \begin{array}{c} 1;\text{ when } v_iv_j \text{ in an edge of } G \\ 0;\text{ when } v_iv_j \text{ in not an edge of } G \end{array} \right\}$$

Here, the number of vertices is 6. The adjacency matrix is

$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b) Evaluate

$$L^{-1}\left\{\log_e \frac{(s+2)}{(s+1)}\right\}$$

Solution: Let

$$F(s) = \log_e \frac{(s+2)}{(s+1)} = \log_e(s+2) - \log_e(s+1)$$

Therefore,

$$F'(s) = \frac{1}{(s+2)} - \frac{1}{(s+1)}$$

So,

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{1}{(s+2)}\right\} - L^{-1}\left\{\frac{1}{(s+1)}\right\} = e^{-2t} - e^{-t}$$

Since,

$$L^{-1}\{F'(s)\} = -tL^{-1}\{F(s)\}$$

we have,

$$-tL^{-1}\{F(s)\} = e^{-2t} - e^{-t}$$

$$or, \quad L^{-1}\{F(s)\} = \frac{e^{-t} - e^{-2t}}{t}$$

Hence,

$$L^{-1}\left\{\log_e \frac{(s+2)}{(s+1)}\right\} = \frac{e^{-t} - e^{-2t}}{t}$$

c) Solve the following differential equation using Laplace transform method.

$$(D^2 - 3D + 2)y = 4t + e^{3t};\ \text{where } y(0) = 0, y'(0) = -1$$

Solution: See Example 12.3 on Page 12.4

11. a) Discuss the convergence of the improper integral:

$$\int \frac{dx}{x(2-x)}$$

Solution:The problem is incorrect.

b) Solve

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x dx = \frac{8}{315}$$

Solution: We have

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x dx = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}; p > -1, q > -1$$

Here, $p = 4$ and $q = 5$, therefore

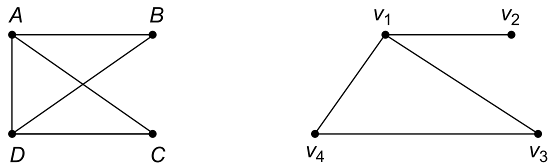
$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x dx = \frac{1}{2} \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{\Gamma\left(\frac{4+5+2}{2}\right)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{5}{2}\right).2!}{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{8}{9.7.5} = \frac{8}{315}$$

c) Examine whether the following two graphs are isomorphic or not:



Solution: Since the number of edges of the two graphs are not same, therefore the two graphs are not isomorphic.