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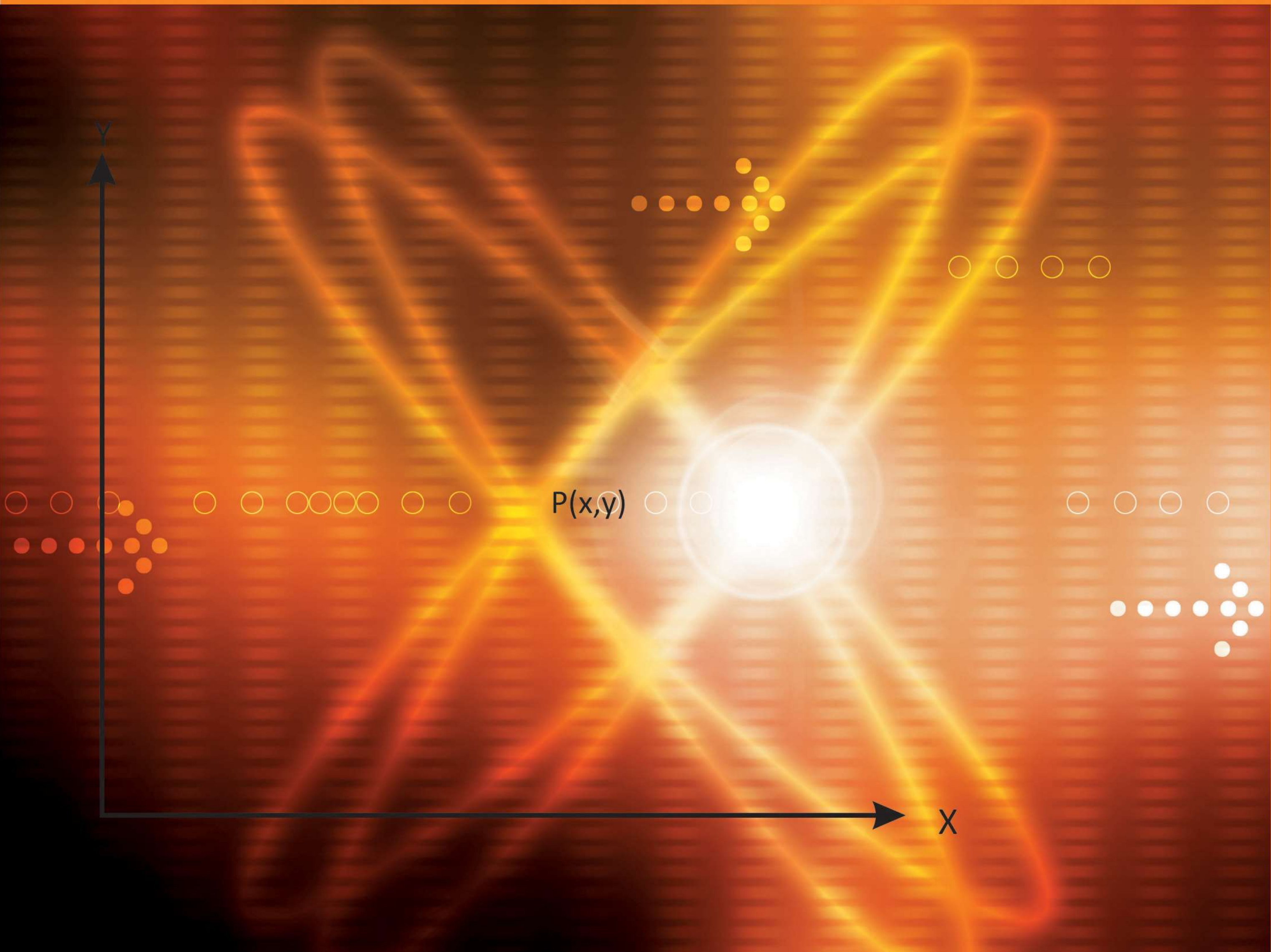

K.N.Saxena
Lalit Yadav

Co-ordinate Geometry

Kavindra Nath Saxena
M.A., M.Sc., L.T.,
Rt. Principal

Lalit Yadav
B.E.(IIT-Roorke)

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Dedicated to the Students

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We would like to thank all of the many people who helped bring this book to print. It's really fascinating for us to write a book, because what begins as a few thoughts ends up as a physical, tangible thing that goes into the hands of many readers. The trip from thought to thing is a long one and involves inputs from lots of people. Thanks goes to all of our editors, art personnel and staff at Silicon Media.

Authors

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1. Geometry and Mensuration (Revision)

1.1 Straight line and Angles

(a) **Straight line :** Two points in space determine a straight line. It has a definite direction. It lies in a plane and divides the plane in two halves.

(b) **Angle :** When a line segment OB, moves from its initial position OB, at O in anti-clockwise direction or in clock-wise direction, in the plane of paper it traces angle B_1OB or B_2OB as in Fig 1(a).

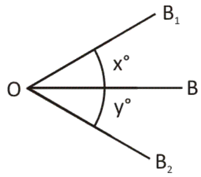

Fig 1(a)

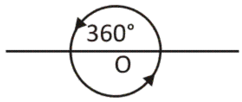

Fig 1(b)

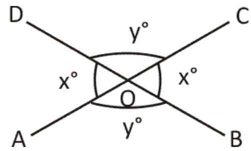

Fig 1(c)

Total angle traced in one complete round is 360^0 Fig 1(b)

(c) **Vertically opposite angles :** When two straight lines intersect at a point then vertically opposite angles so formed. $\angle$BOD = $\angle$COA, are equal. $\angle$COB = $\angle$AOD (Fig. 1C)

(d) When a straight line meets another straight line then sum of adjacent angels so formed, is equal to 180^0. In Fig, 1c angles x and y are adjacent angles and $x + y = 180^0$

(e) acute angle, $0 < x < 90^0$ is acute angle

angle y; $90^0 < y < 180^0$ is obtuse angle

angle z; $180^0 < z < 360^0$ is reflex angle

(f) **Complementary and supplementary angles**

(i) if angles $x + y = 90^0$ then these are complementary angles.

(ii) if $x + y = 180^0$, then these are supplementary angles, complementary angle of θ is $(90^0 – \theta^0)$ and supplementary angle is $(180^0 - \theta)$

1.2 Parallel lines

Lines having same direction are parallel. Distance between two parallel lines is same every where. Parallel lines meet at infinity. Parallel lines posses some properties. In Fig(2), AB, CD, EF are parallel lines. Transverse PT and L Y intersect them.

(a) **Corresponding angles :** (1,5); (2,6), (4,8) (3,7) (1,9) (2,10) (4,12), (3, 11) are equal.

(b) **Alternate angles :** (3,5); (2,8); (6,12), (7,9) are alternate angles and these are equal.

(c) **Interior angles :** Angles (3,8), (2,5), (7,12), (6,9) are interior angles. Sum of interior angles. 3 + 8; 2 + 5; 7 + 12; 6 + 9, is equal to two right angles.

MN is transverse cut by parallel lines AB, CD, QR is transverse cut on PT transverse. Similarly are MK and RS are transverse cuts.

(i) MN : NK = QR : RS

MK : NK = QS : RS .... and so on.

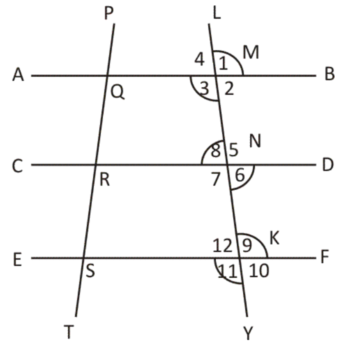

Fig 2

1.3 Triangle

Three points not in a straight line, when joined by line segments, a triangle is formed. A triangle has three sides and 3 angles.

(a) Sum of interior angles of a triangle is equal to two right angles (180^0)

$\angle A + \angle B + \angle C = 180^0$

(b) In triangle ABC (Fig 3a), BC is base and A is vertex. Exterior angle $\angle ACK$, which results on extension of BC is equal to the sum of remaining two Interior Opposite angles $B\hat{A}C$ and $A\hat{B}C$.

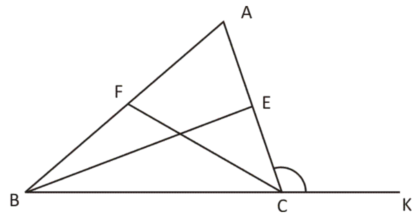

Fig 3

(c) **Median :** The line segment joining a vertex to the mid point of opposite side is called median. In Fig (3a) BE and CF are medians, these are 3 medians of a triangle.

(d) **Altitude :** Perpendicular from a vertex on opposite side is called altitudes. In a triangle number of altitudes is 3.

(e) The sum of two sides of a triangle is greater than the third side. In Fig 3(a)

$(AC + AB) > BC, (BC + CA) > AB,$ so on.

(f) Area of a triangle = $\frac{1}{2}(\text{base}) \times \text{altitude}$.

(g) **Equilateral triangle** : If all sides of a triangle are equal, it is called, equilateral triangle. Its

(i) all angles are equal, each is 60^0

(ii) all medians are equal,

(iii) all altitudes equal, here median is altitude.

Altitude = $\frac{\sqrt{3}}{2}$(sides)

(iv) area of equilateral triangle = $\frac{\sqrt{3}}{4}(\text{side})^2$

If altitude is given then area = $\frac{1}{\sqrt{3}}(\text{altitude})^2$

(h) **Isosceles Triangle:** If two sides of a triangle are equal, then it is called isosceles triangle. Angles opposite to equal sides are equal. The line segment, joining vertex to mid point, of unequal side is perpendicular on it.

(i) **Scalene Traingle :** If all sides of a triangle are unequal, it is called scalene triangle.

(j) **Right angled Triangle** : If a angle of a triangle is of 90^0, then it is called right angled triangle.

(k) The side opposite to right angle is called hypotenuse.

(i) $(\text{hypotenuse})^2$ = sum of the squares of remaining sides of right angled triangle

(ii) The straight line joining right angle vertex to mid point of hypotenuse is equal to half of hypotenuse.

(l) The straight line joining mid points of two sides of a triangle is parallel to third side and equal to half of it.

1.4 Centroid, Circumcentre, Incentre and ortho-centre of a triangle.

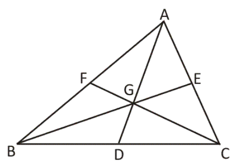

Fig 3(b)

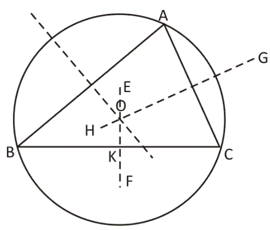

Fig 3(c)

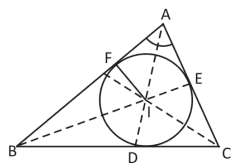

Fig 3(d)

(a) Fig 3(b) shows that medians AD, BE and CF meet at point G. i.e. medians are concurrent. Point G is called centroid of the triangle. It divides each median in ratio of 2 : 1 i.e. AG : GD = BG : GE = CG : GF = 2 : 1

(b) Fig 3(c). Right bisector of sides BC, AC and AB are drawn. These meet in O. O is called circumcentre of triangle OA = OB = OC = R, radius of circum circle of triangle, In triangle OKC, $CK = \frac{1}{2}BC = \frac{1}{2}a$, and $\angle KOC = \frac{1}{2}\angle BOC = \frac{1}{2}.2A = A$ (angle subtended by arc BMC on center O is thrice of angle subtended by it at any point A on the remaining circumference of circle).

From right angled triangle COK; $\frac{KC}{OC}=\frac{a/2}{R}=\sin A$

$\therefore R=\frac{a}{2\sin A}$, Thus $R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}$

(c) In Fig 3(d) angles of the triangle have been bisected. Angle bisectors are concurrent i.e. meet in one point I, called incentre i.e. centre of circle inscribed in the triangle. Perpendicular on sides are ID, IE and IF and ID = IE = IF = r, radius of the circle inscribed in the triangle.

$$\Delta ABC=\Delta ICA+\Delta IAB+\Delta IBC$$

$$\Delta=\frac{1}{2}a.r+\frac{1}{2}b.r+\frac{1}{2}c.b$$

$$=\frac{1}{2}r.(a+b+c)=r.s$$

$$\therefore r=\frac{\Delta}{s}=\frac{\text{area of triangle}}{\text{Semi Perimeter of triangle}}$$

(d) **Orthocentre** – In Fig (3e) AD, BE and CF are perpendicular from vertices on opposite sides. These meet in one point O1. O1 is called orthocentre.

(e) (i) If the triangle is right angle triangle then orthocentre coincides with the right angle vertex.

(ii) If triangle is obtuse angle triangle then circumcentre and orthocentre of triangle, both fall outside the triangle.

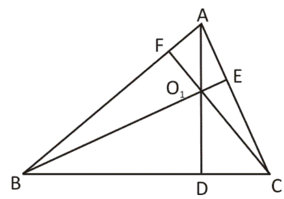

Fig 3(e)

1.5 Relation, between Centroid, Circumcentre and Orthocentre of a triangle

In Fig. 4, AD is median of triangle, ABC; right bisectors of side BC and AC meet in O, O is circumcentre, O_1 is orthocentre. O O_1 meet median AD in G. OB is joined.

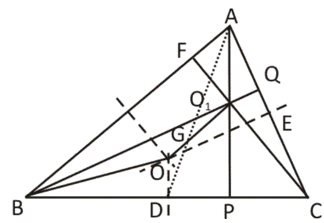

Fig 4

In right angled triangle BOD,

BD = a /2 and $B\hat{O}D$ = A; OB = R radius of circum-circle

OD = R cos A ....(i)

From right angled triangle O_1AQ,

$AQ = AO_1 \cos(90 - C) = AO_1 \sin C$ .... (ii)

And from triangle ABQ,

A Q = AB cos A = c cos A ....(iii)

From (2) and (3) c cos A = A O_1 sin C

$$\therefore\ AO_1 = \frac{c}{\sin C}.\cos A = 2R\cos A \qquad \text{........(iv)} \qquad \left(\because R = \frac{c}{2\sin c}\right)$$

from (i) and (iv) A O_1 = 2OD.

Now AP and OD are perpendicular on BC

$\therefore AP || OD \Rightarrow O_1A || OD$

Δs; AO_1G and GOD are similar $\therefore O_1G:GO = AO_1:OD = AG:GD = 2:1$

$\therefore$ G is centroid.

This proves that orthocentre, centroid and circumcentre of a triangle are collinear and centroid divides O_1O in the ratio of 2 : 1.

Note :

(1) In equilateral triangle, orthocentre, centroid, incentre and circumcentre all coincide.

(2) In isosceles triangle all these four lies on the altitude of unequal side, O, G, O_1 incentre I all lie on AD.

(3) In right angled triangle, orthocentre is at right angle vertex circumcentre at mid point of hypotenuse. Thus orthocentre centroid and circumcentre all lie on median bisecting hypotenuse.

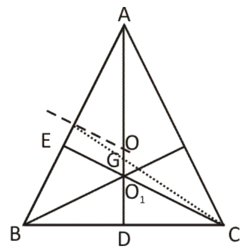

Fig 5

1.6 Congruence of triangles

Two triangle ABC and PQR shall be congruent if each covers the other – completely. There are 3 cases of congruency.

1. **S.S.S. case**, If three sides of a triangle are equal to the corresponding three sides of the other triangle, then the two triangles are congruent.

2. **S.A.S. case**, If two sides of a triangle and the angle between them (included angle) are equal to the two corresponding sides and angle between them of another triangle then the two triangles are congruent.

3. **S.A.A. case**, If one side and any two angles of a triangle are equal to the corresponding side and two angles of other triangle then the two triangles are congruent.

4. **Right angled triangles**: Two right angled triangles are congruent if (i) one side and one angle of one triangle is equal to the corresponding side and corresponding angle of the other triangle (ii) Two sides of one be equal to the corresponding two side of other.

1.7 Similar Triangles

If the angles of a triangle are equal to the three angles of other triangle then the two triangles are similar. Similar triangles are alike in shape but not in size.

(i) The ratios of the corresponding sides of two similar triangles are equal.

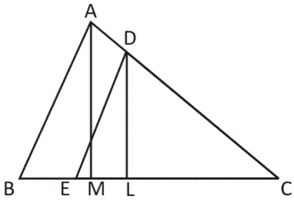

Fig 6(i)

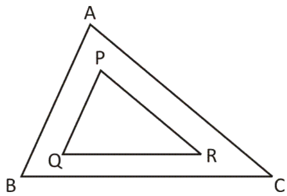

Fig 6(ii)

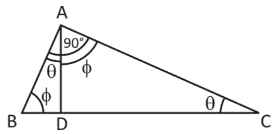

Fig 6(iii)

(ii) In similar triangles ratio of altitudes = ratio of corresponding sides.

(iii) Ratio of area of similar triangles is equal to square of the ratio of corresponding sides

a) Fig. 6(i) Δs ABC and DEC are similar AB || DE. Altitudes $\frac{DL}{AM} = \frac{DC}{AC}$

b) Fig 6(ii) AC || PR, QR || BC and AB || PQ

$\therefore$ As ABC and PQR are similar.

c) Fig 6 (iii) Δ ABC is right angled triangle, $\angle$A = 90^0. AD is perpendicular on BC. Here Δs , ABC, ABD and ADC are similar.

In triangles ABD and ADC

$\angle$ABD = $\angle$DAC = ϕ, $\angle$BAD = $\angle$ACD = θ,
$\angle$ADB = $\angle$ADC

1.8 Apollonius Theorem

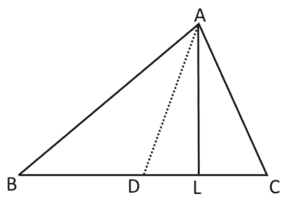

Fig 7

If AD is median of triangle ABC then $AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$

Proof : In Fig. 7, AD is median of Δ ABC and AL is perpendicular on BC, Right angled triangle ABL $AB^2 = BL^2 + AL^2$

In right angled Δ ALC; $AC^2 = AL^2 + LC^2$

$$\therefore AB^2 + AC^2 = 2AL^2 + BL^2 + LC^2$$

$$= 2AL^2 + (BD + DL)^2 + (CD - DL)^2$$

$$(\because BD + DC) \therefore = 2AL^2 + (BD + DL)^2 + (BD - DL)^2$$

$$= 2AL^2 + 2DL^2 + 2BD^2 + 2BD.DL - 2BD.DL$$
$$= 2\left(AL^2 + DL^2\right) + 2BD^2$$
$$= 2AD^2 + 2BD^2 = 2\left(AD^2 + BD^2\right)$$

1.9 Property of angle bisector of an angle of a triangle

(a) See Fig. 8(a) BD is bisector of angle B and it meets opposite side AC in D.

AD : DC = a : c

i.e. angle bisector divides the third side in ratio of its arms.

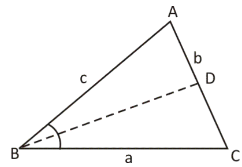

Fig 8(a)

(b) In Fig. 8(b) AM and BK are angle bisectors of angles A and B of ΔABC. These meet in I, which is incentre of triangle.

In co-ordinate you shall see that if (x_1, y_1) and (x_2, y_2) be coordinates of two points P and Q and point R divides PQ internally in ratio m : n then R is $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

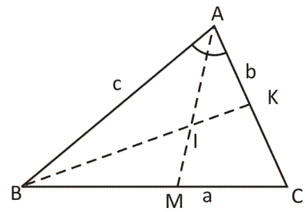

Fig 8(b)

Now if co-ordinates of A, B, C (Fig 8(b)) are (x_1, y_1), (x_2, y_2), (x_3, y_3) bisector of angle B meets AC in K, CK : KA = a : c.

$\therefore$ Co-ordinates of k are $\left[\frac{ax_1 + cx_3}{a+c}, \frac{ay_1 + cy_3}{a+c}\right]$

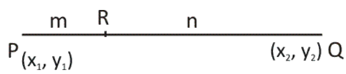

Fig 8(c)

$$\frac{AK}{KC} = \frac{c}{a} \Rightarrow AK = \frac{c}{a}(KC) = \frac{c}{a}(b - AK) \Rightarrow AK = \frac{bc}{a+c}.$$

Bisector of angle A of ΔABK, meets BK in I

$\therefore$ BI : IK = c : AK = c : $\frac{b+c}{a+c}$

$$\therefore \text{abscissa I} = \left[\frac{c.\left(\frac{ax_1 + cx_3}{a+c}\right) + \frac{bc}{a+c}.x_2}{c + \frac{bc}{a+c}}\right] = \frac{ax_1 + bx_2 + cx_3}{a+b+c}$$

$$\therefore \text{Co-ordinates of incentre are } \left[\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right]$$

1.10 Quadrilateral

If four points in a plane (no three being in one line) be joined in order then the Fig. so formed is a quadrilateral. It has 4 vertices 4 sides, 4 angles, 2 diagonals, sum of angle is 360^0.

(b) **Trapezium** : If two opposite sides of a quadrilateral are parallel (but not equal) then it is trapezium. In Fig. 9(a), AD || BC but AD $\neq$ BC, then ABCD is a trapezium. Straight line PQ is parallel to these parallel sides and divides DC in ratio m : n, then

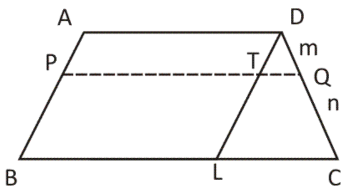

Fig 9(a)

$$PQ = \frac{m}{m+n}(\text{BC longer side}) + \frac{n}{m+n}(\text{AD shorter side})$$

Proof : DL has been drawn parallel to BC.

$\therefore \Delta$ s DTQ and DLC are similar $\Rightarrow \frac{TQ}{LC} = \frac{m}{m+n}$

$$\therefore TQ = \frac{mLC}{m+n} = \frac{m(BC - AD)}{m+n}$$

Now PQ = PT + TQ = AD + $\frac{m(BC - AD)}{m+n}$

$$= \frac{m}{m+n}BC + \frac{n}{m+n}AD$$

c) **Parallelogram** : If two opposite sides of a quadrilateral are parallel and equal then it is a parallelogram. If opposite sides of a quadrilateral are parallel then it is a paralleogram.

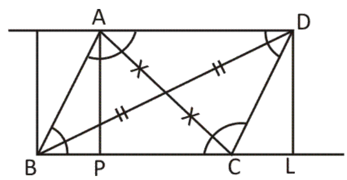

Fig 9(b)

(i) Opposite sides are equal.

(ii) Opposite angles are equal

(iii) Sum of two adjacent angles = 180^0

(iv) Diagonals bisect each other

(v) Each diagonal divides parallelogram in two congruent triangles.

In Fig 9(b) AP is perpendicular from A on BC. AP is height of parallelogram.

Area of parallelogram = $\frac{1}{2}\text{base} \times \text{height}$

d) **Rectangle :** A parallelogram whose all angles are equal is called rectangle. Each angle is 90^0. If Fig 9(b) APLD is a rectangle,PL is length and AP is breadth of rectangle area of rectangle = length x breadth. Diagonals of rectangle are equal each = $\sqrt{l^2+b^2}$

e) **Square –** If a rectangle has all its side equal then it is a square. Its diagonals are equal and intersected right angles. If a is side of square, then diagonal = $a\sqrt{2}$, area = a^2 square units.

e) **Rhombus :** If adjacent sides of a parallelogram are equal, then it is rhombus. Diagonals not equal, but bisect at right angle.

Side of rhombus = $\frac{1}{2}\sqrt{\text{sum of squares of two diagonal}}$

$$=\frac{1}{2}\sqrt{d_1^2+d_2^2}$$

1.11 Polygon

In a plane closed figures having more than 4 sides are called polygon. If all sides of a polygon are equal it is called regular polygon.

(i) If number of sides is 5, it is called pentagon

(ii) If number of sides is 6, it is called hexagon

(iii) If number of sides is 7, it is called septagon

(iv) If number of sides is 8, it is called octagon.

Properties :

(i) Sum of interior angles of a polygon = $(2n-4) \times 90^0$, where n is number of sides.

(ii) Diagonals of a polygon are = $\frac{n(n-1)}{2}-n=\frac{n(n-3)}{2}$

(iii) Exterior angle of a regular polygon = $\frac{360^0}{n}$

(iv) Interior angle of regular pentagon = $\frac{(10-4)}{5}\times 90^0=108^0$, of regular hexagon 120^0, of regular octagon = 135^0

1.12 Regular hexagon

In Fig. 10, ABCDEF is a regular hexagon. O is center of circle drawn circumscribing it. Let side AB =a. Interior angle

$A = \frac{(12-4)}{6} \times 90 = 120^0$. If center O is joined to vertices of hexagon, we get 6 equilateral Δ s as $\angle BAO = \frac{1}{2}(120^0) = 60^0$ and $\angle BOA = \frac{360^0}{6} = 60^0$.

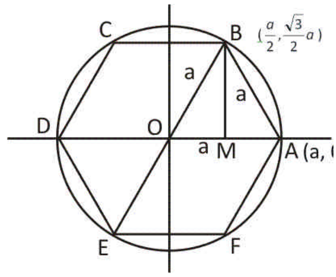

Fig 10

(i) area of hexagon = $6 \times \frac{\sqrt{3}}{4} a^2 = \frac{3\sqrt{3}}{2}(\text{side})^2$

(ii) If O is taken as origin, OA as x-axis and perpendicular on OA at O as y-axis then co-ordinates of vertices are A, (a, 0), B$\left(\frac{a}{2}, \frac{\sqrt{3}}{2}a\right)$, C $\left(-\frac{a}{2}, \frac{\sqrt{3}}{2}a\right)$, d, (–a,0) E $\left(-\frac{a}{2}, -\frac{a\sqrt{3}}{2}\right)$, F $\left(\frac{a}{2}, -\frac{a\sqrt{3}}{2}\right)$

(iii) Radius of circumcircle of hexagon = a = side of hexagon.

(iv) OK is perpendicular from O, on side EF, OK is radius of incircle of hexagon.

1.13 Circle

Circle is locus of a point, which moves such that its distance from a fixed point is always same constant. The fixed point is center of circle and constant distance radius of circle.

Properties:

(1) one and only one circle can pass through 3 points not in the same line.

(2) Perpendicular from center of circle on its any chord bisects the chord.

(3) Equal chords of circle are equidistant from its center.

(4) Angle subtended by an arc of circle at its center is twice of that angle which this arc subtends at any point of remaining circumference.

(5) Angles of same segments are equal.

(6) Angle in semi-circle is right angle.

(7) A quadrilateral inscribed in a circle is called cyclic quadrilateral sum of its two opposite angles is 180^0.

(8) In Figure 11; chords AB and CD intersect at N inside the circle, while chords PQ and RS intersect at T outside the circle

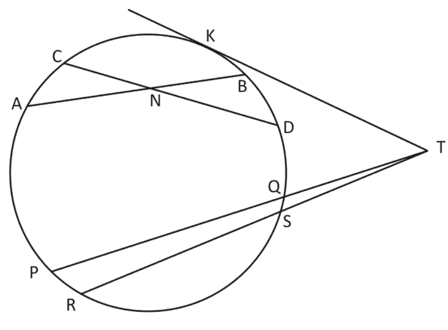

Fig 11

AN . NB = CN . ND

i.e. rectangle contained by the segments of one (AB) is equal to the rectangle contained by the segments of other, CD.

And PT .TQ = RT . TS, .i.e. product of segments of one is equal to the product of the segments of other.

(9) In Fig. 11, TK is tangent of circle and TQP is secant. $TK^2 = TQ . TP$

(10) The angle between a tangent and a chord through point of contact of tangent is equal to the angle in alternate segments.

(11) Two tangents can be drawn to a circle from an external points and these are equal. The straight line joining two points of contact is called chord of contact. It is perpendicular to straight line joining external point to center of circle.

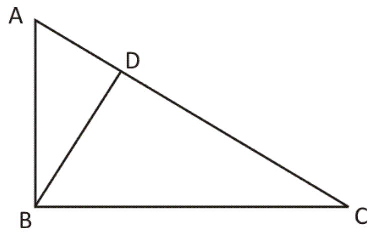

Fig 12

(12) In Fig. 12 ABC is right angle triangle, BD is perpendicular from B on hypotenuse AC. Circle drawn on BC or AB as diameter shall pass through D. For circle drawn on BC, AC shall be a secant and AB shall be tangent.

$\therefore AD.AC = AB^2$

Similarly for circle drawn on AB as diameter CA shall be secant and CD shall be tangent and

$CD. CA = CB^2$

1.14 Intersection of two circles and common tangents

Let O_1 and O_2 be centers of a circle and r_1, r_2 be their radius $r_1 > r_2$.

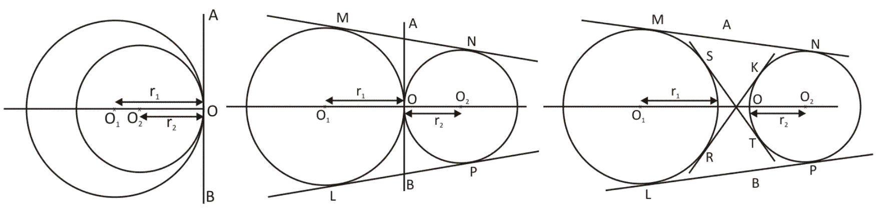

Fig 13 (a) **Fig 13 (b)** **Fig 13 (c)**

1) If $O_1O_2 < (r_1 - r_2)$ the circle with radius r_2 shall lie in circle with radius r_1 without touching it.

2) If $O_1O_2 = r_1 - r_2$ then circle with radius r_2 shall touch internally circle with center O_1 at point of contact one common tangent possible as in Fig 13(a).

3) If $O_1O_2 > (r_1 - r_2)$ but $< (r_1 + r_2)$ the two circles shall intersect and two direct common tangents can be drawn. Length of these common tangents = $\sqrt{(O_1O_2)^2 - (r_1 - r_2)^2}$

4) If $O_1O_2 = r_1 + r_2$ then circles shall touch each other externally. Three common tangents, two direct common tangents and one at point of contact of circles as in Fig 13(b).

5) If $O_1O_2 > (r_1 + r_2)$, circles shall not intersect neither touch each-other. 4 common tangents, 2 direct common tangents and two transverse tangents can be drawn.

Length of transverse tangent = $\sqrt{(O_1O_2)^2 - (r_1 + r_2)^2}$

1.15 When 3 circles touch each other

Fig. (14) O_1, O_2, O_3 are centers of three circles that touch each other externally; r_1, r_2, r_3 are their radius. O_1, O_2, O_3 is a triangle whose sides are $(r_1 + r_2)$, $(r_2 + r_3)$ and $(r_3 + r_1)$. Points of contact of these circles are D, E and F. Common tangents to circles at these points shall be perpendicular to sides O_1O_2 and O_2O_3 and O_3O_1 respectively. These meet in one point O. i.e. common tangents are congruent. The point O is called radical center i.e. a point from which tangents to these circles are equal. OD = OE = OF and these are perpendicular on sides. Therefore O is incentre of triangle $O_1O_2O_3$.

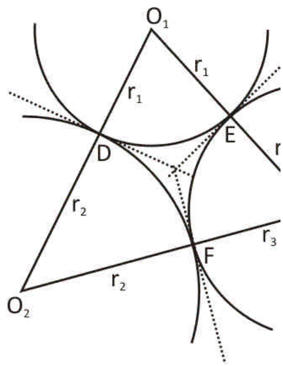

Fig 14

$\therefore OD = $ radius of in circle $= \dfrac{\text{area of } \Delta O_1O_2O_3}{\text{semiperimeter}}$ and $s = r_1 + r_2 + r_3$ and $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\therefore \Delta = \sqrt{(r_1 + r_2 + r_3).r_1r_2r_3}$

$$\therefore OD^2 = \frac{(r_1 + r_2 + r_3).r_1r_2r_3}{(r_1 + r_2 + r_3)^2} = \frac{r_1r_2r_3}{r_1 + r_2 + r_3}$$

Solved Examples

Example 1 : A vertical pole, 8 feet high is divided in two parts at O in such a way that the two parts subtend equal angles at a point on ground, 6 feet away from the pole. Calculate the lower part of pole.

Sol. : Fig (15) AB is pole. P point on ground

$\angle APO = \angle OPB \Rightarrow OP$ bisects $\angle APB$

given AB = 8 ft, PB = 6 ft

$\therefore AP = \sqrt{64+36} = 10'$

and AO : OB = 10 : 6

$\therefore OB = \frac{3}{5} AO.$

$= \frac{3}{3+5} AB = \frac{3}{8} AB$

$= \frac{3}{8} \times 8 = 3'$

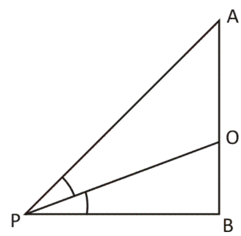

Fig 15

Example 2 : In an isosceles triangle of base 16 cm and of area 120 sq cm. A circle is drawn inside it touching all sides. Find its radius.

Sol. : In Fig. 16, BC is base of isosceles triangle and AD is its altitude.

$\therefore \text{area} = \frac{1}{2} BC.AD$

$\Rightarrow \frac{1}{2}.16.AD = 120$

$\therefore AD = 15cm.$

$\therefore AB^2 = 15^2 + 8^2, \Rightarrow AB = 17cm$

BD and BF are tangents from B on incircle. $\therefore BF = BD = 8cm$.

In right angle triangle AFO, AF = 17 – 8 = 9, AO = 15 –r

$\therefore (15-r)^2 = 9^2 + r^2 \Rightarrow 225 - 30r = 81$

$\Rightarrow 30r = 144 \quad \therefore r = \frac{24}{5} = 4\frac{4}{5} cm$.

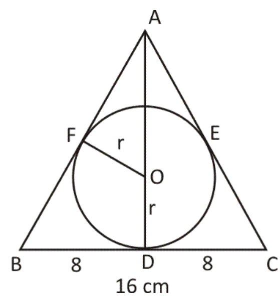

Fig 16

Example 3 : A circle is inscribed in an equilateral triangle of side a. Calculate area of a square which is inscribed in this circle.

Sol.: The center of the circle inscribed in equilateral triangle coincides with centroid of triangle and median is altitude.

$\therefore$ radius r of circle = $\frac{1}{3}(\text{altitude})=\frac{1}{3}.\frac{\sqrt{3}}{2}a=\frac{\sqrt{3}}{6}a$. Square inscribed in the circle has its diagonal along the diameter of circle.

$\therefore$ If x is side of square $x^2+x^2=(2r)^2=\left(\frac{\sqrt{3}}{3}a\right)^2=\frac{1}{3}a^2$

$\therefore$ area of square = x^2 = $\frac{a^2}{6}$ sq. units.

Example 4 : D and E are points on sides BC and AC of triangle ABC; BD = 2DC and AE = 3EC. If AD and BE meet in P, then calculate BP / BE.

Sol. : Given BD = 2DC, AE = 3EC.

$\therefore$ BD : DC = 2 : 1; AE : EC = 3 : 1

Now draw EF parallel to DC.

It meets AD in F.

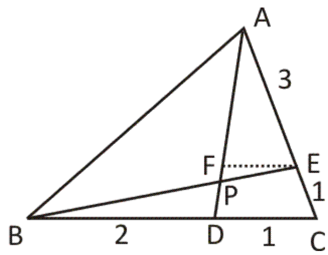

Fig 17

Δs AFE and ADC are similar

$\therefore \frac{FE}{DC}=\frac{AE}{DC}=\frac{3}{4}\Rightarrow FE=\frac{3}{4}DC=\frac{3}{4}x$ (Let DC = x)

Δs, FEP and BPD are similar

$\therefore \frac{BP}{PE}=\frac{BD}{EF}=\frac{2x}{(3/4)x}=8/3$.

$\therefore$ BP : BE = 8: (3 + 8) = 8 : 11

Example 5 : Two adjacent side of a quadrilateral are 2 and 5 angle between them 60^0. If third side is 3, then find 4^{th} side.

Sol. : In Fig. 18, AB = 2, BC = 5, AD = 3, AC = x, DC = y.

In triangle ABC

$\cos B=\cos 60=\frac{1}{2}=\frac{4+25-x^2}{2.2.5}$

$\Rightarrow 29-x^2=10\Rightarrow x^2=19$

ABCD is cyclic quadrilateral

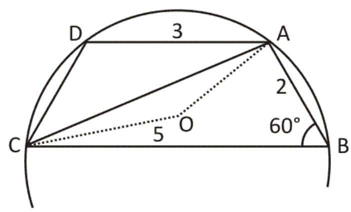

Fig 18

$\therefore \angle ADC=180^0-60^0=120^0$

$\therefore$ In ΔDAC, $\cos 120^0 = -\frac{1}{2} = \frac{9+y^2-x^2}{2.3y}$

$\Rightarrow -3y = 9 + y^2 - 19 \Rightarrow y^2 + 3y - 10 = 0$

$\Rightarrow (y+5)(y-2) = 0 \quad \therefore y = 2$

Example 6 : P is a point inside a circle whose center is O. Find locus of mid points of chords of circle through P. (b) what will be the locus if point P lies outside circle.

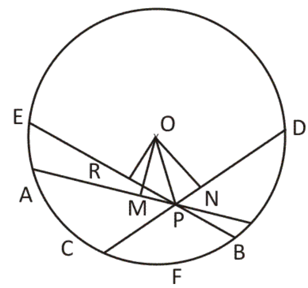

Fig 19(a)

Sol. : APB, CPD, EPF chords of circles pass through P. OM, ON and OR are perpendiculars from center O on them.

$\therefore \angle OMP = \angle ONP = \angle ORN = 90^0$

$\therefore$ M, N, R mid points of chords, lies on the circle drawn on OP as diameter.

(b) PAB, PCP, PEF secant give AB, CD, EF chords. Lines joining their mid points to center OM, ON, OR are perpendicular on them

$\therefore \angle OMP = \angle ONP = \angle ORP = 90^0$

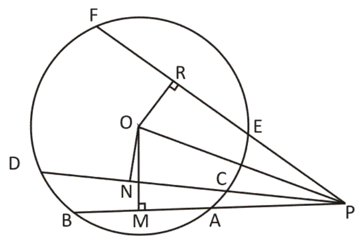

Fig 19(b)

Practice Worksheet (Foundation Level) – 1(a)

1. The bisector of an angle of a triangle bisects third side. The triangle is :

   (a) Scalene (b) isosceles

   (c) equilateral (d) right angled.

2. A square is inscribed in a circle of radius 4cm and then a circle is inscribed in this square. The radius of circle is :

   (a) $2\sqrt{2}$ cm (b) $3\sqrt{2}$ cm (c) 2.5 cm (d) $4\sqrt{2}$

3. ABC is a right angled triangle, $\angle B = 90^0$, BD is perpendicular on AC. If AD = 9cm , DC = 7 cm. Then AB is :

   (a) 15 cm (b) 12cm (c) 10 cm (d) 8 cm

4. In question 3, AB : BC =

   (a) 3 : 2 (b) $3 : \sqrt{7}$ (c) $4 : \sqrt{7}$ (d) $2 : \sqrt{7}$

5. ABC is a triangle. P, Q, R are points on AB, AC and BC such that PQ parallel BC and QR parallel AB; AP : PB = 2 : 3 . Now from R parallel is drawn to AC which meets AB in S. then AS / AB =

   (a) 3/5 (b) 2/5 (c) 5/8 (d) 3/8

6. In question 5; PS =

   (a) 0.25 AB (b) 0.3 AB (c) 0.2 AB (d) 0.4 AB

7. D and E are points on BC and AC sides of triangle ABC, such that BD = 3 DC and AE = 2EC. AD and BE intersect in P, then BP / BE =

   (a) $\frac{7}{9}$ (b) $\frac{9}{11}$ (c) $\frac{11}{13}$ (d) $\frac{7}{11}$

8. The base of an isosceles triangle is 10 cm and its one side is 13 cm. A circle is inscribed in it. Radius of circle is :

   (a) 3 cm (b) 3.5 cm (c) 10/3 cm (d) 8/3 cm

9. Side of a regular hexagon is 6 cm. Ratio of areas of its circumcircle and inscribed circle is:

   (a) 4 : 3 (b) $4 : 2\sqrt{3}$ (c) $2 : \sqrt{3}$ (d) 3 : 2

10. Area of hexagon in Q.9 is :

    (a) $24\sqrt{3}$ sq. m (b) $54\sqrt{3}$ sq. m

    (c) $36\sqrt{3}$ sq. m (d) none of these

11. ABC is a triangle, BC = 6, AC = 4 and AB = 5 cm. Bisector of angle B meets AC in E and bisector of angle A meets BE in P, then BP : PE =

(a) 11 : 5 (b) 11 : 6 (c) 11 : 4 (d) 11 : 7

12. Two circles with radii r_1, r_2, ($r_2 > r_1$) intersect in A and B, AB is joined and extended both ways. P is any point on extended AB and PT and PK are tangents from it to the two circles, then

(a) PT = PK (b) PT > PK

(c) PT < PK (d) none of these

13. Centres of two circles of radii 10 and 6 cm are 6 cm apart. Length of direct common tangent is :

(a) $5\sqrt{2}$ (b) $3\sqrt{2}$ (c) $4\sqrt{5}$ (d) $2\sqrt{5}$

14. Length of direct common tangent of two circles is twice the length of transverse common tangents of these circle. If radii of these circles are 5 cm and 3cm. Distance between their centers is:

(a) $2\sqrt{21}$ cm (b) $5\sqrt{2}$ cm (c) $3\sqrt{2}$ cm (d) $3\sqrt{21}$ cm

15. The sides of a triangle are 18, 24 and 30cm. The radius of incircle is :

(a) 4 cm (b) 6cm (c) 3 cm (d) 8cm

16. Diagonals of a rhombus are 6cm and 8cm. Then side of rhombus is :

(a) 6cm (b) 8cm (c) 5 cm (d) 10 cm

17. ABCD is a cyclic quadrilateral AD = DC = x, BC = 2x, $\angle$ADC = 120, then AB is :

(a) $\sqrt{3}x$ (b) x (c) $\frac{3}{2}x$ (d) none of these

18. Radius of two circles are 3 and 5cm, and distance between their center is 10cm. Length of transverse common tangent is:

(a) 6 cm (b) 8 cm (c) 9 cm (d) none of these

19. In triangle ABC, BC = x, CA = 5 and AB = 4 cm. AD is perpendicular from A on BC and 2BD = DC. The bisector at angle A meets BC at E, then DE =

(a) $\frac{x}{8}$ (b) $\frac{2x}{9}$ (c) $\frac{x}{9}$ (d) $\frac{x}{12}$

20. PAB and PCD are two secants of a circle that meet circle in A, B and C, D; PC = CD = x and PA = 2AB. Then AB is :

(a) $\frac{1}{2}x$ (b) $\frac{1}{3}x$ (c) $\frac{1}{\sqrt{3}x}$ (d) $\frac{2}{3}x$

21. ABCD is a quadrilateral $\angle ADB = \angle ACB$ then angle bisectors of these angles and right bisector of AB are :

(a) concurrent (b) angle bisectors bisect AB

(c) angle bisectors of angles meet right bisector at different points

(d) none of these

22. ABCDEF is a regular hexagon. AB + AC + AD + AE + AF = n AB then n is :

(a) $(3+3\sqrt{3})$AB (b) $(3+\sqrt{3})$AB

(c) $(3+2\sqrt{3})$AB (d) none of these

23. If R and r be radii of circumcircle and incircle of a regular pentagon then r/R =

(a) $\frac{\sqrt{5}-1}{4}$ (b) $\frac{\sqrt{5}+1}{4}$ (c) $\frac{\sqrt{5}}{4}$ (d) none of these

24. AB and CD are chords of a circle. AB goes through centre and CD meets AB in P and is perpendicular on AB. If CP = 5 cm and PB = 3cm. Then radius of circle is :

(a) 6cm (b) 19/3 cm (c) 7 cm (d) 17/3 cm

25. Diagonals of a parallelogram are 10 and 8 cm and included angle is 60^0. Then side of parallelogram opposite to this angle is

(a) 9cm (b) $2\sqrt{21}$ cm (c) $4\sqrt{6}$ cm (d) $3\sqrt{7}$ cm

(Hint : apply $\cos\theta = \frac{a^2+b^2-c^2}{2.a.b}$, θ angle between a and b)

Surface and Volume of solids

1.16 Prism

A right prism is a solid whose top and bottom faces are parallel to each other and identical polygons, lateral faces are perpendicular to base. Base is bottom face. Distance between face and base is called height of the right prism.

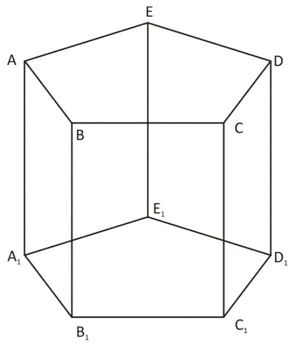

Fig 20

(a) Lateral surface = perimeter of base x height

(b) Total surface = Lateral surface + 2 x area of base.

(c) Volume of right prism = area of base x height.

1.17 Cuboid – (Rectangular solid) :

(a) A right prism, whose base is rectangle is called **Cuboid**. If a and b the sides of rectangle base and c be height of cuboid then:

(a) Total surface of cuboid = 2 (ab+ bc + ca) sq. u.

(b) Volume of cuboid = abc cu. unit

(c) It has 8 vertices, four diagonals and 6 faces diagonals are equal and one diagonal = $\sqrt{a^2+b^2+c^2}$

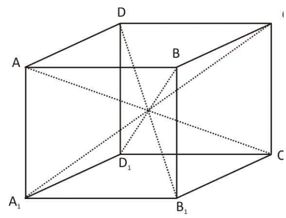

Fig 21

(b) **Cube:** If the base of a right Prism is a square and height equal to side of base, then it is called cube. Here a = b = c

(a) Total surface area of cube = $6a^2$ sq. unit

(b) Volume = a^3 cu. unit

Diagonal = $a\sqrt{3}$ unit

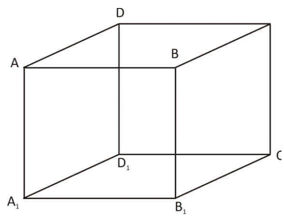

Fig 22

(c) **Parallelopiped** : If any pair of opposite sides of a cuboid are turned parallelograms then it becomes parallelopiped. Thus parallelopiped is a right prism whose base is parallelogram. The opposite faces of the parallelopied all can be parallelograms.

In Fig. 23 parallelogram ABCD is the base of paralleopiped. If $\angle$DAB = α and coterminous edges AB, AD and AA' are a, b and c; and DD' make angle θ with the vertical, then volume of paralleopiped = (ab sinα). C cos θ.

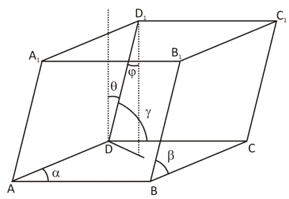

Fig 23

$= \text{area of base} \times \text{height of prism.}$

$\text{Total surface} = 2(ab\sin\alpha + bc\sin\beta + ac\sin\gamma)$

1.18 Cylinder

If base of a right prism is circle then it is cylinder. In the Fig, ABCD is cylinder, radius of base = r, height = h.

OO’ is called axis of cylinder. If rectangle OO’CD rotates an axis OO’, cylinder shall be formed. Lateral surface area of cylinder = $2\pi rh$ sq. units

Total surface area = Lateral surface + areas at top and bottom

= $2\pi rh + 2\pi r^2 = 2\pi r(h+r)$ sq. unit

Volume of cylinder = area of base x height

= $\pi r^2 h$ cu. unit

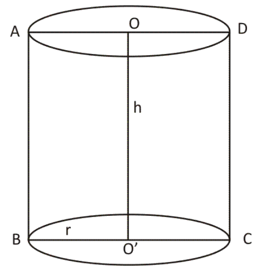

Fig 24

Hollow Cylinder

R = Outer Radius

r = Inner Radius

h = height of cylinder

Total lateral surface = $2\pi h(R+r)$

Total surface = $2\pi(R+r)h + 2\pi(R^2 - r^2)$

Volume = $\pi(R^2 - r^2)h$ cu. unit

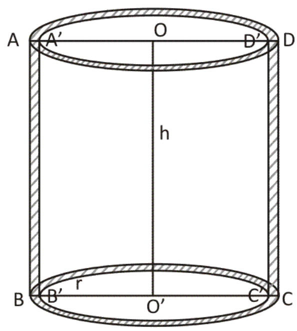

Fig 25

1.19 Pyramid

A solid whose base is polygon and sides are triangles with a common vertex are called a pyramid. Perpendicular from common vertex on the base is height of pyramid.

Volume of cylinder = $\frac{1}{3}$ (area of base) x height

In pyramid, the side triangles are congruent. If the base is regular polygon, then

Lateral surface = $\frac{1}{2}$ (perimeter of base) x slant height

Slant height is length of straight line joining vertex of pyramid with the mid point of side of base regular hexagon. It is perpendicular on the side.

1.20. Tetrahydon

If the base of a pyramid is triangle then it is called tetrahedron. If the base is an equilateral triangle then it is called regular tetrahedron. If h is height, l slant height and a base side then

(a) total surface area = $3.\frac{1}{2}a\times l+\frac{\sqrt{3}}{4}a^2$

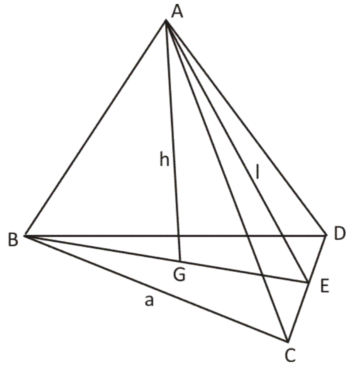

Fig 26

(b) Volume = $\frac{1}{3}.\left(\frac{\sqrt{3}}{4}a^2\right).h$

(c) Relation between h, l and a is

$$l^2=(GE)^2+h^2,=\left(\frac{1}{3}BE\right)^2+h^2$$

$$=\left(\frac{1}{3}.\frac{a\sqrt{3}}{2}\right)^2+h^2=\frac{a^2}{12}+h^2$$

(d) If all the four faces of tetrahedron are congruent then height of this pyramid H is

$$H^2=AE^2-GE^2=\left[\left(\frac{\sqrt{3}}{2}a\right)^2-\left(\frac{1}{3}\frac{\sqrt{3}}{2}a\right)^2=\frac{8}{9}.\left(\frac{\sqrt{3}}{2}.a\right)^2\right]=\frac{2}{3}a^2$$

$$\Rightarrow H=\sqrt{\frac{2}{3}}a$$

1.21. Cone

A right pyramid with a circular base is called a cone In the Fig. O is vertex of cone, c center of circular base; r its radius; h = OC is height of cone and OB = *l* is slant height of cone $\angle COB=\alpha$ is called semivertical angle of cone.

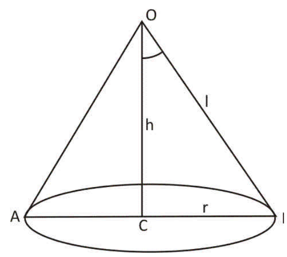

Fig 27

1. Surface area of cone = πrl

2. Total surface of cone = $\pi rl.\ \pi r^2=\pi r(l+r)$

3. Volume of cone = $\frac{1}{3}\pi r^2h$

Frustum of Cone

When a plane parallel to the base of a cone cuts the cone, frustum of cone result. In Fig. ABCD is frustum of cone. O and O' are centers of base and upper end r_1 and r_2 are their radii.

Height OO' = h.

DC is slant height of frustum of cone.

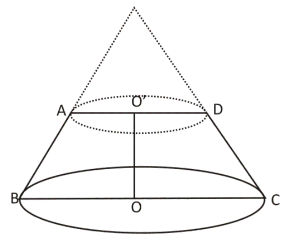

Fig 28

1. Lateral surface area of frustum of cone = $\pi(r_1 + r_2)l$
2. Total surface area = $\pi(r_1 + r_2)l + \pi(r_1^2 + r_2^2)$
3. Volume of frustum of cone = $\frac{\pi}{3}h(r_1^2 + r_1r_2 + r_2^2)$

$$= \frac{1}{3}h\left(A_1 + \sqrt{A_1A_2} + A_2\right)$$

where A_1 and A_2 are area of the base and upper face

4. Slant height of frustum of cone = $\sqrt{h^2 + (r_1 - r_2)^2}$

1.22. Sphere

When a circle revolves at one of its diameter, sphere is generated. Mid point of this diameter is centre of sphere and half of it is radius of sphere. All points on the surface of sphere are at equal distance from center. If a is radius of sphere, then

1. Volume of sphere = $\frac{4}{3}\pi r^3$ cu. u
2. Surface area of sphere = $4\pi r^2$ sq.u

When a sphere is divided in two equal parts by a plane which goes through its center, hemisphere results.

1. Curved surface area of hemisphere = $2\pi r^2$ sq.u
2. Total surface area of hemisphere = $2\pi r^2 + \pi r^2 = 3\pi r^2$sq.v

3. Volume of hemisphere = $\frac{2}{3}\pi r^3$ cu.u

If a sphere is cut in four equal parts i.e. when a hemisphere is bisected.

4. Volume of each part = $\frac{1}{3}\pi r^3$cu.u
5. Total surface = $2\pi r^2$

Spherical shell

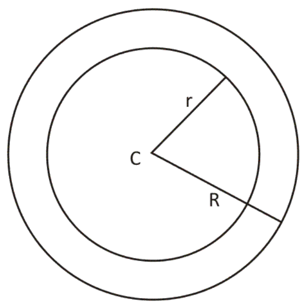

Fig 29

In the Fig, R is radius of outer surface of shell and r is inner radius of the shell. (R–r) is thickness of shell.

6. Volume = $\frac{4}{3}\pi(R^3 - r^3)$cu.u

7. Total surface area = outer surface area + inner surface area = $4\pi\left(R^2 + r^2\right)$cu. unit

8. If d is density of the metal of this shell.

   Total wt. = $\frac{4}{3}\pi\left(R^3 - r^3\right)\times d$

Solved Examples

Example 7 : 16000 small similar balls of radius 0.2 cm. are melted to form (a) a sphere or (b) a cone of height 48 cm or (c) a cylinder of height 24 cm. Find their radii?

Sol. : Radius of iron ball = 2/10 cm.

$$\therefore \text{Volume of 16000 balls} = (16000)\frac{4}{3}\pi\left(\frac{2}{10}\right)^3 \text{cu. cm}$$

(a) Let radius of sphere be R

$$\therefore \frac{4}{3}\pi R^3 = (16000)\left(\frac{4}{3}\pi\right)\left(\frac{2}{10}\right)^3.$$

$$\Rightarrow R^3 = 16\times4\times8 \qquad \Rightarrow R = 8\text{cm}.$$

(b) Let radius of cone be R, height is 48

$$\therefore \frac{1}{3}\pi R^2.48 = \frac{4}{3}\pi\left(\frac{2}{10}\right)^3\times16000$$

$$\Rightarrow R^2 = \frac{32}{3} \qquad \Rightarrow R = \frac{\sqrt{32}}{3}\text{ cm}$$

(c) Height of cylinder = 24 cm, let radius be R

$$\therefore \qquad \pi R^2\times24 = \frac{4}{3}\pi\left(\frac{2}{10}\right)^3\times16000$$

$$R^2 = \frac{4\times8\times16}{3\times24}$$

$$\Rightarrow R = 8/3\text{cm}$$

Example 8: Diameter of a metallic sphere is 60 cm. It is melted and drawn into a uniform wire (solid) of diameter 0.8 cm. Find length of wire.

Sol. : Let l be the length of wire.

Volume of wire = volume of sphere

$$\therefore \pi\left(\frac{4}{10}\right)^2 l = \frac{4}{3}.\pi(30)^3$$

$$\Rightarrow l = \frac{4\times30\times30\times30\times10\times10}{4\times4\times3}$$

$$= 9000\times25 = 2250000\text{ cm}$$

$$= 2250\text{m} = 2.25\text{k.m}$$

Example 9 : A uniform wire of certain length is melted and recast into a wire of cross section 5% length. How much % its length be increased.

Sol. : Let r be the radius and l, length of wire In new wire cross section $\pi r^2 - \frac{95}{100}\pi r^2$

Volume in both cases is same.

$\therefore \frac{95}{100}\pi r^2 l_1 = \pi r^2 l \Rightarrow l_1 = \frac{100}{95} l.$

Increase in length = $\frac{100}{95} l - l = \frac{5}{95} l$

Increase % = $\frac{5}{95} \times 100 = \frac{100}{19} = 5\frac{5}{19}\%$

Example 10 : A sphere of radius 6 cm is dropped slowly into a right circular cylinder half full of water. Sphere completely immersed in water. Water level rises by 2cm. Calculate radius of cylinder.

Sol. : Let r be the radius of the cylinder.

Water level rises by 2cm,

$\therefore$ Volume of cylinder 2 cm high = volume of sphere

$\therefore \pi r^2.2 = \frac{4}{3}\pi 6^3$

$\Rightarrow r^2 = 4 \times 6 \times 6 \Rightarrow r = 12cm$

Example 11: A well of 6m inside diameter is dug 15 m deep. Earth taken out from it is spread all round it to a width of 3m to form an embankment. Find height of embankment.

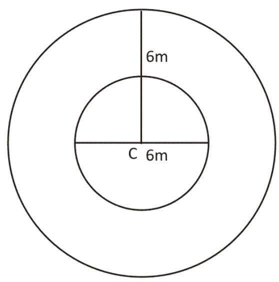

Fig 30

Sol. : The earth spread round the well forms a hollow cylinder of internal and external radii of 3m and 6m. If its height is h, then.

$\pi(6^2 - 3^2)h = \text{Earth dugout}$

$= \pi 3^2 \times 15$

$\therefore \quad h = \frac{3 \times 3 \times 15}{9 \times 3} = 5m$

Example 12 : A uniform canal is 300 m wide at top and 200 m wide at bottom and is 5m deep. Water flows at 10 km/hour. How much area it will irrigate in 30 minutes if 8cm of standing water is required for irrigation. Water of canal is 4m deep.

Sol. : Vertical section of canal is trapezium ABCD, AD = 30 dm, BC = 20 d.m. CK is perpendicular on AD and MN is water level in canal

Δ s, KCD and PCN are similar. Let PN = x

KD = 50m, and KC = 5m, PC = 4m

$\therefore \frac{x}{50} = \frac{4}{5} \Rightarrow x = 40m$.

$\therefore$ MN = 200 + 80 = 280 m

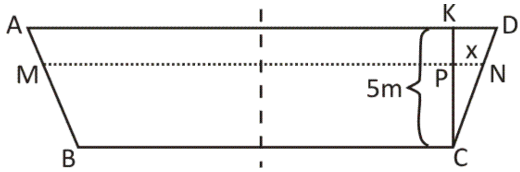

Fig 31

Volume of water goes out in $\frac{1}{2}$ hour = volume of pyramid whose base is trapezium MBCN and height is 5km = 5000 m. (Water flows at 10 km/hour)

If area irrigated is x sq. m then

$$x \times \frac{8}{100} = \text{Volume of pyramid}$$

$$= \frac{1}{2}(200 + 280) \times 5000$$

$$\therefore x = \frac{100 \times 480 \times 5000}{2 \times 8} \text{ sq.m}$$

$$= 15000000 \text{ sq. m} = \frac{15 \times 10^6}{10^6} = 15 \text{ sq.km}$$

Example 13 : A funnel is made by joining frustum of a hollow cone with one end of a hollow cylinder. (See Fig) The free end of cylinder is then closed. Calculate volume of water it will hold. Also calculate total internal surface.

Sol. : Volume of water

= volume of frustum of cone + volume of cylinder

$$= \frac{1}{3}\pi h_1\left(r_1^2 + r_1 r_2 + r_2^2\right) + \pi r_2^2 h_2$$

$$= \frac{1}{3}\pi 12(36 + 6 + 1) + \pi.1.10$$

$$= (4 \times 43 + 10)\pi = 182 \times \frac{22}{7} = 572 \text{cm}^3$$

Total inner surface = Surface of frustum + later surface cylinder + base

$$= \pi(r_1 + r_2)l_1 + 2\pi r h_2 + \pi r_2^2$$

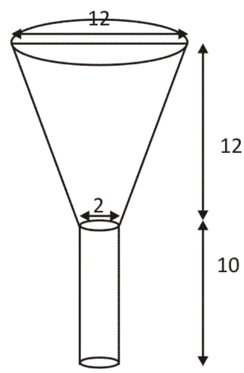

Fig 32

$$=\pi(6+1)13+2\pi.1.10+\pi.1$$
$$=\pi(91+20+1)=112\pi$$
$$=112x\frac{22}{7}=352\,cm^2$$

Example 14 : A conical vessel of radius 12 cm and height 16 cm, is full of water. A sphere is gently lowered in it and as soon as it touches sides, it gets just completely immersed in water. What percentage of cone water overflows?

Sol. : In Fig. A is vertex of cone, BC is base and O is center of base. The sphere on being immersed in water, touches cones slant surface at D and E and center of base O_1. Let r be radius of sphere and θ semi vertical angle of cone. O_1 is center of sphere.

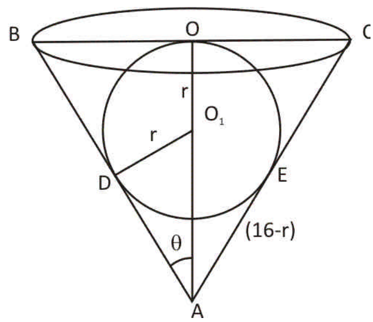

Fig 33

$\therefore O_1D$ is $\perp$ on AB.

From right angled triangle AOB; $\tan\theta=\frac{12}{16}=\frac{3}{4}$

$\therefore \sin\theta=3/5;$ from right angled triangle O_1DA

$$r/AO_1=\sin\theta \Rightarrow r=O_1A\sin\theta=(16-r).\frac{3}{5}$$

$$\therefore 5r=48-3r\Rightarrow r=6cm.$$

Volume of sphere = $\frac{4}{3}\pi.6^3cm^3$

Total volume of cone = $\frac{1}{3}.\pi.12^2.16.$

$\therefore$ % of flow water = $\left[\left(\frac{4}{3}\pi.6^3\right)/\frac{1}{3}\pi.12^2.6\right]\times 100$

$$=\frac{75}{2}=37.5\%$$

Example 15 : A plane parallel to the base of a cone cuts it in two parts. The volume of cone removed is $\frac{1}{8}$ of the total volume of cone. Find total surface of frustum of cone left. Radius of cone is 15 cm and semi-vertical angle 30^0.

Sol. : See Fig. Cone ADC is removed. Let kr be its radius.

(b) Δs ADC' and OAC are similar.

$\therefore OC' = kh, OD = OL = KOA$.

Volume of ODL = $\frac{1}{8}$ volume OAB.

$\therefore \frac{1}{3}\pi(kr)^2.(kh) = \frac{1}{8}\left(\pi r^2 h.\frac{1}{3}\right)$

$\therefore k^3 = \frac{1}{8} \Rightarrow k = \frac{1}{2}$

$\therefore \sin 30 = \frac{1}{2} = \frac{r}{l} \qquad \therefore l = 2r = 30;$

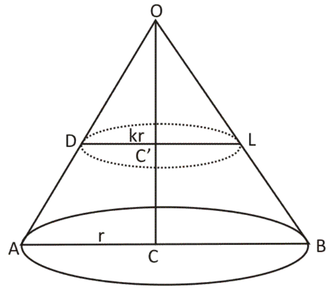

Fig 34

For frustum $r_1 = 15$, $r_2 = 15/2$, $l = 15$; $h = \frac{\sqrt{3}}{2}(15)$

Total surface = $\pi(r_1 + r_2)l' + \pi(r_1^2 + r_2^2)$

$= \pi\left[\left(15 + \frac{15}{2}\right) \times 15 + 225 + \frac{225}{4}\right]$

$= \pi\left[450 + \frac{450 + 225}{4}\right] = 618.75 cm^2$

Example 16 : Side of base of regular tetrahedron is $16\sqrt{3}$ cm and height is 15cm. Find its volume and total surface.

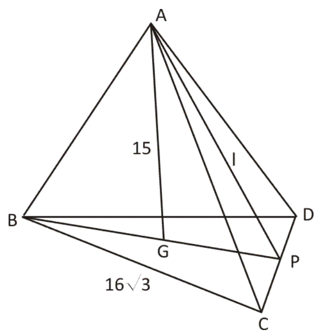

Fig 35

Sol. : In Fig., BCD is base-equilateral triangle of side $16\sqrt{3}$. BP is its median = $\frac{\sqrt{3}}{2}(16\sqrt{3})$ = 24 cm. (G is centroid of Base triangle)

GP = $\frac{1}{3}.24 = 8$

AP is median of Δ ACD, $AP^2 = AG^2 + GP^2$

$\Rightarrow AP^2 = 15^2 + 8^2 \Rightarrow AP = 17 cm.$

(a) Volume = $\frac{1}{3}\left(\frac{\sqrt{3}}{4}.3.16^2\right) \times 15$

= $960\sqrt{3}$ cm3

(b) Total surface area = 3 x area of $\Delta ACD +$ area Δ BCD

$$= 3.\frac{1}{2}CD.AP + \frac{\sqrt{3}}{4}\left(16\sqrt{3}\right)^2 \quad = \frac{3}{2}\left(16\sqrt{3}\right)\times 17 + 192\sqrt{3} = 600\sqrt{3}cm^2$$

Example 17 : The base of a tetrahedron is a right angled triangle, sides forming right angle are 12 and 16 cm. Perpendicular from vertex on this base is 16 cm and meets at right angle corner. Calculate volume and total surface.

Sol. : See Fig., volume = $\Delta ABC \times OA$

$$= \frac{1}{3}.\left[\frac{1}{2}.16.12\right].16$$

$$= 512cm^3$$

Total surface = $\Delta OAB + \Delta OAC + \Delta OBC + \Delta ABC$

Δ s, OAB, OAC, CAB are right angled Δ

In Δ OBC; OC= 16 $\sqrt{2}$, OB = 20, BC = 20

Total surface area =

$$\frac{1}{2}.16\times 12 + \frac{1}{2}16\times 16 + 8\sqrt{2}.4\sqrt{17} + \frac{1}{2}16.12$$

$$= \left(320 + 32\sqrt{34}\right)cm^2$$

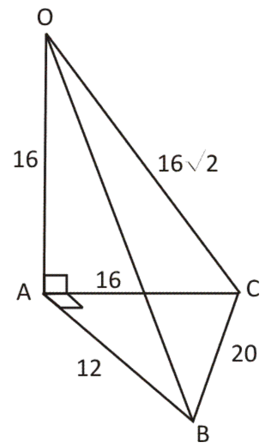

Fig 36

Practice Worksheet (Foundation Level) – 1 (b)

1. The areas of three faces of a cuboid are 48, 60 and 80 cm^2. Diagonal of cuboid it.

(a) $10\sqrt{3}$ cm (b) $10\sqrt{2}$ cm (c) $8\sqrt{3}$ cm (d) $6\sqrt{5}$ cm

2. Angle between the diagonals of a cube is :

(a) $2\sin^{-1}\frac{1}{\sqrt{3}}$ (b) $\frac{\pi}{2}$ (c) $\sin^{-1}\frac{\sqrt{2}}{3}$ (d) $\frac{\pi}{4}$

3. The sides of a cuboid are 12, 6 and 4cm, angle that diagonal makes with base is :

(a) $\sin^{-1}\frac{4}{7}$ (b) $\sin^{-1}\frac{6}{7}$ (c) $2\sin^{-1}\frac{2}{7}$ (d) none of these

4. Angle between diagonals of cuboid of Q.3 is:

(a) $2\sin^{-1}\frac{3}{7}$ (b) $2\sin^{-1}\frac{4}{7}$ (c) $2\sin^{-1}\frac{2}{7}$ (d) none of these

5. A solid sphere of 30 cm radius is melted and then cast into similar balls of diameter 0.6 cm. No. of balls obtained is :

(a) 10^5 (b) 10^6 (c) 10^9 (d) 10^4

6. A hemispherical bowl of diameter 38 cm is full of liquid number 1. Bottles which are in shape of frustum of a cone, radius of top and bottom being 2 cm and 3 cm are filled with this liquid. Number of bottles needed, when height of each bottle is 3.8 cm are.

(a) 190 (b) 1900 (c) 380 (d) 760

7. A conical vessel of radius 15 cm and height 15cm and height 25 cm is full of water upto 20 cm height. This water is then poured in a cylinder of diameter 24 cm. Height of water in cylinder is

(a) $6\frac{2}{3}$cm (b) $3\frac{1}{3}$cm (c) 4cm (d) $1\frac{2}{3}$cm

8. A cube is melted and liquid purified and in process it loses 27.1% of its volume. It is now cast in cube again. What fraction of the old side is side of new cube.

(a) $\frac{7}{10}$ (b) $\frac{8}{10}$ (c) $\frac{9}{10}$ (d) $\frac{729}{1000}$

9. 8 and 15 cm are two sides of a right angled triangle, Angle between them is 90^0. A boy rotates this Δ at axis 8 side and another boy rotates it at axes 15 side. Ratio between volumes of the solids form is:

(a) 15 : 8 (b) 8 : 15 (c) 17 : 15 (d) 15 : 17

10. The perimeters of ends of a frustum of a cone are 48 and 36 cm and height is 17.6 cm. Volume of frustum is :

(a) 2318.4 cm^3 (b) 323.2 cm^3

(c) 646.4 cm^3 (d) 1292.8 cm^3

11. A plane parallel to the base of a cone cuts the cone and the volume of the cone so removed is $\frac{1}{27}$ of the original cone. The plane divides the height of cone in the ratio of :

(a) 1 : 3 (b) 1 : 2

(c) 2 : 3 (d) none of these

12. The ends of circular rod are conical. Total length of rod is 16 m Diameter of rod is 7 m. Total volume is 147 πm^3 . Find height of cones at its end.

13. Find volume of largest right circular cone that can be cut from a cube of side 14 cm.

14. The difference between outside and inside lateral surface of a metallic cylindrical pipe of length 14 cm is 44 sq. cm. Volume of pipe metal is 99 cm^3. Calculate its inner and outer radii.

15. A solid toy is in form of a hemisphere surmounted by a circular cone. Height of cone is 4 cm and diameter of base is 4 cm. If a right circular cylinder circumscribes this toy, then calculate how much more space it will cover.

16. A cylindercal bucket 32 cm high and of 18 cm radius if full of sand. It is emptied on the ground in such a way that a cone of height 24 cm is formed. Calculate radius of base of cone.

17. A tent is in the shape of a cylinder and is surmounted by a cone. Diameter of tent is 16 cm, total height 6m, height of cylindrical portion 2m. Calculate cost of canvas of tent at Rs. 10 per sq. meter.

18. A reservoir is in the shape of frustum of a cone. It is 8m across the top and 4 m across at bottom. Capacity of reservoir is 336 m^3. Calculate slant height of tank.

19. A cone is scooped from a hemisphere of radius 21 cm; base of cone completely covering hemisphere. Height of cone is such that its volume is equal to $\frac{1}{2}$ volume of hemisphere. Find its height.

20. Base of a tetrahedron is isosceles triangle of sides $12\sqrt{3}$, 36, $12\sqrt{3}$. Its volume is 1620 cm^3. Find its height.

21. The base of a right prism is regular hexagon a side 6 cm. Its height is 21 cm. It is scraped and turned into a cylinder of same height and maximum volume. What % of volume is reduced

22. A damroo is of shape given below. The ends are moulded by thin skin, rest is made of wood one cm thick. Calculate its cost. If cost of wood is Rs. 100000.00 per 1 m^3. Moulding of skin with skin is Rs 10 .00 per square decimeter.

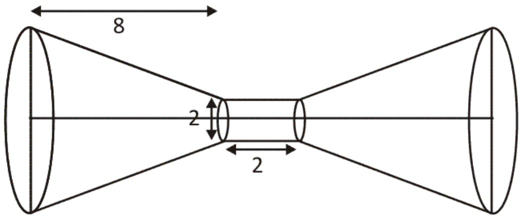

Fig 36

1.24 Image of a point

We can find image of a point in a line, also image of a point in a point. Plane mirror rule is applied. The image of a point in plane mirror is as far behind the mirror as the object is infront and the straight line joining the object to image is perpendicular to the surface of mirror. In case of image of a point is the straight line, straight line serves as surface of mirror.

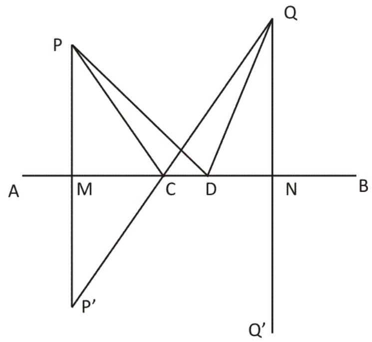

Fig 38

(a) In Fig. 38, AB is the straight line P is above the line P' is image of P in straight line AB. PP' is perpendicular to AB and P'M = PM, (PP' meets AB in M). Similarly image of a point Q is Q' and QN = Q'N.

P'Q is joined. It meets AB in C. If a man wants to go from P to a point in AB and then from that point Q, then PCQ is the shortest route. We shall prove it.

PM = PM' $\Rightarrow$ PC = P'C $\Rightarrow$ PC + CQ = P'Q.

If any other point say D is taken on AB, then PD + DQ = P'D + DQ and (P'D + DQ) > P'Q. (sum of two sides of a triangle is greater than the third side)

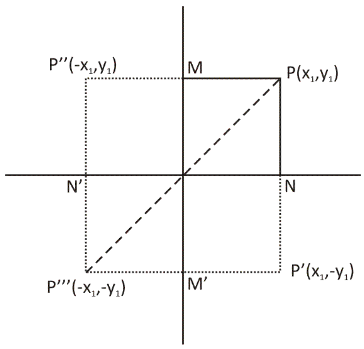

Fig 39

(b) In Fig, 39 x'ox is x-axis yoy' is y-axis. P is point (x_1, y_1) in 1st quadrant.

(i) image of P in x-axis is P', $(x_1 - y_1)$

(ii) Image of P in y-axis is P'' $(-x_1 - y_1)$

(iii) Image of P'' in x-axis and image of P' in y-axis is P''' $(-x_1, -y_1)$

(iv) $(-x_1, -y_1)$ in also image of P in point O.

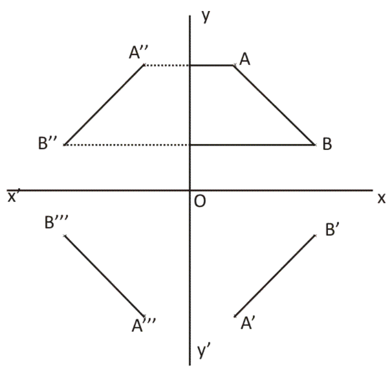

Fig 40

(c) In Fig. 40 A'B' is image of AB in x-axis; A"B" is its image in y-axis, A'''B''' is image of AB in origin, image of A'B' in y axis, image of A"B" in x-axis.

(d) In Fig. 41 AB is x = y image of $P(x_1, y_1)$ is P' which is (y_1, x_1). PP' is perpendicular on AB. Abscissa of P is ordinate of P' and ordinate of P is abscissa of P'.

In case y = – x, image falls in 3rd quadrant. Ordinate of P is abscissa of P” with sign changed. Abscissa of P is ordinate of P” with sign changed.

Image of $y^2 = x$ in $x – y = 0$ is $x^2 = y$

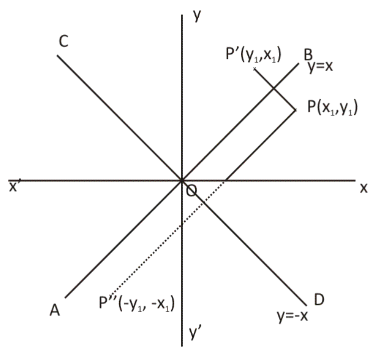

Fig 41

Solved Examples

Example 18 : AB is a rod. Two towns P and Q are situated at distance a and b from it on the same side. A pumping station S has to be installed on this road such that PS + SQ is shortest distance. If distance between foot of perpendicular from P and Q be C then find position of S.

Sol.: In Fig. 42 P' is image of P in the road AB. Join P'Q. It meets road in S, S should be position of pumping station.

If MS = x, then SN = c – x

Δ s P'MS and QSN are similar.

$$\therefore \frac{x}{c-x} = \frac{a}{b} \Rightarrow bx = ac - ax$$

$$\therefore x = \frac{ac}{a+b}$$

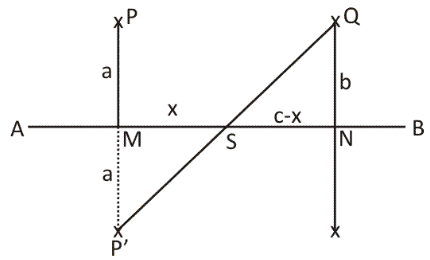

Fig 42

Example 19 : Find volume of the greatest cone that can be inscribed in a sphere of radius r.

Sol. : Fig is the vertical section of sphere through its center O; PQR is cone; height of cone OC shall pass through O. POL and base QCP is perpendicular.

$\therefore$ if R be radius of sphere, r radius of cone and CL = h, then,

r^2 = h (2R –h)

Volume of cone, V = $\frac{1}{3}\pi r^2$.PC

$$= \frac{1}{3}\pi h(2R-h)(2R-h)$$

for maximum value $\frac{dv}{dh} = 0$

i.e. $(2R-h)^2 + h.2(2R-h)(-1) = 0$

$$\Rightarrow (2R-h)(2R-3h) = 0$$

$h \neq 2R; \quad \therefore h = 2/3R$

$$\frac{d^2r}{dh} = -(2R-3h) - 3(2R-h) = 0 - 3\left(2R - \frac{2}{3}R\right) = -4R = -vc$$

$$\therefore V = \frac{\pi}{3}.\frac{2}{3}R\left(2R - \frac{2}{3}R\right)^2 = \frac{32\pi}{81}R^3$$

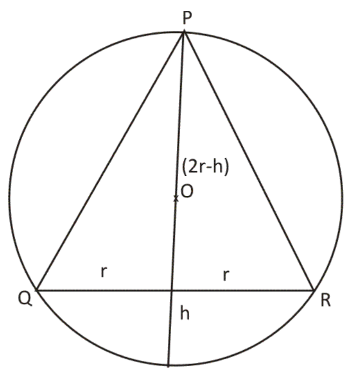

Fig 43

Practice Worksheet (Foundation Level) – 1(c)

1. Image of (4, –5) in origin is:

(a) (4, 5) (b) (–4 –5) (c) (–4, 5) (d) none of these

2. The image of curve $y^2 = x$ in $x = y$ is:

(a) $y^2 + x = 0$ (b) $x^2 = y$ (c) $x^2 + y = 0$ (d) none of these

3. P is in 1st quadrant, Q is in second quadrant, image of P in x –axis coincides with the image of Q in origin, then relation between P and Q is :

(a) P is image of Q in x-axis (b) Q is image of P in x-axis

(c) No relation between P and Q (d) none of these

4. P point is (4,3). Its image in x-axis is P_1 and in y-axis P_2, area of triangle PP_1P_2 is:

(a) 48 cm^2 (b) 24 cm^2 (c) 36 cm^2 (d) none of these

5. Volume of the solid generated when ΔPP_1P_2 of Q.4 is rotated on P_1P_2 axis is :

(a) $\frac{384}{5}\pi cm^3$ (b) $\frac{512}{15}\pi cm^3$ (c) $\frac{1048}{15}\pi cm^3$ (d) none of these

6. Area of quadrilateral formed by point P and its images in x-axis, y-axis and origin. P is (4, –3):

(a) 24 cm^2 (b) 48 cm^2 (c) 64 cm^2 (d) 60 cm^3

7. P_1 is image of point P(2, 1) in x = y and P_2 is the image in x + y = 0. Area of triangle PP_1P_2 is :

(a) 4 (b) 6 (c) 8 (d) none of these

8. P is (2, 4) and Q is (4,2), P'Q' is image of PQ in y-axis. Area of quadrilateral P'Q' QP is :

(a) 12 (b) $12\sqrt{2}$ (c) 24 (d) none of these

9. The sides of a triangle are 4,5,6 cm. Length of median bisecting third side is :

(a) $\frac{\sqrt{23}}{2}$ (b) $\frac{\sqrt{27}}{2}$ (c) $\frac{\sqrt{25}}{2}$ (d) $\frac{\sqrt{19}}{2}$

10. The two sides of a triangle are 12 and 18 cm and the median bisecting third side is 15; then third side is :

(a) 9 cm (b) 8 cm (c) 6cm (d) 10 cm

11. Centre of a circle is (0,0) and radius 5 cm. Through point (3,2) chords of circle are drawn. Locus of mid point of these chord is

(a) a straight line (b) Θ on diameter (0,0) and (3,2)

(c) Θ with center (3,2) (d) Θ with center (3/2, 1)

12. Centre of circle is (2,4) and radius 5. P is (8, 10). Through P chords of circle are drawn. Locus of mid points of these chords is a circle whose center is :

(a) (9, 5) (b) (5, 7) (c) (6,6) (d) none of these

13. Center of a circle is (2,1) radius 4, From point P, (8,4) secant PAB is drawn and tangent PT drawn. If PA = 9 cm, PB = 12 cm, then PT=

(a) $6\sqrt{3}$ (b) 8 (c) $8\sqrt{3}$ (d) none of these

14. x – axis divides the line joining (5,4) and (–1, –3) in the ratio of m : n then m/n =

(a) 4/3 (b) 5/4 (c) 6/7 (d) 3/4

15. Sides of a triangle are 5,5,6 radius of incircle is:

(a) 2 (b) 1.6 (c) 1.5 (d) 1.75

✣ --------------------- ✣

2. Points in a plane and Straight Lines

2.1 Points in a plane

Points on a line (straight line) are in one dimension and represent set of real numbers with reference to a point O on it denoting zero.

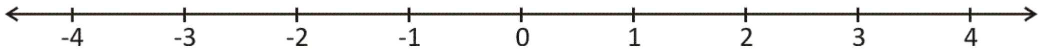

Fig 1

A plane has two dimensions. A point in plane can only be located with reference to two straight lines. These lines are not parallel but intersect. Angle between them has been taken 90^o for convenience. Angle can be acute as well obtuse.

In figure (2) below XOX' and YOY' are two reference lines called x-axis and y-axis. These intersect at right angles at O. O is called origin. The two axes divide the plane in four parts called quadrant. YOX is first quadrant, YOX' is second quadrant, X'OY' is third quadrant and Y'OX is fourth quadrant.

Set R is the set of real numbers. Set R gives set of ordered pairs (a, b); ($a \in$ R, $b \in$ R) and is called Cartician. product. Elements of set ($R \times R$) are points represented on this plane. Points P (x, y), Q ($-x_1$, y_1) R ($-x_2$, $-y_2$) and S (x_3, $-y_3$) are points situated in first, second, third and fourth quadrants respectively.

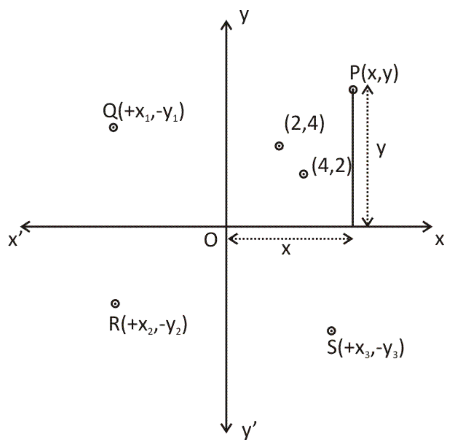

Fig. 2

In figure 2, point P is (x, y). PM is perpendicular on x-axis. OM = x, is called abscissa of the point P and PM = y, is called the ordinate of point P. If co-ordinates of a point T are (a, b) then a is abscissa and b ordinate of the point T. (a, b) is an ordered pair so point (2, 4) and point (4, 2) are two different points (see figure 2)

(a) An algebraic relation between ordered pairs is a sub-set of $R \times R$ and its graph is either a straight line or a curve.

(i) $y = x + 1$ gives points

$(-5, -4), (-4, -3) \dots (1, 2), (2, 3), \dots$

If we plot them and join them, we get a straight line.

(ii) $x^2 + y^2 = 25$ gives points

$(0, \pm 5), (2, \pm\sqrt{21}), (3, \pm 4), (4, \pm 3), (5, 0)\dots$, when these are plotted and joined, we get a circle.

(b) **Graph of a curve**: When equation of a curve is given, to draw its graph, we take x as independent variable and y as dependent variable and express the equation of curve y = f(x). We give values to x and find respective values of y; plot (x, y) and join them. The curve is traced. If independent variable takes values from set A, then A is called the domain of x; values of y constitute the range.

2.2 Theorems related to points:

You know

(a) Distance between two points (x_1, y_1) and (x_2, y_2) = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

(b) Distance of point (x, y) from point origin = $\sqrt{x_1^2 + y_1^2}$

(c) Co-ordinate of point dividing line segment AB, A [(x_1, y_1)], B (x_2, y_2) internally in the ratio m : n is

$$\left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right]$$

In case of external division the co-ordinates of this point are

$$\left[\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right]$$

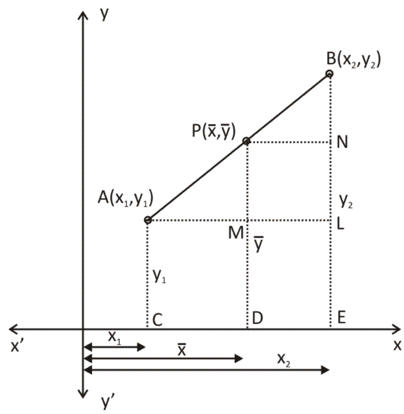

Fig. 3(a)

Proof: Points A and B are (x_1, y_1) and (x_2, y_2). Join them. Point P $(\bar{x}, \bar{y})$ in AB divides AB in the ratio of m : n.

AC, PD and BE are perpendiculars from A, P and B on x-axis. AL and PN are drawn parallel to x-axis meet BE in L. PD in M.

(i) From right angled triangle ABL

$AB^2 = AL^2 + BL^2 = (CE)^2 + (BE - LE)^2$

$= (OE - OC)^2 + (BE - AC)^2 = (x_2 - x_1)^2 + (y_2 - y_1)_2$

(ii) Δs PAM and BPN are similar and

AM = CD = $\bar{x} - x_1$, PN = ML = DE = $x_2 - \bar{x}$

PM = PD – MD = PD – AC = $\bar{y} - y_1$, BN = BE – PD = $y_2 - \bar{y}$

From similar triangles $\frac{AP}{PB} = \frac{AM}{PM} = \frac{PM}{BM} = \frac{m}{n}$

$$\therefore \frac{AM}{PN} = \frac{\bar{x} - x_1}{x_2 - \bar{x}} = \frac{m}{n} \Rightarrow \bar{x} = \frac{mx_2 + nx_1}{m+n}$$

and $\frac{PM}{BN}=\frac{\bar{y}-y_1}{y_2-\bar{y}}=\frac{m}{n}\Rightarrow\bar{y}=\frac{my_2+ny_1}{m+n}$

$\therefore$ Co-ordinates of point P which divides AB internally in the ratio of m : n are $\left[\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right]$

If point P divides AB externally in the ratio of m : n, then from figure.

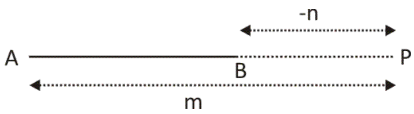

Fig 3(b)

Ratio is m : n. Co-ordinates of P are none $\left[\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n}\right]$

(d) **Centroid of a Triangle**: Point of intersection of medians is centroid and it divides each median in the ratio of 2 : 1. In figure (4). G is centroid of Δ ABC, AD is median.

$\therefore$ D is $\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)$

G, divides AD, A(x_1, y_1), D $\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)$

In the ratio of 2 : 1. If G is $(\bar{x},\bar{y})$ then

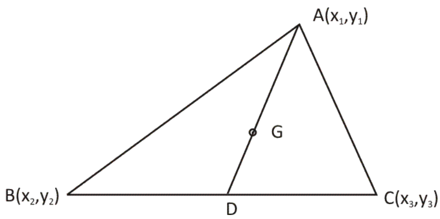

Fig. 4

$$\bar{x}=\frac{2\cdot\left(\frac{x_2+x_3}{2}\right)+1.y_3}{2+1}=\frac{x_2+x_3+x_1}{3}=\frac{x_1+x_2+x_3}{3}$$

Similarly $\bar{y}=\frac{y_1+y_2+y_3}{3}$

$\therefore$ Centroid is $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

(e) **Area of a triangle:** The co-ordinates of vertices of triangle ABC are A (x_1, y_1), B (x_2, y_2), C (x_3, y_3)

AM, CK an BN are perpendiculars from A, C and B on x-axis. From figure (5)

Δ ABC = Trapezium [AMKC + CKNB – AMNB]

$= \frac{1}{2}[(y_1-y_3)(x_3-x_1)+(y_3+y_2)(x_2-x_3)-(y_1+y_2)(x_2-x_1)]$

$$=\frac{1}{2}\left[x(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

$$=\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

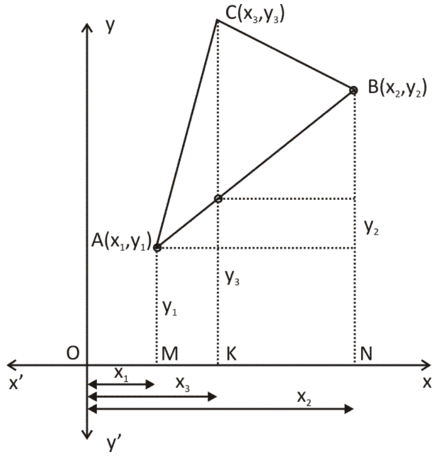

Fig. 5

Solved Examples

Example 1: If (x, y) lies on the line segment joining (a, 0) and (0, b) then prove $\frac{x}{a}+\frac{y}{b}=1$

Sol. In the figure point P (x, y) is on the line segment, joining A (a, 0) and B (0, b) PM is perpendicular on x-axis PM = y, OM = x, MA = a – x

Δs, BOA and PMA are similar

$\therefore \frac{PM}{BO}=\frac{AM}{OA} \Rightarrow \frac{y}{b}=\frac{a-x}{a}$

$\Rightarrow \frac{x}{a}+\frac{y}{b}=1$

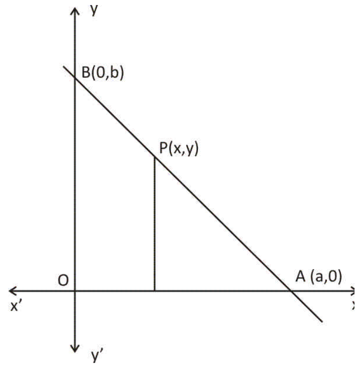

Fig. 6

Example 2: The vertices of triangle ABC are A (6, 3), B (–3, 5) and C (5, –3). If a point P lies on the median AD such that $\Delta PBC/\Delta ABC = \frac{2}{3}$ then find P.

Sol. In the figure 7, AD is median, AM and PN are perpendiculars on BC.

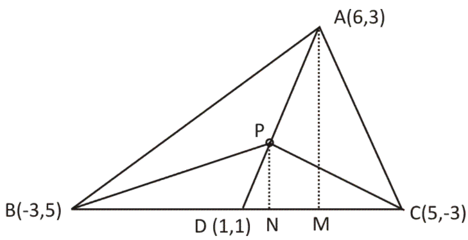

Fig. 7

$\therefore \frac{\Delta PBC}{\Delta ABC}=\frac{\frac{1}{2}BC\cdot PN}{\frac{1}{2}BC\cdot AM}$

$\Rightarrow \frac{PN}{AM}=\frac{2}{3}$ ... (i)

Δs ADM and PDN are similar,

$\therefore \frac{AM}{PN}=\frac{AD}{PD}=\frac{3}{2} \Rightarrow P$ divides AD in the ratio of 1 : 2; D is mid point of BC = (1, 1).

$\therefore$ P is $\left(\frac{12+1}{3},\frac{6+1}{3}\right)$ i.e. P is $\left(\frac{13}{3},\frac{7}{3}\right)$

Example 3: In what ratio does (–3, 7) divide the join of (–5, 11) and (4, –7) internally?

Sol: Let P (–3, 7) divide AB [A(–5, 11), B (4, –7)] in the ratio of m : n.

$\therefore -3=\frac{4m+(-5n)}{m+n}$ and $\frac{-7m+11n}{m+n}=7$

$\therefore$ –3m – 3n = 4m – 5n $\Rightarrow$ 7m = 2n

$\Rightarrow \frac{m}{n} = \frac{2}{7}$ i.e. (–3, 7) divides AB in the ratio 2 : 7

We shall get the same result from other relation.

Example 4: The vertices of a triangle are (3, 5), (–4, 3) and (0, –4). Find centroid and circumcentre.

Sol. Let A (3, 5), B (–4, 3) and C (0, –4)

(i) Centre of Gravity CG = $\left(\frac{3-4+0}{3}, \frac{5+3-4}{3}\right) \Rightarrow \left(-\frac{1}{3}, \frac{4}{3}\right)$

(ii) Let the circumcentre O, be (x, y)

$\therefore$ OA = OB = OC $\Rightarrow (OA)^2 = (OB)^2 = (OC)^2$

$(OA)^2 = (OB)^2 \Rightarrow (x-3)^2 + (y-5)^2 = (x+4)^2 + (y-3)^2$

$\Rightarrow -6x - 10y + 34 = 8x - 6y + 25$

$\Rightarrow 14x + 4y = 9$ ... (1)

and $(OB)^2 = (OC)^2 \Rightarrow (x+4)^2 + (y-3)^2 = x^2 + (y+4)^2$

$\Rightarrow 14y - 8x = 9$ ... (2)

Solving (1) and (2) $x = \frac{15}{38}, y = \frac{33}{38}$

$\therefore$ Circumcentre is $\left(\frac{15}{38}, \frac{33}{38}\right)$

Example 5: The centre of two circles are (5, 2) and (1, 5). The radius of first circle is 3 units. If these touch externally then find point of contact.

(ii) If the radius of first circle be 8 units and circles touch internally then find point of contact.

Sol. Centres O_1 (5, 2), O_2 (1, 5)

$\therefore O_1O_2 = \sqrt{16+9} = 5$

Circles touch externally

$\therefore r_1 + r_2 = O_1O_2 = 5$

$r_1 = 3, \therefore r_2 = 2$

Point of contact P shall divide O_1O_2 internally in the ratio of 3 : 2

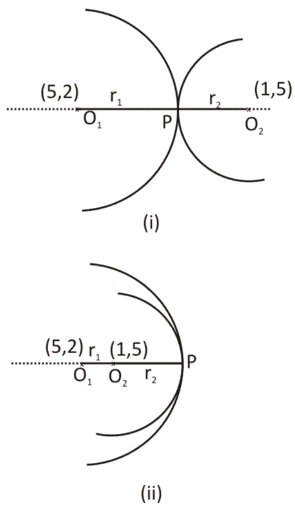

Fig. 8

$\therefore$ P is $\left(\frac{3+10}{5}, \frac{15+4}{4}\right) \Rightarrow \left(\frac{13}{5}, \frac{19}{5}\right)$

(ii) Circles touch internally

$\therefore r_1 - r_2 = O_1O_2 = 5$

$\therefore r_2 = r_1 - 5 = 8 - 5 = 3$

$\therefore$ Point of contact P shall divide O_1O_2 externally in the ratio of 8 : 3.

P is $\left(\frac{8-15}{5}, \frac{40-6}{5}\right)$ i.e. P is $\left(-\frac{7}{5}, \frac{34}{5}\right)$

Example 6: A point moves such that the sum of distances from points (ae, 0) and (–ae, 0) is always 2a. Find locus of the point.

Sol : Let the moving point P be (h, k)

From given condition $\sqrt{(h-ae)^2+k^2} + \sqrt{(h+ae)^2+k^2} = 2a$

$\Rightarrow h^2 - 2aeh + k^2 + h^2 + 2aeh + k^2 + 2a^2e^2 + 2\sqrt{(h-ae)^2+k^2}\sqrt{(h+ae)^2+k^2} = 4a^2$

$\Rightarrow [h^2 + k^2 + a^2(e^2 - 2)]^2 = [h^2 + k^2 - 2aeh + a^2e^2] \times [h^2 + k^2 + 2aeh + a^2e^2]$

$\Rightarrow h^4 + k^4 + a^4(e^2-2)^2 + 2h^2a^2(e^2-2) + 2k^2a^2(e^2-2) + 2h^2k^2$

$= h^4 + 2h^2k^2 + 2aeh^3 + a^2e^2h^2 + k^4 + 2aehk^2 + k^2a^2e^2 - 2aeh^3 - 2aehk^2 - 4a^2e^2h^2 - 2a^3e^3h + a^2e^2h^2 + a^2e^2k^2 + 2a^3e^3h + a^4e^4$

$\Rightarrow -4a^4e^2 + 4a^4 - 4h^2a^2 - 4k^2a^2 = -4a^2e^2h^2$

$\Rightarrow 4h^2 - 4e^2h^2 + 4k^2 = 4a^2 - 4a^2e^2$

$\Rightarrow h^2(1 - e^2) + k^2 = a^2(1 - e^2)$

$\Rightarrow \frac{h^2}{a^2} + \frac{k^2}{a^2(1-e^2)} = 1$

Since (h, k) is any point.

Locus $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

Example 7: ABCD is a rhombus Diagonal AC and BD intersect at M; BD = 2AC; If D and M be (1, 1) and (2, –1) respectively then A is ...

Sol. Diagonals bisect each other

$\therefore$ If B is (x, y) then

$\frac{x+1}{2}=2; \frac{y+1}{2}=-1$

$\Rightarrow$ B is (3, –3)

AC = $\frac{1}{2}$BD $\Rightarrow$ AM = $\frac{1}{2}$DM

Let A be (x_1, y_1) $\therefore$ $(x_1 - 2)^2 + (y_1 + 1)^2 = \frac{1}{4}\left((2-1)^2+(-1-1)^2\right)$

$\Rightarrow x_1^2+y_1^2-4x_1+2y_1=-\frac{15}{4}$ ... (1)

and AB = AD (rhombus)

$\therefore (x_1 - 3)^2 + (y_1 + 3)^2 = (x_1 - 1)^2 + (y_1 - 1)^2$

$\Rightarrow x_1^2+y_1^2-6x_1+6y_1+18=x_1^2+y_1^2-2x_1-2y_1+2$

$\Rightarrow x_1 = 2y_1 + 4$ ... (2)

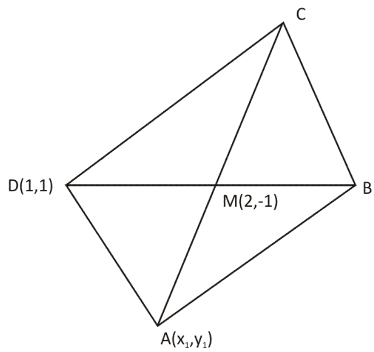

Fig. 9

Solving (1) and (2)

$(2y_1 + 4)^2 + y_1^2 - 4(2y_1 + 4) + 2y_1 = -\frac{15}{4}$

$\Rightarrow 16y_1^2 + 64y_1 + 64 + 4y_1^2 - 32y_1 - 64 + 8y_1 = -15$

$\Rightarrow 4y_1^2 + 8y_1 + 3 = 0 \Rightarrow (2y_1 + 3)(2y_1 + 1) = 0$

$\therefore y_1 = -\frac{3}{2}, -\frac{1}{2}$ and $\therefore x_1 = 1, 3$

$\therefore$ A is $\left(1, -\frac{3}{2}\right)$ or $\left(3, -\frac{1}{2}\right)$

Example 8: Prove if (a, b) , (a + 3, b + 4), (a – 1, b + 1) and (a – 4, b + 3) are vertices of a rhombus or of a square.

Sol. Let A (a, b), B (a + 3, b + 4), C (a – 1, b + 7) and D (a – 4, b + 3).

AB = $\sqrt{3^2+4^2}=5$, BC = $\sqrt{4^2+3^2}=5$,

CD = $\sqrt{3^2+4^2}=5$, DA = $\sqrt{(-4)^2+3^2}=5$

AB = BC = CD = DA It is square or rhombus.

$AC^2 = 1^2 + 7^2 = 50$ and $BD^2 = 7^2 + 1^2 = 50$

Diagonals are equal. $\therefore$ ABCD is a square.

Example 9: The two opposite vertices of a square are (4, 4) and (2, –2). Find co-ordinates of other vertices.

Sol. ABCD is a square; A is (4, 4), C is (2, –2). If B is (x, y) then $(x-4)^2 + (y-4)^2 = (x-2)^2 + (y+2)^2$

$\Rightarrow -8x - 8y + 32 = -4x + 4y + 8 \Rightarrow -4x - 12y = -24$

$\therefore x + 3y = 6$ ... (i)

and $AB^2 + BC^2 = AC^2$ and $AC^2 = 2^2 + 6^2 = 40$

$\therefore x^2 - 8x + 16 + x^2 - 4x + 4 + y^2 - 8y + 16 + y^2 + 4y + 4 = 40$

$\Rightarrow x^2 + y^2 - 6x - 2y = 0$

$\Rightarrow (6-3y)^2 + y^2 - 6(6-3y) - 2y = 0$ from (i)

$\Rightarrow 10y^2 - 20y = 0 \qquad \Rightarrow y(y-2) = 0$

$\therefore y = 0, y = 2 \Rightarrow x = 6, x = 0$

$\therefore$ Two other vertices are (0, 2), (6, 0)

Example 10: A (x, y), B (–1, 2) and C (3, –2) are vertices of triangle ABC. The medians AD and BE are 4 and $\sqrt{10}$ respectively the $\angle A$ is ...

Sol. D is mid point of BC is (1, 0) and E is $\left(\frac{3+x}{2}, \frac{-2+y}{2}\right)$

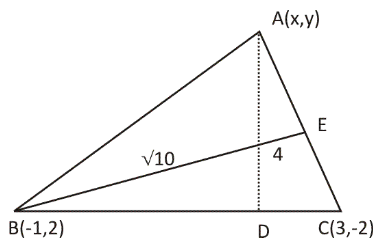

Fig. 10

$AD^2 = (x-1)^2 + y^2 = 16$

$\Rightarrow x^2 + y^2 - 2x = 15$ ... (ii)

$BE^2 = \left(-1 - \frac{3+x}{2}\right)^2 + \left(2 - \frac{y-2}{2}\right)^2 = 10$

$\Rightarrow \quad (-5-x)^2 + (6-y)^2 = 40$

$\Rightarrow \quad x^2 + y^2 + 10x - 12y + 61 = 40$

$\Rightarrow \quad (15 + 2x) + 10x - 12y + 61 = 40$ from (i)

$\Rightarrow \quad 12x - 12y + 36 = 0 \quad \Rightarrow \quad x - y = -3$

From (i) $(y-3)^2 + y^2 - 2(y-3) - 15 = 0$

$\Rightarrow \quad y(y-4) = 0 \quad \Rightarrow \quad y = -4$ or O

$\therefore \quad x = 1$, or -3

$\therefore \quad$ Vertex A is (1, 4) or (–3, 0)

Example 11: A (–1, –1), B (5, –1), C (3, 7) and D (–1, 5) are the vertices of a quadrilateral. ABCD; P, Q, R and S are mid point of AB, BC, CD and DA. Area of PQRS is ...

Sol. When mid points of sides of a quadrilateral are joined, a parallelogram results

$\therefore$ PQRS is a parallelogram

The diagonals of a parallelograms divides it into two equal parts.

$\therefore$ area of PQRS = 2 ΔPQR

P is (2, –1), Q = (4, 3), R (1, 6)

$$\therefore \text{ Required area } = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 3 & 1 \\ 1 & 6 & 1 \end{vmatrix} = 2(-3) + 1.\ 3 + 1\ .\ (21) = 18 \text{ sq. units.}$$

Practice Worksheet (Foundation Level) – 2(a)

1. What point on x-axis is equidistant from points (6, 5) and (–4, 3)?

2. Prove that points A (0, 5), B (–2, –2), C (5, 0) and D (7, 7) are vertices of a rhombus.

3. The vertices of a triangle are (2a, 4a), (2a, 6a) and ($2a + \sqrt{3}a$, 5a). Prove that triangle is equilateral and its area is $\sqrt{3}a^2$ sq. units.

4. Prove that points P (at^2, 2at); Q $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ and S (a, 0) are collinear and $\frac{1}{SP}+\frac{1}{SQ}$ is constant **(I.I.T.)**

5. A circle passes through (0, 0), (–3, 3) and (5, 5). Its centre is

   (a) (4, 1) (b) (1, 4) (c) (2, 3) (d) (–1, 5)

6. Points (7, a), (–5, 2) and (3, 6) are collinear then a is

   (a) 8 (b) 6 (c) –2 (d) 4

7. The vertices of a triangle are (7, 3), (0, –4) and (8, 2). Centre of its circumcircle is

   (a) (4, 1) (b) (4, –1) (c) (3, –3) (d) (1, 4)

8. The vertices of a triangle are (0, 4), (3, 0) and (3, 4). Its in centre is

   (a) (3, 2) (b) (2, 3) (c) (–2, 3) (d) (2, –3)

9. The centroid of a triangle is (2, 3) and circumcentre is (3, 2), then its orthocentre is

   (a) $\left(\frac{3}{2}, \frac{7}{2}\right)$ (b) $\left(\frac{3}{2}, \frac{9}{2}\right)$ (c) $\left(\frac{5}{3}, \frac{9}{2}\right)$ (d) (0, 5)

10. O is the origin and $y^2 = 6x$ is a curve. P is any point on this curve. The locus of mid point of OP is

    (a) $y^2 = 4x$ (b) $y^2 = 3x$ (c) $y^2 = 2x$ (d) $y^2 = \frac{4}{3}x$

11. The co-ordinates of the mid points of sides of a triangle are P (2, 4), Q (0, –2) and R (4, –2). Its centroid ΔPQR is

    (a) (4, 0) (b) (2, 0) (c) (3, 0) (d) (0, 2)

12. A diagonal of a square is along y = x. If one vertex of the other diagonal is (6, 4) then its other vertex is

    (a) (5, 5) (b) (4, 6) (c) (6, 6) (d) (8, 6)

13. In question 12, area of square is

    (a) 8 sq. u. (b) $4\sqrt{2}$ sq. u. (c) $6\sqrt{2}$ sq. u. (d) 4 sq. u.

14. Centres of two circles are (–2, 3) and (3, –9), radius of first circle is 8. Circles touch each other externally. The point of contact is

(a) $\left(\frac{34}{13}, \frac{87}{13}\right)$ (b) $\left(\frac{31}{13}, \frac{-1}{13}\right)$ (c) $\left(\frac{14}{13}, -\frac{57}{13}\right)$ (d) $\left(\frac{-21}{13}, \frac{69}{13}\right)$

15. A, B and C are collinear, A is (3, 4), B is (7, 7) and AC = 15 units. The co-ordinates of C are

(a) (18, 12) (b) (15, 13) (c) (15, 12) (d) (15, 10)

16. The locus of the point which is always equidistant from points (a + b, a – b) and (a – b, a + b) is

(a) x – y = 0 (b) x + y = 0 (c) ax = by (d) $y = \frac{b}{a}x$

17. A is (2, 3) and B is (–2, 5). Point P moves such that $\angle$APB is always 90^o. Locus of P is

(a) $x^2 + y^2 - 8y + 11 = 0$ (b) $x^2 + y^2 - 8x + 11 = 0$

(c) $x^2 + y^2 + 4x - 8y + 11 = 0$ (d) $x^2 + y^2 - 4x - 8y + 11 = 0$

18. Centre of the circle inscribed in a triangle whose vertices are (–36, 7), (20, 7) and (0, –8) is

(a) (4, 0) (b) (1, 0) (c) (–1, 0) (d) (–2, 1)

19. ABCD is a parallelogram; A is (3, 1), B is (–1, 0) and mid point of AC is $\left(\frac{5}{2}, \frac{3}{2}\right)$ then find vertex D and area of parallelogram.

20. A rod of length 10 cm, slides between two rods perpendicular to each other and in vertical plane. Find locus of point which divides the moving rod in the ratio of 3 : 2.

21. The vertices of a quadrilateral are (–1, 1), (4, –1) (5, 4) and (1, 6). Its area is ...

2.3 Equations of straight lines

A straight line is parallel to any axis or inclined to them. The inclination of straight line is measured with x-axis in anti-clockwise direction. In figure 11 straight line AB is inclined at an angle θ with x-axis while CD is inclined at an angle ϕ with x-axis. θ is acute angle and ϕ is obtuse angle incline. Straight line LM is parallel to x-axis i.e. inclined at 0^o with x-axis. Straight line GK is parallel to y-axis i.e. inclined at 90^o with x-axis.

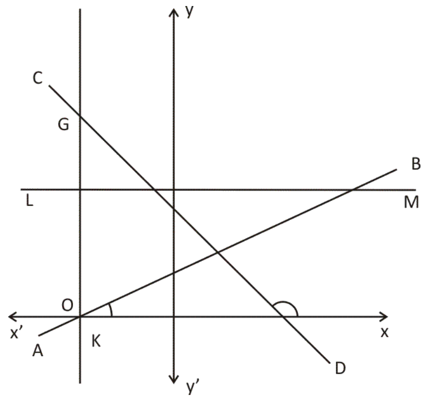

Fig. 11

2.4 Slope or Gradient of a straight line:

If a straight line makes an angle θ with x-axis then tan θ is called gradient or slope of the line. It is denoted by m.

When $0 \leq \theta < 90$ slope is positive

When $90 < \theta < 180$ slope is negative

The slope of y-axis is not defined as tan 90^o is not defined.

2.5 Standard Equations of straight lines

You have already studied the following standard equation of straight lines in class XI.

(a) **Slope-intercept form:** This equation is $y = mx + c$

where m is the gradient of the straight line and c is the intercept on y-axis by the line. If the intercept is above x-axis, it is positive, if below x-axis it is negative. From this equation we conclude

(i) $y = mx$ is the general equation of all lines passing through origin.

(ii) $y = \lambda, \lambda \in R$ is the general equation of all lines which are parallel to x-axis. $(m = 0)$

(iii) $x = \lambda, \lambda \in R$ is the general equation of all lines parallel to y-axis. $\left(\frac{1}{m} = 0\right)$

(iv) $y = 0$ is the equation of x-axis

$x = 0$ is the equation of y-axis

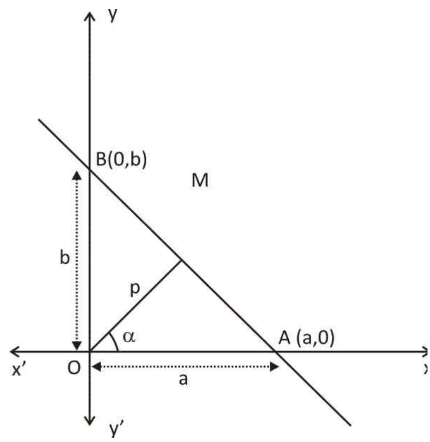

Fig. 12

(b) **Double Intercept form:** If a straight line intercepts of length a and b from x-axis and y-axis respectively then its equation is

$$\frac{x}{a}+\frac{y}{b}=1$$

In figure (12) OA = a, OB = b Co-ordinate of A are (a, 0) and of B (0, b).

(c) **Normal or perpendicular form** : $x \cos \alpha + y \sin \alpha = p$ is the equation of straight line in normal form; p is length of perpendicular from origin on the line and α is the inclination of this perpendicular with x-axis (see figure 12)

(d) **Equation of straight line through two given points (x_1, y_1) and (x_2, y_2) is**

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}$$

From figure 13, it is clear that $\frac{y_2-y_1}{x_2-x_1}$ is the gradient of this line.

$\therefore$ $y - y_1 = m (x - x_1)$ is the equation of all lines passing through (x_1, y_1). If m is given, then this equation shall given the definite straight line.

(e) The equation $y - y_1 = \tan \theta (x - x_1)$ can be written as $\frac{y-y_1}{\sin\theta}=\frac{x-x_1}{\cos\theta}$

This is the equation of straight line in symmetric form or parametric form or distance form. If Q is a point on this line at distance d from (x_1, y_1) then we can calculate the co-ordinates of Q with the help of this equation. $\frac{y-y_1}{\sin\theta}=\frac{x-x_1}{\cos\theta}=d$, then point Q is $(d \cos \theta + x_1, d \sin \theta + y_1)$

2.6 Equation of a straight line in General form:

A linear equation in x and y represents a straight line i.e. $ax + by + c = 0$ is general equation of a straight line. It can be reduced to the standard forms given above.

(a) $ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x-\frac{c}{a}$ slope-intercept form

Please note $m=-\frac{a}{b}=-\frac{\text{coefficient of } x}{\text{coefficient of } y}$

(b) $ax + by + c = 0 \Rightarrow \frac{x}{-\frac{c}{a}}+\frac{y}{-\frac{c}{b}}=1$ intercept form.

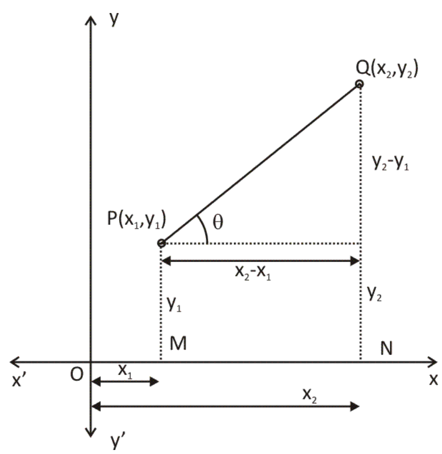

Fig. 13

(c) $ax + by + c = 0 \Rightarrow -\frac{ax}{\sqrt{a^2+b^2}} + \frac{-by}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}$ is normal or symmetric form.

$\cos\alpha = \frac{-a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{-b}{\sqrt{a^2+b^2}}$, $p = \frac{c}{\sqrt{a^2+b^2}}$

If C = 0 then y = $-\frac{a}{b}x$ i.e. lines passes through origin.

If a = 0, y = $-\frac{c}{b}$ straight line is parallel to x-axis

If b = 0, x = $-\frac{c}{a}$ straight line is parallel to y-axis.

2.7 Angle between two straight lines:

From figure 14, the angle between the straight lines AB and CD is ϕ and $\phi = \theta_1 - \theta_2$

$$\therefore \tan\phi = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2}$$

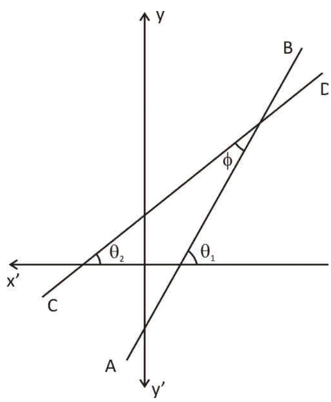

Fig. 14

(a) If $m_1 - m_2 = 0$ i.e. $m_1 = m_2$ i.e. lines are parallel $\phi = 0$

(b) $\cot\phi = \frac{1 + m_1 m_2}{m_1 - m_2} = 0 \Rightarrow 1 + m_1 m_2 = 0 \Rightarrow m_1 \cdot m_2 = -1$

$\therefore$ If the product of the gradients of two lines be equal to –1, then lines are perpendicular to each other. Therefore $L_1 = 0$ and $L_2 = 0$ shall be perpendicular if product of their gradients = –1

(c) If $L_1 = 0$ is $ax + by + c = 0$, $m_1 = -\frac{a}{b}$ for $L_2 = 0$ to be perpendicular to $L_1 = 0$, gradient m_2 of L_2 should be equal to = $\frac{b}{a}$.

$\Rightarrow$ that is in perpendicular straight line $L_2 = 0$, coefficient of x is b and of y is –a

$\therefore L_2 = 0$ is $bx - ay = \lambda$ ... (i)

If we compare it with ax + by + c = 0, we find co-efficients of x and y have been interchanged and sign of one of these co-efficient has been changed

$\therefore$ Perpendicular line to $3x - 4y = 5$ is $4x + 3y = \lambda$

2.8 Straight line through the point of intersection of two lines

(a) If $L_1 = 0$ and $L_2 = 0$ be two straight line then $L_1 + \lambda L_2 = 0$ are straight lines which pass through the point of intersection of $L_1 = 0$ and $L_2 = 0$. The value of λ will give the particular line. λ shall be calculated from the data given.

(b) If $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ be three straight lines then if $\lambda L_1 + \mu L_2 + \nu L_3 = 0$ then the three given lines are concurrent i.e. pass through one point.

(c) Given $a_1x + b_1 + c_1 = 0$, $a_2x + b_1y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ three straight lines.

These lines shall be concurrent (meet in one point) if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

2.9 To find the equation of a line through a given point (x_1, y_1) which makes angle a **with a given line y = mx + c.**

In figure 15, P is point (x_1, y_1) and AB is straight line y = mx + c. Straight lines PL and PM make equal angles α with this straight line. In fact these lines are inclined at α and $-\alpha$ with AB. If M is the slope of these lines then,

$\tan(\pm\alpha) = \dfrac{M-m}{1+mM}$ and it gives two values of M. (i) $M = \dfrac{m+\tan\alpha}{1-m\tan\alpha}$ (ii) $\dfrac{m-\tan\alpha}{1+m\tan\alpha}$

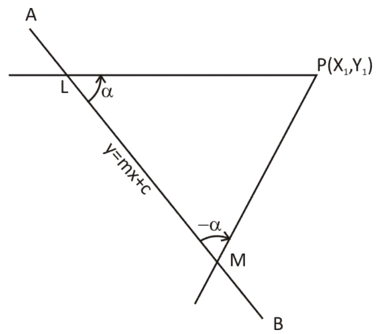

Fig. 15

∴ Straight lines are

$$y - y_1 = \frac{m+\tan\alpha}{1-m\tan\alpha}(x - x_1)$$

and $$y - y_1 = \frac{m-\tan\alpha}{1+m\tan\alpha}(x - x_1)$$

2.10 Distance of a point from a straight line

(a) AB is the straight line. Equation is $x \cos\alpha + y \sin\alpha = p$. In Fig. 16, OM is perpendicular from origin on it. OM = p; Point P is (x_1, y_1) perpendicular from P on straight line AB is PQ. PQ is distance of point P from straight line AB.

Equation of straight line through P and parallel to AB is

$x \cos\alpha + y \sin\alpha = p'$

$p' = ON$, ⊥ from origin on it, $ON = p'$

$PQ = MN = ON - OM = p' - p = x_1\cos\alpha + y_1 \sin\alpha - p$

(b) If the straight line is ax + by + c = 0

In normal form it is $\frac{ax+by}{\sqrt{a^2+b^2}} = -\frac{c}{\sqrt{a^2+b^2}}$

$\therefore$ Distance of (x_1, y_1) from straight line $= \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$

$\sqrt{a^2+b^2}$ = sq. roots of sum of squares of co-efficients x and y.

(c) Distance from origin of the line ax + by + c = 0 is $\frac{c}{\sqrt{a^2+b^2}}$

(d) Distance between two parallel lines is equal to the difference of the perpendicular from origin on them. Care should be taken that while determining the length of perpendiculars from origin, sign of x in both equations be the same (either both positive, or in both –ve).

(i) 2x + 3y = 6 and 6x + 9y + 10 = 0 are parallel lines on the same side of origin

$\therefore\ p_1 = \frac{6}{\sqrt{13}},\ p_2 = \frac{10}{3\sqrt{13}}$ $\quad p_1 > p_2$

$\therefore$ distance between lines = $p_1 - p_2 = \frac{6}{\sqrt{13}} - \frac{10}{3\sqrt{13}} = \frac{8}{3\sqrt{13}}$

(ii) x – 2y + 4 = 0, 6y – 3x + 16 = 0 are parallel lines but lie on opposite sides of origin.

Putting them as x – 2y + 4 = 0, 3x – 6y – 16 = 0

$p_1 = \frac{4}{\sqrt{5}}$, $\quad p_2 = \frac{-16}{3\sqrt{5}}$

$\therefore$ Distance between lines = $p_1 - p_2 = \frac{4}{\sqrt{5}} + \frac{16}{3\sqrt{5}} = \frac{28}{3\sqrt{5}}$

$\therefore$ In solving such questions-

1. The sign of co-efficient of x in both equations should be kept same.
2. p_1 and p_2 should be determined with their sign.
3. $p_1 \sim p_2$ gives the distance between parallel lines.

2.11 Position of a point with reference to a straight line

Let AB be $y = mx + c$. Points P_1 (x_1, y_1) is above the line, point P_2 (x_2, y_2) is on the line

$\therefore$ $y_2 = mx_2 + c$

$y_1 > y_2$ $\therefore$ $y_1 > mx_2 + c$ (x_3, y_3) is P_3 and it is below the line. $y_3 < y_2$

$\therefore$ $y_3 < mx_2 + c$

It follows that point (x_1, y_1) shall be above the line, or on the line, or below the line as $(y_1 - m_1x - c) > 0$ or $(y_1 - m_1x - c) = 0$ or $(y_1 - m_1x - c) < 0$. Point (x_1, y_1) is above straight line $3x - 4y + 5 = 0$

If $(4y_1 - 3x_1 - 5) > 0$ and shall be below this line.

If $(4y_1 - 3x_1 - 5) < 0$

Note: Always take sign of y positive in these questions.

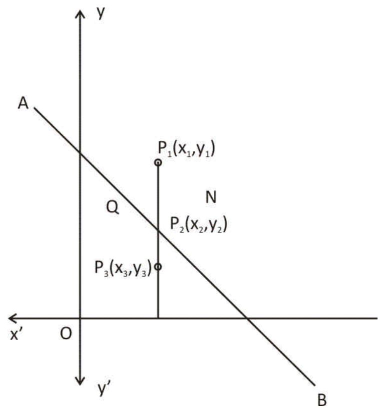

Fig. 17

Solved Examples

Example 12: A straight line passes through (–5, 4) and its segment between the axes is divided at this point in the ratio of 3 : 4. Find its equation.

Sol. Let the straight line be $\frac{x}{a}+\frac{y}{b}=1$. It meets axes in (a, 0) and (0, b) Point (–5, 4) divides the segment joining these points in the ratio of 3 : 4.

points (a, 0) , (0, b) point (–5, 4) ratio 3 : 4

$\therefore \frac{3.0+4a}{7}=-5 \qquad \frac{3b+4.0}{7}=4 \Rightarrow a=-\frac{35}{4}, \; b=\frac{28}{3}$

$\therefore$ Straight lines is $\frac{4x}{-35}+\frac{3y}{28}=1 \Rightarrow 16x-15y+140=0$

Example 13: A straight line cuts off intercept of –2 on y-axis and is equally inclined to axes. Find equation of straight line.

Sol. In figure 18, straight line AB and CD pass through point P (0, –2) and are equally inclined to axes.

Inclination of AB is 45^o and that of CD 135^o. Equation of straight lines are

$y + 2 = \tan 45^o (x – 0)$ and

$y + 2 = \tan 135^o (x – 0)$

i.e. $y + 2 = x$ and $y + x + 2 = 0$.

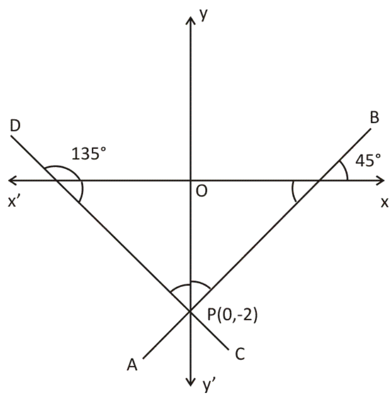

Fig. 18

Example 14: A straight line through A (1, 2) is inclined at 60^o with x-axis. It intersects straight line x + y = 6 in P, find AP

Sol. Equation of straight line through (1, 2) and inclined at 60^o with x-axis is

$y – 2 = \tan 60^o (x – 1) \Rightarrow y – 2 = \sqrt{3} (x – 1)$

It meets $y = –x + 6$ in P.

then $x=\frac{4+\sqrt{3}}{1+\sqrt{3}}, \qquad y=\frac{-4+\sqrt{3}}{1+\sqrt{3}}+6=\frac{2+5\sqrt{3}}{1+\sqrt{3}}$

$\therefore AP^2 = \left(\frac{4+\sqrt{3}}{1+\sqrt{3}}-1\right)^2+\left(\frac{2+5\sqrt{3}}{1+\sqrt{3}}-2\right)^2$

$$=\left(\frac{3}{1+\sqrt{3}}\right)^2+\left(\frac{3\sqrt{3}}{1+\sqrt{3}}\right)^2=\frac{3^2(1+3)}{(1+\sqrt{3})^2}$$

$$=\frac{36}{(1+\sqrt{3})^2}=\left[\frac{6}{\sqrt{3}+1}\right]^2=\left[3(\sqrt{3}-1)\right]^2$$

$\therefore$ AP $= 3(\sqrt{3}-1)$

Aliler Equation of straight line through (1, 2) and inclined at 60^o, in symmetrical form is

$$\frac{x-1}{\cos 60}=\frac{y-2}{\sin 60^o}=r$$

Any point distance r from (1, 2) on this line is

$$[(r\cos 60+1),(r\sin 60+2)]\text{ i.e.}\left(\frac{r}{2}+1,\frac{\sqrt{3}r}{2}+2\right)$$

And this point is on x + y = 6

$$\therefore\ \left(\frac{r}{2}+1+\frac{\sqrt{3}}{2}r+2\right)=6,\qquad \Rightarrow r(1+\sqrt{3})=6$$

$$\therefore\ r=AP=\frac{6}{1+\sqrt{3}}=3(\sqrt{3}-1)$$

Example 15: Find equation of right bisector of line segment of straight line ax + by = c between axes.

Sol. Straight line ax + by = c meets axes in $\left(\frac{c}{a},0\right),\left(0,\frac{c}{b}\right)$. Mid point is $\left[\frac{c}{2a},\frac{c}{2b}\right]$

straight line perpendicular to ax + by + c = 0 is bx – ay = λ

It goes through $\left(\frac{c}{2a},\frac{c}{2b}\right)$ $\therefore$ $\lambda=\frac{bc}{2a}-\frac{ac}{2b}$

$\therefore$ Equation of right bisector is bx – ay = c $(b^2 - a^2)/2ab$

i.e. 2ab (bx – ay) = c $(b^2 - a^2)$

Example 16: Find equation of straight line passing through (a $\cos^3\theta$, a $\sin^3\theta$) and perpendicular to x sec θ + y cosec θ = a **(Roorkee)**

Sol. Equation of given line is

$$\frac{x}{\cos\theta}+\frac{y}{\sin\theta}=a\Rightarrow x\sin\theta+y\cos\theta=\frac{a}{2}\sin 2\theta$$

Equation of straight line perpendicular to it is x cos θ – y sin θ = λ

It passes through ($a\cos^3\theta$, $a\sin^2\theta$)

∴ λ $= a\cos^4\theta - a\sin^4\theta = a(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

$= a\cos 2\theta$

∴ Required straight line is x cos θ – y sin θ = a cos 2 θ

Example 17: A straight line is 5 units away from origin and is perpendicular to $\sqrt{3}$ x – y + 5 = 0. Its equation is...

Sol. Equation of line perpendicular to $\sqrt{3}$ x – y + 5 = 0 is $x+\sqrt{3}y+\lambda=0$

It is 5 units away from origin ⇒ $\frac{\lambda}{\pm\sqrt{1+3}}=5$

∴ λ = ± 10

∴ Equation of straight line is $x+\sqrt{3}y\pm 10=0$

Example 18: Find the equation of the line which joins (4, 1) with the foot of perpendicular from (3, –2) on the straight line 2x – 3y = 1

Sol. Equation of the line through (3, –2) and perpendicular to 2x – 3y = 1 is

3x + 2y = λ = 9 – 4 ⇒ 3x + 2y = 5

Point of intersection of $\left.\begin{matrix}2x-3y=1\\3x+2y=5\end{matrix}\right\}$ is $\left(\frac{17}{13},\frac{7}{13}\right)$

∴ Equation of required line is

$$y-1=\frac{\frac{7}{13}-1}{\frac{17}{13}-4}(x-4)\Rightarrow y-1=\frac{-6}{-35}(x-1)$$

⇒ 35y – 6x = 29

Example 19: p_1 and p_2 are perpendicular from origin on straight line x cos θ + y sin θ == a cos 2θ and x sec θ – y cosec = a, then $4p_2^2+p_1^2=\ldots$

Sol. Straight line x cos θ + y sin θ –a cos 2θ = 0

$$\therefore\ p_1=\frac{a\cos 2\theta}{\sqrt{\cos\theta^2+\sin^2\theta}}=a\cos 2\theta$$

Second line is $\frac{x}{\cos\theta}-\frac{y}{\sin\theta}=a$

$\Rightarrow x \sin\theta - y\cos\theta = a \sin\theta\cos\theta = \frac{1}{2} a \sin 2\theta$

$\perp$ from origin on it $p_2 \frac{\frac{a}{2}\sin 2\theta}{\sqrt{\sin^2\theta + \cos^2\theta}} = \frac{a}{2}\sin 2\theta$

$\therefore 4p_2^2 + p_1^2 = a^2 \sin^2 2\theta + a^2 \cos^2 2\theta = a^2$

$\therefore p_1^2 + 4p_2^2 = a^2$

Example 20: In what ratio the line segment joining (3, 4) and (7, 8) in divided by straight line passing through (–5, 1) and (1, –3).

Sol. Equation of straight line joining (–5, 1) and (1, –3) is

$$y - 1 = \frac{-4}{6}(x + 5) \Rightarrow 2x + 3y = -7 \qquad \text{... (i)}$$

Point dividing line segment of (3, 4) and (7, 8) in ratio k : 1 is $\left(\frac{7k+3}{k+1}, \frac{8k+4}{k+1}\right)$

This point should lie on straight line (i).

$\therefore 2(7k + 3) + 3(8k + 4) = -7(k + 1)$

Solving $45k = -25 \Rightarrow k = \frac{-5}{9}$

$\therefore$ Straight line divides AB externally in the ratio of –5 : 9 (figure 23)

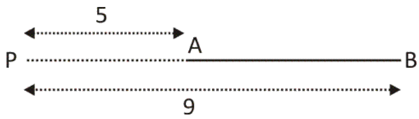

Fig. 19

Example 21: Find equation of straight line which passes through the point of intersection of straight lines 3x + 4y = 7 and x + y = 2 and is

(a) perpendicular to 2y – x = 3

(b) makes equal angle with axes.

Sol. Solving $\left.\begin{matrix} 3x + 4y = 7 \\ x + y = 2 \end{matrix}\right\} \begin{matrix} x = 1 \\ y = 1 \end{matrix}$

(a) Straight line perpendicular to 2y – x=3 is 2x + y = λ, it passes through (1, 1)

$\therefore \lambda = 2 \cdot 1 + 1 = 3$

Equation of this straight line is 2x + y = 3

(b) Make equal angles with axes $\Rightarrow m_1 = \pm 1$

$\therefore$ Equations are $y - 1 = \pm(x - 1)$

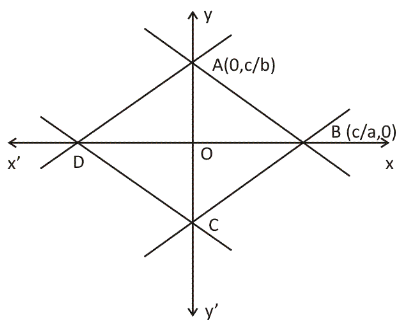

Fig. 20

i.e. $y - x = 0$, $y + x = 2$

Example 22: Prove that area of rhombus formed by straight lines $ax \pm by \pm c = 0$ is $\frac{2c^2}{ab}$.
(I.I.T.)

Sol. Straight lines are

AB, $ax + by - c = 0$; BC, $ax - by - c = 0$; CD, $ax + by + c = 0$; AD, $ax - by + c = 0$

From figure it is clear that area of rhombus

$= 4\ \Delta\ AOB$

$$= 4 \cdot \left(\frac{1}{2} \cdot \frac{c}{a} \cdot \frac{c}{b}\right) = \frac{2c^2}{ab}$$

Example 23: Find area of triangle formed by straight lines $x = 0$, $y = m_1x + c_1$ and $y = m_2x + c_2$.

Sol. $x = 0$ is y-axis and the remaining two lines meet y-axis in A $(0, c_1)$, B $(0, c_2)$ lines

$y = m_1x + c_1$ and $y = m_2x + c_2$ intersect in C. Whose abscissa = $\frac{c_2 - c_1}{m_1 - m_2} = CL$ in figure

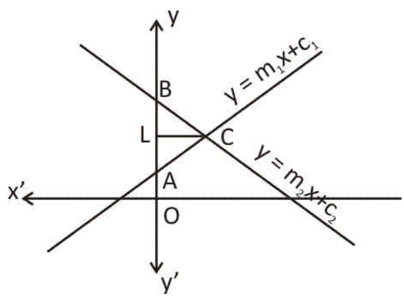

Fig. 21

$\therefore$ Area of ΔCAB

$$= \frac{1}{2} \cdot (AB) \times CL$$

$$= \frac{1}{2}(c_2 - c_1) \cdot \frac{c_2 - c_1}{m_1 - m_2}$$

$$= \frac{1}{2} \frac{(c_2 - c_1)^2}{m_1 - m_2}$$

Example 24: The vertices of a triangle have integral co-ordinates. Prove that it can not be equilateral triangle.

Sol. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the vertices of triangle. $x_1, x_2, x_3, y_1, y_2, y_3 \in I$ i.e. are integral numbers and set I is closed for addition, subtraction and multiplication.

$\therefore$ area of $\Delta = \frac{1}{2}\{x_1(y_2 - y_3)\}$ = Integral number

While we know, area of equilateral triangle is $\frac{\sqrt{3}}{4}(\text{side})^2$ (Irrational number)

$\therefore$ Triangle with vertices of integral numbers cannot be equilateral triangle.

Example 25: Find the orthocentre of the triangle whose vertices are A (2, 3), B (8, –3) and C (5, 6).

Sol. Orthocentre is the point of intersection of perpendiculars dropped from vertices on the opposite sides.

Given A (2, 3), B (8, –3), C (5, 6)

Slope of BC = $\frac{+9}{-3} = -3 \Rightarrow$ Slope of $\perp$ on it = $\frac{1}{3}$

$\therefore$ Equation of perpendicular from A on BC is

$y - 3 = \frac{1}{3}(x - 2) \Rightarrow 3y - x = 7$ ... (i)

Slope of AC = $\frac{3}{3} = 1$ $\therefore$ Slope of $\perp$ on it = –1

$\therefore$ Equation of perpendicular from B on AC is

$y + 3 = -(x - 8) \Rightarrow y + x = 5$ ... (ii)

Solving (i) and (ii) y = 3, x = 2, orthocentre (2, 3)

Note: The orthocentre is the point A itself. This is possible only when triangle ABC is right angled triangle and $\angle A = 90^o$

Example 26: Find the orthocentre of the triangle whose sides are y = 0, y = 2x and y = 6x + 5.

Sol. Let $\left.\begin{array}{r} AB, y = 0 \\ BC, \ y = 2x \\ AC, y = 6x + 5 \end{array}\right\} \begin{array}{l} B(0,0) \\ C\left(-\frac{5}{4}, -\frac{5}{2}\right) \\ A\left(\frac{-5}{6}, 0\right) \end{array}$

Perpendicular on AC from B is 6y + x = 0 ... (i)

Perpendicular on BC from A is $2y + x = -\frac{5}{6}$ ... (ii)

Solving (i) and (ii) orthocentre is $y = \frac{5}{24} \Rightarrow \left(-\frac{5}{4}, \frac{5}{24}\right)$

Example 27: Assuming co-ordinate geometry prove that altitude of a triangle are concurrent. **(I.I.T.)**

Sol. Let the three sides of triangle be

BC, $y = m_1x + c_1$

CA, $y = m_2x + c_2$

AB, $y = m_3x + c_3$

A point is $\left(\dfrac{c_3 - c_2}{m_2 - m_3}, \dfrac{m_2c_3 - c_2m_3}{m_2 - m_3}\right)$

Equation of perpendicular from A on BC is

$$m_1y + x = m_1\left(\frac{m_2c_3 - c_2m_3}{m_2 - m_3}\right) + \left(\frac{c_3 - c_2}{m_2 - m_3}\right)$$

$\Rightarrow m_1(m_2 - m_3)y + (m_2 - m_3)x = m_1\, m_2\, c_3 - m_1\, m_3\, c_2 + c_3 - c_2$ ... (i)

Equation of perpendicular from B on AC is

$m_2\,(m_3 - m_1)y + (m_3 - m_1)x = m_2\, m_3\, c_1 - m_2\, m_1\, c_3 + c_1 - c_3$ ... (ii)

Equation of perpendicular from C on AB is

$m_3\,(m_1 - m_2)y + (m_1 - m_2)x = m_3\, m_1c_2 - m_3\, m_2\, c_1 + c_2 - c_1$ ... (iii)

Adding (i), (ii) $0 \cdot y + 0 \cdot x = 0$

$\therefore$ Altitudes are concurrent

Example 28: Relation between co-efficients of equation ax + by + c = 0 is a + b + c = 0. Prove that straight line passes through a fixed point.

Sol. $a + b + c = 0 \Rightarrow c = -a - b$

$\therefore$ $ax + by + c = ax + by - a - b = 0$

$\Rightarrow a(x - 1) + b(y - 1) = 0$

$\Rightarrow (x - 1) + \dfrac{b}{a}(y - 1) = 0 \Rightarrow L_1 + \lambda\, L_2 = 0$

$\therefore$ Straight line passes through point of intersection of x – 1 = 0 and y – 1 = 0 i.e. (1, 1) a constant.

Example 29: Straight line bx + ay = ab moves in such a way that $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$ where c is a constant. Find the locus of foot of perpendicular from origin on it.

Sol. Equation of straight line bx + ay = ab ... (1)

Equation of perpendicular on it is ax – by = 0 ... (2)

Solving (1) and (2) point of intersection is $\left(\dfrac{ab^2}{a^2 + b^2}, \dfrac{ba^2}{a^2 + b^2}\right)$. If (h, k) is foot of perpendicular on line, then

$$h = \frac{ab^2}{a^2+b^2}, k = \frac{ba^2}{a^2+b^2}$$

$$h^2 + k^2 = \frac{a^2b^2(b^2+a^2)}{(a^2+b^2)^2} = \frac{a^2b^2}{a^2+b^2}$$

From the condition given $\frac{a^2+b^2}{a^2b^2} = \frac{1}{c^2}$

$\therefore$ $h^2 + k^2 = c^2 \Rightarrow$ Locus $x^2 + y^2 = c^2$

Example 30: The base of an equilateral triangle x + y = 2 and its vertex is (2, 1). Find equations of sides of the triangle **(I.I.T. 87)**

Sol. Angles of equilateral triangle are 60^o each.

Lines through A should makes angles 60^o and -60^o with x + y = 2

Slope of given line m = –1.

In figure, if slope of AB be m then $\tan(\pm 60) = \frac{m+1}{1-m}$

(i) $\sqrt{3} - \sqrt{3}M = m + 1 \Rightarrow m = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

(ii) $-\sqrt{3} + \sqrt{3}m = m + 1 \Rightarrow m = \frac{1+\sqrt{3}}{\sqrt{3}-1}$

$\therefore$ Equations of sides are

$$y - 1 = \frac{\sqrt{3}-1}{\sqrt{3}+1}(x-2) \text{ and } y - 1 = \frac{1+\sqrt{3}}{\sqrt{3}-1}(x-2)$$

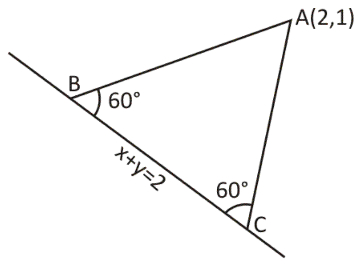

Fig. 22

Example 31: The two opposite vertices of a square are (3, 4) and (1, 2). Find co-ordinates of the other two vertices. What shall be these vertices if the two opposite vertices be (3, 4) and (1, –2).

Sol. Let A be (3, 4) and C (1, 2)

Mid point of AC is (2, 3)

Equation of other diagonal, perpendicular on it is, $y - 3 = -\frac{3-1}{4-2}(x-2) \Rightarrow y + x = 5$

Slope $AC = \frac{4-2}{3-1} = 1$, 45^o of BD = –1, 135^o

$\therefore$ Sides are parallel to axes.

$\therefore$ Other two vertices are (3, 2) and (1, 4)

(b) If A is (3, 4) and B (1, –2), then mid point M is (2, 1)

Equation of other diagonal BD is $y-1=-\frac{1}{3}(x-2)$

i.e. BD is 3y + x = 5 ... (i)

Now if B is (x_2, y_2) then BM = AM

$\Rightarrow (x_2-2)^2+(y_2-1)^2=(3-2)^2+(4-1)^2=10$

$\Rightarrow x_2^2+y_2^2-4x_2-2y_2-5=0$ ... (ii)

B lies on AB $\therefore 3y_2+x_2=5$

$\therefore$ From (ii) $(-3y_2+5)^2+y_2^2-4(-3y_2+5)-2y_2-5=0$

$\Rightarrow 10y_2^2-30y_2^2+12y_2-2y_2+25-20-5=0$

$\Rightarrow 10y_2^2-20y_2=0$

$\therefore y_2=2$, or 0

$y_2=0$ gives $x_2=5$ and $y_2=2$ gives $x_2=-1$

$\therefore$ Other vertices are (5, 0) and (–1, 2)

Example 32: Points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on y = 2x + c. Find c and remaining vertices. **(I.I.T. 81)**

Sol. Let A be (1, 3) and C (5, 1), AC is diagonal.

Mid point M is (3, 2) (BD is not perpendicular to AC. It being rectangle)

M shall lie on y = 2x + c

$\therefore 3\cdot 2+c=2\Rightarrow c=-4$

$\angle ABC=90^o$ $\therefore$ If B is (x_1, y_1) then

$$\left(\frac{y_1-3}{x_1-1}\right)\times\left(\frac{y_1-1}{x_1-5}\right)=-1$$

$\Rightarrow x_1^2+y_1^2-6x_1-4y_1+8=0$ and $y_1=2x_1-4$

$\therefore x_1^2+(2x_1-4)^2-6x_1-4(2x_1-4)+8=0$

$\Rightarrow 5x_1^2-30x_1+40=0\Rightarrow x_1^2-6x_1+8=0$

$\Rightarrow (x_1-4)(x_1-2)=0 \therefore x_1=4$, or 2

$y_1=2x_1-4\Rightarrow y_1=4$ or 0

$\therefore$ Other vertices are (4, 4) and (2, 0)

Example 33: A ray of light is sent along the line x – 2y = 3, upon reaching the line 3x – 2y = 5, it gets reflected from it. Find the equation of line along reflected ray.

Sol. AO is the incident ray x – 2y = 3

It makes angle $(-\alpha)$ with 3x – 2y – 5 = 0. Then reflected ray OA' shall make angle $+\alpha$ with it. O is the point of intersection of x – 2y = 3 and 3x – 2y = 5. O is (1, –1)

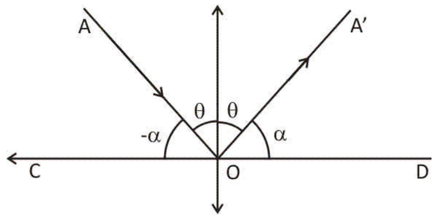

Fig. 23

Slope of AO $m_1 = \frac{1}{2}$ and slope of CD = $\frac{3}{2}$

$$\therefore \tan(-\alpha) = -\tan\alpha = \frac{\frac{3}{2}-\frac{1}{2}}{1+\frac{3}{4}} = \frac{4}{7} \Rightarrow \tan\alpha = -\frac{4}{7}$$

For acute angle $\tan\alpha = \frac{4}{7}$

Let the slope of OA' be M, then

$$\frac{M-\frac{3}{2}}{1+\frac{3}{2}M} = \frac{4}{7} \Rightarrow M = \frac{29}{2}$$

$\therefore$ Equation of reflected ray is $y+1=\frac{29}{2}(x-1)$

$$29x - 2y = 31$$

Example 34: The straight line joining A (2, 0) and B (3, 1) is rotated about A through 15^o in anticlockwise direction. Find equation of line in new position.

Sol. Slope of line AB = $\frac{0-1}{2-3} = 1$ i.e. $\theta = 45^o$

If now the straight line is rotated in anticlockwise direction through 15^o, the slope of line then shall became 60^o.

Equation of straight line now is y – 0 = tan 60 (x –2)

$\therefore$ i.e. $y-\sqrt{3}x+2\sqrt{3}=0$

Example 35: The straight line L_1 is parallel to L_2 and at a distance of 2 units from it. L_1, meets 3x – 4y = 2 in (2, 1) at right angles. Find equation of L_2.

Sol. Straight line L_1 is perpendicular to $3x - 4y = 2$

$\therefore$ Equation of L_1 is $4x + 3y = \lambda$

It meets line in (2, 1)

$\therefore$ Equation L_1 is $4x + 3y = 2 \cdot 4 + 3 \cdot 1 = 11$

Equation of L_1 in normal form is $\frac{4}{5}x + \frac{3}{5}y = \frac{11}{5}$

L_2 is parallel L_1 and is distance of two units away from it.

$\therefore$ Equation of L_2 is $\frac{4}{5}x + \frac{3}{5}y = \frac{11}{5} \pm 2$

i.e. $4x + 3y = 1$ or $4x + 3y = 21$

Example 36: Area of square formed by lines $|x| + |y| = 1$ is

(a) 4 sq. u. (b) 2 sq. u. (c) 8 sq. u. (d) 16 sq. u.

Sol. Sides of square are $x + y = 1$, $x - y = 1$,

$-x - y = 1$, $-x + y = 1$,

these meet axes in (1, 0), (0, 1), (–1, 0), (0, –1)

These point when joined give square.

area = $4 \Delta AOB$

$= 4 \cdot \frac{1}{2} \cdot 1 \cdot 1$

= 2 sq. u.

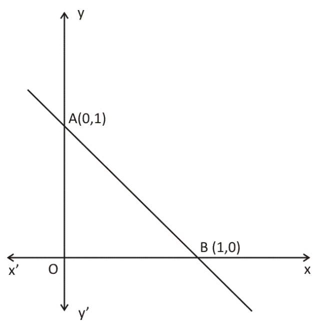

Fig. 24

Example 37: The base AB of triangle ABC is fixed, while vertex c moves in such a way that $\frac{CA}{CB} = \lambda$ (a constant), then locus of C is

(a) Circle (b) ellipse (c) parabola (d) hyperbola

Sol. Let A be (a, 0), B (–a, 0) and C (h, k), then from condition given.

$(h - a)^2 + k^2 = \lambda^2 (h + a)^2 + k^2$

$\Rightarrow h^2 + k^2 - 2ah + a^2 = \lambda^2 (h^2 + k^2 + 2ha + a^2)$

$\Rightarrow (1 - \lambda^2)(h^2 + k^2) - 2ah(1 + \lambda^2) + a^2(1 - \lambda^2) = 0$

$\Rightarrow h^2 + k^2 - 2ah \cdot \left(\frac{1+\lambda^2}{1-\lambda^2}\right) + a^2 = 0$

Locus $x^2 + y^2 - \left(\frac{1+\lambda^2}{1-\lambda^2}\right)2ax + a^2 = 0$

It is circle. (a) Correct.

Example 38: The straight line y = $\frac{3}{2}$ meets parallel lines x + 2y = 6 and 2x + 4y = 10 in A and B. The projection of AB on x + 2y = 5 is

(a) $\frac{2}{\sqrt{5}}$ (b) $\frac{4}{\sqrt{10}}$ (c) $\frac{3}{\sqrt{5}}$ (d) $\frac{1}{2}$

Sol. Slope of L_2, x + 2y = 6 is $-\frac{1}{2}$ and $y = \frac{3}{2}$ is parallel to x-axis. In figure, $\phi = 120^o$, $\theta = 60^o$.

A is $\left(3, \frac{3}{2}\right)$, B $\left(2, \frac{3}{2}\right)$

AB = 1, projection of AB on line 2x + 4y = 10 is AB cos θ.

= AB cos θ = $1 \cdot \frac{1}{2} = \frac{1}{2}$

(d) correct

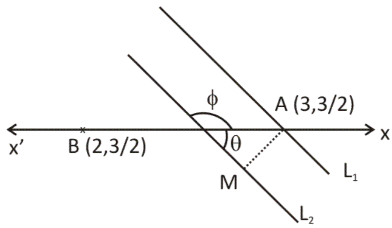

Fig. 25

Example 39: The straight line x + 2y = 6 meets axes in A and B. The line through (5, 8) and perpendicular to it meets the above line and axes in E, C, D. Area of OCEB is

(a) 6 sq. u. (b) 5 sq. u. (c) 4 sq. u. (d) 8 sq. u.

Sol. Equation of straight line is x + 2y = 6 ... (1)

∴ A is (6, 0), B is (0, 3)

slope of straight line $-\frac{1}{2}$

Equation of straight line perpendicular to and passing through (5, 8) is

y – 8 = 2 (x – 5) ⇒ y – 2x + 2 = 0 ... (2)

point of intersection of (1) and (2), E is (2, 2)

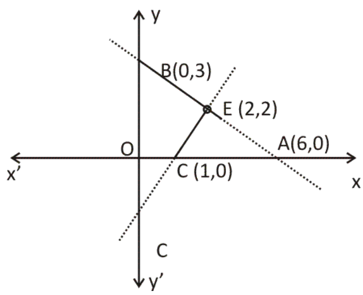

Fig. 26

It meets x-axis in C, C is (1, 0), meets y-axis in D, D is (0, –2).

Area of quadrilateral OCEB

= trapezium BOLE – Δ ECL (see figure)

$= \frac{1}{2}(OB + EL) \cdot OL - \frac{1}{2} CL \cdot EL$

$= \frac{1}{2}(3+2) \cdot 2 - \frac{1}{2} \cdot (2-1) \cdot 2 = 4 \text{sq.u.}$

Example 40: A straight line is such that its segment between lines 5x – y = 4 and 3x + 4y = 4 is bisected at (1, 5). Obtain its equation

Sol. Equation of line through (1, 5) is $\frac{x-1}{\cos\theta} = \frac{y-5}{\sin\theta} = r$ or –r ... (i)

If M and N are intersection points of this straight line with 5x – y = 4 and 3x + 4y = 4 respectively. Then points (r cos θ + 1, r sin θ + 5) and (–r cos θ + 1, –r sin θ + 5) must lie on them respectively.

$\therefore$ 5 r cos θ + 5 – r sin θ – 5 = 4 ⇒ 5 (5 cos θ – sin θ) = 4 ... (2)

and –3r cos θ + 3 – 4 r sin θ + 20 = 4

⇒ r (3 cos θ + 4 sin θ) = 19 ... (3)

$\therefore \ r = \frac{4}{5\cos\theta - \sin\theta} = \frac{19}{3\cos\theta + 4\sin\theta}$

$\therefore$ Equation (1) is $\frac{x-1}{35} = \frac{y-5}{83}$ ⇒ 83x – 35y + 92 = 0

Example 41: A line meets axes in A (7, 0) and B (0, 5). A variable line PQ, perpendicular to AB meets axes in P and Q. AQ and BP meet in R. Find locus of R. **(I.I.T. 90)**

Sol. Equation of AB is $\frac{x}{7} + \frac{y}{5} = 1 \Rightarrow 5x + 7y = 35$

Any straight line ⊥ to it is 7x – 5y = λ

It meets axes in $\left(\frac{\lambda}{7}, 0\right)$ and $\left(0, \frac{-\lambda}{5}\right)$ i.e. in P and Q respectively.

Equation AQ $y = \frac{-\lambda/5}{-7}(x-7) \Rightarrow \lambda = \frac{35y}{x-7}$

Equation BP $y - 5 = \frac{-5}{\frac{\lambda}{7} - 0}(x - 0) \Rightarrow \lambda = \frac{35x}{5-y}$

From (1) and (2) locus of R is $\frac{35y}{x-7} = \frac{35x}{5-y} \Rightarrow y(y-5) + x(x-7) = 0$

$\Rightarrow x^2 + y^2 - 7x - 5y = 0$

Practice Worksheet (Foundation Level) – 2 (b)

1. Find the equation of straight line through (1, 2) and inclined at $\sin^{-1}\left(\frac{3}{5}\right)$ with x-axis.
2. Find the equation of a straight line whose sum of intercepts an axes is 14 and which passes through (3, 4).
3. A straight line passes through (–5, 4) and its line segment between axes is divided by this point in the ratio of 1 : 2. Find its equation.
4. Find the equation of the line parallel to 2x + 5y - 7 = 0 and bisecting the line segment AB whose A is (2, 7) and B is (–4, 1)
5. A line passes through A (–2, 5), and is inclined at 30^o with x-axis. Find the equation of the line perpendicular to its and cutting it at B where AB = 5.
6. If the intercepts of a line on axes are of length a and b and p, is the perpendicular from origin on it, then prove $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.
7. The intercepts on OX and OY of a line are in the ratio of 4 : 3 and it passes through (–4, –9). Find its equation.
8. The slope of a line is 2 and perpendicular from origin on it $\sqrt{5}$. Its equation is ...
9. A straight line meets axes in A and B. The area of triangle OAB is $24\sqrt{3}$ sq. u. The perpendicular p from origin on it is inclined at 60^o with x-axis. Then p is equal to...
10. In what ratio line joining (1, 2) and (4, 3) is divided by the line joining (2, 3) and (4, 1).
11. The vertices of a triangle are (–1, 5), (2, –3) and (5, 3). Find its

    (a) Centroid (b) Circumcentre (c) orthocentre
12. Show that points (a, 2a), (–2a, –a) and (–a, 0) are collinear.
13. Show that in a isosceles triangle whose base is 6 cm and height 4 cm, the distance of orthocentre from base is 2.25 cm.
14. A ray of light travelling along 3x – 4y = 12 gets reflected from x-axis. The reflected ray travels along...
15. Show that the orthocentre of the triangle whose sides are $x + ay = a^2$, $x + by = b^2$ and $x + cy = c^2$ is (1, a + b + c + abc).
16. The equation of diagonal AC of a square is 8x – 15y = 0 and vertex B is (1, 2). Find equation of AB and BC.
17. The set of lines ax + by + c = 0 when 3a + 2b + 4c = 0 are concurrent at point ...

18. The straight line through the point of intersection of x – 3y + 1 = 0 and 2x + 5y – 9 = 0 and at a distance $\sqrt{5}$ from origin is ...

19. Vertices of a triangle are $(at_1t_2, a(t_1 + t_2))$, $(at_2t_3, a(t_2 + t_3))$ and $(at_3t_1, a(t_3 + t_1))$. Show that its orthocentre is $[-a, a(t_1 + t_2 + t_3 + t_1 t_2 t_3)]$

20. Two vertices of a triangle are (5, –1) and (–2, 3). If orthocentre of triangle is at origin, find the third vertex.

21. If lines 2x – 3y + a = 0, 3x – 4y – 13 = 0 and 8x – 11y – 33 = 0 are concurrent then α is

(a) 6 (b) 7 (c) –3 (d) –7

22. Perpendicular from points $\left(\pm\sqrt{a^2-b^2},0\right)$ are dropped on the line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$. The product of these perpendicular is

(a) a b (b) b^2 (c) $a^2 - b^2$ (d) a^2

23. The distance between parallel lines 3x + 4y = 5 and 12 y + 9x + 10 = 0 is

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{5}{3}$ (d) $\frac{4}{3}$

24. The point of intersection of line 3x + 4y = 5 with the straight line parallel to 5x – 12y = 13 and at a distance of + 2 from it is

(a) $\left(\frac{27}{7},-\frac{23}{14}\right)$ (b) $\left(\frac{3}{7},\frac{13}{14}\right)$ (c) $\left(\frac{9}{7},\frac{2}{7}\right)$ (d) $\left(\frac{1}{3},1\right)$

25. The straight line L is perpendicular to 5x – y = 1. The area of triangle formed by straight line and axes is 5. Equation of L is

(a) $5y + x \pm\sqrt{2} = 0$ (b) $5y+x\pm\sqrt{5}=0$

(c) $5y+x\pm5\sqrt{2}=0$ (d) $5y + x \pm 5 = 0$

26. The area of a triangle is 5 sq. units, the two vertices are (2, 1) and (3, –2) and the third lies on y = x + 3. The third vertex is

(a) $\left(-\frac{17}{2},-\frac{11}{2}\right)$ (b) $\left(\frac{7}{2},\frac{13}{2}\right)$ (c) $\left(-\frac{7}{2},-\frac{1}{2}\right)$ (d) $\left(-\frac{1}{2},\frac{5}{2}\right)$

27. A ray of light traveling along y = 4, strikes a plane mirror placed along x – y = 0 and gets reflected. The reflected ray again strikes a plane mirror placed along x + y = 12. Now the equation of reflected ray is ...

(a) y + 4 = 0 (b) y + 8 = 0 (c) y = 12 (d) y = 8

28. A ray of light travelling along y + $\sqrt{3}$x = 4 strikes a mirror placed along x – y = 0 and gets reflected. The reflected ray travels along

(a) $\sqrt{3}y - x = 4$ (b) $\sqrt{3}y - x = 0$

(c) $\sqrt{3}y + x = 4$ (d) $\sqrt{3}y + x = 0$

29. Straight lines x + y – 4 = 0, 3x + y – 4 = 0 and x + 3y – 4 = 0 form a triangle. The triangle is

(a) Scalene (b) equilateral

(c) isosceles (d) right angled

30. A ray of light is sent along the line x – 2y + 3 = 0 and it gets reflected from x + y = 0. The reflected ray travels along.

(a) 2x – y + 3 = 0 (b) 2x + y + 3 = 0

(c) 2x + y – 3 = 0 (d) x + 2y – 3 = 0

31. The diagonals of a parallelogram whose sides are ℓx + my + n = 0, ℓx + my + n' = 0, mx + ℓy + n = 0 and mx + ℓy + n' = 0 include an angle

(a) $\frac{\pi}{3}$ (b) $\tan^{-1}\frac{2\ell m}{\ell^2+m^2}$

(c) $\frac{\pi}{2}$ (d) none of these

32. The angle between the line through (α, β) and the other through (4, 5) is always θ. The locus of the point of intersection of these lines is

(a) an arc of a circle (b) an arc of ellipse

(c) straight line (d) rectangular hyperbola

[Hint: Angle in the same segment are equal]

33. Equation of straight line through (–3, –8) and having an intersept of length 3 between parallel lines 4x + 3y = 12 and 4x + 3y = 3 is

(a) 7x + 24y + 213 = 0 (b) 7x – 24y = 171

(c) 9x – 24y = 165 (d) x + 3 = 0

[Hint: Straight line $\frac{x+3}{\cos\theta} = \frac{y+8}{\sin\theta}$. (i) = r (ii) r + 3, straight point satisfy 4x + 3y = 3 other point 4x + 3y = 12]

34. If the line segment joining A (3, 2), B (6, 5) is rotated at A through 15° in the anti-clock direction, then co-ordinates of B in new position are

(a) $\left[\frac{\sqrt{3}}{2}(\sqrt{2}+1), \frac{1}{2}(4+3\sqrt{6})\right]$ (b) $\left[\frac{3}{2}(2+\sqrt{2}), \frac{1}{2}(4+3\sqrt{6})\right]$

(c) $\left[\frac{\sqrt{3}}{2}(\sqrt{2}-1), \frac{1}{2}(4+3\sqrt{3})\right]$ (d) none of these

35. The image of $x^2 + y^2 - 4x - 6y + 9 = 0$ in $x + y = 0$ is

(a) $x^2 + y^2 + 4y + 6x + 9 = 0$ (b) $x^2 + y^2 + 4x + 6y + 9 = 0$

(c) $x^2 + y^2 + 4y - 6x + 9 = 0$ (d) $x^2 + y^2 - 4x + 6y + 9 = 0$

36. The straight line L is perpendicular to $4x - 3y = 5$, the area of triangle formed by L and co-ordinate axes is 6 sq. u. equation of L is

(a) $6x + 8y + 15 = 0$ (b) $6x + 8y - 15 = 0$

(c) $3x + 4y \pm 12 = 0$ (d) $3x + 4y \pm 15 = 0$

37. The orthocentre of a triangle is $\left(\frac{12}{11}, \frac{30}{11}\right)$ and its circumcentre is $\left(\frac{43}{22}, \frac{47}{22}\right)$. Its centroid is

(a) $\left(\frac{7}{3}, \frac{5}{3}\right)$ (b) $\left(\frac{5}{3}, \frac{11}{3}\right)$ (c) $\left(\frac{5}{3}, \frac{7}{3}\right)$ (d) $\left(\frac{11}{3}, \frac{5}{3}\right)$

38. L_1; $5x - y = 1$; L_2; $x + y = 1$ intersect in A and straight line L_3; $x + 5y + 18 = 0$ meets L_1 and L_2 in B and C, respectively, then $4[AB^2 + BC^2] - 6AC^2 =$

(a) $\left(\frac{65}{6}\right)^2$ (b) $-\left(\frac{65}{6}\right)^2$ (c) $\left(\frac{65}{9}\right)^2$ (d) $\left(\frac{65}{12}\right)^2$

39. A straight line through (2, 2) intersects the lines $y + \sqrt{3}x = 0$ and $\sqrt{3}x - y = 0$ at points A and B. So that the triangle OAB is equilateral. The equation of the line is ...

(a) $x - 2 = 0$ (b) $y - 2 = 0$

(c) $x + y = 0$ (d) $x + y = 4$

40. The straight lines $\lambda x + y + 1 = 0$ and $x + 2y + 3 = 0$ meet axes in A and B and other in C and D, respectively. OC > OA and OD > OB. If ABDC is concyclic then λ is equal to

(a) 2 (b) –2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

41. Vertex of a triangle is (1, 1); mid points of two sides of the triangle through this vertex are (–1, 2) and (3, 2) then centroid is

(a) $\left(\frac{1}{3}, \frac{7}{3}\right)$ (b) $\left(1, \frac{7}{3}\right)$ (c) $\left(-\frac{2}{3}, \frac{7}{3}\right)$ (d) $\left(-1, \frac{7}{3}\right)$

42. Straight line L_1, $bx + ay = ab$ meets axes in P and Q. L_2 is perpendicular on L_1 and meets axes in R and S. Locus of point of intersection of PS and QR is

(a) $x^2 + y^2 - ax - by = 0$ (b) $x^2 + y^2 + ax + by = 0$

(c) $x^2 + y^2 - bx - ay = 0$ (d) $x^2 + y^2 + bx + ay = 0$

43. Points A is (0, 2), B (3, 4). The line segment AB is rotated at A through 45^o in anticlockwise direction. The equation of AB in new position is

(a) $y = 7x - 14$ (b) $y = 5x + 2$

(c) $y = 9x + 18$ (d) $y = 3x - 6$

44. A straight line through the point of intersection of $3x + 4y = 13$ and $4x - 3y = 9$ cuts an intercept of -4 from x axis. Its equation is.

(a) $y - x = 4$ (b) $5y - x = 4$

(c) $7y - x = 4$ (d) $9y - x = 4$

45. A straight line through A (2, 3) meets the line $2x + y = 7$ in B. If AB = $2\sqrt{10}$ then Equation of line is

(a) $y + 3x = 9$ (b) $y - 3x + 3 = 0$

(c) $3y - x = 7$ (d) $y + 2x = 7$

46. What is represented by equation $(x^2 - a^2)^2 + (y^2 - b^2)^2 = 0$

47. L_1; $4x - 3y = 18$ and L_2, $x - 2y = 7$ straight lines intersect at P. L is rotated at P through 45^o in clockwise direction, while L_2 is rotated at P in anti-clockwise direction at P, the angle between two lines in new positions is

(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}(2)$ (c) $\tan^{-1}\left(\frac{2}{3}\right)$ (d) $\tan^{-1}\left(\frac{3}{2}\right)$

48. The straight line joining (–3, 2) and (7, –6) meets y-axis in P. At P it is rotated through 45^o in anticlockwise direction. The equation of line in new position is

(a) $x - 9y = \frac{18}{5}$ (b) $x - 5y = 2$ (c) $x + 9y = \frac{18}{5}$ (d) $x - 15y = 6$

2.12 Equation of angle bisector

The locus of a point which moves in such a way that its distance from two arms of the angle is always equal, is called the bisector of that angle. Perpendiculars dropped from any point of it to the two arms of angle are equal.

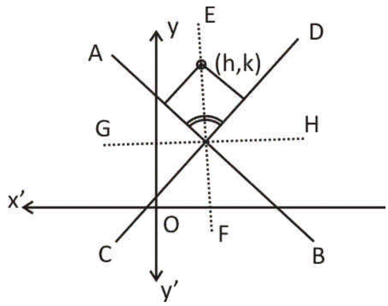

Fig. 28

Let the equations of two arms of an angle be $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$.

If (h, k) is any point on angle bisector then

$$\frac{a_1h + b_1k + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2h + b_2k + c_2}{\sqrt{a_2^2 + b_2^2}}$$

as $\sqrt{\ }$ has $\pm\sqrt{\ }$ sign, so equation of angle bisectors is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Thus two bisectors of an angle are these . see figure EF and GH are angle bisectors. The two bisectors are perpendicular to each other.

Note: Keep sign of constant term (c_1, c_2) + ve. While writing equation of bisectors in above form, then + sign shall give the equation of that angle bisector which contains origin. '–' sign shall give the equation of other angle bisector.

Solved Examples

Example 42: The sides of triangle are 3x + 4y + 2 = 0, 3x – 4y + 12 = 0 and 4x – 3y = 0 find incentre of triangle.

Sol. You know that incentre of a triangle is point of intersection of angle bisectors of angles of triangle

Let AB, 3x + 4y + 2 = 0; BC, 3x – 4y + 12 = 0; CA 4x – 3y = 0 Equation of bisector of ∠ B is $\frac{3x+4y+2}{5} = \pm\frac{3x-4y+12}{5}$. From figure it is clear that bisector contains origin.

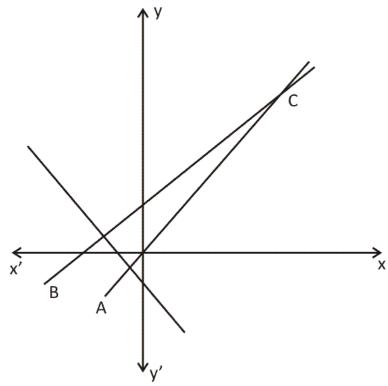

Fig. 29

∴ 3x + 4y + 2 = 3x – 4y + 12

⇒ 8y = 10 ⇒ y = $\frac{5}{4}$ ... (i)

From figure 29, it is clear that bisector of ∠C has positive slope (here origin cannot decide as CA passes through origin)

Equation of angle bisectors of ∠C are 3x – 4y + 12 = ± (4x – 3y)

For positive slope '–' sign shall be taken

∴ angle bisector is 7x – 7y + 12 = 0 ... (ii)

Solving (i) and (ii) incentre is $\left(-\frac{13}{28}, \frac{5}{4}\right)$

Note: You will get the same result by applying formula $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

Example 43: Vertices of a triangles are A (1, 1) B (4, –2) and C (5, 5). Find equation of perpendicular from C to the interior bisector of ∠A. **(Roorkee 94)**

Sol : Equation of AB y + 2 = $\frac{3}{-3}(x-4)$

i.e. x + y = 2

Equation AC, y – 5 = $\frac{-4}{-4}(x-5) \Rightarrow y - x = 0$

AB (x + y = 2) meets x-axis in (2, 0) and is inclined at 45° with x-axis, angle between them is 90° angle bisector shall be inclined to them at 45° & -45° slope of bisector is + ve. Bisector is parallel to x-axis. Equation of bisector of ∠A are

$$\frac{x+y-2}{\sqrt{2}} = \pm\frac{y-x}{\sqrt{2}}$$

Equation of bisector is y = 1

Equation of perpendicular on it from (5, 5) is x = 5

Example 44: A straight line of length c, meets axes in A and B. Rectangle OAPB is completed. Show that locus of foot of perpendicular from P on AB is $x^{2/3} + y^{2/3} = a^{2/3}$

(I.I.T.)

Sol. The straight line of length C meets axes in A and B.

Let A be (a, 0), B (0, b) $\therefore$ $c^2 = a^2 + b^2$

Equation of AB is bx + ay = ab ... (1)

and point P is (a, b)

equation of perpendicular from P on AB is

$ax - by = \lambda = a^2 - b^2 \Rightarrow ax - by = a^2 - b^2$ ... (2)

Point of intersection of (1) and (2) is $\left[\frac{a^3}{a^2+b^2}, \frac{b^3}{a^2+b^2}\right] \Rightarrow \left[\frac{a^3}{c^2}, \frac{b^3}{c^2}\right]$

$\therefore$ If this point is (h, k) then $h = \frac{a^3}{c^2}, k = \frac{b^3}{c^2}$

$$\therefore\ h^{2/3} + k^{2/3} = \left(\frac{a^3}{c^2}\right)^{2/3} + \left(\frac{b^3}{c^2}\right)^{2/3} = \frac{a^2+b^2}{c^{4/3}}$$

$$= \frac{c^2}{c^{4/3}} = c^{2/3}$$

$\therefore$ Locus is $x^{2/3} + y^{2/3} = c^{2/.3}$

Equation 45: The vertices of triangle OBC are (0, 0), (–3, –1) and (–1, –3). Find equation of line parallel to BC and intersecting OB and OC and whose perpendicular distance from origin is $\frac{1}{2}$ **(I.I.T.)**

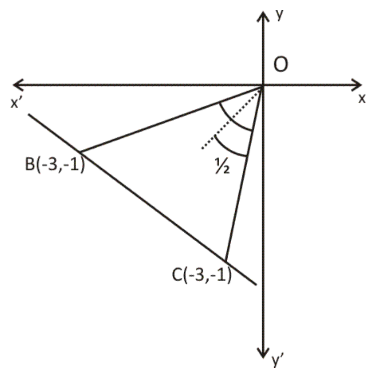

Fig. 30

Sol. B is (–3, –1), C (–1, –3)

equation BC, y + 1 = $\frac{-2}{2}$(x + 3)

$\Rightarrow$ x + y + 4 = 0

Equation of straight line || BC

$x + y = \lambda$

$\perp$ from origin on it = $\frac{1}{2}$

$\therefore \ \frac{\lambda}{\pm\sqrt{2}} = \frac{1}{2} \Rightarrow \lambda = \frac{1}{\pm\sqrt{2}}$

Since the lines intercepts OB and OC are in 3rd quadrant. $\therefore \ \lambda = -\frac{1}{\sqrt{2}}$

$\therefore$ Equation is $\sqrt{2}(x+y)+1=0$

Example 46: Straight lines 3x + 4y = 5 and 4x – 3y = 15 intersect in A. On these lines points B and C are so taken, such that AB = AC. Find possible equation of BC through point (1, 2). **(I.I.T.)**

Sol. Point of intersection of 3x + 4y = 5 and 4x – 3y = 15 is (3, –1) angle between lines is 90^0 as $\left(-\frac{3}{9}\times\frac{4}{3}=-1\right)$ and as AB = AC line shall make angle 45^o and -45^o with them. The straight line also goes through P, (1, 2). It is in 1st quadrant, therefore angle with 3x + 4y = 5 is + ve.

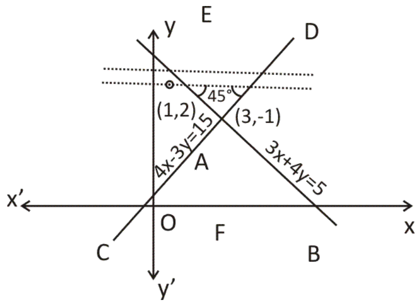

Fig. 31

$\therefore$ tan $45^o = 1 = \frac{m+\frac{3}{4}}{1-\frac{3}{4}m}$

$\Rightarrow 4 - 3m = 4m + 3$

$\Rightarrow 7m = 1 \Rightarrow m = \frac{1}{7}$

equation BC is $y - 2 = \frac{1}{7}(x-1)$

$\Rightarrow 7y - x = 13$

Example 47: Determine α, for which (α, α^2) lies inside the triangle having sides 2x + 3y = 1, x + 2y = 3 and 5x – 6y = 1 **(I.I.T.)**

Sol. Let AB 2x + 3y = 1, BC, 2y + x = 3, CA 6y – 5x = –1

See figure, y in all equations have been kept + ve.

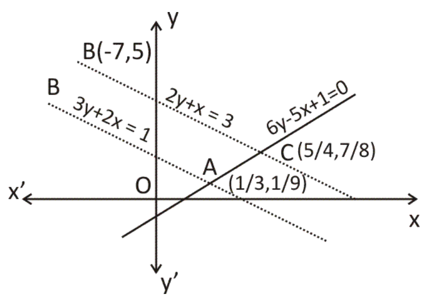

Fig. 32

(α, α^2) shall be inside the triangle if it is above AB, above AC but below BC

$\therefore\ 3\alpha^2 + 2\alpha - 1 \geq 0, \Rightarrow (3\alpha - 1)(\alpha + 1) \geq 0$

$\therefore\ \alpha \notin \left(-1, \frac{1}{3}\right)$

and $6\alpha^2 - 5\alpha + 1 \geq 0 \Rightarrow (2\alpha - 1)(3\alpha - 1) \geq 0$

$\therefore\ \alpha \notin \left(\frac{1}{3}, \frac{1}{2}\right)$

and $2\alpha^2 + \alpha - 3 \leq 0 \Rightarrow (2\alpha + 3)(\alpha - 1) \leq 0$

$\alpha \in \left(-\frac{3}{2}, 1\right)$

$\therefore\ \alpha \in \left(-\frac{3}{2} - 1\right) \cup \left(\frac{1}{2}, 1\right)$

Example 48: Let L_1 be a straight line passing through origin. L_2 straight line is $x + y = 1$. If intercepts of circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 be equal, then find equation of L_1.

Sol. Equation of circle is $x^2 + y^2 - x + 3y = 0$

L_2 is $x + y = 1$

$\therefore\ y = 1 - x$

$\therefore\ x^2 + (1 - x)^2 - x + 3(1 - x)$

$= 0$

$\Rightarrow x^2 - 3x + 2 = (x - 2)(x - 1) = 0$

$\therefore\ L_2$ meets circle in $(2, -1), (1, 0)$

$\therefore$ Length of intercept on $L_2 = \sqrt{(2-1)^2 + 1^2} = \sqrt{2}$

Let Equation of L_1 be $y = mx$, it meets circle

$\therefore\ x^2 + m^2x^2 - x + 3m\,x = 0$

$\Rightarrow (1 + m^2)x^2 - (1 - 3m)x = 0$

$\therefore\ x = 0$, and $x = \frac{1-3m}{1+m^2}$

$\Rightarrow y = 0 \qquad y = \frac{m(1-3m)}{1+m^2}$

Length of intercept, $\left(\sqrt{2}\right)^2 = \frac{(1+m^2)(1-3m)^2}{(1+m^2)^2}$

$\Rightarrow 2(1 + m^2) = (1 - 3m)^2$

$\Rightarrow 7m^2 - 6m - 1 = 0$

$\Rightarrow (m - 1)(7m + 1) = 0 \Rightarrow m = 1, m = -\frac{1}{7}$

$\therefore$ Equation of L_1 is y=x or 7y+x=0.

Equation 49: A straight line through A (–5, –4) meets the lines x + 3y + 2 = 0, 2x + y + 4 = 0 and x – y – 5 = 0 in points B, C and D respectively. If $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$ then find equation of line. **(I.I.T.)**

Sol. Equation of line through (–5, –4) is $\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = k$ where k is the distance of a point on it. Let it be B.

$\therefore$ [(AB cos θ – 5), (AB sin θ – 4)] is the point on it and on straight line x + 3y + 2 = 0

$\therefore$ AB cos θ – 5 + 3 AB sin θ – 12 + 2 = 0

$\therefore \frac{15}{AB} = \cos\theta + 3\sin\theta$ ... (1)

Similarly for straight line 2x + y + 4 = 0

2AC cos θ – 10 + AC sin θ – 4 + 4 = 0.

$\Rightarrow \frac{10}{AC} = 2\cos\theta + \sin\theta$ ... (2)

Similarly $\frac{6}{AD} = \cos\theta - \sin\theta$

Given condition $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$

$\Rightarrow (\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$

$\Rightarrow 5\cos^2\theta + 10\sin^2\theta + 10\sin\theta\cos\theta = \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta$

$\Rightarrow 5\sin^2\theta + 12\sin\theta\cos\theta + 4 = 0$

Dividing by $\cos^2\theta$

$5\tan^2\theta + 12\tan\theta + 4\sec^2\theta = 0$

$\Rightarrow 5\tan^2\theta + 12\tan\theta + 4(1 + \tan^2\theta) = 0$

$\Rightarrow 9\tan^2\theta + 12\tan\theta + 4 = (3\tan\theta + 2)^2 = 0$

$\therefore$ tan θ = $-\frac{2}{3}$, Straight line y + 4 = $-\frac{2}{3}$ (x + 5)

Example 50: Let A (2, 0), B $\left(1, \frac{1}{\sqrt{3}}\right)$ and origin be the vertices of a triangle. Let R be the region consisting of those points, P inside ΔOAB which satisfy d(P, OA) ≤ minor {d (P, OB), d(P, AB)} where d denotes the distance of point P to the corresponding line.

Sol. In figure 33, BD is perpendicular to x-axis,

$BD = \frac{1}{\sqrt{3}} \Rightarrow \angle BOD = \angle BAO = 30^o$

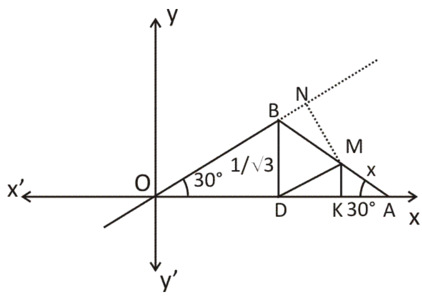

Fig. 33

$\therefore$ ΔOBA is isosceles Δ.

Let M be a point on BA, AM = x, BM = AB – x and $AB^2 = 1 + \frac{1}{3} = \frac{4}{3}$

$\therefore$ BM = $\frac{2}{\sqrt{3}} - x$

$\angle MBN = 60^o$ $\therefore$ MN = BM sin 60^o

$\Rightarrow MN = \left(\frac{2}{\sqrt{3}} - x\right) \cdot \frac{\sqrt{3}}{2} = 1 - \frac{\sqrt{3}}{2}x$

Condition to be satisfied.

(distance of P from OA) ≤ distance of P from OB – distance of P from AB

$\therefore$ MK = x sin 30^o = $\frac{x}{2} \le 1 - \frac{\sqrt{3}}{2}x$

from = sign, $\frac{x}{2} = 1 - \frac{\sqrt{3}}{2}x \Rightarrow x = (\sqrt{3} - 1)$

BD is perpendicular on AD. Point D also satisfies the condition.

Required area = ΔDMA = $\frac{1}{2}$ AD · MK = $\frac{1}{2} \cdot (\sqrt{3} - 1)\sin 30^o = \frac{\sqrt{3} - 1}{4}$ sq . unit

Example 51: A variable line of slope 4, intersects hyperbola xy = 1 at two points. Find locus of point which divides this line segment in the ratio of 1 : 2 **(Roorkee)**

Sol. Let straight line be y = mx + c ... (1)

It meets hyperbola $\therefore$ x(mx+c)=1

Let it meets in (x_1, y_1) and (x_2, y_2) $\Rightarrow mx^2 + cx - 1 = 0$

$\therefore\ x_1 + x_2 = -\dfrac{c}{m}$, $x_1 x_2 = -\dfrac{1}{m}$ ... (2)

Points also lie on straight line $y_1 = 4x_1 + c$, $y_2 = 4x_2 + c$

$\therefore\ y_1 + y_2 = 4(x_1 + x_2) + 2c$ and $y_1 - y_2 = m(x_1 - x_2)$

From (2) $x_1 - x_2 = \sqrt{\dfrac{c^2}{16}+1} = \dfrac{1}{4}\sqrt{c^2+16}$ ... (3)

From (2) and (4) $x_1 = \dfrac{1}{2}\left(\dfrac{\sqrt{c^2+16}-c}{4}\right)$ ... (4)

From (4) $2y_1 = c + 4\cdot\dfrac{1}{4}\sqrt{c^2+16} = c + \sqrt{c^2+16}$

Let (h, k) be the point which divides (x_1, y_1)

and (x_2, y_2) in the ratio of 1 : 2

$$\therefore\ h = \frac{x_2 + 2x_1}{3} = \frac{x_1 + x_2 + x_1}{3} = \frac{-\frac{c}{4} + \frac{1}{8}\left(\sqrt{c^2+16} - c\right)}{3} \quad \text{... } (\alpha)$$

$$\text{and } k = \frac{y_2 + 2y_1}{3} = \frac{y_1 + y_2 + y_1}{3} = \frac{c + \frac{1}{2}(c + \sqrt{c^2+16})}{3} \quad \text{... } (\beta)$$

From (α) $24h = -3c + \sqrt{c^2+16}$

From (β) $6k = 2c + c + \sqrt{c^2+16}$

$\therefore\ 24h - 6k = -6c \Rightarrow c = k - 4h$.

Putting this value of c in (β)

$6k - 3k + 12h = \sqrt{(k-4h)^2 + 16}$

$128h^2 + 8k^2 - 80kh - 16 = 0$

$16h^2 + k^2 - 10kh - 2 = 0$

Locus $16x^2 + y^2 - 10xy - 2 = 0$

Example 52: For the point A (x_1, y_1) and B (x_2, y_2) of the co-ordinate plane, a new distance d(AB) is defined by $d(AB) = |x_1 - x_2| + |y_1 - y_2|$. Let O (0, 0) and P (3, 2). Prove that set of points in the 1st quadrant which are equidistant to the new distance from O and P, consists of union of a line segment of finite length and an infinite ray – sketch the set in a labeled diagram. **(I.I.T. 2001)**

Sol. $d(AB) = |x_1 - x_2| + |y_1 - y_2|$

O is origin and P is (3, 2) and point Q moves such that d(OQ) = d (PQ) since (x, y) Q lies in 1st quadrant.

$|x - 0| = x$ and $|y - 0| = y$

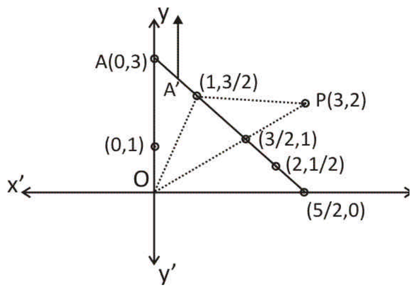

Fig. 33

(i) $x = 0$

$\therefore 0 + y = 3 + |y - 2|$

$\Rightarrow (y-3)^2 - (y-2)^2$

$\Rightarrow y = \frac{5}{2}$

i.e. point is $\left(0, \frac{5}{2}\right)$ but it does not satisfy condition.

(ii) $x = \frac{1}{2}, \frac{1}{2} + y = \frac{5}{2} + |y - 2|$

$\Rightarrow (y-2)^2 = (y-2)^2$ which means y can have any value when $x = \frac{1}{2}$ i.e. it is a ray parallel to y-axis.

(iii) $x = 1, y + 1 = 2 + |y - 2| \Rightarrow (y-1)^2 = (y-2)^2$

$\therefore y = \frac{3}{2}$ point is $\left(1, \frac{3}{2}\right)$

(iv) $x = \frac{3}{2}, y + \frac{3}{2} = \frac{3}{2} + |y - 2| \Rightarrow y^2 = y^2 - 4y + 4 \Rightarrow$ point $\left(\frac{3}{2}, 1\right)$

(v) $x = 2, 2 + y = 1 + |y - 2| \Rightarrow (y+1)^2$
$= (y-2)^2$

point is $\left(2, \frac{1}{2}\right)$

(vi) $x = \frac{5}{2}, \frac{5}{2} + y = \frac{1}{2} + |y - 2|$

$\Rightarrow (2 + y)^2 = (y-2)^2 \Rightarrow y = 0,$

point $\left(\frac{5}{2}, 0\right)$

(vii) $x = 3, (y+3)^2 = (y-2)^2 \Rightarrow y = -\frac{1}{2}$ Point not in 1st quadrant

In the figure A'B line segments of fixed length and at $x = \frac{1}{2}$. A ray from A' upwards parallel to y-axis.

$\therefore$ Graph consists of a line segment + ray.

Example 53: The base of a triangle passes through fixed point (a, b) and its sides are respectively bisected at right angles by straight lines $y^2 - 4xy - 5x^2 = 0$. Find locus of vertex.

Sol. Let vertex A be (h, k) of triangle ΔABC, B & C are (x_1, y_1) and (x_2, y_2), figure 34. (x_1, y_1), (a, b) and (x_2, y_2) are collinear

$$\therefore \begin{vmatrix} a & b & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$\Rightarrow a(y_1 - y_2) - b(x_1 - x_2) + (x_1y_2 - x_2y_1) = 0$ ... (α)

(ii) $y^2 - 4xy - 5x^2 = (y - 5x)(y + x) = 0$

Let $y - 5x = 0$ be $\perp$ to AB, $\therefore \frac{k - y_1}{h - x_1} = -\frac{1}{5}$

$\Rightarrow x_1 + 5y_1 = h + 5k$

Mid point of AB, $\left(\frac{h + x_1}{2}, \frac{k + y_1}{2}\right)$ lies on $y - 5x = 0$

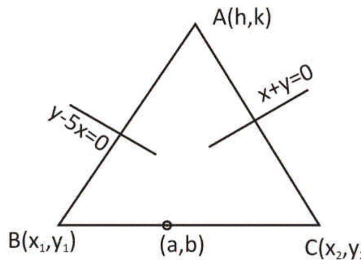

Fig. 34

$\therefore k + y_1 - 5h - 5x_1 = 0 \Rightarrow y_1 - 5x_1 = 5h - k$ ... (2)

Solving (1) and (2), $x_1 = \frac{5k - 12h}{13}, y_1 = \frac{5h + 12k}{13}$

(iii) $x + y = 0$ is $\perp$ AC $\Rightarrow \frac{k - y_2}{h - x_2} = 1 \Rightarrow k - h = y_2 - x_2$ ... (3)

Mid point $\left(\frac{h + x_2}{2}, \frac{k + y_2}{2}\right)$ lies on $x + y = 0$

$\therefore x_2 + h + y_2 + k = 0 \Rightarrow h + k = -(x_2 + y_2)$ ... (4)

Solving (3) and (4) $x_2 = -k$, $y_2 = -h$ ... (5)

(iv) Now $y_1 - y_2 = \frac{5h + 12k}{13} + h = \frac{18h + 12k}{13}$ ... (6)

and $x_1 - x_2 = \frac{5k - 12h}{13} + k = \frac{18k - 12h}{13}$ ... (7)

$$x_1y_2 - y_1x_2 = (-h)\left(\frac{5k-12h}{13}\right) + k\cdot\left(\frac{5h+12k}{13}\right)$$

$$= \frac{-5hk + 12h^2 + 5kh + 12k^2}{13}$$

$$= \frac{12}{13}(h^2 + k^2) \quad \text{... (8)}$$

Putting values from (6), (7), (8) in (α)

$$a\left(\frac{18h+12k}{13}\right) - b\left(\frac{18k-12h}{13}\right) + \frac{12}{13}(h^2+k^2) = 0$$

$\Rightarrow a(18x + 12y) - b(18y - 12x) + 12(x^2 + y^2) = 0$

Example 54: A variable line L passes through B (2, 5), and intersects the lines $2x^2 - 5xy + 2y^2 = 0$ in P and Q. Find locus of R on L, such that the distances BP, BR and BQ are in Harmonic. **(Roorkee)**

Sol. Any line through (2, 5) is L, $y - 5 = m(x - 2) \Rightarrow \frac{x-2}{\cos\theta} = \frac{y-5}{\sin\theta} = r$

Lines $2x^2 - 5xy + 2y^2 = 0$

$\Rightarrow (2x - y)(x - 2y) = 0$

Lines are $2x - y = 0$ and $x - 2y = 0$

Let distance of P from B (2, 5) be r_1

$\therefore$ P is $(r_1 \cos\theta + 2, r_1 \sin\theta + 5)$

It is on line $2x - y = 0$

$\Rightarrow 2r_1 \cos\theta + 4 - r_1 \sin\theta - 5 = 0$

$$\therefore\ r_1 = \frac{1}{2\cos\theta - \sin\theta} \quad \text{... (1)}$$

Let distance Q be r_2 and it is on line $x - 2y = 0$

$(r_2 \cos\theta + 2) - 2(r_2 \sin\theta + 5) = 0$

$$r_2 = \frac{8}{\cos\theta - 2\sin\theta} \quad \text{... (2)}$$

If distance of R from B (2, 5) is r,

then r_1, r, r_2 are in H.P.

$$\therefore\ r=\frac{2r_1r_2}{r_1+r_2}=\left[\frac{\dfrac{8\cdot 2\cdot 1}{(2\cos\theta-\sin\theta)(\cos\theta-2\sin\theta)}}{\dfrac{\cos\theta-2\sin\theta+16\cos\theta-8\sin\theta}{(2\cos\theta-\sin\theta)(\cos\theta-2\sin\theta)}}\right]=\frac{16}{17\cos\theta-10\sin\theta}$$

If (h,k) be the point R

$h = r\cos\theta + 2 \Rightarrow \quad h - 2 = r\cos\theta$

$k = r\sin\theta + 5 \Rightarrow \quad k - 5 = r\sin\theta$

$\therefore 16 = 17\,(r\cos\theta) - 10(r\sin\theta)$

$\Rightarrow 16 = 17\,(h - 2) - 10\,(k - 5)$

$\Rightarrow 17h - 34 - 10k + 50 = 16$

$\therefore$ so locus is $\quad 17x - 10y = 0$

Example 55: The co-ordinates of points at unit distance from straight lines $3x - 4y + 1 = 0$ and $8x + 6y + 1 = 0$ are ...

Sol. We know that perpendicular dropped from any point of the angle bisector of an angle on its two arms are equal. This mean in question points lie on both sides of the angle bisectors of the angle between thse two lines.

Now $\dfrac{3x-4y+1}{\pm 5}=1 \quad \Rightarrow 3x - 4y = 4,\ 3x - 4y = -6$

and $\dfrac{8x+6y+1}{\pm 10}=1 \Rightarrow 8x+6y=9, 8x+6y=-11$

Point 1 $\left.\begin{matrix}3x-4y=4\\8x+6y=9\end{matrix}\right\}$ point$\left(\dfrac{6}{5},-\dfrac{1}{10}\right)$

Point 2 $\left.\begin{matrix}3x-4y=4\\8x+6y=-11\end{matrix}\right\}$ point$\left(-\dfrac{2}{5},-\dfrac{13}{10}\right)$

Point 3 $\left.\begin{matrix}3x-4y=-6\\8x+6y=9\end{matrix}\right\}$ point$\left(0,\dfrac{3}{2}\right)$

Point 3 $\left.\begin{matrix}3x-4y=-6\\8x+6y=-11\end{matrix}\right\}$ point$\left(-\dfrac{8}{5},\dfrac{3}{10}\right)$

Example 56: A line is such that the sum of perpendiculars from a number of points on it is zero. Prove that the line always passes through a fixed point.

Sol. Let the straight line be $ax + by + c = 0$. ... (1)

Let (x_i, y_i) be points, i = 1, 2, 3, ..., n.

Sum of perpendicular from (x_i, y_i) on straight line

$$=\sum\frac{ax_i+by_i+c}{\sqrt{a^2+b^2}}=0 \Rightarrow =\sum(ax_i+by_i+c)=0$$

$$=a\sum x_i+b\sum y_i+cn=0 \qquad \ldots (2)$$

Multiplying (1) by n and subtracting (2) from it

$a(nx-\sum x_i)+b(ny-\sum y_i)=0$

$\Rightarrow (nx-\sum x_i)+\frac{b}{a}(ny-\sum y_i)=0;$

$\sum x_i$ and $\sum y_i$ are constant. Equation is $P+\lambda Q=0$

$\therefore$ Line passes through $x=\frac{\sum x_i}{n}$, $y=\frac{\sum y_i}{n}$; $\left(\frac{\sum x_i}{n}, \frac{\sum y_i}{n}\right)$

Example 57: The vertices of a triangle are A (x_1, y_2), B (x_2, y_2) and (x_3, y_3). Prove orthocentre is

$$\left[\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C}, \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right]$$

Sol. In figure O_1 is orthocentre. AD $\perp$ on BC and BE $\perp$ AC. From right angled triangle BAD and CAD

AD = BD tan B = DC tan C ... (i)

$\therefore \frac{BD}{DC}=\frac{\tan C}{\tan B}$ ... (ii)

$\therefore$ D divides BC in the ratio of tan C : tan B

$\therefore$ D is $\left[\frac{x_3\tan C+x_2\tan B}{\tan B+\tan C}, \frac{y_3\tan C+y_2\tan B}{\tan B+\tan C}\right]$

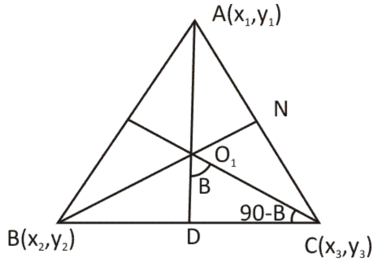

Fig. 35

(ii) From right angled triangle ODC

OD = DC · cot B = AD cot C · cot B ...from (i)

$$=\frac{AD}{\tan B\tan C}$$

$\therefore O_1A = AD - O_1D = AD\left[1-\frac{1}{\tan B\tan C}\right]$

$$=AD\left[\frac{\tan B\tan C-1}{\tan B\tan C}\right]$$

O_1 divides AD in ratio · (tan B tan C – 1) : 1 ... (2)

In Δ tan A + tan B + tan C = tan A · tan B · tan C

$$\Rightarrow 1+\frac{\tan B+\tan C}{\tan C}=\tan B\tan C \quad \text{... (3)}$$

from (3), (2) is $\frac{\tan B+\tan C}{\tan A}:1$

$\therefore$ ratio is tan B + tan C : tan A

$$\therefore \text{ abscissa of } O_1 \text{ is } \bar{x}=\frac{(\tan B+\tan C)\cdot\frac{x_3\tan C+x_2\tan B}{\tan B+\tan C}+x_1\tan A}{\tan A+\tan B+\tan C}$$

$$\therefore p_1 \text{ is } \left[\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C},\frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right]$$

Practice Worksheet (Foundation Level) – 2 (c)

1. (a) Find equation of straight line joining points P (2, 5) and Q (3, 4).
   (b) Find co-ordinates of point which divides its segment between axes in ratio 3 : 4
   (c) The straight line moves at P in anticlockwise direction through 45^o. Write its equation in new position.
   (d) Find co-ordinates of point on original line distance 6 units away from P.
   (e) Write equation of right bisector of PQ.
   (f) Find locus of point R which moves so that $\angle PRQ$ is always 45^o.
   (g) Find angle between PQ and straight line 5x – 12y = 13
   (h) Write equation of bisector of acute angle between PQ and 5x – 12y = 13.
   (i) Find equation of straight line parallel to 5x – 12y = 13 and a distance of 6 units (i) below it (ii) above it.
   (j) Straight line PQ, 5x – 12y = 1 and x = y form a triangle, find its centroid.
   (k) Find circumcentre of Δ in (j)
   (l) Find orthocentre of Δ in (j)
   (m) Find area of Δ in (j)
2. Find equation of straight line which go through (0, a) and are at distance a from point (2a, 2a). Find also equation of straight line joining the foot of perpendicular dropped from (2a, 2a) on them.
3. Find inclinations of two straight lines that go through (1, 2) and whose distance of points of intersection with x + y = 4 from (1, 2) is $\frac{1}{\sqrt{3}} \cdot 6$
4. A straight line moves in such a way that the sum of the reciprocal of its segments on axes is always same constant. Prove that the straight line goes through a stationary point.
5. Straight lines $p_1x + q_1y = 1$, $p_2x + q_2y = 1$ and $p_3x + q_3y = -1$ are concurrent, prove that points (p_1, q_2) (p_2, q_2) and (p_3, q_3) are collinear.
6. Find equations of straight lines that go through (4, 5) and make equal angles with straight lines 3x = 4y + 7 and 5y = 12x + 6.
7. Find equation of straight line that goes through point of intersection of straight lines 3x + 4y = 1 and x + 3y + 2 = 0 and makes equal angle with axes.
8. Find points on x-axis which are a distance away from straight line $\frac{x}{a} + \frac{y}{b} = 1$.

9. Find condition so that points (a, b), (c, d) and (a – c, b – d) be collinear.

10. Find a and b if equations 3x – 4y = 8 and 2ax + 3by + 12 = 0 represent the same straight line.

11. Straight lines 3x – 4y + 18 = 0 meets parallel lines 4x + 3y + 49 = 0 and 4x + 3y – 1 = 0 in B and A respectively. The angle bisectors of ∠A and ∠B meets in C and D. Quadrilateral ACBD is formed. Find (a) point of intersection of AB and CD, (b) angle between diagonals AB and CD, (c) area of quadrilateral ACBD.

12. A and B are (3, 5) and (11, 3) respectively. Find point P, so that orthocentre of ΔPAB lie on P area of triangle PAB is 17 units.

13. Two points P (a, 0) and Q (–a, 0) are in straight line R is a variable point such that (∠RPQ – ∠RQP) is always 2α. Prove locus of R is $x^2 + y^2 - 2ay \cot 2\alpha - a^2 = 0$.

14. A variable line is at a distance p from origin. It meets axes in A and B. Rectangle OAPB is completed. Prove that locus of P is $\frac{1}{x^2}+\frac{1}{y^2}-\frac{1}{p^2}$

15. Prove that four straight lines $\frac{x}{a}+\frac{y}{b}=1, \frac{x}{b}+\frac{y}{a}=1$, $\frac{x}{a}+\frac{y}{b}=2$ and $\frac{x}{b}+\frac{y}{a}=2$ enclose a rhombus whose area is $\frac{a^2b^2}{(a^2-b^2)}$

16. Straight lines y + 3x = 4, ay = x + 10 and 2y + bx = 9 are three sides of a rectangle taken in order. The fourth side passes through (1, –2). Find a, b and equation of fourth side.

17. Ends of a straight rod of length ℓ, moves on two mutually perpendicular lines in a vertical plane. Locus of point which divides the rod in the ratio of 2 : 3 is

(a) $\frac{x^2}{9}+\frac{y^2}{4}=\frac{\ell^2}{25}$ (b) $\frac{x^2}{4}+\frac{y^2}{9}=\frac{\ell^2}{25}$

(c) $\frac{x^2}{2}+\frac{y^2}{3}=\frac{\ell^2}{25}$ (d) $4x^2 + 9y^2 = \ell^2$

18. The vertices of a triangle are A (0, 0), B (1, 4) C (5, 1). Find equation of straight line perpendicular to BC and at a distance 1 unit from origin towards C.

(a) 3x – 4y = 5 (b) 4x – 3y + 5 = 0

(c) 6x – 8y = 9 (d) 8x – 6y = 9

19. Straight lines PA, PB and PC of slope $\frac{1}{2}$, 1 and 3 pass through (6, 6). Point Q is (1, 5). Straight lines through Q meets these lines in M, N and R respectively such that MN = NR, then equation of line through Q is

(a) $3x + 2y = 17$ (b) $x + 3y = 16$

(c) $4x + 3y = 19$ (d) $5y - x = 24$

20. A variable line through the point of intersection of $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ meets the axes in A and B. The locus of mid point of AB is

(a) $(x + y)ab = 2xy (a + b)=0$ (b) $(x + y)ab + 2xy (a + b) = 0$

(c) $2 (x + y)ab + xy (a + b)=0$ (d) $2(x + y)ab - xy (a + b) = 0$

21. The equation of right bisector of the line segment joining point of intersection of $x + 2 = 0$, $y - 2 = 0$, and $3x + 4y = 8$ and $2x - 9 = y$ is

(a) $2y - 2x + 1 = 0$ (b) $2y - 4x = 3$

(c) $2y - 4x + 3 = 0$ (d) $3x - 2y + 3 = 0$

22. Straight line x = a and y = b intersect in A and straight lines $\frac{x}{a}+\frac{y}{b}=5$ and $3bx - 2ay = 0$ intersect in B. Line through (–a, –b) meets AB in P, such that $\frac{AP}{PB}=\frac{3}{2}$. The equation of thin line is

(a) $bx - ay = 0$ (b) $by + ax = a^2 + b^2$

(c) $12ax - 13bx = 6ab$ (d) $13ay - 16bx = 3ab$

23. The vertex of a triangle is A(1, 6) and base lies along $2x + y = -2$. If length of median AD is 5. The C.G. of the triangle is

(a) $\left(-1,\frac{10}{3}\right)$ (b) (–1, 3) (c) $\left(-\frac{7}{3},6\right)$ (d) $\left(-\frac{5}{3},6\right)$

24. The sides of a triangle are $2x + y = 6$, $y = 0$ and $3x - 8y + 5 = 0$. find set for α such that point (α^2, α) lies inside the triangle.

(a) $\left[-2,\frac{5}{3}\right]$ (b) [1, 3] (c) $\left[1,\frac{3}{2}\right]$ (d) [–2, 3]

25. AD is perpendicular from A on base BC of ΔABC and AD = DC = 2BD. IF BDC and DA be taken as x-axis and y-axis then co-ordinates of A are (0, 6). The co-ordinate of point of intersection of AD and angle bisector of $\angle B$ are

(a) $[0,3(\sqrt{5}-1)]$ (b) $\left[0,\frac{3}{2}(\sqrt{5}-1)\right]$

(c) $\left[0,\frac{2}{3}(\sqrt{5}-1)\right]$ (d) $\left[0,\frac{\sqrt{3}}{2}(\sqrt{5}-1)\right]$

26. x-axis bisects the angle A of quadrilateral ABCD. While y-axis is along diagonal BD. Side DC is parallel to x-axis and equal to AD. If AO = 4, and $\angle ADO = 30^o$; then area of quadrilateral is

(a) $36\sqrt{3}$ sq. u. (b) $48\sqrt{3}$ sq. u.

(c) $24\sqrt{3}$ sq. u. (d) 60 sq. u.

27. In question 10, the equation of BC is

(a) $y-\sqrt{3}x-4\sqrt{3}=0$ (b) $y-\sqrt{3}x+4\sqrt{3}=0$

(c) $y+\sqrt{3}x-4\sqrt{3}=0$ (d) $y+\sqrt{3}x+4\sqrt{3}=0$

28. The co-ordinate of mid points of AB, BC and CA of triangle ABC are (–1, 2), (2, 1) and (4, 4) the orthocentre of triangle is:

(a) $\left(\frac{1}{11},\frac{65}{11}\right)$ (b) $\left(-\frac{1}{11},\frac{85}{11}\right)$ (c) $\left(-\frac{7}{13},10\right)$ (d) none of these

Practice Worksheet (Competition Level)

PART-A

1. A point moves such that the sum of its distances from two fixed points (ae, 0) and (–ae, 0) is always 2*a*. Prove that the equation of its locus is $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$.

2. The co-ordinates of *A*, *B*, *C* are (6, 3), (–3, 5) and (4, -2) respectively and *P* is any point(*x*, *y*). Show that the ratio of the areas of the triangles PBC and ABC is $\left|\frac{x+y-2}{7}\right|$.

3. The ends of a rod of length l move on two mutually perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1:2.

4. The straight line L is perpendicular to the line 5x – y = 1. The area of the triangle formed by the line L and co-ordinate axes is 5. Find the equation of the line.

5. A moving line is lx + my + n = 0, where l, m, n are connected by the relations al + bm + cn = 0, and a, b, c are constant. Show that the line passes through a fixed point.

6. If the straight line through the point P (3, 4) makes an angle $\pi/6$ with the x-axis and meets the line 12x + 5y + 10 = 0 at Q find the length of PQ.

7. (a) A ray of light is sent along the line x – 2y + 5 = 0 ; upon reaching the line 3x-2y+7=0, the ray is reflected from it. Find the equation of the line containing the reflected ray.

   (b) The ray of light is sent along the line x – 2y – 3 = 0. Upon reaching the line 3x-2y-5=0= 0, the ray is reflected from it. Find the equation of the line containing the reflected ray.

8. Two consecutive sides of a parallelogram are 4x + 5y = 0 and 7x + 2y = 0. IF the equation to one diagonal is 11x + 7y = 9, find the equation of the other diagonal.

9. Find the equations of lines which pass through the point of intersection of the lines 4x - 3y - 1 = 0, 2x - 5y + 3 = 0 and are equally inclined to the axes.

10. Find the locus of a point whose sum of the distances from the origin and the line x = 2 is 4 units. Sketch the path.

11. Find the equation of the straight line which bisects the obtuse angle between the lines x – 2y + 4 = 0 and 4x – 3y + 2 = 0.

12. Given vertices A (1, 1), B (4, –2) and C (5, 5) of a triangle, find the equation of the perpendicular dropped from C to the interior bisector of the angle A.

13. (a) The opposite angular points of a square are (3, 4) and (1, –1). Find the co-ordinates of the other two vertices.

(b) The extremities of the diagonal of a square are (1, 1), (–2, –1). Obtain the other two vertices and the equation of the other diagonal.

14. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line y = 2x + c. Find c and the remaining vertices.

15. One side of a rectangle lies along the line 4x + 7y + 5 = 0. Two of its vertices are (–3, 1) and (1, 1). Find the equations of the other three sides.

16. The sides AB, BC, CD, DA of a quadrilateral have the equations x + 2y = 3, x = 1, x – 3y = 4, 5x + y + 12 = 0 respectively. Find the angle between the diagonals AC and BD.

17. The vertex A of triangle ABC is given to be (1, 3) and the medians BE and CF are x – 2y + 1 = 0 and y – 1 = 0. Determine the equations of its sides.

18. In the triangle, the side AB is 5x – 3y + 2 = 0 and the altitude AD and BE are 4x – 3y + 1 = 0 and 7x + 2y – 22 = 0 respectively. Determine the equation of sides CA and CB and also the altitude CF through C.

19. In a triangle ABC, we are given the vertex A(4,-1) and x-1=0 and x–y–1=0and the bisectors of angles of B &C Respectively, find the co-ordinates of the vertices B and C and equations of the side AB and AC.

20. Straight line 3x + 4y = 5 and 4x – 3y = 15 intersect at point A. Point B and C are chosen on the two lines such that AB = AC. Determine the possible equations of the line BC through the point (1, 2).

21. The equations of a perpendicular bisectors of the sides AB and AC of triangle ABC are x – y + 5 = 0 andx + 2y = 0 respectively. If the point A is (1, –2) find the equation of the line BC.

22. The equation for the base of an equilateral triangle is x + y = 2 and the vertex is (2, –1). Find the length and the equations of the sides of the triangle.

23. The vertices B, C of a triangle ABC lie on the lines 4y = 3x and y = 0 respectively and the side BC passes through the point P (0, 5). If ABOC is a rhombus where O is the origin and the point P is inside the rhombus, then find the co-ordinates of A.

24. One diagonal of the square is the portion of the line 7x +5y=35 intercepted between the axes obtain the extremities of the other diagonal.

25. A variable straight line is drawn through o to cut two fixed straight lines L_1 and L_2 in A_1 and A_2. A point A is taken on the variable line such that $\frac{m+n}{OA} = \frac{m}{OA_1} + \frac{n}{OA_2}$.

show that locus of P is straight line passing through the point of intersection of L_1 and L_2.

PART-B

1. The vertices of a triangle are A (p, p tan α), B (q, q tan β), C (r, r tan γ). If circumcentre O of triangle ABC is at the origin and H (u, v) be its ortho-centre, then show that

$$\frac{u}{v} = \frac{\cos\alpha + \cos\beta + \cos\gamma}{\sin\alpha + \sin\beta + \sin\gamma}$$

2. Derive the conditions to be imposed on β so that (0, β) should be on or inside the triangle having sides y + 3x + 2 = 0, 3y – 2x – 5 = 0 and 4y + x –14 = 0.

3. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines. 2x + 3y – 1 = 0, x +2y – 3 = 0, 5x – 6y – 1= 0. **[I.I.T-1992, 6]**

4. Let ABC be a triangle with AB = AC. If D is the mid point of BC, E the foot of the perpendicular drawn from D to AC and F the mid point of DE, prove that AF is perpendicular to BE.

5. A line through A (–5, –4) meets the lines x + 3y + 2 = 0, 2x + y + 4 = 0 and x – y – 5 = 0 at the points B, C and D respectively. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$, find the equation of the line. **[I.I.T-1993]**

6. A line is such that its segments between the straight lines 5x – y – 4 = 0 and 3x + 4y – 4 = 0 is bisected at point (1, 5). Obtain its equation.

7. A pair of straight line drawn through the origin form with the line 2x + 3y = 6 an isosceles triangle right angled at the origin. Find the equation of the pair of straight lines and the area of the triangle correct to two places of decimals.

8. The ends A, B of a straight line segment of constant length c slide upon the fixed rectangular axes OX, OY respectively. If the rectangle OAPB be completed, then show that the locus of the foot of the perpendicular drawn from P to AB is $x^{2/3} + y^{2/3} = c^{2/3}$.

9. A rectangle PQRS has its side PQ parallel to a line y = mx and vertices P, Q and S on the lines y = a, x = b and x = –b respectively. Find the locus of vertex R.

10. The line lx + my + n = 0 bisects the angle between a pair of straight lines of which one is px + qy + r = 0. Show that the equation to the other line is $(px + qy + r)(l^2 + m^2) - 2(lp + mq)(lx + my + n) = 0$.

11. A variable straight line passes through the point of intersection of the lines x + 2y = 1 and 2x – y = 1 and meets the co-ordinate axes in A and B. Find the locus of the middle point of AB.

12. A straight line L through the origin meets the lines x +y =1 and x +y =3 at P and Q respectively. Through P and Q two straight lines L_1 and L_2 are drawn parallel to 2x–y = 5 and 3x +y = 5 respectively. Lines $L_1$and L_2 intersect at R, Show that the locus of R, as L varies is a straight line. **[I.I.T-2002, 5]**

13. Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at the point P and make an angle θ with each other. Find the equation of a line L different from L_2 which passes through P and makes the same angle θ with L_1.

14. Equations of the diagonals of a rectangle are y +8x–17=0 and y–8x +7=0 If the area of the rectangle is 8 sq. units Find the equation of the sides of the rectangle.

15. A straight line L with negative slope passes through the point (8, 2) and cuts the positive co-ordinate axes at point P and Q . Find the absolute minimum value of OP +OQ , as L varies, where O is the origin. **[I.I.T-2002, 5]**

16. An isosceles right angled triangle whose sides are 1, 1, v2 lies entirely in first quadrant with ends of hypotenuse on the coordinate axes, if it slides, find the locus of its centriod.

17. A triangle has two lines y = mx and y = m_1x as two of its sides where m and m_1 are the roots of ax^2 +bx +c=0. If H(c, a) is the Ortho-centre of the triangle. Then show that the equation of the third side of the triangle is (a + c)(cx +ay) = ac(a +c–b).

18. Find the equation of the line passing through the point (2, 3) and making intercept of
length 2 units between the lines y +2x =3 and y +2x =5. **[I.I.T-1991, 5]**

19. If line AB of length 2l moves with the end A always on x-axis and end B always on the line y = 6x .Find the equation of locus of mid point of AB.

20. A line cuts the axis at A (7, 0) and the y-axis at B (0, -5). A variable line PQ is drawn perpendicular to AB cutting the x-axis in P and y-axis in Q. If AQ and BP intersect at R, find the locus of R. **[I.I.T-1990, 5]**

21. In a Δ ABC, A $\equiv$ (α, β), B $\equiv$ (1, 2), C $\equiv$ (2, 3) and point A lies on the line y = 2x + 3 where $\alpha, \beta \in$ Integer and area of the triangle is S such that [S] = 2 where [.] denotes greatest integer function. Find all possible co-ordinates of A.

COMPREHENCIVE PASSAGE TYPE PROBLEMS

22. A triangle ABC is given where the vertex A is (1,1) and the orthocentre is (2, 4) . Also sides AB and BC are members of the family of the lines ax +by +c=0 where a, b, c are in A.P.

i) The vertex B is

a) (2, 1) b) (1, –2) c) (–1, 2) d) none of these

ii) The vertex C is

a) (4, 16) b) (17,-4) c) (4, –17) d) (-17, 4)

iii) The triangle ABC is a

a) Obtuse angled triangle b) Right angled triangle

c) Acute angled triangle d) equilateral triangle

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23. **A**(0, 3) , B(-2,0) and C (6, 1) be the vertices of the triangle and **M(**β, β+1) be the moving point

---

i) M lies on the curve

a) y = x +1 b) $y = x^2$ c) x = y +1 d) none of these

ii) If M and A lie on the same side of BC then

a) β>2 b) β<2 c) $\beta > -\frac{6}{7}$ d) $\beta < \frac{3}{4}$

iii) If M lies within ΔABC if

a) $-\frac{6}{7} < \beta < 4$ b) $-4 < \beta < -\frac{6}{7}$ c) $-\frac{6}{7} < \beta < \frac{3}{2}$ d) none of these

Assignments (Objective problems)

Level-1

1. The in-centre of the triangle with vertices (1, √3), (0, 0), (2, 0) is

a) $\left(0, \frac{\sqrt{3}}{2}\right)$ b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$ c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ d) $\left(1, \frac{1}{\sqrt{3}}\right)$

2. Let A (2, –3) and B (–2, 1) be the vertices of the triangle ABC. If the centroid of this triangle moves on the line 2x+3y =1, then locus of the vertex C is the line

a) 2x +3y =9 b) 2x–3y =7 c) 3x +2y =5 d) 3x–2y =3

3. If the non-zero numbers a, b, c are in H.P. then the straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ always passes through affixed point is

a) (1,-2) b) (1, –1/2) c) (–1, 2) d) (–1, –2)

4. The line parallel to x-axis and passing through the intersection of the line ax+2by+ 3b=0 and bx–2ay-3a=0 where (a, b) ≠(0, 0) is

a) Above the x-axis at a distance of 3/2 from it

b) above the x-axis at a distance of 2/3 from it

c) Below the x-axis at a distance of 3/2 from it

d) below the x-axis at a distance of 2/3 from it

5. A straight line through the point (2, 2) intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. The equation to the line AB so that the triangle OAB is equilateral is

a) $x - 2 = 0$ b) $y - 2 = 0$ c) $x + y - 4 = 0$ d)none of these

6. The in-centre of the triangle formed by the lines x = 0 y = 0 and 3x +4y =12 is at

a) (1/2, 1/2) b) (1, 1) c) (1, ½) d) (½, 1)

7. If (a, a^2) falls inside the angle made by the lines $y = \frac{x}{2}$, x>0 and y=3x, x>0, then a belongs to

a) $\left(\frac{1}{2}, 3\right)$ b) $\left(-3, -\frac{1}{2}\right)$ c) $\left(0, \frac{1}{2}\right)$ d) $(3, \infty)$

8. Ortho-centre of the triangle whose vertices are (0, 0), (3, 4) (4,0) is **[I.I.T Sc-2003]**

a) $\left(3, \frac{7}{3}\right)$ b) $\left(3, \frac{5}{4}\right)$ c) (5,-2) d) $\left(3, \frac{3}{4}\right)$

9. The points $\left(0, \frac{8}{3}\right)$, (1, 3) and (82, 30) are vertices of

a) An acute angled triangle b) an isosceles triangle

c) An obtuse angled triangle d) none of these

10. Circum centre of the triangle whose vertices are (2,-1) (3, 2) and (0, 3) is

a) (–1, 1) b) (–1, 1) c) (1, 1) d) none of these

11. The locus of the mid point of the portion intercepted between the axes by the line $x\cos\alpha + y\sin\alpha = p$, where p is constant, is

a) $x^2+y^2=p^2$ b) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$

c) $x^2+y^2=\frac{4}{p^2}$ d) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{2}{p^2}$

12. The equation of the line passing through the intersection of the lines $x-3y+1=0$ and $2x+5y-9=0$ and at distance $\sqrt{5}$ from the origin is

a) $2x-y=5$ b) $x+2y=5$ c) $2x+y=5$ d) $x+2y=1$

13. Locus of the centroid of the triangle whose vertices are (acost, asint), (bsint, –bcost) and (1, 0) where t is the parameter is

a) $(3x-1)^2+(3y)^2=a^2+b^2$ b) $(3x+1)^2+(3y)^2=a^2+b^2$

c) $(3x+1)^2+(3y)^2=a^2-b^2$ d) $(3x-1)^2+(3y)^2=a^2-b^2$

14. Three straight lines $2x+11y-5=0$, $4x-3y-2=0$ and $24x+7y-20=0$

a) form a triangle b) are only concurrent

c) are concurrent with one line bisecting the angle between the other two

d) none of these

15. If the lines $x+ay+a=0$, $bx+y+b=0$ and $cx+cy+1=0$ (a, b, c being distinct$\neq 1$) are concurrent, then the value of $\frac{a}{a-1}+\frac{b}{b-1}+\frac{c}{c-1}$ is

a) -1 b) 0 c) 1 d) none of these

16. Let P (–1, 0) , Q (0,0) and R(3,3v3) be three point . Then the equation of bisector of angle PQR is **[I.I.T Sc-2002]**

a) $\frac{\sqrt{3}}{2}x+y=0$ b) $x+\sqrt{3}y=0$

c) $\sqrt{3}x+y=0$ d) $x+\frac{\sqrt{3}}{2}y=0$

17. Let PS be the median of the triangle with vertices P(2,2), Q(6,-1) and R(7,3) . The equation of the line passing through (1,-1) and parallel to PS is **[I.I.T Sc-2000]**

a) $2x-9y-7=0$ b) $2x-9y-11=0$

c) $2x+9y-11=0$ d) $2x+9y+7=0$

18. The number of integer values of m, for which the x coordinate of the point of intersection of the lines $3x+4y=9$ and $y=mx+1$ is also a integer, is **[I.I.T Sc-2001]**

a) 2 b) 0 c) 4 d) 1

19. The ratio in which the line $3x-2y+5=0$ divides the join of (6,-7) and (-2, 3) is

a) 1:1 b) 7:37 c) 37:7 d) none of these

20. The vertices of a triangle are A (–1, –7), B (5, 1), and C(1,4) .The equation of the bisector of angle∠ABC is

a) x–7y + 2 =0 b) x–7y + 6 =0

c) 7x– y + 4 = 0 d) none of these

21. The co-ordinates of those points on the line 3x + 2y = 5 which are equidistant from the lines 4x + 3y - 7 = 0 and 2y - 5 = 0 are

a) $\left(-\frac{1}{14},\frac{73}{28}\right)$ b) $\left(\frac{11}{16},\frac{47}{32}\right)$

c) $\left(\frac{1}{14},\frac{67}{28}\right)$ d) $\left(\frac{1}{16},\frac{77}{32}\right)$

22. If (-6, -4) and (3, 5) are the extremities of the diagonal of a parallelogram and (-2, 1) is its third vertex, then its fourth vertex is

a) (-1, 0) b) (0, –1) c) (–1, 1) d) none of these.

23. The coordinates of the middle points of the sides of a triangle are (3, 2) (4, 3) and (2, 2) then the coordinates of its centriod are

a) $\left(3,\frac{7}{3}\right)$ b) (3, 3) c) (4, 3) d) none of these.

24. If A (cosα, sin α), B(sinα, -cosα), C(2, 1) are the vertices of a ΔABC, then as α varies the locus of its centriod is

a) $x^2 + y^2 - 2x - 4y + 1 = 0$ b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$

c) $x^2 + y^2 - 2x - 4y + 3 = 0$ d) none of these.

25. The nearest point on the line 3x + 4y = 25 from the origin is

a) (-4, 5) b) (3, -4) c) (3, 4) d) (3, 5)

Level-2

1. Let ax + by +c=0 be a variable straight line, where a, b, c are the 1st , 3rd and 7th term of some increasing A.P. then the variable straight line always passes through the point which lies on

a) $x^2 + y^2 = 13$ b) $x^2 + y^2 = 5$

c) $y^2 = \frac{9}{2}x$ d) 3x + 4y = 9

2. Line L has intercepts a and b on the coordinate axes, when the axes are rotated through a given angle keeping the origin fixed , the same line has intercepts p and q . then

a) $a^2 + b^2 = p^2 + q^2$ b) $1/a^2 + 1/b^2 = 1/p^2 + 1/q^2$

c) $a^2 + p^2 = b^2 + q^2$ d) $1/a^2 + 1/p^2 = 1/b^2 + 1/q^2$

3. The area of the triangle is 5. Two of its vertices are (2, 1) and (3,–2), the third vertex lying on y = x +3 . The coordinate of the third vertex can be

a) (-3/2, 3/2) b) (3/2, –3/2)

c) (7/2, 13/2) d) (–1/4, 11/4)

4. The equation of a straight line passing through the point (4, 5) and equally inclined to the lines 3x = 4y +7 and 5y =12x + 6 is

a) 9x-7y =1 b) 9x+7y =71

c) 7x+9y =73 d) 7x-9y+17=0

5. The diagonals of a parallelogram PQRS are along the lines x +3y = 4 and 6x–2y =7. the PQRS must be

a) Rectangle b) square c) cyclic quadrilateral d) rhombus

6. If the one vertex of an equilateral triangle of side a lies at the origin and the other lies on the line x-√3y = 0 the co-ordinate of the third vertex are

a) (0, a) b) $\left(\frac{\sqrt{3}a}{2},-\frac{a}{2}\right)$ c) (0,–a) d) $\left(-\frac{\sqrt{3}a}{2},\frac{a}{2}\right)$

7. The point (4, 1) undergoes the following three transformations successively:

I. Reflection about the line y = x

II. Translation through a distance 2 unit along the positive direction of x-axis

III. Rotation through an angle of π/4 about the origin in the anti-clockwise direction. The final position of the point is given by the co-ordinates

a) (1/√2, 7/√2) b) (–√2, 7√2)

c) (–1/√2,7/√2) d) (√2,7√2)

8. Three lines px + qy +r = 0, qx + ry + p = 0 and rx +py + q=0 are concurrent if

a) p + q + r =0 b) $p^2 + q^2 + r^2$= pq + qr + rp

c) p^3+ q^3+ r^3=3pqr d) none of these

9. The ortho-centre of the triangle formed by the lines x + y =1, 2x +3y = 6, and 4x–y +9=0 lies in the quadrant number

a) I b) II c) III d) none of these

10. ABC is an equilateral triangle. If the coordinates of the base are B(1,3) and C(–2,7), the coordinate of vertex A can

a) $\left(2\sqrt{3}-\frac{1}{2},\frac{3\sqrt{3}}{2}+5\right)$ b) $\left(-2\sqrt{3}-\frac{1}{2},-\frac{3\sqrt{3}}{2}+5\right)$

c) $\left(-2\sqrt{3}+\frac{1}{2},-\frac{3\sqrt{3}}{2}-5\right)$ d) $\left(2\sqrt{3}+\frac{1}{2},\frac{3\sqrt{3}}{2}-5\right)$

11. All points lying inside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy

a) $3x+2y \geq 0$ b) $2x+y-13 \geq 0$

c) $2x-3y-12 \leq 0$ d) $-2x + y \geq 0$

12. A straight line through the origin O meets the parallel lines 4x+2y=9 and 2x+y +6=0 at appoints P and Q respectively. Then the point O divides the segment PQ in the ratio

a) 1:2 b) 3:4 c) 2:1 d) 4:3 **[I.I.T Sc-2002]**

13. Triangle is formed by the coordinates (0, 0) (0, 21) and (21, 0). Find the number of integral coordinates strictly inside the triangle (integral coordinates of both x and y)

a) 190 b) 105 c) 231 d) 205 **[I.I.T Sc-2003]**

14. Let PQR be the aright angled isosceles triangle, right angled at P (2, 1). If the equation of the line QR is 2x+ y=3, then the equation representing the pair of lines PQ and PR is

a) $3x^2-3y^2+8xy+20x+10y+25=0$ b) $3x^2-3y^2+8xy-20x-10y+25=0$

c) $3x^2-3y^2+8xy+10x+15y+20=0$ d) $3x^2-3y^2-8xy-10x-15y-20=0$

15. If the vertices P, Q, R of a triangle PQR are rational points, which of the following points of triangle PQR is (are) always rational point(s)?

a) centroid b) in-centre c) circum-centre d) orthocentre

16. If x_1, x_2, x_3 as well as y_1,y_2, y_3 are in G.P. with the same common ratio, then the points (x_1,y_1), (x_2,y_2) (x_3,y_3)

a) lie on a straight line b) lie on the ellipse

c) lie on the circle d) are the vertices of triangle

17. Area of the parallelogram formed by the lines y = mx , y = mx +1, y = nx and y = nx+1 equals **[I.I.T Sc-2001]**

a) $\frac{|m+n|}{(m-n)^2}$ b) $\frac{2}{|m+n|}$ c) $\frac{1}{|m+n|}$ d) $\frac{1}{|m-n|}$

18. If $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{c}+\frac{y}{d}=1$ intersect the axes at four con-cyclic points and a +b= c +d then these lines always intersect on the curve (a, b, c, d>0)

a) y = x b) y = –x c) y = ±x d) none of these

19. Equation of circle touching the lines $|x-2|+|y-3|=4$ will be

a) $(x-2)^2+(y-3)^2=12$ b) $(x-2)^2+(y-3)^2=4$

c) $(x-2)^2+(y-3)^2=16$ d) $(x-2)^2+(y-3)^2=8$

20. The circum centre of the triangle formed by the lines xy +2x +2y +4=0 and x + y +2=0 is

a) (0, 0) b) (-2,-2) c) (-1,-1) d) (-1,-2)

21. The point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ lies on the line

a) x - y = 0 b) (x + y) (a + b) = 2ab

c) (lx + my) (a + b) = (/ + m)ab d) (lx - my) (a - b) = (1 - m)ab

22. The ends of the base of an isosceles triangle are at (2a, 0) and (0, 2a). The equation of one of its sides is x = 2a. The equation of the other side is

a) x + 2y - a = 0 b) x + 2y = 2a

c) 3x + 4y - 4a = 0 d) none of these

23. The equations of two sides of a square whose area is 25 square units are 3x - 4y = 0 and 4x + 3y = 0. The equations of the other two sides of the square are

a) 3x - 4y ± 25 = 0 and 4x + 3y ± 25 = 0 b) 3x - 4y ± 5 = 0 and 4x + 3y ± 5 = 0

c) 3x - 4y ± 5 = 0 and 4x + 3y ± 25 = 0 d) none of these.

24. If u = $a_1x + b_1y + c_1 = 0$ and v = $a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, then u + kv = 0 represents

a) a family of concurrent lines b) a family of parallel lines

c) u = 0 d) v = 0

25. Points on the line x + y = 4 that lie at a unit distance from the line 4x + 3y - 10 = 0 are

a) (3, 1) and (-7, 11) b) (-3, 7) and (2, 2)

c) (-3, 7) and (-7, 11) d) none of these.

✣ -------------------- ✣

3. Change of Axes

3.1 Transfer of axes :

To solve a question some times we need to transfer axes, to some point and to rotate the axes, keeping origin same through an angle in the anticlock direction. Some times we need both types of transfer of axes.

3.2 Changing origin to some desired point, keeping axes parallel to original axes.

Let the desired point be O' (h, k).

P is a point (x,y) original axes and (X, Y) changed axes i.e. when origin is transferred to (h,k)

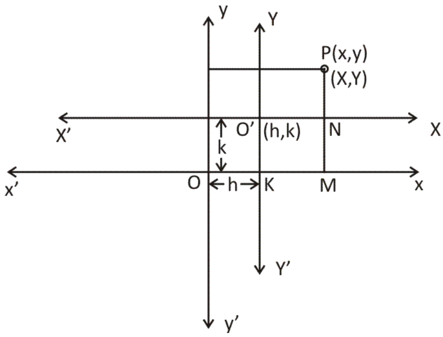

Fig 1

O'N = X, PN = Y

OM = x, PM = y.

$\therefore$ x = OK + KM = OK + O'N = h + X

$\Rightarrow$ x = h + X

$\Rightarrow$ X = x – h ....(α)

PM= PN + NM = Y + k

$\Rightarrow$ y = Y + k $\Rightarrow$ Y = y – k ....(β)

$\therefore$ When only origin is changed to any point (h, k) then the point (x, y) with reference to original axis become (x – h, y – k) with reference to changed axes.

(a) If the locus of a point with reference to original axes is f(x, y) = 0 then when origin is changed to (h, k) it become f(X + h, Y +k) = 0 with reference to changed axes.

(b) The intercepts of straight lines $\frac{x}{a}+\frac{y}{b}=1$ on axes are a and b. After changing origin to (h, k) the equation of straight line becomes. $\frac{X+h}{a}+\frac{Y+k}{b}=1 \Rightarrow \frac{X}{a}+\frac{Y}{b}=1-\frac{h}{a}-\frac{k}{b}=\frac{ab-bh-ak}{ab}$. Intercepts on changed axes are

$$\frac{ab-bh-ak}{b},\frac{ab-bh-ak}{a}$$

It can be proved also intercepts of the line AB on changed axes. (origin O') are O'M and O'N
O'M = LT = OA – OL – TA = a – h – l

Δ s, MTA and BOA are similar TM = O'L = k

$$\therefore \frac{TA}{MT}=\frac{OA}{OB}$$

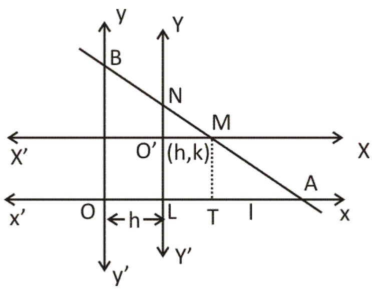

Fig 2

$$\frac{l}{k}=\frac{a}{b}\Rightarrow l=\frac{ak}{b}$$

$$\therefore O'M=a-h-\frac{ak}{b}=\frac{ab-bk-ak}{b}$$

In the similar manner we can prove NO' = $\frac{ab'-hb-ak}{a}$

(c) To find the equation of the tangent to circle $x^2 + y^2 - 6x + 8y - 24 = 0$ which is inclined at 60^0 with x-axis.

The circle is $(x - 3)^2 + (y + 4)^2 = 7^2$

Now changing origin to (3, –4)

The equation of circle with reference to changed origin is $X^2 + Y^2 = 7^2$

Any tangent to it is $Y=mX\pm 7\sqrt{1+m^2}$

Tangent is inclined at 60^0 to the x-axis

Equation of tangent with reference to original axes is $y+4=\sqrt{3}(x-3)\pm 14$

Solved Examples

Example 1: The origin is changed from (0,0) to (1, –1) write the equation $3x^2 - 2xy - 5y^2 + 7x - 9y + 2 = 0$ with reference to changed axes.

Sol. : Now x = X + 1, y = Y –1

$\therefore$ New equation is

$$3(X+1)^2 - 2(X+1)(Y-1) - 5(Y-1)^2 + 7(X+1) - 9(Y-1) + 2 = 0$$

$$\Rightarrow 3X^2 - 2XY - 5Y^2 + 15X - Y + 18 = 0$$

Example 2 : To what point should origin be changed so that the first degree terms of equation $x^2 + 3xy + 4y^2 - 4x - 6y + 5 = 0$. **Vanish**

Sol. Let origin be changed to (h, k) Then equation with reference to new axes shall be

$$(X+h)^2 + 3(X+h)(Y+k) + 4(Y+k)^2 - 4(X+h) - 6(Y+k) + 5 = 0$$

Co-efficient of X is 2h + 3k – 4

Coefficient of Y is 8k + 3 h –6

Putting these equal to O and then solving the two equation h = 2, k = 0

$\therefore$ origin should be transferred to (2, 0)

New equation shall be $X^2 + 3XY + 4Y^2 + 1 = 0$

3.3 Origin kept at (0, 0) and axes rotated through an angle θ

In fig(3), new axes are OX and OY. Point P is (x, y) and with reference to new axes (X, Y)

$\therefore$ OM = X and perpendicular from P on OX, PM = Y

Drawn MR perpendicular on PQ, which is perpendicular on OX.

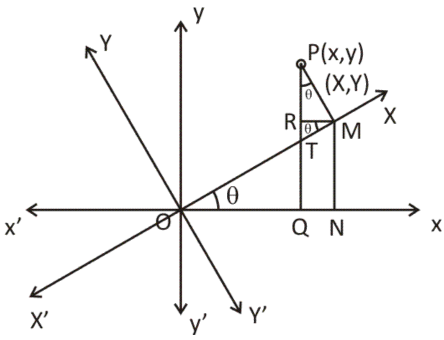

Fig 3

$\angle MPR = 90 - \angle PMR$
and RMO = alternate $\angle MOQ = \theta$

$\therefore \angle MPR = 90 - (90 - \theta) = \theta$

$\therefore RM = PM \sin\theta = Y \sin\theta$, and $PR = Y \cos\theta$

$\therefore x = OQ = ON - QN = OM \cos\theta - RM$

$= OM \cos\theta - Y \sin\theta = X \cos\theta - Y \sin\theta$ .... (1)

$y = PQ = PR + RQ = PR + MN$

$y = Y\cos\theta + X \sin\theta$ .... (2)

$\therefore$ If we put in the given equation $x = X \cos\theta - Y \sin\theta$ and $y = Y \cos\theta + X \sin\theta$ we shall get the transformed equation in X and Y i.e. with referenced to changed axes

(b) from (1) and (2) we get

$X = x \cos\theta + y \sin\theta$ ....(3)

$Y = y \cos\theta - x \sin\theta$ ....(4)

$\therefore$ If the equation is given in X and Y i.e. with reference to changed axes then with the help of (3) and (4) we can transfer it with reference to original axes in x and y.

Solved Examples

Example 3 : If axes are rotated through 45^0 in the anticlockwise direction, then the equation of straight line x – y = 0 and x + y = 0 are transferred to Y = 0, X = 0. Prove.

Sol. (i) $(X\cos 45^0 - Y\sin 45^0) - (Y\cos 45^0 + X\sin 45^0) = 0$,

is transferred equation of

$x - y = 0$ i.e. $-2Y\sin 45 = 0 \Rightarrow Y = 0$

(ii) $(X\cos 45^0 - Y\sin 45^0) + (Y\cos 45^0 + X\sin 45^0) = 0$

is transferred equation of x + y = 0

i.e. $X\frac{1}{\sqrt{2}} + X\frac{1}{\sqrt{2}} = 0 \Rightarrow x = 0$

Example 4 : Axes are rotated through 30^0 in the anticlockwise direction. Find the transformed equation of $y = \sqrt{3}x + 6$

Sol. : Put $x = X\cos 30^0 - Y\sin 30^0 = \frac{\sqrt{3}X}{2} - \frac{1}{2}Y$ and $y = Y\cos 30^0 + X\sin 30^0 = \frac{\sqrt{3}}{2}Y + \frac{1}{2}X$

$\therefore$ Transformed equation is $\frac{\sqrt{3}}{2}Y + \frac{1}{2}X = \sqrt{3}\left(\frac{\sqrt{3}X}{2} - \frac{1}{2}Y\right) + 6$

$$\Rightarrow \sqrt{3}Y + X = 3X - \sqrt{3}Y + 12$$
$$\Rightarrow 2\sqrt{3}Y = 2X + 12$$
$$\Rightarrow Y = \frac{1}{\sqrt{3}}X + 2\sqrt{3}$$

Example 5 : When axes have been rotated through 15^0 in the anticlockwise direction the transformed equation is 3x + 4y = 12 what was the original equation?

Sol. : Transformed equation is 3X + 4Y = 12

Now putting $X = x\cos 15^0 + y\sin 15^0$

And $Y = y\cos 15^0 - x\sin 15^0$

In the given equation we shall get the

Original equation. Therefore original equation is

$$3(x\cos 15^0 + y\sin 15^0) + 4(y\cos 15^0 - x\sin 15^0) = 12$$
$$x[3\cos 15^0 - 4\sin 15^0] + y[3\sin 15^0 + 4\cos 15^0] = 12$$
$$\Rightarrow x[3(\sqrt{3}+1) - 4(\sqrt{3}-1)] + y[3(\sqrt{3}-1) + 4(\sqrt{3}+1)] = 24\sqrt{2}$$
$$\Rightarrow (7 - \sqrt{3})x + (7\sqrt{3} + 1)y = 24\sqrt{2}$$

Example 6 : The line L has intercepts a and b on axes. Keeping origin fixed, axes are rotated through an angle α then p and q are new intercepts on new axes. Find relation between a, b, p and q.

Sol. : Original equation of line is $\frac{x}{a}+\frac{y}{b}=1$ when axes rotated through an angle a .

Equation becomes $\frac{X\cos\alpha - Y\sin\alpha}{a}+\frac{Y\cos\alpha + X\sin\alpha}{b}=1$

$$\Rightarrow X\left(\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}\right)+Y\left(\frac{\cos\alpha}{b}-\frac{\sin\alpha}{a}\right)=1$$

$$\therefore \frac{1}{p}=\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}$$

and $\frac{1}{q}=\frac{\cos\alpha}{b}-\frac{\sin\alpha}{a}$

Squaring and adding $\frac{1}{p^2}+\frac{1}{q^2}=\frac{\cos^2\alpha+\sin^2\alpha}{a^2}+\frac{\sin^2\alpha+\cos^2\alpha}{b^2}$

$$\Rightarrow \frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

Practice Worksheet (Foundation Level) – 3

1. The co-ordiantes of P, Q, R and S are (8, 9), (–15, 7), (–3, –4) and (2, –3). Now origins is transferred to (5, 5). Find new co-ordinates of these points.

2. On transferred origin at (2, –2) the equations of some lines are x + 3= 0, x + y=5, $x^2 + 3xy + 2y^2 = 0$. What were their original equation.

3. The origin is transferred to (1, 1) keeping direction of axes the same. The equation 7x + 3y = 20 then becomes.

   (a) 7x + 3y = 20 (b) 7x + 3y = 10

   (c) 7x + 3y = 0 (d) 7x + 3y + 20 = 0

4. The equation $3x^2 + 2xy + y^2 + 14x + 6y + 19 = 0$ loses its first degree terms when origin in transferred to a certain point. The point is

   (a) (–2, –1) (b) (–2, 1) (c) (2, –1) (d) (2, 1)

5. The axes have been transferred to origin at (–3, 2) and now the co-ordinates of three points are (5, 2), (–3, 4), and (3, –2). Find original co-ordinates of these points.

6. The equation of a circles $x^2 + y^2 - 2x - 4y - 4 = 0$ Now axes are rotated through 30^0 in anticlockwise direction. Find the new equation.

7. Keeping origin fixed axes are rotated through 60^0 in anticlockwise. Find the intercepts of line x – y = 1 on new axes.

8. Axes are rotated through 30^0, keeping origin fixed. The equation of a straight line on changed axes is 4x + 2y = 5. Its original equation ....

9. The vertices of a squares are A(3, 2), B(3, –2) C(7, –2) and D(7, 2). Origin now is transferred to (–2,2). The equation of diagonal AC, with reference to new axes is :

   (a) x + y = 5 (b) y – x = 5 (c) x –y = 5 (d) none of these

10. L_1 is $\frac{x}{3}+\frac{y}{4}=1$ and L_2 is $x-\frac{1}{2}y=1$. Origin is changed to (–1,2). Then point of intersection of the two lines with reference to new axes is :

    (a) (2, 1) (b) (2, 2) (c) (–2, –1) (d) (–2, –2)

11. Keeping origin fixed axes are rotated in clockwise direction through an angle of 30^0. The equation of straight line ax + by = 1 with reference to new axes is :

    (a) $(\sqrt{3}a-b)X+(\sqrt{3}b+a)Y=1$ (b) $(\sqrt{3}a-b)X+(\sqrt{3}b+a)Y=2$

    (c) $(\sqrt{3}a+b)X+(\sqrt{3}b-a)Y=2$ (d) $(\sqrt{3}a+b)X+(\sqrt{3}b+a)Y=2$

12. At what point the origin be shifted so that the equation of circle $x^2 + y^2 + 4x + 6y - 12 = 0$ is transferred as $X^2 + Y^2 = 25$

(a) (–2, –3) (b) (2, –3) (c) (2, 3) (d) (–2, 3)

13. The equation of circle is $x^2 + y^2 - 6x - 8y = 0$ keeping origin fixed, axes are rotated in anticlockwise direction till present diameter through origin coincides with old x-axis. Then equation of circle with reference to new axes is :

(a) $X^2 + Y^2 = 10x$ (b) $X^2 + Y^2 = \frac{1}{5}(48Y - 14X)$

(c) $X^2 + Y^2 = \frac{1}{5}(14X - 48Y)$ (d) none of these

14. In question 13 axes were rotated through an angle of

(a) $\tan^{-1}\left(-\frac{4}{3}\right)$ (b) $\tan^{-1}\left(\frac{4}{3}\right)$ (c) $\tan^{-1}\left(\frac{3}{4}\right)$ (d) $\tan^{-1}\left(\frac{3}{2}\right)$

15. Through what degree the co-ordinates axes be rotated at origin in anticlockwise direction so that the equation $x^2 + 3xy + y^2 - 3x + 4y = 0$ loses its xy term in new equation :

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

16. By rotating axes at fixed origin in anticlockwise direction, through 45^0, show that the following curves represent different conics.:

(a) $x^2 - 14xy + y^2 - 24 = 0$ (b) $x^2 - 2xy + y^2 - 8\sqrt{2}\,y = 0$

(c) $13x^2 - 10xy + 13y^2 - 72 = 0$

✣ --------------------- ✣

4. Pair of Straight Lines

4.1. Homogeneous Equations :

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is general second degree equation. It can represent a pair of straight line or some other curve, under different conditions. Relations between co-efficients of x^2, y^2, xy, x, y and constant terms decide the nature of curves.

$$ax^2 + 2hxy + by^2 = 0$$

is homogeneous equation of second degree. In this equation sum of indices of every term is 2 which is degree of this equation.

$$4x^3 + 6x^2y + 8xy^2 + 3y^3 = 0$$

is homogeneous equation of third degree. Here sum of indices of every term is 3, which is degree of equation.

4.2. Straight lines 3x – 4y = 0, 4x + 3y = 0 2x – y = 0 all pass through origin.

(i) (3x – 4y) (4x + 3y) = 0 is combined equation of two lines that pass through origin. It is $12x^2 - 7xy - 12y^2 = 0$. It is second degree homogeneous equation.

(ii) (3x – 4y) (4x + 3y) (2x – y) = 0 is combined equation of three straight lines that pass through origin. This equation is

$$24x^3 - 26x^2y + 31xy^2 - 12y^3 = 0$$

It is homogeneous equation of third degree.

∴ Homogeneous equation of degree 4 shall represent 4 straight lines that pass through origin, and so on.

Let homogeneous equation $ax^2 + 2hxy + by^2 = 0$ represent straight line $y - m_1x = 0$, $y - m_2x = 0$ that pass through origin.

Combined equation of these lines $(y - m_1x)(y - m_2x) = 0$

$$\Rightarrow y^2 - (m_1 + m_2)xy + m_1m_2y^2 = 0 \qquad \text{............}(\beta)$$

This equation and $x^2 + 2\frac{h}{a}xy + \frac{b}{a}y^2 = 0$ should be same.

$$\therefore m_1 + m_2 = -\frac{2h}{a};$$

$$m_1m_2 = \frac{b}{a}$$

These relation of slopes of two lines must always exist.

4.3. Auxillary Equation of slopes :

In straight line that pass through origin $\frac{y}{x}$ represent the slope of straight line.

$\therefore$ from $ax^2 + 2hxy + by^2 = 0$

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

i.e. $bm^2 + 2hm + a = 0$

This equation is auxillary equation of slopes and see from this equation also

$$m_1 + m_2 = -2h/b;\ m_1 m_2 = \frac{a}{b}$$

4.4. Angle between two lines represented by equation $ax^2 + 2hxy + by^2 = 0$

Let lines be $y - m_1x = 0$ and $y - m_2x = 0$.

$$\therefore m_1 + m_2 = -\frac{2h}{b},\ m_1 m_2 = \frac{a}{b}$$

If straight lines are inclined at θ_1 and θ_2 with x axis then angle between them is $\theta_1 - \theta_2$

$\therefore$ angle θ between lines give.

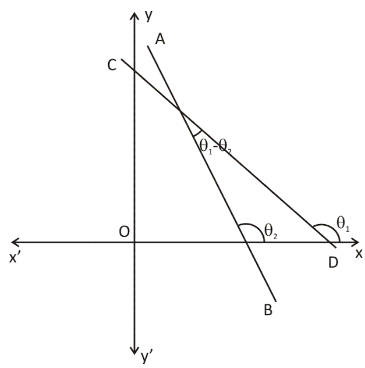

Fig 1

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2} = \sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}} \Big/ \left[1 + \frac{a}{b}\right]$$

$$= \frac{2\sqrt{h^2 - ab}}{a + b}$$

(a) If $h^2 - ab = 0$, then $\theta = 0$, lines are parallel.

(b) If $a + b = 0$. Then $\cot\theta = 0$; lines are perpendicular.

(c) Equation of two straight lines that go through origin and which are perpendicular to each other is $x^2 + \lambda xy - y^2 = 0$

Solved Examples

Example 1 : Find joint equation of two lines that go through origin and one is parallel to $3x + 4y = 3$ and the other perpendicular to $4x – 5y +15 = 0$.

Sol. : Straight line through origin and parallel to

$3x + 4y = 3$ is $3x + 4y = 0$

and straight line through origin and perpendicular to $4x – 5y + 15 = 0$ is $5x + 4y = 0$.

$\therefore$ Combined equation is $(3x+ 4y)(5x + 4y) = 0$

i.e. $15x^2 + 32xy + 16y^2 = 0$

Example 2 : Find joint equation of two lines that form equilateral triangle with $x = 3$.

Sol. : In fig. Straight line (i) is inclined at 30^0 with x – axis while straight line 2 in inclined at $– 30^0$ with x-axis.

ΔOAB is equilateral

$\therefore$ combined equation is

$$\left(y-\frac{1}{\sqrt{3}}x\right)\left(y+\frac{1}{\sqrt{3}}x\right)=0 \quad \text{i.e. } 3y^2 – x^2 = 0$$

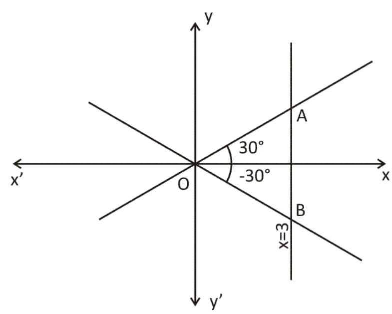

Fig 2

Example 3 : Find combined equation of two straight lines that make equal angles of 30^0 with straight lines $2x – y = 0$.

Sol. : Given straight line $y = 2x$, $\tan\theta = 2$, where θ is the angle this line makes with x-axis. If one straight line makes angle $\theta + 30^0$ with x-axis then other straight line should make $\theta - 30^0$ with x-axis, then the two shall make equal angle 30^0 with $2x – y = 0$.

$$\tan\left(\theta+30^0\right)=\frac{\tan\theta+\tan30^0}{1-\tan\theta\tan30^0}=\frac{2+\frac{1}{\sqrt{3}}}{1-\frac{2}{\sqrt{3}}}=\frac{2\sqrt{3}+1}{\sqrt{3}-2}$$

$$=-\left(8+5\sqrt{3}\right)$$

and $$\tan(\theta-30)=\frac{2-\frac{1}{\sqrt{3}}}{1+\frac{2}{\sqrt{3}}}=5\sqrt{3}-8$$

$\therefore$ straight line $\left[y+\left(8+5\sqrt{3}x\right)\right]\left[y-\left(5\sqrt{3}-8\right)x\right]=0$

$\Rightarrow y^2 + 16xy - 11x^2 = 0$

Example 4 : Combined equation of straight line through origin making equal angle ϕ with x – y = 0 are $\left(y - \frac{1+\tan\phi}{1-\tan\phi}x\right)\left(y - \frac{1-\tan\phi}{1+\tan\phi}x\right) = 0$

$\Rightarrow y^2 - 2xy\sec 2\phi + x^2 = 0$

Example 5 : Find angle between straight lines

(a) $y^2 - 3xy = 0$

(b) $3x^2 - 4xy = 0$,

(c) $(a - b)x^2 + 2ab\,xy + (b - a)\,y^2 = 0$

(d) $x^2 + 2xy\cot\theta + y^2 = 0$

Sol. :

(a) Straight lines $y^2 - 3xy = y(y - 3x) = 0$, y = 0 is x axis, y – 3x = 0 is inclined at $\tan^{-1} 3$ with x axis.

$\therefore$ angle between the two straight line = $\tan^{-1} 3$.

(b) $3x^2 - 4xy = x(3x - 4y) = 0$

x = 0 is y –axis. If 3x – 4y = 0 is inclined at $\tan^{-1}\frac{3}{4}$ with x-axis then it is inclined at $90 - \tan^{-1}\frac{3}{4}$ with y-axis.

$\therefore$ angle between lines is $\tan(90 - \theta) = \cot\theta = 4/3$

$\therefore$ angle is $\tan^{-1}(4/3)$

(c) Equation $(a - b)x^2 + 2ab\,xy + (b - a)^2 = 0$.

Here (a – b) + (b – a) = 0 straight lines are perpendicular to each other. Angle between them = 90^0

(d) Equation $x^2 + 2xy\cot\theta + y^2 = 0$

$a = 1, b = 1, h = \cot\theta$

$\therefore \tan\phi = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{\cot^2\theta - 1}}{2} = \sqrt{\cot^2\theta - 1}$

$= \sqrt{\cos 2\theta}.\operatorname{cosec}\theta$ and , $\cos\phi = \cos^{-1}(\tan\theta)$

Example 6 : Find equation of lines through (3, –4) perpendicular to $4x^2 - xy - 3y^2 = 0$.

Sol. : Given equation is $4x^2 - xy - 3y^2 = 0$ auxiliary equation of slope is

$3m^2 + m - 4 = 0 \Rightarrow (m-1)(3m+4) = 0$

$\Rightarrow m = 1, m = -4/3$

Lines through (3, –4) and perpendicular to given lines are y + 4 = –(x – 3) and $y + 4 = \frac{3}{4}(x-3)$ combined equation is (x + y + 1) (3x – 4y –25) = 0

$\Rightarrow 3x^2 - xy - 4y^2 - 22x - 29y - 25 = 0$

Example 7 : Find relation between a, b and h, co-efficient of equation $ax^2 + 2hxy + by^2 = 0$, if slope of one line of this equation is square of the slope of other line.

Sol. : Given equation

$ax^2 + 2hxy + by^2 = 0$ ....(1)

Auxillary equation of slopes is

$bm^2 + 2hm + a = 0$ ....(2)

The slope of lines are m1 and m_1^2

$\therefore \left(m_1 + m_1^2\right) = -\frac{2h}{b}$ and $m_1.m_1^2 = m_1^3 = \frac{a}{b}$

and $\left(m_1 + m_1^2\right)^3 = m_1^6 + m_1^3 + 3m_1^3\left(m1 + m_1^2\right)$

$$\therefore \left(\frac{-2h}{b}\right)^3 = \frac{a^2}{b^2} + \frac{a}{b} + 3\frac{a}{b}\left(-\frac{2h}{b}\right)$$

$$\Rightarrow -8h^3 = a^2b + ab^2 - 6abh$$

$$8h^3 + ab(a+b) = 6abh$$

Example 8 : Find condition so that the acute angle between lines given by equation $ax^2 + 2hxy + by^2 = 0$ is equal to acute angle between lines $3x^2 - xy - 2y^2 = 0$

Sol. : Given equation $ax^2 + 2hxy + by^2 = 0$ angle between lines given by it.

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

angle ϕ between lines given by equation $3x^2 - xy - 2y^2 = 0$ is

$$\tan\phi = \frac{2\sqrt{\frac{1}{4} + 6}}{3-2} = 5$$

Given $\theta = \phi$ $\therefore 2\sqrt{h^2 - ab} = 5(a+b)$ $\Rightarrow 4\left(h^2 - ab\right) = 25(a+b)^2$

Practice Worksheet (Foundation Level) – 4(a)

1. The slope of 3 lines are $\frac{1}{2}, 1, -2$ and these pass through origin. Write combined equation.

2. Write the combined equation of straight line 3x+ 4y = 0 and the line through origin perpendicular to it.

3. Write the slopes of straight lines given by equation $3x^2y - 5xy^2 + 2y^3 = 0$

4. Two lines pass through origin, write their combined equation if

   (a) they bisect the angles between axes.

   (b) One is parallel to ax – by = c and the other perpendicular to $3ax + 4by = c^2$.

   (c) Each is inclined at 60^0 with x-axis.

5. Write the combined equation of straight lines through (0, 2) each making angles of 30^0 with y –axis.

6. Find angles between straight lines given by following equations :

   (a) $3x^2 - 7xy + 4y^2 = 0$ (b) $x^2 - 2xy\,\mathrm{cosec}\theta + y^2 = 0$

   (c) $a^2x^2 + 5abxy + 4b^2y^2 = 0$ (d) xy = 0

   (e) $x^2 + 2xy\tan\theta + y^2 = 0$ (f) 2xy – 3y = 0

7. Find joint equation of straight line which pass through origin and make :

   (a) angle 60^0 with line $x + \sqrt{3}\,y = 0$

   (b) angle 45^0 with line 3x –y = 0

   (c) angle 30^0 with line ax +by = 0

8. Find k if slopes of lines $3x^2 + kxy + y^2 = 0$ differ by ½.

9. Find the condition (a) if straight line 4x – 3y = 0 coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$.

   (b) If straight line ax+by=0 is perpendicular to one of the lines given by equation $ax^2+2hxy+by^2 = 0$

10. Find the angle between the lines given by equation $(x^2 + y^2)\sin^2 a$ $= (x\cos\theta - y\sin\theta)^2$.

11. Find the condition that of the slope of one line given by equation $ax^2 + 2hxy + by^2 = 0$ is n times the slope of other line ($0 < n < \alpha$).

12. Find the equation if one of the straight line given by equation $ax^2 + 2hxy + by^2 = 0$ is the angle bisector of angle between axes.

13. Find combined equation of straight lines through (2, – 3) inclined at 30^0 and 60^0 with x-axis.

14. Straight line ax + by +c = 0 meets axes in A and B. The combined equation of lines through A and B, passing origin is ....

15. Find combined equation of lines through (2, –3) perpendicular to line given by equation $3x^2 + xy - 2y^2 = 0$

16. Angle between the lines $ax^2 + 2hxy + by^2 = 0$ is 60^0. Express h in terms of a and b.

4.5. To find equation of lines through origin perpendicular to $ax^2 + 2hxy + by^2 = 0$.

Equation is $ax^2 + 2hxy + by^2 = 0$ ....(1)

Let it represents line $y - m_1x = 0,\ y - m_2x = 0$ ....(2)

Then $m_1 + m_2 = \frac{-2h}{b}, m_1m_2 = \frac{a}{b}$ ....(3)

Straight lines perpendicular to lines (2) are

$m_1y + x = 0, \quad m_2y + x = 0$

$\therefore$ combined equation $(m_1y + x)(m_2y + x) = 0$

$\Rightarrow m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$

Putting value from (3)

$$\frac{a}{b}y^2 - \frac{2h}{b}xy + x^2 = 0 \qquad \Rightarrow bx^2 - 2hxy + ay^2 = 0$$

4.6 Equation of the angle bisectors of the angles between lines $ax^2 + 2hxy + by^2 = 0$

Let equation $ax^2 + 2hxy + by^2 = 0$ ....(1)

Represent two lines $y - m_1x = 0,\ y - m_2x = 0$ ....(2)

Then $m_1 + m_2 = -\frac{2h}{b}; m_1m_2 = \frac{a}{b}$ ....(3)

Equation of bisectors are $\frac{y - m_1x}{\sqrt{1+m_1^2}} = \pm\frac{y - m_2x}{\sqrt{1+m_2^2}}$ ....(4)

$$\Rightarrow \left(\frac{y-m_1x}{\sqrt{1+m_1^2}} - \frac{y-m_2x}{\sqrt{1+m_2^2}}\right)\left(\frac{y-m_1x}{\sqrt{1+m_1^2}} + \frac{y-m_2x}{\sqrt{1+m_2^2}}\right) = 0$$

$$\Rightarrow \frac{y^2 - 2m_1xy + m_1^2x^2}{1+m_1^2} - \frac{y^2 - 2m_2xy + m_2^2x^2}{1+m_2^2} = 0$$

$$\Rightarrow y^2 - 2m_1xy + m_1^2x^2 + m_2^2y^2 - 2m_1m_2^2xy + m_1^2m_2^2x^2 - y^2$$
$$+ 2m_2xy - m_2^2x^2 - m_2^2y^2 + 2m_2m_1^2xy - m_1^2m_2^2x^2 = 0$$

$$\Rightarrow (m_1^2 - m_2^2)x^2 - (m_1^2 - m_2^2)y^2 = xy[2m_1 + 2mm_2^2 - 2m_2 - 2m_2m_1^2]$$

$$= 2xy(m_1 - m_2)(1 - m_1m_2)$$

$$\Rightarrow (m_1 + m_2)(x^2 - y^2) = 2xy(1 - m_1m_2)$$

$$\Rightarrow x^2 - y^2 = 2xy\left(\frac{1 - m_1m_2}{m_1 + m_2}\right)$$

$$= 2xy\left(1-\frac{a}{b}\right)\bigg/\left(-\frac{2h}{b}\right)$$

$$\Rightarrow\ x^2 - y^2 = \frac{(a-b)}{2h}2xy$$

$$\Rightarrow\ \frac{x^2 - y^2}{a-b} = \frac{xy}{h}$$

Note : Equation of angle bisector of line xy = 0;

Here, a = 0, b = 0, h = ½

Equation $\frac{x^2 - y^2}{xy} = \frac{a-b}{h} \Rightarrow x^2 - y^2 = 0$

4.7 To find the condition that one of the lines given by ax^2 + 2hxy +by^2 = 0 coincides with one of the lines given by $a'x^2 + 2h'xy + b'y^2 = 0$

Let y = mx be the common line

$\therefore$ substituting this value of y in the equations of the lines given -

$ax^2 + 2hmx^2 + bm^2x^2 = 0$

$\Rightarrow bm^2 + 2hm + a = 0$

Similarly bm^2 + 2h'm + a' = 0

By cross multiplication - $\frac{m^2}{2ha' - 2h'a} = \frac{m}{ab' - ba'} = \frac{1}{2h'b - 2hb'}$

$$\Rightarrow m = \frac{2ha' - 2h'a}{ab' - ba'} = \frac{ab' - ba'}{2h'b - 2hb'}$$

$$\Rightarrow (ab' - ba')^2 = 4(ha' - h'a)(h'b - hb')$$

4.8 To find the condition that one line of equation $ax^2 + 2hxy + by^2 = 0$ **be perpendicular to the one line given by equation** $a'x^2 + 2h'xy + b'y^2 = 0$

Let y = mx be the line of equation ....(1)

$\therefore ax^2 + 2hmx^2 + m^2 bx^2 = 0$

$bm^2 + 2hm + a = 0$ ....(α)

Line $\perp$ to y = mx is $y = -\frac{1}{m}x. \Rightarrow x = -my$

$\therefore a'm^2y^2 - 2h'my^2 + b'y^2 = 0$

$\Rightarrow a'm^2 - 2h'm + b' = 0$ ......(β)

By cross multiplication from (α) and (β)

$$\frac{m^2}{2hb'+2h'a}=\frac{m}{aa'-bb'}=\frac{1}{-(2h'b+2ha')}$$

$$\therefore\ m=\frac{2hb'+2h'a}{aa'-bb'}=\frac{aa'-bb'}{-(2h'b+2ha')}$$

$$\Rightarrow (aa'-bb')^2+4(hb'+h'a)(h'b+ha')=0$$

4.9 General equation of second degree : $ax^2+2hxy+by^2+2gx+2fy+c=0$ **is the general equation of second degree in x and y. It can represent a pair of straight line or a circle or ellipse, or parabola or hyperbola. When co-efficeints a, b, c, f, g, h satisfy certain conditions.**

Condition that it represents a pair of straight lines

Equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ ....(1)

$$\Rightarrow a^2x^2+2haxy+aby^2+2gax+2fay+ca=0$$

$$\Rightarrow a^2x^2+2ax(hy+g)=-a(by^2+2fy+c)$$

$$\Rightarrow a^2x^2+2ax(hy+g)+(hy+g)^2=(hy+g)^2-a(by^2+2fy+c)$$

$$=y^2(h^2-ab)+2y(gh-af)+(g^2-ac)$$

$$\Rightarrow ax+hy+g=\pm\sqrt{y^2(h^2-ab)+2y(gh-af)+(g^2-ac)}$$

Equation (1) shall represent two straight line if $y^2(h^2-ab)+2y(gh-af)+(g^2-ac)$ is a perfect square

i.e. $4(gh-af)^2=4(h^2-ab)+(g^2-ac)$ simplifying $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. It can be denoted by Δ.

$\therefore$ General equation of second degree shall represent two straight lines of

$$\Delta=abc+2fgh-af^2-bg^2-ch^2=0$$

(a) If $\Delta\neq 0$ and a = b, h = 0 it represent a circle.

(b) If $\Delta\neq 0$ and $h^2 = ab$ it represent a parabola

(c) If $\Delta\neq 0$ and $h^2 < ab$ it represent an ellipse.

(d) If $\Delta\neq 0$ and $h^2 > ab$ it represent an hyperbola.

Solved Example

Example 9 : Find λ if equation $\lambda xy - 8x + 9y - 12 = 0$ represents a pair of straight lines.

Sol. : a = b = 0, $h = \frac{\lambda}{2}, c = -12, g = -4, f = \frac{9}{2}$

Equation represents two straight lines

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow 0 + 2.\frac{9}{2}.(-4)\frac{\lambda}{2} - 0 - 0 + 12\frac{\lambda^2}{4} = 0$

$\Rightarrow -18\lambda + 3\lambda^2 = 0 \;\; \Rightarrow \lambda = 0$ or $\lambda = 6$

$\lambda \neq 0$ because then the equation shall be reduced to single degree. $\therefore \lambda = 6$

Example 10 : Find angle between lines given by equation $3x^2 + 8xy - 3y^2 - 40x + 30y - 75 = 0$.

Sol. : Angle between straight lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be the same as angle between parallel lines to them through origin.

$\therefore$ angle between given lines = angle between straight lines $3x^2 + 8xy - 3y^2 = 0$.

a = 3, b = – 3, i.e. a + b = 0, $\therefore$ angle is 90^0.

Example 11 : Find angles that straight lines given by equation $y^2 - 4xy + x^2 + 8y - (16 + 2\sqrt{3})x + 15 = 0$ make with x –axis.

Sol. : Inclination of straight line y = mx +c is given by straight line y = mx

$\therefore$ Inclination with x-axis of the straight lines given by equation shall be given by $y^2 - 4xy + y^2 = 0$ ant it is $y^2 - 4xy + 4x^2 = 3x^2$

$\Rightarrow (y - 2x)^2 = \pm(\sqrt{3}x)^2$

$\Rightarrow y = (2 + \sqrt{3})x, y = (2 - \sqrt{3})x$

angles 15^0, 75^0.

Example 12 : Straight lines given by equation $(\tan^2\alpha + \cos^2\alpha)x^2 - 2xy\tan\alpha + y^2\sin^2\alpha = 0$ make angles θ_1 and θ_2 with x-axis, prove ($\tan\theta_1 \sim \tan\theta_2$) = 2

Sol. : Given equation $(\tan^2\alpha + \cos^2\alpha)x^2 - 2xy\tan\alpha + y^2\sin^2\alpha = 0$. Let the straight line be $y - m_1x = 0, y - m_2x = 0$

$\therefore m_1 + m_2 = \frac{2\tan\alpha}{\sin^2 x} = 2\tan\alpha\,\mathrm{cosec}^2\alpha$

$= 2\tan\alpha(1+\cot^2\alpha) = 2(\tan\alpha + \cot\alpha)$

$m_1.m_2 = \frac{\tan^2\alpha + \cos^2\alpha}{\sin^2\alpha} = \sec^2\alpha + \cot^2\alpha$

$= 1 + \tan^2\alpha + \cot^2\alpha$

$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$

$= 4(\tan\alpha + \cot\alpha)^2 - 4(1 + \tan^2\alpha + \cot^2\alpha)$

$= 8 - 4 = 4 \qquad \therefore (\tan\theta_1 \sim \tan\theta_2) = 2$

Example 13 : Find condition if straight lines $x^2 - 2pxy - y^2 = 0$ bisect angles between straight lines $x^2 - 2qxy - y^2 = 0$.

Sol. : Given equation of straight lines $x^2 - 2qxy - y^2 = 0$ ....(1)

Equation of angle bisectors is $\frac{x^2 - y^2}{2} = \frac{xy}{-q}$

$\Rightarrow -qx^2 - 2xy + qy^2 = 0$ ....(2)

But angle bisectors are $x^2 - 2pxy - y^2 = 0$ ....(3)

(2) and (3) are same $\therefore \frac{-q}{1} = \frac{1}{p} = \frac{q}{-1} \Rightarrow pq = -1$

Example 14 : Find combined equation of two straight line that pass through (0, 3) and are inclined at complementary angles with x-axis.

Sol. : If m = $\tan\theta$, then $\tan(90-\theta) = \cot\theta = \frac{1}{m}$.

$\therefore$ straight lines are y – 3 = mx, y – 3 = $\frac{1}{m}x$,

Combined equation $[(y-3) - mx]\left[(y-3) - \frac{1}{m}x\right] = 0$

$\Rightarrow (y-3)^2 + x^2 - x(y-3)\left(m + \frac{1}{m}\right) = 0$

and $m + \frac{1}{m} = \frac{m^2+1}{m} = \frac{1+\tan^2\theta}{\tan\theta} = \frac{\sec^2\theta}{\tan\theta} = 2\operatorname{cosec}2\theta$

$\therefore$ Equation is $y^2 + x^2 - 2xy\operatorname{cosec}2\theta + 6x\operatorname{cosec}2\theta - 6y + 9 = 0$

where θ is angle, any one straight line makes with x-axis.

Example 15 : If in question of previous example straight lines pass through (3, –4) and are inclined at supplementary angles with x-axis. Then equation of straight line shall be, y + 4 = (x – 3), and y + 4 = – m (x – 3) because tan $(180^0-\theta)=-\tan\theta$.

Combined Equation is $[(y+4)-m(x-3)][(y+4)+m(x-3)]=0$

$\Rightarrow (y+4)^2-m^2(x-3)^2=0 \Rightarrow (y+4)^2-\tan^2\theta(x-3)^2=0$

θ is the inclination of either line with x-axis.

Example 16 : Prove that product of perpendicular dropped from (x_1, y_1) on straight lines $ax^2 + 2hxy + by^2 = 0$ is $\left(ax_1^2+2hx_1y_1+by_1^2\right)\Big/\sqrt{(a-b)^2+4h^2}$

Sol. : Let given lines be $y - m_1x = 0$, $y - m_2x = 0$

$$\therefore m_1+m_2=-\frac{2h}{b}, m_1m_2=\frac{a}{b}$$

and $p_1=\dfrac{y_1-m_1x_1}{\sqrt{1+m_1^2}}$ and $p_2=\dfrac{y_1-m_2x_1}{\sqrt{1+m_2^2}}$

$$\Rightarrow p_1p_2=\frac{(y_1-m_1x_1)(y_1-m_2x_1)}{\sqrt{1+m_1^2+m_2^2+m_1^2m_2^2}}$$

$$=\frac{y_1^2-(m_1+m_2)x_1y_1+m_1m_2x_1^2}{\sqrt{1+(m_1+m_2)^2-2m_1m_2+m_1^2m_2^2}}$$

$$=\left(y_1^2+\frac{2h}{b}x_1y_1+\frac{a}{b}x_1^2\right)\Big/\sqrt{1+\frac{4h^2}{b^2}-\frac{2a}{b}+\frac{a^2}{b^2}}$$

$$=\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}$$

Example 17 : Prove that each of the lines is at distance a from point (x_1, y_1) of the equation. $(x_1y-xy_1)^2=a^2\left(x^2+y^2\right)$ **(R – 89)**

Sol. : The given equation is homogeneous in x and y. Let one line be y = mx distance from (x_1, y_1) of this line is a

$$\therefore a=\frac{y_1-mx_1}{\sqrt{1+m^2}}$$

or $\quad a^2\left(1+m^2\right)=(y_1-mx_1)^2$

put $\quad m=\frac{y}{x}$ we get $a^2\left(1+\frac{y^2}{x^2}\right)=\left(y_1-\frac{y}{x}x_1\right)^2$

or $\quad a^2\left(x^2+y^2\right)=(xy_1-yx_1)^2$

which is the given equation, hence proved.

Example 18: From point A(1, 1), straight line AL and AM are drawn perpendicular to straight lines $3x^2+7xy+2y^2=0$. Find area of quadrilateral ALOM, O is origin.

Sol. : $3x^2+7xy+2y^2=(x+2y)(3x+y)=0$

A, (1, 1); AL $\perp$ x + 2y = 0

$\Rightarrow 2x-y=2-1=1$

AM $\perp$ 3x + y = 0

$\Rightarrow x-3y=-2$

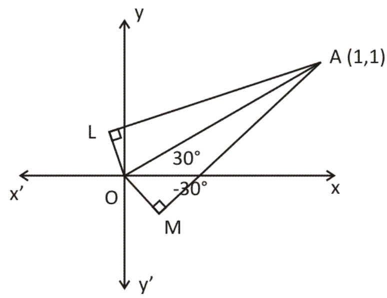

Fig 3

Now $\left.\begin{matrix}2x-y=1\\x+2y=0\end{matrix}\right\}L(2/5,-1/5)$

$\left.\begin{matrix}3x+y=0\\x-3y=-2\end{matrix}\right\}M\left(-\frac{1}{5},\frac{3}{5}\right)$

area AMOL = right angled Δ AMO + right angled Δ AOL

$=\frac{1}{2}.AM.OM+\frac{1}{2}OL.AL$

and $AM=\sqrt{\left(\frac{6}{5}\right)^2+\left(\frac{2}{5}\right)^2}=\frac{1}{5}\sqrt{40}$

OM = $\sqrt{\left(\frac{1}{5}\right)^2+\left(\frac{3}{5}\right)^2}=\frac{1}{5}\sqrt{10}$

$AL=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{6}{5}\right)^2}=\frac{1}{5}\sqrt{45}$

$OL=\frac{1}{5}\sqrt{5}$

$\therefore$ area $=\frac{1}{2}.\frac{1}{5}\sqrt{40}.\frac{1}{5}\sqrt{10}+\frac{1}{2}\frac{1}{5}.\sqrt{45}.\frac{\sqrt{5}}{5}$.

$=\frac{2}{5}+\frac{3}{10}=\frac{7}{10}$ sq, units.

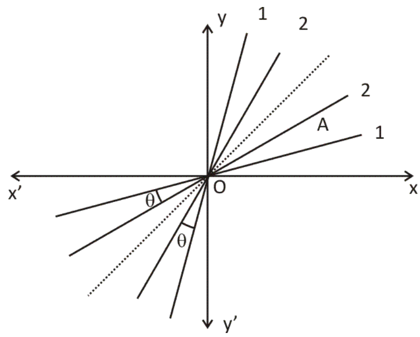

Fig 4

Example 19 : If pair of straight lines $ax^2 + 2hxy + by^2 = 0$ make equal angles with the lines $ax^2 + 2h'xy + b'y^2 = 0$, then prove $(a – b)h' = (a' – b')h$.

Sol.: This is only possible when angle bisectors of two pairs are identical (1), (1) are lines given by 1st pair and 2, 2 lies are given by 2nd pair. They make equal angles θ, when the angle bisector of two pairs are the same.

$$\left.\begin{aligned}\therefore h(x^2-y^2)=(a-b)xy\\ h'(x^2-y^2)=(a'-b')xy\end{aligned}\right\}\Rightarrow \frac{h}{h'}=\frac{a-b}{a'-b'}$$

$$\therefore h(a'-b')=h'(a-b)$$

Example 20 : The area of the triangle formed by the lines $x – y = 3$ and $x^2 + 4xy + y^2 = 0$ is :

(a) $3\sqrt{3}$ squ. (b) $\frac{1}{2}.3\sqrt{2}$ squ. (c) $6\sqrt{3}$ squ. (d) $4\sqrt{3}$ squ

Sol. : Equation $x^2 + 4xy + y^2 = 0$

$$y^2 + 4xy + 4x^2 = 3x^2$$

$$\therefore\ y+2x=\pm\sqrt{3}x \Rightarrow y=x\left(\sqrt{3}-2\right),\ y=-\left(\sqrt{3}+2\right)x$$

(i) angle between these lines, $\tan\theta=\frac{2\sqrt{4-1}}{1+1}=\sqrt{3}\Rightarrow\theta=60^0$

(ii) angle between $x – y = 3$ and $y = \left(\sqrt{3}-2\right)x$

$$\tan\phi=\frac{1-\sqrt{3}+2}{1+\sqrt{3}-2}=\frac{3-\sqrt{3}}{\sqrt{3}-1}=\sqrt{3}$$

$\therefore\ \phi=60^0$. $\therefore$ triangle is equilateral perpendicular, p, from origin on the straight line $x – y = 3$ side of equilateral Δ $p=\frac{3}{\sqrt{2}}$ and it is altitude of triangle. Area = $\frac{p^2}{\sqrt{3}}=\frac{9}{2\sqrt{3}}=\frac{3\sqrt{3}}{2}$ sq. units

Note : Point of intersection of $x – y = 3$ and $y=\left(\sqrt{3}-2\right)x$ is $x=\frac{3}{3-\sqrt{3}}=\frac{3+\sqrt{3}}{2}$ and $y=\left(\sqrt{3}-2\right)x$ is $x=\frac{3}{3-\sqrt{3}}=\frac{3+\sqrt{3}}{2}$ and $y=\frac{\sqrt{3}-3}{2}$

$$\therefore\ \text{side } OA^2 = \frac{\left(3+\sqrt{3}\right)^2+\left(\sqrt{3}-3\right)^2}{4}=\frac{24}{4}=6.$$

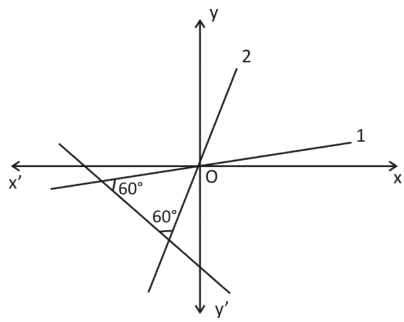

Fig 5

Area = $\frac{\sqrt{3}}{4}(\text{side})^2 = \frac{3\sqrt{3}}{2}$

Example 21 : Find area and internal angles of the triangle formed by straight line $y^2 - 4xy + x^2 = 0$ and $x + y + 4\sqrt{6} = 0$.

Sol. : $y^2 - 4xy + x^2 = 0 \Rightarrow y^2 - 4xy + 4x^2 = 3x^2$

$\Rightarrow (y-2x)^2 = \left(\sqrt{3}x\right)^2 \Rightarrow y = \left(\sqrt{3}+2\right)x, y = -\left(\sqrt{3}-2\right)x$ and, angle between them $\tan\theta = \frac{2\sqrt{4-1}}{2} = \sqrt{3}$

$\therefore \theta = 60^0$

and angle between $y = x\left(\sqrt{3}+2\right)$ and $x + y + 4\sqrt{6} = 0$

$\tan\phi = \frac{\sqrt{3}+2+1}{1-\left(\sqrt{3}+2\right)} = \frac{\sqrt{3}+3}{-\left(\sqrt{3}+1\right)}$

$= -\frac{\sqrt{3}\left(\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)} = -\sqrt{3}$

acute angle $\phi = 60^0$

Δ is equilateral $\perp$ from (0, 0) on line $x + y + 4\sqrt{6} = 0$

$p = \frac{4\sqrt{6}}{\sqrt{2}} = 4\sqrt{3}$

$\therefore$ area = $p^2 / \sqrt{3} = \frac{48}{\sqrt{3}} = 16\sqrt{3}$

Example 22 : If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines then prove distance between them is $2\sqrt{\left(g^2 - ac\right)/\left(a^2 + ab\right)}$

Sol : We know for this equal to represent two parallel lines

$h^2 = ab$ and $bg^2 = f^2$ ....(1)

$\therefore \quad ax^2 + 2hxy + by^2 = ax^2 + 2\sqrt{ab}xy + by^2 = 0$

$\Rightarrow \left(\sqrt{a}x + \sqrt{b}y\right)^2 = 0$ ....(2)

$\therefore$ Parallel lines are $\left(\sqrt{a}x + \sqrt{b}y + A\right)\left(\sqrt{a}x + \sqrt{b}y + B\right) = 0$

given equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore$ Equating co-efficients of x and y

$$\left.\begin{aligned}\sqrt{a}(A+B)=2g\\ \sqrt{b}(A+B)=2f\end{aligned}\right\}\Rightarrow A+B=\frac{2(g+f)}{\sqrt{a}+\sqrt{b}} \qquad \text{..........(3)}$$

and AB = c.

Now $(A-B)^2 = \dfrac{4(g+f)^2}{\left(\sqrt{a}+\sqrt{b}\right)^2}-4c=\dfrac{4\left(g^2+f^2+2gf\right)}{\left(\sqrt{a}+\sqrt{b}\right)^2}-4c$ ....(4)

And $g^2+f^2+2gf=g^2+\dfrac{b}{a}g^2+2g^2\dfrac{\sqrt{b}}{\sqrt{a}}$ from (1)

$$=\frac{g^2}{a}\left(a+b+2\sqrt{ab}\right)=\frac{g^2}{a}\left(\sqrt{a}+\sqrt{b}\right)^2$$

$\therefore$ from (4), $(A-B)^2=\dfrac{4g^2}{a}-4c=\dfrac{4\left(g^2-ac\right)}{a}$

Distance between parallel lines is $d=\dfrac{A-B}{\sqrt{a+b}}\Rightarrow d^2=\dfrac{(A-B)^2}{a+b}=\dfrac{4}{a}\left(\dfrac{g^2-ac}{a+b}\right)$

$$=2\frac{\sqrt{g^2-ac}}{a(a+b)}$$

Example 23 : Show that the straight lines $(A^2 - 3B^2)x^2 + 8\,Abxy + (B^2 - 3A^2)y^2 = 0$ and Ax + By +C =0 form an equilateral triangle of area $\dfrac{c^2}{\sqrt{3}\left(A^2+B^2\right)}$.

Sol. : Let, lines given by equation $(A^2 - 3B^2)\,x^2 + 8ABxy + (B^2 - 3A^2)\,y^2 = 0$ be OL and OM in the fig. and LM be Ax + By + c = 0 Let Δ OLM be equilateral. Slope of LM = – A/B= m of OL = m_1

$\therefore\ \tan(\pm 60)=\dfrac{m_1-m}{1+mm_1}=\dfrac{m_1+A/B}{1-m_1(A/B)}=\pm\sqrt{3}$

$\therefore\quad 3(B-m_1A)^2=(m_1B+A)^2$ ....(2)

Since equation of OL is $y = m_1x$

$\therefore$ putting $m_1=\dfrac{y}{x}$ in (2)

$$3\left(B-\frac{y}{x}A\right)^2=\left(\frac{y}{x}B+A\right)^2$$

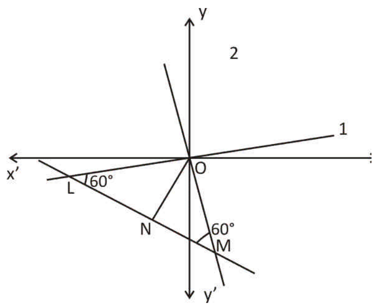

Fig 6

which on simplification gives, $\left(B^2-3A^2\right)y^2+8ABxy+\left(A^2-3B^2\right)x^2=0$ which is the given equation

$\therefore$ triangle formed by lines is equilateral

area = $\frac{1}{2}p.L.M=p.NM$

$$=p.p\cot 60=\frac{1}{\sqrt{3}}p^2$$

and $p=\frac{c}{\sqrt{A^2+B^2}}$

$\therefore$ area = $\frac{c^2}{\sqrt{3}\left(A^2+B^2\right)}$

Example 24 : If the pair of lines $ax^2 + 2(a + b)\,xy + by^2 = 0$ lie along the diameters of a circle and divide the circle in four sector such that area of one is thrice the area of the other then :

(a) $3a^2 + 2ab + 3b^2 = 0$ (b) $3a^2 + 10ab + 3b^2 = 0$

(c) $3a^2 - 2ab + 3b^2 = 0$ (d) $3a^2 - 10ab + 3b^2 = 0$

Sol. : From fig. 7 acute angle $\theta=\frac{360}{8}=45^0$.

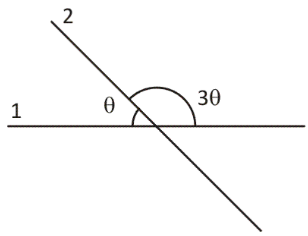

Fig 7

$\therefore$ angle between $y - m_1x = 0$ and $y - m_2x = 0$ is 45^0.

$$\therefore\ \frac{m_1-m_2}{1+m_1m_2}=1$$

$\Rightarrow (m_1-m_2)=1+m_1m_2$ ...(1)

and from the straight lines

$ax^2+2(a+b)xy+by^2=0$ ...(2)

$m_1+m_2=\frac{-2(a+b)}{b}$, $m_1m_2=\frac{a}{b}$ ...(3)

from (3) $(m_1-m_2)^2=\frac{4(a+b)^2-4ab}{b^2}=\frac{4\left(a^2+ab+b^2\right)}{b^2}$

$\therefore$ from (1) $\frac{4\left(a^2+ab+b^2\right)}{b^2}=\left(1+\frac{a}{b}\right)^2$

$\Rightarrow\ 4a^2 + 4ab + b^2 4 = a^2 + b^2 + 2ab$

$\Rightarrow\ 3a^2 + 2ab + 3b^2 = 0$

Example 25 : The equation of straight line AB is x = 2. It is rotated about its point P(2, 5) in the anticlock wise direction through an angle of 60^0 and takes position A.P. a boy further rotates it in anticlock wise direction through 60^0 more. Now new position of straight line is A_2P. Find combined equation of straight line A_1P and A_2P.

Sol. : x = 2 is a straight line parallel to y – axis and meets x-axis in (2, 0); point P(2, 5) when rotated through 60^0 in anti-clock-wise direction, it makes angle 150^0 in position A_1P

Equation $y - 5 = \tan 150^0 (x-2)$

$\Rightarrow y - 5 = -\frac{1}{\sqrt{3}}(x-2)$

When further rotated through 60^0, its inclination with axis become 30^0.

$\therefore$ Equation PA_2 $y - 5 = \frac{1}{\sqrt{3}}(x-2)$

Combined equation $\left[\sqrt{3}(y-5) + (x-2)\right]\left[\sqrt{3}(y-5) - (x-2)\right] = 0$

$\Rightarrow 3(y^2 - 10y + 25) - (x^2 - 4x + 4) = 0$

$\Rightarrow x^2 - 3y^2 - 4x + 30y - 71 = 0$

Example 26 : ABC is right angled isosceles triangle Its base BC is inclined at 15^0 with x-axis. Point B is on x-axis, $\angle A = 90^0$, and point A is $(5, \sqrt{12})$. Find combined equation of AB and AC and also area of Δ.

Sol. : In fig. $\angle A = 90^0$, AB = AC., $\angle ABC = \angle ACB = 45^0$,

AD is perpendicular from A on x-axis. D is (5, 0) In Δ ABD, $\angle ABD = 60^0$.

$\therefore$ If BD = x, then AB = 2x. and AD = $\sqrt{3}x = \sqrt{12}$ $\therefore x = 2$.

Co-ordinate of B (5 –2,0) i.e. (3, 0)

Equation AB $y = \frac{\sqrt{12}}{2}(x-3)$

$\Rightarrow y - \sqrt{3}x + 3\sqrt{3} = 0$

AC is $\perp$ AB. Equation of AC

$\Rightarrow \sqrt{3}y + x = \sqrt{3}.\sqrt{12} + 5$

$\Rightarrow \sqrt{3}y + x - 11 = 0$

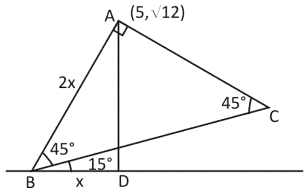

Fig 8

$\therefore$ combined equation is $\left(\sqrt{3}y + x - 11\right)\left(y - \sqrt{3}x + 3\sqrt{3}\right) = 0$

$$\sqrt{3}y^2 - 3xy + 9y + xy - \sqrt{3}x^2 + 3\sqrt{3}x - 11y + 11\sqrt{3}x - 33\sqrt{3} = 0$$

$$\Rightarrow \sqrt{3}y^2 - 2xy - \sqrt{3}x^2 + 14\sqrt{3}x - 2y - 33\sqrt{3} = 0$$

(ii) area = $\frac{1}{2}.AB.AC = \frac{1}{2}AB^2$

$$= \frac{1}{2}\left[(5-3)^2 + \left(\sqrt{12} - 0\right)^2\right] = \frac{1}{2}.16$$

= 8 sq. units

Practice Worksheet (Foundation Level) – 4(b)

1. If equation $6x^2 + 11xy - l\,y^2 + x + 31y - 15 = 0$ represents two straight lines then λ is equal to……
2. The angle bisectors of straight lines $ax^2 - by^2 = 0$ are …….
3. Product of perpendicular from (a, b) on lines $x^2 - xy + y - x = 0$ is ……….
4. The angle between the lines given by equation $x^2 - xy + y - x = 0$ is ……..
5. The two pair of lines $(x+3y)^2 - 6(x+3y) = 0$ and $4x^2 - 4xy + y^2 - 6x + 3y = 0$ form a parallelogram. The equation of diagonal passing through origin is ……
6. The area of triangle formed by straight lines $4x^2 - y^2 = 0$ and y = a is ………
7. Equation lxy + mx + ny + c = 0 shall represents two lines if …………..
8. The equation of angle bisectors of the angles of angle bisector of $x^2 + 2hxy + by^2 = 0$ are …….
9. Equation of lines through (4, 5) parallel to the lines $3x^2 - 5xy + 2y^2 - 2x + y - 1 = 0$ are …..
10. The area of triangle formed by straight lines $5a^2 x^2 - abxy - 4b^2y^2 = 0$ and ax + by = 1 is …..
11. If $8x^2 + 8xy + py^2 + 26x + qy + 15 = 0$ represents a pair of parallel lines, then

    (a) p = ………. (b) q = …………….
12. If pairs of straight lines $ax^2 + 2hxy - 9y^2 = 0$ and $bx^2 + 2gxy - by^2 = 0$ be such that each pair bisects the angles between the other pairs, then find relation between a, b, h and g.
13. Find value of a and c when equation $ax^2 + 3xy - 2y^2 + 13y - 6x + c = 0$ represent two straight lines perpendicular to each other.
14. Find condition that slope of a line given by $ax^2 + 2hxy + by^2 = 0$ is complementary to the slope of other line.
15. The distance of point $P(x_1, y_1)$ from each of the two lines passing through origin is p. Find equation of pair of straight lines.
16. The slope of a line given by equation $3x^2 - 4xy + \lambda y^2 = 0$ is 3 times the slope of other line. Find λ.
17. The equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ where $\lambda \in R$, represents a pair of straight lines If θ is the angle between them, then find value of $\sin q$.
18. The angles between straight lines $ax^2 + 2hxy + by^2 = 0$ and between lines $a'x^2 + 2h'xy + b'y^2 = 0$ are complementary; prove $(a+b)^2(a'+b')^2 = 16(h^2 - ab)(h'^2 - a'b')$

19. A line meets axes in A(7, 0) and B(0, 5). A variable line PQ perpendicular to AB, meets axes in P and Q. If AQ and BP intersect in R, prove locus of R is $x^2 + y^2 - 5y - 7x = 0$.

20. Prove, angle between lines $(x^2 + y^2)(p^2 + q^2 - r^2) = (px + qy)^2$ shall be $p/2$ if $p^2 + q^2 = 2r^2$

21. Angle between the lines $x^2 + 2xy \cot\theta + y^2 = 0$ is

(a) $\tan^{-1}\frac{1}{2}$ (b) $\cos^{-1}(\tan\theta)$ (c) $\sin^{-1}(\cot\theta)$ (d) $\tan^{-1}2$

22. Angle between the lines $x^2 - 2xy \operatorname{cosec} q + y^2 = 0$ is :

(a) $\tan^{-1}(\cot\theta)$ (b) $90+\theta$ (c) $90-\theta$ (d) $\theta/2$

23. The Equation of two pairs of opposite sides of a rectangle are $x^2 - 8x + 12 = 0$ and $y^2 - 20y + 96 = 0$ Equation of its diagonals are :

(a) $y-x=6$, $x-y+6=0$ (b) $y+x=6$, $x+y=14$

(c) $x-y+6=0$, $x+y=14$ (d) none of these

24. The equation of angle bisectors of the lines $x^2\cos^2\theta - xy\sin 2\theta = y^2\sin^2\theta$ are

(a) $x^2 - 2xy\operatorname{cosec}2\theta + y^2 = 0$ (b) $x^2 + 2xy\operatorname{cosec}2\theta - y^2 = 0$

(c) $x^2 - 2xy\operatorname{cosec}2\theta - y^2 = 0$ (d) none of these

25. The locus of point equidistant from straight lines $y^2 + 5xy - 6x^2$ is

(a) $5(x^2 - y^2) + 14xy = 0$ (b) $5(x^2 - y^2) = 14xy$

(c) $(x^2 - y^2) + 3xy = 0$ (d) $(x^2 + y^2) - 3xy = 0$

26. The straight line represented by the equation $x^2 + xy - 6y^2 + 7x + 31y - 18 = 0$ are :

(a) $x+3y+2=0$, $x-2y-9=0$ (b) $x+3y-2=0$, $x-2y+9=0$

(c) $x-3y+2=0$, $x+2y-9=0$ (d)none of these

27. If equation $9x^2 + \lambda xy + y^2 + 6x + 2y - 8 = 0$ represents two parallel lines then λ =

(a) 6 (b) –6 (c) 4 (d) 3

28. Distance between two parallel lines given by equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ is :

(a) $\frac{7\sqrt{5}}{5}$ (b) $\frac{7\sqrt{5}}{10}$ (c) $\frac{13}{\sqrt{10}}$ (d) $\frac{13}{2\sqrt{5}}$

29. The product of perpendicular dropped form (4, 5) on lines $y^2 + 4xy + x^2 = 0$ is :

(a) $\left(\frac{11}{2}\right)^2$ (b) $\left(\frac{13}{2}\right)^2$ (c) 36 (d) none of these

30. Equations $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$...($\alpha$) and $8x^2 + 10xy - 12y^2 - 4x + 14y - 4 = 0$....($\beta$) represents pair of straight line, then :

(a) a straight line given by α is || to one given by β

(b) a straight line given by α is $\perp$ to one given by β

(c) of the two lines given by α one is parallel and other perpendicular to the lines given by β.

(d) Exists no relation between lines given by α and β

31. If equation $x^2 + y^2 + 2gx + 2fy + 1 = 0$ represents a pair of straight line then,

(a) $f^2 - g^2 = 1$ (b) $f^2 + g^2 = 1$

(c) $g^2 - f^2 = 1$ (d) $g^2 - f^2 = \frac{1}{4}$

32. The sum of slopes of straight lines given by equation $4x^2 + 2hxy - 9y^2 = 0$ is equal to twice the product of slopes then h =

(a) 2 (b) 4 (c) –4 (d) 1

33. Equation of pair of straight lines perpendicular to the lines of $lx^2 + 2mny + ny^2 = 0$ is:

(a) $ly^2 - 2mxy + nx^2 = 0$ (b) $lx^2 - 2mxy - ny^2 = 0$

(c) $ly^2 - 2mxy - nx^2 = 0$ (d) none of these

34. The acute angle between the lines given by equation $\lambda y^2 + (1+\lambda^2)xy + \lambda x^2$, when $\lambda = \tan\theta$ is :

(a) 2θ (b) $90 + \theta$ (c) $90 - 2\theta$ (d) none of these

35. The angle between straight line of equation $ax^2 + 2hxy + by^2 = 0$ is $p/3$ then $4h^2$ =

(a) $3a^2 + 10ab + 3b^2$ (b) $3a^2 - 10ab + 3b^2$

(c) $3a^2 + 8ab + 3b^2$ (d) none of these

36. The equation of angle bisectors of the angles between lines $(ax + by)^2 = 3(bx - ay)$ is

(a) one parallel to $ax + by + c = 0$ (b) one $\perp$ to $ax + by + c = 0$

(c) one parallel to $ax - by + c = 0$ (d) one $\perp$ to $ax - by + c = 0$

37. Find value of a and b, when lines given by equation $ax^2 - 7xy + by^2 = 0$ are perpendicular to $4x - 3y = 0$ and $4y + 3x = 0$.

(a) $a = -12$ $b = 12$ (b) $a = 12$, $b = -12$

(c) $a = b = 12$ (d) $a = b = -12$

38. If lines given by $a^2x^2 + bcy^2 = (b + c)\,axy$, are coincident then :

(a) = 0 (b) $a = b$ (c) $b = c$ (d) $a = c$

4.10 Equations of line joining origin to the point of intersection of two curves or a curve and a straight line.

Let f (x, y) = 0 be the equation of a curve in second degree and L = 0 be the equation of line. We know homogeneous equation in second degree represents two lines passing through origin.

$\therefore$ If with the help of L = 0, we can make second degree equation f(x,y) = 0 homogeneous, we shall get the equation of line joining origin to the point of intersection of f(x,y) = 0 and L = 0

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ....(1)

be equation of the curve

and $lx + my + n = 0$ ....(2)

be the equation of straight line.

From (2) we have $\frac{lx+my}{-n}=1$

Multiplying first degree terms of equation (2) by $\frac{lx+my}{-n}$ and the constant term by $\left(\frac{lx+my}{-n}\right)^2$, equation (1) will become homogeneous in second degree as follows.

$$ax^2+by^2+2hxy+(2gx+2fy)\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right)^2=0$$

this is homogeneous in second degree.

$\therefore$ represents two line passing through origin and because this equation has been obtained from (1) with the help of (2), so the lines pass through the points of intersection of (1) and (2).

4.11 Find the equation of two lines joining the origin with the point of intersection of two curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ and also find the condition that they are at right angle.

Given curves are $ax^2 + 2hxy + by^2 + 2gx = 0$ ....(1)

$a'x^2 + 2h'xy + by^2 + 2g'x = 0$ ....(2)

Our aim is to get a second degree homogeneous equation with the help of these two questions. Single degree terms must go.

It can be done by multiplying (1) by g' and (2) by g and then subtracting/

i.e. $ag'x^2+2hg'xy+bg'y^2+2gg'x=0$

$$a'gx^2+2h'gxy+b'gy^2+2gg'x=0$$

Subtracting; $(ag'-a'g)x^2 + 2(hg - h'g)xy + (bg'-b'g)y^2 = 0$

It is homogeneous second degree hence represents 2 straight lines through origin. Lines shall be perpendicular if $(ag'-a'g) + (bg'-b'g) = 0$

$\Rightarrow g(a'+b') = g'(a+b)$.

Example 27 : The straight lines joining origin to the points of intersection of curve $x^2 + y^2 + x - 2y + \lambda = 0$ with the line $x + y = 1$ include an angle of 90^0. Find λ?

Sol. : Curve is $x^2 + y^2 + x - 2y + \lambda = 0$ ....(1)

Straight line $x + y = 1$

$\therefore x^2 + y^2 + x(x + y) - 2y(x + y) + \lambda(x + y)^2 = 0$

is homogeneous in second degree, represent 2 straight line through origin. The straight lines include angle 90^0.

$\therefore \quad (1 + 1 + \lambda) + (1 - 2 + \lambda) = 0 \qquad \Rightarrow \lambda = -1/2$

Example 28 : Show that the straight lines joining origin with the point of intersection of y =- x – 2 and $5x^2 + 12xy - 8y^2 + 8x - 4y + 12 = 0$ are equally inclined to axes.

Sol. : Straight line $y = x - 2 \Rightarrow \dfrac{x-y}{2} = 1$

Making curve equation homogeneous with it

$$5x^2 + 12xy - 8y^2 + 8\left(\frac{x-y}{2}\right)x - 4y\left(\frac{x-y}{2}\right) + 12\left(\frac{x-y}{2}\right)^2 = 0$$

$$\Rightarrow 5x^2 + 12xy - 8y^2 + 4x^2 - 6xy + 2y^2 + 3x^2 - 6xy + 3y^2 = 0$$

$$\Rightarrow 12x^2 - 3y^2 = 0 \quad \Rightarrow y = \pm 2x = 0$$

Lines are equally inclined to axes y

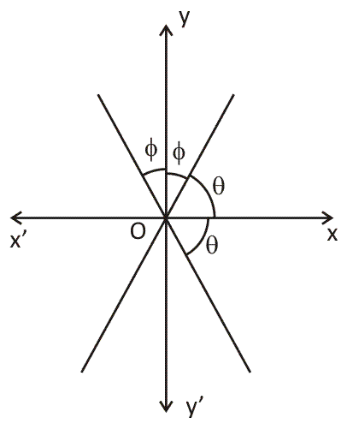

Fig 9

Example 29 : Find equation of straight lines joining origin with point in section of circles $x^2 + y^2 - 4x - 2y - 4 = 0$, $x^2 + y^2 - 6x + 8y + 5 = 0$.

Sol. To get a homogeneous equation from two circles directly not possible circle intersect.

Linear equation of common chord can be obtained

$x^2 + y^2 - 4x - 2y - 4 = 0$ ....(1)

$x^2 + y^2 - 3x + 4y + 5/2 = 0$ ....(2)

Solving $$-x-6y-\frac{13}{2}=0$$

$$\Rightarrow \frac{2x+12y}{-13}=13$$

Now making equation (1) homogeneous with its help.

$$x^2+y^2-4x\left(\frac{2x+12y}{-13}\right)-2y\left(\frac{2x+12y}{-13}\right)-4\left(\frac{2x+12y}{-13}\right)^2=0$$

or $$169(x^2+y^2)+52x(2x+12y)+26y(2x+12y)$$
$$-4(4x^2+48xy+144y^2)=0$$

$$\Rightarrow 257x^2+484xy-75y^2=0$$

This is the required equation of straight lines

Example 30 : Prove that the two straight lines of the equation $Ax^3+3Bx^2y+3Cxy^2+Dy^3=0$ shall be perpendicular if $A^2+3AC+3BD+D^2=0$?

Sol. : Let the two perpendicular lines be $x^2+\lambda xy-y^2=0$ and the third line be Ax – Dy = 0 combined equation is

$$(x^2+\lambda xy-y^2)(Ax-Dy)=0$$

$$\Rightarrow Ax^3+(\lambda A-D)x^2y-(A+\lambda D)xy^2+Dy^3=0$$

comparing with $Ax^3+3Bx^2y+3cxy^2+Dy^3=0$

$\lambda A-D=3B$ and $A+\lambda D=-3c$

Eliminating λ we get

$$A^2+3AC+3BD+D^2=0$$

Example 31 : Find the area of the triangle formed by $ax^2+2hxy+by^2=0$ and $lx+my+n=0$

Sol. : Given equation $ax^2+2hxy+y^2=0$ ....(1)

Let it represent $y-m_1x=0$, $y-m_2x=0$

$$\therefore m_1+m_2=-\frac{2h}{b},\ m_1m_2=\frac{a}{b} \qquad ....(2)$$

Now
$$\left.\begin{array}{ll} OA \text{ is} & y-m_1x=0 \\ OB & y-m_2x=0 \\ AB & lx+my=-n \end{array}\right\} \begin{array}{c} O,(0,0) \\ B\left[\dfrac{-n}{l+mm_2},\dfrac{-m_2n}{l+mm_2}\right] \end{array}$$

and $A\left[\frac{-n}{l+mm_1}, \frac{-m_1 n}{l+mm_1}\right]$

$$\text{area} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = \frac{1}{2}\left[\frac{m_1 n^2}{(l+mm_1)(l+mm_2)} - \frac{m_2 n^2}{(\ )(\ )}\right]$$

$$= \frac{1}{2}\left[\frac{n^2(m_1 - m_2)}{l^2 + lm(m_1 + m_2) + m^2 m_1 m_2}\right]$$

and $m_1 - m_2 = \sqrt{\frac{4h^2}{b^2} - 4\frac{a}{b}} = \frac{2}{b}\sqrt{h^2 - ab}$

Dr = $l^2 - \frac{2hlm}{b} + \frac{am^2}{b} = \left[bl^2 - 2hlm + am^2\right]/b$

$\therefore$ area = $\left(n^2\sqrt{h^2 - ab}\right)/\left(bl^2 - 2hlm + am^2\right)$.

Example 32 : Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines and xy = 0 represent another pair of straight lines. Then prove that the equation of third pair of straight lines which pass through those points where first pair of straight line met co-ordinate axes (i.e.) xy = 0 is $ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4}{c}fgxy = 0$ Prove.

Sol. : Given pair of straight line xy = 0 ....(1)

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ....(2)

Equation (2) represents a pair of straight lines

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ ....(3)

The curve going through the point of intersection of equation (1) and (2) is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda xy = 0$ ....(4)

This equation shall represents two straight line

$$abc + 2\left(h + \frac{\lambda}{2}\right)gf - af^2 - bg^2 - c\left(h + \frac{\lambda}{2}\right)^2 = 0$$

$$\Rightarrow \left(abc + 2fgh - af^2 - bg^2 - ch^2\right) + \lambda gf - c\frac{\lambda^2}{4} - c\lambda h = 0$$

$$\Rightarrow \lambda fg - c\frac{\lambda^2}{4} - hc\lambda = 0 \qquad \lambda \neq 0$$

$$\Rightarrow \frac{\lambda c}{4} = fg - hc \Rightarrow \lambda = 4(fg - hc)/c$$

Putting this values of λ in equation (4)

$$ax^2+2hxy+by^2+2gx+2fy+c+\frac{4(fg-hc)}{c}xy=0$$

$$ax^2+2hxy+by^2+2gx+2fy+c+\frac{4fg}{c}xy=0$$

Example 33: Prove that chords of the curve $2x^2-3y^2-4x+7y=0$ which subtend a right angle at origin always pass through a fixed point.

Sol : Let ax + by =1 ....(1)

Be the chord of curve $2x^2-3y^2-4x+7y=0$ ....(2)

i.e. the angle between the lines joining origin with the points of intersection of (1) and (2) is 90^0.

Making (2) homogeneous with the help of (1)

$$2x^2-3y^2-4x(ax+by)+7y(ax+by)=0$$

Co-efficient of x^2 is 2+4a ; Co-efficient of y^2 is -3+7b

Sum of these co-efficient = 0

$\therefore$ 2 + 4a – 3 + 7b = 0

$$\Rightarrow b=\frac{-4a+1}{7}$$

$\therefore$ Equation of chords is ax + $\frac{-4a+1}{7}$y =1

$$\Rightarrow 7ax-4ay=y-7=0$$

$$\Rightarrow (y-7)+a(7x-4y)=0$$

$\Rightarrow P+\lambda Q=0$ i.e. line passes through the point of intersection of the curve y-7=0

7x-4y=0 i.e. straight line (chord) always passes through a fixed point (4, 7).

Example 34 : If (x_1,y_1) be the orthocentre of the triangle formed by lines $ax^2+2hxy+by^2=0$ and lx+my+n=0 then prove

$$\frac{x_1}{l}=\frac{y_1}{m}=\frac{(-n)(a+b)}{am^2+bl^2-2hlm}$$

Sol : Let equation of sides of triangle ABC be

BC, $y-m_1x = 0$

CA, $y-m_2x = 0$

AB, lx + my + n = 0

Where C is (0,0) and A is $\left(\frac{(-n)}{l+mm_1},\frac{(-m_2n)}{l+mm_2}\right)$

Equation of perpendicular from C on AB

$mx-ly=0$ ....(α)

Equation of perpendicular from A on BC

$$m_1y+x+\frac{n(1+m_1m_2)}{l+m_2m}$$ ....(β)

Orthocentre is point of intersection of (α) and (β)

$$\frac{x(l+m_2n)}{-ln(1+m_1m_2)}=\frac{y(l+m_2m)}{-mn(1+mm_2)}=\frac{1}{mm_1+l}$$

$$\Rightarrow\frac{x}{l}=\frac{y}{m}=-\frac{n(1)+m_2m_1}{(l+m_2m)(mm_1+l)}$$

$$=\frac{-n(1+m_1m_2)}{ml(m_1+m_2)+m^2(m_1m_2)+l^2}$$

But $m_1m_2=\frac{a}{b}$ and $m_1+m_2=-\frac{2h}{b}$ ortho center is (x_1, y_1)

$$\therefore\frac{x_1}{l}=\frac{y_1}{m}=\frac{-n\left(1+\frac{a}{b}\right)}{\frac{-2hml}{b}+\frac{a}{b}m^2+l^2}=\frac{-n(a+b)}{am^2+bl^2-2hlm}$$

Example 35 : if lines of equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be equidistant from origin. Prove $f^4-g^4=c(bf^2-ag^2)$

Sol. : Let lines be $y-m_1x+c_1=0,\ y-m_2x+c_2=0$

$\therefore y^2-(m_1+m_2)xy+m_1m_2x^2-(c_1m_2+c_2m_1)x+(c_1+c_2)y+c_1.c_2=0$ .....(α)

Given equation $ax^2+2hxy-by^2+2gx+2fy+c=0$ ....(β)

Comparing. $(m_1+m_2)=-\frac{2h}{b};m_1m_2=\frac{a}{b}$

$$c_1m_2+c_2m_1=-\frac{2g}{b},c_1+c_2=\frac{2f}{b};c_1c_2=\frac{c}{b}$$

condition $\frac{c_1}{\sqrt{1+m_1^2}} = \frac{c_2}{\sqrt{1+m_2^2}}$ equidistanct from origin

$$\Rightarrow c_1^2\left(1+m_2^2\right) - c_2^2\left(1+m_1^2\right) = 0$$

$$\Rightarrow c_1^2 - c_2^2 + c_1^2 m_2^2 - c_2^2 m_1^2 = 0$$

$$\Rightarrow (c_1 - c_2)(c_1 + c_2) + (c_1 m_2 + c_2 m_1)(c_1 m_2 - c_2 m_1) = 0$$

(i) $(c_1 - c_2)^2 = (c_1 + c_2)^2 - 4c_1 c_2 = \frac{4h^2}{b^2} - \frac{4c}{b}$ ....(A)

(ii) $(c_1 m_2 - c_2 m_1)^2 = (c_1 m_2 + c_2 m_1)^2 - 4c_1 c_2 m_1 m_2 = \frac{4g^2}{b^2} - \frac{4ac}{b^2}$

$\therefore$ from (A) $\frac{-2f}{b}\sqrt{\frac{4f^2 - 4cb}{b^2}} = \frac{-2g}{b}\sqrt{\frac{4g^2 - 4ac}{b^2}}$

$$\Rightarrow f\sqrt{f^2 - cb} = g\left(\sqrt{g^2 - ac}\right)$$

$$\Rightarrow f^2\left(f^2 - cb\right) = g^2\left(g^2 - ac\right) \Rightarrow f^4 - g^4 = c\left(bf^2 - ag^2\right)$$

Practice Worksheet (Foundation Level) – 4(c)

1. Equation $(x-1)^2+4(x-1)(y+2)+p(y+2)^2=0$ represents a pair of straight lines them p is =

(a) 3 (b) 5 (c) –3 (c) – 5

2. Straight lines $x^2 - 4xy + y^2 = 0$ and $2x + 3y = 1$ form a triangle OAB. Median OD is :

(a) $7x + 8y = 0$ (b) $7x - 8y = 0$ (c) $5x+ 7y = 0$ (d) $7x + 6y = 0$

3. Point of intersection of lines given by equation $12x^2-7xy-12y^2-41x+38y+22=0$ is:

(a) (1, 2) (b) (–1, 2) (c) (2, 1) (d) (2, –1)

4. Straight lines joining origin with the point of intersection of $x^2 + hxy - y^2 + gx + fy = 0$ and $fx - gy = \lambda$ are

(a) equally inclined to x-axis (b) mutually perpendicular

(c) coincident (d) none of these

5. If intercept of $lx + m y = 1$ by the curve $x^2 + y^2 + 2gx + 2fy + c = 0$, subtends a right angle at the origin then :

(a) $c(l^2+m^2)+2(gl+fm+1)=0$ (b) $c(l^2+m^2)=2(gl+fm+1)$

(c) $c(l^2+m^2)=gl+fm+1$ (d) $c(l^2+m^2)+(gl+fm+1)=0$

6. $ax^3-bx^2y-cxy^2+dy^3=0$ represents three straight lines through origin, two of them are perpendicular then :

(a) $a^2+ac-bd+d^2=0$ (b) $a^2-ac+bd+d^2=0$

(c) $d^2-bd-ac+a^2=0$ (d) $d^2+bd-ac-a^2=0$

7. The slopes of straight lines given by equation $ax^2+2hxy+by^2=0$ are in the ratio 1 : 3, then

(a) $8h^2 = 9ab$ (b) $4h^2 = 8ab$

(c) $8h^2 = 3ab$ (d) $3h^2 = 4ab$

8. The acute angle between the lines given by equation $(x+y-1)^2=4x^2$ is :

(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}2$ (c) $\tan^{-1}3/2$

9. Equations of lines through origin perpendicular to lines $5(x-2)^2+4(x-2)y-3y^2=0$ is

(a) $3x^2 - 4xy - 5y^2 = 0$ (b) $3x^2 + 4xy - 5y^2 = 0$

(c) $5y^2 - 4xy - 3x^2 = 0$ (d) $5x^2 - 4xy + 3y^2 = 0$

10. The equation $x^3 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ represents a pair of straight lines then λ =

(a) 4 (b) 3 (c) 2 (d) 1

11. Inclination of straight lines given by equation $ax^2 - hxy + by^2 + 4x - 4y + 4 = 0$ with x – axis are complementary. If inclination of one straight line with x – axis is $22\frac{1}{2}^0$, then :

(a) (a+b) (a+b+4) = 4h (b) (a+b)(a+b – 4) = 4 h

(c) (a+b) (a –b + 4) = 4h (d) (a – b) (a – b + 4) = 4h

12. One of the straight lines given by equation $ax^2 + 2hxy - 6y^2 = 0$ is inclined at 45^0 with x-axis and other is inclined at 120^0 with x-axis, the h = ..., a = ...

13. Lines given by equation $x^2 + 3xy + 2y^2 + 4x + 5y + 3 = 0$ intersect in point ....

14. Angle between straight lines $x^2 + hxy - 6y^2 - 3x + 14y - 4 = 0$ is then h = ...

15. The area of triangle formed by straight lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is ....

16. The area of triangle formed by straight lines $5a^2x^2 + abxy - 4b^2y^2 = 0$ and $ax - by = 1$ is ....

17. Two straight lines through origin include an angle of $\pi/2$. One is inclined at q with x-axis. The combined equation of straight line is ....

18. Equation $3y^2 - xy - 4x^2 + 4x - 18y + 24 = 0$ gives a pair of straight lines. Equation $y^2 - xy - 2x^2 - 4x - 2y = 0$ gives a pair of straight line. Equation $y^2 - xy - 2x^2 - 4x - 2y = 0$ gives another pair of straight lines. If one line be common to them, then calculate angle between the remaining two lines.

19. Find equation of lines through (1,2) and perpendicular to lines $3x^2 - xy - 4y^2 = 0$.

20. Two straight lines intersect in (a, –a) at right angles. If one of them is inclined at 45^0 with x-axis then find combined equation of straight line.

21. Find internal angles of triangles formed by straight lines $y^2 - 4xy + x^2 = 0$ and $y + x + 4\sqrt{6} = 0$. Give co-ordinate of vertices and area of the triangle.

22. $L_1 = ax + by + c = 0$, L_2 is $lx + my + n = 0$ and these intersect in P and include angle θ. Find line L_3, different from L_2 which passes through P and makes angle θ with L_1.

23. Show that straight lines $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ lie along sides of a square. Find area of this square.

24. Prove that the angle bisectors of straight lines $(ax + by)^2 = 3(bx - ay)^2$ are respectively parallel and perpendicular to $ax + by + c = 0$.

25. If angle between straight lines $4x^2 + hxy - 3y^2 + 4x + 7y - 2 = 0$ be $\tan^{-1}(7)$ then h = ...

26. Angle between straight lines $ax^2 + 4hxy + by^2 = 0$ be 66^0 , then $16h^2$ = ...

27. If a straight lines given by equation $ax^2 + 2hxy + by^2 = 0$ be coincident with one of the lines given by equation $a'x^2 + 2h'xy + b'y^2 = 0$ and other lines of the two include angle 90, then prove

$$\frac{ha'b'}{b'-a'} = \frac{h'ab}{b-a} = \frac{1}{2}\sqrt{-aa'bb'}$$

28. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent two parallel lines then prove distance between them is $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$.

29. Prove that the equation of perpendicular lines from (x_1, y_1) on $ax^2 + 2hxy + by^2 = 0$ is $b(x - x_1)^2 - 2h(x - x_1)(y - y_1) + a(y - y_1)^2 = 0$

30. Prove that the perpendicular dropped from origin on straight lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ when multiplied give result $\frac{c}{\sqrt{(a-b)^2 + 4h^2}}$

31. If $4x - 3y = 0$ coincides with one of straight lines through origin represented by $ax^2 + 2hxy + by^2 = 0$ then prove $16b + 24h + 9a = 0$.

32. Find point equation of pair of straight lines which pass through origin and make equilateral triangle with (a) $x = \lambda$ (b) $y = 5$

33. Find p and q if straight lines $3x^2 - 8xy + qy^2 + 2x + 14y + p = 1$ represent pair of perpendicular lines.

❖ --------------------- ❖

5. Circle

5.1 Definition

The locus of a point, which moves in plane, in such a way that its distance from a fixed point is always the same constant, is circle. If (h, k) is the fixed point and ‘a’ the fixed constant distance then locus is

$$(x-h)^2+(y-k)^2=a^2 \qquad \text{..............(1)}$$

This is the standard equation of a circle. The fixed point (h, k) is called centre of the circle and fixed distance a is called radius of circle.

$$x^2+y^2=a^2 \qquad \text{...............(2)}$$

is equation of circle whose centre is origin and radius a.

5.2 General equation of a circle : Equation (1) on expansion is

$$x^2+y^2-2hx-2ky+\left(h^2+k^2-a^2\right)=0 \qquad \text{...............(2)}$$

This is a second degree equation . There is

(i) no term of xy in this equation. And

(ii) co-efficient of x^2 = co-efficient of y^2 = 1

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{............... (3)}$$

is the general equation of circle.

Comparing it with (a) we see.

(a) its centre is (–g, –f) and $h^2 + k^2 - c = a^2$

$\therefore$ $(\text{radius})^2 = (-g)^2 + (-f)^2 - c \Rightarrow r=\sqrt{g^2+f^2-c}$

5.3 Equation of circle touching axes :

(a) In fig. (a) c is centre of circle. Co-ordinates of c are (α,β). Circle touches y-axis

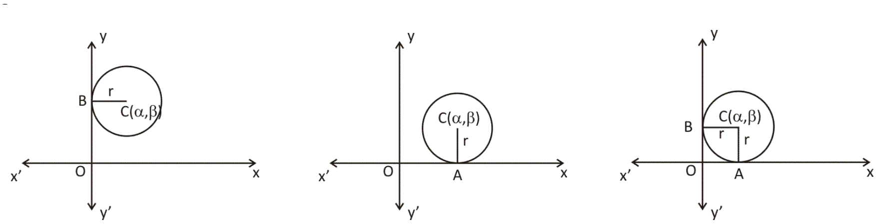

Fig 1(a) **Fig 1(b)** **Fig 1(c)**

radius r = α. Equation of circle is

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2$$

$\Rightarrow x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$

(b) In fig. 1 (b) circle touch x-axis at A. Centre C is (α, β), radius r = β here;

$\therefore$ Equation Θ, $(x-\alpha)^2 + (y-\beta)^2 = \beta^2$

$$\Rightarrow x^2 + y^2 - 2x\alpha - 2\beta y + \alpha^2 = 0$$

(c) In fig. 1 (c) circle touches both axes. X-axis at A and y axis at B. Centre C is (α, β) this circle is in first quadrant $\alpha = \beta = r$

$\therefore$ Equation is $(x-\alpha)^2 + (y-\alpha)^2 = \alpha^2$

$$\Rightarrow x^2 + y^2 - 2\alpha x - 2\alpha y + \alpha^2 = 0$$

(i) In 2nd quadrant centre of such circle shall be $(-\alpha, \alpha)$; Equation $x^2 + y^2 + 2\alpha x - 2\alpha y + \alpha^2 = 0$

(ii) In 3rd quadrant centre of such circle shall be $(-\alpha, -\alpha)$; Equation $x^2 + y^2 + 2\alpha x + 2\alpha y + \alpha^2 = 0$

(iii) In 4th quadrant centre is $(\alpha, -\alpha)$ Equation is $x^2 + y^2 - 2\alpha x + 2\ \ y + \alpha^2 = 0$.

5.4 Equation of circle drawn on the line joining (x_1, y_1) and (x_2, y_2) as diameter.

Let A be (x_1, y_1), B(x_2, y_2) and P any point (x, y) on this circle.

$\angle$APB = 90 (angle in semicircle is 90^0)

$$\therefore \left(\frac{y-y_1}{x-x_1}\right) \times \left(\frac{y-y_2}{x-x_2}\right) = -1$$

$$\Rightarrow (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

Solved Examples

Example 1 : A (3,1), B(14, –1) and C(11, 5) are three points. Find the equation of the circle passing through them.

Sol. : Let circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(1)

It passes through three given points.

Point (3, 1) $10 + 6g + 2f + c = 0$ ....(2)

Point (14, –1) $197 + 28g - 2f + c = 0$ ....(3)

Point (11, 5) $146 + 22g + 10f + c = 0$ ....(4)

(3) – (2) $22g - 4f + 187 = 0$ ....(5)

(3) – (4) $6g - 12f + 51 = 0$ ....(6)

Solving (5) and (6) $g = -\frac{17}{2}, f = 0$.

Putting values of g and f in 2, C = 41

$\therefore$ Equation of circle is $x^2 + y^2 - 17x + 41 = 0$

Note : Circle is circumcircle of triangle ABC.

Example 2 : Find equation of circles whose centres are on x-axis and who touch straight line $x + y\sqrt{3} = 6 + 2\sqrt{3}$ y and $x - y\sqrt{3} = 6 - 2\sqrt{3}$. What shall be its equation if centre lies on y-axis?

Sol. Given straight line $x + y\sqrt{3} = 6 + 2\sqrt{3}$

$x - y\sqrt{3} = 6 - 2\sqrt{3}$ (Let its centre be (α,0) and radius be r)

$$\therefore \frac{\alpha - 6 - 2\sqrt{3}}{2} = \pm \frac{\alpha - 6 + 2\sqrt{3}}{2} = |r|$$

+sign shall cancel α

$$\therefore \alpha - 6 - 2\sqrt{3} = -\alpha + 6 + 2\sqrt{3} \Rightarrow \alpha = 6$$

$$r = \frac{6 - 6 + 2\sqrt{3}}{2} = r = \sqrt{3}$$

Equation is $(x-6)^2 + y^2 = 3 \Rightarrow x^2 + y^2 - 12x + 33 = 0$

(ii) When centre is on y-axis (0, β) then

$$\frac{\beta\sqrt{3} - 6 - 2\sqrt{3}}{2} = \pm\left(\frac{-\beta\sqrt{3} - 6 + 2\sqrt{3}}{2}\right) = |r|$$

+sign; $\beta = 2$, and then r = 3

Equation $x^2 + (y-2)^2 = 9 \Rightarrow x^2 + y^2 - 4y - 5 = 0$

Example 3 : How many circles are possible that touch the three lines x + y = 1, y = x +1 7x – y = 6. Find their centres and radii.

Sol. : Given straight lines are

$$\left.\begin{array}{l} AB\ \ x+y=1 \\ BC\ \ y=x+1 \end{array}\right\} B(0,1)$$

$$\left.\begin{array}{l} BC\ \ y=x+1 \\ CA\ \ 7x-y=6 \end{array}\right\} C\left(\frac{7}{6},\frac{13}{6}\right) > A\left(\frac{7}{8},\frac{1}{8}\right)$$

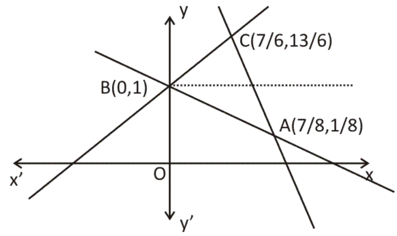

Fig 2

Thus straight line form triangle ABC.

Four circles are possible. Incricle of Δ ABC and three escribed circles.

(i) Angle bisectors of angle A $\frac{x+y-1}{\sqrt{2}} = \pm\frac{7x-y-6}{5\sqrt{2}}$

+ sign gives external, -sign gives internal.

$\therefore$ Internal, $y = -3x + \frac{11}{4}$, External $y = \frac{1}{3}x - \frac{1}{6}$

(ii) Angle bisectors of angle internal B, y = +1, External x=0.

(iii) Angle bisector of angle C,

$$\frac{y-x-1}{\sqrt{2}} = \pm\frac{7x-y-6}{5\sqrt{2}}$$ (Slope of internal bisector is (+ve))

Internal $y = 2x - \frac{1}{6}$, External $y = -\frac{1}{2}x + \frac{11}{4}$

(a) In circle : Bisectors y = +1, $y = 2x - \frac{1}{6}$ centre $\left(\frac{7}{12},1\right)$ $r = \perp$ on any side.

$$= \frac{7/12+1-1}{\sqrt{2}} = \frac{7}{12\sqrt{2}}$$

(b) Eseribed circle opposite to A;

External angle bisector of B is x = 0, of C $y = -\frac{1}{2}x + \frac{11}{4}$, centre $\left(0,\frac{11}{4}\right)$, radius $r_1 = \dfrac{0+\frac{11}{4}-1}{\sqrt{2}} = \dfrac{7}{4\sqrt{2}}$

(c) Escribed circle opposite to B. Inter bisector $\angle B$; y = 1, external bisector of $\angle C$

$= y = -\frac{1}{2}x + \frac{11}{4}$ centre $\left(\frac{7}{2}, 1\right)$, radius $r_2 = \dfrac{7.\frac{7}{2}-1-6}{5\sqrt{2}} = \dfrac{35}{10\sqrt{2}} = \dfrac{7}{2\sqrt{2}}$

(d) Excribed circle opposite to C. External angle bisector of $\angle B$, x = 0, Internal angle bisector of $\angle C$ = y = 2x $-\frac{1}{6}$. Centre $\left(0, -\frac{1}{6}\right)$, $r_3 = \left|\dfrac{-\frac{1}{6}-0-1}{\sqrt{2}}\right| = \dfrac{7}{6\sqrt{2}}$

$\therefore$ centres are, $\left(\frac{7}{12}, 1\right), \left(0, \frac{11}{4}\right), \left(\frac{7}{2}, 1\right), \left(0, -\frac{1}{6}\right)$ and radii

$r = \dfrac{7}{12\sqrt{2}}, r_1 = \dfrac{7}{4\sqrt{2}}, r_2 = \dfrac{7}{2\sqrt{2}}, r_3 = \dfrac{7}{6\sqrt{2}}$

Example 4 : Find equation of three diameters of the circle $x^2 + y^2 + 4x - 6y + 4 = 0$. (i) passing through origin (ii) parallel to x-axis (iii) parallel to y-axis (iv) parallel to x – y = 10.

Sol. : Equation of given Θ $x^2 + y^2 + 4x + 6y + 4 = 0$ centre is (–2, –3). Diameters shall passed through it. $\therefore$ equation of diameter is

$$y + 3 = m (x + 2) \qquad \text{.............(i)}$$

(i) Diameter passes through origin (0,0)

$\therefore$ 0 + 3= m (0 + 2) $\Rightarrow$ 3x – 2y = 0 $\Rightarrow$ m= $\frac{3}{2}$

Equation y+3 = $\frac{3}{2}$ (x+2) $\Rightarrow$ 3x-2y = 0

(ii) Diameter || to x – axis m = 0, y = – 3

(iii) Diameter || to y axis $\frac{1}{m}$ =0 $\Rightarrow x = -2$.

(iv) Diameter || to x – y = 10, m = 1, equation x – y = 1

Example 5 : Find equation of circle which goes through origin and cuts of intercepts 2a and 2b on axes.

Sol : In fig. (3) OA = 2a, OB = 2b. intercepts cut off by circle point A is (2a, 0), B is (0, 2b)

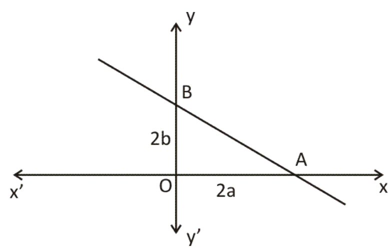

Fig 3

$\therefore (x-2a).(x-0)+(y-0)(y-2b)=0$

$\Rightarrow x^2+y^2-2ax-2by=0$ is equation of circle.

Example 6 : A circle touches y-axis at (0, $\sqrt{3}$) and goes through (–1, 0). Find centre and radius of this circle.

Sol. : Circle touches y-axis at (0, $\sqrt{3}$)

$\therefore$ If centre is (α,β) then $|\alpha|$ = radius and $\beta=\sqrt{3}$. It passes through (–1,0). This means centre is in second quadrant.

$\therefore$ Centre is $\left(-\alpha,\sqrt{3}\right)$, $r=|\alpha|$

Equation $x^2 + y^2 + 2a\,x - 2\sqrt{3}\,y + 3 = 0$

Point on Θ is (–1,0) $\Rightarrow 1+2\alpha+3=0 \therefore \alpha=-2$

$\therefore$ Centre is $\left(-2,\sqrt{3}\right)$ and radius = 2

Example 7 : Find equation of circle which touches x axis and the straight line 4x – 3y + 4 = 0. Its centre lies on x – y – 1 = 0 and is in the third quadrant. **(Dhanbad 92)**

Sol. : Circle touches x-axis, and centre is in third quadrant.

Let centre be $(-\alpha,-\beta)$; $\therefore |\beta|=r$

Equation $x^2 + y^2 + 2\alpha x + 2\alpha\ \beta+\alpha^2=0$ ....(1)

Straight line 4x – 3y + 4 = 0 touches the circle

$\therefore$ perpendicular from $(-\alpha,-\beta)$ on it= radius $-4\alpha + 3\beta + 4 = \underline{+} 5\beta$.

+ sign given $\beta=-2\alpha+2$; –sign, $\beta=\frac{\alpha}{2}-\frac{1}{2}$ centre lies on x –y – 1 = 0

$\Rightarrow -\alpha+\beta-1=0$.

$\therefore \beta=1+\alpha$, $\therefore$ (i) $1+\alpha=-2\alpha+2 \Rightarrow \alpha=1/3$.

$\beta=1+\frac{1}{3}=\frac{4}{3}$

(ii) $1+\alpha=\frac{\alpha}{2}-\frac{1}{2} \Rightarrow \alpha=-3,\ \beta=-2$

This value not admissible as centre $(-\alpha,-\beta)$ now is (3,2) . 1^{st} quadrant.

$\therefore$ Circle is $\left(x+\frac{1}{3}\right)^2+\left(y+\frac{4}{3}\right)^2=\left(\frac{4}{3}\right)^2$

$$\Rightarrow x^2+y^2+\frac{2}{3}x+\frac{8}{3}y+\frac{1}{9}=0$$

Example 8 : Find equation of circle whose radius is 3 and centre lies on y = x –1 and which passes through (7,3) **Roorkee – 88.**

Sol. : Let the circle be $(x-\alpha)^2+(y-\beta)^2=9$. It passes through (7, 3)

$$\Rightarrow (7-\alpha)^2+(3-\beta)^2=9$$

$$\Rightarrow \alpha^2+\beta^2-14\alpha-6\beta+49=0$$

(α,β) lies on y = x – 1 $\Rightarrow \beta=\alpha-1$

$$\therefore \alpha^2+(\alpha-1)^2-14\alpha-6(\alpha-1)+49=0$$

$$\Rightarrow \alpha^2-11\alpha+28=0 \Rightarrow (\alpha-7)(\alpha-4)=0$$

(i) $\alpha=7,\ \beta=\alpha-1=6$ (ii) $\alpha=4, \beta=3$

$\therefore$ Circle are $(x-7)^2+(y-6)^2=9$

and $(x-4)^2+(y-3)^2=9$

Example 9 : Prove that the quadrilateral whose sides are 5x + 3y = 9, x = 3y, 2x – y = 0 and x + 4y +2 = 0 is concyclic. Find equation of circum circle. **(R – 80)**

Sol. : Let

$$\left.\begin{matrix}AB & 5x+3y=9\\ BC & x=3y\end{matrix}\right\} B\text{ is}\left(\frac{3}{2},\frac{1}{2}\right)$$

$$CD\ \ 2x-y=0\}\ C\text{ is}(0,0)$$

$$DA\ \ x+4y+2=0\}\ D\text{ is}\left(-\frac{2}{9},-\frac{4}{9}\right)$$

and point A is $\left(\frac{42}{17},-\frac{19}{17}\right)$ Slope of AB = $-\frac{5}{3}$, of BC, $\frac{1}{3}$; of CD, 2; of AD $-\frac{1}{4}$.

$$\tan B=\frac{\frac{1}{3}+\frac{5}{3}}{1-\frac{5}{9}}=\frac{6\times 9}{3\times 4}=\frac{9}{2}$$

$$\tan D=\frac{2+\frac{1}{4}}{1-\frac{1}{2}}=-\frac{9}{2}=\tan(\pi-B)$$

$\therefore$ B+D = 180^0 Quadrilateral is cyclic

C is (0, 0) circle shall pass through origin.

Circle is $x^2 + y^2 + 2gx + 2fy = 0$.

$$\left.\begin{array}{l}\text{Point B }\left(\frac{3}{2},\frac{1}{2}\right) \Rightarrow \frac{5}{2}+3g+f=0 \\ \text{Point D}\left(-\frac{2}{9},-\frac{4}{9}\right) \Rightarrow g+2f=\frac{5}{9}\end{array}\right\}$$

Solving the equations $g = -\frac{10}{9}, f = \frac{5}{6}$

$\therefore$ Equation of circum circle is $9(x^2+y^2)-20x+15y=0$

Example 10 : Two circles of radii 5 units touch each other at (1, 2) . If the equation of common tangent is 4x + 3y = 10. Find equation of circles.

Sol. : Line joining centres passes through (1,2)

Equation of line is $\frac{y-2}{\sin\theta}=\frac{x-1}{\cos\theta}$ ....(1)

And it is perpendicular to tangent $\tan\theta=\frac{3}{4}$ centres are at 5 and –5 unit distances from (1, 2)

$$\therefore \frac{y-2}{\sin\theta}=\frac{x-1}{\cos\theta}=\pm 5$$

(i) +sign, $x = 5\cos\theta+1=5.\frac{4}{5}+1=5;\ y=5\times\frac{3}{5}+2=5$

centre of this circle is (5, 5)

(ii) –sign, x = – 4 + 1 = – 3 , y = –3 + 2 = – 1

Circles are $(x-5)^2 + (y-5)^2 = 5^2$

$(x+3)^2+(y+1)^2=5^2$

Practice Worksheet (Foundation Level) – 5(a)

1. Find condition between f, g and c so that $x^2+y^2+2gx+2fy+c=0$ represent a circle.
2. Find values of a, h and k if equation $a\,x^2+hxy+2\,y^2+16x-8y+k=0$ represent a circle of radius 5.
3. Find equation of circle that passes through (-2,1), (5,-6) and (4,1).
4. Find circumcentre of the circumscribed circle of ? whose sides are y=x, y=2x and y=3x+2.
5. A circle touches straight line 2x-y-4=0 and its centre is (1,-3). Find its equation.
6. Find equation of circle which passes through (1,-2) and (4,-3) and whose centre lies on 3x+4y=7.
7. Find the equation of circle which touches x-axis at distance of 3 units from origin and cuts an intercept of 6 units on (positive direction) of y axis.
8. The sides of a square are 3x+4y=0, 3x+4y = 20, 4x-3y=0 and 4x-3y=20. Find equation of circle circumscribing it.
9. Find concentric circle of circle $x^2+y^2-8x+12y+15=0$, which goes through point (5,4).
10. Points O (0,0); A(0,2); B(0,4) are fixed points while P is a moving point. If $PO^2+PA^2+PB^2=35$ then prove locus of P is a circle and find its equation.
11. y-2=m(x-1) and y-4=t(x-3) are two straight lines where m and t are variable and mt=k where k is a constant. Find the locus of point of intersection of the two lines. If it is a circle then find value of k.
12. Radius of a circle is 3; circle goes through (7, 3) and its centre lies on y – x + 1 = 0. Find equation of the circle.
13. A circle touches y axis at a distance of 4 units (positive direction) and cuts of an intercept of 6 units from x-axis. Find equation of the circle.
14. y = mx is the chord of the circle whose radius is a and whose diameter is along x-axis (positive direction). If circle goes through origin then find the equation of circle drawn on this chord as diameter.
15. One end of a diameter of circle $x^2 + y^2 - 2x + 6y - 15 = 0$ is (1, 2). Find the other end.
16. Find equation of chord of circle $x^2 + y^2 - 6x - 8y = 0$ whose mid point is (3,3). Prove that its length is $4\sqrt{6}$.
17. y = 2x is a chord of circle $x^2 + y^2 - 10x = 0$ find equation of circle whose diameter is this chord.

18. Find equation of circle passing through $A(-1, -3)$ and touching the line $4x + 3y - 12 = 0$ at point $P(3,0)$.

19. Find equation of the circle whose centre lies on $x - y = 5$ and which touches the lines $2x + y = 5$ and $x - 2y = 4$.

20. $3x + 4y = 16$ and $x - 2y = 2$ are two diameter of the circle and straight-line $3x - 4y + 7 = 0$ touches it. Find equation of circle

21. The two diameters of a circle are $x + y = 1$ $2x + 3y = 5$ and circle cuts off an intercept of 6 units on the line $3x - 4y - 2 = 0$. Find equation of circle.

5.5 Length of intercepts of axes by a circle :

If a circle meets x –axis in two points A and B then length of intercept AB can be calculated by putting y = 0 in the equation of the circle. If circle is

$x^2+y^2+2gx+2fy+c=0$

then y = 0 gives $x^2+2gx+c=0$

If the two points are $(x_1, 0)$ and $(x_2, 0)$ then

$x_1 + x_2 – 2g$, $x_1 x_2 = c$

$\therefore$ Length of intercept $x_1 \sim x_2$ = $\sqrt{4g^2-4c}=2\sqrt{g^2-c}$

Similarly by putting x = 0 in the equation of circle, $y^2-2fy+c=0$ shall given the length of intercept on y-axis as $2\sqrt{f^2-c}$

a. If circle touch x-axis then $g^2 – c = 0 \Rightarrow g^2=c$

b. If circle touch y-axis then $f^2-c=0 \Rightarrow f^2=c$

5.6 Length of chord of a circle :

Let y = mx + c be the chord and circle $S_1 = 0$. It meets the circle in point (x_1, y), (x_2, y_2)

$\therefore$ Length of chord = $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Points (x_1, y_1) and (x_2, y_2) lie on y = mx + c

$$\therefore \left.\begin{aligned} y_1 &= mx_1+c \\ y_2 &= mx_2+c \end{aligned}\right\} \Rightarrow (y_1-y_2)=m(x_1-x_2)$$

$\therefore$ Length of chord = $\sqrt{(x_1-x_2)^2+m^2(x_1-x_2)^2}$

$=(x_1-x_2)\sqrt{1+m^2}$

So find $x_1 – x_2$; put y = mx + c in the equation of given circle. It shall result in 2nd degree equation in x. Get $x_1 + x_2$ and $x_1 x_2$. Calculate $x_1 – x_2$.

Note : This formula is applicable for all curves. In case of circle to calculate the length of the rod, we can utilize the property of circle "Perpendicular from centre on the chord bisects the chord". If p is length of perpendicular from centre on the chord, then length of chord = $2\sqrt{r^2-p^2}$.

5.7 Equation of common chord of two circles :

If $S_1 = 0$ and $S_2 = 0$ are the equations of two circles and if in these equations co-efficients of x^2 and y^2 are unity, then

$S_1 - S_2 = 0$ is the equation of common chord.

5.8 Equation of a circle through the point of intersections of two circles

If $S_1 = 0$, $S_2 = 0$ are two circles then $S_1 = \lambda S_2 = 0, \lambda \in Ra$ is the equation of circles that pass through the points of intersection of circles S_1 and S_2.

i. If $S_1 = 0$ is equation of a circle and $L = 0$ equation of a straight line intersecting it, then $S + \lambda L = 0$ is the equation of all circles that pass through the point of intersection of circles $S_1=0$ and $L= 0$

5.9 Circles touching each other :

When two circles touches each other internally or externally the point of contact and the centres of circles are collinear. i.e. are in a straight line

(a) When the contact is internal, then O_1, O_2 and point of contact P are in a straight line. If r_1, r_2 are radii ($r_1 > r_2$), then point of contact P divides O_1O_2 externally in the ratio of $r_1 : r_2$ but if the circles touch each other externally then point of contact P divides, O_1O_2 internally in the ratio of $r_1 : r_2$.

Tangent at the point of contact to S_1 is also tangent to S_2. i.e. it is common tangent.

5.10 Position of a point with respect to a circle.

Let circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Its centre C is $(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$. Point $P(\alpha, \beta)$ shall be in side the circle if $PC < r$ i.e. $(-g-\alpha)^2 + (-f-\beta)^2 < g^2 + f^2 - c$

$\Rightarrow g^2 + \alpha^2 + 2gh + f^2 + \beta^2 + 2f\beta < g^2 + f^2 - c$

$\Rightarrow \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c < 0$

(ii) Point P, (α, β) shall be out side the circle, if $CP > \sqrt{g^2 + f^2 - c}$

i.e. $\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c > 0$

(iii) Point P (α, β) shall be on the circle if $CP = r$

i.e. $\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c = 0$.

$\therefore$ To find the position of a point with respect to a circle, put the point in the equation of the circle, it shall results in a constant. If it is negative. The point is inside the circle; if it is = 0, point is on the circle and if it is positive i.e. > 0 point is outside the circle.

Solved Examples

Example 11 : A circle touches y axis at (0,3) and cuts an intercept of 8 units from x-axis. Find its equation.

Sol. : Let (α,β) be the centre of the circle. It touches y –axis at (0, 3)

$\therefore\ \alpha$ =radius of circle, $\beta=3$.

$\therefore$ Equation of circle is $x^2 + y^2 - 2\alpha x - 6y + 9 = 0$

It meets x – axis $\therefore\ x^2-2\alpha x+9=0$

$\therefore\ x_1+x_2=2\alpha$, $x_1 x_2 = 9$

$\therefore x_1 - x_2 = \sqrt{4\alpha^2-36}=8$

$\Rightarrow \alpha^2-9=16 \quad \Rightarrow \alpha=\pm 5$

$\therefore$ Circle is $x^2+y^2\pm 10x-6y+9=0$

Example 12: Find equation of circle which cuts an intercept of 6 on straight line 2x – 5y + 18 = 0. Given centre of circle is $(3,-1)$

Sol. : Perpendicular from centre $(3,-1)$ of the circle on the chord is p,

$$p=\frac{2.3+5.1+18}{\sqrt{29}}=\sqrt{29}$$

Length of chord is $2\sqrt{r^2-p^2}$

$\therefore\ 6=2\sqrt{r^2-p^2} \Rightarrow 9=r^2-29$

$\therefore\ r^2=38$

Equation of circle is $(x-3)^2+(y+1)^2=38$

Example 13 : Find the equation of circle which passes through A (4,3) and B(2, 5) and touches y-axis and also find P on y-axis such that angle APB has largest magnitude.

Sol. : Circle touch y-axis. Its equation is $x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$ ....(1)

Point (4, 3) $\left.\begin{array}{ll} \text{point}(4,3) & 25-8\alpha-6\beta+\beta^2=0 \\ \text{point}(2,5) & 29-4\alpha-10\beta+\beta^2=0 \end{array}\right\}$ ....(2)

Subtracting $4 + 4\alpha-4\beta=0$

$\therefore \beta=\alpha+1$

from (2) $25-8\alpha-6(1+\alpha)+(1+\alpha)^2=0$

$\Rightarrow \alpha^2-12\alpha+20=0 \Rightarrow (\alpha-10)(\alpha-2)=0$

$\therefore$ (i) $\alpha=10, \beta=11$

(ii) $\alpha=2, \beta=3$

Circles $x^2 + y^2 - 4x - 6y + 9 = 0$ ....(α)

And $x^2 + y^2 - 20x - 22y + 121 = 0$........ ($\beta$)

From fig (4) it is clear from circle (α) for max. $\angle$APB, P is (0,3) and at this point Θ touches y-axis.

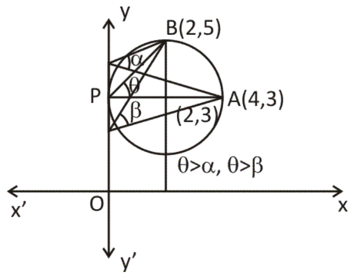

Fig 4

Example 14 : A circle passes through the point (2, 0) and its centre is the limit of the point of intersection of lines 3x + 5y = 1 and $(2 + c)x + 5c^2y = 1$ as C tends to one. **(I.I.T.)**

Sol. : Point of intersection of $\left.\begin{array}{r}3x+5y=1\\(2+c)x+5c^2y=1\end{array}\right\}$ is

$x=\dfrac{c+1}{3c+2}, y=\dfrac{1-3x}{5}$

$z=\lim_{c\to 1}\dfrac{C+1}{3C+2}=\dfrac{2}{5}, y=\left(\dfrac{1}{5}-\dfrac{6}{25}\right)=-\dfrac{1}{25}$

$\therefore$ Centre of circle is $\left(\dfrac{2}{5},-\dfrac{1}{25}\right)$

It passes through (2, 0) $\therefore \left(2-\dfrac{2}{5}\right)^2+\left(\dfrac{1}{25}\right)^2=r^2$

$\therefore$ circle is $x^2+y^2-\dfrac{4}{5}x+\dfrac{2}{25}y+\dfrac{4}{25}+\left(\dfrac{1}{25}\right)^2=\left(\dfrac{8}{5}\right)^2+\left(\dfrac{1}{25}\right)^2$

$\Rightarrow 25(x^2+y^2)-20x+2y-60=0$

Example 15 : Find equation of the chord of a circle whose mid point is (3,3). Circle is $x^2 + y^2 - 6x - 8y = 0$. Find length of chord too.

Sol. : Circle $x^2 + y^2 - 6x - 8y = 0$.

Centre (3, 4); radius 5.

Slope of line joining centre (3, 4) with mid point M of chord $\tan\theta=\dfrac{4-3}{3-3} \Rightarrow \cot\theta=0$. Line is parallel to y-axis.

$\therefore$ chord shall be parallel to x-axis.

$\therefore$ Equation of chord y – 3 = 0

p = CM = $\sqrt{0^2+(4-3)^2}=1$

$\therefore$ Length of chord = $2\sqrt{r^2-p^2}=2\sqrt{25-1}=4\sqrt{6}$

Example 16 : Show that the locus of the point, which moves such that the ratio of its distances from two fixed points is constant is a circle. Also show that the circle cannot pass through these points. IIT 90

Sol. : Let the fixed points be A(a, 0), B(–a,0). Let P, (h, k) be the moving point

Given $\frac{PA}{PB}=\lambda \Rightarrow PA^2=\lambda^2 PB^2$

$\Rightarrow (h-a)^2+k^2=\lambda^2\left[(h+a)^2+k^2\right]$

$\Rightarrow h^2+k^2-\left(\frac{1+\lambda^2}{1-\lambda^2}\right)2ah+a^2=0$

Locus is $x^2+y^2-\left(\frac{1+\lambda^2}{1-\lambda^2}\right)2ax+a^2=0$, which is equation of circle.

Point (a, 0) or (–a, 0) do not satisfy it $2a^2 \pm 2a^2\left(\frac{1+\lambda^2}{1-\lambda^2}\right)\neq 0$. Circle does not pass through these points.

Example 17 : Find locus of mid point of chords of the circle $x^2 + y^2 = r^2$ which pass through (h, k).

Sol. : Equation of chord through (h, k)

y – k = m (x – h) ....(1)

Equation of perpendicular from centre (0,0) on this chord

$y=-\frac{1}{m}x$ ....(2)

The locus of mid points of chords shall be the locus of point of intersection of (1) and (2) eliminating m $y-k=-\frac{x}{y}(x-h)$

$\Rightarrow x^2+y^2-xh-yk=0$

Note : Centre (0,0), point (h, k) equation of circle on this line segments as diameter is x(x–h) +y (y–k) = 0 $\Rightarrow x^2 + y^2 - xh - yk = 0$.

Example 18 : A point moves such that the sum of the squares of its distances from the sides of a square of unit side is equal to 9. Show that its locus is a circle whose centre coincides with the centre of square. Find also its radius.

Sol. : Let one vertex of square be at origin and its two sides be along OX and OY. Let P(h, k) be the moving point.

$\therefore\ k^2 + (1-h)^2 + (1-k)^2 + h^2 = 9$

$\Rightarrow 2k^2 + 2h^2 - 2h - 2k = 7$

Locus $x^2 + y^2 - x - y - 7/2 = 0$

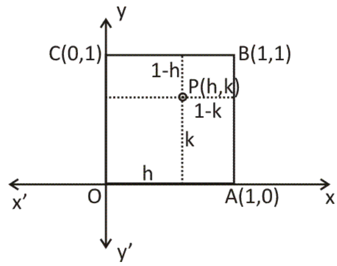

Fig 5

It is a circle, centre $\left(\frac{1}{2},\frac{1}{2}\right)$ which is also the centre of square. Radius = $\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{7}{2}} = \lambda$.

Example 19 : Find locus of the centre of the circle which touches $x^2 + y^2 = a^2$ and $x^2 + y^2 - 4ax = 0$ externally.

Sol. : Let the centre of the circle be (h, k), radius = r.

Give circle $x^2 + y^2 = a^2$, centre (0, 0), radius = a, other circle $x^2 + y^2 - 4ax = 0$, centre (2a, 0), radius = 2a. Since the circle touches both given circle externally.

$\therefore$ a + r = distance between centres

$= \sqrt{h^2+k^2}$ ....(α)

and $2a + r = \sqrt{(h-2a)^2 + k^2}$ .... (β)

from (α) and (β) $a + \sqrt{h^2+k^2} = \sqrt{(h-2a)^2+k^2}$ thus r has been eliminated.

Squaring $a^2 + h^2 + k^2 + 2a\sqrt{h^2+k^2} = h^2 - 4ah + 4a^2 + k^2$

$\Rightarrow 3a^2 - 4ah = 2a\sqrt{h^2+k^2}$

$\Rightarrow (3a-4h)^2 = 4(h^2+k^2)$

Locus is $12x^2 - 24ax - 4y^2 + 9a^2 = 0$.

Example 20 : If circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ then prove $2g'(g - g') + 2f'(f - f') = c - c'$.

Sol. : Equation of common chord of two circle is $2x(g-g') + 2y(f-f') + c - c' = 0$ The centre of second circle must lie on it.

$\therefore\ -2g'(g-g') - 2f'(f-f') + c - c' = 0$

$\Rightarrow 2g'(g-g') + 2f'(f-f') + c - c' = 0$

Example 21 : Consider a family of circle passing through two fixed points A (3,7), B(6, 5) show that the chords in which circle. $x^2 + y^2 - 4x - 6y + 3 = 0$ cuts members of family are concurrent. Find this point.

Sol. : Centres of circles c_1, c_2, c_3 must lie on right bisector of AB. Mid point of AB, M is $\left(\frac{9}{2}, 6\right)$.

$\therefore$ Equation right bisector of AB is, $y - 6 = \frac{3}{2}\left(x - \frac{9}{2}\right)$

$\Rightarrow 6x - 4y = 3$ ....(1)

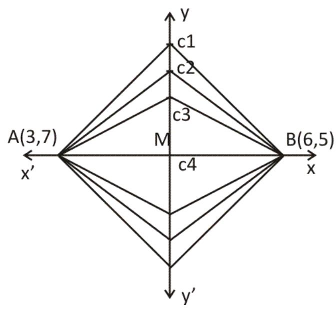

Fig 6

Let (h, k) be centre of one of these circles.

$\therefore (x-h)^2 + (y-k)^2 = (h-6)^2 + (k-5)^2$

$\Rightarrow x^2 + y^2 - 2xh - 2yk + 12h + 10k - 61 = 0$ ....(2)

(h, k) lies on bisector $\Rightarrow 6h - 4k = 3$

$\Rightarrow k = \frac{6h-3}{4}$ and putting this value in (2)

$x^2 + y^2 - 2xh - 2h\left(\frac{6h-3}{4}\right) + 12h + 10\left(\frac{6h-3}{4}\right) - 61 = 0$ ....(3)

$\Rightarrow$ given circle is $x^2 + y^2 - 4x - 6y - 3 = 0$ ....(4)

common chord of (3) and (4) is

$-2xh + 4x - y\left(\frac{6h-3}{2}\right) + 6y - \frac{131}{2} + 27h = 0$

$\Rightarrow 4xh - 8x + 6hy - 3y - 12y - 54h + 131 = 0$

$\Rightarrow (8x + 15y - 131) - h(4x + 6y - 54) = 0$

$\therefore$ Chords are concurrent. Point is point of intersection of

$$\left.\begin{aligned} 8x + 15y - 131 = 0 \\ 4x + 6y - 54 = 0 \end{aligned}\right\} \Rightarrow x = 2, y = \frac{23}{3}, \left(2, \frac{22}{3}\right)$$

Example 22 : If circle C_1, $x^2 + y^2 = 16$ intersects circle C_2 of radius, in such a manner that common chord is of maximum length and has slope of ¾ then centre of circle 2 is .....

Sol. : Circle C_1, $x^2 + y^2 = 16$. Centre (0,0), r = 4

Let C_2 be $(x - h)^2 + (y - k)^2 = 25$

$\therefore$ common chord is $C_1 - C_2 = 0$

$2xh + 2ky = h^2 + k^2 - 9$

Slope of this chord is given and is ¾

i.e. $-\frac{h}{k} = \frac{3}{4}$ ....(1)

Length of perpendicular from (0, 0) on this chord is p, if then

$p = \frac{h^2 + k^2 - 9}{2\sqrt{h^2 + k^2}}$ ....(2)

Length of chord = $2\sqrt{r^2 - p^2} = 2\sqrt{16 - p^2}$

For maximum length p = 0

$\therefore$ from (2) $h^2 + k^2 = 9$ and from (1) $h = -\frac{3}{4}k$ $\therefore \frac{9}{16}k^2 + k^2 = 9$

$\Rightarrow \left(\frac{5}{4}k\right)^2 = 9 \Rightarrow \frac{5}{4}k = \pm 3 \Rightarrow k = \pm\frac{12}{5}$ and $h = -\frac{3}{4}k = \mp\frac{9}{5}$

$\therefore$ Centre of circle C_2 is $\left(-\frac{9}{5}, \frac{12}{5}\right)$ or $\left(\frac{9}{5}, -\frac{12}{5}\right)$

Example 23 : If $\left(m_i, \frac{1}{m_i}\right), m_i > 0$ and $i = 1,2,3,4$ are four points on a circle then $m_1 . m_2 . m_3 . m_4$ =

(a) 0 (b) 1 (c) –1 (c) 2

Sol. : Let Θ be $x^2 + y^2 + 2gx + 2fy + x = 0$

Then $m^2 + \frac{1}{m^2} + 2gm + 2f\frac{1}{m} + c = 0$

$\Rightarrow m^4 + 2hm^3 + cm^2 + 2fm + 1 = 0$

from this equation $m_1 . m_2 . m_3 . m_4 = 1$

Example 24 : Prove that the length of common chord of circles. $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ is $\sqrt{4c^2 - 2(a-b)^2}$

Sol. : Circles are $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0$

$x^2 + y^2 - 2bx - 2ay + a^2 + b^2 - c^2 = 0$

Equation of common chord is x – y = 0

Perpendicular from centre (a, b) on it is $p = (a-b)/\sqrt{2}$

Length of Common Chord = $2\sqrt{r^2 - p^2}$

$$= 2\sqrt{c^2 - \frac{(a-b)^2}{4}} == \sqrt{4c^2 - (a-b)^2}$$

Example 25 : A ball moving round a circle $x^2 + y^2 - 2x - 4y - 20 = 0$ leaves it tangentially at point (4,-2). After getting reflected from a line, it passes through the centre of circle. Find equation of line if its perpendicular distance from point P is 5/2.

Sol. Let Θ be $x^2 + y^2 - 2x - 4y - 20 = 0$

$\therefore$ Center (1,2) , r = 5.

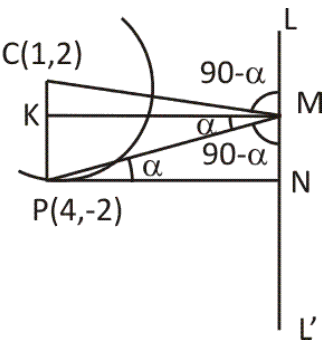

Fig 7

In Fig 7, C is the centre of Θ, P is th point (4,-2), PM is the tangent. Ball travels in this direction after leaving the circle, strikes straight line LL' (plane mirror) at M and returns along MC.

The Normal ($\perp$ to LL') at M should bisect $\angle CMP$.

Let $\angle KMP = \alpha \Rightarrow \angle CMP = 2\alpha \Rightarrow \angle PML' = 90 - \alpha$

PM(tangent) $\perp$ PC

$$\therefore \frac{PM}{CP} = \cot 2\alpha$$

$$\therefore PM = CP\cot 2\alpha = 5\cot 2\alpha \qquad \text{..........(i)}$$

from right angle triangle PNM (PN=5/2)

$$\therefore \frac{PM}{PN} = \sec\alpha \Rightarrow PM = \frac{5}{2}\sec\alpha \qquad \text{...........(ii)}$$

from (i) and (ii)

$$\frac{5}{2\cos\alpha} = \frac{5\cos 2\alpha}{2\sin\alpha\cos\alpha} \Rightarrow 1 = \frac{\cos 2\alpha}{\sin\alpha}$$

$$\Rightarrow 2\sin^2\alpha + \sin\alpha - 1 = 0$$

$$\Rightarrow (\sin\alpha + 1)(2\sin\alpha - 1) = 0$$

$$\therefore \sin\alpha = \frac{1}{2} \Rightarrow \alpha = 30^\circ,\ \sin\alpha \neq -1$$

Slope of CP = $\frac{-2-2}{4-1} = -\frac{4}{3} \Rightarrow$ Slope of PM = $\frac{3}{4}$

And if slope of LL' =m, then

$$\tan(90-\alpha)=\tan 60^\circ=\frac{m-\frac{3}{4}}{1+\frac{3}{4}m}=\sqrt{3}$$

$$\sqrt{3}(4+3m)=4m-3\Rightarrow m=\frac{4\sqrt{3}+3}{4-3\sqrt{3}}=-\frac{4\sqrt{3}+3}{3\sqrt{3}-4}$$

Equation of LL' $y=+\frac{4\sqrt{3}+3}{4-3\sqrt{3}}x+c$

$$y=mx+c$$

$\perp$ from (4, –2) on it is $\frac{5}{2}$.

$$\therefore \frac{5}{2}=\frac{M.4-(-2)+c}{\sqrt{1+M^2}}$$

$$\sqrt{1+M^2}=\left(1+\frac{57+24\sqrt{3}}{43-24\sqrt{3}}\right)^{\frac{1}{2}}=\left(\frac{100}{43-24\sqrt{3}}\right)^{\frac{1}{2}}=\frac{10}{4-3\sqrt{3}}$$

$$\therefore C=\frac{5}{2}.\left(\frac{10}{4-3\sqrt{3}}\right)-4\left(\frac{4\sqrt{3}+3}{4-3\sqrt{3}}\right)-2=\frac{-10\sqrt{3}}{4-3\sqrt{3}}$$

$\therefore$ Equation LL' $y=\frac{4\sqrt{3}+3}{4-3\sqrt{3}}x+\frac{5-10\sqrt{3}}{4-3\sqrt{3}}$

$$\Rightarrow y\left(4-3\sqrt{3}\right)=\left(4\sqrt{3}+3\right)x-5\left(2\sqrt{3}-1\right)$$

Example 26 : Find equation of circle whose diameter is common chord of circles. $x^2+y^2+4x+3y+2=0$, $x^2+y^2+2x+3y+1=0$.

Sol. : Common chord of given circle $S_1 - S_2 = 0$.

$2x + 1 = 0$ i.e. $P = 0$

and equation of the circle passing through point of intersection of $S_1 = 0$, $S_2 = 0$ is $S_1 + \lambda P = 0$.

i.e. $x^2+y^2+4x+3y+2+\lambda(2x+1)=0$

$$\Rightarrow x^2+y^2+x(4+2\lambda)+3y+2\lambda=0$$

Its centre is $\left[-(2+\lambda),-\frac{3}{2}\right]$. It must lie on common chord $\Rightarrow -2(2+\lambda)+1=0$

$\therefore \lambda = -\frac{3}{2}$ $\therefore$ Circle required is

$x^2 + y^2 + (4 - 3)x + 3y + 2\left(-\frac{3}{2}\right) = x^2 + y^2 + x + 3y - 3 = 0$

Example 27 : Show that chords of curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend right angle at origin are concurrent.

Sol. : Equation of curve $3x^2 - y^2 - 2x + 4y = 0$ ....(1)

Let chord be $y = mx + c$ ....(2)

Making (1) homogeneous with the help of equation (2)

$3x^2 - y^2 - 2x\left(\frac{y-mx}{c}\right) + 4y\left(\frac{y-mx}{c}\right) = 0$ angle between these lines is 90^0.

$\therefore$ coefficient of x^2 + Co-efficient of $y^2 = 0$ $\Rightarrow 3 + \frac{2m}{c} + \left(-1 + \frac{4}{c}\right) = 0$

$\therefore$ $C = -(m + 2)$

Equation of chord is $y = mx - (m + 2)$

$\Rightarrow (y+2) - m(x-1) = 0$ and this is of type $P + \lambda Q = 0$

$\therefore$ Chords always pass through point of intersection of $y + 2 = 0$ and $x - 1 = 0$ i.e. through $(1, -2)$.

Example 28 : The straight line $x\cos\alpha + y\sin\alpha = p$ cuts the circle $x^2 + y^2 = a^2$ in M and N. Find equation of circle drawn on MN as diameter. **(R –81)**

Sol. : Equation of circle passing through the points of intersection of given circle and the given line is

$(x^2 + y^2 - a^2) + \lambda(x\cos\alpha + y\sin\alpha - p) = 0$. Centre of circle is $\left(-\frac{\lambda}{2}\cos\alpha, -\frac{\lambda}{2}\sin\alpha\right)$.

If MN is diameter of the circle then centre must lie on it

$\Rightarrow -\frac{\lambda}{2}\cos^2\alpha - \frac{\lambda}{2}\sin^2\alpha = p$

$\Rightarrow \lambda = -2p$

$\therefore$ circle is $x^2 + y^2 - a^2 - 2p(x\cos\alpha + y\sin\alpha) + 2p^2 = 0$

Example 29 : Find locus of the mid point of chords that subtend an angle of $\frac{2\pi}{3}$ at the centre of circle $4x^2+4y^2-12x+4y+1=0$.

Sol. : In the (fig. 8), AB is chord of circle $\angle BCA = 2\pi/3$ CM is perpendicular from centre C on it. M is mid point of AB. Let it be (h, k).

From right angled triangle CM/AC = cos $\pi/3$

i.e. CM = r / 2.

Circle is $4x^2 + 4y^2 - 12x + 4y + 1 = 0$

Centre is $\left(\frac{3}{2},-\frac{1}{2}\right)$ $r=\sqrt{\frac{9}{4}+\frac{1}{4}-\frac{1}{4}}=\frac{3}{2}$

$\therefore$ CM = $\frac{3}{2}.\frac{1}{2}=\frac{3}{4}$

and $\therefore$ $\left(h-\frac{3}{2}\right)^2+\left(k+\frac{1}{2}\right)^2=\frac{9}{16}$

$\Rightarrow h^2+k^2-3h+k+\frac{31}{16}=0$

$\therefore$ Locus mid point is $x^2+y^2-3x+y+\frac{31}{16}=0$

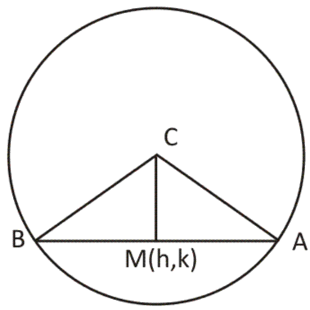

Fig 8

Practice Worksheet (Foundation Level) – 5 (b)

1. Find the equation of circle which goes through (–3, –6) and touches both axes.

2. Find the equation of circle which goes through origin and cuts intercepts of length 2(a+b) and 2(a–b) from axes.

3. Find the equation of the circle which goes through (a, b) and (b, a) and origin.

4. Find the equation of the circle which touches x = – 4, y = 0 and x = 6.

5. A is (2, 5) and B is (6, 5) a circle passes through A and B the centre of circle is 6 cm away from AB. Find the equation of the circle.

6. A circle goes through origin and points of intersection of the straight line 3x + 4y = 12 with axes. Its equation is **(MNR 90)**

   (a) $x^2 + y^2 - 3x - 4y = 0$ (b) $x^2 + y^2 - 4x - 3y = 0$

   (c) $(x - 2)^2 + (y + 3/2)^2 = 0$ (d) $x^2 + y^2 + 4x + 3y = 0$

7. Points (2, 0), (0,1), (4,5) and (0, c) lies on a circle then c is : **(MNR 82)**

   (a) 1 (b) 14 (c) 7 (d) 14/3

8. OP and OQ are tangents to the circle from origin. Circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Centre is C. The circle circumscribing the quadrilateral OPCQ is :

   (a) $x^2 + y^2 - gx - fy = 0$ (b) $x(x + g) + y(y + f) = 0$

   (c) $x^2 + y^2 - 2gx - 2fy = 0$ (d) $x(x - g) + y(y + f) = 0$

9. The equation of the circle which passes through (5, 4) and is concentric with $x^2 + y^2 - 8x + 12y + 15 = 0$ is :

   (a) $x^2 + y^2 - 8x + 12y + 7 = 0$ (b) $x^2 + y^2 - 8x + 12y - 49 = 0$

   (c) $x^2 + y^2 - 8x + 12y + 49 = 0$ (d) none of these

10. A circle goes through (1, 5) and (–2, –3) and its centre lies on 2x – 3y + 4 = 0 equation of circle is :

    (a) $x^2 + y^2 + x - 2y - 17 = 0$ (b) $x^2 + y^2 - x - 2y - 15 = 0$

    (c) $x^2 + y^2 + x + 2y - 36 = 0$ (d) $x^2 + y^2 + x + y - 8 = 0$

11. For circles $x^2 + y^2 + 6x - 4y - 3 = 0$ and $x^2 + y^2 - 4x + 20y + \lambda = 0$ to touch each other externally λ should be :

    (a) 8.4 (b) 31 (c) 23 (d) 29

12. x + y = 4 is a chord of circle $x^2 + y^2 - 3(x + y) + 2 = 0$. The equation of circle drawn on this chord as diameter is

    (a) $x^2 + y^2 + 4x - 4y + 6 = 0$ (b) $x^2 + y^2 + 4x + 4y + 6 = 0$

    (c) $x^2 + y^2 - 4x - 4y + 6 = 0$ (d) $x^2 + y^2 - 4x - 4y - 6 = 0$

13. Locus of mid points of chords through (2, –2) of the circle $x^2 + y^2 = 16$ is :

(a) $x^2 + y^2 + 2x + 2y = 0$ (b) $x^2 + y^2 – 2x – 2y = 0$

(c) $x^2 + y^2 – 2x + 2y = 0$ (d) $x^2 + y^2 + 2x – 2y = 0$

14. The equation of circle passing through the points of intersection of circles $x^2 + y^2 – 6x + 2y – 4 = 0$ and $x^2 + y^2 + 2x – 4y = 0$ and having its centre at y = x is

(a) $7(x^2 + y^2) – 10x – 10y – 12 = 0$ (b) $5(x^2 + y^2) – 10 x – 10y – 12 = 0$

(c) $7(x^2 + y^2) – 10x – 10y + 12 = 0$ (d) $7(x^2 + y^2) +10x + 10y + 12 = 0$

15. Abscissa of two points A and B are roots of equation $x^2 – 2ax – b^2 = 0$ and ordinates are roots of equation $x^2 + 2px + q^2 = 0$. Equation of circle on AB as diameter is :

(IIT 84)

(a) $x^2 + y^2 + 2ax + 2py – b^2 –q^2 = 0$

(b) $x^2 + y^2 – 2ax + 2py + q^2 – b^2 = 0$

(c) $x^2 + y^2 – 2ax – 2py + q^2 – b^2 = 0$

(d) none of these

16. A circle touches y axis and is concentric with $x^2 + y^2 –4x – 6y + 4 = 0$. Its equation is :

(a) $x^2 + y^2 – 4x – 6y + 4 = 0$ (b) $x^2 + y^2 – 4x – 6y – 9 = 0$

(c) $x^2 + y^2 – 4x - 6y + 9 = 0$ (d) $2x^2 + 2y^2 – 8x – 12y + 4 = 0$

17. C_1 is concentric with circle C_2 ; $x^2 + y^2 – 6x + 12y + 15 = 0$ and area of C_1 is just double to the area of C_2. Equation of C_1 is :

(a) $x^2 + y^2 – 6x –12y – 75 = 0$ (b) $x^2 + y^2 – 6x + 12y – 15 =0$

(c) $2x^2 + 2y^2 – 12x – 24y – 15 = 0$ (d) $x^2 + y^2 – 6x + 12y – 60 = 0$

18. A circle touches y-axis at (0, 2) and cuts of intercepts of lengths 3 from positive direction of x-axis. Its equation is :

(a) $x^2 + y^2 + 5x – 6y + 6 = 0$ (b) $x^2 + y^2 – 5x –4y + 4 = 0$

(c) $x^2 + y^2 - 5x + 4y + 4 = 0$ (d) $x^2 + y^2 - 5x – 4y + 4 = 0$

19. The equation of the circle whose two diameters are 2x – 3y + 12 = 0 and x + 4y = 5 and whose area is 154 sq units is :

(a) $x^2 + y^2 – 4x + 6x – 36 = 0$ (b) $x^2 + y^2 + 4y – 6x – 36 = 0$

(c) $x^2 + y^2 – 6x – 4y – 36 = 0$ (d) $x^2 + y^2 + 6x + 4y – 36 = 0$

20. A circle whose radius is a, passes through origin and its diameter is along x-axis. y = mx is a chord of this circle. The equation of circle described on this chord as diameter is :

(a) $(1 + m^2)(x^2 + y^2) + 2a(x + my) = 0$ (b) $(1 + m^2)(x^2 + y^2) - 2a(x + my) = 0$

(c) $(1 + m^2)(x^2 + y^2) - 2a(x - my) = 0$ (d) $(1 + m^2)(x^2 + y^2) + 2a(x - my) = 0$

21. Two circles $x^2 + y^2 - 4x + 6y + \lambda = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other at (3,–1) externally then λ is :

(a) 8 (b) 6 (c) 11 (d) 15

22. Sides of a triangle are x = 0, y = 0 and lx + my = 1. The centre of the circle circumscribing it is :

(a) $\left(\frac{l}{2}, \frac{m}{2}\right)$ (b) $\left(\frac{m}{2}, \frac{l}{2}\right)$ (c) $\left(\frac{1}{2l}, \frac{1}{2m}\right)$ (d) $\left(\frac{2}{l}, \frac{2}{m}\right)$

23. A circle passes through points of intersection of straight line $\lambda x - y + 4 = 0$ and $x - 2y + 2 = 0$ with axes. Then λ =

(a) ± 2 (b) –2 (c) $-\frac{1}{2}$ (d) 2

24. The equation of circle which passes through point of intersection of circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ and passes through (1, 1) is :

(a) $x^2 + y^2 - 3x + 1 = 0$ (b) $x^2 + y^2 + 3x - 5 = 0$

(c) $x^2 + y^2 - 5x + 3 = 0$ (d) $x^2 + y^2 + 5x - 7 = 0$

25. From point (8,9) secants are drawn of the circle $x^2 + y^2 = 36$. Locus of mid points of the chords so formed is :

(a) $x^2 + y^2 - 8x - 9y = 0$ (b) $x^2 + y^2 + 8x + 9y = 0$

(c) $x^2 + y^2 - 8x - 9y + 36 = 0$ (d) $x^2 + y^2 + 8x + 9y + 16 = 0$

26. Point (4, 5) is inside the circle $x^2 + y^2 - 6x - 8y = 0$. Chords of circle pass through this point. Locus of their mid point is:

(a) $(x - 4)(x - 3) + (y - 4)(y - 5) = 0$ (b) $x^2 + y^2 - 7x - 9y + 30 = 0$

(c) $x^2 + y^2 - x - y + 19 = 0$ (d) $x^2 + y^2 - 7x + 9y + 18 = 0$

27. The equation of circle whose diameter is the common chords of circles $x^2 + y^2 + 4x - 6y - 4 = 0$ and $x^2 + y^2 - x + 6y + 2 = 0$ is :

(a) $3(x^2 + y^2) + 4x + 6y - 4 = 0$ (b) $13(x^2 + y^2) + 12x + 18y - 4 = 0$

(c) $13(x^2 + y^2) + 7x - 6y - 6 = 0$ (d) $x^2 + y^2 + 9x - 18y - 10 = 0$

28. The equation of a circle whose centre lies on $\frac{x}{a} - \frac{y}{b} = 2$ and which passes through the point of intersection of $x^2 + y^2 - 2ax = 0$ and $x^2 + y^2 - 2by = 0$ is

(a) $x^2 + y^2 - ax + 3by = 0$ (b) $x^2 + y^2 + ax - 3by = 0$

(c) $x^2 + y^2 - 3ax + by = 0$ (d) $x^2 + y^2 + 3ax - 2by = 0$

29. The locus of mid points of the chords of the circle $x^2 + y^2 - 6x + 8y = 0$ which subtend a right angle at the centre of the circle is :

(a) $x^2 + y^2 - 6x + 8y + \frac{25}{2} = 0$ (b) a concentric circle

(c) $x^2 + y^2 + 6x - 8y - \frac{25}{2} = 0$ (d) $x^2 + y^2 + 8x - 6y + \frac{25}{2} = 0$

5.11 Parametric Co-ordinates

(a cos α, a sin α) is any point on circle $x^2 + y^2 = a^2$, as it satisfies the equation. In co-ordinates of point there is only one variable α thus when co-ordiantes of a point of a curve are expressed in terms of one variable, these are called parametric co-ordiantes.

5.12 Tangent

If a straight line meets a curve in two coincident points it is called tangent to curve at that, point.

(a) Θ is $x^2 + y^2 + 2gx + 2fy + c = 0$ a straight line meets it in P and Q. P is (x_1, y_1) Q is (x_2, y_2) Equation straight line

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

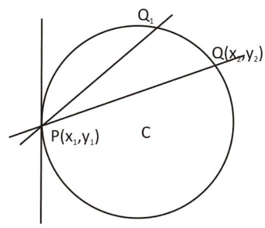

Fig 9

Points lie on Θ

$$\therefore\ x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{............(2)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad \text{...........(3)}$$

Subtracting, $(x_1^2 - x_2^2) + (y_1^2 - y_2^2) + 2g(x_1 - x_2) + 2f(y_1 - y_2) = 0$

$$\Rightarrow (y_1 - y_2)(y_1 + y_2 + 2f) = -(x_1 - x_2)(x_1 + x_2 + 2g)$$

$$\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}$$

Now equation (1) is $y - y_1 = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}.(x - x_1)$ as Q approaches P i.e. $(x_2, y_2) \rightarrow (x_1, y_1)$ the straight line becomes tangent.

$\therefore$ for tangent $y - y_1 = -\frac{2x_1 + 2g}{2y_1 + 2f}(x - x_1)$

$$\Rightarrow (y - y_1)(y_1 + f) + (x_1 + g)(x - x_1) = 0$$

$$\Rightarrow xx_1 + yy_1 + gx + fy - x_1^2 - y_1^2 - gx_1 - fy_1 = 0$$

But from (2) $x_1^2 + y_1^2 + gx_1 + fy_1 = -c - gx_1 - fy_1$

$\therefore$ Tangent at (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(b) The equation of tangent can also determined with the help of calculus.

Given curve $x^2 + y^2 + 2gx + 2fy + c = 0$

Differentiating with respect to x $2x+2y.\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}=-\frac{(g+x)}{f+y}$

at (x_1, y_1) $\quad m=-\frac{(g+x_1)}{(f+y_1)}$

$\therefore$ Equation of tangent at (x_1, y_1) of $(y-y_1)=-\frac{(g+x_1)}{(f+y_1)}(x-x_1)$

$\Rightarrow (x-x_1)(g+x_1)+(y-y_1)(f+y_1)=0$

Which on simplification as done in (a) gives tangent at (x_1, y_1)

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)=0$$

(c) Equation of tangent at (x_1, y_1) of the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

(d) In general if in the equation of curve
x^2 is replaced by xx_1
y^2 is replaced by yy_1
$2x$ is replaced by $x + x_1$
$2y$ is replaced by $y + y_1$

then the resultant equation is the equation of tangent to that curve at (x_1, y_1)

(e) If point is (a cos α, y sin α) then tangents is $x\cos\alpha+y\sin\alpha=a$ to circle $x^2+y^2=a^2$. Similarly $(x-h)\cos\alpha+(y-k)\sin\alpha=a$ is any tangent Θ $(x-h)^2 + (y-k)^2 = a^2$

5.13 To find the condition so that y = mx +c is tangent to circle $x^2 + y^2 = a^2$.

Circle has the special property, the radius through the point of contact of tangent is perpendicular tangent. Thus if perpendicular from centre on a straight line is equal to radius of circle then that straight line is tangent to circle

straight line y = mx + c, centre of Θ is (0, 0)

$\therefore \frac{-c}{\sqrt{1+m^2}}=\pm a \Rightarrow c=\pm a\sqrt{1+m^2}$

$\therefore y=mx\pm a\sqrt{1+m^2}$ is any tangent to the circle.

(ii) Point of contact – If (x_1, y_1) is point of contact then tangent is $xx_1 + yy_1 - a^2 = 0$ comparing with $mx - y + \sqrt{1+m^2}.a=0$

$$x_1 = \frac{-am}{\sqrt{1+m^2}}, y_1 = \frac{a}{\sqrt{1+m^2}}$$

$\therefore$ point of contact of this tangent is $\left(-\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$

5.14 Normal

The normal at any point of curve is perpendicular to the tangent at that point of the curve. In case of circle all normals pass through the centre of the circle. Therefore equation of the line joining point on circle to the centre of circle shall given the equation of normal of that point.

5.15 Chord of contact

P (x_1, y_1) is a point outside circle $x^2 + y^2 = a^2$ PA and PB are tangents to the circle. Point of contact are A and B. The line joining these point of contact is called chord of contact with respect to point P.

Let A be (x', y') and B(x", y"). Tangents at these points are

$xx' + yy' = a^2$ ....(1)

$xx'' + yy'' = a^2$ ....(2)

Both pass through (x_1, y_1)

$\therefore x_1x' = y_1y' = a^2, \quad x_1x'' + y_1y'' = a^2$ ....(α)

from equation (α) it is clear that (x' , y'), (x", y") lie on $xx_1 + yy_1 = a^2$

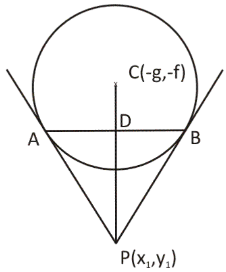

Fig 10

$\therefore$ Equation of chord of contact of (x_1, y_1) w.r. to circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

5.16 Pole and Polar

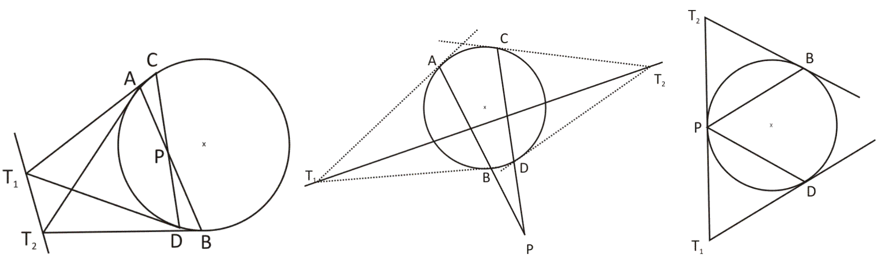

Fig 11(i) **Fig 11(ii)** **Fig 11(iii)**

In fig. 11(i), (ii) and (iii) point P has taken, inside, outside and on the circle respectively. In every figure chords of circle AB and CD pass through P, or are drawn from P. In every fig tangents have been drawn at the ends of these cards and tangents meet at T_1 and T_2.

The polar of a point P with respect to a circle is the locus of point of intersection of tangents drawn at the ends of those chords that pass through P.

(a) In fig (i) P is inside the circle, polar T_1T_2 is outside the circle.

(b) In fig. (ii) P is outside the circle, polar T_1T_2 is chord of contact of P.

(c) In fig. (iii) P is on the circle, polar is tangent at P.

The equation of polar of (x_1, y_1) is $xx_1 + yy_1 = a^2$

Point P is called pole of polar T_1T_2.

5.17 Length of tangent :

Let circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre is $(-g,-f), r = \sqrt{g^2 + f^2 - c}$

PT is tangent from point P (x_1, y_1) to this circle. CT shall be perpendicular on PT.

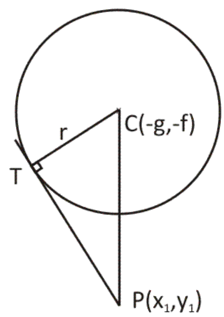

Fig 12

$\therefore PT^2 = PC^2 - r^2$

$= (x_1 + g)^2 + (y_1 + f)^2 - g^2 - f^2 + c$

i.e. $PT^2 = x_1^2 + y_1^2 + 2x_1g + 2y_1f + c$

i.e. if the equation of a circle is give such that all terms are on the L.H.S. and R.H.S. is zero then to find the (length of tangent)^2 from a point P, substitute the point for x and y and remove zero.

5.18 Power of a point with respect to a circle

If P, (x_1, y_1) is a point on side the circle S = 0 and PAB is a secant meeting it in A and B the PA. PB is called power of point P, with respect to the circle S = 0 and you know PA. PB = PT^2 where PT is tangent to the circle.

5.19 Properties of Polar :

1) If polar of a point P with respect to circle S = 0 passes through a point Q, these polar of Q with respect to this circle shall pass through P.

2) The chord of the circle is parallel to the polar of its mid point.

Let (h, k) be the mid point of a chord of circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(1)

Polar of (h, k) is $xh + yk + g(x + h) + f(y + k) + c = 0$

Slope of this polar = $-\left(\frac{h+g}{k+f}\right)$ .... (α)

Slope of line joining mid point (h, k) to the centre of circle $(-g,-f)$ is $\frac{k+f}{h+g}$.

But this line is perpendicular to the chord.

$\therefore$ slope of chord = $-\left(\frac{h+g}{k+f}\right)$ which is (α)

$\therefore$ Chord is parallel to the polar of its mid point.

3) The line joining P to the centre is perpendicular to the polar of P. If G is point of intersection of OP and polar of P then OP. OG = $(\text{radius})^2$.

Let $x^2 + y^2 = a^2$ be the circle and $P(x_1, y_1)$

Polar of P is $xx_1 + yy_1 = a^2$

Slope of polar = $-\frac{x_1}{y_1}$ and slope of $OP = \frac{y_1}{x_1}$ product of slope = –1

$\therefore$ OP is perpendicular to the polar of P.

(ii) OP = $\sqrt{x_1^2 + y_1^2}$ and point of intersection of OP and polar of P is G.

$\therefore$ OG is equal to perpendicular dropped from centre on polar of P.

$\therefore OG = \frac{a^2}{\sqrt{x_1^2 + y_1^2}} \Rightarrow OP.\ OG = a^2 = (\text{radius})^2$

Solved Examples

Example 30 : Find equation of tangent and normal of circle $x^2 + y^2 - 2lx = 0$ at point $[l(1+\cos\alpha), l\sin\alpha]$

Sol. Given circle $x^2 + y^2 - 2lx = 0$, tangent at (x_1, y_1) $xx_1 + yy_1 - l(x + x_1) = 0$ at $[l(1+\cos\alpha), l\sin\alpha]$ $x(l+l\cos\alpha)+yl\sin\alpha-l(x+l+l\cos\alpha)=0$

$\Rightarrow x\cos\alpha + y\sin\alpha = l(1+\cos\alpha)$

(ii) Normal, slope of tangent = $\cot\alpha$

slope of normal = $\tan\alpha$

Equation $(y - l\sin\alpha) = \tan\alpha[x - l - l\cos\alpha]$

$\Rightarrow y\cos\alpha - l\sin\alpha\cos\alpha = x\sin\alpha - l\sin\alpha - l\sin\alpha\cos\alpha$

$\Rightarrow x\sin\alpha - y\cos\alpha = l\sin\alpha$

Example 31: Find the equation of tangent to the circle $(x-3)^2 + (y+2)^2 = 25$ which is

(a) inclined at 60^0 with x-axis

(b) equally inclined with axes.

Sol. : Any tangent to $(x-3)^2 + (y+2)^2 = 25$ is $y + 2 = m(x-3) \pm 5\sqrt{1+m^2}$

(a) $m = \tan 60^0 = \sqrt{3}$, tangent is $y + 2 = \sqrt{3}(x-3) \pm 5.2$

i.e. $y - \sqrt{3}x = \pm 10 - 3\sqrt{3} - 2$

(b) Equally inclined to axes means $m = \pm 1$

$\therefore$ tangent $y + 2 = \pm(x-3) \pm 5\sqrt{2}$

Example 32 : For what value of k the straight line $4x + 3y + k = 0$ is tangent to circle $2x^2 + 2y^2 = 5$.

Sol. : Circle $x^2+y^2=\frac{5}{2}$ circle (0,0), r = $\sqrt{5}/2$ straight line shall be tangent if $\perp$ from centre (0,0) on it is equal to radius of circle. Straight line is $4x + 3y + k = 0$

$\therefore \frac{k}{\pm 5}=\sqrt{\frac{3}{2}} \quad \Rightarrow k=\pm\frac{5\sqrt{5}}{\sqrt{2}}$

Example 33 : Find the equation of circle whose centre is (2,0) and are tangent to it is $\sqrt{3}y-x-2=0$.

Sol. : $\sqrt{3}y-x-2=0$ is tangent

$\therefore$ perpendicular from (2, 0) on it = radius of circle

$$\therefore r = \left|\frac{2}{\sqrt{3+1}}\right| = 1$$

$\therefore$ Circle is $(x-2)^2 + y^2 = 1 \Rightarrow x^2 + y^2 - 4x + 3 = 0$

Example 34 : Find equation of tangent to the circle $x^2 + y^2 - 2x + 8y - 23 = 0$ from external point (8, –3).

Sol. : Circle is $(x-1)^2 + (y+4)^2 = 40 = \left(2\sqrt{10}\right)^2$ ....(1)

Any tangent is $y+4 = m(x-1) + 2\sqrt{10}\sqrt{1+m^2}$ ....(2)

It passes though (8, –3).

$$\therefore -3+4 = m(8-1) + 2\sqrt{10}.\sqrt{1+m^2}$$

$$\Rightarrow (7m-1)^2 = 40\left(\sqrt{1+m^2}\right)^2$$

$$\Rightarrow 9m^2 - 14m - 39 = 0$$

$$\Rightarrow (9m+13)(m-3) = 0 \Rightarrow m = -\frac{13}{9}; \text{and } 3$$

Equation of tangent $y+4 = -\frac{13}{9}(x-1) + 2\sqrt{10}\sqrt{1+169/81}$

$$\Rightarrow 9y + 13x = +13 - 36. + 2\sqrt{10}\sqrt{250}$$
$$\Rightarrow 9y + 13x = 77$$

and $y+4 = 3(x-1) + 2\sqrt{10}\sqrt{10}$

$$\Rightarrow y - 3x = 13$$

Example 35 : Show that $y - x + \sqrt{2} = 0$ is tangent to circle $x^2 + y^2 - 2x - 2y + 1 = 0$. Find point of contact as well.

Sol. : Equation of circle $x^2 + y^2 - 2x - 2y + 1 = 0$ ....(1)

Its centre is (1, 1) and radius = 1

Perpendicular from (1, 1) on $y - x + \sqrt{2} = 0$ ....(2)

$$p = \frac{\sqrt{2}}{\sqrt{2}} = 1 = \text{radius}$$

$\therefore$ straight line is tangent to the circle. Let (x_1, y_1) be point of contact. Tangent at (x_1, y_1) is

$$xx_1 + yy_1 - (x+x_1) - (y+y_1) + 1 = 0$$

$$\Rightarrow x(x_1 - 1) + y(y_1 - 1) + (1 - x_1 - y_1) = 0$$ ....(3)

our line $-x + y + \sqrt{2} = 0$ ....(2)

Comparing co-efficient $\frac{x_1 - 1}{-1} = \frac{y_1 - 1}{1} = \frac{1 - x_1 - y_1}{\sqrt{2}}$

$\Rightarrow x_1 + y_1 = 2$ and $x_1 + y_1(\sqrt{2}+1) = \sqrt{2}+1$

Solving $x = \frac{\sqrt{2}+1}{\sqrt{2}}$, $y = \frac{\sqrt{2}-1}{\sqrt{2}}$

$\therefore$ point of contact is $\left(\frac{\sqrt{2}+1}{\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}}\right)$

Other Method : The equation of straight line y = x $-\sqrt{2}$ putting this value of y in the equation of circle

$$x^2 + (x-\sqrt{2})^2 - 2x - 2(x-\sqrt{2}) + 1 = 0$$

$$\Rightarrow 2x^2 - (4+2\sqrt{2})x + (2\sqrt{2}+3) = 0$$

$$\Rightarrow (\sqrt{2}x)^2 - 2.\sqrt{2}x(1+\sqrt{2}) + (\sqrt{2}+1)^2 = 0$$

$$\Rightarrow [\sqrt{2}x - (\sqrt{2}+1)]^2 = 0$$

Equation is perfect square. If shows that two values of x are coincident. Straight line is tangent.

Point of contact $x = (\sqrt{2}+1)/\sqrt{2}$

From straight line point is $\left(\frac{\sqrt{2}+1}{\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}}\right)$

Example 36 : Show that the circle $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other. Find equation of tangent at this point.

Sol. Given circle are $x^2 + y^2 - 4x + 6y + 8 = 0$ ....(1)

And $x^2 + y^2 - 10x - 6y + 14 = 0$ ....(2)

Centres are (2, –5) and (5, 3) radii $r_1 = \sqrt{5}, r_2 = 2\sqrt{5}$

$\therefore$ Distance between centre = $\sqrt{3^2 + 36} = 3\sqrt{5}$

Now $r_1 + r_2 = 3\sqrt{5}$ = distance between centres

$\therefore$ Circles touch each other, externally.

And $S_1 - S_2$ = common chord $\Rightarrow 6x + 12y - 6 = 0$ becomes tangent as it is perpendicular on the line joining centres.

$\therefore$ Equation of common tangents is x + 2y = r

Example 37 : Prove that there are two points on x – axis such that the tangents from them to the circle $x^2 + y^2 - 10x - 8y + 31 = 0$ contain a right angle.

Sol. : From P tangent to circle whose centre is C are PA and PB. $\angle PAC = \angle PBC = 90^0$. Given $\angle APB = 90^0$.

$\therefore$ APBC is a square $\Rightarrow PA = AC =$ radius.

The point P is on x axis. Let it be $(x_1, 0)$ Length of tangent from $(x_1, 0)$ = radius of circle.

$$\therefore x_1^2 - 10x_1 + 31 = \left(\sqrt{25+16-31}\right)^2 = 10$$

$$\Rightarrow x_1^2 - 10x_1 + 21 = 0$$
$$\Rightarrow (x_1 - 7)(x_1 - 3) = 0$$

$\therefore$ (7, 0) and (3, 0) are two such points.

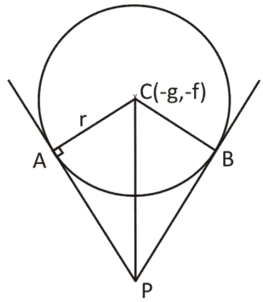

Fig 13

Example 38 : Find pole of straight line 2x + y + 12 = 0 with respect to circle $x^2 + y^2 - 4x + 3y - 1 = 0$.

Sol. : Equation of circle, $x^2 + y^2 - 4x + 3y - 1 = 0$ ....(1)

Let (x_1, y_1) be the pole of $2x + y + 12 = 0$ ....(2)

$$\therefore \text{polar is } xx_1 + yy_1 - 2(x + x_1) + \frac{3}{2}(y + y_1) - 1 = 0$$

$$\Rightarrow x(x_1 - 2) + y(y_1 + \frac{3}{2}) + \left(\frac{3}{2}y_1 - 2x_1 - 1\right) = 0 \qquad \text{..........(3)}$$

Equation (2) and (3) represent the same line

$$\therefore \frac{x_1 - 2}{2} = \frac{y_1 + \frac{3}{2}}{1} = \frac{\frac{3}{2}y_1 - 2x_1 - 1}{12}$$

(i) (ii) (iii)

$\therefore$ from (i) and (ii) $x_1 - 2y_1 = 5$ .....(iv)

$\therefore$ from (ii) and (iii) $16x_1 - 3y_1 = 22$ .....(v)

$\therefore$ from (iv) and (v) $x_1 = 1, y_1 = -2$, pole $= (1, -2)$

Example 39 : Prove that pole of straight line $\frac{x}{a} + \frac{y}{b} = 1$ with respect to circle $x^2 + y^2 = c^2$ is on a fixed line.

Sol.: Equation of circle is $x^2 + y^2 = c^2$ ....(i)

Let the pole of straight line bx+ay-ab = 0 ....(ii)

bc(x_1,y_1) $\therefore$ polar is $xx_1 + yy_1 - c^2 = 0$ ....(iii)

(ii) and (iii) are same line

$\therefore \frac{x_1}{a} = \frac{y_1}{b} = \frac{c^2}{ab}$

$\Rightarrow x_1 = \frac{c^2}{a}, \ y_1 = \frac{c^2}{b}, \ \frac{y_1}{x_1} = \frac{a}{b}$

i.e. (x_1,y_1) lies on the $y = \frac{a}{b}x$, a fixed line.

Example 40 : Find the equation of the chord of circle $x^2 + y^2 - 6x - 8y = 24$ whose mid-point is (-1,2).

Sol. Equation of circle $x^2 + y^2 - 6x - 8y - 24 = 0$

The chord should be parallel to the polar of its mid point. Polar of (-1,2) is

-x+2y-3(x-1)-4(y+2)=24

slope of polar is $\frac{4}{-2}$ =-2

$\therefore$ Equation of the chord $y - 2 = -2(x+1) \Rightarrow y + 2x = 0$

Example 41 : If Polar of P with respect to circle $x^2 + y^2 = a^2$ touches circle $(x-\alpha)^2 + (y-\beta)^2 = b^2$, then find the locus of P.

Sol : Equation of circle $x^2 + y^2 = a^2$ ....(1)

Let P be (h,k), polar $xh + yk - a^2 = 0$ ....(2)

It touches circle $(x-\alpha)^2 + (y-\beta)^2 = b^2$

$\therefore$ perpendicular from (α,β) on it = b

i.e. $\frac{h\alpha + k\beta - a^2}{\sqrt{h^2 + k^2}} = b$

$\Rightarrow \left(h\alpha + k\beta - a^2\right)^2 = b^2\left(h^2 + k^2\right)$

Locus $\left(\alpha x + \beta y - a^2\right)^2 = b^2\left(x^2 + y^2\right)$

Example 42: Find power of (3, 1) with respect to circle $x^2 + y^2 + 6x + y - 8 = 0$

Sol. : Let P be (3, 1) and PT be tangent to the given circle We have to find PT^2

$PT^2 = 9 + 1 + 1 + 18 - 8 = 21$

$\therefore$ Power of P (3, 1) is 21.

Example 43 : Find area of triangle formed by tangents and chord of contact of point (4, 5) w.r. to circle $x^2 + y^2 - 6x + 4y + 4 = 0$.

Sol. Equation Θ $x^2 + y^2 - 6x + 4y + 4 = 0$ centre (3, –2) radius = 3 In fig. (14), PA and PB are tangents from point P (4,5) on circle and AB is chord of contact.

Equation AB, $4x + 5y - 3(x + 4) + 2(y + 5) + 4 = 0$

$$x + 7y + 2 = 0$$

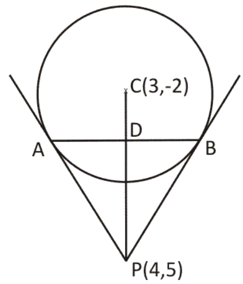

Fig 14

This straight line joining a centre C to point P is perpendicular on AB and bisect AB. BD = AD

Area of Δ PBA = $\frac{1}{2}AB.PD = BD.PD$

$PD = \perp$ from P on AB = $\frac{4 + 35 + 2}{\sqrt{50}} = \frac{41}{5\sqrt{2}}$

CD = $\perp$ from C on AB = $\left|\frac{3 - 14 + 2}{5\sqrt{2}}\right| = \frac{9}{5\sqrt{2}}$

$BD^2 = BC^2 - CD^2 = r^2 - CD^2 = 9 - \frac{81}{50} = \frac{369}{50}$

area = $\sqrt{\frac{369}{50}} \times \frac{41}{5\sqrt{2}} = \frac{41\sqrt{369}}{50}$ sq. unit.

Example 44. OP and OQ are tangents from origin on circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Find equation of circle circumscribing the quadrilateral OPCQ; C is centre of circle.

Sol. : C is centre of circle (–g, –f) and OP is tangent

$\therefore$ $CP \perp OP$ i.e. $\angle OPC = 90^0$ $\Rightarrow$ OC is diameter, O is origin

$\therefore$ circle is $x(x + g) + y(y + f) = 0$.

Example 45 : Find equation of circle which touches circle $x^2 + y^2 + 6x - 6y + 17 = 0$ externally, and to which the straight lines $x^2 - 3xy - 3x + 9y = 0$ are normals.

Sol. : Normals are $x^3 - 3xy - 3x + 9y = 0$

$\Rightarrow (x - 3y)(x - 3) = 0$

$\Rightarrow x = 3y$ and $x = 3$

Point of intersection (3, 1) is centre of Θ.

Given circle is $x^2 + y^2 - 6x + 6y - 17 = 0$, Centre O_2 is (3, –3) and radius $r_2 = \sqrt{9+9-17} = 1$ O_1 is (3, 1); $O_1O_2 = 4$, Circles touch externally $\therefore$ $O_1O_2 = 4 = 1 + r_1 \Rightarrow r_1 = 3$

$\therefore$ Circle is $(x-3)^2 + (y-1)^2 = 3^2$

$\Rightarrow x^2 + y^2 - 6x - 2y + 1 = 0$

Example 46 : Obtain the equation of straight lines passing through the point A(2, 0) and making angle 45^0 with tangent at A of the circle $(x + 2)^2 + (y - 3)^2 = 25$. Find the equation of circle, each of radius 3 and whose centres are on these lines at a distance of $5\sqrt{2}$ from A.

Sol. : Given circles is $x^2 + y^2 + 4x - 6y - 12 = 0$ tangent at (2, 0)

$2x + 2(x + 2) - 3y - 12 = 0$ ....(1)

$\Rightarrow 4x - 3y = 8$ ....(2)

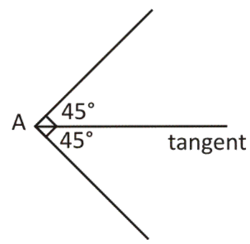

Fig 15

Let equation of line through A be

$y = m(x - 2)$ ....(3)

Slope of tangent is 4/3.

$$\therefore \tan 45 = 1 = \frac{m - 4/3}{1 + \frac{4m}{3}} \Rightarrow 3 + 4m = 3m - 4 \Rightarrow m = -7$$

Slope of other line be + 1/7 as two lines are $\perp$. Writing equation (3) as $\frac{y}{\sin\theta} = \frac{x-2}{\cos\theta} = \pm 5\sqrt{2}$

shall give the co-ordinates of centre of circle which are at distance $5\sqrt{2}$ from A(2,0)

$$\therefore x = \pm 5\sqrt{2}\cos\theta + 2, \quad y = \pm 5\sqrt{2}\sin\theta$$

(i) Taking + sign, $\tan\theta = -7 \Rightarrow \cos\theta = -\frac{1}{\sqrt{50}}, \sin\theta = \frac{7}{\sqrt{50}}$

$\therefore$ centres are $5\sqrt{2}.\left(-\frac{1}{\sqrt{50}}\right) + 2, 5\sqrt{2}.\frac{7}{\sqrt{50}} \Rightarrow (1,7)$ and –sign. $(3,-7)$.

(ii) $m = \frac{1}{7} \Rightarrow \sin\theta = \frac{1}{\sqrt{50}}, \cos\theta = \frac{7}{\sqrt{50}}$

centre +sign (7 + 2, 1) i.e. (9, 1)

– sign (–5, –1)

radius of all circles are 3.

$\therefore$ Circles are (i) $(x+1)^2+(y-7)^2=9$

(ii) $(x-3)^2+(y-7)^2=9$

(iii) $(x-9)^2+(y-1)^2=9$

(iv) $(x+5)^2+(y+1)^2=9$

Example 47 : The locus of point of intersection of tangents to the circle, x = a cos θ , y = a sin θ at points whose parametric angles differ by $\pi/3$. Is ....

Sol. : Circle is $x=a\cos\theta,\ y=a\sin\theta \Rightarrow x^2+y^2=a^2$. The parametric angles of two points differ by $\frac{\pi}{3}$.

Let them be $\alpha,\ \alpha+\pi/3$

$\therefore$ points $(x\cos\alpha, y\sin\alpha)$ and $\left(x\cos\left(\alpha+\frac{\pi}{3}\right), y\sin\left(\alpha+\frac{\pi}{3}\right)\right)$

tangents are $x\cos\alpha+y\sin\alpha=a$ ....(1)

and $x\cos\left(\alpha+\frac{\pi}{3}\right)+y\sin\left(\alpha+\frac{\pi}{3}\right)=a$ ....(2)

$\Rightarrow x\left[\cos\alpha\cos\frac{\pi}{3}-\sin\alpha\sin\frac{\pi}{3}\right]+y\left[\sin\alpha\cos\frac{\pi}{3}+\cos\alpha\sin\frac{\pi}{3}\right]$

$\Rightarrow x\left[\frac{1}{2}\cos\alpha-\frac{\sqrt{3}}{2}\sin\alpha\right]+y\left[\frac{1}{2}\sin\alpha+\frac{\sqrt{3}}{2}\cos\alpha\right]=a$

$\Rightarrow \frac{1}{2}(x\cos\alpha+y\sin\alpha)-\frac{\sqrt{3}}{2}(x\sin\alpha-y\cos\alpha)=a$

$\Rightarrow \frac{1}{2}a-\frac{\sqrt{3}}{2}(x\sin\alpha-y\cos\alpha)=a$ from (1)

$\Rightarrow x\sin\alpha-y\cos\alpha=-\frac{a}{\sqrt{3}}$ ....(3)

Squaring and adding (1) and (3)

$\Rightarrow 3(x^2+y^2)=4a^2$

Example 48 : Find the locus of the centre of circle which touch x-axis and circle x^2 + y^2 – 6x – 6y + 14 externally.

Sol. : Given circle $x^2 + y^2 - 6x - 6y + 14 = 0$

Centre is (3,3), radius = 2

Circle which touches x – axis is $x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 = 0; r = \beta$

The two circles touch externally $\therefore r_1 + r_2 = O_1O_2$

$\therefore (\alpha - 3)^2 + (\beta - 3)^2 = (2 + \beta)^2$

Locus $x^2 - 6x - 10y + 14 = 0$

Example 49 : Origin (A) is the centre of circle C_1 and its radius is 5cm. Circle C_2 and C_3 with radii 4 cm and 3cm. Touch circle C_1 and also axis of x to the right of A. Find equation of any two common tangents to C_2 and C_3.

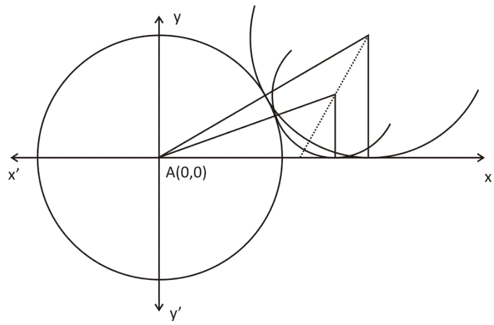

Fig 16

Sol. : Centre of C_1 is (0, 0). Circles C_2 and C_3 touch C_1 but do not touch each other. They intersect.

$\therefore$ Two direct common tangents are possible.

Both touch x-axis $\Rightarrow$ one common tangent is x-axis It is y = 0.

B, Centre of C_2 $\left(\sqrt{9^2 - 16}, 4\right)$ i.e. $\left(\sqrt{65}, 4\right)$

BC produced meets x axis in P. Thus tangents are drawn from P.

Equation BC $y - 3 = \dfrac{4-3}{\sqrt{65} - \sqrt{55}}\left(x - \sqrt{55}\right)$

Put y = 0, $\therefore\ x = \sqrt{55} - 3\left(\sqrt{65} - \sqrt{55}\right)$

$= 4\sqrt{55} - 3\sqrt{65} = \lambda(\text{say})$

$\therefore$ P is $(\lambda, 0)$. Equation of straight line through P is y =m(x–λ); If it is tangent then perpendicular from $\left(\sqrt{65}, 4\right)$ on it is equal to 4.

$$\therefore\ 4 = \frac{m\left(\sqrt{65} - \lambda\right) - 4}{1 + m^2} = \frac{m\left(\sqrt{65} - 4\sqrt{55} + 3\sqrt{65}\right) - 4}{1 + m^2}$$

$$\Rightarrow 4\sqrt{1 + m^2} = 4\left[m\left(\sqrt{65} - \sqrt{55}\right) - 1\right]$$

$$\Rightarrow 1 + m^2 = m^2\left[120 - 10\sqrt{143}\right] + 1 - 2\left(\sqrt{65} - \sqrt{55}\right)m$$

$$\Rightarrow m = 0 \text{ and } m = 2\left[\sqrt{65} - \sqrt{55}\right] / \left[119 - 10\sqrt{43}\right]$$

$\therefore 2^{nd}$ tangent is $y=\left[\frac{2\sqrt{65}-\sqrt{55}}{119-10\sqrt{143}}\left[x-\left(4\sqrt{55}-3\sqrt{65}\right)\right]\right]$

Example 50 : If pole of a straight line with respect to circle $x^2 + y^2 = c^2$ is on circle $x^2 + y^2 = 9c^2$ then polar shall touch $x^2 + y^2 = \frac{1}{9}c^2$.

Sol : Let y=mx+c be a straight line and let (h.k) be the pole of this line with respect to circle $x^2 + y^2 = c^2$.

$\therefore$ polar is $xh+yk=c^2$

pole (h,k) lies on the circle $x^2 + y^2 = 9c^2$

$\therefore$ polar is $h^2 + k^2 = 9c^2$

Polar shall touch circle $x^2 + y^2 = \frac{1}{9}c^2$,

if perpendicular from its centre (0,0) on it = $\frac{c}{3}$

i.e. $\frac{c^2}{\sqrt{h^2+k^2}} = \frac{c}{3} = 9c^4 = c^2(h^2 + k^2)$

i.e. $(h^2 + k^2) = 9c^2$ which is given.

Thus polar touches $x^2 + y^2 = \frac{1}{9}c^2$

Example 51 : a,b are constant and c variable. Prove that the pole of straight line $\frac{x}{a}+\frac{y}{b}=1$ with respect to circle $x^2 + y^2 = c^2$ lies on a fixed straight line.

Sol : Given straight line bx+ay-ab=0 ....(1)

Let (h,k) be its pole with respect to $x^2 + y^2 = c^2$

$\therefore$ polar is $hx+ky-c^2=0$ ....(2)

from (1) and (2) $\frac{h}{b}=\frac{k}{a}=\frac{c^2}{ab}$

$\therefore$ $h=\frac{c^2}{a}$, $k=\frac{c^2}{b}$ and $\frac{h}{k}=\frac{b}{a}$

Locus $x=\frac{b}{a}y$, which is fixed line as a and b are constants.

Example 52 : Find pole of 2x+y+12=0 with respect to circle $x^2 + y^2 - 4x + 3y - 1 = 0$.

Sol : Given circle $x^2 + y^2 - 4x + 3y - 1 = 0$ ....(1)

Given Straight line $2x + y + 12 = 0$ ....(2)

Let (h,k) be the pole, then polar is

$xh+yk-2(x+h)+\frac{3}{2}(y+k)-1=0$

$x(h-2) + y(k+\frac{3}{2}) + (\frac{3}{2}k - 2h - 1) = 0$

comparing (2) and (3)

$$\frac{h-2}{2}=\frac{k+3/2}{1}=\frac{\frac{3}{2}k-2h-1}{12}$$

(i) (ii) (iii)

from (i) and (ii) $h-2k = 5$

from (i) and (iii) $16h - 3k = 22$

$\Rightarrow$ $h=1, k=-2$

$\therefore$ Pole of straight line is (1,-2).

Example 53 : Prove that polar of a given point with respect to circle $x^2 + y^2 - 2kx + c^2 = 0$ (k is a variable) always go through a fixed point irrespective of any value of k.

Sol : Given circle $x^2 + y^2 - 2kx + c^2 = 0$ ....(1)

Polar of (x1, y1) $xx_1 + yy_1 - k(x+x_1) + c^2 = 0$

$\Rightarrow$ $xx_1 + yy_1 + c^2 - k(x+x_1) = 0$

$\Rightarrow$ $P - \lambda Q = 0$

$\therefore$ The Polar always goes through the point of intersection of $xx_1 + yy_1 + c^2 = 0$ and $(x+x_1)=0$

$\therefore$ point $\left[-x_1, \frac{x_1^2-c^2}{y_1}\right]$ which is a fixed point.

Example 54 : The line y=x touches the circle at P and $OP = 4\sqrt{2}$. O is the origin. The point (-10,2) is inside the circle and the length of the chord x+y = 0 is $6\sqrt{2}$. Find equation of the circle. **IIT 84**

Sol : Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Its centre is (-g, -f), radius = $\sqrt{g^2+f^2-c}$

Given $OP = 4\sqrt{2}$

$OP^2 = OC^2 - r^2$ (c, centre)

$\Rightarrow$ $32 = (g^2+f^2)-(g^2+f^2-c) = c$

$\Rightarrow$ $c = 32$ ....(1)

and perpendicular from (-g, -f) on x-y = 0 is equal its radius.

$\therefore$ $-g + f = \sqrt{2}\sqrt{g^2+f^2-c}$

$\Rightarrow$ $g^2 + f^2 - 2gf = 2g^2 + 2f^2 - 2c$

$\Rightarrow$ $g^2 + f^2 + 2gf = 2c$

$\therefore$ $(g + f)^2 = 2 . 32$

$\Rightarrow$ $g + f = \pm 8$ ....(2)

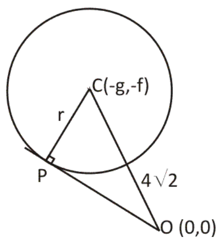

Fig 17

Length of Chord = $2\sqrt{r^2-p^2}$

Equation x + y = 0

p = perpendicular from (-g, -f)

$p = \frac{g+f}{\sqrt{2}}$

$\therefore$ $6\sqrt{2} = 2\sqrt{(g^2+f^2-c)-\frac{(g+f)^2}{2}}$

$\Rightarrow$ $36.2 = 2\ (2g^2+2f^2-2c-g^2-f^2-2gf)$

$\Rightarrow$ $36 = (g^2+f^2-2gf)-64$

$\Rightarrow$ $g-f=\pm10$ ....(3)

from (2) and (3) $g + f = \pm 8$ and $g-f=\pm10$

i) + + g = 9, f = -1 ii) - - g = -9, f = 1

i) - + g = -1, f = 9 ii) + - g = 1, f = -9

Since point (-10, 2) is inside the circle, centre of circle is (-9,1), r = $5\sqrt{2}$

$\therefore$ Equation $(x + 9)^2 + (y - 1)^2 = 82 - 32 = 50$

Practice Worksheet (Foundation Level) – 5(c)

1. $\therefore$ x = 2, x = 6, y = 1, y = – 3 are tangents to a circle, find its equation

2. Find the equation of the tangent to the circle $(x-3)^2+(y+2)^2=25$, inclined at 60^0 to x –axis.

3. Prove that 3x – y – 4 = 0 touches the circle $x^2 + y^2 - 8x + 4y + 10 = 0$. Find point of contact.

4. Find equations of tangents to circle $x^2 + y^2 - 6y = 0$, which are equally inclined to axes.

5. $y^2 - 3x^2 = 0$ are two normals of a circle and 3x + 4 $\pm$15 = 0 are two parallel tangents. Find equation of circle.

6. Find equation of tangent of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are perpendicular to 4x + 3y + 5 = 0.

7. Find the angle between the tangents drawn at (4,3) and (3, 2) of the circle $x^2 + y^2 - 7x - 5y + 18 = 0$. Also find point of intersection of these.

8. $\sqrt{3}y-x=0$ and $y-\sqrt{3}x=0$ are tangent to the circle. Find the locus of centres of these circles.

9. In question 8, find the equation of circle whose radius is 4.

10. For what value of c the straight lines 6x + 8y + c = 0 shall touch the circle $x^2 + y^2 = 4$.

11. Straight line $\sqrt{3}x+y=1$ touches circle $(x-1)^2+(y-1)^2=3/4$, prove. Find point of contact.

12. The vertices of an equilateral triangle are (0, 2) and (0, –2) and the third lies on the positive side of x-axis. Find equation of its circum-circle.

13. The equation of a circle whose centre is at origin and which passes through the vertices of an equilateral triangle, whose median is of length a is .... **(I.I.T. 92)**

14. x + y = 1 is chord of circle $x^2 + y^2 = 4$. Find equation of circle whose diameter is this chord.

15. Find area of triangle formed by x-axis, normals and tangent at ($\sqrt{3}$,1) of circle $x^2 + y^2 = 4$.

16. One end of a diameter of circle $x^2 + y^2 - 6x + 8y = 0$ is (6, 0). Find other end.

17. Find number of common tangents to the circle $x^2 + y^2 + 2x + 8y - 23 = 0$ and $x^2 - y^2 - x - 10y + 9 = 0$.

18. Find the condition that tangents from external point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ include angle of 90^0.

19. Find pole of straight line $9x + y - 28 = 0$ with respect to the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$.

20. $3x + 4y - 43 = 0$ is tangent to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$. Find equation of tangent parallel to it.

21. The centre of a circle is (α, β) and the circle goes through origin. The equation of tangent at origin is:

(a) $\alpha x + \beta y = 0$ (b) $\alpha x - \beta y = 0$ (c) $\alpha x \pm \beta y = 0$ (d) $\alpha x + 2\beta y = 0$

22. If the straight line $lx + my + 2$ touches $x^2 + y^2 = a^2$, then locus of (l, m) is

(a) $a(x^2 + y^2) = 4$ (b) $a^2(x^2 + y^2) = 4$

(c) $axy = 4$ (d) $a^2(x^2 - y^2) = 4$

23. The pole of the line $2ax + by = a^2$ with respect to circle $x^2 + y^2 + 2ax = 0$ is:

(a) $\left(\frac{a}{3}, \frac{b}{3}\right)$ (b) $\left(\frac{a}{3}, -\frac{b}{3}\right)$ (c) $\left(-\frac{a}{3}, \frac{b}{3}\right)$ (d) $\left(-\frac{a}{3}, -\frac{b}{3}\right)$

24. The polar of a point with respect to the circle $x^2 + y^2 - 2kx + c^2 = 0$; $k \in R$

(a) always passes through a fixed point

(b) lies outside the circle

(c) lies inside the circle

(d) is at a constant distance from the centre of circle

25. Circle $x^2 + y^2 - 12x - 3y - 18$ and $x^2 + y^2 + 4x + 9y + 18 = 0$

(a) touch each other (b) have one common tangent

(c) have three common tangents (d) radii are in the ratio of 3 : 1

26. The centres of two circles are (1, 1) and (4, 5) and their radii are 1 and 4. They touch each other (i) point of contact is ………… prove (ii) equation of common tangent is $3x + 4y = 12$.

27. The polar of a point on the circle $x^2 + y^2 = a^2$ with respect to circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$, then a, b, c are in

(a) A. P. (b) G.P. (c) H.P. (d) $a > b > c$

28. If straight line $y = x + \sqrt{2}$ is tangent to the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ then point of contact is

(a) $\left(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}-1}{2}\right)$ (b) $\left(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2}\right)$

(c) $\left(\frac{\sqrt{2}-1}{\sqrt{2}}, \frac{\sqrt{2}+1}{\sqrt{2}}\right)$ (d) $\left(\frac{\sqrt{2}-1}{2}, \frac{\sqrt{2}-1}{2}\right)$

29. The locus of mid points of chord through (h, k) of the circle $x^2 + y^2 = a^2$ is:

(a) $x^2+y^2+xh+yk=0$ (b) $x^2-y^2-xh-yk=0$

(c) $x^2+y^2-xh-yk=0$ (d) $x^2+y^2+xy-yk=0$

30. The equation of the circle of radius 3 which touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ internally at (5, 7) is $25(x^2 + y^2) - 230y - 160x + 560 = 0$ prove.

31. If the tangent at (1, –2) to the circle $x^2 + y^2 = 5$ touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$, then point of contact on this circle is

(a) (3, 1) (b) (–3, –1) (c) (–3, 1) (d) (3, –1)

32. Distances from origins of centres of three circle are in G.P. Circles are $x^2 + y^2 + 2\lambda x - c^2 = 0$ (λ is a variable). Then length of tangents drawn from any point P (on circle $x^2 + y^2 = c^2$) to these circles are in :

(a) A.P. (b) G.P. (c) H.P. (d) None of these

33. From origin chords of circle $(x + 2)^2 + y^2 = 4$ are drawn. Locus of their mid point is:

(a) $x^2 + y^2 + 2x = 0$ (b) $x^2 + y^2 + 2y = 0$

(c) $x^2 + y^2 - 2x = 0$ (d) $x^2 + y^2 - 2y = 0$

34. The equation of circle which passes through (1, 2) and (3, 4) and whose one tangent is $3x + y - 3 = 0$ is

(a) $x^2 + y^2 - 8x - 2y + 7 = 0$ (b) $x^2 + y^2 - 4x - y + 1 = 0$

(c) $x^2 + y^2 - 6x - 4y + 9 = 0$ (d) none of these

35. The equation of the circle which passes through (–1, 3) and touch straight line $x + y = 2$ and $(x - y) = 2$ is :

(a) $x^2 + y^2 + 6x - 4 = 0$ (b) $x^2 + y^2 + 8x - 2 = 0$

(c) $x^2 - y^2 - 10x = 0$ (d) $x^2 + y^2 - 2x - 12 = 0$

36. Circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ shall touch each other if :

(a) $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$ (b) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$

(c) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$ (d) a ~ b = 0

37. The equation of circle whose centre is (3, –1) and which cuts off a chord of 6cm. Length on 2x – 5y + 18 = 0 is :

(a) $x^2 + y^2 - 6x + 2y - 20 = 0$ (b) $x^2 + y^2 - 6x + 2y - 38 = 0$

(c) $x^2 + y^2 - 6x + 2y - 28 = 0$ (d) $x^2 + y^2 - 6x + 2y - 18 = 0$

5.20 Equation of pair of tangents drawn from an external point to the circle

If S=0 is circle $x^2+y^2 = a^2$ and external point P is (x_1, y_1) then equation of pair of tangents is $SS' = T^2$

Where $S = (x^2 + y^2 - a^2)$, $S' = (x_1^2 + y_1^2 - a^2)$ and $T = xx_1 + yy_1 - a^2$

Proof : Let (h, k) be any point on tangent from P (x_1, y_1) to the circle $x^2 + y^2 - a^2 = 0$ Equation of line joining (h, k) and (x_1, y_1) is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$$

$$\Rightarrow x(k - y_1) - y(h - x_1) + (hy_1 - kx_1) = 0$$

since this line is tangent perpendicular from centre (0, 0) on it should be equal to a.

i.e. $$\frac{hy_1 - kx_1}{\sqrt{(k - y_1)^2 - (h - x_1)^2}} = \pm a$$

$$\Rightarrow (hy_1 - kx_1)^2 = a^2\left[(k - y_1)^2 + (h - x_1)^2\right]$$

$$\Rightarrow h^2(y_1^2 - a^2) + k^2(x_1^2 - a^2) - a^2(x_1^2 + y_1^2)$$

$$= 2hkx_1y_1 - 2a^2hx_1 - 2a^2ky_1$$

$$\Rightarrow h^2(x_1^2 + y_1^2 - a^2) + k^2(x_1^2 + y_1^2 - a^2) - a^2(x_1^2 + y_1^2 - a^2)$$

$$= 2hkx_1y_1 - 2a^2hx_1 - 2a^2ky_1 + h^2x_1^2 + k^2y_1^2 + a^4$$

$$= (h^2 + k^2 - a^2)(x_1^2 + y_1^2 - a^2) = (hx_1 + ky_1 - a^2)^2$$

$\therefore$ Locus $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$

i.e. $SS' = T^2$

5.21 Common tangents

In the fig. below the two circles do not intersect. $O_1O_2 > (r_1 + r_2)$ four common tangents are possible, two direct common tangents and two transverse tangents. In this figure, direct common tangents have been drawn.

AB and CD are direct common tangents. A and B are point of contacts on two circles O_1A and O_2B are radii perpendicular to tangent.

Δs O_1AP and O_2BP are similar.

$\therefore$ $\frac{O_1P}{O_2P} = \frac{r_1}{r_2} (r_1 > r_2)$ i.e. P divides the line of centre of two circles O_1O_2 externally in the ratio of $r_1 : r_2$.

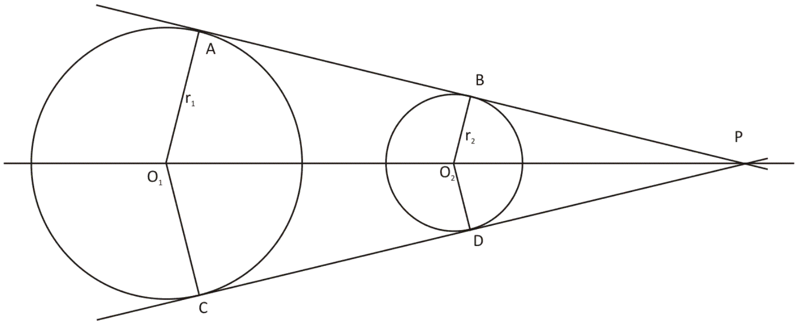

Fig 18

Given two circles, find P on the line of centres which divides externally O_1O_2 in the ratio of $r_1 : r_2$ ($r_1 > r_2$). Through P (x_1, y_1) take a line $y - y_1 = m(x - x_1)$. Drop perpendicular on it from centre of a circle and equate it equal to radius of that circle. m shall be determined It shall give two values of m and so two common tangents.

5.22 Transverse Tangents

AB and CD are transverse tangents, a transverse tangent cuts the line of centres in P. From similar triangles O_1CP and O_2DP it is clear that P divides the line of centres internally in the ratio of $r_1 : r_2$. Rest process is the same.

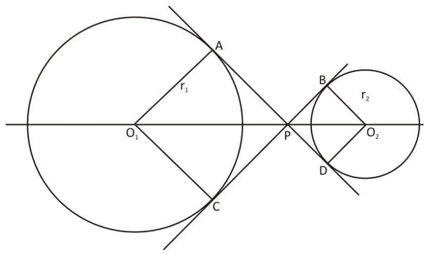

Fig 19

5.23 Angle of intersection of two curves

The slope of a curve of any point is equal to the slope of tangent at that point. In Fig 20 curves c_1 and c_2 intersect in P. Tangents to the curves at P are PT_1 and PT_2. They include angle θ.

This is the angle of intersection of the two curves.

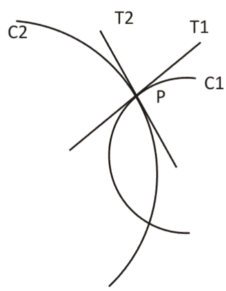

Fig 20

(i) Given two curve, first find the point of intersection of the two curves.

(ii) Find value of $\frac{dy}{dx}$ for both curves at this point. It gives m_1 and m_2.

(iii) Apply formula $\tan\theta = \frac{m_1 \sim m_2}{1 + m_1 m_2}$

Solved Examples

Example 55 : Find common tangents of circles $x^2 + y^2 - 2x - 6y + 9 = 0$, $x^2 + y^2 + 6x - 2y + 1 = 0$.

Sol. : Circle I, $x^2 + y^2 - 2x - 6y + 9 = 0$

O_1, centre (1, 3), $r_1 = 1$

Circle II $x^2 + y^2 + 6x - 2y + 1 = 0$,

O_2 centre is (–3, 1) ; $r_2 = 3$;

$O_1O_2 = \sqrt{4^2+4} = 2\sqrt{5} > (r_1 + r_2)$

(b) Direct Tangents : Let P divide O_2O_1 in the ratio of 3 : 1 externally, O_1 (1,3), O_2 (-3,1),

$$\therefore x_1 = \frac{3+3}{3-1} = 3, \quad y_1 = \frac{9-1}{2} = 4, \quad P(3,4)$$

any straight line through P is y – 4 = m(x – 3) perpendicular from centre (1,3) on it.

$$\frac{3-m+3m-4}{\sqrt{1+m^2}} = \frac{2m-1}{\sqrt{1+m^2}} = 1 \Rightarrow 3m^2 - 4m = 0$$

$\Rightarrow m = 0, m = 4/3.$

(i) m =0, tangent is y – 4 = 0

(ii) m = 4/3 tangent is $y - 4 = \frac{4}{3}(x-3) \Rightarrow 3y - 4x = 0$

(b) Transverse Tangents : Let P divide O_2O_1 in the ratio of 3 : 1 internally,

$$x_2 = \frac{3-3}{4} = 0; \ \frac{9+1}{4} = y_2 \ \text{P is} \left(0, \frac{5}{2}\right)$$

Straight line through (x_2, y_2) $y - \frac{5}{2} = mx$

Perpendicular from centre (1, 3) on it is 1

$$\therefore \frac{3 - \frac{5}{2} - m}{\pm\sqrt{1+m^2}} = 1 \qquad \Rightarrow \left(\frac{1-2m}{2}\right)^2 = (1+m^2)$$

$$1 - 4m + 4m^2 = 4 + 4m^2 \Rightarrow -4m = 3, m = \alpha$$

$\therefore$ tangents $y - \frac{5}{3} = -\frac{3}{4}x \Rightarrow 4y + 3x = 10$ and x = 0.

Example 56 : Find angle between the tangents drawn from origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$.

Sol. : Equation of circle $x^2 + y^2 + 20(x + y) + 20 = 0$

Tangents are $SS' = T^2$

$$(x^2 + y^2 + 20x + 20y + 20).(0 + 20) = (x.0 + y.0 + 10x + 10y + 20)^2$$

$$\Rightarrow 20(x^2 + y^2 + 20x + 20y + 20) = 10^2 (x + y + 2)^2$$

$$\Rightarrow (x^2 + y^2 + 20x + 20y + 20) = 5(x + y + 2)^2$$

$$\Rightarrow 4x^2 + 10xy + 4y^2 = 0 \Rightarrow x^2 + 5/2xy + y^2 = 0$$

$$\therefore \tan\theta = \frac{2\sqrt{\frac{25}{16} - 1}}{1 + 1} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$$\therefore \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

Example 57. Find angle of intersection of the curve $x^2 + 2y^2 = 4$ and $x^2 + y^2 - 4x - 2 = 0$.

Sol. : From circle $x^2 + y^2 - 4x - 2 = 0$

$y^2 = -x^2 + 4x + 2$

$\therefore$ from other curve $x^2 + 2(-x^2 + 4x + 2) - 4 = 0$

$\Rightarrow x^2 - 8x = 0 \Rightarrow x = 0, x = 8$.

(i) $x = 0$, $y = \pm\sqrt{2}$ (ii) $y = \sqrt{-30}$, $x = 8$ imaginary

$\therefore$ points of intersection are $(0, \sqrt{2})$, $(0, -\sqrt{2})$

$$x^2 + 2y^2 = 4 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y}; \text{ at } (0, \sqrt{2})\ m_1 = 0$$

$$x^2 + y^2 - 4x = 2 \Rightarrow \frac{dy}{dx} = \frac{2 - x}{y}, \text{ at } (0, \sqrt{2}),\ m_2 = \sqrt{2}$$

$$\therefore \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sqrt{2}}{1} \Rightarrow \theta = \tan^{-1}(\sqrt{2})$$

Example 58 : Find angle of intersection of circles $x^2 + y^2 = 4$, $x^2 + y^2 - 4x - 6y - 3 = 0$

Sol. : Circle $x^2 + y^2 = 4$, centre (0, 0), $r_1 = 2$ circles $x^2 + y^2 = 4$, $x^2 + y^2 - 4x - 6y - 3 = 0$, centre (2, 3), $r_2 = 4$

P is point of intersection of circles. Radius O_1P is $\perp$ on tangent at P to circle I O_2P is also $\perp$ to tangent at P of circle II.

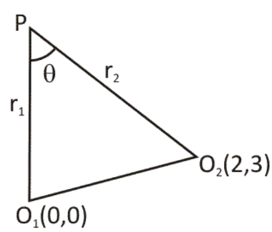

Fig 21

$\therefore$ Angle of intersection $= O_1PO_2 = \theta$

and cos θ $\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2} = \dfrac{4+16-13}{2.2.4} = \dfrac{7}{16}$

$\therefore \theta = \cos^{-1}(7/16)$

Example 59. Straight line AB is divided at C such that AC = 3CB. Circles are described on AC and CB as diameter. A common direct tangent EF meets AB produced in D. Show that BD is equal to radius of smaller circle.

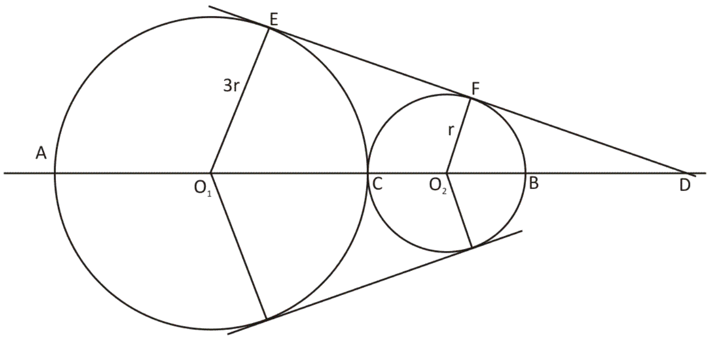

Fig 22

Sol. : In fig. 22. ΔsO_1ED and O_2FD are similar triangle

$\therefore O_1D = 3O_2D$

$\Rightarrow (O_1B + BD) = 3(O_2B + BD)$

$\Rightarrow 5r + BD = 3(r + BD) \Rightarrow BD = r$

Example 60: Find common tangents of circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 8x = 0$.

Sol. : Centres O_1 (1,0), O_2 (–4,0)

Radii $r_1 = 1$ $r_2 = 4$

$r_1 + r_2 = 5$ and $O_1 O_2 = 5$

Circles touch each other externally, 3 tangents; one tangent is $S_1 - S_2 = 0$ $\Rightarrow x = 0$ (y-axis)

Let P divide O_2O_1 in the ratio of 4 : 1

Externally i.e. (–4, 0), (1, 0)

$x = \dfrac{4+4}{4-1} = \dfrac{8}{3}$ and y = 0

straight line through $\left(\dfrac{8}{3}, 0\right)$ is $y = m\left(x - \dfrac{8}{3}\right)$

$\Rightarrow 3y - 3mx + 8m = 0$

perpendicular from (1, 0) on it is equal to 1

$\therefore \frac{-m+8/3m}{\sqrt{1+m^2}}=1 \Rightarrow 25m^2=9\left(1+m^2\right) \Rightarrow m=\pm 3/4$

$\therefore$ common tangents are $y=\pm\frac{3}{4}\left(x-\frac{8}{3}\right)$

i.e. $4y-3x+8=0$, and $4y+3x-8=0$.

Example 61: PA and PB are tangents from P(4, 2) to circle $x^2 + y^2 = a^2$. Find equation of circumcircle of triangle PAB.

Sol. : AB is chord of contact of point P(4, 2) circle is $x^2 + y^2 = a^2$

$\therefore$ Equation AB, $4x + 2y = a^2$

This will be common chord of given circle and required circle.

$\therefore$ Required Θ is $(x^2 + y^2 - a^2) + \lambda\ (4x + 2y - a^2) = 0$

If passes through (4, 2) $\therefore 20-a^2+\lambda\left(20-a^2\right) \Rightarrow \lambda=-1$

$\therefore$ Circumcircle is $x^2 + y^2 - a^2 - 4x - 2y + a^2 = 0$

$\Rightarrow x^2+y^2-4x-2y=0$

Example 62 : Two circles pass through points (a, 5a) and (4a, a) and touch y-axis as well. Find angle of intersection of these circles.

Sol. : Let Θ be $x^2 + y^2 + 2gx + 2fy + c = 0$,

It touches y axis $\therefore c=f^2$

$\therefore$ circle is $x^2+y^2+2gx+2fy+f^2=0$ ....(1)

It passes through (a, 5a) and (4a, a)

$\therefore f^2+10af+2ag+26a^2=0$ ....(2)

$f^2+2af+8ag+17a^2=0$ ....(3)

eliminating g from (2) and (3)

$3f^2+38af+27a^2=0$

$\Rightarrow (3f+29a)(f+3a)=0$

$\therefore f=-29/3$ and $f_2=-3a$

from (2) and (3). $g_1=\frac{-205}{18}a,\ g_2=-\frac{5a}{2}$

Differentiating the circle equation w.r to x

$$2x + 2y\frac{dy}{dx} + 2g + 2f.\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x+g}{y+f}\right) \text{ at } (a, 5a) \quad m_1 = -\frac{187}{84}$$

and for (g_2, f_2) $m_2 = 3/4$

$$\therefore \tan\theta \frac{3/4 + \frac{187}{84}}{1 - \frac{561}{336}} = -\frac{1000}{225} = -\frac{40}{9}$$

for acute angle $\theta = \tan^{-1}\left(\frac{40}{9}\right)$

Example 63 : Two parallel tangents to a given circle are cut by a third tangent at P and Q. If C is centre of circle then show that $\angle PCQ = 90^0$.

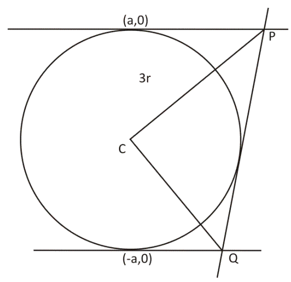

Fig 23

Sol. : Let the circle be $x^2 + y^2 = a^2$

Tangents at (a, 0) and (–a, 0) are y = a

and y = – a

Let third tangent be $x\cos\alpha + y\sin\alpha = a$

$$\therefore \text{ P is } \left[\frac{a(1-\sin\alpha)}{\cos\alpha}, a\right]$$

$$\text{Q is } \left[\frac{a(1+\sin\alpha)}{\cos\alpha}, -a\right]$$

$$\text{Slope of CP x slope of CQ} = \left[\frac{a\cos\alpha}{a(1-\sin\alpha)} \times \frac{-a\cos\alpha}{(1+\sin\alpha)}\right]$$

$$= \frac{-\cos^2\alpha}{\cos^2\alpha} = -1 \therefore \angle PCQ = 90^0$$

Example 64 : Consider curve $ax^2 + 2hxy + by^2 = 1$ and point P not on the curve. A line drawn from P intersects the curve in Q and R. If the product PQ. PR is independent of the slope of line, then show that the curve is a circle.

Sol. : Equation of curve $ax^2 + 2hxy + by^2 = 1$

Let P be (0, 0), a point not on the curve

Let straight line through P be y = mx.

$\therefore$ from curve $ax^2 + 2hmx^2 + bm^2x^2 = 1$

$$\therefore x^2 = \frac{1}{a+2hm+bm^2} \Rightarrow x = \pm\frac{1}{\sqrt{a+2hm+bm^2}}$$

Straight line y = mx $\therefore y = \pm\frac{m}{\sqrt{a+2hm+bm^2}}$

Let Q (x_1, y_1) $\therefore PQ = \sqrt{x_1^2+y_1^2} = \sqrt{\frac{1+m^2}{a+2hm+bm^2}}$

R be (x_2, y_2) $\therefore PR = \sqrt{x_2^2+y_2^2} = \sqrt{\frac{1+m^2}{a+2hm+bm^2}}$

As $(-)^2 = +$

PQ. PR = $\frac{1+m^2}{a+2hm+bm^2}$ and according to question it is free from m (slope) $\Rightarrow$ a + 2hm + bm^2 should be divisible by $1 + m^2$ $\therefore$ h = 0 and a = b

i.e. curve $a(x^2 + y^2) = 1$ i.e. circle.

Example 65 : Find area of triangle formed by tangents from (x_1, y_1) to circle $x^2 + y^2 = a^2$ and its chord of contact.

Sol. : Equation of circle $x^2 + y^2 = a^2$ equation chord of contact of (x_1, y_1)

$$xx_1 + yy_1 = a^2$$

In the fig. AB is chord of contact $\perp$ from P on AB is PD, from O is OD.

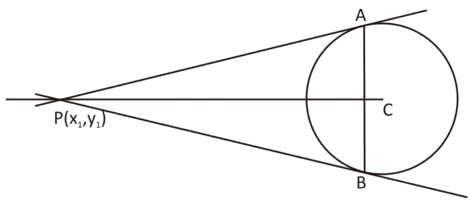

Fig 24

$$PD = \frac{x_1^2+y_1^2-a^2}{\sqrt{x_1^2+y_1^2}}, OD = \frac{a^2}{\sqrt{x_1^2+y_1^2}}$$

$$AD^2 = AO^2 - OD^2 = a^2 - \frac{a^4}{x_1^2+y_1^2} = \frac{a^2(x_1^2+y_1^2-a^2)}{x_1^2+y_1^2}$$

$$\text{area } \Delta PAB = \frac{1}{2}AB.PD = AD.PD = a\sqrt{\frac{x_1^2+y_1^2-a^2}{x_1^2+y_1^2}}.\frac{x_1^2+y_1^2-a^2}{\sqrt{x_1^2+y_1^2}} = \frac{a(x_1^2+y_1^2-a^2)^{3/2}}{x_1^2+y_1^2}$$

Example 66 : Find locus of foot of perpendicular from origin upon and chord of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which subtends an angle of 90^0 at origin.

Sol. : Given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(1)

Let ax + by = 1 be any chord which subtend a right angle at origin. Making (1) homogeneous with the help of this straight line equation.

$x^2+y^2+2gx(ax+by)+2fy(ax+by)+c(ax+by)^2=0$ This is combined equation of straight line joining origin with the points of intersection of circle and straight line these include angle 90^0.

$\therefore$ co-efficient of x^2 + co-efficient of y^2 = 0

$\therefore 1+2ga+ca^2+1+2fb+b^2c=0$

$\Rightarrow 2+2(ga+fb)+c(a^2+b^2)=0$ ....(2)

Line perpendicular from origin on chord is

$bx-ay=0 \Rightarrow \frac{x}{a}=\frac{y}{b} \Rightarrow \frac{x^2}{ax}=\frac{y^2}{by}=\frac{x^2+y^2}{ax+by}$

$\Rightarrow \frac{x^2+y^2}{1}=\frac{x}{a}=\frac{y}{b}$

and $\therefore$ $a=\frac{x}{x^2+y^2}, b=\frac{y}{a^2+b^2}$

Putting these value of and b in (2)

$2+2g\left(\frac{x}{x^2+y^2}\right)+2f.\frac{y}{x^2+y^2}+c\frac{x^2+y^2}{(x^2+y^2)^2}=0$

$\Rightarrow 2(x^2+y^2)+2gx+2fy+c=0$

$\Rightarrow x^2+y^2+gx+fy+\frac{c}{2}=0$

Practice Worksheet (Foundation Level) – 5(d)

1. Find direct and transverse common tangents of circles $x^2 + y^2 – 2x = 0$ and $x^2 + y^2 + 8x=0$. Do the circle touch each other?

2. Find common tangents of circles $x^2 + y^2 = 9$ and $x^2 + y^2 – 2(x\sqrt{3}+y) + 3 = 0$.

3. Prove that common tangents of circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 – 6x = 0$ form an equilateral triangle.

4. Find locus of mid points of chord of circle $x^2 + y^2 – 4x – 6y – 12 = 0$ drawn from a point (6, 6) on it.

5. Find the locus of the point, tangents from which to circle $x^2 + y^2 – 6x + 8y = 0$ include an angle of 60.

6. Find condition that two circles $x^2 + y^2 + 2g_1x + 2f_1y = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y = 0$ touch each other.

7. From point P(–a, b) tangents PA, PB are drawn to circle $x^2 + y^2 = a^2$ (b>a); find equation of circum circle of triangle PAB.

8. Find equation of circle that lies in third quadrant and touches straight lines $y – \sqrt{3}x = 0$ and $\sqrt{3}y - x = 0$ and is of radius 5.

9. Circles $x^2 + (y – a)^2 = a^2$ and $(x – 2\sqrt{ab})^2 + (y – b)^2 = b^2$ touch.

   (a) x – axis (b) y-axis (c) one-another

10. If circle $x^2 + y^2 = 2$ and $x^2 + y^2 – 6x – 6y + 10 = 0$ touch each other then find equation of common tangent and point of contact.

11. A circle cuts intercepts of 2p and 2q on x and y axes respectively. Find locus of its centre.

12. Circles go through points (a, 0) and (–a, 0). Prove that these circles intersect the circle $x^2 + y^2 + px + a^2 = 0$ (for all values of p) at an angle of $\pi/2$.

13. From point (3, 4) chord of circle $x^2 + y^2 – 4x – 6y – 7 = 0$ are drawn. Find locus of mid points of these chords.

14. C_1 and C_2 are two concentric circles and radius of C_2 is twice that of C_1. From a point P on C_2 tangents PA and PB are drawn to C_1. The centroid of ΔPAB lies on

   (a) C_1 (b) Inside C_1 (c) outside C (d) none of these

15. Side of a regular hexagon is 4cm. Circles have been inscribed and circumscribed. Sum of their radii are:

   (a) $2\left(\sqrt{3}+3\right)$ (b) $2\left(\sqrt{3}+2\right)$ (c) $2\left(\sqrt{3}+4\right)$ (d) $2\left(4-\sqrt{3}\right)$

16. Straight line x + y + a = 0 is tangent to circle $x^2 + y^2 – 4x – 2y – 3 = 0$ and is chord of length 8 of its concentric circle C_2. The equation of C_2 is :

(a) $x^2+y^2-4x-2y-19=0$ (b) $2x^2+2y^2-8x-4y-19=0$

(c) $x^2+y^2-4x-2y-25=0$ (d) $2x^2+2y^2-8x-4y-21=0$

17. $2x - y = 1$ and $3x - y = 3$ are two normals of a circle and $3x + 4y - 3 = 0$ is tangent to it. Equation of circle is :

(a) $x^2 + y^2 - 6x - 4y + 12 = 0$ (b) $x^2 + y^2 - 4x - 6y + 4 = 0$

(c) $x^2 + y^2 + 6x - 4y + 4 = 0$ (d) $x^2 + y^2 + 4x + 6y + 9 = 0$

18. AB is fixed base of triangle ABC. If C moves such that CA = 2 CB, then locus of C is :

(a) an ellipse (b) arc of circle

(c) a straight line (d) circle.

5.24 Orthogonal Circles :

When two circle intersect each other at right angle these circles are called orthogonal circles.

You have seen that angle of intersection between two circles, θ is given by.

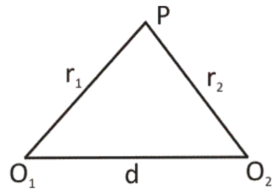

Fig 25

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

where d is distance between two centres O_1 and O_2.

At points of intersection P, angle between two curves is angle between the tangents to two curves at this point. Radii of circles are perpendicular to these tangents so angle between radii is equal to angle between tangents, hence

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}; \theta = 90^0 \Rightarrow r_1^2 + r_2^2 - d^2 = 0$$

$\Rightarrow r_1^2 + r_2^2 = d^2$ If circles are

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

then $\left(g_1^2 + f_1^2 - c_1\right) + \left(g_2^2 + f_2^2 - c_2\right) = \left(g_1 \sim g_2\right)^2 + \left(f_1 \sim f_2\right)^2$

$\Rightarrow -c_1 + c_2 = -2g_1g_2 - 2f_1f_2$

$\therefore\ 2\left(g_1g_2 + f_1f_2\right) = c_1 + c_2$

This is the condition for the two curves to intersect orthogonally.

Solved Examples

Example 67: Two circles pass through (0, a) and (0, –a) and touch the line y = mx +c. Prove that these will intersect orthogonally if $c^2 = a^2 (2 + m^2)$.

Sol. : Let the circles be $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\text{Point}\begin{matrix}(0,a) \\ (0,-a)\end{matrix}\quad \left.\begin{matrix}a^2+2fa+c=0 \\ a^2-2fa+c=0\end{matrix}\right\} \Rightarrow \begin{matrix}f=0 \\ c=-a^2\end{matrix}$$

Now circle is $x^2 + y^2 + 2gx – a^2 = 0$

It shall touch y = mx +c, if perpendicular from (–g, 0) on it is equal to $r = \sqrt{g^2 - c} = \sqrt{g^2 + a^2}$

$$\Rightarrow \frac{-mg + c}{\sqrt{1+m^2}} = \pm\sqrt{g^2 + a^2}$$

$$\Rightarrow (c - mg)^2 = (1 + m^2)(g^2 + a^2)$$

$$\Rightarrow g^2 + 2mcg + a^2(1 + m^2) - c^2 = 0$$

the equation gives two values of g; g_1 and g_2

$g_1g_2 = a^2(1 + m^2) - c^2$

circles $x^2 + y^2 + 2g_1x – a^2 = 0$, $x^2 + y^2 + 2g_2x – a^2 = 0$ shall intersect orthogonally if

$2[g_1g_2 + 0] = - a^2 - a^2$

$\Rightarrow$ $2[a^2(1 + m^2) - c^2] = - 2a^2$

$\Rightarrow$ $a^2(1 + m^2) = c^2 - a^2$

$\Rightarrow$ $c^2 = a^2(2 + m^2)$

Example 68 : Two circles cut orthogonally. Prove that polar of any point P on first circle with respect to the other circle shall pass through the other end of diameter of first circle through P.

Sol : Let circles be $x^2 + y^2 = a^2$ ....(1)

and $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(2)

Polar of $P(x_1,y_1)$ with respect to circle (2) is

$xx_1 + yy_1 + g(x + x_1) + f(y + x_1) + c = 0$ ....(3)

$P(x_1,y_1)$ is on circle $x^2 + y^2 = a^2$. The other end of diameter of circle (1) through P is $(-x_1,-y_1)$.

Putting this value in equation of polar

$$-x_1^2 - y_1^2 + g.0 + f.0 + c = 0$$

$$\Rightarrow x_1^2 + y_1^2 = c \qquad ....(4)$$

Circles cut orthogonally

$\therefore$ 2(g.0 + f.0) = c - a^2

$\Rightarrow c = a^2$ $\therefore$ Eqn (4) is $\Rightarrow x_1^2 + y_1^2 = a^2$

$\therefore$ polar passes through the other end of diameter through P.

Example 69 : Two circles are S=0 and S'=0 and their radius are r and r'. Show that circles $\frac{S}{r} \pm \frac{S'}{r'} = 0$ will intersect orthogonally.

Sol : Let centre of two circles be (a,0) and (-a,0)

$\therefore$ S=0 is $(x - a)^2 + y^2 = r^2$

$\therefore$ S'=0 is $(x + a)^2 + y^2 = r'^2$

Now $\frac{S}{r} \pm \frac{S'}{r'} = 0 \Rightarrow r'S + S'r = 0$

$$r'(x^2 + y^2 - 2ax + a^2 - r^2) + r(x^2 + y^2 + 2ax + a^2 - r'^2) + = 0$$

$$(r'+r)(x^2 + y^2 + a^2) - 2ax(r'-r) + rr'(r + r') = 0$$

$$\Rightarrow x^2 + y^2 - 2ax\frac{r'-r}{r+r'} + a^2 - rr' = 0 \qquad ....(1)$$

and similarly $\frac{S}{r} - \frac{S'}{r'} = 0$ shall give

$$x^2 + y^2 - 2ax\left(\frac{r+r'}{r'-r}\right) + a^2 + xr' = 0 \qquad ....(2)$$

Now $g_1 = -\frac{a(r'-r)}{r+r'}, g_2 = \frac{a(r'+r)}{r'-r}, f_1 = f_2 = 0$

$c_1 = a^2 - rr'$, $c_2 = a^2 + rr'$

Now $2[g_1g_2 + f_1f_2] = 2[a^2 + 0] = 2a$

And $c_1 + c_2 = a^2 - rr' + a^2 + rr' = 2a^2$

$\therefore 2[g_1g_2 + f_1f_2] = c_1 + c_2$ $\therefore$ Circles cut orthogonally.

Example 70 : Find equation of circle which passes through (2, 5) and cuts circles $x^2 + y^2 - 9x + 14 = 0$ and $x^2 + y^2 + 15x + 14 = 0$ orthogonally.

Sol. : Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(1)

Given circles $x^2 + y^2 - 9x + 14 = 0$ ....(2)

$x^2 + y^2 + 15x + 14 = 0$ ....(3)

Circle (1) cuts Circles (2) and (3) orthogonally.

$$2\left[-g.\frac{9}{2}+0\right]=c+14 \Rightarrow -9g=c+14$$

and $2\left[\frac{15}{2}g+0\right]=c+14 \Rightarrow 15g=c+14$

and these given g = 0, c = – 14

Circle passes through (2, 5)

$\therefore 4+25+4g+10f+c=0$

$\Rightarrow 29+0+10f-14=0 \Rightarrow f=-3/2$

$\therefore$ Θ is $x^2 + y^2 - 3y - 14 = 0$

5.25 Radical axis

The locus of point, tangents from which to the two circles are equal is called radical axes of the two circles.

If $S_1 = 0$, $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

And $S_2 = 0$ $x^2 + y^2 + 2g_2x + 2f_2 y + c_2 = 0$

Be two circles. Let P(h, k) be the point, tangent from which the two circles are equal.

i.e. $PT_1 = PT_2 \Rightarrow (PT_1)^2 = (PT_2)^2$

$$h^2 + k^2 + 2g_1h + 2f_1k + c_1 = h^2 + k^2 + 2g_2h + 2f_2k + c_2$$

$$\Rightarrow 2h(g_1 - g_2) + 2k(f_1 - f_2) + c_1 - c_2 = 0$$

Locus $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

This is clearly $S_1 – S_2 = 0$

This is also the equation of common chord, when circles intersect. Thus in case of circles intersecting each other common chord is radial axis.

If circles touch each other externally then common tangent at this point is the radical axis of two circles.

5.26 Radical Centre

The radical axes of three circles, taken two at a time, meet at one point. This point is called radical centre. Length of tangent from this point of the three circles is equal. Let circles be $S_1 = 0$, $S_2 = 0$, $S_3 = 0$.

Radical axes of S_1, S_2 $R_1 = S_1 – S_2 = 0$ ....(1)

of S_2, S_3 $R_2 = S_2 – S_3 = 0$ ....(2)

of S_3, S_1 $R_3 = S_3 – S_1 = 0$ ....(3)

Adding $R_1 + R_2 + R_3 = 0$ i.e. radical axes are concurrent.

Note : If $S_1 = 0$ is a circle and L = 0 straight line intersecting it. Then $S_1 + l L= 0$ shall give all circles for different value of l (all numerical) that have the same radical axis L = 0

Solved Examples

Example 71 : Prove that the locus of the centre of a circle which cuts two given circles orthogonally, is the radical axis of these circles.

Sol. : Let the circles be $x^2 + y^2 + 2g_1x + 2f_1y + c_2 = 0$ ....(1)

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ .... (2)

and circle $(x-h)^2 + (y-k)^2 = a^2$ ....(3)

cuts them orthogonally

$\therefore 2[g_1(-h)+f_1(-k)=c_1+h^2+k^2-a^2]$

and $2[g_2(-h)+f_2(-k)]=c_2+h^2+k^2-a^2$

on substracting $2h(g_1-g_2)+2k(f_1-f_2)=c_2-c_1$

$\therefore$ Locus of centre of circle (3) is

$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0$

Which is radical axis of first two circles.

Example 72 : Find radical centre of circle

$3x^2+3y^2+4x-6y-1=0, 2x^2+2y^2-3x-2y-4=0$ and $2x^2+2y^2-x+y-1=0$

Sol. : Circle $x^2+y^2+\frac{4}{3}x-2y-\frac{1}{3}=0$ ....(1)

$x^2+y^2-\frac{3}{2}x-y-2=0$ ....(2)

$x^2+y^2-\frac{1}{2}x+\frac{1}{2}x-\frac{1}{2}=0$ ....(3)

Radical axis of (1) or (2) $\left(\frac{4}{3}+\frac{3}{2}\right)x-(2-1)y-\frac{1}{3}+2=0$

$\Rightarrow 17x-6y+10=0$ ....(4)

Radical axes of (2) or (3) $2x + 3y + 3 = 0$ ....(5)

Solving (4) and (5)

$x=-\left(\frac{16}{21}\right), y=-\left(\frac{31}{63}\right)$ $\therefore$ Radical centre = $\left(-\frac{16}{21},-\frac{31}{63}\right)$

Example 73 : From a point P, tangents are drawn to three circles $x^2 + y^2 + x - 3 = 0$, $3x^2 + 3y^2 + 5x + 3y = 0$ and $4x^2 + 4y^2 + 8x + 7y + 9 = 0$ are of equal length. Find equation of circle through P which touches $x + y = 5$ at $(6, -1)$ (R – 1992)

Sol. : Circles are $x^2 + y^2 + x - 3 = 0$ ....(1)

$$x^2 + y^2 + \frac{5}{3}x + y = 0 \qquad ....(2)$$

$$x^2 + y^2 + 2x + \frac{7}{4}y + \frac{9}{4} = 0 \qquad ....(3)$$

Point P is radical centre of these circles.

$$\left.\begin{array}{ll} (2)-(1) & \frac{2}{3}x + y + 3 = 0 \\ (3)-(1) & x + \frac{7}{4}y + \frac{21}{4} = 0 \end{array}\right\} \text{Solving } x = 0, y = -3; P(0,-3)$$

Let circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

It passes through (0, –3) and (6, –1)

$\therefore 9 - 6f + c = 0$, and $37 + 12g - 2f + c = 0$.

Subtracting $12g + 4f + 28 = 0 \Rightarrow 3g + f + 7 = 0$

x + y = 5 touches at (6, –1) ....(α)

tangent at (6, –1) $6x - y + g(x + 6) + f(y - 1) + c = 0$

$$x(6+g) + y(f-1) + 6g - f + c = 0$$

Given tangent $x + y - 5 = 0$

$$\therefore \qquad \frac{6+g}{1} = \frac{f-1}{1} = \frac{6g-f+c}{-5}$$

$\therefore 6 + g = f - 1 \Rightarrow g - f = -7$ ....(β)

solving (α) and (β) $g = -7/2, f = 7/2$

and $-30 - 5g = 6g - f + c$

$$\therefore c = -30 - 11g + f$$

$$= -30 + \frac{77}{2} + \frac{7}{2} = 12$$

$\therefore$ Circle is $x^2 + y^2 - 7x + 7y + 12 = 0$

Example 74 : Find locus of a point, whose shortest distance from circle $x^2 + y^2 - 2x + 6y - 6 = 0$ is equal to distance from straight line $x - 3 = 0$

Sol. : Shortest distance of a point from circle is its distance from centre of circle - radius of circle

Given circle $x^2 + y^2 - 2x + 6y - 6 = 0$

Center (1, –3), r = 4. Let point be (h, k) Its distance from x – 3 = 0 is $\frac{h-3}{\pm 1}$

Its distance from centre of Θ is $\frac{\pm 1}{\sqrt{(h-1)^2+(k+3)^2}}$

$\therefore$ from the condition given.

$$\sqrt{(h-1)^2+(k+3)^2}-4=\frac{h-3}{\pm 1}$$

$\therefore \quad (h-1)^2+(k+3)^2=(h+1)^2$ or $(-h+7)^2$

$k^2-4h+6k+9=0 \qquad$ or $k^2+6k+12h-39=0$

$\therefore$ Locus $y^2-4x+6y+9=0$ or $y^2+12x+6y-39=0$

Example 75 : The circle $x^2 + y^2 = 1$ cuts x-axis in P and Q. Another circle with centre Q and Variable radius cuts the first circle in R above x-axis and the line segment PQ is S. Find max area of ΔQSR.

Sol. : Circle is $x^2 + y^2 = 1$ ....(1)

Centre (0, 0), r = 1

$\therefore P(-1,0), Q(1,0)$

Equation of Θ with centre Q and radius r is

$(x-1)^2+y^2=r^2$ ....(2)

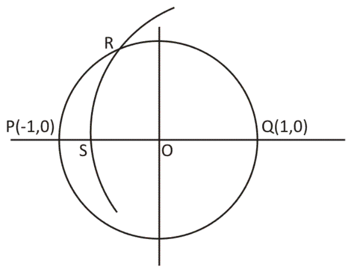

Fig 26

R is point of intersection of these two circles

$$-2x=r^2-2 \Rightarrow x=\frac{2-r^2}{2}$$

and $y^2=1-x^2=1-\frac{4-4r^2+r^4}{4}=\frac{4r^2-r^4}{4}$

$\therefore y=\frac{r}{2}\sqrt{4-r^2}$ + sign taken as R is above x-axis.

Area $\Delta RSQ=\frac{1}{2}.SQ.\left(\frac{r}{2}\sqrt{4-r^2}\right)$

$SQ=r \quad \therefore$ Area $A=\frac{1}{4}r^2\sqrt{4-r^2}$

$$\frac{dA}{dr}=\frac{1}{4}.\left[2r\sqrt{4-r^2}.-\frac{r^2(-2r)}{2\sqrt{4-r^2}}\right]=\frac{r\left(8-3r^2\right)}{4\sqrt{4-r^2}}$$

for maximum area $\frac{dA}{dr} = 0$.

$\therefore 3r^2 - 8 = 0 \Rightarrow r = \sqrt{8/3}$

$\therefore A = \frac{1}{4}.\frac{8}{3}.\sqrt{4-8/3} \quad = \frac{4\sqrt{3}}{9}$ sq units.

Example 76 : 5x + 12y – 10 = 0 and 5x – 12y – 40 = 0 touch circle C_1 of diameter 6 units and which lies in 1^{st} quadrant. Find equation of circle C_2 which is concentric with C_1 and cuts intercepts of 8 on these lines. **(I.I.T. 86)**

Sol. : Centre of circle shall lies on the angle bisectors of straight line 5x + 12y – 10 = 0 and 5x – 12y = 40

i.e. on $\frac{5x+12y-10}{13} = \pm\frac{5x-12y-40}{13}$

(i) +sign. 24y + 30 = 0; $y = -\frac{5}{4}$, rejected, not in 1^{st} quadrant. (ii) –sign 10x = 50, x = 5.

$\therefore$ Centre of circle is (5, λ), Radius = 3 given.

Perpendicular from centre on 5x + 12y – 10 = 0 equal to 3

$\Rightarrow 25 + 12\lambda - 10 = \pm(13.3) = \pm 39$

$\Rightarrow \lambda = 2$ or $\lambda = -\frac{54}{12} = -\frac{9}{2}$ rejected not in 1^{st} quadrant C' – centre is (5, 2)

circle C_1 is $(x – 5)^2 + (y – 2)^2 = 3^2$

$\Rightarrow x^2 + y^2 - 10x - 4y + 20 = 0$

Circle c_2 is concentric $\Rightarrow x^2 + y^2 - 10x - 4y + \lambda = 0$. If p is perpendicular from centre on chord.

$5x + 12y - 10 = 0$ then $p = \frac{25+24-10}{13} = 3$

$\frac{1}{2}$ of length of chord is 4

$\therefore r^2 = 4^2 + 3^2 = 5^2$

$\therefore$ circle c_2 is $(x-5)^2 + (y-2)^2 = 5^2$

$x^2 + y^2 - 10x - 4y + 4 = 0$

Example 77 : AB is diameter of a circle. CD is the chord of circle parallel to AB and equal to half of it. The tangent at B meets the line AC produced in E Then AE/AB =

(a) 1.5 (b) 1.0 (c) 2.0 (d) 2.5

Sol. : Let A be (a, 0), B(–a, 0)

$\Rightarrow$ r = a; CD =a; LD = a/2

$\therefore OL = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}}{2}a$

$\therefore$ C is $\left(\frac{a}{2}, \frac{\sqrt{3}}{2}a\right)$

Tangent at β is $x = -a$ ....(1)

Equation AC is $y = \frac{\sqrt{3}/2a}{\frac{a}{2} - a}(x - a)$

$y + \sqrt{3}(x - a) = 0$ ....(2)

E is point of intersection of (1) and (2)

$\left(-a, 2\sqrt{3}a\right)$

$\therefore AE = \sqrt{4a^2 + 12a^2} = 4a$ and AB = 2a

$\therefore AE/AB = 2.0 = 2$

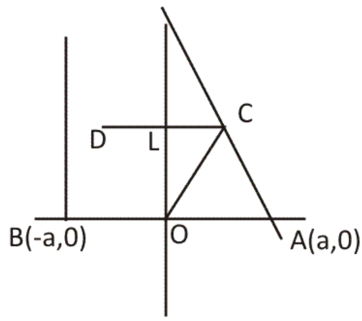

Fig 27

Example 78 : Find equation of circle which passes through (2, 8) and touches the lines 4x – 3y – 24 = 0, 4x + 3y – 42 = 0 and whose abscissa of centre is ≤ 8.

Sol. : Centre of circle shall lie on angle bisector of 4x – 3y – 24 = 0, 4x + 3y – 42 = 0

i.e. on (4x – 3y – 24) = ± (4 x + 3y) – 42

(1) + sign y = +3 (ii) –sign x = 33/4

abscissa of centre is ≤ 8 $\therefore x = 33/4$ is

rejected. Now centre is (λ,3)

radius = perpendicular from (λ,3) on 4x – 3y – 24 = 0

$r = \frac{4\lambda - 9 - 24}{\pm 5} = \frac{4\lambda - 33}{\pm 5}$, Circle goes through (2, 8)

$$\therefore (2-\lambda)^2 + 5^2 = r^2 = \frac{(4\lambda - 33)^2}{25}$$

$$\Rightarrow 9\lambda^2 + 164\lambda - 364 = 0 \Rightarrow (\lambda - 2)(9\lambda + 182) = 0$$

$\lambda = 2, r = 5$ Circles is $(x-2)^2 + (y-3)^2 = 25$

Practice Worksheet (Foundation Level) – 5 (e)

1. From a point tangents are drawn to circle $x^2 + y^2 = 1$, $x^2 + y^2 + 8x + 15 = 0$ and $x^2 + y^2 + 10y+24=0$. Find the point if they are of equal length.

2. Find radical centre of circles $x^2 + y^2 + 4x – 7 = 0$, $2x^2 + 2y^2 + 3x + 5y – 9 = 0$ and $x^2 + y^2 + y = 0$

3. Find the equation of the circle which goes through (2a, 0) and whose radical axis with circle $x^2 + y^2 = a^2$ is $x = \frac{a}{2}$

4. Find equation of circle which intersects circles $x^2 + y^2 = a^2$, $x^2 + (y – b)^2 = a^2$, and $(x–c)^2 + y^2 = a^2$ orthogonally.

5. The triangle OAB is right angled triangle at O and lies in the first quadrant. Squares OALM on side OA, OBPQ on side OB are constricted externally. The point of intersection of lines AP and BL lies on

   (a) angle bisector of $\angle AOB$

   (b) on line joining mid point of AB with O

   (c) on the altitude through O

   (d) none of these

6. The equation of the circle which touches circle $x^2 + y^2 – 6x + 6y + 9 = 0$ externally and to which lines $x^2 – 3xy – 3x + 9y = 0$ are normals is:

   (a) $(x – 3)^2 + (y + 1)^2 = 1$ (b) $(x – 3)^2 + (y – 1)^2 = 9$

   (c) $(x – 3)^2 + (y – 1)^2 = 1$ (d) $(x + 3)^2 + (y + 1)^2 = 9$

7. A circle touches x – axis and the line 4x – 3y + 4 = 0. The centre of circle lies in 3rd quadrant in x – y = 1. Then centre of circle is :

   (a) (–2, 3) (b) (–3, –4) (c) $\left(-\frac{2}{3}, \frac{-5}{3}\right)$ (d) $\left(-\frac{1}{3}, -\frac{4}{3}\right)$

8. Point a point on the circumference of a circle of radius r. Chord RQ is parallel to the tangent at P. Max. area of ΔPRQ is

   (a) $2r^2$ (b) $\frac{3}{2}r^2$ (c) $\frac{3\sqrt{3}}{2}r^2$ (d) $\frac{3\sqrt{3}}{4}r^2$

9. The pole of straight line $\frac{x}{p}+\frac{y}{q}=1$ with respect to circle $x^2 + y^2 = a^2$ moves along

   (a) a circle (b) a straight line

   (c) parabola (d) no curve

10. The radical centre of circles $(x-2)^2+(y-3)^2=16$; $(x+1)^2+(y-4)^2=36$ and $(x-4)^2+y^2=9$ is:

(a) $\left(\frac{19}{7},-\frac{1}{7}\right)$ (b) $\left(\frac{19}{7},\frac{1}{7}\right)$ (c) $\left(-\frac{19}{7},\frac{1}{7}\right)$ (d) $\left(-\frac{19}{7},-\frac{1}{7}\right)$

11. Three circles with centres (6,5), (6,2) and (10, 5) and radii 1, 2 and 3 respectively touch each other externally. Their radical centre is :

(a) (4, 7) (b) (4, 5) (c) (7, 4) (d) (5, 4)

12. p, q are radii of two concentric circles p > q. AB is tangent to the second circle and chord of first circle. Then $AB^2 =$

(a) 4 pq (b) 4q(2p –q) (c) 4q(p + q) (d) p^2+q^2

13. The circle $x^2+y^2-4x-6y-12=0$ cuts on intercepts AB on x-axis. The area of triangle formed by tangents at A and B is :

(a) $\frac{32}{3}$ sq units (b) $\frac{64}{3}$ sq units (c) $\frac{96}{5}$ sq unit (d) $\frac{48}{5}$ sq unit

14. From a point A (0, 3) on circle $x^2+y^2-4x-6y+9=0$ a chord AB is drawn BA is extended outward upto M. If AM = 2 AB. Then locus of M is :

(a) $x^2+y^2-8x-6y+9=0$ (b) $x^2+y^2+8x-8y+15=0$

(c) $x^2+y^2+8x-6y+9=0$ (d) $x^2+y^2-8x+6y+15=0$

15. Line $\lambda x+y=1$ and $x+3y-5=0$ meet axes in A and B; and C and D. If quadrilateral ABCD is concyclic then l =

(a) 3 (b) –3 (c) $\frac{1}{3}$ (d) –2

16. If circle C_1, $x^2+y^2=23$ intersects another circle C_2 of radius 8, in such a manner that common chord is of maximum length and has slope 4/5. The centre of C_2 is :

(a) (–4, 5) (b) (4, –5) (c) (–4, –5) (d) (4, 5)

17. The centres of two circles C_1 and C_2 lie on y = 4; c_1 is $x^2+y^2-8y=0$. Common chord of c_1 and c_2 subtend supplementary any angles at the centre of two circles. If chord subtends angle of 60^0 at C_1, find equation of C_2.

18. Find equation of the circle which passes through (2, 1) and which touches the circle $x^2+y^2-4x+4y-9=0$ at (3, 2)

19. From origin chords are drawn of the circles $(x-1)^2+y^2=1$. Find locus of the mid points of the chords. **(IIT 85)**

20. Let $S=x^2+y^2+2gx+2fy+c=0$ be a given circle. Find locus of foot of perpendicular drawn from origin on any chord, which subtends a right angle at the origin.

21. A circle passes through (–2,4) and through the point of intersection of circle $x^2 + y^2 - 2x - 6y + 6 = 0$ and straight line $3x + 2y - 5 = 0$. Find its equation and prove that it cuts the circle orthogonally.

22. A circle passes through A(2, 5) and B(6, 2) and its centre is 5 units away from A B. Find equation of circle.

23. The extremities of a diagonal of rectangle are (–4, 4) and (6, –6). Circle circumscribing the rectangle cuts intercept AB on y axis. Find the area of triangle formed by AB and tangents at A and B.

24. Tangents are drawn from (2,3) to the circle $x^2 + y^2 - 2x = 0$. Find the angle included by them.

25. Find angle between the tangents drawn from origin to the circle $x^2 + y^2 + 10(x + y) + 10 = 0$.

Practice Worksheet (Competition Level)

PART-A

1. A circle has radius 3 units and its centre lies on the line $y = x - 1$. Find the equation for the circle if it passes through (7, 3).

2. Two vertices of an equilateral triangle are (–1, 0) and (1, 0) and its third vertex lies above x-axis. Find the equation of its circumcircle.

3. The abscissae of the two points A and B are the roots of the equation $x^2 + 2x - a^2 = 0$ and the ordinates arethe roots of the equation $y^2 + 4y - b^2 = 0$. Find the equation of the circle with AB as diameter. Also find the coordinates of the centre and the length of the radius of the circle.

4. Find the equation of the circles passing through the point (-4, 3) and touching the lines $x + y = 2$ and $x - y = 2$.

5. From the point A (0, 3) on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that $AM = 2AB$. Find the equation of the locus of M.

6. One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points (–3, 4) and (5, 4) respectively, then find the area of the rectangle.

7. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. Find the equation for the circle through their points of intersection and the point (1, 1).

8. If the circle C_1, $x^2 + y^2 = 16$, intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to 3/4, then show that the co-ordinates of the centre of the circle C_2 are either (9/5, –12/5) or (–9/5, 12/5).

9. Show that the equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is $x^2 + y^2 + 32x + 4y + 235 = 0$.

10. A square is inscribed in the circle $x^2 + y^2 - 10x - 6y + 30 = 0$. One side of the square is parallel to $y = x + 3$, then determine the coordinates of its vertices.

11. Two circle, each of radius 5 units, touch each other at (1, 2) If the equation of their common tangent is $4x + 3y = 10$, find the equations of the circles.

12. Prove that the length of the common chord of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$ is $\sqrt{\frac{1}{2}(a+b)^2 - 4c}$.

13. The circle $x^2 + y^2 - 2x + 2y - 14 = 0$ cuts chords of length $2\sqrt{3}$ from lines belong to the family of lines $(x + 5y - 22) - \lambda (x - 8y + 30) = 0$. Determine the equations of the lines of the family.

14. Two chords of length 5 are drawn from any point (3, 4) on the circle $4x^2 + 4y^2 - 24x - 7y = 0$. Prove that their equations are given by $y - 4 = \pm\frac{4}{3}(x - 3)$.

15. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5).

16. Determine the locus of the centres of the circles which touches the two circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 4ax$ and are external to both.

17. Let S_1 be a circle passing through A (0, 1) B (–2, 2) and S_2 is the circle with radius $\sqrt{10}$ units such that AB is the common chord of S_1 and S_2. Find the equation of S_2.

18. a) How are the points (0, 1), (3, 1) and (1, 3) situated with respect to the circle

    $x^2 + y^2 - 2x - 4y + 3 = 0$?

    b) Show that the line (x–1)cos? +(y–1)sin?=1 touches a circle for all values of ?. Find the circle.

19. a) Tangents PQ, PR are drawn to the circle $x^2 + y^2 = a^2$ from a given point P(h, k). find the equation of the circum-circle of the triangle PQR.

    b) Tangents OP and OQ are drawn from origin O to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre C. Prove that the centre of circle circum-scribing Δ OPQ is $(-g/2, -f/2)$ and the area of quadrilateral OPCQ is $\sqrt{c(g^2 + f^2 - c)}$.

    c) Tangents TP and TQ are drawn from a point T to the circle $x^2 + y^2 = a^2$. If the point T lies on the line px + qy = r. find locus of centre of the circum-circle of triangle TPQ.

20. a) If from any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, Tangents are drawn to the circle $x^2 + y^2 + 2gx + 2fy + c \sin^2 \alpha + (g^2 + f^2) \cos^2 \alpha = 0$, show that angle between the tangents is 2α.

    b) Show that the circle $x^2 + y^2 + 2g x + 2f y + c = 0$. will bisect the circumference of the circle $x^2 + y^2 + 2g' x + 2f' y + c' = 0$ if $2g'(g - g')+2f'(f - f')=c - c'$.

21. If $(m_i, 1/m_i)$, $m_i > 0$, i = 1, 2, 3, 4, are four distinct points on a circle, then show that $m_1 m_2 m_3 m_4 = 1$.

22. Find the area of the triangle formed by the tangents from the point (h, k) to the circle $x^2 + y^2 = a^2$ and their chord of contact.

23. Prove that the centres of the circle passing through the point (0, 0), (1,0) and touching the circle $x^2+y^2=9$ are (1/2, ±v2).

24. a) A is the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. If the tangents drawn at the points B (1, 7) and D (4, -2) on the circle meet at the point C, then find the area of the quadrilateral ABCD.

b) Find the area of the quadrilateral formed by a pair of tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x + 2y - 11 = 0$ and a pair of its radii.

25. a) Find the equation of the circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having size just sufficient to contain the circle $x(x - 4) + y(y - 3) = 0$.

b) Find the equation of the circle which touches the circle $x^2 + y^2 - 6x + 6y + 17 = 0$ externally and to which the lines $x^2 - 3xy - 3x + 9y = 0$ are normal.

PART-B

1. Tangent are drawn from P (6, 8) to the circle $x^2+y^2= r^2$. Find the radius of the circle such that Area of the triangle formed by the tangents and chord of contact is maximum. **[I.I.T-2003,4]**

2. a) Let a circle be given by $2x(x - a) + y(2y - b) = 0$, $(a \neq 0, b \neq 0)$. Find the condition on a and b if two chords each bisected by the x-axis, can be drawn to the circle from (a, b/2).

b) Find the intervals of values of a for which the line $y + x = 0$ bisect two chords from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle **[I.I.T-1996,5]**

3. AB is a diameter of a circle, CD is a chord parallel to AB and 2CD = AB. The tangent at B meets the line AC produced at E. Prove that AE = 2AB.

4. Find the coordinates of the point at which the circles $x^2 + y^2 - 4x - 2y = 4$ and $x^2 + y^2 - 12x - 8y = -48$ touch each other. Also find the equations of common tangents touching the circles in distinct points.

5. A circle of constant radius r passes through origin O and cuts the axes of co-ordinates in points A and B Prove that the locus of the foot of perpendicular from O to AB is $(x^2 + y^2)^2\left(\frac{1}{x^2}+\frac{1}{y^2}\right) = 4r^2$.

6. Find the locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$.

   (i) Which subtend a right angle at the centre?

   (ii) Which pass through a given point (x_1, y_1);

7. Two rods of length a and b slide along the axes which are rectangular in such a manner that their ends are concyclic .Prove that the locus of centre of circle passing through these ends is the curve $4(x^2-y^2) = a^2-b^2$.

8. (a) From the origin chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. Find the equation of the locus of the middle points of these chords.

(b) Through a fixed point (h, k) secants are drawn to the circle $x^2 + y^2 = a^2$. Show that the locus of mid points of the secants intercepted by the given circle is $x^2 + y^2 = hx + ky$.

9. Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines.

10. Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord, which subtends a right angle at the origin.

11. Obtain the equations of the straight lines passing through the point A(2, 0) and making an angle 45^o with the tangent at A to the circle $(x + 2)^2 + (y - 3)^2 = 25$. Find the equations of the circles each of radius 3, whose centre are on these straight lines at a distance $5\sqrt{2}$ units from A.

12. Distances from the origin to the centres of the three circles $x^2 + y^2 - 2\lambda x = c^2$ (where c is constant and λ is variable) are in G.P. Prove that the lengths of tangents drawn from any point on the circle $x^2 + y^2 = c^2$ to the three circles are also in G.P.

13. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of the sides along the coordinate axes. If the locus of the circum-centre of the triangle is $x + y - xy + \lambda\sqrt{(x^2 + y^2)} = 0$, find λ.

14. Find the pole of the straight line $9x + y - 28 = 0$ with respect to the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$.

15. a) Find the equation of a circle which touches the line $x + y = 5$ at the point (- 2, 7) and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally.

b) Prove that the two circles which pass through the points (0, a) and (0, − a) and touch the line $y = mx + c$ will cut orthogonally if $c^2 = a^2 (2 + m^2)$.

16. Find the equation of tangents to the circle $x^2 + y^2 - 4x - 8y + 16 = 0$ at the point $(2 + \sqrt{3}, 3)$ If the circle rolls up along this tangents by 2 units then find its equation in the new position.

17. A triangle has two of its sides along the axes its third side touches the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. Prove that the locus of the circum-centre of the triangle is $a^2 - 2a (x + y) + 2xy = 0$.

18. If $4l^2 - 5m^2 + 6l + 1 = 0$, Prove that the line $lx + my + 1 = 0$ touches a fixed circle.

19. a) Let $2x^2 + y^2 - 3xy = 0$ be the equation of the pair of tangents drawn from origin O to circle of radius 3 with the centre in the first quadrant . If A is one of the points of contact, find the length OA. **[I.I.T-2001, 5]**

b) Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of centre C . **[I.I.T-2001, 5]**

20. Find the equation of a circle which is coaxial with the circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ and $x^2 + y^2 + 4x + 2y + 1 = 0$. It is given that the centre of the circle to be determined lies on the radical axis of these circles.

21. a) A circle touches the line y = x at a point P such that OP = $4\sqrt{2}$, where O is the origin. The circle contains the point (-10, 2) in its interior and the length of its chord on the line x + y = 0 is $6\sqrt{2}$. Determine the equation of the circle. **[I.I.T-1990, 5]**

b) Consider a family of circles passing through two fixed points A (3, 7) and B (6, 5). Show that chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

22. a) C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point P on C_2, Tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. **[I.I.T-1998, 5]**

b) Let C be any circle with centre $(0, \sqrt{2})$. Prove that at most two rational points can be there on C. (A rational point is a point both of whose coordinates are rational numbers). **[I.I.T-1997, 5]**

23. a) The extremities of a diagonal of a rectangle are (–4, 4) and (6, –1). A circle circumscribes the rectangle and cuts an intercept AB on the y-axis. Find the area of the triangle formed by AB and the tangents to the circle at A and B.

b) A circle passes through the vertex C of a rectangle ABCD and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units, find the area of the rectangle ABCD.

COMPREHENCIVE PASSAGE TYPE PROBLEMS

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24. A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively .The PQ is given by the equation √3 x + y–6 =0 and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further it is given that the origin and the centre C are on the same side of the line PQ.

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i) The equation of the circle C is

a) $\left(x-2\sqrt{3}\right)^2+(y-1)^2=1$

b) $\left(x-2\sqrt{3}\right)^2+\left(y+\frac{1}{2}\right)^2=1$

c) $\left(x-\sqrt{3}\right)^2+(y+1)^2=1$

d) $\left(x-\sqrt{3}\right)^2+(y-1)^2=1$

ii) Points E and F are given by

a) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\sqrt{3}, 0\right)$

b) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\sqrt{3}, 0\right)$

c) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

d) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

iii) Equations of the sides QP, RP are

a) $y=\frac{2}{\sqrt{3}}x+1,\ y=-\frac{2}{\sqrt{3}}x-1$

b) $y=\frac{1}{\sqrt{3}}x$, y =0

c) $y=\frac{\sqrt{3}}{2}x+1, y=-\frac{\sqrt{3}}{2}x-1$

d) $y=\sqrt{3}x$, y = 0

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25. A circle C whose radius is one unit, touches the x-axis at point A. The centre Q of C lies in the first quadrant. The tangent from origin O to the circle touches it at T and the point P lies on tangent such that ΔOAP is a right angled triangle at A and its perimeter is 8 units.

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i) The length QP is

a) $\frac{1}{2}$ b) $\frac{4}{3}$ c) $\frac{5}{3}$ d) none of these

ii) Equation of the circle C is

a) $\left\{x-\left(2+\sqrt{3}\right)\right\}^2+(y-1)^2=1$ b) $\left\{x-\left(\sqrt{2}+\sqrt{3}\right)\right\}^2+(y-1)^2=1$

c) $\left(x-\sqrt{3}\right)^2+(y-2)^2=1$ d) none of these

iii) Equation of tangent OT is

a) $x-\sqrt{3}y=0$ b) $x-\sqrt{2}y=0$

c) $y-\sqrt{3}x=0$ d) none of these

REASONING TYPE QUESTIONS

26. a) Consider $L_1: 2x+3y+p-3=0$

$L_2: 2x+3y+p+3=0$

Where p is a real number, and C: $x^2+y^2+6x-10y+30=0$

STATEMENT-1: If line $\mathbf{L_1}$ is is chord of circle C, then line $\mathbf{L_2}$ is not always a diameter of circle C

And

STATEMENT-2: If line $\mathbf{L_1}$ is diameter of circle C, then line $\mathbf{L_2}$ is not a chord of circle C

a) **STATEMENT-1** is true, **STATEMENT-2** is true; **STATEMENT-2** is the correct explanation **STATEMENT-1**

b) **STATEMENT-1** is true, **STATEMENT-2** is true; **STATEMENT-2** is not the correct explanation **STATEMENT-1**

c) **STATEMENT-1** is true, **STATEMENT-2** is False

d) **STATEMENT-1** is False, **STATEMENT-2** is true **[I.I.T-2008, 3]**

27. Tangent are drawn from the point (17, 7) to the circle $x^2+y^2=169$

STATEMENT-1: The tangents are mutually perpendicular.

because

STATEMENT-2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2+y^2=338$

a) **STATEMENT-1** is true, **STATEMENT-2** is true; **STATEMENT-2** is the correct explanation **STATEMENT-1**

b) **STATEMENT-1** is true, **STATEMENT-2** is true**;** **STATEMENT-2** is not the correct explanation **STATEMENT-1**

c) **STATEMENT-1** is true, **STATEMENT-2** is False

d) **STATEMENT-1** is False, **STATEMENT-2** is true **[I.I.T-2007, 3]**

OBJECTIVE PROBLEMS

Level-1

1. The points (2, 3), (0, 2), (4, 5) and (0, t) are con-cyclic if the value of t is

a) 2 b) 1 c) 17 d) 19

2. The straight line y = mx + c cuts the circle $x^2 + y^2 = a^2$ at real points if

a) $\sqrt{a^2(1+m^2)} \le |c|$ b) $\sqrt{a^2(1-m)^2} \le |c|$

c) $\sqrt{a^2(1+m^2)} \ge |c|$ d) $\sqrt{a^2(1-m^2)} \ge |c|$

3. The locus of the centre of a circle of radius 2 which rolls on the outside of the circle $x^2 + y^2 + 3x - 6y - 9 = 0$ is

a) $x^2 + y^2 + 3x - 6y + 5 = 0$ b) $x^2 + y^2 + 3x - 6y - 31 = 0$

c) $x^2 + y^2 + 3x - 6y + \frac{22}{4} = 0$ d) None of these.

4. The circle described on the line joining the points (0, 1), (a, b) as diameter cuts the x-axis at points whose abscissa are roots of the equation

a) $x^2 + ax + b = 0$ c) $x^2 - ax + b = 0$

b) $x^2 + ax - b = 0$ d) $x^2 - ax - b = 0$

5. Four distinct points (2k, 3k), (1, 0), (0, 1) and (0, 0) lie on a circle when

a) all are integral values of k b) 0 < k < 1

c) k < 0 d) For two values of k.

6. A square inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ which its sides parallel to the coordinate axes. The co-ordinates of its vertices are.

a) (–6, –9), (–6, 5), (8, -9), (8, 5) b) (-6, 9), (-6, -5), (8, -9), (8, 5)

c) (-6, -9), (-6, 5), (8, 9), (8, 5) d) (-6, -9), (-6, 5), (8, -9), (8, -5)

7. The point diametrically opposite to the point P(1, 0) on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is **[AIEEE-2008]**

a) (-3, 4) b) (-3, -4) c) (3, 4) d) (3, -4)

8. One of the diameter of the circle $x^2 + y^2 - 12x + 4y + 6 = 0$ is given by,

a) $x + y = 0$ b) $x + 3y = 0$ c) $x = y$ d) $3x + 2y = 0$

9. The length of the cord cut off by $y = 2x + 1$ from the circle $x^2 + y^2 = 2$ is

a) $\frac{5}{6}$ b) $\frac{6}{5}$ c) $\frac{6}{\sqrt{5}}$ d) $\frac{\sqrt{5}}{6}$.

10. The coordinates of middle point of the chord $2x - 5y + 18 = 0$ cut of by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$ is

a) (1, 4) b) (2, 4) c) (4, 1) d) (1, 1)

11. Equation of the circle which touches the line $3x + 4y = 7$ and passes through (1, -2) and (4, -3) is

a) $x^2 + y^2 - 94x + 18y + 55 = 0$ b) $15x^2 + 15y^2 - 94x + 18y + 55 = 0$

c) $15x^2 + 15y^2 + 94x + 18y + 55 = 0$ d) $x^2 + y^2 - 94x - 18y + 55 = 0$

12. The equations of the circle which touch both the axes and the line x = a are

a) $x^2 + y^2 \pm ax \pm ay + \frac{a^2}{4} = 0$ b) $x^2 + y^2 + ax \pm ay + \frac{a^2}{4} = 0$

c) $x^2 + y^2 - ax \pm ay + \frac{a^2}{4} = 0$ d) None of these.

13. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cuts the co-ordinate axes in concyclic points then

a) $a_1a_2 = b_1b_2$ b) $a_1b_1 = a_2b_2$

c) $a_1b_2 = a_2b_1$ d) None of these.

14. If a circle passes through the points of intersection of the co-ordinate axes with the line $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$ then the value of λ is

a) 2 b) 1/3 c) 6 d) 3

15. The condition that the chord $x\cos\alpha + y \sin \alpha - p = 0$ of $x^2 + y^2 - a^2 = 0$ may subtend a right angle at the centre of the circle is

a) $a^2 = 2p^2$ b) $p^2 = 2a^2$ c) $a = 2p$ d) $p = 2a$

16. If the distances from the origin to the centres of three circles $x^2 + y^2 + 2\lambda_i x - c^2 = 0$ (i = 1, 2, 3) are in G.P. then the length of the tangent drawn to them from any point on the circle $x^2 + y^2 = c^2$ are in

a) A.P. b) G.P. c) H.P. d) None of these.

17. The co-ordinates of the point on the circle $x^2 + y^2 - 12x - 4y + 30 = 0$ which is farthest from the origin are

a) (9, 3) b) (8, 5) c) (12, 4) d) None of these.

18. The circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ intersect at an angle of

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$.

19. If the chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touch the circle $x^2 + y^2 = c^2$, then a, b, c are in

a) A.P. b) G.P. c) H.P. d) None of these.

20. If (2, 5) is an interior point of the circle $x^2 + y^2 - 8x - 12y + p = 0$ and the circle neither cuts a nor touches any one of the co-ordinate axes then:

a) $p \in (36, 47)$ b) $p \in (16, 47)$

c) $p \in (16, 36)$ d) None of these.

21. The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is

a) $4 \le x^2 + y^2 \le 64$ b) $x^2 + y^2 \le 25$

c) $x^2 + y^2 \ge 25$ d) $3 \le x^2 + y^2 \le 9$

22. The radius of the circle passing throughout the point (6, 2) and having x + y = 6 as its normal and x + 2y = 4 as its diameter is

a) 10 b) $2\sqrt{5}$ c) $5\sqrt{2}$ d) $4\sqrt{5}$

23. The locus of the mid points of the chords of the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ which subtends an angle of $\frac{\pi}{3}$ radians at its centre is

a) $(x + 2)^2 + (y - 3)^2 = 6.25$ b) $(x - 2)^2 + (y + 3)^2 = 6.25$

c) $(x + 2)^2 + (y - 3)^2 = 18.75$ d) $(x + 2)^2 + (y + 3)^2 = 18.75$

24. The number of common tangents to the circle $x^2 + y^2 - x = 0$ and $x^2 + y^2 + x = 0$ is

a) 2 b) 1 c) 4 d) 3

25. A circle passes through the origin and has its centre on y = x. If it cuts $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, the equation of the circle is

a) $x^2 + y^2 - x - y = 0$ b) $x^2 + y^2 - 6x - 4y = 0$

c) $x^2 + y^2 - 2x - 2y = 0$ d) $x^2 + y^2 + 2x + 2y = 0$

Level-2

1. A circle touches the x-axis and also touches the circle with centre (0, 3) and radius 2. The locus of the centre of the circle is

   a) A circle   b) A Parabola   c) An Ellipse   d)A Hyperbola

2. The equation of the line passing through the point (–2, 11) and touching the curve $x^2+y^2=25$ is

   a) $4x+3y=25$ b) $3x+4y=38$

   c) $24x-7y+125=0$ d) $7x+24y-230=0$

3. The locus of the mid points of the chords of the circle $x^2+y^2=4$ which subtend a right angle at the centre is

   a) $x+y=2$ b) $x^2+y^2=1$ c) $x^2+y^2=2$ d) $x-y=0$

4. The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2+y^2=a^2$ which touches the circle $x^2+y^2-2ax=0$ is

   a) $y^2=a(a-2x)$ b) $x^2=a(a-2y)$

   c) $x^2+y^2=(x-a)^2$ d) $x^2+y^2=(y-a)^2$

5. If the equation of one tangent to the circle with centre at (2, –1) from the origin is $3x+y=0$, then the equation of the other tangent through the origin is

   a) $3x-y=0$ b) $x+3y=0$

   c) $x-3y=0$ d) $x+2y=0$

6. A line meets the coordinate axes in A and B. A circle is circumscribed about the triangle OAB. If m and n are the distances of the tangent to the circle at the origin from the point A and B respectively, the diameter of the circle is

   a) $m(m+n)$ b) $m+n$

   c) $n(m+n)$ d) $(m+n)/2$

7. The equation of a circle with centre (4, 3) and touching the circle $x^2+y^2=1$ is

   a) $x^2+y^2+8x-6y+9=0$ b) $x^2+y^2+8x+6y-11=0$

   c) $x^2+y^2-8x-6y-11=0$ d) $x^2+y^2+8x+6y-9=0$

8. The equation of the tangents drawn from the origin to the circle $x^2+y^2-2px-2qy+q^2=0$ are perpendicular if

   a) $p=q$ b) $p^2=q^2$ c) $q=-p$ d) $p^2+q^2=1$

9. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ with its sides parallel to the axes of coordinates. The coordinates of the vertices are

a) (8, 5) b) (8, –9) c)(–6, 5) d) (–6, –9)

10. A variable chord is drawn through the origin to the circle $x^2+y^2-2ax = 0$. The locus of the centre of circle drawn on this chord as diameter is

a) $x^2 + y^2 + ax = 0$ b) $x^2 + y^2 + ay = 0$

c) $x^2 + y^2 - ax = 0$ d) $x^2 + y^2 - ay = 0$

11. If OA and OB are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and C is the centre of circle, the area of the quadrilateral OACB is

a) $\frac{1}{2}\sqrt{c(g^2 + f^2 - c}$ b) $\sqrt{c(g^2 + f^2 - c}$

c) $c\sqrt{g^2 + f^2 - c}$ d) $\sqrt{\frac{g^2 + f^2 - c}{c}}$

12. If a circle passes through the point (a, b) and cuts the circle $x^2+y^2= k^2$ orthogonally, the equation of the locus of its centre is

a) $2ax + 2by = a^2 + b^2 + k^2$ b) $ax + by = a^2 + b^2 + k^2$

c) $x^2 + y^2 + 2ax + 2by + k^2 = 0$ d) $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - k^2 = 0$

13. The equation of the circle which passes through the origin, has its centre on the line x + y = 4and cuts the circle $x^2+y^2-4x+2y+4 = 0$ orthogonally, is

a) $x^2 + y^2 - 2x - 6y = 0$ b) $x^2 + y^2 - 6x - 3y = 0$

c) $x^2 + y^2 - 4x - 4y = 0$ d) none of these

14. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2+y^2-6x-8y = 24$ is

a)0 b)1 c)3 d)4

15. The angle between the pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. The equation of locus of the point P is

a) $x^2+y^2 + 4x - 6y + 4 = 0$ b) $x^2+y^2 + 4x - 6y - 9 = 0$

c) $x^2+y^2 + 4x - 6y - 4 = 0$ d) $x^2+y^2 + 4x - 6y + 9 = 0$

16. The locus of the centre of a circle which touches externally the circle $x^2+y^2-6x-6y+14 = 0$ and also touches the y axis is given by

a) $x^2-6x-10y+14=0$ b) $x^2-10x-6y+14=0$

c) $y^2-6x-10y+14=0$ d) $y^2-10x-6y+14=0$

17. The locus of mid point of the chords of the circle $x^2+y^2-2x-2y-2=0$ which makes an angle of 120^o at the centre, is

a) $x^2+y^2-2x-2y+1=0$ b) $x^2+y^2+x+y-1=0$

c) $x^2+y^2-2x-2y-1=0$ d) none of these

18. A line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2+y^2=r^2$ at A and B. Then PA.PB is equal to

a) $(\alpha+\beta)^2-r^2$ b) $\alpha^2+\beta^2-r^2$

c) $(\alpha-\beta)^2+r^2$ d) None of these.

19. If two distinct chords, drawn from the point (p, q)on the circle on the circle $x^2+y^2=px+qy$ are bisected by the x-axis , then

a) $p^2=q^2$ b) $p^2=8q^2$ c) $p^2<8q^2$ d) $p^2>8q^2$

20. Let L_1 be the straight line passing through the origin and L_2 be the straight line $x+y=1$. If the intercept made by the circle $x^2+y^2-x+3y=0$ on the line L_1 and L_2 are equal, then which of the following equation can represent L_1?

a) $x+y=0$ b) $x-y=0$ c) $x+7y=0$ d) $x-7y=0$

21. If a>2b>0then the positive value of m for which $y=mx-b\sqrt{1+m^2}$ is common tangent to $x^2+y^2=b^2$ and $(x-a)^2+y^2=b^2$ is **[I.I.T Sc.2002]**

a) $\frac{2b}{\sqrt{a^2-4b^2}}$ b) $\frac{\sqrt{a^2-4b^2}}{2b}$ c) $\frac{2b}{a-2b}$ d) $\frac{b}{a-2b}$

22. The triangle PQR is inscribed in the circle $x^2+y^2=25$ .If Q and R have the coordinate (3, 4) and (–4, 3) respectively, then Angle QPR is equal to

a) 90^o b) 60^o c) 45^o d) 30^o **[I.I.T Sc.2000]**

23. Let AB be the chord of the circle $x^2+y^2=a^2$ subtending a right angle at the centre. Then the locus of centroid of the triangle PAB as P moves on the circle is

a) a parabola b) a circle

c) an ellipse d) a pair of straight lines

24. Let PQ and RS be the tangents at the extremities of the diameter PR of circle of radius r. If PS and RQ intersect at appoint X on the circumference of the circle, then 2r equals **[I.I.T Sc.2001]**

a) $\sqrt{PQ.RS}$ b) $\frac{PQ+RS}{2}$ c) $\frac{2PQ.RS}{PQ+RS}$ d) $\sqrt{\frac{PQ^2+RS^2}{2}}$

25. The radius of the circle $3x^2+3y^2+?xy+9x+(?-6)y+3=0$ is

a) 3/2 b) 2/3 c) $\frac{1}{2}\sqrt{17}$ d) none

✣ --------------------- ✣

6. Parabola

6.1 Conic Section :

S is a fixed point and z_1 z_2 is a fixed straight line. If a point P, moves in such a way that its distance from fixed point S bears a constant ratio with its distance from fixed line z_1z_2 i.e.

$$\frac{PS}{PM} = \text{constant}$$

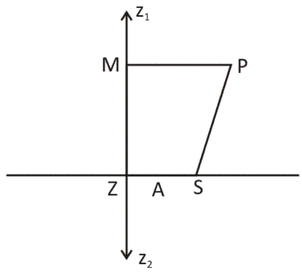

Fig 1

Then locus of the point is a conic section. The constant quantity is called eccentricity and denoted by e.

(1) If e = 1, i.e. PS = PM then locus is Parabola

(2) If e < 1, i.e. PS < PM then locus is Ellipse

(3) If e> 1, i.e. PS > PM then locus is Hyperbola.

Now we can define a Parabola.

6.2 Definition – Parabola

The locus of a point, which moves in such a way that its distance from a fixed point is always equal to its distance from a fixed straight line is Parabola. The fixed point is called focus and denoted by S, and the fixed line is called directrix and is denoted by z' z'' or z_1z_2.

6.3 Equation of parabola

In fig. 2 z_1z_2 is directrix, S is focus. Perpendicular from S on directrix is SZ. Mid point of SZ is Q. ZA = AS, A is on parabola.

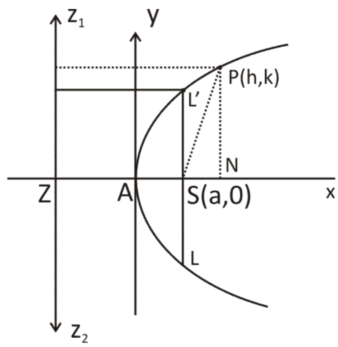

Fig 2

Let A be origin AS x – axis and on it perpendicular YA, y – axis. Let SA = a

$\therefore$ focus S is (a, 0), vertex A (0,0) and equation of directirx z_1z_2, x + a = 0. Now let P(h, k) be any point on parabola and PM is perpendicular from P on directrix.

By definition of parabola PS = PM

$$\Rightarrow \sqrt{(h-a)^2 + (k-0)^2} = h + a \qquad \Rightarrow k^2 = 4ah$$

$$\therefore \text{ Locus } y^2 = 4ax$$

This is the standard equation of parabola. Perpendicular on axis at focus S meets parabola in L and L'_1. LL'_1 is called Latus rectum of parabola. Please note.

(i) Vertex A is (0, 0)

(ii) Focus S is (a, 0)

(iii) Axis ZS is $y = 0$

(iv) Directrix is $x + a = 0$

(iv) Equation of latus rectum LL'_1 is $x = a$

Its length $LL' = 2\,SL = 2\,LT$

$= 2\,ZS = 2.2a \quad = 4a$

End point are L, $(a, 2a)$; $L'(a, -2a)$

(v) Focal distance of a point P : If point P on parabola $y^2 = 4ax$ is (x_1, y_1), then its distance from focus S, is called focal distance PS.
From fig 2.

$PS = PM = ZN = ZA + AS = a + x_1$

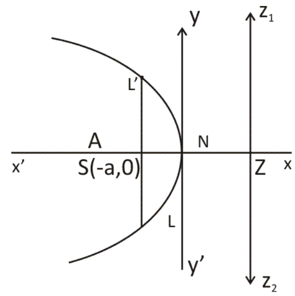

Fig 3

6.4 Other standard equations of parabola.

(a) $y^2 = -4ax$

i) Vertex A, (0,0)

ii) Focus S $(-a, 0)$

iii) Axis $Y = 0$

iv) Directrix $x - a = 0$

v) Equation latus rectum $x + a = 0$

Length, 4a; End points $(-a, 2a)$, $(-a, -2a)$

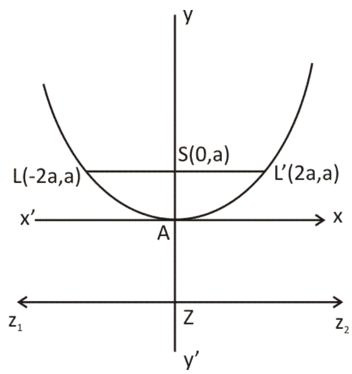

Fig 4a

(b) $X^2 = 4ay$

(i) Vertex A (0,0)

(ii) Focus S (0, a)

(iii) axis $x = 0$

(iv) Directrix $y + a = 0$

(v) L.R. 4a

Equation $y - a = 0$. End points $(2a, a)$ $(-2a, a)$

(c) $x^2 - 4ay$

i) Vertex A (0, 0)

ii) Focus $(0, -a)$

iii) Directrix $y - a = 0$

iv) Latus rectum $y + a = 0$

Length 4a, end points $(-2a, -a)$, $(2a, -a)$

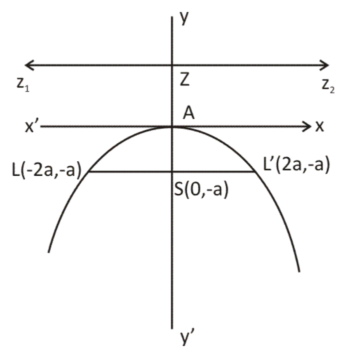

Fig 4b

Note : 1. Given the equation of parabola you may be asked to write down co-ordinates of vertex, focus, equation of axis, directrix and latus rectum.

2. Given vertex and focus you may be asked to write the equation of parabola.

3. Some times equation is not given in standard form. We first reduce it to

(a) $(y-l)^2 = p(x-m)$ (b) $(y-l)^2 = -p(x-m)$

or (c) $(x-m)^2 = \pm p(y-l)$ and then change origin to desired place and further reduce it to standard from.

Solved Examples

Example 1 : Find equation of parabola whose focus is (3, –4) and directrix x + y = 4.

Sol. : Let (h, k) be any point on parabola,

$$\therefore (h-3)^2+(k+4)^2=\left(\frac{h+k-4}{\sqrt{2}}\right)^2$$

$$\Rightarrow 2h^2+2k^2-12h+16k+50=h^2+k^2-8k-8h+2hk+16$$

$$\Rightarrow h^2+k^2-2hk-4h+24k+34=0$$

$\therefore$ Equation of parabola a is

$x^2 + y^2 – 2xy – 4x + 24y + 34 = 0$

Note: This is second degree equation, but $\Delta \neq 0$ and $h^2 = ab$.

Example 2 : Find equation of parabola whose vertex is (2, 2) and focus (4, 2).

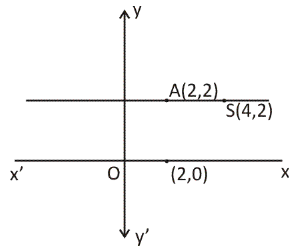

Fig 5

Sol. : Axis of parabola is parallel to x-axis and at a distance of 2units on its positive side. Now if we change axes to (2, 2) the equation of parabola on changed axes is

$Y^2 = 4.2\ x = 8\ X$

And here $Y = y – 2.\ X = x – 2$

Equation of parabola is $(y – 2)^2 = 8(x – 2)$

Example 3 : Vertex of a parabola is (0, a) and length of latus-rectum is 8a. Find its equation when (i) axis is parallel to x-axis. (ii) axis is parallel to y-axis.

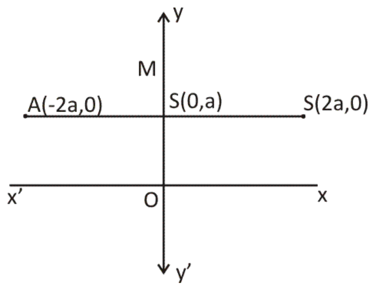

Fig 6

Sol. : (i) When axes is parallel to x-axis, the focus can be on either side of A at a distance 2a from it

$\therefore$ Equation of $(y –a)^2 = 8ax$

or $(y – a)^2 = – 8ax$

(ii) When axis is parallel to y-axis

Equation is $x^2 = \pm 8a\ (y –a)$.

Example 4 : Find equation of parabola whose vertex is (2, 1) and focus (1, –1).

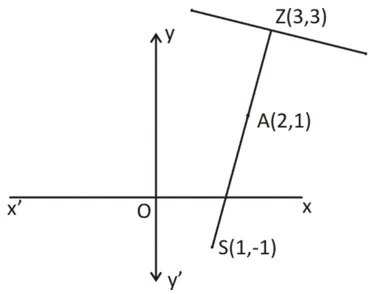

Fig 7

Sol. : Vertex A(2, 1), focus S(1, –1)

$\therefore$ AS is not parallel to any axes. Axis of parabola is AS

SA produced meets directrix in Z. If Z is (α, β) then A being mid point of SZ

$$\frac{\alpha+1}{2}=2, \frac{\beta-1}{2}=1 \Rightarrow \alpha=3, \beta=3.$$

Directrix passes through (3, 3) and is perpendicular to SZ, its equation is

$$y-3=-\frac{1}{2}(x-3) \Rightarrow x+2y-9=0$$

Focus (1, –1); $\therefore$ Equation of parabola is

$$(x-1)^2+(y+1)^2=\frac{(x+2y-9)^2}{5}$$

$$\Rightarrow 4x^2-4xy+y^2+8x+46y-71=0$$

Example 5 : Axis of a parabola is 3x – 4y + 5 = 0 and its vertex is (1, 2). Find equation of parabola if length of L.R. is 8.

Sol. : Axis 3x – 4y + 5 = 0, slope $\tan\theta = 3/4$

$\therefore \cos\theta = 4/5, \ \sin\theta = 3/5$

Latus –rectum is 8 $\Rightarrow a=2$.

Let us find the co-ordinates of points 2 units away on right and left of vertex.

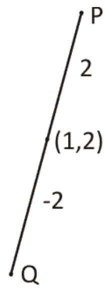

Fig 8

$$\therefore \frac{x-1}{\cos\theta}=\frac{y-2}{\sin\theta}=\pm 2$$

P is $\left[2.\frac{4}{5}+1, 2\frac{3}{5}+2\right]$ and Q is $\left[-\frac{8}{5}+1, -\frac{6}{5}+2\right]$

It P is focus $\left(\frac{13}{5}, \frac{16}{5}\right)$ and directrix goes through $\left(-\frac{3}{5}, \frac{4}{5}\right)$, Q and is perpendicular to axis.

$\therefore$ Directrix is $4x+3y=-\frac{12}{5}+\frac{12}{5}=0$

Equation of parabola is $\left(x-\frac{13}{5}\right)^2+\left(y-\frac{16}{5}\right)^2=\left(\frac{4x+3y}{25}\right)^2$

$$\Rightarrow 9x^2-24xy+16y^2-130x-160y+425=0$$

(ii) if Q is focus and directrix is through P perpendicular to axis.

Directrix then is $4x+3y=\frac{52}{5}+\frac{48}{5}=20$

Equation of parabola is $\left(x+\frac{3}{5}\right)^2+\left(y-\frac{4}{5}\right)^2=\frac{(4x+3y-20)^2}{25}$

$\Rightarrow 9x^2-24xy+16y^2+190x+80y-375=0$

Example 6 : Find the vertex, focus, directrix, Latus rectum and axis of parabola $y^2+6y-2x+5=0$.

Sol. : Equation of parabola $y^2+6y-2x+5=0$

$\Rightarrow (y-3)^2=2x+4=2(x+2)$

Transferring origin to (–2, –3), $Y^2 = 2x$

On changed axes	On original aexs
(i) vertex (0, 0)	(i) vertex (– 2, –3)
(ii) Focus $\left(\frac{1}{2},0\right)$	(ii) Focus $\left(-\frac{3}{2},-3\right)$
(iii) Directrix $X+\frac{1}{2}=0$	(iii) Directrix $x+2+\frac{1}{2}=0$
(iv) Axis Y = 0	(iv) Axis y + 3 = 0
(v) Equation L.R. $X-\frac{1}{2}=0$	(b) L.R. $x+\frac{3}{2}=0$

Example 7 : The axis of a_1 parabola is parallel to y axis and its vertex is (2,3). Find equation of parabola if it passes through (4,5)

Sol. : Axis is parallel to y-axis and passes through (2, 3) $\therefore$ axis is x = 2

Parabola is $(x-2)^2 = p(y-3)$

It passes through (4,5)

$\therefore\ 2^2=p\times 2\Rightarrow p=2$

$\therefore$ Equation is $(x-2)^2=2(y-3)$

Example 8 : The focal distance of a point P on parabola $y^2 = -2x$ is $\frac{5}{2}$. Find co-ordinates of P.

Sol. : Parabola $y^2 = -2x$ lies in left of y-axis.

point P is $(-x_1, y_1)$ or $(-x_1, -y_1)$

Focal distance = $|-x_1| + a = |-x_1| + \frac{1}{2} = \frac{5}{2}$

$\therefore\ |-x_1| = 2,\ y_1^2 = 4 \Rightarrow y_1 = \pm 2$

$\therefore$ Point P is $(-2,2)$ or $(-2,-2)$

Example 9 : The axis of a parabola is along 5x – 12y + 2 = 0 and vertex is (2,1). If latus rectum is 4 units. Find equation of parabola.

Sol. : This type of question has been solved in an example. Now we shall do this question by transferring axes.

Let origin be at (2, 1) and X –axis along 5x – 12y + 2 = 0. Equation of parabola then is $Y^2 = 4X$.

Slope of straight line 5x – 12y + 2 = 0 in $\tan\theta = 5/12$

$\therefore\ \cos\theta = \frac{12}{13}, \sin\theta = \frac{5}{13}$

Now origin is still at (2, 1) and axes have been transferred in anticlockwise direction through angle θ.

$\therefore$ Equation is $(Y'\cos\theta - X'\sin\theta)^2 = 4(X'\cos\theta + Y'\sin\theta)$

$$\Rightarrow \left(\frac{12}{13}Y' - \frac{5}{13}X'\right)^2 = 4\left(\frac{12}{13}X' + \frac{5}{13}Y'\right)$$

Now transfer it to origin axis at (0, 0)

$$\left[\frac{12}{13}(y-1) - \frac{5}{13}(x-2)\right]^2 = 4\left[\frac{12}{13}(x-2) + \frac{5}{13}(y-1)\right]$$

$$\Rightarrow (12y - 5x - 2)^2 = 52[12x + 5y - 29]$$

$$\Rightarrow 25x^2 - 120xy + 144y^2 - 604x - 308y + 1512 = 0$$

(ii) Other parabola $Y^2 = -4X$

$$[12y - 5x - 2]^2 = -52(12x + 5y - 29)$$

$$25x^2 - 120xy + 144y^2 + 644x + 212y - 1504 = 0$$

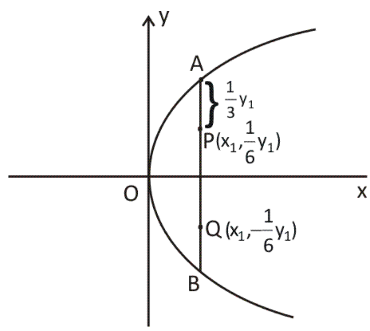

Fig 8

Example 10 : Find the locus of the point which while moving trisects the double ordinate of parabola $y^2=12x$

Sol. : Let AB be one of the double ordinate of parabola $y^2 = 12x$. $A(x_1, y_1)$

Point P and Q trisect this double ordinate.

If P is (h, k), then h = x_1 and

$k = \frac{1}{6}y_1$, (x_1, y_1) is on parabola

$\therefore y_1^2 = 12x_1$

$\Rightarrow (6k)^2 = 12h \ \Rightarrow k^2 = \frac{1}{3}h$

$\therefore$ Locus of this point is parabola $y^2 = \frac{1}{3}x$, (x_1 and $-\frac{1}{6}y_1$) also satisfy this equation.

Example 11: Find vertex, focus, and equations of axis, directrix and latus rectum of parabola $y=ax^2+bx+c$.

Sol. : Parabola $ax^2 + bx + c = y$

$$\Rightarrow x^2 + \frac{b}{a}x = \frac{1}{a}(y-c)$$

$$\Rightarrow \left(x + \frac{b}{2a}x\right)^2 = \frac{y}{a} - \frac{c}{a} + \frac{b^2}{4a^2}$$

$$= \frac{y}{a} + \frac{b^2-4ac}{4a^2}$$

$$= \frac{1}{a}\left[y + \frac{b^2-4ac}{4a}\right]$$

Changing origin to $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$

Equation of parabola $X^2 = \frac{1}{a}Y$

Transferred axes	Original axes
(i) Vertex (0,0)	$\left(-\frac{b}{2a}, -\frac{(b^2-4ac)}{4a}\right)$
(ii) Focus $\left(0, \frac{1}{4a}\right)$	$\left(-\frac{b}{2a}, -\frac{(4ac-b^2)}{4a}\right)$
(iii) Directrix $Y = -\frac{1}{4a}$	$y + \frac{b^2-4ac+1}{4a} = 0$

(iv) L.R. $Y=\frac{1}{4a}$	$y+\frac{b^2-4ac+1}{4a}=0$
(iv) Axis X=0	$x+\frac{b}{2a}=0$

Example 12 : P is any point on parabola $y^2 = 4ax$ PM is perpendicular on directrix from P. S is focus. If P, M, S are vertices of an equilateral triangle then find P.

Sol. : Equation of parabola $y^2 = 4ax$

Directrix $x + a = 0$,

Let P be (x_1, y_1) $\therefore M(-a, y_1)$ S is $(a, 0)$

PM = SP by definition of parabola

MS = PS triangle equilateral

$\therefore 4a^2+y_1^2=a^2+x_1^2+2ax_1$

But $y_1^2=4ax_1$ $\therefore$ $4a^2+4ax_1=a^2+x_1^2+2ax_1$

$\Rightarrow x_1^2-2ax_1-3a^2=0$

$\Rightarrow (x_1-3a)(x_1+a)=0$

$x_1\neq -a$ $\therefore x_1=3a$, $\Rightarrow y_1=\pm 2\sqrt{3}a$

$\therefore$ P is $(3a, \pm 2\sqrt{3}a)$

Practice Worksheet (Foundation Level) – 6(a)

1. Focus of parabola is (2, 3) and directrix is $x - 4y + 3 = 0$. Find its equation.
2. The focus and vertex of a parabola are (5, 3) and (1, 3). Find its equation.
3. The vertex and focus of a parabola are (a, b) and (a, 3b). Find its equation
4. The axis of a parabola is along x –axis and distances of vertex and focus from origin on positive side of x axis are a and a'. Find its equation.
5. The vertex of parabola is at (2, 3); and focus at (2, –5). Find its equation.
6. The focal distance of a point on parabola $y^2 + 20x = 0$ is 10. Find the point.
7. Find vertex, focus, length of latus rectum, and equation of latus rectum and directrix of

   (a) $(y-2)^2 = 3(x+1)$ (b) $4y^2 + 12x - 20y + 67 = 0$

   (c) $x^2 + 4x + 4y = 0$ (d) $x^2 - 2ax + 2ay = 0$
8. The focus of a parabola is (–8, –2) and directrix is $y = 2x - 9$. Find equation of parabola.
9. Find equation of parabola whose focus is (2, 3); axis parallel to y-axis and directrix goes through (2, 5)
10. P is a point on parabola $y^2 = 8x$ and perpendicular from P on directrix is PM. If P, M and focus S are vertices of an equilateral triangle, then find P.
11. Find locus of mid points of chords, which go through vertex of parabola.

    (a) $y^2 = 4ax$ (b) $y^2 = 8x$ (c) $x^2 = -8y$
12. An equilateral triangle has been inscribed in parabola (a) $y^2 = 4ax$ (b) $y^2 = 32x$ and one of its vertex is at origin. Find the length of the side of equilateral triangle.
13. Find the position of points (a) (3, 4) (b) (–2,5) and (c) (2, 6) with respect to parabola $y^2 = 18x$.
14. Find equation of lines joining vertex to the ends of latus rectum of parabola. $(y-3)^2 = 16(x-2)$. Find also angle between them.
15. Find equation of parabola which goes through point (–3, 5) and whose vertex is (–4, 3), axis is parallel to x-axis.
16. Find equation of parabola whose vertex is at (3, 2) and focus (4, 3).
17. A double ordinate of parabola a $y^2 = 4ax$ is 8a. Lines join vertex to the ends of this double ordinate. The angle included by them is

    (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

18. The axis of a parabola is along x – y = 6, its latus rectum is of length $8\sqrt{2}$, vertex is (3, –3) Parabola faces in 1st quadrant. Its equation is :

(a) $x^2-2xy+y^2-4x-28y+36=0$ (b) $x^2-2xy+y^2-28x-4y+36=0$

(c) $x^2-2xy+y^2-12x-24y+36=0$ (d) $x^2-2xy+y^2+12x-4y+36=0$

19. The focal chord of parabola $y^2 = 4ax$ is inclined at 45^0 with x-axis. The angle subtended by this chord at the vertex is

(a) $\tan^{-1}\left(-\frac{2\sqrt{2}}{3}\right)$ (b) $\tan^{-1}\left(-\frac{2}{3}\right)$ (c) $\tan^{-1}\left(-\frac{4\sqrt{2}}{3}\right)$ (d) $\tan^{-1}\left(\frac{4\sqrt{2}}{5}\right)$

20. The equation of latus rectum of a parabola is 3x – 4y + 8 = 0 and its vertex is $\left(\frac{3}{2},0\right)$ its equation is:

(a) $x^2-2xy+y^2+18x-102y+17=0$

(b) $16x^2+24xy+9y^2+102x-236y-189=0$

(c) $16x^2+24xy+9y^2+202x-136y-389=0$

(d) $16x^2+24xy+9y^2-102x+236y-389=0$

21. The axes of two parabolas are along x + y = 0 and x – y = 0 and these lie in the 1st and 2nd quadrant mostly. They have a common vertex. If distances between directrix and latus-rectum of each be $8\sqrt{3}$ then their equations are :

(a) $x^2-2xy+y^2-32x+32y=0$, $x^2+2xy+y^2+32x-32y=0$ (b) $x^2-2xy+y^2-32x-32y=0$, $x^2+2xy+y^2+32x-32y=0$

(c) $x^2-2xy+y^2+32x+32y=0$, $x^2+2xy+y^2-32x-32y=0$ (d) $x^2\pm y^2-32x-32y=0$

6.5 Tangent :

Let PQ be chord of parabola $y^2 = 4ax$; P is (x_1, y_1) , $Q(x_2, y_2)$

Equation PQ $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ ....(1)

P and Q are on parabola

$$\therefore \left.\begin{aligned} y_1^2 &= 4ax_1 \\ y_2^2 &= 4ax_2 \end{aligned}\right\} \begin{aligned} &y_2^2 - y_1^2 = 4a(x_2 - x_1) \\ &\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1} \end{aligned} \quad ....(2)$$

Now (1) becomes $y - y_1 = \frac{4a}{y_1 + y_2}(x - x_1)$

If PQ is rotated at P in such a way that Q coincides with $P \Rightarrow x_2 \to x_1, y_2 \to y_1$ Now PQ is tangent at (x_1, y_1)

$\therefore$ Equation of tangent $y - y_1 = \frac{4a}{2y_1}(x - x_1)$

$$\Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$
$$\Rightarrow yy_1 - 4ax_1 = 2ax - 2ax_1$$
$$\Rightarrow yy_1 = 2a(x + x_1)$$

Note : Here too y^2 is put as yy_1 and $2x$ as $x + x_1$.

6.6 To find condition so that y = mx + c is tangent to parabola y^2 = 4ax.

Straight line $y = mx + c$; Curve $y^2 = 4ax$

putting value of y from straight line equation in curve equation.

$$(mx + c)^2 = 4ax \Rightarrow m^2x^2 + x(2mc - 4a) + c^2 = 0$$

Line should be tangent if roots of this equation are equal.

i.e. $(2mc - 4a)^2 - 4m^2c^2 = 0$

$$\Rightarrow (mc - a) = 0 \Rightarrow mc = a$$

$$\Rightarrow c = \frac{a}{m}$$

$\therefore y = mx + \frac{a}{m}$ is tangent to parabola.

Point of contact: let point of contact be (x_1, y_1)

$\therefore$ tangent $\Rightarrow yy_1 - 2ax - 2ax_1 = 0$

This tangent $my - m^2x - a = 0$

Comparing co-efficients $\frac{y_1}{m} = \frac{2a}{m^2} = 2x_1$

$\therefore x_1 = \frac{a}{m^2}, y_1 = \frac{2a}{m}$ points $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Note : $y = mx + \frac{a}{m}$ is any tangent to parabola $y^2 = 4ax$ and its point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

6.7 Tangents from an external point

Parabola $y^2 = 4ax$; Ex. Point (x_1, y_1) any tangent $y = mx + \frac{a}{m}$. It goes through (x_1, y_1)

$\Rightarrow y_1 = mx_1 + \frac{a}{m}$

$\Rightarrow m^2x_1 - my_1 + a = 0$

It is quadratic in m, gives two values of m. Thus two tangents result from on external point.

Example: Find equation of tangents from (–2a, –a) of parabola $y^2 = 4ax$.

Sol. : $y = mx + \frac{a}{m}$ is any tangent of $y^2 = 4ax$

It passes through (–2a, – a)

$\therefore -a = -2am + \frac{a}{m}$

$\Rightarrow 2m^2 - m - 1 = 0$

$\Rightarrow (m-1)(2m+1) = 0$

$\Rightarrow m = 1, m = -1/2$

$\therefore$ tangents (i) $y = x + a$

(ii) $y = -\frac{1}{2}x - 2a \Rightarrow 2y + x + 4a = 0$

Note : Equation of pair of tangents from an external point, like circles is given by SS' = T^2. In this case, equation is

$(y^2 - 4ax).(a^2 + 8a^2) = [-y - 2x + 4a]^2 \ a^2$

$\Rightarrow 9(y^2 - 4ax) = [y + 2x - 4a]^2$

$\Rightarrow 9(y^2 - 4ax) = [y^2 + 4x^2 + 4xy - 16ax - 8ay + 16a^2]$

$\Rightarrow 8y^2 - 4xy - 4x^2 - 20ax + 8ay - 16a^2 = 0$

$\Rightarrow 2y^2 - xy - x^2 - 5ax + 2ay - 4a^2 = 0$

It y – x – a = 0 and 2y + x + 4a = 0 are multiplied then we obtained this equation as well.

$\therefore$ when need be SS' = T^2 be applied

6.8 Chord of contact

If from an external point P, PA and PB are tangents to parabola $y^2 = ax$, then AB is called chord of contact of point P.

Let P be (x'y'), $A(x_1, y_1)$, $B(x_2, y_2)$

$$\left.\begin{array}{l}\text{Tangent at A } \; yy_1 = 2a(x + x_1) \\ \text{Tangent at B } \; yy_2 = 2a(x + x_2)\end{array}\right\}$$

Both pass through (x', y')

$$\therefore \left.\begin{array}{l} y'y_1 = 2a(x' + x_1) \\ y'y_2 = 2a(x' + x_2)\end{array}\right\} \qquad (\alpha)$$

From these 2 equations it is clear that both (x_1, y_1), (x_2, y_2) lie on $yy' = 2a(x + x')$. This is equation of chord of contact of point P.

Note : There is similarity with the equation of tangent at point (x', y'). But in case of tangent point (x', y') is on the parabola. While in case of chord of contact the point is outside parabola.

6.9 Length of a chord

Apply general formula $(x_1 \sim x_2)$ $\sqrt{1 + m^2}$. Where m is the slope of chord and x_1, x_2 are abscissa of the two end points of the chord. By substituting value of y from the equation of chord, into the equation of parabola quadratic equation in x is obtained.

It gives values of $x_1 + x_2$ and $x_1 x_2$; $x_1 \sim x_2$ is calculated from these..

Solved Examples

Example 13. (a) Find equation of tangent at a point whose abscissa is double of its ordinate. (b) inclined at 30^0 with x-axis. Parabola is $y^2 = 6x$ (c) at the ends of latus-rectum.

Sol. : (a) Let p be the ordinate of the point then 2p is abscissa $\therefore p^2 = 6(2p) \Rightarrow p = 12$ point is (24, 12), tangent 12y = 3(x + 24) $\Rightarrow x - 4y + 24 = 0$

(b) $y = mx + \frac{3}{2m}$ is any tangent to parabola $y^2 = 6x$ (here $a = \frac{6}{4}$). Given $m = \tan 30^0 = \frac{1}{\sqrt{3}}$

$\therefore$ tangent is $y = \frac{1}{\sqrt{3}}x + \frac{3\sqrt{3}}{2}$

$\Rightarrow \sqrt{3}y - x = 9/2$

(c) Ends of latus-rectum are $\left[\frac{3}{2}, 3\right], \left[\frac{3}{2}, -3\right]$ tangents $\pm 3y = 3(x + 3/2)$

$\Rightarrow y = x + \frac{3}{2},\ y + x + \frac{3}{2} = 0$

Note : See these intersect at right angles. Point of intersection is $\left(-\frac{3}{2}, 0\right)$. Axis of parabola meets directrix at this point.

Example 14: Find c if y = mx +c is tangent to parabola $y^2 = 4a(x + a_1)$

Sol. : Putting value of y from straight line equation in to parabola equation

$(mx + c)^2 = 4ax + 4a^2$

$\Rightarrow m^2x^2 + 2mcx + c^2 = 4ax + 4a^2$

$\Rightarrow m^2x^2 + x(2mc - 4a) + (c^2 - 4a^2) = 0$

The line shall be tangent if roots of this equation are equal

$\Rightarrow 4(mc - 2a)^2 - 4m^2(c^2 - 4a^2) = 0$

$\Rightarrow 4a^2 - 4amc + 4a^2m^2 = 0$

$\Rightarrow amc = a^2 + a^2m^2$

$\Rightarrow c = \frac{a}{m} + am$

Example 15 : Prove that 7x + 6y – 13 = 0 is tangent to parabola $y^2 - 7x - 8y + 14 = 0$. Find point of contact as well.

Sol. Straight line 7x + 6y – 13 = 0 $\Rightarrow x = \frac{13-6y}{7}$ putting this value of x in the equation of parabola

$$y^2 - (13-6y) - 8y + 14 = 0$$
$$\Rightarrow y^2 - 2y + 1 = 0 \qquad \Rightarrow (y-1)^2 = 0$$

roots of equation are equal

$\therefore$ straight line is tangent point of contact y = 1, $x = \frac{13-6}{7} = 1$ i.e. (1, 1)

Example 16 : Find equation of tangent to parabola y^2 – 4y – 2x + 2 = 0 (a) which is parallel to 2x- 3y + 5 = 0 (b) which is perpendicular to 3x + 4y = 0 (c) which is equally inclined to axes.

Sol. : Equation of parabola. $y^2 - 4y - 2x + 2 = 0 \Rightarrow (y-2)^2 = 2(x+1) \Rightarrow a = \frac{1}{2}$

Any tangent is $y - 2 = m(x+1) + \frac{1}{2m}$

(a) || to 2x – 3y + 5 = 0 $\Rightarrow m = \frac{2}{3}$ tangent $y - 2 = \frac{2}{3}(x+1) + \frac{3}{4} \Rightarrow 8x - 12y + 41 = 0$

(b) $\perp$ to 3x + 4y = 0 $\therefore m = -\left(-\frac{1}{3/4}\right) = \frac{4}{3}$

tangent $y - 2 = \frac{4}{3}(x+1) + \frac{3}{8}$

$\Rightarrow 32x - 24y + 89 = 0$

(c) Equally inclined to axes, $m = \pm 1$ (45^0 or 133^0)

$\therefore$ tangent $y - 2 = (x+1) + \frac{1}{2} \Rightarrow 2x - 2y + 7 = 0$

and $y - 2 = -(x+1) - \frac{1}{2} \Rightarrow y + x = \frac{1}{2}$

Example 17 : Find the equation of tangent of the parabola y^2 + 2y – 3x – 2 = 0 at point where abscissa = ordinate.

Sol. : Let the point be (p, p)

$$\therefore p^2 + 2p - 3p - 2 = 0 \Rightarrow p^2 - p - 2 = 0$$
$$\Rightarrow (p-2)(p+1) = 0$$

$\therefore$ point is (2, 2) and (–1, –1)

Equation parabola $y^2+2y-3x-2=0$

i) tangent at (2, 2) $2y+(y+2)-\frac{3}{2}(x+2)-2=0$

$\Rightarrow 6y-3x+4-6-4=0 \ \Rightarrow x-2y+2=0$

ii) tangent at (–1,–1) $-y+(y-1)-\frac{3}{2}(x-1)-2=0$

$\Rightarrow -1-\frac{3}{2}x+\frac{3}{2}-2=0 \Rightarrow x+1=0$

Example 18 : Find equations of tangents from (3, 10) to parabola $y^2 = 12x$.

Sol. : Any tangent to parabola $y^2 = 12x$ is $y=mx+\frac{3}{m}$.

It goes through (3,10) $\Rightarrow 10=3m+\frac{3}{m}$

$\Rightarrow 3m^2-10m+3=0 \quad \Rightarrow (3m-1)(m-3)=0$

$\therefore m=1/3$ and m = 3

$\therefore$ tangent $y=\frac{1}{3}x+9$, $y = 3x + 1$

Note : We can do it with the help of $SS' = T^2$ also

i.e. $(y^2-12x)(100-36)=(10y-6x-18)^2$

$\Rightarrow 3x^2-10xy+3y^2+82x-30y+27=0$

$\Rightarrow (3x-y+1)(x-3y+27)=0$

tangents are $3x-y+1=0, x-3y+27=0$

Example 19 : Prove that the locus of point of intersection of two perpendicular tangents of parabola $y^2 = 4ax$ is directrix of parabola.

Sol. : Let a tangent be $y=mx+\frac{a}{m}$ ....(1)

$\therefore$ perpendicular tangent is $y=-\frac{1}{m}x-am$ ....(2)

From (1) and (2) $0=x\left(m+\frac{1}{m}\right)+a\left(\frac{1}{m}+m\right)$

$\Rightarrow x+a=0$

This is the equation of directrix. Hence proved.

Note : This is an important property of parabola.

Example 20 (a) : Prove that the portion of a tangent to parabola $y^2 = 4ax$ between directrix and point of contact subtends a right angle at focus.

Sol. : Let $P(x_1, y_1)$ be the point of contact. Tangent at this point is $yy_1 = 2a(x + x_1)$. It meets directrix $x + a = 0$ at M.

M is $\left(-a, \frac{2ax_1 - 2a^2}{y_1}\right)$

Slope of $PS = \frac{y_1}{x_1 - a}; = m_1$

Slope of $MS = \frac{2ax_1 - 2a^2}{2ay_1} = \frac{x_1 - a}{-y_1} = m_2$

$$m_1 \times m_2 = \left(\frac{y_1}{x_1 - a}\right)\left(\frac{x_1 - a}{-y_1}\right) = -1$$

$\therefore$ PM subtends an angle of 90^0 at focus.

This is a property of parabola.

Example 20(b) : Find length of chord $y = x\sqrt{2} - 4a\sqrt{2}$ of parabola $y^2 = 4ax$.

Sol. : Equation of chord $y = x\sqrt{2} - 4a\sqrt{2}$, parabola $y^2 = 4ax$ putting value of y from chord equation into parabola equation

$$\left(x\sqrt{2} - 4a\sqrt{2}\right)^2 - 4ax = 0 \Rightarrow (x - 4a)^2 - 2ax = 0$$
$$\Rightarrow x^2 - 10ax + 16a^2 = 0$$
$$\therefore \Rightarrow (x - 8a)(x - 2a) = 0 \Rightarrow x_1 \sim x_2 = 6a$$

Length = $(x_1 \sim x_2)\sqrt{1 + m^2} = 6a\sqrt{1 + 2} = 6a\sqrt{3}$

Example 21 : From an external point (x_1, y_1) tangents are drawn to parabola $y^2 = 4ax$. Find area of the triangle formed by tangents and chord of contact.

Sol. : Equation parabola $y^2 = 4ax$

Point P is (x_1, y_1). Chord of contact $yy_1 - 2ax - 2ax_1$. Putting value of y from it into the equation of parabola.

$$\left(\frac{2ax+2ax_1}{y_1}\right)^2=4ax$$

$$\Rightarrow 4a^2x^2+4a^2x_1^2+8a^2xx_1-4axy_1^2=0$$

$$\Rightarrow ax^2+\left(2ax_1-y_1^2\right)x+ax_1^2=0$$

$$\therefore \bar{x}_1+\bar{x}_2=\frac{y_1^2-2ax_1}{a},\ \bar{x}_1\bar{x}_2=x_1^2$$

$$\therefore \bar{x}_1-\bar{x}_2=\frac{1}{a}\sqrt{\left(y_1^2-2ax_1\right)^2-4a^2x_1^2}$$

$$=\frac{y_1}{a}\sqrt{y_1^2-4ax_1}$$

Length of chord = $\left(\bar{x}_1-\bar{x}_2\right)\sqrt{1+m^2}$

$$=\frac{y_1}{a}.\sqrt{y_1^2-4ax_1}\sqrt{1+\left(\frac{2a}{y_1}\right)^2}=\frac{y_1}{a}.\frac{1}{y_1}\sqrt{\left(y_1^2-4ax_1\right)\left(y_1^2+4a^2\right)}$$

perpendicular from (x_1, y_1) on chord of contact $p=\frac{y_1^2-4ax_1}{\sqrt{y_1^2+4a^2}}$

$\therefore$ area Triangle = $\frac{1}{2}.\frac{1}{a}.\frac{y_1^2-4ay_1}{\sqrt{y_1^2+4a^2}}\sqrt{\left(y_1^2-4ax_1\right)\left(y_1^2+4a^2\right)}$

$$=\frac{1}{2a}.\left(y_1^2-4ax_1\right)^{3/2}$$

Example 22 : Find locus of point,. Tangents from which to parabola y^2 = 4ax include an angle of $\tan^{-1}$ 2.

Sol. Let the point be (x_1, y_1) Equation of pair of tangents from (x_1, y_1) on parabola $y^2=4ax$ is; $\left(y^2-4ax\right)\left(y_1^2-4ax_1\right)=\left(yy_1-2ax-2ax_1\right)^2$

Coefficient of x^2 is $-4a^2$

Coefficient of y^2 is $y_1^2-4ax_1-y_1^2=-4ax_1$

Coefficient of xy is $4ay_1$ $h = 2ay_1$

$$\tan\theta=2=\frac{2\sqrt{4a^2y_1^2-16a^3x_1}}{-4a^2-4ax_1}$$

$$\Rightarrow 2=\frac{\sqrt{y_1^2-4ax_1}}{-\left(a+x_1\right)}\Rightarrow 4\left(a+x_1\right)^2=y_1^2-4ax_1$$

$\therefore$ locus is $y^2 = 4(a+x)^2 + 4ax$

Example 23 : Find equation of common tangents to parabola $y^2 = 2x$ and $x^2 = 16y$.

Sol. : Any tangent of parabola $y^2 = 2x$ is $y = mx + \frac{1}{2m}$ ....(1)

Putting value of y from it in $x^2 = 16y$ ....(2)

$x^2 - 16mx - \frac{8}{m} = 0$ ....(3)

If (1) is tangent of parabola (2) then roots of equation (3) must be equal.

i.e. $(-16m)^2 - 4.\left(-\frac{8}{m}\right) = 0$

$\Rightarrow 256m^2 + \frac{32}{m} = 0 \quad \Rightarrow 8m^2 + 1 = 0 \quad \Rightarrow m = -\frac{1}{2}$

$\therefore$ Common tangent is $y = -\frac{1}{2}x - \frac{2}{2}$

$\Rightarrow x + 2y + 2 = 0$

Example 24 : Two straight lines are perpendicular to each other. One touches parabola $y^2 = 4a(x + a)$ and the other touches parabola $y^2 = 4p(x+p)$. Prove that point of intersection of these lines lie on $x + a - p = 0$

Sol. : Parabola $y^2 = 4a(x + a)$ any tangent to it is

$y = m(x+a) + \frac{a}{m}$ ....(1)

Other parabola $y^2 = 4p(x+p)$

Tangent of it is $y = M(x+p) + \frac{p}{M}$ ....(2)

Straight line (1) and (2) are perpendicular $\Rightarrow M = \frac{-1}{m}$

Now equation (2) is $y = -\frac{1}{m}(x+p) - mp$ ....(3)

From (1) and (3) $m(x+a) + \frac{a}{m} = -\frac{1}{m}(x+p) - mp$

$\Rightarrow \left(m + \frac{1}{m}\right)x + a\left(m + \frac{1}{m}\right) + p\left(m + \frac{1}{m}\right) = 0$

$\Rightarrow x + a + p = 0$

$\therefore$ points of intersection of (1) and (2) lies on it.

Example 25 : Find angle of intersection of parabola $y^2 = 2ax$ and $x^2 + y^2 = 3a^2$.

Sol. : Point of intersection of $y^2 = 2ax$ and $x^2 + y^2 = 3a^2$

$x^2 + 2ax - 3a^2 = 0 \qquad \Rightarrow (x+3a)(x-a) = 0$

$x \neq -3a, x = a \qquad \therefore y^2 = 2a^2, \quad y = \pm\sqrt{2}a$

$\therefore$ points of intersection $\left(a, \sqrt{2}a\right), \left(a, -\sqrt{2}a\right)$

Differentiating curves with respect to x.

(i) $\quad 2y\frac{dy}{dx} = 2a, \qquad \therefore m_1 = \frac{2a}{2.\left(\sqrt{2}a\right)} = \frac{1}{\sqrt{2}}$

(ii) $\quad 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}, m_2 = -\frac{a}{\sqrt{2}a} = -\frac{1}{\sqrt{2}}$

$\therefore$ Angle of intersection θ is $\tan\theta = \dfrac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}$

$$\theta = \tan^{-1} 2\sqrt{2}$$

Example 26 : Find c, if $y = mx + c$ touches $x^2 = 4ay$.

Sol. : Equation parabola $x^2 = 4ay$; of line $y = mx + c$, straight line shall touch parabola if roots of equation $x^2 - 4a(mx + c) = 0$ are equal.

i.e. $\quad 16a^2m^2 - 4.4ac = 0 \Rightarrow c = am^2$

$\therefore y = mx + am^2$ is any tangent.

Point of contact : Let it be (x_1, y_1).

$\therefore$ tangent $\quad xx_1 - 2ay - 2ay_1 = 0$

out tangent $\quad mx - y + am^2 = 0$

comparing $\frac{x_1}{m} = \frac{-2a}{-1} = \frac{-2ay_1}{am^2}$

$\Rightarrow x_1 = 2am, \ y_1 = -am^2$

$\therefore$ Point of contact $\left(2am, -am^2\right)$

Practice Worksheet (Foundation Level) – 6 (b)

1. Find equation of tangent of parabola $y^2 = 8x$ (a) whose inclination is 30^0. (b) which passes through (–2,3)

2. Find equation of tangent of parabola $y^2 = 6x$

   (a) perpendicular to $3x - y = 7$

   (b) cutting equal intercepts with different signs on axes.

3. If $x - y + 1 = 0$ is tangent to parabola $y^2 - 4x - 4y + 12 = 0$, Find point of contact.

4. Find condition so that straight line $x \cos \alpha + y \sin\alpha = p$ is tangent to parabola $y^2 = 8x$.

5. The focal distance of a point P on parabola $y^2 = 8x$ is 8. Find equation of tangent at this point.

6. Tangent to parabola $x^2 + 4y = 0$ meets y-axis at (0, 4). Find point of contact of this tangent.

7. The vertex and focus of a parabola are (4, 0) and (8, 0). Find equation of tangent at the ends of latus rectum.

8. If length of a focal chord of parabola $y^2 = 2x$ is 8 then find inclination of the chord.

9. The length of a focal chord of parabola whose vertex is (1,1) and focus (4, 4) is $48\sqrt{2}$. Find its inclination with x-axis.

10. Tangent at point P on parabola $y^2 = 3x$ meets x –axis at M(–3, 0) then find PM.

11. Find length of common chord of parabola $y^2 = 8x$ and $x^2 = y$.

12. Find common tangent of parabolas $y^2 = 4bx$ and $x^2 = 4by$.

13. Find common tangent of $y^2 = 4x$ and $x^2 + y^2 + 4x = 0$

14. Find the angle between tangents drawn from (1, 2) to the parabola $y^2 + 4x = 0$.

15. The focus and vertex of a parabola are (4,3), (2, 1). Find equation of directrix.

16. Find co-ordinates of the point, at which tangent at (–2, –2) of parabola $x^2 - 4x + 6y = 0$ meets the latus rectum.

17. Find length of common chord of parabola $y^2 = 3ax$ and $x^2 = 3ay$.

18. Find the area of the triangle formed by tangents from (–2, 2) to parabola $y^2 = 16x$ and its chord of contact.

19. Prove that straight line $lx + my + n = 0$ will be touching parabola $y^2 - 4ax$ if $ln = am^2$.

20. Find length and mid point of the chord of parabola $y^2 = 4x$ cut off by straight line $2x + y = 3$.

21. Find length of that portion of straight line 5x – 7y + 36 = 0 which is cut off by parabola $y^2 = -10x$

22. Find locus of the foot of perpendicular dropped from focus on the tangents of parabola $y^2 = 4ax$.

23. Prove that the chord of contact of two perpendicular tangents goes through focus of parabola.

24. Prove that the ordinates of point of intersection of two tangents are the arithmetic mean of the ordinates of the point of contact of those two tangents.

25. Find length of focal chord inclined at θ. Equation parabola $y^2 = 4ax$, Focal chord $y = m(x - a)$

$$\therefore [m(x-a)]^2 = 4ax$$

$$\Rightarrow m^2x^2 - x(2am^2 + 4a) + a^2m^2 = 0$$

$$\therefore x_1 + x_2 = \frac{2a(m^2+2)}{m^2}, x_1x_2 = a^2$$

$$(x_1 - x_2)^2 = \left[4a^2(m^4 + 4m^2 + 4) - 4a^2m^4\right]/m^4$$

$$= 16a^2(m^2+1)/m^4 \Rightarrow x_1 - x_2 = \frac{4}{m^2}a\sqrt{m^2+1}$$

Length of chord = $\frac{4a}{m^2}\sqrt{1+m^2}\sqrt{1+m^2} = \frac{4a}{m^2}(1+m^2)$

$$= \frac{4a(1+\tan^2\theta)}{\tan^2\theta} = \frac{4a\sec^2\theta}{\tan^2\theta} = 4a\operatorname{cosec}^2\theta$$

6.10 Normal :

Tangent at (x_1, y_1) is $yy_1 = 2a(x - x_1)$ slope of tangent is $\frac{2a}{y_1} \Rightarrow$ slope of normal $-\frac{y_1}{2a}$

$\therefore$ Equation of normals is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

(b) In terms of slope : put $-\frac{y_1}{2a} = m$ $\therefore y_1 = -2am$

and $x_1 = \frac{y_1^2}{4a} = \frac{4a^2m^2}{4a} = am^2$

$\therefore$ Normal is $y - (-2am) = m(x - am^2)$

i.e. $y = mx - 2am - am^3$ .... (α)

It meets parabola at $(am^2, -2am)$

(α) is the equation of any normal to parabola $y^2 = 4ax$

(c) Let it pass through (h, k)

$\therefore k = mh - 2am - am^3$

$\Rightarrow am^3 + m(2a - h) + k = 0$ ....(β)

This is cubic in m.

$\therefore$ Three normals can be drawn from an external point to the parabola. Suppose these meet in points whose ordinates are y_1, y_2 and y_3

from $m_1 + m_2 + m_3 = 0$

$\Rightarrow 2am_1 + 2am_2 + 2am_3 = 0$

$\Rightarrow y_1 + y_2 + y_3 = 0$

(d) For all the three normals to exist values of m_1 m_2 and m_3 should be real.

$\therefore$ In equation (β) (h – 2a)m should exist, and it should be +ve, so that –(h –2a) should be negative i.e. h > 2a.

$\therefore$ all three normals from an external point (h, k) shall exist if h > 2a.

Solved Examples

Example 27. Find the equation of normals to the parabola $x^2 - 6x + 8y - 15 = 0$, inclined at $\tan^{-1}(-2)$ with x-axis.

Sol. : Let the point on parabola $x^2 - 6x + 8y - 15 = 0$ be (x_1, y_1)

Tangent at $(x_1\, y_1)$ is $xx_1 - 3(x + x_1) + 4(y + y_1) - 15 = 0$

Slope of tangent is $-\frac{x_1 - 3}{4}$

Slope of normal = $\frac{4}{x_1 - 3}$

But normal is inclined at $\tan^{-1}(-2)$

$\therefore \frac{4}{x_1 - 3} = -2 \Rightarrow 4 = -2x_1 + 6 \Rightarrow x_1 = 1$

$\therefore$ From equation of parabola $1 - 6 + 8y_1 - 15 = 0$

$\Rightarrow y_1 = 5/2 \qquad \therefore \text{normal } y - \frac{5}{2} = -2(x - 1)$

$\Rightarrow y + 2x = \frac{9}{2}$

Example 28 : Find equation of normals drawn from (15, 12) on parabola $y^2 = 4x$.

Sol. Equation parabola $y^2 = 4x$ any normal is $y = mx - 2m - m^3$

It passes through (15, 12)

$\therefore 12 = 15m - 2m - m^3$

$\Rightarrow m^3 - 13m + 12 = 0 \Rightarrow (m - 1)(m^2 + m - 12) = 0$

$\Rightarrow (m - 1)(m + 4)(m - 3) \Rightarrow m = 1, 3, -4$

(i) $m = 1$, normals $y = x - 3$

(ii) $m = -4$ normals is $y = -4x + 72 \Rightarrow y + 4x = 72$

(iii) $m = 3$ normals is $y = 3x - 33$.

Example 29 : A normal chord subtends a right angle at the vertex of parabola $y^2 = 6x$ find its inclination.

Sol. : Equation parabola $y^2 = 6x$, $a = 3/2$ any normal is $y = mx - 3m - \frac{3}{2}m^3$ ....(1)

Vertex is at origin. Making equation of parabola homogeneous with the help of (1)

$$y^2 - 6x\left(\frac{mx - y}{3m + 3/2m^3}\right) = 0 \qquad \text{.........(2)}$$

co-efficient of $x^2 = \frac{-6m}{6m + 3m^3}$

co-efficient of $y^2 = +1$

Since angle between lines joining origin with the points of intersection of chord and parabola are 90^0

$\Rightarrow 1 - \frac{12m}{6m + 3m^3} = 0$

$\Rightarrow 3m^3 - 6m = 0 \Rightarrow m^2 = 2 \Rightarrow m = \sqrt{2}$

$\therefore \tan\theta = \sqrt{2}, \quad \theta = \tan^{-1}(\sqrt{2})$

Example 30 : Prove that distance between a tangent and parallel normal of parabola y^2 = 4ax is a $\sec^2\theta \operatorname{cosec}\theta$. Where θ is inclination of either.

Sol. : Equation of parabola $y^2 = 4ax$

any tangent of it $y = mx + \frac{a}{m}$

parallel normal is $y = mx - 2am - am^3$

The normal meets parabola at $(am^2, -2am)$

$\therefore$ Distance between tangent and parallel normal = perpendicular from $(am^2 - 2am)$ on tangent.

$= \frac{am^3 + 2am + a/m}{\sqrt{1+m^2}} = \frac{a(m^4 + 2m^2 + 1)}{m\sqrt{1+m^2}}$

$= \frac{a(1+m^2)^{3/2}}{m} = \frac{a(1 + \tan^2\theta)^{3/2}}{\tan\theta}$

$= a\sec^3\theta.\frac{\cos\theta}{\sin\theta} = a\sec^2\theta.\operatorname{cosec}\theta$

Example 31 : Prove that $y = x\sqrt{2} - 4a\sqrt{2}$ is normal of parabola $y^2 = 4ax$ and that its length is $6a\sqrt{3}$.

Sol. : Any normal of parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$

Our line $y = x\sqrt{2} - 4a\sqrt{2}$.

$\therefore m = \sqrt{2}$ and $-2am - am^3 = -2\sqrt{2}a - 2\sqrt{2}a = -4\sqrt{2}a$

which is constant in the line, $\therefore$ normal

Length putting value of y from chord in parabola equation

$$\left(x\sqrt{2}-4a\sqrt{2}\right)^2-4ax=0\Rightarrow x^2-10ax+16a^2=0$$

$$\therefore x_1+x_2=10a,\ x_1x_2=16a^2$$

$$x_1-x_2=\left(\sqrt{100-64}\right)a$$

$$\therefore \text{length} = 6a\sqrt{1+m^2}=6a\sqrt{1+2}=6a\sqrt{3}$$

Example 32 : If a normal of parabola $y^2 = 4x$ is inclined at ϕ with axis, then prove it will cut the parabola at $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$ again.

Sol. : Let normal at P (x_1, y_1), $y^2 = mx - 2am - am^3$ meets the parabola at Q (x_2, y_2) again. Clearly $y_1 = -2am$ and $m = \tan\phi$

Putting value of x from normal into equation of parabola.

$$y^2=4a\left(\frac{y+2am+am^3}{m}\right)$$

$$\Rightarrow y^2-\frac{4ay}{m}-4a\left(2a+am^2\right)=0$$

$$\therefore y_1+y_2=\frac{4a}{m}\Rightarrow y_2=\frac{4a}{m}-y_1=\frac{4a}{m}+2am$$

$$\therefore y_2=\frac{4a+2am^2}{m}$$

Slope of the tangent at Q, (x_2, y_2) = $\frac{2a}{y_2}=\frac{2am}{4a+2am}=\frac{m}{2+m^2}$

Angle between normal and this tangents is

$$\tan\theta=\frac{m-\frac{m}{2+m^2}}{1+\frac{m^2}{2+m^2}} \qquad =\frac{m+m^3}{2+2m^2}=\frac{m}{2}$$

$$=\frac{1}{2}\tan\phi \qquad \therefore \theta=\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$$

Example 33 : PNP' is a double ordinate of parabola $y^2 = 4ax$. Normal is drawn at P and a line is drawn parallel to x-axis through P. Prove that locus of point of intersection of this line and normal is $y^2 = 4a(x - 4a)$.

Sol. : Let P be (x_1, y_1) then P' is $(x_1, -y_1)$

Line through P parallel x- axis is $y = -y_1$ ....(1)

Normal at P $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

$\Rightarrow 2a(y - y_1) + (x - x_1)y_1 = 0$

If (h,k) is point of intersection, then

$\Rightarrow 2ay + y_1x = 2ay_1 + x_1y_1 = 2ay_1 + \frac{y_1^2}{4a}.y_1$ ....(2)

from (2) $2ak + y_1h = 2ay_1 + \frac{y_1^3}{4a}$ ....(3)

and from (1), $k = -y_1$ ....(4)

$\therefore$ from (3) $2ak - kh = -2ak - \frac{k^3}{4a}$

$\Rightarrow 4a = h - \frac{k^2}{4a}$

$\Rightarrow k^2 = 4a(h - 4a)$

$\therefore$ Locus $y^2 = 4a(x - 4a)$

Example 34 : Prove that $y = x - c$ is normal to $y^2 = 8x$. Find respective point on parabola and also length of normal.

Sol. : Equation parabola $y^2 = 8x$ $\therefore a = 2$

Any normal is $y = mx - 2.2.m - 2m^3$ ....(1)

Slope of given line $y = x - 6$ is 1.

$\therefore$ Equation (1) reduces to $y = x - 4 - 2 = x - 6$

Hence given straight line is normal of parabola, point is $(am^2 - 2am)$; i.e. (2, –4)

(ii) Length, putting value of y from straight line into the curve. $(x - 6)^2 = 8x$

$\Rightarrow x^2 - 20x + 36 = 0 \Rightarrow (x - 18)(x - 2) = 0$

$\therefore$ Length = $(x_1 \sim x_2)\sqrt{1 + m^2} = 16\sqrt{1+1} = 16\sqrt{2}$

Example 35 : Find equation of tangent and normal at the upper end of latus rectum of parabola $y^2 = 4a(x - a)$

Sol. : Equation parabola $y^2 = 4a(x - a)$

$\therefore$ Vertex, (a, 0), focus (2a, 0)

$\therefore$ upper end of latus rectum is (2a, 2a)

Tangent, y.2a = 2a(x + 2a) $-4a^2$

$\Rightarrow y = x$

slope of normal is –1, equation; y – 2a = – (x – 2a)

$y + x = 4a$

Example 36 : If y = (x – 11) cos θ –cos3θ is normal to $y^2 = 16x$, then θ is

(a) $\pi/2$ (b) $\pi/3$ (c) can have any value (d) had a definite value

Sol. : Any normal to $y^2 = 16x$ is $y = mx - 8m - 4m^3$ ....(1)

Given normal $y = -11\cos\theta + x\cos\theta - \cos 3\theta$

$\therefore$ slope = cosθ $\therefore$; m = cosθ;

and –11 cos θ – cos 3θ = –11 cos θ + 3 cos θ –4 $\cos^3\theta$ = –8cos θ – 4 $\cos^3\theta$

$\therefore$ comparing with (1) m = cos θ

$\therefore$ y = x cos θ – 11 cos θ – cos 3 θ is normal

whatever be, the value of θ.

Example 37: Find equation of normal of parabola $(x - 2)^2 = 4y$ which meets axes in (2, 4)

Sol. Equation parabola $(x - 2)^2 = 4y$

$\therefore$ a = 1, axis is x – 2 = 0

If (x_1, y_1) is a point on parabola then tangent is $xx_1 - 2(x - x_1) - 2(y + y_1) + 4 = 0$

Its slope is $\frac{x_1 - 2}{2}$ $\therefore$ Slope of Normal $-\frac{2}{x_1 - 2}$

Equation of normal is $y - y_1 = -\frac{2}{x_1 - 2}(x - x_1)$

$\Rightarrow (y - y_1)(x_1 - 2) + 2(x - x_1) = 0$

It goes through (2, 4) $\Rightarrow (4 - y_1)(x_1 - 2) + 2(2 - x_1) = 0$

$\Rightarrow x_1y_1 - 2y_1 - 2x_1 + 4 = 0 \Rightarrow (x_1 - 2)(y_1 - 2) = 0$

$\therefore x_1 = 2, y_1 = 2$

(i) $x_1 = 2$ gives x = 2 normal

(ii) $y_1 = 2$, point on parabola is $x = 2 \pm 2\sqrt{2}$ and $x = 2 - 2\sqrt{2}$ and $2 + 2\sqrt{2}$ are symmetric points; normals are

$$y - 2 = -\frac{2}{(2 \pm 2\sqrt{2}) - 2}(x - 2 - 2\sqrt{2})$$

$$\Rightarrow y - 2 = \frac{-2}{\pm 2\sqrt{2}}(x - 2 + 2\sqrt{2})$$

$\sqrt{2}y + x = 2 + 4\sqrt{2}$ and $\sqrt{2}y - x = 4\sqrt{2} - 2$

Example 38: Find locus of point of intersection of normals of parabola $y^2 = 4ax$ that include angle 90^o.

Sol. Any normal of parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$ ... (i)

Normal ⊥ to it is $y = -\frac{x}{m} + \frac{2a}{m} + \frac{a}{m^3}$ ... (ii)

$$\therefore mx - 2am - am^3 = -\frac{x}{m} + \frac{2a}{m} + \frac{a}{m^3}$$

$$\Rightarrow x\left(m + \frac{1}{m}\right) = 2a\left(m + \frac{1}{m}\right) + a\left(m^3 + \frac{1}{m^3}\right)$$

$$\Rightarrow x = 2a + a\left(m^2 - 1 + \frac{1}{m^2}\right) \quad \text{... (iii)}$$

From (i) $y = 2am + a\left(m^3 - m + \frac{1}{m}\right) - 2am - am^3$

$$= a\left(\frac{1}{m} - m\right)$$

$$\therefore \quad y^2 = a^2\left(\frac{1}{m^2} + m^2 - 2\right) = a^2\left(\frac{1}{m} + m^2 - 1\right) - a^2$$

$= a(x - 2a) - a^2 = a(x - 3a)$ from (iii)

$\therefore$ Locus is $y^2 = a(x - 3a)$

Example 39: Find c, if $y = mx + c$ is tangent to $x^2 = 4ay$.

Sol. Equation of parabola $x^2 = 4ay$

Straight line is $y = mx + c$

Substituting value of y from line in parabola equation $x^2 - 4a(mx + c) = 0$

For line to be tangent, roots of this equation should be equal i.e. $16a^2m^2 - 4 \cdot 4ac = 0$

$\Rightarrow m^2 = \dfrac{c}{a}$

If $c = am^2$ then given line is tangent.

Point of contact: Let it be (x_1, y_1)

Tangent $xx_1 - 2ay - 2ay_1 = 0$ ... (i)

our tangent $y - mx - am^2 = 0$ ... (ii)

Comparing $\dfrac{x_1}{-m} = -2a = \dfrac{2y_1}{m^2}$

$\therefore$ $x_1 = 2am$, $y_1 = -am^2$, point of contact $[2am - am^2]$

Example 40: Find equation of normal of parabola $x^2 = 4ay$ in terms of slope.

Sol. Tangent at (x_1, y_1) of $x^2 = 4ay$ is $xx_1 - 2ay - 2ay_1$

Slope of tangent $\dfrac{x_1}{2a}$ $\Rightarrow$ slope of normal $-\dfrac{2a}{x_1}$

Equation of normal $y - y_1 = -\dfrac{2a}{x_1}(x - x_1)$

Now $m = -\dfrac{2a}{x_1} \Rightarrow x_1 = -\dfrac{2a}{m} \Rightarrow y_1 = \dfrac{x_1^2}{4a} = \dfrac{a}{m^2}$

$\therefore$ Normal is $y - \dfrac{a}{m^2} = m\left(x + \dfrac{2a}{m}\right)$

$\Rightarrow$ $y = mx + 2a + \dfrac{a}{m^2}$ point $\left[-\dfrac{2a}{m}, \dfrac{a}{m^2}\right]$

Example 41: Two straight lines are perpendicular to each other, one touches parabola $y^2 = 4a(x + a)$ and the other touches $y^2 = 4p(x + p)$. Prove that point of intersection of these lines lies on $x + a + p = 0$

Sol. Tangent to parabola $y^2 = 4a(x + a)$ is

$y = m(x + a) + \dfrac{a}{m}$ ... (1)

Tangent of $y^2 = 4p(x + p)$ is $y = m(x + p) + \dfrac{p}{m}$ ... (2)

Straight lines (i) and (ii) are perpendicular to each other $\therefore$ $M = -\dfrac{1}{m}$

$\therefore$ (2) becomes $y = -\frac{1}{m}(x + p) - mp$ ... (2')

From (1) and (2') m (x + a) + $\frac{a}{m}$ = $-\frac{1}{m}$(x + p) – mp

$\Rightarrow x\left(m + \frac{1}{m}\right) + a\left(m + \frac{1}{m}\right) + p\left(\frac{1}{m} + m\right) = 0$

$\Rightarrow$ x + a + p = 0

Practice Worksheet (Foundation Level) – 6(c)

1. Find equation of normal at the ends of latus rectum of parabola $y^2 = 8x$.
2. The abscissa of a point on parabola $4y^2 + x = 0$, is -4. Find equation of normal at this point.
3. Find equation of normal of parabola $(x – 4)^2 = 8y$ which passes through focus.
4. Find equation of normal of parabola $(y – 2)^2 = 4x$ which goes through focus.
5. Find equation of normal of parabola $x^2 = 4py$ in terms of its slope.
6. Find point on parabola $x^2 = 4ay$, at which $2y – x = 12a$ is normal.
7. The normal of parabola $(y – 2)^2 = 4(x + 1)$ is inclined at 45^o with axis. Find its equation.
8. Show that $y = mx + 2a + \frac{a}{m^2}$ is normal to parabola $x^2 = 4ay$. Find its point on parabola.
9. Find length of normal chord of parabola $y^2 = 4ax$, inclined at 30^o with axis.
10. Find the point of intersection of normal drawn at (2, 2) of parabola. $y^2 – 8y + 4x + 4 = 0$ and the directrix.
11. The straight line $2x + 4y = 9$ is normal at point (p, q) of parabola $y^2 = 8x$, find p and q.
12. Find equation of normal at (4, 6) of parabola $y^2 = 9x$. At what point does it meet the tangent at (1, 3).
13. Find condition so that $ax + by = 1$ be normal to $y^2 = 6x$.
14. Normals are drawn from point (3p, 0) to parabola $y^2 = 4px$. Find angle between any two of them.
15. Prove that $y = x\sqrt{3} - 5a\sqrt{3}$ is normal of parabola $y^2 = 4ax$ and that its length is $\frac{32}{3}a$.
16. Find condition so that $\ell x + my + n = 0$ be normal to parabola $y^2 = 4ax$.
17. The normal at point P on parabola meets the axis at G. Ordinate of P is PN. Prove that GN is equal to semi latus rectum.
18. Show that for every value of θ, $y = (x – 11) \cos \theta – \cos 3\theta$ will be normal of parabola $y^2 = 16x$. Find the point on parabola and write equation of tangent on it.
19. PNP' is double ordinate of parabola $y^2 = 4ax$. Show that the locus of point of intersection of normal at P and the straight line drawn parallel to axis through P' is $y^2 = 4a (x – 4a)$.
20. Two normals of parabola $y^2 = 4ax$ are inclined at axis at $\tan^{-1} 1$ and $\tan^{-1} 2$. If these meet at point P, then prove the third normal from P is inclined at acute angle $\tan^{-1} 3$.

Pole and Polar

6.11 Polar

If from a point P (outside, on or inside parabola) chords of parabola are drawn then the locus of the point of intersection of tangents drawn at the ends of these chords is called polar of P and P is called pole of the polar.

In figure P is point outside parabola. PQR is one chord. Tangents at Q and R meet at T_1. Locus of T_1 is the polar of P. Let T_1 be (h, k) then QR is chord of contact of T_1

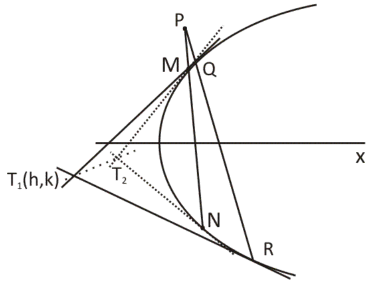

Fig 9

$yk = 2a(x + h)$ ... (1)

QR goes through P · (x_1, y_1)

$\therefore y_1 k = 2a (x_1 + h)$ ... (2)

From (1), (2) it is clear that point (h, k) satisfies $yy_1 = 2a (x + x_1)$

For every value of h and k.

$\therefore$ Locus of T_1 i.e. polar is $yy_1 = 2a(x + x_1)$

Note: When point is outside parabola, then polar of P is chord of contact of P. When P is on parabola, polar of P is tangent at P and when P is inside then polar is only polar and it is outside the parabola.

6.12 Other Properties of Polar

(1) If polar of P with respect to a parabola passes through Q, then polar of Q shall pass through P.

Let $y^2 = 4ax$ be the parabola and let P be (x_1, y_1) and Q (x_2, y_2)

Polar of P is $yy_1 = 2a (x + x_1)$ ... (i)

It passes through Q $\therefore y_2y_1 = 2a (x_2 + x_1)$ ... (ii)

from (ii) it is clear that (x_1, y_1) lies on $yy_2 = 2a(x + x_2)$ i.e. on the polar of Q.

(2) The point of intersection of polars of P and Q is the pole of straight line PQ.

Polars of P (x_1, y_1), Q (x_2, y_2) with respect to parabola $y^2 = 4ax$ are

$yy_1 = 2a (x + x_1)$ and $yy_2 = 2a (x + x_2)$ ... (1)

If these intersect at point (h, k) then

$ky_1 = 2a (h + x_1)$ and $ky_2 = 2a(h + x_2)$

From relation (2) it is clear that (x_1, y_1), (x_2, y_2) lie on $ky = 2a (x + h)$ and it is the equation of polar of their point of intersection.

(3) The chord of parabola is parallel to the polar of its mid point.

Let y = mx + c be chord of parabola $y^2 = 4ax$; which meets parabola in (x_1, y_1) and (x_2, y_2)

If (h, k) is its mid point, then. $x_1 + x_2 = 2h$, $y_1 + y_2 = 2k$.

Putting value of x in terms of y from straight line into equation of parabola.

$$y^2 - 4a\left(\frac{y-c}{m}\right) = 0$$

$$\Rightarrow my^2 - 4ay + 4ac = 0$$

$$\therefore\ y_1 + y_2 = \frac{4a}{m} \text{ and this } = 2k.$$

$$\therefore\ m = \frac{2a}{k}$$

Polar of (h, k) is yk = 2a(x + h)

Slope of polar is $\frac{2a}{k}$, which is the slope of chord. Thus chord is parallel to the polar of its mid point.

(4) The equation of chord of parabola $y^2 = 4ax$ whose mid point is (h, k) is

$yk - 2a(x + h) = k^2 - 4ah$

Suppose the chord is inclined at angle θ with the axis (x-axis); (h, k) is its mid point.

Equation of chord is $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$

$\therefore\ x = r\cos\theta + h, \qquad y = r\sin\theta + k$

point (x, y) is on parabola $y^2 = 4ax$

$\therefore\ (r\sin\theta + k)^2 = 4a(r\cos\theta + h)$

$\Rightarrow r^2\sin^2\theta + 2r(k\sin\theta - 2a\cos\theta) + k^2 - 4ah = 0$

The two values of r are equal in magnitude but opposite in signs.

$\therefore\ r_1 + r_2 = 0$ in this equation.

$$\therefore\ k\sin\theta - 2a\cos\theta = 0 \Rightarrow \tan\theta - \frac{2a}{k}$$

$$\therefore \text{ Equation of chord is } y - k = \frac{2a}{k}(x-h)$$

or $yk - 2ax - 2ah = k^2 - 4ah$.

$\therefore\ yk - 2a(x + h) = k^2 - 4ah$

Solved Examples

Example 35: Find pole of straight line $\ell x + my + n = 0$ with respect to $y^2 = 4ax$.

Sol. Let (x_1, y_1) be the pole.

Polar is $yy_1 - 2ax - 2ax_1 = 0$

$my + \ell x + n = 0$

$\therefore \frac{y_1}{m} = \frac{-2a}{\ell} = -\frac{2ax_1}{n}$

$\therefore x_1 = \frac{n}{\ell}, y_1 = \frac{-2am}{\ell}$

$\therefore$ Pole is $\left(\frac{n}{\ell}, -\frac{2am}{\ell}\right)$

Example 36: Find the pole of the latus rectum of parabola $y^2 = 4a(x - a)$

Sol. Equation parabola $y^2 = 4a(x - a)$ ... (1)

Focus $(2a, a)$, Equation of latus rectum $x - 2a = 0$ ... (2)

Suppose (h, k) is its pole, then

Polar is $yk - 2ax - 2ah + 4a^2 = 0$ ... (3)

Comparing (2) and (3)

$\frac{0}{k} = \frac{1}{-2a} = \frac{2a}{2ah - 4a^2} = \frac{1}{h - 2a}$

$\therefore$ pole $(0, 0)$

Example 37: Prove that the locus of poles of tangents of parabola $y^2 = 4ax$ with respect to parabola $y^2 = 4bx$ is $y^2 = \frac{4b^2}{a}x$.

Sol. Any tangent of parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ ... (1)

Let its pole with respect to $y^2 = 4bx$ be (x_1, y_1)

Then polar is $2bx - yy_1 + 2bx_1 = 0$ ... (2)

Comparing Co-efficients of (1) and (2)

$$\frac{2b}{m} = \frac{y_1}{1} = \frac{2bmx_1}{a}$$

$\therefore x_1 = \frac{a}{m^2}, y_1 = \frac{2b}{m} \text{ or } m = \frac{2b}{y_1}$

$\therefore\ x_1 = \dfrac{ay_1^2}{4b^2}$

$\therefore$ Locus $y^2 = \dfrac{4b^2}{a}x$

Example 38: Find the equation of that chord of parabola $y^2 = 12\,x$ whose mid point is (2, –3)

Sol. Polar of point (2, –3) w.r.t. to $y^2 = 12x$ is $-3y = 6x + 12$

Slope of this polar is –2. This will be the slope of the chord.

$\therefore$ Equation of chord is $y + 3 = -2(x - 2)$

$\Rightarrow 2x + y = 1$

6.13 Diameter

The locus of mid point of parallel chords of a parabola is called diameter.

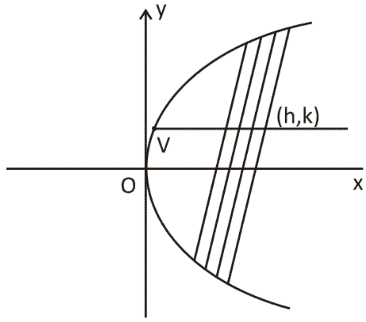

Fig 10

Parabola $y^2 = 4ax$ ... (1)

and parallel chords are $y = mx + \lambda$ ... (2)

Let mid point be (h, k)

from (1) and (2)

$$y^2 = 4a\left(\frac{y-\lambda}{m}\right) = 0 \quad \text{... (3)}$$

This gives two value of y, namely y_1, y_2 [chord meets parabola in (x_1, y_1), (x_2, y_2)]

From (3) $\quad y_1 + y_2 = +\frac{4a}{m} = 2k$

$\therefore\ K = \frac{2a}{m}$, which gives equation of diameter as; $y = \frac{2a}{m}$; and it is a straight line parallel to axis.

Corollary: The tangent drawn at the end of diameter is parallel to the chords, which are bisected by the diameter.

In the figure the diameter of parallel chords $y = mx + \lambda$ meets the parabola in v. The slope of a tangent at (x_1, y_1) on parabola.

$y^2 = 4ax$ is $\quad \frac{2a}{y_1} = \frac{2a}{\text{ordinate of point}}$

Equation of diameter is $y = \frac{2a}{m}$

$\therefore$ Slope of tangents at v is $\frac{2a}{\text{ordinate of point v}} = \frac{2a}{2a/m} = m$ slope of parallel chords.

$\therefore$ The tangent drawn at the end of diameter is parallel to the chords bisected by diameter.

Solved Examples

Example 39: Prove that the tangents drawn at the ends of a chord of parabola meet at the diameter which bisects those chords.

Sol. Equation of parabola $y^2 = 4ax$ ... (1)

Let ends of chords be (x_1, y_1) and (x_2, y_2)

Equation of tangents at these points, $yy_1 = 2a(x + x_1)$ ... (2)

$yy_2 = 2a(x + x_2)$ ... (3)

The ordinate of point of intersection of (2) and (3) is $y = \frac{2a(x_1 - x_2)}{y_1 - y_2}$ ... (4)

Slope of this chord is $\frac{y_1 - y_2}{x_1 - x_2}$ and therefore equation of diameter,

$$y = \frac{2a}{\text{slope}} = \frac{2a(x_1 - x_2)}{y_1 - y_2} \quad \text{... (5)}$$

Clearly, this diameter passes through the point of intersection of tangents.

Practice Worksheet (Foundation Level) – 6 (d)

1. Find the equation of polar of point (4, 3) with respect to parabola $x^2 - 4x - 3y - 8 = 0$.
2. Find point of intersection of the polars of points (–2, –3) and (3, 5) with respect to parabola $y^2 = 8x$.
3. Find pole of straight line $x + 5y - 5 = 0$ with respect to parabola $x^2 = 10y$.
4. Find the equation of the chord of parabola $y = x^2 - 8x + 4y + 3 = 0$, whose mid point is (2, –1). Find the length of this chord.
5. Prove that the locus of poles of focal chords of parabola $y^2 = 4ax$ is directrix of parabola.
6. Prove that the locus of the poles of chords of parabola $y^2 = 4ax$, which are at a distance of b units from vertex is $y^2 + 4a^2\left(1 - \frac{x^2}{b^2}\right) = 0$.
7. Find the locus of poles of tangents of parabola $y^2 = 4ax$ with respect to circle $x^2 + y^2 - 2ax = 0$.
8. From point P, perpendicular, on its polar with respect to $y^2 = 4ax$, touches the parabola $x^2 = 4by$. Prove locus of P is $2ax + by + 4a^2 = 0$
9. Find the locus of the poles of normals chords of parabola $y^2 = 4ax$.
10. Find pole of $ax + by + c = 0$ with respect to parabola $y^2 = 8(x + 2)$.
11. Find the locus of mid points of chords of parabola $(y - 3)^2 = 2(x - 1)$ which are perpendicular to $3x - 4y = 5$.
12. Find pole of straight line $3x + 4y = 12$ with respect to parabola $(y - 4)^2 = 8(x - 1)$.

6.14 Parametric Co-ordinates:

We have seen that $y = mx + \frac{a}{m}$ is tangent to parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ and $y = mx - 2am - am^3$ is normal to this parabola at $(am^2, -2am)$.

The co-ordinates of the point have been expressed in one variable, but not in uniform one variable. Therefore $(at^2, 2at)$ are parametric co-ordinates of any point 't' on parabola $y^2 = 4ax$.

(a) Equation of tangent at 't' is $2at \cdot y = 2a(x + at^2)$

$$\Rightarrow ty = x + at^2$$

(b) Normal at t is $\frac{1}{t}(y - 2at) + (x - at^2) = 0$

$$\Rightarrow y + tx = 2at + at^3$$

(c) Point of intersection of tangent at t_1 and t_2 is

$$\left.\begin{aligned} yt_1 &= x + at_1^2 \\ yt_2 &= x + at_2^2 \end{aligned}\right\} (at_1t_2, a(t_1 + t_2))$$

(d) Equation of line joining points t_1 and t_2 is

$$y - 2at_1 = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)}(x - at_1^2)$$

$$= \frac{2}{t_2 + t_1}(x - at_1^2)$$

$$\Rightarrow y(t_2 + t_1) = 2(x + at_1t_2)$$

6.15 Sub-tangent and Sub-normal:

In the figure 11, P is a point on the parabola. Tangent at P, PT meets axis at T and Normal at P, meets axis in G. PN is ordinate of P. Then

TN = Sub-tangent

NG = Sub-normal

(a) Let P be (x_1, y_1), tangent at P is $yy_1 = 2ax + 2ax_1$

Putting $y = 0$, $x = -x_1$ $\Rightarrow AT = -x_1$

and $|AT| = x_1$; Sub-tangent TN = $|TA|$ + AN = $x_1 + x_1 = 2x_1$

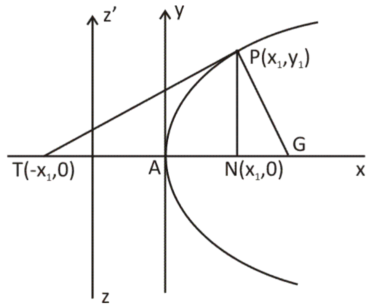

Fig 11

(b) Normal at P $\frac{2a}{y_1}(y - y_1) + (x - x_1) = 0$

Putting y = 0 $\quad -2a + x - x_1 = 0$

$\therefore$ G is $(2a + x_1, 0)$

Sub-normal = NG = AG – AN = $2a + x_1 - x_1$

= 2a (constant)

$\therefore$ Sub-normal of every point on parabola is 2a = semi-Latus-rectum = constant.

6.16 Properties of Parabola:

1. The tangent and normal at any point of parabola bisect the angle between focal distance and perpendicular from point on directrix internally and externally respectively.

In the figure, P is point (x_1, y_1) on parabola PF is focal distance, PM = perpendicular from P on directrix.

Tangent at P, PT meets axis in T; T is $(-x_1, 0)$

By definition of parabola $PM = PF = a + x_1$ and $TF = |TA| + AF = x_1 + a$.

In triangle PTF, PF = TF $\Rightarrow \angle PTF = \angle TPF = \theta$

And $\angle MPT$ = alternate $\angle PTF = \theta$

$\therefore \angle MPT = \angle TPF \Rightarrow$ PT bisect $\angle MPF$

2. The segment of every tangent between directrix and point of contact subtends a right angle at focus (Proved in example 20).

3. If point t1, $(at_1^2 + 2at)$ is one end of a focal chord then $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$ i.e. $-\frac{1}{t_1}$ is its other end. Let two ends of a focal chord PQ be t1 and t2

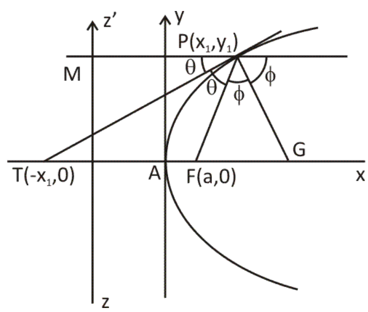

Fig 12

$\therefore$ slope of PS = $\frac{2at_1}{a(t_1^2-1)}$, of QS = $\frac{2at_2}{a(t_2^2-1)}$

PSQ is one straight line $\Rightarrow \frac{2at_1}{a(t_1^2-1)} = \frac{2at_2}{a(t_2^2-1)}$

$\Rightarrow t_1(t_2^2-1) - t_2(t_1^2-1) = 0$

$\Rightarrow t_1t_2^2 - t_1 - t_2t_1^2 - t_2 = 0$

$\Rightarrow t_1t_2(t_2 - t_1) + (t_2 - t_1) = 0$

$\Rightarrow (t_1t_2 + 1)(t_2 - t_1) = 0$

$\because t_1 \neq t_2 \qquad \therefore t_1t_2 = -1,\ t_2 = -\dfrac{1}{t_1}$

$\therefore$ ends of focal chord are t_1 and $-\dfrac{1}{t_1}$

4. Tangents at the ends of focal chord meet at right angles on the directrix.

If ends of focal chord are t_1 and t_2, then point of intersection of tangents at these points is $[at_1t_2, a(t_1+t_2)]$.

And we know $t_1t_2 = -1$ $\qquad \therefore x = at_1t_2 = -a$

and x=-a is a directrix.

Slope of the tangents are $\dfrac{1}{t_1}$ and $\dfrac{1}{t_2}$ and $\dfrac{1}{t_1}.\dfrac{1}{t_2} = \dfrac{1}{t_1t_2}$ = -1

$\therefore$ tangents meet at right angles on directrix.

5. The foot of perpendicular from focus on any tangents falls on the tangents drawn at vertex of parabola.

Any tangent to parabola $y^2 = 4ax$ is $yt = x + at^2$ ... (1)

Equation of perpendicular from focus (a, 0) on it is

$y - 0 = -t(x - a) \Rightarrow yt = -t^2x + at^2$ ... (2)

From (1) and(2) $x + at^2 = -t^2x + at^2 \Rightarrow x = 0$ and x = 0 is tangent at vertex of parabola.

6. Tangents drawn from an external point to the parabola $y^2 = 4ax$ subtends equal angles at focus.

Let P be the external point of the parabola $y^2 = 4ax$ and PM and PN be tangents from it to parabola. Let M be t_1 and N, t_2 $\qquad \therefore P = [at_1t_2, a(t_1 + t_2)]$

Equation of MS is $y = \dfrac{2at_1}{a(t_1^2-1)} \cdot (x-a) \qquad \Rightarrow 2t_1x - (t_1^2 - 1)y - 2at_1 = 0$

Length of perpendicular p_1 from P on it.

$$p_1 = \frac{2t_1at_1t_2 - (t_1^2-1)a(t_1+t_2) - 2at_1}{\sqrt{4t_1^2 + (t_1^2-1)^2}}$$

$$= \frac{a(t_1^2t_2 - t_1^3) + a(t_2 - t_1)}{t_1^2+1} = a(t_2 - t_1)$$

Similarly length of perpendicular from P on NS, $p_2 = a(t_1 - t_2)$

$\Rightarrow |p_1|^2 = |p_2| \Rightarrow \angle MSP = \angle NSP$.

Solved Example

Example 40: One end of focal chord of parabola $y^2 = 4ax$ is $(at^2, 2at)$. Find its length.

Sol. The other end of focal chord of $y^2 = 4ax$ is $\left(\frac{a}{t^2} - \frac{-2a}{t}\right)$

$$\therefore \text{ Length} = \sqrt{\left(at^2 - \frac{a}{t^2}\right)^2 + \left(2at + \frac{2a}{t}\right)^2}$$

$$= a\sqrt{\left(\frac{t^4 - 1}{t^2}\right)^2 + 4\left(\frac{t^2+1}{t}\right)^2} = a\left(\frac{t^2+1}{t}\right)\sqrt{\frac{(t^2-1)^2}{t^2} + 4}$$

$$= a\left(\frac{t^2+1}{t}\right)\sqrt{\frac{(t^2+1)^2}{t^2}} = a\left(\frac{t^2+1}{t}\right)^2 = a\left(t + \frac{1}{t}\right)^2$$

Example 41: Circle is drawn on the focal distance of point 't' of parabola $y^2 = 4ax$ as diameter. Find the length of segment cut on normal at t by this circle.

Sol. In the figure 13 the line segment cut off by the circle on PS as diameter is PK. If θ is angle between normal at P and PS then $PK = PS \cos\theta$

Normal is $y + tx = 2at + at^2$; slope $= -t$

and slope PS = $\frac{2t}{t^2 - 1}$ $\quad \therefore \tan\theta = \frac{\left(-t - \frac{2t}{t^2-1}\right)}{\left(1 - \frac{2t^2}{t^2-1}\right)}$

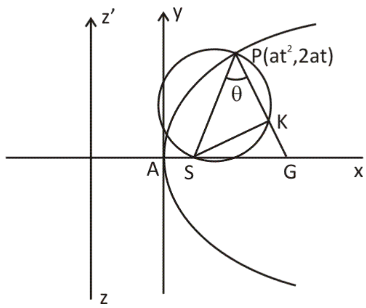

Fig 13

$$\therefore \cos\theta = \frac{1}{\sqrt{1+t^2}}$$

$$\therefore PS\cos\theta = (at^2 + a) \cdot \frac{1}{\sqrt{1+t^2}} = a\sqrt{1+t^2}$$

Example 42: Prove that the locus of mid point of the portion of normal of $y^2 = 4ax$ intercepted between curve and x-axis is another parabola. Find its vertex and length of Latus rectum.

Sol. Equation of normal of parabola $y^2 = 4ax$ at P $(at^2, 2at)$ is $y + xt = 2at + at^3$

It meets $y = 0$ in G, $(2a + at^2, 0)$

Mid point of PG is $(h, k) \Rightarrow h = a + at^2$, and $k = at$

from $k = at$, $h = a + a\cdot\left(\frac{k}{a}\right)^2 \Rightarrow k^2 = ah - a^2$

$\therefore$ Locus of mid point is $y^2 = a(x - a)$

It is parabola, vertex (a, 0) Latus rectum = a.

Example 43: Prove that the harmonic means of segments of focal chord cut off by x-axis is equal to semi-latus rectum.

Sol. Let ends of focal chord PQ be P(t) and $Q\left(\frac{-1}{t}\right)$. Focus S is (a, 0)

$\therefore$ PS = $\sqrt{(at^2 - a)^2 + (2at)^2} = a(1 + t^2)$

QS = $\sqrt{(\frac{a}{t^2} - a)^2 + (\frac{-2a}{t})^2} = \frac{a(1+t^2)}{t^2}$

HM = $\frac{2a^2(1+t^2)^2}{at^2\left[(1+t^2) + \frac{(1+t^2)}{t^2}\right]} = \frac{2a^2(1+t^2)^2}{at^2(1+t^2)\left(1+\frac{1}{t^2}\right)} = 2a$ = semi latus rectum

Example 44: If normal at P (t_1) meets the parabola again at Q (t_2) then prove $t_2 = -t_1 - \frac{2}{t_1}$

Sol. Normal at t_1 of parabola $y^2 = 4ax$ is $y + t_1x = 2at_1 + at_1^3$

It goes through Q. $\therefore$ $2at_2 + t_1at_2^2 = 2at_1 + at_1^3$

$\Rightarrow 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow a(t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$ $\quad \because t_1 \neq t_2 \Rightarrow [2 + t_1(t_2 + t_1)] = 0$

$t_1 + t_2 = -\frac{2}{t_1}$ $\quad \Rightarrow t_2 = -t_1 - \frac{2}{t_1}$

Example 45: Normals of parabola $y^2 = 4x$ drawn from an external point meets axis of parabola in points whose co-ordinate are in A.P. Show that locus of point is $27y^2 = 2(x - 2)^3$

(Roorkee 1995)

Sol. $y = mx - 2m - m^3$ is normal of $y^2 = 4x$. It goes through (h, k)

$\therefore k = mh - 2m - m^3$

$\Rightarrow m^3 + m(2 - h) + k = 0$ ... (1)

If gives three values of m namely m_1, m_2, m_3 from it

$m_1 + m_2 + m_3 = 0 \Rightarrow m_1 + m_3 = -m_2$ ... (2)

and $m_2(m_1 + m_3) + m_1m_3 = 2 - h$ ... (3)

$m_1 m_2 m_3 = -k \Rightarrow m_1 m_3 = -\frac{k}{m_2}$ ... (4)

from (2), (3), (4) $-m_2^2 = (2-h) + \frac{k}{m_2}$ ... (α)

Normal meets axis in points whose abscissa are $2 + m_1^2$, $2 + m_2^2$, $2 + m_3^2$ and ordinates (O)

Co-ordinates are in A. P. $\Rightarrow 4 + 2m_2^2 = 4 + m_1^2 + m_3^2$

$\Rightarrow 2m_2^2 = (m_1 + m_3)^2 - 2m_1 m_3 = m_2^2 + \frac{2k}{m_2} \Rightarrow m_2^3 = 2k$ ... (β)

From (α) $h - 2 = m_2^2 + \frac{k}{m_2}$

$\Rightarrow m_2(h-2) = m_2^3 + k^2$

$\therefore$ Cube of it gives.

$m_2^3(h-2)^3 = m_2^9 + 3m_2^6 k + 3m_2^3 k^2 + k^3$

Putting value of m_2^3 from (β)

$2k(h-2)^3 = 8k^3 + 12k^3 + 6k^3 + k^3 \Rightarrow 2k(h-2)^3 = 27k^3$

$\therefore 27y^2 = 2(x-2)^3$.

Example 46: Prove that the length of focal chord is inversely proportional to the square of its distance from vertex.

Sol. PQ is the focal chord. PQ = $4a\operatorname{cosec}^2\theta$ where θ is its inclination with x-axis.

In figure AN is perpendicular from vertex on PQ.

AN = AS $\sin\theta \Rightarrow \sin\theta = \frac{AN}{AS} = \frac{AN}{a}$

$\therefore$ PQ = $4a\operatorname{cosec}^2\theta$ =

$\frac{4a}{\sin^2\theta} = \frac{4a\cdot a^2}{(AN)^2} \propto \frac{1}{(AN)^2}$ ($4a^3$ is constant)

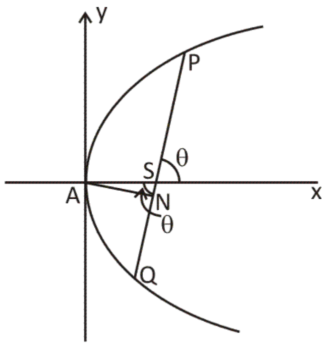

Fig 14

Example 47: Normals at P and Q of parabola $y^2 = 4ax$ meet at R on the parabola. Show that the product of ordinates of P and Q is $8a^2$

Sol. Let P, Q and R_1 be t_1, t_2 and t_3. Normal at P meets at R on parabola $\Rightarrow t_3 = -t_1 - \frac{2}{t_1}$

Normal at Q meets at R on parabola $\Rightarrow t_3 = -t_2 - \frac{2}{t_2}$

$$\therefore -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$$\Rightarrow\ t_1 - t_2 + \frac{2}{t_1 t_2}(t_2 - t_1) = (t_1 - t_2)\left(1 - \frac{2}{t_1 t_2}\right) = 0$$

$t_1 \neq t_2 \quad \therefore\ 1 + \frac{2}{t_1 t_2} = 0 \Rightarrow t_1\, t_2 = 2$

Product of ordinates of P and Q = $4a^2 t_1\, t_2 = 8a^2$

Example 48: PQ is chord of contact of point T. With respect to parabola $y^2 = 4ax$ and PQ is also normal at P. Show that directrix bisects TP.

Sol. T is pole of PQ with respect to $y^2 = 4ax$

Let P be $(am^2, -2am)$. Normal at P is

$y = mx - 2am - am^3$ ... (i)

If T is (h, k); its polar is

$ky = 2ax + 2ah$ ... (ii)

Comparing (i) and (ii) $K = \frac{2a}{m} = \frac{2ha}{-2am - am^3}$

$\therefore\ h = -2a - am^2$, and $k = \frac{2a}{m}$

Mid point of PT, $\left(\frac{am^2 - 2a - am^2}{2}, -2am + \frac{2a}{m}\right)$

If it is $\bar{x}, \bar{y}$ then $\bar{x} = -a$ i.e. it is on directrix.

Example 49: If any chord of parabola $y^2 = 4ax$ subtends a right angle at the vertex, prove tangents at its ends meet on $x + 4a = 0$

Sol. Let PQ be the chord and $P(t_1)$ and $Q(t_2)$; Vertex A is (0, 0)

Slope of AP = $\frac{2at_1}{at_1^2} = \frac{2}{t_1}$

Slope of AQ = $\frac{2}{t_2}$

Point of intersection of tangents at P and Q is $[at_1t_2, a(t_1 + t_2)]$

$\angle PAQ = 90^o \quad \Rightarrow \frac{4}{t_1\, t_2} = -1 \quad \Rightarrow t_1\, t_2 = -4$

$\therefore$ If $h = a\, t_1\, t_2 \Rightarrow h = -4a$

$\therefore$ Locus of point of intersection is $x + 4a = 0$

Example 50: Two straight line are perpendicular to each other. One line touches the parabola $y^2 = 4a(x + a)$ and the other touches the parabola $y^2 = 4p\,(x + p)$. Prove that the locus of point of intersection of these lines is $x + a + p = 0$.

Sol. Any tangent to $y^2 = 4a(x + a)$ is

$$y = m(x + a) + \frac{a}{m} \quad \text{... (1)}$$

Any tangent to parabola $y^2 = 4p\,(x + p)$ and perpendicular to (1) is

$$y = -\frac{1}{m}(x + p) - pm \quad \text{... (2)}$$

From (1) and (2), $O = x\left(m + \frac{1}{m}\right) + p\left(\frac{1}{m} + m\right) + a\left(\frac{1}{m} + m\right)$

$\Rightarrow x + a + p = 0$

This is the locus of point of intersection.

Example 51: The ordinates of three points P, Q and R on parabola $y^2 = 4ax$ are in G.P.. Then show that tangents at P and R meet at the ordinate of Q.

Sol. Let P be $(at^2, 2at)$, Q $(ar^2t^2, 2art)$ and R $(ar^4t^2, 2ar^2t)$

Tangent at P; $ty = x + at^2$ ... (1)

Tangent at R; $r^2 \pm y = x + ar^4t^2$ ... (2)

Subtracting $t(1 - r^2)y = at^2(1 - r^4)$

$\Rightarrow y = at\,(r^2 + 1)$

From (1) $x = ty - at^2$

$= at^2(r^2 + 1) - at^2$

$= ar^2t^2$

Which is ordinate of Q.

Example 52: Find locus of the point, tangents drawn from which to parabola $y^2 = 4ax$ include and angle of $\tan^{-1} 2$.

Sol. Let the point be (x_1, y_1)

Equation of pair of tangents drawn from it to parabola are $(y^2 - 4ax)\,(y_1^2 - 4ax_1) = (yy_1 - 2ax - 2ax_1)^2$

In this equation co-efficient of x^2 is $-4a^2$; co-efficient of y^2 is $y_1^2 - 4ax_1 - y_1^2 = -4ax_1$; co-efficient of xy is $4ay_1$

$\therefore$ tan θ = 2 = $\dfrac{2\sqrt{4a^2y_1^2 - 16a^3x_1}}{-4a(a+x_1)}$

$\Rightarrow 16(x_1 + a^2) \cdot a^2 = 4a^2 (y_1{}^2 - 4ax_1)$

$\therefore$ Locus $y^2 = 4(x + a)^2 + 4ax$

Example 53: Prove that normal of parabola $y^2 = 4px$, at point ordinate = abscissa, subtend a right angle at focus of parabola.

Sol. Let point be A (4p, 4p) on parabola $y^2 = 4px$

$\therefore$ slope of tangent at $A = \dfrac{2p}{y_1} = \dfrac{2p}{4p} = \dfrac{1}{2}$

Normal at A is $\dfrac{1}{2}(y-4p)+(x-4p)=0$

$\Rightarrow y = 12p - 2x$

It meets parabola $\therefore$ $(12p - 2x)^2 - 4px = 0$

$\Rightarrow 36p^2 - 13px + x^2 = 0$

$\Rightarrow (x - 4p)(x - 9p) = 0 \Rightarrow x = 4p, 9p$

x = 4p gives point (4p, 4p); other point is (9p, ± 6p)

Normal shall meet below x-axis, $\therefore$ point (9p – 6p), A is (4p, 4p), B (9p, –6p);

Slope AS = m_1 = $\dfrac{4p}{4p-p} = \dfrac{4}{3}$

Slope BS = m_1 = $\dfrac{6p}{9p-p} = \dfrac{-3}{4}$

$m_1m_2 = -1$ $\therefore$ $\angle ASB = 90^o$

Example 54: Find locus of mid points of normal chords of parabola $y^2 = 4ax$

Sol. Any normal $y = mx - 2am - am^3$ ... (1)

Let its mid point be (h, k)

Chord is parallel to the polar of (h, k)

Polar yk = 2a(x + h) slope $\dfrac{2a}{K} \Rightarrow m = \dfrac{2a}{K}$

(h, k) is also on normal.

$\therefore K = \dfrac{2a}{K}\cdot h - 2a\cdot\left(\dfrac{2a}{K}\right) - a\left(\dfrac{2a}{K}\right)^3$

$\Rightarrow \frac{K^2}{2a} = h - 2a - \frac{4a^3}{K^2}$

Locus is $\frac{y^2}{2a} + \frac{4a^2}{y^2} = x - 2a$

Example 55: The normal of parabola $y^2 - 8x - 2ay + 8b + a^2 = 0$ at x = 2b meets (i) axis of parabola at ... (ii) x-axis at ...

Sol. Equation of parabola $(y - a)^2 = 8(x - b)$

Changing origin at (b, a), Equation of Parabola $y^2 = 8x$

When vertex is origin and latus rectum 4a, the normal $x = x_1$ meets axis of parabola at $(x_1 + 2a, 0)$

The point x = 2b on changed axis is x = b.

Normal shall meet axis of parabola (changed axes) at (b + 4, 0)

On original axis at (b + b + 4, a) i.e. (2b + 4, a)

(ii) Normal shall meet x axis in $\left[\sqrt{\frac{2}{b}}a + 2b + 4,\ 0\right]$

Example 56: If two normals drawn from a point to parabola $y^2 = 4ax$ are coincident, then prove that locus of point is $27ay^2 = 4(x - 2a)^3$

Sol. Any normal to parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$

Let it pass through (h, k) the point from which normals are drawn

$\therefore$ $k = mh - 2am - am^2$

$\Rightarrow am^3 + m(2a - h) + k = 0$ ... (α)

Let slopes of normals be m_1, m_2, m_3

$\therefore$ $m_1 + m_2 + m_3 = 0$

But $m_1 = m_2$ given

$\therefore$ $m_3 = -2m_1$ ... (1)

Product $m_1 m_2 m_3 = -\frac{K}{a} = m_1^2 \cdot m_3$

and from (i) $m_1^2(-2m_1) = \frac{-K}{a} \Rightarrow m_1^3 = \frac{K}{2a}$ ... (β)

From (α) $m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{2a - h}{a}$

i.e. $m_1^2 + 2m_1m_3 = \frac{2a-h}{a}$

i.e. $m_1^2 + 2m_1(-2m_1) = \frac{2a-h}{a}$

$3m_1^2 = \frac{h-2a}{a}$ ... (2)

Now $(3m_1^2)^3 = \frac{1}{a^3}(h-2a)^3 = 27m_1^6 = 27(m_1^2)^3$

$\therefore$ From (β) $27\left(\frac{K^2}{4a^2}\right) = \frac{1}{a^3}(h-2a)^3$

$\therefore$ Locus is $27\,ay^2 = 4(x-2a)^3$

Example 57: A is a fixed point on parabola $y^2 = 4ax$; Normal at A meets parabola at B. If AB subtends a right angle at vertex, find slope of AB.

Sol. Let A be $(at_1^2, 2at_1)$ and B $(at_2^2, 2at_2)$

Since AB is normal to parabola

$\therefore\ t_2 = -t_1 - \frac{2}{t_1}$ ... (α)

O is vertex, slope OA, $m_1 = \frac{2}{t_1}$ and slope of OB is $m_2 = \frac{2}{t_2}$

AB subtends right angle at vertex $\Rightarrow \frac{4}{t_1 t_2} = -1$

from (α) $t_1 t_2 = -t_1^2 - 2 \Rightarrow -4 = -t_1^2 - 2$

$\therefore\ t_1 = \pm\sqrt{2}$

Slope of normal is $-t_1 = \mp\sqrt{2}$

Example 58: Through vertex O of parabola $y^2 = 4ax$ OP and OQ drawn at right angles. Show that for all values of P, PQ cuts axis of parabola in a fixed point. Find also locus of mid point of PQ

Sol. Let OP be $y - mx = 0$ then OQ is $my + x = 0$

From parabola $(mx)^2 = 4ax \Rightarrow x = \frac{4a}{m^2}, y = \pm\frac{4a}{m}$

If P is $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$ then Q is $(4am^2, -4am)$

Equation of PQ; $$y-\frac{4a}{m}=\frac{-4a\left(m+\frac{1}{m}\right)}{4a\left(m^2-\frac{1}{m^2}\right)}\left(x-\frac{4a}{m^2}\right)$$

$$\Rightarrow y-\frac{4a}{m}=\frac{-1}{m-\frac{1}{m}}\left(x-\frac{4a}{m^2}\right)$$

It meets y = 0 (axis of parabola) then

$$4a\left(m-\frac{1}{m}\right)=m\left(x-\frac{4a}{m^2}\right)$$

$x=4a-\frac{4a}{m^2}+\frac{4a}{m^2} \Rightarrow x = 4a.$ (fixed point)

Now equation of PQ is $y - 0 = m(x - 4a)$

If (h, k) is mid point M, then

Slope of PQ = slope of polar of M (h, k)

Slope of polar of (h, k) is $\frac{2a}{k}$

$\therefore$ Locus mid point $k=\frac{2a}{k}(h-4a)$

$y^2 = 2a (x - 4a)$

Example 59: C_1 and C_2 are parabolas $x^2 = y - 1$ and $y^2 = x - 1$. Let P be any point on C_1 and Q any point on C_2. Let P_1, Q_1 be reflections of P and Q in straight line y = x. Prove that P_1 lies on C_2 and Q_1 lies on C_1 and PQ $\geq$ min (PP', QQ'). hence or otherwise determine points P_0 and Q_0 on parabolas C_1 and C_2 respectively such that $P_0 Q_0 \leq PQ$ for all values of P on C_1 and Q on C_2.

Sol. If in a equation of a curve if we replace y for x and x for y, then it becomes image of that curve in x = y.

$\therefore$ Image of point P on C_1 shall be P_1 on C_2 and of point Q_1 on C_2 shall be image of Q of C_2. If P is (2, 5) then it is (5, 2) on C_2 (ii) PP_1 and QQ_1 are perpendicular on y = x.

$\therefore$ PQ_1QP_1 is a trapezium and $PQ_1 = P_1Q$.

$\therefore$ Diagonal PQ > PP' – QQ'

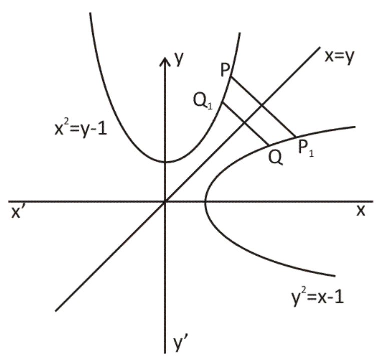

Fig 15

Now if P_1 coincides with Q then P shall coincide with Q_1. Then $PP_1 - QQ' = 0 \Rightarrow PQ = PP_1 - QQ_1$

$\therefore PQ \geq (PP_1 - QQ_1)$

(iii) Distance of point (x_1, y_1) on curve $x^2 = y - 1$ from y = x, equal to perpendicular from it online.

$$p = \frac{x_1 - y_1}{\sqrt{2}} = \frac{x_1 - x^2 - 1}{\pm\sqrt{2}} = \frac{x^2 - x + 1}{\sqrt{2}} = \frac{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}{\sqrt{2}}$$

For minimum p, $x = \frac{1}{2}$ and $p = \frac{3}{4\sqrt{3}}$

$\therefore$ For $P_0Q_0 \leq PQ$, $P_0Q_0 = 2p = \frac{3}{2\sqrt{2}}$

Points on parabolas are $P_0\left(\frac{1}{2}, \frac{5}{4}\right)$, $Q_0\left(\frac{5}{4}, \frac{1}{2}\right)$ (ends of Latus rectum).

Example 60: Find radius and centre of the smaller of two circles that touch parabola.

$75y^2 = 64(5x - 3)$ at $\left(\frac{6}{5}, \frac{8}{5}\right)$ and x-axis.

Sol. Circle touches x axis

$\therefore \Theta$ is $x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 = 0$ ... (1)

Parabola and circle have common tangent at $\left(\frac{6}{5}, \frac{8}{5}\right)$. Equation of tangent of circle.

$$\frac{6}{5}x + \frac{8}{5}y - \alpha\left(x + \frac{6}{5}\right) - \beta\left(y + \frac{8}{5}\right) + \alpha^2 = 0$$

$$\Rightarrow x\left(\frac{6}{5} - \alpha\right) + y\left(\frac{8}{5} - \beta\right) + \alpha^2 - \frac{6\alpha}{5} - \frac{8\beta}{5} = 0 \quad \text{... (2)}$$

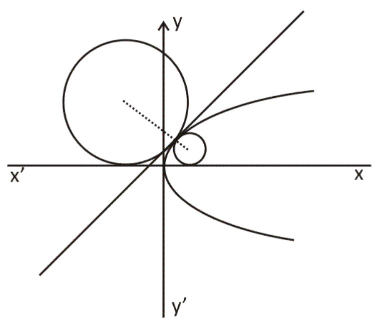

Fig 16

Equation os parabola $75y^2 = 64(5x - 3)$

Tangent at $\left(\frac{6}{5}, \frac{8}{5}\right)$; $\frac{8}{5}y = \frac{32}{15}\left(x + \frac{6}{5}\right) - \frac{192}{75}$

$$\Rightarrow y = \frac{4}{3}x \quad \text{... (3)}$$

(2) and (3) should be same

$$\therefore \frac{4}{\frac{6}{5}-\alpha} = \frac{-3}{\frac{8}{5}-\beta} = \frac{0}{\alpha^2 - \frac{6\alpha}{5} - \frac{8\beta}{5}}$$

$$\therefore \frac{32}{5} - 4\beta = -\frac{18}{5} + 3\alpha \Rightarrow \beta = \frac{10-3\alpha}{4}$$

and $\alpha^2 - \frac{6\alpha}{5} - \frac{8}{5}\left(\frac{10-3\alpha}{4}\right) = 0$ $\alpha^2 - 4 = 0 \Rightarrow \alpha = \pm 2$

$$\therefore \beta = \frac{10 \mp 6}{4},\ \beta = 4, 1$$

Since radius of Θ is β; radius smaller circle 1, centre (2, 1)

Example 61: Show that locus of a point that divides a chord of slope 2 of parabola $y^2 = 4ax$ internally in the ratio of 1 : 2 is a parabola.

Sol. Let t_1 and t_2 be the ends of chord and (h, k) divides it in the ratio 1 : 2.

$$\therefore h = \frac{a(t_2^2 + 2t_1^2)}{3}, k = \frac{2a(t_2 + 2t_1)}{3}$$

Slope of chord is 2; $2 = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} \Rightarrow t_1 + t_2 = 1$

$$\therefore K = \frac{2a}{3}(1 + t_1) \Rightarrow 1 + t_1 = \frac{3k}{2a}$$

and $h = \frac{a}{3}\left[(1-t_1)^2 + 2t_1^2\right] = \frac{a}{3}\left[3t_1^2 - 2t_1 + 1\right]$

$$= \frac{a}{3}\left[3(t_1^2 + 2t_1 + 1) - 8(t_1 + 1) + 6\right]$$

$$= \frac{a}{3}\left[\frac{27k^2}{4a^2} - \frac{24k}{2a} + 6\right]$$

$$= \frac{1}{4a}\left[9k^2 - 16ak + 8a^2\right]$$

$$\frac{4ah}{9} = \left[\left(k - \frac{8}{9}a\right)^2 + \frac{8}{81} \cdot a^2\right]$$

$\Rightarrow \left(y - \frac{8}{9}a\right)^2 = \frac{4a}{9}\left(x - \frac{2a}{9}\right)$ which is equation of parabola.

Practice Worksheet (Foundation Level) – 6 (e)

1. The focal distance of (6, – 1) on parabola $y^2 - 6y - 4x + 17 = 0$ is ...
2. The tangents at the ends of focal chord of parabola meet on ... and include an angle of ...
3. The polar of mid point of a chord of parabola with respect to this parabola is ... to chord.
4. The sub-normal of any point of parabola is equal to ...
5. The point on parabola $(y - 4)^2 = 6(x + 3)$ for which sub-tangent is equal to sub-normal is ...
6. The normal of parabola $(y + 2)^2 - 4(x - 3) = 0$ is inclined at 45^o to x-axis, meets

   (a) axis of parabola at ... (b) meets x-axis at ...
7. The axis of parabola is $3x - 4y = 7$ and vertex is (1, –1), latus rectum 20. If its open mouth is towards $3^{rd}/4^{th}$ quadrant, finds its equation.
8. $2x + 4y = 9$ is normal to parabola $y^2 = 8x$ at point (p^2, q^2) then p = ..., q = ...
9. Find the point at which normal to parabola $(y - a)^2 = 12(x + b)$ at x = 2b, meets the axis of parabola.
10. Normal at (x_1, y_1) point on parabola $y^2 + 6x + 4y + 16 = 0$ is inclined at 45^o with its axis, then find the point.
11. The point A on parabola $y^2 = 6x$ is such that its geometric mean of sub-tangent and sub-normal is equal to latus-rectum of parabola. Find co-ordinates of A.
12. Find common tangent of $y^2 = 8x$ and $x^2 + y^2 + 4x = 0$.
13. Angle between curves $y^2 = 6x$ and $x^2 + y^2 - 8x = 0$ is ...
14. Find the equation of tangents drawn from point (4, 10) to parabola to $y^2 = 9x$.
15. Tangents are drawn from (–1, 2) to parabola $y^2 = 4x$. Find the area of triangle formed by tangents and chord of contact.
16. C_1 and C_2 are two parabolas whose focii are same and latus rectum equal, but axes in opposite direction. These intersect. Find angle between them when axes lie on x-axis.
17. Find length of chord $y = x\sqrt{3} - 5p\sqrt{3}$ of parabola $y^2 = 4px$.
18. Point P is (2, 4) and is on parabola $(y - 2)^2 = 4(x - 1)$. Normal at P meets axis of parabola in G and the tangent at vertex A in G'; rectangle AG'QG is completed. Find Q and area of rectangle.
19. P is a point on parabola $y^2 = 4ax$. PN is ordinate. The tangent at P meets axis in T. A is vertex and S is focus of parabola. Find SP^2.

20. Find locus of foot of perpendicular from focus of parabola $y^2 = 8(x - 3)$ on its tangent.

21. A normal chord of parabola $y^2 = 6x$ makes an angle 90^o at the vertex. Find its inclination with x-axis.

22. Perpendicular from P on its polar with respect to $y^2 = 4bx$ touches $x^2 = 4ay$. Find locus of P.

23. P is a point on parabola $y^2 = 4x$, PM is perpendicular from P on directrix, S is focus. Find P if P, M, S are vertices of an equilateral triangle.

24. Find point on parabola $y = (x - 3)^2$ at which tangent is parallel to chord joining (3, 0) and (4, 1).

25. Parabolas $y^2 = 4x$, $x^2 = 4y$ divide the square region bounded by x = 4 and y = 4 and co-ordinate axes in points s_1, s_2 and s_3 from top to bottom. Prove that $S_1 : S_2 : S_3 = 1 : 1 : 1$.

26. The focal distance of point (p, q) on parabola $y^2 = 8a(x - b)$ is

(a) p + 2a + b (b) p + 2a – b (c) p + a + b (d) p – b + a

27. The point on parabola $(y + 4)^2 = 6(x - 3)$ for which sub-tangent is equal to sub-normal is

(a) $\left(\frac{9}{2}, -1\right)$ (b) $\left(\frac{3}{2}, -1\right)$ (c) $\left(\frac{9}{2}, -7\right)$ (d) $\left(\frac{3}{2}, -1\right)$

28. LL is latus rectum of parabola $(y - 1)^2 = 3x$. Tangent and normal are drawn at L, then

(a) Their inclination to axes are supplementary.

(b) Latus rectum bisects the angle between them.

(c) Tangents inclination with LL' is complementary of normal's inclination to it.

(d) The angle made by tangent with LL' is equal to half the angle which normal makes with LL'.

29. The length of sub-normal of any point on any parabola is equal to

(a) $\frac{1}{4}$ latus-rectum (b) $\frac{1}{2}$ latus rectum

(c) $\frac{1}{3}$ latus rectum (d) none of these

30. 2x + 4y = 9 is normal to parabola $y^2 = 8x$ at point, $\left(\frac{p}{q}, \frac{q}{p}\right)$ then $\frac{q}{p} =$

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) none of these

31. From point P (3a, 0) on axis of parabola $y^2 = 4ax$, normals of parabola are drawn then

(a) A normal is parallel to y-axis

(b) Angle between two normals is 90^o

(c) A normal is inclined at 90^o to y-axis.

(d) Two normals are equally inclined to axis

32. $x \cos \alpha + y \sin \alpha = p$ shall touch parabola $x^2 = 4ay$ if

(a) $a^2 \cos^2 \alpha + p \sin \alpha = 0$ (b) $a \cos^2 \alpha + p \sin \alpha = 0$

(c) $a^2 \cos^2 \alpha - p \sin \alpha = 0$ (d) $a \cot \alpha - p \tan \alpha = 0$

33. The portion of tangent between point of contact and directrix subtend a focus on angle of

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

34. The chord x = 3 of parabola $(y - 2)^2 = 6x$ subtend at vertex of parabola an angle of

(a) $\tan^{-1}\sqrt{2}$ (b) $\tan^{-1}2\sqrt{2}$ (c) $\tan^{-1}(-2\sqrt{2})$ (d) $\frac{\pi}{2}$

35. Normal drawn at point (2, 2) of parabola $y^2 - 8y + 4x + 4 = 0$, meets directrix in

(a) (4, 0) (b) (6, 0) (c) (2, 0) (d) (–4, 0)

36. Normal at (8, 2) of parabola $y^2 - 6x + 8y + 28 = 0$ meets its axis in

(a) (9, 0) (b) (7, 4) (c) (10, 0) (d) (11, –4)

37. The normal at (4, 8) of parabola $y^2 = 16x$ meets the parabola again at

(a) (36, –24) (b) (9, –6) (c) (24, –16)

38. Tangents drawn from (–2, 3) to parabola $y^2 = 6x$ include an angle of

(a) $\tan^{-1}3\sqrt{\frac{3}{7}}$ (b) $\tan^{-1}2\sqrt{21}$ (c) $\tan^{-1}\frac{2\sqrt{7}}{3}$ (d) none of thse

39. Straight line ax + b = 0 is directrix of parabola $y^2 + kx = 0$, then k is

(a) $-\frac{b}{4a}$ (b) $\frac{b}{4a}$ (c) $-\frac{4b}{a}$ (d) $\frac{-b}{2a}$

40. The length of focal chord of parabola $y^2 = 4(x - 1)$ inclined at 30^o with its axis is

(a) 16 units (b) $8\sqrt{2}$ u. (c) $16\sqrt{3}$ u.

41. Equation of directrix of parabola $(y - a)^2 = p(x + b)$ is

(a) $x = -\frac{p}{4}$ (b) x + b = 0 (c) $x + b = \frac{p}{4}$ (d) $x + b + \frac{p}{4} = 0$

42. x – y = 0 is axis of a parabola, whose focus is (4, 4) and latus-rectum 8. Its mouth is towards first quadrant. Equation of its directrix is

(a) x + y = $4\sqrt{2}$ (b) x + y = $4(2-\sqrt{2})$

(c) x + y = $2(4-\sqrt{2})$ (d) x + y = $2(2-\sqrt{2})$

43. In question 42, equation of parabola is

(a) $(x-y)^2 + 8\sqrt{2}xy + (8-4\sqrt{2})^2 = 0$

(b) $(x-y)^2 - 8\sqrt{2}(x+y) - (8-4\sqrt{2})^2$

(c) $x^2 - 2xy + y^2 - 16(x+y) + 96 - 64\sqrt{2} = 0$

(d) none of these

44. The harmonic mean of two segments of focal chord of parabola $y^2 = 12(x - 3)$ inclined at 30^o is

(a) 6 (b) 4 (c) $4\sqrt{3}$ (d) $6\sqrt{3}$

45. AB is focal chord of parabola $y^2 = 4bx$ and OP is perpendicular from vertex on it, then

(a) $AB \propto \frac{1}{OP}$ (b) $AB \propto \frac{1}{(OP)^2}$ (c) $AB \propto \frac{1}{\sqrt{OP}}$ (d) none of these

46. P is (1, 0) and Q is a point on parabola $y^2 = 8x$. Locus of mid point of PQ is

(a) $x^2 - 4y + 2 = 0$ (b) $x^2 + 4y + 2 = 0$

(c) $x^2 + y^2 + 4x + 2 = 0$ (d) $y^2 - 4x + 2 = 0$

47. If x – y + 7 = 0 be tangent to parabola $(y - 4)^2 = p(x + 1)$, then p is

(a) 4 (b) 6 (c) 8 (d) –6

48. 2x – 3y + K = 0 is normal to parabola $(y - 3)^2 = 4(x + 2)$ then K is

(a) $\frac{-73}{9}$ (b) $\frac{89}{7}$ (c) $\frac{73}{9}$ (d) $\frac{-37}{9}$

49. If y_1, y_2, y_3 are ordinate of three points A, B and C on parabola $y^2 = 6x$, then area of triangle ABC is

(a) $\frac{1}{8}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$ (b) $\frac{1}{16}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$

(c) $\frac{1}{12}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$ (d) $\frac{1}{8}(y_1 + y_2)(y_2 + y_3)(y_3 + y_1)$

50. The point of intersection of polars of point (–1, –2) w.r.t. parabolas $(y-3)^2 = 4x$ and $x^2 = 4y$ is

(a) (22, –9) (b) (–14, 9) (c) $\left(-4, \frac{10}{3}\right)$ (d) $\left(4, \frac{9}{2}\right)$

51. P is a point on parabola $y^2 = 4ax$, tangent and normal drawn at P meet axes in T and G respectively. MPM' is drawn parallel to x-axis PN is ordinate, then

(a) $M\hat{P}T = N\hat{P}G$ (b) $M\hat{P}T = S\hat{P}G$

(c) $S\hat{P}G = M\hat{P}G$ (d) $M\hat{P}T$ and $S\hat{P}G$ are complementary

52. SY is perpendicular from focus on tangent at p of parabola $y^2 = 4ax$. A is vertex. Prove that $SY^2 = AS \cdot SP$.

53. Tangents drawn at P and Q points of $y^2 = 4ax$ meet at T. Prove $ST^2 = SP \cdot SQ$, S is focus

54. Perpendicular dropped from vertex A of parabola $y^2 = 4ax$ on any tangent of parabola meet it at P and meets parabola at Q. Prove AP · AQ is a constant number.

Practice Worksheet (Competition Level)

PART-A

1. a) Find the equation of parabola whose focus is (2, 3) and directrix is the line x – 4y + 3 = 0.

   b) Find the vertex, focus, directrix, latusrectum and tangent at the vertex of the following parabolas

   i) $9y^2 - 16x - 12y - 57 = 0$ ii) $5x^2 + 30x + 2y + 59 = 0$

   c) Find the equation of parabola whose focus is (1,-1) and whose vertex is (2, 1).

2. Find the coordinate of the focus and the vertex and the equations of the axis, directrix and the letus- rectum of the parabola $4y^2 - 6x - 4y = 5$.

3. Show that the condition to the line y = mx +c to be the tangent to the parabola $y^2 = 4a(x+a)$ will be $c = am + \frac{a}{m}$.

4. A tangent to the parabola $y^2 = 8x$ makes an angle of 45^o with the straight line y = 3x + 5. Find the equation of the tangent and the point of contact.

5. Find the equation of common tangents to the parabola $y^2 = 16x$ and the circle $x^2 + y^2 = 8$.

6. If the normals at two points P, Q of the parabola intersect at a third point R on the curve, show that the product of the ordinates of P and Q is $8a^2$.

7. Through the vertex O of the parabola $y^2 = 4ax$ chords OP and OQ are drawn at the right angle to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point Also find the locus of middle point of PQ

8. Prove that the tangent at one extremity of a focal chord of the parabola $y^2 = 4ax$ is parallel to the normal drawn at the other extremity.

9. Find the Locus of the point of intersection of two mutually perpendicular normals to the parabola $y^2 = 4ax$ and Show that abscissa of the point can never be smaller than 3a.what is the ordinate when the abscissa is the smallest?

10. A is the point on the parabola $y^2 = 4ax$. the normal at A cuts the parabola again at the point B. If AB subtends aright angle at the vertex of the parabola, find the slope of AB.

11. Show that the locus of middle points of all the chords of the parabola $y^2 = 4ax$ which are drawn through the vertex is the parabola $y^2 = 2ax$

12. If from the vertex of the parabola $y^2 = 4ax$ a pair of chord be drawn at the right angle to one another and with these chords as adjacent sides a rectangle be drawn,

prove the locus of vertex of farther angle of the rectangle is the parabola $y^2 = 4a(x - 8a)$.

13. The tangents from the point (α, β) to the parabola $y^2 = 4ax$ form a triangle with the chord of contact. Prove that the area of the triangle is $\frac{(\beta^2 - 4a\alpha)^{\frac{3}{2}}}{2a}$

14. From the point (–1, 2) tangent lines are drawn to the parabola $y^2 = 4x$. Find the equation of the chord of contact. Also find the area of the triangle formed by the chord of contact and the tangents.

15. Show that the locus of the points of intersection of the tangents to $y^2 = 4ax$ which intercept a constant length l on the directrix is $(y^2 - 4ax)(x + a)^2 = l^2x^2$.

16. Prove that the locus of mid points of the normal chords of parabola $y^2 = 4ax$ is $\frac{y^2}{2a} + \frac{4a^3}{y^2} = x - 2a$

17. If normal to the parabola $y^2 = 4ax$ at three points, P, Q and R meet at A and S be the focus then Show that $SP.SQ.SR = aSA^2$

18. a) Prove that the locus of points such that two of the three normals to the parabola $y^2 = 4ax$ from them coincide is $27ay^2 = 4(x - 2a)^3$.

b) Three normals with the slope m_1, m_2, m_3 are drawn from the point P not on the axis of the parabola $y^2 = 4x$, If locus of P with $m_1m_2 = a$ results being a part of the parabola itself, Find the value of a. **[I.I.T -2003, 4]**

19. a) Find the locus of a point P when three normals drawn from it are such that

i) one bisects the angle between the other two.

ii) Two of them make complementary angles with axis.

b) Find the locus of the point of intersection of those normals to the parabola $x^2 = 8y$ which are at the right angles to each other. **[I.I.T -1997, 5]**

20. Three normals from a point to the parabola $y^2 = 4ax$ meet the axis of the parabola in the points whose abscissa are in A.P. Find the locus of the point.

21. Show that the locus of the middle points of a variable chords of the parabola $y^2 = 4ax$, such that the focal distance of its extremities are in the ratio 2: 1 is $9(y^2 - 2ax)^2 = 4a^2(2x - a)(4x + a)$.

22. Show that if r_1 and r_2 be the length of perpendicular chords of a parabola drawn through the vertex, then $(r_1r_2)^{4/3} = 16a^2(r_1^{2/3} + r_2^{2/3})$.

23. From a point A common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and parabola $y^2 = 4ax$ and find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola.

24. Prove that the length of the chord of contact of the tangents drawn from the point (x_1,y_1) to the parabola $y^2 = 4ax$ is $\frac{1}{a}\sqrt{\{(y^2_1 + 4a^2)(y^2_1 - 4ax_1)\}}$. Hence show that the area of the triangle formed by these tangents and their chord of contact is $\frac{1}{2a}(y^2_1 - 4ax_1)^{3/2}$.

25. Prove that the locus of the middle points of tangents drawn from points on the directrix to the parabola $y^2 = 4ax$ is $y^2(2x + a) = a(3x + a)^2$.

PART-B

1. Find the locus of point of intersection of the two normals to the parabola $y^2 = 4ax$ which at the right angle to one another.

2. If the distribution of weight is uniform, then the rope of the suspended bridge takes the form of parabola. The height of the supporting towers is 20 meter, the distance between these towers is 150 meters and the height of the lowest point of the rope from the road is 3 meter. Find the equation of the parabolic shape of the road, considering the floor of the bridge as x axis and the axis of parabola as y–axis. Find the height of that tower which support the rope and is at a distance of 30 meter from the center of the rope.

3. Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ cannot have a common normal unless $\frac{b}{a-c} > 2$.

4. Equilateral triangles are inscribed in the parabola $y^2 = 4x$; prove that the locus of their centroids is $9y^2 = 4(x-8)$.

5. Find the equation of the circle, which touches the y-axis and the parabola $y^2 = x$ and $2y = 2x^2 - 5x + 1$only at the point where they touch each other and at no other point.

6. Prove that the locus of the point of intersection of tangents drawn at the extremities of the normal chord to the parabola $y^2 = 4ax$ is the curve $y^2(x+2a)+4a^3=0$

7. Find the locus of the mid points of chords of a parabola, which are such that the normals at the ends of these Chords meet on the curve again.

8. The normal at any point P meets the axis in G and the tangent at the vertex in G'. If A be the vertex and the rectangle AGQG' be completed. Prove that the equation of locus of Q is $x^3 = 2ax^2 + ay^2$.

9. Three normals are drawn from (2k, 0) to the curve $y^2 = 4x$. Show that k must be greater than 1. One normal is always the x - axis. Find k for which the other two normals are perpendicular to each other.

10. Show that the locus of the middle points of chords of the parabola $y^2 = 4ax$ which are of constant length 2l is $(y^2 + 4a^2)(4ax - y^2) = 4a^2l^2$.

11. Find the radius and the centre of the smaller of the two circle that touch the parabola $75y^2 = 64(5x - 3)$ at (6/5, 8/5) and the x axis.

12. Prove that the length of the intercepts on the normal at the point P $(at^2, 2at)$ of a parabola $y^2 = 4ax$ made by the circle described on the line joining the focus and P as diameter is $a\sqrt{(1 + t^2)}$.

13. The ordinate of the points P and Q on the parabola $y^2=4ax$ are in the ratio 1:2. Find the locus of the point of intersection of the normals to the parabola at P and Q.

14. A tangent to the parabola $y^2 + 4bx = 0$ meets the parabola $y^2 = 4ax$ at P and Q. Prove that the locus of the middle point of PQ is $y^2 (2a + b) = 4a^2x$.

15. Show that the locus of the point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1:2 is a parabola. Find the vertex of the parabola.

[I.I.T -1995, 5]

16. The angle between a pair of tangent drawn from the point P to the $y^2=4ax$ is 45^o . Show that the locus of point P is hyperbola. **[I.I.T -1998, 10]**

17. Tangent is drawn at any point $(x_1\ y_1)$ on the parabola $y^2 = 4ax$. Now tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of contact pass through a fixed point $(x_{2.}\ y_2)$. Prove that $4\ \frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0.$

18. Two variable perpendicular chords are drawn from the origin 'O' to the parabola $y = x^2 - x$, which meet the parabola at P and Q Rectangle POQR is completed. Find the locus of R.

19. At any point on the parabola $y^2-2y-4x+5 = 0$, a tangent is drawn which meets the directrix at Q. Find the locus of the point R which divides QP externally in the ratio ½:1. **[I.I.T -2004, 4]**

20. Find the locus of the point through which pass three normals to the parabola $y^2 = 4ax$ such that two of them make angles $\alpha$and β respectively with the axis such that $\tan\alpha \tan\beta = 2$.

21. The normals at three points P, Q, R, of the parabola $y^2 = 4ax$ meet in (h, k). Prove that the centroid of Δ PQR lies on the axis at a distance 2 (h − 2a) /3 from the vertex.

COMPREHENCIVE PASSAGE TYPE PROBLEMS

22. Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth Quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangent to the parabola at P and Q intersect the x-axis at S.

i) The ratio of the areas of the triangles PQS and PQR is

a) $1:\sqrt{2}$ b) 1:2 c) 1:4 d) 1:8

ii) The radius of the circum-circle of the triangle PRS is

a) 5 b) $3\sqrt{3}$ c) $3\sqrt{2}$ d) $2\sqrt{3}$

iii) The radius of the in-circle of the triangle PQR is

a) 4 b) 3 c) 8/3 d) 2

[IIT-JEE-07, 4]

23. If a circle is drawn on focal distance of any point P, lying on a parabola, as diameter, it touches the tangent at vertex of the parabola at the point where the tangent to the parabola at point P meets the circle. Now considering a parabola with focus as S(−1, −1). Let $3x - y - 8 = 0$ be the equation of tangent of the parabola a point P (7, 13). Then,

i) The foot of perpendicular from focus upon the tangent to the parabola is

a) $\left(\frac{13}{5}, \frac{1}{5}\right)$ b) (2, −2) c) (−2, 2) d) $\left(\frac{1}{5}, \frac{13}{5}\right)$

ii) Slope of the normal to the circle through the point found in the previous question is

a) 4 b) 8 c) -4 d) -8

iii) Equation of tangent at vertex to the parabola is

a) $x - 8y + 14 = 0$ (b) $8x - y + 14 = 0$

(c) $x + 8y + 14 = 0$ (d) $8x + y + 14 = 0$

iv) Directrix of the parabola is

(a) $x - 8y + 14 = 0$ (b) $8x - y + 14 = 0$

(c) $8x - y + 19 = 0$ (d) $x + 8y + 19 = 0$

v) Equation of axis of the parabola is

(a) $8x - y + 7 = 0$ (b) $8x + y + 9 = 0$

(c) $x + 8y + 7 = 0$ (d) $8x - y + 9 = 0$

MATCH THE COLUMN TYPE PROBLEMS

24.

Column-I	**Column-II**
A) $y^2 = 4ax$	p) $y = mx + 2a + a/m^2$
B) $y^2 = -4ax$	q) $y = mx - 2am - am^3$
C) $x^2 = 4ay$	r) $y = mx + 2am + am^3$
D) $x^2 = -4ay$	s) $y = mx - 2a - a/m^2$

25. Normals are drawn at points P, Q, and R lying on the parabola $y^2 = 4ax$ which intersect at (3, 0) then

Column-I	**Column-II**
A) Area of ΔPQR	p) 2
B) Radius of circumcircle of ΔPQR	q) 5/2
C) Centroid of ΔPQR	r) (5/2, 0)
D) Circumcentre of ΔPQR	s) (2/3, 0)

[IIT-JEE-07, 6]

OBJECTIVE PROBLEMS

Level-1

1. The co-ordinates of the point on the parabola $y^2 = 8x$ which is at minimum distance from the circle $x^2 + (y+6)^2 = 1$ are

a) (2, –4) b) (18, –12) c) (2, 4) d) none of these

2. The condition that the two tangents to the parabola $y^2 = 4ax$ become normal to the circle $x^2 + y^2 - 2ax - 2by + c = o$ is given by

a) $a^2 > 4b^2$ b) $b^2 > 2a^2$ c) $a^2 > 2b^2$ d) $b^2 > 4a^2$

3. The equation of common tangent of the parabolas $y^2 = 32x$ and $x^2 = 108y$ is

a) $x = 0$ b) $2x - 3y - 36 = 0$ c) $2x + 3y + 36 = 0$ d) $2x - 3y + 36 = 0$

4. The line $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$ if

a) $al(l^2 + 2m^2) + m^2n = 0$ b) $al(l^2 + 2m^2) = m^2n$

c) $al(2l^2 + m^2) = -m^2n$ d) $al(2l^2 + m^2) = 2m^2n$

5. If the line $x + y - 1=0$ touches the parabola $y^2= kx$ then the value of k is

a) 4 b) -4 c) 2 d) -2

6. The normal chord of the parabola $y^2 =4ax$ at $(x_{1,} x_1)$ subtends a right angle at the

a) focus b) vertex c) end of the latus rectum d) none of thes

7. The number of tangent(s) to the parabola $y^2 =8x$ through (2, 1) is

a) 1 b) 2 c) 0 d)3

8. The locus of the centre of the circle described on any focal chord of the parabola $y^2=4ax$ as diameter is

a) $x^2= 2a(y- a)$ b) $x^2= -2a(y- a)$

c) $y^2=2a(x - a)$ d) $y^2= -2a(x-a)$

9. The angle subtended at the focus by the normal chord at the point (?, ?) ??0 on the parabola $y^2=4ax$ is

a) p/4 b) $\tan^{-1}\left(\frac{\lambda}{a}\right)$ c) $\tan^{-1}\left(\frac{1}{4}\right)$ d) p/2

10. If the normal drawn at the end points of a variable chord PQ of parabola $y^2= 4ax$ intersect at parabola, then the locus of the point of intersection of the tangent drawn at the points P and Q is

a) $x + a=0$ b) $x-2a =0$ c) $y^2-4x + 6=0$ d) none of these

11. Three normals to the parabola $y^2 = x$ are drawn through a point (c, 0) ,then

a) c=1/4 b) c=1/2 c) c>1/2 d) none of these

12. The equation of the line of the shortest distance between the parabola $y^2= 4x$ and the circle $x^2+y^2-4x-2y+4=0$ is

a) $x + y =3$ b) $x - y =3$ c) $2x + y =5$ d) none of these

13. The equation of the parabola whose axis is parallel to the x-axis, latus rectum 6,and focus (5,3) is

a) $y^2+6x +6y +30=0$ b) $y^2-6x-6y-30=0$

c) $y^2+6x +6y-30=0$ d) $y^2-6x-6y+30=0$

14. The locus of poles of all tangents to the parabola $y^2= 4ax$ w. r. t. the parabola $y^2= 4bx$ is

a) $ay^2= 4b^2x$ b) $ax^2= 4b^2y$

c) $ax = 4b^2y^2$ d) none of these

15. The locus of middle points of chords of the parabola $y^2= 4ax$ which subtend a right angle at the vertex of the parabola is

a) $y^2= 2a(x–4a)$ b) $y^2= a(x–4a)$

c) $y^2= 2a(x – a)$ d) none of these

16. Tangent to the curve $y = x^2+6$ at a point P (1, 7) touches the circle $x^2+y^2+16x+12y+c =0$at appoint Q then the coordinates of Q are **[I.I.T.Sc-2005]**

a) (-6,-11) b) (-9,-13)

c) (-10, -15) d) (-6,-7)

17. The equation of the tangent to the parabola $y^2= 8x$ is $y = x +2$ the point on this line from which the other tangent to the parabola is perpendicular to given tangent is

a) (–1, 1) b) (0, 2) c) (2, 4) d) (-2, 0) **[AIEEE-2007]**

18. A parabola has the origin as its focus and the line x =2 as is directrix. Then the vertex of the parabola is at

a) (1, 0) b) (0, 1) c) (2, 0) (d) (0, 2)

19. The slope of a tangent to the parabola $y^2 = 9x$ which passes through the point (4, 10) is

a) 9/4 b) 1/4 c) 3/4 (d) 1/3

20. Equation of parabola having it's focus at S(2, 0) and one extremity of it's latus rectum as (2, 2) is

a) $y^2 = 4 (3 – x)$ b) $y^2 = 4 (1 – x)$

c) $y^2 = 8 (3 – x)$ d) $y^2 = 8(3 – x)$

21. Circle drawn with diameter being any focal chord of the parabola $y^2 – 4x – y – 4 = 0$ will always touch a fixed line, whose equation is

a) 16 + 33x = 0 b) 32x + 13 = 0

c) 13 x + 32 = 0 d) 16x + 33 = 0

22. Radius of the circle touching $y^2= x$ at (1, 1) and having directrix of $y^2= x$ as its normal is

a)½ b) 3/2 c) 5/2 d)$5\sqrt{5}/4$

23. The focal chord of $y^2=16x$ is tangent to $(x–6)^2 +y^2 =2$, then the possible values of the lope of this chord are **[I.I.T.Sc-2003]**

a) 1, –1 b) –1/2, 2 c) –2, ½ d)½, 2

Level-2

1. If b and c are the lengths of the segments of any focal chord of a parabola $y^2= 4ax$ then the length of semi letus rectum is

a) $\frac{b+c}{2}$ b) $\frac{bc}{b+c}$ c) $\frac{2bc}{b+c}$ d) $\sqrt{bc}$

2. The locus of mid points of the focal chord of the parabola $y^2= 4ax$ is

a) $y^2= a(x- a)$ b) $y^2= 2a(x- a)$

c) $y^2= 4\,a(x- a)$ d) none of these

3. The length of focal chord of the parabola $y^2= 4ax$ making an angle of θ with the axis of parabola is

a) $4a\text{cosec}^2\theta$ b) $4a\sec^2\theta$

c) $a\text{cosec}^2\theta$ d) none of these

4. The locus of the point of intersection of the perpendicular tangents to the parabola $x^2=4ay$ is

a) $y = a$ b) $y = -a$ c) $x = a$ d) $x = -a$

5. The common tangent(s) of $y = x^2$ and $y= -x^2+4x-4$ is (are) **[I.I.T.-2006]**

a) $y =4(x-1)$ b) $y = -4(x-1)$ c) $y=0$ d) $y=30x-20$

6. If the normal at two points P and Q of the parabola $y^2= 4ax$ intersect at the third point R on the curve then the product of the ordinates of P and Q is

a) $4a^2$ b) $2a^2$ c) $-4a^2$ d) $8a^2$

7. If $y =2x +3$ is a tangent to the parabola, $y^2=24x$, then its distance from the parallel normal is

a) 5v5 b) 10v5 c)15v5 d)none of these

8. An equilateral triangle is inscribed in the parabola $y^2= 4ax$ whose one vertex is at the vertex of parabola. The length of its side is

a) $4a\sqrt{3}$ b) $2a\sqrt{3}$ c) $16a\sqrt{3}$ d) $8a\sqrt{3}$

9. If a chord which is normal to the parabola $y^2= 4ax$ at one end subtends a right angle at the vertex, then its slope is

a) 1 b) $\sqrt{3}$ c) $\sqrt{2}$ d) 2

10. If the line $x =1$ is the directrix of the parabola $y^2 - kx +8=0$, then one of the values of k is **[I.I.T.Sc-2000]**

a)1/8 b)8 c)4 d)1/4

11. Equation of the parabola whose axis is y =x, distance from origin to the vertex isv2 distance from origin to the focus is 2v2 is (focus and vertex lie in the first quadrant):

a) $(x+y)^2 = 2(x+y-2)$ b) $(x-y)^2 = 8(x+y-2)$

c) $(x-y)^2 = 4(x+y-2)$ d) $(x-y)^2 = 4(x+y-2)$ **[I.I.T.-2006]**

12. The locus of the mid point of the line segment joining the focus to a moving point on the parabola y^2=4ax is another parabola with directrix **[I.T.Sc-2002]**

a) x = – a b)x = – a/2 c)x = 0 d)x = a/2

13. The length of the letus rectum of the parabola $169\{(x-1)^2 + (y-3)^2\} = (5x-12y+17)^2$ is

a) 12/13 b) 14/13 c)28/13 d)none of these

14. The equation of the common tangent to the curve y^2=8x and xy = –1 is

a) 3y=9x+2 b) y=2x+1

c) 2y = x+8 d) y = x+2 **[I.I.T.Sc-2002]**

15. The straight line x +y = a touches the parabola y = x–x^2 if a =

a) 0 b) 1 c) –1 d)none of these

16. The angle between the tangents drawn from the point (1, 4) to the parabola y^2= 4x is

a) p/6 b) p/4 c) p/3 d) p/2 **[I.I.T.Sc-2004]**

17. The vertex of the parabola y^2=8x is at the centre of the circle and parabola cuts the circle at the ends of its letus rectum. Then the equation of the circle is

a) $x^2 + y^2 = 4$ b) $x^2+ y^2=20$

c) $x^2+ y^2= 8$ d) none of these

18. From the focus of the parabola y^2=8x, tangents are drawn to the circle $(x-6)^2 + y^2 = 4$, then the equation of circle through the focus and point of contact of tangents is

a) $x^2 + y^2 + 8x - 12 = 0$ b) $x^2 + y^2 - 6x + 12 = 0$

c) $x^2 + y^2 - 8x + 12 = 0$ d) $x^2 + y^2 + 6x - 12 = 0$

19. **STATEMENT-1:** The curve $y = -\frac{x^2}{2} + x + 1$ is symmetric with respect to line x =1

And

STATEMENT-2: A parabola is symmetric about its axis **[I.I.T-2007]**

a) **STATEMENT-1** is true, **STATEMENT-2** is true; **STATEMENT-2** is the correct explanation **STATEMENT-1**

b) **STATEMENT-1** is true, **STATEMENT-2** is true; **STATEMENT-2** is not the correct explanation **STATEMENT-1**

c) **STATEMENT-1** is true, **STATEMENT-2** is False

d) **STATEMENT-1** is False, **STATEMENT-2** is true

20. Consider the two curves C_1: $y^2 = 4x$, C_2: $x^2 + y^2 - 6x + 1 = 0$ then,

a) C_1 & C_2 touch each other only at one point

b) C_1 & C_2 touch each other exactly at two points

c) C_1 & C_2 intersect (but do not touch) each other exactly at two points

d) C_1 & C_2 neither intersects nor touches each other touch each other exactly at two points **[I.I.T.-2008]**

21. A focal chord of the parabola $y^2 = 4x$ is inclined at an angle of $\pi/4$ with positive x-axis then the slope of the normal drawn at the ends of chord will satisfy the equation

a) $m^2 - 2m - 1 = 0$ b) $m^2 + 2m - 1 = 0$

c) $m^2 - 1 = 0$ d) none of these

22. If two circles $x^2 + y^2 - 6x - 6y + 13 = 0$ and $x^2 + y^2 - 8y + 9 = 0$ intersect at A & B. The Focus of the parabola whose directrix is line AB and vertex at (0, 0) is

a) $\left(\frac{3}{5}, \frac{1}{5}\right)$ b) $\left(-\frac{3}{5}, \frac{1}{5}\right)$ c) $\left(-\frac{3}{5}, -\frac{1}{5}\right)$ d) $\left(\frac{3}{5}, -\frac{1}{5}\right)$

23. If the focus of the parabola $(y - k)^2 = 4(x - h)$ always lies between $x+y=1$ & $x + y = 3$ then

a) $0 < h + k < 2$ b) $0 < h + k < 1$

c) $1 < h + k < 2$ d) $1 < h + k < 3$

✣ --------------------- ✣

7. Ellipse

7.1 Definition :

Ellipse is the locus of a point, which moves such that its distance from a fixed point bears a constant ratio of its distance from a fixed straight line and this constant is less than unity. In the fig (1) P is the moving point, S is the fixed point and z'z z' is fixed straight line PM is perpendicular from P on z'zz'

$$\frac{SP}{PM} = e = \text{constant and } e < 1$$

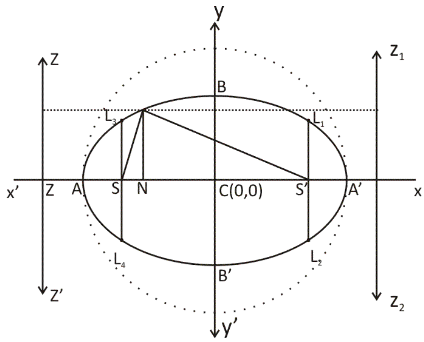

Fig 1

e is called eccentricity of ellipse. Fixed point is called focus and fixed straight line directrix. In the fig. SZ is perpendicular from S on z'z'. In ZS, A is a point such that AS = eAz i.e. it divides ZS in the ratio of 1 : e internally. If we extend ZS then there is one point A' on it such that ZA' : SA' = 1 : e i.e. A' divides ZS externally in the ratio of 1 : e. Both A and A' are on ellipse be origin, A A', x axis and let AA'= 2a

$$\therefore AC = CA' = a$$

The equation of directrix is $x = -\frac{a}{e}$ and S is (–ae, 0)

Now $\quad SA + SA' = e(CZ - AC + A'C + CZ)$

$$\Rightarrow 2a = e(CZ - AC + A'C + CZ)$$

$$= 2eCZ \qquad \because AC = A'C$$

$$\therefore cz = \frac{a}{e}$$

and $\quad SA' = SA = e(CZ + CA') - e(CZ - CA)$

$$= 2e.CA = 2ae$$

and $\quad SA' - SA = (CS + CA') - (CA - CS) = 2ae$

$$\therefore CS = ae$$

$$\therefore S \text{ is } (-ae, 0), S' \text{ is } (ae, 0)$$

By symmetry S' is another focus and z_1z_2 is another directrix.

If $P(x_1, y_1)$ is a point on the ellipse then by definition.

$$\sqrt{(x_1 + ae)^2 + y_1^2} = ePM = eNZ$$

$$= e(cz - cN)$$

$$= e\left(\frac{a}{e} + x_1\right)$$

$$\therefore x_1^2 + y_1^2 + 2aex_1 + a^2e^2 = a^2 + 2aex_1 + e^2x_1^2$$

or $x_1^2(1-e^2) + y_1^2 = a^2(1-e^2)$

or $\frac{x_1^2}{a^2} + \frac{y_1^2}{a^2(1-e^2)} = 1$

Thus equations of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = a^2(1 - e^2)$

7.2 Characteristics:

(1) C is centre of ellipse (0, 0)

(2) AA' is major axis of ellipse = 2a

A is (–a, 0) and A'(a, 0)

(3) BB' is perpendicular on AA' through c

BB' is called minor axis of ellipse = 2b.

B is (0, b) and B'(0, –b)

(4) z z_1 is axis of ellipse. Equation y = 0

(5) z'z z' and z_1z_2 are directrix of ellipse equations $x + \frac{a}{e} = 0$, $x - \frac{a}{e} = 0$.

(6) L_1L_2 and L_3 L_4 are perpendicular chords of ellipse through foci. These are called latus-rectum. Length = $\frac{2b^2}{a}$

$$L_1\left(ae, \frac{b^2}{a}\right), L_2\left(ae, -\frac{b^2}{a}\right), L_3\left(-ae, \frac{b^2}{a}\right), L_4\left(-ae, -\frac{b^2}{a}\right)$$

Equation of latus rectums are $x = \pm ae$

(7) Focal Distance : Distance of point on ellipse from its focus is called focal distance of that point. In the fig P is (x_1, y_1)

$$SP = ePM = ezN = e(|CZ| - CN)$$

$$= e\left(\frac{a}{e} - (-x_1)\right) = a + ex_1$$

(8) Sum of focal distances of a point : Ellipse has two focus and two directrix. So there are two focal distances of a point. From fig.

$$PS + PS' = e(PM + PM')$$
$$= e(PM + PM') = e(MM')$$
$$= e(zz_1)$$
$$= e.2cz$$
$$= 2e.\frac{a}{e} = 2a$$

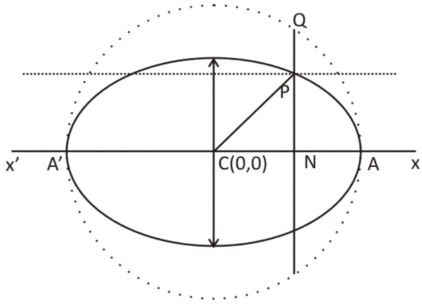

Fig 2

This property of ellipse gives another definition of ellipse. Ellipse is the locus of a point whose sum of distances from two fixed points is always the same constant. Fixed points are foci and sum is major axis of ellipse.

(9) Circle drawn on major axis as diameter is called auxillary circle. Its equation is $x^2 + y^2 = a^2$.

(10) Relation between a, b and e is

$$b^2 = a^2 (1 - e^2) \Rightarrow e^2 = \frac{a^2 - b^2}{a^2}$$

(11) In fig (2) P is a point on ellipse, PN is its ordinate NP on extension meets auxillary circle in Q. QN is perpendicular on A'A

$$\therefore QN^2 = A'N.AN \qquad ....(\alpha)$$

Now if P is (x_1, y_1) then Q is (x_1, y')

$\therefore$ from (a) $(y')^2 = (a + x_1)(a - x_1) = a^2 - x_1^2$ ....(β)

and from ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \frac{y_1^2}{b^2} = \frac{a^2 - x_1^2}{a^2}$ ....(γ)

from (β) and (g) $\frac{y_1^2 a^2}{b^2} = (y')^2 \Rightarrow y' = \frac{a}{b} y_1$

$\therefore$ If Q is (a cos ϕ, a sinϕ) then P is (a cos ϕ, b sinϕ); (a cosϕ, b sinϕ) are parametric co-ordinates of a point of ellipse.

7.3 When in equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, b > a then 2a is minor axis, 2b major axis.

(i) x = 0, axis of ellipse

(ii) $e^2 = (b^2 - a^2) / b^2$

(iii) foci $(0, \pm be)$

(iv) Directrix $y = \pm\frac{b}{e}$

(v) Length of latus-rectum $\frac{2a^2}{b}$, ends $\left(\pm\frac{a^2}{b}, \pm be\right)$

(vi) Focal distance of a point $(x_1, y_1) = b + ey_1$

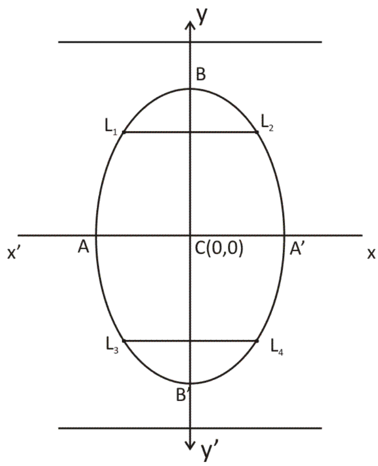

Fig 3

7.4 Radius Vector

The line joining centre with a point on ellipse is called radius – vector of that point of ellipse. In fig (2) CP is radius vector of point P.

If P is $(a \cos \phi, b \sin \phi)$ then

$$CP^2 = r^2 = a^2 \cos^2 \phi + b^2 \sin^2 \phi.$$

At A, r= a, at B, r = b

7.5 Position of a point with respect to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The ellipse is $b^2x^2 + a^2y^2 = a^2b^2$. The point (x_1, y_1) is inside, or on or outside ellipse if

$$(b^2x^2 + a^2y^2 - a^2b^2) < 0 \text{ or } 0 \text{ or } > 0$$

Solved Example

Example 1 : Find eccentricity co-ordinate of foci, length of latus-rectum and equation of directrix, of ellipse $4x^2 + 9y^2 = 16$.

Sol. : Equation of ellipse $\frac{x^2}{4}+\frac{y^2}{(16/9)}=1$

$$a^2=4, b^2=\frac{16}{9}, c^2=\frac{4-16/9}{4}=\frac{5}{9}$$

$$\therefore e=\sqrt{5}/3$$

(i) foci $(\pm ae,0)$ i.e. $\left(\frac{2\sqrt{5}}{3},0\right),\left(-\frac{2\sqrt{5}}{3},0\right)$

(ii) Length L.R. $\frac{2b^2}{a}=\frac{2.16}{2.9}=\frac{16}{9}$

(iii) Directrix $x=\pm\frac{a}{e}=\pm\frac{2.3}{\sqrt{5}}\Rightarrow\sqrt{5}x\mp 6=0$

Example 2 : Find eccentricity foci, length of L.R. its equation and directrix of $25x^2 + 16y^2 = 400$.

Sol. : Equation of ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ $\quad b > a$

(i) $e^2=\frac{b^2-a^2}{b^2}=\frac{25-16}{25}=\frac{9}{25}, e=3/5$

(ii) foci $(0,\pm be)$ i.e. $(0,\pm 3)$

(iii) L.R. $=\frac{2a^2}{b}=\frac{32}{5}$ Equation $y=\pm be\Rightarrow y=\pm 3$

(iv) Directrix $y=\pm\frac{b}{e}$ i.e. $y=\pm\frac{25}{3}$

Example 3 : Find foci, eccentricity, and equations of directrix and L.R. of ellipse $4x^2 + 9y^2 - 24x - 72y + 144 = 0$

Sol. : Ellipse $4x^2 - 24x + 9y^2 - 72y + 144 = 0$

Or $\quad 4(x^2-6x+9)+9(y^2-8y+16)=36$

i.e. $\quad \frac{(x-3)^2}{9}+\frac{(y-4)^2}{4}=1$

$$e^2 = \frac{9-4}{9} = \frac{5}{9} \text{ i.e. } e = \sqrt{5}/3; ae = \sqrt{5}$$

Transferring origin to (3, 4) equation of ellipse is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Transferred origin	Original origin
(i) foci $\left(\pm\sqrt{5},0\right)$	$\left(\pm\sqrt{5}+3,4\right)$
(ii) Equation L.R. X = $\pm\sqrt{5}$	$x - 3 = \pm\sqrt{5} \Rightarrow x = 3\pm\sqrt{5}$
(iii) Directrix X = $\pm\frac{9}{\sqrt{5}}$	$x-3=\pm\frac{9}{\sqrt{5}} \Rightarrow x = 3\pm\frac{9}{\sqrt{5}}$
(iv) axis Y = 0	Y – 4 = 0

Example 4 : Find equation of ellipse whose focus is (3, 4) eccentricity 2/3 and directrix 3x + 4y = 5

Sol. : Let (h, k) be any point on ellipse

$$\therefore (h-3)^2 + (k-4)^2 = \left(\frac{2}{3}\right)^2\left(\frac{3h+4k-5}{5}\right)^2$$

is equation of ellipse. Simplifying

$$225\left[h^2 + k^2 - 6h - 8k + 25\right] = 4\left[9h^2 + 16k^2 + 24hk - 30h - 40k + 25\right]$$

$$\Rightarrow 189h^2 - 96hk + 161k^2 - 1230h - 1640k + 6525$$

$\therefore$ Ellipse $189x^2 - 96xy + 161y^2 - 1230x - 1640y + 6525 = 0$

Example 5 : Find eccentricity of ellipse if length of latus rectum is equal to semi-major axis.

Sol. : Given $\frac{2b^2}{a} = a \Rightarrow 2b^2 = a^2$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{2b^2 - b^2}{2b^2} = \frac{1}{2} \Rightarrow e = \sqrt{\frac{1}{2}}$$

Example 6 : Ends of chord PQ of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are $P(\phi_1)$, $Q(\phi_2)$. Find condition if (i) PQ subtends a right angle at vertex (ii) PQ subtends a right angle at centre (iii) PQ passes through a focus.

Sol. : P is $a\cos\phi_1$, $b\sin\phi_1$, Q $(a\cos\phi_2, b\sin\phi_2)$ vertex A is (–a, 0),

slope PA = $m_1 = \frac{b\sin\phi_1}{a(\cos\phi_1+1)}$

Slope of PQ, $m_2 = \frac{b\sin\phi_2}{a(\cos\phi_2+1)}$

$\angle$PAQ = $90^0 \Rightarrow m_1.m_2 = -1 \Rightarrow \frac{b^2\sin\phi_1\sin\phi_2}{a^2(\cos\phi_1+1)(\cos\phi_2+1)} = -1$

$$\Rightarrow \frac{4b^2\sin\frac{\phi_1}{2}\cos\frac{\phi_1}{2}.\sin\frac{\phi_2}{2}.\cos\frac{\phi_2}{2}}{4a^2\cos^2\frac{\phi_1}{2}.\cos^2\frac{\phi_2}{2}} = -1$$

$$\Rightarrow \frac{b^2}{a^2}\tan\frac{\phi_1}{2}.\tan\frac{\phi_2}{2} = -1$$

$$\Rightarrow \tan\frac{\phi_1}{2},\tan\frac{\phi_2}{2} + \frac{a^2}{b^2} = 0$$

(ii) Centre C is (0, 0) slope PC = $m_1 = \frac{b\sin\phi_1}{a\cos\phi_1}$

Slope QC = $m_2 = \frac{b\sin\phi_2}{a\cos\phi_2}$

$\angle PCQ = 90 \qquad \therefore \frac{b^2}{a^2}\frac{\sin\phi_1\sin\phi_2}{\cos\phi_1\cos\phi_2} = -1$

i.e. $\tan\phi_1.\tan\phi_2 = -\frac{a^2}{b^2}$

(iii) If PQ goes through focus S, then

Slope of PS = slope of SQ

$$\Rightarrow \frac{b\sin\phi_1}{a(\cos\phi_1-e)} = \frac{b\sin\phi_2}{a(\cos\phi_2-e)}$$

$$\Rightarrow \sin\phi_1\cos\phi_2 - \sin\phi_2\cos\phi_1 = e(\sin\phi_1 - \sin\phi_2)$$

$$\Rightarrow \sin(\phi_1-\phi_2) = e(\sin\phi_1 - \sin\phi_2)$$

$$\Rightarrow 2\sin\left(\frac{\phi_1-\phi_2}{2}\right)\cos\left(\frac{\phi_1-\phi_2}{2}\right) = e.2\cos\frac{\phi_1+\phi_2}{2}.\sin\frac{\phi_1-\phi_2}{2}$$

$$\Rightarrow \frac{\cos(\phi_1-\phi_2)/2}{\cos(\phi_1+\phi_2)/2} = e$$

$$\Rightarrow \frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1-\phi_2}{2}\right)+\cos\left(\frac{\phi_1+\phi_2}{2}\right)}=\frac{e-1}{e+1}$$

$$\Rightarrow \frac{2\sin\phi/2\sin\phi_2/2}{2\cos\phi_1/2\cos\phi_2/2}=\frac{e-1}{e+1}$$

$$\Rightarrow \tan\frac{\phi_1}{2}.\tan\frac{\phi_2}{2}=\frac{e-1}{e+1}$$

Example 7 : P, is a variable point on ellipse $b^2x^2 + a^2y^2 = a^2b^2$ with foci S_1 and S_2. If A is the area of triangle PS_1S_2, then find maximum value of A.

Sol. : $$A=\frac{1}{2}\begin{vmatrix} ae & 0 & 1 \\ -ae & 0 & 1 \\ a\cos\phi & b\sin\phi & 1 \end{vmatrix} =\frac{1}{2}[-eab\sin\phi-aeb\sin\phi]$$

$$=|eab\sin\phi|$$

For maximum A, $\sin f$ should be maximum = 1

$\therefore$ Maximum value of A = e ab sq. units

Practice Worksheet (Foundation Level) – 7(a)

1. Find equation of ellipse if

(a) major axis is $\sqrt{ab}$ and minor axis $\sqrt{a/b}$ a>b

(b) distance between foci is 4, and between directrix is 9.

(c) Latus rectum is 5 and eccentricity 2/3

(d) Focus (–1, 1) directrix x – y + 4 = 0 and $e = \frac{1}{\sqrt{2}}$

2. The ends of a chord 8 m long are tied to separate pegs A and B, 4m apart. A boy with a pencil traces the ellipse by moving the pencil in contact of the string and keeping it tight. The ellipse traced is …………

3. The eccentricity of ellipse when length of latus rectum is equal to half of minor axis is ……..

4. The straight lines joining one end of minor axis with the two foci include an angle of 90^0, eccentricity of ellipse is ………..

5. The focal distance of one end of minor axis is k and distance between two foci is 2h, then major and minor axes are …………

6. Eccentricity of ellipse $4x^2 + 3y^2 = 4$, is ……..

7. Find eccentricity, foci and latus rectum of ellipse $5x^2 + 4y^2 = 2$.

8. Find eccentricity, foci directrix and latus rectum of ellipse $4x^2 + 9y^2 - 4x - 6y - 34 = 0$.

9. The distance between two foci is half of the distance between two directrix and latus rectum is 4, then equation of ellipse is ……………

10. The minor axis of an ellipse is equal to half of the major axis and axes of ellipse are y = 0, x = 0 find eccentricity of ellipse.

11. AA' and BB' are major and minor axes of an ellipse of length 2a and 2b; but B" is image of B in x + y = 0 and A" image of A in x – y = 0, AB" is :

(a) $\sqrt{a^2+b^2}$ (b) $2\sqrt{a^2+b^2}$ (c) $\sqrt{2(a^2+b^2)}$ (d) a + b

12. PN is ordinate of point P on ellipse $16x^2 + 25y^2 = 400$ and Q is the corresponding point on auxillary circle. QN = $\sqrt{3}$ PN then eccentricity of ellipse is

(a) $\sqrt{3}/2$ (b) $1/\sqrt{3}$ (c) $\sqrt{2/3}$ (d) 1/3

13. In an ellipse distance between two foci is the geometric mean of the length of two axes. Then :

(a) $e^4 + e^2 - 2 = 0$ (b) $e^4 + e^2 - 1 = 0$

(c) $e^4 - e^2 + 1 = 0$ (d) none of these

14. The line joining point P on auxillary circle of ellipse to its centre is inclined at 30^0 with major axis. Ellipse is $x^2 + 2y^2 = 4$, corresponding point P' on ellipse.

(a) $\left(\sqrt{3}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(\sqrt{3}, \frac{1}{2}\right)$ (c) $\left(\sqrt{3}, -\frac{1}{\sqrt{2}}\right)$ (d) $\left(\sqrt{3}, \sqrt{2}\right)$

15. The radius vector of point $\phi(60^0)$ on ellipse is $\sqrt{5/2}$. If eccentricity of ellipse is $\frac{1}{\sqrt{2}}$ then its equation is :

(a) $3x^2 + 4y^2 = 12$ (b) $4x^2 + 8y^2 = 16$

(c) $x^2 + 2y^2 = 4$ (d) $2x^2 + 5y^2 = 10$

7.6 Tangent :

(a) at point (x_1, y_1)

If we proceed, as done in circle, its equation is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

(b) at point ϕ, Point is $(a\cos\phi, b\sin\phi)$

Tangent is $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$

It is used as equation of any tangent of ellipse

(c) Equation in terms of m; Straight line y = mx + c shall be tangent to ellipse $b^2x^2 + a^2y^2 = a^2b^2$ if roots of equation $b^2x^2+a^2(mx+c)^2-a^2b^2=0$

$$\Rightarrow (b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$$

are equal i.e. $4a^4m^2c^2=4(b^2+a^2m^2)a^2(c^2-b^2)$

$$\Rightarrow a^2m^2c^2=b^2c^2-b^4+a^2m^2c^2-a^2m^2b^2$$

$$\Rightarrow b^2c^2=b^4+a^2b^2m^2$$

$$\Rightarrow c=\pm\sqrt{a^2m^2+b^2}$$

Tangent is $y=mx+c\sqrt{a^2m^2+b^2}$

This equations is also used as equation of any tangent, when needed.

(d) Point of contact : Let (x_1, y_1) be the point of contact

$\therefore$ tangent $\quad b^2xx_1+a^2yy_1-a^2b^2=0$ ....(1)

Our tangent is $\quad mx-y+\sqrt{a^2m^2+b^2}=0$ ....(2)

comparing, $\quad \frac{b^2x_1}{m}=\frac{-a^2y_1}{1}=\frac{-a^2b^2}{\sqrt{a^2m^2+b^2}}$

$$\therefore x_1=\frac{-a^2m}{\sqrt{a^2m^2+b^2}},\ y_1=\frac{b^2}{\sqrt{a^2m^2+b^2}}$$

$\therefore$ point of contact is $\left[\frac{-a^2m}{\sqrt{a^2m^2+b^2}},\frac{b^2}{\sqrt{a^2m^2+b^2}}\right]$

7.7 Length of chord.

Length of chord y = mx + c is $|x_1 \sim x_2| \sqrt{1+m^2}$

Where m is slope of chord and $|x_1 \sim x_2|$ is determined from quadratic equation in x obtained with the help of line and curve.

7.8 Equation of pair of tangents from an external point to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Like, circle and parabola.

Equation of pair tangents from (x_1, y_1) to ellipse is $SS' = T^2$

Where $S=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right), S'=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

And $T=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)$

Note : When point is substituted in equation of any tangent $y=mx+\sqrt{a^2m^2+b^2}$, it gives a quadratic equation in m and gives two values of m, namely m_1, m_2. Two tangents are thus obtained.

7.9 Director circle :

It is the locus of a point; tangents from which of ellipse, include an angle of 90^0 i.e. it is the locus of point of intersection of two mutually perpendicular tangents of ellipse.

Any tangent of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Is $y=mx+\sqrt{a^2m^2+b^2}$ ....(1)

Tangent perpendicular to this tangent is

$y=-\frac{x}{m}+\sqrt{\frac{a^2}{m^2}+b^2}$ ....(2)

(1) and (2) $y-mx=\sqrt{a^2m^2+b^2}$

$$my+x=\sqrt{a^2+b^2m^2}$$

Squaring and adding the two equations

$$y^2(1+m^2)+x^2(1+m^2)=a^2(1+m^2)+b^2(1+m^2)$$

$$\Rightarrow x^2+y^2=a^2+b^2$$

This is equation of director circle. Which is concentric with ellipse and auxillary circle.

7.10 Chord of contact :

If from an external point P, PA and PB are tangents of ellipse then AB is called chord of contact.

Let P be (h, k) and ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

PA and PB are tangents. Thus A and B are points of contact.

If A is (x_1, y_1) tangent is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

B is (x_2, y_2) tangent is $\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1$

Both pass through P, (h, k)

$$\therefore \quad \frac{x_1h}{a^2}+\frac{y_1k}{b^2}=1 \quad ....(1)$$

$$\frac{x_2h}{a^2}+\frac{y_2k}{b^2}=1 \quad ....(2)$$

From (1) and (2) it is clear that (x_1, y_1) and (x_2, y_2) lie upon $\frac{hx}{a^2}+\frac{yk}{b^2}=1$

This is the equation of chord of contact of point (h, k)

Solved Example

Example 8 : Find locus of point of intersection of tangents of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at points ϕ and $90^0+\phi$.

Sol. : Equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$;

point f is (a cos ϕ, b sinϕ) tangent $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$.

Point (90+ϕ) is (-a Sinϕ, b cos ϕ) tangent $\frac{-x\sin\phi}{a}+\frac{y\cos\phi}{b}=1$

Squaring and adding both equations.

$$\frac{x^2}{a^2}\left(\cos^2\phi+\sin^2\phi\right)+\frac{y^2}{b^2}\left(\sin^2\phi+\cos^2\phi\right)=1+1$$

Locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$

Example 9 : The equation of ellipse is $25(x-3)^2+16y^2=400$ (a) Find its centre and foci (b) How should the axes be transferred so that ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$.

Sol. : Given equation $25(x-3)^2+16y^2=400$

i.e. $\frac{(x-3)^2}{16}+\frac{y^2}{25}=1, \quad b>a$

transforming origin to (3, 0) equation is

$$\frac{x^2}{16}+\frac{y^2}{25}=1$$

$$e^2=\frac{25-16}{25}=\frac{9}{25}, e=\frac{3}{5}, \text{ be }=5,\frac{3}{5}=3$$

on changed axes, Centre (0, 0), focus (0, 3)

on original axes centre (3, 0) focus (3, $\pm$3)

To get the equation of ellipse transferred to $\frac{x^2}{25}+\frac{y^2}{16}=1$, first transfer origin to (3, 0) and then rotate axes in anticlockwise direction through an angle of 90^0.

Example 10 : Find length and equation of focal radii drawn to point $(4\sqrt{3},5)$ on ellipse $25x^2+16y^2=1600$.

Sol. : Equation of ellipse $\frac{x^2}{64}+\frac{y^2}{100}=1$ b > a

$\therefore e^2=\frac{100-64}{100}=\frac{36}{100}\Rightarrow e=\frac{3}{5},be=10.\frac{3}{5}=6$

foci (0, $\pm$6), point on ellipse $(4\sqrt{3}, 5)$

Length focal radii $\sqrt{\left(4\sqrt{3}\right)^2+1^2}$ and $\sqrt{\left(4\sqrt{3}\right)^2+11^2}$

$\therefore r_1=7,r_2=13$

Equation $y-5=\frac{1}{-4\sqrt{3}}\left(x-4\sqrt{3}\right)\Rightarrow x+4\sqrt{3}y=24\sqrt{3}$

And $y-5=\frac{11}{4\sqrt{3}}\left(x-4\sqrt{3}\right)\Rightarrow 4\sqrt{3}y-11x+24\sqrt{3}=0$

Example 11 : Find c, if y = 4x + c touches ellipse $x^2 + 2y^2 = 8$

Sol. : y = 4x + c touches $x^2 + 2y^2 = 8$

$\therefore x^2+2(4x+e)^2-8=0$

Should give equal values of x.

Equation is $33x^2 + 16cx + (2c^2 - 8) = 0$

$\therefore (16c)^2-4.33.\left(2c^2-8\right)=0$

$256c^2-264c^2+32.33=0$

$\Rightarrow 8c^2-32.33=0 \quad \Rightarrow c=\pm 2\sqrt{33}$

Example 12 : Find the condition that the straight line lx + my + n = 0 is tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol. : $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$ is any tangent to ellipse $b^2x^2 + a^2y^2 = a^2b^2$

Given straight lines $lx+my=-n$

Comparing co-efficients of the two $\frac{\cos\phi}{al}=\frac{\sin\phi}{bm}=\frac{1}{-n}$

$\therefore \cos\phi = -\frac{al}{n}$, $\sin\phi = -\frac{bm}{n}$

and $\cos^2\phi + \sin^2\phi = \frac{a^2l^2}{n^2} + \frac{b^2m^2}{n^2} = 1$

$\therefore$ conditions is $a^2l^2 + b^2m^2 = n^2$

Example 13 : Find condition that $x\cos\alpha + y\sin\alpha = p$ touches ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; Find point of contact as well.

Sol. : Tangent at (x_1, y_1) of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ ....(1)

out straight line $x\cos\alpha + y\sin\alpha = p$ ....(2)

Comparing coefficient $\frac{x_1}{a^2\cos\alpha} = \frac{y_1}{b^2\sin\alpha} = +\frac{1}{p}$

$\therefore$ point of contact is $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$

condition $\left(\frac{a^2\cos\alpha}{p}\right)^2 \Big/ a^2 + \left(\frac{b^2\sin\alpha}{p}\right)^2 \Big/ b^2 = 1$

$\Rightarrow a^2\cos^2\alpha + b2\sin^2\alpha = p^2$

Example 14 : Prove that $y = \frac{1}{\sqrt{3}}x + \sqrt{3}$ touches the ellipse $2x^2 + 3y^2 = 6$; Find point of contact.

Sol. : Substitute value of y from straight line into the equation of ellipse we get

$2x^2 + 3\left(\frac{x+3}{\sqrt{3}}\right)^2 - 6 = 0 \Rightarrow 3(x^2 + 2x + 1)^2 = 0$

$\Rightarrow (x+1)^2 = 0$ perfect square. Straight line is tangent, point of contact x = –1, $\left(-1, \pm\frac{2}{\sqrt{3}}\right)$

Example 15 : Find point of intersection of tangents drawn at the ends of latus-rectum of ellipse $4x^2+3y^2=12$.

Sol. : Ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1,\ e^2=\frac{4-3}{4}$ i.e. e = ½

Foci are (0, $\pm 2.\frac{1}{2}$) i.e. (0, ± 1); $\frac{a^2}{b}=3/2$

Ends of latus rectum in 1st and 2nd quadrant are $\left(\frac{3}{2},1\right)$ and $\left(-\frac{3}{2},1\right)$ [b > a]

Tangents at these points are

$$\left.\begin{aligned}4(3/2)x+3y=12\\-4(3/2)x+3y=12\end{aligned}\right\}\Rightarrow \text{ point of intersection (0, 4), b > a } \therefore \frac{b}{e}=\frac{2}{1/2}=4$$

$\therefore y=4$ is the equation of directrix. Thus tangents meet at the point of intersection of axis and corresponding directrix It is a property of parabola.

Example 16 : The orbit of earth is an ellipse with e = 1/60. Sun is at one of it focus. The major axis is 186 x 10^6 mile. Find longest and shortest distance of earth from sun.

Focus is $\left(\frac{186}{2}\times 10^6\right)\frac{1}{60}\Rightarrow$ foci are $\left(\pm\frac{31}{2}\times 10^5,0\right)$

longest distance $a+ae=93\times 10^6+\frac{31}{2}.10^5$

$=\frac{61\times 31}{2}\times 10^5$ miles

Shortest distance = a – ae = $\frac{59\times 31}{2}\times 10^5$ miles

Example 17 : Find equation of tangents of ellipse 3x^2 + 2y^2 = 6 inclined at 60^0 with x – axis. Find point of contact as well.

Sol. : Equation of ellipse $\frac{x^2}{2}+\frac{y^2}{3}=1$ any tangent $y=mx\pm\sqrt{a^2m+b^2}=mx\pm\sqrt{2m^2+3}$

$m=\tan 60=\sqrt{3}\Rightarrow$ tangents $y=\sqrt{3}x\pm\sqrt{9}$

$\therefore$ tangents are $y=\sqrt{3}x+3, y=\sqrt{3}x-3$

point of contact is $\left[\frac{-a^2m}{\sqrt{a^2m^2+b^2}},\frac{b^2}{\sqrt{a^2m^2+b^2}}\right]$

$\therefore$ points are $\left(-\frac{2\sqrt{3}}{3},1\right),\left(\frac{-2\sqrt{3}}{-3},-1\right)$

Example 18 : Find locus of mid point of segment of the tangent of ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ between axes.

Sol. : Equation of ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ any tangent is $\frac{x\cos\phi}{b}+\frac{y\sin\phi}{a}=1$

It meets axes in A and B.

$\therefore$ A is $\left(\frac{b}{\cos\phi},0\right)$, B is $\left(0,\frac{a}{\sin\phi}\right)$

Let (h, k) be mid point of segment AB

$\therefore h=\frac{b}{2\cos\phi}\Rightarrow \cos\phi=\frac{b}{2h}$ and $k=\frac{a}{2\sin\phi}\Rightarrow \sin\phi=\frac{a}{2k}$

$\sin^2\phi+\cos^2\phi=1\Rightarrow \frac{a^2}{4k^2}+\frac{b^2}{4h^2}=1$

locus is $b^2y^2+a^2x^2=4x^2y^2$

Example 19 : Prove that product of the perpendiculars dropped from foci on any tangent of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is constant and equal to square of semi minor axis.

Sol. : $y=mx+\sqrt{a^2m^2+b^2}=a^2b^2$ is any tangent of ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Foci are $(\pm ae,0)$

$$p_1=\frac{aem+\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}},p_2=\frac{-aem+\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}$$

$$\therefore p_1p_2=\frac{a^2m^2+b^2-a^2e^2m}{(1+m^2)}=\frac{a^2m^2+b^2-m^2(a^2-b^2)}{1+m^2}$$

$$=\frac{b^2(1+m^2)}{1+m^2}=b^2=(\text{semi minor axis})^2$$

Example 20 : Chord PQ of ellipse $x^2/a^2+y^2/b^2=1$ passes through focus s. If eccentricity angle of P and Q be ϕ_1,ϕ_2 Prove.

(a) SP. S'P = $a^2\sin^2\phi_1 + b^2\cos^2\phi_1$

(b) Perimeter of triangle SPQ = 4a

(c) $(a^2-b^2)\cos^2\frac{1}{2}(\phi_1+\phi_2)=a^2\cos^2\frac{(\phi_1-\phi_2)}{2}$

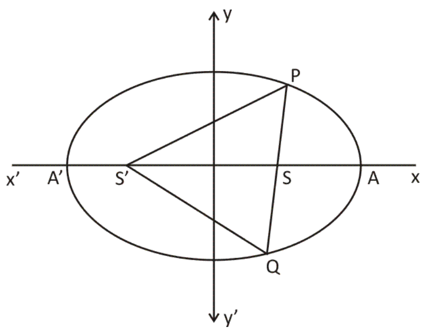

Fig 4

Sol.: (a) SP and S'P are focal distances.

$\therefore$ SP . S'P = $(a + e\,a\cos\phi_1)(a - e\,a\cos\phi_1)$

$= a^2 - e^2a^2\cos^2\phi_1 = a^2 - (a^2 - b^2)\cos^2\phi_1$

$= a^2(1-\cos^2\phi_1) + b^2\cos^2\phi_1 = a^2\sin\phi_1 + b^2\cos^2\phi_1$

(b) Perimeter $\Delta S'PQ = S'P + PQ + S'Q$

$$= (S'P + PS) + (SQ + S'Q)$$
$$= 2a + 2a = 4a$$

$$= 2a + 2a = 4a$$

(c) Slope of PS = slope of SQ

$$\therefore \frac{b\sin\phi_1}{a(\cos\phi_1 - e)} = \frac{b\sin\phi_2}{a(\cos\phi_2 - e)}$$

$$\Rightarrow \sin\phi_1.(\cos\phi_2) - \cos\phi_1\sin\phi_2 = e(\sin\phi_1 - \sin\phi_2)$$

$$\Rightarrow \sin(\phi_1 - \phi_2) = e(\sin\phi_1 - \sin\phi_2)$$

$$\Rightarrow 2\sin\frac{\phi_1-\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2} = e2\cos\frac{\phi_1+\phi_2}{2}\sin\frac{\phi_1-\phi_2}{2}$$

$$\Rightarrow \quad \cos\frac{\phi_1-\phi_2}{2} = e\cos\frac{\phi_1+\phi_2}{2}$$

$$\Rightarrow \quad e^2\cos^2\frac{\phi_1+\phi_2}{2} = \cos^2\frac{\phi_1-\phi_2}{2}$$

$$\Rightarrow \quad (a^2 - b^2)\cos^2\frac{\phi_1+\phi_2}{2} = a^2\cos^2\frac{\phi_1-\phi_2}{2}$$

Example 21 : The tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at ϕ is chord of its auxillary circle and this chord subtends a right angle at centre. Prove eccentricity of ellipse is $\frac{1}{\sqrt{1+\sin^2\phi}}$.

Sol. : Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(1)

Any tangent is $\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1$ ....(2)

Auxiallyar circle is $x^2 + y^2 = a^2$

This tangent which is chord of auxillary circle subtends a right angle at centre of circle.

∴ Making equation of circle homogeneous with the help of tangent equation we get

$$x^2+y^2-a^2\left(\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}\right)^2=0 \qquad \text{............(3)}$$

co-efficient of x^2 is $1-\cos^2\phi$

co-efficient of y^2 is $1-\frac{a^2}{b^2}\sin^2\phi$

angle between lines is 90^0 $\quad\therefore A + B = 0$

i.e. $\quad 1-\cos^2\phi+1-\frac{a^2}{b^2}\sin^2\phi=0$

$$\Rightarrow \sin^2\phi-\frac{1}{1-e^2}\sin^2\phi=-1$$

$$\Rightarrow e^2\sin^2\phi=1-e^2 \Rightarrow e^2\left(1+\sin^2\phi\right)=1$$

$$\therefore e=\frac{1}{\sqrt{1+\sin^2\phi}}$$

Example 22 : Find length of chord $y=\sqrt{2}x+2$ cutoff by ellipse $x^2 + 4y^2 = 16$

Sol. : Ellipse $x^2 + 4y^2 = 16$, chord $y=\sqrt{2}x+2$

$$\therefore \quad x^2+4\left(\sqrt{2}x+2\right)^2=16$$

$$\Rightarrow 9x^2+16\sqrt{2}x=0$$

$$\therefore x_1=0,\ x_2=-\frac{16}{9}\sqrt{2}$$

$$\therefore \quad |x_1-x_2|=\frac{16}{9}\sqrt{2}$$

Length = $\left(x_1\sim x_2\right)\sqrt{1+m^2}=\frac{16}{9}\sqrt{2}.\sqrt{3}=\frac{16}{9}\sqrt{6}$

Example 23: Find equation of tangents from (1, 2) to ellipse $3x^2 + 2y^2 = 5$, and also angle between them:

Sol: Ellipse $3x^2 + 2y^2-5 = 0$, point (1, 2). Equation of pair of tangents from it is $\left(3x^2+2y^2-5\right)(3+8-5)\ =[3.1x+2.2y-5]^2$

$$\Rightarrow 18x^2+12y^2-30=9x^2+16y^2+24xy-30x-40y+25$$

$$\Rightarrow 9x^2-24xy-4y^2+30x+40y-55=0$$

In it co-efficient of x^2 is 9; of xy, –24, of y^2, –4

$$\therefore \tan\theta = \frac{2\sqrt{144+36}}{9-4} = \frac{2\sqrt{180}}{5} \Rightarrow \theta = \tan^{-1}\left(\frac{12\sqrt{5}}{5}\right)$$

Example 24 : PN is ordinate of a point P on ellipse. The tangent at P meets x –axis in T. Show that circle drawn on NT as diameter intersects auxillary circle orthogonally.

Sol. : Let point P on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, b e (x_1, y_1)

$\therefore$ N is $(x_1, 0)$, Tangent is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

It meets x-axis in T $\Rightarrow$ T is $\left(\frac{a^2}{x_1},0\right)$

$\therefore$ Equation of circle on NT as diameter is $(x-x_1)\left(x-\frac{a^2}{x_1}\right)+y^2 = 0$

$$x^2 - x\left(x_1+\frac{a^2}{x_1}\right)+y^2+a^2=0$$

$$\therefore g_1 = -\frac{1}{2}\left(x_1+\frac{a^2}{x_1}\right); f_1 = 0, c_1 = a^2$$

auxillary circle $x^2 + y^2 = a^2$, $g_2 = 0$, $f_2 = 0$, $c_2 = -a^2$

$\therefore 2(g_1g_2 + f_1f_2) = 0 = c_1 + c_2$

$\therefore$ Circles intersect at right angle.

Example 25 : The portion of straight line $x\cos\alpha + y\sin\alpha = c$ intersected by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ subtends a right angle at centre of ellipse. Show that line touches concentric circle of radius $ab/\sqrt{a^2+b^2}$.

Sol: Making equation of curve (ellipse) homogeneous with the help of equation of straight line joining points of intersection with origin is obtained.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(\frac{x\cos\alpha+y\sin\alpha}{c}\right)^2=0$$

co-efficient of x^2 is $\frac{1}{a^2}-\frac{\cos^2\alpha}{c^2}=0$

co-efficient of y^2 is $\frac{1}{b^2}-\frac{\sin^2\alpha}{c^2}=0$

Since angle between lines is 90^0

$$\therefore \frac{1}{a^2}-\frac{\cos^2\alpha}{c^2}+\frac{1}{b^2}-\frac{\sin^2\alpha}{c^2}=1$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$$

From here, $c^2=\dfrac{a^2b^2}{a^2+b^2}$

Now straight line is $x\cos\alpha+y\sin\alpha=\dfrac{ab}{\sqrt{a^2+b^2}}$

Which is tangent to the circle $x^2+y^2=\dfrac{a^2b^2}{a^2+b^2}$

Straight line touched concentric circle of radius $\dfrac{ab}{\sqrt{a^2+b^2}}$

Example 26: Prove that portion of tangent between curve and directrix subtends an angle of 90^0 at the corresponding focus of ellipse.

Sol : Equation ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$,

Let P be ($a\cos\alpha, y\sin\alpha$), Equation any tangent $\dfrac{x\cos\alpha}{a}+\dfrac{y\sin\alpha}{b}=1$

Tangent meets directrix $x=\dfrac{a}{e}\Rightarrow \dfrac{a}{e}\dfrac{\cos\alpha}{a}+\dfrac{y\sin\alpha}{b}=1$

$$\Rightarrow y=\frac{b}{e\sin\alpha}(e-\cos\alpha)$$

$\therefore$ Tangent meets directrix at T $\left(\dfrac{a}{e},\dfrac{b(e-\cos\alpha)}{e\sin\alpha}\right)$

Slope of SP $=\dfrac{b(e-\cos\alpha)}{e\sin\alpha}=m_1$

Slope of TS $=\dfrac{b(e-\cos\alpha)}{e\sin\alpha\left(\dfrac{a}{e}-ae\right)}$

$$=\frac{b}{a}\frac{(e-\cos\alpha)}{\sin\alpha(1-e^2)}$$

$$\therefore\ m_2=\frac{ba}{a^2\left(1-e^2\right)}.\frac{e-\cos\alpha}{\sin\alpha}=\frac{ba}{b^2}.\frac{e-\cos\alpha}{\sin\alpha}=\frac{a}{b}.\frac{e-\cos\alpha}{\sin\alpha}$$

$$\therefore m_1xm_2=\frac{b(e-\cos\alpha)}{e\sin\alpha}x\frac{a}{b}.\frac{e-\cos\alpha}{\sin\alpha}=-1$$

$$\therefore \angle PST=90^0$$

Example 27: The tangent of ellipse $b^2x^2+a^2y^2=a^2b^2$ meets ellipse $\frac{x^2}{a}+\frac{y^2}{b}=a+b$ in P and Q. Prove that tangents at P and Qare perpendicular to each other. If they meet in T, find the locus of T.

Sol: Any tangent of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \qquad ....(1)$$

It meets $\frac{x^2}{a}+\frac{y^2}{b}=a+b \qquad ....(2)$

In P and Q and tangents at P and Q to ellipse (2) meet in T.

$\therefore$PQ is the chord of contact of T. If T is (h,k), chord of contact is $\frac{xh}{a}+\frac{yk}{b}=a+b$

(1) and (3) are same lines. Comparing co-efficients $\frac{\cos\phi}{h}=\frac{\sin\phi}{k}=\frac{1}{a+b}$

$$\Rightarrow \cos\phi=\frac{h}{a+b},\ \sin\phi=\frac{k}{a+b}$$

and $\cos^2\phi+\sin^2\phi=\frac{h^2}{(a+b)^2}+\frac{k^2}{(a+b)^2}=1$

$\therefore$Locus T , $\quad x^2+y^2=(a+b)^2$

and $(a+b)^2=\left(\sqrt{a(a+b)}\right)^2+\left(\sqrt{b(a+b)}\right)^2$

$$\therefore x^2+y^2=\left(\sqrt{a(a+b)}\right)^2+\left(\sqrt{b(a+b)}\right)^2$$

which is equivalent of director circle of ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$

Hence, angle between tangents is 90^0.

Example 28: Tangents from P to ellipse $b^2x^2 + a^2y^2 = a^2b^2$ make angle θ_1 and θ_2 with x-axis. Find locus of P if

(a) $\tan\theta_1 + \tan\theta_2 = c$ (b) $\tan\theta_1 . \tan\theta_2 = c$ c-constant

Sol: Ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$

any tangent of it is $y = mx + \sqrt{a^2m^2 + b^2}$

Let it pass through point P (h,k)

$\therefore k = mh + \sqrt{a^2m^2 + b^2}$

$\Rightarrow (k - mh)^2 = a^2m^2 + b^2$

$\Rightarrow (k - mh)^2 = a^2m^2 + b^2$

$\Rightarrow m^2(h^2 - a^2) - 2mhk + k^2 - b^2 = 0$

from it $m_1 + m_2 = \frac{2kh}{h^2 - a^2}$, $m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2}$

$\therefore \tan\theta_1 + \tan\theta_2 = \frac{2kh}{h^2 - a^2} = \text{constant } c$

$\therefore \tan\theta_1 . \tan\theta_2 = \frac{k^2 - b^2}{h^2 - a^2} = \text{constant } c$

Locus (i) $c(x^2 - a^2) = 2xy$

(ii) $c(x^2 - a^2) = y^2 - b^2$

Practice Worksheet (Foundation Level) – 7 (b)

1. S is the focus of ellipse, the ordinate through S meets the ellipse in P and auxillary circle in Q; angle QCS is 60^0, CQ is 5, C is centre, then co-ordinates of P are .... And equation of ellipse is ....

2. P is a point of ellipse and its corresponding point on auxillary circle is Q. PN is ordinate of P; PN QN = 2.5 : 3, eccentricity of ellipse is ....

3. The ordinate of point P on ellipse and of corresponding point Q an auxillary circle are 1 and $\sqrt{2}$ respectively. If distance of Q from centre C be 2 then find equation of ellipse and its eccentricity.

4. Find the condition that straight line bx + ay = p touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

5. Find the length of the chord of ellipse cut off from the line y = x + 1. Ellipse is $\frac{1}{4}x^2+y^2=1$. Find also its mid point.

6. The tangent of point ϕ of ellipse. $16x^2 + 11y^2 = 256$ touches the circle $x^2 + y^2 - 2x = 15$. Find ϕ.

7. Tangents equally inclined to axes of ellipse $3x^2 + 4y^2 = 12$ are ....

8. Find the point of ellipse for which tangent at that point is inclined at 60^0 with x-axis. The ellipse is $9x^2 + 16y^2 = 144$.

9. Tangent at point P of ellipse $3x^2 + 4y^2 = 12$ meets major axis produced in T. Perpendicular from centre C on this tangent is CQ. Find P if CQ = $\frac{1}{2}$RT;RT is intercept of tangent on axes.

10. Prove that the locus of the foot of perpendicular from focus on any tangent is

    [Hint : $y=mx+\sqrt{a^2m^2+b^2}$ is any tangent equation of perpendicular from focus on it is $y=-\frac{1}{m}(x-ae)$. Eleminate m from these two.]

11. Find point of intersection of two tangents of ellipse $x^2 + 4y^2 = 4$, one inclined at 30^0 with x-axis and the other inclined at 150^0 with x-axis.

12. Tangents are drawn at points $\phi=30^0$ and 120^0 of ellipse $4x^2 + y^2 = 4$. Find point of intersection.

13. Tangent are drawn at $\phi=30^0$ and 60^0 of ellipse $3x^2 + 4y^2 = 12$. Find their point of intersection.

14. Find locus of foot of perpendicular drawn from centre of ellipse on its any tangent ellipse is $3x^2 + 4y^2 = 12$.

15. Tangents from a point P, outside ellipse $x^2 + 4y^2 = 4$ make angles θ_1 and θ_2 with x-axis. If $\tan\theta_1 - \tan\theta_2 = 2$, find locus of P.

16. P_1 and P_2 are two points on ellipse and Q_1, Q_2 are their respective points on auxiallary circle. Prove straight lines P_1P_2 and Q_1, Q_2 meet on major axis.

17. The tangent of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point. P meets axes in M and N. Prove that $(PM)^2+(PN)^2=2\left(a^2\sin^2\phi+b^2\cos^2\phi\right)\left(2\operatorname{cosec}^2\phi-1\right)$

18. Prove that parabola $y^2 = 2ax$, and ellipse $2x^2 + y^2 = 4a^2$ cut each other at right angle.

19. Prove that locus of foot of perpendicular from centre on any tangent of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\left(x^2+y^2\right)^2=a^2x^2+b^2y^2$.

20. Find equation of common tangents of $9x^2 + 4y^2 = 36$ and $x^2 = 6(y - 1)$.

21. The tangent of ellipse $x^2 + 2y^2 = 4$ meets minor and major axes in Q and P. Find locus of point dividing PQ in the ratio of 2 : 1.

22. An ellipse is described by using a string of length ℓ tried to two pin A and B, (AB = m < l). Find eccentricity of this ellipse.

23. On minor axis of ellipse $16x^2 + 25y^2 = 400$ two points are taken equidistant from centre at distance 3. Perpendicular are dropped from these points on any tangents of ellipse. Find sum of squares of these tangents.

24. The chord of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ joining ϕ_1 and ϕ_2 passes through a focus. Find the value of $\tan\frac{1}{2}\phi_1.\tan\frac{1}{2}\phi_2$.

25. Find equation of chord of ellipse when eccentric angle of its end differ by $\pi/2$.

26. Prove that chord of contact of tangents drawn from any point of directrix of ellipse, $b^2x^2 + a^2y^2 = a^2b^2$ passes through corresponding focus.

27. Prove that the tangents from (x_1, y_1) to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ include angle $\tan^{-1}\left[\left(\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1}\right)\Big/\left(x_1^2+y_1^2-a^2-b^2\right)\right]$

28. Find equation of tangents of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are equally inclined to axes. Find point of contact as well. Also prove that length of perpendicular dropped from centre on any one of them is $\sqrt{(a^2+b^2)/2}$.

29. Find length of chord cut off by x + 3y – 5 = 0 of ellipse $4x^2 + 9y^2 = 36$

30. Find condition that straight line px + qy = n touches the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

31. Tangents of ellipse $4x^2 + 9y^2 = 36$, equally inclined to axes are :

(a) $x \pm y = 3\sqrt{2}$ (b) $x \pm y = \pm 2\sqrt{3}$

(c) $y \pm x + \sqrt{13} = 0$ (d) none of these

32. Length of perpendicular from centre of ellipse $16x^2 + 12y^2 = 192$, on its tangent inclined at 60^0 to y axis is.

(a) $2\sqrt{5}$ (b) $\sqrt{5}$ (c) $\sqrt{15}$ (d) 8

33. P is point on ellipse $16x^2 + 9y^2 = 144$; S_1 and S_2 are foci. Maximum area of ΔPS_1S_2.

(a) 12 sq. m (b) 16 sq.m. (c) 9 sq.u. (d) none of these

34. The tangents at the ends of a latus rectum of ellipse meet major axes at the point

(a) where directrix meets it. (b) at $\left(\frac{a}{e},0\right)$

(c) where corresponding directrix meets it. (d) none of these

35. The chord PQ of ellipse $16x^2 + 25y^2 = 400$ passes through S'. The perimeter of Δ PSQ is:

(a) 28 (b) 20 (c) 16 (d) none of these

36. The portion of tangent between point of contact and directrix of ellipse subtends at the corresponding focus on angle of

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\tan^{-1}\left(\frac{b}{a}\right)$ (d) $\tan^{-1}\left(\frac{a}{b}\right)$

37. B is an end of minor, axis of ellipse, F and F' are foci. If $\angle FBF' = 90^0$ then eccentricity is :

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{2}}$

38. A tangent to ellipse $16x^2 + 25y^2 = 400$ is inclined at 45^0 with x-axis Its distance from nearest focus is :

(a) $\frac{3+\sqrt{41}}{\sqrt{2}}$ (b) $\frac{3-\sqrt{41}}{\sqrt{2}}$ (c) $\frac{\sqrt{41}-3}{\sqrt{2}}$ (d) none of these

39. Tangent of ellipse $9x^2 + 16y^2 = 144$ is inclined at 30^0 with x-axis. Its distance from nearest focus is:

(a) $\frac{\sqrt{43}-\sqrt{7}}{\sqrt{3}}$ (c) $\frac{\sqrt{7}-\sqrt{43}}{\sqrt{3}}$ (c) $\frac{\sqrt{7}-5}{\sqrt{3}}$ (d) $\frac{\sqrt{7}-3}{\sqrt{3}}$

40. The major axes of two ellipse are equal, eccentricities are $\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}}$ then ratio of minor axes is :

(a) $2 : \sqrt{3}$ (b) $\sqrt{3} : 2$ (c) $\sqrt{3} : \sqrt{2}$ (d) 3 : 2

41. Locus of point of intersection of straight lines $\frac{tx}{a}-\frac{y}{b}+t=0$ and $\frac{x}{a}+\frac{ty}{b}+1=0$ is :

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (b) $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ (c) $\frac{x^2}{a}+\frac{y^2}{b}=1$ (d) none of these

42. The portion of straight line $x\cos\alpha+y\sin\alpha=p$ cut off by ellipse $4x^2 + 9y^2 = 36$ subtends a right angle at centre of ellipse then p =

(a) $\pm\frac{4}{\sqrt{3}}$ (b) $\pm\frac{3}{\sqrt{13}}$ (c) $\pm\frac{6}{\sqrt{13}}$ (d) $\pm\frac{2\sqrt{3}}{\sqrt{13}}$

43. p_1 and p_2 are lengths of perpendicular dropped from foci of ellipse $2x^2 + 3y^2 = 6$ on tangent at $\left(\sqrt{2},\frac{\sqrt{2}}{\sqrt{3}}\right)$ of ellipse. Then $p_1 \times p_2$.

(a) 2 (b) 3 (c) $2\sqrt{2}$ (d) none of these

44. Equation chord of ellipse $b^2x^2 + a^2y^2 + a^2b^2$ when eccentric angles of its ends differ by $p/2$:

(a) $\left(\frac{x}{a}+\frac{y}{b}\right)\cos\phi+\left(\frac{y}{b}-\frac{x}{a}\right)\sin\phi=1$

(b) $\left(\frac{x}{a}+\frac{y}{b}\right)\cos\phi+\left(\frac{x}{a}-\frac{y}{b}\right)\sin\phi=1$

(c) $b(\cos\phi-\sin\phi)x+a(\cos\phi+\sin\phi)=ab$

(d) none of these

45. Equation of ellipse when S'BS is equilateral triangle is :

(a) $3x^2 + 4y^2 = 12a^2$ (b) $3x^2 + 4y^2 = 3a^2$

(c) $3x^2 + 4y^2 = 4a^2$ (d) none of these

46. Tangents drawn from (–15, –7) to ellipse $9x^2 + 25y^2 = 225$:

(a) $\begin{matrix} x-4y-13=0 \\ 4x-5y+25=0 \end{matrix}$ (b) $\begin{matrix} x+4y=13 \\ 5x+4y=25 \end{matrix}$

(c) $\begin{matrix} 4x+y-13=0 \\ 5x+5y-25=0 \end{matrix}$ (d) $\begin{matrix} x-4y+13=0 \\ 4x-5y-25=0 \end{matrix}$

47. Tangents from a point P to ellipse $b^2x^2 + a^2y^2 = a^2b^2$ make angles θ_1 and θ_2 with major axis locus of P, Q $\tan\theta_1 + \tan\theta_2 = c$ is :

(a) $cx^2 + 2xy = ca^2$ (b) $cx^2 - 2xy = ca^2$

(c) $cx + xy = ca$ (d) none of these

7.11 Normal :

Tangent at (x_1, y_1) on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y-y_1=-\frac{x_1b^2}{a^2y_1}(x-x_1)$ Slope of tangent is $\frac{-x_1b^2}{y_1a^2}$

$\therefore$ Slope of normal is $\frac{a^2y_1}{x_1b^2}$

$\therefore$ Equation of normal at (x_1, y_1) is

$$y-y_1=\frac{a^2}{b^2}.\frac{y_1}{x_1}(x-x_1) \quad ....(\alpha)$$

$$\Rightarrow \frac{b^2(y-y_1)}{y_1}=\frac{a^2}{x_1}(x-x_1)$$

$\therefore$ Normal at $(x_1\ y_1)$ is $\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{y_1}$

(b) Normal at ϕ point $(a\cos\phi, b\sin\phi)$

Normal is $\frac{a^2(x-a\cos\phi)}{a\cos\phi}=\frac{b^2(y-b\sin\phi)}{b\sin\phi}$

$$\Rightarrow \frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2$$

$$\Rightarrow ax\sec\phi-by\,\text{cosec}\phi=a^2-b^2$$

(c) Normal in terms of m :

From Equation (α) normal is

$$y=\frac{a^2y_1}{b^2x_1}.x-y_1\left(\frac{a^2}{b^2}-1\right)$$

Now $m=\frac{a^2}{b^2}\frac{y_1}{x_1}\Rightarrow\frac{ay_1}{b^2m}=\frac{x_1}{a}$ ....(i)

Point (x_1, y_1) is on ellipse

$\therefore \frac{y_1^2}{b^2}=1-\frac{x_1^2}{a^2}=1-\frac{a^2y_1^2}{b^4m^2}$ from (i)

$$\Rightarrow \frac{y_1^2}{b^2}+\frac{a^2y_1^2}{b^4m^2}=1$$

$$\Rightarrow y_1 = b^2m / \sqrt{a^2 + b^2m^2}$$

$$\text{and } y_1\left(\frac{a^2}{b^2} - 1\right) = \frac{b^2m.\left(a^2 - b^2\right)}{b^2\sqrt{a^2 + b^2m}} = \frac{m\left(a^2 - b^2\right)}{\sqrt{a^2 + b^2m^2}}$$

$\therefore$ Equation of normal is $y = mx - \dfrac{m\left(a^2 - b^2\right)}{\sqrt{a^2 + b^2m^2}}$

point is $\left(\dfrac{a^2}{\sqrt{a^2 + b^2m^2}}, \dfrac{b^2m}{\sqrt{a^2 + b^2m^2}}\right)$

7.12 Polar :

Locus of point of intersection of tangents at the ends of chords drawn through a fixed point is called polar of fixed point with respect to ellipse.

Equation of polar of (x_1, y_1) with respect to ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$

Note : There is similarity in equation of tangent at (x_1, y_1) chord of contact of (x_1, y_1) and polar of (x_1, y_1) with respect to the same ellipse.

(a) When point is out side ellipse, the equation gives equation of chord of contact. The polar of the point is chord of contact.

(b) When point is on ellipse, polar is tangent at (x_1, y_1)

(c) When point is inside ellipse, the equation of polar is equation of polar only.

Corollary I : The polar of focus (ae, 0) is $\dfrac{xae}{a^2} = 1$ i.e. $x = \dfrac{a}{e}$ (directrix)

Corollary II : If polar of point P passes through point Q, then polar of Q shall pass through P.

Corollary III : The point of intersection of polars of P and Q with respect to same ellipse is the pole of straight line PQ.

Collary IV : Chord of ellipse is parallel to the polar of its mid point.

Solved Examples

Example 29: A normal inclined at 45^0 with x-axis meets axes in A and B of ellipse $3x^2 + 4y^2 = 12$. Find area of ΔCAB.

Sol. : Ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$

And Normal is $y=mx+\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}} \Rightarrow y=mx-\frac{m(4-3)}{\sqrt{4+3m^2}}$

m = 1, $\therefore$ Normal is $y=x-\frac{1}{\sqrt{7}}$ meet axes is A and B

$\therefore$ A is $\left(\frac{1}{\sqrt{7}},0\right), B=\left(0,-\frac{1}{\sqrt{7}}\right)$

area $\Delta CAB=\frac{1}{2}CA\times CB=\frac{1}{2}.\frac{1}{\sqrt{7}}.\frac{1}{\sqrt{7}}=\frac{1}{14}$ sq. u

Example 30. Find point of intersection of normals drawn at the ends of latus-rectum of ellipse $9x^2 + 16y^2 = 144$. Find also angle between these normals.

Sol. : Ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

$\therefore e^2=\frac{16-9}{16} \Rightarrow e=\frac{\sqrt{7}}{4}; ae=\sqrt{7}$ and $\frac{b^2}{a}=\frac{9}{4}$

$\therefore$ Ends of a latus rectum are $\left(\sqrt{7},\frac{9}{4}\right),\left(\sqrt{7}-\frac{9}{4}\right)$

Normal at $\left(\sqrt{7},\frac{9}{4}\right)$ is $\frac{16}{\sqrt{7}}\left(x-\sqrt{7}\right)=\frac{9}{9/4}\left(y-\frac{9}{4}\right)$

$\Rightarrow 16x-4\sqrt{7}y=-9\sqrt{7}+16\sqrt{7}=7\sqrt{7}$ ....(1)

and normal at $\left(\sqrt{7},-\frac{9}{4}\right)$ is $\frac{16}{\sqrt{7}}\left(x-\sqrt{7}\right)=-4\left(y+\frac{9}{4}\right)$

$\Rightarrow 16x+4\sqrt{7}y=7\sqrt{7}$ ....(2)

Point of intersection, y = 0, x = $\frac{7\sqrt{9}}{16};\left(\frac{7\sqrt{7}}{16},0\right)$

Slopes of normals are $\frac{16}{4\sqrt{7}}$ and $-\frac{16}{4\sqrt{7}}$

$\therefore$ angle between them $\tan\theta = \dfrac{32}{4\sqrt{7}\left(1-\dfrac{256}{112}\right)} = \dfrac{-8\sqrt{7}}{9}$

$\therefore \theta = \tan^{-1}\left(\dfrac{-8\sqrt{7}}{9}\right)$

Example 31 : Find point of intersection of straight line 2x + y = 3 with the curve $4x^2 + y^2 = 5$, obtain equations of normals at these points. Find also point of intersection of these normals. (R)

Sol. : Ellipse $4x^2 + y^2 = 5$, straight line y = 3 –2x

$\therefore 4x^2 + (3-2x)^2 - 5 = 0 \quad \Rightarrow \quad 8x^2 - 12x + 4 = 0$

$\Rightarrow 2x^2 - 3x + 1 = 0 \Rightarrow (2x-1)(x-1) = 0$

$\therefore\ x = 1/2, 1,\ y = 3-2x,\ y = 2, 1$

points are $\left(\dfrac{1}{2}, 2\right), (1,1)$

normals $\dfrac{5}{4}\left(\dfrac{x-1/2}{1/2}\right) = 5\dfrac{(y-2)}{2} \Rightarrow x - \dfrac{1}{2} = y - 2$

and $\dfrac{5}{4}\left(\dfrac{x-1}{1}\right) = 5\left(\dfrac{y-1}{1}\right) \Rightarrow x - 1 = 4y - 4$

$\therefore$ Norms are $y - x = 3/2$ and $4y - x = 3$

point of intersection 3y = 3 – 3 /2, $y = \dfrac{1}{2}, x = -1$

points $\left(-1, \dfrac{1}{2}\right)$

Example 32 : Find equation that $lx + my + n = 0$ is normal to ellipse

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (b) $16x^2 + 9y^2 = 144$

Sol. : Any normal to ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is

$ax \sec\phi - by \operatorname{cosec}\phi + (b^2 - a^2) = 0$

our line $lx + my + n = 0$

$$\therefore \frac{a\sec\phi}{l}=\frac{-b\operatorname{cosec}\phi}{m}=\frac{b^2-a^2}{n}$$

$$\therefore \cos\phi=\frac{an}{l(b^2-a^2)},\sin\phi\frac{-bn}{m(b^2-a^2)}$$

$$\therefore \cos\phi=\frac{-an}{l(b^2-a^2)},\sin\phi=\frac{-bn}{m(b^2-a^2)}$$

$$=\frac{-an}{l(a^2-b^2)} \qquad =\frac{bn}{m(a^2-b^2)}$$

$$\therefore \frac{a^2n^2}{l^2}+\frac{b^2n^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$$

(ii) Ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$

any normal. $3x\sec\phi - 4y\operatorname{cosec}\phi = -7$

line equation $lx + my = -n$

$$\therefore \frac{3}{l\cos\phi}=\frac{-4}{m\sin\alpha}=\frac{7}{n}\Rightarrow \cos\phi=\frac{3n}{7l},\sin\phi=\frac{-4n}{7m}$$

$$\therefore \frac{9}{l^2}+\frac{16}{m^2}=\frac{49}{n^2}$$

Example 33 : The normal at any point. P on ellipse $b^2x^2+a^2y^2=a^2b^2$ meets the axes in G and g. Prove (a) PG : Pg = $b^2 : a^2$ (b) $a^2\,(CG)^2 + b^2\,(Cg)^2 = (a^2-b^2)^2$

Sol. : Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Let P be $(a\cos\phi, b\sin\phi)$

Normal is $ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2$

$$\therefore G \text{ is } \left(\frac{a^2-b^2}{a}\cos\phi, 0\right) \text{ g is } \left(0, \frac{a^2-b^2}{-b}\sin\phi\right)$$

(a) $PG = \sqrt{\left(a-\frac{a^2-b^2}{a}\right)^2\cos^2\phi+b^2\sin^2\phi}=\sqrt{\frac{b^4}{a^2}\cos^2\phi+b^2\sin^2\phi}$

$$=\frac{b}{a}\sqrt{b^2\cos^2\phi+a^2\sin^2\phi}$$

$Pg = \sqrt{a^2\cos^2\phi+\left(b+\frac{a^2-b^2}{b}\right)^2\sin^2\phi}=\sqrt{a^2\cos^2\phi+\frac{a^4}{b^2}\sin^2\phi}$

$$=\frac{a}{b}\sqrt{b^2\cos^2\phi+a^2\sin^2\phi}$$

$$\therefore PG:Pg=\frac{b}{a}:\frac{a}{b}=b^2:a^2$$

(b) $a^2(CG)^2=a^2\left(\frac{a^2-b^2}{a}\right)^2\cos^2\phi=\left(a^2-b^2\right)^2\cos^2\phi$

$$b^2(cg)^2=b^2\left(\frac{a^2-b^2}{-b}\right)^2\sin^2\phi=\left(a^2-b^2\right)^2\sin^2\phi$$

$$\therefore\ a^2(CG)^2+b^2(Cg)^2=\left(a^2-b^2\right)^2\left(\cos^2\phi+\sin^2\phi\right)=\left(a^2-b^2\right)^2$$

Example 34 : Tangent and normal at point P on ellipse meet minor axis in T and G. Find angle subtended by TG at focus of ellipse.

Sol. : Let P, be point ϕ, on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Tangent at P is $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$ ....(i)

Normal at P is $\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2$ ....(ii)

Tangent meets minor axis of T, $\left(0,\frac{b}{\sin\phi}\right)$

Normal meets minor axis in G $\left(0,\frac{b^2-a^2}{b}\sin\phi\right)$

Focus S' is (ae, 0)

Slope S' T = $\frac{b}{\sin\phi(-ae)}=m_1$

Slope of S'G = $\frac{\left(b^2-a^2\right)\sin\phi}{b(-ac)}=\frac{ae\sin\phi}{b}=m_2$

$$m_1\times m_2=\frac{b}{\sin\phi(-ae)}\times\frac{ae\sin\phi}{b}=-1$$

$\therefore$ TG subtends a right angle at focus.

Example 35 : Find pole of chord of ellipse $9x^2 + 16y^2 = 144$, whose mid point is (2,3)

Sol. : Equation of ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

Equation of chord is $y-3=m(x-2)$ ....(i)

It shall be parallel to the polar of its mid point (2,3) i.e. parallel to the polar of its mid point (2,3) i.e. parallel to $\frac{x}{8}+\frac{y}{3}=1$

m of chord = slope of this polar = $-\frac{3}{8}$

$\therefore$ Equation chord is $y-3=-\frac{3}{8}(x-2)$

$$\Rightarrow 3x+8y=30 \qquad ....(\alpha)$$

Let (x_1, y_1) be its pole. Polar is

$$9xx_1 + 16yy_1 = 144 \qquad ....(\beta)$$

(α) and (β) should be same

$$\therefore \qquad \frac{9x_1}{3}=\frac{16y_1}{8}=\frac{144}{30}=\frac{24}{5}$$

$$\therefore \qquad x_1=\frac{8}{5},\quad y_1=\frac{12}{5}$$

Example 36 : P is a point on ellipse whose centre is C. Find angle between CP and normal at P.

Sol. : Let ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and point P, ϕ

Normal at P is $ax\sec\phi-$ by cosec ϕ = a^2-b^2

Slope of normal is $m_1=\frac{a}{b}\frac{\sin\phi}{\cos\phi}=\frac{a}{b}\tan\phi$

Slope of CP = $\frac{b}{a}\tan\phi$

If angle between normal and CP be θ.

Then $$\tan\theta=\frac{(a^2-b^2)\tan\phi}{ab(1+\tan^2\phi)}$$

$$\Rightarrow \tan\theta=\frac{(a^2-b^2)}{2ab}.\frac{2\tan\theta}{1+\tan^2\theta}=\frac{a^2-b^2}{2ab}.\sin 2\phi$$

Example 37 : P and Q are corresponding points on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its auxillary circle respectively. Normal at P meets CQ in R, C is centre of ellipse. Prove CR = (a + b)

Sol. : Let P be $(a\cos\theta, b\sin\phi)$ then Q is (a cos θ, asin ϕ)

Normal at P is $ax\sec\theta - by\ \mathrm{cosec}\theta = a^2-b^2$ ....(i)

Equation of CQ is $y = x\tan\phi$ ....(ii)

Putting value of y from (ii) in (i)

$$ax\sec\theta - \frac{b.x\tan\phi}{\sin\theta} = a^2-b^2$$

$$(ax-bx)\sec\theta = a^2-b^2$$

$$\Rightarrow x = (a+b)\cos\phi, y = (a+b)\sin\phi$$

$$\therefore (CR)^2 = (a+b)^2(\cos^2\phi+\sin^2\phi) \Rightarrow CR = (a+b)$$

Example 38 : Normal at P (ϕ) of ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ meets the curve again at Q(2ϕ). Prove $\cos\phi = -2/3$

Sol. : Ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$, ϕ is ($\sqrt{14}\cos\theta, \sqrt{5}\sin\phi$)

Normal is $\frac{\sqrt{14}.x}{\cos\theta} - \frac{\sqrt{5}y}{\sin\theta} = 14-5=9$ ....(i)

This meet ellipse again in $(\sqrt{14}\cos 2\theta, \sqrt{5}\sin 2\phi)$

$$\therefore \frac{\sqrt{14}.\sqrt{14}\cos 2\theta}{\cos\theta} - \frac{\sqrt{5}.\sqrt{5}\sin 2\theta}{\sin\phi} = 9$$

$$\therefore 14(2\cos^2\theta - 1) - 5.2\cos^2\phi = 9\cos\phi$$

$$\Rightarrow 18\cos^2\theta - 9\cos\phi - 14 = 0$$

$$\Rightarrow 18\cos^2\theta - 21\cos\phi + 12\cos\phi - 14 = 0$$

$$\Rightarrow (6\cos\phi - 7)(3\cos\theta + 2) = 0$$

$\cos\phi \neq 7/6$ $\quad \therefore \cos\theta = -2/3$.

Example 39 : If tangent and normal at any point P on ellipse meet major axis in A and B respectively and if AB =a, prove eccentricity of ellipse is given by $e^2\cos^2\theta + \cos\theta = 1$

Sol. : Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, P (a cos$q$, b sin$q$)

$\therefore$ Tangent $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

It meets major axis in A, $\left(\frac{a}{\cos\theta},0\right)$

Normal at P is $ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2$

It meets major axis in B, $\left(\frac{(a^2-b^2)\cos\theta}{a},0\right)$ i.e. $(ac^2\cos\theta,0)$.

A and B both are on x-axis

$$\therefore AB=\frac{a}{\cos\theta}-ac^2\cos\theta=a$$

$$\Rightarrow c^2\cos^2\theta+a\cos\theta=a$$

Example 40. The normal drawn at the end of a latus-rectum of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the end of minor axis prove $e^4 + e^2 - 1 = 0$.

Sol. : Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ End L_1 of latus rectum $\left(ae,\frac{b^2}{a}\right)$

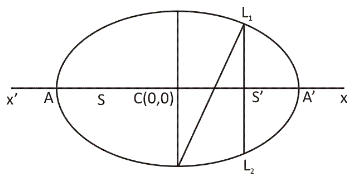

Fig 5

Equation of normal $\frac{a^2(x-ae)}{ae}=\frac{b^2\left(y-\frac{b^2}{a}\right)}{b^2/a}$

It passes through end B' (0, –b) of minor axis

$$\therefore -a^2=a\left(-b-b^2/a\right)$$

$$\Rightarrow -a^2=-ab-b^2 \Rightarrow a^2=ab+b^2$$

$$\Rightarrow a^2=ab+a^2\left(-e^2\right)\Rightarrow ab=a^2e^2$$

$$\Rightarrow b^2=a^2e^4 \Rightarrow a^2\left(1-e^2\right)=a^2e^4$$

$$\Rightarrow e^4+e^2-1=0$$

Example 41 : Prove that the tangent and normal at a point of ellipse bisects the angle between focal radii of the point.

Sol. : In the fig. point P is ($a\cos\phi$, $b\sin\phi$) S and S' are foci; C centre. PT is tangent PN is normal. If normal bisects the angle SPS' them S'M : SM = PS' : PS

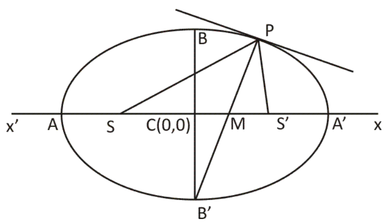

Fig 6

$$\Rightarrow SM : S'M = a(1+e\cos\phi):(1-e\cos\phi)a$$

$$=(1+e\cos\phi):(1-e\cos\phi)$$

Normal at P $\quad ax\sec\phi - by\,\mathrm{cosec}\phi = a^2-b^2$

It meets major axis in M

M is $\left(\dfrac{a^2-b^2}{a}\cos\theta,0\right)$ i.e. $\left[ae^2\cos\theta,0\right]$

$$MS' = ae - ae^2\cos\theta = ae(1-e\cos\phi)$$
$$SM = ae^2\cos\theta-(-ae)=ae(1+e\cos\theta)$$
$$\therefore SM : S'M = (1+e\cos\theta):(1-e\cos\theta)$$

$\therefore$ Normal bisects the angle S'PS.

Example 42 : Two points are taken on major axis of ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, equidistant from centre. The eccentric angles of the ends of chords through these points are α,β,γ and δ. Prove that $\tan\dfrac{\alpha}{2}.\tan\dfrac{\beta}{2}.\tan\dfrac{\gamma}{2}.\tan\dfrac{\delta}{2}=1$.(IIT)

Sol. : Let P(x_1, 0) and Q ($-x_1$, 0) be the two point on major – axis of ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, equidistance from its centre c AB chord goes through P and MN chord goes through Q.

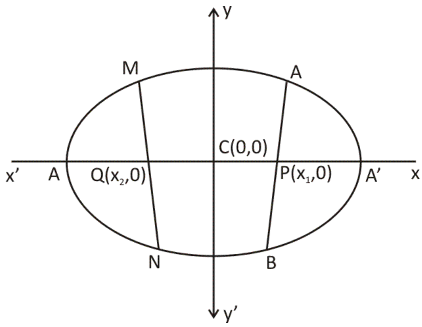

Fig 7

(a) Slope of AP = Slope BP

$$\frac{b\sin\alpha}{a\cos\alpha-x_1}=\frac{b\sin\beta}{a\cos\beta-x_1}$$

$$\Rightarrow a\sin\alpha\cos\beta - a\sin\beta\cos\alpha = x_1(\sin\alpha-\sin\beta)$$
$$\Rightarrow a\sin(\alpha-\beta)=x_1(\sin\alpha-\sin\beta)$$
$$\Rightarrow a.2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}=2x_1\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow x_1 = a\cos\left(\frac{\alpha-\beta}{2}\right)\Big/\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \frac{a\left(\cos\frac{\alpha}{2}.\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}.\sin\frac{\beta}{2}\right)}{\cos\frac{\alpha}{2}.\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}.\sin\frac{\beta}{2}}$$

$$x_1 = a\left(\frac{1+\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}\right) \qquad \text{.....................(A)}$$

for the second chord, slope of MQ = slope of NQ

$$\frac{b\sin r}{a\cos r + x_1} = \frac{b\sin\delta}{a\cos\delta + x_1}$$

$$\Rightarrow a\sin(r-\delta) = -x_1(\sin r - \sin\delta)$$

$$\Rightarrow -x_1 = a\left(\frac{1+\tan\frac{r}{2}.\tan\frac{\delta}{2}}{1-\tan\frac{r}{2}.\tan\frac{\delta}{2}}\right)$$

$$\therefore\ a\left[\frac{1+\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}\right] + a\left[\frac{1+\tan\frac{\gamma}{2}.\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}.\tan\frac{\delta}{2}}\right] = 0$$

on simplification $\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}.\tan\frac{\gamma}{2}.\tan\frac{\delta}{2} = 1$

Example 43 : A circle is concentric with ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that common tangent is inclined at an angle of $\tan^{-1}\left(\sqrt{\frac{r^2-b^2}{a^2-r^2}}\right)$ where r is radius of circle.

Sol. : Ellipse $x^2 / a^2 + y^2 / b^2 = 1$, circle $x^2 + y^2 = r^2$ any tangent of ellipse $y = mx + \sqrt{a^2m^2+b^2}$. It shall be tangent to circle if perpendicular from centre on it = radius of circle

i.e. $$\frac{\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}} = r \Rightarrow a^2m^2 + b^2 = r^2\left(1+m^2\right)$$

$$\Rightarrow m^2\left(a^2-r^2\right) = r^2 - b^2$$

$$\Rightarrow m = \sqrt{\frac{r^2-b^2}{a^2-r^2}} \Rightarrow \theta = \tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$$

Example 44 : Find angle of intersection of curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and circle $x^2 + y^2 = ab$

Sol. : Curves are $b^2x^2 + a^2y^2 = a^2b^2$ and $x^2 + y^2 = ab$.

For point of intersection $b^2x^2 + a^2\left(ab - x^2\right) = a^2b^2$

$$\Rightarrow x^2(b^2 - a^2) = a^2b(b-a)$$

$$\Rightarrow x = \pm a\sqrt{b/(a+b)}, y = \pm b\sqrt{a/(a+b)}$$

one, point of intersection is $\left(\frac{a\sqrt{b}}{\sqrt{a+b}}, \frac{b\sqrt{a}}{\sqrt{a+b}}\right)$

Differentiating curves with respect ot x

(i) $2b^2x + 2a^2y.\frac{dy}{dx} = 0, \frac{dy}{dx} = -\frac{b^2}{a^2}.\frac{x}{y}$

at the point $m_1 = -\frac{b^2}{a^2}.\frac{a\sqrt{b}}{b\sqrt{a}} = -\left(\frac{b}{a}\right)^{3/2}$

(ii) other curve $2x + 2y.\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

at point $\frac{dy}{dx} = -\sqrt{a}/\sqrt{b}$

$$\therefore \tan\theta \frac{-\sqrt{\frac{a}{b}}\left(1 - \frac{b^2}{a^2}\right)}{1 + b/a} = \frac{-a\sqrt{a}(a^2 - b^2)}{a^2\sqrt{b}(a+b)}$$

$$= -\frac{(a-b)}{\sqrt{ab}}$$

acute angle $\theta = \tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$

Example 45 : ABC is an equilateral triangle is inscribed in circle $x^2 + y^2 = a^2$. Centre of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ coincides with the centres of circle. Perpendicular S from ABC on major axis of ellipse meet ellipse in P, Q and R and these lie on the same side of major axis as A, B and C. Prove that normals at P, Q and R are concurrent.

Sol. : Let a be radius of circle; x side of equilateral triangle altitude AD = $\frac{3}{2}a$.

$\therefore$ side of triangle

$$\frac{\sqrt{3}}{2}x = \frac{3}{2}a \Rightarrow x = \sqrt{3}a$$

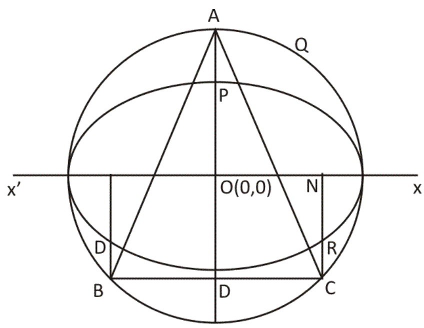

Fig 8

co-ordiantes of A, B, C are (0, a), $\left(-\frac{\sqrt{3}}{a}a,-\frac{a}{2}\right),\left(\frac{\sqrt{3}}{2}a,-\frac{a}{2}\right)$

P is (0, b), Q is $\left(-\frac{\sqrt{3}}{2}a,-\frac{b}{2}\right)$,R$\left(\frac{\sqrt{3}}{2}a,-\frac{b}{a}\right)$

$\left(\because \frac{3/4a^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow\frac{y^2}{b^2}=\frac{1}{4},y=\pm\frac{1}{2}b\right)$

Normal at P is x = 0, y-axis ....(i)

Normal at Q is $\frac{a^2\left(x+\frac{\sqrt{3}}{2}a\right)}{-\sqrt{3}/2a}=\frac{b^2(y+b/2)}{-b/2}$

i.e. $\frac{2}{\sqrt{3}}ax+a^2=2by+b^2$ ....(ii)

Normal at R $\frac{2}{\sqrt{3}}ax-a^2=-2by-b^2$ ....(iii)

Solving (ii) and (iii) x = 0, $y=\frac{a^2-b^2}{2b}$ normals are concurrent.

Example 46 : Perpendicular are drawn from foci of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ on polar of point P and their products is c^2. Prove locus of P is $b^4 x^2 (e^2 + a^2e^2) + e^2 a^4 y^2 = a^4b^4$.

Sol. : Let point P be (x_1, y_1). Polar of (x_1, y_1) is

$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

$\Rightarrow b^2x_1x+a^2y_1y-a^2b^2=0$

p_1 and p_2 are perpendicular on it from foci (ae, 0) and (–ae,0)

$p_1p_2=\frac{(b^2aex_1-a^2b^2)(-b^2aex_1-a^2b^2)}{b^4x_1^2+a^4y_1^2}=c^2$

$\Rightarrow c^2(b^4x_1^2+a^4y_1^2)=-(b^4a^2e^2x_1^2-a^4b^4)$

$\Rightarrow c^2(b^4x_1^2+a^4y_1^2)+b^4a^2e^2x_1^2=a^4b^4$

$\Rightarrow b^4x_1^2(c^2+a^2e^2)+c^2a^4y_1^2=a^4b^4$

locus $b^4x^2(c^2+a^2e^2)+c^2a^4y_1^2=a^4b^4$

Practice Worksheet (Foundation Level) – 7 (c)

1. Find equation of normal at (3, –2) of ellipse $4x^2 + 9y^2 = 72$.
2. Find equation of normal at the end of latus-rectum (fourth quadrant) of ellipse $4x^2 + 9y^2 = 36$.
3. Find pole of straight line $12x + 7y + 16 = 0$ with respect to ellipse $4x^2 + 7y^2 = 8$.
4. Find equation of the chord of ellipse $x^2 + 4y^2 = 36$ whose mid point is (2, 1).
5. Find locus of pole of normals of ellipse $9x^2 + 36y^2 = 36$.
6. Find pole of straight line $lx + my = 1$ with respect to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$.
7. Find locus of mid points of chords that pass through focus (ae, 0) of ellipse $b^2x^2 + a^2y^2 = a^2b^2$.
8. L_1 is end of latus-rectum of ellipse $3x^2 + 4y^2 = 12$, in first quadrant. A is end of major axis in this quadrant. Find pole of L_1A.
9. If polar of (3, 4) with respect to $4x^2 + 9y^2 = 36$ touches the circle $x^2 + y^2 = p^2$, then find p.
10. C is centre of ellipse $b^2 x^2 + a^2 y^2 = a^2b^2$. P is a point on it and its ordinate is PN. Perpendicular from centre on polar of P is Ck; PM is perpendicular from P on its polar. PM produced meets major axis in R and polar meets major axis in T. Prove

    (a) CK. PR = b^2 (b) CK.RM = product of perpendicular from foci on polar

    (c) CR = e^2 CN (c) CT . CN = a^2
11. Straight line x = 3 intersects ellipse $16x^2 + 25y^2 = 400$ in points P and Q of eccentric angles ϕ_1, ϕ_2. Find (a) $\phi_1 + \phi_2$ (b) point of intersection of normals at P and Q.
12. (4, 3) is a point on the auxillary circle of a ellipse. Normal at this point meets ellipse in P and Q; PN is ordinate; the corresponding point of P on auxillary circle is P_1; If P divides P_1N in the ratio of 2 : 3, find equation of ellipse.
13. If normal of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at P(ϕ_1) meets ellipse again at Q(ϕ_2). Prove $e^2 \cos\phi_1 = \sin(\phi_1 - \phi_2)/[\sin\phi_1 + \sin\phi_2]$
14. Find locus of point P if its polar with respect to ellipse $4x^2 + 3y^2 = 12$ touches circle $x^2 + y^2 + 2x = 0$
15. Find common tangents of ellipse $3x^2 + y^2 = 12$ and $x^2 + 4y^2 = 4$.
16. Find condition if $px + qy = k$ is normal to ellipse $b^2x^2 + a^2y^2 = a^2b^2$
17. Find angle of intersection of ellipses $3x^2 + y^2 = 12$ and $x^2 + 4y^2 = 15$.
18. Find locus of point of intersection of two perpendicular tangents of ellipse $4x^2 + y^2 = 12$.

19. A rod of length 14 feet slides between smooth vertical wall and horizontally floor, maintaining end point in contact. Find locus of point which divides the rod in ratio of 3 : 4

20. A tangent of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ meets the tangents drawn at extremities of major axis in P and Q. Prove that circle drawn on PQ as diameter passes through foci of ellipse.

21. Prove that locus of pole w.r.t. to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of any tangent of auxillary circle is the curve $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}$

22. Prove that locus of poles of normal chords of ellipse $9x^2+16y^2=25$ is $\frac{256}{9x^2}+\frac{81}{16y^2}=\frac{49}{25}$.

23. PN is ordinate of a point P on ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and it meets auxiallary circle in Q. Show that point of intersection of normals at P and Q lie on $x^2+y^2=(a+b)^2$

24. Circle of radius r has its centre, the centre of ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Prove that common tangent of these two curves is inclined at $\tan^{-1}\left[\left(\sqrt{r^2-b^2}\right)/\sqrt{\left(a^2-r^2\right)}\right]$ with major axis. Also prove that the length of common tangent is $\sqrt{\left\{\left(r^2-b^2\right)-\left(a^2-r^2\right)\right\}}/r$.

25. Show that area of the triangle formed by joining point ϕ_1, ϕ_2 and ϕ_3 of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=$ is $2ab\sin\frac{\phi_1-\phi_2}{2}\sin\frac{\phi_2-\phi_3}{2}.\sin\frac{\phi_3-\phi_1}{2}$.

26. Normal at point P of ellipse $x^2 + 4y^2 = 4$ is $4x-\frac{2}{\sqrt{3}}y=3$ then point P is :

(a) $\left(1,\frac{\sqrt{2}}{3}\right)$ (b) $\left(1,\frac{-\sqrt{3}}{2}\right)$ (c) $\left(1,\frac{\sqrt{3}}{2}\right)$ (d) $\left(-1,-\frac{\sqrt{3}}{2}\right)$

27. The pole of chord of ellipse $9x^2+16y^2=144$ when mid point of chord is (2, 1) is :

(a) $\left(\frac{72}{13},\frac{36}{13}\right)$ (b) $\left(\frac{36}{13},\frac{72}{13}\right)$ (c) $\left(\frac{36}{13},\frac{18}{13}\right)$ (d) $\left(-\frac{36}{13},\frac{18}{13}\right)$

28. Normal at (3, –2) of ellipse $4x^2 + 9y^2 = 72$ meets axes in M and N. Area of triangle MCN (C, centre of ellipse) is :

(a) $\frac{25}{6}$ (b) $\frac{25}{3}$ (c) $\frac{50}{3}$ (d) $\frac{25}{12}$

29. The normal at point $P\left(\frac{2}{\sqrt{3}}, \sqrt{2}\right)$ of ellipse $3x^2 + 4y^2 = 12$ meets axes in M and N then PM : PN is:

(a) $2\sqrt{2} : \sqrt{3}$ (b) $\sqrt{3} : \sqrt{2}$ (c) 4 : 3 (d) 3 : 4

30. Straight line y = mx +c shall intersect ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in real points if :

(a) $a^2m^2 < (c^2 - b^2)$ (b) $a^2m^2 \geq (c^2 - b^2)$

(c) $a^2m^2 \geq (a^2 - b^2)$ (d) $c \geq b$

31. If tangents from point P to ellipse $x^2 + 4y^2 = 4$ include an angle of 90^0, then locus of P is:

(a) $x^2 - y^2 = 5$ (b) $xy = 5$

(c) $x^2 + y^2 = 5$ (d) $y = 5x$

32. Distance of tangent drawn at P point of ellipse from centre of ellipse is d, F1, F2 are foci, $(PF_1 - PF_2)^2 =$

(a) $4a^2\left(1 - \frac{b^2}{a^2}\right)$ (b) $a^2\left(1 - \frac{b^2}{a^2}\right)$

(c) $4a^2\left(2 - \frac{b^2}{a^2}\right)$ (d) None of these

33. Tangents are drawn from (2, 1) to ellipse $3x^2 + 2y^2 = 5$ include an angle of :

(a) $\tan^{-1}\frac{4\sqrt{3}}{5}$ (b) $\tan^{-1}\frac{2\sqrt{270}}{5}$

(c) $\tan^{-1}\frac{2\sqrt{390}}{5}$ (d) none of these

34. P is a point on ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ such that CS = SP where C is centre and S focus of ellipse, then P is :

(a) $\left(\frac{10}{3}, \frac{4\sqrt{5}}{3}\right)$ (b) $\pm\frac{10}{3}, \frac{4\sqrt{5}}{3}$

(c) $\left(\frac{10}{3}, \frac{5\sqrt{2}}{3}\right)$ (d) $\left(3, \frac{5\sqrt{3}}{3}\right)$

35. P, (ϕ) is a point on ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and $\angle S'PS = 90^0$, then $p^2\sin\phi + a^2\cos^2\phi$ =

(a) a^2e^2 (b) $a^2 - b^2$ (b) $a^2 + b^2$ (d) $2b^2 - a^2$

36. P,(ϕ) is a point on ellipse $x^2/a^2 + y^2/b^2 = 1$ and tangent at P meets major axis in T, then :

(a) PT^2 = TA . TA' (b) PT^2 > PA.PA'

(c) PT^2 < TA . TA' (d) none of these

37. Angle between curve $7x^2 + y^2 = 1$ and $y^2 = 6x$ is :

(a) $\tan^{-1}\left(-\frac{56}{37}\right)$ (b) $\tan^{-1}\left(-\frac{3}{2}\right)$ (c) $\tan^{-1}\left(-\frac{4\sqrt{42}}{15}\right)$ (d) $\tan^{-1}\left(-\frac{4\sqrt{7}}{15}\right)$

38. Locus of point of intersection of tangents at ϕ and $90^0 + \phi$ points of ellipse $4x^2 + 9y^2 = 36$ is

(a) $x^2 + 3y^2 = 24$ (b) $4x^2 + y^2 = 36$

(c) $4x^2 + 9y^2 = 72$ (d) $9x^2 + 4y^2 = 144$

39. Normals at points whose eccentric angles are p /6 and p /3 of ellipse $x^2 + 4y^2 = 12$ meet at :

(a) $\left[\sqrt{3}-1, 2+2\sqrt{3}\right]$ (b) $\left[\left(\sqrt{3}-1\right), 2\left(1-\sqrt{3}\right)\right]$

(c) $\left[\sqrt{3}-1, \sqrt{3}+1\right]$ (d) $\left[\sqrt{3}+1, \sqrt{3}-3\right]$

40. P_1 and P_2 are two points on ellipse and M_1, M_2 are corresponding points on auxiallary circle. P_1P_2 and M_1M_2 meet on :

(a)minor axis (b) directrix (c) major axis (d) none of these

7.13 Sub-tangent and Sub-nomral :

In the fig. PN is ordinate of point P on ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Tangent at P meets major axis in T and normal at P meets major axis in G.

NT = Length of sub-tangent

GN = sub-normal.

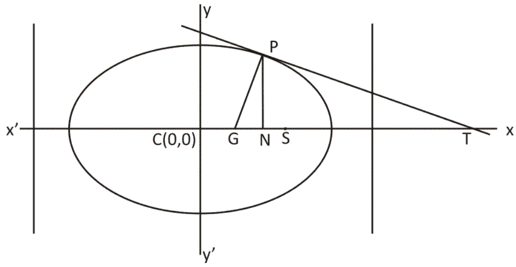

Fig 9

(a) Let $P(x_1, y_1)$, tangent $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

$y = 0 \Rightarrow$ T is $\left(\frac{a^2}{x_1},0\right)$, and N is $(x_1, 0)$

Sub-Tangent NT = $\frac{a^2}{x_1}-x_1=\frac{a^2-x_1^2}{x_1}$

(b) Normal at P is $\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{y_1}$

It meets x axis, y = 0, $G=\left(\frac{a^2-b^2}{a^2}x_1,0\right)\Rightarrow\left(e^2x_1,0\right)$

$\therefore G.N.=x_1-e^2x_1=x_1\left(1-e^2\right)=x_1\left(1-e^2\right)=x_1\frac{b^2}{a^2}$

Subnormal = $x_1\frac{b^2}{a^2}=x_1\left(1-e^2\right)$

7.14 Diameter

The locus of mid points of parallel chords of ellipse is called diameter of ellipse. Let the parallel chords of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be y = m x+λ. Where λ is a variable, In fig. PQ, P'Q', P"Q" are parallel chords. Put, from equation of line value of y in terms of x into the equation of ellipse

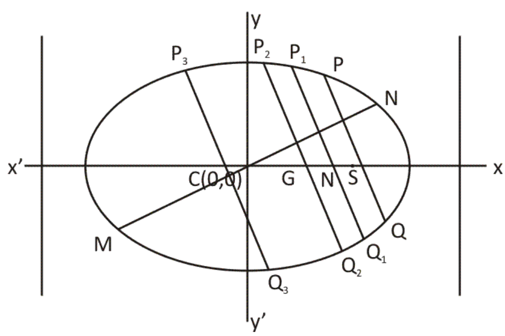

Fig 10

$$\frac{x^2}{a^2}+\frac{(mx+\lambda)^2}{b^2}=1\Rightarrow b^2x^2+\left(m^2x^2+2m\lambda x+\lambda^2\right)a^2=a$$

$$\Rightarrow x^2\left(a^2m^2+b^2\right)+2a^2m\lambda x+a^2\left(\lambda^2-b^2\right)=0$$

$$\therefore x_1+x_2=-\frac{2a^2m\lambda}{a^2m^2+b^2},x_1x_2=\frac{\left(\lambda^2-b^2\right)a^2}{a^2m^2+b^2}$$

If mid point of PQ is (h, k)

then $h = \frac{x_1 + x_2}{2} = \frac{a^2 m\lambda}{a^2 m^2 + b^2}$ ....(1)

point (h, k) lie on the chord $\Rightarrow k = mh + \lambda$

$$\Rightarrow k = mh - \frac{h(a^2m^2 + b^2)}{a^2 m} = -\frac{b^2}{a^2 m}h \quad \text{from (1)}$$

$\therefore$ Equation of diameter is $y = -\frac{b^2}{a^2 m}x$

Diameter passes through centre of ellipse.

7.15 Conjugate Diameter :

Two diameters are conjugate if each diameter bisects the chords parallel to the other diameter.

The locus of mid points of chords y = mx + λ, is the diameter $y = -\frac{b^2}{a^2 m}x$,

Let $m_1 = \frac{-b^2}{a^2 m}$

Locus of mid points of parallel chords to this diameter is $y = m_1 x + \lambda$ is $y = -\frac{b^2}{a^2 m_1}x$

If the two diameters are conjugate then $-\frac{b^2}{a^2 m_1} = m \Rightarrow mm_1 = -\frac{b^2}{a^2}$

$\therefore$ Two diameter $y = m_1x$ and $y = m_2x$ shall be conjugate if $m_1m_2 = -\frac{b^2}{a^2}$

7.16 The tangents at the end of diameter is parallel to the lines bisected by the diameter.

Let y = mx be a diameter of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This bisect the chords $y = m_1x + \lambda$

$$\therefore m = -\frac{b^2}{a^2 m_1} \Rightarrow m_1 = -\frac{b^2}{a^2 m} \quad \text{..............(1)}$$

If diameter meets the ellipse in (x_1, y_1) then

$y_1 = mx_1 \Rightarrow m = \frac{y_1}{x_1}$ ....(2)

Tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

$\Rightarrow y = -\frac{b^2}{y_1}.\frac{x_1}{a^2}x + \frac{b^2}{y_1}$

and from (2) $y = -\frac{b^2}{a^2}.\frac{1}{m}x + \frac{b^2}{y_1}$ from (2)

and from (1) it reduce to y = $m_1x + \frac{b^2}{y_1}$ i.e. tangent is parallel to the chords bisected by the diameter.

7.17 Tangents drawn at the ends of a chord of ellipse intersect at the diameter bisecting chords parallel to this chord.

Let the equation of chord be y = mx + c and suppose tangents drawn at its ends meet in (x_1, y_1)

$\therefore y = mx + c$ is chord of contact of point (x_1, y_1) ....(1)

Equation chord of contact $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ ....(2)

Equation (1) and (2) are same

$\therefore \frac{y_1}{b^2} = \frac{-x_1}{a^2m} \Rightarrow y_1 = -\frac{b^2}{a^2}x_1$

and it proves x_1, y_1 lie on $y = -\frac{b^2}{a^2}x$ which is the equation of diameter of lines parallel to y= mx+c

7.18 Eccentric angles of ends of conjugate diameters differ by a right angle.

Let PP' and QQ' be the two conjugate diameters and eccentric angles of P and Q be ϕ_1, ϕ_2 respectively. Diameters pass through centre C.

Slope of CP = $\frac{b}{a}\tan\phi_1$ and of CQ = $\frac{b}{a}\tan\phi_2$ Diameter are conjugate $\therefore m_1m_2 = -\frac{b^2}{a^2}$

$$\Rightarrow \frac{b^2}{a^2}\tan\phi_1.\tan\phi_2 = -\frac{b^2}{a^2}$$

$$\Rightarrow \sin\phi_1.\sin\phi_2 = -\cos\phi_1.\cos\phi_2$$

$$\Rightarrow \cos(\phi_1 - \phi_2) = 0$$

$$\therefore \phi - \phi_2 = 90^0$$

$\therefore$ If P is (a cos ϕ ,b sin ϕ), then Q is a cos (90 + ϕ), b sin(90+ ϕ) i.e. $(-a\sin\phi, b\cos\phi)$

7.19 Sum of squares of semi-conjugate diameters is constant :

Let CP and CQ be the two semi conjugate diameters; P is (acos ϕ, b sin ϕ), Q(–a sin ϕ, bcos ϕ)

$CP^2 + CQ^2 = a\cos^2\phi + b^2\sin^2\phi + a^2\sin^2\phi + b^2\cos^2\phi$

$= a^2 + b^2 =$ constant

7.20 To find the area of parallelogram formed by tangents draw at the ends of two conjugate diameters.

If fig. PCP' and QCQ' are conjugate diameter. ABCD is the parallelogram formed by tangents drawn at the ends of these. This parallelogram is divided into four equal parts by the diameters.

$\therefore$ area of || gram ABCD = 4. II gram CQBP

= 4.BP ($\perp$ from c on BP)

= 4 . BP . CE

= 4 CQ . CE $(\because CQ = BP)$ ......(1)

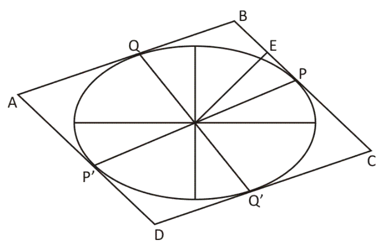

Fig 12

Equation of tangents P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$$\therefore CE = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin\theta}} \qquad .....(2)$$

Q is $(-a\sin\phi, b\cos\phi)$, because P is (a cos ϕ, b sin ϕ)

$$\therefore\ CQ = \sqrt{a^2\sin^2\theta + b^2\cos^2\phi}$$

$\therefore$ from (1) 4CR .CQ = 4 ab = 2a . 2b

= product of axes.

$\therefore$ area of || gram ABCD = 4 ab = constant.

Solved Examples

Example 47 : Find point on ellipse for which sub normal is equal to the square of semi-minor axis of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol. : Given sub normal $\frac{b^2}{a^2}x_1=b^2 \Rightarrow x_1=a^2$

$\therefore \frac{y_1^2}{b^2}=1-\frac{a^4}{a^2}=1-a^2 \Rightarrow y_1=\pm b\sqrt{1-a^2}$

$\therefore$ point is $\left(a^2, \pm b\sqrt{1-a^2}\right)$

Example 48 : Find equation of ellipse if sub-tangent of point (3, 12/5) is 16/3.

Sol.: Given point (3, 12/5), subtangent $=\frac{a^2-x_1^2}{x_1}$

On CF, $y=\frac{b}{a}x$ is $y=-\frac{a}{b}x+\lambda$

i.e. $by+ax=\frac{-b^2}{e}+\frac{a^2}{e}=\frac{-a^2\left(1-e^2\right)}{e}+\frac{a^2}{e}$

i.e. by + ax = a^2e

This intersect y = 0 in (ae, 0) i.e. focus

$\therefore$ Orthocentre of triangle CEF lies at focus of ellipse.

$\therefore \frac{a^2-9}{3}=\frac{16}{3} \Rightarrow a^2=25$

and point is an ellipse $\Rightarrow \frac{9}{25}+\frac{144}{25b^2}=1$

$\Rightarrow 9b^2+144=25b^2 \Rightarrow b^2=9$

$\therefore$ Ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$

Example 49 : Prove that product of focal distance of a point P, is equal to the square of that semi-diameter which is parallel to the tangents at P.

Sol. : Let P be (x_1, y_1), Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore$ tangent at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Diameter parallel to it is $y=-\frac{x_1}{a^2}.\frac{b^2}{y_1}$ ....(1)

It shall meet the ellipse

$$\therefore \frac{x^2}{a^2}+\frac{x_1^2}{a_1^2}.\frac{b^4}{y_1^2}.\frac{1}{b^2}x^2=1$$

$$x^2\left(1+\frac{b^2}{a^2}.\frac{x_1^2}{y_1^2}\right)=a^2$$

$$\Rightarrow x=\left(a^2y_1\right)/\sqrt{a^2y_1^2+b^2x_1^2} \text{ ....(2)}$$

and ax (x_1, y_1) is an ellipse $b^2x_1^2+a^2y_1^2=a^2b^2 \Rightarrow \sqrt{a^2y_1^2+b^2x_1^2}=ab$ .....(3)

from (2) x = ayu1/ b, and then from (1) $y=-\frac{x_1}{a^2}.\frac{b^2}{y_1}.\frac{ay_1}{b}$

$$\Rightarrow y=-\frac{b}{a}x_1$$

$\therefore$ Length of semi-diameter = $\sqrt{\frac{a^2}{b^2}y_1^2+\frac{b^2}{a^2}x_1^2}$

$$=\sqrt{\frac{a^2}{b^2}\left(1-\frac{x_1^2}{a^2}\right)b^2+\frac{b^2}{a^2}x_1^2}=\sqrt{a^2-x_1^2+\frac{b^2}{a^2}x_1^2}$$

(semi-diameter)^2 = $a^2-x_1^2+\frac{b^2}{a^2}x_1^2=a^2-x_1^2+\left(1-e^2\right)x_1^2$

$$=a^2-e^2x_1^2=\left(a+ex_1\right)\left(a-ex_1\right)$$

= product of focal distances.

Example 50 : Semi-conjugate diameter of ellipse are extended to directrix. Prove that orthocentre of triangle so formed is at the focus.

Sol. : CP and CQ are semi-conjugate diameters. Equations is

$$y=+\frac{b}{a}x \text{ and } y=-\frac{b}{a}x$$

These on extension meets directrix $x=\frac{a}{e}$, in $E\left(\frac{a}{e},-\frac{b}{e}\right),F\left(\frac{a}{e},\frac{b}{e}\right)$

CL perpendic7ualr from C on EF, y = 0 Equation of perpendicular from E$\left(\frac{a}{e},-\frac{b}{e}\right)$.

Example 51 : D = 0 is diameter which bisects lines parallel to 2y – 3x = l of ellipse $3x^2 + 4y^2 = 12$. Find angle subtended by diameter at focus.

Sol. : Ellipse $3x^2 + 4y^2 = 12$ $e^2 = \frac{4-3}{4} \Rightarrow e = 1/2$

$\therefore$ focus is $\left(2\frac{1}{2},0\right)$ i.e. (1, 0)

Diameter of parallel lines 2y – 3x = λ is $y = -\frac{3}{4}.\frac{2}{3}x$

i.e. $y = -\frac{1}{2}x$. It meets ellipse in P and Q.

$$3x^2 + 4.\left(\frac{1}{4}x^2\right) = 12 \Rightarrow x = \pm\sqrt{3}, y = \mp\frac{1}{2}\sqrt{3}$$

Slope of PS $m_1 = \frac{-\sqrt{3}}{2(\sqrt{3}+1)}$, Slope QS, $m_2 = \frac{-\sqrt{3}}{2(\sqrt{3}-1)}$

$$\therefore \tan\theta = -\frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1}\right)\Big/ 1 + \frac{3}{4(3-1)}$$

$$= -\frac{\sqrt{3}}{2}.\frac{2}{2}.\frac{8}{11} = -\frac{4\sqrt{3}}{11}$$

acute angle $\theta = \tan^{-1}\left(\frac{4\sqrt{3}}{11}\right)$

Example 52: Prove that equation of straight line joining ends of semi-conjugate diameters is $x\cos\alpha + y\sin\alpha = p$ if $b^2\sin^2\alpha + a^2\cos^2\alpha = 2p^2$

Sol. : Ends of semi-conjugate diameters are (a cos θ, b sinϕ) and (–a sinθ, b cos θ). If line joining them is x cosα + y sin α = p then points should satisfy it.

$\therefore$ a cosθ cosα + b sinθ sin α = p ....(1)

–a sin ϕ cosα + b cosθ sinα = p ....(2)

Squaring (1) and (2) and adding

$$a^2\cos^2\alpha\left(\cos^2\theta+\sin^2\theta\right)+b^2\sin^2\alpha\left(\sin^2\theta+\cos^2\theta\right)+2ab\sin\theta\cos\theta\cos\alpha\sin\alpha$$
$$-2ab\sin\theta\cos\theta\sin\alpha\cos\alpha$$
$$\Rightarrow a^2\cos^2\alpha+b^2\sin^2\alpha=2p^2$$

Practice Worksheet (Foundation Level) – 7 (d)

1. Which of the following statements are correct?

   (a) If distance between foci : Length of latuse rectum is 1 : 2 then e = $\sqrt{2}-1$

   (b) The chord of ellipse is parallel to the polar of its mid point.

   (c) Eccentric angles of ends of two semi-conjugate diameter differ by $\pi/3$

   (d) Tangents at ends of a latus-rectum of ellipse meet on corresponding directrix at the point, where major axis meets it.

   (e) Portion of tangent between point of contact and directrix subtend a right angle at focus.

   (f) The perpendicular droped from focus on any tangents meets on auxillary circle.

   (g) The product of perpendicular from foci on any tangent is equal to twice the square of semi-minor axis.

   (h) The point of intersection of polars of P and Q with respect to ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is the pole of PQ.

   (i) S and S' are foci of ellipse, Normal at a point P bisect the angle SPS'

   (j) If PCQ is a chord of ellipse whose centre is C, then tangents at P and Q' intersect at major axis.

2. The normal at any point of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ meets axes in M and N. Prove

   $a^2CM^2 + b^2CN^2 = \left(a^2 - b^2\right)^2$

3. Prove that maximum value of tan q , where q is angle between radius vector CP and normal at P of ellipse is $\left(\frac{1}{4\sqrt{3}}\right)$.

4. The tangent of ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ meets ellipse $\frac{x^2}{3}+\frac{y^2}{2}=5$ in P and Q. Prove that tangents at P and Q meet at right angles.

5. The straight line 2x + y = 3 cuts the ellipse $4x^2 + y^2 = 5$ in P and Q. Show that normals at P and Q include an angle of $\tan^{-1}(3/5)$.

6. Find locus of point of intersection of tangents drawn at the end of semi conjugate diameters of ellipse $b^2x^2 + a^2y^2 = a^2b^2$

7. Prove that straight lines 3x + y = 0 and 4y = x are conjugate diameters of $3x^2 + 4y^2 = 5$.

8. Show that straight lines 7x + 6y – 5 = 0 and 3x – 5y + 4 = 0 are parallel to conjugate diameters of ellipse $7x^2 + 10y^2 = 70$.

9. Find locus of point tangents from which of ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ include an angle of 90^0.

10. Show that straight line x = 6/ $\sqrt{13}$, subtends a right angle at the centre of ellipse $4x^2 + 9y^2=36$.

11. CP and CQ are two semi conjugate diameters prove that locus of the foot of perpendicular dropped from centre on PQ is $2(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

12. Prove that tangents drawn at the ends of a chord of ellipse meet on diameter that bisects the chords.

13. Tangents are drawn from (α,β) to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and chord of contact touches the circle $x^2 + y^2 = c^2$. Prove (α,β) lie on ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$.

14. If portion of straight line x cos$a$ +y sin a =c intercepted by ellipse $b^2x^2 + a^2y^2 = a^2b^2$ subtends a right angle at the centre of ellipse, than prove it touches a concentric circle of radius $ab/\sqrt{a^2+b^2}$.

15. Two points on minor axis are ……. At distance 2, from centre in opposite direction of ellipse $12x^2 + 16y^2 = 192$. Perpendicular are dropped from these points on any tangent of ellipse. Prove that the sum of squares of these perpendiculars is equal to twice of square of semi major axes.

16. Prove that the product of focal distances of a point is equal to square of that semi-diameter which is parallel to tangent drawn at that point.

17. The chords of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ pass through fixed point (x_1, y_1). Prove that locus of mid points of these chords is $\frac{x(x-x_1)}{a^2}+\frac{y(y-y_1)}{b^2}=0$

18. If the straight lines joining ends of two conjugate diameters of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be $\frac{lx}{a}+\frac{my}{b}=n$ then prove $l^2 + m^2 = 2n^2$.

19. Prove that circle drawn on focal distance of a point as diameter touches auxiallary circle.

20. Prove that locus of poles of tangents of $b^2x^2 + a^2y^2 = a^2b^2$ with respect to parabola $y^2 = 4ax$ is $b^2y^2 = 4a^2 (x^2 – a^2)$

21. The straight lines lx + my = n meets the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at points whose eccentric angles differ by p /2. Prove then $a^2l^2 + b^2m^2 = 2n^2$.

22. Prove that two perpendicular diameters of an ellipse are its major and minor axes only.

23. P is a point on ellipse whose foci are S and S'. Prove $\tan\frac{1}{2}PSS'.\tan\frac{1}{2}PS'S = \frac{1-e}{1+e}$.

24. Chords of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches parabola $ay^2 = -2b^2x$, then prove locus of their poles is $ay^2 = 2b^2x$.

25. Find the pole of the chord of ellipse $9x^2 + 4y^2 = 36$, when (1, 2) is mid point of chord.

26. Find equation of normal of ellipse $4x^2 + 9y^2 = 36$ which is parallel to $3x - 4y = 7$.

27. CP and CQ are two conjugate diameters tangent at their ends include an angle of ....(ellipse $4x^2 + 3y^2 = 12$)..........

28. P and Q are ends of conjugate diameters of ellipse $b^2x^2 + a^2y^2$; P' and Q' are corresponding points on auxillary circle at P tangent is drawn and at Q' normal is drawn. Tangent and normal intersect at ......

29. Angle between two diameters is 90^0. The product of gradients of their respective chords (ellipse $4x^2 + 9y^2 = 36$) is ....

30. $D_1=0$, $D_2=0$ two diameters of ellipse $x^2+4y^2=4$ of chords $y=\frac{1}{\sqrt{3}}x+\lambda$ and $y=\sqrt{3}x+\alpha(\lambda,\alpha \text{ are variables})$. The acute angle between diameters is ....

31. The angle between diameters of ellipse $2x^2 + 3y^2 = 12$ is 90^0. If D1 = 0 is diameters of parallel chords who cut equal intercepts on positive sides of axes, then angle between respective chords is ....

32. $bx + ay = \lambda$ and $bx - ay = \gamma$ are two sets of parallel chords of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ angle between two diameters is

(a) $\tan^{-1}\frac{2b}{a(a^2-b^2)}$ (b) $\tan^{-1}\frac{2a}{a(a^2-b^2)}$

(c) $\tan^{-1}\frac{2ab}{(a^2+b^2)}$ (d) $\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$

33. $lx+y=0$ and $x-my=0$ are two conjugate diameters of ellipse $3x^2 + 4y^2 = 5$, then

(a) $4l = 3m$ (b) $4l + 3m = 0$

(c) $3l + 4m = 0$ (d) None of these

34. $l^2x+y=0$ and $mx-ly=0$ are two conjugate diameters of ellipse $x^2 + 4y^2 = 4$ then

(a) $4l\, m + 1 = 0$ (b) $4l\, m - 1 = 0$ (c) $4l + m = 0$ (d) $l + 4m = 0$

35. Two diameters of ellipse $2x^2 + 3y^2 = 6$ bisects chords parallel to $2x + 3y = \lambda$ and $4x - 3y = \gamma$. Angle between diameters is :

(a) $\tan^{-1}\sqrt{3}$ (b) $\tan^{-1}\sqrt{2}$ (c) $\tan^{-1}2$ (d) $\tan^{-1}3$

36. The point on ellipse $3x^2 + 4y^2 = 12$, whose sub-tangent is equal to twice its abscissa is :

(a) $\left(\frac{2}{\sqrt{3}}, \sqrt{2}\right)$ (b) $\left(\frac{2}{\sqrt{3}}, -2\right)$ (c) $\left(-\frac{2}{\sqrt{3}}, -\sqrt{2}\right)$ (d)

37. The sub-tangent of point $\left(\frac{5}{2}, 2\sqrt{3}\right)$ on ellipse is $\frac{15}{2}$, eccentricity of ellipse is :

(a) $\frac{3}{5}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{2}{\sqrt{5}}$

38. Sum of squares of semi-conjugate diameter of ellipse $3x^2 + 4y^2 = 1$ is :

(a) $\frac{7}{12}$ (b) $\frac{7}{\sqrt{12}}$ (c) $\frac{5}{6}$ (d) $\frac{3}{4}$

39. The locus of point of intersection of tangents drawn at the ends of semi-conjugate diameters of ellipse $16x^2 + 25y^2 - 400$ is :

(a) $16x^2 + 25y^2 = 0$ (b) $16x^2 + 25y^2 = 800$ (c) $16x^2 + 25y^2 = 1200$ (d) $16x^2 + 2y^2 = 600$

40. Locus of poles of normal chords of $2x^2 + 3y^2 = 6$ is

(a) $\frac{27}{x^2} + \frac{8}{y^2} = 1$ (b) $\frac{8}{x^2} + \frac{27}{y^2} = 1$ (c) $\frac{9}{x^2} + \frac{4}{y^2} = 1$

41. Normal at $A(\phi = 60)$ of ellipse $3x^2 + 4y^2 = 12$, meets axes of ellipse in M and N. area of triangle CMN is :

(a) $\frac{1}{8}$sq.u (b) $\frac{1}{4}$sq.u (c) $\frac{1}{32}$sq.u (d) $\frac{1}{16}$sq.u

42. The angle between tangents drawn from (2,2) to the ellipse $3x^2 + 4y^2 = 4$ is :

(a) $\tan^{-1}\left(\frac{3\sqrt{2}}{5}\right)$ (b) $\tan^{-1}\left(\frac{12\sqrt{2}}{7}\right)$ (c) $\tan^{-1}\left(\frac{10\sqrt{2}}{17}\right)$ (d) $\tan^{-1}\left(\frac{16\sqrt{2}}{7}\right)$

43. Angle of intersection of curves $9x^2 + 4y^2 = 26$ and $x^2 + y^2 = 6$ is :

(a) $\tan^{-1}\frac{1}{2\sqrt{3}}$ (b) $\tan^{-1}\frac{1}{3\sqrt{2}}$ (c) $\tan^{-1}\frac{1}{\sqrt{6}}$

44. The pole of the straight line joining positive ends of axes of ellipse $3x^2 + 4y^2 = 12$ is :

(a) $\left(2,\sqrt{3}\right)$ (b) $\left(2,\sqrt{5}\right)$ (c) $\left(3,\sqrt{2}\right)$ (d) $\left(\sqrt{2},\sqrt{3}\right)$

45. From a point P $\left(\phi=45^0\right)$ on ellipse $3x^2 + 4y^2 = 16$ normal and tangent are drawn. The normal meets y axis in M and tangent meets x-axis in N. Area of triangle PMN is :

(a) $\frac{3\sqrt{3}}{4}$ (b) $\frac{4\sqrt{3}}{3}$ (c) $\frac{14\sqrt{3}}{9}$ (d) $\frac{16\sqrt{3}}{9}$

46. The polar of (5,7) with respect to ellipse $2x^2 + y^2 = 6$, passes through (2, –2) then polar of (2, –2) shall pass through :

(a) $\left(\frac{7}{2},\frac{5}{2}\right)$ (b) (10, 14) (c) (7, 5) (d) none of these

47. Two points are taken on minor axis, on opposite sides of centre at distance $\sqrt{2}$ from centre. Perpendicular are dropped from points on any tangent of ellipse $5x^2 + 7y^2 = 35$, sum of the squares of perpendicular is :

(a) 10 (b) 7 (c) $7\sqrt{2}$ (d) 14

48. The foci and directrix of ellipse $9x^2 + 4y^2 - 36x + 16y + 16 = 0$ are

(a) $(2, \pm\sqrt{5})$ (b) $(2, \pm\sqrt{5}-2)$ (c) $(0, \pm\sqrt{5}-2)$ (d) none of these

$y + 2 = \pm\frac{9}{\sqrt{5}}$ $y + 2 \pm\frac{9}{\sqrt{5}} = 0$ $y = \pm\frac{9}{\sqrt{5}}$

49. Tangent to ellipse $2x^2 + 3y^2 = 1$ are equally inclined to axes. Product of perpendicular dropped from foci on it is :

(a) $\frac{1}{6}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

Practice Worksheet (Competition Level)

1. Find the centre, the length of axes and the eccentricity of the ellipse $2x^2+3y^2-4x-12y+13=0$.

2. Find the condition that the line lx + my + n = 0 may be a

a) Tangent b) normal

c) A chord whose eccentric angles differ by a right angle, to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3. Find the value of θ If the tangent at the point $\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x^2 + y^2 - 2x = 15$.

4. Th e sum of the ordinates of two points P and Q on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ is 3. Prove that the locus of the point of intersection of tangent at P and Q is $9x^2 + 25y^2 = 150y$

5. Prove that the locus of the mid points of the tangents to the ellipse intercepted between the axes is $a^2y^2 + b^2x^2 = 4x^2y^2$

6. Find the equation of tangents to the ellipse $9x^2+16y^2=144$ which pass through the point (2, 3)

7. The tangent and the normal at any point P of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect its major axis in points T and T ' respectively, so that TT'= a Show that the eccentric angle θ of the point P is given by $e^2\cos^2\theta+\cos\theta-1=0$

8. P,Q are two points on the ellipse $x^2 + 4y^2 = 4$ such that the PQ touches the fixed circle $x^2+ y^2-2x = 0$ if a, ß be the eccentric angle of P and Q prove that seca + secß = 2

9. From a point O on the circle $x^2 + y^2 = d^2$ tangents OP and OQ are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that the locus of the mid point of the chord PQ describes the curve $x^2+y^2=d^2\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}\right]^2$

10. A tangent to the ellipse $x^2+4y^2=4$ meets the ellipse $x^2+2y^2=6$ at P and Q. Prove that the tangents at the P and Q of the ellipse $x^2+2y^2=6$ are at the right angles. **[I.I.T. 1997, 5]**

11. If the tangent drawn at the point (t^2, 2t) on the parabola $y^2 = 4x$ is same as the normal drawn at a point $\left(\sqrt{5}\cos\varphi, 2\sin\varphi\right)$ on the ellipse $4x^2 + 5y^2 = 20$ find the value of t and ϕ.

12. Show that the tangents at the extremities of all chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which subtend a right angle at the centre intersect on the ellipse $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$.

13. Tangents are drawn from any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the circle $x^2 + y^2 = r^2$; prove that chords of the contact are the tangents to the ellipse $a^2x^2 + b^2y^2 = r^4$

14. Find the coordinates of those points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ tangents at which make equal angles with the axes. Also prove that the length of the perpendicular from the centre on either of these is $\sqrt{\left\{\frac{1}{2}(a^2 + b^2)\right\}}$.

15. If p is the length of perpendicular from the focus S of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the tangent at p, then show that $\frac{b^2}{p^2} = \frac{2a}{SP} - 1$.

16. Show that there are always two tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from a point (x', y') outside the ellipse and The slopes of the two tangents are given by $a^2m^2 + b^2 = (y' - mx')^2$. Hence obtain the locus of a point from which the two tangents to the ellipse are inclined at an angle α.

17. If S and S_1 are the foci of an ellipse and P a point on it, if e is the eccentricity of the ellipse, prove that $\tan\frac{1}{2} PSS_1 \tan\frac{1}{2} PS_1S = \frac{(1 - e)}{(1 + e)}$.

18. Find where the line 2x + y =3 cuts the curve $4x^2 + y^2 = 5$. Obtain the equations of the normals at the points of intersection and determine the coordinates of point where these normals cut each other.

19. Find the coordinates of all the points of intersection of the ellipse $x^2/4 + y^2/9 = 1$ and the circle $x^2 + y^2 = 6$. Write down the equations of the tangents to the ellipse and circle at one of the points of intersection and find the angle between them.

20. Show that locus of the middle points of the normal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=\left(a^2-b^2\right)^2$

21. The tangent at a points P on an ellipse intersects the major axis in T and N is the foot of the perpendicular from P to same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

22. Any ordinates NP of an ellipse meets the auxiliary circle in Q, prove that the locus of the intersection of the Normals at P and Q is the circle $x^2 + y^2 = (a + b)^2$.

23. If P is any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ whose ordinate is y', prove that the angle between the tangent at P and the focal distance of P is $\tan^{-1}(b^2/aey')$.

24. Tangents at right angles are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that the locus of the middle points of the chords of contact is the curve $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2+b^2}$

25. a) Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are two foci of the ellipse then show that $\left(PF_1-PF_2\right)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$.

b) An ellipse has eccentricity ½ and one focus at the point P (1/2, 1). Its one diretrix is the common tangent, nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$ find the equation of the ellipse in the standard form. **[I.I.T. 96]**

c) Find the equation of the common tangent in 1st quadrant to the circle $x^2+y^2=16$and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the coordinate axes. **[I.I.T. 2005, 4]**

COMPREHENCIVE PASSAGE TYPE PROBLEMS

26. Consider an ellipse $\frac{x^2}{4}+y^2=\alpha$ (α is the parameter>0) and the parabola y^2 = 8x. If the common tangent to the ellipse and parabola meets the coordinate axes at A and B respectively then

i) Locus of the mid point of AB is

a) $y^2=-2x$ b) $y^2=-x$ c) $y^2=-x/2$ d) $\frac{x^2}{4}+\frac{y^2}{2}=1$

ii) If the eccentric angle of appoint on the ellipse where the common tangents meets, is $\frac{2\pi}{3}$, then α is equal to

a) 4 b) 5 c) 26 d) 36

iii) If two of the three normals drawn from the point (h, 0) on the ellipse to the parabola y^2 =8x are perpendicular then

a) h =2 b) h =3 c) h= 4 d) h=6

OBJECTIVE QUESTIONS

Level-1

1. The centre of the ellipse $14x^2-4xy+11y^2-44x-58y+71=0$ is at

a) (2, 3) b) (2,-3) c) (-2,3) d) (-2,-3)

2. Tangents are drawn from a point P (3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ touching the ellipse in Q and R. The angle between PQ and PR is

a) 45^o b) 60^o c) 90^o d) 30^o

3. The equation(s) of the tangent(s) to the ellipse $9(x-1)^2+4y^2=36$ parallel to the letus rectum, is (are)

a) y =3 b) y = –3 c) x =3 d) x = –3

4. If the chords joining two points whose eccentric angles are a and ß cuts the major axis of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a distance c from the centre, then tana/2. tanß/2 is equal to

a) 1 b) c c) $\frac{c+a}{c-a}$ d) $\frac{c-a}{c+a}$

5. The condition that the line lx+my = n may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

a) $\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)=1$ b) $\frac{n^2}{(a^2+b^2)^2}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)=1$

c) $\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)=2$ d) none

6. If the normal at the point P (?) to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersects it again at the point Q (2?), the cos? is equal to

a) 2/3 b) -2/3 c) ¾ d) none of these

7 The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, and having its centre (0,3) is

a) 4 b) 3 c) v12 d) 7/2

8. In an ellipse the distance between its foci is 6 and its minor axis is 8. then its eccentricity is

a) 4/5 b) 1/v52 c) 3/5 d) ½

9. If S and S' are two foci of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a<b) and P(x_1, y_1) a point on it the SP +S'P is equal to

a) 2a b) 2b c) a+ex_1 d) b+ey_1

10. If the axes of an ellipse coincide with the coordinate axes and it passes through the point (4,-1)and touches the line x +4y –10=0 then its equation is

a) $\frac{x^2}{16}+\frac{y^2}{15}=1$ b) $\frac{x^2}{80}+\frac{y^2}{5/4}=1$

c) $\frac{x^2}{20}+\frac{y^2}{5}=1$ d) $\frac{x^2}{5}+\frac{y^2}{16}=1$

11. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2+y^2=9$. let Pand Q be the points (1,2) and (2,1) respectively . then

a) Q lies inside C but outside E
b) Q lies outside both C and E
c) P lies inside both C and E
d) P lies inside C but outside E

12. A point moves in a plane so that its distance PA and PB from two fixed point A and B in the plane satisfy the relation PA +PB = k (k?0), then the locus of P is

a) a parabola
b) an ellipse
c) a hyperbola
d) a branch of hyperbola

13. The equation of the ellipse whose focus is (1,-1) directrix is the line x –y–3 = 0 and eccentricity is ½ is

a) $7x^2 +2xy +7y^2 -10x +10y+7=0$
b) $7x^2+2xy+7y^2 +7=0$
c) $7x^2+2xy+7y^2 +10x-10y-7=0$
d) none of these

14 Equation x = acos?, y = bsin? (a>b)represent a conic section whose eccentricity e is given by

a) $e^2 = \frac{a^2+b^2}{a^2}$
b) $e^2 = \frac{a^2+b^2}{b^2}$
c) $e^2 = \frac{a^2-b^2}{a^2}$
d) $e^2 = \frac{a^2-b^2}{b^2}$

15. If the focal distance of an end of the minor axis of the ellipse (referred to the axes as the axes of x and y respectively) is k and the distance between the foci is 2h then its equation is

a) $\frac{x^2}{k^2}+\frac{y^2}{h^2}=1$
b) $\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1$
c) $\frac{x^2}{k^2}+\frac{y^2}{h^2-k^2}=1$
d) $\frac{x^2}{k^2}+\frac{y^2}{k^2+h^2}=1$

16. The locus of the mid point of the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

a) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{ex}{a}$
b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{ex}{a}$
c) $x^2+ y^2 = a^2 +b^2$
d) none

17. From a point on 4x+3y =25 two mutually perpendicular tangents are drawn to $\frac{x^2}{16}+\frac{y^2}{9}=1$, and from their points of contact normals are drawn which intersect at P. The coordinate of P is

a) (0, 0) b) (2, 3/2) c) (–2, –3/2) d) none o these

18. A focus of an ellipse is at the origin. The directrix is the line x = 4 and the eccentricity is ½.Then the length of the semi-major axis is **[AIEEE-08]**

a) 2/3 b) 4/3 c) 5/3 d) 8/3

19. Tangents are drawn to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ at the end of the letusrectum. The area of the quadrilateral so formed is **[I.I.T Sc.03]**

a) 27 b) 27/2 c) 27/4 d) 27/55

20. The minimum area of the triangle formed by the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the coordinate axes is **[I.I.T Sc.05]**

a) ab sq. units b) $\frac{a^2+b^2}{2}$ sq. units

c) $\frac{(a+b)^2}{2}$ sq. units d) $\frac{a^2+ab+b^2}{3}$ sq. units

Level-2

1. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the point (-3, 1) and has the eccentricity $\sqrt{\frac{2}{5}}$ then the major axis is

a) $\frac{8\sqrt{5}}{2}$ b) $\frac{8\sqrt{2}}{5}$ c) $\frac{8\sqrt{2}}{\sqrt{3}}$ d) $\frac{8\sqrt{3}}{5}$

2. If the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose centre is C , meets the major and minor axis at P and Q respectively then $\frac{a^2}{CP^2}+\frac{b^2}{CQ^2}$ is equal to

a) 1 b) –1 c)a +b d) none of these

3. Find the locus of the point of intersection of the straight lines 3tx–y +3v3t=0, 3x+ty–3v3=0

a) $9x^2+y^2=27$ b) $3x^2+y^2=9$ c) $\frac{x^2}{3}+y^2=9$ d) none of these

4. A point on the ellipse $x^2+3y^2=37$ where the normal is parallel to line 6x–5y=2 is

a) (5,-2) b) (5, 2) c) (–5, 2) d) (–5,–2)

5. A tangent to the ellipse $16x^2 + 9y^2 = 144$ making equal intercepts on both the axes is

a) y = x +5 b) y = x –5

c) y = –x+5 d) y = –x–5

6. If the tangent to the ellipse $x^2+4y^2=16$ at the point P(ϕ) is a normal to the circle $x^2+y^2-8x-4y=0$ then ϕ is equal to

a) p/2 b) p/4 c) 0 d) –p/4

7. An ellipse having foci (3, 1) and (1, 1) passes through the point (1, 3) has the eccentricity

a) v2-1 b) $\sqrt{3}-1$ c) $\frac{\sqrt{2}-1}{2}$ d) $\frac{\sqrt{3}-1}{2}$

8. S and T are the foci of an ellipse B is an end of the minor axis. If STB is an equilateral triangle, the eccentricity of the ellipse is

a) 1/4 b)1/3 c) ½ d) 2/3

9. If P is point on the ellipse $\frac{x^2}{16}+\frac{y^2}{20}=1$ whose foci are S and S' . Then PS'+PS is

a) 4v5 b) 4/v5 c) 10 d) 4

10. The length of the common chord of the ellipse $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$ and the circle $(x-1)^2+(y-2)^2=1$

a) 2 b) v3 c) 4 d) none of these

11. If tangents are drawn to the ellipse $x^2+2y^2=2$, then the locus of the mid-point of the intercepts made by the tangents between the coordinate axes is **[I.I.T Sc.04]**

a) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ b) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

c) $\frac{x^2}{2}+\frac{y^2}{4}=1$ d) $\frac{x^2}{4}+\frac{y^2}{2}=1$

12. The equation of the chord of contact of pair of tangents drawn to the ellipse $4x^2+9y^2=36$ from the point (m, n) where m.n = m + n m, n being non-zero positive integers is

a) 2x +9y =18 b) 2x +2y =1

c) 4x +9y = 18 d) none of these

13. If the normal at $\left(2, \frac{3\sqrt{3}}{2}\right)$ meet the major axis of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at Q and S and S′ are foci of the given ellipse , then SQ: S′Q is

a) $\frac{8-\sqrt{7}}{8+\sqrt{7}}$ b) $\frac{4+\sqrt{7}}{4-\sqrt{7}}$ c) $\frac{8+\sqrt{7}}{8-\sqrt{7}}$ d) $\frac{4-\sqrt{7}}{4+\sqrt{7}}$

14) The locus of the point of intersection of tangents to the ellipse at two points, sum of whose eccentric angle angles is constant is a/an

a) Parabola b) circle c) ellipse d)straight line

15) The locus of the point which is such that the chord of contact of the tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ forms a triangle of constant area with the coordinate axes is

a) A straight line b) a hyperbola

c) an ellipse d) a circle

16) The maximum distance of the centre of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ from the chord of contact of mutually perpendicular tangents of the ellipse is

a) $\frac{144}{5}$ b) $\frac{16}{5}$ c) $\frac{9}{5}$ d) none of these

17) If the line x +2 y +4=0 cutting the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the points whose eccentric angles are 30^o and 60^o subtends right angle t the origin then its equation is

a) $\frac{x^2}{8}+\frac{y^2}{4}=1$ b) $\frac{x^2}{16}+\frac{y^2}{4}=1$

c) $\frac{x^2}{4}+\frac{y^2}{16}=1$ d) none of these

18) Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1<0$, $y_2<0$, be the end points of the letus rectum of the ellipse $x^2+4y^2=4$ the equation of the parabolas with letus rectum PQ are **[I.I.T- 08]**

a) $x^2+2\sqrt{3}y=3+\sqrt{3}$ b) $x^2-2\sqrt{3}y=3+\sqrt{3}$

c) $x^2+2\sqrt{3}y=3-\sqrt{3}$ d) $x^2-2\sqrt{3}y=3-\sqrt{3}$

19. The locus of the point whose chord of contact with respect to the ellipse $x^2+2y^2=1$ subtends a right angle at the centre of the ellipse is

a) $x^2+4y^2=3$ b) $y^2 = 4x$

c) $2x^2+y^2=1$ d) none of these

20. If normals are drawn to the ellipse $x^2+2y^2=2$ from the point (2, 3). Then the co normal points on the curve

a) $xy+2x-3y=0$ b) $xy+3x-4y=0$

c) $2xy+3x-4y=0$ d) none of these

✣ --------------------- ✣

8. Hyperbola

8.1. Definition:

The locus of a point which moves such that its distance from a fixed point bears a constant ratio to its distance from a fixed line and this constant is greater than unity, is called hyperbola. The fixed point is called focus and the fixed line directrix.

In this figure (1) S is a fixed point and z'zz', the fixed point line. P is a point on hyperbola and PM is perpendicular from P on z'zz'.

PS = ePM and e > 1

SZ is perpendicular from S on z'zz'. In SZ, A is a point touch that AS = eZA. i.e. A divides ZS in the ratio of 1 : e internally. A is on hyperbola. On SZ produced there is another point A' which divides ZS externally in the ratio of 1 : e therefore A' is also on the curve.

AS = eAZ and A'S = eA'Z

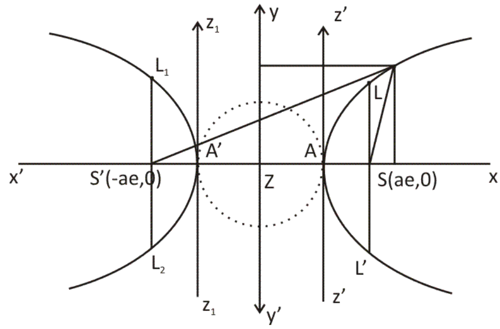

Fig 1

Let AA' = 2a and its mid point C be origin. Perpendicular to AA' through C, y-axis. A'CZAS is x-axis.

Now AS + A'S = e (AZ + A'Z) = eAA' = 2ae

$\Rightarrow$ (CS – CA) + (CS + CA') = 2CS = 2ae, (CA = CA')

$\therefore$ CS = ae $\Rightarrow$ focus (ae, 0)

again A'S – AS = e(A'Z – AZ)

$\Rightarrow$ (CS + CA') – (CS – CA) = e (A'C + CZ) – e(CA –CZ)

$\Rightarrow$ CA' + CA = e(CZ + CZ) [A'C = AC]

$\Rightarrow$ 2a = 2e CZ $\Rightarrow$ CZ = $\frac{a}{e}$

$\therefore$ Equation of directix Z'ZZ' is $x = \frac{a}{e}$

If P is any point on Hyperbola (x_1, y_1) then

PS2 = e2 (PM)2 $\Rightarrow$ $(x_1 - ae)^2 + y_1^2 = e^2 (x_1 - \frac{a}{e})^2$

$\Rightarrow x_1^2 + y_1^2 - 2ae\, x_1 + a^2e^2 = e^2x_1^2 + a^2 - 2ae\, x_1$

$\Rightarrow x_1^2 (1 - e^2) + y_1^2 = a^2(1 - e^2)$

$\Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{a^2(1-e^2)} = 1$

$\therefore$ Locus $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(1-e^2)$

This is the equation of hyperbola.

a) By symmetry about y-axis there is one more focus S', (-ae, 0) and another directorix Z_1Z_2, $x = -\frac{a}{e}$.

b) AA' is called Transverse Axis whose length is 2a. Its mid point C is called Centre of hyperbola; C is (0,0). Every chord of hyperbola through C is bisected at C. Through C, perpendicular to AA' is y-axis. Points distance $\sqrt{a^2(e^2-1)}$ i.e. b, above C and below C on y axis are named B and B'. BB' is called Conjugate axis, its length BB'=2b.

c) Like parabola, ellipse, perpendicular chord to axis through focus is called Latus Rectum. If fig(1), LL', L_1L_2 are latus rectum.

$L_1L_2 = LL' = \frac{2b^2}{a}$, Equation are $x = \pm ae$

8.2. Focal distance of a Point.

In the fig(1), P is (x_1, y_1). PN is ordinate, PM $\perp$ to directix.

By definition SP = e PM = eZN

$= e\,(CN - CZ) \quad = e(x_1 - \frac{a}{e})$

$= ex_1 - a$

$\therefore$ Focal distance of point $(x_1, y_1) = ex_1 - a$

Similarly $S'P = a + ex_1$

And $S'P - SP = 2a$

i.e. the difference of the distances of point P from two focus is equal to transverse axis = 2a (a constant).

Hyperbola can now be defined as locus of a point which moves such that its difference of distances from two fixed points is always same (constant).

8.3 For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(a) Axis y = 0

(b) Centre (0, 0)

(c) focii $(\pm ae, 0)$

(d) Equation of directrix $x = \pm\frac{a}{e}$

(e) Transverse axis 2a, Conjugate axis

(f) eccentricity $e^2 = \frac{a^2 + b^2}{a^2}$

(g) Length of Latus rectum = $\frac{2b^2}{a}$

equation of latus rectum $x = \pm ae$

ends of latus rectum $\left(\pm ae, \pm\frac{b^2}{a}\right)$

(h) Circle drawn on transverse axis as diameter is called anxillvary circle. Its equation is $x^2 + y^2 = a^2$

8.4 Conjugate Hyperbola:

The hyperbola whose transverse axis is BB' and conjugate axis AA' is called conjugate hyperbola.

Its Equation is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

(a) focii lie on y-axis $(0 \pm be)$

(b) Directrix parallel to x-axis $y = \pm\frac{b}{e}$

(c) eccentricity $e^2 = \frac{b^2 + a^2}{b^2}$

(d) L.R. = $\frac{2a^2}{b}$, equation $y = \pm be$ ends $\left(\pm\frac{a^2}{b}, \pm be\right)$

(e) Transverse axis = 2b, Conjugate axis = 2a.

(f) Axis of hyperbola, x = 0

8.5 Asymptote:

An asymptote a curve is the line which meets the curve in two point

$\therefore$ Asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$

i.e. $\frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$.

8.6 Relation between equations of hyperbola, asymptotes and conjugate hyperbola.

(i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 is equation of hyperbola.

(ii) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ is equation of asymptotes.

It differs from the equation of hyperbola by a constant 1.

(iii) $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ is equation of conjugate hyperbola it can be written as $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

This differs from equation of asymptotes by the same constant 1.

$\therefore$ If $3x \pm 4y = 0$ are equation of asymptotes

then $\frac{x^2}{16}-\frac{y^2}{9}=1$ is equation of hyperbola and $\frac{x^2}{16}-\frac{y^2}{9}=-1$ is equation of conjugate hyperbola.

8.7 Some Properties of hyperbola

(a) In the figure (2) LL' and MM' are asymptotes.

The conjugate hyperbola is shown in dotted lines H_1BH_2 and $H_1'B'H_2'$. Its vertices are B and B'

(b) Through B an B' lines parallel to x axis are drawn and through A and A' lines parallel to y-axis are drawn. Rectangle $k_1k_2k_3k_4$ is completed. k_1, k_2, k_3 and k_4 are points on asymptotes.

(c) With centre C and radius ek_1 an arc has been drawn. It passes through focus S and S'.

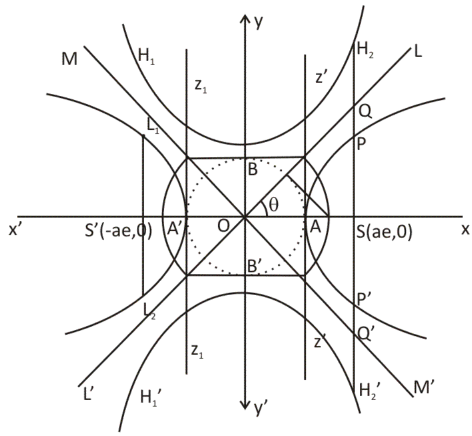

Fig 2

Solved Examples

Example 1: Find equation of hyperbola whose transverse axis is twice of the conjugate axis and which passes through $\left(4, \frac{\sqrt{7}}{2}\right)$.

Sol. Given a = 2b $\Rightarrow$ hyperbola $\frac{x^2}{4b^2} - \frac{y^2}{b^2} = 1$. It passes through $\left(4, \frac{\sqrt{7}}{2}\right)$,

$$\therefore \frac{16}{4b^2} - \frac{7}{4b^2} = 1$$

i.e. $9 = 4b^2 \Rightarrow b^2 = \frac{9}{4}$, $a^2 = 9$

Hyperbola is $\frac{x^2}{9} - \frac{4y^2}{9} = 1$

Example 2: The product of gradient of two lines is k and both pass through fixed point (ℓ , 0), k > 1. Show that the locus of points of intersection of these lines is hyperbola.

Sol. Let lines be $y = m_1 (x - \ell)$ ... (1)

and $y = m_2 (x - \ell)$ ... (2)

For the locus of point of intersection of these lines you need to eliminate m_1 and m_2.

It is given $m_1m_2 = k$

$\therefore$ Multiplying (i) and (ii), we get

$$y^2 = k(x^2 - \ell^2) = x^2 - \frac{y^2}{k} = \ell^2$$

$$\Rightarrow \frac{x^2}{\ell^2} - \frac{y^2}{k\ell^2} = 1$$, what is hyperbola

Example 3: Find eccentricity, focii vertices and equation of directrix and latus-rectum of hyperbola $4x^2 - 9y^2 - 8x - 6y - 33 = 0$

Sol. Equation of hyperbola is

$$4(x^2 - 2x + 1) - 9\left(y^2 + \frac{2}{3}y + \frac{1}{9}\right) = 36$$

$$\Rightarrow 4(x-1)^2 - 9\left(y + \frac{1}{3}\right)^2 = 36$$

$$\Rightarrow \frac{(x-1)^2}{9}+\frac{\left(y+\frac{1}{3}\right)^2}{4}=1$$

$$e^2=\frac{9+4}{9}\Rightarrow e=\frac{\sqrt{13}}{3},\ ae=3\cdot\frac{\sqrt{13}}{3}=\sqrt{13}$$

Transferring origin at $\left(1,-\frac{1}{3}\right)$

Hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$

	Changed axes	Original axes
(i)	Vertex $(\pm 3, 0)$	$\left(\pm 3+1,-\frac{1}{3}\right)$
(ii)	Focii $(\pm\sqrt{13},0)$	$\left(\pm\sqrt{13}+1,-\frac{1}{3}\right)$
(iii)	Equation of directrix $x=\pm\frac{9}{\sqrt{13}}$	$x-1=\pm\frac{9}{\sqrt{13}}\Rightarrow x=1\pm\frac{9}{\sqrt{13}}$
(iv)	Equation of Latus rectum $x=\pm\sqrt{13}$	$x-1=\pm\sqrt{13}\Rightarrow x=1+\sqrt{13}$
(v)	Length of L R. = $\frac{8}{3}$	Length L.R. $=\frac{2b^2}{a}=\frac{8}{3}$

Example 4: Find equation of hyperbola whose focii are (–3, 2) and (5, 2) and transverse axis is 6.

Sol. The focii S and S' and fixed points. In figure S'S axis of hyperbola is parallel to x-axis and its equation is y = 2 and a = 3

Centre C = (1, 2), es = ae = 3e = 4 ⇒ e = $\frac{4}{3}$

$\therefore$ $b^2 = a^2 (e^2 - 1) = 9\left(\frac{16}{9}-1\right) = 7$

$\therefore$ Hyperbola is $\frac{(x-1)^2}{9}-\frac{(y-2)^2}{7}=1$

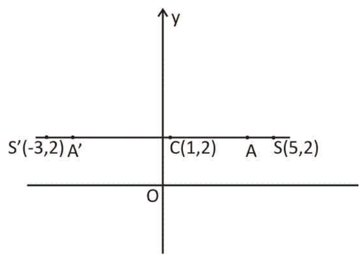

Fig 3

Example 5: Find equation of hyperbola whose focii arc (4, 3) and (–4, –3) and transverse axis is 8.

Sol. The focii S and S' fixed points and from definition of hyperbola if P be any point on it. Then PS' – PS = constant = transverse axis

$$\therefore\ \sqrt{(x+4)^2+(y+3)^2}-\sqrt{(x-4)^2+(y-3)^2}=8$$

$$\Rightarrow\ \sqrt{(x+4)^2+(y+3)^2}=8+\sqrt{(x-4)^2+(y-3)^2}$$

Squaring

$$x^2 + y^2 + 8x + 6y + 25 = 64 + x^2 + y^2 - 8x - 6y + 25 + 2\cdot 8\cdot\sqrt{(x-4)^2+(y-3)^2}$$

$$\Rightarrow 4x + 3y - 16 = 4\sqrt{x^2+y^2-8x-6y+25}$$

Squaring again

$$16x^2 + 9y^2 + 24xy - 96y - 128x + 256 = 16x^2 + 16y^2 - 128x - 96y + 400$$

Equation is $7y^2 - 2xy + 144 = 0$

Alliler

Question can also done as follows.

The centre of hyperbola is origin and 2a = 4 – (–4) = 8

CS = ae = 5, a = 4

$$\therefore\ e=\frac{5}{4},\qquad b^2=16\left(\left(\frac{5}{4}\right)^2-1\right)=9$$

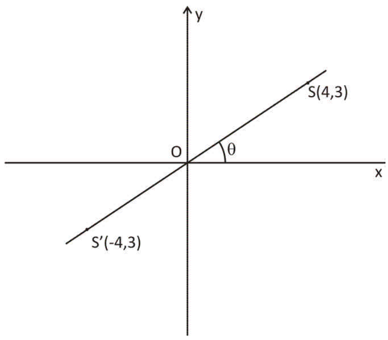

Fig 4

Hyperbola on charge axes is $\frac{x^2}{16}-\frac{y^2}{9}=1$ axes are taken along SS' and perpendicular to it through e.

If axes here have bee rotated through θ in the anticlockwise direction. $\tan\theta=\frac{3}{4}\Rightarrow\sin\theta=\frac{3}{5}$ and $\cos\theta=\frac{4}{5}$

and $X = x\cos\theta + y\sin\theta = \frac{4x+3y}{5}$

$Y = y\cos\theta - y\sin\theta = \frac{4y-3x}{5}$

$$\therefore \text{ Equation}\quad 9\left(\frac{4x+3y}{5}\right)^2-16\left(\frac{4y-3x}{5}\right)^2=144\qquad \Rightarrow 7y^2 - 24xy + 144 = 0$$

Example 6: The asymptotes of a hyperbola are $3x + 4y + 5 = 0$ and $(x - 2y + 3) = 0$. The hyperbola passes through $(1, -1)$. Find its equation and also equation of conjugate hyperbola.

Sol. Equation of hyperbola is $(3x + 4y + 5)(x - 2y + 3) = \lambda$

It passes through $(1, -1)$ $\therefore$ $4 \cdot 6 = \lambda = 24$

$\Rightarrow$ Hyperbola is $(3x + 4y + 5)(x - 2y + 3) = 24$

Conjugate hyperbola is $(3x + 4y + 5)(x - 2y + 3) = -24$

Example 7: Show that $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$ represents a hyperbola. Find equation of asymptotes and conjugate hyperbola.

Sol. Given equation is $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$ ... (1)

$a = 3, b = -2, c = -8, g = \frac{5}{2}, f = \frac{11}{2}, h = -\frac{5}{2}$

$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 48 - \frac{275}{4} - \frac{363}{4} + \frac{25}{2} + 50 \neq 0$

and $h^2 - ab = \frac{25}{4} + 6 > 0$

$\therefore$ Equation represents a hyperbola.

Practice Worksheet (Foundation Level) – 8 (a)

1. The transverse axis of a hyperbola is twice the conjugate then eccentricity of hyperbola is ...

2. Find equation of hyperbola, when distance between two focii is 8 and distance between two directrix is 6

3. A asymptotes of a hyperbola are $2x - 3y = 4$ and $3x + 4y + 2 = 0$ and it goes through point (1, –2). Find equation of hyperbola.

4. Distance between two vertices of a central conic is 3 and distance between two focii is 4. Find equation of conic.

5. Find vertex, focii and equation of directrix and asymptotes of hyperbola $16x^2 - 9y^2 - 32x + 18y - 137 = 0$

6. The distance between two focii of a central conic is twice the distance between directrix, then conic is

   (a) ellipse, (b) hyperbola, (c) rectangular hyperbola (d) none of these

7. Focii of two central conics are $(\pm 6, 0)$ and their eccentricities are reciprocal of each other. Find equation of conic when difference of their eccentricities is $\frac{5}{6}$.

8. Prove $x = \frac{(e^t + e^{-t})}{2}, y = \frac{(e^t - e^{-t})}{2}$ represent a hyperbola. Find its equation.

9. Find asymptotes of hyperbola $2x^2 + 5xy - 2y^2 + 4x + 6y = 0$

10. e and e' are eccentricities of a hyperbola and its conjugate, find value of $\frac{1}{e^2} + \frac{1}{e'^2}$

11. Focii of a hyperbola are (–2, 3) and (4, 3) latus-rectum is of length $\frac{7}{2}$. Calculate eccentricity of hyperbola.

12. The equation of hyperbola in Q. 11 ...

13. The asymptotes of hyperbola $xy - 4x - 3y - 4 = 0$ are ...

14. The asymptotes of a hyperbola are $x \cos \theta - y \sin \theta = 0$ and $x \sin \theta + y \cos \theta = 0$. Find the equation of hyperbola if it passes through (1, 1).

15. The asymptotes of a hyperbola are inclined at 60^o and 120^o with x-axis , the axis of hyperbola centre is at origin. If It passes through (2, 4). Then its equation is ...

16. The straight line $x + 2y = c$ cuts the hyperbola $x^2 - 2y^2 - 2x + 8y - 15 = 0$ in two points. The locus of mid point of this segment is ...

17. The circle $x^2 + y^2 = 3$ meets the hyperbola $x^2 - 2y^2 = 2$ in points A, B, C and D. Area of quadrilateral ABCD is ...

18. The circle $x^2 + y^2 = 16$ meets the hyperbola $xy = c$ in points (x_i, y_i), $i = 1, 2, 3, 4$ then if $\{x_1x_2 + x_1x_2x_3x_4 = 0$ then c is equal to ...

19. The distance between two focii of a hyperbola is 10 and the difference of focal distances of a point is 8. Find equation of hyperbola and its eccentricities.

20. The ratio of eccentricities of two hyperbola $3x^2 - 5y^2 = 15$ and $4y^2 - 5x^2 = 20$ is

21. PFQ is focal chord of hyperbola $9x^2 - 16y^2 = 144$ such that PF = 3FQ then inclination of chord with x-axis is ...

    [Hint: Focus is (5, 0) and $\frac{x-5}{\cos\theta} = \frac{y}{\sin\theta} = 3r$ and $-r$ and both points are on hyperbola.]

22. The angle subtended by latus-rectum of hyperbola $9x^2 - 16y = 144$ at its nearest vertex is ...

23. The vertices of an ellipse and hyperbola are $(\pm 4, 0)$

8.8 Parametric co-ordinates, x = a sec f, y = b tan f satisfy equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

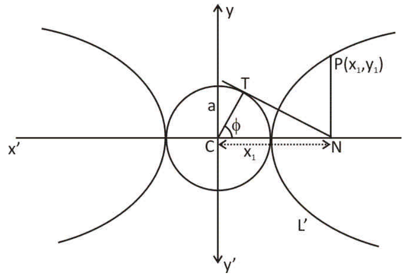

Fig 5

$\therefore$ Any point hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a sec ϕ, b tan ϕ and these are parametric co-ordinates.

In figure (5), P is a point (x_1, y_1) on hyperbola.

PN is ordinate of P; NT is tangent from N on auxiliary circle join CT. Let $\angle TCN = \phi$

$\therefore$ CT = a, CN = x_1 = CT sec ϕ = a sec ϕ

and from right angled triangle CTN, NT = a tan ϕ and as (a sec ϕ, b tan ϕ) satisfy equation of hyperbola.

PN : NT = b tan ϕ : a tan ϕ = b : a.

This proves that ratio between ordinate of any point on hyperbola and the length of tangent from (its ordinate foot on axis) on auxiliary circle is b : a (constant ratio)

8.9 Tangent and Normal:

Hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(A) (a) equation tangent at (x_1, y_1) $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

(b) Equation tangent at ϕ $\frac{x\sec\phi}{a}-\frac{y\tan\phi}{b}=1$

(c) y = mx + c shall be tangent to this hyperbola, if $c=\pm\sqrt{a^2m^2-b^2}$

$\therefore$ Equation of any tangent of hyperbola in terms of its slope is $y=mx+\sqrt{a^2m^2-b^2}$

Point of contact $\left[\frac{-a^2m}{\sqrt{a^2m^2-b^2}},\frac{-b^2}{\sqrt{a^2m^2-b^2}}\right]$

(B) (a) Normal at (x_1, y_1) $\frac{a^2(x-x_1)}{x_1}+\frac{b^2(y-y_1)}{y_1}=0$

(b) Normal at ϕ, Slope of tangent $\frac{b}{a}-\frac{\sec\theta}{\tan\theta}$

Normal, $\frac{b\sec\theta}{a\tan\theta}(y-b\tan\theta)+(x-a\sec\theta)=0$

$\Rightarrow$ ax cos θ + by cot ϕ – $a^2 + b^2$

8.10 Chord of contact Pole and Polar

(a) If PA and PB are tangents from point P(x_1, y_1) to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then equation of AB, the chord of contact of point P (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

(b) Length of any chord of hyperbola is $=(x_1 \sim x_2)\sqrt{1+m^2}$

m is the slope of the chord and ($x_1 \sim x_2$) is to be determined by reducing the equation of hyperbola into a quadratic equation x, with the help of given equation of chord.

(c) Polar of point P (x_1, y_1) with respect to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is, $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

If point P is on the hyperbola the polar becomes tangent, if it is outside hyperbola polar becomes chord of contact. If P is inside hyperbola it is only polar. Point P is called the pole of polar.

(d) Chord of hyperbola is parallel to the polar of its mid point.

8.11 Disector Circle

Disector circle is the locus of point of intersection of intersection of perpendicular to hyperbola.

Equation of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Any tangent $y-mx=\sqrt{a^2m^2-b^2}$ ... (1)

Perpendicular tangent $my+x=\sqrt{a^2-b^2m^2}$ ... (2)

Squaring (1) and (2) and adding

$x^2(1+m^2)+y^2(1+m^2)=a^2(1+m^2)-b^2(1+m^2)$

$\Rightarrow x^2+y^2=a^2-b^2$

Note: The dissector circle of hyperbola $x^2-y^2=a^2$ is point circle, i.e. C centre at origin $x^2-y^2=a^2$ is rectangular hyperbola (a = b). The director circle of rectangular hyperbola is point circle.

Solved Example

Example 9: Find equation of tangent of hyperbola $4x^2 - 9y^2 = 1$ which is (a) parallel to straight line 4y = 5x + 7 (b) equally inclined to axes.

Sol. Equation of hyperbola $\frac{x^2}{\frac{1}{4}} - \frac{y}{\frac{1}{9}} = 1$

$\therefore$ Equation of any tangent is $y = mx \pm \sqrt{\frac{1}{4}m^2 - \frac{1}{9}}$

Slope of given line $= \frac{5}{4} = m$

$\therefore$ Tangent is $y = \frac{5}{4}x \pm \sqrt{\frac{1}{4} \cdot \frac{25}{16} - \frac{1}{9}}$

$\Rightarrow 4y = 5x \pm \frac{\sqrt{161}}{6}$

Point of contact $\left[\frac{\left(-\frac{1}{4} \cdot \frac{5}{4}\right) \times 24}{\sqrt{161}}, \frac{-\frac{1}{9} \times 24}{\sqrt{161}}\right]$

i.e. $\left[\frac{-15}{2\sqrt{161}}, \frac{-8}{3\sqrt{161}}\right]$ and $\left[\frac{15}{2\sqrt{161}}, \frac{-8}{\sqrt{161}}\right]$

(b) Equally inclined to axes, m = 1 or –1

Tangents y = $x \pm \frac{1}{6}\sqrt{5}$

and $y + x = \pm \frac{1}{6}\sqrt{5}$

Example 10: Find the condition that straight line $\ell x + my + n = 0$ touches hyperbola $b^2x^2 = a^2y^2 = a^2b^2$

Sol. Any tangent of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} - 1 = 0$

our line $\ell x + my + n = 0$

Comparing $\frac{\sec\theta}{a\ell} = -\frac{\tan\theta}{bm} = -\frac{1}{n}$

$\Rightarrow \sec\theta = -\frac{a\ell}{n}; \tan\theta = \frac{bm}{n}$

$\sec^2\theta - \tan^2\theta = 1 \ \therefore \frac{a^2\ell^2}{n^2} - \frac{b^2m^2}{n^2} = 1$

$\therefore \ a^2\ell^2 - b^2m^2 = n^2$

Example 11: Find the point of intersection of tangents drawn at the ends of a latus rectum of hyperbola $9x_2 - 16y^2 - 144$

Sol. Equation of hyperbola $9x^2 - 16y^2 = 144$, $a^2 = 16$, $b^2 = 9$

$\therefore \ e^2 = \frac{16+9}{16} \Rightarrow e = \frac{5}{4}$

$\therefore$ ae = $4 \cdot \frac{5}{4}$ = 5 $\therefore$ focus (5, 0)

$\frac{b^2}{a} = \frac{9}{4}$ $\therefore$ Ends of L. Rectum $\left(5, \pm\frac{9}{4}\right)$

Tangents are 45x – 36y = 144 and 45x + 36y = 144

Point of intersection 90x = 288 $\Rightarrow x = \frac{16}{5}$

Point of intersection is $\left(\frac{16}{5}, 0\right)$ and $\frac{a}{e} = \frac{4}{\frac{5}{4}} = \frac{16}{5}$

$\therefore$ Tangents meet at directrix at the point it meets rransverse axis of hyperbola.

$\therefore$ It follow that latus rectum is the chord of contact of point of intersection of transverse axes and corresponding directrix.

Example 12: Show that 5x – 6y + 13 = 0 is tangent to hyperbola $x^2 - y^2 + 8x - 6y + 18 = 0$. Write also co-ordinates of point of contact.

Sol. Equation of hyperbola $x^2 - y^2 + 8x - 6y + 18 = 0$ ... (1)

equation of straight line 5x – 6y + 13 = 0 ... (2)

From (2) $y = \frac{5x+13}{6}$, putting this value of y in (1)

$x^2 - \left(\frac{5x+13}{6}\right)^2 + 8x - (5x + 13) + 18 = 0$

$\Rightarrow 36x^2 - 25x^2 - 130x - 169 + 108 + 186 = 0$

$\Rightarrow 11x^2 - 22x + 11 = 0 \Rightarrow (x - 1)^2 = 0$

Roots are equal, therefore straight line is tangent point of contact x = 1, y = $\frac{5+13}{6} = 3$; (1, 3)

Example 13: Find locus of mid point of the intercept of tangent, of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ between the axes.

Sol. Any tangent of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is bx sec θ – 2y tanθ = ab

It meets axes in $M\left(\frac{a}{\sec\theta}, 0\right), N\left(0, \frac{-b}{\tan\theta}\right)$

If (h, k) is its mid point then $h = \frac{a}{\sec\theta},\ k = \frac{-b}{2\tan\theta}$

$\therefore \Rightarrow \sec\theta = \frac{a}{2h}, \tan\theta = \frac{-b}{2k}$

$\therefore \frac{a^2}{4h^2} - \frac{b^2}{4k^2} = 1$,

$\therefore$ Locus $a^2y^2 - b^2x^2 = 4x^2y^2$

Example 14: Prove that the product of perpendiculars dropped from focii of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ on any tangent of it is $-b^2$

Sol. Any tangent of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \Rightarrow$ bx secθ – ay tan θ = ab

perpendiculars from (ae, 0) and (–ae, 0) are

$$p_1 = \frac{aeb\sec\theta - ab}{\sqrt{b^2\sec^2\theta + a^2\tan\theta}}, p_2 = \frac{-(aeb\sec\theta + ab)}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$$

$$\therefore -p_1 \cdot p_2 = \frac{-(a^2e^2b^2\sec^2\theta - a^2b^2)}{b^2\sec^2\theta + a^2\tan^2\theta} = \frac{-b^2(a^2e^2\sec^2\theta - a^2)}{b^2\sec^2\theta + a^2\tan^2\theta}$$

$$=-\frac{b^2[(a^2+b^2)\sec^2\theta-a^2]}{b^2\sec^2\theta+a^2\tan^2\theta}$$

$$=-\frac{b^2[b^2\sec^2\theta+a^2\tan^2\theta]}{b^2\sec^2\theta+a^2\tan^2\theta}=-b^2$$

Example 15: Find equations of tangent from (1, 2) on hyperbola $3x^2 - 4y^2 = 12$

Sol. Equation hyperbola $\frac{x^2}{4}-\frac{y^2}{3}=1$

Any tangent of it $y = mx+\sqrt{4m^2-3}$

It passes through (1, 2) $\therefore\ 2 = m+\sqrt{4m^2-3}$

$\therefore\ m^2 - 4m + 4 = 4m^2 - 3 \Rightarrow 3m^2 + 4m - 7 = 0$

$\Rightarrow (3m + 7)(m - 1) = 0 \Rightarrow m_1 = \frac{7}{3}$ and 1

$\therefore$ tangent $y = x + 1$ and $3y + 7x = 13$

Example 16: Find equations of normal at the ends of L. rectum of the right side of y-axis of hyperbola $9x^2 - 16y^2 = 144$

Sol. $a^2 = 16, b^2 = 9,\ e^2 = \frac{a^2+b^2}{a^2} = \frac{25}{16}$ $\therefore\ ae = 5; \frac{b^2}{a} = \frac{9}{4}$

$\therefore$ Ends of L . Rectum are $\left(5, \pm\frac{9}{4}\right)$

Normals $\frac{16(x-5)}{5}+\frac{9\left(y\mp\frac{9}{4}\right)}{\pm\frac{9}{4}} = 0$

$\Rightarrow \frac{16(x-5)}{5}\pm 4\left(y\mp\frac{9}{4}\right) = 0 \Rightarrow 16x-80\pm 20\left(y\mp\frac{9}{4}\right) = 0$

$\Rightarrow 16x + 20y = 125$ and $16x - 20y = 125$

Point of intersection is $\left(\frac{125}{16}, 0\right)$

Note: If hyperbola is $b^2 x^2 - a^2 y^2 = a^2 b^2$ normals at the end of L.R. $\left(ae, \pm\frac{b^2}{a}\right)$ are x + ey = $e\left(\frac{a^2+b^2}{a}\right)$ and x – ey = $\frac{e(a^2+b^2)}{a}$

Example 17: Find the equation of chord of hyperbola $25x^2 - 16y^2 = 400$ whose mid point is (6, 2)

Sol. Equation of hyperbola $\frac{x^2}{16} - \frac{2y}{25} = 1$

$\Rightarrow$ Polar of (6,2) $\frac{6x}{16} - \frac{2y}{25} = 1 \Rightarrow 75x - 16y = 200$

Slope of chord = slope of this polar = $\frac{75}{16}$

$\therefore$ Equation of chord $y - 2 = \frac{75}{16}(x - 6)$

$\Rightarrow$ 75x – 16y = 418

Example 18: Prove that the locus of poles of normal chords of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $a^6y^2 - b^6x^2 = (a^2 + b^2)^2 \cdot x^2y^2$

Sol. Any normal of $\frac{x^2}{a^2} - \frac{y^2}{b^2} =$ is

$ax \cos\theta + by \cot\theta = a^2 + b^2$ ... (1)

Let its pole be (x_1, y_1), then its polar is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{... (2)}$$

Comparing (1) and (2) $\frac{x_1 \sec\theta}{a^3} = \frac{-y_1 \tan\theta}{b^3} = \frac{1}{a^2 + b^2}$

$\Rightarrow \sec\theta = \frac{a^3}{(a^2+b^2)x_1}$, $\tan\theta = -\frac{b^3}{y_1(a^2+b^2)}$

$\therefore \frac{a^6}{(a^2+b^2)^2 x_1^2} - \frac{b^6}{y_1^2(a^2+b^2)^2} = 1$

Locus $a^6y^2 - b^6x^2 = x^2y^2 (a^2 + b^2)^2$

Example 19: A tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets transverse axis in Q and tangent at vertex in R. Find locus of mid point of QR.

Sol. Any tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$$

It meets y = 0 in Q, $\therefore$ Q is (a cos θ, 0)

Tangent at vertex is x = a $\therefore$ R is $\left(a,\frac{(1-\cos\theta)b}{\sin\theta}\right)$

Let mid point of QR be (h, k)

$\therefore$ 2h = a(1 + cos θ), $2k=\frac{b(1-\cos\theta)}{\sin\theta}$

$\Rightarrow \cos\theta=\frac{2h-a}{a}$ and $4kh=\frac{ab(1-\cos^2\theta)}{\sin\theta}$

$\Rightarrow 4kh=ab\sin\theta \Rightarrow \sin\theta=\frac{4kh}{ab}$

$\therefore \left(\frac{2h-a}{a}\right)^2+\frac{16k^2h^2}{a^2b^2}=1$

$\therefore$ locus $16x^2y^2+b^2(2c-a)^2=a^2b^2$

$\Rightarrow 16x^2y^2+4x^2b^2-4ab^2x=0$

$\Rightarrow x(4y^2+b^2)=ab^2$

Example 20 : Show that the chords of the curve $3x^2-y^2-2x+4y=0$, which suntends a right angle at origin are concurrent.

Sol : Let chord be y = mx+c ....(1)

Hyperbola is $3x^2-y^2-2x+4y=0$ ....(2)

making (2) homogeneous with the help of (1)

$$3x^2-y^2-2x\left(\frac{y-mx}{c}\right)+4y\left(\frac{y-mx}{c}\right)=0 \quad ....(3)$$

These represent two straight line joining origin with the point of intersection of (1) and (2)

Since angle between these lines is 90^0

$\therefore$ Co-efficient of x^2 + Co-efficient of y^2 = 0

$$\therefore 3+\frac{2m}{c}+\left(-1+\frac{4}{c}\right)=0 \Rightarrow (m+2)=-c$$

$\Rightarrow$ (y+2)=m(1-x) = 0

m is a variable. Chords pass through point of intersection of y+2=0 and x = 1, i.e. all such chords are concurrent, pass through (1,-2).

Example 21: C is the centre of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The tangent at any point on it meets the asymptotes in Q and R. Prove that $CQ.CR=a^2+b^2$

Sol: Asymptotes of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

are $\frac{x}{a}+\frac{y}{b}=0$ and $\frac{x}{a}-\frac{y}{b}=0$

any tangent is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

$\therefore \frac{x}{a}=\frac{y}{b}$ gives $Q\left[\frac{a}{\sec\theta-\tan\theta},\frac{b}{\sec\theta-\tan\theta}\right]$

and $\frac{x}{a}=-\frac{y}{b}$ gives $Q\left[\frac{a}{\sec\theta+\tan\theta},\frac{-b}{\sec\theta+\tan\theta}\right]$

$$\therefore CQ.CR = \left(\frac{\sqrt{a^2+b^2}}{\sec\theta-\tan\theta}\right).\left(\frac{\sqrt{a^2+b^2}}{\sec\theta+\tan\theta}\right)=\frac{a^2+b^2}{\sec^2\theta-\tan^2\theta}=c^2+b^2$$

Example 22: Prove that the segment of tangent to the hyperbola, intercepted between asymptotes is bisected at the point of contact and area of triangle formed by tangent and asymptotes is constant.

Sol: Tangent at (x_1,y_1), $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ ....(1)

Asymptotes $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$ ....(2)

From (i), (ii) $\frac{xx_1}{a^2}-\frac{y_1}{b^2}.\frac{b}{a}x=1 \Rightarrow x=\frac{a^2b}{bx_1-ay_1}$

$\therefore$ Point of Intersection of (1) and $y = \frac{b}{a}x$; P is $\frac{a^2b}{bx_1 - ay_1}, \frac{ab^2}{bx_1 - ay_1}$

$\therefore$ Similarly from $y = -\frac{b}{a}x$; Q is $\frac{a^2b}{bx_1 + ay_1}, \frac{-ab^2}{bx_1 + ay_1}$

Let mid point of PQ be (h,k)

$$\therefore\ 2h = \frac{a^2b}{bx_1 - ay_1} + \frac{a^2b}{bx_1 + ay_1} = \frac{a^2b(2bx_1)}{b^2x_1^2 - a^2y_1^2}$$

point (x_1, y_1) is on hyperbola, $\therefore\ b^2x_1^2 - a^2y_1^2 = a^2b^2$

$$\therefore\ 2h = \frac{a^2b(2bx_1)}{a^2b^2} = 2x_1 \Rightarrow h = x_1$$

Similarly $2k = \frac{a^2b(2ay_1)}{a^2b^2} = 2y_1 \Rightarrow k = y_1$

$\therefore$ Mid point of PQ is point of contact of tangent.

(ii) $$PQ^2 = \left(\frac{a^2b}{bx_1 - ay_1} - \frac{a^2b}{bx_1 + ay_1}\right)^2 + \left(\frac{ab^2}{bx_1 - ay_1} - \frac{ab^2}{bx_1 + ay_1}\right)^2$$

$$= \frac{a^4b^2(2ay_1)^2}{a^4b^4} + \frac{a^2b^4(2bx_1)^2}{a^4b^4}$$

$$= 4 \cdot \frac{a^2}{b^2}y_1^2 + 4 \cdot \frac{b^2}{a^2} \cdot x_1^2 = \frac{4}{a^2b^2}(a^4y_1^2 + b^4x_1^2)$$

$$\therefore\ PQ = \frac{2}{ab}\sqrt{a^4y_1^2 + b^4x_1^2}$$

Perpendicular from c (0, 0) on tangent $xx_1b^2 = yy_1a^2 = a^2b^2$

$$p = \frac{a^2b^2}{\sqrt{x_1^2b^4 + y_1^2a^4}} \Rightarrow \Delta CPQ = \frac{1}{2}p \cdot PQ$$

$\therefore$ Area of triangle = $\frac{1}{2}\frac{a^2b^2 2\sqrt{a^4y_1^2 + b^4x_1^2}}{\left[\sqrt{x_1^2b^4 + y_1^2a^4}\right](ab)}$

= ab = constant.

Example 23: Prove that the locus of the mid points of chords of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ which pass through a fixed point (α, β) is a hyperbola whose centre is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

Sol. Equation of chords through (α, β) is $y - \beta = m(x - \alpha)$, $m \in R$

Let (h, k) be the mid point of this chord.

$\therefore$ m = slope of polar of (h, k) w.r.t. to hyperbola; Polar $\frac{xh}{a^2} - \frac{yk}{b^2} = 1 \Rightarrow m = \frac{h}{k} \cdot \frac{b^2}{a^2}$ (h, k) also lies on chord.

$$\therefore\ k - \beta = \frac{h}{k} \cdot \frac{b^2}{a^2}(h - \alpha)$$

$\Rightarrow a^2 k^2 - a^2 k\beta = h^2 b^2 - h\,\alpha b^2$

$\therefore$ Locus of mid point (h, k) is

$b^2 x^2 - a^2 y^2 - b^2 x\,\alpha + a^2 \beta y = 0$

$$\Rightarrow b^2\left[x^2 - x\alpha + \left(\frac{\alpha}{2}\right)^2\right] - a^2\left[y^2 - \beta y + \left(\frac{\beta}{2}\right)^2\right] = \frac{\alpha^2 + \beta^2}{4}$$

$$\Rightarrow \frac{\left(x - \frac{\alpha}{2}\right)^2}{a^2} - \frac{\left(y - \frac{\beta}{2}\right)^2}{b^2} = \frac{\alpha^2 + \beta^2}{4a^2b^2}$$

Locus is hyperbola whose centre is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

Example 24: Prove that the intercept of tangent of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ between point of contact and directrix subtends a right angle at the focus.

Sol. Tangent at P (a sec θ, b tan θ) of parabola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \qquad \text{... (1)}$$

Focus F is (ae, 0), slope PF = $\frac{b\tan\theta}{a(\sec\theta - x)}$

$$\therefore\ m_1 = \frac{(b\tan\theta)}{a(\sec\theta - e)}$$

tangent meets directrix $x = \frac{a}{e}$ in Q.

$$\therefore\ \frac{\sec\theta}{e} - 1 = \frac{y}{b}\tan\theta \Rightarrow y = \frac{b}{\tan\theta}\left(\frac{\sec\theta - e}{e}\right)$$

$\therefore$ Q is $\left(\frac{a}{e}, \frac{b(\sec\theta - e)}{e\tan\phi}\right)$

Slope of FQ = m_2 = $\frac{b(\sec\theta - e)}{\tan\theta\left(\frac{a}{e} - ae\right)}$

$= \frac{-b(\sec\theta - e)}{a\tan\theta(e^2 - 1)} = \frac{-b(\sec\theta - e)}{a\tan\theta \cdot b^2} \cdot a^2$

$= -\frac{a}{b} \cdot \frac{\sec\theta - e}{\tan\theta}$

$\therefore$ m_1 m_2 = $\left[\frac{b\tan\theta}{a(\sec\theta - e)}\right]\left[-\frac{a}{b}.\frac{(\sec\theta - e)}{\tan\theta}\right] = -1$

i.e. the intercept of tangent between point of contact and directrix subtends a right angle at focus.

Example 25: Show that tangents at the ends of focal chord meet (intersect) on corresponding directrix.

Sol. Let ϕ_1 P and ϕ_2 Q be ends of focal chord.

$\therefore$ Slope of PS = slope of QS

$\therefore$ $\frac{b\tan\theta_1}{a\sec\theta_1 - ae} = \frac{b\tan\theta_2}{a\sec\theta_2 - ac}$

$\Rightarrow \tan\theta_1 \sec\theta_2 - e\tan\theta_1 = \sec\theta_1 \tan\theta_2 - e\tan\theta_2$

$\Rightarrow \sec\theta_1 \tan\theta_2 - \tan\theta_2 \sec\theta_2 = e(\tan\theta_2 - \tan\theta_1)$ ... (1)

Equations of tangents at P and Q are

$bx\sec\theta_1 - ay\tan\theta_1 = ab$ ... (2)

$bx + \sec\theta_2 - ay\tan\theta_2 = ab$ ... (3)

Multiplying (2) by $\tan\theta_2$ and (3) by $\tan\theta_1$ and then subtracting.

$bx\sec\theta_1\tan\theta_2 - bx\sec\theta_2\tan\theta_1 = ab(\tan\theta_2 - \tan\theta_1)$

$\Rightarrow bx(\sec\theta_1\tan\theta_2 - \sec\theta_2\tan\theta_1) = ab(\tan\theta_2 - \tan\theta_1)$

$\therefore$ from (1) $ebx(\tan\theta_2 - \tan\theta_1) = ab(\tan\theta_2 - \tan\theta_1)$

$\Rightarrow x = \frac{a}{e}$ i.e. tangents meet at directrix.

Example 26: The normal at P of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ meets transverse axis in G and conjugate axis in g. CF is perpendicular from centre e on normal. Prove

(a) $PF \cdot PG = CB^2 = b^2$ (b) $PF \cdot Pg = CA^2 = a^2$

Sol. Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Equation of normal at (a sec θ, b tan θ) $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ ... (1)

It meets axes in G and g.

$\therefore$ G is $\left(\frac{a^2+b^2}{a\cos\theta}, 0\right)$; g, $\left(0, \frac{(a^2+b^2)\tan\theta}{b}\right)$

CF perpendicular on normal.

$$y = \frac{b\sec\theta}{a\tan\theta}x \Rightarrow \frac{bx}{\tan\theta} - \frac{ay}{\sec\theta} = 0$$

PF = perpendicular from P on CF

$$= \frac{\left[\frac{b \cdot a\sec\theta}{\tan\theta} - \frac{a \cdot b\tan\theta}{\sec\theta}\right]}{\sqrt{\frac{b^2}{\tan^2\theta} + \frac{a^2}{\sec^2\theta}}}$$

$$= \frac{ab[\sec^2\theta - \tan^2\theta]}{\tan\theta\sec\theta} \cdot \left(\frac{\tan\theta \cdot \sec\theta}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}\right)$$

$$= \frac{ab}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$$

$$(PG)^2 = \left[\frac{a^2+b^2}{a^2} \cdot a\sec\theta - a\sec\theta\right]^2 + b^2\tan^2 gq$$

$$= [e^2 a\sec\theta - a\sec\theta]^2 + b^2\tan^2\theta$$

$$= [a\sec\theta(e^2-1)]^2 + b^2\tan^2\theta = \frac{b^2}{a^2}[b^2\sec^2\theta + a^2\tan^2\theta] \quad \text{... (2)}$$

$$(Pg)^2 = \left[a^2\sec^2\theta + \tan^2\theta\left(b - \frac{a^2+b^2}{b}\right)^2\right]$$

$$= a^2\sec^2\theta + \frac{a^4}{b^2}\tan^2\theta \quad \text{... (3)}$$

$$\therefore PF \cdot PG = \frac{ab\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}{\sqrt{b^2\sec^2\theta + \tan^2\theta}} \cdot \frac{b}{a} = b^2$$

$$PF \cdot Pg = ab \times \frac{a}{b} = a^2$$

Example 27: P and Q are two such points on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ that CP $\perp$ CQ. Prove $\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} + \frac{1}{b^2}$

Sol. Let CP be y = mx, It meets hyperbola

$$\therefore \frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1 \Rightarrow x^2(b^2 - a^2m) = a^2b^2$$

$$\Rightarrow x^2 = \frac{a^2b^2}{b^2 - a^2m^2} \text{ and } y^2 = m^2x^2 = \frac{m^2a^2b^2}{b^2 - a^2m^2}$$

$$CP^2 = x^2 + y^2 = \frac{a^2b^2(1+m^2)}{b^2 - a^2m^2}$$

Putting $-\frac{1}{m}$ for m in it we get CQ^2

$$CQ^2 = \frac{a^2b^2(1+m^2)}{(b^2m^2 - a^2)}$$

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{b^2 - a^2m^2 + b^2m^2 - a^2}{a^2b^2(1+m^2)} = \frac{1}{a^2} - \frac{1}{b^2}$$

$$= \frac{(1+m^2)(b^2 - a^2)}{a^2b^2(1+m^2)}$$

Practice Worksheet (Foundation Level) – 8 (b)

1. Find equation of tangent and normal at $\left(-2, \frac{2\sqrt{7}}{3}\right)$ of hyperbola $4x^2 - 9y^2 - 16x = 20$.

2. Find equation of tangents equally inclined to axes of hyperbola $3x^2 - 4y^2 = 12$.

3. Find locus of poles of focal chords of hyperbola $x^2 - 4y^2 = 4$

4. Prove that polar of any point of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to hyperbola $\frac{x^2}{a} - \frac{y^2}{b} = ab$ touches circle $x^2 + y^2 = a^2b^2$.

5. Prove that locus of the poles of tangents of $x^2 + y^2 = 9$ with respect to hyperbola $3x^2 - 4y^2 = 1$ is ellipse $81x^2 + 144y^2 = 1$

6. Find equation of hyperbola, whose focus is (a, 0) eccentricity is $\frac{5}{4}$ and directrix $4x - y = a$.

7. Find locus of point of intersection of $\ell x + my = \ell mt$ and $\ell x - my = \frac{\ell m}{t}$.

8. Find equation of asymptotes and conjugate hyperbola of hyperbola $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$

9. Find equation of asymptotes of $(x^2 - y^2) \sin 2\theta + 2 xy \cos 2\theta = a^2$

10. Find equation of asymptotes of hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find general equation of all those hyperbolas which have these asymptotes.

11. P is a point on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Tangent at P, latus-rectum and one asymptote are concurrent. S is focus, prove SP is parallel to other asymptote.

12. Prove that locus of poles of tangents of circle $x^2 + y^2 = 9$ with respect to hyperbola $3x^2 - 4y^2 = 1$ is an ellipse.

13. Perpendiculars from centre(i), on any tangent at P of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ meets it in Q. Find locus of Q.

14. Prove that locus of poles of tangent of circle drawn on SS' as diameter (S, S' are focii) of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$

15. a, b are variable quantities. The polar of (a, b) with respect to hyperbola $2x^2 - 3y^2 = b$, always passes through a fixed point. The fixed point is ...

16. Prove that the segment of tangent of hyperbola $3x^2 - 4y^2 = 12$ between point of contact and directrix subtends at the focus an angle of $\frac{\pi}{2}$.

17. A tangent of hyperbola $2x^2 - 3y^2 = 6$ meets axes in P and Q. Prove that locus of point which divides PQ in the ratio of 1 : 2 is $12y^2 - 2x^2 = 9x^2y^2$.

18. Prove that the latus rectum on the right side of y-axis of hyperbola $5x^2 + 4y^2 = 20$ makes on corresponding vertex an angle of $2\tan^{-1}\left(\frac{5}{2}\right)$.

19. Find length of chord of hyperbola $5x^2 - 4y^2 = 20$ whose mid points is (4, 2).

20. PFQ is a focal chord of hyperbola $9x^2 - 16y^2 = 144$. F is focus PF = 3 FQ. Find length of this chord.

21. Eccentricity of hyperbola $2x^2 - 3y^2 = 1$ is

(a) $\sqrt{\frac{5}{3}}$ (b) $\sqrt{\frac{5}{2}}$ (c) $\sqrt{\frac{3}{2}}$ (d) none of these

22. If x + y = 5 touches hyperbola $x^2 - 2y^2 = 50$ then point of contact is

(a) (10, 5) (b) (10, –5) (c) $(8, \sqrt{7})$ (d) $(8, -\sqrt{7})$

23. $\ell x + my = 1$ shall touch $3x^2 - 2y^2 = 4$, if

(a) $4\ell^2 - 18m^2 = 9$ (b) $4\ell^2 - 12m^2 = 9$

(c) $2\ell^2 - 3m^2 = 6$ (d) $4\ell^2 - 6m^2 = 3$

24. A normal of hyperbola $3x^2 - 4y^2 = 12$ meets transverse axis is in (7, 0). Inclination of normal is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{3\pi}{4}$

25. Equation of tangent of hyperbola $4x^2 - 5y^2 = 1$ inclined at 60^o to x-axis is

(a) $y = \sqrt{3}x \pm \sqrt{11}$ (b) $y = \sqrt{3}x \pm \frac{1}{5}\sqrt{11}$

(c) $y = \sqrt{3}x - \sqrt{\frac{11}{5}}$ (d) $y = \sqrt{3}x \pm \frac{1}{2}\sqrt{\frac{11}{5}}$

26. The equation of tangent of hyperbola $bx^2 - ay^2 = ab$ which goes through centre of hyperbola is

(a) $y = \frac{b}{a}x$ (b) $y = \frac{a}{b}x$ (c) $\sqrt{a}y \pm \sqrt{b}x = 0$ (d) none of these

27. The straight line x cos α + y sin α = p shall touch the hyperbola $9x^2 - 16y^2 = 144$ if

(a) $16\cos^2\alpha - 9\sin^2\alpha = p^2$ (b) $9\sin^2\alpha - 16\cos^2\alpha = p^2$

(c) $4\cos\alpha - 3\sin\alpha = p$ (d) $9\cos^2\alpha - 16\sin^2\alpha = p^2$

28. The straight line $4x - \sqrt{3}y = 6$ touch hyperbola $4x^2 - 3y^2 = 12$, point of contact is

(a) $\left(2, -\frac{2}{\sqrt{3}}\right)$ (b) $\left(-2, -\frac{2}{\sqrt{3}}\right)$ (c) $\left(2, \frac{2}{\sqrt{3}}\right)$ (d) $\left(-2, \frac{2}{\sqrt{3}}\right)$

29. Tangents at the ends of a latus-rectum of hyperbola $2x^2 - 3y^2 = 1$ meet at

(a) $\left(\sqrt{\frac{6}{5}}, 0\right)$ (b) $\left(\sqrt{\frac{5}{6}}, 0\right)$ (c) $\left(\sqrt{\frac{10}{3}}, 0\right)$ (d) $\left(\sqrt{\frac{3}{10}}, 0\right)$

30. Acute angle between asymptotes of hyperbola $4x^2 - 3y^2 + 8x + 6y - 11 = 0$ is

(a) $\tan^{-1}\left(\frac{4}{3}\right)$ (b) $\tan^{-1}\left(4\sqrt{3}\right)$ (c) $\tan^{-1}\left(\frac{4}{\sqrt{3}}\right)$ (d) $\tan^{-1}(2\sqrt{3})$

31. Tangents drawn from (2, 3) to hyoerbola $4x^2 - 5y^2 = 20$ are

(a) y = x + 1 (b) y = 2x + 7 (c) y + x = 3 (d) y + 13x = 29

32. Tangents at the ends of focal chords of hyperbola $2x^2 - 3y^2 = 6$ meet at

(a) $\sqrt{5}x - 3 = 0$ (b) $\sqrt{5}x + 3 = 0$ (c) $\sqrt{5}x - \sqrt{3} = 0$ (d) $\sqrt{5}x + 3\sqrt{3} = 0$

33. Equation of chord of hyperbola $9x^2 - 16y^2 = 144$ whose mid point is $\left(6, \frac{3}{2}\right)$ is

(a) 4x – 9y = 1 (b) 4x – 9y = 32 c) 9x – 16y = 20 d) 9x – 4y = 48

34. The length of tangent drawn at $\left(1, \frac{1}{\sqrt{3}}\right)$ of hyperbola $2x^2 - 3y^2 = 1$ between asymptotes is

(a) $\sqrt{\frac{10}{3}}$ (b) $\sqrt{\frac{14}{3}}$ (c) $\sqrt{\frac{38}{3}}$ (d) $\sqrt{5}$

35. The tangent at L_1 end of Latus rectum (1^{st} quadrant) and normal at vertex of hyperbola $3x^2 - 4y^2 = 12$ meet at

(a) centre (b) asymptote (c) directrix (d) none of these

36. The distance between focus and corresponding directrix of hyperbola $2x^2 - 3y^2 = 6$ is

(a) $\frac{3}{\sqrt{5}}$ (b) $\frac{3}{\sqrt{10}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{1}{\sqrt{5}}$

37. Tangents from (3, 1) on hyperbola $\frac{x^2}{16}-\frac{y^2}{12}=1$ are

(a) $y - x = \pm 2$ (b) $7y + 13x = \pm 46$ (c) $7y - 13x = \pm 46$ (d) $x + y = \pm 2$

38. The locus of poles of tangents of circle $x^2 + y^2 = 9$ with reference to hyperbola $3x^2 - 4y^2 = 12$ is

(a) $9x^2 + 16y^2 = 16$ (b) $9x^2 - 16y^2 = 12$

(c) $9x^2 + 16y^2 = 144$ (d) $3x^2 + 4y^2 = 15$

39. Perpendicular dropped from centre of $\frac{x^2}{15}-\frac{y^2}{16}=1$ on its tangent meet at Q. Locus of Q is

(a) $x^2 + y^2 = 25x^2 - 16y^2$ (b) $x^2 + y^2 = 16y^2 - 25x^2$

(c) $x^2 + y^2 = 5x^2 - 4y^2$ (d) none of these

40. Eccentricity of conic $9x^2 - 4y^2 - (8x - 16y - 43) = 0$

(a) $\frac{\sqrt{13}}{3}$ (b) $\frac{\sqrt{13}}{2}$ (c) $\frac{\sqrt{5}}{2}$ (d) none of these

8.11 Diameter

Locus of mid points of parallel chords of hyperbola is called diameter. If y = mx + λ are parallel chords then as in ellipse equation of diameter is $y = \frac{b^2}{a^2 m} x$

The slope of this diameter is $m_1 = \frac{b^2}{a^2 m}$, chords parallel to this diameter are $y = m_1 x + \lambda$

Equation of locus of mid points of these chords is $y = \frac{b^2}{a^2 m_1} x = m_2 x$

i.e. Slope of this diameter is m_2, and $m_2 = \frac{b^2}{a^2 m_1} \Rightarrow m_1 m_2 = \frac{b^2}{a^2}$

$\therefore$ If $y_1 = m_1 x$ and $y = m_2 x$ be two such diameters that each bisects the chords parallel to the other. Then these are called conjugate diameter and the product of their slopes i.e.

$$m_1 \cdot m_2 = \frac{b^2}{a^2}$$

(b) Let y = mx be a diameter of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore\ x^2\left[\frac{1}{a^2} - \frac{m^2}{b^2}\right] = 1 \Rightarrow x^2 = \frac{a^2 b^2}{b^2 - a^2 m^2}$$

The diameter shall meet the hyperbola in real points only when $a^2 m^2 < b^2$

The numerical value of m be less than $\frac{b}{a}$

(c) $y = m_1 x$ and $y = m_2 x$ shall be conjugate diameter if $m_1 m_2 = \frac{b^2}{a^2}$

Now if numerical value of m_1 is less than $\frac{b}{a}$ then numerical value of $m_2 > \frac{b}{a}$.

$\therefore$ If $y = m_1 x$ and $y = m_2 x$ are two conjugate diameters, and $y = m_1 x$ meets the hyperbola in real points then $y = m_2 x$ shall meet the hyperbola in imaginary points.

(d) The equation of conjugate diameter is obtained by putting $-a^2$ for a^2 and $-b^2$ for b^2 and because $\left(\frac{-b^2}{-a^2}\right) = \frac{b^2}{a^2}$, therefore two diameters which are conjugate for hyperbola are also conjugate diameters of conjugate hyperbola. And if one of them meets hyperbola in real points then the other shall meet conjugate hyperbola in real points.

8.12 Rectangular hyperbola

When a = b i.e. when transverse axis is equal to conjugate axis. Then equation of hyperbola is $x^2 - y^2 = a^2$. This is the equation of rectangular hyperbola. Asymptotes $x^2 - y^2 = 0$ i.e. x + y = 0, x – y = 0. These intersect at right angles.

Eccentricity of $x^2 - y^2 = a^2$ is $e^2 = \frac{2a^2}{a^2} = 2$

$\Rightarrow e = \sqrt{2}$; focii, $(\pm\sqrt{2}a, 0)$

directrix $x = \pm\frac{a}{\sqrt{2}}$, Vertex $(\pm a, 0)$Latus rectum is 2a.

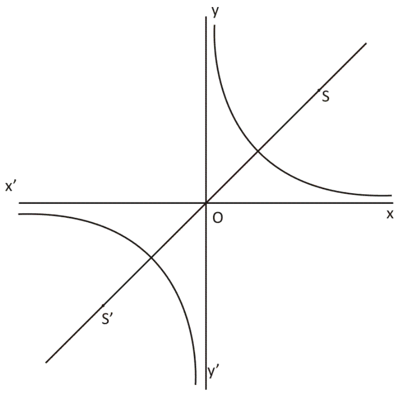

Fig 6

If axes of this hyperbola be along asymptotes x – y = 0, x + y = 0 then asymptotes shall be x = 0, y = 0 and equation of hyperbola shall be xy = constant = c^2 (suppose)

Now if axes be rotated through -45^o we must get original equation of hyperbola.

i.e. $\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)\left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y\right) = c^2$

$\Rightarrow \quad x^2 - y^2 = 2c^2$

But original equation is $x^2 - y^2 = a^2 \Rightarrow c^2 = \frac{a^2}{2}$

$\therefore$ Equation of rectangular hyperbola with asymptotes as axes is $xy = c^2$, $\left(c^2 = \frac{a^2}{2}\right)$

(a) Vertex (c, c) and (–c, c)

(b) Focii $(\sqrt{2}c, \sqrt{2}c)$ and $(-\sqrt{2}c, -\sqrt{2}c)$

(c) Equation of directrix $x + y = \sqrt{2}c, x + y = \sqrt{2}c$

(d) Equation of Latus rectum $x + y = \pm 2\sqrt{2}c$

(e) Any point t on $xy = c^2$ is $\left(ct, \frac{c}{t}\right)$

(f) Line joining t_1 and t_2 is $x + y\, t_1 t_2 - c(t_1 + t_2) = 0$

(g) If the line is focal chord then $\sqrt{2} = \frac{t_1 + t_2}{1 + t_1 t_2}$

(h) When $t_1 = t_2$ the line becomes tangent.

$\therefore$ Equation of tangent at t is $x + t^2 y - 2ct = 0$

(i) Tangent at (x_1, y_1) is $\frac{x}{x_1} + \frac{y}{y_1} = 2 \left(m = -\frac{y_1}{x_1} \right)$

(j) Equation of normal at t,

Slope of tangent is $-\frac{1}{t^2} \Rightarrow$ of normal t^2

$\therefore$ Normal is $y - \frac{c}{t} = \frac{t^2}{4}(x - ct)$

i.e. $xt^3 - yt - ct^4 + c = 0$

(k) Normal at (x_1, y_1) is $y - y_1 = \frac{x_1}{t_1}(x - x_1)$

i.e. $xx_1 - yy_1 = x_1^2 - y_1^2$

Solved Examples

Example 28: Find equation of tangents of xy = 4 which are equally inclined to axes.

Sol. Any tangent is $x + yt^2 - 2ct = 0$ (c = 2)

Tangent equally inclined means $m_1 = 1$ or –1. If m = 1 then equation gives $-\frac{1}{t^2} = 1$

i.e. $t^2 = -1$ not possible $\therefore$ $m \neq 1$

$m = -1$, then $-\frac{1}{t^2} = -1$ i.e. $t = \pm 1$

$\therefore$ tangents are $x + y = 1 \pm 4$

Note: If hyperbola is x y + 4 = 0

$\therefore$ Point on hyperbola is $\left(2t - \frac{2}{t}\right)$

$\frac{dy}{dx} = -\frac{y}{x}$ $\therefore$ at point $\frac{dy}{dx} = \frac{1}{t^2}$

$\therefore$ Tangent $y + \frac{2}{t} - \frac{1}{t^2}(x - 2t)$

Now if $\frac{1}{t^2} = 1$ i.e. $t = \pm 1$, angle 45^o is valid

But if $\frac{1}{t^2} = -1$, not valid, 135^o not valid.

$\therefore$ Tangents are $y \pm 2 = 1 \cdot (x \mp 2)$

$\Rightarrow y = x \pm 4$

Example 29: Show that locus of mid points of normal chords of rectangular hyperbola is $x^2 - a^2$ is $(y^2 - x^2)^3 = 4a^2 x^2 y^2$

Sol. Equation of chord with mid point (h, k) is $y - k = m(x - h)$

Slope m of it is slope of polar of (h, k) = $\frac{h}{k}$

$\therefore$ Chord is $y - k = \frac{h}{k}(x - h) \Rightarrow hx + ky + (k^2 - h^2) = 0$ ... (1)

Equation of normal is $x \cos\theta + y \cot\theta = 2a$ ... (2)

If (1) and (2) are same then, comparing $h \sec\theta = -k \tan\theta = \frac{h^2 - k^2}{2a}$

$\therefore \sec\theta = \dfrac{h^2 - k^2}{2ah}$; $\tan\theta = -\dfrac{(h^2+k^2)}{2ak}$

and $\sec^2\theta - \tan^2\theta = 1 \therefore (h^2-k^2)^2\left[\dfrac{1}{4a^2h^2} - \dfrac{1}{4a^2k^2}\right] = 1$

$\therefore$ Locus $(y^2 - x^2)^3 = 4a^2 x^2 y^2$

Example 30: Find the equation of normal of hyperbola $xy = c^2$ inclined at 60^o to x-axis.

Sol. The equation of normal at point t is $xt^3 - yt - ct^4 + c = 0$

Slope $t^2 = \sqrt{3} \Rightarrow t^4 = 3,\ t^3 = (3)^{\frac{3}{4}},\ t = (3)^{\frac{1}{4}}$

$\therefore$ Normal is $(\sqrt{3}x - y)(3)^{\frac{1}{4}} = c\,(3-1) = 2c$

Example 31: Prove that the foot of perpendicular from focus S on asymptote on auxiliary circle and also lies on corresponding directrix.

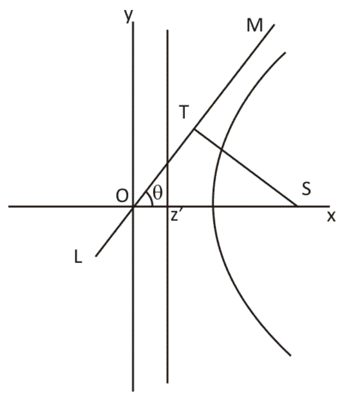

Fig 7

Sol. In the figure (7) ST is perpendicular fro focus and asymptote LM.

Let $\angle TCS = \theta$

Asymptote is $\dfrac{x}{a} - \dfrac{y}{b} = 0$

$\Rightarrow \tan\theta = \dfrac{b}{a}$ and $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$

From right angled triangle CTS

$CT = CS\cos\theta = ae \cdot \dfrac{a}{\sqrt{a^2+b^2}} = \dfrac{ae \cdot a}{ae}$

$= a$

$\therefore$ T lies on auxiliary circle.

(ii) $CZ \cdot CS = \dfrac{a}{e} \cdot ae = a^2 = eT^2$

$\therefore \angle TZS = 90^o$, $\left(\because \dfrac{CT}{CS} = \dfrac{CZ}{CT}\right)$

$\therefore$ T lies on directrix as well.

$\therefore$ Foot of perpendicular drawn focus on any asymptote lies on auxiliary circle and also on corresponding directrix.

Example 32: Find the equation of normal of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which is parallel to the asymptote?

Sol. Equation of normal is ax cos θ + by cot θ = $a^2 + b^2$

and equation of other asymptote is $\frac{x}{a}+\frac{y}{b}=1$

$\therefore$ slope $-\frac{a}{b}\cdot\frac{\cos\theta}{\cot\theta}=-\frac{b}{a}\Rightarrow \sin\theta=\frac{b^2}{a^2}$

$\cos\theta = \frac{\sqrt{a^4-b^4}}{a^2}$ $\quad \cot\theta=\frac{\sqrt{a^4-b^4}}{b^2}$

$\therefore$ Normal, $\frac{ax\sqrt{a^4-b^4}}{a^2}+by\frac{\sqrt{a^4-b^4}}{b^2}=a^2+b^2$

$\Rightarrow \frac{x}{a}+\frac{y}{b}=\frac{a^2+b^2}{\sqrt{a^4-b^4}}=\sqrt{\frac{a^2+b^2}{a^2-b^2}}$

Example 33: Show that the normal to hyperbola xy = c^2 at point t_1 meets the curve again at t_2 then $t_1^3\cdot t_2=-1$

Sol. Equation of normal of hyperbola xy = c^2 at t_1 is $xt_1^3+yt_1=c(t_1^4-1)$

It passes through $\left(ct_2,\frac{c}{t_2}\right)$

$\therefore\ ct_2t_1^3-\frac{c}{t}\cdot t_1=c(t_1^4-1)$

$\Rightarrow 1+t_2t_1^3-\frac{t_1}{t_2}(t_1^3\cdot t_2+1)=0$

$\Rightarrow (1+t_2t_1^3)\left(1-\frac{t_1}{t_2}\right)=0$

$t_1\neq t_2 \quad \therefore\ t_2\cdot t_1^3+1=0$.

Example 34: Prove that conjugate diameters of hyperbola and conjugate hyperbola are the same. Also prove that if one diameter meets hyperbola in real points then its conjugate shall meets conjugate hyperbola in real points.

Sol. Equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... (i) equation of conjugate hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ ... (ii)

diameters are conjugate if

from (i) $m_1 \cdot m_2 = \frac{b^2}{a^2}$, from (ii) $m_1 m_2 = \frac{-b^2}{-a^2} = \frac{b^2}{a^2}$

$\therefore$ Conjugate diameter of the two hyperbola are the same.

(ii) Diameter y = m_1x meets hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in real points $x^2\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)=1$

$$\Rightarrow x = \frac{\pm ab}{\sqrt{b^2 - a^2 m_1 y_1^2}}$$

$\therefore\ b^2 > a^2 m_1^2$ i.e. $|m_1| < \frac{b}{a}$

and if $m_1 < \frac{b}{a}$, $m_2 > \left|\frac{b}{a}\right|$

y = m_2x shall meet conjugate hyperbola then $x^2\left(\frac{m_2^2}{b^2}-\frac{1}{a^2}\right)=1 \Rightarrow x^2 = \frac{a^2b^2}{a^2m_2^2 - b^2}$

It shall meet in real points if $a^2m_2^2 - b^2 > 0$

$\Rightarrow |m_1| > \frac{b}{a}$

$\therefore$ Of the two conjugate diameters one meets hyperbola in real points and the other meets conjugate hyperbola in real points.

Example 35: Prove that the segment of tangent of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ intercepted between asymptotes is bisected at point of contact. Find area of triangle formed by tangent and asymptote.

Sol. With asymptotes as axes equation of hyperbola is $xy = \frac{a^2+b^2}{4}$ or xy = c^2 where

$c^2 = \frac{a^2+b^2}{4}$

If angle between asymptotes is 2α, then tan $\alpha = \frac{b}{a}$. Tangent at (x_1, y_1) of xy = c^2 is

$\frac{x}{x_1}+\frac{y}{y_1}=2$ and as point is on curve $x_1y_1 = c^2$

Tangent meets axes (asymptote) at $(2x_1, 0)$, $(0, 2y_1)$

Mid points of this intercept (x_1, y_1), which is point of contact of tangent.

(ii) Area = $2x_1y_1 \tan 2\alpha = 2x_1y_1 \dfrac{2\tan\alpha}{1+\tan^2\alpha}$

$$= 2c^2 \cdot \frac{2ab}{a^2+b^2} = 2\cdot\left(\frac{a^2+b^2}{4}\right)\left(\frac{2ab}{a^2+b^2}\right)$$

= ab = constant

Example 36: The pair of conjugate diameter meet hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ in P and conjugate hyperbola in D. Prove $CP^2 = CD^2 = a^2 - b^2$.

Sol. Let PCP' be one diameter, P $(a \sec\theta, b\tan\theta)$

$\therefore$ Equation CP $y = \dfrac{b}{a}x\sin\theta$ (Equation diameter)

and from definition of conjugate diameter, slope of other diameter $m_2 = \dfrac{b^2}{a^2m_1} = \dfrac{b^2\cdot a}{a^2b\sin\theta}$

$$\therefore\ m_2 = \left(\frac{b}{a}\right)\frac{1}{\sin\theta}$$

$\therefore$ Equation of other diameter $y = \dfrac{b}{a}\cdot\dfrac{x}{\sin\theta}$

It meets conjugate diameter DCD' in D

Point D is $(b\sec\theta, a\tan\theta)$; c (0, 0)

$$\left.\begin{aligned}\therefore CP^2 &= a^2\sec^2\theta + b^2\tan^2\theta\\ CD^2 &= b^2\sec^2\theta + a^2\tan^2\theta\end{aligned}\right\} \quad \begin{aligned}&\therefore CP^2 - CD^2\\ &= (a^2-b^2)(\sec^2\theta - \tan^2\theta)\end{aligned}$$

$\therefore\ CP^2 - CD^2 = a^2 - b^2$

Example 37: Prove that from a point (α, β) four normals can be drawn to rectangular hyperbola $xy = c^2$ and if co-ordinates of foot of normals be (xi, yi); i = 1, 2, 3, 4, then prove (a) $\sum xi = \alpha$, (b) $\sum yi = \beta$, (c) $x_1\cdot x_2\cdot x_3\cdot x_4 = y_1\cdot y_2\cdot y_3\cdot y_4 = -c^4$

Sol. Equation of normal of $xy = c^2$ is $xt^3 - yt = c(t^4 - 1)$ ... (1)

It passes through (α, β)

$\therefore\ \alpha t^3 - \beta t = c(t^4 - 1)$

$\Rightarrow ct^4 - \alpha t^3 + \beta t - c = 0$

It gives four values of t, namely t_1, t_2, t_3, t_4 and $\sum t_1 = \frac{\alpha}{c} \Rightarrow ct_1 + ct_2 + ct_3 + ct_4 = \alpha$

$\Rightarrow \sum xi = \alpha$

(ii) and $\sum t_1t_2t_3 = -\frac{\beta}{c}$ and $t_1\, t_2\, t_3\, t_4 = -1$

$$\therefore \frac{\sum t_1t_2t_3}{t_1\, t_2\, t_3\, t_4} = \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = +\frac{\beta}{c}$$

$$\therefore \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3} + \frac{c}{t_4} = \beta \Rightarrow \sum yi = \beta$$

(iii) $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = c^4 \cdot t_1\, t_2\, t_3\, t_4 = -c^4$

$$y_1 \cdot y_2 \cdot y_3 \cdot y_4 = \frac{c^4}{t_1t_2t_3t_4} = \frac{c^4}{-1} = -c^4$$

$$\therefore x_1 \cdot x_2 \cdot x_3 \cdot x_4 = y_1 \cdot y_2 \cdot y_3 \cdot y_4 = c^4$$

Example 38: A circle circle cuts rectangular hyperbola $xy = c^2$ in A, B, C and D. If these points be respectively t_1, t_2, t_3, t_4 then prove

(i) $t_1\, t_2\, t_3\, t_4 = 1$ (ii) the centre of mean positions of points bisect the distance between two centres of the curves.

Sol. Hyperbola $xy = c^2$, point $\left(ct, \frac{c}{t}\right)$

Let circle be $x^2 + y^2 + 2gx + 2fy + k = 0$

Solving $c^2t^4 + \frac{c^2}{t^2} + 2g \cdot ct + 2f \cdot \frac{c}{t} + k = 0$

$\Rightarrow c^2\, t^2 + 2gct^3 + kt^2 + 2fct^2 + c^2 = 0$

This is equation of 4 degree in t, hence gives four values of t, namely t_1, t_2, t_3, t_4.

$\therefore t_1 + t_2 + t_3 + t_4 = -\frac{2g}{c}$

$\sum t_1t_2 = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{k}{c^2}$

$\sum t_1t_2t_3 = -\frac{2f}{c}$

Mean centre of four points of intersection is

$$\left(c\frac{\sum t_1}{4}, \frac{c}{4}\sum\frac{1}{t_1}\right) \Rightarrow \left(\frac{c}{4}\left(\frac{-2g}{c}\right), \frac{c}{4}\left(-\frac{2f}{c}\right)\right) \Rightarrow \left(-\frac{g}{2}, -\frac{f}{2}\right)$$

Centre of hyperbola is (0, 0) and of circle (–g, –f)

$\therefore$ mid point is $\left(-\frac{g}{2}, -\frac{f}{2}\right)$

$\therefore$ Centre of mean positions of points

Bisects the distance between centres of the curves.

Example 39: Prove that the chords of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ that touch the conjugate hyperbola are bisected at the point of contact.

Sol. Equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and conjugate hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. Any point on the conjugate hyperbola is (a tan θ, b sec θ). Equation of tangent at it is $\frac{y\sec\theta}{b} - \frac{x\tan\theta}{a} = 1$

Slope of this tangent = $\frac{b\tan\theta}{a\sec\theta}$ ... (1)

and point of contact is (a tan θ, b sec θ)

If this point is mid point of chord of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ then its slope is equal to the slope of polar of this mid point with respect to hyperbola.

Polar is $\frac{x\tan\theta}{a} - \frac{y\sec\theta}{b} = 1$

Slope $\frac{b\tan\theta}{a\sec\theta}$ and it is equal to that of tangent. Hence chords of hyperbola which touch conjugate hyperbola are bisected at the point of contact.

Example 40: A normal of hyperbola $b^2x^2 - a^2y^2 = 1$ meets axes in Q and R; QL and RL are drawn at right angles to axes meet in L. Rectangle QORL is completed. Prove that locus of L is $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$. Find locus of mid point of QR.

Sol. Hyperbola $b^2x^2 - a^2y^2 = a^2b^2$

any normal ax cos θ + by cot θ = $a^2 + b^2$

It meets axes in Q and R

$\therefore$ Q is $\left[\frac{a^2+b^2}{a\cos\theta}, 0\right]$ and R is $\left[0, \frac{a^2+b^2}{b\cot\theta}\right]$

Perpendicular on axes at Q and R meet in L

$\therefore$ L is $\left[\frac{a^2+b^2}{a\cos\theta}, \frac{a^2+b^2}{b\cot\theta}\right]$, Let it be (h, k)

$\therefore\ h = \frac{a^2+b^2}{a\cos\theta} \Rightarrow \sec\theta = \frac{ab}{a^2+b^2}$

Similarly $k = \frac{a^2+b^2}{b\cot\theta} \Rightarrow \tan\theta = \frac{bk}{a^2+b^2}$

and $\sec^2\theta - \tan^2\theta = 1 \Rightarrow \frac{a^2h^2 - b^2k^2}{(a^2+b^2)^2} = 1$

$\therefore$ Locus is $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$

Let mid point of RQ be (p, q)

$\therefore\ 2p = \frac{a^2+b^2}{a\cos\theta}$, $2q = \frac{a^2+b^2}{b\cot\theta}$

$\Rightarrow\ \sec\theta = \frac{2ap}{a^2+b^2}, \tan\theta = \frac{2bq}{a^2+b^2}$

$\therefore$ Locus (p, q) $4(a^2x^2 - b^2q^2) = (a^2 + b^2)^2$

Example 41: Prove that conjugate diameter CP and CD of rectangular hyperbola are equal and that their inclination with x-axis is complementary.

Sol. Equation of hyperbola $x^2 - y^2 = a^2$

For conjugate diameters $m_1 \cdot m_2 = \frac{a^2}{b^2} = \frac{a^2}{a^2} = 1$

$\therefore\ m_2 = \frac{1}{m_1} \Rightarrow \tan\theta_1 = \cot\theta_2 = \tan(90 - \theta_2)$

$\Rightarrow \theta_1 = 90 - \theta_2$ i.e. θ_1 and θ_2 are complementary

(ii) If P is (a sec θ, a tan θ) then D shall be on conjugate hyperbola $y^2 - x^2 = a^2$

and $\therefore$ D is (a tan θ, a sec θ)

$\therefore\ CP = \sqrt{a^2\sec^2\theta + a^2\tan^2\theta}, CD = \sqrt{a^2\tan^2\theta + a^2\sec^2\theta}$

$\therefore$ CP = CD

Example 42: Find equation of common tangent of hypobals $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Sol. Any tangent of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=mx+\sqrt{a^2m^2-b^2}$ ... (i)

putting this value of y in equation of second hyperbola,

$b^2(mx+\sqrt{a^2m^2-b^2})^2-a^2x^2-a^2b^2=0$

or $x^2(b^2m^2-a^2)+2mb^2\sqrt{a^2m^2-b^2}+b^2a^2m^2-b^4-a^2b^2=0$

The straight line (tangent) (1) it shall touch second hyperbola if roots of (2) are equal.

i.e. $4m^2b^4(a^2m^2-b^2)=4b^2(b^2m^2-a^2)(a^2m^2-b^2)$

or $a^2(a^2+b^2)m^2=a^2(a^2+b^2) \Rightarrow m \pm 1$

$\therefore$ Common tangent is $y=\pm x+\sqrt{a^2-b^2}$

Example 43: Prove that straight lines 3x + 4y = 18 and 4x + 9y = 15 are parallel to conjugate diameters of hyperbola $2x^2-3y^2=12$

Sol. Given straight lines $\left.\begin{matrix} 3x+4y=18 \\ 8x+9y=15 \end{matrix}\right\} \begin{matrix} m_1=-3/4 \\ m_2=-8/9 \end{matrix}$

and $m_1m_2=\left(-\frac{3}{4}\right)\left(-\frac{8}{9}\right)=\frac{2}{3}$ and $\frac{b^2}{a^2}=\frac{2}{3}$ for conjugate diameters

(Product of the two slopes of diameters - a diameter and its conjugate)

$\therefore$ Straight lines are parallel to conjugate diameters.

Practice Worksheet (Foundation Level) – 8 (c)

1. A and B are two fixed points and AB = 6. Find the locus of a point P which moves in such a way that (PA – PB) is always 4.

2. Find eccentricity focii, directrix and ℓ (latus rectum) of hyperbola $4y^2 - 5x^2 + 30x - 8y - 21 = 0$

3. The focii of a hyperbola are focii of ellipse $3x^2 + 4y^2 = 12$ and its eccentricity is reciprocal of the eccentricity of ellipse. Find equation of hyperbola.

4. If the tangent at the point (h, k) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the circle $x^2 + y^2 = a^2$ at points whose ordinates are y_1 and y_2. Then prove that $\frac{1}{y_1} + \frac{1}{y_2} = \frac{2}{k}$.

5. The straight line 2x + y = 4 touches the hyperbola whose focii are (± 3, 0). Find equation of hyperbola.

6. Two tangents of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are inclined at θ_1 and θ_2 to axis. If $\cot\theta_1 \cot\theta_2 = 6$, then find the locus of point of intersection of those tangents.

7. m is a variable. Prove that lines $\sqrt{3}x - y = 4\sqrt{3}\,m$ and $\sqrt{3}mx + my = 4\sqrt{3}$ describe a hyperbola. Find its eccentricity.

8. A variable tangent to the hyperbola $4x^2 - 9y^2 = 36$ meets axes in P and Q. Find the locus of mid point of PQ.

9. Find the locus of point of intersection of two tangents drawn to $b^2x^2 - a^2y^2 = a^2b^2$. If $\cot\alpha + \cot\beta = 4$ where α and β are slopes of tangents.

10. M and N are two points on hyperbola $b^2x^2a^2y^2 = a^2b^2$ such that MN passes through centre c. R is any other point on hyperbola. Prove that the product of slopes of MR and NR is $\frac{b^2}{a^2}$.

11. Show that the locus of poles of normal chords of $x^2 - 4y^2 = 4$ is $64y^2 - x^2 = 25x^2y^2$.

12. Prove that curves $x^2 + y^2 = 10$ and $xy = 4$ intersect at an angle of $\tan^{-1}\left(\frac{3}{4}\right)$.

13. Prove that the locus of poles of tangents of circle $x^2 + y^2 = 9$ with respect to hyperbola $3x^2 - 4y^2 = 12$ is $9x^2 + 16y^2 = 16$

14. Find locus of point of intersection of two perpendicular tangents of hyperbola $9x^2 - 16y^2 = 144$.

15. G is a point on transverse axis of hyperbola $b^2x^2 - a^2y^2 = a^2b^2$. GL is perpendicular on asymptote. PG is normal at point P on hyperbola. Prove that PL is perpendicular to conjugate axis.

16. $3x^2 - 4y^2 = p^2$ is conjugate hyperbola of $a^2y^2 - b^2x^2 = 1$. Find relation between a and b

17. A focal chord of hyperbola $9x^2 - 16y^2 = 144$ is inclined at 60^o with the axis. Find angle between tangents drawn at its ends

18. Find angle between the tangent at one end of focal chord of Q. No. 17 and normal at the other end.

19. The distance between two focii of a hyperbola is equal to the diameter of director circle of ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and its conjugate axis is equal to its minor axis. Find equation of hyperbola and its eccentricity.

20. Eccentricities of hyperbola and its conjugate are equal. Find its eccentricity.

21. Find mid point and length of chord $3x - y = 5$ of hyperbola $3x^2 - 2y^2 = 6$.

22. Find length of common chord of parabola $y^2 = 4ax$ and hyperbola $x^2 - y^2 = 5a^2$.

23. Find angle of intersection between hyperbola $3x^2 - 2y^2 = 1$ and circle $x^2 + y^2 = 7$

24. Eccentricity of a hyperbola is $\frac{5}{3}$, calculate eccentricity of its conjugate hyperbola.

25. The equation of hyperbola is $xy = 16$. Now axes are rotated through 45^o in clockwise direction. Write equation of hyperbola on new axes.

26. Equation of tangent of hyperbola $xy + 16 = 0$ inclined at 45^o is

(a) $x - y = \pm 2$ (b) $y - x = \pm 8$ (c) $x - y = \pm 4$ (d) $x - y = \pm 6$

27. The equation of normal of hyperbola $xy = 4$ at point whose ordinate is twice the abscissa is

(a) $x - 2y + 3\sqrt{2} = 0$ (b) $x + 2y - 5\sqrt{2} = 0$

(c) $2x - y - 4\sqrt{2} = 0$ (d) $x - 2y - 3\sqrt{2} = 0$

28. The area of triangle formed by a tangent and asymptotes of hyperbola $xy = 4$ is 8. The point of contact is

(a) (4, 1) (b) (1, 4) (c) (2, 2) (d) $(2\sqrt{2}, 2)$

29. The normal of hyperbola $xy = 4$ which passes through focus is

(a) $y = x + 1$ (b) $x - y = 0$ (c) $x + y = 0$ (d) $x - y = 1$

30. The length of perpendicular from focus on the tangent at $\left(1, \frac{1}{\sqrt{3}}\right)$ of hyperbola $2x^2 - 3y^2 = 1$ is

(a) $\frac{\sqrt{10}-\sqrt{3}}{\sqrt{21}}$ (b) $\frac{\sqrt{10}-3}{\sqrt{21}}$ (c) $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{21}}$ (d) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{15}}$

31. The length of chord of hyperbola $3x^2 - 2y^2 = 6$ which meets x-axis at an angle of 120^o is

(a) 8 (b) 16 (c) $8\sqrt{3}$ (d) 12

32. The equation of normal of hyperbola xy = 16 inclined at $\tan^{-1} 3$ with x-axis is

(a) $3\sqrt{3}x - \sqrt{3}y = 32$ (b) $3x - y = 18\sqrt{3}$

(c) $\sqrt{3}x - \frac{1}{\sqrt{3}}y = 32$ (d) $3x - y = 12\sqrt{3}$

33. 5x + 3y = 0 is a diameter of hyperbola $2x^2 - 3y^2 = 1$. Equation of conjugate diameter is

(a) 2x – 5y = 0 (b) 2x + 5y = 0

(c) 10x + 9y = 0 (d) none of these

34. Focii and directrix of hyperbola $3x^2 - 4y^2 - 12x + 24y - 36 = 0$ are

(a) $\left(\pm\sqrt{7}, 0\right), x = 2 \pm \frac{\sqrt{7}}{4}$ (b) $\left(\pm\sqrt{7}, 0\right), x = 2 \pm \frac{4}{\sqrt{7}}$

(c) $\left(\pm\sqrt{5}, 0\right), x = 2 \pm \frac{\sqrt{5}}{4}$ (d) $\left(\pm\sqrt{5}, 0\right), x = 2 \pm \frac{4}{\sqrt{5}}$

35. Straight line $y = mx + \sqrt{4m^2 - 3}$ touches hyperbola $3x^2 - 4y^2 = 12$ at point (2 sec θ, $\sqrt{3}$ tan θ), then sin θ =

(a) $\frac{\sqrt{3}}{2m}$ (b) $\frac{2}{3m}$ (c) $\frac{\sqrt{3m}}{4}$ (d) $\frac{2\sqrt{m}}{3}$

36. Eccentricity of a hyperbola is $\frac{3}{2}$ and distance between vertex and corresponding focus is 1; equation of hyperbola is

(a) $4x^2 - 5y^2 = 20$ (b) $5x^2 - 4y^2 = 10$

(c) $5x^2 - 4y^2 = 20$ (d) none of these

37. The vertices of a hyperbola are (2, 2) and (–2, –2) and focii (3, 3), (–3, –3). Equation of asymptotes are

(a) $xy = 0$ (b) $x \pm y = 0$

(c) $(2\sqrt{5}+4)x - 2(\sqrt{5}-4)y = 0$ (d) $\sqrt{10}x \pm \sqrt{8}y = 0$

38. The conjugate axis of hyperbola is $\frac{2}{3}$ times the transverse axis. If its latus rectum is $\frac{4\sqrt{3}}{3}$ then equation of hyperbola is

(a) $4x^2 - 6y^2 = 12$ (b) $9x^2 - 4y^2 = 27$

(c) $3x^2 - 2y^2 = 6$ (d) $4x^2 - 9y^2 = 27$

39. Perpendicular dropped from centre c of hyperbola $3x^2 - 4y^2 = 12$ on any tangent of hyperbola meets at Q. Locus of Q is

(a) $(x^2 + y^2)^2 = (4x^2 - 3y^2)$ (b) $2(x^2 + y^2)^2 = (3x^2 - 9y^2)$

(c) $(x^2 + y^2)^2 = (3x^2 - 4y^2)$ (d) $(x^2 + y^2)^2 = (3x^2 + 4y^2)$

40. Perpendiculars dropped from focii on any tangent of hyperbola are p_1 and p_2. Then $p_1 \cdot p_2 =$

(a) 3 (b) 4 (c) –3 (d) –4

41. The locus of poles of tangents of hyperbola $x^2 - y^2 = 4$ with respect to parabola $y^2 = 8x$, is

(a) $4x^2 + y^2 = 4$ (b) $4x^2 + y^2 = 12$

(c) $4x^2 + y^2 = 16$ (d) none of these

42. Asymptotes of hyperbola are $x + 2y + 1 =- 0$ and $2x + 3y - 1 = 0$. Hyperbola passes through (2, –3). The equation of hyperbola is

(a) $2x^2 + 7xy + 6y^2 + x + y - 19 = 0$ (c) $2x^2 + 5xy + 3y^2 - x - y - 18 = 0$

(c) $2x^2 + 7xy + 6y^2 + x + 4y - 10 = 0$ (d) none of these

43. Focal chord PQS of hyperbola is such that PS : PQ = 2 : 1. Length of this focal chord is

(a) $12\sqrt{2}$ (b) $15\sqrt{2}$ (c) $\frac{15\sqrt{2}}{2}$ (d) $\frac{9\sqrt{2}}{2}$

44. Inclination of chord in Q. No. 43 with axes is

(a) $\tan^{-1} 4$ (b) $\tan^{-1}\sqrt{15}$ (c) $\tan^{-1}\sqrt{19}$ (d) $\tan^{-1}\sqrt{17}$

45. The area of triangle formed by tangent at $\left(5, \frac{8}{3}\right)$ of hyperbola $4x^2 - 9y^2 = 36$ and its asymptotes is

(a) 12 sq. units (b) 6 sq. units (c) $\frac{200}{3}$ sq. units (d) $\frac{100}{3}$ sq. units

Practice Worksheet (Competition Level)

1. a) Find the equation of the hyperbola whose directrix is $2x+y=1$, focus (1,2) and eccentricity $\sqrt{3}$.

   b) Find the coordinate of the foci, vertices, eccentricity, directices and latus-rectum for the hyperbolas

   i) $9y^2-4x^2=36$ ii) $x^2+2x-y^2+5=0$ iii) $x^2-2x-3y^2=8$

   c) Show that the equation $9x^2-16y^2-18x+32y-151=0$ represent a hyperbola. Find the coordinate of the centre, length of the axes, eccentricity, latus-rectum, coordinate of the foci and vertices, equations of the directrices

   d) Find the equation of the hyperbola whose foci are (8, 3), (0, 3) and eccentricity= 4/3.

   e) Find equation of hyperbola whose eccentricity is 5/4, whose focus is (3, 0) and whose directrix is $4x-3y=3$.

2. a) Prove that the line $y = x+2$ touches the hyperbola $5x^2-9y^2=45$, Find also its point of c contact.

   b) Find the equation of tangents drawn from (–2, –1) to the hyperbola $2x^2-3y^2=6$.

   c) Find the equation of tangents to the hyperbola $x^2-4y^2=36$ which are perpendicular to the line $x-y+4=0$.

3. Prove that the straight lines $\frac{x}{a}-\frac{y}{b}=m$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{m}$, where a and b are given real numbers and m is parameter, always meet on a hyperbola.

4. Find the locus of the mid points of the chords of the circle $x^2+y^2=16$, which are tangent to the hyperbola $9x^2-16y^2=144$.

5. Prove that the straight line $lx+my+n=0$ will be normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $\frac{a^2}{l^2}=\frac{b^2}{m^2}+\frac{(a^2+b^2)^2}{n^2}$.

6. a) Find the equation of common tangents to the parabola $y^2=8x$ and the hyperbola $3x^2-y^2=3$.

   b) Determine the equations of the common tangents to the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

7. Show that the locus of the mid points of the portions of the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intercepted between the axes is $4x^2y^2=a^2y^2-b^2x^2$.

8. If e and e' are the eccentricities of the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ then prove that $\frac{1}{e^2}+\frac{1}{e'^2}=1$

9. The tangent at the point P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets one of the directrices in F. Show that PF subtends a right angle at the corresponding focus.

10. a) Show that the product of the lengths of the perpendiculars drawn from foci on any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is b^2.

b) For any point on the rectangular hyperbola $x^2-y^2=a^2$ Prove $SP.S'P=CP^2$, where S and S' are foci and C is the centre of the hyperbola.

11. Show that normal to the hyperbola $xy = c^2$ at the point t_1 meets the curve again at the point t_2 such that $t_1^3t_2=-1$.

12. a) Prove that the locus of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a circle $x^2+y^2=a^2-b^2$.

b) If two points P and Q on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, whose centre C be such that CP is perpendicular to CQ, a<b, then prove that $\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}-\frac{1}{b^2}$.

13. a) If the tangent at the point (p, q) on the $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the auxiliary circle in the points whose ordinates are y_1 and y_2 then prove that q is the harmonic mean of y_1 and y_2.

b) Find the equation of the chord of hyperbola $25x^2-16y^2=400$ which is bisected at the point (6, 2).

14. Show that the locus of the middle points of the normal chords of the rectangular hyperbola $x^2-y^2=a^2$ is $(y^2-x^2)^3=4a^2x^2y^2$.

15. A normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axis in M and N and the line MP and NP are drawn perpendicular to the axis meeting at P. Prove that locus of P is the hyperbola $a^2x^2-b^2y^2=\left(a^2+b^2\right)^2$.

16. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Find the locus of the point which divides the line segment between these two points in the ratio of 1:2. **[I.I.T-97, 5]**

17. Show that locus of points of intersection of tangents at the extremities of the normal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{a^6}{x^2}-\frac{b^6}{y^2}=\left(a^2+b^2\right)^2$

18. Tangents are drawn from the point (x_1, y_1) and (x_2, y_2) to the rectangular hyperbola $xy = c^2$. The normals at the points of contact meet at the point (h, k). Prove that $h\left[\frac{1}{x_1}+\frac{1}{x_2}\right]=k\left[\frac{1}{y_1}+\frac{1}{y_2}\right]$

19. Prove that locus of point of intersection of tangents to the hyperbola, which meet at the constant angle β, is the curve $\left(x^2+y^2+b^2-a^2\right)^2=4\cot^2\beta\left(a^2y^2-b^2x^2+a^2b^2\right)$

20. If the hyperbola be rectangular and its equation be $xy = c^2$, Prove that the locus of the middle points of the chords of constant length 2d is $\left(x^2+y^2\right)\left(xy-c^2\right)=d^2xy$

21. Chords of the hyperbola $x^2-y^2 = a^2$ touch the parabola $y^2= 4ax$. Prove that the locus of the middle point is the curve $y^2(x- a) = x^3$

22. A variable tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the transverse axis and the tangent at the vertex (a, 0) at Q and R. Show that the locus of the middle point of QR is $x(4y^2+b^2)= ab^2$.

23. a) The normal at the three points P, Q, R on the rectangular hyperbola, intersect at a point S on the curve ; prove that the centre of the hyperbola is the centroid of the triangle of PQR..

    b) If the normal at four points P (x_i, y_i), i =1, 2, 3, 4 on the rectangular hyperbola $xy = c^2$ meet at the point Q (h, k), prove that

    (i) $x_1+x_2+x_3+x_4= h$

    (ii) $y_1+y_2+y_3+y_4= k$

    (iii) $x_1x_2x_3x_4 = y_1y_2y_3y_4= -c^4$

24. a) A variable chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is tangent to the circle $x^2+y^2=c^2$.

    Prove that the locus of its mid point is $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$

b) If a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, show that the locus of its middle point is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2}-\frac{y^2}{b^2}$.

COMPREHENCIVE PASSAGE TYPE PROBLEMS

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25. Let a hyperbola whose centre is at origin. A line x + y =2 touches the hyperbola at P(1,1)and the asymptotes at A and B such that AB =6√2 units(you can use the concept that in case of hyperbola portion of the tangents intercepted between the asymptotes is bisected at the point of contact

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i) Equation of the asymptotes are

a) $5xy+2x^2+2y^2=0$ b) $3x^2+4y^2+6xy=0$

c) $2x^2+2y^2-5xy=0$ d) none of these

ii) Angle subtended by AB at the centre of the hyperbola is

a) $\sin^{-1}\frac{4}{5}$ b) $\sin^{-1}\frac{2}{5}$ c) $\sin^{-1}\frac{3}{5}$ d) none of these

iii) Equation of the tangent to the hyperbola at (-1, 7/2) is

a) 5x+2y =2 b) 3x+2y = 4
c) 3x+2y = 11 d) none of these

Objective Problems

Level-1

1. The eccentricity of hyperbola whose latus –rectum is 8and conjugate axis is equal to half the distance between the foci is

a) $\frac{4}{3}$ b) $\frac{4}{\sqrt{3}}$ c) $\frac{2}{\sqrt{3}}$ d) none of these

2. If the foci of the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, with the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ then the value of b^2 is **[AIEEE-03]**

a) 1 b)5 c)7 d)9

3. Let P(asec?, btan?)and Q(asecϕ, btanϕ) where ?+ϕ= p/2, be the two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h, k) is the point of intersection of normals at P and Q then K is equal to **[I.I.T-1999]**

a) $\frac{a^2+b^2}{a}$ b) $-\frac{a^2+b^2}{a}$ c) $\frac{a^2+b^2}{b}$ d) $-\frac{a^2+b^2}{b}$

4. The Area of the triangle formed by the lines x – y =0, x + y =0 and any tangent to the hyperbola $x^2–y^2=a^2$ is

a)a^2 b)$2a^2$ c)$3a^2$ d)$4a^2$

5. The locus of the point of intersection of two tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if the sum of the slope is constant λ is;

a) $(x^2–a^2) = \lambda x$ b) $(x^2–a^2) = \lambda y$

c) $\lambda(x^2–a^2) = 2xy$ d) none of these

6. If x =9 is the chord of contact of the hyperbola $x^2–y^2=9$, then the equation of the corresponding pair of tangents is

a)$9x^2–8y^2+18x–9=0$ b) $9x^2–8y^2–18x+9=0$

c) $9x^2–8y^2–18x–9=0$ d) $9x^2–8y^2+18x+9=0$

7. Number of points outside the hyperbola $\frac{x^2}{25}-\frac{y^2}{36}=1$ from where two perpendicular tangents can be drawn to the hyperbola is (are)

a) 3 b) 2 c)1 d)0

8. The equation $16x^2–3y^2–32x–12y–44=0$ represents a hyperbola, which one of the following is incorrect

a) The length of whose transverse axes is $2\sqrt{3}$

b) the length of whose conjugate axes is 8

c) Whose centre is at (1, –2)

d) whose eccentricity is $\sqrt{19}$

9. Eccentricity of the hyperbola $\frac{x^2}{a}+\frac{y^2}{a^2}=1$,(a<0) is

a) $\sqrt{1-a}$ b) $\sqrt{1+a}$ c) $\sqrt{1+\frac{1}{a}}$ d) $\sqrt{1-\frac{1}{a}}$

10. For the hyperbola $\frac{x^2}{\cos^2\theta}-\frac{y^2}{\sin^2\theta}=1$, which of the following remains constant with change in ?? **[I.I.T Sc-03]**

a) abscissa of the vertices b) abscissa of the foci

c) eccentricity d) directrix

11. Equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy=c^2$ **[AIEEE-02]**

a) $\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$ b) $\frac{x}{x_1-x_2}+\frac{y}{y_1-y_2}=1$

c) $\frac{x}{y_1+y_2}+\frac{y}{x_1+x_2}=1$ d) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$

12. Locus of the point P(α, β) moving under the condition that the line $y=\alpha x+\beta$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is **[AIEEE-05]**

a) a parabola b) a hyperbola c) an ellipse d)a circle

13. Length of the letus rectum of the hyperbola xy–3x–4y +8=0

a) 4 b) $4\sqrt{2}$ c) 8 d) None of these

14. The number of values of c such that the line y=2x+c touches the hyperbola $x^2-y^2=1$ is

a)1 b)2 c)4 d) infinite

15. The angle between the tangents drawn from any point on the circle $x^2+y^2=3$ to the hyperbola $\frac{x^2}{4}-y^2=1$ is

a) p/3 b) p/4 c)p/2 d)p/6

16. The value(s)of m for which y = mx+6 is a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{49}=1$ is

a) $\sqrt{\frac{17}{20}}$ b) $\sqrt{\frac{20}{17}}$ c) $-\sqrt{\frac{20}{17}}$ d) $-\sqrt{\frac{17}{20}}$

17. The curve represented by $x = ae^?$, $y = be^{-?}$, $? \in R$ is

a) a hyperbola b) an ellipse

c) a parabola d) a circle

18. Equation of the latus rectum of the hyperbola $(10x-5)^2 + (10y-2)^2 = 9(3x+4y-7)^2$ is

a) y–1/5= –3/4(x–1/2)
b) x–1/5= –3/4(y–1/2)
c) y +1/5=-3/4(x+1/2)
d) x +1/5=-3/4(y +1/2)

19. Asymptotes of the hyperbola x y = 4x +3y are

a) x =3, y = 4
b) x = 4, y =3
c) x =2 , y = 6
d) x =6, y =2

20. A common tangent to $9x^2-16y^2=144$ and $x^2+y^2=9$ is

a) $y=\frac{3}{\sqrt{7}}x+\frac{15}{\sqrt{7}}$
b) $y=3\sqrt{\frac{2}{7}}x+\frac{15}{\sqrt{7}}$
c) $y=2\sqrt{\frac{3}{7}}x+15\sqrt{7}$
d) none of these

21. Let e(k) be the eccentricity of $(x-3)(y+2)=k^2$, then the value of e(2)–e(3) is

a) 3 b) –5 c) 0 d) none of these

Level-2

1. Equation of the rectangular hyperbola whose asymptotes are x =3 and y=5 and passing through (7, 8) is

a) xy-3y +5x +3=0
b) xy +3y +5x +3=0
c) xy–3y +5x-3=0
d) xy–3y-5x +3=0

2. A normal to the parabola $y^2= 4ax$ with slope m touches the rectangular hyperbola $x^2-y^2=a^2$ if

a) $m^6 +4m^4 -3m^2+1=0$
b) $m^6 +4m^4 +3m^2+1=0$
c) $m^6 -4m^4 +3m^2-1=0$
d) $m^6 -4m^4 -3m^2+1=0$

3. If e the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and ? be the angle between then asymptotes, then cos?/2 is equal to

a)1/e
b) 1/√e
c)-1/e
d)none of these

4. The locus of the mid point of the portion of a line of constant slope 'm' between two branches of a rectangular hyperbola xy =1 is

a) $y - mx = 0$ b) $y + mx = 0$

c) $my + x = 0$ d) $y = x$

5. A circle cuts the rectangular hyperbola $xy = 1$ in the points (x_r, y_r) r=1,2,3,4 then the values of $x_1x_2x_3x_4$ and $y_1y_2y_3y_4$ respectively, are

a) –1, –1 b) –1, 1

c) 1,–1 d) 1, 1

6. The locus of the middle points of the chords of the hyperbola $3x^2-2y^2+4x-6y=0$ parallel to $y=2x$ is

a) $3x-4y = 4$ b) $3y - 4x + 4 = 0$

c) $4x-4y = 3$ d) $3x-4y = 2$

7. The point on the hyperbola $\frac{x^2}{24}-\frac{y^2}{18}=1$ which nearest to the line $3x+2y-1=0$ is

a) (6, 3) b) (-6, 3)

c) (6,-3) d) (-6,-3)

8. The equation $(x-\alpha)^2 + (y-\beta)^2 = k(lx + my + n)^2$ represents

a) a parabola for $k=(l^2+m^2)^{-1}$ b) an ellipse for $0<k<(l^2+m^2)^{-1}$

c) a hyperbola for $k>(l^2+m^2)^{-1}$ d) a point circle for k=0

9. The angle between the hyperbolas $xy = c^2$ and $x^2-y^2=a^2$ is

a) Independent of c b) independent of a

c) Always p/2 d) none of these

10. If (asec?, btan?) and $(a\sec\phi, b\tan\phi)$ are the ends of the focal chord of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equals

a) $\frac{e-1}{e+1}$ b) $\frac{1-e}{1+e}$ c) $\frac{1+e}{1-e}$ d) $\frac{e+1}{e-1}$

11. A hyperbola , having the transverse axis of length 2sin?, is confocal with the ellipse $3x^2+4y^2=12$. then its equation is **[I.I.T -07, 3]**

a) x^2cosec^2? –y^2sec^2?=1 b) x^2sec^2? –y^2cosec^2?=1

c) x^2sin^2? –y^2cos^2?=1 d) x^2cos^2? –y^2sin^2?=1

12. If e_1 is the eccentricity of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ & e_2 is the eccentricity of the hyperbola passing through the foci of the ellipse and $e_1e_2=1$ then the equation of the hyperbola is **[I.I.T -06, 5]**

a) $\frac{x^2}{9}-\frac{y^2}{16}=1$

b) $\frac{x^2}{16}-\frac{y^2}{9}=-1$

c) $\frac{x^2}{9}-\frac{y^2}{25}=1$

d) none of these

13. If the line $2x+\sqrt{6}y=2$ touches the hyperbola $x^2-2y^2=4$ then the point of contact is

a) $(-2, \sqrt{6})$

b) $(-5, 2\sqrt{6})$

c) $(1/2, 1/\sqrt{6})$

d) $(4, -\sqrt{6})$

14. If the circle $x^2+y^2=a^2$ intersects the hyperbola $xy=c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ & $S(x_4, y_4)$ then

a) $x_1+x_2+x_3+x_4=0$

b) $y_1+y_2+y_3+y_4=0$

c) $x_1 . x_2 . x_3 . x_4=c^4$

d) $y_1 . y_2 . y_3 . y_4=c^4$

15. The product of the length of perpendicular from any point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to its asymptotes is equal to

a) $\frac{ab}{a+b}$

b) $\frac{a^2b^2}{a^2+b^2}$

c) $\frac{2a^2b^2}{a^2+b^2}$

d) $\frac{a^2{y_1}^2-b^2{x_1}^2}{a^2+b^2}$

16. e is the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (a>b)and ? is the angle between its asymptotes

a) $\sin\frac{\theta}{2}=\frac{\sqrt{e^2-1}}{e}$

b) $\sin\frac{\theta}{2}=\frac{e}{\sqrt{e^2-1}}$

c) $\tan\theta=\frac{2\sqrt{e^2-1}}{2-e^2}$

d) $\tan\theta=\sqrt{\frac{e^2-1}{e^2+1}}$

17. If the tangent and the normal to the rectangular hyperbola $xy=c^2$ at a point cuts off intercepts a_1 and a_2 on the x-axis and b_1, b_2 on the y-axis, then $a_1a_2+b_1b_2$ is equal to

a) 3

b) 1

c) 2

d) none of these

18. Consider the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Area of the triangle formed by the asymptotes and the tangent drawn to it at (a, 0) is

a) ab/2 b) ab c) 2ab d) 4ab

19. Centre of the hyperbola $x^2 + 4y^2 + 6xy + 8x - 2y + 7 = 0$

a) (1, 1) b) (0, 2) c) (2, 0) d) none of these

20. Equation of the hyperbola passing through the point (1, –1) and having Asymptotes x +2y+3=0 and 3x+4y+5=0 is

a) $3x^2+10xy+8y^2+14x+22y+7=0$ b) $3x^2-10xy+8y^2+14x+22y+7=0$

c) $3x^2-10xy+8y^2-14x+22y+7=0$ d) none of these

21. Consider a branch of the hyperbola $x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$ with vertex at the point A Let B be the end points of its letus rectum If C is the focus of the hyperbola nearest to the point A , then the area of the triangle ABC is **[I.I.T -08, 3]**

a) $1-\sqrt{\frac{2}{3}}$ b) $\sqrt{\frac{3}{2}}-1$ c) $1+\sqrt{\frac{2}{3}}$ d) $\sqrt{\frac{3}{2}}+1$

✣ --------------------- ✣

Miscellaneous Problems

Example 1: Obtain equations of lines passing through A (2, 0) and making equal angles of 45^o with tangent at A to circle, $(x + 2)^2 + (y - 3)^2 = 25$. Find equation of circles of radius 3 whose centres lie on lines at distance $5\sqrt{2}$ from A.

Sol. Equation of circle $(x + 2)^2 + (y - 3)^2 = 25$ ... (1)

Any tangent of it is, $y - 3 = m(x + 2) + 5\sqrt{1+m^2}$

It passes through (2, 0), $\therefore -3 = 4m + 5\sqrt{1+m^2} \Rightarrow 9 + 16m^2 + 24m = 75(1 + m^2)$

$\Rightarrow (3m - 4)^2 = 0 \qquad \Rightarrow m = \frac{4}{3}$

Lines make 45^o and -45^o angle with tangent

$$\therefore\ M_1 = \frac{\frac{4}{3}+1}{1-\frac{4}{3}} = -7 \text{ and } M_2 = \frac{\frac{4}{3}-1}{1+\frac{4}{3}} = \frac{1}{7}$$

Equation of lines $y = -7(x - 2)$ and $y = \frac{1}{7}(x - 2)$ ... (2)

(A) Centres of circles lies on positive sides.

$\tan\theta = -7$, gives $\cos\theta = \frac{-1}{5\sqrt{2}}$, $\sin\theta = \frac{7}{5\sqrt{2}}$ and

$\tan\theta = \frac{1}{7}$, gives $\cos = \frac{7}{5\sqrt{2}}$, $\sin\theta = \frac{1}{5\sqrt{2}}$

$\therefore$ Centres (a) $\frac{x-2}{\cos\theta} = \frac{y}{\sin\theta} = 5\sqrt{2}$ and $\frac{x-2}{\cos\theta} = \frac{y}{\sin\theta} = 5\sqrt{2}$

From (a) $x = 2 + 5\sqrt{2}\left(-\frac{1}{5\sqrt{2}}\right) = 1$

$y = 5\sqrt{2}\sin\theta = 5\sqrt{2}\cdot\frac{7}{5\sqrt{2}} = 7$, centre (1, 7)

From (b) $x = 2 + 5\sqrt{2}\cos\theta = 2 + 5\sqrt{2}\cdot\frac{7}{5\sqrt{2}} = 9$, $y = 1$

$\therefore$ Circles are $(x - 1)^2 + (y - 7)^2 = 9$

and $(x - 9)^2 + (y - 1)^2 = 9$

(B) On negative side $(x - 3)^2 + (y + 7)^2 = 9$

$$(x + 5)^2 + (y + 1)^2 = 9$$

Example 2: A ray of light coming along y = b from positive direction of x-axis strikes mirror $y^2 = 4ax$ (it is intersection of mirror with x = y plane). Find equation of reflected ray and show that it passes through the focus of mirror.

Sol. y = b incident ray meets $y^2 = 4ax$ at $\left(\frac{b^2}{4a}, b\right)$ and at this point inclination of normal with x-axis is $\left[\frac{-y}{2a}\right]$

i.e. $-\frac{b}{2a}$ and y = b is parallel to x-axis.

∴ If α is the angle between inclined ray y = b and this normal. Then tan α = $-\frac{b}{2a}$ and reflected ray shall make angle 2α with x-axis i.e. with y = b.

$$\text{and } \tan 2\alpha = -\frac{\frac{-2b}{2a}}{\left(1-\frac{b^2}{4a^2}\right)} = -\frac{4ab}{4a^2-b^2}$$

∴ Equation of reflected ray is

$$y - b = \frac{-4ab}{4a^2-b^2}\left(x - \frac{b^2}{4a}\right)$$

It meets x-axis i.e. y = 0, then $-b = \frac{-4ab}{4a^2-b^2}\left(x - \frac{b^2}{4a}\right) \Rightarrow 4a^2 - b^2 = 4ax - b^2$

⇒ x = a i.e. it goes through (a, 0) which is focus of parabola.

Example 3: Circles pass through (3, 7) and (6, 5). Find equation of this family of circles. Show that all those chords in which circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the family of circles, are concurrent. Find this point of concurrency.

Sol. The family of circles pass through two points (3, 7) and (6, 5)

∴ Centres of all these circles shall lie on the right bisector of segment joining them.

Equation of right bisector is $y - 6 = \frac{3}{2}\left(x - \frac{9}{2}\right)$

i.e. $6x - 4y = 3$ ... (1)

If (h, k) is the centre of circle of this family then its equation is $(x - h)^2 + (y - k)^2 = (h - 6)^2 + (k - 5)^2$

$\Rightarrow x^2 + y^2 - 2xh - 2ky + 12h + 10k - 61 = 0$

and (h, k) lie on bisector $\Rightarrow 6h - 4k = 3$

$\therefore \Theta$ is $x^2 + y^2 - 2x\left(\frac{3+4k}{6}\right) - 2ky + 12\left(\frac{3+4k}{6}\right) + 10k - 61 = 0$ ... (2)

Given circle $x^2 + y^2 - 4x - 6y - 3 = 0$ ... (3)

Common chords of circles are

$x\left(4 - \frac{3+4k}{3}\right) + (6 - 2k)y + 3 + 18k - 55 = 0$

$\Rightarrow x\left(3 - \frac{4k}{3}\right) + 6y - 2ky + 18k - 52 = 0$

$\Rightarrow (3x + 6y - 52) - k\left(\frac{4}{3}x + 2y - 18\right) = 0$

This is of type $P + \lambda Q = 0$.

$\therefore$ Chords pass through the point of intersection of $3x + 6y - 52 = 0$ and $4x + 6y - 54 = 0$.

i.e. $\left(2, \frac{23}{3}\right)$ and it is point of concurrency

Example 4: If a triangle is inscribed in rectangular hyperbola $xy = c^2$ then its orthocentre shall lie on inside or outside of the curve.

Sol. Let $\left(ct, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ be co-ordinates of vertices A, B and C of the triangle.

Equation of AB $(x + yt_1t) - c(t_1 + t_2) = 0$ ... (1)

Equation of BC $(x + yt_2t_3) - c(t_2 + t_3) = 0$ ... (2)

Perpendicular from C on AB and from A on BC are $t_1t_2x - y = ct_1t_2t_3 - \frac{c}{t_3}$ ... (3)

$t_2t_3x - y = ct_1t_2t_3 - \frac{c}{t_1}$ ... (4)

Orthocentre is point of intersection of (3) & (4)

Subtracting $t_2(t_1 - t_3)x = c\left(\frac{1}{t_1} - \frac{1}{t_3}\right)$

$\Rightarrow x = -\dfrac{c}{t_1t_2t_3}$

and from (3) $\quad -\dfrac{c}{t_3} - y = ct_1t_2t_3 - \dfrac{c}{t_3}$

$\Rightarrow y = -c\, t_1t_2t_3$

$\therefore$ Orthocentre is $\left(-\dfrac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

and this is a point on hyperbola.

Example 5: A variable line of slope 4, intersects the hyperbola xy = 1 at two points. Find locus of point which divides this segment in the ratio of 2 : 3.

Sol. Let y = 4x + c be the variable line

$\therefore x(4x + c) = 1 \Rightarrow 4x^2 + xc - 1 = 0$

$\therefore x_1 + x_2 = -\dfrac{c}{4}, x_1x_2 = -\dfrac{1}{4}$

$\therefore x_1 - x_2 = \sqrt{\dfrac{c^2}{16} - 1} = \dfrac{1}{4}\sqrt{c^2 - 16}$

$\therefore x_1 = \dfrac{1}{2}\left[\dfrac{-c + \sqrt{c^2 - 16}}{4}\right]$

Let point (h, k) divide the line segment in the ratio of 2 : 3.

$\therefore h = \dfrac{2x_2 + 3x_1}{5} = \dfrac{2(x_1 + x_2) + x_1}{5}$

$\therefore h = \dfrac{1}{5}\left[-\dfrac{c}{2} - \dfrac{c}{8} + \dfrac{\sqrt{c^2 - 16}}{8}\right] = \dfrac{1}{5}\left[\dfrac{\sqrt{a^2 - 16}}{8} - \dfrac{5c}{8}\right]$ ... (α)

Point (h, k) lies on straight line too $\quad \therefore k - 4h = c$

Putting this value of c in (α)

$40h = \sqrt{(k - 4h)^2 - 16} - 5(k - 4h)$

$\Rightarrow 20h + 5k = \sqrt{(k - 4h)^2 - 16}$

$\Rightarrow 25(4h + k)^2 = (k - 4h)^2 - 16$

Locus; $25(4x + y)^2 = (y - 4x)^2 - 16$

$\Rightarrow 48x^2 + 26xy + 3y^2 + 2 = 0$

Example 6: Find points on straight line 4x – 3y = 6, tangents from which to circle $x^2 + y^2 - 6x - 4y + 4 = 0$ include an angle of $\tan^{-1}\frac{24}{7}$ then, find co-ordinates of these points and equations of tangents.

Sol. Given circle $x^2 + y^2 - 6x - 4y + 4 = 0$

$\Rightarrow (x-3)^2 + (y-2)^2 = 9 \Rightarrow X^2 + Y^2 = 9$ ... (1)

Straight line $4x - 3y = 6 \Rightarrow 4(x-3) - 3(y-2) = 4X - 3Y = 0$ ... (2)

Equation (1) and (2) written when origin transferred to (3, 2). Let (X_1, Y_1) be a point on straight line.

equation pair of tangents $SS' = T^2 \Rightarrow$

$(X^2 + Y^2 - 9)(X_1^2 + Y_1^2 - 9) = (XX_1 + YY_1 - 9)^2$

Co-efficients of $X^2 = X_1^2 + Y_1^2 - 9 - X_1^2 = Y_1^2 - 9$

Co-efficients of $Y^2 = X_1^2 + Y_1^2 - 9 - Y_1^2 = X_1^2 - 9$

Co-efficients of $XY = -2X_1Y_1$

$\therefore \tan\theta = \frac{24}{7} = 2\frac{\sqrt{X_1^2Y_1^2 - (X_1^2-9)(Y_1^2-9)}}{X_1^2 + Y_1^2 - 18}$

$= \frac{6\sqrt{X_1^2 + Y_1^2 - 9}}{(X_1^2 + Y_1^2 - 18)}$

$\therefore 4(X_1^2 + Y_1^2 - 18) = 7\sqrt{X_1^2 + Y_1^2 - 9}$ ... (3)

From (2) $Y = \frac{4}{3}X$, putting this value in (3)

$4\left(\frac{25}{9}X_1^2 - 18\right) = 7\sqrt{\frac{25}{9}X_1^2 - 9} \Rightarrow 4|p-9| = 7\sqrt{p}$

$\Rightarrow 4p - 7\sqrt{p} - 36 = 0 \Rightarrow (\sqrt{p} - 4)(4\sqrt{p} + 9) = 0$

$p = \frac{25}{9}X_1^2 - 9$; p = 16, $a\sqrt{p} = -\frac{9}{4}$.

$\therefore \frac{25}{9}X_1^2 - 9 = 16 \Rightarrow X_1^2 = 9 \Rightarrow X_1 = \pm 3$

$\therefore X_1 = 3, Y_1 = 4$ or $X_1 = -3, Y_1 = -4$

$\therefore$ Points are (6, 6) and (0, –2) at original origin any tangent to circle is

Equation of pair of tangents from (6, 6)

$16(x^2 + y^2 - 6x - 4y + 4) = (6x + 6y - 3x - 18 - 2y - 12 + 4)^2$

$= (3x + 4y - 26)^2$

$= 9x^2 + 16y^2 + 24xy - 15x - 208y + 676$

$\Rightarrow 7x^2 - 24xy + 60x + 144y - 612 = 0$

$\Rightarrow x(7x - 24y + 102) - 6(7x - 24y + 102) = 0$

$\Rightarrow (x - 6)(7x - 24y + 102) = 0$

$\therefore$ Tangents $x = 6$, $7x - 24y + 102 = 0$

(ii) Point (0, –2). Equation pair of tangents $(x^2 + y^2 - 6x - 4y + 4)(4 + 8 + 4) = (-2y - 3x - 2y + 4 + 4)^2$

$\Rightarrow 16(x^2 + y^2 - 6x - 6y + 4) = (-3x - 4y + 8)^2 = 16y^2 + 9x^2 + 24xy - 48x - 64y + 64$

$\Rightarrow 7x^2 - 24xy - 48x = 0$

$\Rightarrow x(7x - 24y - 48) = 0$

$\therefore$ Common tangents are

$x = 0$, $7x - 24y = 48$

and $x = 6$, $7x - 24y + 102 = 0$

Miscellaneous Problems

1. Prove that the four lines $\frac{x}{m}+\frac{y}{n}=1, \frac{x}{n}+\frac{y}{m}=1$, $\frac{x}{m}+\frac{y}{n}=2$ and $\frac{x}{n}+\frac{y}{m}=2$ enclose a rhombus whose area is $\frac{m^2n^2}{(m^2+n^2)}$

2. A line is such that the sum of the perpendiculars from a number of points (n) is zero. Prove that line always passes through a fixed point.

3. At what angle in anticlockwise direction, the axes be rotated so as to get rid of xy term in equation $x^2+2\sqrt{3}xy-y^2=0$.

4. The diagonal of a square is the portion of the line $\frac{x}{p}+\frac{y}{q}=1$, intercepted between the axes. Find co-ordinates of ends of other diagonal.

5. The line segment A; (3, 0) and B; (6, 3) is rotated about A through 15^6 in anticlockwise direction. B goes to C in new position. D is image of C in y-axis and E is image of C in x-axis. Find equation of DE.

6. The sides y – 2x = 5, ay + x = 10, 3y – bx = 12 from three sides of a rectangle. The fourth sides goes through $\left(0, \frac{5}{2}\right)$. Find a, b and equation of fourth side.

7. A point moves such that the sum of squares of its distances from n points is constant. Prove that locus of point is a circle.

8. Two circles cut orthogonally. Prove that the polar of any point P on first circle with respect to other circle shall pass through the other end of diameter through P of circle.

9. Find the locus of point whose shortest distance from circle $x^2 + y^2 - 2x + 6y - 6 = 0$ is equal to its distance from line $x - 3 = 0$.

10. What is the equation of ellipse whose vertices are focii of hyperbola $4x^2 - 5y^2 = 20$ and whose focii are vertices of this hyperbola.

11. From origin chords of circle $(x + 2)^2 + y^2 = 4$ are drawn. Find

    (a) Locus of mid points of these chords.

    (b) Locus of poles of these chords.

    (c) Locus of point of intersection, of tangents drawn at the ends of these chords.

12. Show that the length of normal chord of parabola $y^2 = 4ax$, which is inclined at 60^o to x-axis is $\frac{32}{3}a$.

13. Find equation of tangent of (a) $(y - 2)^2 = 9(x + 1)$ inclined at 60^o with x-axis, (b) of $xy + 4 = 0$ equally inclined to axes, (c) of $x^2 + y^2 - 4x - 6y = 12$, perpendicular to $3x + 4y = 5$.

14. Prove that every circle passing through (a, 0) (–a, 0) shall cut the circle $x^2 + y^2 + qx + a^2 = 0$ orthogonally, what ever be the value of q.

15. Find P : if polar of P with respect to circles $x^2 + y^2 + 6y + 5 = 0$ and $x^2 + y^2 + 2x + 8y + 5 = 0$ are coincident.

16. Find locus of mid points of chords of circle $x^2 + y^2 - 4x + 6y - 3 = 0$ which subtend an angle of 72^o at the centre.

17. Find locus of mid points of chords of parabola $(y - 3)^2 = 2(x - 1)$, which are perpendicular to $3x - 4y = 5$.

18. The sub-tangent of a point $\left(1, \frac{3}{2}\right)$ for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is 3. Find eccentricity of ellipse and also length of sub-normal.

19. From origin lines are drawn joining points of intersection of circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 2x + 4y = 6$. Show that the equation of these lines is $3x^2 + 16xy + 15y^2 = 0$ and angle between them is $\tan^{-1}\left(\frac{\sqrt{19}}{9}\right)$.

20. Prove that common normal of parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ is x-axis only unless $b > 2(a - c)$.

21. Prove that two parabolas with equal latus rectum and same focus but axes in opposite directions cut each other at right angles.

22. The line joining ϕ_1 and ϕ_2 on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtends a right angle at (–a, 0); prove that $b^2 \tan\frac{\phi_1}{2} \cdot \tan\frac{\phi_2}{2} + a^2 = 0$

23. The normals at points P, Q, R of parabola $y^2 = 4ax$ meet at point M (h, k). Prove that (a) sum of squares of the sides of triangle PQR is $2(h - 2a)(h + 10a)$ (b) area of triangle PQR is $4(h - a)\sqrt{ah - 2a^2}$.

24. The co-ordinates of a point are $[a \tan(\theta + \alpha), b \tan(\theta + \beta)]$ where θ is a variable. Show that the locus of point is a hyperbola.

25. Find points of contact of common tangents of hyperbola $x^2 - y^2 = 3a^2$ and $xy = 2a^2$.

26. Two points P and Q are equidistant from the centre of ellipse on major axis. Eccentric angles of the ends of chords through them are α, β, γ and δ. Prove $\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} \cdot \tan\frac{\gamma}{2} \cdot \tan\frac{\delta}{2} = 1$.

27. The normals at P and Q on the parabola $y^2 = 4ax$ meet at R on the parabola. Prove that product of their ordinate digits is $8a^2$.

28. Prove that the locus of point of intersection of straight lines $\sqrt{3}x - y - 4\sqrt{3}p = 0$ and $\sqrt{3}px + py - 4\sqrt{3} = 0$, for different values of p is a hyperbola of eccentricity 2.

29. Find point on straight line $4x - 3y - 6 = 0$, tangent from which to circle $x^2 + y^2 + 6x - 4y + 4 = 0$ include an angle of $\tan^{-1}\left(\frac{24}{7}\right)$ between them. Find co-ordinates of all such points and equation of tangents. (Roorkee)

30. Given base of a triangle and the ratio of lengths of the remaining two sides. Prove vertex lies on a circle.

31. From point A (2, 2) straight lines AL and AM are drawn at right angle to the straight lines $3x^2 + 7xy + 2y = 0$. Find combined equation of AL , AM. Find area of quadrilateral ALOM where O is origin.

32. The equation of a circle is $2x(x - a) + y(2y - b) = 0$, $a \neq 0$, $b \neq 0$; Find condition on a, b of its chords drawn through $\left(a, \frac{b}{2}\right)$ are bisected by x-axis.

33. A is a fixed point on parabola $y^2 = 4ax$. The normal at A meets the parabola at B. If AB subtends a right angle at vertex, then find slope of AB. (I.I.T.)

34. Prove that the locus of point of intersection of lines $\sqrt{3}x - y - 4\sqrt{3}p = 0$ and $\sqrt{3}p + py - 4\sqrt{3}$ is a hyperbola of eccentricity 2.

35. Find angle of intersection of parabola $y^2 = 4ax$ and hyperbola $x^2 - y^2 = 5a^2$.

36. Straight line y = x meets the lines $\sqrt{3}x - y = \sqrt{3}$ and $3x - \sqrt{3}y + 5 = 0$ in A and B. Find AB and its inclination with parallel lines.

37. The two vertices of a triangle are A (2, 5) B (4, –11) and the third lies on straight line $3x + 4y + 8 = 0$. Prove that the locus of its centroid is $9x + 12y + 14 = 0$

38. Point A is (3, 5) and B (–2, –7). The line segment AB meets axes in P and Q, then AP : PQ : QB is ...

39. ABC is a right angled triangle, $\angle B = 90^o$, BD is perpendicular on AC and DE is perpendicular on AB. Find AE : EB.

40. (1, 5) is orthocentre of triangle and (3, 2) is its centroid. Find circumcentre of triangle.

41. $\ell x - my + 2n = 0$ is a straight line and $\ell + m = n$. Show that the line always passes through a fixed point. Find the fixed points.

42. The straight line $ax - by + c = 0$ is perpendicular to $3x + 4y = \lambda$ and is at a constant distance of 4 units from origin. Find a : b : c.

43. A plane mirror is placed along $x + y = 0$. A ray of light traveling along $x - 3y = 4$ strikes it. Find the reflected ray and its direction.

44. Straihgt line L_1 cuts intercepts of -3 and 4 from axes. Straight line L_2 cuts intercepts of -1 and -2 from axes. Straight line L_3 passes through the point of intersection of L_1 and L_2 and is perpendicular $3x - 4y = \lambda$. Find equation of L_3.

45. The straight line at $3x - 2y = 5$ rotates at (3, 2) and finally rests at a position. That the area of triangle formed by the line and old axes is $\frac{3}{2}$ sq. units. Find the slope of line in this position.

46. Points A and B are $(-1, -1)$ and $(5, 5)$ and AB is chord of a circle whose one diameter is along $4x - 3y + 12 = 0$. Find centre and radius of the circle.

47. The centres of circles $x^2 + y^2 - ax = 0$; $x^2 + y^2 + ax - 2y - 1 = 0$ and $x^2 + y^2 - 2bx + 4y - 1 = 0$ are collinear. Find relation between a and b.

48. $3y^2 - x^2 = 0$ are tangents to circle whose radius is 3. Find equation of circles.

49. The equation of base and altitude of an equilateral triangle are $3x - 4y = 1$ and $4x + 3y = 18$ respectively. If the side of triangle is $10\sqrt{3}$. Find equation of circle circumscribing it.

50. Find locus of mid points of chords that subtend an angle of 120^o at the centre of circle $x^2 + y^2 - 8x - 4y - 5 = 0$.

51. The chord of contact of (a, b) with respect to circle $x^2 + y^2 = p^2$ subtends a right angle at the centre if ...

52. $L_1 = 0$ is a straight line inclined at 135^o with x-axis. L_2 is $x\text{-}y = 1$ intercepts of these two lines on circle $x^2 + y^2 - 3x + 2y = 0$ are equal. Find equation of L_1.

53. The length of focal chord inclined at 60^o to axis of parabola $(y - 2)^2 = p(x + 1)$ is $\frac{32}{3}$. Find p.

54. From point $(-1, 3)$ tangents to parabola $y^2 = 6x$ are drawn. Find area of triangle formed by tangents and chord of contact/

55. Tangents drawn at the ends of a chord of parabola $y^2 = 8x$ meet at $x + 8 = 0$. Find angle subtended by this chord at the vertex of parabola.

56. The normal at point ϕ of ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ meets the ellipse again at point 2ϕ. Find value of $\cos\phi$.

57. p is length of perpendicular from centre of ellipse $16x^2 + 25y^2 = 400$, on a tangent at point P of it. F_1 and F_2 are two cell of ellipse. Prove that $(PF_1 - PF_2)^2 = 100\left(1-\frac{16}{p^2}\right)$.

58. Prove that common chords of ellipses. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2y}{q}=0$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}+\frac{2y}{q}=0$ subtend at origin an angle of $\frac{\pi}{2}$.

59. Prove that chords of curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at origin pass through (1, –2).

60. APB is a tangent of ellipse $\frac{x^2}{9}+\frac{y^2}{4}$ = 1 at point P (θ). The auxillary circle cuts it in a nd B. If AB subtends a right angle at the centre then prove $\sqrt{1+\sin^2\theta}=e^{-1}$.

61. CK is perpendicular from centre c on the polar of point P, of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. PM is perpendicular from P on the polar and it on extension meets the major axis in R. Prove that CK · PR = b^2.

62. CP and CD are two semi-conjugate diameters of ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Find the locus of the foot of pe4rpendicular from centre on PD.

63. Find common tangents of hyperbola $3x^2 - y^2 = 3$ and $y^2 = 8x$.

64. A tangent is drawn from P (t^2, 2t) of parabola $y^2 = 4x$ is the same as normal at point ϕ of ellipse $4x^2 + 5y^2 = 20$. Find t and ϕ.
(Roorkee)

65. Chords of hyperbola $x^2 - y^2 = a^2$ touch $y^2 = 4ax$. Find locus of mid points of chords.

66. Find acute angle between curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$.

[Hint: sec x should be greater than 1 and less than 3. ⇒ curves are $y = x^2 - 1$, $y = 3 - x^2$]

67. Find locus of poles of any tangent of auxiliary circle with respect to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

68. A variable line of a slope 4 intersects the hyperbola xy = 1 at two point. Find locus of point which divides this line segment in the ratio of 1 : 2 (ii) bisects it. (Roorkee)

69. The vertex and focus of a hyperbola are (2, 3) and(3, 3). Find (a) equation of (i) directrix (ii) asymptotes.

70. The asymptotes of hyperbola are $16x^2 - 9y^2 - 64x - 54y - 17 = 0$. Find equation of conjugate hyperbola and its eccentricity.

71. PSQ is focal chord of parabola $y^2 = 4ax$ and PS = 3SQ, then inclination of chord is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

72. Straight line x cos θ + y sin θ = p is tangent to circle $x^2 + y^2$ 2ax cos θ + 2ay sin θ = 0 then p is

(a) 0, –2a (b) 0, 2a (c) a, –a (d) a, –2a

73. The radius and centre of circle x = –3 + 5 cos θ, y = 4 – 5sin θ is

(a) (3, 4), 5 (b) (–3, –4), 5 (c) (–3, 4), 5 (d) (3, –4), 8

74. (2, 3) is one end of diameter of circle $x^2 + y^2 - x - 4y + 1 = 0$, the other end of diameter is

(a) (–1, 4) (b) (1, 4) (c) (1, –1) (d) (–1, 1)

75. Distance between lines $\sqrt{3}x - y = \sqrt{3}$ and $3x - \sqrt{3}y + 5 = 0$ is

(a) $\frac{8}{\sqrt{3}}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{2\sqrt{3}}{3}$ (d) $\frac{5\sqrt{3}}{3}$

76. The vertices of a triangle (2, 4), (5, 4) and (2, 8). The orthocentre of triangle is

(a) (3, 5) (b) (4, 3) (c) (2, 4) (d) (4, 2)

77. ABCD is a rhombus and AC = $\frac{1}{2}BD$. If Co-ordinates of A and C are (5, 1) and (1, 5) then co-ordinates of B and D are

(a) (8, 8), (–2, 2) (b) (–1, –1), (7, 7) (c) (2, 1), (4, 5) (d) (1, 1), (5, 5)

78. The equation of parallel sides of a rhombus are x – 2y = 6 and 3x – 6y = 10 and a diagonal is 3x – 4y – 7 = 0. The equation of second diagonal is

(a) 3x – 4y = 7 (b) 4x + 3y = 38 (c) $4x + 3y + \frac{119}{6}$ (d) None of these

79. The two vertices of a triangle are A (2, 5), B (4, –11) an the third lies on straight line 3x + 4y – 8 = 0. The locus of centroid of triangle is

(a) 9x + 12y = 2 b) 3x – 4y = 11

(c) 5x + 3y = 9 d) 9x + 12y – 14 = 0

80. Straight lines ax + by = ab, x + 2y = 5 and 3x – 4y = 10 are concurrent, then

(a) $\frac{4}{b} - \frac{1}{2a} = 1$ b) $\frac{1}{2a} - \frac{4}{b} = 1$

(c) $\frac{4}{b} + \frac{1}{2a} = 1$ d) $\frac{4}{a} + \frac{1}{2b} = 1$

81. Vertices of a triangle are A (2, 5), B (–2, –3) and C (4, 1), its circumcentre is

(a) $\left(\frac{1}{2},\frac{5}{4}\right)$ b) $\left(-\frac{1}{2},\frac{5}{4}\right)$

(c) $\left(-\frac{1}{3},\frac{1}{3}\right)$ d) $\left(-\frac{1}{2},\frac{7}{4}\right)$

82. Straight line $\frac{x}{a}+\frac{y}{b}=1$ meets axes in A and B and straight line $\frac{x}{a}+\frac{y}{b}+1=0$ meets axes in C and D. Area of quadrilateral ABCD is

(a) 4ab (b) 2ab (c) $2\sqrt{2}$ ab (d) ab

83. The vertices of a triangle are (–1, 3), (7, 3) and (3, 0). Incentre of triangle is

(a) $\left(\frac{5}{3},2\right)$ (b) $\left(\frac{8}{3},4\right)$ (c) $\left(3,\frac{5}{3}\right)$ (d) $\left(3,\frac{7}{2}\right)$

84. Co-ordinates of a point satisfy equations x + y = sin θ and $x^2 - y^2$ = sin 2x. Then locus of point is

(a) $3x^2 + 3y^2 - 8xy + 4 = 0$ (b) $5x^2 + 5y^2 + 6xy - 4 = 0$

(c) $5x^2 + 5y^2 - 10xy - 4 = 0$ (d) $3x^2 + 3y^2 - 10xy - 4 = 0$

85. The straight line $\ell x + my = \ell m$ rotates in such a manner that $\frac{1}{\ell^2}+\frac{1}{m^2}=\frac{1}{n^2}$ where n is a constant. The locus of foot of perpendicular from origin on it is

(a) $x^2+y^2=\ell^2+m^2$ (b) $x^2 + y^2 = n^2$ (c) $2x^2 + y^2 = n^2$ (d) $xy = n^2$

86. The straight line through the point of intersection of $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ meets axes in A and B. Locus of mid point of AB is

(a) 2(a + b)xy + ab(x + y) = 0 (b) (a + b)xy + 2ab (x + y) = 0

(c) 2(a + b)xy – ab(x + y) = 0 (d) (a + b)xy – ab(x + y) = 0

87. The base of an equilateral triangle is $\sqrt{3}x$ + y = 1 and its vertex is (2, 1). Equation of other sides are

(a) $y - \sqrt{3}x = 1 + \sqrt{3}$, $y - 1 = 4$ (b) $y+\sqrt{3}x=1+2\sqrt{3}, y+1=0$

(c) $y-\sqrt{3}x=1-2\sqrt{3}, y-1=0$ (d) $y-\sqrt{3}x=1-2\sqrt{3}, y+1=0$

88. The point of intersection of lines bx + ay = ab and ax – by = ab is always at a constant distance C from origin, then

(a) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{2}{c}$ b) $\frac{1}{a}+\frac{1}{b}=\frac{2}{c}$

(c) $a^2 + b^2 = 2c^2$ d) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{2}{c^2}$

89. The slope of a line of equation $ax^2 + 2hxy + by^2 = 0$ is three times the slope of other line, then

(a) $3h^2 = 4ab$ (b) $4h^2 = 3ab$ (c) $8h^2 = 9ab$ (d) $6h^2 = 5ab$

90. The pair of lines $ax^2 + 2hxy + b^2y = 0$ shall be equally inclined with the pair of lines $a_1x^2 + 2h_1xy + b_1y^2 = 0$ if

(a) $(a – b)h = (a_1 – b_1)h_1$ b) $(a + b)h_1 = (a_1 + b_1)h$

(c) $abh_1 = a_1b_1h$ d) $(a – b)h_1 = (a_1 – b_1)h$

91. The vertex of a triangle is (2, 1) and mid points of sides through it are (–1, 4) and (4, 3). The centroid of triangle is

(a) $\left(\frac{5}{3},\frac{8}{3}\right)$ (b) $\left(\frac{14}{3},4\right)$ (c) $\left(\frac{4}{3},\frac{12}{3}\right)$ (d) $\left(0,\frac{13}{3}\right)$

92. A square is inscribed in a circle of radius R. In the square is inscribed and in this circle square is inscribed again. The side of square is

(a) $\sqrt{2}R$ (b) $\sqrt{3}R$ (c) R (d) $\frac{4}{3}R$

93. A straight line of pair $x^2 – 5xy – 6y^2 = 0$ with negative slope coincides with one of these lines of pair $ax^2 + 2hxy + by^2 = 0$ and the other is perpendicular to the other straight line of pair, then

(a) a = 6b (b) h = 3b c) a = 6h (d) a + b = 2h

94. The straight line $L_1 = 0$ meets $L_2 = 0$ in (2, 4) and $L_3 = 0$ in (5, a). The area of triangle formed by $L_1 = 0$ and axes is $\frac{64}{3}$ sq. units then a is

(a) 2 (b) 14 (c) 16 (d) –2

95. A circle passes through (3, 5) and (2, 3) and its centre lies on 2x + y = 6. Equation of circle is

(a) $x^2 + y^2 – 9x – 10y + 45 = 0$ (b) $x^2 + y^2 – 9x – 6y + 23 = 0$

(c) $x^2 + y^2 – x + 6y – 46 = 0$ (d) $x^2 + y^2 – x – 10y + 19 = 0$

96. Circle $x^2 + y^2 - 2\sqrt{3}x + ay = 0$ meets x-axis in A and passes through a point C in first quadrant. O is origin. If triangle COA is equilateral, then a is

(a) 2 (b) ± 2 (c) 4 (d) –2

97. The centre of a circle lies on 4x + 3y = 24 and it passes through (–1, 1). Equation of smallest possible circle is

(a) $x^2 + y^2 - 6x - 8y = 0$ (b) $x^2 + y^2 - 4x - 6y = 0$

(c) $x^2 + y^2 + 6x + 8y = 4$ (d) $x^2 + y^2 - 8x + 6y = 16$

98. A circle touches y-axis and the straight line $\sqrt{3}y - x = 0$. Its centre lies on

(a) $y = \sqrt{3}x$ (b) $\sqrt{3}y + x = 0$ (c) $\sqrt{3}x + y = 0$ (d) x – y = 0

99. The two sides of an equilateral triangle are $\sqrt{3}x - y = \sqrt{3} - 1$ and $\sqrt{3}x + y = \sqrt{3} + 1$ and its side is 6 cm. The centre of circle circumscribing it is

(a) $\left(1, 1 + 2\sqrt{3}\right)$ (b) $\left(1, 1 - 2\sqrt{3}\right)$

(c) $\left(1, 1 - 3\sqrt{3}\right)$ (d) $\left(1, 1 + 3\sqrt{3}\right)$

100. Points A and B are (–1, 1) and (5, 5) and AB is chord of a circle whose one diameter is along 4x – 3y + 12 = 0. The centre and radius of circle is

(a) $\left(3, \frac{81}{97}\right), \frac{\sqrt{122}}{5}$ (b) $\left(-2, \frac{4}{3}\right), \frac{\sqrt{58}}{3}$

(c) $(0, -4), \sqrt{29}$ (d) $(0, 4), \sqrt{26}$

101. Co-ordinates of points A and B are (2, 1) and (6, 1) respectively. Point C moves such that $\angle ACB$ is always 45^o. Locus of P is

(a) $x^2 + y^2 - 4x - 3y + 6 = 0$ (b) $x^2 + y^2 - 8x + 2y + 9 = 0$

(c) $x^2 + y^2 - 8x - 6y + 17 = 0$ (d) $x^2 + y^2 - 4x - 6y + 9 = 0$

102. $3y^2 - x^2 = 0$ are two diameters of a circle which divide circle in 4 parts. If area of minor sector is $\frac{132}{7}$ sq. units then equation of circle is,

(a) $x^2 + y^2 = 25$ (b) $x^2 + y^2 = 9$

(c) $x^2 + y^2 = 16$ (d) $x^2 + y^2 = 36$

103. The side of an equilateral triangle is $4\sqrt{3}$ cm, and its orthocentre (3, 4). The equation of circle, inscribed in the triangle is

(a) $x^2 + y^2 - 6x - 8y + 22 = 0$ (b) $x^2 + y^2 - 6x - 8y + 16 = 0$

(c) $x^2 + y^2 - 6x - 8y + 22 = 0$ (d) $x^2 + y^2 - 6x - 8y + 13 = 0$

104. Angle between tangents drawn from (3, 4) to circle $x^2 + y^2 - 2x - 2y = 0$ is

(a) $\tan^{-1}\left(\frac{2\sqrt{54}}{25}\right)$ (b) $\tan^{-1}\left(\frac{2\sqrt{46}}{25}\right)$

(c) $\tan^{-1}\frac{6}{25}$ (d) $\tan^{-1}\frac{2\sqrt{54}}{21}$

105. Circles $2x^2 + 2y^2 - 4x - 8y + 7 = 0$ and $3x^2 + 3y^2 - 12x + ay + 9 = 0$ intersect orthogonally, then a is

(a) $-\frac{27}{4}$ (b) -9 (c) $\frac{15}{8}$ (d) $\frac{15}{4}$

106. If straight lines joining origin to the points of intersection of $ax^2 + 2h'xy + b'y^2 + 2f'y = 0$ and $ax^2 + 2hxy + by^2 + 2fy = 0$ include an angle of 90^o; then

(a) $f'(a + b) = f(a' + b')$ (b) $f'(a + b) + f(a' + b') = 0$

(c) $f'(a - b) = f(a' - b')$ (d) $f'ab - fa'b' = 0$

107. From point (–2, 3) secants of circle $x^2 + y^2 - 6x - 8y + 16 = 0$ are drawn. The locus of mid points of chords of circle so formed is

(a) $x^2 + y^2 - x - 7y + 6 = 0$ (b) $x^2 + y^2 + x - 7y + 6 = 0$

(c) $x^2 + y^2 - x + 7y + 12 = 0$ (d) $x^2 + y^2 - x - 7y - 18 = 0$

108. The length of common chord of circles $x^2 + y^2 + 4x + 2y - 4 = 0$ and $x^2 + y^2 - 4x + 6y - 12 = 0$ is

(a) $4\sqrt{\frac{6}{5}}$ (b) $4\sqrt{\frac{14}{5}}$ (c) $4\sqrt{\frac{11}{5}}$ (d) $\sqrt{\frac{66}{5}}$

109. Tangents are drawn from (4, 0) to the circle $x^2 + y^2 = 4$; the area of triangle formed by tangents and chord of contact is

(a) $\frac{1}{2}3\sqrt{3}$ (b) $2\sqrt{3}$ (c) $\frac{4}{3}\sqrt{3}$ (d) $3\sqrt{3}$

110. The chord of contact of (a, b) with respect to circle $x^2 + y^2 = p^2$ shall subtend a right angle at the centre of circle if

(a) $a^2 + b^2 = \sqrt{2}\,p^2$(b) $a^2 + b^2 = 4p^2$ (c) $a^2 + b^2 = 2p^2$ (d) $a^2 + b^2 = p^2$

111. Tangents drawn from P on circles $x^2 + y^2 - 4x - 6y + 9 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 + 2x + 4y - 4 = 0$ are equal then P is

(a) $\left(\frac{5}{4},\frac{11}{4}\right)$ (b) $\left(\frac{27}{4},-\frac{11}{4}\right)$ (c) $\left(\frac{27}{4},\frac{11}{4}\right)$ (d) $\left(-\frac{5}{4},-\frac{11}{4}\right)$

112. P is a point on radical axes of circles $x^2 + y^2 + 4x - 6y - 3 = 0$ and $x^2 + y^2 - 2x + 4y - 4 = 0$ and tangents PT to any one of them is $\frac{9}{2}$ in length, then P is

(a) $\left(1,\frac{1}{10}\right)$ (b) $\left(-6,-\frac{7}{2}\right)$ (c) $\left(-\frac{7}{2},-2\right)$ (d) $\left(4,\frac{5}{2}\right)$

113. Direct common tangents of circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 12x + 35 = 0$ are

(a) $\sqrt{3}y+x=8$, $\sqrt{3}y-x+8=0$ (b) $\sqrt{3}y+x+8=0$, $\sqrt{3}y-x-8=0$

(c) $y+\sqrt{3}x=8, y-\sqrt{3}x+8=0$ (d) $y+\sqrt{3}x+8=0, \sqrt{3}x-y-8=0$

114. A circle goes through (2p, 0) and its radical axis with respect to circle $x^2 + y^2 = p^2$ is $x=\frac{p}{2}$ then equation of circle is

(a) $x^2 + y^2 - px + 2p^2 = 0$ (b) $x^2 + y^2 - 2px = 0$

(c) $x^2 + y^2 + 2px = 0$ (d) $x^2 + y^2 + px + p^2 = 0$

115. The slope of common tangent of circle $x^2 + y^2 = p^2$ and parabola $y^2 = 4px$ satisfy the equation

(a) $4m^2 + m + 4 = 0$ (b) $m^4 + m^2 - 1 = 0$

(c) $m^4 + m^2 - 4 = 0$ (d) $4m^4 - 4m^2 + 1 = 0$

116. A circle touches circle $x^2 + y^2 - 6x - 8y = 0$ externally and to which $2x^2 - 6y^2 - xy - 14x - 21y = 0$ are normals. The equation of circle is

(a) $x^2 + y^2 - 6x + 4y + 9 = 0$ (b) $x^2 + y^2 + 4y - 6x + 12 = 0$

(c) $x^2 + y^2 + 6x - 4y + 9 = 0$ (d) $x^2 + y^2 + 6x + 4y + 12 = 0$

117. If circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q, then line $5x + by = a$ passes through P and Q for

(a) Exactly two values of a (b) infinite value of a

(c) no value of a (d) exactly one value of a

118. $L_1 = 0$is straight line inclined at $\frac{3\pi}{4}$ with x-axis, $L_2 = x - y = 1$, intercepts of these lines on circle $x^2 + y^2 - 3x + 2y = 0$ are equal. Equation of L_1 is

(a) $x + y + 1 = 0$ (b) $x + y = 4$

(c) $x + y = 2$ (d) $x + y + 3 = 0$

119. A circle through (a, b) cuts circle $x^2 + y^2 = p^2$ orthogonally. The locus of its centre is

(a) $2ax + 2by - (a^2 + b^2 + p^2) = 0$ (b) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 + p^2) = 0$

(c) $2ax + 2by + (a^2 - b^2 + p^2) = 0$ (d) $x^2 + y^2 - 3ax - 4by - (a^2 - b^2 + p^2) = 0$

120. The tangent at (1, –2) to circle $x^2 + y^2 = 5$ is also tangent to circle $x^2 + y^2 - 8x + 6y + 20 = 0$. The point of contact on this circle is

(a) (3, –1) (b) (0, 5) (c) (1, 3) (d) (–1, 7)

121. Focus and directrix of parabola $y^2 - 6y + 4x + 17 = 0$, are

(a) (–3, 3), $x + 3 = 0$ (b) (–3, 3), $x + 1 = 0$

(c) (3, 3), $x = 1$ (d) (1, 3), $x + 4 = 0$

122. Axis of parabola is $x + y = 0$, focus (–4, 4) and vertex (–2, 2). Equation of directrix is

(a) $x - y + 12 = 0$ (b) $x - y = -6$ (c) $x - y = 0$ (d) $y - x = 4$

123. The focal distance of a point on parabola $(y - 2)^2 = 12(x - 3)$ is equal to Latus rectum of parabola. The point is

(a) $12, 6\sqrt{3}+2$ (b) $(12, 4\sqrt{3}+2)$

(c) $(12, -6\sqrt{3}+2)$ (d) $(9, 6\sqrt{2}+2)$

124. An equilateral triangle has been described in parabola $y^2 = 6x$. Its one vertex is at the vertex of parabola. Its area is

(a) 216 (b) $108\sqrt{3}$ (c) $216\sqrt{3}$ (d) $144\sqrt{3}$

125. The equation of normal of $(y - 2)^2 = 8(x + 1)$ perpendicular to $x + 2y = 3$ is

(a) $y + 2x + 10 = 0$ (b) $y - 2x + 20 = 0$ (d) $y + 2x - 8 = 0$ (d) $y - 2x + 16 = 0$

126. The length of focal chord inclined at 60^o to x-axis of parabola $(y - 2)^2 = p(x + 1)$ is $\frac{32}{3}$ then p is equal to

(a) $\frac{16}{3}$ (b) $\frac{8}{3}$ (c) 3 (d) $\frac{4}{3}$

127. Point P (9, 6) is on parabola $y^2 = 4x$ and PQ is focal chord. Line through P inclined at 135^o with x-axis and line through Q inclined at 45^o with x-axis meet at

(a) $\left(\frac{71}{9}, \frac{64}{9}\right)$ (b) $\left(\frac{71}{3}, \frac{64}{3}\right)$ (c) $\left(\frac{71}{9}, \frac{-64}{9}\right)$ (d) $\left(\frac{17}{3}, \frac{64}{9}\right)$

128. Tangents from origin to parabola $y^2 = 4a(x - a)$ are drawn. They include an angle of

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\tan^{-1}\frac{1}{2}$ (d) $\frac{\pi}{4}$

129. Locus of foot of perpendicular dropped from focus on the tangent of parabola $y^2 = 4a(x - a)$ is,

(a) $y^2 - (x - 2a)^2 = 0$ (b) $x - a = 0$

(c) $ay^2 = (x - 2a)^2$ (d) $y^2 + (x - 2a)^2 = 0$

130. Length of common chord of parabola $y^2 = ax$ and $x^2 = ay$ is

(a) $\frac{3\sqrt{2}}{2}a$ (b) $3\sqrt{2}a$ (c) 2a (d) $\left(\frac{3\sqrt{2}}{4}\right)a$

131. Tangents at point t_1 and t_2 of parabola $y^2 = 16x$ are drawn. Then ordinate of point of intersection is

(a) Geometric mean of ordinate of t_1 and t_2

(b) Harmonic mean of ordinate of t_1 and t_2

(c) Arithmetic mean of ordinates of t_1 and t_2

(d) Square root of sum of ordinates of points t_1 and t_2

132. Tangents drawn from an external point to parabola $y^2 = 4ax$ make at the focus angles θ_1 and θ_2, then

(a) $\frac{\pi}{2} \leq (\theta_1 + \theta_2) < \pi$ (b) $\theta_2 = 90 - \theta_1$

(c) $\theta_1 = \theta_2$ (d) $(\theta_1 + \theta_2) \leq \pi$

133. The point on parabola $y = (x - 3)^2$ tangent at which is parallel to chord joining (3, 0) and (4, 1) is

(a) $\left(\frac{5}{2}, \frac{1}{4}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (c) $\left(\frac{9}{2}, \frac{9}{4}\right)$ (d) (6, 9)

134. The perpendicular dropped from P on its polar with respect to parabola $y^2 = 4px$ touches parabola $x^2 = 4ay$. The locus of P is

(a) $2px - ay + 4p^2 = 0$ (b) $xy + 2ap = 0$

(c) $2px + ay + 4p^2 = 0$ (d) $2px - ay - 4p^2 = 0$

135. The locus of mid points of chords that subtend a right angle at vertex of parabola $y^2 = 4ax$ is

(a) $y^2 = 2a(x - 4a)$ (b) $y^2 = 2a(x + 4a)$

(c) $y^2 = 4a(x + 2a)$ (d) $y^2 = a(x - 4a)$

136. The locus of mid points of focal chords of parabola $y^2 = 16x$ is

(a) $y^2 = 8(x + 4)$ (b) $y^2 = 8(x - 4)$

(c) $y^2 = 4(x + 4)$ (d) $y^2 = 4(x - 4)$

137. ℓ is length of focal chord of parabola $y^2 = 8x$ and p is the perpendicular from vertex on the chord then,

(a) $\ell \propto \frac{1}{p}$ (b) $\ell \propto \frac{1}{p^2}$ (c) $\ell \propto \frac{1}{\sqrt{p}}$ (d) $\ell \propto \sqrt{p}$

138. The pole of chord of ellipse $9x^2 + 16y^2 = 144$, whose mid point is (2, 1) is

(a) $\left(\frac{72}{13}, -\frac{36}{13}\right)$ (b) $\left(\frac{72}{13}, \frac{36}{13}\right)$

(c) $\left(\frac{36}{13}, -\frac{18}{13}\right)$ (d) $\left(\frac{36}{13}, \frac{18}{13}\right)$

139. The point on ellipse for which sub-tangent is equal to half the abscissa of point on ellipse $4x^2 + 4y^2 = 36$ is

(a) $\left(\pm 6, \frac{4}{\sqrt{3}}\right)$ (b) $\left(\sqrt{6}, \frac{4}{\sqrt{3}}\right)$

(c) $\left(\pm\sqrt{6}, \pm\frac{2}{\sqrt{3}}\right)$ (d) $\left(\pm\sqrt{6}, \pm\sqrt{3}\right)$

140. s and S' are focii of ellipse $16x^2 + 25y^2 = 200$. P is any point on it, then $\tan\frac{1}{2}$ PSS'. $\tan\frac{1}{2}$ PS'S =

(a) $\frac{8}{17}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{9}{41}$

141. Tangents of ellipse $9x^2 + 5y^2 - 30y = 0$ equally inclined to axes are

(a) $y \pm x + 3 \pm \sqrt{14}$ (b) $y \pm x = \pm\sqrt{14}$

(c) $y \pm x = 3 \pm \sqrt{14}$ (d) none of these

142. The portion of straight line $x \cos \alpha + y \sin \alpha = p$ cut off by the ellipse $9x^2 + 16y^2 = 144$ subtends a right angle at the centre of ellipse, then p =

(a) $\frac{12}{5}$ (b) $\pm\frac{12}{5}$ (c) $\pm\frac{6}{5}$ (d) $\pm\frac{9}{5}$

143. Focus of ellipse and hyperbola is same (3, 0) and their eccentricities are reciprocal. The distance between their nearest vertices is $\frac{16}{5}$, then eccentricity of ellipse is

(a) $\frac{2}{3}$ (b) $\frac{3}{5}$ (c) $\frac{2}{5}$ (d) $\frac{3}{4}$

144. The point of intersection of tangents at the ends of latus-rectum of hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, is

(a) (3, 0) (b) (3, –1) (c) $\left(\frac{14}{5}, 0\right)$ (d) $\left(\frac{9}{5}, 0\right)$

145. The number of volues of c such that the straight line y = 2x + c touches the curve $\frac{x^2}{4}+y^2=1$ are

(a) 0 (b) 1 (c) 2 (d) infinite

146. Eccentricity of ellipse $9x^2 + 5y^2 - 30y = 0$, is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $\frac{3}{5}$

147. The locus of mid points of the segments joining ends of two semi-conjugate diameters of ellipse $9x^2 + 16y^2 = 144$, is

(a) $9x^2 + 16y^2 = 36$ (b) $9x^2 + 16y^2 = 72$ (c) $16x^2 + 9y^2 = 72$ (d) $9x^2 - 16y^2 = 36$

148. P and Q are two points on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. CP is perpendicular on CQ, C is centre then, $\frac{1}{CP^2}+\frac{1}{CQ^2}=$

(a) $\frac{1}{a^2}+\frac{1}{b^2}$ (b) $\left(\frac{1}{a}+\frac{1}{b}\right)^2$ (c) $\frac{1}{a^2}-\frac{1}{b^2}$ (d) $\left(\frac{1}{a}-\frac{1}{b}\right)^2$

149. Tangents are drawn from (5, –1) to ellipse $x^2 + 3y^2 = 8$. They include an angle of

(a) $\tan^{-1}\frac{8\sqrt{15}}{23}$ (b) $\tan^{-1}\frac{8\sqrt{30}}{23}$

(c) $\tan^{-1}\frac{4\sqrt{30}}{23}$ (d) $\tan^{-1}\frac{2\sqrt{70}}{3}$

150. The chords of ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ pass through a fixed point (a, b). The locus of mid points of chords is

(a) $9x(x - a) + 16y(y - b) = 0$ (b) $16x(x - a) + 9y(y - b) = 0$

(c) $9x(x - a) - 16y(y - b) = 0$ (d) $16(x^2 + y^2) - 9(x + y) = 0$

151. The focii of a hyperbola are (–4, 4) and (6, 4) and transverse axis is 8. equation of hyperbola is

(a) $\frac{(x-1)^2}{16}-\frac{(y-4)^2}{9}=1$ (b) $\frac{(x-1)^2}{9}-\frac{(y-4)^2}{16}=1$

(c) $\frac{(x+1)^2}{16}-\frac{(y-4)^2}{9}=1$ (d) $\frac{(x-1)^2}{9}-\frac{(y+4)^2}{16}=1$

152. Locus of poles of tangents of $x^2 - y^2 = a^2$ with respect to parabola $y^2 = 4ax$ is

(a) $x^2 - 4y^2 = 4a^2$ (b) $4x^2 + y^2 = 4a^2$

(c) $4x^2 + y^2 = a^2$ (d) $x^2 - 4y^2 = a^2$

153. The two asymptotes of a hyperbola are $x - 2y + 1 = 0$ and $2x + 3y - 5 = 0$. Hyperbola passes through (5, 2). Equation of conjugate hyperbola is

(a) $2x^2 - xy - 6y^2 - 3x + 13y - 27 = 0$ (b) $2x^2 - xy - 6y^2 - 3x + 13y + 17 = 0$

(c) $2x^2 - xy - 6y^2 + 3x - 13y - 27 = 0$ (d) none of these

154. S_1 and S_2 are focii of ellipse, these are joined to B, end of minor axis and $\angle S_1BS_2 = 90^o$, eccentricity of ellipse is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{2}$

155. The ends of a focal chord of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are ϕ_1 and ϕ_2, then $\tan\frac{1}{2}\phi_1 \cdot \tan\frac{1}{2}\phi_2 =$

(a) $\frac{1+e}{1-e}$ (b) $\frac{1}{e}$ (c) $\frac{e-1}{e+1}$ (d) $\frac{1-e}{1+e}$

156. The locus of point of intersection of tangents at ϕ_1 and $(90 - \theta)$ of ellipse $x^2 + y^2 = 4$ is

(a) $x - \sqrt{2}y = 0$ (b) $\sqrt{2}x - y = 0$

(c) $\sqrt{2}x + y = 0$ (d) $x + \sqrt{2}y = 0$

157. The normal drawn at point P ($\phi = 45^o$) of ellipse $x^2 + 2y^2 = 4$ meets axes in M and N then PM : PN =

(a) $1:\sqrt{2}$ (b) 1 : 2 (c) 2 : 3 (d)3 : 4

158. PK is normal at point P on ellipse $3x^2 + 4y^2 = 12$, C is a central of ellipse. Maximum value of tan $\angle$CPK is

(a) $\frac{1}{4}$ (b) $\frac{1}{4\sqrt{3}}$ (c) $\frac{1}{2\sqrt{3}}$ (d) $\frac{2}{5\sqrt{3}}$

159. P (4, 3) and Q (–3, 2) are two points. Polars of these points with respect to ellipse $3x^2 + 4y^2 = 12$ intersect in R then polar of R with respect to ellipse.

(a) goes through P (b) lies along PQ

(c) passes through Q (d) is rt. bisector of PQ

160. P is a point on ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and PN is its ordinate. Polar of P with respect to ellipse meets major axis in T. Then CT · CN =

(a) ab (b) a^2 (c) b^2 (d) $a^2 - b^2$

161. Equation of director circle of ellipse $\frac{x^2}{a}+\frac{y^2}{b}$ = a + b is

(a) $x^2 + y^2 = ab(a + b)$ (b) $x^2 + y^2 = ab(a^2 + b^2)$

(c) $x^2 + y^2 = 1$ (d) $(a + b)^2 (x^2 + y^2) = a^2 + b^2$

162. tangents drawn from an external point P of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are inclined at θ_1 and θ_2 with x-axis. Locus of P when $\tan\theta_1 \cdot \tan\theta_2 = p$, is

(a) $p(x^2 - y^2) = a^2 - b^2$ (b) $px^2 - y^2 = pa^2 - b^2$

(c) $py^2 - x^2 = pb^2 - a^2$ (d) $p(x^2 + y^2) = a^2 - b^2$

163. The locus of mid point of chords $3x - 4y = \lambda$(λ is variable) of ellipse $3x^2 + 4y^2 = 12$ is

(a) $x + y = 0$ (b) $y - x = 0$

(c) $3x - 4y = 0$ (d) none of these

164. The angle between diameters of chords $3x + 4y = \lambda$ and $8x - 6y = \alpha$ (λ, α are variables) of ellipse $9x^2 + 16y^2 = 144$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\tan^{-1}\frac{7}{12}$ (d) $\tan^{-1}\frac{12}{7}$

165. The locus of point of intersection of tangents drawn at the ends of semi-conjugate diameters of ellipse $3x^2 + 4y^2 = 12$ is

(a) $\frac{x^2}{4}-\frac{y^2}{3}=2$ (b) $\frac{x^2}{4}+\frac{y^2}{3}=2$

(c) $\frac{x^2}{3}+\frac{y^2}{4}=2$ (d) none of these

166. Equation of directrix and eccentricity of hyperbola $x^2 - 4y^2 + 2x - 8y - 7 = 0$ are

(a) $\frac{\sqrt{5}}{2}, x+1\pm\frac{4}{5}=0$ (b) $\frac{\sqrt{5}}{2}, x-1\pm\frac{4}{\sqrt{5}}=0$

(c) $\frac{\sqrt{5}}{2}, x+1\pm\frac{6}{5}=0$ (d) $\frac{\sqrt{5}}{2}, x-1\pm\frac{2}{5}=0$

167. A circle touches two circles externally. Locus of its centre is

(a) Circle (b) ellipse (c) hyperbola (d) straight line

168. p_1 and p_2 are length of perpendicular dropped from focii of hyperbola $2x^2 - 3y^2 = 6$ on any tangent of it, then $p_1 \times p_2$ =

(a) 3 (b) 2 (c) –3 (d) –2

169. The ordinates of points P and Q are in the ratio of 1 : 2. The locus of the point of intersection of normals at P and Q points of parabola $y^2 = 12x$ is

(a) $12(x + 6)^3 = 343y^2$ (b) $12(x - 6)^3 = 343y^2$

(c) $6(x - 6)^3 = 343y^2$ (d) $12(x - 6)^3+343y^2=0$

170. $x = \frac{e^t + e^{-t}}{2}, y = \frac{e^t - e^{-t}}{2}$ represent a

(a) circle (b) ellipse

(c) hyperbola (d) rectangular hyperbola

171. The locus of poles of tangents of circle $x^2 + y^2 = 9$ with respect to hyperbola $3x^2 - 4y^2 = 1$ is

(a) $81x^2 + 144y^2 = 108$ (b) $81x^2 - 144y^2 = 1$

(c) $81x^2 + 144y^2 = 1$ (d) $144x^2 + 81y^2 = 108$

172. Locus of intersection of normals of parabola $x^2 = 8y$ which are perpendicular to each other is

(a) $x^2 = 2(y - 6)$ (b) $x^2 = 4(y + 3)$

(c)$x^2 = 4(y - 3)$ (d) $x^2 = 2(y + 6)$

173. The angle of intersection of parabola $y^2 = 4ax$ and hyperbola $x^2 - y^2 = 5a^2$ is

(a) $\tan^{-1}\left(\frac{3}{5}\right)$ (b) $\tan^{-1}\frac{3\sqrt{5}}{4}$

(c) $\tan^{-1}\frac{3\sqrt{5}}{5}$ (d) $\frac{3\pi}{8}$

174. The abscissa of point of intersection of lines, one going through positive vertex of hyperbola xy = 16 inclined at 15^o to transverse is, and the other through nearest focus inclined at 75^o with x-axis is

(a) $2(\sqrt{6} + \sqrt{2} - \sqrt{3})$ (b) $2(1 + \sqrt{6} + \sqrt{2} + \sqrt{3})$

(c) $2(1 + \sqrt{6} + \sqrt{2} - \sqrt{3})$ (d) $2(\sqrt{6} + \sqrt{3} - \sqrt{2} + 1)$

✣ --------------------- ✣

Answers

Practice Worksheet (Foundation Level) – I(a)

1 b,c	2) a	3) b	4) b	5) a	6) c	7) b
8) c	9) a	10) b	11) c	12) a	13) d	14) a
15) b	16) d	17) b	18) a	19) c	20) c	21) a
22) c	23) a	24) d	25) b			

Practice Worksheet (Foundation Level) – I(b)

1) b	2) c	3) c	4) c	5) b	6) a	7) a
8) c	9) a	10) b	11) b	12) 3cm	13) $178\frac{2}{3}cm^3$	

14) 2.5, 2cm 15) $\frac{40}{3}\pi\, cm^3$ 16) 36 cm) 17) Rs. 3520/- 18) $\sqrt{85}$ m 19) 21 cm

20) $15\sqrt{3}$ cm 21) $\left(\frac{1134\sqrt{3}-1782}{1134\sqrt{3}}\right)\times 100\%$ 22) Rs 61.14%

Practice Worksheet (Foundation Level) – 1(c)

1) a	2) b	3) a,b	4)b	5) a	6) b	7) b
8) a	9) a	10) c	11) b	12) b	13) a	14) d
15) c						

Practice Worksheet (Foundation Level) – 2 (a)

1) $\left(\frac{9}{5},0\right)$ 5) b 6) a 7) b 8) b 9) d

10) b	11) b	12) b	13) d	14) c	15) b
16) a	17) a	18) c	19) (6, 3); 5 sq. units		

20) $9x^2 + 4y^2 = 144$ 21) 25)5 sq. units

Practice Worksheet (Foundation Level) – 2 (b)

1) $3x - 4y + 5 = 0$ 2) $4x + 3y = 24$ or $x + y = 7$ 3) $8x - 5y + 60 = 0$

4) $2x + 5y = 18$ 5) $\sqrt{3}x + y = 15 - 2\sqrt{3}$ 7) $3x - 4y = 24$

8) $y - 2x = \pm 5$ 9) 6 units 10) 1 : 1

11) $\left(2,\frac{5}{3}\right), \left(\frac{15}{14},\frac{17}{14}\right)$ 14) $3x + 4y = 12$

16) $23x - 7y = 9, 7x + 23y = 53$ 17) $\left(\frac{3}{4}, \frac{1}{2}\right)$ 18) $2x + y = 5$

20) (–4, –7)	21) d	22) b	23) c	24) a	25) c
26) b	27) d	28) c	29) c	30) a	31) c
32) a	33) a, d	34) b	35) a	36) c	37) c
38) b	39) b	40) a	41) b	42) a	43) b
44) c	45) d	46) straight lines $x \pm a = 0$		47) b	48) a

Practice Worksheet (Foundation Level) – 2 (c)

1) (a) $x + y = 7$ (b) (4, 4) (c) $y = 7$ (d) $(-3\sqrt{2}+2, 3\sqrt{2}+5)$ (e) $y - x = 2$

(f) $x^2 + y^2 - 4x - 8y + 19 = 0$ (g) $\tan^{-1}\left(\frac{17}{7}\right)$

(h) $(13-5\sqrt{2})x + (13+12\sqrt{2})y = 91 - 13\sqrt{2}$ (i) (i) $5x - 12y = 91$, (ii) $5x - 12y + 65 = 0$

(j) $\left(\frac{39}{14}, \frac{25}{14}\right)$ (k) $\left(\frac{17}{7}, \frac{13}{14}\right)$ (l) $\left(-\frac{7}{2}, \frac{-7}{2}\right)$

(m) $\frac{153}{28}$ sq. units 2) $y = a, 3y - 4x = 3a, 2x + y = 5a$ 3) $15^o, 75^o$

6) $7x + 9y = 73, 9x - 7y = 1$ 7) $5x - 5y = 18, 5x + 5y = 4$ 8) $\left(\pm\frac{a}{b}\sqrt{a^2+b^2}, 0\right)$

9) $ad = bc$ 10) $a = -\frac{9}{4}, b = 2$ 11) (a) (–6, 0), (b) $\frac{\pi}{2}$, (c) 50 sq. units

12) (8, 8) or (6, 0) 16) $a = 3, b = 6, x - 3y = 7$

17) a	18) b	19) b, d	20) a	21) c	22) d
23) c, a	24) c	25) b	26) b	27) a	28) b
29) a	30) b	31) a	32) a	33) c	34) c
35) a	36) b				

PART-A

3) $9x^2 + 36y^2 = 4l^2$ 4. $x + 5y = \pm 5\sqrt{2}$ 5. (a/c, b/c) 6. $\frac{132}{12\sqrt{3}+5}$ 7. a) $29x - 2y + 33 = 0$ b) $29x - 2y - 31 = 0$

8. y = x 9. y = x ; x + y = 2 10. $y^2 = 4(x+1)$, x=2, $y^2 = 12(x-3)$ x=2 11. 12. x =5

13.a) $\left(-\frac{1}{2},\frac{5}{2}\right)$ and $\left(\frac{9}{2},\frac{1}{2}\right)$ b) $\left(-\frac{3}{2},\frac{3}{2}\right)$ and $\left(\frac{1}{2},-\frac{3}{2}\right)$ 14. c= –4, (4, 4) and (2,0) 15) 7x–4y +25=0, 4x +7y=11, 7x-4y =3 16) 90^o

17) x +2y-7=0, x–4y–1=0, x –y+2=0 18) 3x +4y –22=0, 2x–7y–5=0, 3x +5y–23=0

19) (1, 5), (–4, –5) x-2y–6=0, 2x +y–7=0 22) $\sqrt{\frac{2}{3}}$, $(2-\sqrt{3})x-y-5+2\sqrt{3}=0$, $(2+\sqrt{3})x-y-5-2\sqrt{3}=0$

23) (–3, 9) 24) (6, 6) (1, –1)

Part-B

2) $\frac{5}{3} \le \beta \le \frac{7}{2}$ 3) $\left(-\frac{3}{2},-1\right) \cup \left(\frac{1}{2},1\right)$ 5) 2x + 3y + 22 = 0 6) 83x–35y +92=0

7) x –5y =0, 5x +y =0 , 2.76 sq units 9) $x(1-m^2)+my-am-mb^2-b=0$

10) $(a^2+b^2)(lx+my+n) = 2(al+bm)(ax+by+c)$

11) 10xy = x +3y 14) a) x +2y-7=0, x-4y-1=0 x- y +2=0

17) 14x +23y – 40=0 18) 7x +y–9=0, x–7y +13=0

12) x- 3y+5=0 18) 3x+4y=18, x-2=0 19) $9x^2+10y^2-6xy=9l^2$

20) $x^2 + y^2 - 7x + 5y = 0$ 21) (-7,-11), (-6,-9), (2, 7), (3, 9)

Level-1

1) d	2) a	3) a	4) c	5) b	6) b	7) a
8) d	9) d	10) c	11) b	12) c	13) a	14) c
15) c	16) c	17) d	18) a	19) c	20) a	
21) a, d	22) a	23) a	24) d	25) c		

Level-2

1) a, c	2) b	3) a, c	4) a, c	5) b, d	6) a, b, c, d	7) c
8) a, b, c 9)	10) a, b	11) a, c	12) b	13) a	14) b	
15) a, c, d	16) a	17) d	18) a	19) d	20) c	
21) a, b, c	22) d	23) a	24) c, d 25) a			

Practice Worksheet (Foundation Level) – 3

1. (3, 4), (–20, 2), (–8, –9), (–3, –8)

2. $x + 1 = 0$, $x + y = 5$, $x^2 + 3xy + 2y^2 + 2x + 2y = 0$

3. b

4. a

5. (2, 4) (–6 ,6), (0,0)

6. $x^2 + y^2 - (\sqrt{3} + 2)x + (1 - 2\sqrt{3})y = 4$

8. $(2\sqrt{3}-1)x+(2+\sqrt{3}y)=5$

9. a

10. b

11. b

12. a

13. c

14. a

15.c

16. (a) Hyperbola $4x^2 - 3x^2 = 12$

(b) Parabola $(y - 2)^2 = 4(x + 1)$

(c) ellipse $4x^2 + 9y^2 = 36$

Practice Worksheet (Foundation Level) – 4(a)

1) $2y^3 + y^2x - 5x^2y + 2x^3 = 0$ 2) $12x^2 + 7xy - 12y^2 = 0$ 3) $\frac{\pi}{4}, \tan^{-1}(3/2); 0$

4) (a) $x^2 - y^2 = 0$ (b) $4abx^2 - (3a^2 + 4b^2)xy + 3aby^2 = 0$ (c) $y^2 - 3x^2 = 0$

5) $(y - 2)^2 - 3x^2 = 0$ 6) (a) $\tan^{-1}\left(\frac{1}{7}\right)$ (b) $92 - \theta$ (c) $\tan^{-1}\left(\frac{3ab}{a^2+4b^2}\right)$

(d) $\frac{\pi}{2}$ (e) $\cos^{-1}(\cot\theta)$ (f) $\frac{\theta}{2}$

7) (a) $x^2 - \sqrt{3}xy = 0$ (b) $2y^2 + 3xy - 2x^2 = 0$ (c) $(3b^2 - a^2)y^2 + 8ab - (b^2 - 3a^2)x^2 = 0$

8) $k = \pm\frac{y}{1}$ 9) 9a+ 24h + 16b = 0 9 (b) $a^3 + 2hab + b^3 = 0$

10) $\pm 2\alpha$ 11) $4nh^2 = ab(n+1)^2$ 12) $4h^2 = (a+b)^2$

13) $\sqrt{3}(x^2+y^2)-4xy-(12+4\sqrt{3})x+4(8+6\sqrt{3})y+24+13\sqrt{3}=0$ 14) xy = 0

15) $3y^2 - xy - 2x^2 + 5x + 20y + 25 = 0$ 16) $h=\frac{1}{2}\sqrt{3a^2+10ab+3b^2}$

Practice Worksheet (Foundation Level) – 4(b)

1) 10 2) xy = 0 3) (a –b) (a –1) / $\sqrt{2}$ 4) $\pi/4$

5) 5y –3x = 0 6) $\frac{1}{2}a^2$ 7) mn = cl 8) $(a-b)(x^2-y^2)+4xy-h=0$

9) $3x^2 - 5xy + 2y^2 + x - 2 = 0$ 10) $\frac{9}{4ab}$ 11) p = 2, q = 13 12) ab + hg = 0

13) a = 2, c = –20 14) a =b, $h \neq b$ 15) $(xy_1 - yx_1)^2 = p^2(x^2+y^2)$

16) 1	17) 1/ $\sqrt{10}$	21) b	22) a,c	23) c
24) b	25) a	26) b	27) a	28) b
29) a	30) a,b,c,	31) b	32) c	33) a
34) c	35) a	36) c, d	37) a	38) c

Practice Worksheet (Foundation Level) – 4(c)

1) d	2) b	3) c	4) b	5) a	6) c
7) d	8) b	9) c	10) c	11) a	

12)a=6$\sqrt{3}$, h=3(1–$\sqrt{3}$) 13) (1, –2) 14) h = $\pm$ 1

15) $\sqrt{h^2-ab}\,/(bl^2-2hlm+am^2)$ 16) $\frac{1}{36ab}$ 17) $x^2+2xy\cot 2\theta-y^2=0$

18) $\tan^{-1}\left(\frac{2}{11}\right)$ 19) $3y^2+xy-4x^2+6x-13y+10=0$

20) $y^2-x^2+2ax+2ay=0$

21) Triangle is equilateral, area 16$\sqrt{3}$ sq. units vertices (0, 0), $[-2\sqrt{2}(\sqrt{3}-1), -2\sqrt{2}(1+\sqrt{3})]$, $[-2\sqrt{2}(\sqrt{3}+1)-2\sqrt{2}(\sqrt{3}-1)]$

22) L_1 shall be angle bisector of angle between L_2 and L_3)

23) $\frac{1}{25}$ sq. units 25) h = $\pm$1 26) $9a^2 + 10ab + 9b^2$

32) (a) $3y^2 = x^2$ (b) $y^2 - 3x^2 = 0$ 33) p = –7, q = – 3

Practice Worksheet (Foundation Level) – 5(a)

1. $(g^2 + f^2 - c) > 0$ 2. a = 2, h = 0, k = – 10 3. circle $x^2 + y^2 - 2x + 6y - 15 = 0$

4. (3 –4) 5. $x^2 + y^2 - 2x + 6y + \frac{49}{5} = 0$

6. $15x^2 + 15y^2 - 94x + 18y + 55 = 0$ 7. $x^2 + y^2 - 6x + 6\sqrt{2}y + 9 = 0$

8. $5(x^2 + y^2) - 28x - 4y = 0$ 9. $x^2 + y^2 - 8x + 12y - 49 = 0$

10.(0, 2), 3. 11. (y – 2) (y – 4) – k (x – 1) (x – 3) = 0; k = – 1

12. $(x - 7)^2 + (y - 6)^2 = 9$ 13. $x^2 + y^2 \pm 10x - 8y + 16 = 0$

14. $(1 + m^2)(x^2 + y^2) - a(x + my) = 0$ 15. (1, –8) 16. y = 3

17. $x^2 + y^2 - 2x - 4y = 0$ 18. $x^2 + y^2 - 2x + 3y - 3 = 0$

19. $5x^2 + 5y^2 - 40x + 10y + 81 = 0$, $5x^2 + 5y^2 - 30x - 44 = 0$

20. $(x - 4)^2 + (y - 1)^2 = 9$ 21. $(x + 2)^2 + (y - 3)^2 = 25$

Practice Worksheet (Foundation Level) – 5(b)

1. $x^2 + y^2 + 6x + 6y + 9 = 0$, $x^2 + y^2 + 30x + 30y + 225 = 0$

2. $x^2 + y^2 - 2(a + b)x - 2(a - b) = 0$

5. $x^2 + y^2 - 8x - 22y + 97 = 0$, $x^2 + y^2 - 8x + 2y - 23 = 0$

6. b 7. d 8. b 9. b 10. a 11. c

12. c 13. c 14. a 15. b 16. c 17. b

18. b 19. a 20. b 21. a 22. c 23. d

24. a 25. a 26. a 27. b 28. c 29. a, b

Practice Worksheet (Foundation Level) – 5(c)

1. $x^2 + y^2 - 8x + 2y + 13 = 0$ 2. $y + 2 = \sqrt{3}(x - 3) \pm 10$ 3. (1, –1)

4. $y \pm x = \pm 3 + 3\sqrt{2}$ 5. $x^2 + y^2 = 9$

6. 3x – 4y + 8 = 0, 3x – 4y – 42 = 0 7. Tangents are parallel, no point of intersection

8. x – y = 0 9. $[x - 4(\sqrt{3} + 1)]^2 + [y - 4(\sqrt{3} + 1)^2] = 16$

9. $[x + 4(\sqrt{3} + 1)]^2 + [y + 4(\sqrt{3} + 1)^2] = 16$

$[x - 4(\sqrt{3} - 1)]^2 + [y - 4(\sqrt{3} - 1)^2] = 16$

$\left[x+4\left(\sqrt{3}-1\right)\right]^2+\left[y+4\left(\sqrt{3}-1\right)^2\right]=16$

10. $c=\pm 2a$ 11. $\left(\frac{1}{4}, 1-\frac{\sqrt{3}}{4}\right)$ 12. $3\left(x^2+y^2\right)-4\sqrt{3}x-12=0$

13. $9(x^2 + y^2) = 4a^2$ 14. x2 + y2 – x – y – 3 = 0 15. $\frac{2}{3}\sqrt{3}$

16. (0, – 8) 17. 2 18. $x^2+y^2=2a^2$ 19. (3, –1)

20. 3x + 4y + 7 = 0 21. a 22. b 23. c

24. a, d 25. a,d 26. $\left(\frac{8}{5}, \frac{9}{5}\right)$

27. b 28. c 29. c 31. d 32. b 33. a

34. a 35. b 36. c 37. c

Practice Worksheet (Foundation Level) – 5(d)

1. Circle touch, Direct tangents 4y + 3x – 8 = 0 and 4y – 5x + 8 = 0; Transvers tangent is one only x = 0. 2. Circle touch each other internally; y + $\sqrt{3}x-6=0$

3. common tangents are $\sqrt{3}y=\pm(x+3)$ and x=0

4. $x^2 + y^2 - 8x - 9y + 30 = 0$ 5. $x^2 + y^2 - 6x + 8y - 75 = 0$

6. g_1/g_2 = f1/f2 7. x2 + y2 + ax + ay = 0

8. $\left[x+5\left(\sqrt{3}+1\right)\right]^2+\left[y+5\left(\sqrt{3}+1\right)\right]^2=25$ 9. (a) 10. x + y = 2, (1, 1)

13. $x^2 + y^2 - 5x - 7y + 18 = 0$ 14. a

15. b 16. a = 1; $x^2 + y^2 - 4x - 2y - 19 = 0$

17. b 18. d

Practice Worksheet (Foundation Level) – 5(e)

1. (–2, –5/2) 2. (2, 1) 3. $x^2 + y^2 - 2ax = 0$ 4. $x^2 + y^2 - cx - by + a^2 = 0$

5. c 6. c 7. d 8. d 9. b

10. b 11. c 12. b 13. b 14. a

15. c 16. a,b

17. $\left(x\pm\frac{8}{\sqrt{3}}\right)^2+(y-4)^2=16/3$ 18. $5x^2+5y^2-28x-12y+43=0$

19. $x(x-1) + y^2 = 0$

20. $2(x^2+y^2)+2gx+2fy+c=0$ 21. $x^2+y^2+4x-2y=0$

22. $(x-7)^2+\left(y-\frac{15}{2}\right)^2=\frac{125}{4}$, $(x-1)^2+\left(y+\frac{1}{2}\right)^2=\frac{125}{4}$ 23. 343 sq units 24. $\tan^{-1}\left(\frac{3}{4}\right)$

25. $\tan^{-1}\left(\frac{4}{3}\right)$

Part-A

1. $x^2+y^2-14x-12y+76=0$, $x^2+y^2-8x-6y+16=0$

2. $x^2+y^2-\frac{2y}{\sqrt{3}}-1=0$ 3. $x^2+y^2+2x+4y-(a^2+b^2)=0$

4. $x^2+y^2+2(10\pm3\sqrt{6})x+(55\pm24\sqrt{6})=0$ 5. $x^2+y^2+8x-6y+9=0$

6. 32 units 7. $x^2+y^2-3x+1=0$ 10. (5, 5), (5, 1), (7, 3), (3, 3)

11. $(x-5)^2+(y-5)^2=25$, $(x+3)^2+(y+1)^2=25$

13. $2x-3y+8=0$, $3x+2y-14=0$ 14. $x^2+y^2-6x-2y+1=0$

16. $12x^2-4y^2-24ax+9a^2=0$ 17. $x^2+y^2+2x-3y+2\pm\sqrt{7}(x+2y-2)=0$

18. a) (0, 1) on the circle, (3, 1) outside the circle and (1, 3) inside the circle

b) $(x-1)^2+(y-1)^2=1$

19. a) $x^2+y^2-xh-yk=0$ c) $2px+2qy=r$ 22. $\frac{a(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$

24. a) 75 sq units b) $8\sqrt{6}$ sq units

25) a) $(x+3)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{15}{2}\right)^2$ b) $(x-3)^2+(y-1)^2=9$

PART-B

1. 5units 2. $a^2>2b^2$ 3. $x^2+y^2+6x-3y+45=0$

4. $x^2+y^2-6x-2y+1=0$ 5. 75sq units

6. (14, 10) , $y-10=\frac{12\pm2\sqrt{6}}{15}(x-14)$ 7. $a>2$ or $a<-2$

13. $x^2+y^2-x=0$ 14. $(x-5)^2+(y-2)^2=25$

15. $x^2+y^2+gx+fy+\frac{1}{2}c=0$

16. y +7x – 14 = 0 , 7y – x +2 = 0 centers of the circle are (1,7), (3,-7) (9,1) and (-5, -1) 17. 8 sq m.

19. $\lambda = 1$ 20. (3,-1) 21. $x^2+y^2+7x-11y+38=0$

23. $(x-3)^2+\left(y-4-\sqrt{3}\right)^2=4$ 24. i) d ii) aiii) d

25. i) c ii) aiii) a 27. a

Answers to Objective Assignments

Level-1

1) c	2) c	3) b	4) b	5) d	6) a	7) b
8) b	9) c	10) a	11) b	12) c	13) a	14) a
15) a	16) b	17) a	18) d	19) b	20) a	21) a
22) b	23) c	24) d	25) d			

Level-2

1) b	2) a, c	3) c	4) a, c	5) c	6) b	7) c
8) a, b, c	9) a, b, c, d	10) c	11) b	12) a	13) c	14) b
15) d	16) d	17) a	18) b	19) d	20) b, c	21) a
22) c	23) b	24) a	25) a	26) a		

Practice Worksheet (Foundation Level) – 6 (a)

1. $16x^2 + 8xy + y^2 - 74x - 78y + 212 = 0$ 2. $y^2 - 6y - 16x + 25 = 0$

3. $(x - a)^2 = 8b(y - b)$ 4. $y^2 = 4(a' - a)(x - a)$

5. $(x - 2)^2 + 32(y - 3) = 0$ 6. $(-5, \pm 10)$

7. (a) (–1, 2), $\left(-\frac{1}{4},2\right)$,3, 4x + 1 = 0, 4x + 7 = 0

(b) $\left(-\frac{7}{2},\frac{5}{2}\right)$, $\left(-\frac{1}{4},2\right)$, 2, $x+\frac{17}{4}=0$, $x+\frac{1}{4}=0$

(c) (–2, 1), (–2, 0), 4, y = 0, y = 2

(d) $\left(a,\frac{a}{2}\right)$, (a, 0), 2a, y = 0, y = a

8. $x^2 + 4x + 4y^2 + 116x + 2y + 259 = 0$ 9. $x^2 - 4x + 4y - 12 = 0$

10. $(6, 4\sqrt{3})$ 11. (i) $y^2 = 2ax$, (ii) $y^2 = 4x$, (iii) $x^2 = -4y$

12. $8a\sqrt{3}, 64\sqrt{3}$ 13. (a) inside, (b) outside, (c) on the parabola

14. $y - 2x + 1 = 0$, $y + 2x = 7$, $\tan^{-1}\left(-\frac{4}{3}\right)$ 15. $y^2 - 4x - 6y - 7 = 0$

16. $x^2 - 2xy + y^2 - 10x - 6y + 41 = 0$ 17. c 18. b

19. c 20. b 21. b

Practice Worksheet (Foundation Level) – 6 (b)

1. $2y - x - 8 = 0$, $\sqrt{3}\,y - x = 6$, $y + 2x + 1 = 0$ 2. $6y + 2x + 27 = 0$, $2y - x = 3$

3. (3, 4) 4. $2 \sin \alpha \cdot \tan \alpha + p = 0$

5. $x - \sqrt{3}y + 6 = 0$, $x + \sqrt{3}y + 6 = 0$ 6. (4, –4), (–4, –4)

7. $x \pm y = 0$ 8. $\frac{\pi}{6}$ 9. $\frac{5\pi}{12}$ 10. $3\sqrt{5}$

11. $2\sqrt{5}$ 12. $x + y + b = 0$

13. $2\sqrt{3}y - x = 8$, $2\sqrt{2}y + x + 8 = 0$ 14. $\frac{\pi}{2}$

15. $x + y + 1 = 0$ 16. $\left(-\frac{9}{8}, -\frac{5}{6}\right)$ 17. $3\sqrt{2}\,a$ 18. 27 sq. cm.

20. $\sqrt{35}$, (2, –1) 21. $\frac{(22\sqrt{74})}{5}$ 22. y-axis

Practice Worksheet (Foundation Level) – 6 (c)

1. $y - x + 6 = 0$, $y + x + 6 = 0$ 2. $y - 8x = 33$, $y + 8x + 33 = 0$

3. $x - 4 = 0$ 4. $y = 2$

5. $y = mx + 2p + \frac{p}{m^2}$ 6. (–4a, 4a)

9. $\frac{32\sqrt{3}}{3}a$ 10. (4, 0)

11. $p = \frac{1}{2}$, $q = 2$ 12. $3y + 4x = 34$

13. $\left(\frac{59}{17},\frac{114}{17}\right)$

13. $6ab^2 + 3a^3 - 2b^2 = 0$

14. (a) $\frac{\pi}{4}$, (b) $\tan^{-1} 2$

16. $a\ell^3 + 2a\ell^3 + 2a\ell m^2 + m^2 n = 0$

18. $(4\cos^2\theta, -8\cos\theta)$; $x + y\cos\theta + 4\cos^2\theta = 0$

Practice Worksheet (Foundation Level) – 6(d)

1. $4x - 3y - 41 = 0$

2. $\left(\frac{1}{8},\frac{5}{2}\right)$

3. $(-1, -1)$

4. $y = x - 3$, $2\sqrt{26}$

7. $x^2 + y^2 - ax = 0$

9. $y^2 (x + 2a) + 4a^3 = a$

10. $\left(\frac{c-4a}{a},-\frac{4b}{a}\right)$

11. $4y = 5$

12. $\left(\frac{11}{3},-\frac{10}{3}\right)$

Practice Worksheet (Foundation Level) – 6(e)

1. 5
2. Directrix, $\frac{\pi}{2}$
3. parallel
4. semi-latus rectum
5. $(6, -2)$, $(8, 0)$
5. $\left(-\frac{3}{2},1\right)$
7. $9x^2 - 24xy + 16y^2 + 358x + 256y - 51 = 0$
8. $p=\frac{1}{4}, q=\sqrt{2}$
9. $(2b + 6, a)$
10. $\left(-\frac{7}{2},1\right)$
11. $(6, 6)$
12. $y=\frac{1}{\sqrt{3}}x+2\sqrt{3}$, $\sqrt{3}y+x+6=0$
13. $\tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)$
14. $x - 4y + 36 = 0$, $9x - 4y + 4 = 0$
15. $8\sqrt{2}$ sq. u.
16. $\frac{\pi}{2}$
17. $\frac{32}{3}p$
18. $(4, 5)$, 9 sq. u.
19. $SP^2 = ST^2$
20. $x = 3$
21. $\tan^{-1}\sqrt{2}$
22. $2bx + ay + 4b^2 = 0$
23. $(3, \pm 2\sqrt{3})$
24. $\left(\frac{7}{2},\frac{1}{4}\right)$
26. b
27. a, c
28. a, b, c
29. b
30. d
31. b, c, d
32. b
33. c
34. c
35. a
36. d

37. a	38. b	39. c	40. a
41. d	42. b	43. b	44. a
45. b	46. d	47. c	48. c
49. c	50. b	51. a, c, d	

Answers to Subjective Assignments
Part-A

1) a) $16x^2+y^2+8xy-74x-78y+212=0$

b) i) vertex $\left(\frac{-61}{16},\frac{2}{3}\right)$, Focus $\left(\frac{-485}{144},\frac{2}{3}\right)$ Directrix $x=\frac{-613}{144}$, L.R.=16/9, 16x + 61=0 ii) vertex (-3,-7) , Focus $\left(-3,\frac{-71}{10}\right)$ Directrix $y=-\frac{69}{10}$, L.R.=2/5, y= –7

c) $4x^2+y^2-4xy+8x+46y-71=0$

2) vertex $\left(-1,\frac{1}{2}\right)$, Focus $\left(-\frac{5}{8},\frac{1}{2}\right)$, Directrix $x=\frac{-11}{8}$

4) $2x+y+1=0$, $x-2y+8=0$, (1/2, –2), (8, 8)

5) $y=\pm(x+4)$

7) $y^2=2(x-4a)$

9) $y^2=a(x-3a)$,(3a, 0)

10) $m=\pm\sqrt{2}$

14) $y=x-1$, $8\sqrt{2}$

18) $\alpha=2$

19) a) i) $27ay^2=(x-5a)^2(x-2a)$ ii) $y^2=a(x-a)$ b) $x^2-2y+12=0$

20) $27ay^2\ 2(x-2a)^3$

23) $\frac{15a^2}{4}$

Part-B

1. $y^2=a(x-3a)$

2. $x^2=\frac{5625}{17}(y-3)$, 5.72 meter

5) $(x-1)^2+(y+1)^2+\frac{\sqrt{5}-1}{2}(x+2y+1)=0$

9. $k=\frac{3}{2}$

11. (2,1) radius1

13. $y^2=\frac{12}{343}(x-6)^3$

15. $\left(\frac{2}{9},\frac{8}{9}\right)$

18. $x^2-3x-y+4=0$

19. $(y-1)^2(x+1)+4=0$

20. $y^2=4ax$

22. i)c, ii) b, iii) d

23. i)b, ii) b, iii) c, iv) d, v) a   24. A-q, B-r, C-p, D-s   25. A-p, B-q, C-s D-r

Answers to Objective Assignments

Level-1

1) a	2) d	3) c	4) a	5) b	6) a	7) c
8) c	9) d	10) b	11) c	12) a	13) d	14)
15) a	16) d	17) d	18) a	19) a, b	20) a	21) d
22) d	23) a					

Level-2

1) c	2) b	3) a	4) b	5) a, c	6) d	7) c
8) d	9) c	10) c	11) b	12) c	13) c	14) d
15) b	16) c	17) b	18) c	19) a	20) b	21) b
22) b	23) a					

Practice Worksheet (Foundation Level) – 7(a)

1. (a) $4x^2 + 4b^2 y^2 = ab$ (b) $5x^2 + 9y^2 = 45$ (c) $20x^2 + 36y^2 = 405$
   (d) $3x^2 + 2xy + 3y^2 - 8 = 0$

2. $3x^2 + 4y^2 = 48$ 3. $\sqrt{3}/2$ 4. $\frac{1}{2}$ 5. $2k, 2\sqrt{k^2 - h^2}$

6. $\frac{1}{2}$ 7. $\frac{1}{\sqrt{5}}$; $\left(0, \pm\frac{1}{\sqrt{10}}\right); \frac{4\sqrt{2}}{5}$

8. $\sqrt{5}/3; \left(\pm\sqrt{5}+\frac{1}{2}, \frac{1}{3}\right), x = \pm\frac{9}{5}+\frac{1}{2}; \frac{8}{3}$

9. $x^2 + 2y^2 = 16$ 10 $\sqrt{3}/2$ 10. $\sqrt{3}/2$ 11. (a)   12. (c)

13. a   14. a   15. c

Practice Worksheet (Foundation Level) – 7(b)

1. $\left(\frac{5}{2}, \frac{15}{4}\right); 3x^2 + 4y^2 = 75$ 2. $\frac{\sqrt{11}}{6}$ 3. $x^2 + 2y^2 = 4, -l = \frac{1}{\sqrt{2}}$

4. $p = 2ab/\sqrt{2}$ 5. $\frac{8\sqrt{2}}{5}$, Mid. Point $\left(-\frac{4}{5}, \frac{1}{5}\right)$ 6. 60^0

7. $x + y = \pm\sqrt{7}$, $y - x = \pm\sqrt{7}$ 8. $\left(-\frac{16}{\sqrt{19}}, \frac{3\sqrt{3}}{\sqrt{19}}\right), \left(\frac{16}{\sqrt{19}}, -\frac{3\sqrt{3}}{\sqrt{19}}\right)$ 9. $\left(\frac{4}{\sqrt{7}}, \pm\frac{3}{\sqrt{7}}\right)$

11. (0, 2) 12. $\left(\frac{\sqrt{3}-1}{2}, \sqrt{3}+1\right)$

13. $\left(2\sqrt{3}-1, \frac{3-2\sqrt{3}}{2}\right)$ 14. $(x^2+y^2)^2 = 4x^2+3y^2$

15. $x^4 - 4y^4\ \ 9x^2 + 20 = 0$ 20. y = 2x + 5, y + 2x = 5 21. $8x^2 + 4y^2 = 9x^2\ y^2$

22. m/l 23. 50

24. $\frac{e-1}{e+1}$ 25. $\left(\frac{x}{a}+\frac{y}{b}\right)\cos\phi + \left(\frac{y}{b}-\frac{x}{a}\right)\sin\phi$

28. $y = x \pm \sqrt{a^2+b^2}, y + x = \pm\sqrt{a^2+b^2}, \left(\frac{-a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right)\left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right)$

29. $\frac{8}{3}\sqrt{2}$ 30. $a^2p^2 + b^2q^2 = n^2$ 31. c 32. b

33.a 34. c 35. b

36. b 37. d 38. c 39. a

40. b 41. d 42. c 43. a

44. a 45. b, a 46. a 47. b

Practice Worksheet (Foundation Level) – 7 (c)

1. 3x + 2y = 5 2. $9x - 3\ \sqrt{5}y + 5\sqrt{5} = 0$ 3. $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ 4. x + 2y = 4

5. $x^2 + 64y^2 = 9x^2y^2$ 6. $(4a^2l, 4b^2m)$ 7. $b^2x^2 + a^2y^2 - b^2\sqrt{a^2-b^2}\,x = 0$

8. $\left(\frac{4}{3}, \frac{3}{4}\right)$ 9. $3/\sqrt{10}$ 11. (a) 0 (b) $\left(\frac{27}{25}, 0\right)$

12. $\frac{x^2}{25} + \frac{y^2}{9} = 1$ 14. $3y^2 - 32x + 48 = 0$ 15. $x = \pm 2$ 16. $\frac{a^2}{p^2} + \frac{b^2}{q^2} = \frac{(a^2-b^2)}{k^2}$

17. $\tan^{-1}\left(\frac{11}{7}\right)$ 18. $x^2 + y^2 = 15$ 19. $9x^2 + 16y^2 = 576$ 26. c

27. a 28. d 29. d 30. b

31. c 32 33. b 34. b

35. a 36. c 37. a 38. c

39. b 40. c

Practice Worksheet (Foundation Level) – 7 (d)

1. a, b, d, e, f, g, h, i correct.

6. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$

9. x2 + y2 = 28

25. $\left(\frac{36}{25},\frac{72}{25}\right)$

26. $3x - 4y = \bullet\frac{10}{\sqrt{5}}$

27. $\theta=\tan^{-1}\left(\frac{-4\sqrt{3}}{\sin 2\phi}\right)$ where ϕ is eccentric angle of P.

28. Infinite	29. $-\frac{16}{81}$	30. $\tan^{-1}\frac{8}{17\sqrt{3}}$	31. $\tan^{-1}\left(\frac{13}{5}\right)$
32. α	33. a	34. b	35. d
36. a, c	37. a	38. a	39.b
40. a	41. d	42. b	43. c
44. a	45. b	46. d	47. d
49. d			

Answers to Objective Assignments

1. Centre (1,2), e=1/√3, √2, 2/√3

2. a) $a^2l^2+b^2m^2=n^2$ b) $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$ c) $a^2l^2+b^2m^2=2n^2$

3. 60^o

6) y +x =5, y =0

11) 0, $\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ t =0 , $\pm\frac{1}{\sqrt{5}}$

14) $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}},\pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$

18) (1/2, 2) , (1,1), (-1,1/2)

19) $\left(\pm\sqrt{\frac{12}{5}},\pm\sqrt{\frac{18}{5}}\right)$, $\tan^{-1}\frac{1}{\sqrt{6}}$

25. b) $3x^2+4y^2-2x-8y+4=0$ c) $y=-\frac{2x}{\sqrt{3}}+4\sqrt{\frac{7}{3}}$ AB= $\frac{14}{\sqrt{3}}$

Level-1

Objective:

1) a	2) c	3) a, b	4)c, d	5)a	6)b	7) a	8)c
9)b	10)b, c	11)d	12) b	13)a	14)c	15)b	16) b

17) 18) 19) a 20) a

Level-2

1)c	2)a	3)a	4)b, d	5)a, b, c, d	6)a, c	7)a
8)c	9)a	10)d	11)a	12) c	13) a	14)d
15) b	16) b	17) b	18) b, c	19) a	20)	

Practice Worksheet (Foundation Level) – 8 (a)

1. $\frac{\sqrt{5}}{2}$ 2. $x^2 - 3y^2 = 12$ 3. $(3x + 4y + 2)(2x - 3y - 4) + 12 = 0$

4. $\frac{4}{9}x^2 - \frac{7y^2}{7} = 1$

5. (i) (14, 1), (–2, 11), (ii) (6, 1), (–4, 1), (iii) $x = \frac{14}{5}$, $x = \frac{-4}{5}$, (iv) $4x + 3y = 7$, $4x - 3y = 1$

6. c   7. $\frac{x^2}{81} + \frac{y^2}{45} = 1$, $\frac{x^2}{16} - \frac{y^2}{20} = 1$ 8. $x^2 - y^2 = 1$

9. $2x^2 + 5xy - 2y^2 + 4x + 6y + \frac{80}{41} = 0$ 10. 1   11. $\frac{4}{3}$

12. $\frac{16(x-1)^2}{81} - \frac{16(y-3)^2}{63} = 1$ 13. $x = 3$

14. $(x^2 - y^2)\sin 2\theta + 2xy\cos 2\theta = 2\cos 2\theta$

15. $y^2 - 3x^2 = 4$ 16. $x + y = 3$ 17. $\frac{8\sqrt{2}}{3}$ sq. u.   18. 4

19. $\frac{5}{4}$, $\frac{x^2}{16} - \frac{y^2}{9} = 1$ 20. $2\sqrt{2} : 3$

21. $\tan^{-1}\sqrt{21}$ 22. $2\tan^{-1}\left(\frac{9}{4}\right)$ 23. $e^2 : 1$

24. 0	25. a	26. a, c	27. d	28. a	29. b
30. c	31. b	32. c			

Practice Worksheet (Foundation Level) – 8(b)

1. $8x + 3\sqrt{7}y + 2 = 0$, $3\sqrt{7}x - 8y + \frac{34\sqrt{7}}{3} = 0$ 2. $y - x = \pm$, $y + x = \pm 1$ 3. $x = \frac{4}{\sqrt{5}}$

6. $128x^2 - 200xy - 247y^2 + 244\,ax + 50ay - 247a^2 = 0$ 7. $\frac{x^2}{m^2} - \frac{y^2}{\ell^2} = 1$

8. $3x^2 - 5xy - 2y^2 + 5x + 11y - 12 = 0$; $3x^2 - 5xy - 2y^2 + 5x + 11y - 16 = 0$

9. $x \sin\theta + y \cos\theta\ x \cos\theta - y \sin\theta = 0$

10. $(2x + y + 2)\,(x + 2y + 1)\ \ 0$, $(2x + y + 2)\,(x + 2y + 1) = \lambda$

13. $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$ 15. $\left(0, -\frac{1}{3}\right)$ 19. approximately 8

20. 6	21. a	22. b	23. d	24. d	25. d
26. c	27. a	28. c	29. d	30. b	31. a, d
32. a	33. d	34. b	35. c	36. c	37. a, d
38. a	39. a	40. b			

Practice Worksheet (Foundation Level) – 8 (c)

1. $x^2 - 4y^2 = 1$

2. $\frac{3}{2}$; (6, 1), (0, 1); $x - 3 = \pm\frac{4}{3}$; 5

3. $12x^2 - 4y^2 = 3$ 5. $4x^2 - 5y^2 = 20$ 6. $x^2 - 6y^2 = a^2 - 6b^2$

7. $e = 2$, $3x^2 - y^2 = 48$

8. $9y^2 - 4x^2 = 4x^2y^2$

9. $zy^2 = xy - 2b^2$

14. $x^2 + y^2 = 7$

16. $3a^2 = 4b^2$

17. $\tan^{-1}\left(\frac{20}{3\sqrt{3}}\right)$

18. $\tan^{-1}(10\sqrt{3})$

19. $9x^2 - 16y^2 = 144$, $e = \frac{5}{4}$

20. $\sqrt{2}$

21. $\frac{4\sqrt{6}}{3}$, (2, 1)

22. $4\sqrt{5}a$

23. $\tan^{-1}(10\sqrt{3})$

24. $\frac{5}{4}$	25. $x^2 - y^2 = 32$	26. b	27. a	28. c	29. b
30. a	31. b	32. a	33. b	34. b	35. a
36. c	37. a	38. d	39. a	40. c	41. c
42. a	43. c	45. d	46. b		

Answers to Subjective assignment

1. a) $7x^2 - 2y^2 - 2x + 12xy + 14y - 22 = 0$

6. a) 2x– y +1=0, 2x +y +1=0 b) $y = \pm x \pm \sqrt{a^2 - b^2}$ 16) $16x^2 + y^2 + 10xy - 2 = 0$

Answers to objective assignments

Level-1

1) c	2) c	3) d	4) a	5) c	6) b	7) d
8) d	9) d	10) b	11) a	12) b	13) b	14) b
15) c	16) a, d	17) a	18) a	19) a	20) b	21) c

Level-2

1) d	2) b	3) a	4) b	5) d	6) a	7) c
8) a, b, c, d	9) a, b, c	10) b	11) a	12) a	13) d	
14) a, b, c, d	15) b, d	16) a, c	17) d	18) b	19) d	
20) a	21) b					

Miscellaneous Practice Worksheet (Foundation Level) –

3. $\frac{\pi}{6}$ 4. $\left[\frac{1}{2}(p+q), \frac{1}{2}(p+q)\right], \left[\frac{1}{2}(p-q), \frac{1}{2}(p-q)\right]$

5. $(1+\sqrt{2})y + \sqrt{3}x = 0$ 6. a = 2, b = 6, 2y + x = 5

9. $y^2 - 4x + 6y + 9 = 0$ 10. $4x^2 + 9y^2 = 36$

11. (i) $x^2 + y^2 + 2x = 0$ (ii) (0, 0) (iii) x = 2

13. (a) $y = \sqrt{3}x + 2 + \frac{7\sqrt{3}}{4}$ (b) $y - x \pm 4 = 0$ (c) 4x – 3y = 24 and 4x – 3y + 26 = 0

15. (1, –2) and (2, –1) 16. $(x-2)^2 + (y+3)^2 = (\sqrt{5}+1)^2$ 17. $y = \frac{9}{4}$

18. $e = \frac{1}{2}; \frac{3}{4}$ 25. $\left(\frac{3\sqrt{3}}{\sqrt{8}}a, -\frac{\sqrt{3}a}{\sqrt{8}}\right)$, common tangent $3x + y = 2\sqrt{6}a$ $\left(\frac{\sqrt{6}a}{3}, \sqrt{6}a\right)$

29. (6, 6); (0, 2)

31. L,$\left(\frac{4}{5}, -\frac{2}{5}\right)$, M,$\left(-\frac{2}{5}, \frac{6}{5}\right)$; $2(x-2)^2 - 7(x-2)(y-2) + 3(y-2)^2 = 0$ 32. $a^2 > 2b^2$

33. $\tan^{-1}(\sqrt{2})$, $\frac{14}{5}$ sq. unit 35. $\tan^{-1}\left(\frac{\sqrt{5}}{5}\right)$ 36. $\frac{\sqrt{2}}{3}(3+\sqrt{3})$, $\frac{\pi}{12}$

38. 25 : 11 : 24 39. $c^2 : a^2$ 40. $\left(4, \frac{1}{2}\right)$

41. (–2, 2) 42. 4 : 30 : 20 43. $3x - y = 4$

44. $5(4x + 3y) + 12 = 0$ 45. $\tan^{-1}\left(\frac{1}{3}\right)$ and $\tan^{-1}\left(\frac{4}{3}\right)$

46. (0, 4), $\sqrt{26}$ 47. $5a - 2b = 0$

48. $x^2 + y^2 \pm 12x + 27 = 0$, $x^2 + y^2 \pm 4\sqrt{3}\,y + 3 = 0$

49. $x^2 + (y - 6)^2 = 100$, $(x - 6)^2 + (y + 2)^2 = 100$ 50. $x^2 + y^2 - 8x - 4y + \frac{55}{4} = 0$

51. $a^2 + b^2 = 2p^2$ 52. $x + y + 1 = 0$ 53. p = 8 54. $5\sqrt{30}$ sq. u.

55. $\frac{\pi}{2}$ 56. $-\frac{2}{3}$ 62. $2(x^2 + y^2) = a^2x^2 + b^2y^2$

63. $y - 2x = 1$, $y + 2x + 1 = 0$ 64. $t = \pm\frac{1}{\sqrt{5}}$, $\phi = \text{tam}^{-1}\, 2$

65. $y^2(x - a) = x^3$ 66. $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ 67. $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$

68. (i) $16x^2 + y^2 + 10xy = 1$ (ii) $4x + y = 0$

69. $x = \pm\frac{4}{3}$; $5x + 2y - 6 = 0$, $5x - 2y + 6 = 0$ 70. $\frac{(y+3)^2}{9} - \frac{(x-2)^2}{16} = 1, e = \frac{5}{4}$

71. d	72. a	73. c	74. d	75. b	76. c	77. b	78. c	79. a
80. c	81. b	82. b	83. c	84. b	85. b	86. c	87. c	88. d
89. a	90. d	91. c	92. c	93. d	94. a, d	95. d	96. d	97. a
98. a	99. a, b	100. d	101. b	102. d	103. a	104. a	105. d	106. a
107. a	108. d	109. d	110. c	111. d	112. d	113. a	114. b	115. b
116. b	117. c	118. a	119. a	120. a	121. b	122. c	123. a	124. b
125. b	126. d	127. a	128. a	129. d	130. a	131. c	132. a, d	133. b
134. c	135. a	136. b	137. b	138. b	139. c	140. b	141. c	142. b
143. b	144. a	145. c	146. a	147. b	148. c	149. c	150. a	151. a
152. b	153. b	154. b	155. c	156. a	157. b	158. b	159. b	160. b
161. c	162. b	163. a	164. d	165. b	166. a	167. c	168. d	169. d
170. d	171. c	172.a	173. c	174. c	175. b	176. d	177. a	178. a
179. b								

About the Book

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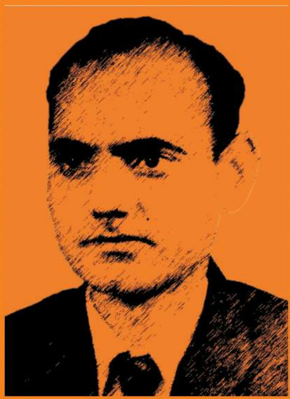
The book has been designed keeping in mind the present trend of IITJEE and other competitive examinations. For smart students who are raring to ride on the tides of this adventurous IITJEE, here is a unique book to satisfy their appetite! The book presented henceforth has been crafted whilst keeping in mind the present trend and style of the examination. The examination style has changed over the years and as such students need to prepare themselves accordingly. The book tries to explain the concepts in such an easy and comprehendible manner that a student will not have to go any further than this. Every lesson has been elaborated with related illustrations, core objectives and solved examples (subjective). Apart from objective questions, subjective answers have been included in order to lend an in-depth understanding. In order to help the student grasp the fundamentals, we have made it a point to illustrate and elucidate things starting from scratch. This will enable the student to get a better understanding of the topic.

About the Authors



Kavindra Nath Saxena, M.A., M.Sc., L.T., Rt. Principal, Govt. Inter College, Gochar, Chamouli, has taught Mathematics to Intermediate Classes upto Dec 1972 and during Dec 1980-82.
During this period he wrote three Mathematics books for Intermediate classes - Algebra (1968), Co-ordinate (1976) and Dynamics (1971). He also worked as Deputy-Secretary, Intermediate Board for 2 years and dealt services for one year at Head Quarters, Allahabad. Also had the privilege for setting board paper continuously for 5 years. After retirement in June 1982, he remained in constant touch with the subject and coached students for competitive examinations. After through study he has written these books for the benefits of students preparing for competitions.

Lalit Yadav, a graduate in engineering from world famous University of Roorkee (Now IIT-Roorkee), has been into shaping the future of students for past 15+ years and has gained expertise in mathematics. Having the reputation for excellence in coaching, motivating, and instructing students through challenging math curricula for competitive examinations, he has designed lesson plans and provided data driven instruction in mathematics as Head at many Institutes in North India providing coaching to the students for IIT-JEE examinations.
He has also monitored academic performance and provided additional attention to students in need. Also served as an advisor to students and liaison to the parents and directed team discipline efforts on a daily basis.

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