Control Engineering

Yusheng Wei
Zongli Lin

Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay

Birkhäuser

Birkhäuser

Control Engineering

More information about this series at http://www.springer.com/series/4988

Yusheng Wei • Zongli Lin

Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay

Birkhäuser

Yusheng Wei
Electrical and Computer Engineering
University of Virginia
Charlottesville, VA, USA

Zongli Lin
Electrical and Computer Engineering
University of Virginia
Charlottesville, VA, USA

ISSN 2373-7719 ISSN 2373-7727 (electronic)
Control Engineering
ISBN 978-3-030-53428-8 ISBN 978-3-030-53429-5 (eBook)
https://doi.org/10.1007/978-3-030-53429-5

Mathematics Subject Classification: 34K06, 93D05, 93D09, 93D15, 93D20, 93D21

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To Our Families

Preface

Differential equations have been used to model dynamic systems. Such modeling has met great success primarily because solutions of differential equations precisely predict the actual behaviors of the modeled systems. However, the analytic solution of a dynamical system is in general quite challenging to obtain. Various methods have been developed to determine and analyze the behaviors of dynamical systems without obtaining their explicit solutions. Among these methods is the Lyapunov function based analysis. Over the course of its development, the Lyapunov function based analysis has shown its universal applicability to the study of a wide range of systems such as linear and nonlinear systems, time-invariant and time-varying systems, and deterministic and stochastic systems.

Time delay systems refer to those dynamic systems whose change of current state depends on the past values of its state and/or input. Such lagging phenomena have been frequently observed in engineering practice. Input delay of a system emerges whenever the transmission of the control signal from the controller to the actuator of the system takes a certain amount of time. This lagging effect in the input can be caused by long-distance transmission of the control signal or time-consuming computation of a control algorithm carried out by the controller. A fundamental problem in the control of time delay systems is the problem of stabilization. The importance of such a problem is obvious from the observation that feedback laws designed without consideration of the delay typically fail to stabilize when the value of the delay grows large.

There are two commonly followed paths to achieving the stabilization of linear systems with input delay. The first path extensively involves the Lyapunov function based analysis. We pick a Lyapunov function for the open loop system. By designing a feedback law that makes the time derivative of the Lyapunov function along the trajectory of the closed-loop system negative definite, we achieve closed-loop stability. Typically, a stability criterion obtained from such an analysis is in the form of linear matrix inequalities (LMIs). Thanks to effective computation techniques, the solution of LMIs is an easy task. The Lyapunov function based method possesses considerable advantages in dealing with time-varying features of a time delay system, including time-varying delays and other time-varying system parameters.

Uncertainties in the system, such as external noises, stochastic nature of the system, and even uncertainties in the delay, can also be readily handled.

The second path to achieving stabilization is more straightforward. To directly compensate the input delay, we express the control input as the product of a feedback gain matrix and the state of the system at a future time ahead of the current time by the same amount of the delay. Such a feedback law results in a stable closed-loop system free of delay. By predicting the future state as the sum of the zero input solution and the zero state solution of the system, the feedback law allows implementation from the causality point of view. Such a form of feedback is referred to as the predictor feedback. However, the term in the predictor feedback that corresponds to the zero state solution is distributed because the zero state solution is a convolution between the state transition matrix and the input term. This causes difficulty in the implementation of the predictor feedback.

A possible way to avoid such difficulty is to truncate the distributed term from the predictor feedback law. The resulting feedback law is referred to as the truncated predictor feedback law. The delay-dependent state transition matrix in the truncated predictor feedback law prevents its application to control scenarios when the exact knowledge of the delay is not available. A delay independent truncated predictor feedback law results when the delay-dependent transition matrix in the truncated predictor feedback law is further dropped. It has been shown that the truncated predictor feedback law compensates an arbitrarily large delay in a linear system with all its open loop poles in the closed left-half plane when the feedback gain matrix is designed by the use of the low gain design techniques and is parametrized in a single low gain parameter. Conversely, the delay independent truncated predictor feedback law compensates an arbitrarily large delay in a linear system only when all its open loop poles are at the origin or in the open left-half plane. The key to achieving stabilization is the parameterization of the feedback gain matrix by a single constant low gain parameter. For a given, arbitrarily large, delay, the closed-loop stability is guaranteed by tuning the feedback parameter to a small enough value.

By either the truncated predictor feedback law or the delay independent truncated predictor feedback law, the stabilization requires some knowledge of the delay. In particular, the state transition matrix of the truncated predictor feedback law contains the explicit value of the delay. An upper bound of the delay is required for the determination of a stabilizing feedback parameter in the delay independent truncated predictor feedback law. In the absence of any knowledge of the delay, the design of a time-varying feedback parameter whose value is updated by an adaptive algorithm is then crucial for arriving at a stabilizing feedback law. Even when we have the knowledge of an upper bound of the delay, the design of a time-varying feedback parameter in the delay independent truncated predictor feedback law helps to result in stronger closed-loop performance in terms of a smaller overshoot and a higher convergence rate.

This book focuses on the design of truncated predictor based feedback laws for general, possibly exponentially unstable, linear systems. The first part of the book is dedicated to the design of truncated predictor based feedback laws with a constant feedback parameter. It is established through examples that these feedback laws cannot stabilize general, possibly exponentially unstable, linear systems with a sufficiently large delay. Admissible delay bounds that guarantee closed-loop stability are established. The second part of the book is dedicated to the design of time-varying feedback parameters in the truncated predictor based feedback laws. Such designs are motivated by the desire to improve the closed-loop performance of systems under the truncated predictor feedback laws with a constant feedback parameter and to enable the feedback laws to accommodate the unknown delay. In particular, a family of time-varying feedback parameters in the delay independent truncated predictor feedback law improves the closed-loop performance with a smaller overshoot and a higher convergence rate. Moreover, we manage to not require any knowledge of the delay in the regulation of linear systems by equipping the delay independent truncated predictor feedback law with a delay independent update algorithm for its feedback parameter.

The organization of the book is as follows: Chapter 1 introduces time delay systems and recalls some fundamental concepts and design methods. Time delay as a frequent scene in almost every aspect of engineering practice is illustrated through examples. For the study of asymptotic behaviors of a closed-loop system with delay, stability definitions are presented. At the end of the chapter, we introduce the basic design of the predictor feedback law for the stabilization of linear systems with input delay, along with a discussion on the difficulty associated with its implementation.

Chapter 2 introduces the design of the truncated predictor feedback law for continuous-time linear systems. Low gain feedback design techniques are employed to parameterize the feedback gain matrix of the truncated predictor feedback law by a single constant low gain parameter. For a system with all its open loop poles in the closed left-half plane, the truncated predictor feedback law compensates an arbitrarily large delay as long as the low gain parameter is chosen small enough. The original truncated predictor feedback law is parameterized by using an eigenstructure assignment based low gain feedback design technique. An alternative approach by using an algebraic Riccati equation based low gain design technique is also reviewed. Both state and output feedback designs are covered.

Chapter 3 develops the discrete-time counterparts of the results in Chap. 2. It is shown that the truncated predictor feedback law compensates an arbitrarily large delay in a discrete-time linear system with all its open loop poles on or inside the unit circle. Both an eigenstructure assignment based and an algebraic Riccati equation based low gain feedback design techniques for the parameterization of the feedback gain matrix are reviewed. As in Chap. 2, both state and output feedback designs are presented.

Chapter 4 generalizes the results in Chaps. 2 and 3 to a general linear system that is possibly exponentially unstable. For an exponentially unstable linear system, no feedback law can achieve stabilization when the input delay is large enough. We construct state and output feedback laws of the form of a truncated predictor feedback law and establish conditions on the delay, the system, and controller parameters under which the closed-loop system is asymptotically stable. Based on these conditions, the design of the feedback law that allows maximum delay is determined. Both continuous-time and discrete-time systems are considered.

In Chaps. 5 and 6, we focus on the design of delay independent truncated predictor based feedback laws. In the absence of the exact knowledge of the delay, the truncated predictor feedback law is no longer implementable due to its delay-dependent term. The removal of the delay-dependent term from the truncated predictor feedback law results in the delay independent truncated predictor feedback law. It is shown through an example that the delay independent feedback cannot compensate an arbitrarily large delay in a linear system with purely imaginary open loop poles. Admissible delay bounds with stability guarantee are then established for general, possibly exponentially unstable, linear systems. However, for a system with all its open loop poles at the origin or in the open left-half plane, the delay independent truncated predictor feedback law compensates an arbitrarily large delay as long as the low gain feedback design technique is applied to parameterize the feedback gain matrix. Such a low gain nature of the delay independent truncated predictor feedback law leads to a large overshoot and a low convergence rate of the closed-loop system. A time-varying feedback parameter design is then proposed to improve the closed-loop performance under the constant feedback parameter design. To comprehensively study the stabilizing effects of the delay independent truncated predictor feedback, we examine the stabilization of continuous-time and discrete-time linear systems, respectively, in Chaps. 5 and 6, by either state or output feedback.

The time-varying feedback parameter design in Chap. 5 requires an upper bound of the delay to be known for the stabilization. We manage to not require any knowledge of the delay in feedback designs in Chap. 7. In the absence of any knowledge of the delay, a control scheme is proposed that equips the delay independent truncated predictor feedback law with an updated algorithm, which is also delay independent, for the feedback parameter. This control scheme allows ease of implementation because only current state, and no knowledge of the delay, is required. Discrete-time counterpart of such an adaptation scheme is also developed in Chap. 8.

This monograph was typeset by the authors using LaTeX. All simulation and numerical computation were carried out in MATLAB.

The authors would like to thank the National Science Foundation for its generous support that has led to most of the results contained in this book.

Charlottesville, VA, USA Yusheng Wei
Charlottesville, VA, USA Zongli Lin

Contents

Notation

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
$\mathbb{R}_0^+$	The set of nonnegative real numbers
$\mathbb{R}^n$	The set of real vectors of dimension n
$\mathbb{R}^{n\times m}$	The set of real matrices of dimensions $n \times m$
$\mathbb{C}$	The set of complex numbers
j	The imaginary unit $\sqrt{-1}$
$\mathrm{Re}(\cdot)$ $(\mathrm{Im}(\cdot))$	The real (imaginary) part of a complex number
$\lvert\cdot\rvert$	The absolute value of a scalar, the Euclidean norm of a vector, or the norm of a matrix induced by a vector Euclidean norm
0	A zero scalar, vector, or matrix of appropriate dimensions
I (I_n)	An identity matrix of appropriate dimensions (of dimensions $n \times n$)
$I[a, b]$	The set of integers within the interval $[a, b]$, where $a, b \in \mathbb{R}$ and $a \leq b$. Either side of the interval can be open if a or b is replaced by ∞
$\dot{x}(t)$	The first-order derivative of $x(t) : \mathbb{R} \to \mathbb{R}^n$ with respect to time t
$C([t_1, t_2], \mathbb{R}^n)$	Or $C[t_1, t_2]$ for brevity, the set of $\mathbb{R}^n$-valued continuous functions on $t \in [t_1, t_2]$
$D([k_1, k_2], \mathbb{R}^n)$	Or $D[k_1, k_2]$ for brevity, the set of $\mathbb{R}^n$-valued functions on $k \in I[k_1, k_2]$, where $k, k_1, k_2 \in \mathbb{Z}$
$\lvert\lvert f\rvert\rvert_C$	The continuous norm $\sup_{t\in[t_1,t_2]} \lvert f(t)\rvert$ of $f \in C([t_1, t_2], \mathbb{R}^n)$
$\lvert\lvert f\rvert\rvert_D$	The discrete norm $\max_{k\in I[k_1,k_2]} \lvert f(k)\rvert$ of $f \in D([k_1, k_2], \mathbb{R}^n)$
$L_2([t_1, t_2], \mathbb{R}^n)$	Or $L_2[t_1, t_2]$ for brevity, the set of $\mathbb{R}^n$-valued square integrable functions on $t \in [t_1, t_2]$
$AC([t_1, t_2], \mathbb{R}^n)$	Or $AC[t_1, t_2]$ for brevity, the set of $\mathbb{R}^n$-valued absolutely continuous functions f on $t \in [t_1, t_2]$ with $\dot{f} \in L_2([t_1, t_2], \mathbb{R}^n)$

$\|\|f\|\|_{AC}$	The absolutely continuous norm $\sup_{t\in[t_1,t_2]} \|f(t)\|^2+\int_{t_1}^{t_2} \|\dot{f}(t)\|^2 \mathrm{d}s$ of $f \in AC([t_1, t_2], \mathbb{R}^n)$
$C^k([t_1, t_2], \mathbb{R}^n)$	Or $C^k[t_1, t_2]$ for brevity, the set of $\mathbb{R}^n$-valued functions having continuous kth order time derivative on $t \in [t_1, t_2]$
$PC([t_1, t_2], \mathbb{R}^n)$	Or $PC[t_1, t_2]$ for brevity, the set of $\mathbb{R}^n$-valued piecewise continuous functions on $t \in [t_1, t_2]$
MF	The set of multivariate functions $f(x, t) : [0, 1] \times \mathbb{R} \to \mathbb{R}^n$
$f_x(x, t)$	$\frac{\partial}{\partial x} f(x, t)$ for $f(x, t) \in MF$
$f_{xt}(x, t)$	$\frac{\partial^2}{\partial t \partial x} f(x, t)$ for $f(x, t) \in MF$
$\|\|f(t)\|\|$	$\sqrt{\int_0^1 \|f(x, t)\|^2 \mathrm{d}x}$ for $f(x, t) \in MF$
L^1	$\left\{x(t) \;\middle\vert\; x(t) : [0, \infty) \to \mathbb{R}^n \text{ and } \int_0^\infty \|x(t)\| \mathrm{d}t < \infty\right\}$
x_t	The restriction of $x(s) : \mathbb{R} \to \mathbb{R}^n$ to $s \in [t - \tau, t]$, for some $\tau \in \mathbb{R}_0^+$
$\dot{x}_t$	The restriction of $\dot{x}(s) : \mathbb{R} \to \mathbb{R}^n$ to $s \in [t - \tau, t]$, for some $\tau \in \mathbb{R}_0^+$
$x_t(\theta)$	$x(t + \theta)$, where $x(s) : \mathbb{R} \to \mathbb{R}^n$ and $\theta \in [-\tau, 0]$, for some $\tau \in \mathbb{R}_0^+$
x_k	The restriction of $x(p) : \mathbb{Z} \to \mathbb{R}^n$ to $p \in I[k - r, k]$, for some $r \in \mathbb{N}$
$x_k(l)$	$x(k + l)$, where $x(p) : \mathbb{Z} \to \mathbb{R}^n$ and $l \in I[-r, 0]$, for some $r \in \mathbb{N}$
$\mathrm{tr}(\cdot)$	The trace of a square matrix
$\det(\cdot)$	The determinant of a square matrix
$\lambda(\cdot)$	The set of eigenvalues of a square matrix
$\lambda_{\min}(\cdot)$ ($\lambda_{\max}(\cdot)$)	The minimum (maximum) eigenvalue of a real symmetric matrix
v^{T} (A^{T})	The transpose of a vector v (a matrix A)
$A > B$ ($A \geq B$)	$A - B$ is positive definite (semi-definite), where A and B are real symmetric matrices
$A < B$ ($A \leq B$)	$A - B$ is negative definite (semi-definite), where A and B are real symmetric matrices

Chapter 1
Introduction

1.1 Introduction to Time Delay Systems

The phenomenon of time delay is a commonplace in almost every scientific discipline. Time delay refers to the amount of time it takes for the matter, energy, or information in a dynamic system to transfer from one place to another or to make their full impact on the system after their emergence. Such a lagging effect causes the change of current state of the system to rely on past values of its state and/or input. For instance, the economic model in [127] reveals that economic growth relies on population growth and technological advancement. In particular, the population growth does not take effect on the economic growth until it transitions to the labor growth, which potentially takes a couple of decades. Similarly, the technological advancement does not boost the economy until the productivity of the workforce is improved through technology innovations. Other examples of time delay in the study of biology, physics, mathematics, and engineering are many, and we will mention a few in the following subsection as examples.

1.1.1 Examples of Time Delay Systems

1.1.1.1 A Predator–Prey Model

In biology studies, the Lotka–Volterra equations are differential equations that describe predator–prey interactions in a natural ecosystem. A typical set of the Lotka–Volterra equations takes the following form (see [98]):

$$\dot{x}_1(t) = \alpha x_1(t)\left(1 - \frac{x_1(t)}{K}\right) - \beta x_1(t)x_2(t), \tag{1.1}$$

$$\dot{x}_2(t) = -\gamma x_2(t) + \omega x_1(t-\tau)x_2(t-\tau), \tag{1.2}$$

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5_1

where $x_1(t)$ and $x_2(t)$ are the populations of the prey and the predator, respectively, and α, K, β, γ, and ω are positive constants determined by the characteristics of the prey and the predator and the natural ecosystem in which they inhabit. The first term on the right-hand side of (1.1) suggests that even in the absence of predation, the population of the prey cannot exceed K, which represents the carrying capacity of the ecosystem for the prey population. The second term on the right-hand side of the equation describes the negative effect of the predation on the population of the prey. Basically, the predation causes the population of the prey to decrease, and the rate of this decrease is proportional to the number of interactions between the prey and the predator, which can be characterized by the product of the population of the prey and that of the predator.

The right-hand side of (1.2) also contains two terms. The first term suggests that, in the absence of the prey, the population of the predator decreases exponentially toward zero. The second term indicates that the positive effect of the predation on the population of the predator does not occur instantly. The positive constant τ represents the time it takes for the predation to show its impact on the growth of the predator population. Compared to the predator–prey model without consideration of the delay τ (see [45]), the differential equations (1.1) and (1.2) are more accurate in describing the actual population dynamics of the two species.

1.1.1.2 The Distribution of Primes

In the study of the distribution of primes, mathematicians formulated the asymptotic behavior of the prime counting function $\pi(x)$ with respect to the prime x in the Prime Number Theorem.[1]The prime countering function $\pi(x)$ is the number of prime numbers that are not greater than the prime number x. It turns out that such a theorem can be proved from the perspective of time delay systems [96, 111]. Reference [96] defined a smooth curve $y(x)$ that best fits the actual variation of $\pi(x)$. By using a probability argument for the distribution of primes, [96] obtained

$$2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right) = 0, \tag{1.3}$$

where the first- and second-order derivatives with respect to x are defined as $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, respectively. Reference [111] defined two functions of x, v, and w, as

$$2^v = \ln x$$

and

[1]The Prime Number Theorem: $\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1$.

$$w(v) = \frac{\mathrm{d}y}{\mathrm{d}x} \log x - 1,$$

which, together with (1.3), imply that

$$\frac{\mathrm{d}x}{\mathrm{d}v} = \alpha x \log x, \tag{1.4}$$

$$\frac{\frac{\mathrm{d}w}{\mathrm{d}v}}{1 + w(v)} = \alpha \left(1 - 2^{v-1} \frac{\mathrm{d}y}{\mathrm{d}x} \left(x^{\frac{1}{2}}\right)\right), \tag{1.5}$$

$$1 + w(v - 1) = 2^{v-1} \frac{\mathrm{d}y}{\mathrm{d}x} \left(x^{\frac{1}{2}}\right), \tag{1.6}$$

where $\alpha = \ln 2$. Then, the following time delay system is obtained from (1.5) and (1.6),

$$\frac{\mathrm{d}w}{\mathrm{d}v} = -\alpha w(v - 1)(1 + w(v)). \tag{1.7}$$

According to [111], the solution to (1.7) satisfies that $w(v) \to 0$ as $v \to \infty$. Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x}$ approaches $1/\log x$ as x goes to infinity, which coincides with the statement of the Prime Number Theorem. This example shows that the study of time delay systems facilitates the development of number theory.

1.1.1.3 A Traffic Flow Model

The efficiency of a transportation system relies on the smooth flow of traffic. A simple mathematical model for traffic flow can be established by considering n number of automobiles that move along a straight road (see [32]). Denote the position of the ith automobile at time t as $x_i(t)$ and let the $(i + 1)$th automobile move in front of the ith one, $i \in I[1, n]$. The following sequence of equations describes the movement of the automobiles in terms of their positions:

$$\ddot{x}_i(t) = k(\dot{x}_i(t - \tau_i) - \dot{x}_{i+1}(t - \tau_i)), \quad i \in I[1, n]. \tag{1.8}$$

Basically, the ith equation implies that the acceleration of the ith automobile at time t is proportional to the relative velocity between the ith and $(i+1)$th automobiles at a past time instant $t - \tau_i$, with the coefficient of proportionality $k < 0$. The constant $\tau_i \in \mathbb{R}_0^+$ represents the time it takes for the driver of the ith automobile to sense the traffic condition, determines a necessary action to accelerate or decelerate the automobile, and regulates the gas paddle or the brake based on his/her decision. These time delays are caused by the lagged behaviors of the drivers, which cannot be neglected, especially in modeling fast moving traffic.

1.1.2 Delay Differential Equations

The study of delay phenomena scatters among different scientific disciplines. A systematic study of time delay and its effects within a unified framework relies on modeling time delay systems by delay differential equations (also known as functional differential equations, differential-difference equations, equations with dead time, after-effects, or deviating arguments). A typical delay differential equation is given by

$$\dot{x}(t) = f(t, x_t, \dot{x}_t), \tag{1.9}$$

where $f : \mathbb{R} \times C^1([-\tau, 0], \mathbb{R}^n) \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a general nonlinear functional, x_t and $\dot{x}_t$ are respectively the restrictions of the state $x(s)$ and its derivative $\dot{x}(s)$ to $s \in [t - \tau, t]$, and $\tau \in \mathbb{R}_0^+$ is the amount of the delay. The equation indicates that the change of the current state $x(t)$ depends on the values of the state over the time interval $[t - \tau, t]$, rather than solely on the current value of the state $x(t)$.

Equation (1.9) belongs to the group of delay differential equations of neutral type whose right-hand side depends on the derivative of x_t. In contrast to the neutral type is the retarded type of delay differential equations whose right-hand side does not depend on the derivative of x_t. A retarded equation takes the form of

$$\dot{x}(t) = f(t, x_t), \tag{1.10}$$

where $f : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a general nonlinear functional . The focus of this book is on the study of delay differential equations of retarded type. We refer to the rich literature on delay differential equations of neutral type (see [31, 32, 43, 52, 55], and the references therein).

We note that the expression for a general delay free ordinary differential equation is given by

$$\dot{x}(t) = f(t, x(t)), \tag{1.11}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a general nonlinear function. On the other hand, f in (1.9) or (1.10) has two arguments, the time t and a function $x(s)$ restricted to $s \in [t - \tau, t]$. It is thus natural that delay differential equations are also referred to as functional differential equations.

The way x_t appears in f in (1.9) or (1.10) determines whether the delay is distributed or discrete (pointwise). As the names suggest, if f depends on x_t over a time interval $[t_1, t_2]$, where $t - \tau \le t_1 \le t_2 \le t$, then the delay is distributed. For example, the following scalar equation with distributed delay describes the circumnutation of a plant to geotropic movements [46],

$$\dot{\alpha}(t) = -k \int_1^{\infty} f(n) \sin \alpha(t - nt_0) \mathrm{d}n, \tag{1.12}$$

where $\alpha(t)$ denotes the angle of the plant with the plumb line, k is some constant, $f(n)$ is an exponentially decaying function, and t_0 denotes the geotropic reaction time of the plant. If, on the other hand, f depends only on the value of x_t at distinct time instants, then the delay is discrete or pointwise. Examples of delay differential equations with discrete delay are (1.2), (1.7), and (1.8).

1.1.3 The Initial Condition, the Cauchy Problem, and the Step Method

The initial condition of a delay differential equation is defined differently in comparison with that of an ordinary differential equation without delay. We take Eq. (1.7) for illustration of its initial condition. We first rewrite the equation as

$$\dot{x}(t) = -\alpha x(t-1)(1 + x(t)). \tag{1.13}$$

Let $x(t)$ start its evolution from $t = 0$. The value of $x(t-1)$ is required for computing $\dot{x}(t)$, $t \in [0, 1]$. As time elapses beyond $t = 1$, no information of $x(t)$ on $t \leq 0$ is required. In order to depict the complete evolution of $x(t)$ on $t \in [0, \infty)$, it is necessary for (1.13) to take

$$x_0 = \phi(\theta), \quad \theta \in [-1, 0], \tag{1.14}$$

as its initial condition, where ϕ is typically assumed to be a piecewise continuous function. Similarly, the initial condition of (1.9) or (1.10) can be defined by

$$x_0 = \psi(t_0 + \theta), \quad \theta \in [-\tau, 0], \tag{1.15}$$

where the state of the equation starts its evolution from $t_0 \in \mathbb{R}$ and ψ is a piecewise continuous function. On the other hand, the initial condition of an ordinary differential equation without delay is defined by $x_0 = x(t_0)$ if the state of the equation starts its evolution from t_0. The distributed feature of the initial condition of a delay differential equation implies that the equation is infinite-dimensional. In contrast to delay differential equations, ordinary differential equations without delay are finite-dimensional .

Closely related to the initial condition of a delay differential equation is the Cauchy problem for the equation. The Cauchy problem, also referred to as the initial value problem, is to determine whether the equation admits a unique solution given an initial condition. For both the neutral type equation (1.9) and the retarded type equation (1.10), comprehensive analysis was established in [43] that addresses the problem. Sufficient conditions on the general, possibly nonlinear, functional f that guarantees the existence and uniqueness of the solution were established in [37, 41, 43, 52], and [32]. Most of these sufficient conditions do not involve the explicit solution. When f allows a delay differential equation to be explicitly solved, we can address the Cauchy problem by directly solving the equation.

Consider Eq. (1.13) for example. Given the initial condition $\phi(\theta) = c \in \mathbb{R} \setminus \{0\}$, $\theta \in [-1, 0]$, $x(t)$ satisfies the ordinary differential equation without delay on $t \in [0, 1]$,

$$\dot{x}(t) = -\alpha c(1 + x(t)), \tag{1.16}$$

and thus,

$$x(t) = (1 + c)\mathrm{e}^{-\alpha ct} - 1, \quad t \in [0, 1]. \tag{1.17}$$

Forwarding the time interval of interest to $t \in [1, 2]$, we see that $x(t)$ satisfies the ordinary differential equation without delay,

$$\dot{x}(t) = -\alpha\Big((1 + c)\mathrm{e}^{-\alpha c(t-1)} - 1\Big)(1 + x(t)), \tag{1.18}$$

which is obtained by using the solution of $x(t)$ on $t \in [0, 1]$ in Eq. (1.13). From (1.18), we obtain

$$x(t) = (1 + c)\mathrm{e}^{\alpha(t-1-c)+\frac{1+c}{c}\left(\mathrm{e}^{-\alpha c(t-1)}-1\right)} - 1, \quad t \in [1, 2]. \tag{1.19}$$

Following the same solution procedure for each of the time interval $[k, k+1]$, $k \in \mathbb{N}$, in a successive manner, we obtain the explicit solution of $x(t)$ on $t \in [0, \infty)$, which obviously implies the existence and uniqueness of the solution. Such a method to obtain the explicit solution to a delay differential equation is referred to as the step method, which was originally proposed in [12]. The step method is applicable to both equations of neutral type and retarded type with initial conditions that are possibly time-varying.

The existence and uniqueness of the solution to a delay differential equation are only byproducts of the use of the step method. In most circumstances, the step method provides more intricate properties of the solution. For example, the solution of a retarded type equation becomes smoother as time progresses, while it is not the case for a neutral type equation. Such conclusions were reached in [12] by employing the step method. Moreover, the solution to a neutral type equation obtained by the use of the step method facilitates the stability analysis of the time delay system described by the equation (see [55]).

1.2 Stability of Time Delay Systems

1.2.1 Stability Definitions

Major interest in the study of a time delay system is in the evolution of the state of the system as time tends to infinity. It is often desirable to find an equilibrium point in the state space and that the state stays within a small neighborhood of the point after some finite time, if such a point exists. Otherwise, a diverging state implies that the system is unstable. We here present stability definitions for system (1.10). It is assumed that given the initial condition $x_{t_0} = \psi(t_0 + \theta) \equiv 0$, $\theta \in [-\tau, 0]$, $x(t) \equiv 0$, $t \geq t_0$, is a trivial solution of the system. We made this assumption without loss of generality because the stability of a nontrivial solution $\bar{x}(t)$ of the system is equivalent to that of the trivial solution $\tilde{x}(t) \equiv 0$ of the following system

$$\dot{\tilde{x}}(t) = f(t, (\tilde{x} + \bar{x})_t) - f(t, \bar{x}_t), \tag{1.20}$$

which is the delay differential equation governing the evolution of $\tilde{x}(t) = x(t) - \bar{x}(t)$ (see [41]).

Definition 1.1 The trivial solution $x(t) \equiv 0$ of system (1.10) is stable if for every $\epsilon > 0$ and every $t_0 \geq 0$, there exists $\delta(\epsilon, t_0) > 0$ such that

$$||x_{t_0}||_C \leq \delta \;\text{ implies }\; |x(t)| \leq \epsilon, \quad t \geq t_0.$$

Definition 1.2 The trivial solution $x(t) \equiv 0$ of system (1.10) is uniformly stable if δ in Definition 1.1 is independent of t_0.

Definition 1.3 The trivial solution $x(t) \equiv 0$ of system (1.10) is attractive if there exists $\delta(t_0) > 0$ such that

$$||x_{t_0}||_C \leq \delta \;\text{ implies }\; \lim_{t \to \infty} x(t) = 0.$$

Definition 1.4 The trivial solution $x(t) \equiv 0$ of system (1.10) is asymptotically stable if it is both stable and attractive.

Definition 1.5 The trivial solution $x(t) \equiv 0$ of system (1.10) is uniformly asymptotically stable if it is uniformly stable and there exists $\delta > 0$, independent of t_0, such that, for every $\epsilon > 0$, there exists $T = T(\delta, \epsilon)$ such that

$$||x_{t_0}||_C \leq \delta \;\text{ implies }\; |x(t)| \leq \epsilon, \quad t \geq t_0 + T.$$

Definition 1.6 The trivial solution $x(t) \equiv 0$ of system (1.10) is globally uniformly asymptotically stable if δ in Definition 1.5 can be any positive number.

In the case of a linear system, we often refer to stability (asymptotic stability) of its trivial solution as stability (asymptotic stability) of the system.

1.2.2 Lyapunov Stability Theorems

A straightforward approach to analyzing the stability of a time delay system is to obtain the analytic solution of the system by employing the step method. However, the step method ceases to work when the analytic solution is not available. Even when an analytic solution is available, it is in many instances challenging to express the solution as a simple function of $t \in \mathbb{R}$, as seen in the example for the illustration of the step method in Sect. 1.1.3. Therefore, stability analysis that does not involve the analytic solution is preferable. Similar to Lyapunov function based stability analysis for systems without delay, Lyapunov functional based stability analysis for time delay systems does not require an analytic solution. By picking a positive definite Lyapunov functional $V(t, x_t)$ and taking its time derivative along the system trajectory, we conclude that the system is stable if this time derivative is negative definite. The following theorem on the stability of system (1.10) lays foundation for such Lyapunov functional based stability analysis.

Theorem 1.1 (The Krasovskii Stability Theorem) *Suppose f in (1.10) maps bounded sets in $\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n)$ to bounded sets in $\mathbb{R}^n$, $u, v, w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous and nondecreasing functions, $u(s)$, $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. The trivial solution of system (1.10) is uniformly stable if there exists a continuously differentiable functional $V(t, \phi) : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_0^+$ such that*

$$u(|\phi(0)|) \le V(t, \phi) \le v(||\phi||_C), \tag{1.21}$$

and the time derivative of $V(t, \phi)$ along the trajectory of the system satisfies

$$\dot{V}(t, \phi) \le -w(|\phi(0)|). \tag{1.22}$$

If $w(s) > 0$ for $s > 0$, then the trivial solution is uniformly asymptotically stable. If in addition $\lim_{s\to\infty} u(s) = \infty$, then the trivial solution is globally uniformly asymptotically stable.

It was pointed out in [32] that the inclusion of the time derivative of x_t as an additional argument of the Lyapunov functional in Theorem 1.1 might make the stability conditions in the theorem easier to satisfy. The following theorem extends Theorem 1.1 by allowing such an additional argument in the Lyapunov functionals.

Theorem 1.2 (An Extension of the Krasovskii Stability Theorem) *Suppose f in (1.10) maps bounded sets in $\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n)$ to bounded sets in $\mathbb{R}^n$, $u, v, w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous and nondecreasing functions, $u(s)$, $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. The trivial solution of system (1.10) is uniformly stable if there exists a continuously differentiable functional $V(t, \phi, \dot{\phi}) : \mathbb{R} \times AC([-\tau, 0], \mathbb{R}^n) \times L_2([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_0^+$ such that*

$$u(|\phi(0)|) \le V\left(t, \phi, \dot{\phi}\right) \le v(||\phi||_{AC}), \tag{1.23}$$

and the time derivative of $V(t, \phi)$ along the trajectory of the system satisfies

$$\dot{V}(t, \phi, \dot{\phi}) \leq -w(|\phi(0)|). \tag{1.24}$$

If $w(s) > 0$ for $s > 0$, then the trivial solution is uniformly asymptotically stable. If in addition $\lim_{s\to\infty} u(s) = \infty$, then the trivial solution is globally uniformly asymptotically stable.

In the application of the Lyapunov–Krasovskii Theorem or its extension, the choice of a delicate Lyapunov functional $V(t, x_t)$ that satisfies all the stability conditions is the key. Typically, the time derivative of a Lyapunov functional along the trajectory of a time delay system depends on x_t in one form or another. This makes the majorization of the time derivative by a negative definite term that depends solely on the current state $x(t)$ of the system difficult (see (1.22) or (1.24)). To overcome such difficulty, the following stability theorem takes an approach differently from Theorem 1.1 or 1.2. By picking a positive definite Lyapunov function $V(t, x(t))$, we can conclude the stability of the system if $\dot{V}(t, x(t))$ along the system trajectory is negative definite under a condition on the evolution of $V(s, x(s))$ over the time interval $s \in [t - \tau, t]$. This condition facilitates the majorization of the time derivative of the Lyapunov function by a negative definite term that depends solely on the current state $x(t)$.

Theorem 1.3 (The Razumikhin Stability Theorem) *Suppose f in (1.10) maps bounded sets in $\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n)$ to bounded sets in $\mathbb{R}^n$, $u, v, w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous and nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, $u(0) = v(0) = 0$, and v is strictly increasing. The trivial solution of system (1.10) is uniformly stable if there exists a continuously differentiable function $V(t, x(t)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_0^+$ such that*

$$u(|x|) \leq V(t, x) \leq v(|x|), \tag{1.25}$$

and the time derivative of $V(t, x(t))$ along the trajectory of the system satisfies

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \ \ \textit{whenever} \ \ V(t+\theta, x(t+\theta)) \leq V(t, x(t)), \ \ \theta \in [-\tau, 0]. \tag{1.26}$$

If in addition, $w(s) > 0$ for $s > 0$, and there exists a continuous and nondecreasing function $p(s) > s$ for $s > 0$ such that condition (1.26) is strengthened to

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \ \textit{whenever} \ V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))), \ \ \theta \in [-\tau, 0], \tag{1.27}$$

then the trivial solution is uniformly asymptotically stable. If in addition $\lim_{s\to\infty} u(s) = \infty$, then the trivial solution is globally uniformly asymptotically stable.

1.3 Control Systems with Time Delays

1.3.1 Input and State Delays

In a control system, time delay can take place in the input and/or state of the system. One form of input delay is induced in the implementation of a digital controller in continuous-time control systems. A digital controller typically consists of a computer that generates an input signal according to a control algorithm, an A/D converter before the computer, and a D/A converter after the computer. Given a complex control algorithm that demands heavy computation, the computer generates the input signal $u(t - \tau)$ in a non-negligible time $\tau \in \mathbb{R}^+$. Moreover, transformation between digital and analog signals carried out by the A/D and D/A converters also adds to the delay. Another form of input delay is induced by long-distance transmission of the input signal between controllers and controlled plants. This form of input delay typically appears in control of large networks where the controller and the controlled plant are located far apart. In general, all the time consumption related to the generation, processing, and transmission of the input signal can be modeled as input delay.

State delay is also typical in control systems. One form of state delay appears in open loop systems. Examples of this form of state delay were given in Sect. 1.1.1, where the time delay systems are autonomous because no external input is applied to the systems. Another form of state delay appears in closed-loop systems when their open loop systems are subject to state feedback and input delay simultaneously. In this case, the controller utilizes the current state $x(t)$ to generate the current input signal $u(t)$. However, the actual input signal injected to the open loop systems lags behind the current input signal by the amount of the input delay. Therefore, the actual input appears as a function of past state in the closed-loop system.

The mathematical description for a control system with input and state delays is given by

$$\begin{cases} \dot{x}(t) = f(t, x_t, u_t), \\ y(t) = w(t, x(t)), \end{cases} \tag{1.28}$$

where $f : \mathbb{R} \times C([-\sigma, 0], \mathbb{R}^n) \times C([-\tau, 0], \mathbb{R}^m) \to \mathbb{R}^n$ is a general nonlinear functional, $w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^q$ is a general nonlinear function, and $\tau, \sigma \in \mathbb{R}_0^+$ are the amount of the input delay and the state delay, respectively. A typical state feedback law takes the form of

$$u(t) = z(t, x_t, u_t), \tag{1.29}$$

where $z : \mathbb{R} \times C([-\tilde{\sigma}, 0], \mathbb{R}^n) \times C([-\tilde{\tau}, 0], \mathbb{R}^m) \to \mathbb{R}^m$, for some $\tilde{\sigma}, \tilde{\tau} \in \mathbb{R}_0^+$, is an appropriate functional and might contain linear operations such as addition, multiplication by a constant and integration, as well as nonlinear operations.

On the other hand, the output of a control system typically consists of physical quantities that can be directly measured. Therefore, output feedback laws that employ only the measurement of the output are more practical than state feedback laws. A common output feedback law design for time delay systems first seeks to utilize the measurement of u_t and $y(t)$ to construct an observer whose state $\hat{x}(t)$ approaches $x(t)$ asymptotically. Then, based on the observed state $\hat{x}(t)$, a feedback law is constructed whose structure replicates that of a state feedback law. If the state feedback law (1.29) achieved a certain control objective, the output feedback law

$$\begin{cases} \dot{\hat{x}}(t) = g\big(t, \hat{x}_t, u_t, y(t)\big), \\ u(t) = z(t, \hat{x}_t, u_t), \end{cases} \tag{1.30}$$

potentially achieves the same control objective, where $g : \mathbb{R} \times C([-\tilde{\sigma}, 0], \mathbb{R}^n) \times C([-\tilde{\tau}, 0], \mathbb{R}^m) \times \mathbb{R}^q \to \mathbb{R}^n$, for some $\tilde{\sigma} \in \mathbb{R}_0^+$, is a proper functional that guarantees

$$\lim_{t\to\infty} \big(x(t) - \hat{x}(t)\big) = 0.$$

While most control systems behave in a nonlinear manner when the state of the system is far away from the equilibrium point, near the equilibrium point, their dynamics can be efficiently approximated by a linear system. Such an approximation can be obtained by Jacobian linearization of the control systems at the equilibrium point. The mathematical description for a linear system with input and state delays is as follows:

$$\begin{cases} \dot{x}(t) = \int_{-\sigma}^{0} A(t, s)x(t+s)\mathrm{d}s + \int_{-\tau}^{0} B(t, s)u(t+s)\mathrm{d}s, \\ y(t) = C(t)x(t), \end{cases} \tag{1.31}$$

where $\tau, \sigma \in \mathbb{R}_0^+$ represent the input delay and the state delay, respectively, $A(t, s) : \mathbb{R} \times [-\sigma, 0] \to \mathbb{R}^{n\times n}$, $B(t, s) : \mathbb{R} \times [-\tau, 0] \to \mathbb{R}^{n\times m}$ are the dynamics matrix and control matrix of the system, respectively, and are of bounded variation with respect to s, and $C(t) : \mathbb{R} \to \mathbb{R}^{q\times n}$ is the sensor matrix of the system. System (1.31) is time-varying and the delay is of distributed type. The time-invariant version of system (1.31) is given by

$$\begin{cases} \dot{x}(t) = \int_{-\sigma}^{0} A(s)x(t+s)\mathrm{d}s + \int_{-\tau}^{0} B(s)u(t+s)\mathrm{d}s, \\ y(t) = Cx(t), \end{cases} \tag{1.32}$$

where $A(s) : [-\sigma, 0] \to \mathbb{R}^{n\times n}$, $B(s) : [-\tau, 0] \to \mathbb{R}^{n\times m}$ are the dynamics matrix and the control matrix of the system, respectively, and are of bounded variation, and $C \in \mathbb{R}^{q\times n}$ is the sensor matrix of the system. Furthermore, the discrete delay version of system (1.32) is given by

$$\begin{cases} \dot{x}(t) = \sum_{i=0}^{l} A_i x(t - \sigma_i) + \sum_{j=0}^{k} B_j u(t - \tau_j), \\ y(t) = Cx(t), \end{cases} \tag{1.33}$$

where $A_\text{I} \in \mathbb{R}^{n\times n}$, $i \in I[0, l]$, and $B_j \in \mathbb{R}^{n\times m}$, $j \in I[0, k]$, are the dynamics matrices and control matrices of the system, respectively, and $\tau_j \in \mathbb{R}_0^+$, $j \in I[0, k]$, and $\sigma_\text{I} \in \mathbb{R}_0^+$, $i \in I[0, l]$, are the lengths of the input and state delays, respectively. The attention of this book is restricted to linear time-invariant systems with a single input delay,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t - \tau), \\ y(t) = Cx(t), \end{cases} \tag{1.34}$$

and its stabilization via either state or output feedback. Occasionally in this book, the constant delay τ in system (1.34) is allowed to be time-varying but bounded, namely, $\tau(t) : \mathbb{R} \to [0, D]$, where $D \in \mathbb{R}_0^+$ is an upper bound on the delay.

1.3.2 An Overview of Stabilization of Time Delay Systems

The study of control systems with time delay spans a wide range of problems, including stabilization, trajectory tracking, output regulation, disturbance rejection, performance improvement, robust control, and adaptation to accommodate unknown system parameters.

Central to these problems is the problem of stabilization. The stabilization of a time delay system can be accomplished through two different paths. The first path extensively involves Lyapunov functional based stability analysis. We pick a positive definite Lyapunov functional and compute its time derivative along the trajectory of the system. By designing feedback that renders the time derivative negative definite, we achieve closed-loop stability (see [47]). It turns out that such analysis is effective for systems with either input or state delay (see [31, 32, 41, 67, 82–86, 120] and references therein). Furthermore, the Lyapunov functional based stability analysis has shown its universal applicability to the study of various time delay systems, including time-varying, nonlinear, and stochastic systems with delays allowed to be time-varying or even unknown (see [11, 13–15, 43, 47, 58, 59, 91] and [97], for a small sample of the literature).

The second path to the stabilization of a time delay system is more straightforward. Given a linear system with a single constant input delay, a feedback law completely cancels the effect of the delay by multiplying a feedback gain matrix with the state of the system at the future time ahead of the current time by the amount of the delay. Thanks to the linearility of the system, the future state can be explicitly predicted as the sum of the zero input solution and the zero state solution of the system. Such a feedback law is referred to as the predictor feedback law [69] due to the prediction of the future state. The spectrum of the closed-loop system under the predictor feedback is finite and can be arbitrarily assigned in the complex plane by an appropriate choice of the feedback gain matrix. Therefore, such a predictor feedback design is also referred to as finite spectrum assignment [68]. An alternative design that also leads to the predictor feedback law involves a key step of transforming the original open loop system with delay into an open loop system free of delay, and is referred to as the model reduction technique [7].

The predictor feedback laws for linear systems with input delay were generalized to stabilize linear systems with input and state delays in [53, 54] and [56], where the state of the systems at a future time is predicted by the use of the variation-of-constants formula (see [12] for the formula). Such feedback laws designed by the use of the variation-of-constants formula were further generalized to stabilize neutral type systems with input delay in [55]. Besides these generalizations, the combinated use of the finite spectrum assignment or the model reduction technique with control techniques for nonlinear systems such as cross-term forwarding, backstepping, and/or recursive methods induced various prediction methods for the stabilization of linear and nonlinear systems with input and state delays (see [9, 48–50, 59] and [11]). Predictor based feedback laws were also developed for linear systems with input and state delays [117, 126].

In all these predictor based feedback designs, a distributed delay term appears in the resulting predictor based feedback laws. Such a distributed nature of the feedback laws originates from predicting the state of the system at a future time by the use of the variation-of-constants formula . It was pointed out that these distributed delay terms would cause difficulty in their implementation (see [95] and [78]). Recently, a sequential predictors approach to the stabilization of linear systems with input delay was proposed that manage to observe the state of the system at the future time ahead of the current time by the amount of the delay without using the variation-of-constants formula (see [13, 73, 74], and [16]). The sequential predictors are a sequence of dynamic predictors that observe the future state of the system in a progressive manner. Specifically, the state of the first predictor observes that of the system at the future time ahead of the current time by a small amount. Repeating such an observation manner, the state of the next predictor observes that of the previous predictor at the future time ahead of the current time by the same small amount. Given sufficiently large number of sequential predictors, the state of the final predictor would observe the state of the system at the future time ahead of the current time by the amount of the delay. The method of sequential prediction has an advantage over other prediction methods because the sequential predictors only contain discrete delay terms that can be readily implemented.

In the remainder of the book, we will focus our attention on predictor based feedback designs for linear systems with input delay. The development of our feedback laws is inspired by predictor feedback design methods, while the stability analysis of the resulting closed-loop system is carried out by employing Lyapunov functional based methods. The scope of the book is not limited to basic stabilization. It expands to more challenging problems such as performance improvement and adaptation to accommodate unknown system parameters. The proposed feedback laws for stabilization will be modified to address each of these challenging problems effectively. In the following section, we review some predictor feedback laws and their roles in the stabilization of linear systems with input and state delays.

1.4 Predictor Feedback

1.4.1 Linear Systems with a Single Input Delay

For a linear time-invariant system with a single input delay,

$$\dot{x}(t) = Ax(t) + Bu(t - \tau), \tag{1.35}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\tau \in \mathbb{R}_0^+$, and (A, B) is controllable, the feedback law

$$u(t) = Fx(t + \tau), \tag{1.36}$$

where F is a feedback gain matrix such that $A + BF$ is Hurwitz, completely cancels the effect of the input delay. Under such a feedback law, the closed-loop system

$$\dot{x}(t) = (A + BF)x(t) \tag{1.37}$$

is free of delay and is asymptotically stable. Also, the spectrum of the closed-loop system is finite and can be arbitrarily assigned on the complex plane by an appropriate choice of F. We note that feedback law (1.36) cannot be directly implemented because it requires the future state of the system $x(t + \tau)$ to be known. However, since system (1.35) is linear, the explicit solution of the future state can be obtained as the sum of the zero input solution and the zero state solution of the system,

$$\begin{aligned} x(t + \tau) &= \mathrm{e}^{A\tau} x(t) + \int_t^{t+\tau} \mathrm{e}^{A(t+\tau-s)} Bu(s - \tau)\mathrm{d}s \\ &= \mathrm{e}^{A\tau} x(t) + \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} Bu(s)\mathrm{d}s. \end{aligned} \tag{1.38}$$

Then, the feedback law (1.36) is expressed as

$$u(t) = F\mathrm{e}^{A\tau} x(t) + F \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} Bu(s)\mathrm{d}s, \tag{1.39}$$

which is referred to as the predictor feedback law due to the prediction of the future state [69].

The predictor feedback law consists of two terms. The first term is a static feedback term that is readily implementable. The second term is a distributed delay term that requires the input to be integrated over the past time interval $[t-\tau, t]$. A straightforward method to implement the distributed delay term is to approximate the term by a finite sum. By using the backward rectangular rule to approximate the distributed delay term, we arrive at the following feedback law:

$$u(t) = F\mathrm{e}^{A\tau}x(t) + F\frac{1}{N}\sum_{i=0}^{N-1}\mathrm{e}^{A\tau\left(1-\frac{i}{N}\right)}Bu\left(t-\tau\left(1-\frac{i}{N}\right)\right), \tag{1.40}$$

where $N \in \mathbb{N}\setminus\{0\}$ is the number of the integration steps used by the backward rectangular approximation. The feedback law (1.40) can be readily implemented and, intuitively, would achieve stabilization if N is sufficiently large. However, in [95], it was shown through an example, where the system is given by

$$\dot{x}(t) = x(t) + u(t-1), \tag{1.41}$$

and the feedback gain is given by $F = -2$, that the feedback law (1.40) for system (1.41) fails to stabilize no matter how large N is. The use of other numerical integration methods such as the composite trapezoidal rule and the Simpson rule to approximate the distributed delay term by a finite sum may encounter similar difficulty in achieving stabilization.

The cause of the instability of the closed-loop system consisting of system (1.35) and the feedback law (1.40) was examined in [26] and [78]. The characteristic equation of the closed-loop system is given by

$$\det\left(\begin{bmatrix} sI-A & -B\mathrm{e}^{-s\tau} \\ -F\mathrm{e}^{A\tau} & I-\frac{F}{N}\sum_{i=0}^{N-1}\mathrm{e}^{-\tau\left(1-\frac{i}{N}\right)(sI-A)}B \end{bmatrix}\right) = 0, \tag{1.42}$$

which is the characteristic equation of a delay differential equation of neutral type. According to [26] and [78], it is the neutral feature of this characteristic equation that leads to instability.

In [78], a safe implementation method was proposed to resolve the implementation problem induced by the finite sum approximations of the distributed delay term in the feedback law (1.39). By defining an augmented state vector

$$z(t) = \begin{bmatrix} x(t) \\ u(t-\tau) \end{bmatrix}, \tag{1.43}$$

we obtain a new open loop system with $z(t)$ as its state,

$$\dot{z}(t) = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} \dot{u}(t-\tau) = \tilde{A}z(t) + \tilde{B}\dot{u}(t-\tau). \tag{1.44}$$

Inspired by the design of the predictor feedback law (1.39), we let

$$\dot{u}(t) = \tilde{F}z(t+\tau), \tag{1.45}$$

where $\tilde{F}$ is a feedback gain matrix such that $\tilde{A} + \tilde{B}\tilde{F}$ is Hurwitz. The existence of such an $\tilde{F}$ is guaranteed because the controllability of the pair $(\tilde{A}, \tilde{B})$ is implied by that of the pair (A, B). Clearly, the closed-loop system

$$\dot{z}(t) = \left(\tilde{A} + \tilde{B}\tilde{F}\right) z(t) \tag{1.46}$$

is asymptotically stable. The spectrum of the closed-loop system with $z(t)$ as its state is finite and can be arbitrarily assigned on the complex plane by an appropriate choice of $\tilde{F}$. Because $x(t)$ is part of $z(t)$, system (1.35) is stabilized. We rewrite (1.45) as

$$\begin{aligned} \dot{u}(t) &= \tilde{F}z(t+\tau) \\ &= \tilde{F}\begin{bmatrix} x(t+\tau) \\ u(t) \end{bmatrix}. \end{aligned} \tag{1.47}$$

Substituting the explicit solution of $x(t+\tau)$ given by (1.38) in (1.47), we arrive at the following dynamic predictor based feedback law:

$$\dot{u}(t) = \tilde{F}\begin{bmatrix} \mathrm{e}^{A\tau}x(t) + \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s \\ u(t) \end{bmatrix}. \tag{1.48}$$

Let $\tilde{F} = \left[\tilde{F}_x \ \ \tilde{F}_u\right]$. The feedback law (1.48) can be written as

$$\dot{u}(t) = \tilde{F}_x \mathrm{e}^{A\tau}x(t) + \tilde{F}_x \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s + \tilde{F}_u u(t). \tag{1.49}$$

By using the backward rectangular rule to approximate the distributed delay term in (1.49), we arrive at

$$\dot{u}(t) = \tilde{F}_x \mathrm{e}^{A\tau}x(t) + \frac{\tilde{F}_x}{N}\sum_{i=0}^{N-1} \mathrm{e}^{A\tau\left(1-\frac{i}{N}\right)}Bu\left(t - \tau\left(1 - \frac{i}{N}\right)\right) + \tilde{F}_u u(t). \tag{1.50}$$

It was shown in [78] that (1.50) asymptotically stabilizes system (1.35) as long as the number of the integration steps N in (1.50) is sufficiently large. The use of other numerical integration methods to approximate the distributed delay term in (1.49) by a finite sum was also shown to be effective in the stabilization of system (1.35).

The characteristic equation of the closed-loop system consisting of (1.35) and (1.50) is given by

$$\begin{aligned}&\det\left(\begin{bmatrix} sI-A & -Be^{-s\tau} \\ -\tilde{F}_x e^{A\tau} & sI-\tilde{F}_u-\frac{\tilde{F}_x}{N}\sum_{i=0}^{N-1} e^{-\tau\left(1-\frac{i}{N}\right)(sI-A)}B \end{bmatrix}\right) \\ &= \det\left(sI-\begin{bmatrix} A & Be^{-\tau s} \\ \tilde{F}_x e^{A\tau} & \tilde{F}_u+\frac{\tilde{F}_x}{N}\sum_{i=0}^{N-1} e^{-\tau\left(1-\frac{i}{N}\right)(sI-A)}B \end{bmatrix}\right) \\ &= 0, \end{aligned} \tag{1.51}$$

which is the characteristic equation of a delay differential equation of retarded type. According to [78], it is the retarded feature of the characteristic equation of the closed-loop system that guarantees the safe implementation of the feedback law (1.50).

1.4.2 Linear Systems with Multiple Input Delays

The predictor feedback law (1.39) cannot be directly generalized to stabilize linear systems with multiple input delays. However, through the use of model reduction technique [7], a predictor based feedback law can be constructed for the following linear systems with multiple input delays:

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{k} B_i u(t-\tau_i), \tag{1.52}$$

where $\tau_i \in \mathbb{R}_0^+$, $i \in I[0,k]$, are the lengths of the input delays. We first construct an auxiliary signal

$$y(t) = x(t) + \sum_{i=0}^{k} \int_{t-\tau_i}^{t} e^{A(t-\tau_i-s)} B_i u(s) \mathrm{d}s. \tag{1.53}$$

In view of (1.52), we have

$$\dot{y}(t) = Ay(t) + \left(\sum_{i=0}^{k} e^{-A\tau_i} B_i\right) u(t), \tag{1.54}$$

which is an open loop system free of delay. Assume that the pair $\left(A, \sum_{i=0}^{k} e^{-A\tau_i} B_i\right)$ is controllable. Under the following feedback law:

$$u(t) = F_m y(t), \tag{1.55}$$

where F_m is such that $A + \left(\sum_{i=0}^{k} \mathrm{e}^{-A\tau_i} B_i\right) F_m$ is Hurwitz, the closed-loop system

$$\dot{y}(t) = \left(A + \sum_{i=0}^{k} \mathrm{e}^{-A\tau_i} B_i F_m\right) y(t) \tag{1.56}$$

is asymptotically stable. Moreover, the spectrum of the closed-loop system is finite and can be freely assigned on the complex plane by an appropriate choice of F_m. Since

$$x(t) = y(t) - \sum_{i=0}^{k} \int_{t-\tau_i}^{t} \mathrm{e}^{A(t-\tau_i-s)} B_{\mathrm{I}} F_m y(s) \mathrm{d}s, \tag{1.57}$$

system (1.52) is stabilized by the feedback law (1.55). In view of (1.53), we rewrite (1.55) as

$$u(t) = F_m x(t) + F_m \sum_{i=0}^{k} \int_{t-\tau_i}^{t} \mathrm{e}^{A(t-\tau_i-s)} B_{\mathrm{I}} u(s) \mathrm{d}s. \tag{1.58}$$

The core step in designing (1.58) is to define the auxiliary signal $y(t)$, which helps to transform the open loop system with multiple input delays (1.52) to the open loop system free of delay (1.54). Thus, such a method of designing (1.58) is referred to as the model reduction technique [7], which also applies to linear systems with a single input delay.

In particular, when system (1.52) simplifies to a linear system with a single input delay (1.35), the feedback law (1.58) simplifies to

$$u(t) = F_m x(t) + F_m \int_{t-\tau}^{t} \mathrm{e}^{A(t-\tau-s)} B u(s) \mathrm{d}s, \tag{1.59}$$

where $A + \mathrm{e}^{-A\tau} B F_m$ is Hurwitz. Note that the controllability of the pair $(A, \mathrm{e}^{-A\tau} B)$ is equivalent to that of the pair (A, B). Let F be such that $A + BF$ is Hurwitz and $F_m = F\mathrm{e}^{A\tau}$. Then,

$$A + \mathrm{e}^{-A\tau} B F_m = \mathrm{e}^{-A\tau}(A + BF)\mathrm{e}^{A\tau} \tag{1.60}$$

is also Hurwitz. Therefore, the feedback law (1.59) is the same as the predictor feedback law (1.39), recalled below,

$$u(t) = F\mathrm{e}^{A\tau} x(t) + F \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} B u(s) \mathrm{d}s. \tag{1.61}$$

From this perspective, the feedback law (1.58) is of predictor type and can be considered an extension of the predictor feedback law (1.39) for linear systems with a single input delay.

1.4.3 Linear Systems with Input and State Delays

We consider the following linear systems with input and state delays:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\sigma) + Bu(t-\tau), \tag{1.62}$$

where $\tau, \sigma \in \mathbb{R}_0^+$ are the amount of the input delay and the state delay, respectively. Assume that there exist two feedback gain matrices F_0 and F_1 such that the time delay system

$$\dot{x}(t) = (A_0 + BF_0)x(t) + (A_1 + BF_1)x(t-\sigma) \tag{1.63}$$

is asymptotically stable. Basically, such matrices F_0 and F_1 can be obtained by solving stability conditions of system (1.63), which are typically in the form of linear matrix inequalities (LMIs) and thus can be numerically solved in an efficient way.

Define the fundamental matrix $K(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ of system (1.62) by

$$\dot{K}(t) = A_0 K(t) + A_1 K(t-\sigma), \quad t \geq 0, \tag{1.64}$$

with the initial condition

$$\begin{cases} K(\theta) = I, & \theta = 0, \\ K(\theta) = 0, & \theta < 0. \end{cases} \tag{1.65}$$

Then, the variation-of-constants formula for the prediction of the state of the system at a future time $t+\theta$ for any given $\theta \in \mathbb{R}_0^+$ is

$$\begin{aligned} x(t+\theta) = K(\theta)x(t) &+ \int_{-\sigma}^{0} K(\theta-\sigma-s)A_1 x(t+s)\mathrm{d}s \\ &+ \int_{-\theta}^{0} K(-s)Bu(t+s+\theta-\tau)\mathrm{d}s, \end{aligned} \tag{1.66}$$

from which we readily obtain

$$x(t+\tau) = K(\tau)x(t) + \int_{-\sigma}^{0} K(\tau-\sigma-s)A_1 x(t+s)\mathrm{d}s + \int_{-\tau}^{0} K(-s)Bu(t+s)\mathrm{d}s, \tag{1.67}$$

and if $\tau \geq \sigma$,

$$x(t+\tau-\sigma) = K(\tau-\sigma)x(t) + \int_{-\sigma}^{0} K(\tau-2\sigma-s)A_1 x(t+s)\mathrm{d}s + \int_{-\tau}^{-h} K(-s-\sigma)Bu(t+s)\mathrm{d}s. \tag{1.68}$$

We design the following feedback law for system (1.62),

$$u(t) = F_0 x(t+\tau) + F_1 x(t+\tau-\sigma), \tag{1.69}$$

under which the closed-loop system is given by (1.63) and is thus asymptotically stable. By using (1.67) and (1.68), we rewrite the feedback law as

$$\begin{aligned} u(t) = {} & F_0 K(\tau)x(t) + F_0 \int_{-\sigma}^{0} K(\tau-\sigma-s)A_1 x(t+s)\mathrm{d}s + F_0 \int_{-\tau}^{0} K(-s)Bu(t+s)\mathrm{d}s \\ & + F_1 K(\tau-\sigma)x(t) + F_1 \int_{-\sigma}^{0} K(\tau-2\sigma-s)A_1 x(t+s)\mathrm{d}s \\ & + F_1 \int_{-\tau}^{-\sigma} K(-s-\sigma)Bu(t+s)\mathrm{d}s, \quad \text{if } \tau \geq \sigma, \end{aligned} \tag{1.70}$$

or

$$\begin{aligned} u(t) = {} & F_0 K(\tau)x(t) + F_0 \int_{-\sigma}^{0} K(\tau-\sigma-s)A_1 x(t+s)\mathrm{d}s + F_0 \int_{-\tau}^{0} K(-s)Bu(t+s)\mathrm{d}s \\ & + F_1 x(t+\tau-\sigma), \quad \text{if } \tau < \sigma. \end{aligned} \tag{1.71}$$

If there is no state delay in system (1.62), that is $\sigma = 0$, then

$$K(t) = \mathrm{e}^{(A_0+A_1)t}$$

and F_0 and F_1 can be chosen such that $A_0 + A_1 + B(F_0 + F_1)$ is Hurwitz. Thus, the feedback law (1.70) becomes

$$\begin{aligned} u(t) = {} & (F_0+F_1)\mathrm{e}^{(A_0+A_1)\tau}x(t) \\ & + (F_0+F_1)\int_{t-\tau}^{t} \mathrm{e}^{(A_0+A_1)(t-s)}Bu(s)\mathrm{d}s, \end{aligned} \tag{1.72}$$

which is the same as the predictor feedback law (1.39) if we let $A = A_0 + A_1$ and $F = F_0 + F_1$ in (1.39). From this perspective, the design of the predictor based feedback law (1.70) for linear systems with input and state delays can be considered an extension of the predictor feedback design for linear systems with a single input delay.

1.5 Discrete-Time Systems with Delay

As counterparts of continuous-time systems with delay, discrete-time systems with delay have been well studied in the past few decades (see [1–3, 25, 34, 35, 37], and [119], for a small sample of the literature). The initial condition of a continuous-time system with delay, which is given in Sect. 1.1.3, implies that the system is infinite-dimensional. In contrast, discrete-time systems with delay are finite-dimensional. This is due to the simple structure of the initial condition of a discrete-time system with delay, which is defined at a finite number of time instants. Therefore, control problems for discrete-time systems with delay are considerably less challenging than those for their continuous-time counterparts. For instance, the exact stability criterion for continuous-time linear time-invariant systems with constant state delays has yet to be established, while such a criterion for their discrete-time counterparts was identified in [37] by transforming the systems with delays to systems without delay.

In this section, we introduce delay difference equations that parallel delay differential equations in the continuous-time setting. The initial condition of delay difference equations is defined to illustrate the finite-dimension nature of discrete-time systems with delay. Stability definitions of delay difference equations that parallel those of delay differential equations are formulated. As a powerful tool for stability analysis of a delay difference equation, the Razumikhin Stability Theorem in the discrete-time setting is introduced. The control design for discrete-time systems with delay is focused on stabilization due to its fundamental importance. The predictor feedback for the stabilization of continuous-time systems with delay has its parallel in the discrete-time setting. The intuition and expression of the predictor feedback for discrete-time linear systems with delay are presented.

1.5.1 Delay Difference Equations

Discrete-time systems evolve at discrete time instants. Typically, those isolated time instants are equally spaced, and the separation distance is referred to as the sampling period. As a result of the sampling feature, the time variable of discrete-time systems is defined to take values from the set of integers, and the elements of the time variable are referred to as time steps, or simply, steps. We refer to the textbook [8] for engineering examples of discrete-time systems.

A discrete-time system without delay can be described by a difference equation, where the value of the state at the next time step depends solely on the value of the current state, i.e.,

$$x(k+1) = f(k, x(k)), \tag{1.73}$$

where $k \in \mathbb{Z}$ represents the time step, $x \in \mathbb{R}^n$ denotes the state vector of the system, and $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ describes the system dynamics. When the delay appears in the system such that the value of the state at the next time instant depends on the state at past time instants, the system can be represented by the following delay difference equation:

$$x(k+1) = f(k, x_k), \tag{1.74}$$

where x_k is the restriction of $x(s)$ to $s \in [k-r, k]$, for some $r \in \mathbb{N}$, r represents the amount of the delay, and $f : \mathbb{Z} \times D([k-r, k], \mathbb{R}^n) \to \mathbb{R}^n$ is a functional. The well definedness of the delay difference equation (1.74) relies on its initial condition. If Eq. (1.74) starts its evolution from the time step k_0 and the initial condition is taken to be

$$x(s) = \phi(s), \quad s \in I[k_0 - r, k_0], \tag{1.75}$$

then $x(k)$, $k \geq k_0$, can be uniquely determined by the functional f. It can be readily observed from (1.75) that the initial condition of a delay difference equation is defined at a finite number of time instants, whereas the initial condition of a delay differential equation is defined over a continuous time interval. In contrast to the infinite-dimension nature of delay differential equations, delay difference equations are finite-dimensional, which significantly reduces the complexity in their mathematical treatment, including but not restricted to stability analysis and control design.

1.5.2 Stability of Delay Difference Equations

Central to the analysis of discrete-time systems with delay is the determination of their stability. We follow the presentation of Sect. 1.2 to present definitions of the stability of system (1.74), the Lyapunov method of analyzing its stability, and a Razumikhin-type stability theorem as the discrete-time counterpart of Theorem 1.3. Without loss of generality, we assume that given the zero initial condition $x(s) = 0$, $s \in I[k_0 - r, k_0]$, $x(k) \equiv 0$, $k \in I[k_0, \infty)$, is a trivial solution of system (1.74). If, on the other hand, a nontrivial solution $\bar{x}(k)$, $k \in I[k_0, \infty)$, exists, then the stability of such a nontrivial solution is equivalent to the stability of the trivial solution of the following system:

$$\tilde{x}(k+1) = f(k, \tilde{x}_k + \bar{x}_k) - f(k, \bar{x}_k), \tag{1.76}$$

where $\tilde{x}(k) = x(k) - \bar{x}(k)$.

Definition 1.7 The trivial solution $x(k) \equiv 0$ of system (1.74) is stable if for every $\epsilon > 0$ and every $k_0 \geq 0$, there exists $\delta(\epsilon, k_0) > 0$ such that

$$||x_{k_0}||_D \le \delta \text{ implies } |x(k)| \le \epsilon, \quad k \in I[k_0, \infty).$$

Definition 1.8 The trivial solution $x(k) \equiv 0$ of system (1.74) is uniformly stable if δ in Definition 1.7 is independent of k_0.

Definition 1.9 The trivial solution $x(k) \equiv 0$ of system (1.74) is attractive if there exists $\delta(k_0) > 0$ such that

$$||x_{k_0}||_D \le \delta \text{ implies } \lim_{k\to\infty} x(k) = 0.$$

Definition 1.10 The trivial solution $x(k) \equiv 0$ of system (1.74) is asymptotically stable if it is both stable and attractive.

Definition 1.11 The trivial solution $x(k) \equiv 0$ of system (1.74) is uniformly asymptotically stable if it is uniformly stable and there exists $\delta > 0$, independent of k_0, such that, for every $\epsilon > 0$, there exists $T = T(\delta, \epsilon) \in \mathbb{N}$ such that

$$||x_{k_0}||_D \le \delta \text{ implies } |x(k)| \le \epsilon, \quad k \in I[k_0 + T, \infty).$$

Definition 1.12 The trivial solution $x(k) \equiv 0$ of system (1.74) is globally uniformly asymptotically stable if δ in Definition 1.11 can be any positive number.

The Lyapunov based method for stability analysis of continuous-time systems with delay can be naturally adapted for stability analysis of their discrete-time counterparts. We define a positive definite Lyapunov functional $V(k, x_k)$ for system (1.74). By computing the forward difference of the Lyapunov functional $\Delta V(k, x_k) = V(k+1, x_{k+1}) - V(k, x_k)$ along the trajectory of the system, we conclude that the system is asymptotically stable (stable) if this forward difference is a negative definite (semi-definite) function of $x(k)$. Such an approach is referred to as the Krasovskii's approach. However, the Krasovskii's approach requires the majorization of the forward difference by a negative definite term depending solely on $x(k)$, but not on the restriction of $x(s)$ to $s \in [k-r, k]$. This elevates the difficulty level in majorizing terms of the forward difference.

On the other hand, the Razumikhin's approach overcomes such difficulty by taking a slightly different route. We first define a Lyapunov function $V(k, x(k))$ and then take its forward difference along the trajectory of the system. If the forward difference is negative definite (semi-definite) under a condition on the evolution of the Lyapunov function on the time interval $s \in [k-r, k]$, then the system is concluded to be asymptotically stable (stable). In what follows, we present the Razumikhin Stability Theorem in the discrete-time setting as a powerful tool for the stability analysis of discrete-time systems with delay.

Theorem 1.4 (The Razumikhin Stability Theorem for Discrete-Time Systems) *Suppose f in (1.74) maps bounded sets in $\mathbb{Z} \times D([-r, 0], \mathbb{R}^n)$ to bounded sets in $\mathbb{R}^n$, $u, v, w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous and nondecreasing functions, $u(z)$ and $v(z)$ are positive for $z > 0$, $u(0) = v(0) = 0$, and v is strictly increasing. The trivial* *solution of system (1.74) is uniformly stable if there exists a continuous function* $V(k, x(k)) : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}_0^+$ *in x such that*

$$u(|x|) \leq V(k, x) \leq v(|x|), \tag{1.77}$$

and the forward difference of $V(k, x(k))$ *along the trajectory of the system satisfies*

$$\Delta V(k, x(k)) \leq -w(|x(k)|), \tag{1.78}$$

whenever

$$V(k+s, x(k+s)) \leq V(k, x(k)), \quad s \in I[-r, 0]. \tag{1.79}$$

If, in addition, $w(z) > 0$ *for* $z > 0$*, and there exists a continuous and nondecreasing function* $p(z) > z$ *for* $z > 0$ *such that condition (1.78) is strengthened to*

$$\Delta V(k, x(k)) \leq -w(|x(k)|), \tag{1.80}$$

whenever

$$V(k+s, x(k+s)) \leq p(V(k, x(k))), \quad s \in [-r, 0], \tag{1.81}$$

then the trivial solution is uniformly asymptotically stable. If, in addition, $\lim_{z\to\infty} u(z) = \infty$*, then the trivial solution is globally uniformly asymptotically stable.*

1.5.3 Predictor Feedback

We are concerned with the stabilization problem of the following discrete-time linear time-invariant system with input delay:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k-r), \\ \quad y(k) = Cx(k), \end{cases} \tag{1.82}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ are the state, the input, and the output of the system, respectively, $r \in \mathbb{N}$ is the amount of the input delay, and A, B, and C are constant matrices of appropriate dimensions. It is assumed that (A, B) is stabilizable and (A, C) is detectable. In some cases, the constant delay r is allowed to be time-varying but bounded, i.e., $r(k) \in I[0, R]$, $k \in \mathbb{Z}$, where $R \in \mathbb{N}$ is an upper bound of the delay.

Feedback designs for system (1.82) can be accomplished by following two different paths. The first path is based on Lyapunov stability analysis. We take a positive definite Lyapunov function/functional and compute its forward difference along the trajectory of the system. Those feedback designs that make the forward difference negative definite are the desired ones, which achieve closed-loop stability. Lyapunov based analysis is applicable not only to such linear time-invariant systems as (1.82), but also to a wide range of systems including nonlinear systems, time-varying systems, and even stochastic systems with time-varying or unknown delays (see [1, 2, 33, 34], and [35], for a small sample of literature). The resulting stability conditions are often expressed in the form of linear matrix inequalities (LMIs), which can be efficiently solved through numerical computations carried out on MATLAB.

The second path is the predictor feedback design as discussed for continuous-time systems with delay. Take system (1.82) with a single constant input delay for example. The predictor feedback law is the product of a feedback gain matrix and the predicted state of the system at the future time step ahead of the current time step by the amount of the delay. Thanks to the linearility of the system, the explicit expression of the predicted future state can be written as the sum of the zero input solution and the zero state solution of the system. Such a feedback law cancels the effect of the input delay, no matter how large the delay is. The spectrum of the closed-loop system under the predictor feedback law can be arbitrarily assigned on the complex plane by choosing an appropriate feedback gain matrix. The finite spectrum assignment and the model reduction technique for the stabilization of continuous-time systems with input and state delays can be also adapted so as to deal with their discrete-time counterparts.

We here illustrate the predictor feedback law and its stabilizing capacity on system (1.82). The feedback law

$$u(k) = Fx(k + r), \tag{1.83}$$

where F is a feedback gain matrix and assigns the eigenvalues of $A + BF$ inside the unit circle in the complex plane, cancels the effect of the input delay. Under such a feedback law, the closed-loop system is given by

$$x(k + 1) = (A + BF)x(k), \tag{1.84}$$

which is asymptotically stable. However, the feedback law (1.83) is not directly implementable because it requires the future state $x(t + r)$ to be known. As a result of the linearility of system (1.82), we can write out the explicit expression of the future state of system (1.82) as follows:

$$x(k + r) = A^r x(k) + \sum_{i=k-r}^{k-1} A^{k-1-i} Bu(i), \tag{1.85}$$

where the first term on the right-hand side is the zero input solution of the system and the second term is its zero state solution. Therefore, the feedback law (1.83) becomes

$$u(k) = FA^r x(k) + F \sum_{i=k-r}^{k-1} A^{k-1-i} Bu(i), \tag{1.86}$$

which can be readily implemented.

In contrast to the predictor feedback law (1.39) for continuous-time systems, whose implementation by employing finite sum approximation may fail to achieve closed-loop system stability, the predictor feedback law (1.86) for discrete-time systems is free from such implementation problems. Such a relief in implementing the predictor feedback law for discrete-time systems with delay is a direct consequence of their finite-dimension nature. This is another example from which we can see that control problems for discrete-time systems with delay are generally less challenging than those for their continuous-time counterparts.

The use of predictor based feedback to stabilize continuous-time linear systems with multiple input delays and with both input and state delays, as seen in Sects. 1.4.2 and 1.4.3, can be adapted to the stabilization of their discrete-time counterparts. Like the predictor feedback law for system (1.82) with a single input delay, these predictor based feedback laws for discrete-time systems with input and state delays also contain finite summation terms that involve only past input.

1.6 Notes and References

This chapter covered basic elements of the analysis and control design for time delay systems. Examples of time delay systems were presented to show that the study of them is relevant to the development of many scientific and engineering fields. To unify all the models in these examples into a single framework, we presented the concept of the delay differential equation, along with the definition of its initial condition. We refer to Krasovskii's [57], Bellman and Cooke's [12], Halanay's [42], and Hale's [43] works for the early development of the mathematical theory on the delay differential equation. A primary interest in the study of time delay systems is to investigate their asymptotic behaviors, which relies on rigorous definitions of stability. Following the celebrated monograph [41], Sect. 1.2 formulated basic elements of the stability of time delay systems, including stability definitions and Lyapunov stability theorems for stability analysis. For brevity in presentation, we omitted the proofs of the Lyapunov stability theorems. Readers are referred to [43], [41], or [32] for these proofs.

Apart from Lyapunov functional based methods, the stability analysis of time delay systems has been extensively carried out by following frequency domain approaches. Unlike Lyapunov functional based stability analysis which typically leads to sufficient stability conditions with considerable conservativeness, the frequency domain analysis tends to result in necessary and sufficient stability conditions (see [17, 19, 24, 36, 38, 39, 41, 65, 66, 75, 80, 87], and [127]). However, the Laplace transform as a key element of the frequency domain approaches limits their applications to the stability analysis of linear time-invariant systems with constant delays.

Sections 1.3 and 1.4 focus on the control design for time delay systems. Among various design techniques, we emphasized on the predictor feedback design. Dating back to Smith's predictor [89] in 1950s and Mayne's predictor feedback law in 1960s [69], the development of the predictor based feedback design technique has left a long historical trail, just as the Lyapunov functional based feedback design has. The two celebrated papers, Manitius and Olbrot's [68] and Artstein's [7] papers, formulated predictor feedback in the context of the finite spectrum assignment and model reduction techniques, respectively. Predictor based feedback laws have since also been developed for linear systems with input and state delays [29, 30, 53, 56], nonlinear systems with input delay [72], and neutral systems with input delay [52].

In Sects. 1.4.2 and 1.4.3, we omitted discussions on the implementation of the distributed delay terms in the predictor-type feedback laws (1.58), (1.70), and (1.71). As extensions of the predictor feedback law for linear systems with a single input delay, these feedback laws with distributed delay terms naturally encounter similar implementation problems encountered in the predictor feedback law for a linear system with a single input delay when the terms are approximated by finite sums. In [54], Kharitonov proposed a dynamic feedback law for linear systems with input and state delays that can be safely implemented when its distributed delay terms are approximated by finite sums. Specifically, the stabilization by the approximated feedback law is achieved as long as the number of the integration steps used by finite sum approximations is sufficiently large. Readers are referred to [54] for the detailed construction of the dynamic feedback law, the approximated feedback law, and a proof of the stability of the resulting closed-loop system.

Section 1.5 covers the analysis and control design for discrete-time linear systems with delay. Similarities between the notions of discrete-time linear systems and those of continuous-time linear systems facilitate the exposition of delay difference equations, their stability definitions, and the Razumikhin Stability Theorem in the discrete-time setting. On the other hand, control design for discrete-time linear systems emphasizes on predictor feedback.

Chapter 2
Truncated Predictor Feedback for Continuous-Time Linear Systems

2.1 Introduction

A difficulty arises in the implementation of a predictor feedback law with the distributed delay term in it. To overcome this implementation difficulty, the authors of [63] proposed to truncate the distributed delay term from the predictor feedback law, implementing only the remaining static state feedback term. This leads to a truncated predictor feedback (TPF) law, which can be readily implemented. In this chapter, we introduce the intuition, expression, and construction of the TPF law and its output feedback counterpart. In particular, the construction of the feedback gain matrix of the truncated predictor feedback law is first carried out by using an eigenstructure assignment based low gain feedback design technique. An alternative design is then presented by using the Lyapunov equation based low gain design technique. Stabilization of linear systems with input delay by the TPF is established under both the eigenstructure assignment based TPF and the Lyapunov equation based TPF.

2.2 The Eigenstructure Assignment Based Design

This section examines the asymptotic stabilizability of linear systems with delayed input by TPF laws. By explicit construction of stabilizing feedback laws, it is shown that a stabilizable and detectable linear system with an arbitrarily large delay in the input can be asymptotically stabilized by either a truncated predictor state feedback law (also referred to as the state feedback TPF law) or a truncated predictor output feedback law (also referred to as the output feedback TPF law) as long as the open loop system is not exponentially unstable (i.e., all the open loop poles are on the closed left-half plane). A simple example will show that such results would not be true if the open loop system is exponentially unstable.

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5_2

Consider the following linear system with a time delay in the input:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t-\tau), \\ y(t) = Cx(t), \\ x(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurement output, and $\tau \geq 0$ is the amount of time delay in the control input. It is also assumed that the pair (A, B) is stabilizable and the pair (A, C) is detectable.

Control problems for linear time delay systems in the form of (2.1) or in a variety of other forms have been a subject of extensive research (see, for example, [5, 6, 18, 20–23, 41, 44, 71, 76, 77, 79, 81, 93, 112–115, 118], and the references therein). Various stability and stabilizability conditions were identified and stabilizing feedback laws constructed. In particular, two special classes of (2.1) were studied in [28, 71] and [70], respectively. Both [71] and [28] showed that an oscillator system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -x_1(t) + u(t-\tau), \end{cases} \tag{2.2}$$

with an arbitrarily large delay τ, is globally asymptotically stabilizable by bounded state feedback laws.

In [70], the authors established global asymptotic stabilizability by bounded state feedback of a chain of integrators with an arbitrarily large delay in the input,

$$\begin{cases} \dot{x}_i(t) = x_{i+1}(t), \quad i \in I[1, n-1], \\ \dot{x}_n(t) = u(t-\tau). \end{cases} \tag{2.3}$$

While (2.2) and (2.3) are both special classes of (2.1), the feedback laws that were constructed for stabilizing them are very different and the proofs of the closed-loop stability involve different techniques. The feedback laws in [71] and [28] both involve a saturation function and both require the explicit knowledge of the amount of the delay. The asymptotic stability of the closed-loop system in [71] was established through establishing the asymptotic stability of a system under a distributed control law, while [28] resorted to the analysis of trajectories. The feedback law in [70] does not require explicit knowledge of the amount of delay but involves a set of nested saturation functions. The result of [70] was extended to open loop systems with all poles located on the closed left-half plane [115], where L_p stabilization is also considered.

These results in the literature in a way indicate the complexity in the stabilization of systems in the form of (2.1). The objective of this section is to reexamine asymptotic stabilizability of system (2.1). By explicit construction of TPF laws, we will show that system (2.1), with an arbitrarily large finite delay, is asymptotically stabilizable by either linear state or output feedback as long as the open loop system is not exponentially unstable (*i.e.*, all the open loop poles are in the closed left-half plane). Key to establishing our results is the low gain feedback design technique. The potential of the low gain feedback design has been well demonstrated in [61]. In contrast to the bounded control laws in [28, 70, 71], which are all nonlinear, TPF laws we are to construct are linear.

2.2.1 Low Gain Feedback Design

The eigenstructure assignment based low gain feedback design for multiple input systems is developed from the design for single input systems.

Consider the following single input linear system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}, \tag{2.4}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Assume that all eigenvalues of A are on the imaginary axis. Let $F(\varepsilon) : (0, 1] \mapsto \mathbb{R}^{1\times n}$ be the unique state feedback gain such that

$$\lambda(A + BF(\varepsilon)) = -\varepsilon + \lambda(A), \quad \varepsilon \in (0, 1]. \tag{2.5}$$

Then, we have the following lemmas [61] on the properties of the resulting closed-loop system. Explicit construction of all the matrices involved in these lemmas can be found in [61].

Lemma 2.1 *Consider system (2.4) and let F be as given by (2.5). Then, there exists a nonsingular transformation matrix $Q(\varepsilon) \in \mathbb{R}^{n\times n}$ such that*

$$\begin{aligned} Q^{-1}(\varepsilon)(A + BF(\varepsilon))Q(\varepsilon) &= J(\varepsilon) \\ &= \text{blkdiag}\left\{J_0(\varepsilon), J_1(\varepsilon), \cdots, J_l(\varepsilon)\right\}, \end{aligned}$$

where

$$J_0(\varepsilon) = \begin{bmatrix} -\varepsilon & 1 & & \\ & \ddots & \ddots & \\ & & -\varepsilon & 1 \\ & & & -\varepsilon \end{bmatrix}_{n_0 \times n_0},$$

and for each $i \in I[1, l]$

$$J_i(\varepsilon) = \begin{bmatrix} J_i^\star(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & J_i^\star(\varepsilon) & I_2 \\ & & & J_i^\star(\varepsilon) \end{bmatrix}_{2n_i \times 2n_i},$$

$$J_i^\star(\varepsilon) = \begin{bmatrix} -\varepsilon & \beta_i \\ -\beta_i & -\varepsilon \end{bmatrix},$$

with $\beta_i > 0$ *for all* $i \in I[1, l]$ *and* $\beta_i \neq \beta_j$ *for* $i \neq j$.

Lemma 2.2 *Consider system (2.4) and let* F *be given by (2.5). Let* $J(\varepsilon)$ *be as given in Lemma 2.1. Let*

$$S(\varepsilon) = \text{blkdiag}\{S_0(\varepsilon), S_1(\varepsilon), S_2(\varepsilon), \cdots, S_l(\varepsilon)\},$$

where

$$S_0(\varepsilon) = \text{diag}\left\{\varepsilon^{n_0-1}, \varepsilon^{n_0-2}, \cdots, \varepsilon, 1\right\},$$

and for each $i \in I[1, l]$,

$$S_i(\varepsilon) = \text{blkdiag}\left\{\varepsilon^{n_i-1} I_2, \varepsilon^{n_i-2} I_2, \cdots, \varepsilon I_2, I_2\right\}.$$

Then,

1. $S(\varepsilon) J(\varepsilon) S^{-1}(\varepsilon) = \varepsilon \tilde{J}(\varepsilon) \triangleq \varepsilon \text{blkdiag}\left\{\tilde{J}_0, \tilde{J}_1(\varepsilon), \cdots, \tilde{J}_l(\varepsilon)\right\}$, *where*

$$\tilde{J}_0 = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix}_{n_0 \times n_0},$$

and for each $i \in I[I, l]$,

$$\tilde{J}_i(\varepsilon) = \begin{bmatrix} \tilde{J}_i^\star(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & \tilde{J}_i^\star(\varepsilon) & I_2 \\ & & & \tilde{J}_i^\star(\varepsilon) \end{bmatrix}_{2n_i \times 2n_i},$$

$$\tilde{J}_i^\star(\varepsilon) = \begin{bmatrix} -1 & \beta_i/\varepsilon \\ -\beta_i/\varepsilon & -1 \end{bmatrix},$$

with $\beta_i > 0$ *for all* $i \in I[I, l]$ *and* $\beta_i \neq \beta_j$ *for* $i \neq j$;

2. *The unique positive definite solution* $\tilde{P}$ *to the Lyapunov equation*

$$\tilde{J}^{\mathrm{T}}(\varepsilon)\tilde{P} + \tilde{P}\tilde{J}(\varepsilon) = -I$$

is independent of ε.

Lemma 2.3 *Consider system (2.4) and let* F *be as given by (2.5). Let* $Q(\varepsilon)$, l, *and* n_i *for* $i = 0$ *to* l, *be as defined in Lemma 2.1. Let the scaling matrix* $S(\varepsilon)$ *be as defined in Lemma 2.2. Then, there exist* $\gamma, \alpha, \beta, \vartheta > 0$, *all independent of* ε, *such that, for all* $\varepsilon \in (0, 1]$,

$$|F(\varepsilon)| \leq \gamma\varepsilon,$$

$$\left|F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)\right| \leq \alpha\varepsilon,$$

$$\left|F(\varepsilon)AQ(\varepsilon)S^{-1}(\varepsilon)\right| \leq \beta\varepsilon,$$

$$|Q(\varepsilon)| \leq \vartheta,$$

$$\left|Q^{-1}(\varepsilon)\right| \leq \vartheta.$$

2.2.2 Truncated Predictor State Feedback Design

For system (2.1) with all eigenvalues of A on the closed left-half plane, we construct a family of linear state feedback laws as follows.

Step 1. Find nonsingular transformation matrices T_{S} and T_{I} such that the pair (A, B) is transformed into the following block diagonal control canonical form,

$$T_{\mathrm{S}}^{-1}AT_{\mathrm{S}} = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_l & 0 \\ 0 & 0 & \cdots & 0 & A_0 \end{bmatrix}, \tag{2.6}$$

$$T_{\rm s}^{-1}BT_{\rm I} = \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1l} & * \\ 0 & B_2 & \cdots & B_{2l} & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_l & * \\ B_{01} & B_{02} & \cdots & B_{0l} & * \end{bmatrix}, \tag{2.7}$$

where A_0 contains all the open left-half plane eigenvalues of A, for each $i \in I[i, l]$, all eigenvalues of A_i are on the $j\omega$ axis and (A_i, B_i) is controllable as given by,

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{in_i} & -a_{i(n_i-1)} & -a_{i(n_i-2)} & \cdots & -a_{i1} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and finally, $*$'s represent submatrices of less interest.

We note that the existence of the above canonical form was shown in [110]. The software realization can be found in [62].

Step 2. For each (A_i, B_i), let $F_i(\varepsilon) \in \mathbb{R}^{1\times n_i}$ be the state feedback gain such that

$$\lambda(A_i + B_i F_i(\varepsilon)) = -\varepsilon + \lambda(A_i), \quad \varepsilon \in (0, 1].$$

Note that $F_i(\varepsilon)$ is unique.

Step 3. Construct a family of low gain state feedback laws as

$$u(t) = F(\varepsilon)\mathrm{e}^{A\tau}x(t), \tag{2.8}$$

where the low gain matrix $F(\varepsilon)$ is given by

$$F(\varepsilon) = T_{\rm I} \begin{bmatrix} F_1(\varepsilon) & 0 & \cdots & 0 & 0 & 0 \\ 0 & F_2(\varepsilon) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & F_{l-1}(\varepsilon) & 0 & 0 \\ 0 & 0 & \cdots & 0 & F_l(\varepsilon) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} T_{\rm s}^{-1}. \tag{2.9}$$

The low gain feedback law (2.8) is a truncated version of the predictor feedback law (1.39), whose distributed delay term resulting from the zero state solution of the predicted future state has been discarded. Therefore, the feedback law (2.8) is referred to as the TPF law. The key here is to parameterize the feedback gain matrix of the TPF law through the eigenstructure assignment based low gain feedback design. The theorem below establishes that such a linear state feedback law (2.8) asymptotically stabilizes system (2.1) as long as all eigenvalues of A are on the closed left-half plane, no matter how large the delay is.

Theorem 2.1 *Consider the closed-loop system comprising of the plant (2.1) and the state feedback TPF law (2.8). Let all eigenvalues of A be on the closed left-half plane. Then, for any given arbitrarily large* $\tau \geq 0$*, there exists* $\varepsilon^* > 0$ *such that, for each* $\varepsilon \in (0, \varepsilon^*]$*, the closed-loop system is asymptotically stable.*

Proof Without loss of generality, assume that the pair (A, B) is already in the form of (2.6)–(2.7). Under the state feedback law (2.8), the closed-loop system is given by

$$\dot{x}(t) = Ax(t) + BF(\varepsilon)\mathrm{e}^{A\tau}x(t-\tau), \tag{2.10}$$

from which we have

$$\mathrm{e}^{A\tau}x(t-\tau) = x(t) - \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}x(s-\tau)\mathrm{d}s. \tag{2.11}$$

Substitution of (2.11) in (2.10) results in

$$\dot{x}(t) = (A + BF(\varepsilon))x(t) - BF(\varepsilon) = \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}x(s-\tau)\mathrm{d}s. \tag{2.12}$$

Partitioning the state x according to the structure of (2.6)–(2.7), $x = \left[x_1^{\mathrm{T}}\, x_2^{\mathrm{T}} \cdots x_l^{\mathrm{T}}\, x_0^{\mathrm{T}}\right]^{\mathrm{T}}$, $x_i \in \mathbb{R}^{n_i}$, $i \in I[1, l]$, we rewrite the state equation (2.12) as follows:

$$\begin{cases} \dot{x}_1(t) = (A_1 + B_1F_1(\varepsilon))x_1(t) + \displaystyle\sum_{j=2}^{l} B_{1j}F_j(\varepsilon)x_j(t) \\ \qquad\qquad -B_{\mathrm{R}1}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}x(s-\tau)\mathrm{d}s, \\ \dot{x}_2(t) = (A_2 + B_2F_2(\varepsilon))x_2(t) + \displaystyle\sum_{j=3}^{l} B_{2j}F_j(\varepsilon)x_j(t) \\ \qquad\qquad -B_{\mathrm{R}2}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}x(s-\tau)\mathrm{d}s, \\ \quad\vdots \\ \dot{x}_l(t) = (A_l + B_lF_l(\varepsilon))x_l(t) - B_{\mathrm{R}l}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}x(s-\tau)\mathrm{d}s, \\ \dot{x}_0(t) = A_0x_0(t) + \displaystyle\sum_{j=1}^{l} B_{0j}F_j(\varepsilon)x_j(t) \\ \qquad\qquad -B_{\mathrm{R}0}F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}x(s-\tau)\mathrm{d}s, \end{cases} \tag{2.13}$$

where for each $i \in I[1, l]$, B_{Ri} is the ith row of the right-hand side of (2.7) and B_{R0} is the last row.

Now for each $i \in I[1, l]$, let $Q_i(\varepsilon)$, $S_i(\varepsilon)$, $\tilde{J}_i(\varepsilon)$, $\tilde{P}_i$, γ_i, α_i, β_i, and ϑ_i be the matrices $Q(\varepsilon)$, $S(\varepsilon)$, $\tilde{J}(\varepsilon)$, and $\tilde{P}$ and the constants γ, α, β, and ϑ as defined in Lemmas 2.1–2.3, but for the triple $(A_i, B_i, F_i(\varepsilon))$. Define a state transformation as $\tilde{x} = \left[\tilde{x}_1^{\mathrm{T}}, \tilde{x}_2^{\mathrm{T}}, \cdots, \tilde{x}_l^{\mathrm{T}}, \tilde{x}_0^{\mathrm{T}}\right]^{\mathrm{T}}$, where $\tilde{x}_0 = x_0$, and for each $i \in I[1, l]$, $\tilde{x}_i = S_i(\varepsilon)Q_i^{-1}(\varepsilon)x_i$. It follows from Lemmas 2.1 and 2.2 that, under this state transformation, the state equation (2.13) can be written as

$$\begin{cases} \dot{\tilde{x}}_1(t) = \varepsilon \tilde{J}_1(\varepsilon)\tilde{x}_1(t) + \displaystyle\sum_{j=2}^{l} S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{1j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(t) - S_1(\varepsilon) \\ \qquad \times Q_1^{-1}(\varepsilon)B_{R1}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau} \times Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s, \\ \dot{\tilde{x}}_2(t) = \varepsilon \tilde{J}_2(\varepsilon)\tilde{x}_2(t) + \displaystyle\sum_{j=3}^{l} S_2(\varepsilon)Q_2(\varepsilon)B_{2j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(t) - S_2(\varepsilon) \\ \qquad \times Q_2^{-1}(\varepsilon)B_{R2}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s, \\ \qquad \vdots \\ \dot{\tilde{x}}_l(t) = \varepsilon \tilde{J}_l(\varepsilon)\tilde{x}_l(t) - S_l(\varepsilon)Q_l^{-1}(\varepsilon)B_{Rl}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau} \\ \qquad \times Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s, \\ \dot{\tilde{x}}_0(t) = A_0\tilde{x}_0(t) + \displaystyle\sum_{j=1}^{l} B_{0j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(t) \\ \qquad -B_{R0}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s, \end{cases} \tag{2.14}$$

where

$$Q(\varepsilon) = \mathrm{blkdiag}\,\{Q_1(\varepsilon), Q_2(\varepsilon), \cdots, Q_l(\varepsilon), I\}$$

and

$$S(\varepsilon) = \mathrm{blkdiag}\{S_1(\varepsilon), S_2(\varepsilon), \cdots, S_l(\varepsilon)I\}.$$

Let us choose a Lyapunov function

$$V(\tilde{x}) = \sum_{i=1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}}\tilde{P}_i\tilde{x}_i + \tilde{x}_0^{\mathrm{T}}\tilde{P}_0\tilde{x}_0 \triangleq \tilde{x}^{\mathrm{T}}\tilde{P}\tilde{x},$$

where $\tilde{P}_0 > 0$ is such that

$$A_0^{\mathrm{T}}\tilde{P}_0 + \tilde{P}_0A_0 = -I,$$

$\kappa > 0$ is a constant whose value is to be determined later, and

$$\tilde{P} = \text{blkdiag}\left\{\kappa \tilde{P}_1, \kappa^2 \tilde{P}_2, \cdots, \kappa^l \tilde{P}_l, \tilde{P}_0\right\}.$$

The existence of such a $\tilde{P}_0$ is due to the fact that A_0 is Hurwitz.

The derivative of V along the trajectory of the closed-loop system (2.14) can be evaluated as follows:

$$\begin{aligned}\dot{V}(\tilde{x}(t)) = &\sum_{i=1}^{l}\Bigg[\varepsilon\kappa^i \tilde{x}_i^{\mathrm{T}}(t)\left(\tilde{P}_i \tilde{J}_i(\varepsilon) + \tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_i\right)\tilde{x}_i(t) + 2\sum_{j=i+1}^{l}\kappa^i \tilde{x}_i^{\mathrm{T}}(t)\tilde{P}_i S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{ij} \\ &\times F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon)\tilde{x}_j(t)\Bigg] + \tilde{x}_0^{\mathrm{T}}(t)\left(A_0^{\mathrm{T}}\tilde{P}_0 + \tilde{P}_0 A_0\right)\tilde{x}_0(t) + 2\sum_{j=1}^{n}\tilde{x}_0^{\mathrm{T}}(t)\tilde{P}_0 B_{0j} \\ &\times F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon)\tilde{x}_j(t) - 2\left[\sum_{i=1}^{l}\kappa^i \tilde{x}_i^{\mathrm{T}}(t)\tilde{P}_i S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} + \tilde{x}_0^{\mathrm{T}}\tilde{P}_0 B_{\mathrm{R}0}\right] \\ &\times F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} B F(\varepsilon)\mathrm{e}^{A\tau} Q(\varepsilon) S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s. \end{aligned} \tag{2.15}$$

Recall that

$$\tilde{P}_i \tilde{J}_i(\varepsilon) + \tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_i = -I$$

and

$$A_0^{\mathrm{T}}\tilde{P}_0 + \tilde{P}_0 A_0 = -I.$$

Also, it follows from Lemma 2.3 that the matrices defining the $(\tilde{x}_i, \tilde{x}_j)$ cross terms, $i \in I[0, l-1]$, $j \in [i+1, l]$, are all of order ε. It is then straightforward to verify that there exist $\kappa > 0$ and $\varepsilon_1^* \in (0, 1]$ such that

$$\begin{aligned}\dot{V}(\tilde{x}(t)) \le &-\frac{\varepsilon}{2}\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t) - 2\left[\sum_{i=1}^{l}\kappa^i \tilde{x}_i^{\mathrm{T}}(t)\tilde{P}_i S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} + \tilde{x}_0^{\mathrm{T}}\tilde{P}_0 B_{\mathrm{R}0}\right] F(\varepsilon) \\ &\times \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} B F(\varepsilon)\mathrm{e}^{A\tau} Q(\varepsilon) S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s, \quad \varepsilon \in (0, \varepsilon_1^*]. \end{aligned} \tag{2.16}$$

Recalling the special structural of $\tilde{J}_i(\varepsilon)$, we have

$$\begin{aligned}&\left|S(\varepsilon) Q^{-1} A Q(\varepsilon) S^{-1}(\varepsilon)\right| \\ &\le \sum_{i=1}^{l}\left|S_i(\varepsilon) Q_i^{-1}(\varepsilon) A_i Q_i(\varepsilon) S_i^{-1}(\varepsilon)\right| + |A_0|\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{l} \left| S_i(\varepsilon) Q_i^{-1}(\varepsilon)(A_i + B_i F_i(\varepsilon)) Q(\varepsilon) S_i^{-1}(\varepsilon) - S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_i F_i(\varepsilon) Q_i(\varepsilon) S_i^{-1}(\varepsilon) \right| \\
&\quad + |A_0| \\
&\le \sum_{i=1}^{l} \left(\varepsilon \left| \tilde{J}_i(\varepsilon) \right| + \varepsilon \alpha_i \vartheta_i |B_i| \right) + |A_0| \delta, \quad \varepsilon \in (0, \varepsilon_1^*],
\end{aligned}
$$

for some $\delta > 0$. Thus, by noting that

$$
\mathrm{e}^{A\tau} = I + A\tau + \frac{1}{2!} A^2 \tau^2 + \frac{1}{3!} A^3 \tau^3 + \cdots
$$

and, for any $i \ge 1$,

$$
\begin{aligned}
&\left| S(\varepsilon) Q^{-1}(\varepsilon) A^i Q(\varepsilon) S^{-1}(\varepsilon) \right| \\
= &\left| S(\varepsilon) Q^{-1}(\varepsilon) A Q^{-1} S^{-1}(\varepsilon) S(\varepsilon) Q^{-1}(\varepsilon) A \cdots A Q(\varepsilon) S^{-1}(\varepsilon) \right| \\
\le &\delta^i,
\end{aligned} \tag{2.17}
$$

we have,

$$
\begin{aligned}
&\left| F(\varepsilon) \mathrm{e}^{A\tau} Q(\varepsilon) S^{-1}(\varepsilon) \right| \\
&= \left| F(\varepsilon) Q(\varepsilon) S^{-1}(\varepsilon) S(\varepsilon) Q^{-1}(\varepsilon) \mathrm{e}^{A\tau} Q(\varepsilon) S^{-1}(\varepsilon) \right| \\
&\le \left| F(\varepsilon) Q(\varepsilon) S^{-1}(\varepsilon) \right| \left| S(\varepsilon) Q^{-1} \mathrm{e}^{A\tau} Q(\varepsilon) S^{-1}(\varepsilon) \right| \\
&\le \left(\sum_{i=1}^{l} \left| F_i(\varepsilon) Q_i(\varepsilon) S_i^{-1}(\varepsilon) \right| \right) \mathrm{e}^{\delta\tau} \\
&\le \varepsilon \left(\sum_{i}^{l} \alpha_i \right) \mathrm{e}^{\delta\tau}.
\end{aligned} \tag{2.18}
$$

Hence, inequality (2.16) can be continued as follows:

$$
\dot{V}(\tilde{x}(t)) \le -\frac{\varepsilon}{2} \lambda_{\max}^{-1}(\tilde{P}) V(\tilde{x}(t)) + \varepsilon^2 \varpi V^{\frac{1}{2}}(\tilde{x}(t)) \int_{t-\tau}^{t} \left| \mathrm{e}^{A(t-s)} \right| V^{\frac{1}{2}}(\tilde{x}(s-\tau)) \mathrm{d}s, \quad \varepsilon \in (0, \varepsilon_1^*],
$$

for some $\varpi > 0$, independent of ε. In arriving at the above inequality, we have used the inequality

$$
\begin{aligned}
V(\tilde{x}(t)) &= \tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{x}(t) \\
&\le \lambda_{\max}(\tilde{P})\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t).
\end{aligned}
$$

Now, let $\eta > 1$ be any constant. If $V(\tilde{x}(t+\theta)) < \eta V(\tilde{x}(t)),\ \theta \in [-\tau, 0]$, then

$$
\begin{aligned}
\dot{V}(\tilde{x}(t)) &\le -\frac{\varepsilon}{2}\lambda_{\max}^{-1}(\tilde{P})V(\tilde{x}(t)) + \varepsilon^2\varpi\eta\int_0^\tau \left|\mathrm{e}^{As}\right| \mathrm{d}s\, V(\tilde{x}(t)) \\
&= -\varepsilon\left[\frac{1}{2}\lambda_{\max}^{-1}(\tilde{P}) - \varepsilon\varpi\eta\int_0^\tau \left|\mathrm{e}^{As}\right|\mathrm{d}s\right]V(\tilde{x}(t)),
\end{aligned}
$$

where the following estimate was used:

$$
\begin{aligned}
V^{\frac{1}{2}}(\tilde{x}(s-\tau)) &\le \sqrt{\eta}V^{\frac{1}{2}}(\tilde{x}(t-\tau)) \\
&\le \eta V^{\frac{1}{2}}(\tilde{x}(t)), \quad s \in [t-\tau, t].
\end{aligned}
$$

It is clear that, for any given $\tau > 0$ and $\eta > 1$, there exists $\varepsilon^* \in (0, \varepsilon_1^*]$ such that, for each $\varepsilon \in (0, \varepsilon^*]$,

$$
\dot{V}(\tilde{x}(t)) \le -\mu(\varepsilon)V(\tilde{x}(t)), \ \text{if } V(\tilde{x}(t+\theta)) < \eta V(\tilde{x}(t)), \quad \theta \in [-\tau, 0],
$$

for some positive scalar $\mu(\varepsilon)$. It thus follows from the Razumikhin Stability Theorem (Theorem 1.3) that the closed-loop system (2.10) is asymptotically stable. □

To examine the conservativeness of result of Theorem 2.1, we consider the following simple system, whose open loop system is exponentially unstable,

$$
\begin{cases}
\dot{x} = \alpha x + u(t-\tau), & \alpha > 0, \\
u = -kx, & k > 0.
\end{cases}
\tag{2.19}
$$

The following result from [79] shows that system (2.19) is not asymptotically stable for any $k > 0$ if τ is large enough and thus the condition of Theorem 2.1 is tight. This result also indicates that asymptotic stabilization of systems whose open loop poles are not all in the closed left-half plane is possible for a smaller time delay.

Proposition 2.1 *System (2.19) is asymptotically stable if and only if $k > \alpha$ and*

$$
\tau < \frac{\arccos\left(\frac{\alpha}{k}\right)}{\sqrt{k^2 - \alpha^2}} < 1.
$$

On the other hand, it is well known [21] that one of the necessary condition for a system of the form

$$
\dot{x}(t) = Ax(t) + Bx(t-\tau)
$$

is asymptotically stable independent of τ is that all eigenvalues of A have negative real parts. The result in Theorem 2.1 establishes that the real part of eigenvalues of

A being non-positive alone is sufficient for achieving asymptotic stabilization for an arbitrarily large bounded delay τ.

2.2.3 Truncated Predictor Output Feedback Design

For system (2.1) with all eigenvalues of A on the closed left-half plane, we construct the following output feedback TPF law:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t-\tau) - L(y(t) - C\hat{x}(t)), \\ u(t) = F(\varepsilon)\mathrm{e}^{A\tau}\hat{x}(t), \end{cases} \tag{2.20}$$

where $F(\varepsilon)$ is as given by (2.9) and $L \in \mathbb{R}^{n\times p}$ is such that all eigenvalues of $A+LC$ have negative real parts. We note that such a matrix L exists as the pair (A, C) is detectable.

The theorem below establishes that the output feedback law (2.20) asymptotically stabilizes system (2.1) as long as all eigenvalues of A are on the closed left-half plane.

Theorem 2.2 *Consider the closed-loop system comprising of the plant (2.1) and the linear output feedback law (2.20). Let all eigenvalues of A be on the closed left-half plane. Then, for any given arbitrarily large* $\tau \geq 0$*, there exists* $\varepsilon^* > 0$ *such that, for each* $\varepsilon \in (0, \varepsilon^*]$*, the closed-loop system is asymptotically stable.*

Proof Under the linear output feedback law (2.20), the closed-loop system is given by,

$$\begin{cases} \dot{x}(t) = Ax(t) + BF(\varepsilon)\mathrm{e}^{A\tau}\hat{x}(t-\tau), \\ \dot{\hat{x}}(t) = A\hat{x}(t) + BF(\varepsilon)\mathrm{e}^{A\tau}\hat{x}(t-\tau) - L(y(t) - C\hat{x}(t)), \end{cases} \tag{2.21}$$

which, in the new state $(x, e) = (x, x - \hat{x})$, can be written as,

$$\begin{cases} \dot{x}(t) = Ax(t) + BF(\varepsilon)\mathrm{e}^{A\tau}x(t-\tau) - BF(\varepsilon)\mathrm{e}^{A\tau}e(t-\tau), \\ \dot{e}(t) = (A + LC)e(t), \end{cases} \tag{2.22}$$

which in turn implies that

$$\begin{aligned} \mathrm{e}^{A\tau}x(t-\tau) = x(t) &- \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}x(s-\tau)\mathrm{d}s \\ &+ \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}e(s-\tau)\mathrm{d}s, \end{aligned} \tag{2.23}$$

and

$$e(t-\tau) = \mathrm{e}^{-(A+LC)\tau} e(t). \tag{2.24}$$

Substitution of (2.23) and (2.24) into (2.22) results in

$$\begin{cases} \dot{x}(t) = (A+BF)x(t) - BF(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} x(s-\tau)\mathrm{d}s + BF(\varepsilon) \\ \qquad \times \displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} e(s-\tau)\mathrm{d}s - BF(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau} e(t), \\ \dot{e}(t) = (A+LC)e(t). \end{cases} \tag{2.25}$$

Without loss of generality, assume that the pair (A, B) is already in the form of (2.6)–(2.7). Partitioning the state x according to the structure of (2.6)–(2.7) as

$$x = \begin{bmatrix} x_1^{\mathrm{T}} \; x_2^{\mathrm{T}} \; \cdots \; x_l^{\mathrm{T}} \; x_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \qquad x_i \in \mathbb{R}^{n_i}, \;\; i \in [1, l].$$

we rewrite the state equation (2.25) as

$$\begin{cases} \dot{x}_1(t) = (A_1 + B_1F_1(\varepsilon))x_1(t) + \displaystyle\sum_{j=2}^{l} B_{1j}F_j(\varepsilon)x_j(t) - B_{\mathrm{R}1}F(\varepsilon) \\ \qquad \times \displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} x(s-\tau)\mathrm{d}s + B_{\mathrm{R}1}F(\varepsilon) \\ \qquad \times \displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} e(s-\tau)\mathrm{d}s - B_{\mathrm{R}1}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau} e(t), \\ \dot{x}_2(t) = (A_2 + B_2F_2(\varepsilon))x_2(t) + \displaystyle\sum_{j=3}^{l} B_{2j}F_j(\varepsilon)x_j(t) - B_{\mathrm{R}2}F(\varepsilon) \\ \qquad \times \displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} x(s-\tau)\mathrm{d}s + B_{\mathrm{R}2}F(\varepsilon) \\ \qquad \times \displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} e(s-\tau)\mathrm{d}s - B_{\mathrm{R}2}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau} e(t), \\ \qquad \vdots \\ \dot{x}_l(t) = (A_l + B_lF_l(\varepsilon))x_l(t) - B_{\mathrm{R}l}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} x(s-\tau)\mathrm{d}s \\ \qquad + B_{\mathrm{R}l}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} e(s-\tau)\mathrm{d}s \\ \qquad - B_{\mathrm{R}l}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau} e(t), \\ \dot{x}_0(t) = A_0x_0(t) + \displaystyle\sum_{j=1}^{l} B_{0j}F_j(\varepsilon)x_j(t) \\ \qquad - B_{\mathrm{R}0}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} x(s-\tau)\mathrm{d}s \\ \qquad + B_{\mathrm{R}0}F(\varepsilon)\displaystyle\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BF(\varepsilon)\mathrm{e}^{A\tau} e(s-\tau)\mathrm{d}s \\ \qquad - B_{\mathrm{R}0}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau} e(t), \\ \dot{e}(t) = (A+LC)e(t), \end{cases} \tag{2.26}$$

where, for each $i \in I[1, l]$, B_{Ri} is the ith row of the right-hand side of (2.7) and B_{R0} is the last row.

Now, for each $i \in I[1, l]$, let $Q_i(\varepsilon)$, $S_i(\varepsilon)$, $\tilde{J}_i(\varepsilon)$, $\tilde{P}_i$, γ_i, α_i, β_i, and ϑ_i be the matrices $Q(\varepsilon)$, $S(\varepsilon)$, $\tilde{J}(\varepsilon)$, and $\tilde{P}$ and the constants γ, α, β, and ϑ as defined in Lemmas 2.1–2.3, but for the triple $(A_i, B_i, F_i(\varepsilon))$. Define a state transformation as

$$\tilde{x} = \left[\tilde{x}_1^{\mathrm{T}}, \tilde{x}_2^{\mathrm{T}}, \cdots, \tilde{x}_l^{\mathrm{T}}, \tilde{x}_0^{\mathrm{T}}\right]^{\mathrm{T}}, \quad \tilde{e} = e,$$

where $\tilde{x}_0 = x_0$, and, for each $i \in I[1, l]$,

$$\tilde{x}_i = S_i(\varepsilon) Q_i^{-1}(\varepsilon) x_i.$$

It follows from Lemmas 2.1 and 2.2 that, under this state transformation, the state equation (2.26) can be written as

$$\begin{cases} \dot{\tilde{x}}_1(t) = \varepsilon \tilde{J}_1(\varepsilon)\tilde{x}_1(t) \\ \qquad +\sum_{j=2}^{l} S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{1j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(t) - S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{R1} \\ \qquad \times F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s + S_1(\varepsilon)Q_1^{-1}(\varepsilon) \\ \qquad \times B_{R1}F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(s-\tau)\mathrm{d}s \\ \qquad - S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{R1}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau}\tilde{e}(t), \\ \dot{\tilde{x}}_2(t) = \varepsilon \tilde{J}_2(\varepsilon)\tilde{x}_2(t) + \sum_{j=3}^{l} S_2(\varepsilon)Q_2(\varepsilon)B_{2j}F_j(\varepsilon)Q_j(\varepsilon) \\ \qquad \times S_j^{-1}(\varepsilon)\tilde{x}_j(t) - S_2(\varepsilon)Q_2^{-1}(\varepsilon)B_{R2}F(\varepsilon) \\ \qquad \times \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s \\ \qquad + S_2(\varepsilon)Q_2^{-1}(\varepsilon)B_{R2}F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon) \\ \qquad \times \tilde{e}(s-\tau)\mathrm{d}s - S_2(\varepsilon)Q_2^{-1}(\varepsilon)B_{R2}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau}\tilde{e}(t), \\ \quad \vdots \\ \dot{\tilde{x}}_l(t) = \varepsilon \tilde{J}_l(\varepsilon)\tilde{x}_l(t) - S_l(\varepsilon)Q_l^{-1}(\varepsilon)B_{Rl}F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon) \\ \qquad \times S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s + S_l(\varepsilon)Q_l^{-1}(\varepsilon)B_{Rl}F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau} \\ \qquad \times Q(\varepsilon)\tilde{S}^{-1}(\varepsilon)e(s-\tau)\mathrm{d}s - S_l(\varepsilon)Q_l^{-1}(\varepsilon)B_{R2}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau}\tilde{e}(t), \\ \dot{\tilde{x}}_0(t) = A_0\tilde{x}_0(t) + \sum_{j=1}^{l} B_{0j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(t) - B_{R0}F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} \\ \qquad \times BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s + B_{R0}F(\varepsilon)\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} \\ \qquad \times BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(s-\tau)\mathrm{d}s - B_{R0}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau}\tilde{e}(t), \\ \dot{\tilde{e}}(t) = (A+LC)\tilde{e}(t), \end{cases} \tag{2.27}$$

where

$$Q(\varepsilon) = \text{blkdiag}\{Q_1(\varepsilon), Q_2(\varepsilon), \cdots, Q_l(\varepsilon), I\}$$

and

$$S(\varepsilon) = \text{blkdiag}\{S_1(\varepsilon), S_2(\varepsilon), \cdots, S_l(\varepsilon), I\}.$$

Let us choose the Lyapunov function

$$\begin{aligned} V(\tilde{x}, \tilde{e}) &= \sum_{i=1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}} \tilde{P}_i \tilde{x}_i + \tilde{x}_0^{\mathrm{T}} \tilde{P}_0 \tilde{x}_0 + \kappa^{l+1} \tilde{e}^{\mathrm{T}} \tilde{Q} \tilde{e} \\ &= \tilde{x}^{\mathrm{T}} \tilde{P} \tilde{x} + \tilde{e}^{\mathrm{T}} \tilde{Q} \tilde{e}, \end{aligned}$$

where $\tilde{P}_0 > 0$ and $\tilde{Q} > 0$ are the solutions to the Lyapunov equations

$$A_0^{\mathrm{T}} \tilde{P}_0 + \tilde{P}_0 A_0 = -I,$$

and

$$\tilde{Q}(A + LC) + (A + LC)^{\mathrm{T}} \tilde{Q} = -I,$$

respectively, $\kappa > 0$ is a constant whose value is to be determined later, and

$$\tilde{P} = \text{blkdiag}\left\{\kappa \tilde{P}_1, \kappa^2 \tilde{P}_2, \cdots, \kappa^l \tilde{P}_l, \tilde{P}_0\right\}.$$

The existence of such $\tilde{P}_0$ and $\tilde{Q}$ is due to the fact that both A_0 and $A + LC$ are Hurwitz.

The derivative of V along the trajectory of the closed-loop system (2.27) can be evaluated as follows:

$$\begin{aligned} &\dot{V}(\tilde{x}(t), \tilde{e}(t)) \\ &= \sum_{i=1}^{l} \Bigg[\varepsilon \kappa^i \tilde{x}_i^{\mathrm{T}}(t) \left(\tilde{P}_i \tilde{J}_i(\varepsilon) + \tilde{J}_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i \right) \tilde{x}_i(t) - 2\kappa^i \tilde{x}_i^{\mathrm{T}}(t) \tilde{P}_i S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} \\ &\quad \times F(\varepsilon) \mathrm{e}^{A\tau} \mathrm{e}^{-(A+LC)\tau} \tilde{e}(t) + 2 \sum_{j=i+1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}}(t) \tilde{P}_i S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{ij} F_j(\varepsilon) Q_j(\varepsilon) \\ &\quad \times S_j^{-1}(\varepsilon) \tilde{x}_j(t) \Bigg] + \tilde{x}_0^{\mathrm{T}}(t) \left(A_0^{\mathrm{T}} \tilde{P}_0 + \tilde{P}_0 A_0 \right) \tilde{x}_0(t) + 2 \sum_{j=1}^{l} \tilde{x}_0^{\mathrm{T}}(t) \tilde{P}_0 B_{0j} F_j(\varepsilon) \end{aligned}$$

$$
\begin{aligned}
&\times Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(t) - 2\tilde{x}_0^{\mathrm{T}}(t)\tilde{P}_0 B_{\mathrm{R}0}F(\varepsilon)\mathrm{e}^{A\tau}\mathrm{e}^{-(A+LC)\tau}\tilde{e}(t)\\
&-2\left[\sum_{i=1}^{l}\kappa^i\tilde{x}_i^{\mathrm{T}}(t)\tilde{P}_i S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{\mathrm{R}i} + \tilde{x}_0^{\mathrm{T}}\tilde{P}_0 B_{\mathrm{R}0}\right]F(\varepsilon)\\
&\times\int_{t-\tau}^{t}\mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s\\
&+2\left[\sum_{i=1}^{l}\kappa^i\tilde{x}_i^{\mathrm{T}}(t)\tilde{P}_i S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{\mathrm{R}i} + \tilde{x}_0^{\mathrm{T}}\tilde{P}_0 B_{\mathrm{R}0}\right]F(\varepsilon)\\
&\times\int_{t-\tau}^{t}\mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(s-\tau)\mathrm{d}s\\
&+\kappa^{l+1}\tilde{e}^{\mathrm{T}}(t)\left((A+LC)^{\mathrm{T}}\tilde{Q}+\tilde{Q}(A+LC)\right)\tilde{e}(t).
\end{aligned}
$$

In view of Lemmas 2.3, the matrices defining the $(\tilde{x}_i, \tilde{x}_j)$, $i \neq j$, and $(\tilde{x}_i, \tilde{e})$ cross terms are all in the order of ε. It is then straightforward to verify that there exist $\kappa > 0$ and $\varepsilon_1^* \in (0, 1]$ such that,

$$
\begin{aligned}
\dot{V}(\tilde{x}(t), \tilde{e}(t)) \le{}& -\frac{\varepsilon}{2}\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t) - \frac{1}{2}\tilde{e}^{\mathrm{T}}(t)\tilde{e}(t) - 2\left[\sum_{i=1}^{l}\kappa^i\tilde{x}_i^{\mathrm{T}}(t)\tilde{P}_i S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{\mathrm{R}i}\right.\\
&\left.+\tilde{x}_0^{\mathrm{T}}\tilde{P}_0 B_{\mathrm{R}0}\right]F(\varepsilon)\int_{t-\tau}^{t}\mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(s-\tau)\mathrm{d}s\\
&+2\left[\sum_{i=1}^{l}\kappa^i\tilde{x}_i^{\mathrm{T}}(t)\tilde{P}_i S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{\mathrm{R}i} + \tilde{x}_0^{\mathrm{T}}\tilde{P}_0 B_{\mathrm{R}0}\right]F(\varepsilon)\\
&\times\int_{t-\tau}^{t}\mathrm{e}^{A(t-s)}BF(\varepsilon)\mathrm{e}^{A\tau}Q(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(s-\tau)\mathrm{d}s,\quad \varepsilon\in(0,\varepsilon_1^*].
\end{aligned}
$$

By using Lemma 2.3 again and (2.18), we easily see that,

$$
\begin{aligned}
\dot{V}(\tilde{x}(t), \tilde{e}(t)) \le{}& -\frac{\varepsilon}{2}\min\left\{\lambda_{\max}^{-1}\left(\tilde{P}\right), \lambda_{\max}^{-1}\left(\tilde{Q}\right)\right\}V(\tilde{x}(t), \tilde{e}(t)) + \varepsilon^2\varpi V^{\frac{1}{2}}(\tilde{x}(t), \tilde{e}(t))\\
&\times\int_{t-\tau}^{t}\left|\mathrm{e}^{A(t-s)}\right|V^{\frac{1}{2}}(\tilde{x}(s-\tau))\mathrm{d}s,\quad \varepsilon\in(0,\varepsilon_1^*],
\end{aligned}
$$

for some $\varpi > 0$, independent of ε.

Now, let $\eta > 1$ be any constant. If $V(\tilde{x}(t+\theta), \tilde{e}(t+\theta)) < \eta V(\tilde{x}(t), \tilde{e}(t))$, $\theta \in [-\tau, 0]$,

$$
\dot{V}(\tilde{x}(t), \tilde{e}(t)) \le -\frac{\varepsilon}{2}\min\left\{\lambda_{\max}^{-1}\left(\tilde{P}\right), \lambda_{\max}^{-1}\left(\tilde{Q}\right)\right\}V(\tilde{x}(t), \tilde{e}(t))
$$

$$+\varepsilon^2\varpi\eta\int_0^\tau \left|\mathrm{e}^{As}\right| \mathrm{d}s V(\tilde{x}(t), \tilde{e}(t)), \quad \varepsilon \in (0, \varepsilon_1^*].$$

It is now clear that there exists an $\varepsilon^* \in (0, 1]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V}(\tilde{x}(t), \tilde{e}(t)) \leq -\sqrt{\varepsilon}V(\tilde{x}(t), \tilde{e}(t)).$$

It then follows from the Razumikhin Stability Theorem (Theorem 1.3) that the closed-loop system (2.21) is asymptotically stable. This completes the proof. □

2.2.4 A Numerical Example

Consider system (2.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

The open loop system has two pairs of repeated poles at $s = \pm j$. Following the proposed design method, we choose $F(\varepsilon) = \begin{bmatrix} -\varepsilon^4 - 2\varepsilon^2 & -4\varepsilon^3 - 4\varepsilon & -6\varepsilon^2 & -4\varepsilon \end{bmatrix}$. Then, the state feedback TPF law (2.8) is given by

$$u(t) = F(\varepsilon)\mathrm{e}^{A\tau}x(t) = \begin{bmatrix} -0.9974\varepsilon^4 + 0.0203\varepsilon^3 - 1.2757\varepsilon^2 + 1.9177\varepsilon \\ -0.4997\varepsilon^4 - 0.9974\varepsilon^3 - 0.8766\varepsilon^2 - 3.5103\varepsilon \\ -0.1199\varepsilon^4 - 0.4591\varepsilon^3 - 4.7861\varepsilon^2 + 1.9177\varepsilon \\ -0.0203\varepsilon^4 - 0.1199\varepsilon^3 - 2.7953\varepsilon^2 + 3.5103\varepsilon \end{bmatrix}^{\mathrm{T}} x(t).$$

To design an output feedback law, we choose $L = \begin{bmatrix} -13 & 50 & 23 & -10 \end{bmatrix}^{\mathrm{T}}$, which places the eigenvalues of $A + LC$ at $\{-1, -2, -3, -4\}$, and obtain the output feedback TPF law (2.20) as follows:

$$\begin{cases} \dot{\hat{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & 50 \\ 0 & 0 & 0 & 24 \\ -1 & 0 & -2 & -10 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t-\tau) - \begin{bmatrix} -13 \\ 50 \\ 23 \\ -10 \end{bmatrix} y(t), \\ u(t) = \begin{bmatrix} -0.9974\varepsilon^4 + 0.0203\varepsilon^3 - 1.2757\varepsilon^2 + 1.9177\varepsilon \\ -0.4997\varepsilon^4 - 0.9974\varepsilon^3 - 0.8766\varepsilon^2 - 3.5103\varepsilon \\ -0.1199\varepsilon^4 - 0.4591\varepsilon^3 - 4.7861\varepsilon^2 + 1.9177\varepsilon \\ -0.0203\varepsilon^4 - 0.1199\varepsilon^3 - 2.7953\varepsilon^2 + 3.5103\varepsilon \end{bmatrix}^{\mathrm{T}} \hat{x}(t). \end{cases}$$

Some simulation results of the resulting closed-loop systems are shown in Figs. 2.1, 2.2, 2.3, and 2.4. These simulation results verify the conclusion of Theorems 2.1 and 2.2. In particular, under the TPF laws, a larger delay entails a smaller value of the low gain parameter to achieve closed-loop stability.

2.3 The Lyapunov Equation Based Design

This section develops an alternative TPF design by adopting a Lyapunov equation based low gain feedback design technique instead of the eigenstructure assignment based low gain feedback design technique presented in Sect. 2.2.

The Lyapunov equation based design shows its advantages over the eigenstructure assignment based design in the following two aspects. TPF laws under the eigenstructure assignment based design have been developed to handle constant input delay, while those under the Lyapunov equation based design have been developed to handle time-varying input delay. Moreover, explicit bounds on the low gain parameter in the case of the Lyapunov equation based design have been established, while no such bounds have been established in the case of the eigenstructure assignment based design. Such bounds can also be established for the eigenstructure assignment based design. However, doing so takes many more steps and may incur more conservativeness, because of the multiple-step nature of the eigenstructure assignment based design.

Consider the following linear system with input delay

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(\phi(t)), \\ y(t) = Cx(t), \end{cases} \tag{2.28}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the state and input, the pair (A, B) and the pair (A, C) are assumed to be stabilizable and detectable, respectively, and $\phi(t) : \mathbb{R}_0^+ \to \mathbb{R}$ denotes the delay function. Here, $\phi(t)$ can be defined in a more standard form

$$\phi(t) = t - d(t), \tag{2.29}$$

where $d(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is the time-varying delay that is continuously differentiable. However, as pointed out in [60], the formalism involving the function $\phi(t)$ turns out to be more convenient because the predictor problem we will consider later requires the inverse function of $\phi(t)$, namely, $\phi^{-1}(t)$. In this section, we will proceed with model (2.28) and assume (2.29) whenever necessary. Some necessary assumptions on $\phi(t)$ is made as follows.

Assumption 2.1 *The function* $\phi : \mathbb{R}_0^+ \to \mathbb{R}$ *is a continuously differentiable, invertible, and exactly known function such that* $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) > 0, t \geq 0$*, and the delay* $d(t)$ *is bounded, namely, there exists a finite, but arbitrarily large, number* $D > 0$

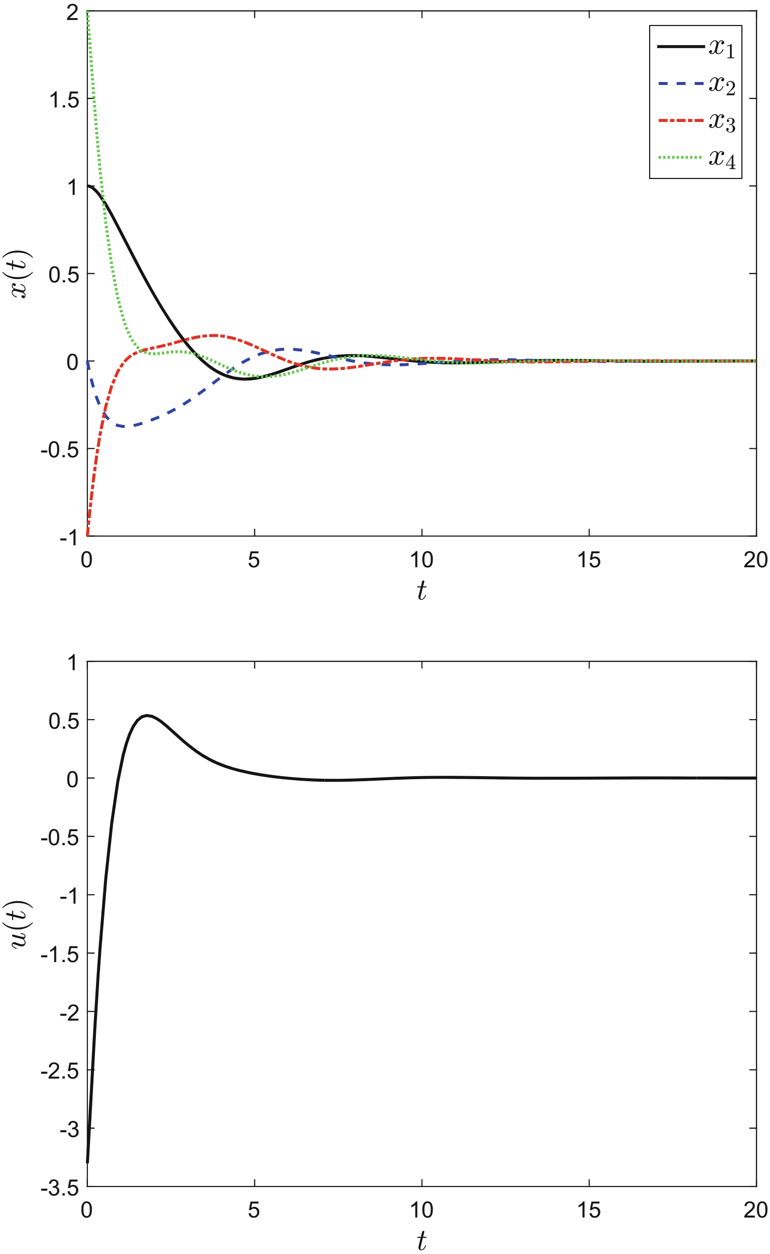


Fig. 2.1 State response and control input under the state feedback TPF law (2.8): $\tau = 0.1$s and $\varepsilon = 0.5$

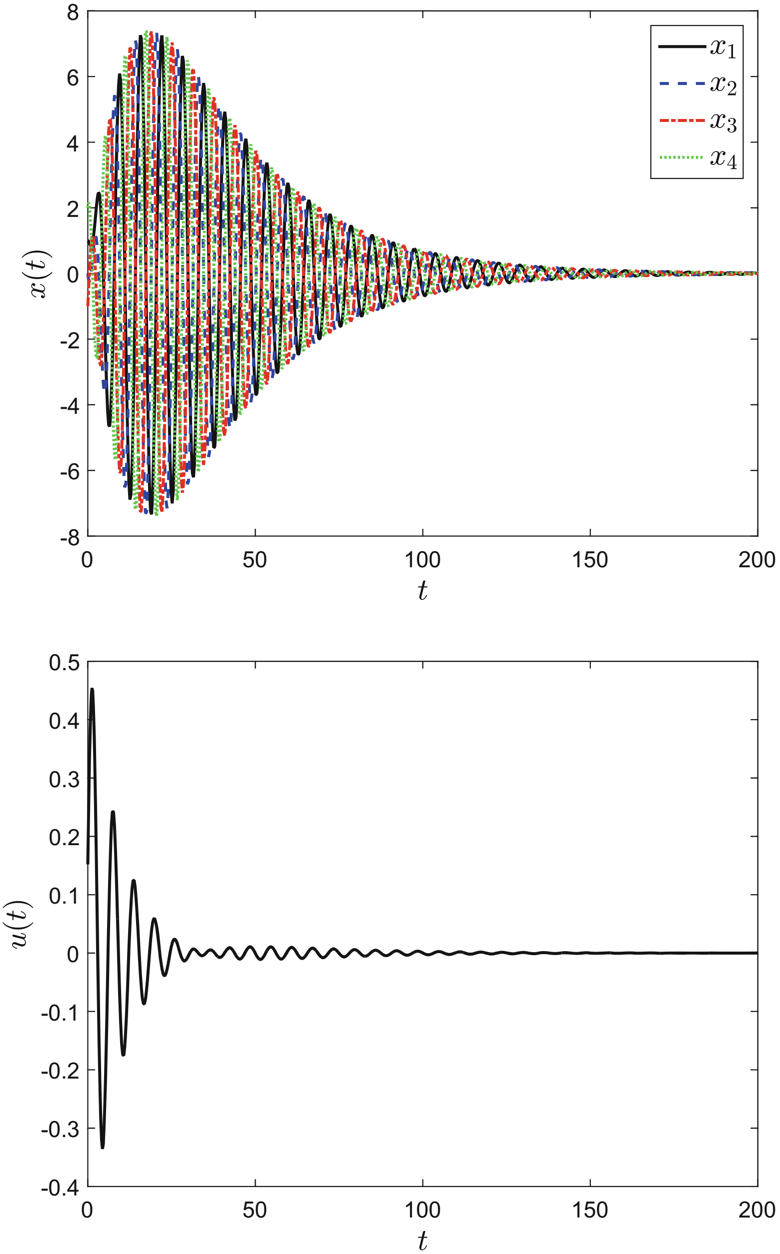


Fig. 2.2 State response and control input under the state feedback TPF law (2.8): $\tau = 2$s and $\varepsilon = 0.05$

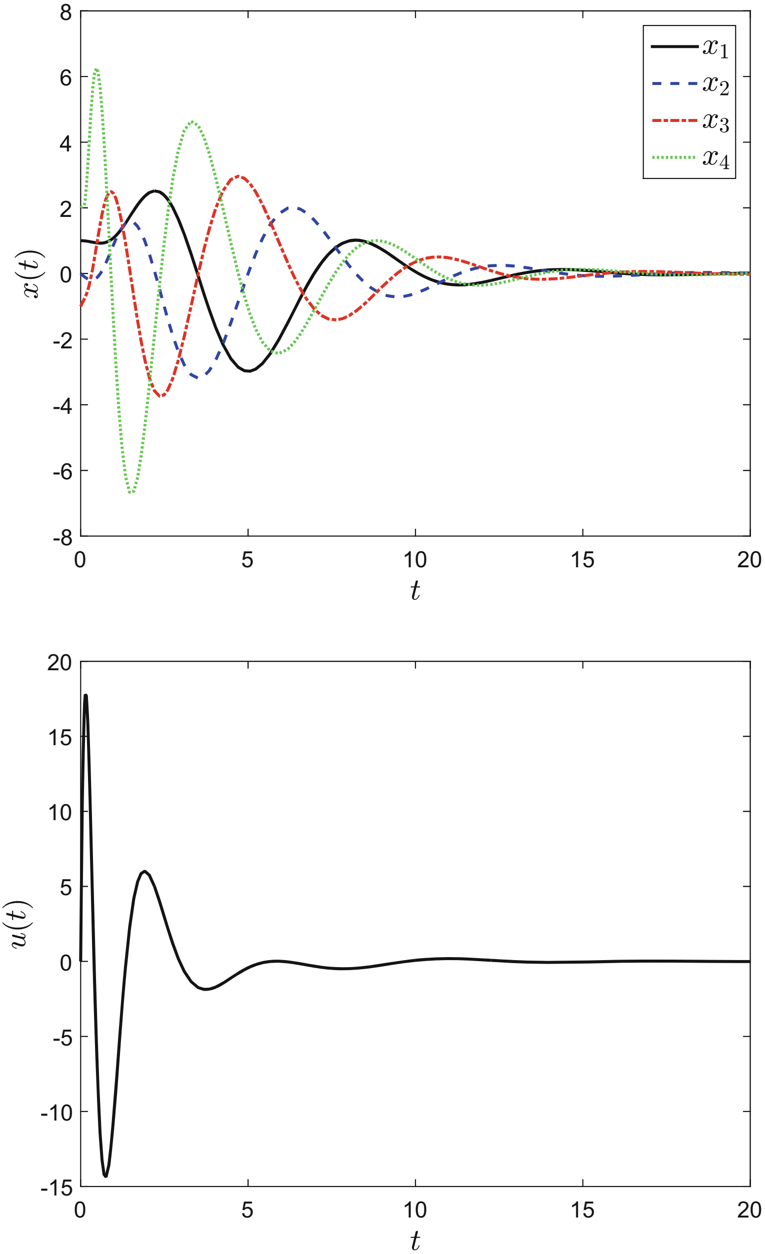


Fig. 2.3 State response and control input under the output feedback TPF law (2.20): $\tau = 0.1$s and $\varepsilon = 0.5$

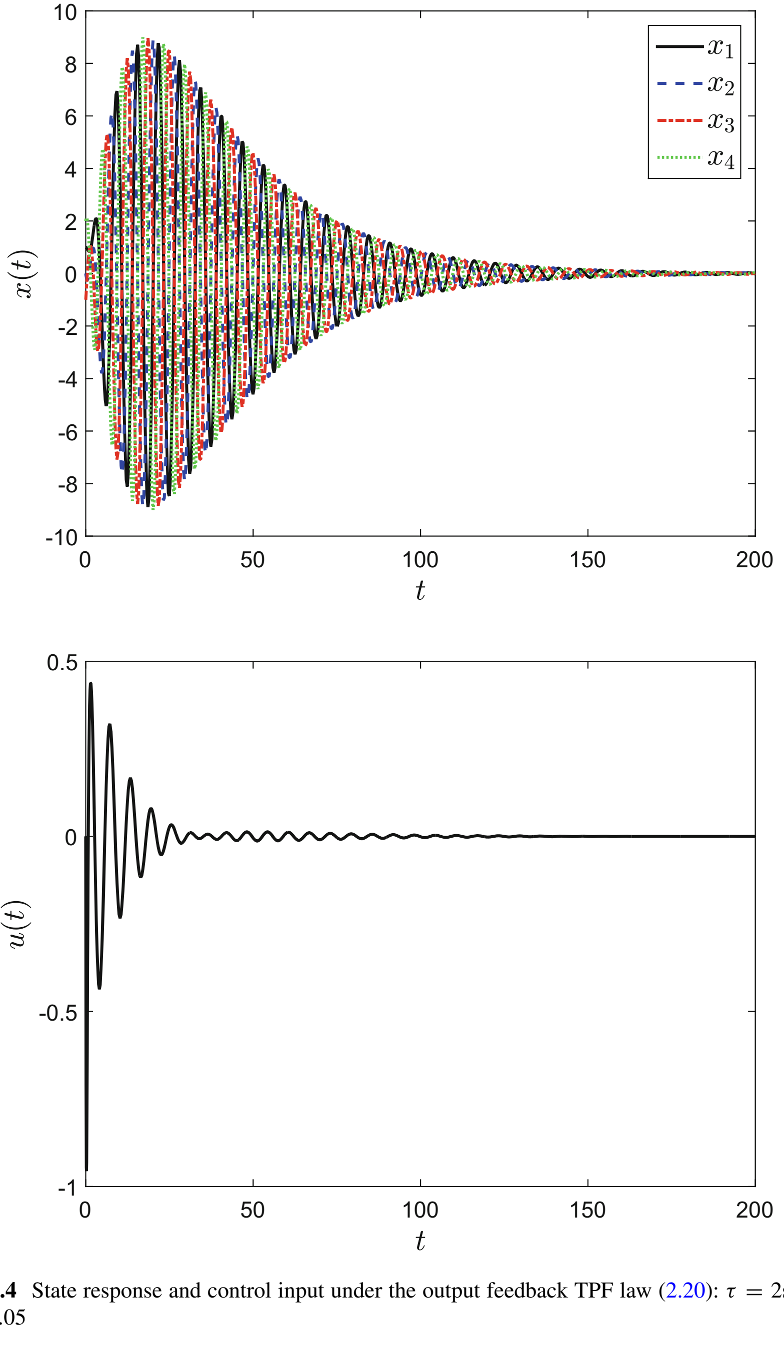


Fig. 2.4 State response and control input under the output feedback TPF law (2.20): $\tau = 2$s and $\varepsilon = 0.05$

such that

$$0 \le d(t) \le D, \quad t \in [0, \infty). \tag{2.30}$$

2.3.1 Low Gain Feedback Design

Consider a controllable pair (A, B), where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We construct a parameterized feedback gain matrix $F(\gamma)$ as follows:

$$F(\gamma) = -B^{\mathrm{T}} P(\gamma), \tag{2.31}$$

where $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation,

$$A^{\mathrm{T}} P(\gamma) + P(\gamma)A - P(\gamma)BB^{\mathrm{T}} P(\gamma) = -\gamma P(\gamma), \quad \gamma > -2\min\{\mathrm{Re}(\lambda(A))\}. \tag{2.32}$$

Note that the inequality on the values of γ is necessary and sufficient for the existence and uniqueness of a positive definite solution to (2.32). To compute $P(\gamma)$, we can first compute the positive definite solution $W(\gamma)$ to the following Lyapunov equation:

$$\left(A + \frac{\gamma}{2} I\right) W(\gamma) + W(\gamma) \left(A + \frac{\gamma}{2}\right)^{\mathrm{T}} = BB^{\mathrm{T}}, \tag{2.33}$$

and then take $P(\gamma) = W^{-1}(\gamma)$. Equation (2.33) results from (2.32) by a substitution of $P(\gamma) = W^{-1}(\gamma)$. We recall the following lemma on the properties of the solution $P(\gamma)$ to (2.32).

Lemma 2.4 ([116, 121, 122]) *Assume that all eigenvalues of A are on the closed right-half plane. The unique positive definite solution $P(\gamma)$ to (2.32) satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} P(\gamma) > 0, \tag{2.34}$$

$$\mathrm{tr}\left(B^{\mathrm{T}} P(\gamma) B\right) = 2\mathrm{tr}(A) + n\gamma, \tag{2.35}$$

$$P(\gamma) BB^{\mathrm{T}} P(\gamma) \le (2\mathrm{tr}(A) + n\gamma) P(\gamma), \tag{2.36}$$

$$\mathrm{e}^{A^{\mathrm{T}} t} P(\gamma) \mathrm{e}^{At} \le \mathrm{e}^{\omega\gamma t} P(\gamma), \tag{2.37}$$

where $\gamma > 0$, $t \ge 0$, and $\omega \ge 2\frac{\mathrm{tr}(A)}{\gamma} + n - 1$. Moreover, the eigenvalues of A and those of $A + BF(\gamma)$ are symmetric with respect to $\mathrm{Re}\{s\} = -\frac{\gamma}{2}$ in the complex plane, i.e.,

$$\lambda(A) + \lambda(A + BF(\gamma)) = -\gamma, \tag{2.38}$$

and $P(\gamma)$ is a rational matrix in γ. When all eigenvalues of A are on the imaginary axis,

$$\lim_{\gamma \to 0^+} P(\gamma) = 0. \tag{2.39}$$

In the case when all eigenvalues of A are on the imaginary axis, property (2.39) in Lemma 2.4 and (2.31) imply that the norm of the feedback gain matrix $F(\gamma)$ goes to zero as γ goes to zero. In this case, such a parametrization of a feedback gain matrix in (2.31) is referred to as the Lyapunov equation based low gain feedback design.

Throughout the book, stability analysis of time delay systems under feedback laws whose feedback gain matrices are constructed by following the Lyapunov equation based low gain feedback design frequently utilizes the following Jensen's Inequality.

Lemma 2.5 ([40]) *For any positive semi-definite matrix $Q \geq 0$, two scalars γ_2 and γ_1 with $\gamma_2 \geq \gamma_1$, and a vector valued function $\omega : [\gamma_1, \gamma_2] \to \mathbb{R}^n$ such that the integrals in the following are well defined, then*

$$\left(\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta)\,\mathrm{d}\beta\right) Q \left(\int_{\gamma_1}^{\gamma_2} \omega(\beta)\,\mathrm{d}\beta\right) \leq (\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta)\, Q\omega(\beta)\,\mathrm{d}\beta. \tag{2.40}$$

2.3.2 Truncated Predictor State Feedback Design

The main idea of predictor feedback is to design the feedback controller

$$u(\phi(t)) = Fx(t), \quad \phi(t) \geq 0, \tag{2.41}$$

such that the closed-loop system consisting of (2.28) and (2.41) is

$$\dot{x}(t) = (A + BF)\,x(t), \quad \phi(t) \geq 0, \tag{2.42}$$

where F is such that $A + BF$ is Hurwitz. The controller (2.41) can also be written as

$$u(t) = Fx\left(\phi^{-1}(t)\right), \quad t \geq 0. \tag{2.43}$$

However, as $\phi^{-1}(t) \geq t, \forall t \geq 0$, the above controller cannot be directly implemented. To overcome this problem, $x\left(\phi^{-1}(t)\right)$ should be predicted based on the current state. By using the system model (2.28) and the variation-of-constants formula [43, 60], it can be obtained that

$$x\left(\phi^{-1}(t)\right) = \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}x(t) + \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)}Bu(\phi(s))\,\mathrm{d}s. \tag{2.44}$$

Substituting the above relation into (2.43) gives the following predicator feedback:

$$u(t) = F\left(\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}x(t) + \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)}Bu(\phi(s))\,\mathrm{d}s\right). \tag{2.45}$$

For easy reference, the first term

$$u_{\mathrm{s}}(t) = F\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}x(t) \tag{2.46}$$

is referred to as the static state feedback term while the second term

$$u_{\mathrm{d}}(t) = F\int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)}Bu(\phi(s))\,\mathrm{d}s \tag{2.47}$$

is referred to as the distributed delay term .

We notice that, since $d(t)$ is bounded, the function $\phi^{-1}(t) - t$, which was referred to as the prediction time, is also bounded. In fact,

$$0 \le \phi^{-1}(t) - t \le D. \tag{2.48}$$

Let the nominal feedback gain F be parameterized as $F = F(\gamma) : \gamma \in (0, 1]$. If $F(\gamma)$ is of order $O(\gamma)$ with respect to γ, namely,

$$\lim_{\gamma\to 0^+} \frac{1}{\gamma}|F(\gamma)| < \infty, \tag{2.49}$$

then the static state feedback term $u_{\mathrm{s}}(t)$ in the predictor feedback law (2.45) is also of order $O(\gamma)$ with respect to γ in view of (2.48). Consequently, control $u(t)$ itself is of order $O(\gamma)$ with respect to γ, namely,

$$\lim_{\gamma\to 0^+} \frac{1}{\gamma}|u(t)| < \infty, \quad t \ge 0. \tag{2.50}$$

As a result, by virtue of (2.48),

$$\lim_{\gamma\to 0^+} \frac{1}{\gamma^2}\left|F(\gamma)\int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)}Bu(\phi(s))\,\mathrm{d}s\right|$$

$$
\begin{aligned}
&= \lim_{\gamma\to 0^+} \left| \left(\frac{1}{\gamma} F(\gamma)\right) \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)} B \left(\frac{u(\phi(s))}{\gamma}\right) \mathrm{d}s \right| \\
&\leq \lim_{\gamma\to 0^+} \frac{1}{\gamma} |F(\gamma)| \int_t^{\phi^{-1}(t)} \left(\left|\mathrm{e}^{A\left(\phi^{-1}(t)-s\right)} B\right| \left(\lim_{\gamma\to 0^+} \frac{1}{\gamma} |u(\phi(s))| \right) \right) \mathrm{d}s, \\
&< \infty,
\end{aligned} \tag{2.51}
$$

namely, the distributed delay term $u_{\mathrm{d}}(t)$ is at least of order $O(\gamma^2)$ with respect to γ. This indicates that, no matter how large the value of D is, the distributed delay term $u_{\mathrm{i}}(t)$ in (2.47) is dominated by the static state feedback term $u_{\mathrm{s}}(t)$ in (2.46) and thus might be safely neglected in $u(t)$ when γ is sufficiently small. As a result, the predictor feedback law (2.45) can be truncated to result in the following TPF law:

$$
u(t) = u_{\mathrm{s}}(t) = F(\gamma)\,\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)} x(t). \tag{2.52}
$$

When the delay in the time delay system (2.28) is constant, say, $\phi(t) = t - d$, where d is a positive constant, then $\phi^{-1}(t) = t + d$, and the TPF (2.52) becomes

$$
u(t) = F(\gamma)\,\mathrm{e}^{Ad} x(t), \tag{2.53}
$$

which coincides with the TPF law (2.8) in Sect. 2.2.2 for linear systems with a constant input delay.

Through the formulation of the TPF law, we have assumed a low gain nature of $F(\gamma)$ with respect to a low gain parameter γ. It is well known that such a feedback gain matrix $F(\gamma)$ can be explicitly constructed for the stabilization of linear systems with all open loop poles on the closed right-half plane. Therefore, we impose the following assumption on system (2.28).

Assumption 2.2 *The matrix pair $(A, B) \in \left(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}\right)$ is controllable with all the eigenvalues of A being on the imaginary axis.*

In what follows, we first prove that the TPF (2.52) can indeed stabilize the time-varying delay system (2.28) under Assumptions 2.1 and 2.2.

Theorem 2.3 *Consider the linear system (2.28) with time-varying delay. Let Assumptions 2.1 and 2.2 be satisfied and $n \geq 2$. Then the TPF*

$$
u(t) = -B^{\mathrm{T}} P(\gamma)\,\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)} x(t), \quad t \geq 0, \tag{2.54}
$$

(globally) asymptotically stabilizes system (2.28), where $u(t)$ is a function that is bounded over $t \in [\phi(0), 0)$, the matrix $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation (2.32) and the parameter γ satisfies

$$
\gamma \in \left(0, \gamma^*\right), \quad \gamma^* = \frac{\delta^*}{D\omega}, \tag{2.55}
$$

with $\omega = n-1$ and δ^* being the unique positive root of the following equation:

$$\frac{\omega^2}{n^3} = \delta \mathrm{e}^{\delta}\left(\mathrm{e}^{\delta}-1\right). \tag{2.56}$$

Proof For simplicity, we denote $F = F(\gamma) = -B^{\mathrm{T}}P(\gamma)$ and $P = P(\gamma)$. Consider an arbitrary initial condition of the time-varying delay system (2.28) as

$$x(t) = \varphi(t), \quad t \in [\phi(0), 0]. \tag{2.57}$$

From the TPF (2.54) we can write

$$u(\phi(t)) = F\mathrm{e}^{A(t-\phi(t))}x(\phi(t)), \ \ t \geq \phi^{-1}(0). \tag{2.58}$$

The closed-loop system can thus be written as

$$\dot{x}(t) = Ax(t) + BF\mathrm{e}^{A(t-\phi(t))}x(\phi(t)), \ \ t \geq \phi^{-1}(0). \tag{2.59}$$

Since $u(t), t \in [\phi(0), 0)$, is a bounded function, the solution in the interval $\left[0, \phi^{-1}(0)\right)$ to the closed-loop system is simply given by

$$\begin{aligned} x(t) &= \mathrm{e}^{At}x(0) + \int_0^t \mathrm{e}^{A(t-s)}Bu(\phi(s))\,\mathrm{d}s \\ &= \mathrm{e}^{At}\varphi(0) + \int_0^t \mathrm{e}^{A(t-s)}Bu(\phi(s))\,\mathrm{d}s, \ \ t \in [0, \phi^{-1}(0)), \end{aligned} \tag{2.60}$$

which is also bounded for any bounded initial condition $\varphi(t), t \in \left[\phi^{-1}(0), 0\right]$. Hence, we need only to consider the asymptotic stability of the closed-loop system with $t \geq \phi^{-1}(0)$, say, asymptotic stability of system (2.59). However, the solution to system (2.59) in the interval $\left[\phi^{-1}(0), \phi^{-1}\left(\phi^{-1}(0)\right)\right)$ is given by

$$x(t) = \mathrm{e}^{A\left(t-\phi^{-1}(0)\right)}x\left(\phi^{-1}(0)\right) + \int_{\phi^{-1}(0)}^t \mathrm{e}^{A(t-s)}BF\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s, \tag{2.61}$$

which is again bounded for any bounded initial condition $\varphi(t), \forall t \in \left[\phi^{-1}(0), 0\right]$, in view of (2.60). Therefore, we need only to consider the asymptotic stability of system (2.59) with $t \geq \phi^{-1}\left(\phi^{-1}(0)\right)$.

With the help of Eq. (2.59) and the variation-of-constants formula [43, 60], we can compute, for all $t \geq \phi^{-1}\left(\phi^{-1}(0)\right)$,

$$x(t) = \mathrm{e}^{A(t-\phi(t))}x(\phi(t)) + \int_{\phi(t)}^t \mathrm{e}^{A(t-s)}BF\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s. \tag{2.62}$$

Then the closed-loop system (2.59) can be rewritten as

$$\begin{aligned}\dot{x}(t) &= (A+BF)x(t) - BF\int_{\phi(t)}^{t} \mathrm{e}^{A(t-s)}BF\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s \\ &= (A+BF)x(t) - BF\lambda(t), \end{aligned} \tag{2.63}$$

where

$$\lambda(t) = \int_{\phi(t)}^{t} \mathrm{e}^{A(t-s)}BF\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s.$$

By virtue of (2.32) and Lemma 2.4, the time derivative of

$$V(x(t)) = x^{\mathrm{T}}(t)Px(t) \tag{2.64}$$

along the trajectory of the system (2.63) satisfies

$$\begin{aligned}\dot{V}(x(t)) &= x^{\mathrm{T}}(t)\left((A+BF)^{\mathrm{T}}P + P(A+BF)\right)x(t) - 2\lambda^{\mathrm{T}}(t)PBFx(t) \\ &= -\gamma V(x(t)) - x^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) - 2\pi^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) \\ &\le -\gamma V(x(t)) - x^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) + x^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) \\ &\quad + \lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\lambda(t) \\ &= -\gamma V(x(t)) + \lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\lambda(t) \\ &\le -\gamma V(x(t)) + n\gamma\lambda^{\mathrm{T}}(t)P\lambda(t). \end{aligned} \tag{2.65}$$

We next simplify the term $\lambda^{\mathrm{T}}(t)P\lambda(t)$. By using the Jensen's inequality in Lemma 2.5, we get

$$\begin{aligned}&\lambda^{\mathrm{T}}(t)P\lambda(t) \\ &= \left(\int_{\phi(t)}^{t} \mathrm{e}^{A(t-s)}BF\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s\right)^{\mathrm{T}} P\left(\int_{\phi(t)}^{t} \mathrm{e}^{A(t-s)}BF\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s\right) \\ &\le (t-\phi(t))\int_{\phi(t)}^{t} x^{\mathrm{T}}(\phi(s))\mathrm{e}^{A^{\mathrm{T}}(s-\phi(s))}PBB^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}(t-s)}P\mathrm{e}^{A(t-s)}BB^{\mathrm{T}}P\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s.\end{aligned}$$

Using Lemma 2.4 again, the above inequality can be continued as

$$\begin{aligned}&\lambda^{\mathrm{T}}(t)P\lambda(t) \\ &\le d(t)\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)}x^{\mathrm{T}}(\phi(s))\mathrm{e}^{A^{\mathrm{T}}(s-\phi(s))}PBB^{\mathrm{T}}PBB^{\mathrm{T}}P\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s\end{aligned}$$

$$\le d(t)\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)}x^{\mathrm{T}}(\phi(s))\,\mathrm{e}^{A^{\mathrm{T}}(s-\phi(s))}PB\mathrm{tr}\left(B^{\mathrm{T}}PB\right)B^{\mathrm{T}}P\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s$$
$$= n\gamma d(t)\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)}x^{\mathrm{T}}(\phi(s))\,\mathrm{e}^{A^{\mathrm{T}}(s-\phi(s))}PBB^{\mathrm{T}}P\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s$$
$$\le (n\gamma)^2 d(t)\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)}x^{\mathrm{T}}(\phi(s))\,\mathrm{e}^{A^{\mathrm{T}}(s-\phi(s))}P\mathrm{e}^{A(s-\phi(s))}x(\phi(s))\,\mathrm{d}s$$
$$\le (n\gamma)^2 d(t)\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)}\mathrm{e}^{\omega\gamma(s-\phi(s))}x^{\mathrm{T}}(\phi(s))\,Px(\phi(s))\,\mathrm{d}s$$
$$= (n\gamma)^2 d(t)\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-\phi(s))}V(x(\phi(s)))\,\mathrm{d}s, \tag{2.66}$$

where $\omega = n - 1$. By using the boundedness assumption on $d(t)$, we further get

$$\begin{aligned}\lambda^{\mathrm{T}}(t)\,P\lambda(t) &\le (n\gamma)^2 D\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-\phi(s))}V(x(\phi(s)))\,\mathrm{d}s\\ &= (n\gamma)^2 D\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s+d(s))}V(x(\phi(s)))\,\mathrm{d}s\\ &\le (n\gamma)^2 D\mathrm{e}^{\omega\gamma D}\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)}V(x(\phi(s)))\,\mathrm{d}s.\end{aligned} \tag{2.67}$$

Substituting (2.67) into (2.65) gives

$$\dot{V}(x(t)) \le -\gamma V(x(t)) + (n\gamma)^3 D\mathrm{e}^{\omega\gamma D}\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)}V(x(\phi(s)))\,\mathrm{d}s. \tag{2.68}$$

Notice that

$$\begin{aligned}\phi(\phi(t)) &= t - d(t) - d(t - d(t))\\ &= t - \tilde{d}(t),\end{aligned} \tag{2.69}$$

where

$$\tilde{d}(t) = d(t - d(t)).$$

Clearly, we have $\left|d'(t)\right| \le 2D$. Hence, under the condition that

$$V(x(t+\theta)) < \eta V(x(t)), \quad \theta \in [-2D, 0], \tag{2.70}$$

where $t \ge \phi^{-1}\left(\phi^{-1}(0)\right)$ and $\eta > 1$ is any given scalar, inequality (2.68) can be continued as

$$\dot{V}\left(x\left(t\right)\right) \leq -\gamma \chi\left(\gamma\right) V\left(x\left(t\right)\right), \tag{2.71}$$

where

$$\chi\left(\gamma\right) \triangleq 1 - \mathrm{e}^{\omega\gamma D}\eta n^3 \gamma^2 D \int_{t-D}^{t} \mathrm{e}^{\omega\gamma(t-s)}\mathrm{d}s \tag{2.72}$$

$$= 1 - \eta n^3 \gamma^2 D \mathrm{e}^{\omega\gamma D} \frac{1}{\omega\gamma}\left(\mathrm{e}^{\omega\gamma D} - 1\right)$$

$$= \frac{\eta n^3}{\omega^2}\left(\frac{\omega^2}{\eta n^3} - \eta\delta\mathrm{e}^{\delta}\left(\mathrm{e}^{\delta} - 1\right)\right), \tag{2.73}$$

in which $\delta = \omega\gamma D$. Notice that

$$f\left(\delta\right) = \delta\mathrm{e}^{\delta}\left(\mathrm{e}^{\delta} - 1\right), \quad \delta \geq 0, \tag{2.74}$$

is a strictly increasing function. Therefore we deduce from Eq. (2.56) that

$$\frac{\omega^2}{n^3} - \delta\mathrm{e}^{\delta}\left(\mathrm{e}^{\delta} - 1\right) > 0, \quad \delta \in \left(0, \delta^*\right), \tag{2.75}$$

and consequently, there exists a number $\eta > 1$ and a sufficiently small number $\varepsilon > 0$ such that

$$\frac{\omega^2}{\eta n^3} - \eta\delta\mathrm{e}^{\delta}\left(\mathrm{e}^{\delta} - 1\right) > \varepsilon, \quad \delta \in \left(0, \delta^*\right). \tag{2.76}$$

With inequality (2.76), we obtain from (2.71) and (2.73) that

$$\dot{V}\left(x\left(t\right)\right) \leq -\gamma\frac{\eta n^3 \varepsilon}{\omega^2} V\left(x\left(t\right)\right), \quad \gamma \in \left(0, \frac{\delta^*}{D\omega}\right). \tag{2.77}$$

The closed-loop system (2.59) is thus asymptotically stable by the Razumikhin Stability Theorem (Theorem 1.3). The proof is complete. □

Remark 2.1 In Theorem (2.3) we have assumed that $n \geq 2$. If $n = 1$, say, the system (2.28) is of the form $\dot{x}\left(t\right) = -u\left(\phi\left(t\right)\right)$, then we get from (2.72) that $\chi\left(\gamma\right) = 1 - \eta\gamma^2 D^2$. Consequently, inequality (2.71) reads

$$\dot{V}\left(x\left(t\right)\right) \leq -\gamma\left(1 - \eta\gamma^2 D^2\right) V\left(x\left(t\right)\right). \tag{2.78}$$

Therefore the asymptotic stability of the closed-loop system is guaranteed provided that $\gamma \in \left(0, \frac{1}{D}\right)$. □

Regarding the convergence rate of the closed-loop system, we have the following result.

Proposition 2.2 *Let γ satisfy (2.55) and $\alpha = \alpha(\delta)$ be defined as*

$$\alpha = \frac{1}{D}\frac{n^3}{\omega^3}\delta\left(\frac{\omega^2}{n^3} - \delta e^{\delta}\left(e^{\delta} - 1\right)\right), \tag{2.79}$$

where $\delta = \omega\gamma D$ and $\omega = n - 1$. Then there exists a constant $c = c(\gamma) > 0$ such that, for any $t \geq 0$,

$$|x(t)| \leq c\left(\sup_{\theta\in[\phi(0),0]}\{|x(\theta)|\} + \sup_{\theta\in[\phi(0),0]}\{|u(\theta)|\}\right)e^{-\frac{\alpha}{2}t}. \tag{2.80}$$

Proof Since γ satisfies (2.55), it follows from (2.56) that

$$\frac{\omega^2}{n^3} - \delta e^{\delta}\left(e^{\delta} - 1\right) > 0, \quad \gamma \in \left(0, \gamma^*\right). \tag{2.81}$$

We next prove (2.80). Define a new state vector $\xi(t)$ as

$$\xi(t) = x(t)\, e^{\frac{\alpha}{2}t}, \quad t \geq \phi(0), \tag{2.82}$$

by which the closed-loop system (2.59) can be transformed as

$$\begin{aligned}\dot{\xi}(t) &= \dot{x}(t)\, e^{\frac{\alpha}{2}t} + \frac{\alpha}{2}x(t)\, e^{\frac{\alpha}{2}t} \\ &= \left(Ax(t) + BFe^{A(t-\phi(t))}x(\phi(t))\right)e^{\frac{\alpha}{2}t} + \frac{\alpha}{2}\xi(t) \\ &= \left(A + \frac{\alpha}{2}I_n\right)\xi(t) + BFe^{A(t-\phi(t))}\xi(\phi(t))\, e^{\frac{\alpha}{2}(t-\phi(t))} \\ &= \left(A + \frac{\alpha}{2}I_n\right)\xi(t) + BFe^{\left(A+\frac{\alpha}{2}I_n\right)(t-\phi(t))}\xi(\phi(t)),\end{aligned} \tag{2.83}$$

where $t \geq \phi^{-1}(0)$. Moreover, by definition of $V(x(t))$ in (2.64), we can compute

$$V(\xi(t)) = e^{\alpha t}V(x(t)). \tag{2.84}$$

Let $t_d > 0$ be a prescribed number that is sufficiently small. It follows that

$$V(x(t)) = e^{-\alpha t}V(\xi(t)) \leq e^{-\alpha t_d}V(\xi(t)), \quad t \geq t_d. \tag{2.85}$$

Then, in view of (2.68), (2.84), and (2.85), the time derivative of $V(\xi(t))$ along the trajectory of system (2.83) satisfies

$$\begin{aligned}
\dot{V}(\xi(t)) &= \mathrm{e}^{\alpha t}\dot{V}(x(t)) + \alpha \mathrm{e}^{\alpha t} V(x(t)) \\
&\leq \alpha \mathrm{e}^{\alpha t} V(x(t)) \\
&\quad + \mathrm{e}^{\alpha t}\left(-\gamma V(x(t)) + (n\gamma)^3 D \mathrm{e}^{\omega\gamma D}\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)} V(x(\phi(s)))\,\mathrm{d}s\right) \\
&= (\alpha-\gamma) V(\xi(t)) + (n\gamma)^3 D \mathrm{e}^{\omega\gamma D}\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)} V(x(\phi(s)))\,\mathrm{d}s \\
&\leq (\alpha-\gamma) V(\xi(t)) \\
&\quad + (n\gamma)^3 D \mathrm{e}^{\omega\gamma D}\mathrm{e}^{-\alpha t_d}\int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)} V(\xi(\phi(s)))\,\mathrm{d}s \\
&\leq (\alpha-\gamma) V(\xi(t)) \\
&\quad + (n\gamma)^3 D \mathrm{e}^{\omega\gamma D}\mathrm{e}^{-\alpha t_d}\int_{t-D}^{t} \mathrm{e}^{\omega\gamma(t-s)} V(\xi(\phi(s)))\,\mathrm{d}s,
\end{aligned} \tag{2.86}$$

where $t \geq \phi^{-1}\left(\phi^{-1}(t_d)\right)$. Therefore, under the condition that

$$V(\xi(t+\theta)) < \eta V(\xi(t)), \quad \theta \in [-2D, 0], \tag{2.87}$$

where $t \geq \phi^{-1}\left(\phi^{-1}(t_d)\right)$ and $\eta > 1$ is a given scalar to be specified later, we can obtain, for any $t \geq \phi^{-1}\left(\phi^{-1}(t_d)\right)$,

$$\begin{aligned}
\dot{V}(\xi(t)) &\leq \left((\alpha-\gamma) + \eta (n\gamma)^3 D \mathrm{e}^{\omega\gamma D}\mathrm{e}^{-\alpha t_d}\int_{t-D}^{t} \mathrm{e}^{\omega\gamma(t-s)}\mathrm{d}s\right) V(\xi(t)) \\
&= \left(\alpha - \frac{1}{D}\frac{n^3}{\omega^3}\delta\left(\frac{\omega^2}{n^3} - \eta \mathrm{e}^{-\alpha t_d}\delta \mathrm{e}^{\delta}\left(\mathrm{e}^{\delta}-1\right)\right)\right) V(\xi(t)).
\end{aligned} \tag{2.88}$$

Notice that, in view of (2.79) and (2.81),

$$\begin{aligned}
&\alpha - \frac{1}{D}\frac{n^3}{\omega^3}\delta\left(\frac{\omega^2}{n^3} - \eta \mathrm{e}^{-\alpha t_d}\delta \mathrm{e}^{\delta}\left(\mathrm{e}^{\delta}-1\right)\right) \\
&= \alpha - \frac{1}{D}\frac{n^3}{\omega^3}\delta\left(\frac{\omega^2}{n^3} - \mathrm{e}^{\alpha t_d}(1-\varepsilon)\mathrm{e}^{-\alpha t_d}\delta \mathrm{e}^{\delta}\left(\mathrm{e}^{\delta}-1\right)\right) \\
&= -\frac{1}{D}\frac{n^3}{\omega^3}\delta \mathrm{e}^{\delta}\left(\mathrm{e}^{\delta}-1\right)\varepsilon,
\end{aligned} \tag{2.89}$$

where ε is a sufficiently small positive number such that

$$\eta = \mathrm{e}^{\alpha t_d}(1-\varepsilon) > 1. \tag{2.90}$$

We then get from (2.88) that, for all $t \geq \phi^{-1}\left(\phi^{-1}(t_d)\right)$,

$$\dot{V}(\xi(t)) \leq -\frac{1}{D}\frac{n^3}{\omega^3}\delta \mathrm{e}^{\delta}\left(\mathrm{e}^{\delta}-1\right)\varepsilon V(\xi(t)). \tag{2.91}$$

Since $V(\cdot)$ is a quadratic function, by the Razumihkin Stability Theorem (Theorem 1.3), we conclude from (2.91) that the ξ-system (2.83) is asymptotically stable, namely, there exists a sufficiently small positive number ϵ and a positive number k such that, for all $t \geq t_0 = \phi^{-1}\left(\phi^{-1}(t_d)\right)$,

$$|\xi(t)| \leq k \sup_{t\in[\phi(0),t_0]}\{|\xi(t)|\}\,\mathrm{e}^{-\epsilon(t-t_0)}, \tag{2.92}$$

which, according to (2.82), implies that, for all $t \geq t_0 = \phi^{-1}\left(\phi^{-1}(t_d)\right)$,

$$\begin{aligned} |x(t)| &\leq k \sup_{\tau\in[\phi(0),t_0]}\{|\xi(\tau)|\}\,\mathrm{e}^{-\epsilon(t-t_0)}\mathrm{e}^{-\frac{\alpha}{2}(t-t_0)} \\ &\leq k' \sup_{\tau\in[\phi(0),t_0]}\{|x(\tau)|\}\,\mathrm{e}^{-\frac{\alpha}{2}(t-t_0)}, \end{aligned} \tag{2.93}$$

where $k' = k\mathrm{e}^{\frac{\alpha}{2}t_0}$. Finally, it follows from (2.60), (2.61), and (2.62) that there exists a positive constant $c = c(\gamma)$ such that

$$\sup_{\tau\in[\phi(0),t_0]}\{|x(\tau)|\} \leq c\left(\sup_{\theta\in[\phi(0),0]}\{|x(\theta)|\} + \sup_{\theta\in[\phi(0),0]}\{|u(\theta)|\}\right). \tag{2.94}$$

Combining (2.93) and (2.94) gives the inequality (2.80). The proof is completed. □

Remark 2.2 It follows from (2.56) that $\alpha(\delta^*) = \alpha(0) = 0$, which indicates that there exists at least one number $\delta^\dagger$ such that $\alpha(\delta)$ is minimized, namely, the estimation of the convergence rate is maximized. Letting $\frac{\mathrm{d}}{\mathrm{d}\delta}\alpha(\delta) = 0$ to give

$$\frac{\omega^2}{n^3} = \delta \mathrm{e}^{\delta}\left(2\left(\mathrm{e}^{\delta}-1\right)+\delta\left(2\mathrm{e}^{\delta}-1\right)\right), \tag{2.95}$$

which has a unique solution $\delta^\dagger \in (0, \delta^*)$. Then $\alpha(\delta)$ is minimized at $\delta = \delta^\dagger$. □

Remark 2.2 implies that, for a given delay bound D, a larger value of γ (or δ) not necessarily leads to a higher convergence rate of the closed-loop system. In our design, we may want to choose the optimal value of γ (denoted by γ_{opt}) such that the convergence rate of the closed-loop system is maximized. Since the convergence rate given in Proposition 2.2 is only an estimation, the optimal value γ_{opt} may not be simply determined by $\gamma_{\mathrm{opt}} = \frac{\delta^\dagger}{\omega D}$ according to Remark 2.2. We notice that determining γ_{opt} is generally a hard problem if the delay $d(t)$ is time-varying. However, if the delay is a constant, we can propose the following numerical method to compute γ_{opt}.

When $d(t) = d$ is a constant, the characteristic quasi-polynomial of the closed-loop system (2.59) is given by

$$\beta(s, \gamma) = \det\left(sI_n - A - BF(\gamma)\,\mathrm{e}^{Ad}\mathrm{e}^{-sd}\right). \tag{2.96}$$

It is well known that, for any fixed γ and any prescribed number l, equation $\beta(s, \gamma) = 0$ has only a finite number of zeros on $\{s : \mathrm{Re}\{s\} \geq l\}$ [43]. The right-most zeros of equation $\beta(s, \gamma) = 0$ can be computed by the efficient software package DDE-BIFTOOL [27].

For a given $\gamma \geq 0$, denote the real part of the right-most roots of the characteristic quasi-polynomial equation $\beta(s, \gamma) = 0$ by

$$\lambda_{\max}(\gamma) = \max\{\mathrm{Re}\{s\} : \beta(s, \gamma) = 0\}. \tag{2.97}$$

It follows that the closed-loop system (2.59) is asymptotically stable if and only if $\lambda_{\max}(\gamma) < 0$. Moreover, it is well known that the convergence rate of the closed-loop system (2.59) is totally determined by $\lambda_{\max}(\gamma)$, namely, the smaller the value of $\lambda_{\max}(\gamma)$, the faster the state converges to the origin [43].

According to Theorem 2.3, there is an interval $\mathscr{I} \subset (0, \infty)$ such that $\lambda_{\max}(\gamma) < 0, \forall \gamma \in \mathscr{I}$. Let $\gamma_{\sup} = \sup_{\gamma \in \mathscr{I}}\{\gamma\}$. By definition, we have $\lambda_{\max}(0) = \lambda_{\max}(\gamma_{\sup}) = 0$, namely, the closed-loop system (2.59) is marginally unstable with $\gamma = 0$ and $\gamma = \gamma_{\sup}$. Moreover, we have $\lambda_{\max}(\gamma) < 0, \gamma \in (0, \gamma_{\sup}) = \mathscr{I}$, i.e., the closed-loop system is asymptotically stable with $\gamma \in (0, \gamma_{\sup}) = \mathscr{I}$. Therefore, by continuity of zeros of quasi-polynomials [88], there exists a value $\gamma_{\mathrm{opt}} \in (0, \gamma_{\sup}) = \mathscr{I}$ such that $\lambda_{\max}(\gamma)$ is minimized with $\gamma = \gamma_{\mathrm{opt}}$. Denote such minimal value by $\lambda_{\max}^{\min}$, namely,

$$\lambda_{\max}^{\min} = \min_{\gamma \in \mathscr{I}}\{\lambda_{\max}(\gamma)\} = \min_{\gamma \in \mathscr{I}}\left\{\max\left\{\mathrm{Re}\{s\} : \beta(s, \gamma) = 0\right\}\right\}. \tag{2.98}$$

Then $\lambda_{\max}^{\min}$ is the maximal convergence rate that the TPF (2.54) can achieve.

We can compute the function $\lambda_{\max}(\gamma)$ defined in (2.97) via the efficient software package DDE-BIFTOOL [27] by choosing

$$\gamma = k\Delta_\gamma, \quad k = 0, 1, \cdots, N, \tag{2.99}$$

where Δ_γ is a sufficiently small number denoting the step size, and N is chosen as the minimal number such that $\lambda_{\max}(N\Delta_\gamma) = 0$. From the computational result of $\lambda_{\max}(\gamma)$, the optimal value γ_{opt} and the maximal convergence rate $\lambda_{\max}^{\min}$ can be obtained accordingly.

2.3.3 Truncated Predictor Output Feedback Design

In this subsection, we discuss the output feedback stabilization of system (2.28) by a Lyapunov equation based TPF law. Without loss of generality, we assume that the stabilizable pair (A, B) has the following block structure:

$$A = \begin{bmatrix} A_{\mathrm{L}} & 0 \\ 0 & A_{\mathrm{R}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{L}} \\ B_{\mathrm{R}} \end{bmatrix}, \tag{2.100}$$

where $A_{\mathrm{L}} \in \mathbb{R}^{n_{\mathrm{L}} \times n_{\mathrm{L}}}$ is Hurwitz, all eigenvalues of $A_{\mathrm{R}} \in \mathbb{R}^{n_{\mathrm{R}} \times n_{\mathrm{R}}}$ are in the closed right-half plane, $n_{\mathrm{L}} + n_{\mathrm{R}} = n$, and the dimensions of B_{L} and B_{R} correspond to those of A_{L} and A_{R}, respectively. Based on this structure, the pair $(A_{\mathrm{R}}, B_{\mathrm{R}})$ is controllable.

We construct the following observer based controller:

$$\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(\phi(t)) - L\big(y(t) - C\widehat{x}(t)\big), \\ u(t) = -B^{\mathrm{T}}\mathcal{P}(\gamma)\,\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}\widehat{x}(t), \end{cases} \tag{2.101}$$

where

$$\mathcal{P}(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & P(\gamma) \end{bmatrix}, \tag{2.102}$$

$P(\gamma)$ is the unique positive definite solution to (2.32) with (A, B) replaced by $(A_{\mathrm{R}}, B_{\mathrm{R}})$, and $L \in \mathbb{R}^{n\times p}$ is such that $A + LC$ is Hurwitz. The controllability of the pair $(A_{\mathrm{R}}, B_{\mathrm{R}})$ guarantees the existence and uniqueness of such a $P(\gamma)$, and the detectability of the pair (A, C) guarantees the existence of such an L.

By defining an error signal $e(t) = x(t) - \hat{x}(t)$, we obtain the dynamic of $e(t)$ as

$$\dot{e}(t) = (A + LC)e(t). \tag{2.103}$$

A combination of the dynamic of the error signal $e(t)$ with the open loop system (2.28) gives the following closed-loop system:

$$\begin{cases} \dot{x}_{\mathrm{L}}(t) = A_{\mathrm{L}}x_{\mathrm{L}}(t) - B_{\mathrm{L}}B_{\mathrm{R}}^{\mathrm{T}}P\mathrm{e}^{A_{\mathrm{R}}d(t)}(x_{\mathrm{R}}(\phi(t)) - e_{\mathrm{R}}(\phi(t))), \\ \dot{x}_{\mathrm{R}}(t) = A_{\mathrm{R}}x_{\mathrm{R}}(t) - B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P\mathrm{e}^{A_{\mathrm{R}}d(t)}(x_{\mathrm{R}}(\phi(t)) - e_{\mathrm{R}}(\phi(t))), \\ \dot{e}(t) = (A + LC)e(t), \end{cases} \tag{2.104}$$

where $x = \begin{bmatrix} x_{\mathrm{L}} & x_{\mathrm{R}} \end{bmatrix}^{\mathrm{T}}$ and $e = \begin{bmatrix} e_{\mathrm{L}} & e_{\mathrm{R}} \end{bmatrix}^{\mathrm{T}}$ are respectively the decompositions of the state and the error signal corresponding to the structure of the (A, B) pair. Based on the closed-loop system and the fact that A_{L} is Hurwitz, we see that the asymptotic stability of the second and third subsystems of the closed-loop system (2.104) implies that of the overall closed-loop system. Therefore, in the following theorem, we establish the stability of

$$\begin{cases} \dot{x}_{\mathrm{R}}(t) = A_{\mathrm{R}}x_{\mathrm{R}}(t) - B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P\mathrm{e}^{A_{\mathrm{R}}d(t)}(x_{\mathrm{R}}(\phi(t)) - e_{\mathrm{R}}(\phi(t))), \\ \dot{e}(t) = (A + LC)e(t). \end{cases} \tag{2.105}$$

Theorem 2.4 *If all eigenvalues of A are on the closed left-half plane, then there exists $\gamma^\star > 0$ such that system (2.105) is asymptotically stable for any $\gamma \in (0, \gamma^*]$.*

Proof We rewrite the dynamic of x_{R} as

$$\dot{x}_{\mathrm{R}}(t) = A_{\mathrm{RC}}x_{\mathrm{R}}(t) + B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P\lambda_{\mathrm{R}}(t) + B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P\mathrm{e}^{A_{\mathrm{R}}d(t)}e_{\mathrm{R}}(\phi(t)), \tag{2.106}$$

where $A_{\mathrm{RC}} = A_{\mathrm{R}} - B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P$ and $\lambda_{\mathrm{R}}(t) = x_{\mathrm{R}}(t) - \mathrm{e}^{A_{\mathrm{R}}d(t)}x_{\mathrm{R}}(\phi(t))$. By applying the variation-of-constants formula to (2.106), we obtain

$$\lambda_{\mathrm{R}}(t) = -\int_{t-d(t)}^{t} \mathrm{e}^{A_{\mathrm{R}}(t-s)}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P\mathrm{e}^{A_{\mathrm{R}}d(s)}(x_{\mathrm{R}}(\phi(s)) - e_{\mathrm{R}}(\phi(s)))\mathrm{d}s. \tag{2.107}$$

We now define a Lyapunov function

$$V(x_{\mathrm{R}}(t), e(t)) = x_{\mathrm{R}}^{\mathrm{T}}Px_{\mathrm{R}} + e^{\mathrm{T}}Re, \tag{2.108}$$

where R is the positive definite solution to the Lyapunov equation

$$(A + LC)^{\mathrm{T}}R + R(A + LC) = -I. \tag{2.109}$$

Then, the time derivative of the Lyapunov function along the trajectory of system (2.105) is

$$\begin{aligned} \dot{V}(x_{\mathrm{R}}(t), e(t)) = {} & x_{\mathrm{R}}^{\mathrm{T}}(PA_{\mathrm{RC}} + A_{\mathrm{RC}}^{\mathrm{T}}P)x_{\mathrm{R}} + 2x_{\mathrm{R}}^{\mathrm{T}}PB_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P\lambda_{\mathrm{R}} \\ & + 2x_{\mathrm{R}}^{\mathrm{T}}PB_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P\mathrm{e}^{A_{\mathrm{R}}d(t)}e_{\mathrm{R}}(\phi(t)) \\ & + e^{\mathrm{T}}(R(A + LC) + (A + LC)^{\mathrm{T}}R)e. \end{aligned} \tag{2.110}$$

In (2.110), we have suppressed all the time variable t for notational brevity, and we will continue doing so in the rest of this proof as long as no ambiguity will occur. We evaluate (2.110) by using the parametric algebraic Riccati equation (2.32), Young's Inequality, Lemma 2.4, (2.109), and (2.103),

$$\begin{aligned} \dot{V} \leq & -\gamma x_{\mathrm{R}}^{\mathrm{T}}Px_{\mathrm{R}} + 2n_{\mathrm{R}}\gamma\lambda_{\mathrm{R}}^{\mathrm{T}}P\lambda_{\mathrm{R}} \\ & + 2n_{\mathrm{R}}\gamma\mathrm{e}^{\omega_{\mathrm{R}}\gamma d(t)}e_{\mathrm{R}}^{\mathrm{T}}\mathrm{e}^{-(A+LC)^{\mathrm{T}}d(t)}P\mathrm{e}^{-(A+LC)d(t)}e_{\mathrm{R}} - e^{\mathrm{T}}e, \end{aligned} \tag{2.111}$$

where $\omega_{\mathrm{R}} \geq n_{\mathrm{R}} - 1$ is any constant.

The term $\lambda_{\mathrm{R}}^{\mathrm{T}}P\lambda_{\mathrm{R}}$ can be evaluated as

$$\begin{aligned}
\lambda_{\mathrm{R}}^{\mathrm{T}} P \lambda_{\mathrm{R}} \leq{} & d(t) \int_{t-d(t)}^{t} (x_{\mathrm{R}}(\phi(s)) - e_{\mathrm{R}}(\phi(s)))^{\mathrm{T}} \mathrm{e}^{A_{\mathrm{R}}^{\mathrm{T}} d(s)} P B_{\mathrm{R}}^{\mathrm{T}} \mathrm{e}^{A_{\mathrm{R}}^{\mathrm{T}}(t-s)} P \mathrm{e}^{A_{\mathrm{R}}(t-s)} B_{\mathrm{R}} \\
& \times B_{\mathrm{R}}^{\mathrm{T}} P \mathrm{e}^{A_{\mathrm{R}} d(s)} (x_{\mathrm{R}}(\phi(s)) - e_{\mathrm{R}}(\phi(s))) \mathrm{d}s \\
\leq{} & d(t) \int_{t-d(t)}^{t} \mathrm{e}^{\omega_{\mathrm{R}} \gamma (t-s)} (n_{\mathrm{R}} \gamma)^2 \mathrm{e}^{\omega_{\mathrm{R}} \gamma d(s)} \\
& \times (x_{\mathrm{R}}(\phi(s)) - e_{\mathrm{R}}(\phi(s)))^{\mathrm{T}} P (x_{\mathrm{R}}(\phi(s)) - e_{\mathrm{R}}(\phi(s))) \mathrm{d}s \\
\leq{} & 2D \mathrm{e}^{2\omega_{\mathrm{R}} \gamma D} (n_{\mathrm{R}} \gamma)^2 \\
& \times \int_{t-D}^{t} \left(x_{\mathrm{R}}^{\mathrm{T}}(\phi(s)) P x_{\mathrm{R}}(\phi(s)) + e^{\mathrm{T}}(\phi(s)) \mathcal{P} e(\phi(s)) \right) \mathrm{d}s,
\end{aligned} \tag{2.112}$$

based on which (2.111) can be continued as

$$\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq{} & -\gamma x_{\mathrm{R}}^{\mathrm{T}} P x_{\mathrm{R}} + 4D(n_{\mathrm{R}}\gamma)^3 \mathrm{e}^{2\omega_{\mathrm{R}}\gamma D} \int_{t-D}^{t} \bigg(x_{\mathrm{R}}^{\mathrm{T}}(\phi(s)) P x_{\mathrm{R}}(\phi(s)) \\
& + e^{\mathrm{T}}(\phi(s)) \mathcal{P} e(\phi(s)) \bigg) \mathrm{d}s + 2 n_{\mathrm{R}} \gamma \mathrm{e}^{\omega_{\mathrm{R}} \gamma d(t)} e_{\mathrm{R}}^{\mathrm{T}} \\
& \times e^{-(A+LC)^{\mathrm{T}} d(t)} P \mathrm{e}^{-(A+LC) d(t)} e_{\mathrm{R}} - e^{\mathrm{T}} e.
\end{aligned} \tag{2.113}$$

The structure of $\mathcal{P}$, combined with the low gain nature of $P(\gamma)$ as seen in Lemma 2.4, implies that there exists $\gamma_1^\star > 0$ such that for any $\gamma \in (0, \gamma_1^\star]$,

$$\mathcal{P}(\gamma) \leq R, \tag{2.114}$$

which further implies that

$$\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq{} & -\gamma x_{\mathrm{R}}^{\mathrm{T}} P x_{\mathrm{R}} + 4D(n_{\mathrm{R}}\gamma)^3 \mathrm{e}^{2\omega_{\mathrm{R}}\gamma D} \int_{t-D}^{t} \bigg(x_{\mathrm{R}}^{\mathrm{T}}(\phi(s)) P x_{\mathrm{R}}(\phi(s)) \\
& + e^{\mathrm{T}}(\phi(s)) R e(\phi(s)) \bigg) \mathrm{d}s + e^{\mathrm{T}} \bigg(-I + 2 n_{\mathrm{R}} \gamma \mathrm{e}^{\omega_{\mathrm{R}} \gamma D} \\
& \times \mathrm{e}^{-(A+LC)^{\mathrm{T}} d(t)} \mathcal{P} \mathrm{e}^{-(A+LC) d(t)} \bigg) e.
\end{aligned} \tag{2.115}$$

Note that for any $D \in \mathbb{R}_0^+$, there exists $\gamma_2^\star \in (0, \gamma_1^\star]$ such that, for any $\gamma \in (0, \gamma_2^\star]$ and any $d(t) \leq D$,

$$-I + 2 n_{\mathrm{R}} \gamma \mathrm{e}^{\omega_{\mathrm{R}} \gamma D} \mathrm{e}^{-(A+LC)^{\mathrm{T}} d(t)} \mathcal{P} \mathrm{e}^{-(A+LC) d(t)} \leq -\gamma R. \tag{2.116}$$

Then, for any $\gamma \in (0, \gamma_2^\star]$,

$$\dot{V}(x_{\text{R}}(t), e(t)) \leq -\gamma V + 4D(n_{\text{R}}\gamma)^3 \mathrm{e}^{2\omega_{\text{R}}\gamma D} \times \int_{t-D}^{t} \left(x_{\text{R}}^{\text{T}}(\phi(s))Px_{\text{R}}(\phi(s)) + e^{\text{T}}(\phi(s))Re(\phi(s))\right) \mathrm{d}s. \tag{2.117}$$

Assume that $V(x_{\text{R}}(t+\theta), e(t+\theta)) < \eta V(x_{\text{R}}(t), e(t))$, $\theta \in [-2D, 0]$, for some $\eta > 1$. Then, we have

$$\dot{V}(x_{\text{R}}(t), e(t)) \leq -\gamma V(x_{\text{R}}(t), e(t)) \left(1 - 4D^2 n_{\text{R}}^3 \gamma^2 \mathrm{e}^{2\omega_{\text{R}}\gamma D}\right). \tag{2.118}$$

It is clear that if

$$\gamma < \min\left\{\gamma_2^\star, \gamma_3^\star\right\} \triangleq \gamma^\star, \tag{2.119}$$

where $\gamma_3^\star$ is the unique positive solution to

$$4D^2 n_{\text{R}}^3 \gamma^2 \mathrm{e}^{2\omega_{\text{R}}\gamma D} = 1, \tag{2.120}$$

then by the Razumikhin Stability Theorem (Theorem 1.3), system (2.105) is asymptotically stable. □

An estimate of the convergence rate of the closed-loop system (2.105) can also be obtained by using the methods in the proof of Proposition 2.2. The details are omitted for brevity.

2.3.4 A Numerical Example

We consider a delayed double oscillator system characterized by (2.28) in which A, B, and C are given by

$$A = \begin{bmatrix} 0 & \omega & 0 & 0 \\ -\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}. \tag{2.121}$$

Here, ω is a positive number. It can be readily verified that (A, B) is controllable and (A, C) is observable. For this system, the unique solution to the parametric algebraic Riccati equation (2.32) can be computed as

$$P = \begin{bmatrix} \frac{\gamma^7}{\omega^4} - \frac{2\gamma^5}{\omega^2} + 8\gamma^3 & \frac{3\gamma^6}{\omega^3} - \frac{4\gamma^4}{\omega} & \frac{3\gamma^5}{\omega^3} - \frac{8\gamma^3}{\omega} & \frac{\gamma^4}{\omega^2} - 4\gamma^2 \\ \frac{3\gamma^6}{\omega^3} - \frac{4\gamma^4}{\omega} & \frac{10\gamma^5}{\omega^2} + 8\gamma^3 & \frac{11\gamma^4}{\omega^2} + 4\gamma^2 & \frac{4\gamma^3}{\omega} \\ \frac{3\gamma^5}{\omega^3} - \frac{8\gamma^3}{\omega} & \frac{11\gamma^4}{\omega^2} + 4\gamma^2 & \frac{14\gamma^3}{\omega^2} + 4\gamma & \frac{6\gamma^2}{\omega} \\ -4\gamma^2 & \frac{4\gamma^3}{\omega} & \frac{6\gamma^2}{\omega} & 4\gamma \end{bmatrix} \tag{2.122}$$

and the matrix exponential is given by

$$\exp(As) = \begin{bmatrix} \cos(\omega s) & \sin(\omega s) & \frac{1}{2}s\sin(\omega s) & \frac{1}{2\omega}\varpi(s) \\ -\sin(\omega s) & \cos(\omega s) & \frac{1}{2\omega}\psi(s) & \frac{1}{2}s\sin(\omega s) \\ 0 & 0 & \cos(\omega s) & \sin(\omega s) \\ 0 & 0 & -\sin(\omega s) & \cos(\omega s) \end{bmatrix} \tag{2.123}$$

in which $\varpi(s) = \sin(\omega s) - \omega s\cos(\omega s)$ and $\psi(s) = \omega s\cos(\omega s) + \sin(\omega s)$.

We consider two cases of the delay function $\phi(t)$:

- In the first case, the delay function is (Example 5.3 in [60])

$$\phi(t) = t - \frac{t+1}{2t+1}, \quad t \geq 0. \tag{2.124}$$

It follows that $D = 1$ and the inverse function of $\phi(t)$ is

$$\phi^{-1}(t) = \frac{t + \sqrt{(t+2)^2 + 1}}{2}. \tag{2.125}$$

Hence, according to Theorem 2.3, the truncated predictor state feedback law (2.54) is given as

$$u(t) = -B^{\mathrm{T}} P(\gamma) \exp\left(A\left(\frac{t+1}{\sqrt{(t+1)^2+1}+t}\right)\right) x(t). \tag{2.126}$$

- In the second case, the delay function is given by

$$\phi(t) = t - 4\frac{t+1}{2t+1}, \quad t \geq 0. \tag{2.127}$$

It follows that $D = 4$ and the inverse function of $\phi(t)$ is

$$\phi^{-1}(t) = \frac{-2t + 3 + \sqrt{4t^2 + 20t + 41}}{4}. \tag{2.128}$$

Hence, according to Theorem 2.3, the truncated predictor state feedback law (2.54) is given as

$$u(t) = -B^{\mathrm{T}} P(\gamma) \exp\left(A\left(\frac{8t+8}{\sqrt{4t^2+20t+41}+2t-3}\right)\right) x(t). \quad (2.129)$$

With a given initial condition

$$x(\theta) = [1\ 0\ -1\ 2]^{\mathrm{T}}, \quad \theta \in [-D, 0], \quad (2.130)$$

and by setting $\omega = 2$, and letting $\gamma = 0.3$ and $\gamma = 0.1$ for the first and second case of the delay function, respectively, the state response and control input under these two cases are shown in Figs. 2.5 and 2.6. It is clear that the systems are indeed stabilized by the truncated predictor state feedback TPF law (2.54).

The stabilization of system (2.121) by the truncated predictor output feedback law (2.101) is also studied through simulation. We pick $L = \begin{bmatrix}-35 & -18.5 & -5 & 25\end{bmatrix}^{\mathrm{T}}$ to assign the eigenvalues of $A + LC$ at $\{-1, -2, -3, -4\}$ in the complex plane. With the given initial condition (2.130) and $\hat{x}(\theta) = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}, \theta \in [-D, 0]$, system (2.121) with the delay function given by either (2.124) or (2.127) is stabilized by the output feedback TPF law (2.101), which is shown in Figs. 2.7 and 2.8. The feedback parameter γ in the first and second case of the delay function is chosen as 0.3 and 0.1, respectively, just as in the simulation for the state feedback case.

2.4 Conclusions

By truncating the distributed delay term of the predictor feedback law that stems from the zero state solution of the predicted future state, we arrived at the truncated predictor feedback (TPF) law, containing only the static state feedback term of the predictor feedback law that stems from the zero input solution of the predicted future state. The safety in discarding the distributed delay term relies on a delicate parametrization of the feedback gain matrix of the TPF law, either by the eigenstructure assignment low gain design or by the Lyapunov equation based low gain design. Stabilization of linear systems with input delay by the TPF justifies the effectiveness of such a parametrization, in which the eigenstructure assignment based approach and the Lyapunov equation based approach were revisited. Both state feedback and output feedback TPF laws were considered.

2.5 Notes and References

The predictor feedback design for linear systems with input delay results in a distributed control law, which is known to incur difficulty in its implementation. Reference [63] overcomes such a difficulty by proposing the method of discarding the distributed delay term, leading to a TPF law. The validity of such a truncation process requires the parameterization of the feedback gain matrix of the TPF law by

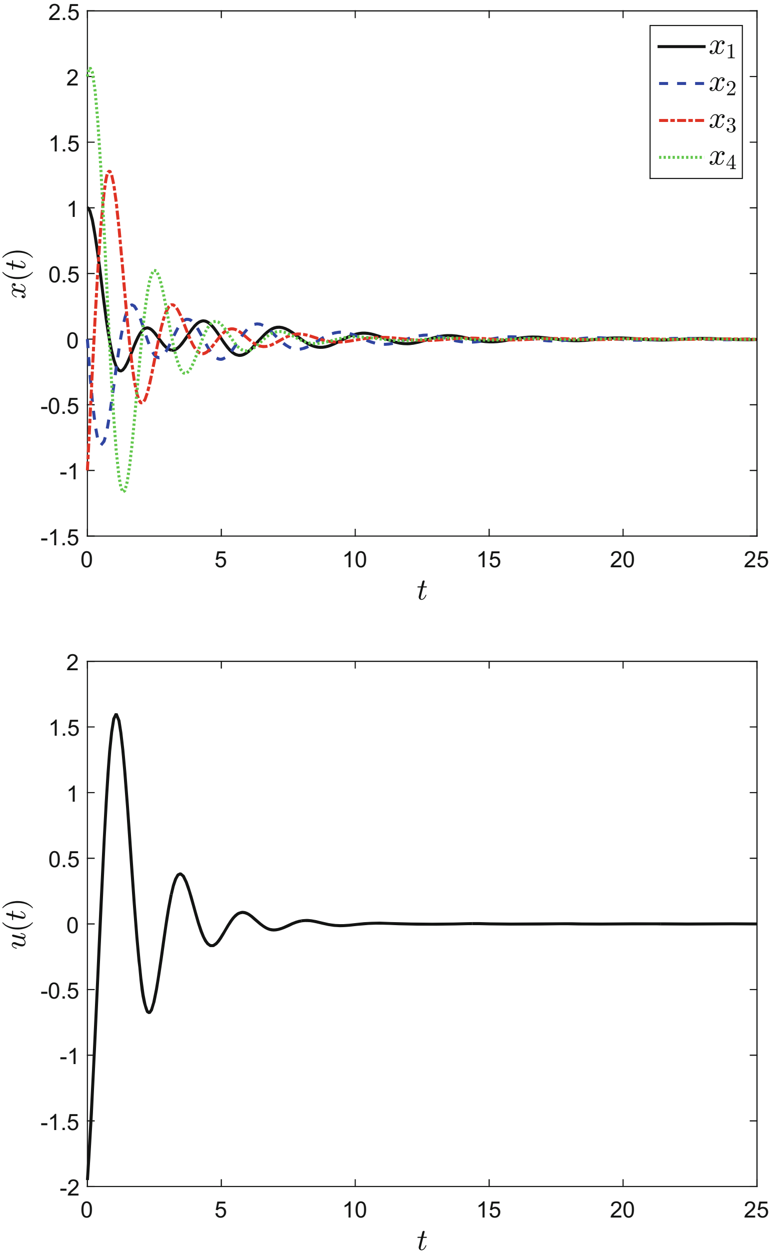
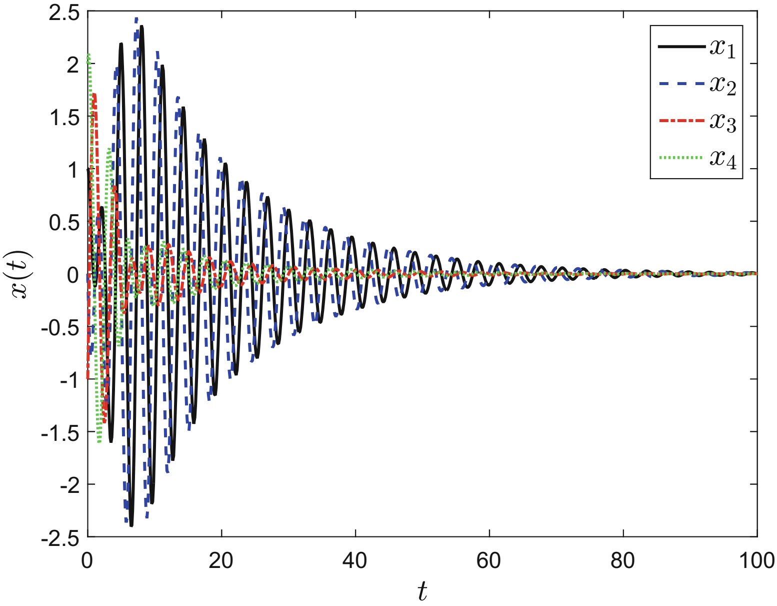


Fig. 2.5 State response and control input under the state feedback TPF law (2.54): $d(t) = \frac{t+1}{2t+1}$ and $\gamma = 0.3$



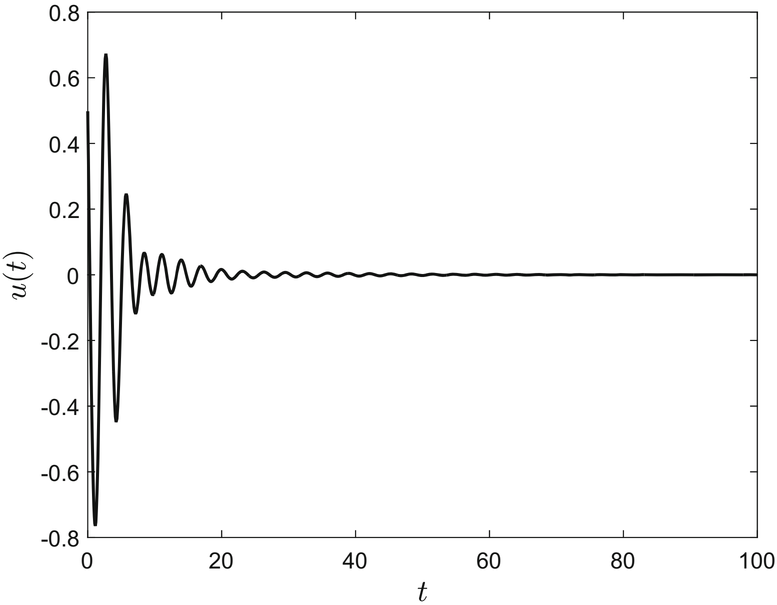
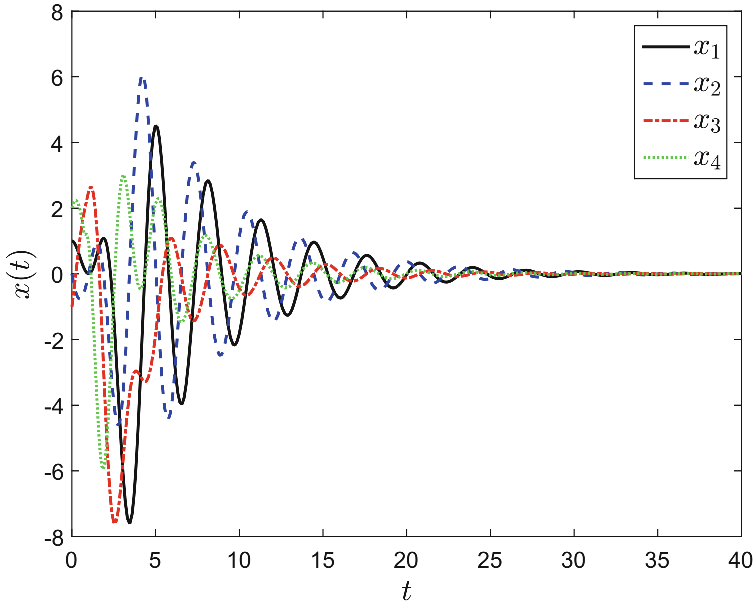


Fig. 2.6 State response and control input under the state feedback TPF law (2.54): $d(t) = 4\frac{t+1}{2t+1}$ and $\gamma = 0.1$



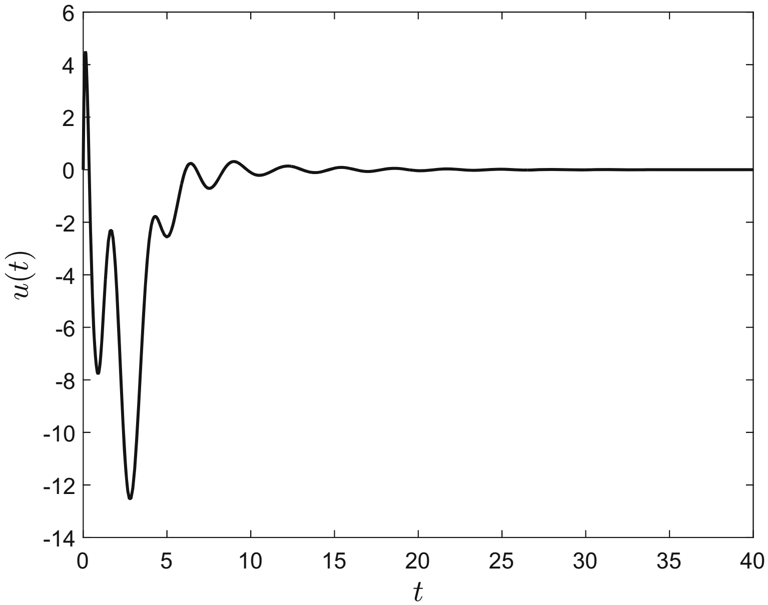
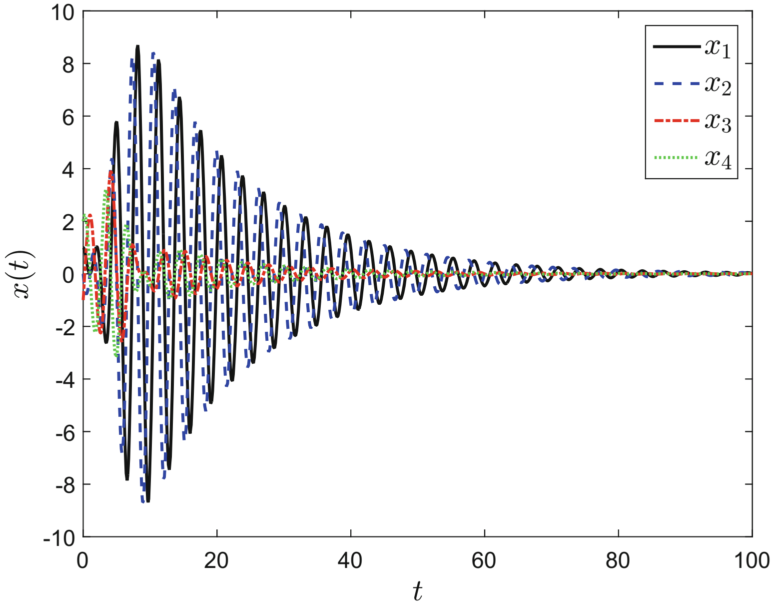


Fig. 2.7 State response and control input under the output feedback TPF law (2.101): $d(t) = \frac{t+1}{2t+1}$ and $\gamma = 0.3$



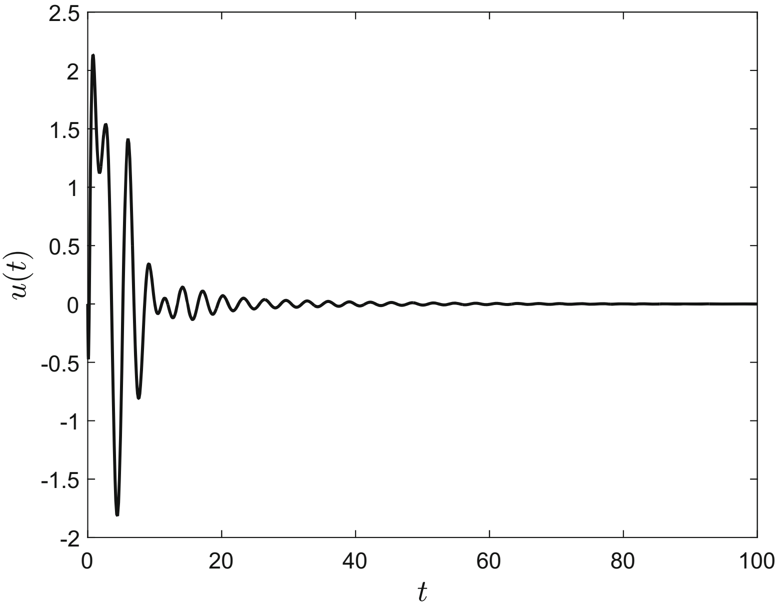


Fig. 2.8 State response and control input under the output feedback law (2.101): $d(t) = 4\frac{t+1}{2t+1}$ and $\gamma = 0.1$

using low gain feedback design techniques. An eigenstructure assignment based low gain design technique was utilized in [63], while an alternative design, a Lyapunov equation based low gain design technique, was employed in [124] and [102]. An advantage of the Lyapunov equation based design over the eigenstructure assignment based design is that the former facilitates the treatment of the time-varying delay. The presentation of Sects. 2.2 and 2.3 mostly follows that of [63] and [124], respectively. However, [124] does not consider the effect of stable poles of the open loop system throughout its output feedback results. In the output feedback setting, it is not without loss of generality to assume that the open loop system does not have stable poles. The contribution of the output signal corresponding to the subsystem with all its open loop poles on the imaginary axis cannot be distinguished from the output signal of the overall system. Therefore, in Sect. 2.3.3, we reformulated the truncated predictor output feedback law for linear systems with open loop poles in the closed left-half plane, along with the corresponding stability analysis of the closed-loop system under our output feedback law.

Chapter 3
Truncated Predictor Feedback for Discrete-Time Linear Systems

3.1 Introduction

Based on the predictor feedback law for the stabilization of discrete-time systems with delay, as introduced in Sect. 1.5, we develop a truncated predictor feedback (TPF) law that simplifies the implementation of its predictor feedback prototype. An eigenstructure assignment based low gain design as well as an alternative Lyapunov equation based design are employed to parameterize the feedback gain matrix of the TPF law in a low gain parameter. The core of low gain feedback designs is that a sufficiently small low gain parameter would compensate an arbitrarily large delay in a discrete-time linear system without exponentially unstable open loop poles.

Consider system (1.82) recalled below,

$$\begin{cases} x(k+1) = Ax(k) + Bu(k-r), \\ y(k) = Cx(k), \end{cases} \tag{3.1}$$

where (A, B) is stabilizable and (A, C) is detectable. The implementation of the predictor feedback law (1.86), recalled below,

$$u(k) = FA^r x(k) + F \sum_{i=k-r}^{k-1} A^{k-1-i} Bu(i), \tag{3.2}$$

requires to measure, store, and extract the input signal $u(s)$ on the time interval $s \in [k-r, k]$ in real time. As the amount of delay grows large, the number of memory units that store the values of past input grows large accordingly. This leads to considerable burden on the implementation of the predictor feedback law. The state feedback TPF law

$$u(k) = FA^r x(k) \tag{3.3}$$

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5_3

requires solely the measurement of the current state. Compared with the predictor feedback law (1.86), the TPF law (3.2) is much easier to implement. In the following two sections, we introduce two methods of parameterizing the feedback gain matrix F of the TPF law in a low gain feedback parameter, and establish the stabilizing effects, in the presence of an arbitrarily large delay, of the TPF law on system (3.1) without exponentially unstable open loop poles.

3.2 The Eigenstructure Assignment Based Design

Our interest in discrete-time systems of the form (3.1) has been motivated by several results on asymptotic stabilization of their continuous-time counterparts (2.1). A general result on stabilizability of system (2.1) was established in [63]. By explicit construction of stabilizing feedback laws, it was shown that system (2.1), with an arbitrarily large finite delay, is asymptotically stabilizable by the truncated predictor state or output feedback law, namely, (2.8) or (2.20), as long as the open loop system is not exponentially unstable (i.e., all the open loop poles are in the closed left-half plane).

This result in a way indicates the complexity in the stabilization of systems with delay in the input such as those in the form of (3.1). The objective of this section is to establish parallel results of [63] in the discrete-time setting. That is, system (3.1), with an arbitrarily large finite delay, is asymptotically stabilizable by either truncated predictor state or output feedback law as long as the open loop system is not exponentially unstable (i.e., all the open loop poles are inside or on the unit circle). Key to establishing this discrete-time result is the Razumikhin Stability Theorem in the discrete-time setting, Theorem 1.4. Simple examples show that these results would not be true if the open loop system is exponentially unstable and thus are not conservative.

3.2.1 Low Gain Feedback Design

The eigenstructure assignment based low gain feedback design for discrete-time systems with multiple inputs is developed from the design for single input systems.

Consider the following linear system with a single input:

$$x(k+1) = Ax(k) + Bu(k), \tag{3.4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{3.5}$$

Assume that all eigenvalues of A are on the unit circle. Let

$$F(\varepsilon) : (0, 1] \to \mathbb{R}^{1\times n}$$

be the unique state feedback gain such that

$$\lambda(A + BF(\varepsilon)) = (1 - \varepsilon)\lambda(A), \quad \varepsilon \in (0, 1]. \tag{3.6}$$

Then, we have the following lemmas [61] on the properties of the resulting closed-loop system. Explicit construction of all the matrices involved in these lemmas can be found in [61].

Lemma 3.1 *Consider system (3.4) and let F be as given by (3.6). Then, there exists a nonsingular transformation matrix $Q(\varepsilon) \in \mathbb{R}^{n\times n}$ such that*

$$\begin{aligned} Q^{-1}(\varepsilon)(A + BF(\varepsilon))Q(\varepsilon) &= J(\varepsilon) \\ &\triangleq \text{blkdiag}\big\{J_{-1}(\varepsilon), J_{+1}(\varepsilon), J_1(\varepsilon), \cdots, J_l(\varepsilon)\big\}, \end{aligned}$$

where

$$J_{-1}(\varepsilon) = \begin{bmatrix} -(1-\varepsilon) & 1 & & \\ & \ddots & \ddots & \\ & & -(1-\varepsilon) & 1 \\ & & & -(1-\varepsilon) \end{bmatrix}_{n_{-1}\times n_{-1}}, \tag{3.7}$$

$$J_{+1}(\varepsilon) = \begin{bmatrix} 1-\varepsilon & 1 & & \\ & \ddots & \ddots & \\ & & 1-\varepsilon & 1 \\ & & & 1-\varepsilon \end{bmatrix}_{n_{+1}\times n_{+1}}, \tag{3.8}$$

and, for each $i \in I[1, l]$,

$$J_i(\varepsilon) = \begin{bmatrix} J_i^\star(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & J_i^\star(\varepsilon) & I_2 \\ & & & J_i^\star(\varepsilon) \end{bmatrix}_{2n_i \times 2n_i}, \tag{3.9}$$

$$J_i^\star(\varepsilon) = (1-\varepsilon)\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}, \tag{3.10}$$

with

$$\alpha_i^2 + \beta_i^2 = 1$$

for all $i \in I[1, l]$ *and*

$$\alpha_i \neq \alpha_j$$

for $i \neq j$.

Lemma 3.2 *Consider system (3.4) and let* F *be as given by (3.6). Let* $J(\varepsilon)$ *be as given in Lemma 3.1. Let*

$$S(\varepsilon) = \text{blkdiag}\{S_{-1}(\varepsilon), S_{+1}(\varepsilon), S_1(\varepsilon), S_2(\varepsilon), \cdots, S_l(\varepsilon)\}, \tag{3.11}$$

where

$$S_{-1}(\varepsilon) = \text{diag}\left\{\varepsilon^{n_{-1}-1}, \varepsilon^{n_{-1}-2}, \cdots, \varepsilon, 1\right\}, \tag{3.12}$$

$$S_{+1}(\varepsilon) = \text{diag}\left\{\varepsilon^{n_{+1}-1}, \varepsilon^{n_{+1}-2}, \cdots, \varepsilon, 1\right\} \tag{3.13}$$

and, for each $i \in I[1, l]$,

$$S_i(\varepsilon) = \text{blkdiag}\left\{\varepsilon^{n_i-1} I_2, \varepsilon^{n_i-2} I_2, \cdots, \varepsilon I_2, I_2\right\}. \tag{3.14}$$

Then,

1.

$$S(\varepsilon)J(\varepsilon)S^{-1}(\varepsilon) = \tilde{J}(\varepsilon) \triangleq \text{blkdiag}\left\{\tilde{J}_{-1}(\varepsilon), \tilde{J}_{+1}(\varepsilon), \tilde{J}_1(\varepsilon), \cdots, \tilde{J}_l(\varepsilon)\right\},$$

where

$$\tilde{J}_{-1}(\varepsilon) = \begin{bmatrix} -(1-\varepsilon) & \varepsilon & & \\ & \ddots & \ddots & \\ & & -(1-\varepsilon) & \varepsilon \\ & & & -(1-\varepsilon) \end{bmatrix}_{n_{-1}\times n_{-1}}, \tag{3.15}$$

$$\tilde{J}_{+1}(\varepsilon) = \begin{bmatrix} 1-\varepsilon & \varepsilon & & \\ & \ddots & \ddots & \\ & & 1-\varepsilon & \varepsilon \\ & & & 1-\varepsilon \end{bmatrix}_{n_{+1}\times n_{+1}}, \tag{3.16}$$

and, for each $i \in I[1, l]$,

$$\tilde{J}_i(\varepsilon) = \begin{bmatrix} J_i^{\star}(\varepsilon) & \varepsilon I_2 & & \\ & \ddots & \ddots & \\ & & J_i^{\star}(\varepsilon) & \varepsilon I_2 \\ & & & J_i^{\star}(\varepsilon) \end{bmatrix}_{2n_i\times 2n_i}, \tag{3.17}$$

$$J_i^{\star}(\varepsilon) = (1-\varepsilon)\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix},$$

with

$$\beta_i > 0$$

for all $i \in I[1, l]$ *and*

$$\beta_i \neq \beta_j$$

for $i \neq j$;

2. *There exists* $\varepsilon^* \in (0, 1]$ *such that the unique positive definite solution* $\tilde{P}(\varepsilon)$ *to the Lyapunov equation*

$$\tilde{J}^{\mathrm{T}}(\varepsilon)\tilde{P}\tilde{J}(\varepsilon) - \tilde{P} = -\varepsilon I \tag{3.18}$$

is bounded over $\varepsilon \in (0, \varepsilon^*]$, *i.e., there exist positive definite matrices* $\tilde{P}_1$ *and* $\tilde{P}_2$ *such that*

$$\tilde{P}_1 \leq \tilde{P}(\varepsilon) \leq \tilde{P}_2, \quad \forall \varepsilon \in (0, \varepsilon^*]. \tag{3.19}$$

Lemma 3.3 *Consider system (3.4) and let* F *be as given by (3.6). Let* $Q(\varepsilon)$, l, *and* n_i *for* $i = 0$ *to* l, *be as defined in Lemma 3.1. Let the scaling matrix* $S(\varepsilon)$ *be as* *defined in Lemma 3.2. Then, there exist $\gamma, \alpha, \beta, \vartheta > 0$, all independent of ε, such that, for all $\varepsilon \in (0, 1]$,*

$$|F(\varepsilon)| \leq \gamma\varepsilon, \tag{3.20}$$

$$\left|F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)\right| \leq \alpha\varepsilon, \tag{3.21}$$

$$\left|F(\varepsilon)AQ(\varepsilon)S^{-1}(\varepsilon)\right| \leq \beta\varepsilon, \tag{3.22}$$

$$|Q(\varepsilon)| \leq \vartheta, \tag{3.23}$$

$$\left|Q^{-1}(\varepsilon)\right| \leq \vartheta. \tag{3.24}$$

3.2.2 Truncated Predictor State Feedback Design

For system (3.1) with all eigenvalues of A on or inside the unit circle, we construct a family of linear state feedback laws as follows.

State Feedback Design

Step 1. Find nonsingular transformation matrices T_{s} and T_{I} such that the pair (A, B) is transformed into the following block diagonal control canonical form,

$$T_{\mathrm{s}}^{-1}AT_{\mathrm{s}} = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_l & 0 \\ 0 & 0 & \cdots & 0 & A_0 \end{bmatrix}, \tag{3.25}$$

$$T_{\mathrm{s}}^{-1}BT_{\mathrm{I}} = \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1l} & * \\ 0 & B_2 & \cdots & B_{2l} & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_l & * \\ B_{01} & B_{02} & \cdots & B_{0l} & * \end{bmatrix}, \tag{3.26}$$

where A_0 contains all the eigenvalues of A that are strictly inside the unit circle, for each $i \in I[1, l]$, all eigenvalues of A_i are on the unit circle and hence (A_i, B_i) is controllable and is given by,

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{in_i} & -a_{i(n_i-1)} & -a_{i(n_i-2)} & \cdots & -a_{i1} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and finally, $*$'s represent submatrices of less interest.

We note that the existence of the above canonical form was shown in [110]. The software realization can be found in [62].

Step 2. For each (A_i, B_i), let $F_i(\varepsilon) \in \mathbb{R}^{1\times n_i}$ be the state feedback gain such that

$$\lambda(A_i + B_i F_i(\varepsilon)) = (1-\varepsilon)\lambda(A_i), \quad \varepsilon \in (0, 1]. \tag{3.27}$$

Note that $F_i(\varepsilon)$ is unique.

Step 3. Construct the family of low gain state feedback laws as

$$u(k) = F(\varepsilon)A^r x(k), \tag{3.28}$$

where the low gain matrix $F(\varepsilon)$ is given by

$$F(\varepsilon) = T_{\mathrm{I}} \begin{bmatrix} F_1(\varepsilon) & 0 & \cdots & 0 & 0 & 0 \\ 0 & F_2(\varepsilon) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & F_{l-1}(\varepsilon) & 0 & 0 \\ 0 & 0 & \cdots & 0 & F_l(\varepsilon) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} T_{\mathrm{S}}^{-1}. \tag{3.29}$$

The theorem below establishes that the state feedback TPF law (3.28) asymptotically stabilizes system (3.1) as long as all eigenvalues of A are on or inside the unit circle.

Theorem 3.1 *Consider the closed-loop system comprising of system (3.1) and the state feedback TPF law (3.28). Let all eigenvalues of A be on or inside the unit circle. Then, for any given arbitrarily large $r \geq 0$, there exists $\varepsilon^* > 0$, such that, for each $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system is asymptotically stable.*

Proof Without loss of generality, assume that the pair (A, B) is already in the form of (3.25)–(3.26). Under the state feedback law (3.28), the closed-loop system is given by

$$x(k+1) = Ax(k) + BF(\varepsilon)A^r x(k-r), \tag{3.30}$$

from which we have

$$A^r x(k-r) = x(k) - \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r x(k+s-2r). \tag{3.31}$$

Substitution of (3.31) in (3.30) results in

$$x(k+1) = (A+BF(\varepsilon))x(k) - BF(\varepsilon)\sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon)A^r x(k+s-2r). \tag{3.32}$$

Partitioning the state x according to the structure of (3.25)–(3.26),

$$x = \begin{bmatrix} x_1^{\mathrm{T}} & x_2^{\mathrm{T}} & \cdots & x_l^{\mathrm{T}} & x_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \quad x_i \in \mathbb{R}^{n_i}, \ \ i \in I[1,l],$$

we rewrite the state equation (3.32) as

$$\begin{cases} x_1(k+1) = (A_1 + B_1 F_1(\varepsilon))x_1(k) + \sum_{j=2}^{l} B_{1j} F_j(\varepsilon) x_j(k) \\ \qquad\qquad -B_{\mathrm{R}1} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon) A^r x(k+s-2r), \\ x_2(k+1) = (A_2 + B_2 F_2(\varepsilon))x_2(k) + \sum_{j=3}^{l} B_{2j} F_j(\varepsilon) x_j(k) \\ \qquad\qquad -B_{\mathrm{R}2} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon) A^r x(k+s-2r), \\ \qquad \vdots \\ x_l(k+1) = (A_l + B_l F_l(\varepsilon))x_l(k) \\ \qquad\qquad -B_{\mathrm{R}l} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon) A^r x(k+s-2r), \\ x_0(k+1) = A_0 x_0(k) + \sum_{j=1}^{l} B_{0j} F_j(\varepsilon) x_j(k) \\ \qquad\qquad -B_{\mathrm{R}0} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon) A^r x(k+s-2r). \end{cases} \tag{3.33}$$

where for each $i \in I[1,l]$, $B_{\mathrm{R}i}$ is the ith row of the right-hand side of (3.26) and $B_{\mathrm{R}0}$ is the last row.

Now for each $i \in I[1,l]$, let $Q_i(\varepsilon)$, $S_i(\varepsilon)$, $\tilde{J}_i(\varepsilon)$, $\tilde{P}_i$, γ_i, α_i, β_i, and ϑ_i be the matrices $Q(\varepsilon)$, $S(\varepsilon)$, $\tilde{J}(\varepsilon)$, and $\tilde{P}$ and the constants γ, α, β, and ϑ as defined in Lemmas 3.1–3.3, but for the triple $(A_i, B_i, F_i(\varepsilon))$. Define a state transformation as,

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1^{\mathrm{T}}, \tilde{x}_2^{\mathrm{T}}, \cdots, \tilde{x}_l^{\mathrm{T}}, \tilde{x}_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{3.34}$$

where $\tilde{x}_0 = x_0$, and, for each $i \in I[1,l]$, $\tilde{x}_i = S_i(\varepsilon) Q_i^{-1}(\varepsilon) x_i$.

It follows from Lemmas 3.1 and 3.2 that under the state transformation (3.34) the state equation (3.33) can be written as

$$\begin{cases} \tilde{x}_1(k+1) = \tilde{J}_1(\varepsilon)\tilde{x}_1(k) + \sum_{j=2}^{l} S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{1j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k) \\ \qquad -S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{\mathrm{R}1}F(\varepsilon)\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r), \\ \tilde{x}_2(k+1) = \tilde{J}_2(\varepsilon)\tilde{x}_2(k) + \sum_{j=3}^{l} S_2(\varepsilon)Q_2(\varepsilon)B_{2j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k) \\ \qquad -S_2(\varepsilon)Q_2^{-1}(\varepsilon)B_{\mathrm{R}2}F(\varepsilon)\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r), \\ \qquad \vdots \\ \tilde{x}_l(k+1) = \tilde{J}_l(\varepsilon)\tilde{x}_l(k) - S_l(\varepsilon)Q_l^{-1}(\varepsilon)B_{\mathrm{R}l}F(\varepsilon) \\ \qquad \times\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r), \\ \tilde{x}_0(k+1) = A_0\tilde{x}_0(k) + \sum_{j=1}^{l} B_{0j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k) \\ \qquad -B_{\mathrm{R}0}F(\varepsilon)\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r). \end{cases} \tag{3.35}$$

where

$$Q(\varepsilon) = \mathrm{blkdiag}\{Q_1(\varepsilon), Q_2(\varepsilon), \cdots, Q_l(\varepsilon), I\} \tag{3.36}$$

and

$$S(\varepsilon) = \mathrm{blkdiag}\{S_1(\varepsilon), S_2(\varepsilon), \cdots, S_l(\varepsilon), I\}. \tag{3.37}$$

Let us choose a Lyapunov function

$$\begin{aligned} V(\tilde{x}) &= \sum_{i=1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}}\tilde{P}_i(\varepsilon)\tilde{x}_i + \tilde{x}_0^{\mathrm{T}}\tilde{P}_0\tilde{x}_0 \\ &\triangleq \tilde{x}^{\mathrm{T}}\tilde{P}(\varepsilon)\tilde{x}, \end{aligned} \tag{3.38}$$

where $\tilde{P}_0 > 0$ is such that

$$A_0^{\mathrm{T}}\tilde{P}_0A_0 - \tilde{P}_0 = -I,$$

$\kappa > 0$ is a constant whose value is to be determined later, and

$$\tilde{P}(\varepsilon) = \text{blkdiag}\left\{\kappa \tilde{P}_1(\varepsilon), \kappa^2 \tilde{P}_2(\varepsilon), \cdots, \kappa^l \tilde{P}_l(\varepsilon), \tilde{P}_0\right\}. \tag{3.39}$$

The existence of such $\tilde{P}_0$ is due to the fact that A_0 is Schur stable. The forward difference of V along the trajectory of the closed-loop system (3.35) can be evaluated as follows:

$$\begin{aligned}
\Delta V(\tilde{x}(k)) = \sum_{i=1}^{l} \Bigg[& -\kappa^i \tilde{x}_i^{\mathrm{T}}(k) \left(\tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_i(\varepsilon)\tilde{J}_i(\varepsilon) - \tilde{P}_i\right)\tilde{x}_i(k) \\
& + 2\sum_{j=i+1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}}(k)\tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_i(\varepsilon)S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{ij}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k) \\
& - 2\kappa^i \left(\sum_{j=i+1}^{l} S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{ij}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k)\right)^{\mathrm{T}} \tilde{P}_i(\varepsilon) \\
& \times S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{\mathrm{R}i}F(\varepsilon)\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon) \\
& \times \tilde{x}(k+s-2r) + k^i \left[\sum_{j=i+1}^{l} S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{ij}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k)\right]^{\mathrm{T}} \\
& \times \tilde{P}_i(\varepsilon)\left[\sum_{j=i+1}^{l} S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{ij}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k)\right]\Bigg] \\
& - \tilde{x}_0^{\mathrm{T}}(k)\left(A_0^{\mathrm{T}}\tilde{P}_0A_0 - \tilde{P}_0\right)\tilde{x}_0(k) + 2\sum_{j=1}^{l}\tilde{x}_0^{\mathrm{T}}(k)A_0^{\mathrm{T}}\tilde{P}_0B_{0j}F_j(\varepsilon)Q_j(\varepsilon) \\
& \times S_j^{-1}(\varepsilon)\tilde{x}_j(k) + \left[\sum_{j=1}^{l} B_{0j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k)\right]^{\mathrm{T}} P_0 \\
& \times \left[\sum_{j=1}^{l} B_{0j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k)\right] \\
& - 2\left[\sum_{i=1}^{l}\kappa^i \tilde{x}_i^{\mathrm{T}}(k)\tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_i(\varepsilon)S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{\mathrm{R}i} + \tilde{x}_0^{\mathrm{T}}A_0^{\mathrm{T}}\tilde{P}_0B_{\mathrm{R}0}\right]F(\varepsilon) \\
& \times \sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r)
\end{aligned}$$

$$+\left[\sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon)\tilde{x}(k+s-2r)\right]^{\mathrm{T}}$$
$$\times\left[\sum_{i=1}^{l} \kappa^i F^{\mathrm{T}}(\varepsilon) B_{\mathrm{R}i}^{\mathrm{T}} (Q_i^{-1})^{\mathrm{T}} S_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) + F^{\mathrm{T}}(\varepsilon) B_{\mathrm{R}0}^{\mathrm{T}}\right.$$
$$\left.\times \tilde{P}_0 B_{\mathrm{R}0} F(\varepsilon)\right]\left[\sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon)\tilde{x}(k+s-2r)\right]. \quad (3.40)$$

Recall that

$$\tilde{J}^{\mathrm{T}}(\varepsilon)\tilde{P}_i(\varepsilon)\tilde{J}_i(\varepsilon) - \tilde{P}_i = -I \quad (3.41)$$

and

$$A_0^{\mathrm{T}}\tilde{P}_0 A_0 - \tilde{P}_0 = -I. \quad (3.42)$$

Also, it follows from Lemma 3.3 that the matrices defining the remaining $(\tilde{x}_i(k), \tilde{x}_j(k))$ terms, other than the terms defined by $\tilde{J}^{\mathrm{T}}(\varepsilon)\tilde{P}_i(\varepsilon)\tilde{J}_i(\varepsilon) - \tilde{P}_i$ and $A_0^{\mathrm{T}}\tilde{P}_0 A_0 - \tilde{P}_0$, are all of order $O(\varepsilon)$ or order $O(\varepsilon^2)$. It is then straightforward to verify that there exist $\kappa > 0$ and $\varepsilon_1^* \in (0, 1]$ such that,

$$\Delta V(\tilde{x}(k)) \le -\frac{\varepsilon}{2}\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k) - 2\left[\tilde{x}_0^{\mathrm{T}} A_0^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0} + \sum_{i=1}^{l}\kappa^i\left[\tilde{x}_i^{\mathrm{T}}(k)\tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_i(\varepsilon)S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{\mathrm{R}i}\right.\right.$$
$$\left.+\left(\sum_{j=i+1}^{l} S_i(\varepsilon)Q^{-1}(\varepsilon)B_{ij}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k)\right)^{\mathrm{T}}\tilde{P}_i(\varepsilon)S_i(\varepsilon)\right]$$
$$\left.\times Q_i^{-1}(\varepsilon)B_{\mathrm{R}i}\right] F(\varepsilon)\sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon)\tilde{x}(k+s-2r)$$
$$+\left[\sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon)\tilde{x}(k+s-2r)\right]^{\mathrm{T}}$$
$$\times\left[\sum_{i=1}^{l}\kappa^i F^{\mathrm{T}}(\varepsilon)B_{\mathrm{R}i}^{\mathrm{T}}(Q_i^{-1}(\varepsilon))^{\mathrm{T}} S_i^{\mathrm{T}}(\varepsilon)S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_{\mathrm{R}i}F(\varepsilon)\right.$$
$$\left.+F^{\mathrm{T}}(\varepsilon)B_{\mathrm{R}0}^{\mathrm{T}}B_{\mathrm{R}0}F(\varepsilon)\right]\left[\sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon)\right.$$
$$\left.\times\tilde{x}(k+s-2r)\right], \quad \varepsilon\in(0,\varepsilon_1^*]. \quad (3.43)$$

Recalling the special structure of $\tilde{J}_i(\varepsilon)$ and Lemma 3.3, we have

$$\begin{aligned}
&\left|S(\varepsilon)Q^{-1}(\varepsilon)AQ(\varepsilon)S^{-1}(\varepsilon)\right| \\
&=\sum_{i=1}^{l}\left|S_i(\varepsilon)Q_i^{-1}(\varepsilon)A_iQ_i(\varepsilon)S_i^{-1}(\varepsilon)\right|+|A_0| \\
&=\sum_{i=1}^{l}\left|S_i(\varepsilon)Q_i^{-1}(\varepsilon)(A_i+B_iF_i(\varepsilon))Q(\varepsilon)S_i^{-1}(\varepsilon) - !S_i(\varepsilon)Q_i^{-1}(\varepsilon)B_iF_i(\varepsilon)Q_i(\varepsilon)S_i^{-1}(\varepsilon)\right|+|A_0| \\
&\leq\sum_{i=1}^{l}\left(\left|\tilde{J}_i(\varepsilon)\right|+\varepsilon\alpha_i\vartheta_i|B_i|\right)+|A_0|\leq\delta,\quad \varepsilon\in(0,\varepsilon_1^*].
\end{aligned}\tag{3.44}$$

Hence, we have

$$\begin{aligned}
&\left|S(\varepsilon)Q^{-1}(\varepsilon)A^iQ(\varepsilon)S^{-1}(\varepsilon)\right| \\
&=\left|S(\varepsilon)Q^{-1}(\varepsilon)AQ^{-1}(\varepsilon)S^{-1}(\varepsilon)S(\varepsilon)Q^{-1}(\varepsilon)A\cdots AQ(\varepsilon)S^{-1}(\varepsilon)\right|\leq\delta^i,
\end{aligned}\tag{3.45}$$

and

$$\begin{aligned}
&\left|F(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\right| \\
&=\left|F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)S(\varepsilon)Q^{-1}(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\right| \\
&\leq\left|F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)\right|\left|S(\varepsilon)Q^{-1}(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\right| \\
&\leq\left(\sum_{i=1}^{l}\left|F_i(\varepsilon)Q_i(\varepsilon)S_i^{-1}(\varepsilon)\right|\right)\delta^r \\
&\leq\varepsilon\left(\sum_{i=1}^{l}\alpha_i\right)\delta^r.
\end{aligned}\tag{3.46}$$

Consequently, the inequality (3.43) can be continued as follows:

$$\begin{aligned}
\Delta V(\tilde{x}(k))\leq&-\frac{\varepsilon}{2}\lambda_{\max}^{-1}\big(\tilde{P}(\varepsilon)\big)V(\tilde{x}(k)) \\
&+\varepsilon^2\varpi_1(r)V^{\frac{1}{2}}(\tilde{x}(k))\sum_{s=0}^{r-1}|A|^{r-s-1}V^{\frac{1}{2}}(\tilde{x}(k+s-2r)), \\
&+\varepsilon^4\varpi_2(r)\left[\sum_{s=0}^{r-1}|A|^{r-s-1}V^{\frac{1}{2}}(\tilde{x}(k+s-2r))\right]^2,\ \ \varepsilon\in(0,\varepsilon_1^*],
\end{aligned}$$

for some $\varpi_1(r), \varpi_2(r) > 0$, both independent of ε. In arriving at the above inequality, we have used the inequality

$$V(\tilde{x}(k)) = \tilde{x}^{\mathrm{T}}(k)\tilde{P}(\varepsilon)\tilde{x}(k) \le \lambda_{\max}(\tilde{P}(\varepsilon))\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k).$$

Now, let $\eta > 1$ be any constant. If $V(\tilde{x}(k+s)) < \eta V(\tilde{x}(k)), s \in I[-r, 0]$, then

$$\begin{aligned}
\Delta V(\tilde{x}(k)) &\le -\frac{\varepsilon}{2}\lambda_{\max}^{-1}\big(\tilde{P}(\varepsilon)\big)V\big(\tilde{x}(k)\big) + \varepsilon^2\varpi_1(r)\eta\left(\sum_{s=0}^{r-1}|A|^{r-s-1}\right)V(\tilde{x}(k)) \\
&\quad +\varepsilon^4\varpi_2(r)\eta^2\left(\sum_{s=0}^{r-1}|A|^{r-s-1}\right)^2 V(\tilde{x}(k)) \\
&= -\varepsilon\left[\frac{1}{2}\lambda_{\max}^{-1}\big(\tilde{P}(\varepsilon)\big) - \varepsilon\varpi_1(r)\eta\left(\sum_{s=0}^{r-1}|A|^{r-s-1}\right)\right. \\
&\quad \left. -\varepsilon^3\varpi_2(r)\eta^2\left(\sum_{s=0}^{r-1}|A|^{r-s-1}\right)^2\right]V(\tilde{x}(k)), \quad \varepsilon \in (0, \varepsilon_1^*],
\end{aligned} \tag{3.47}$$

where the following estimate was used:

$$\begin{aligned}
V^{\frac{1}{2}}(\tilde{x}(k+s-2r)) &\le \sqrt{\eta}V^{\frac{1}{2}}(\tilde{x}(k-r)) \\
&\le \eta V^{\frac{1}{2}}(\tilde{x}(k)), \quad s \in I[-r, 0].
\end{aligned} \tag{3.48}$$

We recall that, by the definition of $\tilde{P}(\varepsilon)$ in (3.39) and because of (3.19), $\lambda_{\max}(\tilde{P}(\varepsilon))$, as a function of ε, is bounded from below by a positive scalar independent of ε. It is clear that, for any given $r \ge 0$ and $\eta > 1$, there exists $\varepsilon^* \in (0, \varepsilon_1^*]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\Delta V(\tilde{x}(k)) \le -\mu(\varepsilon)V(\tilde{x}(k)), \quad \text{if } V(\tilde{x}(k+s)) < \eta V(\tilde{x}(k)), \quad s \in I[-r, 0], \tag{3.49}$$

for some positive scalar $\mu(\varepsilon)$. It thus follows from Lemma 1.4 that the closed-loop system (3.30) is asymptotically stable. □

To examine the conservativeness of the result of Theorem 3.1, we consider the following simple system, whose open loop system is exponentially unstable,

$$\begin{cases} x(k+1) = \beta x(k) + u(k-r), \quad \beta > 1, \\ u(k) = -\alpha x(k). \end{cases} \tag{3.50}$$

We will show that, for large enough delay r, system (3.50) is not asymptotically stable for any choice of the feedback gain α, and thus the condition of Theorem 3.1

is tight. To this end, let us consider the characteristic equation of the closed-loop system,

$$z^{r+1} - \beta z^r + \alpha = 0. \tag{3.51}$$

We assume that there exists at least one solution z_0 to the characteristic equation (3.51) that lies inside the unit circle, i.e., $|z_0| < 1$. Then, as r increases to infinity, both $|z_0|^{r+1}$ and $|z_0|^r$ go to zero. Thus, the first two terms on the left-hand side of the characteristic equation go to zero accordingly, no matter what the value of β is. This leads to a contradiction if $\alpha \neq 0$, which implies that all the roots of the characteristic equation lie outside the unit circle when r is sufficiently large. On the other hand, if $\alpha = 0$, then the characteristic equation has r number of solutions at $z = 0$ and an unstable solution at $z = \beta > 1$, which also implies that the system is unstable. Concluding from the above, we know that, given a sufficiently large r, if $\beta > 1$, then any α as the feedback gain would cause instability of the closed-loop system. Moreover, all the solutions to the characteristic equation of the closed-loop system are unstable because they all lie outside the unit circle.

3.2.3 Truncated Predictor Output Feedback Design

For system (3.1) with all eigenvalues of A on or inside the unit circle, we construct the following family of output feedback laws:

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k-r) - L(y(k) - C\hat{x}(k)), \\ \quad u(k) = F(\varepsilon)A^r\hat{x}(k), \end{cases} \tag{3.52}$$

where $F(\varepsilon)$ is as given by (3.29), and $L \in \mathbb{R}^{n\times p}$ is such that all eigenvalues of $A + LC$ are strictly inside the unit circle. We note that such a matrix L exists as the pair (A, C) is detectable.

The theorem below establishes that the output feedback law (3.52) asymptotically stabilizes system (3.1) as long as all eigenvalues of A are on or inside the unit circle.

Theorem 3.2 *Consider the closed-loop system comprising of system (3.1) and the linear output feedback law (3.52). Let all eigenvalues of A be on or inside the unit circle. Then, for any given arbitrarily large $r \geq 0$, there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system is asymptotically stable.*

Proof Under the linear output feedback law (3.52), the closed-loop system is given by,

$$\begin{cases} x(k+1) = Ax(k) + BF(\varepsilon)A^r\hat{x}(k-r), \\ \hat{x}(k+1) = A\hat{x}(k) + BF(\varepsilon)A^r\hat{x}(k-r) - L(y(k) - C\hat{x}(k)), \end{cases} \tag{3.53}$$

which, in the new state $(x, e) = (x, x - \hat{x})$, can be written as

$$\begin{cases} x(k+1) = Ax(k) + BF(\varepsilon)A^r x(k-r) - BF(\varepsilon)A^r e(k-r), \\ e(k+1) = (A+LC)e(k), \end{cases} \tag{3.54}$$

which in turn implies that

$$\begin{aligned} A^r x(k-r) = x(k) - \sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon)A^r x(k+s-2r) \\ + \sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon)A^r e(k+s-2r) \end{aligned} \tag{3.55}$$

and

$$e(k-r) = (A+LC)^{-r} e(k). \tag{3.56}$$

Substitution of (3.55) and (3.56) into (3.54) results in

$$\begin{cases} x(k+1) = (A+BF)x(k) - BF(\varepsilon)\sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon)A^r x(k+s-2r) \\ \qquad +BF(\varepsilon)\sum_{s=0}^{r-1} A^{r-s-1} BF(\varepsilon)A^r e(k+s-2r) \\ \qquad -BF(\varepsilon)A^r (A+LC)^{-r} e(k), \\ e(k+1) = (A+LC)e(k). \end{cases} \tag{3.57}$$

Without loss of generality, assume that the pair (A, B) is already in the form of (3.25)–(3.26). Partitioning the state x according to the structure of (3.25)–(3.26),

$$x = \begin{bmatrix} x_1^{\mathrm{T}} & x_2^{\mathrm{T}} & \cdots & x_l^{\mathrm{T}} & x_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \quad x_i \in \mathbb{R}^{n_i}, \quad i \in I[1, l],$$

we rewrite the state equation (3.57) as follows:

$$\begin{cases} x_1(k+1) = (A_1 + B_1 F_1(\varepsilon))x_1(k) + \sum_{j=2}^{l} B_{1j} F_j(\varepsilon) x_j(k) - B_{\mathrm{R}1} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B \\ \qquad \times F(\varepsilon) A^r x(k+s-2r) + B_{\mathrm{R}1} F(\varepsilon) \sum_{s=0}^{r-1} A^r B F(\varepsilon) A^r e(k+s-2r) \\ \qquad - B_{\mathrm{R}1} F(\varepsilon) A^r (A+LC)^{-r} e(k), \\ x_2(k+1) = (A_2 + B_2 F_2(\varepsilon))x_2(k) + \sum_{j=3}^{l} B_{2j} F_j(\varepsilon) x_j(k) - B_{\mathrm{R}2} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B \\ \qquad \times F(\varepsilon) A^r x(k+s-2r) + B_{\mathrm{R}2} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r e(k+s-2r) \\ \qquad - B_{\mathrm{R}2} F(\varepsilon) A^r (A+LC)^{-r} e(k), \\ \quad \vdots \\ x_l(k+1) = (A_l + B_l F_l(\varepsilon))x_l(k) - B_{\mathrm{R}l} F(\varepsilon) \sum_{s=0}^{r-1} A^r B F(\varepsilon) A^r x(k+s-2r) \\ \qquad + B_{\mathrm{R}l} F(\varepsilon) \sum_{s=0}^{r-1} A^r B F(\varepsilon) A^r e(k+s-2r) - B_{\mathrm{R}l} F(\varepsilon) A^r (A+LC)^{-r} e(k), \\ x_0(k+1) = A_0 x_0(k) + \sum_{j=1}^{l} B_{0j} F_j(\varepsilon) x_j(k) - B_{\mathrm{R}0} F(\varepsilon) \sum_{s=0}^{r-1} A^r B F(\varepsilon) A^r x(k+s-2r) \\ \qquad + B_{\mathrm{R}0} F(\varepsilon) \sum_{s=0}^{r-1} A^r B F(\varepsilon) A^r e(k+s-2r) - B_{\mathrm{R}0} F(\varepsilon) A^r (A+LC)^{-r} e(k), \\ e(k+1) = (A+LC)e(k), \end{cases} \tag{3.58}$$

where, for each $i \in I[1, l]$, $B_{\mathrm{R}i}$ is the ith row of the right-hand side of (3.26) and $B_{\mathrm{R}0}$ is the last row.

Now, for each $i \in I[1, l]$, let $Q_i(\varepsilon)$, $S_i(\varepsilon)$, $\tilde{J}_i(\varepsilon)$, $\tilde{P}_i$, γ_i, α_i, β_i, and ϑ_i be the matrices $Q(\varepsilon)$, $S(\varepsilon)$, $\tilde{J}(\varepsilon)$, and $\tilde{P}$ and the constants γ, α, β and ϑ as defined in Lemmas 3.1–3.3, but for the triple $(A_i, B_i, F_i(\varepsilon))$. Define a state transformation as, $\tilde{x} = \left[\tilde{x}_1^{\mathrm{T}}, \tilde{x}_2^{\mathrm{T}}, \cdots, \tilde{x}_l^{\mathrm{T}}, \tilde{x}_0^{\mathrm{T}}\right]^{\mathrm{T}}$, $\tilde{e} = e$, where $\tilde{x}_0 = x_0$, and, for each $i \in I[1, l]$, $\tilde{x}_i = S_i(\varepsilon) Q_i^{-1}(\varepsilon) x_i$.

It follows from Lemmas 3.1 and 3.2 that, under this state transformation, the state equation (3.58) can be written as,

$$
\begin{cases}
\tilde{x}_1(k+1) = \tilde{J}_1(\varepsilon)\tilde{x}_1(k) + \displaystyle\sum_{j=2}^{l} S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{1j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k) \\
\quad -S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{\mathrm{R}1}F(\varepsilon) \\
\quad \times \sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r) \\
\quad +S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{\mathrm{R}1}F(\varepsilon) \\
\quad \times \sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r) \\
\quad -S_1(\varepsilon)Q_1^{-1}(\varepsilon)B_{\mathrm{R}1}F(\varepsilon)A^r(A+LC)^{-r}\tilde{e}(k), \\
\tilde{x}_2(k+1) = \tilde{J}_2(\varepsilon)\tilde{x}_2(k) + \displaystyle\sum_{j=3}^{l} S_2(\varepsilon)Q_2(\varepsilon)B_{2j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k) \\
\quad -S_2(\varepsilon)Q_2^{-1}(\varepsilon)B_{\mathrm{R}2}F(\varepsilon)\displaystyle\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r) \\
\quad -S_2(\varepsilon)Q_2^{-1}(\varepsilon)B_{\mathrm{R}2}F(\varepsilon)e^{A\tau}e^{-(A+LC)\tau}\tilde{e}(k), \\
\quad +S_2(\varepsilon)Q_2^{-1}(\varepsilon)B_{\mathrm{R}2}F(\varepsilon)\displaystyle\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r) \\
\quad -S_2(\varepsilon)Q_2^{-1}(\varepsilon)B_{\mathrm{R}2}F(\varepsilon)A^r(A+LC)^{-r}\tilde{e}(k), \\
\quad \vdots \\
\tilde{x}_l(k+1) = \tilde{J}_l(\varepsilon)\tilde{x}_l(k) \\
\quad -S_l(\varepsilon)Q_l^{-1}(\varepsilon)B_{\mathrm{R}l}F(\varepsilon)\displaystyle\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r) \\
\quad +S_l(\varepsilon)Q_l^{-1}(\varepsilon)B_{\mathrm{R}l}F(\varepsilon)\displaystyle\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r) \\
\quad -S_l(\varepsilon)Q_l^{-1}(\varepsilon)B_{\mathrm{R}2}F(\varepsilon)A^r(A+LC)^{-r}\tilde{e}(k), \\
\tilde{x}_0(k+1) = A_0\tilde{x}_0(k) + \displaystyle\sum_{j=1}^{l} B_{0j}F_j(\varepsilon)Q_j(\varepsilon)S_j^{-1}(\varepsilon)\tilde{x}_j(k) \\
\quad -B_{\mathrm{R}0}F(\varepsilon)\displaystyle\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r), \\
\quad +B_{\mathrm{R}0}F(\varepsilon)\displaystyle\sum_{s=0}^{r-1} A^{r-s-1}BF(\varepsilon)A^r Q(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r) \\
\quad -B_{\mathrm{R}0}F(\varepsilon)A^r(A+LC)^{-r}\tilde{e}(k), \\
\tilde{e}(k+1) = (A+LC)\tilde{e}(k),
\end{cases}
\tag{3.59}
$$

where

$$Q(\varepsilon) = \mathrm{blkdiag}\{Q_1(\varepsilon), Q_2(\varepsilon), \cdots, Q_l(\varepsilon), I\},$$

and

$$S(\varepsilon) = \mathrm{blkdiag}\{S_1(\varepsilon), S_2(\varepsilon), \cdots, S_l(\varepsilon), I\}.$$

Let us choose the Lyapunov function

$$V(\tilde{x},\tilde{e}) = \sum_{i=1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}} \tilde{P}_i \tilde{x}_i + \tilde{x}_0^{\mathrm{T}} \tilde{P}_0 \tilde{x}_0 + \kappa^{l+1} \tilde{e}^{\mathrm{T}} \tilde{Q} \tilde{e}$$
$$\triangleq \tilde{x}^{\mathrm{T}} \tilde{P} \tilde{x} + \kappa^{l+1} \tilde{e}^{\mathrm{T}} \tilde{Q} \tilde{e}, \tag{3.60}$$

where $\tilde{P}_0 > 0$ and $\tilde{Q} > 0$ are the solutions to the Lyapunov equations

$$A_0^{\mathrm{T}} \tilde{P}_0 A_0 - \tilde{P}_0 = -I,$$

and

$$(A+LC)^{\mathrm{T}} \tilde{Q} (A+LC) - \tilde{Q} = -I,$$

respectively, $\kappa > 0$ is a constant whose value is to be determined later, and

$$\tilde{P} = \mathrm{blkdiag}\left\{\kappa \tilde{P}_1, \kappa^2 \tilde{P}_2, \cdots, \kappa^l \tilde{P}_l, \tilde{P}_0\right\}.$$

The existence of such $\tilde{P}_0$ and $\tilde{Q}$ is due to the fact that both A_0 and $A+LC$ are Schur stable.

The forward difference of V along the trajectory of the closed-loop system (3.59) can be evaluated as follows:

$$\begin{aligned}
\Delta V(\tilde{x}(k)) = \sum_{i=1}^{l} \Bigg[& -\kappa^i \tilde{x}_i^{\mathrm{T}}(k) \left(\tilde{J}_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i(\varepsilon) \tilde{J}_i(\varepsilon) - \tilde{P}_i(\varepsilon) \right) \tilde{x}_i(k) \\
& + 2 \sum_{j=i+1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}}(k) \tilde{J}_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{ij} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \tilde{x}_j(k) \\
& - 2\kappa^i \tilde{x}_i^{\mathrm{T}}(k) \tilde{J}_i(\varepsilon) \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) A^r (A+LC)^{-1} \tilde{e}(k) \\
& - 2\kappa^i \left(\sum_{j=i+1}^{l} S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{ij} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \tilde{x}_j(k) \right)^{\mathrm{T}} \tilde{P}_i(\varepsilon) \\
& \times S_i(s) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) A^r (A+LC)^{-r} \tilde{e}(k) \\
& - 2\kappa^i \left(\sum_{j=i+1}^{l} S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{ij} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \tilde{x}_j(k) \right)^{\mathrm{T}} \tilde{P}_i(\varepsilon) \\
& \times S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{x}(k+s-2r)
\end{aligned}$$

$$
\begin{aligned}
&- 2\kappa^i \left(\sum_{j=i+1}^{l} S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{ij} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \tilde{x}_j(k) \right)^{\mathrm{T}} \tilde{P}_i(\varepsilon) \\
&\times S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{e}(k+s-2r) \\
&+ 2\kappa^i \tilde{e}^{\mathrm{T}}(k) \left(S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) A^r (A+LC)^{-r} \right)^{\mathrm{T}} \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) \\
&\times B_{\mathrm{R}i} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{x}(k+s-2r) \\
&- 2\kappa^i \tilde{e}^{\mathrm{T}}(k) \left(S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) A^r (A+LC)^{-r} \right)^{\mathrm{T}} \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) \\
&\times B_{\mathrm{R}i} F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{e}(k+s-2r) \\
&+ k^i \left[\sum_{j=i+1}^{l} S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{ij} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \tilde{x}_j(k) \right]^{\mathrm{T}} \tilde{P}_i(\varepsilon) \\
&\times \left[\sum_{j=i+1}^{l} S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{ij} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \tilde{x}_j(k) \right] \\
&+ \kappa^i \left(S_i(s) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) A^r (A+LC)^{-r} \tilde{e}(k) \right)^{\mathrm{T}} \tilde{P}_i(\varepsilon) \\
&\left. \times S_i(s) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} F(\varepsilon) A^r (A+LC)^{-r} \tilde{e}(k) \right] \\
&- \tilde{x}_0^{\mathrm{T}}(k) \left(A_0^{\mathrm{T}} \tilde{P}_0 A_0 - \tilde{P}_0 \right) \tilde{x}_0(k) + 2 \sum_{j=1}^{l} \tilde{x}_0^{\mathrm{T}}(k) A_0^{\mathrm{T}} \tilde{P}_0 B_{0j} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \\
&\times \tilde{x}_j(k) - 2\tilde{x}_0^{\mathrm{T}}(k) A_0^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0} F(\varepsilon) A^r (A+LC)^{-r} \tilde{e}(k) \\
&- 2 \left(\sum_{j=1}^{l} B_{0j} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \tilde{x}_j(k) \right)^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0} F(\varepsilon) A^r (A+LC)^{-r} \tilde{e}(k) \\
&+ \left[\sum_{j=1}^{l} B_{0j} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \tilde{x}_j(k) \right]^{\mathrm{T}} \tilde{P}_0 \left[\sum_{j=1}^{l} B_{0j} F_j(\varepsilon) Q_j(\varepsilon) S_j^{-1}(\varepsilon) \right. \\
&\left. \times \tilde{x}_j(k) \right] + \left(B_{\mathrm{R}0} F(\varepsilon) A^r (A+LC)^{-r} \tilde{e}(k) \right)^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0} F(\varepsilon) A^r (A+LC)^{-r} \tilde{e}(k)
\end{aligned}
$$

$$
\begin{aligned}
& - 2\left[\sum_{i=1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}}(k) \tilde{J}_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} + \tilde{x}_0^{\mathrm{T}} A_0^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0}\right] \\
& \times F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{x}(k+s-2r) \\
& + 2\left[\sum_{i=1}^{l} \kappa^i \tilde{x}_i^{\mathrm{T}}(k) \tilde{J}_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} + \tilde{x}_0^{\mathrm{T}} A_0^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0}\right] \\
& \times F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{e}(k+s-2r) \\
& + \left[F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{x}(k+s-2r)\right]^{\mathrm{T}} \\
& \times \left[\sum_{i=1}^{l} \kappa^i B_{\mathrm{R}i}^{\mathrm{T}} (Q_i^{-1}(\varepsilon))^{\mathrm{T}} S_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} + B_{\mathrm{R}0}^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0}\right] \\
& \times \left[F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{x}(k+s-2r)\right] \\
& + \left[F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{e}(k+s-2r)\right]^{\mathrm{T}} \\
& \times \left[\sum_{i=1}^{l} \kappa^i B_{\mathrm{R}i}^{\mathrm{T}} (Q_i^{-1}(\varepsilon))^{\mathrm{T}} S_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} + B_{\mathrm{R}0}^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0}\right] \\
& \times \left[F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{e}(k+s-2r)\right] \\
& - 2\left[F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{x}(k+s-2r)\right]^{\mathrm{T}} \\
& \times \left[\sum_{i=1}^{l} \kappa^i B_{\mathrm{R}i}^{\mathrm{T}} (Q_i^{-1}(\varepsilon))^{\mathrm{T}} S_i^{\mathrm{T}}(\varepsilon) \tilde{P}_i(\varepsilon) S_i(\varepsilon) Q_i^{-1}(\varepsilon) B_{\mathrm{R}i} + B_{\mathrm{R}0}^{\mathrm{T}} \tilde{P}_0 B_{\mathrm{R}0}\right] \\
& \times \left[F(\varepsilon) \sum_{s=0}^{r-1} A^{r-s-1} B F(\varepsilon) A^r Q(\varepsilon) S^{-1}(\varepsilon) \tilde{e}(k+s-2r)\right] \\
& + \kappa^{l+1} \tilde{e}^{\mathrm{T}}(k) \left((A+LC)^{\mathrm{T}} \tilde{Q} (A+LC) - \tilde{Q}\right) \tilde{e}(k).
\end{aligned} \tag{3.61}
$$

In view of Lemmas 3.3, the matrices defining the $(\tilde{x}_i, \tilde{x}_j)$ and $(\tilde{x}_i, \tilde{e})$ cross terms, other than the terms $\tilde{x}_i^{\mathrm{T}}(k)\left(\tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_i(\varepsilon)\tilde{J}_i(\varepsilon)-\tilde{P}_i(\varepsilon)\right)\tilde{x}_i(k)$, $\tilde{x}_0^{\mathrm{T}}(k)\left(A_0^{\mathrm{T}}\tilde{P}_0A_0-\tilde{P}_0\right)\tilde{x}_0(k)$, and $\tilde{e}^{\mathrm{T}}(k)\left((A+LC)^{\mathrm{T}}\tilde{Q}(A+LC)-\tilde{Q}\right)\tilde{e}(k)$, are all of order $O(\varepsilon)$. It is then straightforward to verify that there exists $\kappa > 0$ and an $\varepsilon_1^* \in (0, 1]$ such that,

$$
\begin{aligned}
\Delta V(\tilde{x}(k),\tilde{e}(k)) \le{}& -\frac{\varepsilon}{2}\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)-\frac{1}{2}\tilde{e}^{\mathrm{T}}(k)\tilde{e}(k)\\
&+\tilde{x}^{\mathrm{T}}(k)M_1(\varepsilon)F(\varepsilon)\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r)\\
&+\tilde{x}^{\mathrm{T}}(k)M_2(\varepsilon)F(\varepsilon)\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r)\\
&+\tilde{e}^{\mathrm{T}}(k)M_3(\varepsilon)F(\varepsilon)\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r)\\
&+\tilde{e}^{\mathrm{T}}(k)M_4(\varepsilon)F(\varepsilon)\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r)\\
&+\left(\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r)\right)^{\mathrm{T}}F^{\mathrm{T}}(\varepsilon)M_5(\varepsilon)\\
&\times F(\varepsilon)\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r)\\
&+\left(\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r)\right)^{\mathrm{T}}F^{\mathrm{T}}(\varepsilon)M_5(\varepsilon)\\
&\times F(\varepsilon)\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r)\\
&+\left(\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{x}(k+s-2r)\right)^{\mathrm{T}}F^{\mathrm{T}}(\varepsilon)M_6(\varepsilon)\\
&\times F(\varepsilon)\sum_{s=0}^{r-1}A^{r-s-1}BF(\varepsilon)A^rQ(\varepsilon)S^{-1}(\varepsilon)\tilde{e}(k+s-2r),
\end{aligned}
\tag{3.62}
$$

where matrices $M_i(\varepsilon)$, $i = 1, 2, \cdots, 6$, are defined in an obvious way and are all of order ε^0.

By using Lemma 3.3 again and (3.46), we can easily see that,

$$\begin{aligned}\Delta V(\tilde{x}(k), \tilde{e}(k)) \leq & -\frac{\varepsilon}{2} \min\left\{\lambda_{\max}^{-1}\left(\tilde{P}\right), \lambda_{\max}^{-1}\left(\kappa^{l+1}\tilde{Q}\right)\right\} V\left(\tilde{x}(k), \tilde{e}(k)\right) \\ & + \varepsilon^2 \varpi_1(r) V^{\frac{1}{2}}(\tilde{x}(k), \tilde{e}(k)) \sum_{s=0}^{r-1} |A|^{r-s-1} \\ & \times V^{\frac{1}{2}}(\tilde{x}(k+s-2r), \tilde{e}(k+s-2r)) + \varepsilon^4 \varpi_2(r) \\ & \times \left[\sum_{s=0}^{r-1} |A|^{r-s-1} V^{\frac{1}{2}}(\tilde{x}(k+s-2r), \tilde{e}(k+s-2r))\right]^2, \varepsilon \in (0, \varepsilon_1^*],\end{aligned}$$

for some $\varpi_1(r), \varpi_2(r) > 0$, both independent of ε.

Now, let $\eta > 1$ be any constant. If $V(\tilde{x}(k+s), \tilde{e}(k+s) < \eta V(\tilde{x}(k), \tilde{e}(k)), s \in I[-r, 0]$,

$$\begin{aligned}\Delta V(\tilde{x}(k), \tilde{e}(k)) \leq & -\frac{\varepsilon}{2} \min\left\{\lambda_{\max}^{-1}\left(\tilde{P}\right), \lambda_{\max}^{-1}\left(\kappa^{l+1}\tilde{Q}\right)\right\} V\left(\tilde{x}(k), \tilde{e}(k)\right) \\ & + \varepsilon^2 \varpi_1(r) \eta \left(\sum_{s=0}^{r-1} |A|^{r-s-1}\right) V(\tilde{x}(k), \tilde{e}(k)) \\ & + \varepsilon^4 \varpi_2(r) \eta^2 \left(\sum_{s=0}^{r-1} |A|^{r-s-1}\right)^2 V(\tilde{x}(k), \tilde{e}(k)), \ \varepsilon \in (0, \varepsilon_1^*].\end{aligned} \tag{3.63}$$

It is now clear that, for any given r, there exists $\varepsilon^* \in (0, 1]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\Delta V(\tilde{x}(k), \tilde{e}(k)) \leq -\sqrt{\varepsilon} V(\tilde{x}(k), \tilde{e}(k)). \tag{3.64}$$

It then follows from the Razumihkin Stability Theorem for discrete-time systems (Theorem 1.4) that the closed-loop system (3.53) is asymptotically stable. This completes the proof. □

3.2.4 A Numerical Example

Consider system (3.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2\sqrt{2} & -4 & 2\sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.65}$$

The open loop system has two pair of repeated poles on the unit circle at $z = \pm\sqrt{2}/2 \pm j\sqrt{2}/2$. Following the proposed design method, we choose

$$F(\varepsilon)=\left[-\varepsilon^4+4\varepsilon^3-6\varepsilon^2+4\varepsilon \quad -2\sqrt{2}\varepsilon^3+6\sqrt{2}\varepsilon^2-6\sqrt{2}\varepsilon \quad -4\varepsilon^2+8\varepsilon \quad -2\sqrt{2}\varepsilon\right].$$

Then, the family of linear state feedback laws (3.28) is given by

$$u(k) = F(\varepsilon)A^r x(k). \tag{3.66}$$

To design an output feedback law, we choose $L = \left[-2\sqrt{2}\ -4\ -2\sqrt{2}\ 3/4\right]^{\mathrm{T}}$, which places the eigenvalues of $A+LC$ at $z = \pm 1/2 \pm j/2$, and obtain the family of linear output feedback laws (3.52) as follows:

$$\begin{cases} \hat{x}(k+1) = \begin{bmatrix} -2\sqrt{2} & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -2\sqrt{2} & 0 & 0 & 1 \\ -1/4 & 0 & -2 & 2\sqrt{2} \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} A^r \hat{x}(k-r) - \begin{bmatrix} -2\sqrt{2} \\ -4 \\ -2\sqrt{2} \\ 3/4 \end{bmatrix} y(k), \\ u(k) = F(\varepsilon)A^r \hat{x}(k). \end{cases} \tag{3.67}$$

Some simulation results of the resulting closed-loop systems are shown in Figs. 3.1, 3.2, 3.3, and 3.4.

3.3 The Lyapunov Equation Based Design

This section presents an alternative low gain feedback design for the parametrization of the feedback gain matrix F of the TPF law (3.3). Like the eigenstructure assignment based low gain feedback design presented in Sect. 3.2, the Lyapunov equation based design enables the TPF law to compensate an arbitrarily large input delay in a discrete-time linear system without exponentially unstable open loop poles.

3.3.1 Low Gain Feedback Design

Consider a controllable pair (A, B) where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We construct a parameterized feedback gain matrix $F(\gamma)$ as follows:

$$F(\gamma) = -\left(I_m + B^{\mathrm{T}}P(\gamma)B\right)^{-1} B^{\mathrm{T}}P(\gamma)A, \tag{3.68}$$

where $P(\gamma)$ is the unique positive definite solution to the following parametric discrete-time algebraic Riccati equation

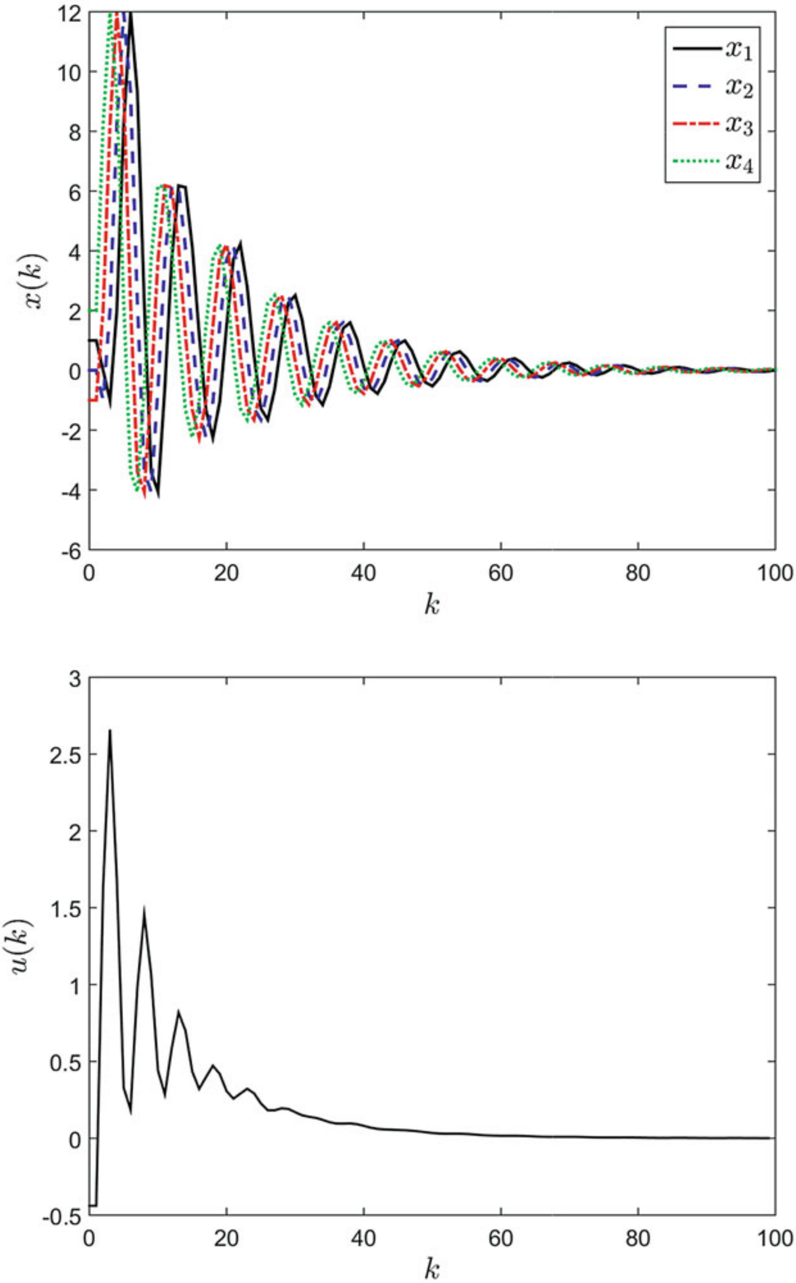


Fig. 3.1 State response and control input under the state feedback TPF law (3.28): $r = 1$ and $\varepsilon = 0.1$

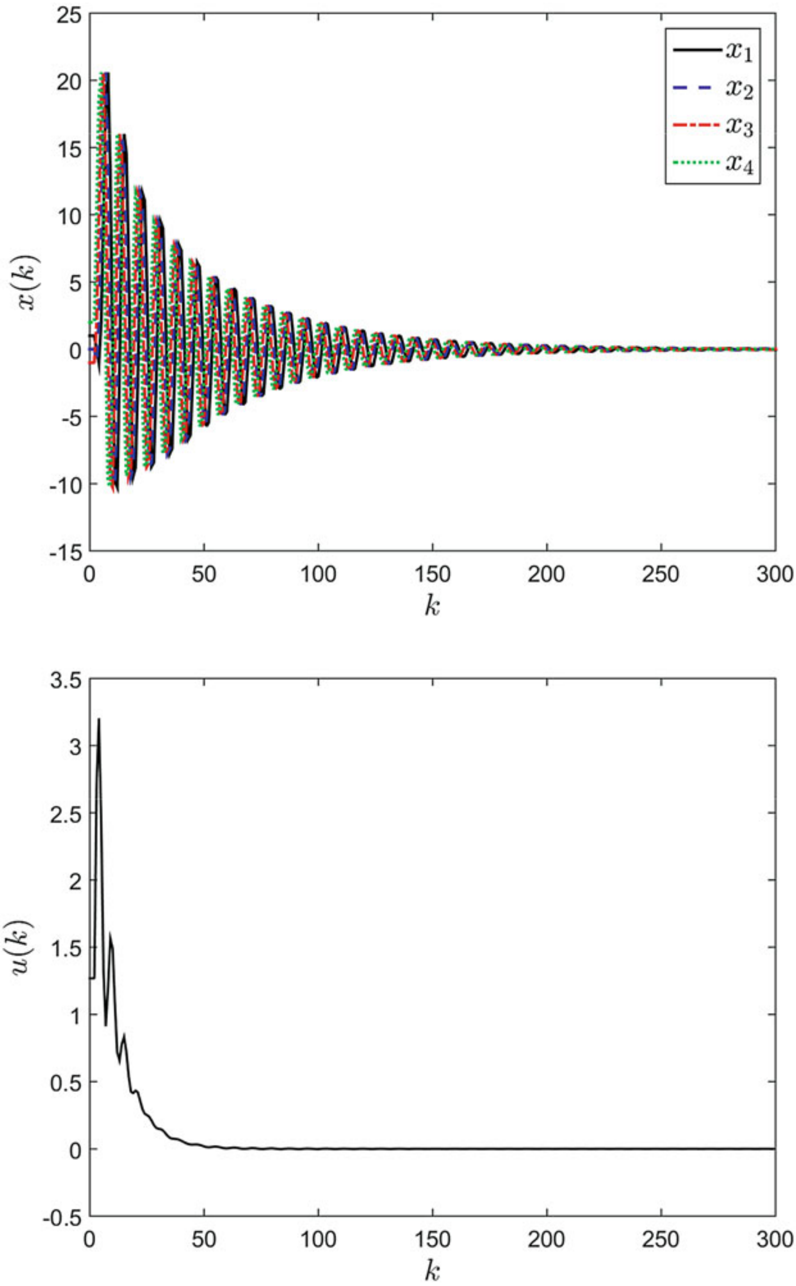


Fig. 3.2 State response and control input under the state feedback TPF law (3.28): $r = 2$ and $\varepsilon = 0.04$

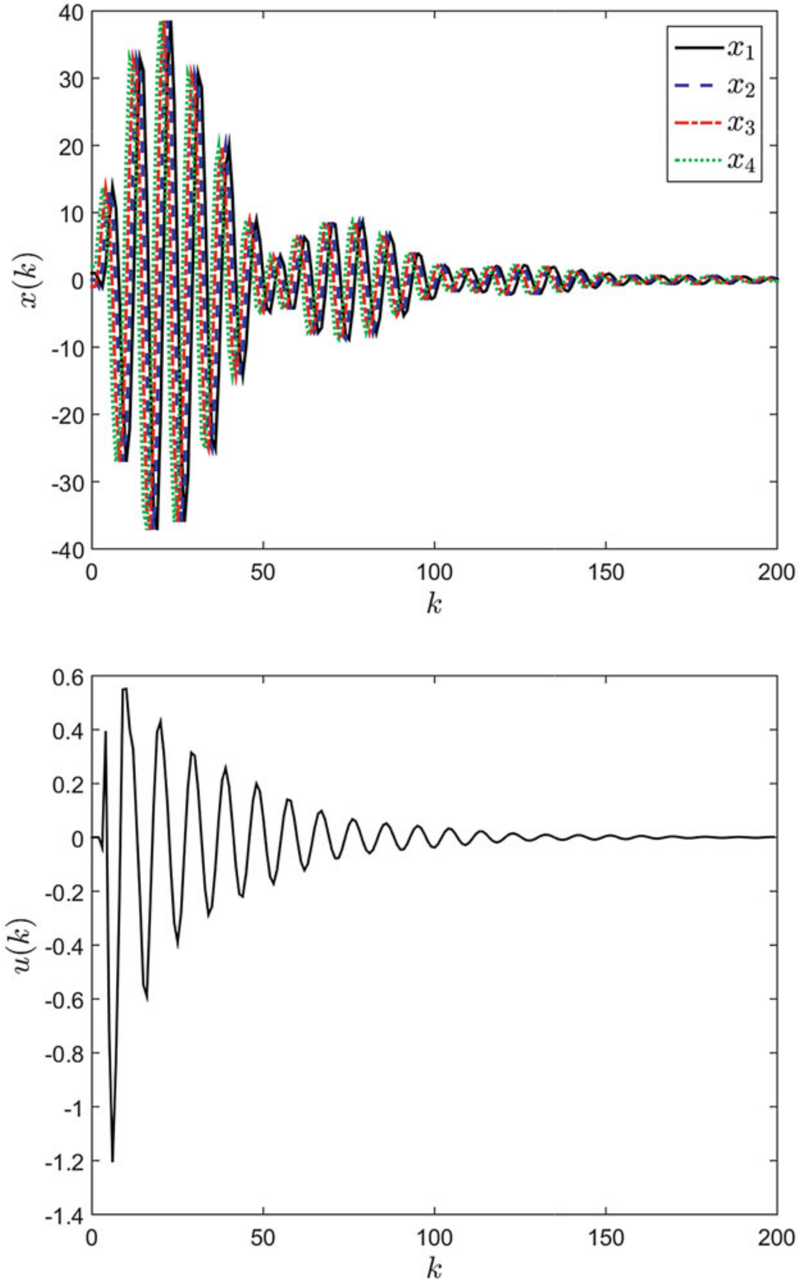
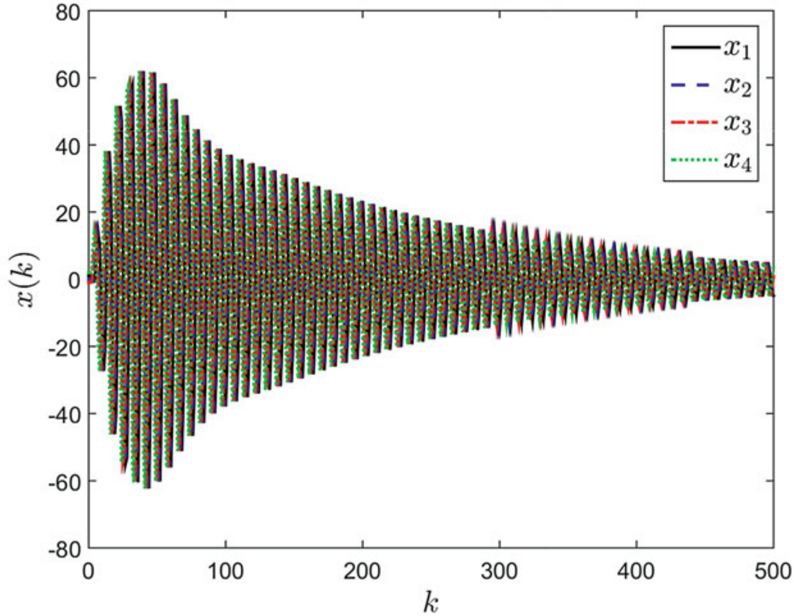


Fig. 3.3 State response and control input under the output feedback TPF law (3.52): $r = 1$ and $\varepsilon = 0.05$



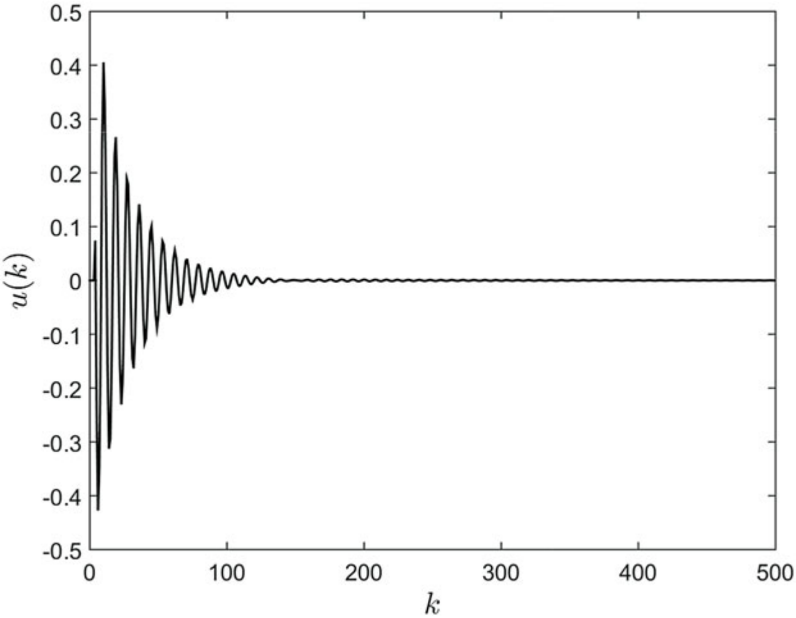


Fig. 3.4 State response and control input under the output feedback TPF law (3.52): $r = 2$ and $\varepsilon = 0.01$

$$A^{\mathrm{T}} P(\gamma) A - P(\gamma) - A^{\mathrm{T}} P(\gamma) B \left(I_m + B^{\mathrm{T}} P(\gamma) B \right)^{-1} B^{\mathrm{T}} P(\gamma) A = -\gamma P(\gamma),$$
$$\gamma \in \left(1 - \min\left\{ |\lambda(A)|^2 \right\}, 1\right). \tag{3.69}$$

According to [123], the range of γ guarantees the existence and uniqueness of a positive definite solution $P(\gamma)$. Also, such a design requires that $\min\{|\lambda(A)|\} \neq 0$, which implies that the open loop system cannot have any poles at the origin.

To compute $P(\gamma)$, we can first compute the positive definite solution $W(\gamma)$ to the following discrete-time Lyapunov equation,

$$W(\gamma) - \frac{1}{1-\gamma} A W(\gamma) A^{\mathrm{T}} = -B B^{\mathrm{T}}. \tag{3.70}$$

By taking $P(\gamma) = W^{-1}(\gamma)$, we obtain the solution of $P(\gamma)$. Equation (3.70) results from Eq. (3.69) by letting $W(\gamma) = P^{-1}(\gamma)$.

When all eigenvalues of A are on the unit circle, we recall the following lemmas on the properties of the solution $P(\gamma)$ to (3.69).

Lemma 3.4 ([100, 125]) *The eigenvalues of A and those of $A + BF(\gamma)$ are reciprocal with respect to $|s| = \sqrt{1-\gamma}$ in the complex plane, i.e.,*

$$\lambda(A)\lambda(A + BF(\gamma)) = 1 - \gamma, \tag{3.71}$$

$P(\gamma)$ is a rational matrix in γ, and $P(\gamma)$ is strictly increasing with respect to γ. If all eigenvalues of A are on or outside the unit circle, the unique positive definite solution $P(\gamma)$ to (3.69) satisfies

$$(A^i)^{\mathrm{T}} P A^i \leq \frac{\det^2(A)}{(1-\gamma)^{(n-1)i}} P, \tag{3.72}$$

$$F^{\mathrm{T}}(\gamma) R(\gamma) F(\gamma) \leq \frac{\det^2(A) - (1-\gamma)^n}{(1-\gamma)^{n-1}} P(\gamma), \tag{3.73}$$

where $i \in \mathbb{N}$, $R(\gamma) = I_m + B^{\mathrm{T}} P(\gamma) B$, and $F(\gamma)$ is given in (3.68). If, in addition, all eigenvalues of A are on the unit circle,

$$\lim_{\gamma \to 0^+} P(\gamma) = 0. \tag{3.74}$$

Lemma 3.5 ([100, 125]) *Assume that all eigenvalues of A are on or outside the unit circle. Then, the unique positive definite solution $P(\gamma)$ to (3.69) satisfies*

$$\left(A^{r-s-1}BR(\gamma)^{-1}B^{\mathrm{T}}P(\gamma)A^{r+1}\right)^{\mathrm{T}} F^{\mathrm{T}}(\gamma)R(\gamma)F(\gamma) \tag{3.75}$$

$$\times A^{r-s-1}BR^{-1}(\gamma)B^{\mathrm{T}}P(\gamma)A^{r+1}$$

$$\leq \frac{\left(\det^2(A)-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(2r-s+1)}}(\det^2(A))^{2r-s-2}P(\gamma), \tag{3.76}$$

with $r \geq 1$, $s \in I[0, r-1]$ *and* $R(\gamma)$ *is defined as in Lemma 3.4.*

Property (3.74) in Lemma 3.4 and (3.68) imply that the feedback gain matrix $F(\gamma)$ goes to zero as γ goes to zero. Such a parametrization of a feedback gain matrix is referred to as the Lyapunov equation based low gain feedback design.

Throughout the book, stability analysis of time delay systems under feedback laws whose feedback gain matrices are constructed by adopting the discrete-time Lyapunov equation based low gain feedback design frequently utilizes the following discrete-time Jensen's Inequality.

Lemma 3.6 ([40]) *For any positive semi-definite matrix* $M \geq 0$, *two integers* r_2 *and* r_1 *with* $r_2 \geq r_1$, *and a vector valued function* $\omega : I[r_1, r_2] \to \mathbb{R}^n$, *then*

$$\left(\sum_{i=r_1}^{r_2} \omega(i)\right)^{\mathrm{T}} M \left(\sum_{i=r_1}^{r_2} \omega(i)\right) \leq (r_2 - r_1 + 1) \sum_{i=r_1}^{r_2} \omega^{\mathrm{T}}(i) M \omega(i).$$

3.3.2 Truncated Predictor State Feedback Design

In this subsection, we present the stabilization of system (3.1) by a truncated predictor state feedback law whose feedback gain matrix is parameterized by the use of the Lyapunov equation based low gain feedback design.

Without loss of generality, we assume that (A, B) of system (3.1) is given in the following form:

$$A = \begin{bmatrix} A_{\mathrm{s}} & 0 \\ 0 & A_{\mathrm{o}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{s}} \\ B_{\mathrm{o}} \end{bmatrix}, \tag{3.77}$$

where $A_{\mathrm{s}} \in \mathbb{R}^{n_{\mathrm{s}} \times n_{\mathrm{s}}}$ contains all eigenvalues of A that are strictly inside the unit circle, $A_{\mathrm{o}} \in \mathbb{R}^{n_{\mathrm{o}} \times n_{\mathrm{o}}}$ contains all eigenvalues of A that are on the unit circle, and $n_{\mathrm{s}} + n_{\mathrm{o}} = n$. The stabilizability of (A, B) then implies that $(A_{\mathrm{o}}, B_{\mathrm{o}})$ is controllable. Clearly, the subsystem $(A_{\mathrm{s}}, B_{\mathrm{s}})$ does not affect the stabilizability of the system. In the following theorem, we can safely omit all stable eigenvalues of A.

Theorem 3.3 *Assume that all eigenvalues of* A *are on the unit circle. Then there exists a positive scalar* $\gamma^* \in (0, 1)$, *where*

$$\gamma^* = \frac{2^{-\left(\frac{(n-1)(2r+1)}{2}+1\right)}}{n\sqrt{nr}}, \tag{3.78}$$

such that the following state feedback TPF law

$$u(k-r) = -F(\gamma)A^r x(k-r), \quad \gamma \in \left(0, \gamma^*\right], \quad k \geq 0, \tag{3.79}$$

asymptotically stabilizes system (3.1), where $F(\gamma)$ is given by (3.68).

Proof With the feedback law (3.79), the resulting closed-loop system is given by

$$x(k+1) = Ax(k) - BFA^r x(k-r). \tag{3.80}$$

We solve (3.80) to give

$$A^r x(k-r) = x(k) + \sum_{s=0}^{r-1} A^{r-s-1} BFA^r x(k+s-2r),$$

with which the closed-loop system (3.80) can be written as,

$$x(k+1) = A_{\mathrm{c}} x(k) - BFq(k), \tag{3.81}$$

where

$$A_{\mathrm{c}} = A + BF, \quad q(k) = \sum_{s=0}^{r-1} A^{r-s-1} BFA^r x(k+s-2r). \tag{3.82}$$

Choose the following Lyapunov function:

$$V(x(k)) = x^{\mathrm{T}}(k)Px(k).$$

Then the forward difference of $V(x(k))$ along the trajectory of system (3.81) can be evaluated as follows:

$$\begin{aligned} \Delta V(x(k)) &\triangleq x^{\mathrm{T}}(k+1)Px(k+1) - x^{\mathrm{T}}(k)Px(k) \\ &= x^{\mathrm{T}}(k)\left(A_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c}} - P\right)x(k) + q^{\mathrm{T}}(k)F^{\mathrm{T}}B^{\mathrm{T}}PBFq(k) \\ &\quad - x^{\mathrm{T}}(k)A_{\mathrm{c}}^{\mathrm{T}}PBFq(k) - q^{\mathrm{T}}(k)F^{\mathrm{T}}B^{\mathrm{T}}PA_{\mathrm{c}}x(k). \end{aligned} \tag{3.83}$$

Therefore, by using (3.69), Eq. (3.83) can be further simplified as

$$\Delta V(x(k)) = x^{\mathrm{T}}(k)\left(-\gamma P - F^{\mathrm{T}}F\right)x(k) + q^{\mathrm{T}}(k)F^{\mathrm{T}}B^{\mathrm{T}}PBFq(k)$$

$$-x^{\mathrm{T}}(k)\,F^{\mathrm{T}}Fq(k) - q^{\mathrm{T}}(k)\,F^{\mathrm{T}}Fx(k)\,. \tag{3.84}$$

By using Young's Inequality, we have

$$-2x^{\mathrm{T}}(k)\,F^{\mathrm{T}}Fq(k) \leq x^{\mathrm{T}}(k)\,F^{\mathrm{T}}Fx(k) + q^{\mathrm{T}}(k)\,F^{\mathrm{T}}Fq(k)\,.$$

Inserting the above inequality into (3.84) gives

$$\Delta V(x(k)) \leq -\gamma x^{\mathrm{T}}(k)\,Px(k) + q^{\mathrm{T}}(k)\,F^{\mathrm{T}}RFq(k)\,. \tag{3.85}$$

By using (3.82) and Lemma 3.6, we obtain

$$\begin{aligned}
&q^{\mathrm{T}}(k)\,F^{\mathrm{T}}RFq(k) \\
&\leq r\sum_{s=0}^{r-1}\left(A^{r-s-1}BFA^{r}x(k+s-2r)\right)^{\mathrm{T}}F^{\mathrm{T}}RF\left(A^{r-s-1}BFA^{r}x(k+s-2r)\right) \\
&= r\sum_{s=0}^{r-1}x^{\mathrm{T}}(k+s-2r)\,U(s,\gamma)\,x(k+s-2r)\,,
\end{aligned}$$

where $U(s,\gamma)$ is defined as the left-hand side of (3.75). Then it follows from Lemma 3.5 that

$$q^{\mathrm{T}}(k)\,F^{\mathrm{T}}RFq(k) \leq r\sum_{s=0}^{r-1}\frac{\left(1-(1-\gamma)^{n}\right)^{3}}{(1-\gamma)^{(n-1)(2r-s+1)}}x^{\mathrm{T}}(k+s-2r)\,Px(k+s-2r)\,. \tag{3.86}$$

Therefore, under the condition

$$V(x(k+s)) < \eta V(x(k))\,,\, s\in I[-2r,0],\quad \eta>1,$$

it follows from (3.86) that

$$q^{\mathrm{T}}(k)\,F^{\mathrm{T}}RFq(k) \leq \eta r\sum_{s=0}^{r-1}\frac{\left(1-(1-\gamma)^{n}\right)^{3}}{(1-\gamma)^{(n-1)(2r-s+1)}}x^{\mathrm{T}}(k)\,Px(k)\,.$$

With this, the inequality in (3.85) reduces to

$$\Delta V(x(k)) \leq -\gamma g(\gamma,\eta)\,x^{\mathrm{T}}(k)\,Px(k)\,, \tag{3.87}$$

where $g(\gamma,\eta)$ is given by

$$g(\gamma,\eta) = 1 - \frac{\eta r\left(1-(1-\gamma)^n\right)^3}{\gamma(1-\gamma)^{(n-1)(2r+1)}} \sum_{s=0}^{r-1}\left((1-\gamma)^{(n-1)}\right)^s$$
$$\geq 1 - \frac{\eta r^2\left(1-(1-\gamma)^n\right)^3}{\gamma(1-\gamma)^{(n-1)(2r+1)}}. \tag{3.88}$$

It is clear that $\lim_{\gamma\to 0^+} g(\gamma,\eta) = 1$ and $\lim_{\gamma\to 1^-} g(\gamma,\eta) = -\infty$. Then, for any $\eta > 1$, there exists $\gamma^*(\eta) \in (0,1)$ such that

$$g(\gamma,\eta) \geq \frac{1}{2}, \quad \forall \gamma \in \left[0, \gamma^*(\eta)\right]. \tag{3.89}$$

Therefore, it follows from (3.87) and (3.89) that

$$\Delta V(x(k)) \leq -\frac{1}{2}\gamma x^{\mathrm{T}}(k) P x(k), \quad \forall \gamma \in \left(0, \gamma^*(\eta)\right],$$

which, in view of the Razumikhin Stability Theorem for discrete-time systems (Theorem 1.4), implies the asymptotic stability of the closed-loop system (3.80).

Therefore, to complete the proof, we need only to show that there exists $\eta > 1$ such that (3.89) is satisfied with $\gamma \in (0, \gamma^*]$. To show this, we note that $1-\gamma < 1$ and it follows that

$$1-(1-\gamma)^n = \gamma \sum_{s=0}^{n-1}(1-\gamma)^s \leq n\gamma. \tag{3.90}$$

We then deduce from (3.88) that

$$g(\gamma,\eta) \geq 1 - \frac{\eta r^2\gamma^2 n^3}{(1-\gamma)^{(n-1)(2r+1)}}. \tag{3.91}$$

Since $n \geq 1$ and $r \geq 1$, we compute from (3.78) that

$$1-\gamma \geq 1-\gamma^* = 1 - \left(2^{\left(\frac{(n-1)(2r+1)}{2}+1\right)} n\sqrt{nr}\right)^{-1} \geq \frac{1}{2}.$$

Thus, if we let $\eta = 2$, the inequality in (3.91) can be continued as

$$g(\gamma,2) \geq 1 - 2r^2\gamma^2 n^3 2^{(n-1)(2r+1)}$$
$$\geq 1 - 2r^2\left(\gamma^*\right)^2 n^3 2^{(n-1)(2r+1)}$$
$$= 1 - \left(2^{\left(\frac{(n-1)(2r+1)}{2}+1\right)} n\sqrt{nr}\right)^{-2} 2r^2 n^3 2^{(n-1)(2r+1)}$$

$$= 1 - \frac{2}{4}$$
$$= \frac{1}{2},$$

which completes the proof. □

3.3.3 Truncated Predictor Output Feedback Design

In this subsection, we propose the stabilization of system (3.1) by a truncated predictor output feedback law whose feedback gain matrix is parameterized by the use of the Lyapunov equation based low gain feedback design. For an output feedback problem, it is no longer without loss of generality to assume that all the eigenvalues of A are located on the unit circle. To see this, assume that (A, B) is in the form of (3.77) and let

$$x(k) = \left[x_{\mathrm{o}}^{\mathrm{T}}(k), x_{\mathrm{s}}^{\mathrm{T}}(k)\right]^{\mathrm{T}},$$

then system (3.1) becomes

$$\begin{cases} x_{\mathrm{s}}(k+1) = A_{\mathrm{s}} x_{\mathrm{s}}(k) + B_{\mathrm{s}} u(k-r), \\ x_{\mathrm{o}}(k+1) = A_{\mathrm{o}} x_{\mathrm{o}}(k) + B_{\mathrm{o}} u(k-r), \\ y(k) = C\left[x_{\mathrm{o}}^{\mathrm{T}}(k)\ x_{\mathrm{s}}^{\mathrm{T}}(k)\right]^{\mathrm{T}}, \end{cases} \tag{3.92}$$

which indicates that the stable substate $x_{\mathrm{s}}(k)$ will appear in the output signal $y(k)$, and the output signal related to the unstable substate $x_{\mathrm{o}}(k)$ cannot be separated from the whole output signal y. Therefore, the stable open loop poles of the system cannot be ignored when we carry out an output feedback design and the corresponding closed-loop stability analysis.

We construct the following observer-based output feedback law for system (3.1):

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k-r) - L(y(k) - C\hat{x}(k)), \\ u(k) = \tilde{F}(\gamma)A^r\hat{x}(k), \end{cases} \tag{3.93}$$

where

$$\tilde{F}(\gamma) = -\left(I + B^{\mathrm{T}}\tilde{P}(\gamma)B\right)^{-1} B^{\mathrm{T}}\tilde{P}(\gamma)A, \quad \tilde{P}(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & P_{\mathrm{o}}(\gamma) \end{bmatrix}, \tag{3.94}$$

and $P_{\mathrm{o}}(\gamma)$ is the unique positive definite solution to the discrete-time parametric algebraic Riccati equation

$$A_o^T P_o(\gamma) A_o - P_o(\gamma) - A_o^T P_o(\gamma) B_o \left(I + B_o^T P_o(\gamma) B_o\right)^{-1} B_o^T P_o(\gamma) A_o = -\gamma P_o(\gamma), \tag{3.95}$$

and L is such that $A + LC$ is Schur stable. The detectability of the pair (A, C) implies that such an L exists. Define an error between the state of the system of the system and the state of the observer as $e(k) = x(k) - \hat{x}(k)$. Then, the dynamic of this error signal is computed as

$$e(k+1) = (A + LC)e(k). \tag{3.96}$$

Consequently, the closed-loop system consisting of the open loop system (3.1) and the output feedback law (3.93) can be written as,

$$\begin{cases} x(k+1) = Ax(k) + B\tilde{F}(\gamma)A^r(x(k-r) - e(k-r)), \\ e(k+1) = (A + LC)e(k). \end{cases} \tag{3.97}$$

Recalling the structure of the pair (A, B) as given in (3.77) and $\tilde{P}(\gamma)$ as in (3.94), we see that the parameterized feedback gain matrix of the output feedback law has the following form:

$$\tilde{F}(\gamma) = -\left(I + B_o^T P_o(\gamma) B_o\right)^{-1} \left[0 \ B_o^T P_o(\gamma) A_o\right], \tag{3.98}$$

from which the closed-loop system can be decomposed into the following three subsystems:

$$\begin{cases} x_s(k+1) = A_s x_s(k) + B_s F_o(\gamma) A_o^r (x_o(k-r) - e_o(k-r)), \\ x_o(k+1) = A_o x_o(k) + B_o F_o(\gamma) A_o^r (x_o(k-r) - e_o(k-r)), \\ e(k+1) = (A + LC)e(k), \end{cases} \tag{3.99}$$

where $F_o(\gamma) = -(I + B_o^T P_o(\gamma) B_o)^{-1} B_o^T P_o(\gamma) A_o$ is defined for notational simplicity. It is clear from A_o in the first subsystem of the closed-loop system that the asymptotic stability of the system

$$\begin{cases} x_o(k+1) = A_o x_o(k) + B_o F_o(\gamma) A_o^r (x_o(k-r) - e_o(k-r)), \\ e(k+1) = (A + LC)e(k) \end{cases} \tag{3.100}$$

implies that of the whole closed-loop system. Therefore, stabilization of system (3.1) by the output feedback TPF law (3.93) is achieved if the stability of system (3.100) is established.

Theorem 3.4 *Assume that A_s does not have eigenvalues at the origin and A_o has all its eigenvalues on the unit circle. Then, there exists a sufficiently small positive constant γ^* such that for each $\gamma \in (0, \gamma^*]$, the closed-loop system (3.100) is asymptotically stable.*

Proof Define a Lyapunov function

$$V(x_{\mathrm{o}}(k), e(k)) = x_{\mathrm{o}}^{\mathrm{T}}(k)P_{\mathrm{o}}x_{\mathrm{o}}(k) + e^{\mathrm{T}}(k)Qe(k), \tag{3.101}$$

where Q is the positive definite solution to the following discrete-time Lyapunov equation,

$$(A + LC)^{\mathrm{T}}Q(A + LC) - Q = -I. \tag{3.102}$$

The existence and uniqueness of such Q are due to the fact that $A + LC$ is Schur stable. Then, we define the forward difference of the Lyapunov function as

$$\Delta V(x_{\mathrm{o}}(k), e(k)) = V(x_{\mathrm{o}}(k+1), e(k+1)) - V(x_{\mathrm{o}}(k), e(k)). \tag{3.103}$$

To facilitate our stability analysis, we rewrite the dynamic of system (3.100) as follows:

$$\begin{cases} x_{\mathrm{o}}(k+1) = A_{\mathrm{oc}}(\gamma)x_{\mathrm{o}}(k) + B_{\mathrm{o}}F_{\mathrm{o}}(\gamma)\lambda(k) - B_{\mathrm{o}}F_{\mathrm{o}}(\gamma)A_{\mathrm{o}}^{r}e_{\mathrm{o}}(k-r), \\ e(k+1) = (A + LC)e(k), \end{cases} \tag{3.104}$$

where $A_{\mathrm{oc}}(\gamma) = A_{\mathrm{o}} + B_{\mathrm{o}}F_{\mathrm{o}}(\gamma)$ and $\lambda(k) = A_{\mathrm{o}}^{r}x_{\mathrm{o}}(k-r) - x_{\mathrm{o}}(k)$. By the first equation of (3.100), we obtain

$$\lambda(k) = -\sum_{s=1}^{r} A_{\mathrm{o}}^{s-1}B_{\mathrm{o}}F_{\mathrm{o}}A_{\mathrm{o}}^{r}\hat{x}_{\mathrm{o}}(k-s-r),$$

from which and the first equation of (3.1), the forward difference $\Delta V(x_{\mathrm{o}}(k), e(k))$ along the trajectory of the closed-loop system can be computed as

$$\begin{aligned} &\Delta V(x_{\mathrm{o}}(k), e(k)) \\ &= x_{\mathrm{o}}^{\mathrm{T}}(k)A_{\mathrm{oc}}^{\mathrm{T}}P_{\mathrm{o}}A_{\mathrm{oc}}x_{\mathrm{o}}(k) + \lambda_{\mathrm{o}}(k)F_{\mathrm{o}}^{\mathrm{T}}B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}\lambda_{\mathrm{o}}(k) \\ &\quad + e_{\mathrm{o}}^{\mathrm{T}}(k-r)(A_{\mathrm{o}}^{r})^{\mathrm{T}}F_{\mathrm{o}}^{\mathrm{T}}B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}A_{\mathrm{o}}^{r}e_{\mathrm{o}}(k-r) + 2x_{\mathrm{o}}^{\mathrm{T}}(k)A_{\mathrm{oc}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}\lambda_{\mathrm{o}}(k) \\ &\quad - 2x_{\mathrm{o}}^{\mathrm{T}}(k)A_{\mathrm{oc}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}A_{\mathrm{o}}^{r}e_{\mathrm{o}}(k-r) - 2\lambda_{\mathrm{o}}^{\mathrm{T}}(k)F_{\mathrm{o}}^{\mathrm{T}}B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}A_{\mathrm{o}}^{r}e_{\mathrm{o}}(k-r) \\ &\quad - e^{\mathrm{T}}(k)e(k) - x_{\mathrm{o}}^{\mathrm{T}}(k)P_{\mathrm{o}}x_{\mathrm{o}}(k). \end{aligned} \tag{3.105}$$

By using Young's Inequality and Eq. (3.95), the forward difference can be evaluated as follows:

$$\begin{aligned} &\Delta V(x_{\mathrm{o}}(k), e(k)) \\ &\le -\gamma x_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}x_{\mathrm{o}} + 2\lambda_{\mathrm{o}}^{\mathrm{T}}F_{\mathrm{o}}^{\mathrm{T}}(I + B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}})F_{\mathrm{o}}^{\mathrm{T}}\lambda_{\mathrm{o}} \\ &\quad + 2e_{\mathrm{o}}^{\mathrm{T}}(k-r)(A_{\mathrm{o}}^{r})^{\mathrm{T}}F_{\mathrm{o}}^{\mathrm{T}}(I + B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}})F_{\mathrm{o}}A_{\mathrm{o}}^{r}e_{\mathrm{o}}(k-r) - e^{\mathrm{T}}(k)e(k). \end{aligned} \tag{3.106}$$

The use of the discrete-time Jensen's Inequality given by Lemmas 3.6 and 3.5 yields

$$\lambda_{\rm o}^{\rm T} F_{\rm o}^{\rm T} \left(I + B_{\rm o}^{\rm T} P_{\rm o} B_{\rm o}\right) F_{\rm o} \lambda_{\rm o} \\ \leq r \sum_{s=1}^{r} \frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r+s+1)}} \hat{x}_{\rm o}^{\rm T}(k-s-r) P_{\rm o} \hat{x}_{\rm o}(k-s-r). \tag{3.107}$$

Moreover, a combined use of Lemmas 3.4 and 3.5 gives

$$(A_{\rm o}^r)^{\rm T} F_{\rm o}^{\rm T} \left(I + B_{\rm o}^{\rm T} P_{\rm o} B_{\rm o}\right) F_{\rm o} A_{\rm o}^r \leq \frac{1-(1-\gamma)^n}{(1-\gamma)^{(n-1)(r+1)}} P_{\rm o}, \tag{3.108}$$

which, along with inequality (3.108), implies

$$\Delta V(x_{\rm o}(k), e(k)) \\ \leq -\gamma x_{\rm o}^{\rm T} P_{\rm o} x_{\rm o} + 2r \sum_{s=1}^{r} \frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r+s+1)}} \hat{x}_{\rm o}^{\rm T}(k-s-r) P_{\rm o} \hat{x}_{\rm o}(k-s-r) \\ + 2\frac{1-(1-\gamma)^n}{(1-\gamma)^{(n-1)(r+1)}} e_{\rm o}^{\rm T}(k-r) P_{\rm o} e_{\rm o}(k-r) - e^{\rm T}(k) e(k). \tag{3.109}$$

By the assumptions that the pair (A, C) is detectable and that $A_{\rm s}$ does not have any eigenvalues at the origin, there exists matrix L such that $A + LC$ is Schur stable and does not have eigenvalues at the origin neither. Therefore, $A + LC$ is invertible. Based on the invertibility of $A + LC$ and the dynamics of the error $e(k)$, we obtain

$$e(k-r) = (A + LC)^{-r} e(k). \tag{3.110}$$

Then, the forward difference can be further evaluated as,

$$\Delta V(x_{\rm o}(k), e(k)) \\ \leq -\gamma x_{\rm o}^{\rm T} P_{\rm o} x_{\rm o} + 2r \sum_{s=1}^{r} \frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r+s+1)}} x_{\rm o}^{\rm T}(k-s-r) P_{\rm o} x_{\rm o}(k-s-r) \\ + e^{\rm T}(k) \Bigg(4r \sum_{s=1}^{r} \frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r+s+1)}} \left((A+LC)^{\rm T}\right)^{-(s+r)} \tilde{P} (A+LC)^{-(s+r)} \\ + 2\frac{1-(1-\gamma)^n}{(1-\gamma)^{(n-1)(r+1)}} \left((A+LC)^{\rm T}\right)^{-r} \tilde{P} (A+LC)^{-r} - I \Bigg) e(k). \tag{3.111}$$

By the properties of $\tilde{P}(\gamma)$ as given in Lemma 3.4, there exists a sufficiently small γ_1^* such that for each $\gamma \in (0, \gamma_1^*]$, the following inequality holds:

$$4r\sum_{s=1}^{r}\frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r+s+1)}}\left((A+LC)^{\mathrm{T}}\right)^{-(s+r)}\tilde{P}(A+LC)^{-(s+r)}$$
$$+2\frac{1-(1-\gamma)^n}{(1-\gamma)^{(n-1)(r+1)}}\left((A+LC)^{\mathrm{T}}\right)^{-r}\tilde{P}(A+LC)^{-r}-I\le-\gamma Q, \quad (3.112)$$

which implies that

$$\Delta V(x_{\mathrm{o}}(k),e(k))$$
$$\le-\gamma V(k)+4r\sum_{s=1}^{r}\frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r+s+1)}}x_{\mathrm{o}}^{\mathrm{T}}(k-s-r)P_{\mathrm{o}}x_{\mathrm{o}}(k-s-r).$$

Under the assumption that $V(x_{\mathrm{o}}(k+s),e)(k+s)<\eta V(x_{\mathrm{o}}(k),e(k))$, $s\in[-2r,0]$, where $\eta>1$, we have

$$\Delta V(x_{\mathrm{o}}(k),e(k))\le\left(-\gamma+4\eta r\sum_{s=1}^{r}\frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r+s+1)}}\right)V(x_{\mathrm{o}}(k),e(k)). \quad (3.113)$$

Note that there exists $\gamma^*\in(0,\gamma_1^*]$ such that, for each $\gamma\in(0,\gamma^*]$,

$$-\gamma+4\eta r\sum_{s=1}^{r}\frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r+s+1)}}\le-\frac{\gamma}{2}. \quad (3.114)$$

Then,

$$\Delta V(x_{\mathrm{o}}(k),e(k))\le-\frac{\gamma}{2}V(x_{\mathrm{o}}(k),e(k)). \quad (3.115)$$

The asymptotic stability of system (3.100) follows from the Razumikhin Stability Theorem for discrete-time systems (Theorem 1.4). □

3.3.4 A Numerical Example

We first consider the state feedback of a discrete-time linear system under the TPF law (3.79) when its feedback gain matrix is parameterized by the use of the Lyapunov equation based low gain feedback design. Consider system (3.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2\sqrt{2} & -4 & 2\sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.116}$$

It can be verified that A has all its eigenvalues on the unit circle, (A, B) is controllable and (A, C) is observable. Take $r = 3$ and the initial condition of the system as

$$x(s) = \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix}^{\mathrm{T}}, \quad s \in I[0, r]. \tag{3.117}$$

We tune the low gain parameter of the TPF law (3.79) to small values until the stabilization of the closed-loop system is achieved. Such a process results in $\gamma = 0.02$. The system performance is illustrated in Fig. 3.5.

We next examine the counterpart of the first simulation in the output feedback setting. Let L assign the eigenvalues of $A + LC$ at $\left\{\frac{1}{2} \pm \frac{1}{2}j, -\frac{1}{2} \pm \frac{1}{2}j\right\}$. The initial condition of the state and the observed state are given by

$$x(s) = \hat{x}(s) = \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix}^{\mathrm{T}}, \quad s \in I[0, r]. \tag{3.118}$$

As in the state feedback case, we tune the low gain parameter to a sufficiently small value $\gamma = 0.01$. The stabilization of the system by the output feedback law (3.93) can be readily seen from Fig. 3.6.

3.4 Conclusions

By truncating the finite summation term of the predictor feedback law for discrete-time systems, a TPF law that simplifies the implementation of the predictor feedback law was formulated. Such a design involves the parameterization of the feedback gain matrix of the TPF law, whether by the eigenstructure assignment based low gain feedback design or the Lyapunov equation based design. The low gain nature of the TPF law is the key to the stabilization of discrete-time linear systems with input delay whose open loop poles are inside or on the unit circle. Both state and output feedback TPF laws were explicitly constructed.

3.5 Notes and References

Parallel to Chap. 2, this chapter introduces the formulation, expression, and construction of the predictor feedback law in the discrete-time setting. The presentation of Sect. 3.2 mostly follows that of [64], except that we provide a different analysis

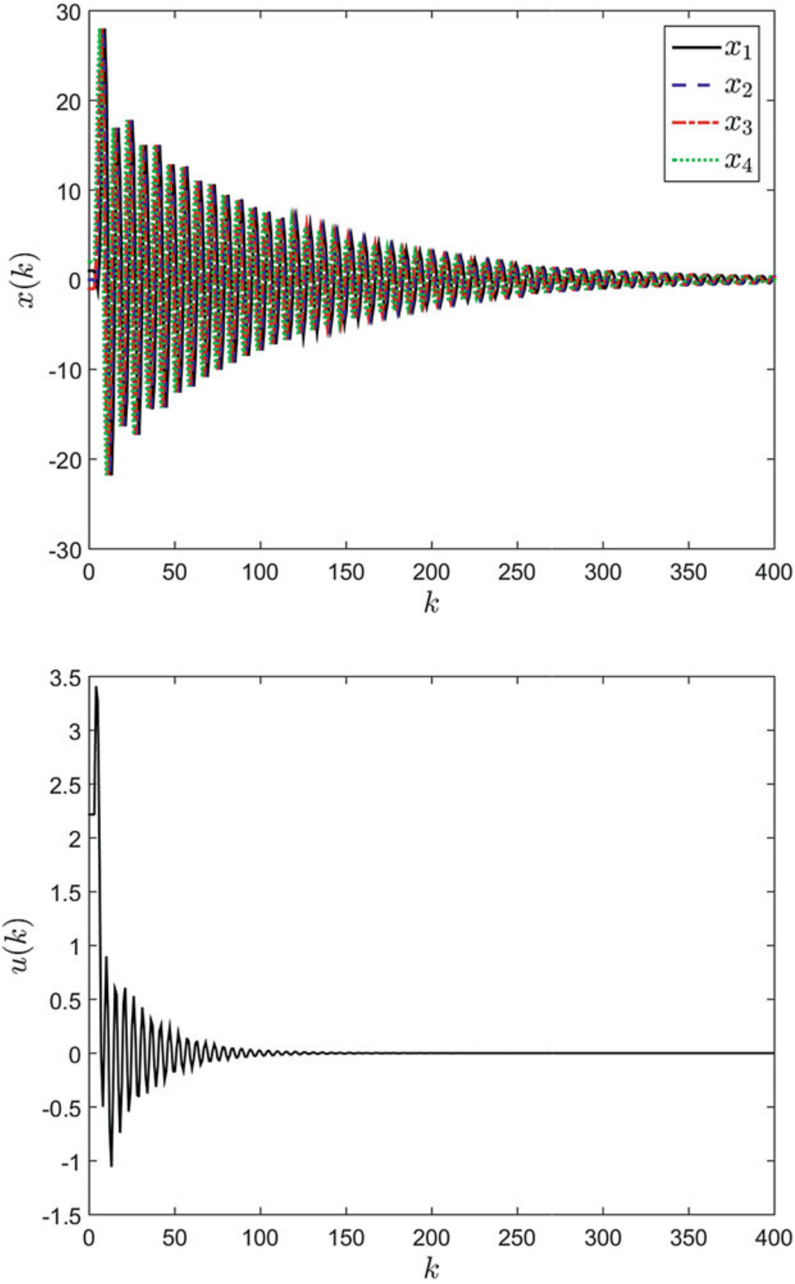


Fig. 3.5 State response and control input under the state feedback TPF law (3.79): $r = 3$ and $\gamma = 0.02$

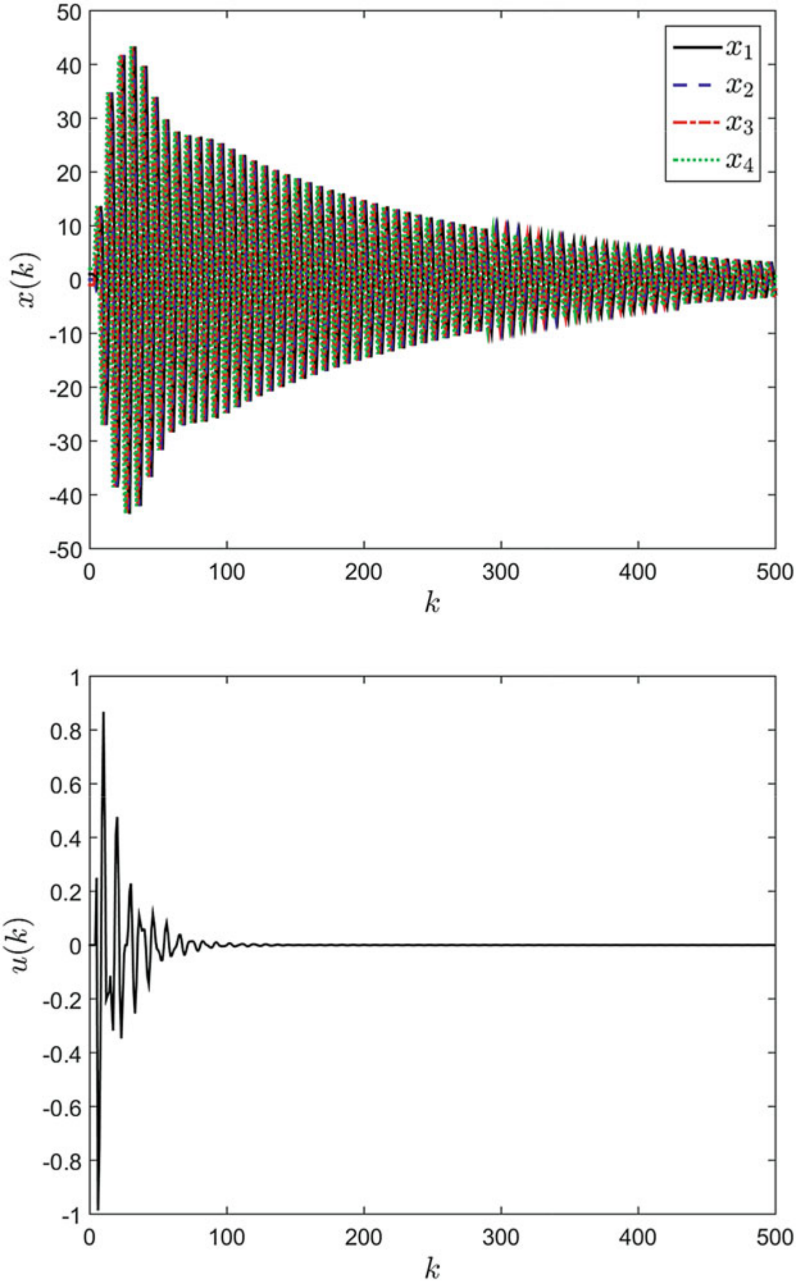


Fig. 3.6 State response and control input under the output feedback TPF law (3.93): $r = 3$ and $\gamma = 0.01$

of the conservativeness of the result of Theorem 3.1. The original analysis in [64] resorts to the method of root locus in order to determine the positions of the closed-loop poles. The analysis provided in this chapter scrutinizes the characteristic equation of the closed-loop system to reach the same conclusion. Also, the state feedback results of Sect. 3.3.2 were drawn from [125].

Chapter 4
Truncated Predictor Feedback for General Linear Systems

4.1 Introduction

Low gain feedback designs, such as the eigenstructure assignment based design and the Lyapunov equation based design, are essential elements of TPF designs. TPF laws whose feedback gain matrices are parameterized in a feedback parameter compensate an arbitrarily large delay in linear systems that are not exponentially unstable. Given any amount of delay, stabilization is achieved as long as the feedback parameter is sufficiently small. As manifested in Chaps. 2 and 3, such a low gain nature of the TPF laws, of both state feedback and output feedback types, is the core of their stabilizing ability.

The low gain feedback designs that have been introduced so far in this book actually do not restrict values of the feedback parameter to be small. A small value of the feedback parameter is required to achieve stabilization for linear systems that are not exponentially unstable and are subject to arbitrarily large bounded input delay. Take for example the Lyapunov equation based feedback design for continuous-time systems (see Sect. 2.3). Such a design is valid as long as a positive definite solution to the parametric algebraic Riccati equation (2.32) exists and is unique, that is, $\gamma > -2\min\{\mathrm{Re}(\lambda(A))\}$. It turns out that, given any feedback parameter within a range, the TPF law stabilizes a general, possibly exponentially unstable, linear system, as long as the delay is not too large. Stability of the closed-loop system can be seen as follows.

Because the stable open loop poles of a linear system typically do not affect the stabilizability of the system, we assume, without loss of generality, that all the open loop poles of the system are in the closed right-half plane. Let $\gamma \in \mathbb{R}^+$ in order to satisfy $\gamma > -2\min\{\mathrm{Re}(\lambda(A))\}$. Recall that the truncated predictor state feedback law is given as follows:

$$u(t) = F(\gamma)\mathrm{e}^{A\tau}x(t), \tag{4.1}$$

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5_4

where $F(\gamma)$ results from the parameterized Lyapunov equation based low gain feedback design. Given any $\gamma \in \mathbb{R}^+$, the Lyapunov equation based design assigns the eigenvalues of $A + BF(\gamma)$ at those positions in the complex plane that are symmetric to the eigenvalues of A with respect to $\text{Re}\{s\} = -\frac{\gamma}{2} < 0$ (see [121]). Since the eigenvalues of A are all in the closed right-half plane, the eigenvalues of $A + BF(\gamma)$ are all in the open left-half plane, which implies that the system is stabilized if the amount of delay is zero. If the amount of delay is nonzero, the overall feedback gain of the TPF law $\bar{F} = F(\gamma)\mathrm{e}^{A\tau}$ satisfies

$$\bar{F} \to F(\gamma) \text{ as } \tau \to 0^+. \tag{4.2}$$

Thus, such an $\bar{F}$ still assigns the eigenvalues of $A + B\bar{F}$ in the open left-half plane when the amount of delay is small. Recall from [67] the fact that the system

$$\dot{x}(t) = Ax(t) + B\bar{F}x(t - \tau) \tag{4.3}$$

with $A + B\bar{F}$ Hurwitz is asymptotically stable as long as the amount of the delay τ is sufficiently small. It then follows that the closed-loop system under the TPF law is asymptotically stable as long as the delay is sufficiently small.

The above examination is based on the robustness of the stability of a stable linear system to a small amount of state delay. This implies that the TPF law is robust to a certain amount of input delay, even though the feedback law itself is delay dependent due to its exponential factor $\mathrm{e}^{A\tau}$. In this chapter, we fully examine such a robustness property of the TPF law in the stabilization of a general linear system that is possibly exponentially unstable. Both state and output feedbacks are considered. All the feedback designs in this chapter adopt the Lyapunov equation based method.

We will deal with continuous-time and discrete-time systems separately, in Sects. 4.2 and 4.3.

4.2 Continuous-Time Systems

Recall the linear system with a time-varying input delay (2.28) from Sect. 2.3,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(\phi(t)), \\ y(t) = Cx(t), \end{cases} \tag{4.4}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the state and input, the pair (A, B) and the pair (A, C) are stabilizable and detectable, respectively, and $\phi(t) : \mathbb{R}_0^+ \to \mathbb{R}$ denotes the delay function. Here, $\phi(t)$ can be defined in a more standard form

$$\phi(t) = t - d(t), \tag{4.5}$$

where $d(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is the time-varying delay that is continuously differentiable. The function $\phi : \mathbb{R}_0^+ \to \mathbb{R}$ is a continuously differentiable, invertible, and exactly known function such that $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) > 0, t \geq 0$, and the delay $d(t)$ is bounded, namely, there exists a finite, yet arbitrarily large, number $D > 0$ such that

$$0 \leq d(t) \leq D, \quad t \in [0, \infty). \tag{4.6}$$

Section 2.3 considers stabilization of system (4.4) by TPF when the system has all its open loop poles on the closed left-half plane. In this section, we remove such a restriction on the open loop poles and solve the stabilization problem for a general linear system that may have exponentially unstable poles. The section is divided into two parts, containing state and output feedback results, respectively.

4.2.1 Truncated Predictor State Feedback Design

As explained in Sect. 2.3, the stable open loop poles of system (4.4) do not affect the stabilizability of the system under the truncated predictor state feedback. Thus, it is without loss of generality to assume that all the open loop poles of the system are in the closed right-half plane.

Recall the Lyapunov equation based truncated predictor state feedback law, also referred to as the state feedback TPF law, from Sect. 2.3,

$$u(t) = -B^{\mathrm{T}} P(\gamma) \mathrm{e}^{A(\phi^{-1}(t)-t)} x(t), \tag{4.7}$$

where $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation

$$A^{\mathrm{T}} P(\gamma) + P(\gamma) A - P(\gamma) B B^{\mathrm{T}} P(\gamma) = -\gamma P(\gamma), \quad \gamma > -2 \min\{\mathrm{Re}(\lambda(A))\}. \tag{4.8}$$

In the case where all eigenvalues of A are in the closed left-half plane, the delay can be arbitrarily large but bounded, and the value of the parameter is required to approach zero as the bound on the delay increases to infinity. As a result, the parametric algebraic Riccati equation (4.8) is a low gain feedback design and the feedback parameter is referred to as the low gain parameter.

The following theorem establishes stabilizability of system (4.4) by the state feedback TPF law (4.7).

Theorem 4.1 *Consider the system (4.4). Assume that all the eigenvalues of A are on the closed right-half plane. Then, if, for each $\gamma > 0$,*

$$D < D^*, \tag{4.9}$$

where D^* is the unique positive solution to the following equation:

$$D^2 \mathrm{e}^{2\gamma\omega D} = \frac{\gamma}{\big(2\mathrm{tr}(A) + n\gamma\big)^3}, \tag{4.10}$$

and ω is defined as in Lemma 2.4, then the state feedback TPF law (4.7) asymptotically stabilizes the system.

Proof Applying the state feedback TPF law (4.7) to system (4.4) results in the closed-loop system,

$$\dot{x}(t) = A_{\mathrm{c}} x(t) + BF\lambda(t), \tag{4.11}$$

where $A_{\mathrm{c}} = A + BF$, $F = F(\gamma) = -B^{\mathrm{T}} P(\gamma)$, and $\lambda(t)$ is defined as

$$\lambda(t) = \mathrm{e}^{Ad(t)} x(\phi(t)) - x(t) \tag{4.12}$$

and $\lambda(t)$ can be obtained by solving the closed-loop system as

$$\begin{aligned} \lambda(t) &= -\int_{\phi(t)}^{t} \mathrm{e}^{A(t-s)} B u(\phi(s)) \mathrm{d}s \\ &= -\int_{\phi(t)}^{t} \mathrm{e}^{A(t-s)} BF \mathrm{e}^{A(s-\phi(s))} x(\phi(s)) \mathrm{d}s. \end{aligned} \tag{4.13}$$

Define a Lyapunov function for the closed-loop system,

$$V(x(t)) = x^{\mathrm{T}}(t) P(\gamma) x(t), \tag{4.14}$$

and compute its time derivative along the trajectory of the closed-loop system as

$$\dot{V}(x(t)) = x^{\mathrm{T}}(t) \left(-\gamma P - PBB^{\mathrm{T}} P\right) x(t) + 2x^{\mathrm{T}}(t) PBF\lambda(t), \tag{4.15}$$

where we have used the parametric Lyapunov based equation (2.32), that is,

$$A^{\mathrm{T}} P(\gamma) + P(\gamma) A - P(\gamma) BB^{\mathrm{T}} P(\gamma) = -\gamma P(\gamma), \quad \gamma > -2\min\{\mathrm{Re}(\lambda(A))\}. \tag{4.16}$$

An estimate of $\dot{V}(x(t))$ is obtained by the use of Young's Inequality and Lemma 2.4,

$$\dot{V}(x(t)) \le -\gamma X^{\mathrm{T}}(t) P x(t) + \big(2\mathrm{tr}(A) + n\gamma\big) \lambda^{\mathrm{T}}(t) P \lambda(t). \tag{4.17}$$

A further estimate of $\dot{V}(x(t))$ relies on the following estimate of $\lambda^{\mathrm{T}}(t) P \lambda(t)$:

$$\lambda^{\mathrm{T}}(t) P \lambda(t) \le d(t) \int_{\phi(t)}^{t} x^{\mathrm{T}}(\phi(s)) \mathrm{e}^{A^{\mathrm{T}} d(s)} F^{\mathrm{T}} B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}(t-s)} P \mathrm{e}^{A(t-s)} BF \mathrm{e}^{Ad(s)} x(\phi(s)) \mathrm{d}s$$

$$\leq D\mathrm{e}^{2\gamma\omega D}\big(2\mathrm{tr}(A)+n\gamma\big)^2\int_{t-D}^{t}x^{\mathrm{T}}(\phi(s))Px(\phi(s))\mathrm{d}s, \tag{4.18}$$

where we have used Lemma 2.5, Lemma 2.4, and the fact that $d(t) \leq D$.

Using (4.18) and (4.17), we have

$$\dot{V}(x(t)) \leq -\gamma x^{\mathrm{T}}(t)Px(t) + \big(2\mathrm{tr}(A)+n\gamma\big)^3 D\mathrm{e}^{2\gamma\omega D}\int_{t-D}^{t}x^{\mathrm{T}}(\phi(s))Px(\phi(s))\mathrm{d}s. \tag{4.19}$$

When $V(x(t+\theta)) \leq \eta V(x(t))$, $\theta \in [-2D, 0]$, for some constant $\eta > 1$, we have

$$\dot{V}(x(t)) \leq \left(-\gamma + \eta D^2\mathrm{e}^{2\gamma\omega D}\big(2\mathrm{tr}(A)+n\gamma\big)^3\right)V(x(t)), \tag{4.20}$$

which implies, by the Razumikhin Stability Theorem (Theorem 1.3), that the closed-loop system is asymptotically stable if

$$D^2\mathrm{e}^{2\gamma\omega D} < \frac{\gamma}{\big(2\mathrm{tr}(A)+n\gamma\big)^3}, \tag{4.21}$$

holds. The left-hand side of inequality (4.21) is strictly increasing with respect to γ and its right-hand side is a positive constant independent of D. Moreover, the left-hand side goes to zero as D goes to zero and goes to infinity as D goes to infinity. Thus, Eq. (4.10) has a unique positive solution. This completes the proof. □

Theorem 4.1 states that, given any feedback parameter $\gamma > 0$, the state feedback TPF law (4.7) stabilizes a general linear system, possibly exponentially unstable, as long as the delay is small enough. An explicit bound on the delay is provided that guarantees stability. From this point of view, the theorem establishes a robustness property of the TPF law to a certain amount of delay. An intuitive explanation of such a property is already given in Sect. 4.1.

Corollary 4.1 *Consider system (4.4). Assume that all the open loop poles are on the imaginary axis. Then, given a time-varying delay with an arbitrarily large upper bound D, the state feedback TPF law (4.7) stabilizes the system as long as $\gamma \in (0, \gamma^\star)$, where $\gamma^\star$ is the unique positive solution to the following nonlinear equation:*

$$\gamma^2\mathrm{e}^{2\gamma\omega D} = \frac{1}{n^3D^2}. \tag{4.22}$$

Proof As all the eigenvalues of A are on the imaginary axis, $\mathrm{tr}(A) = 0$ in (4.21). After a simple manipulation on the inequality (4.21), we obtain

$$\gamma^2\mathrm{e}^{2\gamma\omega D} < \frac{1}{n^3D^2}, \quad \gamma > 0. \tag{4.23}$$

Note that the left-hand side of inequality (4.23) is strictly increasing with respect to γ, goes to zero as $\gamma \to 0^+$, and goes to infinity as $\gamma \to \infty$, whereas its right-hand side is a positive constant. Thus, the nonlinear equation (4.23) has a unique positive solution $\gamma^\star$. This completes the proof. □

Corollary 4.1 includes as a special case the low gain feedback result of the state feedback TPF law (2.41) in the stabilization of a linear system with all its open loop poles on the closed left-half plane. Similar recoveries of the low gain feedback results in Chaps. 2 and 3 will frequently appear throughout the rest of this chapter.

4.2.2 Truncated Predictor Output Feedback Design

Consider system (4.4), recalled as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(\phi(t)), \\ y(t) = Cx(t). \end{cases} \tag{4.24}$$

It is assumed, without loss of generality, that the stabilizable pair (A, B) has the following block structure:

$$A = \begin{bmatrix} A_{\mathrm{L}} & 0 \\ 0 & A_{\mathrm{R}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{L}} \\ B_{\mathrm{R}} \end{bmatrix}, \tag{4.25}$$

where $A_{\mathrm{L}} \in \mathbb{R}^{n_{\mathrm{L}} \times n_{\mathrm{L}}}$ is Hurwitz, all eigenvalues of $A_{\mathrm{R}} \in \mathbb{R}^{n_{\mathrm{R}} \times n_{\mathrm{R}}}$ are in the closed right-half plane, $n_{\mathrm{L}} + n_{\mathrm{R}} = n$, and the dimensions of B_{L} and B_{R} correspond to those of A_{L} and A_{R}, respectively. Based on this structure, the pair $(A_{\mathrm{R}}, B_{\mathrm{R}})$ is controllable.

We recall from Sect. 2.3 the observer based truncated predictor output feedback law (output feedback TPF law) whose feedback gain matrix is parameterized through a parametric Lyapunov equation,

$$\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(\phi(t)) - L\big(y(t) - C\widehat{x}(t)\big), \\ u(t) = -B^{\mathrm{T}}\mathcal{P}(\gamma)\,\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}\widehat{x}(t), \end{cases} \tag{4.26}$$

where the pair (A, B) is in the form of (4.25), $\mathcal{P}(\gamma)$ has the form of (2.102), that is,

$$\mathcal{P}(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & P(\gamma) \end{bmatrix}, \tag{4.27}$$

and L is such that $A + LC$ is Hurwitz. As discussed in Sect. 2.3, it is no longer without loss of generality to assume in the output feedback setting that all the open loop poles of system (4.7) are in the closed right-half plane.

As pointed out in Sect. 2.3, output feedback stabilization of system (4.24) by the output feedback TPF law (4.26) is equivalent to the asymptotic stability of the following closed-loop system:

$$\begin{cases} \dot{x}_{\mathrm{R}}(t) = A_{\mathrm{R}} x_{\mathrm{R}}(t) - B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P \mathrm{e}^{A_{\mathrm{R}} d(t)} (x_{\mathrm{R}}(\phi(t)) - e_{\mathrm{R}}(\phi(t))), \\ \dot{e}(t) = (A + LC) e(t). \end{cases} \tag{4.28}$$

The following theorem establishes the stability of system (4.28) and relaxes the restriction on the eigenvalues of A_{R} that they must locate on the imaginary axis, as imposed in Theorem 2.4.

Theorem 4.2 *For each $\gamma > 0$, there exists $D^{\star} > 0$ such that, if*

$$D < D^{\star},$$

system (4.28) is asymptotically stable.

Proof Define a Lyapunov function that consists of two terms corresponding to the two subsystems of system (4.28), respectively,

$$V(x_{\mathrm{R}}(t), e(t)) = x_{\mathrm{R}}^{\mathrm{T}}(t) P(\gamma) x_{\mathrm{R}}(t) + e^{\mathrm{T}}(t) R e(t), \tag{4.29}$$

where R is the unique positive definite solution to the following Lyapunov equation

$$(A + LC)^{\mathrm{T}} R + R(A + LC) = -\beta I, \tag{4.30}$$

and β is a positive constant whose value is to be determined later. Such R exists because $A + LC$ is Hurwitz. Then, the time derivative of the Lyapunov function along the trajectory of the closed-loop system (4.28) is given by

$$\begin{aligned} \dot{V}(x_{\mathrm{R}}(t), e(t)) = &- x_{\mathrm{R}}^{\mathrm{T}}(t) \left(\gamma P + P B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P\right) x_{\mathrm{R}}(t) + 2 x_{\mathrm{R}}^{\mathrm{T}}(t) P B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P \lambda_{\mathrm{R}}(t) \\ &+ 2 x_{\mathrm{R}}^{\mathrm{T}}(t) P B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P \mathrm{e}^{A_{\mathrm{R}} d(t)} e_{\mathrm{R}}(\phi(t)) - \beta e^{\mathrm{T}}(t) e(t), \end{aligned} \tag{4.31}$$

where $\lambda_{\mathrm{R}}(t)$ is defined by (2.107), that is,

$$\lambda_{\mathrm{R}}(t) = -\int_{t-d(t)}^{t} \mathrm{e}^{A_{\mathrm{R}}(t-s)} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P \mathrm{e}^{A_{\mathrm{R}} d(s)} (x_{\mathrm{R}}(\phi(s)) - e_{\mathrm{R}}(\phi(s))) \mathrm{d}s. \tag{4.32}$$

Using Young's Inequality and Lemma 2.4, we obtain

$$\begin{aligned} \dot{V}(x_{\mathrm{R}}(t), e(t)) \leq &- \gamma x_{\mathrm{R}}^{\mathrm{T}}(t) P x_{\mathrm{R}}(t) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) \lambda_{\mathrm{R}}^{\mathrm{T}}(t) P \lambda_{\mathrm{R}}(t) \\ &+ 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) \mathrm{e}^{\omega_{\mathrm{R}} \gamma d(t)} e_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P e_{\mathrm{R}}(\phi(t)) - \beta e^{\mathrm{T}}(t) e(t). \end{aligned} \tag{4.33}$$

Note that

$$\dot{e}(t) = (A + LC)e(t), \tag{4.34}$$

which yields

$$e(\phi(t)) = (A + LC)^{\phi(t)-t} e(t). \tag{4.35}$$

This implies that

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq &- \gamma x_{\mathrm{R}}^{\mathrm{T}}(t) P x_{\mathrm{R}}(t) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)\lambda_{\mathrm{R}}^{\mathrm{T}}(t) P \lambda_{\mathrm{R}}(t) \\ &+ 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) \\ &\times \mathrm{e}^{\omega_{\mathrm{R}}\gamma d(t)} e^{\mathrm{T}}(t) \left((A + LC)^{\phi(t)-t}\right)^{\mathrm{T}} \mathcal{P}(A + LC)^{\phi(t)-t} e(t) \\ &- \beta e^{\mathrm{T}}(t) e(t).\end{aligned} \tag{4.36}$$

By Lemmas 2.4 and 2.5, we have

$$\begin{aligned}\lambda_{\mathrm{R}}^{\mathrm{T}}(t) P \lambda_{\mathrm{R}}(t) \leq{}& 2D\mathrm{e}^{2w_{\mathrm{R}}\gamma D}\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2 \\ &\times \int_{t-D}^{t} \big(x_{\mathrm{R}}^{\mathrm{T}}(\phi(s)) P x_{\mathrm{R}}(\phi(s)) + e^{\mathrm{T}}(\phi(s))\mathcal{P}e(\phi(s))\big)\, \mathrm{d}s,\end{aligned} \tag{4.37}$$

based on which $x_{\mathrm{R}}(t)\dot{,}\ e(t)(t)$ can be further estimated as

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq{}& -\gamma x_{\mathrm{R}}^{\mathrm{T}}(t) P x_{\mathrm{R}}(t) + 4D\mathrm{e}^{2w_{\mathrm{R}}\gamma D}(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)^3 \\ &\times \int_{t-D}^{t} \big(x_{\mathrm{R}}^{\mathrm{T}}(\phi(s)) P x_{\mathrm{R}}(\phi(s)) \\ &+ e^{\mathrm{T}}(\phi(s))\mathcal{P}e(\phi(s))\big)\mathrm{d}s + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)\mathrm{e}^{\omega_{\mathrm{R}}\gamma d(t)} \\ &\times e^{\mathrm{T}}(t) \left((A + LC)^{\phi(t)-t}\right)^{\mathrm{T}} \mathcal{P}(A + LC)^{\phi(t)-t} e(t) - \beta e^{\mathrm{T}}(t) e(t).\end{aligned} \tag{4.38}$$

For any $\gamma > 0$, there exists a sufficiently large $\beta > 0$ such that

$$\mathcal{P}(\gamma) \leq R, \tag{4.39}$$

$$\gamma I + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)\mathcal{P}(\gamma) < \beta I, \tag{4.40}$$

$$1 \leq \lambda_{\max}(R). \tag{4.41}$$

Fix the value of β. Then, inequality (4.39) implies that

$$\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq - \gamma x_{\mathrm{R}}^{\mathrm{T}}(t) P x_{\mathrm{R}}(t) + 4D\mathrm{e}^{2w_{\mathrm{R}}\gamma D}(2\mathrm{tr}(A_{\mathrm{R}})$$

$$+ n_{\mathrm{R}}\gamma)^3 \int_{t-D}^{t} V(\phi(s))\mathrm{d}s + 2(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)\mathrm{e}^{\omega_{\mathrm{R}}\gamma d(t)}$$
$$\times e^{\mathrm{T}}(t)\left((A+LC)^{\phi(t)-t}\right)^{\mathrm{T}} \mathcal{P}(A+LC)^{\phi(t)-t} e(t) - \beta e^{\mathrm{T}}(t)e(t).$$

The inequality in (4.40) further implies that there exists a sufficiently small $D_1 > 0$ such that, for any $D < D_1$,

$$\gamma I + 2\mathrm{e}^{\omega_{\mathrm{R}}\gamma D}\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)\left((A+LC)^{\phi(t)-t}\right)^{\mathrm{T}} \mathcal{P}(\gamma)(A+LC)^{\phi(t)-t} \le \beta I. \tag{4.42}$$

Here, we have used the boundedness of $\phi(t) - t$. Then, by a simple manipulation of inequality (4.42), we obtain

$$-\beta I + 2\mathrm{e}^{w_{\mathrm{R}}\gamma D}\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)\left((A+LC)^{\phi(t)-t}\right)^{\mathrm{T}} \mathcal{P}(\gamma)(A+LC)^{\phi(t)-t} \le -\gamma I,$$

which implies that

$$\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \le & -\gamma\left(x_{\mathrm{R}}^{\mathrm{T}} P x_{\mathrm{R}} + e^{\mathrm{T}} e\right) + 4D\mathrm{e}^{2w_{\mathrm{R}}\gamma D}(2\mathrm{tr}(A_{\mathrm{R}}) \\
& + n_{\mathrm{R}}\gamma)^3 \int_{t-D}^{t} V(x_{\mathrm{R}}(\phi(s)), e(\phi(s)))\mathrm{d}s \\
\le & -\gamma\left(x_{\mathrm{R}}^{\mathrm{T}} P x_{\mathrm{R}} + \frac{1}{\lambda_{\max}(R)} e^{\mathrm{T}} R e\right) \\
& + 4D\mathrm{e}^{2w_{\mathrm{R}}\gamma D}\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^3 \int_{t-D}^{t} V(x_{\mathrm{R}}(\phi(s), e(\phi(s)))\mathrm{d}s \\
\le & -\frac{\gamma}{\lambda_{\max}(R)} V(x_{\mathrm{R}}(t), e(t)) \\
& + 4D\mathrm{e}^{2w_{\mathrm{R}}\gamma D}\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^3 \int_{t-D}^{t} V(x_{\mathrm{R}}(\phi(s)), e(\phi(s)))\mathrm{d}s,
\end{aligned}$$

where we have used inequality (4.41).

When $V(x_{\mathrm{R}}(t+\theta), e(t+\theta)) < \eta V(x_{\mathrm{R}}(t), e(t)), \theta \in [-2D, 0]$, for some constant $\eta > 1$, we have

$$\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \le \Bigg(& -\frac{\gamma}{\lambda_{\max}(R)} \\
& + 4D^2\eta\mathrm{e}^{2\omega_{\mathrm{R}}\gamma D}\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^3\Bigg) V(x_{\mathrm{R}}(t), e(t)).
\end{aligned}$$

It then follows from the Razumikhin Stability Theorem (Theorem 1.3) that the closed-loop system is asymptotically stable if

$$D^2 \mathrm{e}^{2w_{\mathrm{R}}\gamma D} < \frac{\gamma}{4\lambda_{\max}(R)\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^3}. \tag{4.43}$$

The left-hand side of inequality (4.43) is strictly increasing with respect to D and its right-hand side is a positive constant independent of D. Moreover, its left-hand side goes to zero as D goes to zero and goes to infinity as D goes to infinity. Thus, the equality (4.43) has a unique positive solution on D, which is denoted as D_2. Letting $D^\star = \min\{D_1, D_2\}$ completes the proof. □

Theorem 4.2 shows that the output feedback TPF law (4.26) is robust to a certain amount of delay in the input of a general linear system that is possibly exponentially unstable. This theorem extends such a robustness property of the state feedback TPF law, which is implied by Theorem 4.1, to the output feedback TPF law.

Theorem 4.3 *Consider system (4.28). Assume that all the eigenvalues of A_{R} are on the imaginary axis. Given an arbitrarily large delay bound D, there exists $\gamma^* > 0$ such that, for each $\gamma \in (0, \gamma^*)$, the system is asymptotically stable.*

Proof The same Lyapunov function as in (4.29), along with the estimate (4.38) on its time derivative along the trajectory of system (4.28), are adopted for our stability analysis. In view of the assumption that all the eigenvalues of A_{R} are on the imaginary axis, we rewrite (4.38) as

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq & -\gamma x_{\mathrm{R}}^{\mathrm{T}}(t)Px_{\mathrm{R}}(t) + 4D\mathrm{e}^{2w_{\mathrm{R}}\gamma D} n(_{\mathrm{R}}\gamma)^3 \int_{t-D}^{t} \big(x_{\mathrm{R}}^{\mathrm{T}}(\phi(s))Px_{\mathrm{R}}(\phi(s)) \\ & + e^{\mathrm{T}}(\phi(s))\mathcal{P}e(\phi(s))\big)\,\mathrm{d}s + 2n_{\mathrm{R}}\gamma \mathrm{e}^{\omega_{\mathrm{R}}\gamma d(t)} e^{\mathrm{T}}(t) \left((A+LC)^{\phi(t)-t}\right)^{\mathrm{T}} \\ & \times \mathcal{P}(A+LC)^{\phi(t)-t} e(t) - \beta e^{\mathrm{T}}(t)e(t).\end{aligned} \tag{4.44}$$

Pick a β such that

$$1 < \lambda_{\max}(R). \tag{4.45}$$

For this β, there exists $\gamma_1 > 0$ such that, for each $\gamma \in (0, \gamma_1]$,

$$\begin{aligned} \mathcal{P}(\gamma) \leq R, \\ \gamma I + 2\mathrm{e}^{\omega_{\mathrm{R}}\gamma D} n_{\mathrm{R}}\gamma \left((A+LC)^{\phi(t)-t}\right)^{\mathrm{T}} \mathcal{P}(\gamma)(A+LC)^{\phi(t)-t} \leq \beta I. \end{aligned} \tag{4.46}$$

Then, the estimate on the time derivative of V can be continued as

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq & -\gamma x_{\mathrm{R}}^{\mathrm{T}}(t)Px_{\mathrm{R}}(t) + 4D\mathrm{e}^{2w_{\mathrm{R}}\gamma D}(n_{\mathrm{R}}\gamma)^3 \int_{t-D}^{t} V(x_{\mathrm{R}}(\phi(s)), e(\phi(s)))\mathrm{d}s \\ & + 2n_{\mathrm{R}}\gamma \mathrm{e}^{\omega_{\mathrm{R}}\gamma d(t)} e^{\mathrm{T}}(t) \left((A+LC)^{\phi(t)-t}\right)^{\mathrm{T}} \mathcal{P}(A+LC)^{\phi(t)-t} e(t)\end{aligned}$$

$$
\begin{aligned}
&- \beta e^{\mathrm{T}}(t)e(t) \\
&\leq -\gamma\left(x_{\mathrm{R}}^{\mathrm{T}} P x_{\mathrm{R}} + e^{\mathrm{T}} e\right) + 4D\mathrm{e}^{2\omega\gamma D}(n_{\mathrm{R}}\gamma)^3 \int_{t-D}^{t} V(\phi(s))\mathrm{d}s \\
&\leq -\frac{\gamma}{\lambda_{\max}(R)} V + 4D\mathrm{e}^{2w_{\mathrm{R}}\gamma D}(n_{\mathrm{R}}\gamma)^3 \int_{t-D}^{t} V(x_{\mathrm{R}}(\phi(s)), e(\phi(s)))\mathrm{d}s,
\end{aligned}
\tag{4.47}
$$

where we have employed (4.45).

When $V(x_{\mathrm{R}}(t+\theta), e(t+\theta)) < \eta V(x_{\mathrm{R}}(t), e(t)), \theta \in [-2D, 0]$, for some constant $\eta > 1$, we have

$$
\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq \left(-\frac{\gamma}{\lambda_{\max}(R)} + 4D^2\eta\mathrm{e}^{2\omega_{\mathrm{R}}\gamma D}(n_{\mathrm{R}}\gamma)^3\right) V(x_{\mathrm{R}}(t), e(t)). \tag{4.48}
$$

It then follows from the Razumikhin Stability Theorem (Theorem 1.3) that the closed-loop system is stable if

$$
\gamma^2 \mathrm{e}^{2\omega_{\mathrm{R}}\gamma D} < \frac{1}{4\lambda_{\max}(R) n_{\mathrm{R}}^3 D^2}. \tag{4.49}
$$

There exists $\gamma_2 > 0$ such that, for each $\gamma \in (0, \gamma_2]$, inequality (4.49) holds. Taking $\gamma^* = \min\{\gamma_1, \gamma_2\}$ completes the proof. □

Theorem 4.3 manifests the low gain nature of the output feedback TPF law (4.26) in its compensation of an arbitrarily large bounded delay in a linear system with all its open loop poles in the closed left-half plane.

4.2.3 A Numerical Example

Consider an exponentially unstable linear system (4.4) with

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{4.50}
$$

This system is both controllable and observable with all its open loop poles at $\lambda = \{\pm j, \pm j, 1\}$. We choose the feedback parameter of the TPF laws (4.7) and (4.26) as $\gamma = 0.2$. Given such a γ, the time-varying delay is allowed to have an upper bound $D = 0.5$. Let

$$d(t) = 0.5\frac{t+1}{2t+1}. \tag{4.51}$$

Based on the expression of $d(t)$, we compute

$$\phi^{-1}(t) - t = \frac{t+1}{\sqrt{4t^2+6t+4.25}+2t+0.5} \tag{4.52}$$

which appears in the expression of the TPF laws (4.7) and (4.26). Simulation is run in both the state feedback setting and the output feedback setting. In the case of state feedback, the initial condition of the system is given by

$$x(\theta) = \begin{bmatrix}1 & 0 & -1 & 2 & 0\end{bmatrix}^{\mathrm{T}}. \tag{4.53}$$

In the case of output feedback, the initial condition of the open loop system is taken to be

$$x(\theta) = \begin{bmatrix}1 & 0 & -1 & 2 & 0\end{bmatrix}^{\mathrm{T}}, \quad \hat{x}(\theta) = \begin{bmatrix}0 & 0 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}, \tag{4.54}$$

and the eigenvalues of $A + LC$ are assigned at $\lambda = \{-1, -2, -3, -4, -5\}$. The performance of the system and the control input under the state feedback and output feedback TPF laws are given in Figs. 4.1 and 4.2, respectively.

4.3 Discrete-Time Systems

We consider the following discrete-time linear system with a time-varying input delay:

$$\begin{cases} x(k+1) = Ax(k) + Bu(\phi(k)), \\ \quad\;\; y(k) = Cx(k), \end{cases} \tag{4.55}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^q$ are the state, input and output, respectively, (A, B) is stabilizable and (A, C) is detectable. The time-varying delay function $\phi(k) : \mathbb{N} \to \mathbb{Z}$ is assumed to have the standard form of

$$\phi(k) = k - r(k), \tag{4.56}$$

whose inverse function $\phi^{-1}(k) : \mathbb{Z} \to \mathbb{N}$ exists and is known. Also, $r(k) : \mathbb{N} \to \mathbb{N}$ denotes the time-varying delay that satisfies $r(k) \in I[0, R]$, where $R \in \mathbb{N}$ is an upper bound of the delay.

Note that system (4.55) contains a time-varying input delay. To formulate state and output feedback laws that adapt to a time-varying delay, we introduce the

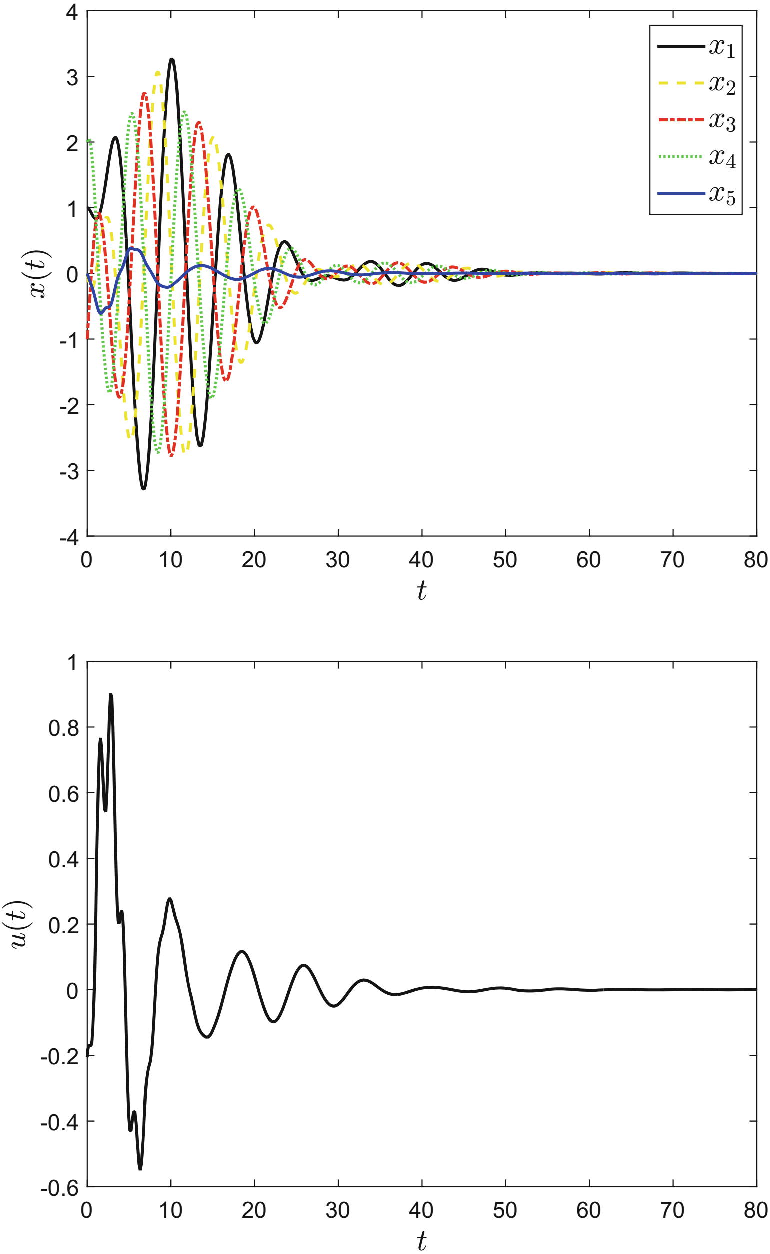


Fig. 4.1 State response and control input under the state feedback TPF law (4.7): $d(t) = 0.5\frac{t+1}{2t+1}$ and $\gamma = 0.2$

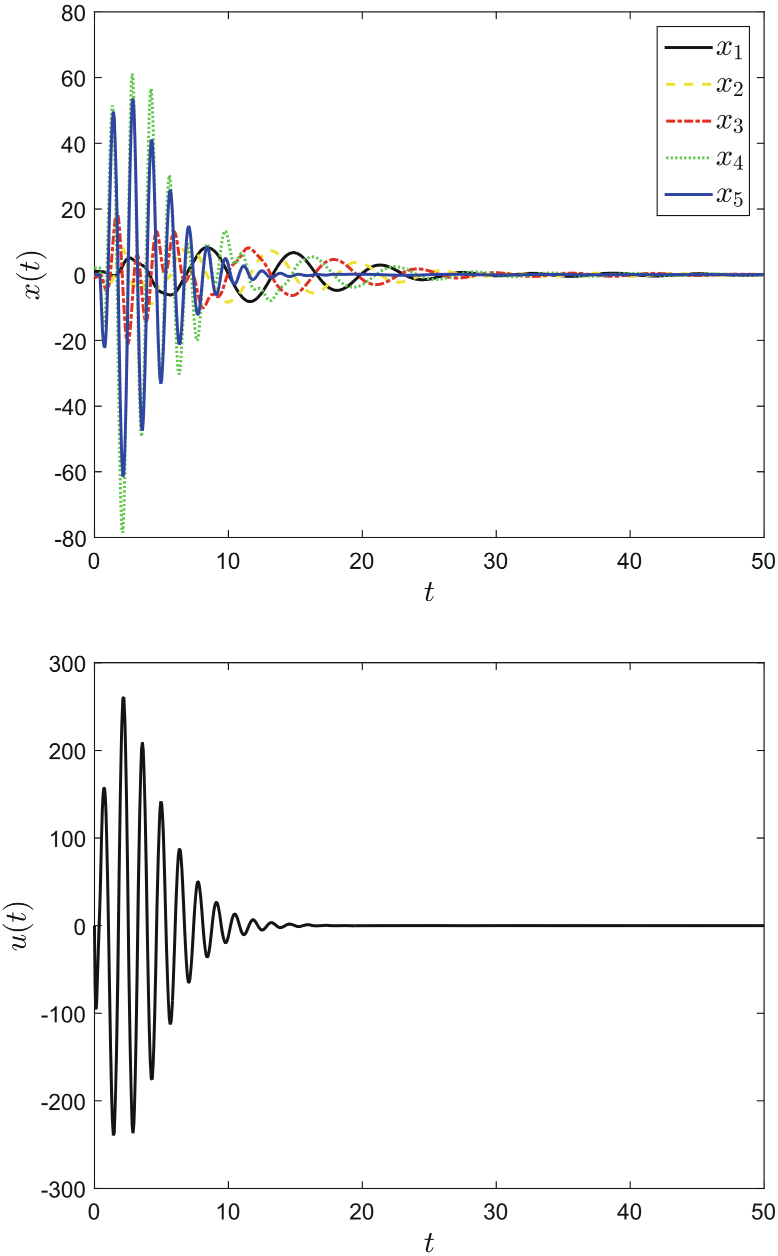


Fig. 4.2 State response and control input under the output feedback TPF law (4.26): $d(t) = 0.5\frac{t+1}{2t+1}$ and $\gamma = 0.2$

predictor feedback laws and the TPF laws in the time-varying delay setting in the following subsections.

4.3.1 Truncated Predictor State Feedback Design

Consider the following feedback law for system (4.55),

$$u(\phi(k)) = Fx(k), \tag{4.57}$$

where F is the feedback gain matrix such that $A + BF$ is Schur stable. Under the feedback law (4.57), the closed-loop system

$$x(k+1) = (A + BF)x(k) \tag{4.58}$$

is asymptotically stable because $A + BF$ is Schur stable.

Since $\phi^{-1}(k)$ exists and is known, we obtain from (4.57) that

$$u(k) = Fx\left(\phi^{-1}(k)\right). \tag{4.59}$$

Recall that $\phi(k) = k - r(k)$ and $r(k) \geq 0$. We have

$$k = \phi\left(\phi^{-1}(k)\right) = \phi^{-1}(k) - r\left(\phi^{-1}(k)\right) \leq \phi^{-1}(k). \tag{4.60}$$

Thus, the right-hand side of (4.59) may contain the state at a future time instant, namely, $x(\phi^{-1}(k))$. This state can be obtained as the solution of system (4.55),

$$x(\phi^{-1}(k)) = A^{\phi^{-1}(k)-k}x(k) + \sum_{s=2k-\phi^{-1}(k)}^{k-1} A^{k-s-1}Bu(s). \tag{4.61}$$

Substitution of (4.61) in (4.59) yields the predictor state feedback law,

$$u(k) = FA^{\phi^{-1}(k)-k}x(k) + F\sum_{s=2k-\phi^{-1}(k)}^{k-1} A^{k-s-1}Bu(s). \tag{4.62}$$

The predictor feedback law consists of two terms, corresponding to the zero input solution and the zero state solution of the system, respectively. The zero state solution only involves past input $u(s)$, $s \leq k-1$. This can be readily verified by observing the fact that

$$2k - \phi^{-1}(k) \leq k. \tag{4.63}$$

Note that the equality sign in (4.63) holds if and only if $r(k) = 0$. In this case, no delay exists in the input and the summation term in (4.62) disappears. Thus, the predictor feedback law reduces to a static state feedback law

$$u(k) = Fx(k).$$

Discarding the summation term of the predictor feedback law results in the TPF law,

$$u(k) = FA^{\phi^{-1}(k)-k}x(k), \tag{4.64}$$

which simplifies the implementation of the predictor feedback law.

If the time-varying delay in system (4.55) is constant, that is, $\phi(k) = k - r$, where $r \in \mathbb{N}$ is a constant, then the predictor feedback law (4.62) and the TPF law (4.64) simplify to (1.86) and (3.3), respectively.

Following the Lyapunov equation based feedback design, we construct the truncated predictor state feedback law whose feedback gain matrix is parameterized in a feedback parameter γ,

$$\begin{aligned} u(k) &= F(\gamma)A^{\phi^{-1}(k)-k}x(k) \\ &= -\left(I_m + B^{\mathrm{T}}P(\gamma)B\right)^{-1} B^{\mathrm{T}}P(\gamma)A^{\phi^{-1}(k)-k+1}x(k), \end{aligned} \tag{4.65}$$

where $P(\gamma)$ is the unique positive definite solution to the discrete-time Riccati equation (3.69), that is

$$\begin{aligned} A^{\mathrm{T}}P(\gamma)A - P(\gamma) - A^{\mathrm{T}}P(\gamma)B\left(I_m + B^{\mathrm{T}}P(\gamma)B\right)^{-1} B^{\mathrm{T}}P(\gamma)A = -\gamma P(\gamma), \\ \gamma \in \left(1 - \min\left\{|\lambda(A)|^2\right\}, 1\right). \end{aligned} \tag{4.66}$$

In the case where all eigenvalues of A are inside or on the unit circle, the delay is allowed to be arbitrarily large but bounded, and the value of the parameter γ is required to approach zero as the bound on the delay increases to infinity. As a result, the parametric algebraic Riccati equation (4.66) is a low gain feedback design and the feedback parameter is referred to as the low gain parameter.

As discussed in Sect. 3.3.2, in asymptotic stabilization of system (4.55) by state feedback, it is without loss of generality to assume that there is no asymptotically stable open loop poles. Thus, in the following theorem on the stabilization of a general discrete-time linear system that is possibly exponentially unstable, we assume, without loss of generality, that the system has all its open loop poles on or outside the unit circle.

Theorem 4.4 *Consider system (4.55). Assume that all the eigenvalues of A are on or outside the unit circle. If, for each* $\gamma \in (0, 1)$*,*

$$R < R^{\star}, \tag{4.67}$$

where $R^{\star}$ is the unique positive solution to the following equation:

$$\frac{R^2(\det(A))^{2R-4}}{(1-\gamma)^{(n-1)(2R+1)}} = \frac{\gamma}{\left(\det^2(A) - (1-\gamma)^n\right)^3}, \tag{4.68}$$

then the system is asymptotically stabilized by the state feedback TPF law (4.65).

Proof Under the state feedback TPF law (4.65), the closed-loop system can be written as follows:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(\phi(k)) \\ &= Ax(k) + BF(\gamma)A^{r(k)}x(\phi(k)) \\ &= A_{\rm c}x(k) + BF(\gamma)\mu(k), \end{aligned}$$

where

$$A_{\rm c} = A + BF(\gamma) \tag{4.69}$$

and

$$\mu(k) = A^{r(k)}x(\phi(k)) - x(k). \tag{4.70}$$

From

$$x(k+1) = Ax(k) + BF(\gamma)A^{r(k)}x(\phi(k)), \tag{4.71}$$

we can recursively obtain

$$x(k) = A^{r(k)}x(\phi(k)) + \sum_{s=1}^{r(k)} A^{s-1}BF(\gamma)A^{r(k-s)}x(k-s-r(k-s)),$$

which implies that

$$\mu(k) = -\sum_{s=1}^{r(k)} A^{s-1}BF(\gamma)A^{r(k-s)}x(k-s-r(k-s)). \tag{4.72}$$

We adopt the Lyapunov function

$$V(x(k)) = x^{\rm T}(k)P(\gamma)x(k), \tag{4.73}$$

where $P(\gamma)$ is the unique positive definite solution to the discrete-time Riccati equation (4.66). Let $\gamma \in (0, 1)$ be such that $\gamma \in (1 - \min\{|\lambda(A)|^2\}, 1)$. Then the forward difference of $V(x(k))$ along the trajectory of the closed-loop system can be evaluated as,

$$\begin{aligned}\Delta V(x(k)) &= V(x(k+1)) - V(x(k)) \\ &= x^{\mathrm{T}}(k)\big(A_{\mathrm{C}}^{\mathrm{T}} P A_{\mathrm{C}} - P\big)x(k) + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}} P B F \mu(k) + \mu^{\mathrm{T}}(k)F^{\mathrm{T}} B^{\mathrm{T}} P B F \mu(k) \\ &= x^{\mathrm{T}}(k)\left(-\gamma P - F^{\mathrm{T}} F\right)x(k) + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}} P B F \mu(k) + \mu^{\mathrm{T}}(k)F^{\mathrm{T}} B^{\mathrm{T}} P B F \mu(k) \\ &\le -\gamma x^{\mathrm{T}}(k) P x(k) + \mu^{\mathrm{T}}(k) F^{\mathrm{T}}\left(I + B^{\mathrm{T}} P B\right) F \mu(k),\end{aligned}$$

where $A_{\mathrm{C}}^{\mathrm{T}} P A_{\mathrm{C}} - P = -\gamma P - F^{\mathrm{T}} F$ and $F = -B^{\mathrm{T}} P A_{\mathrm{C}}$. Also, for notational brevity, we have suppressed the dependence on γ of P and F and will continue to do so in the rest of the proof.

In view of the fact that $F^{\mathrm{T}}(I + B^{\mathrm{T}} P B) F$ is positive semi-definite, (4.72) and Lemmas 3.5 and 3.6, we compute

$$\begin{aligned}&\mu^{\mathrm{T}}(k) F^{\mathrm{T}}\left(I + B^{\mathrm{T}} P B\right) F \mu(k) \\ &\le r(k) \sum_{s=1}^{r(k)} x^{\mathrm{T}}(k - s - r(k-s)) \left(A^{r(k-s)}\right)^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} \left(A^{s-1}\right)^{\mathrm{T}} F^{\mathrm{T}}(I + B^{\mathrm{T}} P B) F \\ &\quad \times A^{s-1} B F A^{r(k-s)} x(k - s - r(k-s)) \\ &\le r(k) \sum_{s=1}^{r(k)} \frac{\left(\det^2(A) - (1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}} (\det(A))^{2r(k-s)+2s-4} x^{\mathrm{T}}(k - s - r(k-s)) P \\ &\quad \times x(k - s - r(k-s)).\end{aligned} \tag{4.74}$$

By (4.74), $\Delta V(x(k))$ can be further evaluated as

$$\begin{aligned}\Delta V(x(k)) \le &- \gamma x^{\mathrm{T}}(k) P x(k) + r(k) \sum_{s=1}^{r(k)} (\det(A))^{2r(k-s)+2s-4} \\ &\times \frac{\left(\det^2(A) - (1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}} x^{\mathrm{T}}(k - s - r(k-s)) P \\ &\times x(k - s - r(k-s)).\end{aligned} \tag{4.75}$$

Notice that $s \le K$ for any $s \in I[1, r(k)]$, and $r(k-s) \le K$. It follows that $k - s - r(k-s) \in I[k - 2K, k]$. Thus, when $V(x(k+z)) < \eta V(x(k))$ for any $z \in I[-2K, 0]$, where $\eta > 1$ is some constant, we have

$$\begin{aligned}\Delta V(x(k)) \leq & -\gamma V(x(k)) + \eta r(k) V(x(k)) \sum_{s=1}^{r(k)} (\det(A))^{2r(k-s)+2s-4} \\ & \times \frac{\left(\det^2(A) - (1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}} \\ = & -\Bigg(\gamma - \eta r(k) \sum_{s=1}^{r(k)} (\det(A))^{2r(k-s)+2s-4} \\ & \times \frac{\left(\det^2(A) - (1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}}\Bigg) V(x(k)) \\ \leq & -\Bigg(\gamma - \eta R^2 (\det(A))^{2R-4} \\ & \times \frac{\left(\det^2(A) - (1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(2R+1)}}\Bigg) V(x(k)). \end{aligned} \tag{4.76}$$

If

$$\frac{R^2 (\det(A))^{2R-4}}{(1-\gamma)^{(n-1)(2R+1)}} < \frac{\gamma}{\left(\det^2(A) - (1-\gamma)^n\right)^3}, \tag{4.77}$$

then the asymptotic stability of the closed-loop system follows from the Razumikhin Stability Theorem for discrete-time systems (Theorem 1.4). Given $\gamma \in (0, 1)$, the left-hand side of inequality (4.77) is a strictly increasing function of R and its right-hand side is a positive constant independent of R. Moreover, its left-hand side goes to zero as R goes to zero and goes to infinity as R goes to infinity. Thus, Eq. (4.68) has a unique positive solution $D^\star$. This completes the proof. □

Like the state feedback TPF law in the continuous-time setting, the state feedback TPF law in the discrete-time setting is also robust to a certain amount of input delay. Theorem 4.4 establishes an admissible bound on the delay in terms of the feedback parameter and parameters of the open loop system that guarantees closed-loop stability.

Corollary 4.2 *Consider the system (4.55). Assume that all the eigenvalues of A are on the unit circle. Given an arbitrarily large delay, there exists* $\gamma^\star \in (0, 1)$ *such that, for each* $\gamma \in (0, \gamma^\star)$*, the system is asymptotically stabilized by the state feedback TPF law (4.65).*

Proof When all the eigenvalues of A are on the unit circle, $\det(A) = 1$, which implies that the stability condition (4.77) simplifies to

$$\frac{\left(1-(1-\gamma)^n\right)^3}{\gamma(1-\gamma)^{(n-1)(2R+1)}} < \frac{1}{R^2}. \tag{4.78}$$

Given any R, the left-hand side of inequality (4.78) is strictly increasing as γ, and goes to zero as γ goes to zero because

$$\begin{aligned}&\lim_{\gamma\to 0^+} \frac{\left(1-(1-\gamma)^n\right)^3}{\gamma}\\ &= \lim_{\gamma\to 0^+} 3\left(1-(1-\gamma)^n\right)^2 n(1-\gamma)^{n-1}\\ &= 0\end{aligned} \tag{4.79}$$

and goes to infinity as γ goes to 1. Thus, there exists a unique $\gamma^\star \in (0, 1)$ such that, for each $\gamma \in (0, \gamma^\star)$, the simplified stability condition (4.78) holds. This completes the proof. □

This corollary recovers the low gain nature of the state feedback TPF law (3.79) in its compensation of an arbitrarily large delay in a discrete-time linear system with all its open loop poles on or inside the unit circle.

4.3.2 Truncated Predictor Output Feedback Design

Without loss of generality, we assume that the pair $(A.B)$ in system (4.55) are in the form of

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_o \end{bmatrix}, \quad B = \begin{bmatrix} B_s \\ B_o \end{bmatrix}, \tag{4.80}$$

where $A_s \in \mathbb{R}^{n_s\times n_s}$ contains all eigenvalues of A that are strictly inside the unit circle, $A_o \in \mathbb{R}^{n_o\times n_o}$ contains all eigenvalues of A that are on or outside the unit circle, and $n_s + n_o = n$. The stabilizability of (A, B) then implies that (A_o, B_o) is controllable.

Unlike the truncated predictor output feedback law (3.93), as given in Sect. 3.3.3, for a discrete-time linear system with a constant delay, a truncated predictor output feedback law that copes with a time-varying delay is to be constructed to achieve asymptotic stabilization of system (4.55). Referring to the state feedback case where the time-varying prediction time $\phi^{-1}(k) - k$ of the state feedback TPF law (4.64) replaces the constant prediction time r to cope with the time-varying input delay, we construct the following observed based truncated predictor output feedback law that handles the time-varying delay $\phi(k)$ in system (4.55),

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(\phi(k)) - L\left(y(k) - C\hat{x}(k)\right), \\ \quad u(k) = \tilde{F}(\gamma)A^{\phi^{-1}(k)-k}\hat{x}(k), \end{cases} \tag{4.81}$$

where

$$\tilde{F}(\gamma) = -\left(I + B^{\mathrm{T}}\tilde{P}(\gamma)B\right)^{-1} B^{\mathrm{T}}\tilde{P}(\gamma)A, \quad \tilde{P}(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & P_{\mathrm{o}}(\gamma) \end{bmatrix}, \tag{4.82}$$

with $P_{\mathrm{o}}(\gamma)$ being the unique positive definite solution to the discrete-time parametric algebraic Riccati equation

$$A_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}(\gamma)A_{\mathrm{o}} - P_{\mathrm{o}}(\gamma) - A_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}(\gamma)B_{\mathrm{o}}\left(I + B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}(\gamma)B_{\mathrm{o}}\right)^{-1} B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}(\gamma)A_{\mathrm{o}} = -\gamma P_{\mathrm{o}}(\gamma), \tag{4.83}$$

and L is such that $A + LC$ is Schur stable. The detectability of the pair (A, C) implies that such L exists.

The two truncated predictor output feedback laws (3.93) and (4.81) are highly similar. Basically, the latter can be considered an extension of the former to cope with a time-varying delay.

Define the estimation error as

$$e(k) = x(k) - \hat{x}(k), \tag{4.84}$$

then the error dynamics can be written as

$$e(k+1) = (A + LC)e(k). \tag{4.85}$$

In terms of the state $x(k)$ and the estimation error $e(k)$, the closed-loop system under the output feedback TPF law (4.81) can be rewritten as

$$\begin{cases} x(k+1) = Ax(k) + B\tilde{F}(\gamma)A^{\phi^{-1}(k)-k}(x(\phi(k)) - e(\phi(k))), \\ e(k+1) = (A + LC)e(k), \end{cases} \tag{4.86}$$

which, in view of the special structure of the pair (A, B) in (4.80), can be expanded as

$$\begin{cases} x_{\mathrm{s}}(k+1) = A_{\mathrm{s}}x_{\mathrm{s}}(k) + B_{\mathrm{s}}F_{\mathrm{o}}(\gamma)A_{\mathrm{o}}^{\phi^{-1}(k)-k}(x_{\mathrm{o}}(\phi(k)) - e_{\mathrm{o}}(\phi(k))), \\ x_{\mathrm{o}}(k+1) = A_{\mathrm{o}}x_{\mathrm{o}}(k) + B_{\mathrm{o}}F_{\mathrm{o}}(\gamma)A_{\mathrm{o}}^{\phi^{-1}(k)-k}(x_{\mathrm{o}}(\phi(k)) - e_{\mathrm{o}}(\phi(k))), \\ e(k+1) = (A + LC)e(k), \end{cases} \tag{4.87}$$

where $F_{\mathrm{o}}(\gamma) = -\left(I + B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}(\gamma)B_{\mathrm{o}}\right)^{-1} B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}(\gamma)A_{\mathrm{o}}$. Notice that the dynamics of the first subsystem is governed by the Schur matrix A_{s}, which implies that the stability of the second and the third subsystems determines the stability of the whole closed-loop system. Thus, in the following theorem, we establish the stability of the system comprising of the second and the third subsystems,

$$\begin{cases} x_{\mathrm{o}}(k+1) = A_{\mathrm{o}}x_{\mathrm{o}}(k) + B_{\mathrm{o}}F_{\mathrm{o}}(\gamma)A_{\mathrm{o}}^{\phi^{-1}(k)-k}(x_{\mathrm{o}}(\phi(k)) - e_{\mathrm{o}}(\phi(k))), \\ e(k+1) = (A+LC)e(k). \end{cases} \tag{4.88}$$

Theorem 4.5 *Consider system (4.88). Assume that A_s is nonsingular. For each $\gamma \in (0,1)$, there exists $R^* > 0$ such that, for each $R \in I[0, R^*)$, the system is asymptotically stable.*

Proof Construct a Lyapunov function as

$$V(x_{\mathrm{o}}(k), e(k)) = x_{\mathrm{o}}^{\mathrm{T}}(k)P_{\mathrm{o}}(\gamma)x_{\mathrm{o}}(k) + e^{\mathrm{T}}(k)Qe(k), \tag{4.89}$$

where Q is the unique positive definite solution to the Lyapunov equation

$$(A+LC)^{\mathrm{T}}Q(A+LC) - Q = -\beta I, \tag{4.90}$$

and β is some positive constant whose values are to be determined. Note that the existence and uniqueness of such Q are guaranteed by the Schur stability of $A+LC$.

We rewrite the first subsystem of (4.88) as

$$x_{\mathrm{o}}(k+1) = A_{\mathrm{oc}}x_{\mathrm{o}}(k) + B_{\mathrm{o}}F_{\mathrm{o}}(\gamma)\lambda_{\mathrm{o}}(k) - B_{\mathrm{o}}F_{\mathrm{o}}(\gamma)A_{\mathrm{o}}^{r(k)}e_{\mathrm{o}}(\phi(k)), \tag{4.91}$$

where

$$A_{\mathrm{oc}} = A_{\mathrm{o}} + B_{\mathrm{o}}F_{\mathrm{o}}(\gamma), \quad \lambda_{\mathrm{o}}(k) = A_{\mathrm{o}}^{r(k)}x_{\mathrm{o}}(\phi(k)) - x_{\mathrm{o}}(k). \tag{4.92}$$

Then, $\lambda_{\mathrm{o}}(k)$ can be computed by the use of the solution of the first subsystem of (4.88) as follows:

$$\lambda_{\mathrm{o}}(k) = -\sum_{s=1}^{r(k)} A_{\mathrm{o}}^{s-1}B_{\mathrm{o}}F_{\mathrm{o}}(\gamma)A_{\mathrm{o}}^{r(k-s)}\hat{x}_{\mathrm{o}}(k-s-r(k-s)), \tag{4.93}$$

where $\hat{x}_{\mathrm{o}}(k) = x_{\mathrm{o}}(k) - e_{\mathrm{o}}(k)$.

The forward difference of $V(x_{\mathrm{o}}(k), e(k))$ along the trajectory of the closed-loop system (4.88) can be evaluated as follows:

$$\begin{aligned} \Delta V(x_{\mathrm{o}}(k), e(k)) &= V(x_{\mathrm{o}}(k+1), e(k+1)) - V(x_{\mathrm{o}}(k), e(k)) \\ &= x_{\mathrm{o}}^{\mathrm{T}}(k)A_{\mathrm{oc}}^{\mathrm{T}}P_{\mathrm{o}}A_{\mathrm{oc}}x_{\mathrm{o}}(k) + 2x_{\mathrm{o}}^{\mathrm{T}}(k)A_{\mathrm{oc}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}\lambda_{\mathrm{o}}(k) \\ &\quad - 2x_{\mathrm{o}}^{\mathrm{T}}(k)A_{\mathrm{oc}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}A_{\mathrm{o}}^{r(k)}e_{\mathrm{o}}(\phi(k)) \end{aligned}$$

$$
\begin{aligned}
&+\lambda_{\mathrm{o}}^{\mathrm{T}}(k)F_{\mathrm{o}}^{\mathrm{T}}B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}\lambda_{\mathrm{o}}(k)\\
&-2\lambda_{\mathrm{o}}^{\mathrm{T}}(k)F_{\mathrm{o}}^{\mathrm{T}}B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}A_{\mathrm{o}}^{r(k)}e_{\mathrm{o}}(\phi(k))\\
&+e_{\mathrm{o}}^{\mathrm{T}}(\phi(k))(A_{\mathrm{o}}^{\mathrm{T}})^{r(k)}F_{\mathrm{o}}^{\mathrm{T}}B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}F_{\mathrm{o}}A_{\mathrm{o}}^{r(k)}e_{\mathrm{o}}(\phi(k))\\
&-\beta e^{\mathrm{T}}(k)e(k)-x_{\mathrm{o}}^{\mathrm{T}}(k)P_{\mathrm{o}}x_{\mathrm{o}}(k)\\
\leq\ &-\gamma x_{\mathrm{o}}^{\mathrm{T}}(k)P_{\mathrm{o}}x_{\mathrm{o}}(k)+2\lambda_{\mathrm{o}}^{\mathrm{T}}(k)F_{\mathrm{o}}^{\mathrm{T}}\left(I+B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}\right)F_{\mathrm{o}}\lambda_{\mathrm{o}}(k)\\
&+2e_{\mathrm{o}}^{\mathrm{T}}(\phi(k))(A_{\mathrm{o}}^{\mathrm{T}})^{r(k)}F_{\mathrm{o}}^{\mathrm{T}}\left(I+B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}\right)F_{\mathrm{o}}A_{\mathrm{o}}^{r(k)}e_{\mathrm{o}}(\phi(k))\\
&-\beta e^{\mathrm{T}}(k)e(k),
\end{aligned}
$$

where

$$
A_{\mathrm{oc}}^{\mathrm{T}}P_{\mathrm{o}}A_{\mathrm{oc}}-P_{\mathrm{o}}=-\gamma P_{\mathrm{o}}-F_{\mathrm{o}}^{\mathrm{T}}F_{\mathrm{o}},\quad F_{\mathrm{o}}=-B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}A_{\mathrm{oc}} \tag{4.94}
$$

and Young's Inequality have been used.

In view of Lemmas 3.4 and 3.6, we have

$$
\begin{aligned}
&\lambda_{\mathrm{o}}^{\mathrm{T}}(k)F_{\mathrm{o}}^{\mathrm{T}}\left(I+B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}\right)F_{\mathrm{o}}\lambda_{\mathrm{o}}(k)\\
\leq\ & r(k)\sum_{s=1}^{r(k)}\frac{\left(\det^2(A_{\mathrm{o}})-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}}(\det(A_{\mathrm{o}}))^{2r(k-s)+2s-4}\hat{x}_{\mathrm{o}}^{\mathrm{T}}(k-s-r(k-s))\\
&\times P_{\mathrm{o}}\hat{x}_{\mathrm{o}}(k-s-r(k-s)).
\end{aligned} \tag{4.95}
$$

On the other hand, by Lemma 3.5, we have

$$
(A_{\mathrm{o}}^{\mathrm{T}})^{r(k)}F_{\mathrm{o}}^{\mathrm{T}}\left(I+B_{\mathrm{o}}^{\mathrm{T}}P_{\mathrm{o}}B_{\mathrm{o}}\right)F_{\mathrm{o}}A_{\mathrm{o}}^{r(k)}\leq\left(1-\frac{(1-\gamma)^n}{\det^2(A_{\mathrm{o}})}\right)\left(\frac{\det^2(A_{\mathrm{o}})}{(1-\gamma)^{n-1}}\right)^{r(k)+1}P_{\mathrm{o}}. \tag{4.96}
$$

Then, the evaluation of $\Delta V(x_{\mathrm{o}}(k),e(k))$ can be continued as

$$
\begin{aligned}
\Delta V(x_{\mathrm{o}}(k),e(k))\leq&-\gamma x_{\mathrm{o}}^{\mathrm{T}}(k)P_{\mathrm{o}}x_{\mathrm{o}}(k)\\
&+2r(k)\sum_{s=1}^{r(k)}\frac{\left(\det^2(A_{\mathrm{o}})-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}}\det(A_{\mathrm{o}})^{2r(k-s)+2s-4}\\
&\times\hat{x}_{\mathrm{o}}^{\mathrm{T}}(k-s-r(k-s))P_{\mathrm{o}}\hat{x}_{\mathrm{o}}(k-s-r(k-s))\\
&+2\left(1-\frac{(1-\gamma)^n}{\det^2(A_{\mathrm{o}})}\right)\left(\frac{\det^2(A_{\mathrm{o}})}{(1-\gamma)^{n-1}}\right)^{r(k)+1}e_{\mathrm{o}}^{\mathrm{T}}(\phi(k))P_{\mathrm{o}}e_{\mathrm{o}}(\phi(k))\\
&-\beta e^{\mathrm{T}}(k)e(k),
\end{aligned} \tag{4.97}
$$

which, in view of $\hat{x}_{\rm o}(k) = x_{\rm o}(k) - e_{\rm o}(k)$ and the special structure of $\tilde{P}$, can be further continued as

$$\begin{aligned}\Delta V(x_{\rm o}(k), e(k)) \leq & -\gamma x_{\rm o}^{\rm T}(k) P_{\rm o} x_{\rm o}(k) \\ & + 4r(k) \sum_{s=1}^{r(k)} \frac{\left(\det^2(A_{\rm o}) - (1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}} \det{}^2(A_{\rm o})^{r(k-s)+s-2} \\ & \times \Big(x_{\rm o}^{\rm T}(k-s-r(k-s)) P_{\rm o} x_{\rm o}(k-s-r(k-s)) \\ & + e^{\rm T}(k-s-r(k-s)) \tilde{P} e(k-s-r(k-s)) \Big) \\ & - \beta e^{\rm T}(k) e(k) + 2\left(1 - \frac{(1-\gamma)^n}{\det^2(A_{\rm o})}\right)\left(\frac{\det^2(A_{\rm o})}{(1-\gamma)^{n-1}}\right)^{r(k)+1} \\ & \times e^{\rm T}(\phi(k)) \tilde{P} e(\phi(k)). \end{aligned} \tag{4.98}$$

Since $A_{\rm s}$ is nonsingular, there exists L that assigns the eigenvalues of $A + LC$ inside the unit circle but not at the origin. Thus, $A + LC$ is also nonsingular. Then, based on the error dynamics, we obtain

$$e(\phi(k)) = (A + LC)^{-r(k)} e(k),$$

from which it follows that

$$\begin{aligned}\Delta V(x_{\rm o}(k), e(k)) \leq & -\gamma x_{\rm o}^{\rm T}(k) P_{\rm o} x_{\rm o}(k) \\ & + 4r(k) \sum_{s=1}^{r(k)} \frac{\left(\det^2(A_{\rm o}) - (1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}} (\det(A_{\rm o}))^{2r(k-s)+2s-4} \\ & \times \Big(x_{\rm o}^{\rm T}(k-s-r(k-s)) P_{\rm o} x_{\rm o}(k-s-r(k-s)) \\ & + e^{\rm T}(k-s-r(k-s)) \tilde{P} e(k-s-r(k-s)) \Big) \\ & + e^{\rm T}(k) \left(2\left(1 - \frac{(1-\gamma)^n}{\det^2(A_{\rm o})}\right)\left(\frac{\det^2(A_{\rm o})}{(1-\gamma)^{n-1}}\right)^{r(k)+1} \right. \\ & \left. \times \left((A+LC)^{\rm T}\right)^{-r(k)} \tilde{P} (A+LC)^{-r(k)} - \beta I \right) e(k). \end{aligned} \tag{4.99}$$

For each $\gamma \in (0, 1)$, there exists a sufficiently large $\beta > 0$ such that

$$\tilde{P}(\gamma) \leq Q, \tag{4.100}$$

$$\gamma I + 2\left(1 - \frac{(1-\gamma)^n}{\det^2(A_o)}\right)\left(\frac{\det^2(A_o)}{(1-\gamma)^{n-1}}\right)\tilde{P}(\gamma) < \beta I, \tag{4.101}$$

$$1 < \lambda_{\max}(Q). \tag{4.102}$$

Fix this β. Inequality (4.100) indicates that $\tilde{P}(\gamma)$ in (4.99) can be replaced by R. Inequality (4.101) implies that there exists $D_1 > 0$ such that, for each $D \in (0, D_1)$,

$$\begin{aligned} & 2\left(1 - \frac{(1-\gamma)^n}{\det^2(A_o)}\right)\left(\frac{\det^2(A_o)}{(1-\gamma)^{n-1}}\right)^{r(k)+1}\left((A+LC)^{\mathrm{T}}\right)^{-r(k)} \\ & \times \tilde{P}(A+LC)^{-r(k)} - \beta I \\ \leq & -\gamma I. \end{aligned} \tag{4.103}$$

The use of (4.100), (4.103), and (4.102) in (4.99) then suggests that

$$\begin{aligned} \Delta V(x_o(k), e(k)) \leq & \Bigg(-\frac{\gamma}{\lambda_{\max}(Q)} \\ & + 4R^2\eta \frac{\left(\det^2(A_o) - (1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(2R+1)}}(\det(A_o))^{4R-4}\Bigg) \\ & \times V(x_o(k), e(k)), \end{aligned} \tag{4.104}$$

when $V(k+s) < \eta V(k)$, $s \in I[-2R, 0]$ holds for some constant $\eta > 1$. Then, if the condition

$$R^2 \frac{(\det(A_o))^{4R-4}}{(1-\gamma)^{(n-1)(2R+1)}} < \frac{\gamma}{4\lambda_{\max}(Q)\left(\det^2(A_o) - (1-\gamma)^n\right)^3} \tag{4.105}$$

holds, system (4.88) is asymptotically stable according to the Razumikhin Stability Theorem (Theorem 1.4). The left-hand side of inequality (4.105) is a strictly increasing function of R. Moreover, it goes to zero as R goes to zero and goes to infinity as R goes to infinity. On the other hand, the right-hand side of the inequality is a positive constant independent of R. Thus, the nonlinear equation

$$R^2 \frac{(\det(A_o))^{4R-4}}{(1-\gamma)^{(n-1)(2R+1)}} = \frac{\gamma}{4\lambda_{\max}(Q)\left(\det^2(A_o) - (1-\gamma)^n\right)^3} \tag{4.106}$$

has a unique positive solution R_2. Taking $R^* = \min\{R_1, R_2\}$ competes the proof. □

Theorem 4.5 suggests that our truncated predictor output feedback law for a general discrete-time linear system stabilizes the system as long as the delay of the system is small enough. An admissible delay bound in terms of open loop parameters and the feedback parameter is established to guarantee the closed-loop stability.

Theorem 4.6 *Consider system (4.88). Assume that A_s is nonsingular and all the eigenvalues of A_o are on the unit circle. Then, given an arbitrarily large R, there exists γ^* such that, for each $\gamma \in (0, \gamma^*)$, the system is asymptotically stable.*

Proof The same Lyapunov function as in (4.89), along with the estimate (4.99) of the forward difference along the trajectory of system (4.88), are adopted for our stability analysis. We rewrite (4.99) as

$$\begin{aligned}\Delta V(x_{\rm o}(k), e(k)) \leq & -\gamma x_{\rm o}^{\rm T}(k) P_{\rm o} x_{\rm o}(k) \\ & + 4r(k) \sum_{s=1}^{r(k)} \frac{(1-(1-\gamma)^n)^3}{(1-\gamma)^{(n-1)(r(k-s)+s+1)}} \\ & \times \Big(x_{\rm o}^{\rm T}(k-s-r(k-s)) P_{\rm o} x_{\rm o}(k-s-r(k-s)) \\ & + e^{\rm T}(k-s-r(k-s)) \tilde{P} e(k-s-r(k-s)) \Big) \\ & + e^{\rm T}(k) \bigg(2\left(1-(1-\gamma)^n\right) \left(\frac{1}{(1-\gamma)^{n-1}} \right)^{r(k)+1} \\ & \times \left((A+LC)^{\rm T} \right)^{-r(k)} \tilde{P} (A+LC)^{-r(k)} - \beta I \bigg) e(k).\end{aligned} \tag{4.107}$$

There exists $\beta > 0$ such that

$$1 < \lambda_{\max}(Q). \tag{4.108}$$

Fix this β. Then, there exists $\gamma_1 > 0$ such that, for each $\gamma \in (0, \gamma_1)$,

$$\tilde{P}(\gamma) \leq Q, \tag{4.109}$$

$$\begin{aligned}& \gamma I + 2\left(1-(1-\gamma)^n\right) \left(\frac{1}{(1-\gamma)^{n-1}} \right)^{r(k)+1} \\ & \left((A+LC)^{\rm T}\right)^{-r(k)} \tilde{P}(\gamma) \, (A+LC)^{-r(k)} < \beta I.\end{aligned} \tag{4.110}$$

Inequalities (4.109) and (4.110) imply that

$$\Delta V(x_{\rm o}(k), e(k)) \leq \left(-\frac{\gamma}{\lambda_{\max}(Q)} + 4R^2 \eta \frac{\left(1-(1-\gamma)^n\right)^3}{(1-\gamma)^{(n-1)(2R+1)}} \right) V(x_{\rm o}(k), e(k)), \tag{4.111}$$

when $V(k+s) < \eta V(k)$, $s \in I[-2R, 0]$, holds for some constant $\eta > 1$. Then, if the condition

$$\frac{\left(1-(1-\gamma)^n\right)^3}{\gamma(1-\gamma)^{(n-1)(2R+1)}} < \frac{1}{4\lambda_{\max}(Q)R^2} \tag{4.112}$$

holds, system (4.88) is asymptotically stable according to the Razumikhin Stability Theorem (Theorem 1.4). In view of the fact that

$$\begin{aligned}&\lim_{\gamma\to 0^+}\frac{\left(1-(1-\gamma)^n\right)^3}{\gamma}\\ &=\lim_{\gamma\to 0^+} 3\left(1-(1-\gamma)^n\right)^2 n(1-\gamma)^{n-1}\\ &=0,\end{aligned} \tag{4.113}$$

there exists $\gamma_2 > 0$ such that, for each $\gamma \in (0, \gamma_2)$, the inequality (4.112) holds. Taking $\gamma^* = \min\{\gamma_1, \gamma_2\}$ completes the proof. □

Again, our analysis on the stabilization of a general discrete-time linear system by truncated predictor output feedback reconstructs the low gain nature of the feedback law when it comes to the stabilization of a discrete-time linear system with all its open loop poles on or inside the unit circle.

4.3.3 A Numerical Example

Consider the discrete-time linear system (4.55) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ -1 & 2\sqrt{2} & -4 & 2\sqrt{2} & 0\\ 0 & 0 & 0 & 0 & 1.1 \end{bmatrix},\quad B = \begin{bmatrix} 0\\0\\0\\1\\1\end{bmatrix},\quad C = \begin{bmatrix}1 \ 0 \ 0 \ 0 \ 1\end{bmatrix}. \tag{4.114}$$

It can be readily verified that the triple (A, B, C) is both controllable and observable. Also, the eigenvalues of A are at

$$\lambda = \left\{\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}j, \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}j, 1.1\right\},$$

four on the unit circle and one outside the unit circle. Thus, we pick a $\gamma = 0.05$. Then, based on such a γ, the time-varying delay of the system is chosen as follows to have a sufficiently small upper bound of $R = 2$,

$$r(k) = \begin{cases} 0, & \text{if} \quad \text{mod}\ (k,3) = 0, \\ 1, & \text{if} \quad \text{mod}\ (k,3) = 1, \\ 2, & \text{if} \quad \text{mod}\ (k,3) = 2, \end{cases}$$

where mod(p, q) denotes the remainder after division of p by q. Set the initial condition of the system as

$$x(k) = \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \end{bmatrix}^{\mathrm{T}}, \quad k \in I[0, R]. \tag{4.115}$$

The state response and the input evolution under the truncated predictor state feedback law are shown in Fig. 4.3.

The truncated predictor output feedback law (4.81) involves the eigenvalue assignment of $A + LC$ inside the unit circle. For this purpose, we design an L such that the eigenvalues of $A + LC$ are assigned at

$$\lambda = \left\{ \frac{1}{2} \pm \frac{1}{2} j, \frac{1}{2} \pm \frac{1}{2} j, 0.9 \right\}.$$

We pick $\gamma = 0.03$ for illustration. Given the same time-varying delay as in the state feedback simulation, the stabilization is achieved. The state response and the input evolution of the system are shown in Fig. 4.4. In the simulation, we have chosen the initial condition of the system and that of the observer as

$$x(k) = \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \hat{x}(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad k \in I[0, R]. \tag{4.116}$$

4.4 Conclusions

A TPF law, simplified from a predictor feedback law, is able to stabilize a general, possibly exponentially unstable, linear system as long as the input delay of the system is small enough. We consider this property of the TPF law its robustness to a certain amount of delay. Such a robustness holds in various control settings as manifested throughout this chapter. One of the many implications of the robustness property is the low gain nature of the TPF law in the stabilization of a linear system that is not exponentially unstable, namely, given an arbitrarily large delay, the feedback law stabilizes the system as long as its feedback parameter is tuned to be sufficiently small. Another noteworthy implication of the robustness property concerns the tuning of the feedback parameter of the feedback law in the stabilization of an exponentially unstable system. There exists an admissible bound on the delay such that, for each delay below this bound, the TPF law would stabilize

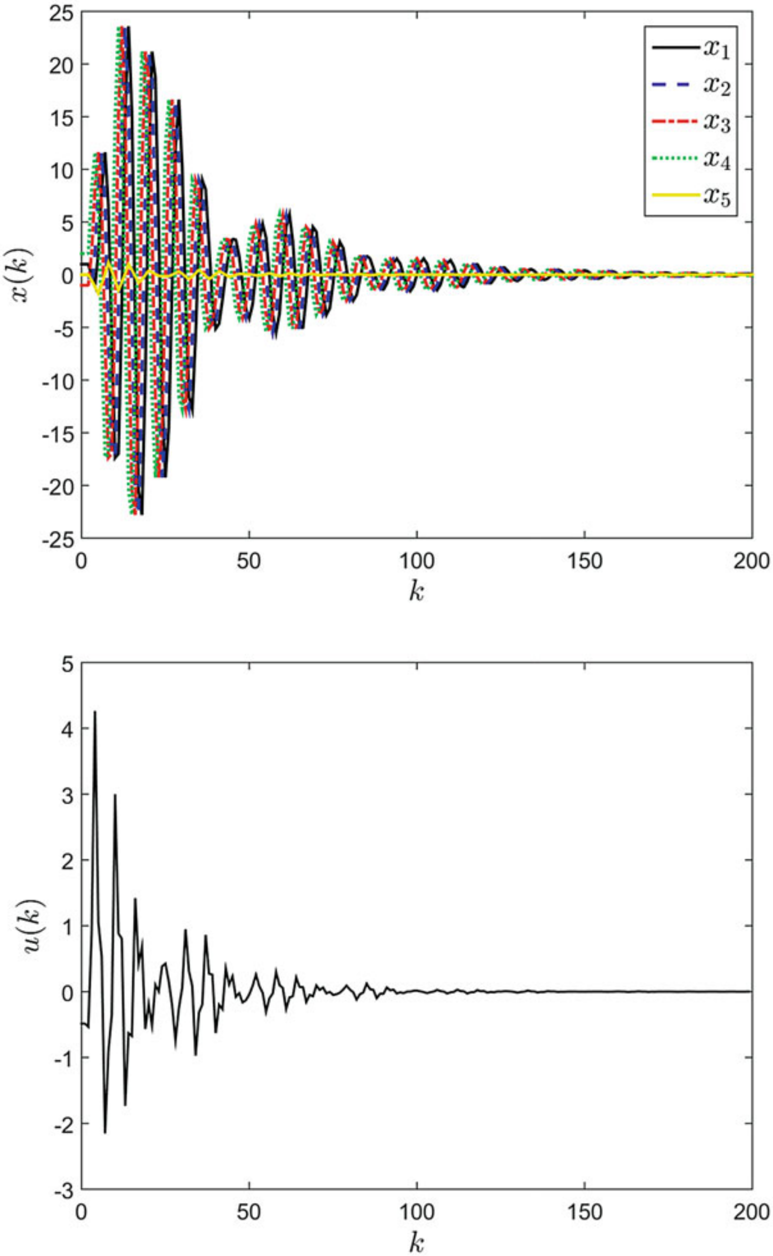


Fig. 4.3 State response and control input under the state feedback TPF law (4.65): $\gamma = 0.05$

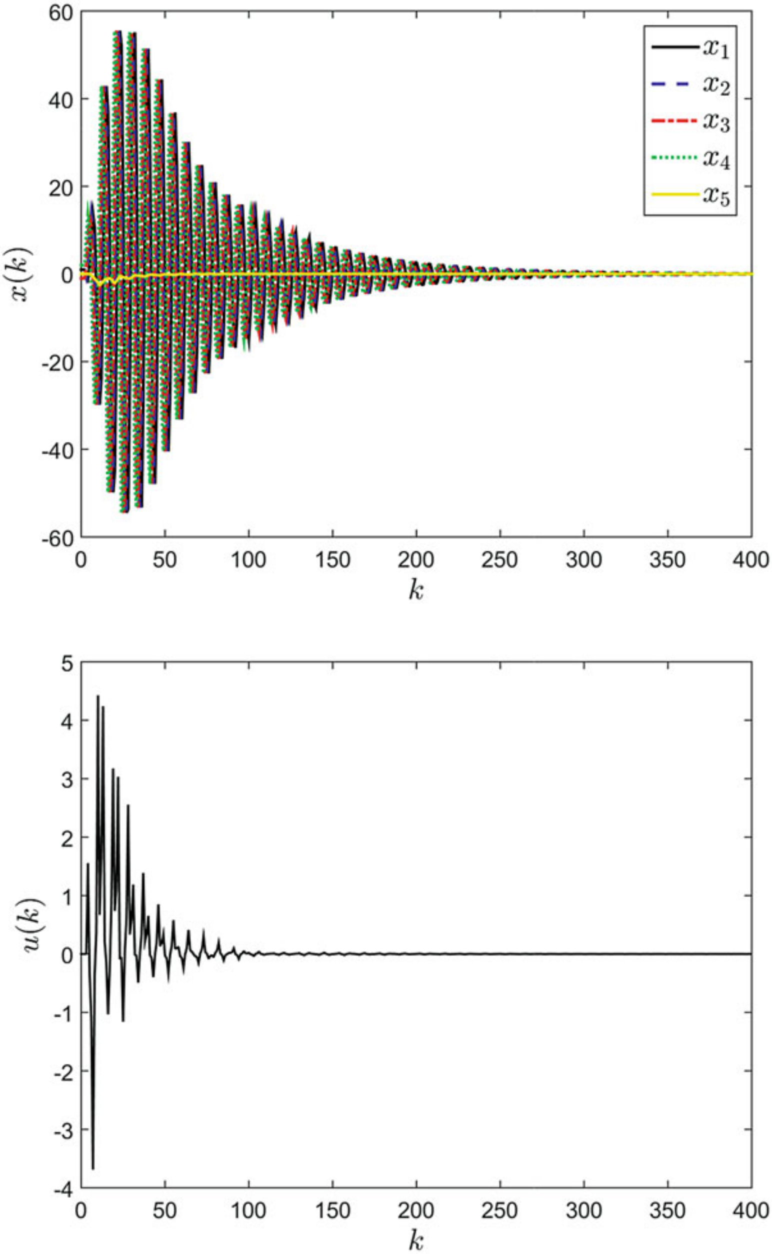


Fig. 4.4 State response and control input under the output feedback TPF law (4.26): $\gamma = 0.03$

the system with any feedback parameter belonging to a certain range. Such a bound on the delay and the range of the feedback parameter will be further elaborated in the following section.

4.5 Notes and References

The stability conditions given in this chapter address the stabilization problem from the perspective that given a feedback parameter, a general, possibly exponentially unstable, linear system is stabilized as long as the delay is small enough. However, in practice, the problem is typically formulated from an opposite perspective. Given a certain amount of delay, is a TPF law able to stabilize a general linear system? The answer to this problem is clear for a linear system that is not exponentially unstable. It lies in the low gain nature of the TPF law. For a linear system that is exponentially unstable, the answer is also clear from a simple examination of the stability conditions established in this chapter. For illustration, we consider the case of the truncated predictor state feedback law in the continuous-time setting. The established delay bound implied by (4.10) is a function of γ and $\mathrm{tr}(A)$. For a given exponentially unstable system, the function of the delay bound solely depends on $\gamma > 0$. Sweeping the value of γ over positive numbers, we note that the delay bound has a maximum that is positive. This observation can be readily made by noting that the delay bound goes to zero as γ goes to zero and as γ goes to infinity. Thus, given any delay that is below this maximum, there exists at least one continuous interval of γ within which stabilization is achieved.

The above discussion on the maximum delay bound of an exponentially unstable system is inspired by [116], in which the method of tuning the feedback parameter of the state feedback TPF law in the stabilization of an exponentially unstable system is also presented. Moreover, the presentation of Sect. 4.3 roughly follows that of [100].

Chapter 5
Delay Independent Truncated Predictor Feedback for Continuous-Time Linear Systems

5.1 Introduction

The truncated predictor feedback design simplifies the predictor feedback design by discarding the distributed delay term. The implementation of the remaining static feedback term of the predictor feedback law requires the exact knowledge of the input delay. The truncated predictor state feedback law for continuous-time linear systems with a constant delay is such an example. The exact value of the delay appears in the exponential factor $\mathrm{e}^{A\tau}$ of the truncated predictor feedback law. Moreover, the determination of the value of the feedback parameter of the truncated predictor feedback law requires, although not the exact value of the delay, an upper bound of the delay to be known. This requirement of the information of the delay, which explicitly or implicitly appears in the truncated predictor feedback law, suggests that the truncated predictor feedback law is delay-dependent. The delay-dependency of the truncated predictor feedback law also manifests itself in the observer based feedback design. To construct an observer whose state asymptotically approaches the state of the open loop system, the dynamics of the observer generally contains the delayed input. This implies that the exact value of the delay is also required in the construction of the observer used in the observer based truncated predictor feedback law.

The truncated predictor feedback design compensates a constant delay by requiring the exact value of the delay. Such compensation becomes trickier when the delay is time-varying. The compensation of a bounded time-varying delay via a truncated predictor feedback design relies on the prediction of the future state of the system at the future time instant $\phi^{-1}(t)$, where $\phi(t)$ represents the past time instant at which the input is injected into the system. Clearly, the implementation of a truncated predictor feedback law in the face of a time-varying delay fails if $\phi(t)$ does not admit an inverse. Consider a standard form of

$$\phi(t) = t - d(t),$$

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5_5

where $d(t)$ is the time-varying delay. The inverse of $\phi(t)$ does not exist whenever

$$d(t) = t - t_1,$$

on some time interval $t \in [t_1, t_2]$. It is noteworthy that such a time-varying delay is physically meaningful. The requirement on the existence of $\phi^{-1}(t)$ restricts the application of truncated predictor feedback in compensating time-varying delays. Therefore, the exact knowledge of time-varying delays is only necessary, but not necessarily sufficient, for the success in the implementation of a truncated predictor feedback law.

Compared to feedback laws whose realization relies heavily on the knowledge of the delay, those that require less such knowledge are preferable. The craving for feedback laws that are delay independent, at least partially if not completely, is driven by the lack of the information of the delay in practice. In practice, the delay is hardly precisely known. Oftentimes, only its upper bound and/or lower bound is known. In the worst case, no knowledge of the delay can be assumed. Basically, the overall objective of the rest of this book is to relax the assumption on the availability of the knowledge of the delay for control design.

In this chapter, we introduce delay independent truncated predictor feedback laws that discard the exponential factor of the truncated predictor feedback laws. The remaining feedback gain of the delay independent truncated predictor feedback laws is parameterized in a feedback parameter, whose value is determined based on the knowledge of an upper bound of the delay. The delay independent truncated predictor feedback laws are less delay dependent than the truncated predictor feedback laws because their implementation no longer requires the exact knowledge of the delay to be known. Besides the basic stabilization problem, we further consider the problem of improving the performance of a closed-loop system under a delay independent truncated predictor feedback law. Such a problem originates from the low gain nature of the delay independent truncated predictor feedback law in the stabilization of a linear system with all its open loop poles at the origin or in the open left-half plane. It has been observed that an excessively small value of the feedback parameter results in a large overshoot and a slow convergence rate of the closed-loop system. The poor closed-loop performance inspires the design of a time-varying feedback parameter whose value is chosen large at the beginning phase of the system evolution and is decreased as needed to facilitate the proof of stability. The large value of the parameter at the starting phase of the system evolution reduces the overshoot and increases the convergence rate. Benefits of the time-varying parameter design in the closed-loop performance are demonstrated in the convergence rate analysis and the numerical studies.

5.2 Delay Independent Truncated Predictor State Feedback Design

In a predictor feedback law for a linear system with input delay, the future state is predicted by the solution of the linear system. The zero input solution contains the transition matrix. The zero state solution gives rise to the distributed nature of the feedback law. In Chap. 2, it is established that when the system is not exponentially unstable, low gain feedback can be designed such that the predictor feedback law, with the distributed delay term truncated, still achieves stabilization in the presence of an arbitrarily large delay. Furthermore, in the absence of purely imaginary poles, the transition matrix in the truncated predictor feedback can be safely dropped, resulting in a delay independent truncated predictor feedback law , which is simply a delay independent linear state feedback. In this section, we first construct an example to show that, in the presence of purely imaginary poles, the linear delay independent truncated predictor feedback in general cannot stabilize the system for an arbitrarily large delay. By using the extended Krasovskii Stability Theorem (Theorem 1.2), we derive a bound on the delay under which a delay independent truncated predictor feedback law achieves stabilization for a general system that may be exponentially unstable.

We consider the asymptotic stabilization problem for the following linear system with time-varying delay in the input:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(\phi(t)), \\ x(\theta) = \phi(\theta),\ \ \theta \in [-D, 0], \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are state and input, respectively. The time-varying delay function $\phi(t) : \mathbb{R}^+ \to \mathbb{R}$ is assumed to take the standard form of

$$\phi(t) = t - d(t), \tag{5.2}$$

where $d(t) : \mathbb{R}^+ \to \mathbb{R}^+$ denotes time-varying delay which is bounded by a finite positive constant D, i.e.,

$$0 \le d(t) \le D,\ \ t \ge 0. \tag{5.3}$$

Only the information on the bound D, but not the delay $d(t)$ itself, will be required in our feedback design and stability analysis. We also assume that the pair (A, B) is stabilizable.

In [63], it is shown that when the open loop system is not exponentially unstable, a parametrized feedback gain matrix $F(\gamma)$ can be designed by the low gain feedback design technique [61] such that the finite-dimensional truncated predictor feedback law

$$u(t) = F(\gamma)e^{Ad}x(t) \tag{5.4}$$

would still asymptotically stabilize system (5.1) for an arbitrarily large constant delay d as long as the low gain parameter γ is tuned small enough. In the absence of purely imaginary poles, the transition matrix in the truncated predictor feedback law (5.4) can be dropped and the feedback law further simplifies to a delay independent truncated predictor state feedback law (also referred to as the delay independent state feedback TPF law),

$$u(t) = F(\gamma)x(t). \tag{5.5}$$

Such a feedback law, parameterized in the low gain parameter γ, is referred to as the delay independent truncated predictor feedback law. The truncated predictor feedback design originally proposed in [63] uses the eigenstructure assignment based low gain feedback design method. The design was simplified in [122], where a parametric Lyapunov equation based low gain feedback design was adopted.

In this section, we will examine the properties of the delay independent truncated predictor feedback for general systems, which may have purely imaginary (nonzero poles on the imaginary axis) or exponentially unstable poles. In particular, we will first construct an example to show that, in the presence of purely imaginary poles, the delay independent state feedback TPF law in general does not have the ability to stabilize the system for an arbitrarily large delay. We then derive, by applying the extended Krasovskii Stability Theorem (Theorem 1.2), a bound on the delay under which a delay independent truncated predictor feedback law achieves stabilization for the system. The expression of this bound indicates that when all the closed right-half plane poles are at the origin, stabilization of the system would be achieved for an arbitrarily large delay as long as the low gain parameter is chosen to be sufficiently small. This observation coincides with the results in both [63] and [122]. Moreover, it will be shown that, for a given delay with an arbitrarily large upper bound, the upper bound of the low gain parameter derived in this chapter that guarantees stability is less conservative than the one given in [122].

5.2.1 Preliminaries

It is shown in [63] that a linear system with all its open loop poles at the origin or in the open left-half plane can be stabilized for an arbitrarily large delay by a delay independent truncated predictor feedback law,

$$u(t) = F(\gamma)x(t), \ \gamma > 0. \tag{5.6}$$

The construction of $F(\gamma)$ was given in [124] by utilizing the Lyapunov equation based low gain design technique [121]. That is, for a controllable pair (A, B), the parametrized feedback gain matrix $F(\gamma)$ in (5.6) is constructed as,

$$F(\gamma) = -B^{\mathrm{T}}P(\gamma), \tag{5.7}$$

where the positive definite matrix $P(\gamma)$ is the solution to the parametric algebraic Riccati equation

$$A^{\mathrm{T}}P(\gamma) + P(\gamma)A - P(\gamma)BB^{\mathrm{T}}P(\gamma) = -\gamma P(\gamma) \tag{5.8}$$

with

$$\gamma > -2\min\{\mathrm{Re}(\lambda(A))\}. \tag{5.9}$$

In the case where all eigenvalues of A are at the origin or in the open left-half plane, the delay is allowed to be arbitrarily large but bounded, and the value of the parameter γ is required to approach zero as the bound on the delay increases to infinity. As a result, the parametric algebraic Riccati equation (5.8) is a low gain feedback design and the feedback parameter is referred to as the low gain parameter.

In this section, we will first show that when system (5.1) is not exponentially unstable but has purely imaginary poles, the delay independent truncated predictor feedback law (5.6) is in general not able to achieve asymptotic stabilization for a large enough delay. We will then derive bound on the delay under which the delay independent truncated predictor feedback law would achieve stabilization for a general system that may be exponentially unstable.

To achieve our objectives, we need some technical preliminaries. We first recall some properties of the solution $P(\gamma)$ of the algebraic Riccati equation (5.8) from [124].

Lemma 5.1 *Assume that (A, B) is controllable and γ satisfies (5.9). Then,*

$$A_c^{\mathrm{T}}(\gamma)P(\gamma)A_c(\gamma) \le \varpi(\gamma)P(\gamma),$$

where

$$A_c(\gamma) = A - BB^{\mathrm{T}}P(\gamma) \tag{5.10}$$

and

$$\varpi(\gamma) = \frac{1}{2}\Big(n\gamma + 2\mathrm{tr}(A)\Big)\Big((n+1)\gamma + 2\mathrm{tr}(A)\Big) - \gamma\mathrm{tr}(A) - \mathrm{tr}\left(A^2\right). \tag{5.11}$$

We next establish the following simple fact and its two corollaries.

Lemma 5.2 *If all eigenvalues of $A \in \mathbb{R}^{n\times n}$ are on the closed right-half plane, then*

$$\mathrm{tr}^2(A) \ge \mathrm{tr}\left(A^2\right). \tag{5.12}$$

Moreover, the equality sign holds if and only if all the eigenvalues of A are real and at most one of them is positive.

Proof Let the eigenvalues of A be $\lambda_1, \lambda_2, \cdots, \lambda_p, \alpha_1 \pm j\beta_1, \alpha_2 \pm j\beta_2, \cdots, \alpha_q \pm j\beta_q$, where

$$p + 2q = n \tag{5.13}$$

and

$$\begin{cases} \lambda_i \geq 0, \ i = 1, 2, \cdots, p, \\ \alpha_i \geq 0, \ i = 1, 2, \cdots, q, \\ \beta_i > 0, \ i = 1, 2, \cdots, q. \end{cases} \tag{5.14}$$

It then follows that

$$\mathrm{tr}(A) = \sum_{i=1}^{p} \lambda_i + 2\sum_{i=1}^{q} \alpha_i \tag{5.15}$$

and

$$\mathrm{tr}\left(A^2\right) = \sum_{i=1}^{p} \lambda_i^2 + 2\sum_{i=1}^{q} \left(\alpha_i^2 - \beta_i^2\right). \tag{5.16}$$

Then, the difference between $\mathrm{tr}^2(A)$ and $\mathrm{tr}\left(A^2\right)$ can be expressed as

$$\begin{aligned} \mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right) &= \left(\sum_{i=1}^{p} \lambda_i\right)^2 + 4\left(\sum_{i=1}^{q} \alpha_i\right)^2 + 4\left(\sum_{i=1}^{p} \lambda_i\right)\left(\sum_{i=1}^{q} \alpha_i\right) \\ &\quad - \sum_{i=1}^{p} \lambda_i^2 - 2\sum_{i=1}^{q} \left(\alpha_i^2 - \beta_i^2\right) \\ &= 2\sum_{1 \leq i < j \leq p} \lambda_i \lambda_j + 4\sum_{i=1}^{p} \lambda_i \sum_{i=1}^{q} \alpha_i + 2\sum_{i=1}^{q} \alpha_i^2 \\ &\quad + 8\sum_{1 \leq i < j \leq q} \alpha_i \alpha_j + 2\sum_{i=1}^{q} \beta_i^2 \\ &\geq 0, \end{aligned} \tag{5.17}$$

from which we can readily conclude by invoking (5.14) that if A has at least one pair of imaginary eigenvalues, then

$$\mathrm{tr}^2(A) > \mathrm{tr}\left(A^2\right). \tag{5.18}$$

Therefore, a necessary condition for the equality sign to hold is that all the eigenvalues of A are real. Under this condition, (5.17) can be simplified as

$$\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right) = \left(\sum\nolimits_{i=1}^{p} \lambda_i\right)^2 - \sum\nolimits_{i=1}^{p} \lambda_i^2, \tag{5.19}$$

where $p \geq 1$. Now, we consider two separate cases, $p = 1$ and $p \geq 2$. If $p = 1$, $\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right) = 0$ for any $\lambda_1 \geq 0$. If $p \geq 2$,

$$\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right) = 2\sum\nolimits_{1 \leq i < j \leq p} \lambda_i \lambda_j, \tag{5.20}$$

from which it readily follows that the necessary and sufficient condition for the equality sign to hold is that at most one of the eigenvalues of A is positive. Combining the two separate cases, we can conclude that the equality sign in (5.12) holds if and only if all eigenvalues of A are real and at most one of them is positive.

□

Corollary 5.1 *If all eigenvalues of $A \in \mathbb{R}^{n \times n}$ are on the closed right-half plane, then the following inequality holds for any real $\gamma > 0$:*

$$\left(\frac{\gamma}{2\Big(2\mathrm{tr}(A) + n\gamma\Big)\left(\Big(2\mathrm{tr}(A) + n\gamma\Big)\left(\mathrm{tr}(A) + \frac{n+3}{2}\gamma\right) - \gamma\mathrm{tr}(A) - \mathrm{tr}\left(A^2\right)\right)}\right)^{\frac{1}{2}}$$
$$< \frac{1}{\sqrt{2}\Big(2\mathrm{tr}(A) + n\gamma\Big)}. \tag{5.21}$$

Proof First we denote the left-hand side and the right-hand side of inequality (5.21) as $D_1(\gamma)$ and $D_2(\gamma)$, respectively. The functions $D_1(\gamma)$ and $D_2(\gamma)$ are well defined because of the assumption that all eigenvalues of A are in the closed right-half plane, Lemma 5.2 and $\gamma > 0$. We notice that $D_1(\gamma)$ can be written as follows:

$$D_1(\gamma) = \left(2\Big(2\mathrm{tr}(A) + n\gamma\Big)^2 \left(\frac{n+3}{2} + \frac{\mathrm{tr}(A)}{\gamma} - \frac{\mathrm{tr}(A) + \dfrac{\mathrm{tr}\left(A^2\right)}{\gamma}}{2\mathrm{tr}(A) + n\gamma}\right)\right)^{-\frac{1}{2}}.$$

In order to show that inequality (5.21) holds, it suffices to show that

$$\frac{n+3}{2} + \frac{\mathrm{tr}(A)}{\gamma} - \frac{\mathrm{tr}(A) + \dfrac{\mathrm{tr}\left(A^2\right)}{\gamma}}{2\mathrm{tr}(A) + n\gamma} > 1,$$

which is equivalent to

$$2n\gamma \mathrm{tr}(A) + \frac{n(n+1)}{2}\gamma^2 + 2\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right) > 0,$$

which can be easily verified by using the facts that $\mathrm{tr}(A) \geq 0$ and $\gamma > 0$, along with Lemma 5.2. □

Corollary 5.2 *If $A \in \mathbb{R}^{n\times n}$ is exponentially unstable with all eigenvalues on the closed right-half plane, then $D_1(\gamma)$ as defined in the proof of Corollary 5.1, where $\gamma > 0$, has a unique maximal value $D_1(\gamma^\star)$ achieved at $\gamma = \gamma^\star$, where*

$$\gamma^\star = \frac{n\left(2\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right) + (n+1)\mathrm{tr}(A)\right) + \sqrt{\Delta}}{n(n+3)\mathrm{tr}(A)}, \tag{5.22}$$

and

$$\Delta = n^2\left(2\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right) + (n+1)\mathrm{tr}(A)\right)^2 + 2n(n+3)\mathrm{tr}^2(A)\left(2\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right)\right). \tag{5.23}$$

Proof We determine the monotonicity of $D_1(\gamma)$, which is equivalent to the monotonicity of $D_1^2(\gamma)$. The derivative of $D_1^2(\gamma)$ with respect to γ is derived as follows:

$$\frac{\mathrm{d}D_2^2(\gamma)}{\mathrm{d}\gamma} = \frac{h(\gamma) - \gamma\dfrac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma}}{h^2(\gamma)},$$

where

$$h(\gamma) = 2\big(2\mathrm{tr}(A) + n\gamma\big)\left(\big(2\mathrm{tr}(A) + n\gamma\big)\left(\mathrm{tr}(A) + \frac{n+3}{2}\gamma\right) - \gamma\mathrm{tr}(A) - \mathrm{tr}\left(A^2\right)\right). \tag{5.24}$$

Then, it suffices to determine the sign of $h(\gamma) - \gamma\dfrac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma}$, whose expression can be obtained as,

$$h(\gamma) - \gamma\frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} = -2n(n+3)\mathrm{tr}(A)\gamma^2 + 2n\left(4\mathrm{tr}^2(A) - 2\mathrm{tr}\left(A^2\right) + (2n+2)\mathrm{tr}(A)\right)\gamma + 4\mathrm{tr}(A)\left(2\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right)\right). \tag{5.25}$$

It can be observed that the right-hand side of (5.25) is a quadratic function of γ. By Lemma 5.2, it has a unique positive root $\gamma^\star$ as long as A is exponentially unstable with all its eigenvalues in the closed right-half plane, where $\gamma^\star$ is defined as in (5.22). Moreover, for each $\gamma \in (0, \gamma^\star)$,

$$h(\gamma) - \gamma \frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} > 0, \tag{5.26}$$

and for each $\gamma \in [\gamma^\star, \infty)$,

$$h(\gamma) - \gamma \frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} \leq 0. \tag{5.27}$$

Therefore, the continuity of $D_1(\gamma)$ with respect to γ implies that $D_1(\gamma)$ reaches its unique maximal value $D_1(\gamma^\star)$ at $\gamma^\star$. □

5.2.2 *Stability Analysis*

We first provide an example to show that for a system that is not exponentially unstable but has purely imaginary poles, delay independent truncated predictor feedback in general is not able to achieve stabilization for a sufficiently large delay.

Example 5.1 Consider system (5.1) with a constant delay

$$d(t) = \tau$$

and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.28}$$

The system is controllable with $\lambda(A) = \{\pm j\}$. Consider the delay independent state feedback TPF law (5.6) with

$$F(\gamma) = -\begin{bmatrix} \gamma^2 & 2\gamma \end{bmatrix}, \tag{5.29}$$

where $\gamma > 0$ by the inequality in (5.9). The characteristic equation of the closed-loop system is given by

$$\det\left(sI - A + BB^{\mathrm{T}}P(\gamma)\mathrm{e}^{-\tau s}\right) = s^2 + 2\gamma s e^{-\tau s} + \gamma^2 e^{-\tau s} + 1 = 0.$$

We first define two real sequences as

$$\begin{cases} \tau_{i,1} = \dfrac{1}{\omega_{\mathrm{R}}} \cos^{-1}\left(\dfrac{\omega_{\mathrm{R}}^2 - 1}{4\omega_{\mathrm{R}}^2 + \gamma^2}\right) + i\dfrac{2\pi}{\omega_{\mathrm{R}}}, & i \in \mathbb{N}, \\ \tau_{i,2} = \dfrac{1}{\omega_{\mathrm{L}}} \cos^{-1}\left(\dfrac{\omega_{\mathrm{L}}^2 - 1}{4\omega_{\mathrm{L}}^2 + \gamma^2}\right) + i\dfrac{2\pi}{\omega_{\mathrm{L}}}, & i \in \mathbb{N}, \end{cases} \tag{5.30}$$

where $\tau_{i,1}$ is defined on $\gamma \in (0, +\infty)$, $\tau_{i,2}$ is defined on $\gamma \in (0, 1)$,

$$\omega_{\mathrm{R}} = \left(1 + 2\gamma^2 + \gamma\sqrt{5\gamma^2 + 4}\right)^{\frac{1}{2}} \tag{5.31}$$

and

$$\omega_{\mathrm{L}} = \left(1 + 2\gamma^2 - \gamma\sqrt{5\gamma^2 + 4}\right)^{\frac{1}{2}}. \tag{5.32}$$

We consider two cases with respect to the value of γ.

(I) Fix a $\gamma \geq 1$. Then, according to [24], the closed-loop system is unstable if

$$\tau > \tau_{0,1}.$$

Note from (5.30) that

$$\tau_{0,1} < \frac{\pi}{2\sqrt{6}},$$

because

$$\omega_{\mathrm{R}} > \sqrt{6}$$

and

$$\cos^{-1}\left(\frac{\omega_{\mathrm{R}}^2 - 1}{4\omega_{\mathrm{R}}^2 + \gamma^2}\right) \leq \frac{\pi}{2}.$$

This implies that the closed-loop system is unstable if

$$\tau > \frac{\pi}{2\sqrt{6}}.$$

(II) Fix a $\gamma \in (0, 1)$. According to [24], because the closed-loop system in the absence of delay is stable and

$$\omega_{\mathrm{R}} > \omega_{\mathrm{L}} > 0,$$

there exists $k \in \mathbb{N} \setminus \{0\}$ such that the sequences in (5.30) satisfy

$$0 < \tau_{0,1} < \tau_{0,2} < \tau_{1,1} < \tau_{1,2} < \ldots < \tau_{k-1,1} < \tau_{k-1,2} < \tau_{k,1} \\ < \tau_{k+1,1} < \tau_{k,2} < \ldots,$$

and the closed-loop system is unstable if

$$\tau \in \cup_{i \in I[0,k-1]}[\tau_{i,1}, \tau_{i,2}] \cup [\tau_{k,1}, \infty).$$

Note from (5.30) that

$$\tau_{i,1} < \frac{\pi}{2} + 2\pi i$$

and

$$\tau_{i,2} > \frac{\pi}{2} + 2\pi i, \quad i \in \mathbb{N},$$

because

$$\omega_{\mathrm{R}} > 1,$$

$$\cos^{-1}\left(\frac{\omega_{\mathrm{R}}^2 - 1}{4\omega_{\mathrm{R}}^2 + \gamma^2}\right) \leq \frac{\pi}{2},$$

$$\omega_{\mathrm{L}} < 1,$$

and

$$\cos^{-1}\left(\frac{\omega_{\mathrm{L}}^2 - 1}{4\omega_{\mathrm{L}}^2 + \gamma^2}\right) \geq \frac{\pi}{2}.$$

Therefore, if

$$\tau = \frac{\pi}{2} + 2\pi l, \quad l \in \mathbb{N},$$

the closed-loop system is unstable.

Combining the two cases, we see that, for any of the delays,

$$\tau = \frac{\pi}{2} + 2\pi l, \quad l \in \mathbb{N},$$

the closed-loop system is unstable for each $\gamma > 0$. This implies that the delay independent truncated predictor feedback law (5.6) fails to stabilize the system with

$$\tau = \frac{\pi}{2} + 2\pi l, \quad l \in \mathbb{N}.$$

□

Unlike the class of linear systems with open loop poles at the origin, linear systems with purely imaginary open loop poles cannot be stabilized by tuning the feedback parameter of the delay independent state feedback law if the delay is too large. However, the stabilization of a general linear system can be achieved by tuning the feedback parameter as long as the delay value is small enough. We now present a theorem on this fact. Before presenting the theorem, we make the following assumption on system (5.1).

For a general system (5.1), the eigenvalues of A can be anywhere on the complex plane. Without loss of generality, let the pair (A, B) be given in the following form:

$$A = \begin{bmatrix} A_{\mathrm{L}} & 0 \\ 0 & A_{\mathrm{R}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{L}} \\ B_{\mathrm{R}} \end{bmatrix},$$

where all eigenvalues of A_{L} are in the open left-half plane and all eigenvalues of A_{R} are in the closed right-half plane. Correspondingly, system (5.1) can be written as

$$\begin{cases} \dot{x}_{\mathrm{L}}(t) = A_{\mathrm{L}} x_{\mathrm{L}}(t) + B_{\mathrm{L}} u(\phi(t)), \\ \dot{x}_{\mathrm{R}}(t) = A_{\mathrm{R}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} u(\phi(t)), \end{cases}$$

where

$$x(t) = \left[x_{\mathrm{L}}^{\mathrm{T}}(t) \;\; x_{\mathrm{R}}^{\mathrm{T}}(t)\right]^{\mathrm{T}}.$$

It is clear that any linear state feedback law that stabilizes x_{R} subsystem would stabilize the entire system. Thus, we will assume, without loss of generality, that all eigenvalues of A are on the closed right-half plane.

Theorem 5.1 *Consider system (5.1). Let (A, B) be controllable and all eigenvalues of A be on the closed right-half plane. If, for each $\gamma > 0$,*

$$D < \left(\frac{\gamma}{2\big(2\mathrm{tr}(A) + n\gamma\big)\left(\big(2\mathrm{tr}(A) + n\gamma\big)\left(\mathrm{tr}(A) + \frac{n+3}{2}\gamma\right) - \gamma\mathrm{tr}(A) - \mathrm{tr}\left(A^2\right)\right)} \right)^{\frac{1}{2}}, \tag{5.33}$$

then the delay independent state feedback TPF law (5.6) asymptotically stabilizes the system.

Proof Under the assumption that all eigenvalues of A are on the closed right-half plane, we let $\gamma > 0$ satisfy (5.9). The closed-loop system consisting of system (5.1) and the delay independent truncated predictor feedback law (5.6) is expressed as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\left(-B^{\mathrm{T}}P\right)x(\phi(t))) \\ &= \left(A - BB^{\mathrm{T}}P\right)x(t) + BB^{\mathrm{T}}P(x(t) - x(\phi(t))) \\ &= A_{\mathrm{C}}x(t) + BB^{\mathrm{T}}P\lambda(t),\end{aligned} \tag{5.34}$$

where

$$\begin{aligned}A_{\mathrm{C}} &= A + BF(\gamma) \\ &= A - BB^{\mathrm{T}}P(\gamma)\end{aligned} \tag{5.35}$$

is defined in Lemma 5.1 and

$$\lambda(t) = x(t) - x(\phi(t)).$$

Consider a Lyapunov functional

$$V(x_t) = V_1(x) + V_2(x_t), \tag{5.36}$$

where

$$V_1(x) = x^{\mathrm{T}}(t)Px(t), \tag{5.37}$$

$$V_2(x_t) = \epsilon \int_{-D}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s\mathrm{d}\theta,$$

and ϵ is some real positive constant whose value is to be determined later. Then the derivative of $V(x_t)$ along the trajectory of the closed-loop system is

$$\dot{V}(x_t) = \dot{V}_1(x) + \dot{V}_2(x_t).$$

On one hand, $\dot{V}_1(x)$ can be evaluated as

$$\begin{aligned}\dot{V}_1(x) &= \dot{x}^{\mathrm{T}}(t)Px(t) + x^{\mathrm{T}}(t)P\dot{x}(t) \\ &= x^{\mathrm{T}}(t)\left(A_{\mathrm{C}}^{\mathrm{T}}P + PA_{\mathrm{C}}\right)x(t) + 2\lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) \\ &= x^{\mathrm{T}}(t)\left(-\gamma P - PBB^{\mathrm{T}}P\right)x(t) + 2\lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t)\end{aligned}$$

$$\begin{aligned}&\le -\gamma x^{\mathrm{T}}(t)Px(t)+\lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\lambda(t)\\&\le -\gamma x^{\mathrm{T}}(t)Px(t)+\big(2\mathrm{tr}(A)+n\gamma\big)\lambda^{\mathrm{T}}(t)P\lambda(t),\end{aligned}\tag{5.38}$$

where we have used Young's Inequality and the properties of $P(\gamma)$ as established in Lemma 5.1. Considering that

$$\lambda(t)=\int_{\phi(t)}^{t}\dot{x}(s)\mathrm{d}s,$$

we can continue (5.38) by using Lemma 2.2 as follows:

$$\begin{aligned}\dot{V}_1(x)&\le -\gamma x^{\mathrm{T}}(t)Px(t)+\big(2\mathrm{tr}(A)+n\gamma\big)\left(\int_{\phi(t)}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}}P\left(\int_{\phi(t)}^{t}\dot{x}(s)\mathrm{d}s\right)\\&\le -\gamma x^{\mathrm{T}}(t)Px(t)+\big(2\mathrm{tr}(A)+n\gamma\big)d(t)\int_{t-d(t)}^{t}\dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s\\&\le -\gamma x^{\mathrm{T}}(t)Px(t)+\big(2\mathrm{tr}(A)+n\gamma\big)D\int_{t-D}^{t}\dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s.\end{aligned}\tag{5.39}$$

On the other hand, we have

$$\begin{aligned}\dot{V}_2(x_t)&=\epsilon\int_{-D}^{0}\Big(\dot{x}^{\mathrm{T}}(t)P\dot{x}(t)-\dot{x}^{\mathrm{T}}(t+\theta)P\dot{x}(t+\theta)\Big)\mathrm{d}\theta\\&=\epsilon\left(D\dot{x}^{\mathrm{T}}(t)P\dot{x}(t)-\int_{t-D}^{t}\dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s\right).\end{aligned}\tag{5.40}$$

Combining (5.39) and (5.40), we have

$$\begin{aligned}\dot{V}(x_t)\le& -\gamma x^{\mathrm{T}}(t)Px(t)+\big(2\mathrm{tr}(A)+n\gamma\big)D\int_{t-D}^{t}\dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s\\&+\epsilon D\dot{x}^{\mathrm{T}}(t)P\dot{x}(t)-\epsilon\int_{t-D}^{t}\dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s.\end{aligned}\tag{5.41}$$

By (5.34), Young's Inequality, Lemma 5.1, and Lemma 2.2, $\dot{x}^{\mathrm{T}}(t)P\dot{x}(t)$ can be evaluated as

$$\begin{aligned}\dot{x}^{\mathrm{T}}(t)P\dot{x}(t)&=\Big(A_{\mathrm{C}}x(t)+BB^{\mathrm{T}}P\lambda(t)\Big)^{\mathrm{T}}P\Big(A_{\mathrm{C}}x(t)+BB^{\mathrm{T}}P\lambda(t)\Big)\\&\le 2x^{\mathrm{T}}(t)A_{\mathrm{C}}^{\mathrm{T}}PA_{\mathrm{C}}x(t)+2\lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}PBB^{\mathrm{T}}P\lambda(t)\\&\le 2\varpi x^{\mathrm{T}}(t)Px(t)+2\big(2\mathrm{tr}(A)+n\gamma\big)^2\lambda^{\mathrm{T}}(t)P\lambda(t)\\&\le 2\varpi x^{\mathrm{T}}(t)Px(t)+2\big(2\mathrm{tr}(A)+n\gamma\big)^2\left(\int_{t-d(t)}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}}P\left(\int_{t-d(t)}^{t}\dot{x}(s)\mathrm{d}s\right)\end{aligned}$$

$$\leq 2\varpi x^{\mathrm{T}}(t)Px(t) + 2\big(2\mathrm{tr}(A) + n\gamma\big)^2 D \int_{t-D}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s, \tag{5.42}$$

where ϖ is as defined in Lemma 5.1.

Substitution of (5.42) into (5.41) results in

$$\begin{aligned}\dot{V}(x_t) \leq {} & (-\gamma + 2\epsilon D\varpi)x^{\mathrm{T}}(t)Px(t) \\ & + \Big(\big(2\mathrm{tr}(A) + n\gamma\big)D + 2\epsilon D^2\big(2\mathrm{tr}(A) + n\gamma\big)^2 - \epsilon\Big) \\ & \times \int_{t-D}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s. \end{aligned} \tag{5.43}$$

Let

$$\epsilon = \frac{\big(2\mathrm{tr}(A) + n\gamma\big)D}{1 - 2D^2\big(2\mathrm{tr}(A) + n\gamma\big)^2} \tag{5.44}$$

be such that the second term in (5.43) is zero. Notice that ϵ is positive if and only if

$$D < \frac{1}{\sqrt{2}\big(2\mathrm{tr}(A) + n\gamma\big)}. \tag{5.45}$$

Hence, under condition (5.45),

$$\dot{V}(x_t) \leq (-\gamma + 2\epsilon D\varpi)x^{\mathrm{T}}(t)Px(t).$$

Furthermore, if

$$-\gamma + 2\epsilon D\varpi < 0, \tag{5.46}$$

then there exists some positive constant $\rho(\gamma)$, depending on γ, such that

$$\begin{aligned}\dot{V}(x_t) &\leq -\rho(\gamma)x^{\mathrm{T}}(t)Px(t) \\ &\leq -\rho(\gamma)\lambda_{\min}(P)\|x(t)\|_2^2 < 0,\ \ x(t) \neq 0,\end{aligned}$$

where $\lambda_{\min}(P)$ denotes the minimal eigenvalue of P and $\|x(t)\|_2$ is the Euclidean norm of $x(t)$. Recalling the structure of $V(x_t)$, we have

$$\begin{aligned}\lambda_{\min}(P)\|x(t)\|_2^2 &\leq V(x_t) \\ &\leq \lambda_{\max}(P) \max_{s\in[-D,0]} \|x_t(s)\|_2^2 + \epsilon D\lambda_{\max}(P)\int_{-D}^{0} \|\dot{x}_t(s)\|_2^2 \mathrm{d}s,\end{aligned}$$

where $\lambda_{\max}(P)$ is the maximum eigenvalue of P. By Theorem 1.1, stabilization of system (5.1) is achieved under the delay independent truncated predictor feedback law (5.6) as long as (5.45) and (5.46) hold.

Substitution of the expressions of ϵ and ϖ into (5.46) gives

$$\gamma > 2D^2 \frac{2\mathrm{tr}(A) + n\gamma}{1 - 2D^2\big(2\mathrm{tr}(A) + n\gamma\big)^2} \left(\frac{1}{2}\big(n\gamma + 2\mathrm{tr}(A)\big)\big((n+1)\gamma + 2\mathrm{tr}(A)\big) \right.$$
$$\left. -\gamma \mathrm{tr}(A) - \mathrm{tr}\left(A^2\right) \right),$$

which is equivalent to (5.33). In view of (5.45), if the delay bound D satisfies

$$D < \min\{D_1, D_2\},$$

where D_1 and D_2 are as defined in the proof of Corollary 5.1, then the closed-loop system is asymptotically stable. Noting that $0 < D_1 < D_2$ for any $\gamma > 0$ by Corollary 5.1, we have

$$\min\{D_1, D_2\} = D_1.$$

This completes the proof. □

Remark 5.1 To examine the conservativeness of Theorem 5.1, we now compare (5.33) with the existing results in the literature. Considering the case where all eigenvalues of the open loop system are at the origin, we have

$$\mathrm{tr}(A) = \mathrm{tr}\left(A^2\right) = 0.$$

Inequality (5.33) then simplifies to

$$D < \frac{1}{n\sqrt{n+3}\gamma}, \quad \gamma > 0,$$

which is equivalent to

$$\gamma < \frac{1}{n\sqrt{n+3}D}, \quad D > 0. \tag{5.47}$$

This implies that for any time-varying delay that is bounded by an arbitrarily large D, the system would be stabilized under the delay independent truncated predictor feedback law as long as γ is chosen sufficiently small. Moreover, the upper bound on γ is given by (5.47). Recall the result of Theorem 2 from [122], which establishes an upper bound of γ as

$$\gamma < \frac{1}{3\sqrt{3}n\sqrt{n}D}, \quad D > 0. \tag{5.48}$$

Comparing (5.47) with (5.48), we can easily observe that, for any system whose eigenvalues are all at the origin, our result on the upper bound of γ is less conservative than that of [122]. Note that the Razumikhin Stability Theorem was adopted to obtain (5.48). We consider the Krasovskii stability analysis an advantage over the Razumikhin stability analysis. □

Remark 5.2 We consider a system whose open loop poles are all on the imaginary axis and there exist at least one pair of nonzero poles. In this case, $\text{tr}(A) = 0$ and $\text{tr}\left(A^2\right) < 0$. Consequently, (5.33) reduces to

$$D < \left(\frac{1}{n\left(n(n+3)\gamma^2 - 2\text{tr}\left(A^2\right)\right)}\right)^{\frac{1}{2}}, \quad \gamma > 0. \tag{5.49}$$

Moreover, the right-hand side of (5.49) is strictly decreasing with respect to γ. Hence, any

$$D < \left(\frac{1}{-2n\text{tr}\left(A^2\right)}\right)^{\frac{1}{2}} \tag{5.50}$$

is a valid bound for some $\gamma > 0$.

In addition, recall that the system in Example 5.1 is unstable under any feedback parameter if

$$\tau = \frac{\pi}{2}$$

On the other hand, by (5.50), the delay bound of the system can be arbitrarily close to $\frac{1}{2\sqrt{2}} < \frac{\pi}{2}$, which requires γ to approach zero. This implies that Theorem 5.1 does not conflict with Example 5.1. □

Remark 5.3 We now consider exponentially unstable systems whose open loop poles are all on the closed right-half plane. By Corollary 5.2, we observe that the delay bound given in (5.33) has the unique maximal value $D_1(\gamma^\star)$ at $\gamma = \gamma^\star$, where $\gamma^\star$ is the unique positive solution to (5.22), $D_1(\gamma)$ is as defined in Corollary 5.1 and $D_1(\gamma^\star)$ is independent of γ. Furthermore, we note that

$$\begin{aligned}\lim_{\gamma\to 0^+} D_1(\gamma) &= \lim_{\gamma\to +\infty} D_1(\gamma) \\ &= 0.\end{aligned} \tag{5.51}$$

Therefore, any delay bound that satisfies $0 < D \leq D_1(\gamma^\star)$ is valid for some $\gamma > 0$. □

Remark 5.4 We note that only the information of the delay bound, not even the information on the derivative of the delay, is involved in (5.33), which guarantees the stabilization of system (5.1) under the delay independent truncated predictor feedback law (5.6). Therefore, Theorem 5.1 can be applied to systems whose input delays are fast-varying, which will be shown in the simulation to be presented in Sect. 5.2.3. It is worth mentioning here that, for the practical application of Theorem 5.1, the delay bound D of the time-varying delay $d(t)$ needs to be known in order to determine an appropriate value of γ for the stability of the closed-loop system. However, no knowledge of the delay itself is needed in the construction of the delay independent state feedback TPF law (5.6). This illustrates the robustness of the feedback law (5.6) with respect to uncertainty in the delay. □

5.2.3 Numerical Examples

In this section, we provide simulation results to demonstrate the theoretical conclusion of Theorem 5.1. We consider three different cases of system (5.1): a system whose open loop poles are all at the origin, a system whose open loop poles are all on the imaginary axis with at least one pair of nonzero poles, and an exponentially unstable system whose open loop poles are all on the closed right-half plane.

Example 5.2 (A System with All Poles at the Origin) Consider system (5.1) with

$$A = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, the system is controllable with all its open loop poles at the origin. For a linear system with all its open loop poles at the origin, the delay independent state feedback TPF law stabilizes the system in the presence of any bounded input delay as long as the value of the low gain feedback parameter in the feedback law is chosen sufficiently small. Consider a fast-varying time delay

$$d(t) = 0.5\left(1 + \sin^2(100t)\right), \tag{5.52}$$

with an upper bound $D = 1$. We pick a feedback parameter $\gamma = 0.3$. Let the initial condition of the system be

$$x(\theta) = \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix}^{\mathrm{T}}, \quad \theta \in [-D, 0]. \tag{5.53}$$

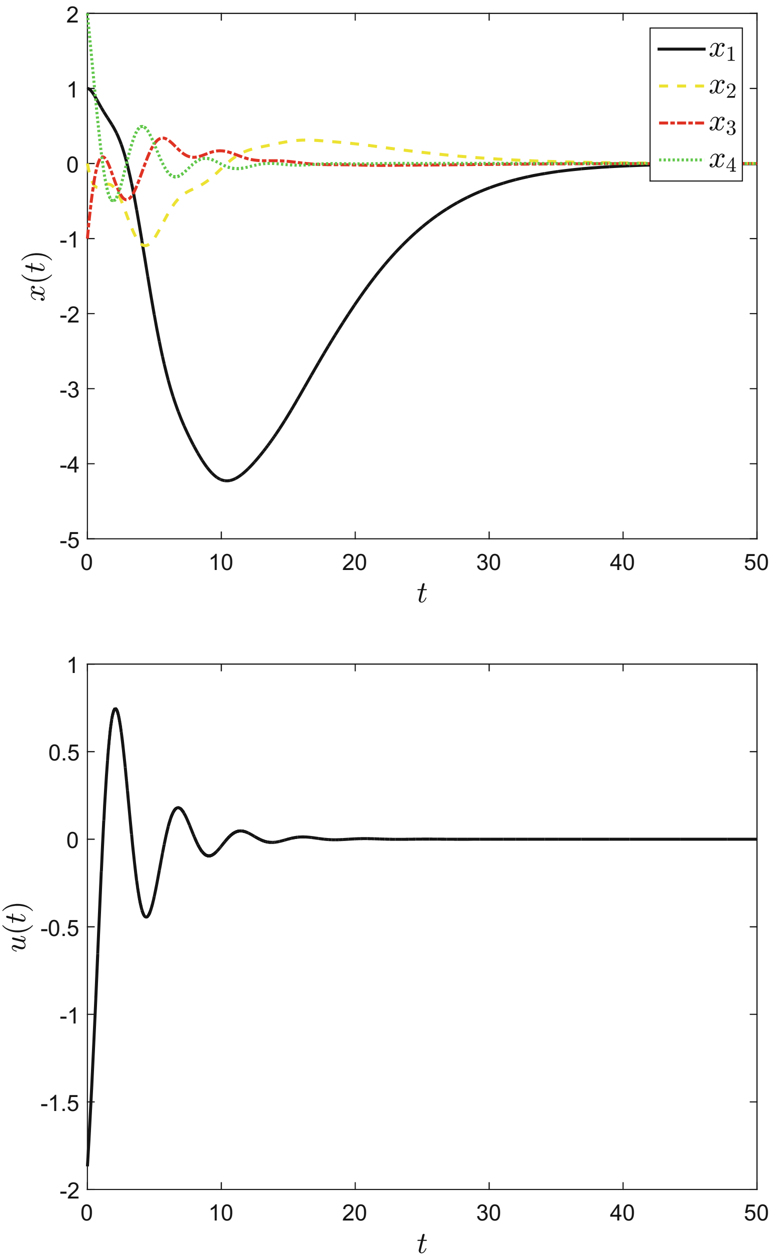


Fig. 5.1 Example 5.2: State response and control input under the delay independent state feedback TPF law (5.6): $\gamma = 0.3$

The state response and control evolution under the delay independent state feedback TPF law are shown in Fig. 5.1. To examine the low gain nature of the feedback law (5.6), we pick a smaller value for γ as 0.1 and observe its effects on the closed-loop performance of the system. Figure 5.2 illustrates the state response and input evolution of the closed-loop system with the same initial condition as given by (5.53). A comparison of the two figures clearly shows that although the stabilization of the system in the presence of a large input delay requires the feedback parameter to be sufficiently small, an excessively small feedback parameter typically leads to poor closed-loop performance in the form of a large overshoot and a small convergence rate. Thus, we need to avoid choosing an unnecessarily small value of the low gain parameter. □

Example 5.3 (A System with All Poles on the Imaginary Axis) Consider system (5.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, (A, B) is controllable with $\lambda(A) = \{\pm j, \pm j\}$. Also, $\text{tr}(A) = 0$ and $\text{tr}\left(A^2\right) = -4$. Let $\gamma = 0.5$ be the feedback parameter, and the corresponding delay bound in (5.49) is given as $D < 0.1291$. For the simulation purpose, let $D = 0.12$. In order to show that the delay independent truncated predictor feedback law (5.6) is applicable to systems with a fast-varying input delay, we choose the time-varying delay as follows:

$$d(t) = 0.06\Big(1 + \sin^2(100t)\Big),$$

from which it can be easily verified that $d(t) \in [0.06, 0.12]$ and $|\dot{d}(t)| \leq 6,\ t \geq 0$. Simulation is run for 20 s with the initial condition

$$x^{\mathrm{T}}(\theta) = \begin{bmatrix} -2 & 1 & 0 & 2 \end{bmatrix}, \quad \theta \in [-0.12, 0].$$

Shown in Fig. 5.3 are the state response $x(t)$ and the control input $u(t)$, for $\gamma = 0.5$. The stability of the closed-loop system is clear. To examine the conservativeness of delay bound in Theorem 5.1, we determine through simulation the tight bound on the constant delay for $\gamma = 0.5$ to be $D = 0.50$. □

Example 5.4 (An Exponentially Unstable System) Consider system (5.1) with a controllable pair

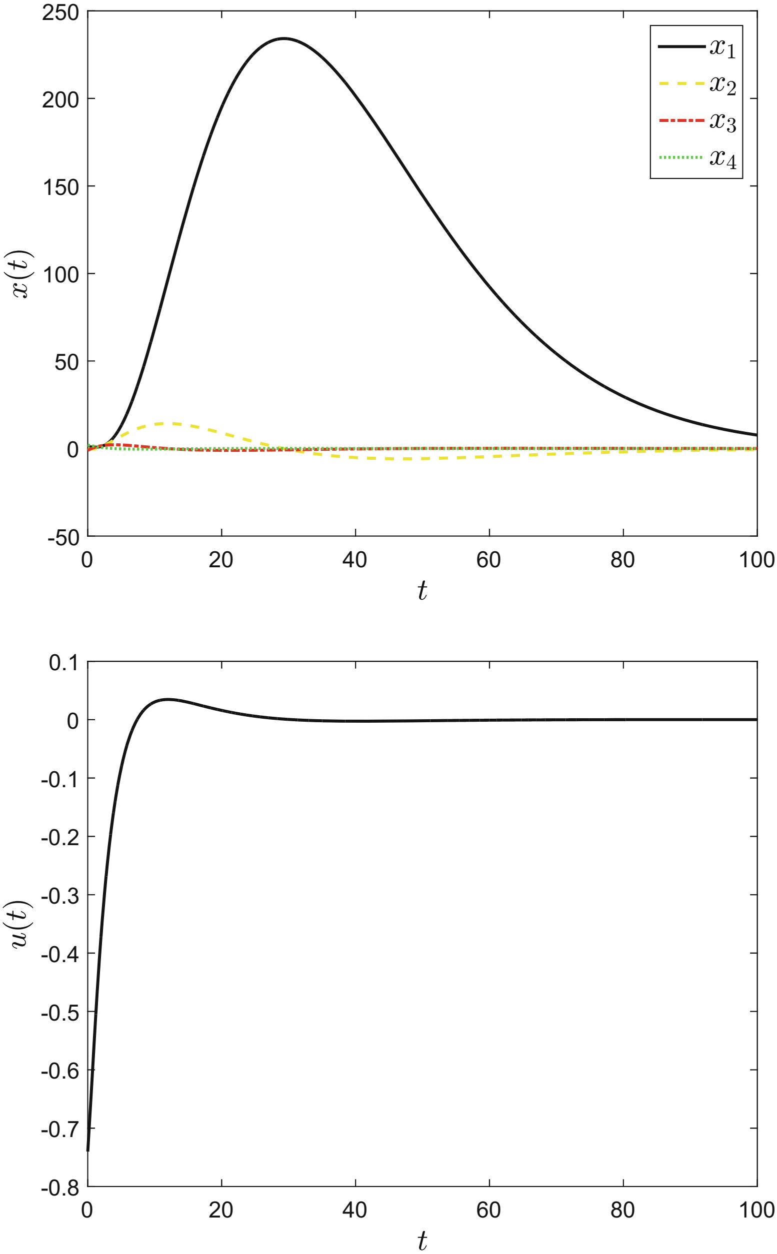


Fig. 5.2 Example 5.2: State response and control input under the delay independent state feedback TPF law (5.6): $\gamma = 0.1$

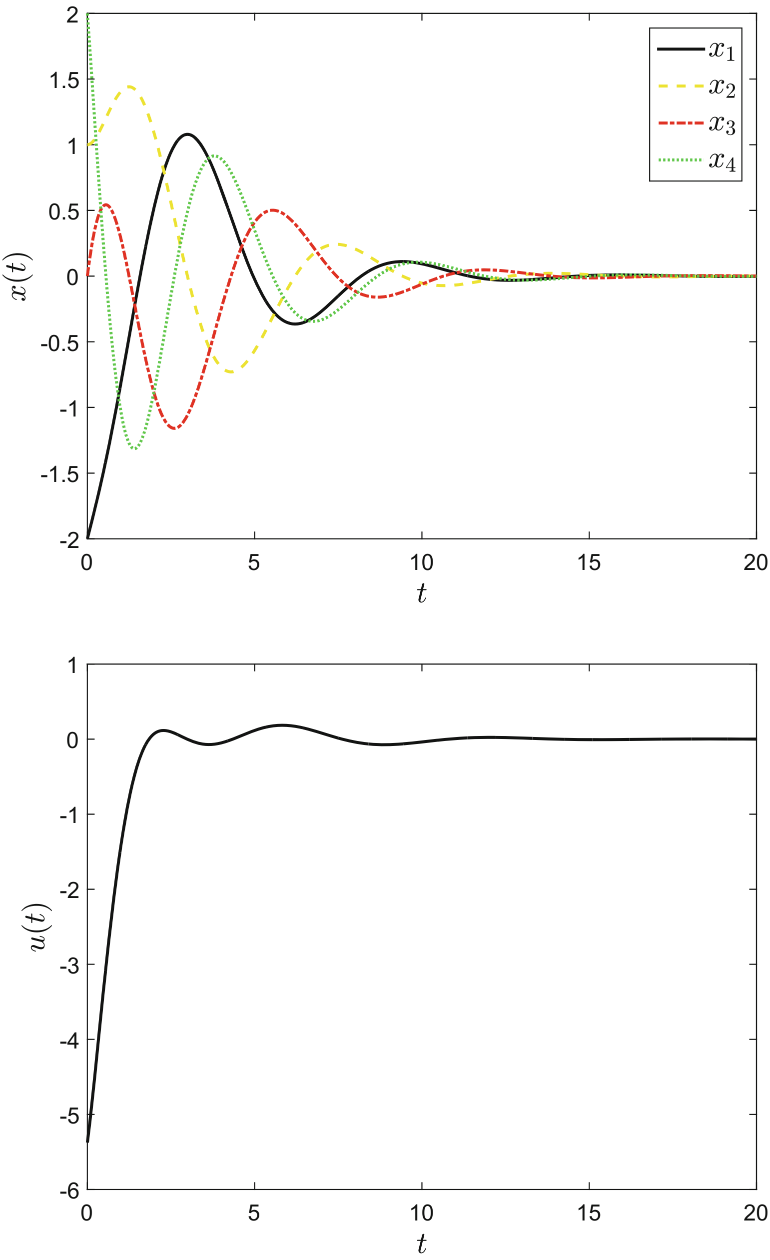


Fig. 5.3 Example 5.3: State response and control input under the delay independent state feedback TPF law (5.6): $\gamma = 0.5$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

It can be verified that $\lambda(A) = \{\pm j, \pm j, 1\}$. Also, $\mathrm{tr}(A) = 1$ and $\mathrm{tr}\left(A^2\right) = -3$. Let $\gamma = 0.1$. The corresponding delay bound in (5.33) is given as $D < 0.1768$. For the simulation purpose, we choose $D = 0.17$, and a fast-varying delay

$$d(t) = 0.085\Big(1 + \sin^2(100t)\Big).$$

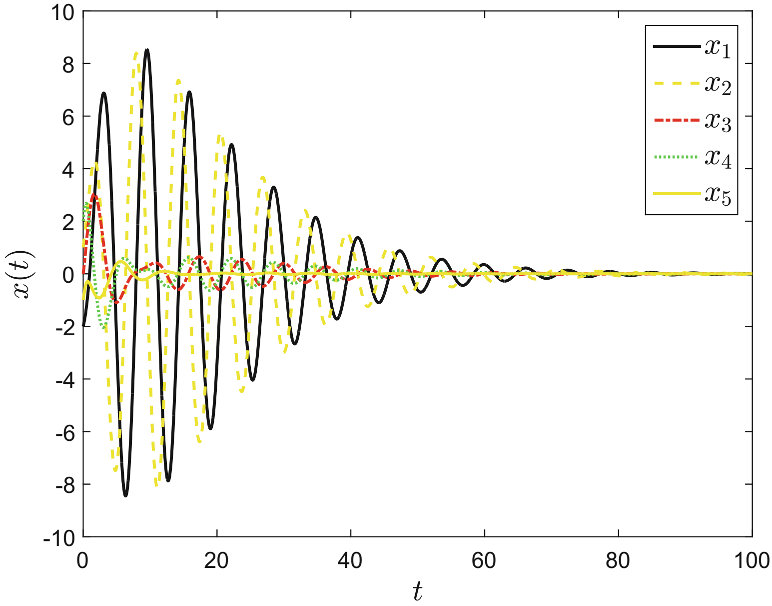
Clearly, $d(t) \in [0.085, 0.17]$ and $|\dot{d}(t)| \le 8.5, t \ge 0$. With the initial condition given by

$$x^{\mathrm{T}}(\theta) = \begin{bmatrix} -2 \; 1 \; 0 \; 2 \; -1 \end{bmatrix}, \quad \theta \in [-0.17, 0],$$

simulation is run for 100 s. Shown in Fig. 5.4 are the state response and the control input, which show that the system is asymptotically stable. As in Example 1, we determine through simulation the tight bound on the constant delay for $\gamma = 0.1$ to be $D = 0.45$. □

5.3 Improvement on the Closed-Loop Performance

Example 5.2 in Sect. 5.2 numerically verifies that a small value of the feedback parameter increases the ability of the delay independent truncated predictor feedback law to stabilize a linear system with all its open loop poles at the origin or in the open left-half plane. However, too small a feedback parameter degrades the closed-loop performance in terms of the overshoot and the convergence rate. Generally speaking, poor closed-loop performance with a large overshoot and a slow convergence rate is the consequence of the application of low gain feedback designs. We will examine the side effects of small values of the feedback parameter on the closed-loop performance in the stabilization of a linear system with all its open loop poles at the origin or in the open left-half plane. Furthermore, we will propose an approach to improving the closed-loop performance under a delay independent truncated predictor feedback law. Specifically, the traditional low gain feedback design with a constant feedback parameter is generalized to a time-varying parameter design. For an unknown delay with a known upper bound, a delay independent truncated predictor feedback law with a time-varying feedback parameter, constructed by using the parametric Lyapunov equation based approach, globally regulates the system to zero as long as the time-varying low gain parameter



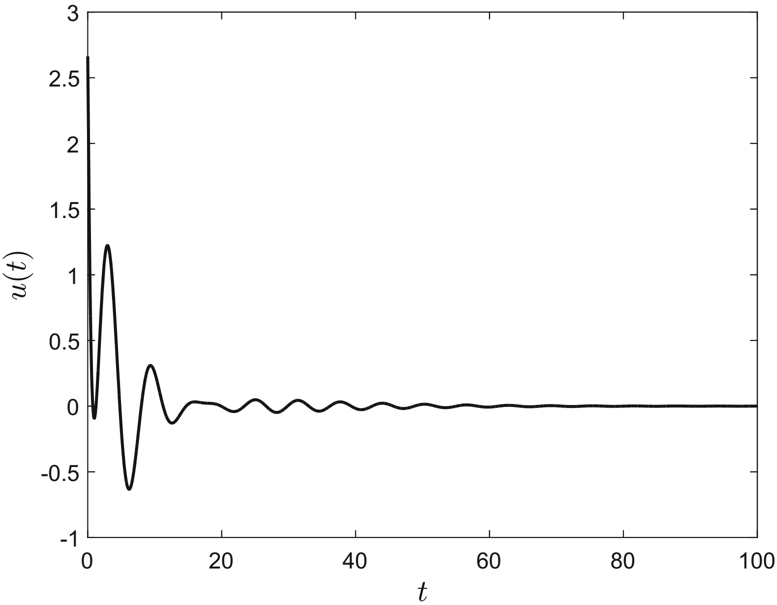


Fig. 5.4 Example 5.4: State response and control input under the delay independent state feedback TPF law (5.6): $\gamma = 0.1$

has a continuous second derivative and approaches a sufficiently small constant with its derivative approaching zero as time goes to infinity. Improvement on the closed-loop performance over the traditional constant parameter design is demonstrated through a convergence rate analysis and then observed in simulation.

The value of the low gain parameter embedded in the feedback gain matrix of the delay independent truncated predictor feedback law is determined by an upper bound of the delay (see [63] and [122]). The upper bound of the feedback parameter derived based on a Lyapunov analysis is conservative due to numerous estimations made throughout the derivation. Inspired by the spirit of the traditional low gain feedback design [61], we propose the concept of time-varying low gain feedback, where the low gain parameter is time-dependent. To improve the closed-loop performance, we design the parameter that starts from a relatively large value, and approaches a sufficiently small positive constant dependent on the known bound on the delay as time goes to infinity. Intuitively, such a time-varying parameter reduces the overshoot and increases the convergence rate in the early stage of the state evolution. As the closed-loop system reaches a state around zero, the value of the parameter is decreased to a sufficiently small constant that would not affect the closed-loop performance while guaranteeing stability.

A fundamental difficulty in stability analysis that involves a feedback law with a time-varying feedback parameter lies in the Lyapunov analysis. Typically, Lyapunov analysis requires $\dot{V}(x(t)) \leq -w(|x(t)|)$, where $V(x(t))$ is a Lyapunov function, $w(\cdot)$ is a positive scalar function, and $|x(t)|$ is the Euclidean norm of the current state of system. In the case of time-varying feedback laws, taking the time derivative of a typical Lyapunov function $V(x(t), t) = x^{\mathrm{T}}(t)P(t)x(t)$, where $P(t)$ is a time-varying positive definite matrix, results in terms involving different time instants such as $x^{\mathrm{T}}(s)P(\theta)x(s)$ with $s \neq \theta$. Majorization of the time derivative by $-w(|x(t)|)$ poses challenges. However, the partial differential equation (PDE) representation of the closed-loop system, which has been extensively explored in [59], provides us with more freedom to construct a Lyapunov functional that facilitates such majorization. Indeed, this PDE representation has been exploited in [14] and [15] to carry out design and stability analysis of adaptive predictor feedback laws. Furthermore, as seen in these references, the Lyapunov analysis is direct, without resorting to the Krasovskii or Razumikhin Stability Theorem. Recent advancements in designing predictor-based feedback laws have been made by employing the PDE method and various backstepping transformations for linear time-invariant systems with distinct input delays [94] and nonlinear systems with multiple input delays [11] or even distributed input delays [10]. In this section, we will also adopt the PDE representation of the closed-loop system and direct stability analysis to obtain design specifications for the feedback parameter that guarantee closed-loop stability.

5.3.1 Time-Varying Low Gain Feedback Design

We consider the regulation of a linear system with delayed input,

$$\dot{X}(t) = AX(t) + BU(t - \tau), \quad t \geq 0, \tag{5.54}$$

where $X \in \mathbb{R}^n$ and $U \in \mathbb{R}^m$ are the state and the input, respectively, and $\tau \in \mathbb{R}^+$ is an unknown constant delay with a known upper bound $\overline{\tau} \geq \tau$. In this section, we adopt capital letters X and U to denote the state and the input of the system because their lower cases are reserved for other purposes. The initial condition is given by

$$X(\theta) = \phi(\theta), \tag{5.55}$$

where $\phi(\theta) \in C[-\tau, 0]$. It is assumed that (A, B) is controllable with all eigenvalues of A at the origin.

Remark 5.5 Alternatively, we can define the initial condition of the delayed system (5.54) as

$$U(\theta) = \psi(\theta) \in C[-\tau, 0] \tag{5.56}$$

and

$$X(0) = X_0 \in \mathbb{R}^n. \tag{5.57}$$

Because we are considering the closed-loop system under a state feedback law, it is more convenient to define the initial condition solely in terms of the state $X(t)$. □

It was pointed out in Theorem 1 of [122] that the delay independent truncated predictor state feedback law (also referred to as the delay independent state feedback TPF law)

$$\begin{aligned} U(t) &= F(\gamma)X(t) \\ &= -B^{\mathrm{T}}P(\gamma)X(t), \end{aligned}$$

with a constant feedback parameter γ, asymptotically stabilizes system (5.54) for an arbitrarily large delay τ if

$$\gamma \in \left(0, \frac{1}{3\sqrt{3}n\sqrt{n}\overline{\tau}}\right], \tag{5.58}$$

where $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation,

$$A^{\mathrm{T}}P(\gamma) + P(\gamma)A - P(\gamma)BB^{\mathrm{T}}P(\gamma) = -\gamma P(\gamma), \quad \gamma > 0. \tag{5.59}$$

Notice that the feedback gain matrix $F(\gamma)$ in the delay independent state feedback TPF law is constant when γ is fixed. Also, the theoretical bound of γ given by (5.58) is considerably smaller compared with the bound observed from simulation.

To overcome this conservativeness in determining the value of γ, we design a time-varying low gain feedback law whose feedback parameter is time-dependent,

$$\begin{aligned} U(t) &= F(\gamma(t))X(t) \\ &= -B^{\mathrm{T}}P(\gamma(t))X(t), \quad t \geq -\tau. \end{aligned} \tag{5.60}$$

Considering the inverse proportionality relationship between the upper bound of γ and $\overline{\tau}$ as shown in (5.58), we design the time-varying feedback parameter as

$$\gamma(t) = \frac{h}{\hat{\tau}(t)}. \tag{5.61}$$

Here h is a positive constant and $\hat{\tau}(t)$ satisfies the following conditions:

$$\hat{\tau}(t) \in C^2[-\tau, \infty), \quad \hat{\tau}(t) > 0, \quad \lim_{t\to\infty} \hat{\tau}(t) = \overline{\tau}, \quad \lim_{t\to\infty} \dot{\hat{\tau}}(t) = 0. \tag{5.62}$$

5.3.2 PDE-ODE Cascade Representation

As pointed out in [59], the delayed input $U(t-\tau)$ in system (5.54) can be considered as the boundary value of

$$u(x,t) = U(t + \tau(x-1)), \quad x \in [0,1] \tag{5.63}$$

at $x = 0$, where $u(x,t)$ is the solution of a transport PDE

$$\tau u_t(x,t) = u_x(x,t), \tag{5.64}$$

with the boundary condition

$$u(1,t) = U(t). \tag{5.65}$$

Thus, system (5.54) is equivalent to the cascade of an ordinary differential equation (ODE) with a transport PDE,

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{5.66}$$

$$\tau u_t(x,t) = u_x(x,t), \tag{5.67}$$

$$u(1,t) = U(t). \tag{5.68}$$

Following the idea in analyzing an adaptive predictor feedback law for linear systems with unknown input delay [15], we define a signal associated with $\hat{\tau}(t)$,

$$\hat{u}(x,t) = U(t + \hat{\tau}(t)(x-1)), \tag{5.69}$$

which can be verified to satisfy the PDE,

$$\hat{\tau}(t)\hat{u}_t(x,t) = (1 + \dot{\hat{\tau}}(t)(x-1))\hat{u}_x(x,t), \tag{5.70}$$

$$\hat{u}(1,t) = U(t). \tag{5.71}$$

In view of the expression for the time-varying low gain feedback law (5.60), we have the difference between $\hat{u}(x,t)$ and $U(t)$ as

$$\begin{aligned}\hat{w}(x,t) &= \hat{u}(x,t) - U(t)\\ &= \hat{u}(x,t) + B^{\mathrm{T}}P(\gamma(t))X(t).\end{aligned} \tag{5.72}$$

To measure the distance between $\hat{\tau}(t)$ and the actual delay τ, we define

$$\tilde{\tau}(t) = \tau - \hat{\tau}(t), \tag{5.73}$$

and correspondingly,

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \tag{5.74}$$

from which, along with (5.60), (5.66), (5.70), (5.71), and (5.72), we obtain the PDE for $\hat{w}(x,t)$ as

$$\begin{aligned}\hat{\tau}(t)\hat{w}_t(x,t) = \hat{w}_x(x,t)\left(1 + \dot{\hat{\tau}}(t)(x-1)\right) &+ \hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t)\\ &+ \hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\Big(\big(A - BB^{\mathrm{T}}P(\gamma(t))\big)X(t)\\ &+ B\tilde{u}(0,t) + B\hat{w}(0,t)\Big),\end{aligned} \tag{5.75}$$

$$\hat{w}(1,t) = 0. \tag{5.76}$$

By (5.66), (5.72) and (5.74), the closed-loop system under the time-varying low gain feedback law (5.60) takes the form,

$$\dot{X}(t) = \big(A - BB^{\mathrm{T}}P(\gamma(t))\big)X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t). \tag{5.77}$$

By (5.67), (5.70), (5.71), (5.72), (5.73), and (5.74), the PDE for $\tilde{u}(x,t)$ is obtained as

$$\tau \tilde{u}_t(x,t) = \tilde{u}_x(x,t) - \frac{\tilde{\tau} + \tau \dot{\hat{\tau}}(t)(x-1)}{\hat{\tau}(t)} \hat{w}_x(x,t), \tag{5.78}$$

$$\tilde{u}(1,t) = 0. \tag{5.79}$$

The constructed Lyapunov functional used in the stability analysis to be carried out in the next subsection also contains $\hat{w}_x(x,t)$ besides $\tilde{u}(x,t)$ and $\hat{w}(x,t)$. Thus, in view of (5.70), (5.72), (5.75), and (5.76), we derive the governing PDE for $\hat{w}_x(x,t)$ as

$$\hat{\tau}(t)\hat{w}_{xt}(x,t) = \hat{w}_{xx}(x,t)\left(1 + \dot{\hat{\tau}}(t)(x-1)\right) + \dot{\hat{\tau}}(t)\hat{w}_x(x,t), \tag{5.80}$$

$$\begin{aligned} \hat{w}_x(1,t) = & -\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t) - \hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\Big((A - BB^{\mathrm{T}}P(\gamma(t)))X(t) \\ & + B\tilde{u}(0,t) + B\hat{w}(0,t)\Big). \end{aligned} \tag{5.81}$$

The following lemma establishes estimates of cross terms between $\tilde{u}(x,t)$, $\hat{w}(x,t)$, $\hat{w}_x(x,t)$, $\hat{w}_{xx}(x,t)$,, and $X(t)$.

Lemma 5.3 *The following properties hold for system (5.54):*

$$\begin{aligned} & -\int_0^1 (1+x)\left(\tilde{\tau} + \tau\dot{\hat{\tau}}(t)(x-1)\right)\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \\ \leq & \left(|\tilde{\tau}| + \frac{1}{2}\tau\left|\dot{\hat{\tau}}(t)\right|\right)\left(\epsilon\,||\tilde{u}(t)||^2 + \frac{1}{\epsilon}\left\|\hat{w}_x(t)\right\|^2\right), \end{aligned} \tag{5.82}$$

for any positive constant ϵ,

$$\begin{aligned} & \int_0^1 (1+x)\left(1 + \dot{\hat{\tau}}(t)(x-1)\right)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \\ \leq & \frac{1}{2}\left(\left|\dot{\hat{\tau}}(t)\right| - 1\right)\left|\hat{w}(0,t)\right|^2 + \left(\left|\dot{\hat{\tau}}(t)\right| - \frac{1}{2}\right)\left\|\hat{w}(t)\right\|^2, \end{aligned} \tag{5.83}$$

$$\begin{aligned} & \int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t)\mathrm{d}x \\ \leq & h^{\frac{1}{2}}\left\|\hat{w}(t)\right\|^2 + \left(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\right)^2 h^{\frac{3}{2}} X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial \gamma}X(t), \end{aligned} \tag{5.84}$$

$$\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\left(\left(A - BB^{\mathrm{T}}P(\gamma(t))\right)X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t)\right)\mathrm{d}x$$

$$\leq \hat{\tau}(t)n\gamma(t)\left\|\hat{w}(t)\right\|^2 + \frac{3}{2}\hat{\tau}(t)n(n+1)\gamma^2(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)$$
$$+3\hat{\tau}(t)n\gamma(t)\left(|\tilde{u}(0,t)|^2 + |\hat{w}(0,t)|^2\right), \tag{5.85}$$

$$\int_0^1 (1+x)\left(1+\dot{\hat{\tau}}(t)(x-1)\right)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x$$
$$\leq |\hat{w}_x(1,t)|^2 + \frac{1}{2}\left(\left|\dot{\hat{\tau}}(t)\right|-1\right)|\hat{w}_x(0,t)|^2 + \left(\left|\dot{\hat{\tau}}(t)\right|-\frac{1}{2}\right)\left\|\hat{w}_x(t)\right\|^2, \tag{5.86}$$

and

$$|\hat{w}_x(1,t)|^2 \leq 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t) + 6\hat{\tau}^2(t)n^2\gamma^2(t)$$
$$\times\left(\frac{n+1}{2}\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t) + |\tilde{u}(0,t)|^2 + |\hat{w}(0,t)|^2\right). \tag{5.87}$$

Proof Equation (5.82): By using Young's Inequality, we obtain

$$-\int_0^1 (1+x)\left(\tilde{\tau}+\tau\dot{\hat{\tau}}(t)(x-1)\right)\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x$$
$$\leq \left(|\tilde{\tau}| + \frac{1}{2}\tau\left|\dot{\hat{\tau}}(t)\right|\right)\int_0^1\left(\epsilon\,|\tilde{u}(x,t)|^2 + \frac{1}{\epsilon}\left|\hat{w}_x(x,t)\right|^2\right)\mathrm{d}x$$
$$= \left(|\tilde{\tau}| + \frac{1}{2}\tau\left|\dot{\hat{\tau}}(t)\right|\right)\left(\epsilon\,\|\tilde{u}(t)\|^2 + \frac{1}{\epsilon}\left\|\hat{w}_x(t)\right\|^2\right),$$

where ϵ is any positive constant.
Equation (5.83): Noting that $\hat{w}(1,t)=0$ and using integration by parts, we have

$$\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x = (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}(x,t)\Big|_0^1$$
$$-\int_0^1 \hat{w}^{\mathrm{T}}(x,t)\hat{w}(x,t)\mathrm{d}x - \int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x,$$

which implies that

$$\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x = -\frac{1}{2}\left(\left|\hat{w}(0,t)\right|^2 + \left\|\hat{w}(t)\right\|^2\right). \tag{5.88}$$

On the other hand,

$$\int_0^1 \left(x^2-1\right)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x = \frac{1}{2}\left|\hat{w}(0,t)\right|^2 - \int_0^1 x\left|\hat{w}(x,t)\right|^2\mathrm{d}x$$

implies that

$$\dot{\hat{\tau}}(t)\int_0^1 \left(x^2-1\right)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \le \left|\dot{\hat{\tau}}(t)\right|\left(\frac{1}{2}|\hat{w}(0,t)|^2 + \left\|\hat{w}(t)\right\|^2\right). \tag{5.89}$$

Thus, (5.83) readily follows from adding (5.88) and (5.89).

Equation (5.84): By (5.61) and Young's Inequality, we have

$$\begin{aligned}
&\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)\mathrm{d}x\\
&= -\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}hB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)\mathrm{d}x\\
&\le 2\int_0^1 \left|h^{\frac{1}{4}}\hat{w}^{\mathrm{T}}(x,t)\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}h^{\frac{3}{4}}B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)\right|\mathrm{d}x\\
&\le h^{\frac{1}{2}}\left\|\hat{w}(t)\right\|^2 + \left(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\right)^2 h^{\frac{3}{2}}X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t).
\end{aligned}$$

Equation (5.85): By Lemma 2.1 and Young's Inequality, we derive

$$\begin{aligned}
&\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\Big(\big(A-BB^{\mathrm{T}}P(\gamma(t))\big)X(t)+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x\\
&= \hat{\tau}(t)\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)B^{\mathrm{T}}P^{\frac{1}{2}}(\gamma(t))P^{\frac{1}{2}}(\gamma(t))\Big(\big(A-BB^{\mathrm{T}}P(\gamma(t))\big)X(t)\\
&\quad +B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x\\
&\le \hat{\tau}(t)\int_0^1 \hat{w}^{\mathrm{T}}(x,t)B^{\mathrm{T}}P(\gamma(t))B\hat{w}(x,t)\mathrm{d}x + \hat{\tau}(t)\int_0^1 \Big(\big(A-BB^{\mathrm{T}}P(\gamma(t))\big)X(t)\\
&\quad +B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)^{\mathrm{T}}P(\gamma(t))\Big(\big(A-BB^{\mathrm{T}}P(\gamma(t))\big)X(t)+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x\\
&\le \hat{\tau}(t)n\gamma(t)\left|\hat{w}(t)\right\|^2 + 3\hat{\tau}(t)\int_0^1 \Big(X^{\mathrm{T}}(t)(A-BB^{\mathrm{T}}P(\gamma(t)))^{\mathrm{T}}P(\gamma(t))\big(A\\
&\quad -BB^{\mathrm{T}}P(\gamma(t)))X(t)+\tilde{u}^{\mathrm{T}}(0,t)B^{\mathrm{T}}P(\gamma(t))B\tilde{u}(0,t)+\hat{w}^{\mathrm{T}}(0,t)B^{\mathrm{T}}P(\gamma(t))B\hat{w}(0,t)\Big)\mathrm{d}x\\
&\le \hat{\tau}(t)n\gamma(t)\left|\hat{w}(t)\right\|^2 + \frac{3}{2}\hat{\tau}(t)n(n+1)\gamma^2(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)\\
&\quad +3\hat{\tau}(t)n\gamma(t)\left(|\tilde{u}(0,t)|^2+\left|\hat{w}(0,t)\right|^2\right).
\end{aligned}$$

Equation (5.86): It can be verified that

$$\int_0^1 (1+x)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x = \left|\hat{w}_x(1,t)\right|^2 - \frac{1}{2}\left|\hat{w}_x(0,t)\right|^2 - \frac{1}{2}\left|\hat{w}_x(t)\right\|^2.$$

On the other hand,

$$\int_0^1 \left(x^2-1\right)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x = \frac{1}{2}\left|\hat{w}_x(0,t)\right|^2 - \int_0^1 x\left|\hat{w}_x(x,t)\right|^2\mathrm{d}x$$

implies that

$$\dot{\hat{\tau}}(t)\int_0^1 \left(x^2-1\right)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x \le \left|\dot{\hat{\tau}}(t)\right|\left(\frac{1}{2}\left|\hat{w}_x(0,t)\right|^2 + \left\|\hat{w}_x(t)\right\|^2\right).$$

Thus, (5.86) holds.
Equation (5.87): We evaluate $\hat{w}_x(1,t)$ by using (5.81), Young's Inequality and Lemma 2.1 as follows:

$$\begin{aligned}
&|\hat{w}_x(1,t)|^2\\
\le\ & 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t) + 2\hat{\tau}^2(t)\left(\left(A-BB^{\mathrm{T}}P(\gamma(t))\right)X(t)\right.\\
&\left.+B\tilde{u}(0,t)+B\hat{w}(0,t)\right)^{\mathrm{T}}P(\gamma(t))BB^{\mathrm{T}}P(\gamma(t))\left(\left(A-BB^{\mathrm{T}}P(\gamma(t))\right)X(t)\right.\\
&\left.+B\tilde{u}(0,t)+B\hat{w}(0,t)\right)\\
\le\ & 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t) + 2n\gamma(t)\hat{\tau}^2(t)\left(\left(A-BB^{\mathrm{T}}P(\gamma(t))\right)X(t)\right.\\
&\left.+B\tilde{u}(0,t)+B\hat{w}(0,t)\right)^{\mathrm{T}}P(\gamma(t))\left(\left(A-BB^{\mathrm{T}}P(\gamma(t))\right)X(t)\right.\\
&\left.+B\tilde{u}(0,t)+B\hat{w}(0,t)\right)\\
\le\ & 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t) + 6n\gamma(t)\hat{\tau}^2(t)\left(X^{\mathrm{T}}(t)\left(A-BB^{\mathrm{T}}P(\gamma(t))\right)^{\mathrm{T}}\right.\\
&\times P(\gamma(t))\left(A-BB^{\mathrm{T}}P(\gamma(t))\right)X(t)+\tilde{u}^{\mathrm{T}}(0,t)B^{\mathrm{T}}P(\gamma(t))B\tilde{u}(0,t)\\
&\left.+\hat{w}^{\mathrm{T}}(0,t)B^{\mathrm{T}}P(\gamma(t))B\hat{w}(0,t)\right)\\
\le\ & 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)\\
&+6\hat{\tau}^2(t)n^2\gamma^2(t)\left(\frac{n+1}{2}\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)+|\tilde{u}(0,t)|^2+\left|\hat{w}(0,t)\right|^2\right).
\end{aligned}$$

□

5.3.3 Direct Stability Analysis

Based on the fact that $\hat{\tau}(t)$ has a continuous second derivative, we can establish the second-order differentiability of the feedback gain matrix $F(\gamma(t))$, the state $X(t)$ and the input $U(t)$. The differentiability of these signals is extensively involved in the construction of the Lyapunov functional and stability analysis to be carried out next.

Lemma 5.4 *The time-varying feedback gain matrix $F(\gamma(t))$ in (5.60) is bounded and has a continuous second derivative on $t \in [-\tau, \infty)$.*

Proof By the time-varying low gain feedback design (5.62), $\hat{\tau}(t) \in C[-\tau, \infty)$ is positive and has a finite limit $\overline{\tau}$, and hence $\hat{\tau}(t)$ is bounded on $t \in [-\tau, \infty)$. Suppose that

$$\inf_{t \geq -\tau} \hat{\tau}(t) = \tau_{\min}, \tag{5.90}$$

and

$$\sup_{t \geq -\tau} \hat{\tau}(t) = \tau_{\max}, \tag{5.91}$$

where $0 < \tau_{\min} \leq \tau_{\max}$. By the inverse proportionality relationship between $\gamma(t)$ and $\hat{\tau}(t)$, $\gamma(t)$ is also bounded, with

$$\inf_{t \geq -\tau} \gamma(t) = h/\tau_{\max} \tag{5.92}$$

and

$$\sup_{t \geq -\tau} \gamma(t) = h/\tau_{\min}. \tag{5.93}$$

The increasing monotonicity of $P(\gamma)$ with respect to γ by Lemma 2.1 implies that $P(\gamma(t))$ is bounded with

$$P(h/\tau_{\max}) \leq P(\gamma(t)) \leq P(h/\tau_{\min}). \tag{5.94}$$

Therefore, the boundedness of $F(\gamma(t))$ follows readily from the construction of

$$F(\gamma(t)) = -B^{\mathrm{T}} P(\gamma(t)).$$

On the other hand, as noted in [121], $P(\gamma)$ is a rational matrix function of γ, which implies that $P(\gamma)$ is infinitely differentiable with respect to the feedback parameter. The first and second derivatives of $F(\gamma(t))$ with respect to t are given as follows:

$$\dot{F}(\gamma(t)) = B^{\mathrm{T}} \frac{\partial P}{\partial \gamma} \frac{h}{\hat{\tau}^2(t)} \dot{\hat{\tau}}(t), \tag{5.95}$$

$$\ddot{F}(\gamma(t)) = B^{\mathrm{T}} \frac{h}{\hat{\tau}^2(t)} \Big(- \frac{\partial^2 P}{\partial \gamma^2} \frac{h}{\hat{\tau}^2(t)} \dot{\hat{\tau}}^2(t) - 2 \frac{\partial P}{\partial \gamma} \frac{\dot{\hat{\tau}}^2(t)}{\hat{\tau}(t)} + \frac{\partial P}{\partial \gamma} \ddot{\hat{\tau}}(t) \Big), \tag{5.96}$$

from which it follows that $F(\gamma(t))$ has a continuous second derivative with respect to t since $\hat{\tau}(t) \in C^2[-\tau, \infty)$. □

Lemma 5.5 *With the initial condition $X(\theta) = \psi(\theta) \in C[-\tau, 0]$, system (5.54) under the time-varying low gain feedback law (5.60) has a unique solution $X(t) \in C[-\tau, \infty)$. Moreover, $U(t) \in C[-\tau, \infty)$ and $X(t), U(t) \in C^1(0, \infty) \cap C^2(\tau, \infty)$, where the notation $C^1(0, \infty) \cap C^2(\tau, \infty)$ denotes the set of functions defined on $(0, \infty)$ that has continuous first-order derivative on $(0, \infty)$ and has continuous second-order derivative on (τ, ∞).*

Proof The proof follows the general idea of the proof of Theorem 3.1 in [12]. Under the time-varying low gain feedback law (5.60), system (5.54) becomes

$$\dot{X}(t) = AX(t) + BF(\gamma(t - \tau))X(t - \tau), \quad t \geq 0. \tag{5.97}$$

Considering the continuity of both $F(\gamma(\theta))$ and $\psi(\theta)$ on $\theta \in [-\tau, 0]$, there exists a unique solution

$$X(t) = \mathrm{e}^{At}\psi(0) + \int_0^t \mathrm{e}^{A(t-s)} BF(\gamma(s - \tau))\psi(s - \tau)\mathrm{d}s,$$

on $t \in [0, \tau]$, which implies that $X(t) \in C[-\tau, \tau]$. Furthermore, the solution $X(t)$ on $t \in [\tau, 2\tau]$ is obtained as follows:

$$X(t) = \mathrm{e}^{A(t-\tau)}X(\tau) + \int_\tau^t \mathrm{e}^{A(t-s)} BF(\gamma(s - \tau))X(s - \tau)\mathrm{d}s, \quad t \in [\tau, 2\tau],$$

which implies that $X(t) \in C[-\tau, 2\tau]$ due to the continuity of $X(t)$ and $F(\gamma(t))$ on $[0, \tau]$. Similarly, the existence and uniqueness of $X(t)$ can be established along the time axis toward positive infinity. The continuity and uniqueness of $X(t)$ on $t \in [-\tau, \infty)$ follow readily.

From the right-hand side of (5.97), we obtain $\dot{X}(t) \in C(0, \infty)$ in view of the fact that $X(t - \tau)$ is continuous on $t \in (0, \infty)$. Taking the time derivative of both sides of (5.97) yields

$$\begin{aligned}\ddot{X}(t) = {} & A^2X(t) + ABF(\gamma(t - \tau))X(t - \tau) + B\dot{F}(\gamma(t - \tau))X(t - \tau) \\ & + BF(\gamma(t - \tau))AX(t - \tau) + BF(\gamma(t - \tau))BF(\gamma(t - 2\tau))X(t - 2\tau),\end{aligned}$$

which implies that $X(t) \in C^2(\tau, \infty)$ since $X(t) \in C[-\tau, \infty)$ and $F(\gamma(t))$ is continuously differentiable on $t \in [-\tau, \infty)$, as established in Lemma 5.4.

Considering the time-varying low gain feedback law

$$U(t) = F(\gamma(t))X(t), \tag{5.98}$$

we know that $U(t) \in C[-\tau, \infty)$ because both $X(t)$ and $F(\gamma(t))$ are continuous on $t \in [-\tau, \infty)$. Also, the first and second derivatives of $U(t)$ take the following form:

$$\dot{U}(t) = \dot{F}(\gamma(t))X(t) + F(\gamma(t))\dot{X}(t), \tag{5.99}$$

$$\ddot{U}(t) = \ddot{F}(\gamma(t))X(t) + 2\dot{F}(\gamma(t))\dot{X}(t) + F(\gamma(t))\ddot{X}(t), \tag{5.100}$$

which, in view of $X(t) \in C[-\tau, \infty) \cap C^1(0, \infty) \cap C^2(\tau, \infty)$ and the continuity of the second derivative of $F(\gamma(t))$, imply that $U(t) \in C^1(0, \infty) \cap C^2(\tau, \infty)$. □

With the time-varying low gain feedback law and the PDE representation of the closed-loop system in hand, we can now establish global regulation of the system by a direct Lyapunov stability analysis.

Theorem 5.2 *There exists a sufficiently small positive constant $h^\star$ such that, for each $h \in (0, h^\star]$, the time-varying low gain feedback law (5.60) globally regulates $X(t)$ and $U(t)$ of system (5.54) to zero, that is,*

$$\lim_{t\to\infty} X(t) = 0, \quad \lim_{t\to\infty} U(t) = 0,$$

for any given $X(\theta) \in C[-\tau, 0]$ and $U(\theta) \in C[-\tau, 0]$.

Proof Inspired by the Lyapunov functional in the stability analysis of linear systems under an adaptive predictor feedback law [15], where a term involving $\tilde{\tau}^2(t)$ is introduced to bound $\tilde{\tau}(t)$, we consider

$$V(x_t, \gamma(t)) = X^{\mathrm{T}}(t)P(\gamma(t))X(t) + b_1\tau\int_0^1 (1+x)\,|\tilde{u}(x,t)|^2\,\mathrm{d}x$$
$$+ b_2\hat{\tau}(t)\int_0^1 (1+x)\left(\left|\hat{w}(x,t)\right|^2 + \left|\hat{w}_x(x,t)\right|^2\right)\mathrm{d}x, \tag{5.101}$$

by discarding the term associated with $\tilde{\tau}^2(t)$ because $\hat{\tau}(t)$ in our case is not an estimate of τ but only provides information of an upper bound of delay to the time-varying low gain feedback law in order to achieve regulation. Here, b_1 and b_2 are two positive constants whose values are to be determined later. Taking the time derivative of V along the closed-loop trajectory gives,

$$\dot{V}(X_t, \gamma(t))$$
$$= 2\dot{X}^{\mathrm{T}}(t)P(\gamma(t))X(t) + X^{\mathrm{T}}(t)\dot{P}(\gamma(t))X(t)$$

$$
\begin{aligned}
&+2b_1\int_0^1(1+x)\tilde{u}^{\mathrm{T}}(x,t)\bigg(\tilde{u}_x(x,t)\\
&-\left(\tilde{\tau}+\tau\dot{\hat{\tau}}(t)(x-1)\right)\frac{\hat{w}_x(x,t)}{\hat{\tau}(t)}\bigg)\mathrm{d}x+2b_2\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\\
&\times\left(\hat{w}_x(x,t)\left(1+\dot{\hat{\tau}}(t)(x-1)\right)+\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)+\hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\right.\\
&\left.\times\left(\left(A-B^{\mathrm{T}}P(\gamma(t))\right)X(t)+B\tilde{u}(0,t)+B\hat{w}(0,t)\right)\right)\mathrm{d}x\\
&+2b_2\int_0^1(1+x)\hat{w}_x^{\mathrm{T}}(x,t)\Big(\hat{w}_{xx}(x,t)\left(1+\dot{\hat{\tau}}(t)(x-1)\right)+\dot{\hat{\tau}}(t)\hat{w}_x(x,t)\Big)\mathrm{d}x\\
&+b_2\dot{\hat{\tau}}(t)\int_0^1(1+x)\left(\left|\hat{w}(x,t)\right|^2+\left|\hat{w}_x(x,t)\right|^2\right)\mathrm{d}x,
\end{aligned}
\tag{5.102}
$$

where (5.70), (5.75), (5.78), and (5.80) are used.

To determine a time domain where $\dot{V}(X_t,\gamma(t))$ is well defined, we first observe from the right-hand side of (5.102) that the highest order derivative is $\hat{w}_{xx}(x,t)$, which can be computed as,

$$
\begin{aligned}
\hat{w}_{xx}(x,t)&=\hat{u}_{xx}(x,t)\\
&=\frac{\partial^2 U}{\partial s^2}\hat{\tau}^2(t)\Big|_{s=t+\hat{\tau}(t)(x-1)}
\end{aligned}
\tag{5.103}
$$

by virtue of (5.72) and (5.69). Recall from Lemma 5.5 that $U(t)$ has a continuous second derivative on $t\in(\tau,\infty)$. Then, $\hat{w}_{xx}(x,t)$ is well defined if

$$
t>\tau+\tau_{\max},\tag{5.104}
$$

where $\tau_{\max}$ is defined in the proof of Lemma 5.4 as the supremum of $\hat{\tau}(t)$ on $t\in[-\tau,\infty)$. This guarantees that

$$
s=t+\hat{\tau}(t)(x-1)>\tau\tag{5.105}
$$

for any $x\in[0,1]$. It then follows from $X(t)\in C^1(0,\infty)$, the continuous differentiability of $P(\gamma(t))$ with respect to t, which is equivalent to that of $F(\gamma(t))$ as indicated in Lemma 5.4, and $\hat{\tau}(t),\gamma(t)\in C^1[-\tau,\infty)$, which is implied by the time-varying low gain feedback design (5.61) and (5.62), that $\dot{V}\in C[\tau+\tau_{\max}+1,\infty)$ is well defined. Let

$$
t_s=\tau+\tau_{\max}+1\tag{5.106}
$$

denote the starting point of the consideration of $\dot{V}(X_t,\gamma(t))$ as a function of t.

With the help of the parametric algebraic Riccati equation (5.59) and the closed-loop representation (5.77), we obtain from (5.102) that

$$
\begin{aligned}
\dot{V}(X_t,\gamma(t)) = {} & X^{\mathrm{T}}(t)\Big(-\gamma(t)P(\gamma(t)) - P(\gamma(t))BB^{\mathrm{T}}P(\gamma(t))\Big)X(t) \\
& + 2X^{\mathrm{T}}(t)P(\gamma(t))B\tilde{u}(0,t) + 2X^{\mathrm{T}}(t)P(\gamma(t))B\hat{w}(0,t) \\
& + X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t) \\
& + 2b_1\int_0^1 (1+x)\tilde{u}^{\mathrm{T}}(x,t)\tilde{u}_x(x,t)\mathrm{d}x \\
& - \frac{2b_1}{\hat{\tau}(t)}\int_0^1 (1+x)\left(\tilde{\tau} + \tau\dot{\hat{\tau}}(t)(x-1)\right)\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \\
& + 2b_2\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\left(1+\dot{\hat{\tau}}(t)(x-1)\right)\mathrm{d}x \\
& + 2b_2\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t)) \\
& \times\left(\left(A - BB^{\mathrm{T}}P(\gamma(t))\right)X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t)\right)\mathrm{d}x \\
& + 2b_2\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t)\mathrm{d}x \\
& + 2b_2\int_0^1 (1+x)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\left(1+\dot{\hat{\tau}}(t)(x-1)\right)\mathrm{d}x \\
& + 2b_2\int_0^1 (1+x)\left|\hat{w}_x(x,t)\right|^2\dot{\hat{\tau}}(t)\mathrm{d}x \\
& + b_2\dot{\hat{\tau}}(t)\int_0^1 (1+x)\left(\left|\hat{w}(x,t)\right|^2 + \left|\hat{w}_x(x,t)\right|^2\right)\mathrm{d}x.
\end{aligned}
$$

Next, we use

$$
\int_0^1 (1+x)\tilde{u}^{\mathrm{T}}(x,t)\tilde{u}_x(x,t)\mathrm{d}x = -\frac{1}{2}(|\tilde{u}(0,t)|^2 + |\tilde{u}(t)\|^2),
$$

Young's Inequality and the properties (5.82)–(5.86) in Lemma 5.3 to estimate $\dot{V}(X_t,\gamma(t))$ as,

$$
\begin{aligned}
\dot{V}(X_t,\gamma(t)) \le {} & -\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t) + 2\,|\tilde{u}(0,t)|^2 \\
& + 2\left|\hat{w}(0,t)\right|^2 + X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t)
\end{aligned}
$$

$$
\begin{aligned}
&- b_1 \left(|\tilde{u}(0,t)|^2 + \|\tilde{u}(t)\|^2\right) + 2b_1 \frac{|\tilde{\tau}| + \frac{1}{2}\tau \left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)} \left(\epsilon \|\tilde{u}(t)\|^2\right. \\
&\left. + \frac{1}{\epsilon} \left\|\hat{w}_x(t)\right\|^2\right) + 2b_2 \left(\frac{1}{2}\left(\left|\dot{\hat{\tau}}(t)\right| - 1\right) \left|\hat{w}(0,t)\right|^2\right. \\
&\left. + \left(\left|\dot{\hat{\tau}}(t)\right| - \frac{1}{2}\right) \left\|\hat{w}(t)\right\|^2\right) \\
&+ 2b_2 \left(\hat{\tau}(t) n\gamma(t) \left\|\hat{w}(t)\right\|^2 + \frac{3}{2}\hat{\tau}(t) n(n+1)\gamma^2(t) X^{\mathrm{T}}(t) P(\gamma(t)) X(t)\right. \\
&\left. + 3\hat{\tau}(t) n\gamma(t) \left(|\tilde{u}(0,t)|^2 + \left|\hat{w}(0,t)\right|^2\right)\right) \\
&+ 2b_2 \left(h^{\frac{1}{2}} \left\|\hat{w}(t)\right\|^2 + \left(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\right)^2 h^{\frac{3}{2}} X^{\mathrm{T}}(t) \frac{\partial P}{\partial \gamma} B B^{\mathrm{T}} \frac{\partial P}{\partial \gamma} X(t)\right) \\
&+ 2b_2 \left(\left|\hat{w}_x(1,t)\right|^2 + \frac{1}{2}\left(\left|\dot{\hat{\tau}}(t)\right| - 1\right) \left|\hat{w}_x(0,t)\right|^2 + \left(\left|\dot{\hat{\tau}}(t)\right| - \frac{1}{2}\right) \left\|\hat{w}_x(t)\right\|^2\right) \\
&+ 4b_2 |\dot{\hat{\tau}}(t) \left\|\hat{w}_x(t)\right\|^2 + 2b_2 |\dot{\hat{\tau}}(t)| \left(\left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2\right). \qquad (5.107)
\end{aligned}
$$

Substitution of $\left|\hat{w}_x(1,t)\right|^2$ in (5.107) by its estimate (5.87) and arranging the terms in (5.107) give

$$
\begin{aligned}
\dot{V}(X_t, \gamma(t)) \leq\ & X^{\mathrm{T}}(t) P(\gamma(t)) X(t) \left(-\gamma(t) + 6b_2 \hat{\tau}^2(t) n^2 (n+1) \gamma^3(t)\right. \\
&\left. + 3b_2 \hat{\tau}(t) n(n+1) \gamma^2(t)\right) + |\tilde{u}(0,t)|^2 \left(2 - b_1 + 6b_2 \hat{\tau}(t) n\gamma(t)\right. \\
&\left. + 12 b_2 \hat{\tau}^2(t) n^2 \gamma^2(t)\right) + \left|\hat{w}(0,t)\right|^2 \left(2 + b_2\left(\left|\dot{\hat{\tau}}(t)\right| - 1\right) + 6b_2 \hat{\tau}(t) n\gamma(t)\right. \\
&\left. + 12 b_2 \hat{\tau}^2(t) n^2 \gamma^2(t)\right) + \|\tilde{u}(t)\|^2 \left(-b_1 + 2b_1 \epsilon \frac{|\tilde{\tau}| + \frac{1}{2}\tau \left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)}\right) \\
&+ X^{\mathrm{T}}(t) \left(\frac{\partial P}{\partial \gamma} \dot{\gamma}(t) + 2b_2 h^{\frac{3}{2}} \left(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\right)^2 \frac{\partial P}{\partial \gamma} B B^{\mathrm{T}} \frac{\partial P}{\partial \gamma}\right. \\
&\left. + 4b_2 \hat{\tau}^2(t) \dot{\gamma}^2(t) \frac{\partial P}{\partial \gamma} B B^{\mathrm{T}} \frac{\partial P}{\partial \gamma}\right) X(t) \\
&+ \left\|\hat{w}_x(t)\right\|^2 \left(\frac{2b_1}{\epsilon} \frac{|\tilde{\tau}| + \frac{1}{2}\tau |\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)} - b_2 + 8b_2 |\dot{\hat{\tau}}(t)|\right) \\
&+ \left\|\hat{w}(t)\right\|^2 \left(-b_2 + 2b_2 \hat{\tau}(t) n\gamma(t) + 2b_2 h^{\frac{1}{2}} + 4b_2 |\dot{\hat{\tau}}(t)|\right) \\
&+ \left|\hat{w}_x(0,t)\right|^2 b_2 \left(\left|\dot{\hat{\tau}}(t)\right| - 1\right). \qquad (5.108)
\end{aligned}
$$

Recall that

$$\gamma(t) = \frac{h}{\hat{\tau}(t)}. \tag{5.109}$$

We have

$$h = \gamma(t)\hat{\tau}(t).$$

Substitution of $\gamma(t)\hat{\tau}(t)$ in (5.108) by h simplifies the estimate of $\dot{V}(X_t, \gamma(t))$ as follows:

$$\begin{aligned}
\dot{V}(X_t, \gamma(t)) \leq\ & \gamma(t) X^{\mathrm{T}}(t) P(\gamma(t)) X(t) \Big(-1 + 6b_2 n^2 (n+1) h^2 \\
& + 3b_2 n(n+1)h\Big) + |\tilde{u}(0,t)|^2 \left(2 - b_1 + 6b_2 nh + 12 b_2 n^2 h^2\right) \\
& + \left|\hat{w}(0,t)\right|^2 \left(b_2 \left(\left|\dot{\hat{\tau}}(t)\right| - 1\right) + 2 + 6b_2 nh + 12 b_2 n^2 h^2\right) \\
& + \|\tilde{u}(t)\|^2 b_1 \left(-1 + 2\epsilon \frac{|\tilde{\tau}| + \frac{1}{2}\tau \left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)}\right) \\
& + X^{\mathrm{T}}(t) \left(-\frac{h}{\hat{\tau}^2(t)} \dot{\hat{\tau}}(t) \frac{\partial P}{\partial \gamma} + 2b_2 \left(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\right)^2 \frac{\partial P}{\partial \gamma} B B^{\mathrm{T}} \frac{\partial P}{\partial \gamma} h^{\frac{3}{2}} \right. \\
& \left. + 4b_2 \left(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\right)^2 \frac{\partial P}{\partial \gamma} B B^{\mathrm{T}} \frac{\partial P}{\partial \gamma} h^2 \right) X(t) \\
& + \left\|\hat{w}_x(t)\right\|^2 \left(\frac{2b_1}{\epsilon} \frac{|\tilde{\tau}| + \frac{1}{2}\tau \left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)} + 8b_2 \left|\dot{\hat{\tau}}(t)\right| - b_2\right) \\
& + \left\|\hat{w}(t)\right\|^2 b_2 \left(-1 + 2nh + 2h^{\frac{1}{2}} + 4\left|\dot{\hat{\tau}}(t)\right|\right) \\
& + \left|\hat{w}_x(0,t)\right|^2 b_2 \left(\left|\dot{\hat{\tau}}(t)\right| - 1\right).
\end{aligned} \tag{5.110}$$

To group terms that involve $\frac{\partial P}{\partial \gamma}$ in (5.110), we consider

$$\begin{aligned}
\frac{\partial P}{\partial \gamma} B B^{\mathrm{T}} \frac{\partial P}{\partial \gamma} &= \left(\frac{\partial P}{\partial \gamma}\right)^{\frac{1}{2}} \left(\frac{\partial P}{\partial \gamma}\right)^{\frac{1}{2}} B B^{\mathrm{T}} \left(\frac{\partial P}{\partial \gamma}\right)^{\frac{1}{2}} \left(\frac{\partial P}{\partial \gamma}\right)^{\frac{1}{2}} \\
&\leq \operatorname{tr}\left(B^{\mathrm{T}} \frac{\partial P}{\partial \gamma} B\right) \frac{\partial P}{\partial \gamma} \\
&= \frac{\partial}{\partial \gamma} \left(\operatorname{tr}\left(B^{\mathrm{T}} P(\gamma) B\right)\right) \frac{\partial P}{\partial \gamma} \\
&= n \frac{\partial P}{\partial \gamma},
\end{aligned}$$

where Lemma 2.1 is used. It then follows that

$$
\begin{aligned}
&-\frac{h}{\hat{\tau}^2(t)}\dot{\hat{\tau}}(t)\frac{\partial P}{\partial \gamma}+2b_2\left(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\right)^2\frac{\partial P}{\partial \gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\left(h^{\frac{3}{2}}+2h^2\right)\\
&\leq h\frac{\partial P}{\partial \gamma}\frac{\left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}^2(t)}\left(1+2b_2\left|\dot{\hat{\tau}}(t)\right|h^{\frac{1}{2}}n+4b_2\left|\dot{\hat{\tau}}(t)\right|hn\right)\\
&\leq \gamma(t)P(\gamma(t))\frac{\tau_{\max}\lambda_{\max}\left(\max_{\gamma\in[h/\tau_{\max},h/\tau_{\min}]}\left\{\frac{\partial P}{\partial \gamma}\right\}\right)}{\lambda_{\min}(P(\frac{h}{\tau_{\max}}))}\frac{\left|\dot{\hat{\tau}}(t)\right|}{\tau_{\min}^2}\\
&\quad\times\left(1+2b_2\left|\dot{\hat{\tau}}(t)\right|h^{\frac{1}{2}}n+4b_2\left|\dot{\hat{\tau}}(t)\right|hn\right),
\end{aligned}\tag{5.111}
$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote respectively the minimum and the maximum eigenvalue of a real symmetric matrix, $\tau_{\min}$ and $\tau_{\max}$ are respectively the infimum and the supremum of $\hat{\tau}(t)$ over $[-\tau,\infty)$, and we have used the boundedness of $\hat{\tau}(t)$, $\gamma(t)$ and $P(\gamma(t))$ as shown in the proof of Lemma 5.4. By denoting

$$
\sigma=\frac{\tau_{\max}\lambda_{\max}\left(\max_{\gamma\in[h/\tau_{\max},h/\tau_{\min}]}\left\{\frac{\partial P}{\partial \gamma}\right\}\right)}{\tau_{\min}^2\lambda_{\min}\left(P\left(\frac{h}{\tau_{\max}}\right)\right)},
$$

and recalling from the design of $\hat{\tau}(t)$ that

$$
\lim_{t\to\infty}\hat{\tau}(t)=\overline{\tau}\tag{5.112}
$$

and

$$
\lim_{t\to\infty}\dot{\hat{\tau}}(t)=0,\tag{5.113}
$$

we see that there exists a sufficiently large time constant $t_0\geq t_s$ such that, for each $t\geq t_0$,

$$
|\dot{\hat{\tau}}(t)|\leq\min\left\{\frac{1}{2b_2h^{\frac{1}{2}}n+4b_2hn},\frac{h}{\sigma}\right\}.
$$

Thus, (5.111) can be continued as follows:

$$
-\frac{h}{\hat{\tau}^2(t)}\dot{\hat{\tau}}(t)\frac{\partial P}{\partial \gamma}+2b_2\left(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\right)^2\frac{\partial P}{\partial \gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\left(h^{\frac{3}{2}}+2h^2\right)\leq 2h\gamma(t)P(\gamma(t)).
$$

The estimate of $\dot{V}(X_t, \gamma(t))$ then takes the form,

$$\begin{aligned}
\dot{V}(X_t, \gamma(t)) \le\ & \gamma(t) X^{\mathrm{T}}(t) P(\gamma(t)) X(t) \left(-1 + 2h + 6b_2 n^2 (n+1) h^2 + 3b_2 n (n+1) h\right) \\
& + |\tilde{u}(0,t)|^2 \left(2 - b_1 + 6b_2 nh + 12 b_2 n^2 h^2\right) \\
& + \left|\hat{w}(0,t)\right|^2 \left(b_2 \left(\left|\dot{\hat{\tau}}(t)\right| - 1\right) + 2 + 6b_2 nh + 12 b_2 n^2 h^2\right) \\
& + \|\tilde{u}(t)\|^2 b_1 \left(-1 + 2\epsilon \frac{|\tilde{\tau}| + \frac{1}{2}\tau \left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)}\right) \\
& + \left\|\hat{w}_x(t)\right\|^2 \left(\frac{2b_1}{\epsilon} \frac{|\tilde{\tau}| + \frac{1}{2}\tau \left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)} + 8b_2 \left|\dot{\hat{\tau}}(t)\right| - b_2\right) \\
& + \left\|\hat{w}(t)\right\|^2 b_2 \left(-1 + 2nh + 2h^{\frac{1}{2}} + 4|\dot{\hat{\tau}}(t)|\right) \\
& + \left|\hat{w}_x(0,t)\right|^2 b_2 \left(\left|\dot{\hat{\tau}}(t)\right| - 1\right).
\end{aligned} \tag{5.114}$$

We next simplify those terms in (5.114) that contain $|\dot{\hat{\tau}}(t)|$. In view of

$$\lim_{t\to\infty} \hat{\tau}(t) = \overline{\tau}$$

and

$$\lim_{t\to\infty} \dot{\hat{\tau}}(t) = 0,$$

both of which are required by the time-varying parameter design, as given in (5.62), there exists a sufficiently large positive constant $t_1 \ge t_0$ such that, for each $t \ge t_1$,

$$\begin{aligned}
0 \le \frac{\tau}{\hat{\tau}(t)} \le 2 \ & \left(\text{or } \frac{|\tilde{\tau}|}{\hat{\tau}(t)} \le 1\right), \\
|\dot{\hat{\tau}}(t)| \le \frac{\tau_{\min}}{\tau}&,
\end{aligned}$$

which together guarantee that

$$\begin{aligned}
\frac{|\tilde{\tau}| + \frac{1}{2}\tau \left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)} &\le 1 + \frac{\tau_{\min}}{2\hat{\tau}(t)} \\
&\le \frac{3}{2}.
\end{aligned} \tag{5.115}$$

Let $\epsilon = \frac{1}{4}$. Then,

$$-1 + 2\epsilon \frac{|\tilde{\tau}| + \frac{1}{2}\tau \left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)} < 0.$$

Then, the term associated with $\|\tilde{u}(t)\|^2$ in (5.114) becomes negative. Also, there exists a sufficiently large positive constant $t_2 \geq t_1$ such that,

$$\left|\dot{\hat{\tau}}(t)\right| \leq \frac{1}{16}, \quad t \geq t_2. \tag{5.116}$$

Thus, with the help of (5.115), we obtain

$$\frac{2b_1}{\epsilon}\frac{|\tilde{\tau}| + \frac{1}{2}\tau\left|\dot{\hat{\tau}}(t)\right|}{\hat{\tau}(t)} + 8b_2\left|\dot{\hat{\tau}}(t)\right| - b_2 \leq 12b_1 - \frac{b_2}{2}. \tag{5.117}$$

To make the right-hand side of (5.117) non-positive, we let b_1 and b_2 satisfy

$$24b_1 \leq b_2. \tag{5.118}$$

It then follows that the term corresponding to $\left\|\hat{w}_x(t)\right\|^2$ in (5.114) is non-positive.

Choose $b_2 > 4$. Then, there exists a sufficiently large constant $t_3 \geq t_2$ such that, for each $t \geq t_3$,

$$\left|\dot{\hat{\tau}}(t)\right| \leq \frac{1}{2} - \frac{2}{b_2}, \tag{5.119}$$

which is equivalent to

$$1 - \left|\dot{\hat{\tau}}(t)\right| - \frac{2}{b_2} \geq \frac{1}{2}. \tag{5.120}$$

Then,

$$b_2\left(\left|\dot{\hat{\tau}}(t)\right| - 1\right) + 2 + 6b_2nh + 12b_2n^2h^2 < 0, \tag{5.121}$$

which is implied by

$$6nh + 12n^2h^2 < \frac{1}{2}, \tag{5.122}$$

leads to the negativeness of the coefficient associated with $\left|\hat{w}(0,t)\right|^2$ in (5.114). Note that the definition of t_3 naturally gives rise to the negativeness of the coefficient associated with $\left|\hat{w}_x(0,t)\right|^2$ in (5.114), and

$$-1 + 2nh + 2h^{\frac{1}{2}} + 4\left|\dot{\hat{\tau}}(t)\right| < 0 \tag{5.123}$$

if

$$2nh + 2h^{\frac{1}{2}} < \frac{1}{2}. \tag{5.124}$$

In view of (5.114), we see that there exists a sufficiently large constant t_3 such that, for each $t \geq t_3$, the coefficients associated with $\|\tilde{u}(t)\|^2$, $\left\|\hat{w}_x(t)\right\|^2$, $\left\|\hat{w}(t)\right\|^2$, $|\hat{w}(0,t)|^2$ and $|\hat{w}_x(0,t)|^2$ in (5.114) are all non-positive as long as

$$\begin{cases} b_2 > 4, \\ 24b_1 \leq b_2, \\ nh(1+2nh) < \frac{1}{12}, \\ nh + h^{\frac{1}{2}} < \frac{1}{4}. \end{cases}$$

It is then clear that $\dot{V}(X_t, \gamma(t)) < 0$ on $t \geq t_3$ if

$$\begin{cases} 2h + 6b_2n^2(n+1)h^2 + 3b_2n(n+1)h < 1, \\ 6b_2nh(1+2nh) < b_1 - 2, \\ nh(1+2nh) < \frac{1}{12}, \\ nh + h^{\frac{1}{2}} < \frac{1}{4}, \\ 24b_1 \leq b_2, \\ b_1 > 2, \\ b_2 > 4 \end{cases} \tag{5.125}$$

holds. We first choose b_1 and b_2 according to the last row of (5.125). Then, substitution of b_1 and b_2 in the rest of (5.125) shows that there exists a sufficiently small $h^\star$ such that, for each $h \in (0, h^\star]$,

$$\dot{V}(X_t, \gamma(t)) < 0, \quad t \geq t_3. \tag{5.126}$$

It is worth mentioning here that $h^\star$ is independent of any information of the delay, including $\overline{\tau}$, but only depends on the dimension of the system n.

To complete the proof, it remains to establish global regulation of both $X(t)$ and $U(t)$. We first demonstrate the square integrability of $X(t)$ over $t \geq 0$. By denoting

$$1 - 2h - 6b_2n^2(n+1)h^2 - 3b_2n(n+1)h = \eta,$$

which is a positive constant as long as h is chosen small enough, we have from (5.114) that

$$\dot{V}(X_t, \gamma(t)) \leq -\eta\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t), \quad t \geq t_3.$$

Then, in view of the boundedness of $\gamma(t)$ and $P(\gamma(t))$, as established in the proof of Lemma 5.4, we have,

$$\eta\lambda_{\min}\left(P\left(\frac{h}{\tau_{\max}}\right)\right)\frac{h}{\tau_{\max}}|X(t)|^2 \le -\dot{V}, \quad t \ge t_3,$$

which leads to

$$\begin{aligned}\int_{t_3}^{\infty}|X(t)|^2\mathrm{d}t &\le \frac{\tau_{\max}(V(X_{t_3},\gamma(t_3)) - V(X_{\infty},\gamma(\infty))}{\eta\lambda_{\min}\left(P\left(\frac{h}{\tau_{\max}}\right)\right)h} \\ &\le \frac{\tau_{\max}V(X_{t_3},\gamma(t_3))}{\eta\lambda_{\min}\left(P\left(\frac{h}{\tau_{\max}}\right)\right)h}.\end{aligned}$$

Recalling that V is continuously differentiable on $t \in [t_s, \infty)$, we see that V is bounded for $t \in [t_s, t_3]$, where $t_3 \ge t_s$ according to the definition of t_3. Then,

$$\int_{t_3}^{\infty}|X(t)|^2\mathrm{d}t < \infty,$$

which, together with the fact that $X(t) \in C[0, t_3]$, as indicated in Lemma 5.4, imply that

$$\begin{aligned}\int_{0}^{\infty}|X(t)|^2\mathrm{d}t &= \int_{0}^{t_3}|X(t)|^2\mathrm{d}t + \int_{t_3}^{\infty}|X(t)|^2\mathrm{d}t \\ &\le t_3 \max_{t\in[0,t_3]}\left\{|X(t)|^2\right\} + \int_{t_3}^{\infty}|X(t)|^2\mathrm{d}t \\ &< \infty,\end{aligned}$$

that is, $X(t)$ is square integrable on $t \in [0, \infty)$.

On the other hand, from the definition of $V(X_t, \gamma(t))$ in (5.101) and the boundedness of $V(X_t, \gamma(t))$ for $t \in [t_s, \infty)$, which can be seen by noting the boundedness of $V(X_t, \gamma(t))$ on $t \in [t_s, t_3]$ and $\dot{V}(X_t, \gamma(t)) < 0$ on $t \ge t_3$, we establish the boundedness of $X(t)$ on $t \in [t_s, \infty)$. Then, $X(t) < \infty$ for $t \in [-\tau, \infty)$ readily follows from the continuity of $X(t)$ on $t \in [-\tau, t_s]$. Thus, (5.97) implies the boundedness of $\dot{X}(t)$ on $t \ge 0$ since $F(\gamma(t))$ is bounded, as indicated in Lemma 5.4. With the square integrability of $X(t)$ and the boundedness of $\dot{X}(t)$, global regulation of $X(t)$, which further implies that

$$\lim_{t\to\infty} U(t) = 0 \tag{5.127}$$

by (5.60) and the boundedness of $F(\gamma(t))$, follows from the Barbalat's lemma. □

Remark 5.6 The requirement for $\hat{\tau}(t)$ to have a continuous second derivative comes from the requirement for the well definedness of the partial derivatives in the PDEs (5.70), (5.75), (5.78), and (5.80), where $\hat{w}_{xt}$ and $\hat{w}_{xx}$ are the highest order derivatives. From the definition of $\hat{w}(x, t)$ in (5.72) and the time-varying low gain feedback law (5.60) it follows that both the second-order partial derivatives contain the same term $\ddot{\hat{\tau}}(s)|_{s=t+\hat{\tau}(x-1)}$. Take $\hat{w}_{xx}(x,t)$ for example,

$$\hat{w}_{xx}(x,t) = \Bigg(B^{\mathrm{T}}\frac{h}{\hat{\tau}^2(s)}\Big(-\frac{\partial^2 P}{\partial\gamma^2(s)}\frac{h}{\hat{\tau}^2(s)}\dot{\hat{\tau}}^2(s) - 2\frac{\partial P}{\partial\gamma(s)}\frac{\dot{\hat{\tau}}^2(s)}{\hat{\tau}(s)} + \frac{\partial P}{\partial\gamma(s)}\ddot{\hat{\tau}}(s)\Big)X(s)$$
$$+2B^{\mathrm{T}}\frac{\partial P}{\partial\gamma(s)}\frac{h}{\hat{\tau}^2(s)}\dot{\hat{\tau}}(s)\dot{X}(s) - B^{\mathrm{T}}P(\gamma(s))\ddot{X}(s)\Bigg)\hat{\tau}^2(t)\Bigg|_{s=t+\hat{\tau}(t)(x-1)},$$

where we have used (5.95), (5.96), (5.100), (5.103), and the fact that

$$F(\gamma(t)) = -B^{\mathrm{T}}P(\gamma(t)). \tag{5.128}$$

Thus, the existence of $\ddot{\hat{\tau}}(t)$ is necessary for that of $\hat{w}_{xx}(x,t)$. Without loss of generality, we assume that $\hat{\tau}(t) \in C^2[-\tau,\infty)$ as in the design of the time-varying low gain feedback laws in Sect. 5.3.1. This requirement on $\hat{\tau}(t)$, which is necessary in our design, facilitates all our derivations in Sects. 5.3.2 and 5.3.3, and in general, cannot be relaxed unless a new Lyapunov analysis on the closed-loop system is established involving at most the first derivative of $\hat{\tau}(t)$. □

Theorem 5.2 reveals a group of time-varying low gain feedback laws which achieve global regulation of the closed-loop system. By the structure of the time-varying feedback parameter in (5.61) and Theorem 5.2, h and $\hat{\tau}(t)$ need to be designed first in order to construct $\gamma(t)$. Thus, for the sake of simplicity, we consider directly designing $\gamma(t)$ without involving h and $\hat{\tau}(t)$.

Corollary 5.3 *There exists a sufficiently small positive constant $\gamma^\star$, which is inversely proportional to the delay bound $\overline{\tau}$, such that the time-varying low gain feedback law (5.60) globally regulates $X(t)$ and $U(t)$ of system (5.54) as long as*

$$\gamma(t) \in C^2[-\tau,\infty),\quad \gamma(t) > 0,\quad \lim_{t\to\infty}\gamma(t) \in (0,\gamma^\star],\quad \lim_{t\to\infty}\dot{\gamma}(t) = 0. \tag{5.129}$$

Proof Given a $\gamma(t)$ satisfying (5.129), we write $\gamma(t)$ in the form of (5.61), where h and $\hat{\tau}(t)$ are selected as

$$h = \overline{\tau}\lim_{t\to\infty}\gamma(t) > 0 \tag{5.130}$$

and

$$\hat{\tau}(t) = \frac{h}{\gamma(t)}. \tag{5.131}$$

It can be readily verified that

$$\lim_{t\to\infty}\hat{\tau}(t) = \overline{\tau}. \tag{5.132}$$

Also, $\gamma(t) \in C^2[-\tau, \infty)$, $\gamma(t) > 0$, and $\lim_{t\to\infty} \dot{\gamma}(t) = 0$ imply that $\hat{\tau}(t) \in C^2[-\tau, \infty)$, $\hat{\tau}(t) > 0$ and $\lim_{t\to\infty} \dot{\hat{\tau}}(t) = 0$, respectively, because

$$\dot{\hat{\tau}}(t) = \frac{-h\dot{\gamma}(t)}{\gamma^2(t)},$$
$$\ddot{\hat{\tau}}(t) = \frac{h}{\gamma^2(t)}\left(\frac{2\dot{\gamma}^2(t)}{\gamma(t)} - \ddot{\gamma}(t)\right).$$

Note from the selection of

$$h = \overline{\tau} \lim_{t\to\infty} \gamma(t) \tag{5.133}$$

that $\lim_{t\to\infty} \gamma(t) \in (0, \gamma^\star]$ is equivalent to $h \in (0, \overline{\tau}\gamma^\star]$. Theorem 5.2 concludes that there exists a sufficiently small positive constant $h^\star$, which is independent of any information of the delay, such that, for any $h \in (0, h^\star]$, the time-varying low gain feedback law

$$U(t) = -B^{\mathrm{T}} P(\gamma(t))X(t) \tag{5.134}$$

globally regulates the system. Thus, there exists a sufficiently small positive constant

$$\gamma^\star = \frac{h^\star}{\overline{\tau}}, \tag{5.135}$$

which is inversely proportional to $\overline{\tau}$, such that the regulation of the closed-loop system is guaranteed if $\lim_{t\to\infty} \gamma(t) \in (0, \gamma^\star]$. □

Remark 5.7 The traditional constant low gain feedback law is a special case of the time-varying low gain feedback law. In particular, Corollary 5.2 concludes that there exists a sufficiently small positive constant $\gamma^\star$, which is inversely proportional to the upper bound of delay $\overline{\tau}$ such that, for each $\gamma \in (0, \gamma^\star]$, the low gain feedback law

$$U(t) = -B^{\mathrm{T}} P(\gamma)X(t) \tag{5.136}$$

globally regulates system (5.54). This observation is consistent with the observation in Remark 5.1. □

Remark 5.8 Dealing with the conservativeness incurred in the Lyapunov stability analysis on the upper bound of feedback parameter γ under the constant parameter low gain feedback design, as given by (5.58), the time-varying parameter design proposed in this section takes a proactive selection of γ at a relatively large value, which corresponds to a fast convergence rate, during the starting phase of the system evolution. After the system reaches a state near the equilibrium point zero, reducing γ to a sufficiently small constant, as required by Corollary 5.3, would not affect the closed-loop transient performance, while guaranteeing stability.

Therefore, such a time-varying parameter design would manifest its merits in the closed-loop performance compared with that of the constant parameter design. □

Remark 5.9 Remark 5.8 provides a guideline for the construction of a time-varying $\gamma(t)$ that outperforms a constant γ in terms of the closed-loop performance, as demonstrated by simulation. A theoretical proof of such an improvement is challenging due to the time-varying design of $\gamma(t)$ and remains to be carried out. □

5.3.4 Convergence Rate Analysis

To illustrate the merits of the time-varying low gain feedback design in comparison with the constant parameter design in terms of the closed-loop performance, we compare the convergence rates of the closed-loop system under a constant parameter feedback with different values of the feedback parameter within the range where exponential stability of the closed-loop system is ensured.

Theorem 5.3 *The delay independent state feedback TPF law with a constant parameter,*

$$U(t) = -B^{\mathrm{T}} P(\gamma) X(t), \tag{5.137}$$

for a sufficiently small $\gamma > 0$, *exponentially stabilizes system (5.54) with*

$$|X(t)|^2 \leq \lambda_{\min}^{-1}(P(\gamma)) \mathrm{e}^{-\frac{\beta}{\zeta} t} V(X_0, 0), \quad t \geq 0, \tag{5.138}$$

where

$$\begin{aligned} \beta &= \min\left\{\gamma\Big(1 - 120n^2(n+1)\overline{\tau}^2\gamma^2 - 60n(n+1)\overline{\tau}\gamma\Big), 20\Big(1 - 2n\overline{\tau}\gamma - 2\overline{\tau}^{\frac{1}{2}}\gamma^{\frac{1}{2}}\Big), 1\right\}, \\ \zeta &= \max\{1, 40\overline{\tau}\}, \end{aligned}$$

and $V(X_t, t)$ *is as given in (5.101) with* $\gamma(t) = \gamma$ *and* $\hat{\tau}(t) = \overline{\tau}$.

Proof The proof follows an analysis similar to that in the proof of Theorem 5.2, except the distinction between a constant feedback parameter and a time-varying parameter. With a Lyapunov functional $V(X_t, \gamma(t))$ given by (5.101), its estimated time derivative along the trajectory of the closed-loop system (5.97), (5.108), for the special case of the constant parameter design takes the form of

$$\begin{aligned} &\dot{V}(X_t, \gamma(t)) \\ &\leq \gamma X^{\mathrm{T}} P X\Big(-1 + 6b_2 n^2(n+1)\overline{\tau}^2\gamma^2 + 3b_2 n(n+1)\overline{\tau}\gamma\Big) \\ &\quad + |\tilde{u}(0,t)|^2\Big(2 - b_1 + 6b_2 n\overline{\tau}\gamma + 12 b_2 n^2 \overline{\tau}^2\gamma^2\Big) \end{aligned}$$

$$
\begin{aligned}
&+|\hat{w}(0,t)|^2\Big(-b_2+2+6b_2n\overline{\tau}\gamma+12b_2n^2\overline{\tau}^2\gamma^2\Big)\\
&+\|\tilde{u}(t)\|^2 b_1\Big(-1+2\epsilon\frac{|\tilde{\tau}|}{\hat{\tau}(t)}\Big)+\left\|\hat{w}_x(t)\right\|^2\left(\frac{2b_1}{\epsilon}\frac{|\tilde{\tau}|}{\hat{\tau}(t)}-b_2\right)\\
&+\left\|\hat{w}(t)\right\|^2 b_2\Big(-1+2n\overline{\tau}\gamma+2\overline{\tau}^{\frac{1}{2}}\gamma^{\frac{1}{2}}\Big)-|\hat{w}_x(0,t)|^2 b_2,
\end{aligned}
\tag{5.139}
$$

where we have replaced h with $\gamma\overline{\tau}$ and all the terms involving $\dot{\gamma}$ or $\dot{\hat{\tau}}(t)$ disappear because of the constant parameter design. Noting that

$$
\frac{|\tilde{\tau}|}{\hat{\tau}(t)}=\frac{\overline{\tau}-\tau}{\overline{\tau}}<1
$$

and taking $\epsilon=\frac{1}{3}$ in (5.139), we arrive at a further estimate of $\dot{V}(X_t,\gamma(t))$,

$$
\begin{aligned}
\dot{V}(X_t,\gamma(t))\leq{}&\gamma X^{\mathrm{T}}PX\left(-1+6b_2n^2(n+1)\overline{\tau}^2\gamma^2+3b_2n(n+1)\overline{\tau}\gamma\right)\\
&+|\tilde{u}(0,t)|^2\left(2-b_1+6b_2n\overline{\tau}\gamma+12b_2n^2\overline{\tau}^2\gamma^2\right)\\
&+\left|\hat{w}(0,t)\right|^2\left(-b_2+2+6b_2n\overline{\tau}\gamma+12b_2n^2\overline{\tau}^2\gamma^2\right)\\
&-\frac{1}{3}\|\tilde{u}(t)\|^2 b_1+\left\|\hat{w}_x(t)\right\|^2(6b_1-b_2)\\
&+\left\|\hat{w}(t)\right\|^2 b_2\left(-1+2n\overline{\tau}\gamma+2\overline{\tau}^{\frac{1}{2}}\gamma^{\frac{1}{2}}\right)-\left|\hat{w}_x(0,t)\right|^2 b_2.
\end{aligned}
\tag{5.140}
$$

With the selection of $b_1>2$ and $b_2>6b_1$, the terms involving $\|\tilde{u}(0,t)\|^2$, $|\hat{w}(0,t)|^2$ or $\left\|\hat{w}_x(t)\right\|^2$ in (5.140) are negative if

$$
6b_2n\overline{\tau}\gamma+12b_2n^2\overline{\tau}^2\gamma^2<b_1-2.
$$

For illustration, we choose $b_1=3$ and $b_2=20$. Then, a sufficiently small γ satisfying

$$
\max\left\{120n^2(n+1)\overline{\tau}^2\gamma^2+60n(n+1)\overline{\tau}\gamma,\,2n\overline{\tau}\gamma+2\overline{\tau}^{\frac{1}{2}}\gamma^{\frac{1}{2}}\right\}<1
$$

results in

$$
\begin{aligned}
\dot{V}(X_t,\gamma(t))&\leq-\beta\Big(X^{\mathrm{T}}PX+\|\tilde{u}(t)\|^2+\left\|\hat{w}(t)\right\|^2+\left\|\hat{w}_x(t)\right\|^2\Big)\\
&\leq-\frac{\beta}{\zeta}V(X_t,\gamma(t)),
\end{aligned}
$$

where we have used

$$V(X_t, \gamma(t)) \le \max\{1, 2b_1\tau, 2b_2\hat{\tau}(t)\}\Big(X^{\mathrm{T}}PX + \|\tilde{u}(t)\|^2 + \big\|\hat{w}(t)\big\|^2 + \big\|\hat{w}_x(t)\big\|^2\Big)$$
$$\le \zeta\Big(X^{\mathrm{T}}PX + \|\tilde{u}(t)\|^2 + \big\|\hat{w}(t)\big\|^2 + \big\|\hat{w}_x(t)\big\|^2\Big).$$

Consequently, an estimate of $V(X_t, \gamma(t))$ readily follows from the comparison lemma,

$$V(X_t, \gamma(t)) \le \mathrm{e}^{-\frac{\beta}{\zeta}t}V(X_0, 0), \quad t \ge 0,$$

which, by the definition of $V(X_t, \gamma(t))$ in (5.101), further implies the exponential convergence of $X(t)$, as expressed in (5.138). □

Remark 5.10 Theorem 5.3 establishes a guaranteed convergence rate of $\frac{\beta}{2\zeta}$ for the state of the closed-loop system under the constant parameter design of the delay independent truncated predictor feedback law. Examination of the guaranteed convergence rate indicates that γ only appears in the expression of β and

$$\lim_{\gamma\to 0}\beta = \lim_{\gamma\to\overline{\gamma}}\beta$$
$$= 0,$$

where

$$\overline{\gamma} = \min\{\gamma_1, \gamma_2\} \tag{5.141}$$

with γ_1 and γ_2 being the unique positive solutions to the nonlinear equations,

$$120n^2(n+1)\overline{\tau}^2\gamma^2 + 60n(n+1)\overline{\tau}\gamma = 1,$$
$$2n\overline{\tau}\gamma + 2\overline{\tau}^{\frac{1}{2}}\gamma^{\frac{1}{2}} = 1,$$

respectively. Thus, there exists a maximum β on the interval $\gamma \in (0, \overline{\gamma})$, corresponding to the fastest convergence rate. □

5.3.5 A Numerical Example

To compare the closed-loop performance of system (5.54) under the traditional constant parameter low gain feedback and the time-varying parameter low gain feedback, we consider system (5.54) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tau = \overline{\tau} = 1, \quad \phi(\theta) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \theta \in [-1, 0],$$

as an example. The values of $\gamma(t)$ is designed to decrease slowly from 0.3. On the other hand, $\lim_{t\to\infty}\gamma(t)$ is computed as 7.6×10^{-4} by employing (5.125), where $b_1 = 3$ and $b_2 = 72$, Corollary 5.3 and the fact that $\overline{\tau} = 1$.

Simulation shows that, regardless of how conservative the bound on $\lim_{t\to\infty}\gamma(t)$ is, it does not prevent us from designing a time-varying low gain feedback law with better closed-loop performance than a traditional constant parameter design. This can be demonstrated by selecting

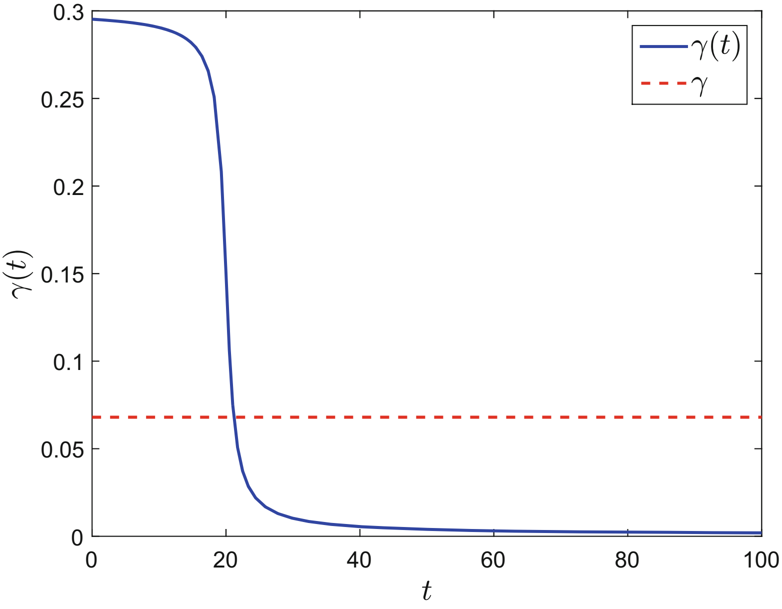
$$\gamma(t) = -0.0953\arctan(t-20) + 0.1504, \quad t \geq -1, \tag{5.142}$$

which satisfies all the design requirements in (5.129). Evolution of the time-varying low gain parameter, the system state, and the control input under the time-varying parameter design is illustrated in Figs. 5.5 and 5.6. The system evolution under the constant parameter design with $\gamma = 0.068$, which is the theoretical upper bound given by (5.58), is also given for comparison. Obviously, this choice of $\gamma(t)$ achieves better closed-loop performance than the choice of a constant γ.

5.4 Delay Independent Truncated Predictor Output Feedback Design

A delay-dependent or delay independent truncated predictor state feedback law stabilizes a general linear system in the presence of a certain amount of input delay. Results in Chap. 4 and Sect. 5.2 provide estimates of the maximum delay bound under which the closed-loop stability can be achieved. In the face of time-varying or unknown delay, delay independent feedback laws are preferable over delay-dependent feedback laws as the former provide robustness to uncertainties in the delay. We present in this section a delay independent observer based output feedback law that stabilizes the system. Our design is of the delay independence nature. We establish an estimate of the maximum allowable delay bound through a Razumikhin-type stability analysis. This delay bound result reveals the capability of the proposed output feedback law in handling an arbitrarily large input delay in linear systems with all open loop poles at the origin or in the open left-half plane. Compared with that of the delay-dependent output feedback laws in the literature, this same level of stabilization result is not sacrificed by the absence of the prior knowledge of the delay.

More specifically, we consider stabilization of a general linear system with time-varying input delay by delay independent output feedback. A state observer is constructed, without resorting to any knowledge of the delay, not even its upper bound. The design of such an observer based output feedback law, seemingly straightforward at the first glance, turns out to be a meticulous one because of the feed of the current input signal rather than the delayed version into the dynamic of the state observer to guarantee the delay independence of the observer dynamics.



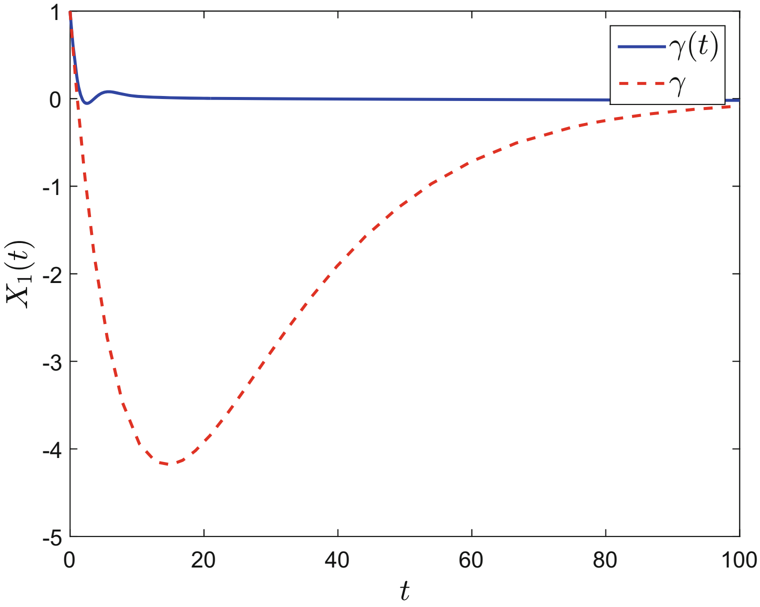


Fig. 5.5 $\gamma(t)$ given by (5.142) and $\gamma = 0.068$, and the corresponding evolution of $X_1(t)$

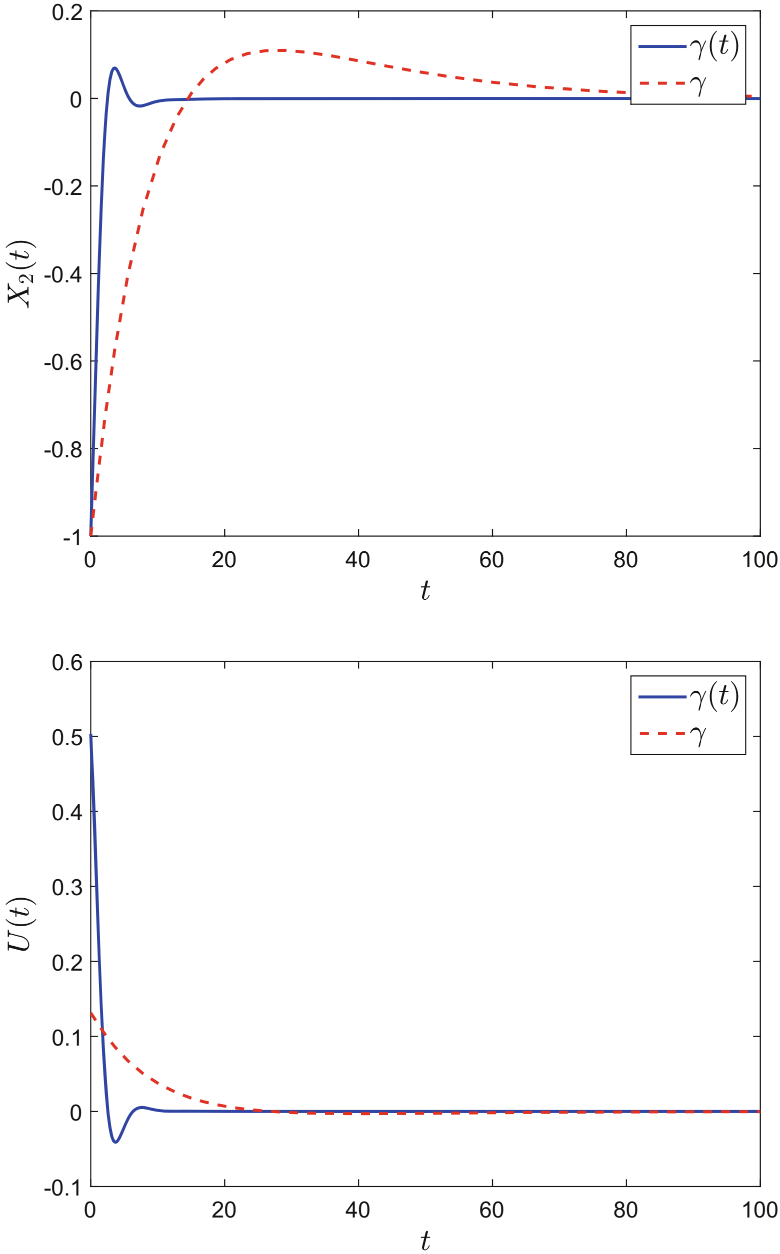


Fig. 5.6 The evolution of $X_2(t)$ and $U(t)$ corresponding to $\gamma(t)$ given by (5.142) and $\gamma = 0.068$

An estimated maximum delay bound for the stability of a general linear system under the influence of the proposed observer-based output feedback law is derived. The expression of the delay bound indicates that, for a class of systems whose open loop poles are at the origin or in the open left-half plane, the proposed delay independent output feedback law would handle an arbitrarily large delay, just as the delay-dependent truncated predictor output feedback laws constructed in [63].

5.4.1 Feedback Design

We consider a linear system with a time-varying delay in the input,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(\phi(t)), \quad t \geq 0, \\ y(t) = Cx(t), \\ x(0) = x_0, \end{cases} \tag{5.143}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$, and $x_0 \in \mathbb{R}^n$ are the state, the input, the output, and the initial state, respectively. The time-varying delay function $\phi(t) : \mathbb{R}^+ \to \mathbb{R}$ is assumed to be expressed by

$$\phi(t) = t - d(t),$$

where the time-varying delay $d(t)$ is continuous and satisfies

$$0 \leq d(t) \leq D, \quad t \geq 0, \tag{5.144}$$

and $D \in \mathbb{R}^+$ represents an upper bound of the delay. We also assume that the pair (A, B) is stabilizable and the pair (A, C) is detectable.

Without loss of generality, it is assumed that the pair (A, B) are in the form of

$$A = \begin{bmatrix} A_{\mathrm{L}} & 0 \\ 0 & A_{\mathrm{R}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{L}} \\ B_{\mathrm{R}} \end{bmatrix},$$

where all eigenvalues of $A_{\mathrm{L}} \in \mathbb{R}^{n_{\mathrm{L}} \times n_{\mathrm{L}}}$ are on the open left-half plane and all eigenvalues of $A_{\mathrm{R}} \in \mathbb{R}^{n_{\mathrm{R}} \times n_{\mathrm{R}}}$ are on the closed right-half plane. The block diagonal form of matrix A indicates that the stabilizability of the pair (A, B) implies the controllability of the pair $(A_{\mathrm{R}}, B_{\mathrm{R}})$.

It was established in [63] that an output feedback law built upon a delay-dependent state observer,

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(\phi(t)) - L\big(y(t) - C\hat{x}(t)\big), \\ u(t) = F(\gamma)e^{Ad(t)}\hat{x}(t), \end{cases} \tag{5.145}$$

stabilizes the system with all open loop poles on the closed left-half plane for an arbitrarily large delay as long as the feedback gain matrix $F(\gamma)$ is designed by the use of the eigenstructure assignment based low gain feedback technique [61] and the value of the low gain parameter γ is sufficiently small. Moreover, a slightly simpler observer-based output feedback law,

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(\phi(t)) - L\left(y(t) - C\hat{x}(t)\right), \\ u(t) = F(\gamma)\hat{x}(t), \end{cases} \tag{5.146}$$

is shown to stabilize a system with all open loop poles at the origin or in the open left-half plane for an arbitrarily large delay if $F(\gamma)$ is constructed by using the low gain technique (see [61]). However, in (5.146), the complete knowledge of the time-varying delay, $d(t)$, is still required in the observer. Thus, to achieve the objective of a delay independent observer-based output feedback law, we propose the following observer-based delay independent truncated predictor output feedback law, also referred to as the delay independent output feedback TPF law,

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L\left(y(t) - C\hat{x}(t)\right), \quad t \geq 0, \\ u(t) = F(\gamma)\hat{x}(t), \\ \hat{x}(\theta) = \psi(\theta), \quad \theta \in [-D, 0], \end{cases} \tag{5.147}$$

where $\hat{x}$ is the state of the observer, $\psi(\theta)$ is the initial condition of $\hat{x}(t)$ and is assumed to be piecewise continuous on $\theta \in [-D, 0]$, $L \in \mathbb{R}^{n\times p}$ is such that all eigenvalues of $A + LC$ have a negative real part,

$$F(\gamma) = -B^{\mathrm{T}} P(\gamma),$$

and

$$P(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & P_{\mathrm{R}}(\gamma) \end{bmatrix}, \tag{5.148}$$

with $P_{\mathrm{R}}(\gamma) \in \mathbb{R}^{n_{\mathrm{R}}\times n_{\mathrm{R}}}$ being the unique positive definite solution to the parametric algebraic Riccati equation

$$A_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\gamma) + P_{\mathrm{R}}(\gamma)A_{\mathrm{R}} - P_{\mathrm{R}}(\gamma)B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\gamma) = -\gamma P_{\mathrm{R}}(\gamma), \tag{5.149}$$

for $\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}$. Note that the controllability of the pair $(A_{\mathrm{R}}, B_{\mathrm{R}})$ guarantees the existence and uniqueness of such P_{R} [122], and the detectability of the pair (A, C) ensures the existence of such L.

By defining an error signal as

$$e(t) = x(t) - \hat{x}(t), \tag{5.150}$$

we obtain the derivative of $e(t)$ from (5.143) and (5.147),

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\ &= Ae(t) + Bu(\phi(t)) - Bu(t) + LCe(t) \\ &= (A + LC)e(t) - BB^{\mathrm{T}}P\left(\hat{x}(\phi(t)) - \hat{x}(t)\right) \\ &= (A + LC)e(t) + BB^{\mathrm{T}}P(\alpha(t) - \beta(t)),\end{aligned}$$

where

$$\alpha(t) = x(t) - x(\phi(t)) \tag{5.151}$$

and

$$\beta(t) = e(t) - e(\phi(t)). \tag{5.152}$$

Then, the closed-loop system consisting of system (5.143) and the output feedback law (5.147) can be written as

$$\begin{cases}\dot{x}(t) = Ax(t) - BB^{\mathrm{T}}P(x(\phi(t)) - e(\phi(t))), \\ \dot{e}(t) = (A + LC)e(t) + BB^{\mathrm{T}}P(\alpha(t) - \beta(t)).\end{cases} \tag{5.153}$$

Decompose the state $x(t)$ as

$$x(t) = \begin{bmatrix} x_{\mathrm{L}}(t) \\ x_{\mathrm{R}}(t) \end{bmatrix},$$

we rewrite the closed-loop system as

$$\begin{cases}\dot{x}_{\mathrm{L}}(t) = A_{\mathrm{L}}x_{\mathrm{L}}(t) - B_{\mathrm{L}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(x_{\mathrm{R}}(\phi(t)) - e_{\mathrm{R}}(\phi(t))), \\ \dot{x}_{\mathrm{R}}(t) = A_{\mathrm{C}}x_{\mathrm{R}}(t) + B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) + e_{\mathrm{R}}(\phi(t))), \\ \dot{e}(t) = (A + LC)e(t) + BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t)),\end{cases} \tag{5.154}$$

where

$$A_{\mathrm{C}} = A_{\mathrm{R}} - B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}} \tag{5.155}$$

and signals $e_{\mathrm{L}}(t) \in \mathbb{R}^{n_{\mathrm{L}}}$, $e_{\mathrm{R}}(t) \in \mathbb{R}^{n_{\mathrm{R}}}$, $\alpha_{\mathrm{L}}(t) \in \mathbb{R}^{n_{\mathrm{L}}}$, $\alpha_{\mathrm{R}}(t) \in \mathbb{R}^{n_{\mathrm{R}}}$, $\beta_{\mathrm{L}}(t) \in \mathbb{R}^{n_{\mathrm{L}}}$ and $\beta_{\mathrm{R}}(t) \in \mathbb{R}^{n_{\mathrm{R}}}$ correspond to the following partitions:

$$e(t) = \begin{bmatrix} e_{\mathrm{L}}(t) \\ e_{\mathrm{R}}(t) \end{bmatrix}, \ \alpha(t) = \begin{bmatrix} \alpha_{\mathrm{L}}(t) \\ \alpha_{\mathrm{R}}(t) \end{bmatrix}, \ \beta(t) = \begin{bmatrix} \beta_{\mathrm{L}}(t) \\ \beta_{\mathrm{R}}(t) \end{bmatrix},$$

respectively.

Clearly, the asymptotic stability of

$$\begin{cases} \dot{x}_{\mathrm{R}}(t) = A_{\mathrm{C}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) + e_{\mathrm{R}}(\phi(t))), \\ \dot{e}(t) = (A + LC)e(t) + B B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t)) \end{cases} \tag{5.156}$$

guarantees that of system (5.154) because A_{L} is Hurwitz. Hence, without loss of generality, we only need to consider the asymptotic stability of system (5.156) for simplicity.

As pointed out throughout Chaps. 2–4, it is no longer without loss of generality to make the assumption on system (5.143) that all the eigenvalues of A are on the closed right-half plane, as is the case for stabilization analysis of truncated predictor based state feedback laws (see [124] or [122]).

5.4.2 Stability Analysis

Theorem 5.4 *For each $\gamma > 0$, there exists $D^* > 0$ such that, for each $D < D^*$, system (5.156) is asymptotically stable.*

Proof We adopt the following Lyapunov function for the closed-loop system (5.156):

$$V(x_{\mathrm{R}}(t), e(t)) = x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}}(\gamma) x_{\mathrm{R}}(t) + e^{\mathrm{T}}(t) R e(t),$$

where $P_{\mathrm{R}}(\gamma)$ is the unique positive definite solution to (5.149) with

$$\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}, \tag{5.157}$$

R is the unique positive definite solution to the Lyapunov equation

$$(A + LC)^{\mathrm{T}} R + R(A + LC) = -\xi I, \tag{5.158}$$

and ξ is some positive constant whose value is to be determined later. By using Young's Inequality and Lemma 2.1, and defining

$$\sigma = \mathrm{tr}\left(B^{\mathrm{T}} R^2 B\right), \tag{5.159}$$

the derivative of $V(x_{\mathrm{R}}(t), e(t))$ along the trajectory of system (5.156) can be evaluated as follows:

$$\begin{aligned} &\dot{V}(x_{\mathrm{R}}(t), e(t)) \\ &= \left(A_{\mathrm{C}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) + e_{\mathrm{R}}(\phi(t)))\right)^{\mathrm{T}} P_{\mathrm{R}} x_{\mathrm{R}}(t) + x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}}\big(A_{\mathrm{c}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} \\ &\quad \times (\alpha_{\mathrm{R}}(t) + e_{\mathrm{R}}(\phi(t)))\big) + \left((A + LC)e(t) + B B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t))\right)^{\mathrm{T}} R e(t) \\ &\quad + e^{\mathrm{T}}(t) R\big((A + LC)e(t) + B B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t))\big) \end{aligned}$$

$$
\begin{aligned}
&= x_{\mathrm{R}}^{\mathrm{T}}(t)(A_{\mathrm{C}}^{\mathrm{T}} P_{\mathrm{R}} + P_{\mathrm{R}} A_{\mathrm{c}}) x_{\mathrm{R}}(t) + 2\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} x_{\mathrm{R}}(t) + 2e_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} x_{\mathrm{R}}(t) \\
&\quad + e^{\mathrm{T}}(t)\big((A+LC)^{\mathrm{T}} R + R(A+LC)\big)e(t) + 2(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t))^{\mathrm{T}} P_{\mathrm{R}} B_{\mathrm{R}} B^{\mathrm{T}} R e(t) \\
&\le -x_{\mathrm{R}}^{\mathrm{T}}(t)\left(\gamma P_{\mathrm{R}} + P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}\right) x_{\mathrm{R}}(t) + 2\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} \alpha_{\mathrm{R}}(t) \\
&\quad + 2e_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} e_{\mathrm{R}}(\phi(t)) + x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} x_{\mathrm{R}}(t) - \xi e^{\mathrm{T}}(t) e(t) + \frac{1}{2} e^{\mathrm{T}}(t) e(t) \\
&\quad + 2(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t))^{\mathrm{T}} P_{\mathrm{R}} B_{\mathrm{R}} B^{\mathrm{T}} R^2 B B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} (\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t)) \\
&\le -\gamma x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} \alpha_{\mathrm{R}}(t) + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) e_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} e_{\mathrm{R}}(t) \\
&\quad + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)\beta_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} \beta_{\mathrm{R}}(t) - \left(\xi - \frac{1}{2}\right) e^{\mathrm{T}}(t) e(t) + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) \\
&\quad \times \sigma\left(\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} \alpha_{\mathrm{R}}(t) + \beta_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} \beta_{\mathrm{R}}(t)\right) \\
&= -\gamma x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)(1+2\sigma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} \alpha_{\mathrm{R}}(t) + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) \\
&\quad \times (1+\sigma)\beta^{\mathrm{T}}(t) P \beta(t) + e^{\mathrm{T}}(t)\left(-\left(\xi - \frac{1}{2}\right) I + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) P\right) e(t).
\end{aligned}
\tag{5.160}
$$

For each $\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}$, there exists a sufficiently large ξ such that

$$
\gamma I + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) P(\gamma) \le \left(\xi - \frac{1}{2}\right) I, \tag{5.161}
$$

$$
P(\gamma) \le R, \tag{5.162}
$$

$$
1 < \lambda_{\max}(R), \tag{5.163}
$$

$$
4(A+LC)^{\mathrm{T}} P(\gamma)(A+LC) \le 3R\left(\varpi_{\mathrm{R}}(\gamma) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2\right), \tag{5.164}
$$

where $\varpi_{\mathrm{R}}(\gamma)$ is given by

$$
\varpi_{\mathrm{R}}(\gamma) = \frac{1}{2}\big(n_{\mathrm{R}}\gamma + 2\mathrm{tr}\,(A_{\mathrm{R}})\big)\big((n_{\mathrm{R}}+1)\,\gamma + 2\mathrm{tr}\,(A_{\mathrm{R}})\big) - \gamma\mathrm{tr}\,(A_{\mathrm{R}}) - \mathrm{tr}\left(A_{\mathrm{R}}^2\right).
$$

Fix this ξ. Based on inequality (5.161), which is equivalent to

$$
-\left(\xi - \frac{1}{2}\right) I + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) P(\gamma) \le -\gamma I, \tag{5.165}
$$

we estimate $\dot{V}$ as

$$
\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \le -\gamma x_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} x_{\mathrm{R}} - \gamma e^{\mathrm{T}} e &+ 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)(1+2\sigma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} \alpha_{\mathrm{R}}(t) \\
&+ 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)(1+\sigma)\beta^{\mathrm{T}}(t) P \beta(t).
\end{aligned}
\tag{5.166}
$$

It follows from the definition of $\alpha_{\mathrm{R}}(t)$ and (5.156) that

$$\alpha_{\mathrm{R}}(t) = \int_{\phi(t)}^{t} \Big(A_{\mathrm{c}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} \alpha_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} e_{\mathrm{R}}(\phi(t))\Big)\mathrm{d}t.$$

By using Lemmas 2.1, 2.2, and Young's Inequality, we evaluate

$$\begin{aligned}
&\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} \alpha_{\mathrm{R}}(t)\\
&\leq (t-\phi(t)) \int_{\phi(t)}^{t} \Big(A_{\mathrm{C}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} \alpha_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} e_{\mathrm{R}}(\phi(t))\Big)^{\mathrm{T}}\\
&\quad \times P_{\mathrm{R}}\Big(A_{\mathrm{C}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} \alpha_{\mathrm{R}}(t) + B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} e_{\mathrm{R}}(\phi(t))\Big)\mathrm{d}t\\
&\leq 3D \int_{\phi(t)}^{t} \Big(x_{\mathrm{R}}^{\mathrm{T}}(t) A_{\mathrm{C}}^{\mathrm{T}} P_{\mathrm{R}} A_{\mathrm{C}} x_{\mathrm{R}}(t) + \alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} \alpha_{\mathrm{R}}(t)\\
&\quad + e_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} e_{\mathrm{R}}(\phi(t))\Big)\mathrm{d}t\\
&\leq 3D \int_{\phi(t)}^{t} \Big(\varpi_{\mathrm{R}}(\gamma) x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2 x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t)\\
&\quad + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2 x_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P_{\mathrm{R}} x_{\mathrm{R}}(\phi(t)) + \big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2 e_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P_{\mathrm{R}} e_{\mathrm{R}}(\phi(t))\Big)\mathrm{d}t\\
&\leq 3D \int_{\phi(t)}^{t} \Big(\Big(\varpi_{\mathrm{R}}(\gamma) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2\Big) x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t)\\
&\quad + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2 \Big(x_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P_{\mathrm{R}} x_{\mathrm{R}}(\phi(t)) + e^{\mathrm{T}}(\phi(t)) P e(\phi(t))\Big)\Big)\mathrm{d}t. \qquad (5.167)
\end{aligned}$$

Similarly, by the definition of $\beta(t)$ and (5.156), we have

$$\beta(t) = \int_{\phi(t)}^{t} \Big((A+LC)e(t) + B B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t))\Big)\mathrm{d}t.$$

Then, by using Lemmas 2.1, 2.2, Young's Inequality, and the fact that

$$B^{\mathrm{T}} P B = B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} B_{\mathrm{R}},$$

we evaluate

$$\begin{aligned}
\beta^{\mathrm{T}}(t) P \beta(t) \leq (t-\phi(t)) \int_{\phi(t)}^{t} &\Big((A+LC)e(t) + B B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}\big(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t)\big)\Big)^{\mathrm{T}}\\
&\times P\Big((A+LC)e(t) + B B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}\big(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t)\big)\Big)\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+\big(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t)\big)^{\mathrm{T}}\\
&\quad\times P_{\mathrm{R}}B_{\mathrm{R}}B^{\mathrm{T}}PBB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}\big(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t)\big)\Big)\mathrm{d}t\\
&\leq 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+2\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^{2}\\
&\quad\times\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)+2\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^{2}\beta_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\beta_{\mathrm{R}}(t)\Big)\mathrm{d}t\\
&\leq 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^{2}\\
&\quad\times\Big(x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)+x_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}x_{\mathrm{R}}(\phi(t))+e_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}e_{\mathrm{R}}(t)\\
&\quad+e_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}e_{\mathrm{R}}(\phi(t))\Big)\Big)\mathrm{d}t\\
&=2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^{2}\\
&\quad\times\Big(x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)+x_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}x_{\mathrm{R}}(\phi(t))+e^{\mathrm{T}}(t)Pe(t)\\
&\quad+e^{\mathrm{T}}(\phi(t))Pe(\phi(t))\Big)\Big)\mathrm{d}t.
\end{aligned}
\tag{5.168}
$$

We employ (5.162) to simplify (5.167) and (5.168) as

$$
\begin{aligned}
\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)\leq 3D\int_{\phi(t)}^{t}&\Big(\Big(\varpi_{\mathrm{R}}(\gamma)+2\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^{2}\Big)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)\\
&+2\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^{2}V(x_{\mathrm{R}}(\phi(t)),e(\phi(t)))\Big)\mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^{\mathrm{T}}(t)P\beta(t)\leq 2D\int_{\phi(t)}^{t}&\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^{2}\\
&\times\Big(V(x_{\mathrm{R}}(t),e(t))+V(x_{\mathrm{R}}(\phi(t)),e(\phi(t)))\Big)\Big)\mathrm{d}t,
\end{aligned}
$$

respectively.

It then follows from (5.166) that

$$
\begin{aligned}
&\dot{V}(x_{\mathrm{R}}(t), e(t)) \\
&\leq -\gamma\big(x_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}} x_{\mathrm{R}} + e^{\mathrm{T}} e\big) + 6D\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)(1+2\sigma) \\
&\quad \times \int_{\phi(t)}^{t} \Big(\Big(\varpi_{\mathrm{R}}(\gamma) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2\Big) x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t) \\
&\quad + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2 V(x_{\mathrm{R}}(\phi(t)), e(\phi(t)))\Big) \mathrm{d}t + 8D\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big) \\
&\quad \times (1+\sigma) \int_{\phi(t)}^{t} \Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}} P (A+LC) e(t) + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2 \\
&\quad \times \Big(V(x_{\mathrm{R}}(t), e(t)) + V(x_{\mathrm{R}}(\phi(t)), e(\phi(t)))\Big)\Big) \mathrm{d}t.
\end{aligned} \tag{5.169}
$$

We simplify (5.169) by using (5.162) and (5.164) as follows:

$$
\begin{aligned}
&\dot{V}(x_{\mathrm{R}}(t), e(t)) \\
&\leq -\frac{\gamma}{\lambda_{\max}(R)} V(x_{\mathrm{R}}(t), e(t)) + 6D\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)(1+2\sigma) \\
&\quad \times \int_{\phi(t)}^{t} \Big(\Big(\varpi_{\mathrm{R}}(\gamma) + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2\Big) V(x_{\mathrm{R}}(t), e(t)) \\
&\quad + 2\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2 V(x_{\mathrm{R}}(\phi(t)), e(\phi(t)))\Big) \mathrm{d}t \\
&\quad + 32D\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^3 (1+\sigma) \int_{\phi(t)}^{t} \Big(V(x_{\mathrm{R}}(t), e(t)) + V(x_{\mathrm{R}}(\phi(t)), e(\phi(t)))\Big) \mathrm{d}t.
\end{aligned}
$$

When $V(x_{\mathrm{R}}(t+\theta), e(t+\theta)) < \eta V(x_{\mathrm{R}}(t), e(t))$, $\theta \in [-2D, 0]$, for a constant $\eta > 1$, we have

$$
\begin{aligned}
&\dot{V}(x_{\mathrm{R}}(t), e(t)) \\
&\leq -\frac{\gamma}{\lambda_{\max}(R)} V(x_{\mathrm{R}}(t), e(t)) + 6D^2\eta\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)(1+2\sigma) \\
&\quad \times \Big(\varpi_{\mathrm{R}}(\gamma) + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2\Big) V(x_{\mathrm{R}}(t), e(t)) \\
&\quad + 64D^2\eta\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^3 (1+\sigma) V(x_{\mathrm{R}}(t), e(t)) \\
&= -V(x_{\mathrm{R}}(t), e(t))\Big(\frac{\gamma}{\lambda_{\max}(R)} - 6D^2\eta\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)(1+2\sigma) \\
&\quad \times \Big(\varpi_{\mathrm{R}}(\gamma) + 4\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^2\Big) - 64D^2\eta\big(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma\big)^3 (1+\sigma)\Big).
\end{aligned}
$$

It then follows that if

$$\frac{\gamma}{\lambda_{\max}(R)} > 6D^2\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)(1+2\sigma)\Big(\varpi_{\mathrm{R}}(\gamma)+4\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^2\Big) \\ +64D^2\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^3(1+\sigma), \tag{5.170}$$

then $\dot{V}(x_{\mathrm{R}}(t), e(t)) < -\rho(\gamma)V(x_{\mathrm{R}}(t), e(t))$ for some positive constant $\rho(\gamma)$, which implies that system (5.156) is asymptotically stable by the Rzumikhin Stability Theorem (Theorem 1.3). By $\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}$, the definitions of σ and $\varpi_{\mathrm{R}}(\gamma)$, and Lemma 2.3, the right-hand side of (5.170) is nonnegative. Thus, $\gamma > 0$, which naturally satisfies $\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}$, is necessary for (5.170) to hold. We further note that the right-hand side of (5.170) is a strictly increasing function of D, and it goes to zero as D goes to zero and goes to infinity as D goes to infinity. On the other hand, the left-hand side of (5.170) is a positive constant independent of D. Therefore, for each $D < D^*$, where D^* is the unique positive solution to the following nonlinear equation,

$$\frac{\gamma}{\lambda_{\max}(R)} = 6D^2\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)(1+2\sigma)\Big(\varpi_{\mathrm{R}}(\gamma)+4\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^2\Big) \\ +64D^2\big(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma\big)^3(1+\sigma), \tag{5.171}$$

system (5.156) is asymptotically stable. This completes the proof. □

Theorem 5.5 *Assume that all the eigenvalues of A_{R} are at the origin. For an arbitrarily large delay bound D, there exists $\gamma^*>0$ such that, for each $\gamma \in (0, \gamma^*)$, system (5.156) is asymptotically stable.*

Proof We adopt the same Lyapunov function (5.157) and the estimate (5.160) of its time derivative along the trajectory of system (5.156) for our stability analysis. In view of the assumption that all the eigenvalues of A_{R} are at the origin, we rewrite (5.160) as

$$\dot{V}(x_{\mathrm{R}}(t), e(t)) \le -\gamma x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t) + 2n_{\mathrm{R}}\gamma(1+2\sigma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) \\ +4n_{\mathrm{R}}\gamma(1+\sigma)\beta^{\mathrm{T}}(t)P\beta(t) + e^{\mathrm{T}}(t)\left(-\left(\beta-\frac{1}{2}\right)I + 4n_{\mathrm{R}}\gamma P\right)e(t). \tag{5.172}$$

By the low gain feedback design,

$$\lim_{t\to\infty} P(\gamma) = 0,$$

and hence there exist $\nu > 0$ and $\gamma_1 > 0$ such that

$$P(\gamma) \le \gamma\nu I, \quad \gamma \le \gamma_1. \tag{5.173}$$

Pick $\xi > \max\left\{\frac{1}{2}, \nu\right\}$. Then, there exists $\gamma_2 > 0$ such that, for each $\gamma \in (0, \gamma_2)$,

$$-\left(\xi - \frac{1}{2}\right) I + 4n_{\mathrm{R}}\gamma P(\gamma) \leq -\gamma R, \tag{5.174}$$

which implies that

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq &- \gamma V(x_{\mathrm{R}}(t), e(t)) + 2n_{\mathrm{R}}\gamma(1 + 2\sigma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) \\ &+ 4n_{\mathrm{R}}\gamma(1 + \sigma)\beta^{\mathrm{T}}(t) P\beta(t).\end{aligned} \tag{5.175}$$

To further estimate the time derivative (5.175) of V, we follow (5.167) and (5.168) to derive

$$\begin{aligned}\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) \leq 3D \int_{\phi(t)}^{t} \Big(&\Big(\varpi_{\mathrm{R}}(\gamma) + 2(n_{\mathrm{R}}\gamma)^2\Big) x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t) \\ &+ 2(n_{\mathrm{R}}\gamma)^2 \Big(x_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P_{\mathrm{R}} x_{\mathrm{R}}(\phi(t)) + e^{\mathrm{T}}(\phi(t)) P e(\phi(t))\Big)\Big) \mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\beta^{\mathrm{T}}(t) P\beta(t) \leq 2D \int_{\phi(t)}^{t} \Big(&e^{\mathrm{T}}(t)(A + LC)^{\mathrm{T}} P(A + LC)e(t) + 4(n_{\mathrm{R}}\gamma)^2 \\ &\times \Big(x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t) + x_{\mathrm{R}}^{\mathrm{T}}(\phi(t)) P_{\mathrm{R}} x_{\mathrm{R}}(\phi(t)) + e^{\mathrm{T}}(t) P e(t) \\ &+ e^{\mathrm{T}}(\phi(t)) P e(\phi(t))\Big)\Big) \mathrm{d}t,\end{aligned}$$

respectively.

There exists $\gamma_3 > 0$ such that, for each $\gamma \in (0, \gamma_3)$,

$$P(\gamma) \leq R, \tag{5.176}$$

by which $\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)$ and $\beta^{\mathrm{T}}(t) P\beta(t)$ can be respectively majorized further as

$$\begin{aligned}\alpha_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) \leq 3D \int_{\phi(t)}^{t} \Big(&\Big(\varpi_{\mathrm{R}}(\gamma) + 2(n_{\mathrm{R}}\gamma)^2\Big) x_{\mathrm{R}}^{\mathrm{T}}(t) P_{\mathrm{R}} x_{\mathrm{R}}(t) \\ &+ 2(n_{\mathrm{R}}\gamma)^2 V(x_{\mathrm{R}}(\phi(t)), e(\phi(t))\Big) \mathrm{d}t,\end{aligned}$$

and

$$\beta^{\mathrm{T}}(t)P\beta(t) \leq 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4(n_{\mathrm{R}}\gamma)^2 \times\Big(V(x_{\mathrm{R}}(t),e(t))+V(x_{\mathrm{R}}(\phi(t)),e(\phi(t)))\Big)\Big)\mathrm{d}t.$$

Inequality (5.175) can then be continued as

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t),e(t)) \leq &-\gamma V(x_{\mathrm{R}}(t),e(t))+6Dn_{\mathrm{R}}\gamma(1+2\sigma)\int_{\phi(t)}^{t}\Big(\Big(\varpi_{\mathrm{R}}(\gamma)\\ &+2(n_{\mathrm{R}}\gamma)^2\Big)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)+2(n_{\mathrm{R}}\gamma)^2V(x_{\mathrm{R}}(\phi(t)),e(\phi(t))\Big)\mathrm{d}t\\ &+8Dn_{\mathrm{R}}\gamma(1+\sigma)\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)\\ &+4(n_{\mathrm{R}}\gamma)^2\Big(V(x_{\mathrm{R}}(t),e(t))+V(x_{\mathrm{R}}(\phi(t)),e(\phi(t)))\Big)\Big)\mathrm{d}t.\end{aligned}\tag{5.177}$$

When $V(x_{\mathrm{R}}(t+\theta),e(t+\theta)) < \eta V(x_{\mathrm{R}}(t),e(t)), \theta \in [-2D,0]$, for a constant $\eta > 1$, we obtain

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t),e(t)) \leq &-\gamma V(x_{\mathrm{R}}(t),e(t))+6D^2n_{\mathrm{R}}\gamma(1+2\sigma)(\varpi_{\mathrm{R}}(\gamma)\\ &+4(n_{\mathrm{R}}\gamma)^2)V(x_{\mathrm{R}}(t),e(t))+8D^2n_{\mathrm{R}}\gamma(1+\sigma)\\ &\times\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+8(n_{\mathrm{R}}\gamma)^2V(x_{\mathrm{R}}(t),e(t))\Big),\end{aligned}\tag{5.178}$$

which, by (5.173), can be continued as,

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t),e(t)) \leq &-\gamma V(x_{\mathrm{R}}(t),e(t))+6D^2n_{\mathrm{R}}\gamma(1+2\sigma)\left(\varpi_{\mathrm{R}}(\gamma)+4(n_{\mathrm{R}}\gamma)^2\right)V(x_{\mathrm{R}}(t),e(t))\\ &+8D^2n_{\mathrm{R}}\gamma^2(1+\sigma)\left(\frac{\nu}{\lambda_{\max}(R)}+8(n_{\mathrm{R}}\gamma)^2\right)V(x_{\mathrm{R}}(t),e(t))\\ =&-\gamma\Big(1-6D^2n_{\mathrm{R}}(1+2\sigma)\left(\varpi_{\mathrm{R}}(\gamma)+4(n_{\mathrm{R}}\gamma)^2\right)-8D^2n_{\mathrm{R}}\gamma(1+\sigma)\\ &\times\left(\frac{\nu}{\lambda_{\max}(R)}+8(n_{\mathrm{R}}\gamma)^2\right)\Big)V(x_{\mathrm{R}}(t),e(t)).\end{aligned}\tag{5.179}$$

Then, there exists $\gamma_4 > 0$ such that, for each $\gamma \in (0,\gamma_4)$,

$$6D^2 n_{\mathrm{R}}(1+2\sigma)\left(\varpi_{\mathrm{R}}(\gamma)+4(n_{\mathrm{R}}\gamma)^2\right)+8D^2 n_{\mathrm{R}}\gamma(1+\sigma)\left(\frac{\nu}{\lambda_{\max}(R)}+8(n_{\mathrm{R}}\gamma)^2\right)<1.$$

Taking $\gamma^* = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ completes the proof. □

5.4.3 Numerical Examples

We consider two examples. The first example illustrates the ability of the delay independent output feedback law (5.147) in stabilizing a linear system with all open loop poles at the origin and in the presence of a large fast-varying input delay. The second example demonstrates the stabilization of an exponentially unstable system with a fast-varying input delay under the delay independent observer based feedback law (5.147).

Example 5.5 (A System with All Open Loop Poles at the Origin) Consider system (5.143) with

$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\quad C=\begin{bmatrix}1&0&0&0\end{bmatrix} \tag{5.180}$$

and initial conditions

$$x(0)=\begin{bmatrix}1&0&-1&2\end{bmatrix}^{\mathrm{T}},\quad \hat{x}(\theta)=\begin{bmatrix}0&0&0&0\end{bmatrix}^{\mathrm{T}},\quad \theta\in[-D,0]. \tag{5.181}$$

To study the nonsensitivity of the controller in handling a fast-varying delay, we consider a delay function,

$$d(t)=1+\sin(10t). \tag{5.182}$$

This delay function has an upper bound of 2 and its time derivative reaches a peak value of 10.

For linear systems whose open loop poles are all at the origin, we note from Theorem 5.5 that, in the face of an arbitrarily large bounded delay, the delay independent observer-based feedback law asymptotically stabilizes the system as long as γ is chosen small enough. With the pole placement of $\lambda(A+LC) = \{-1,-2,-3,-4\}$, the selection of a small $\gamma = 0.3$ completes the design elements of the feedback law (5.147) for simulation of the closed-loop system. Figures 5.7 and 5.8 show the evolution of the system states, the error signal, and the control input.

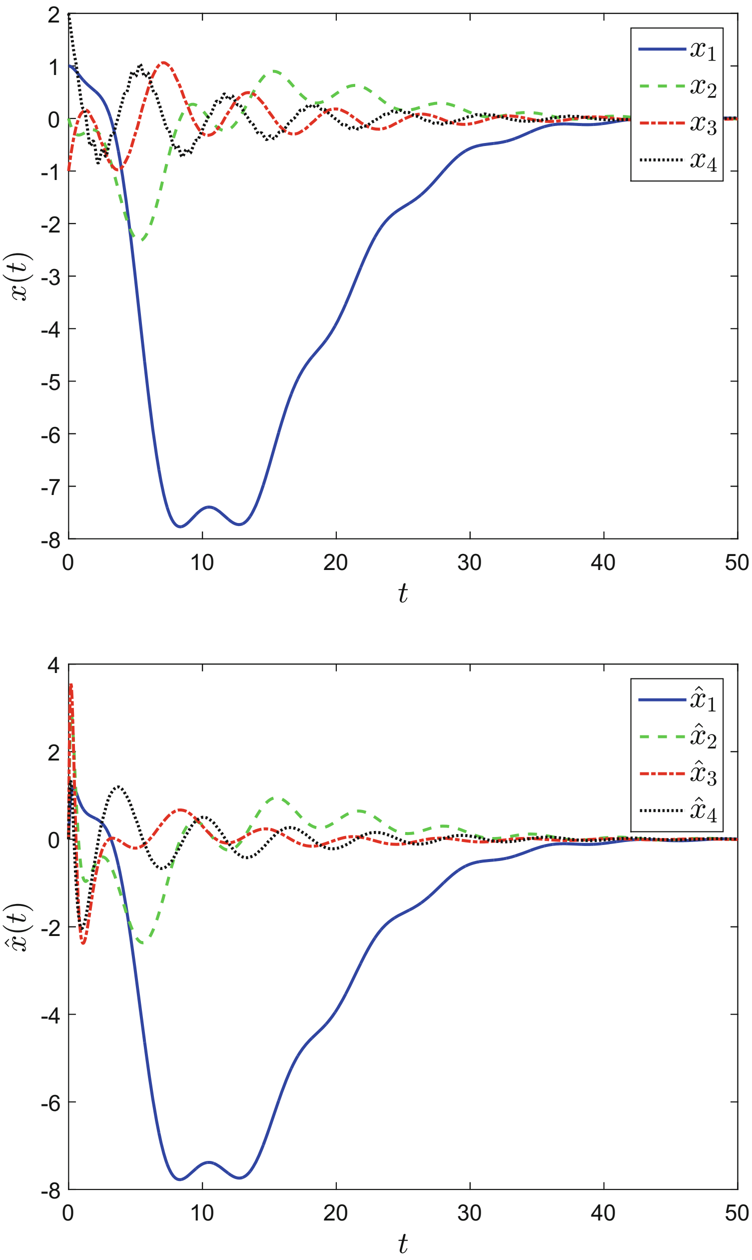


Fig. 5.7 Example 5.5: The evolution of x and $\hat{x}$ in system (5.180) under the delay independent output feedback TPF law (5.147)

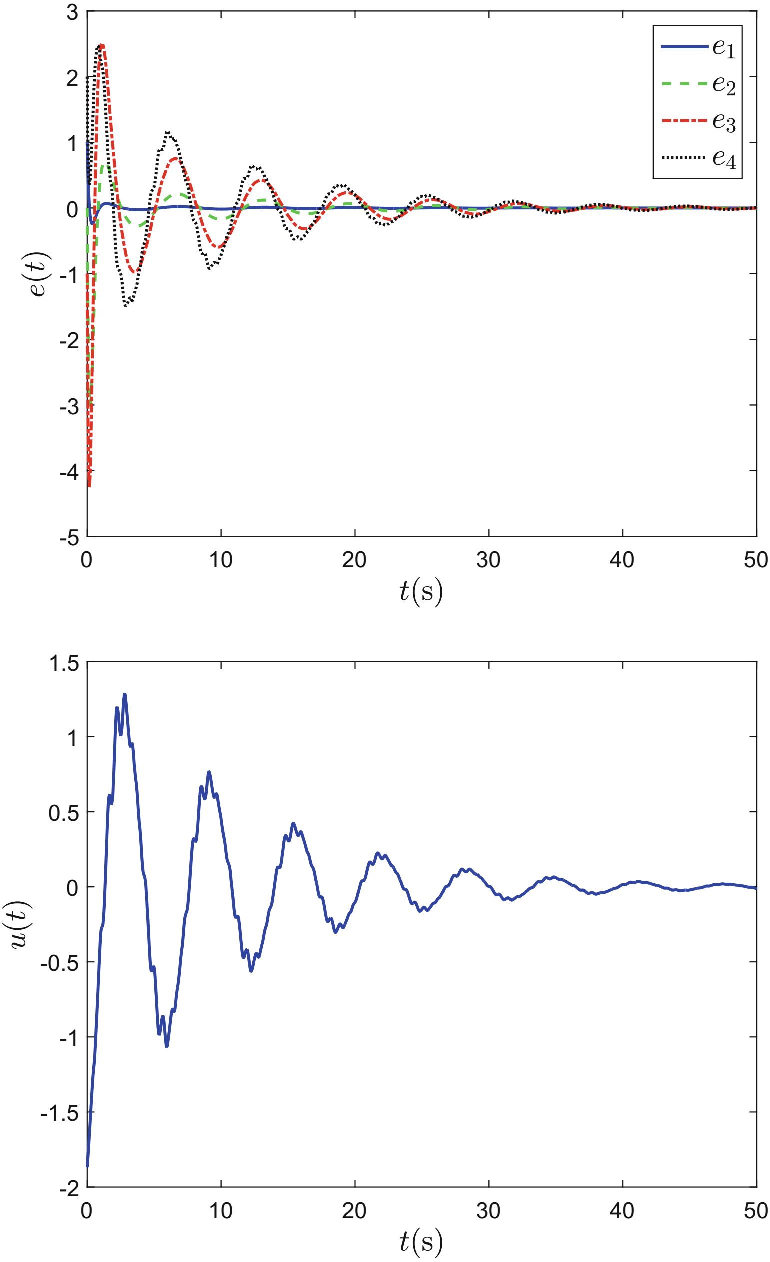


Fig. 5.8 Example 5.5: The evolution of e and u in system (5.180) under the delay independent output feedback TPF law (5.147)

Example 5.6 (An Exponentially Unstable System) Consider system (5.143) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \tag{5.183}$$

and initial conditions

$$x(0) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^{\mathrm{T}}, \quad \hat{x}(\theta) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \theta \in [-D, 0]. \tag{5.184}$$

The time-varying delay is specified to be

$$d(t) = 0.5(1 + \sin(10t)). \tag{5.185}$$

Placing the eigenvalues of $A + LC$ at $\lambda = \{-1, -2, -3\}$ and picking a small low gain parameter $\gamma = 0.1$, we run the simulation and obtain the closed-loop evolution of system (5.183) under (5.147), which, as shown in Figs. 5.9 and 5.10, verifies the stabilizing effect of controller (5.147).

5.5 Conclusions

To overcome the lack of the exact knowledge of the delay, we constructed truncated predictor based feedback laws that do not involve the delay in their expression. We formulated the stability conditions in terms of the robustness of the feedback laws to a certain amount of input delay. Specifically, for each value of the feedback parameter, a general linear system that is possibly exponentially unstable is stabilized as long as the delay is below a certain bound. The proposed feedback laws are not completely delay independent because the design of the feedback parameter requires an upper bound of the delay to be known. Examination of the stability conditions shows that as long as the delay is below some upper bound, there exist some intervals of the feedback parameter such that any feedback parameter within such intervals would stabilize the system.

5.6 Notes and References

We here provide an intuitive explanation to the robustness of the delay independent TPF laws to a certain amount of input delay. Take the delay independent TPF law (5.6) for a general linear system (5.1) for example. Following the reasoning in Sect. 4.1, we assume that all open loop poles of the system are on the closed right-half plane. Such an assumption can be made without loss of generality because the

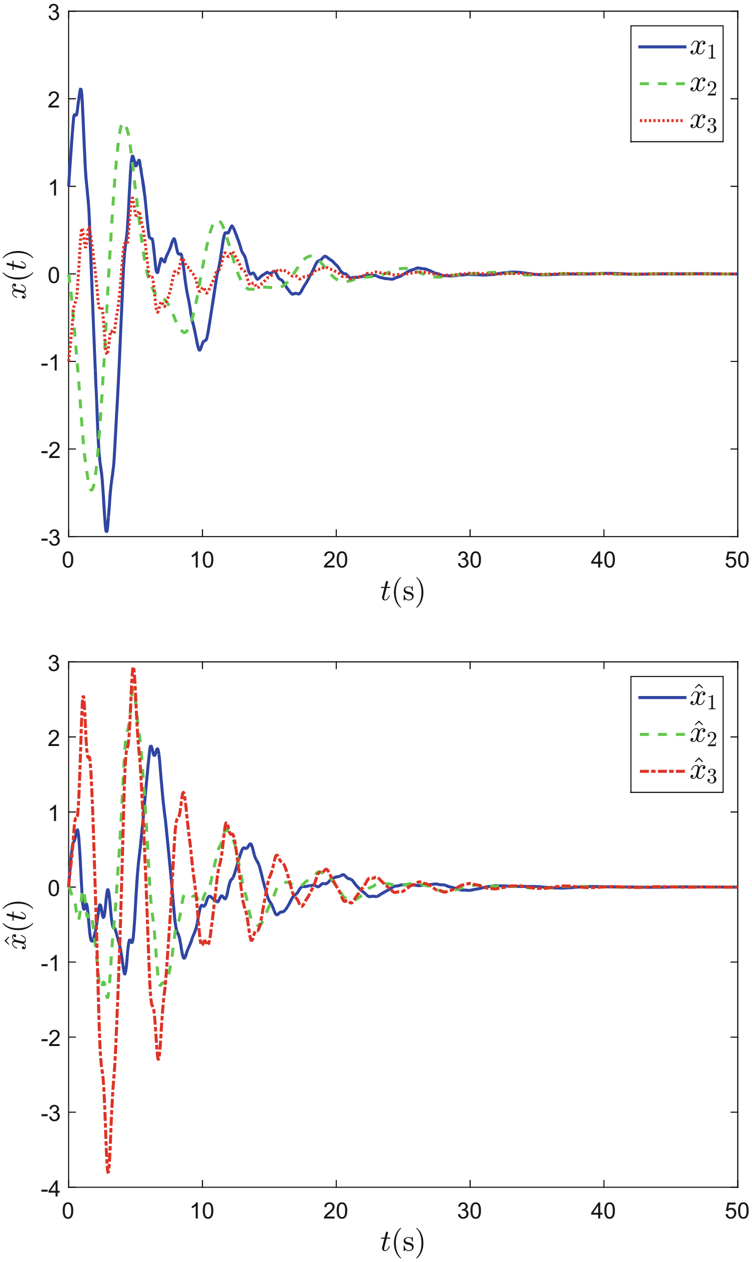


Fig. 5.9 Example 5.6: The evolution of x and $\hat{x}$ in system (5.183) under the delay independent output feedback TPF law (5.147)

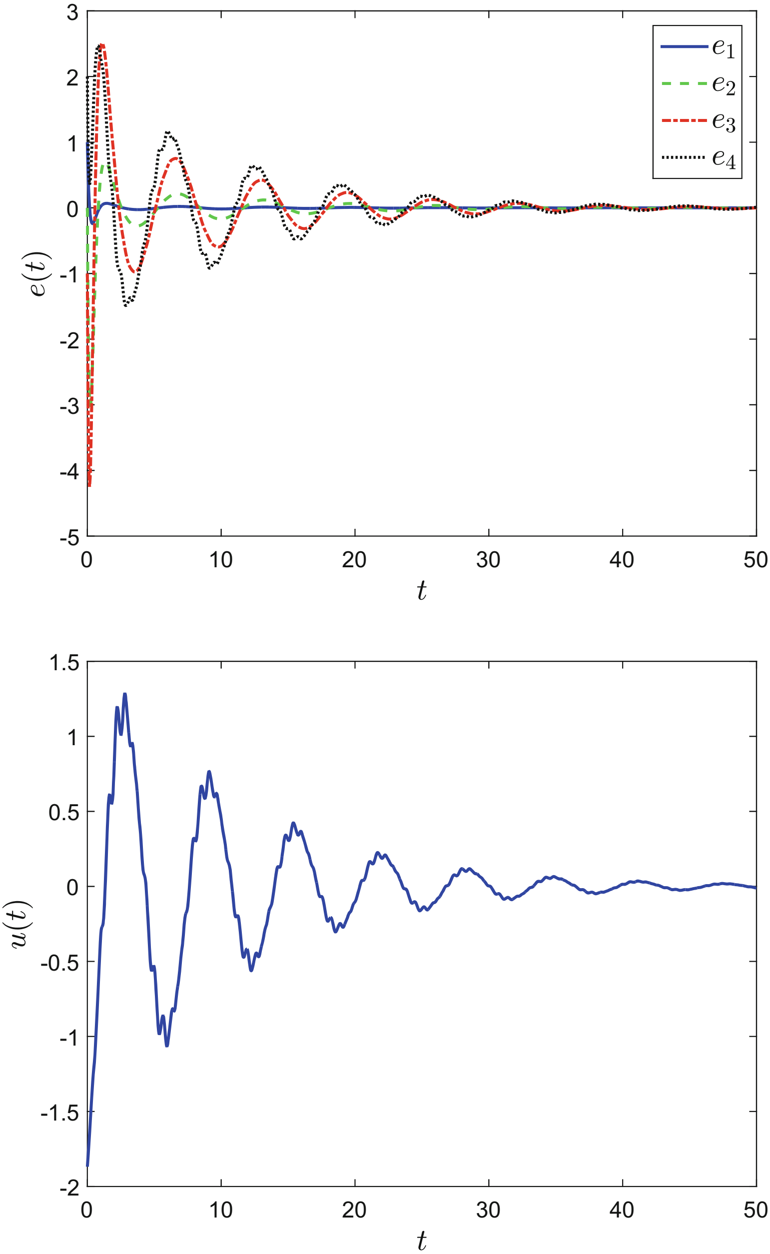


Fig. 5.10 Example 5.6: The evolution of e and u in system (5.183) under the delay independent output feedback TPF law (5.147)

stable open loop poles of the system do not affect the stabilizability of the system. Then, we derive

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t-\tau)\\ &= Ax(t) - BB^{\mathrm{T}}P(\gamma)x(t-\tau),\end{aligned} \tag{5.186}$$

where $P(\gamma)$ is parametrized by the use of the Lyapunov equation based feedback design. Let $\gamma \in \mathbb{R}^+$ be such that $\gamma > -2\min\{\mathrm{Re}(\lambda(A))\}$, which guarantees the existence and uniqueness of a positive definite solution $P(\gamma)$. The Lyapunov equation based design assigns the eigenvalues of $A - BB^{\mathrm{T}}P(\gamma)$ at locations on the complex plane that are symmetric to the eigenvalues of A with respect to $s = -\frac{\gamma}{2} < 0$. Thus, the matrix $A - BB^{\mathrm{T}}P(\gamma)$ is Hurwitz. This implies that the closed-loop system under the delay independent TPF law is asymptotically stable if $\tau = 0$. According to [67], the following system

$$\dot{x}(t) = Ax(t) + BFx(t-\tau) \tag{5.187}$$

with $A+BF$ being Hurwitz is asymptotically stable as long as the value of the delay is small enough. It then follows directly from this fact that the closed-loop system (5.186) is asymptotically stable as long as τ is sufficiently small. The robustness of the delay independent TPF law to a small input delay is thus established. However, such a robustness property is merely an implication of Theorem 5.1, which further establishes an explicit upper bound of the delay that guarantees stability. Similar improvements by establishing explicit upper bounds on the delay in the theorems throughout this chapter imply that our stability results are established quantitatively rather than qualitatively.

Sections 5.2 and 5.3 are presented by following the presentation of [101] and [105], respectively. The only difference lies in the stability analysis of the closed-loop system in the constructed example at the beginning of Sect. 5.2.2. Section 5.4.1 contains stability results that are stronger than those in [104]. In particular, the robustness of the delay independent output feedback law (5.147) is established in Theorem 5.4 for all $\gamma > 0$, whereas in [104], it is only established for γ less than a positive constant.

Chapter 6
Delay Independent Truncated Predictor Feedback for Discrete-Time Linear Systems

6.1 Introduction

As with a continuous-time linear system, the predictor state feedback law for a discrete-time linear system with input delay is the product of a feedback gain matrix with the predicted state at a future time instant ahead of the current time instant by the amount of the delay, which is the sum of the zero input solution and the zero state solution of the system. The zero state solution for a discrete-time system is a finite summation that involves past input. The zero input solution is the product of the transition matrix and the initial state. The truncated predictor feedback (TPF), which results from discarding the finite summation part of the predictor feedback law, reduces implementation complexity. The delay independent TPF law further discards the delay-dependent transition matrix in the TPF law and is thus robust to unknown delays. It was pointed out in [64] and [125] that such a delay independent TPF law stabilizes a discrete-time linear system with all its poles at $z = 1$ or inside the unit circle no matter how large the delay is.

In this chapter, we first construct an example to show that the delay independent truncated predictor state feedback law cannot compensate too large a delay if the open loop system has poles on the unit circle at $z \neq 1$. Then, a delay bound is provided for the stabilizability of a general, possibly exponentially unstable, discrete-time linear system by the delay independent truncated predictor state feedback. Parallel to the continuous-time setting, an observer based output feedback law is also constructed for a general discrete-time linear system that manages to require only an upper bound of the delay to be known. Such a relaxed requirement on the knowledge of the delay is an improvement if we consider the existing truncated predictor output feedback designs, which require the exact knowledge of the delay in their observer dynamics because of the delayed input. Similar to the continuous-time setting, we establish robustness of our proposed output feedback law to a certain amount of time-varying delay. We further reveal the low gain nature of the delay independent output feedback in the stabilization of a discrete-time linear system

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5_6

whose open loop poles are all at $z = 1$ or inside the unit circle. Specifically, the output feedback law with a sufficiently small feedback parameter compensates a time-varying input delay with an arbitrarily large upper bound.

6.2 Delay Independent Truncated Predictor State Feedback Design

We consider asymptotic stabilization of a discrete-time linear system with time-varying delay in the input,

$$\begin{cases} x(k+1) = Ax(k) + Bu(\phi(k)), k \geq 0, \\ \quad x(k) = \psi(k), \; k \in I[-R, 0], \end{cases} \tag{6.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the input, respectively, and the delay function $\phi(k)$ takes the standard form of

$$\phi(k) = k - r(k).$$

The time-varying delay $r(k) : \mathbb{N} \to \mathbb{N}$ satisfies $r(k) \leq R$, where $R \in \mathbb{N}$ is a delay bound. It is assumed that (A, B) is stabilizable.

Recall the predictor feedback law (4.62) for system (6.1)

$$u(k) = FA^{\phi^{-1}(k)-k}x(k) + F \sum_{s=2k-\phi^{-1}(k)}^{k-1} A^{k-s-1}Bu(s), \tag{6.2}$$

which involves the zero input solution and the zero state solution of the system, and a feedback gain F that is designed such that $A + BF$ is Schur stable. Then, the closed-loop system

$$x(k+1) = (A + BF)x(k), \tag{6.3}$$

is exponentially stable. The following truncated predictor state feedback law, also referred to as the state feedback TPF law,

$$u(k) = FA^{\phi^{-1}(k)-k}x(k), \tag{6.4}$$

discards the finite summation part of the predictor state feedback law (6.2) and is easy to implement. Furthermore, the delay independent state feedback TPF law

$$u(k) = Fx(k) \tag{6.5}$$

further discards the delay-dependent transition matrix $A^{\phi^{-1}(k)-k}$ in the truncated predictor state feedback law (6.4). This simplification results in the robustness of (6.5) with respect to time-varying delays and even unknown delays. It was shown in [64] that the state feedback TPF law (6.4) stabilizes system (6.1) with all open loop poles on or inside the unite circle no matter how large the delay r is. Furthermore, the delay independent state feedback TPF law (6.5) stabilizes system (6.1) with all open loop poles at $z = 1$ or inside the unit circle, again, no matter how large the delay r is. The eigenstructure assignment low gain feedback design technique was employed in [64] to construct the feedback gain matrix F, while in [123] and [125], the Lyapunov equation based low gain feedback design was employed.

In this section, we first construct an example to show that the delay independent state feedback TPF law (6.5) cannot compensate an arbitrarily large delay if the open loop system has poles on the unit circle at $z \neq 1$. Then, an admissible delay bound under the feedback law (6.5) is given for a general linear system that may be exponentially unstable. The results extend their continuous-time counterparts in Sect. 5.2.

6.2.1 Preliminaries

The feedback gain matrix of the delay independent state feedback TPF law (6.5) for system (6.1) with a controllable pair (A, B) is constructed following the Lyapunov equation based feedback design (see Sect. 4.3.1),

$$\begin{aligned} u(k) &= F(\gamma)x(k) \\ &= -\left(I + B^{\mathrm{T}}P(\gamma)B\right)^{-1} B^{\mathrm{T}}P(\gamma)Ax(k), \end{aligned} \tag{6.6}$$

where $P(\gamma)$ is the unique positive definite solution to the discrete-time parametric algebraic Riccati equation,

$$A^{\mathrm{T}}P(\gamma)A - P(\gamma) - A^{\mathrm{T}}P(\gamma)B\left(I + B^{\mathrm{T}}P(\gamma)B\right)^{-1} B^{\mathrm{T}}P(\gamma)A = -\gamma P(\gamma), \tag{6.7}$$

with

$$\gamma \in \left(1 - |\lambda(A)|^2_{\min}, 1\right). \tag{6.8}$$

Note that (6.8) is necessary and sufficient for the existence and uniqueness of $P(\gamma)$. In the case where all eigenvalues of A are at the origin or in the open left-half plane, the delay is allowed to be arbitrarily large but bounded, and the value of the parameter γ is required to approach zero as the bound on the delay increases to infinity. As a result, the parametric algebraic Riccati equation (6.7) is a low gain feedback design and the feedback parameter is referred to as the low gain parameter.

We recall the following lemma from [99].

Lemma 6.1 *Let* (A, B) *be controllable with* A *nonsingular. For each* $\gamma \in (1 - |\lambda(A)|^2_{min}, 1)$*, the unique positive definite solution* $P(\gamma)$ *to the parametric algebraic Riccati equation (6.7) satisfies*

$$F^{\mathrm{T}} B^{\mathrm{T}} P B F \leq \varrho P, \tag{6.9}$$

$$F^{\mathrm{T}} (I + B^{\mathrm{T}} P B) F \leq \vartheta P, \tag{6.10}$$

$$(A_{\mathrm{C}} - I)^{\mathrm{T}} P (A_{\mathrm{C}} - I) \leq \varpi P, \tag{6.11}$$

where

$$\begin{aligned}
F &= -\left(I + B^{\mathrm{T}} P B\right)^{-1} B^{\mathrm{T}} P A, \\
A_{\mathrm{C}} &= A + B F, \\
\varrho &= \frac{\left(\det^2(A) - (1-\gamma)^n\right)^2}{(1-\gamma)^{2n-1}}, \\
\vartheta &= \frac{\det^2(A) - (1-\gamma)^n}{(1-\gamma)^{n-1}}, \\
\varpi &= \frac{\det^2(A)}{(1-\gamma)^n} - 1 - n\gamma + 2(n - \mathrm{tr}(A)) + \frac{\left(\det^2(A) - (1-\gamma)^n\right)^2}{\det^2(A)(1-\gamma)^{n-1}}.
\end{aligned}$$

We next establish an inequality relating the determinant and the trace of a class of matrices.

Lemma 6.2 *If* $A \in \mathbb{R}^{n \times n}$ *has all its eigenvalues on or outside the unit circle, then*

$$4\det^4(A) - 6\det^2(A) + \det^{-2}(A) + 1 + 2n - 2\mathrm{tr}(A) \geq 0.$$

Proof We use polar coordinates to denote the eigenvalues of A as

$$\lambda_i = r_i \mathrm{e}^{j\theta_i},$$

where $r_i \geq 1$ and $\theta_i \in (-\pi, \pi]$, for $i \in I[1, n]$. The determinant and the trace of A are respectively expressed as

$$\begin{aligned}
\det(A) &= \prod_{i=1}^{n} r_i \mathrm{e}^{j\theta_i} \\
&= \prod_{i=1}^{n} r_i,
\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}(A) &= \sum_{i=1}^{n} r_i \mathrm{e}^{j\theta_i} \\ &= \sum_{i=1}^{n} r_i \cos\theta_i ,\end{aligned}$$

where we have used the fact that eigenvalues appear in conjugate pairs. We then have

$$\begin{aligned}&4\det^4(A) - 6\det^2(A) + \det^{-2}(A) + 1 + 2n - 2\operatorname{tr}(A) \\ &\geq 4\prod_{i=1}^{n} r_i^4 - 6\prod_{i=1}^{n} r_i^2 + \prod_{i=1}^{n} r_i^{-2} + 1 + 2n - 2\sum_{i=1}^{n} r_i .\end{aligned}$$

Thus, it suffices to show that

$$4\prod_{i=1}^{n} r_i^4 - 6\prod_{i=1}^{n} r_i^2 + \prod_{i=1}^{n} r_i^{-2} + 1 + 2n - 2\sum_{i=1}^{n} r_i \geq 0.$$

Define a multivariate function

$$g(r_1, r_2, \ldots, r_n) = 4\prod_{i=1}^{n} r_i^4 - 6\prod_{i=1}^{n} r_i^2 + \prod_{i=1}^{n} r_i^{-2} + 1 + 2n - 2\sum_{i=1}^{n} r_i ,$$

where $r_i \geq 1,\ i \in I[1,n]$. We observe that $g(r_1, r_2, \ldots, r_n)$ remains unchanged under any permutation of $r_1, r_2, \ldots r_n$. We consider the partial derivative of $g(r_1, r_2, \ldots, r_n)$ with respect to r_n,

$$\frac{\partial g(r_1, r_2, \ldots, r_n)}{\partial r_n} = 16\left(\prod_{i=1}^{n-1} r_i^4\right) r_n^3 - 12\left(\prod_{i=1}^{n-1} r_i^2\right) r_n - 2\left(\prod_{i=1}^{n-1} r_i^{-2}\right) r_n^{-3} - 2. \tag{6.12}$$

In addition,

$$\begin{aligned}\frac{\partial^2 g(r_1, r_2, \ldots, r_n)}{\partial r_n^2} &= 12\prod_{i=1}^{n-1} r_i^2 \left(4\left(\prod_{i=1}^{n-1} r_i^2\right) r_n^2 - 1\right) + 6\prod_{i=1}^{n-1} r_i^{-2} r_n^{-4} \\ &> 0,\end{aligned} \tag{6.13}$$

from which it follows that $\frac{\partial}{\partial r_n} g(r_1, r_2, \ldots, r_n)$ is strictly increasing with respect to r_n. Recalling the fact that $r_i \geq 1$, $i \in I[1, n]$, we get

$$\begin{aligned}\frac{\partial g(r_1, r_2, \ldots, r_n)}{\partial r_n} \Big|_{r_n=1} &= 4 \prod_{i=1}^{n-1} r_i^2 \left(4 \prod_{i=1}^{n-1} r_i^2 - 3 \right) - 2 \prod_{i=1}^{n-1} r_i^{-2} - 2 \\ &\geq 0,\end{aligned}$$

where the equality sign holds if and only if $r_i = 1$ for all $i \in I[1, n-1]$. Therefore,

$$\frac{\partial}{\partial r_n} g(r_1, r_2, \ldots, r_n) \geq 0,$$

and $g(r_1, r_2, \ldots, r_n)$ is nondecreasing with respect to r_n. Since $g(r_1, r_2, \ldots, r_n)$ remains unchanged under any permutation of $r_1, r_2, \ldots$ and r_n, we have

$$\frac{\partial}{\partial r_i} g(r_1, r_2, \ldots, r_n) \geq 0, \quad i \in I[1, n].$$

It follows from (6.12) that

$$\begin{aligned}g(r_1, r_2, \cdots, r_n) &\geq g(1, 1, \cdots, 1) \\ &= 0.\end{aligned}$$

This completes the proof. □

Without loss of generality, we assume that the stabilizable pair (A, B) in system (6.1) takes the following decomposed form:

$$A = \begin{bmatrix} A_{\text{I}} & 0 \\ 0 & A_{\text{O}} \end{bmatrix}, \ B = \begin{bmatrix} B_{\text{I}} \\ B_{\text{O}} \end{bmatrix},$$

where $A_{\text{I}} \in \mathbb{R}^{n_{\text{I}} \times n_{\text{I}}}$ has all its eigenvalues inside the unit circle and $A_{\text{O}} \in \mathbb{R}^{n_{\text{O}} \times n_{\text{O}}}$ has all its eigenvalues on or outside the unit circle. Then, system (6.1) can be written as,

$$\begin{cases} x_{\text{I}}(k+1) = A_{\text{I}} x_{\text{I}}(k) + B_{\text{I}} u(\phi(k)), \\ x_{\text{O}}(k+1) = A_{\text{O}} x_{\text{O}}(k) + B_{\text{O}} u(\phi(k)), \\ x(k) = \psi(k), \quad k \in I[-R, 0], \end{cases}$$

where

$$x(k) = \left[x_{\text{I}}^{\text{T}}(k) \ x_{\text{O}}^{\text{T}}(k) \right]^{\text{T}}$$

and

$$\psi(k) = \left[\phi_{\mathrm{I}}^{\mathrm{T}}(k)\ \phi_{\mathrm{O}}^{\mathrm{T}}(k)\right]^{\mathrm{T}}.$$

The stabilizability of (A, B) implies that $(A_{\mathrm{O}}, B_{\mathrm{O}})$ is controllable. Note that the asymptotic stabilizability of the second subsystem implies that of the whole system because all eigenvalues of A_{I} are inside the unit circle. Thus, without loss of generality, we assume that the pair (A, B) is controllable with all eigenvalues of A on or outside the unit circle.

6.2.2 An Admissible Delay Bound

We first construct an example to show that if system (6.1) has open loop poles on the unit circle at $z \neq 1$, the delay independent state feedback TPF law (6.6) cannot stabilize the system when the delay is large enough.

Consider system (6.1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and a constant delay $r(k) = 2$, $k \in \mathbb{N}$. It can be readily verified that (A, B) is controllable with eigenvalues of A at $z = \pm j$. By solving the parametric algebraic Riccati equation (6.7), we obtain the unique positive definite solution

$$P(\gamma) = \begin{bmatrix} \frac{1}{1-\gamma} - 1 + \gamma & 0 \\ 0 & \frac{1}{(1-\gamma)^2} - 1 \end{bmatrix},$$

where $\gamma \in (0, 1)$ in view of (6.8). Hence, the delay independent state feedback TPF law (6.6) is given by

$$u(k) = -\left[\gamma^2 - 2\gamma\ \ 0\right] x(k),$$

under which the closed-loop system is given by

$$\begin{cases} x_1(k+1) = x_2(k), \\ x_2(k+1) = -x_1(k) + \left(2\gamma - \gamma^2\right) x_1(k-2), \end{cases}$$

The characteristic equation of the closed-loop system is

$$z^4 + z^2 + \gamma^2 - 2\gamma = 0,$$

whose four roots are

$$z_{1,2} = \pm \left(\frac{\sqrt{1 + 4\gamma(2-\gamma)} - 1}{2} \right)^{\frac{1}{2}},$$

$$z_{3,4} = \pm j \left(\frac{\sqrt{1 + 4\gamma(2-\gamma)} + 1}{2} \right)^{\frac{1}{2}}.$$

Note that z_3 and z_4 both lie outside the unit circle for each $\gamma \in (0, 1)$. Thus, the delay independent TPF law (6.6) fails to stabilize the system.

We next establish a delay bound for a general linear system to be stabilizable by the delay independent state feedback TPF law (6.6).

Theorem 6.1 *Let all the eigenvalues of A in the system (6.1) be on or outside the unit circle. If*

$$R < \sqrt{\frac{\gamma}{2\frac{\det^2(A) - (1-\gamma)^n}{(1-\gamma)^{n-1}} \left(\varpi + 4\frac{\left(\det^2(A) - (1-\gamma)^n\right)^2}{(1-\gamma)^{2n-1}} \right)}}, \quad \gamma \in (0, 1), \tag{6.14}$$

where ϖ is as defined in Lemma 6.1, then the delay independent state feedback TPF law (6.6) asymptotically stabilizes system (6.1).

Proof The closed-loop system under the delay independent TPF law (6.6) is given by

$$x(k+1) = A_{\rm c}x(k) - BFx_{\rm d}(k), \tag{6.15}$$

where $A_{\rm c}$ and F are defined as in Lemma 6.1 and

$$x_{\rm d}(k) = x(k) - x(\phi(k)).$$

Consider a Lyapunov function

$$V(x(k)) = x^{\rm T}(k)P(\gamma)x(k).$$

In view of the assumption on system (6.1) that all the eigenvalues of A are on or outside the unit circle, we let $\gamma \in (0, 1)$ satisfy (6.8). By Young's inequality, Lemma 6.1 and (6.15), we evaluate the forward difference of $V(x(k))$ along the trajectory of the closed-loop system (6.15) as follows:

$$\begin{aligned} \Delta V(x(k)) &= V(x(k)) - V(x(k-1)) \\ &= x^{\rm T}(k)\left(-\gamma P - F^{\rm T}F\right)x(k) - 2x^{\rm T}(k)A_{\rm c}^{\rm T}PBFx_{\rm d}(k) \end{aligned}$$

$$
\begin{aligned}
&+x_{\mathrm{d}}^{\mathrm{T}}(k)F^{\mathrm{T}}B^{\mathrm{T}}PBFx_{\mathrm{d}}(k)\\
&\leq -\gamma x^{\mathrm{T}}(k)Px(k)+\frac{\det^2(A)-(1-\gamma)^n}{(1-\gamma)^{n-1}}x_{\mathrm{d}}^{\mathrm{T}}(k)Px_{\mathrm{d}}(k).
\end{aligned}
$$

By the fact that

$$
x_{\mathrm{d}}(k)=\sum_{l=\phi(k)+1}^{k}\big((A_{\mathrm{c}}-I)x(l-1)-BFx_{\mathrm{d}}(l-1)\big),
$$

and Lemmas 3.6 and 6.1, $x_{\mathrm{d}}^{\mathrm{T}}(k)Px_{\mathrm{d}}(k)$ satisfies

$$
\begin{aligned}
x_{\mathrm{d}}^{\mathrm{T}}(k)Px_{\mathrm{d}}(k)&\leq 2r(k)\sum_{l=\phi(k)+1}^{k}\Bigg(\varpi x^{\mathrm{T}}(l-1)Px(l-1)\\
&\quad+\frac{\left(\det^2(A)-(1-\gamma)^n\right)^2}{(1-\gamma)^{2n-1}}x_{\mathrm{d}}^{\mathrm{T}}(l-1)Px_{\mathrm{d}}(l-1)\Bigg)\\
&\leq 2r(k)\sum_{l=\phi(k)+1}^{k}\Bigg(\varpi x^{\mathrm{T}}(l-1)Px(l-1)\\
&\quad+2\frac{\left(\det^2(A)-(1-\gamma)^n\right)^2}{(1-\gamma)^{2n-1}}\big(x^{\mathrm{T}}(l-1)Px(l-1)\\
&\quad+x^{\mathrm{T}}(l-1-r(l-1))Px(l-1-r(l-1))\big)\Bigg).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\Delta V(x(k))&\leq -\gamma x^{\mathrm{T}}(k)Px(k)+2\frac{\det^2(A)-(1-\gamma)^n}{(1-\gamma)^{n-1}}r(k)\\
&\times\sum_{l=\phi(k)+1}^{k}\Bigg(\varpi x^{\mathrm{T}}(l-1)Px(l-1)+2\frac{\left(\det^2(A)-(1-\gamma)^n\right)^2}{(1-\gamma)^{2n-1}}\\
&\times\big(x^{\mathrm{T}}(l-1)Px(l-1)+x^{\mathrm{T}}(l-1-r(l-1))Px(l-1-r(l-1))\big)\Bigg).
\end{aligned}
$$

For $V(x(k+\kappa))<\eta V(x(k))$, $\kappa\in I[-2R,0]$, with some positive constant $\eta>1$, we have

$$
\begin{aligned}
\Delta V(x(k))\leq -\Bigg(&\gamma-2R^2\eta\frac{\det^2(A)-(1-\gamma)^n}{(1-\gamma)^{n-1}}\\
&\left(\varpi+4\frac{\left(\det^2(A)-(1-\gamma)^n\right)^2}{(1-\gamma)^{2n-1}}\right)\Bigg)V(x(k)). \qquad (6.16)
\end{aligned}
$$

In view of (6.14), it follows from (6.16) that

$$\Delta V(x(k)) \leq -\epsilon V(x(k))$$

for some $\epsilon > 0$. The asymptotic stability of the closed-loop system then follows from the Razumikhin Stability Theorem (Theorem 1.4). It remains to verify that the denominator inside the square root on the right-hand side of (6.14) is positive. This can be trivially verified by the use of Lemma 6.2 and the facts that $\det^2(A) \geq 1$ and $\gamma \in (0, 1)$. This completes the proof. □

Corollary 6.1 *Let A in system (6.1) have all its eigenvalues at $z = 1$. Given an arbitrarily large delay bound R, the delay independent state feedback TPF law (6.6) with each $\gamma \in (0, \gamma^*)$ asymptotically stabilizes the system, where γ^* is the smallest positive solution to the following nonlinear equation,*

$$\frac{1-(1-\gamma)^n}{\gamma(1-\gamma)^{n-1}}\left(\frac{(1-(1-\gamma)^n)^2}{(1-\gamma)^{n-1}}\left(\frac{4}{(1-\gamma)^n}+1\right)\right.$$
$$\left.+\frac{1}{(1-\gamma)^n}-1-n\gamma\right)=\frac{1}{2R^2}. \tag{6.17}$$

Proof Since A has all its eigenvalues at $z = 1$, $\mathrm{tr}(A) = n$ and $\det^2(A) = 1$. Thus, (6.14) is equivalent to (6.17) with the equality sign replaced by " $<$ ". Note that the left-hand side of (6.17) approaches zero as $\gamma \to 0$ and goes to infinity as $\gamma \to 1$, which implies that (6.17) has a smallest positive solution γ^*. This completes the proof. □

Remark 6.1 The result of Corollary 6.1 is consistent with a result in [64] or [125], where it is established that a discrete-time linear system with all its open loop poles at $z = 1$ or inside the unit circle can be stabilized by the delay independent state feedback TPF law (6.6) no matter how large the delay is.

6.2.3 Numerical Examples

Example 6.1 (A System with All Poles at $z = 1$) Consider system (6.1) with

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \tag{6.18}$$

Clearly, the open loop system has all its poles at $z = 1$ and is controllable. Corollary 6.1 suggests that the delay independent state feedback TPF law (6.6) is capable of asymptotically stabilizing the system with an arbitrarily large delay bound. Consider a fast-varying delay

$$r(k) = \left\lceil 1 + \sin^2(100k) \right\rceil, \quad k \in \mathbb{N},$$

with an upper bound $R = 2$, where $\lceil \cdot \rceil$ denotes the ceiling function. Let $\gamma = 0.1$. For the initial condition

$$x(k) = \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \quad k \in I[0, R],$$

the state response of the closed-loop system and the control input are shown in Fig. 6.1, from which the stability of the system is clear. □

Example 6.2 (A System with All Poles on the Unit Circle) Consider system (6.1) with

$$A = \begin{bmatrix} 0.987 & 0.1607 & 0 \\ -0.1607 & 0.987 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{6.19}$$

The open loop system has a pair of poles on the unit circle at $z = 0.987 \pm j0.1607$ and another pole at $z = 1$. It can be easily verified that (A, B) is controllable and $\mathrm{tr}(A) = 2.974$. Consider

$$r(k) = \frac{1 + (-1)^k}{2}, \quad k \in \mathbb{N},$$

which is 1 when k is even and is 0 otherwise. Let $\gamma = 0.0078$. Let the initial condition be

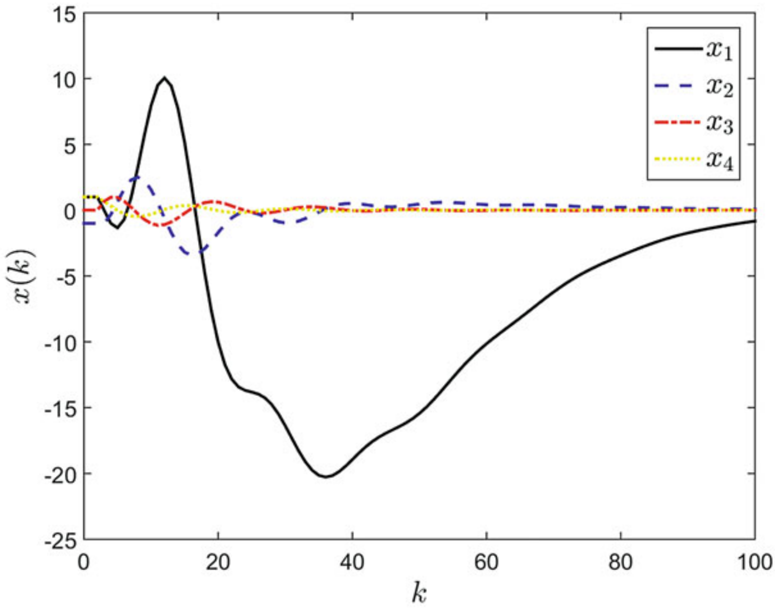
$$x(k) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^{\mathrm{T}}, \quad k \in I[0, 1].$$

The state response and the input signal are shown in Fig. 6.2, which demonstrates the convergence of the closed-loop signals. □

Example 6.3 (An Exponentially Unstable System) Consider system (6.1) with

$$A = \begin{bmatrix} 1 & 0.09 & 0 \\ -0.09 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{6.20}$$

The open loop system has a pair of exponentially unstable poles at $z = 1 \pm j0.09$ and another pole at $z = 1$. It can be readily verified that (A, B) is controllable, $\det(A) = 1.0163$ and $\mathrm{tr}(A) = 3$. We consider



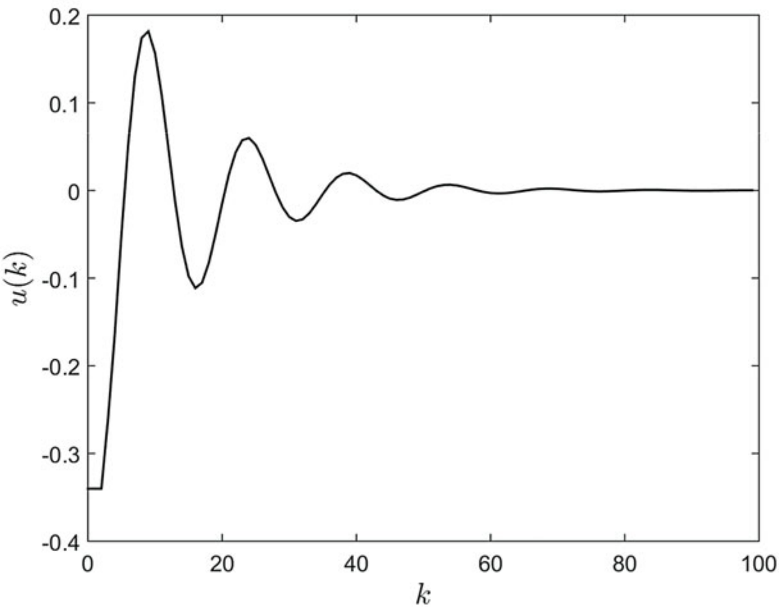


Fig. 6.1 Example 6.1: State response and control input under the delay independent state feedback TPF law (6.6) with $\gamma = 0.1$

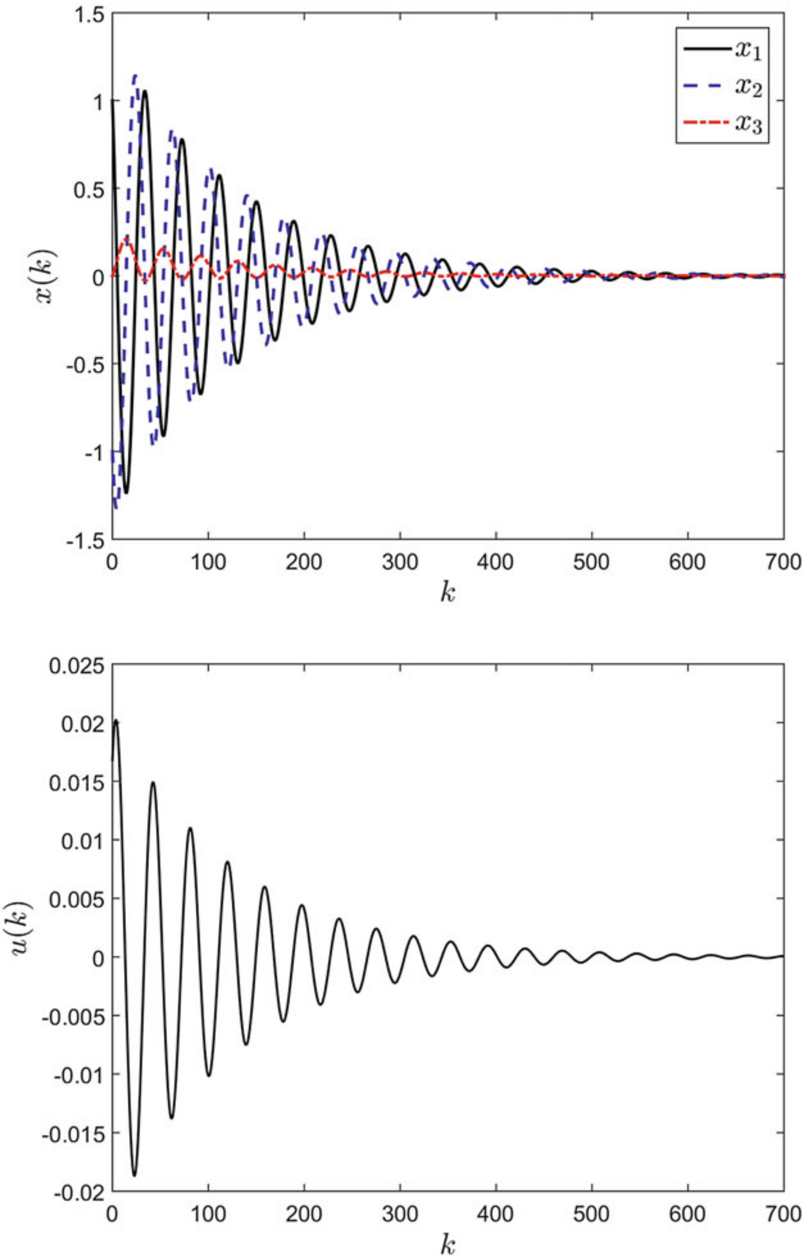


Fig. 6.2 Example 6.2: State response and control input under the delay independent state feedback TPF law (6.6) with $\gamma = 0.0078$

$$r(k) = \frac{1 + (-1)^k}{2}, \quad k \in \mathbb{N}.$$

Let the initial condition be

$$x(k) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^{\mathrm{T}}, \ k \in I[0, 1].$$

Simulation is run with $\gamma = 0.016$, which satisfies (6.14). The convergent state response and the corresponding input signal are shown in Fig. 6.3. □

6.3 Delay Independent Truncated Predictor Output Feedback Design

In this section, we consider stabilization of a general discrete-time linear system with a time-varying input delay by delay independent truncated predictor output feedback. Such feedback can be considered parallel to the delay independent truncated predictor output feedback in the continuous-time setting. With only the knowledge of an upper bound of the time-varying delay, it is established that a general discrete-time linear system is stabilized by an observer based output feedback law as long as the delay does not exceed a certain upper bound. An explicit upper bound is provided that guarantees stabilizability. Moreover, given a time-varying delay with an arbitrarily large upper bound, our output feedback law with a sufficiently small feedback parameter stabilizes a discrete-time linear system with all its open loop poles at $z = 1$ or inside the unit circle. Existing results in the literature on the output feedback design require the exact knowledge of the delay (see [64] and [125]) to achieve stabilization. Our design relaxes the requirement to the extent that only an upper bound of the delay is required to be known. This is why we refer to our feedback as the delay independent truncated predictor output feedback.

6.3.1 Feedback Design

Consider output feedback stabilization for the following discrete-time linear system with a time-varying input delay,

$$\begin{cases} x(k+1) = Ax(k) + Bu(\phi(k)), & k \geq 0, \\ y(k) = Cx(k), \end{cases} \tag{6.21}$$

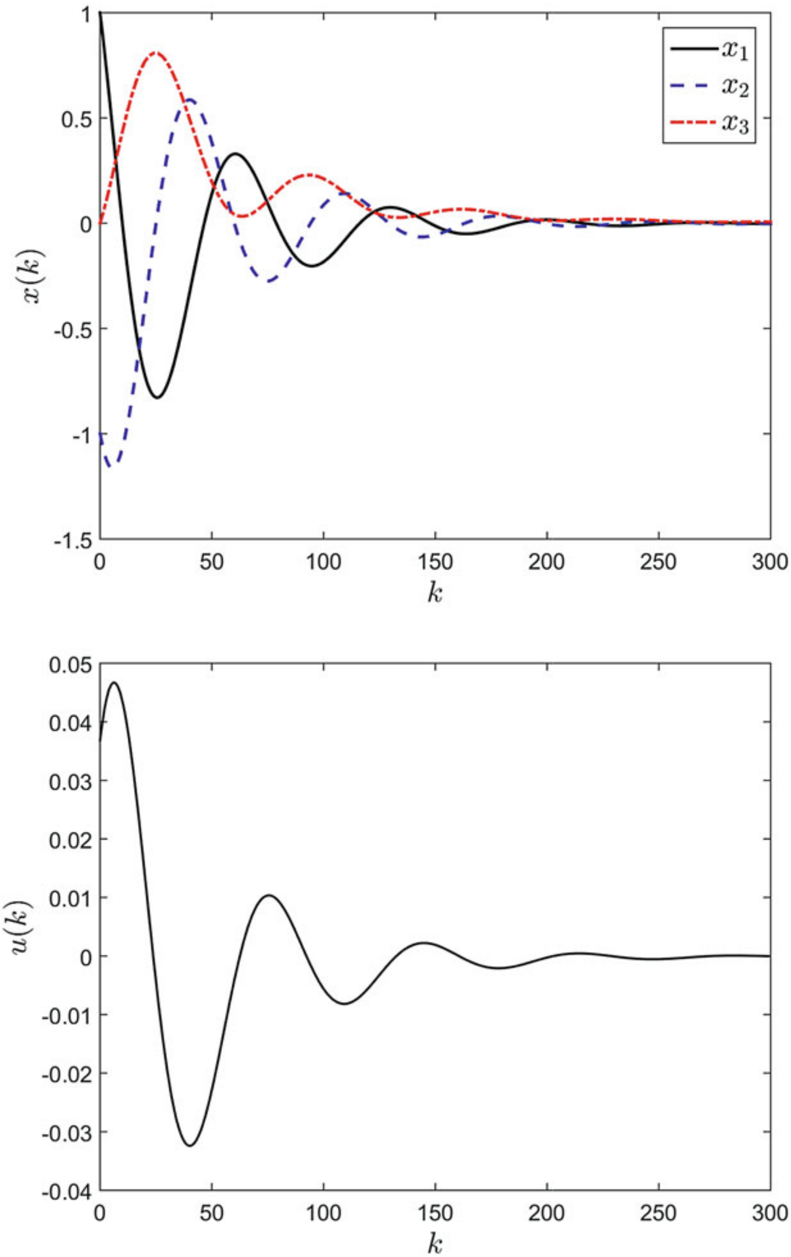


Fig. 6.3 Example 6.3: State response and control input under the delay independent state feedback TPF law (6.6) with $\gamma = 0.016$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ correspond to the state, the input, and the output, respectively, the triple (A, B, C) are both stabilizable and detectable, and $\phi(k)$ takes the standard form of

$$\phi(k) = k - r(k).$$

The time-varying delay $r(k) : \mathbb{N} \to \mathbb{N}$ is assumed to be bounded from above by $R \in \mathbb{N}$, that is,

$$r(k) \leq R, \quad k \in \mathbb{N}.$$

Our feedback design only requires that R be known. It does not require to know how the delay $r(k)$ vary with respect to time.

It was established in [64] that the following observer based output feedback law

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k-r) - L\left(y(k) - C\hat{x}(k)\right), \\ \quad u(k) = F(\gamma)\hat{x}(k), \end{cases} \tag{6.22}$$

where $F(\gamma)$ is constructed by using the eigenstructure assignment based low gain feedback design and L is designed such that $A + LC$ is Schur stable, stabilizes system (6.21) with all open loop poles at $z = 1$ or inside the unit circle no matter how large the constant delay r is. The compensation of an arbitrarily large constant delay is achieved by tuning the low gain parameter γ to be small enough. Alternatively, the use of the Lyapunov equation based low gain feedback design to parametrize $F(\gamma)$ in (6.22) also achieves stabilization (see [125]).

However, the dynamics of the observer of the output feedback law (6.22) involves the delayed input $u(k - r)$. Clearly, the implementation of such a feedback law requires the exact value of the constant delay r. Our goal in this section is to design a delay independent truncated predictor output feedback law that does not involve the delay in its expression even when the delay is time-varying.

Before proceeding to our output feedback design, we assume, without loss of generality, that the triple (A, B, C) in system (6.21) are in the following form:

$$A = \begin{bmatrix} A_{\mathrm{I}} & 0 \\ 0 & A_{\mathrm{o}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{I}} \\ B_{\mathrm{o}} \end{bmatrix}, \quad C = \begin{bmatrix} C_{\mathrm{I}} & C_{\mathrm{o}} \end{bmatrix}, \tag{6.23}$$

where all the eigenvalues of $A_{\mathrm{I}} \in \mathbb{R}^{n_{\mathrm{s}} \times n_{\mathrm{s}}}$ are inside the unit circle and all the eigenvalues of $A_{\mathrm{o}} \in \mathbb{R}^{n_{\mathrm{o}} \times n_{\mathrm{o}}}$ are on or outside the unit circle, and $n_{\mathrm{o}} + n_{\mathrm{s}} = n$.

We propose the following delay independent output feedback law, also referred to as the delay independent output feedback TPF law, for system (6.21),

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) - L\left(y(k) - C\hat{x}(k)\right), \\ \quad u(k) = F(\gamma)\hat{x}(k), \end{cases} \tag{6.24}$$

where L is such that $A + LC$ is Schur stable and the Lyapunov equation based feedback design is adopted to construct $F(\gamma)$ as

$$F(\gamma) = -\left(I + B^{\mathrm{T}} P(\gamma) B\right)^{-1} B^{\mathrm{T}} P(\gamma) A, \tag{6.25}$$

where

$$P(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & P_{\mathrm{o}}(\gamma) \end{bmatrix}, \tag{6.26}$$

and $P_{\mathrm{o}}(\gamma)$ is the unique positive definite solution to the discrete-time parametric Riccati equation,

$$A_{\mathrm{o}}^{\mathrm{T}} P_{\mathrm{o}}(\gamma) A_{\mathrm{o}} - P_{\mathrm{o}}(\gamma) - A_{\mathrm{o}}^{\mathrm{T}} P_{\mathrm{o}}(\gamma) B_{\mathrm{o}} \left(I + B_{\mathrm{o}}^{\mathrm{T}} P_{\mathrm{o}}(\gamma) B_{\mathrm{o}}\right)^{-1} B_{\mathrm{o}}^{\mathrm{T}} P_{\mathrm{o}}(\gamma) A_{\mathrm{o}} = -\gamma P_{\mathrm{o}}(\gamma),$$
$$\gamma \in \left(1 - \min\left\{|\lambda(A_{\mathrm{o}})|^2\right\}, 1\right). \tag{6.27}$$

The existence of such an L is guaranteed by the assumption that the pair (A, C) are detectable. Note that the stabilizability of the pair (A, B) implies the controllability of the pair $(A_{\mathrm{o}}, B_{\mathrm{o}})$, which in turn implies the existence and uniqueness of the solution $P_{\mathrm{o}}(\gamma)$ to the discrete-time parametric Riccati equation the Lyapunov equation (6.27).

Comparing the proposed feedback law (6.24) to the existing feedback law (6.22), we see that our design replaces the delay input $u(k - r)$ with its undelayed version $u(k)$, thus eliminating the requirement for the exact knowledge of the delay. We can see that such an improvement is not trivial if we consider the error signal between the state of the system and the observer state,

$$e(k) = x(k) - \hat{x}(k). \tag{6.28}$$

Under the delay-dependent design (6.22), the error signal exponentially decreases to zero because

$$e(k + 1) = (A + LC)e(k),$$

which naturally leads to the stabilization of system (6.21) in view of the delay independent truncated predictor state feedback design (see Sect. 6.2). However, under our design (6.24), the error signal is governed by

$$e(k + 1) = (A + LC)e(k) + BF(\gamma)\left(\hat{x}(\phi(k)) - \hat{x}(k)\right), \tag{6.29}$$

whose right-hand side contains a difference between $x(\phi(k))$ and $x(k)$. As will become clear in Sect. 6.3.2 next, the presence of this difference drastically complicates the stability analysis.

6.3.2 Stability Analysis

The closed-loop system consisting of system (6.21) and the delay independent truncated predictor output feedback law (6.24) can be written in terms of the state (x, e) as,

$$\begin{cases} x(k+1) = Ax(k) + BF(\gamma)(x(\phi(k)) - e(\phi(k))), \\ e(k+1) = (A+LC)e(k) + BF(\gamma)(x(\phi(k)) - e(\phi(k)) - (x(k) - e(k))). \end{cases} \tag{6.30}$$

Let

$$\begin{cases} \xi(k) = x(k) - x(\phi(k)), \\ \zeta(k) = e(k) - e(\phi(k)). \end{cases} \tag{6.31}$$

Then, the closed-loop system is simplified to

$$\begin{cases} x(k+1) = Ax(k) + BF(\gamma)(x(\phi(k)) - e(\phi(k))), \\ e(k+1) = (A+LC)e(k) + BF(\gamma)(\zeta(k) - \xi(k)). \end{cases} \tag{6.32}$$

Noting the structure of (A, B, C) shown in (6.23), we have

$$F(\gamma) = \begin{bmatrix} 0 & F_{\mathrm{O}}(\gamma) \end{bmatrix},$$

where $F_{\mathrm{O}}(\gamma)$ is defined as

$$F_{\mathrm{O}}(\gamma) = -\left(I + B_{\mathrm{O}}^{\mathrm{T}} P_{\mathrm{O}}(\gamma) B_{\mathrm{O}}\right)^{-1} B_{\mathrm{O}}^{\mathrm{T}} P_{\mathrm{O}} A_{\mathrm{O}}. \tag{6.33}$$

Hence, the closed-loop system can be decomposed as

$$\begin{cases} x_{\mathrm{I}}(k+1) = A_{\mathrm{I}} x_{\mathrm{I}}(k) + B_{\mathrm{I}} F_{\mathrm{O}}(\gamma)(x_{\mathrm{O}}(\phi(k)) - e_{\mathrm{O}}(\phi(k))), \\ x_{\mathrm{O}}(k+1) = A_{\mathrm{O}} x_{\mathrm{O}}(k) + B_{\mathrm{O}} F_{\mathrm{O}}(\gamma)(x_{\mathrm{O}}(\phi(k)) - e_{\mathrm{O}}(\phi(k))), \\ e(k+1) = (A+LC)e(k) + BF_{\mathrm{O}}(\gamma)(\zeta_{\mathrm{O}}(k) - \alpha_{\mathrm{O}}(k)), \end{cases} \tag{6.34}$$

where $x_{\mathrm{I}}(k)$ and $x_{\mathrm{O}}(k)$ are respectively the first n_{I} and the last n_{O} states of $x(k)$, $e_{\mathrm{O}}(k)$, $\xi_{\mathrm{O}}(k)$ and $\zeta_{\mathrm{O}}(k)$ are the last n_{O} states of $e(k)$, $\xi(k)$ and $\zeta(k)$, respectively. The second and the third subsystems describe the dynamics of $x_{\mathrm{O}}(k)$ and $e(k)$, whereas the first subsystem takes $x_{\mathrm{O}}(k)$ and $e_{\mathrm{O}}(k)$ as external signals. Moreover, the internal stability of the first subsystem is determined by the Schur stability of matrix A_{I}. Therefore, the asymptotic stability of the last two subsystems implies that of the whole closed-loop system, and it suffices to consider only the stability of

$$\begin{cases} x_{\mathrm{O}}(k+1) = A_{\mathrm{O}} x_{\mathrm{O}}(k) + B_{\mathrm{O}} F_{\mathrm{O}}(\gamma)(x_{\mathrm{O}}(\phi(k)) - e_{\mathrm{O}}(\phi(k))), \\ e(k+1) = (A+LC)e(k) + BF_{\mathrm{O}}(\gamma)(\zeta_{\mathrm{O}}(k) - \xi_{\mathrm{O}}(k)). \end{cases} \tag{6.35}$$

Theorem 6.2 *Given any $\gamma \in (0, 1)$, there exists $R^\star \in \mathbb{N}$ such that, for each $R < R^\star$, the closed-loop system (6.35) is asymptotically stable.*

Proof Choose a Lyapunov function

$$V(x_o(k), e(k)) = x_o^T(k) P_o(\gamma) x_o(k) + e^T(k) Q e(k),$$

where Q is the unique positive definite solution to the following discrete-time Lyapunov equation,

$$(A + LC)^T Q (A + LC) - Q = -\rho I, \tag{6.36}$$

and ρ is a positive constant whose value is to be determined later.

By defining

$$A_{oc}(\gamma) = A_o + B_o F_o(\gamma),$$

we rewrite the closed-loop system (6.35) as follows:

$$\begin{cases} x_o(k+1) = A_{oc} x_o(k) - B_o F_o(\gamma)(\xi_o(k) + e_o(\phi(k))), \\ e(k+1) = (A + LC)e(k) + B F_o(\gamma)(\zeta_o(k) - \xi_o(k)). \end{cases} \tag{6.37}$$

The forward difference of $V(x_o(k), e(k))$ can be computed as

$$\begin{aligned} \Delta V(x_o(k), e(k)) &= V(x_o(k+1), e(k+1)) - V(x_o(k), e(k)) \\ &= x_o^T(k) \left(-\gamma P_o - F_o^T F_o\right) x_o(k) - \rho e^T(k) e(k) \\ &\quad + 2(\xi_o(k) + e_o(\phi(k)))^T F_o^T F_o x_o(k) \\ &\quad + (\xi_o(k) + e_o(\phi(k)))^T F_o^T B_o^T P_o B_o F_o (\xi_o(k) + e_o(\phi(k))) \\ &\quad + 2(\zeta_o(k) - \xi_o(k))^T F_o^T B Q (A + LC) e(k) \\ &\quad + (\zeta_o(k) - \xi_o(k))^T F_o^T B^T Q B F_o (\zeta_o(k) - \xi_o(k)), \end{aligned}$$

where we have used the discrete-time Riccati equation (6.27) and the discrete-time Lyapunov equation (6.36). Let

$$\sigma = \mathrm{tr}\left(B^T Q^2 B\right). \tag{6.38}$$

The linear relation between Q and ρ in (6.36) suggests that there exists a sufficiently large ρ_1 such that for each $\rho \geq \rho_1$,

$$Q \leq Q^2,$$

and thus

$$\begin{aligned}\Delta V\,(x_{\rm o}(k), e(k)) \leq\; & x_{\rm o}^{\rm T}(k)(-\gamma P_{\rm o} - F_{\rm o}^{\rm T} F_{\rm o})x_{\rm o}(k) - \rho e^{\rm T}(k)e(k)\\ & + 2\xi_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} F_{\rm o} x_{\rm o}(k) + 2e_{\rm o}^{\rm T}(\phi(k)) F_{\rm o}^{\rm T} F_{\rm o} x_{\rm o}(k)\\ & + (\xi_{\rm o}(k) + e_{\rm o}(\phi(k)))^{\rm T} F_{\rm o}^{\rm T} B_{\rm o}^{\rm T} P_{\rm o} B_{\rm o} F_{\rm o}(\xi_{\rm o}(k) + e_{\rm o}(\phi(k)))\\ & + 2(\zeta_{\rm o}(k) - \xi_{\rm o}(k))^{\rm T} F_{\rm o}^{\rm T} B Q (A + LC) e(k)\\ & + \sigma(\zeta_{\rm o}(k) - \xi_{\rm o}(k))^{\rm T} F_{\rm o}^{\rm T} F_{\rm o}(\zeta_{\rm o}(k) - \xi_{\rm o}(k)).\end{aligned}$$

By Young's Inequality and the definitions of $\xi(k)$ and $\zeta(k)$, we obtain

$$\begin{aligned}\Delta V(x_{\rm o}(k), e(k)) \leq\; & -\gamma x_{\rm o}^{\rm T}(k) P_{\rm o} x_{\rm o}(k) - \rho e^{\rm T}(k)e(k)\\ & + 2\xi_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} F_{\rm o} \xi_{\rm o}(k) + 4e_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} F_{\rm o} e_{\rm o}(k)\\ & + 4\zeta_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} F_{\rm o} \zeta_{\rm o}(k) + 2\xi_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} B_{\rm o}^{\rm T} P_{\rm o} B_{\rm o} F_{\rm o} \xi_{\rm o}(k)\\ & + 2e_{\rm o}^{\rm T}(\phi(k)) F_{\rm o}^{\rm T} B_{\rm o}^{\rm T} P_{\rm o} B_{\rm o} F_{\rm o} e_{\rm o}(\phi(k))\\ & + \sigma(\zeta_{\rm o}(k) - \xi_{\rm o}(k))^{\rm T} F_{\rm o}^{\rm T} F_{\rm o}(\zeta_{\rm o}(k) - \xi_{\rm o}(k))\\ & + e^{\rm T}(k)(A + LC)^{\rm T}(A + LC)e(k)\\ & + 2\sigma\zeta_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} F_{\rm o} \zeta_{\rm o}(k) + 2\sigma\xi_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} F_{\rm o} \xi_{\rm o}(k). \qquad (6.39)\end{aligned}$$

Grouping similar terms in the above inequality yields

$$\begin{aligned}\Delta V(x_{\rm o}(k), e(k)) \leq\; & -\gamma x_{\rm o}^{\rm T}(k) P_{\rm o} x_{\rm o}(k) + 2\xi_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T}\left(I + B_{\rm o}^{\rm T} P_{\rm o} B_{\rm o}\right) F_{\rm o}\xi_{\rm o}(k)\\ & + 4\sigma\xi_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} F_{\rm o}\xi_{\rm o}(k) + 4\zeta_{\rm o}^{\rm T}(k)\left(I + B_{\rm o}^{\rm T} P_{\rm o} B_{\rm o}\right) F_{\rm o}\zeta_{\rm o}(k)\\ & + 4\sigma\zeta_{\rm o}^{\rm T}(k) F_{\rm o}^{\rm T} F_{\rm o}\zeta_{\rm o}(k) + 4e_{\rm o}^{\rm T}(k)\left(I + B_{\rm o}^{\rm T} P_{\rm o} B_{\rm o}\right) F_{\rm o} e_{\rm o}(k)\\ & + e^{\rm T}(k)\left(-\rho I + (A + LC)^{\rm T}(A + LC)\right) e(k),\end{aligned}$$

to which we apply Lemma 6.1 to obtain

$$\begin{aligned}\Delta V(x_{\rm o}(k), e(k)) \leq\; & -\gamma x_{\rm o}^{\rm T}(k) P_{\rm o} x_{\rm o}(k) + (2 + 4\sigma)\vartheta_{\rm o}\xi_{\rm o}^{\rm T}(k) P_{\rm o}\xi_{\rm o}(k)\\ & + (4 + 4\sigma)\vartheta_{\rm o}\zeta^{\rm T}(k) P\zeta(k)\\ & + e^{\rm T}(k)\left(-\rho I + (A + LC)^{\rm T}(A + LC) + 4\vartheta_{\rm o} P\right) e(k), \qquad (6.40)\end{aligned}$$

with

$$\vartheta_{\rm o} = \frac{\det^2(A_{\rm o}) - (1-\gamma)^{n_{\rm o}}}{(1-\gamma)^{n_{\rm o}-1}}.$$

Notice that there exists $\rho_2 \geq \rho_1$ such that, for each $\rho \geq \rho_2$,

$$\gamma I + (A + LC)^{\mathrm{T}}(A + LC) + 4\vartheta_{\mathrm{o}} P \leq \rho I,$$

based on which inequality (6.40) can be continued as

$$\begin{aligned}
\Delta V(x_{\mathrm{o}}(k), e(k)) &\leq -\gamma(x_{\mathrm{o}}^{\mathrm{T}}(k) P_{\mathrm{o}} x_{\mathrm{o}}(k) + e^{\mathrm{T}}(k) e(k)) + (2 + 4\sigma)\vartheta_{\mathrm{o}} \xi_{\mathrm{o}}^{\mathrm{T}}(k) P_{\mathrm{o}} \xi_{\mathrm{o}}(k) \\
&\quad + (4 + 4\sigma)\vartheta_{\mathrm{o}} \zeta^{\mathrm{T}}(k) P \zeta(k) \\
&\leq -\frac{\gamma}{\lambda_{\max}(Q)} V(x_{\mathrm{o}}(k), e(k)) + (2 + 4\sigma)\vartheta_{\mathrm{o}} \xi_{\mathrm{o}}^{\mathrm{T}}(k) P_{\mathrm{o}} \xi_{\mathrm{o}}(k) \\
&\quad + (4 + 4\sigma)\vartheta_{\mathrm{o}} \zeta^{\mathrm{T}}(k) P \zeta(k).
\end{aligned}$$

Next, we focus our attention on the terms $\xi_{\mathrm{o}}^{\mathrm{T}}(k) P_{\mathrm{o}} \xi_{\mathrm{o}}(k)$ and $\zeta(k) P \zeta(k)$. Using

$$x_{\mathrm{o}}(l + 1) = A_{\mathrm{oc}} x_{\mathrm{o}}(l) - B_{\mathrm{o}} F_{\mathrm{o}}(\xi_{\mathrm{o}}(l) + e_{\mathrm{o}}(\phi(l)))$$

for $l \in I[k - r(k), k - 1]$ repeatedly, we obtain

$$\xi_{\mathrm{o}}(k) = \sum_{l=1}^{r(k)} (A_{\mathrm{oc}} - I) x_{\mathrm{o}}(k - l) - B_{\mathrm{o}} F_{\mathrm{o}}(\xi_{\mathrm{o}}(k - l) + e_{\mathrm{o}}(\phi(k - l))),$$

which implies that

$$\begin{aligned}
\xi_{\mathrm{o}}^{\mathrm{T}}(k) P_{\mathrm{o}} \xi_{\mathrm{o}}(k) &\leq r(k) \sum_{l=1}^{r(k)} \Big((A_{\mathrm{oc}} - I) x_{\mathrm{o}}(k - l) - B_{\mathrm{o}} F_{\mathrm{o}}(\xi_{\mathrm{o}}(k - l) + e_{\mathrm{o}}(\phi(k - l)))\Big)^{\mathrm{T}} P_{\mathrm{o}} \\
&\quad \times \Big((A_{\mathrm{oc}} - I) x_{\mathrm{o}}(k - l) - B_{\mathrm{o}} F_{\mathrm{o}}(\xi_{\mathrm{o}}(k - l) + e_{\mathrm{o}}(\phi(k - l)))\Big) \\
&\leq 2r(k) \sum_{l=1}^{r(k)} x_{\mathrm{o}}^{\mathrm{T}}(k - l)(A_{\mathrm{oc}} - I)^{\mathrm{T}} P_{\mathrm{o}} (A_{\mathrm{oc}} - I) x_{\mathrm{o}}(k - l) \\
&\quad + \Big(\xi_{\mathrm{o}}(k - l) + e_{\mathrm{o}}(\phi(k - l))\Big)^{\mathrm{T}} F_{\mathrm{o}}^{\mathrm{T}} B_{\mathrm{o}} P_{\mathrm{o}} B_{\mathrm{o}} F_{\mathrm{o}}(\xi_{\mathrm{o}}(k - l) + e_{\mathrm{o}}(\phi(k - l))\Big) \\
&\leq 2R \sum_{l=1}^{R} \varpi_{\mathrm{o}} x_{\mathrm{o}}^{\mathrm{T}}(k - l) P_{\mathrm{o}} x_{\mathrm{o}}(k - l) + \varrho_{\mathrm{o}}(\xi_{\mathrm{o}}(k - l) + e_{\mathrm{o}}(\phi(k - l)))^{\mathrm{T}} \\
&\quad \times P_{\mathrm{o}}(\xi_{\mathrm{o}}(k - l) + e_{\mathrm{o}}(\phi(k - l))) \\
&\leq 2R \sum_{l=1}^{R} \varpi_{\mathrm{o}} x_{\mathrm{o}}^{\mathrm{T}}(k - l) P_{\mathrm{o}} x_{\mathrm{o}}(k - l) + 4\varrho_{\mathrm{o}} x_{\mathrm{o}}^{\mathrm{T}}(k - l) P_{\mathrm{o}} x_{\mathrm{o}}(k - l) \\
&\quad + 4\varrho_{\mathrm{o}} x_{\mathrm{o}}^{\mathrm{T}}(\phi(k - l)) P_{\mathrm{o}} x_{\mathrm{o}}(\phi(k - l)) + 2\lambda_{\mathrm{o}} e^{\mathrm{T}}(\phi(k - l)) P e(\phi(k - l)),
\end{aligned}$$

where we have used Young's Inequality, Lemmas 3.6 and 6.1, and have defined

$$\varpi_{\mathrm{o}} = \frac{\det^2(A_{\mathrm{o}})}{(1-\gamma)^{n_{\mathrm{o}}}} - 1 - n_{\mathrm{o}}\gamma + 2\big(n_{\mathrm{o}} - \mathrm{tr}(A_{\mathrm{o}})\big) + \frac{\big(\det^2(A_{\mathrm{o}}) - (1-\gamma)^{n_{\mathrm{o}}}\big)^2}{\det^2(A_{\mathrm{o}})(1-\gamma)^{n_{\mathrm{o}}-1}},$$

$$\varrho_{\mathrm{o}} = \frac{\big(\det^2(A_{\mathrm{o}}) - (1-\gamma)^{n_{\mathrm{o}}}\big)^2}{(1-\gamma)^{2n_{\mathrm{o}}-1}}.$$

Again, by the linear relation between Q and ρ, note that there exists $\rho_3 \geq \rho_2$ such that, for each $\rho \geq \rho_3$,

$$P(\gamma) \leq Q,$$

which implies that

$$\begin{aligned}\xi_{\mathrm{o}}^{\mathrm{T}}(k)P_{\mathrm{o}}\xi_{\mathrm{o}}(k) \leq 2R\sum_{l=1}^{R}&(\varpi_{\mathrm{o}} + 4\varrho_{\mathrm{o}})x_{\mathrm{o}}^{\mathrm{T}}(k-l)P_{\mathrm{o}}x_{\mathrm{o}}(k-l)\\ &+4\varrho_{\mathrm{o}}x_{\mathrm{o}}^{\mathrm{T}}(\phi(k-l))P_{\mathrm{o}}x_{\mathrm{o}}(\phi(k-l))\\ &+2\lambda_{\mathrm{o}}e^{\mathrm{T}}(\phi(k-l))Qe(\phi(k-l)).\end{aligned} \tag{6.41}$$

On the other hand, by using

$$e(l+1) = (A+LC)e(l) + BF_{\mathrm{o}}(\zeta_{\mathrm{o}}(l) - \xi_{\mathrm{o}}(l))$$

for $l \in I[k-r(k), k-1]$ repeatedly, we obtain

$$\zeta(k) = \sum_{l=1}^{r(k)}(A+LC-I)e(k-l) + BF_{\mathrm{o}}(\zeta_{\mathrm{o}}(k-l) - \xi_{\mathrm{o}}(k-l)),$$

based on which we have

$$\begin{aligned}\zeta^{\mathrm{T}}(k)P\zeta(k) &\leq 2R\sum_{l=1}^{r(k)}e^{\mathrm{T}}(k-l)(A+LC-I)^{\mathrm{T}}P(A+LC-I)e(k-l)\\ &\quad+(\zeta_{\mathrm{o}}(k-l)-\xi_{\mathrm{o}}(k-l))^{\mathrm{T}}F_{\mathrm{o}}^{\mathrm{T}}B^{\mathrm{T}}P_{\mathrm{o}}BF_{\mathrm{o}}(\zeta_{\mathrm{o}}(k-l)-\xi_{\mathrm{o}}(k-l))\\ &\leq 2R\sum_{l=1}^{R}e^{\mathrm{T}}(k-l)(A+LC-I)^{\mathrm{T}}P(A+LC-I)e(k-l)\\ &\quad+2\varrho_{\mathrm{o}}\zeta_{\mathrm{o}}^{\mathrm{T}}(k-l)P_{\mathrm{o}}\zeta_{\mathrm{o}}(k-l) + 2\varrho_{\mathrm{o}}\xi_{\mathrm{o}}^{\mathrm{T}}(k-l)P_{\mathrm{o}}\xi_{\mathrm{o}}(k-l)\end{aligned}$$

$$\begin{aligned}
&\leq 2R\sum_{l=1}^{R} e^{\mathrm{T}}(k-l)(A+LC-I)^{\mathrm{T}}P(A+LC-I)e(k-l)\\
&\quad +4\varrho_{\mathrm{o}}e^{\mathrm{T}}(k-l)Qe(k-l)+4\varrho_{\mathrm{o}}e^{\mathrm{T}}(\phi(k-l))Qe(\phi(k-l))\\
&\quad +4\varrho_{\mathrm{o}}x_{\mathrm{o}}^{\mathrm{T}}(k-l)P_{\mathrm{o}}x_{\mathrm{o}}(k-l)+4\varrho_{\mathrm{o}}x_{\mathrm{o}}^{\mathrm{T}}(\phi(k-l))P_{\mathrm{o}}x_{\mathrm{o}}(\phi(k-l)).
\end{aligned}$$

Once again, by the linear relation between Q and ρ, there exists $\rho_4 \geq \rho_3$ such that, for each $\rho \geq \rho_4$,

$$(A+LC-I)^{\mathrm{T}}P(\gamma)(A+LC-I) \leq Q,$$

which implies that

$$\begin{aligned}
\zeta^{\mathrm{T}}(k)P\zeta(k) &\leq 2R\sum_{l=1}^{R} e^{\mathrm{T}}(k-l)Qe(k-l)\\
&\quad +4\varrho_{\mathrm{o}}e^{\mathrm{T}}(k-l)Qe(k-l)+4\varrho_{\mathrm{o}}e^{\mathrm{T}}(\phi(k-l))Qe(\phi(k-l))\\
&\quad +4\varrho_{\mathrm{o}}x_{\mathrm{o}}^{\mathrm{T}}(k-l)P_{\mathrm{o}}x_{\mathrm{o}}(k-l)+4\varrho_{\mathrm{o}}x_{\mathrm{o}}^{\mathrm{T}}(\phi(k-l))P_{\mathrm{o}}x_{\mathrm{o}}(\phi(k-l)).
\end{aligned}$$

Fix a $\rho \geq \rho_4$. Using the above inequalities on the terms $\xi_{\mathrm{o}}^{\mathrm{T}}(k)P_{\mathrm{o}}\xi_{\mathrm{o}}(k)$ and $\zeta(k)P\zeta(k)$ yields

$$\begin{aligned}
\Delta V(x_{\mathrm{o}}(k),e(k)) \leq &-\frac{\gamma}{\lambda_{\max}(Q)}V(x_{\mathrm{o}}(k),e(k))\\
&+(2+4\sigma)\vartheta_{\mathrm{o}}\Bigg(2R\sum_{l=1}^{R}(\varpi_{\mathrm{o}}+4\varrho_{\mathrm{o}})x_{\mathrm{o}}^{\mathrm{T}}(k-l)\\
&\times P_{\mathrm{o}}x_{\mathrm{o}}(k-l)+4\varrho_{\mathrm{o}}x_{\mathrm{o}}^{\mathrm{T}}(\phi(k-l))P_{\mathrm{o}}x_{\mathrm{o}}(\phi(k-l))\\
&+2\lambda_{\mathrm{o}}e^{\mathrm{T}}(\phi(k-l))Qe(\phi(k-l))\Bigg)+(4+4\sigma)\vartheta_{\mathrm{o}}\\
&\times\Bigg(2R\sum_{l=1}^{R}e^{\mathrm{T}}(k-l)Qe(k-l)+4\varrho_{\mathrm{o}}e^{\mathrm{T}}(k-l)Qe(k-l)\\
&+4\varrho_{\mathrm{o}}e^{\mathrm{T}}(\phi(k-l))Qe(\phi(k-l))+4\varrho_{\mathrm{o}}x_{\mathrm{o}}^{\mathrm{T}}(k-l)P_{\mathrm{o}}x_{\mathrm{o}}(k-l)\\
&+4\varrho_{\mathrm{o}}x_{\mathrm{o}}^{\mathrm{T}}(\phi(k-l))P_{\mathrm{o}}x_{\mathrm{o}}(\phi(k-l))\Bigg).
\end{aligned}$$

Consequently, under the assumption that $V(k+s) \leq \eta V(k)$, $s \in I[-2R, 0]$, for a constant $\eta > 1$, we have

$$\Delta V(x_\text{o}(k), e(k)) \leq \bigg(-\frac{\gamma}{\lambda_{\max}(Q)} + 4\eta R^2(1+2\sigma)\vartheta_\text{o}(\varpi_\text{o} + 10\varrho_\text{o}) + 8\eta R^2(1+\sigma)\vartheta_\text{o}(1+12\varrho_\text{o}) \bigg) V(x_\text{o}(k), e(k)).$$

Thus, by the Razumikhin Stability Theorem (Theorem 1.4), the closed-loop system is asymptotically stable if

$$R < \sqrt{\frac{\gamma}{8\lambda_{\max}(Q)\vartheta_\text{o}(1+\sigma)(1+\varpi_\text{o}+22\varrho_\text{o})}}.$$

This completes the proof. □

Theorem 6.2 suggests that, for any feedback parameter γ, the delay independent output feedback TPF law (6.24) is robust to a certain amount of input delay in a general, possibly exponentially unstable, discrete-time linear system.

Theorem 6.3 *Let all the eigenvalues of A_o be at $z = 1$. Then, given a time-varying delay with an arbitrarily large upper bound, there exists $\gamma^* > 0$ such that, for each $\gamma \in (0, \gamma^*)$, the closed-loop system (6.35) is asymptotically stable.*

Proof The scheme of the proof follows mostly that of Theorem 6.2, with minor differences in the handling of the parameters γ and ρ. Consider the Lyapunov function

$$V(x_\text{o}(k), e(k)) = x_\text{o}^\text{T}(k)P_\text{o}(\gamma)x_\text{o}(k) + e^\text{T}(k)Qe(k),$$

and recall (6.40) as

$$\begin{aligned}\Delta V(k) \leq & -\gamma x_\text{o}^\text{T}(k)P_\text{o}x_\text{o}(k) + (2+4\sigma)\vartheta_\text{o}\xi_\text{o}^\text{T}(k)P_\text{o}\xi_\text{o}(k) \\ & +(4+4\sigma)\vartheta_\text{o}\zeta^\text{T}(k)P\zeta(k) \\ & +e^\text{T}(k)\left(-\rho I + (A+LC)^\text{T}(A+LC) + 4\vartheta_\text{o}P\right)e(k),\end{aligned}$$

with σ and ϑ_o being defined in the proof of Theorem 6.2. Choose a sufficiently large ρ such that

$$\begin{cases} \lambda_{\min}(Q) > 1, \\ \max\{\rho I, Q\} > (A+LC)^\text{T}(A+LC). \end{cases}$$

Then, there exists a sufficiently small $\gamma_1 > 0$ such that, for each $\gamma \in (0, \gamma_1)$,

$$\begin{cases} P(\gamma) < Q, \\ \gamma I + 4\vartheta_{\mathrm{o}} P(\gamma) + (A + LC)^{\mathrm{T}}(A + LC) < \rho I, \\ (A + LC - I)^{\mathrm{T}} P(\gamma)(A + LC - I) < Q, \\ P(\gamma) < \gamma \nu I, \end{cases}$$

where ν is a positive constant that depends on γ_1. All the above four inequalities come from the properties that when all the eigenvalues of A_{o} are at $z = 1$, $P_{\mathrm{o}}(\gamma)$ is a rational function of γ and property (3.74) (see Lemma 3.4).

We adopt the inequality on the term $\xi_{\mathrm{o}}^{\mathrm{T}}(k) P_{\mathrm{o}} \xi_{\mathrm{o}}(k)$, as given by (6.41),

$$\xi_{\mathrm{o}}^{\mathrm{T}}(k) P_{\mathrm{o}} \xi_{\mathrm{o}}(k) \leq 2R \sum_{l=1}^{R} (\varpi_{\mathrm{o}} + 4\varrho_{\mathrm{o}}) x_{\mathrm{o}}^{\mathrm{T}}(k-l) P_{\mathrm{o}} x_{\mathrm{o}}(k-l)$$
$$+ 4\varrho_{\mathrm{o}} x_{\mathrm{o}}^{\mathrm{T}}(\phi(k-l)) P_{\mathrm{o}} x_{\mathrm{o}}(\phi(k-l)) + 2\lambda_{\mathrm{o}} e^{\mathrm{T}}(\phi(k-l)) Q e(\phi(k-l)).$$

Moreover, the term $\zeta^{\mathrm{T}}(k) P \zeta(k)$ can be majorized as

$$\zeta^{\mathrm{T}}(k) P \zeta(k) \leq 2R \sum_{l=1}^{R} \gamma \nu e^{\mathrm{T}}(k-l) Q e(k-l)$$
$$+ 4\varrho_{\mathrm{o}} e^{\mathrm{T}}(k-l) Q e(k-l) + 4\varrho_{\mathrm{o}} e^{\mathrm{T}}(\phi(k-l)) Q e(\phi(k-l))$$
$$+ 4\varrho_{\mathrm{o}} x_{\mathrm{o}}^{\mathrm{T}}(k-l) P_{\mathrm{o}} x_{\mathrm{o}}(k-l) + 4\varrho_{\mathrm{o}} x_{\mathrm{o}}^{\mathrm{T}}(\phi(k-l)) P_{\mathrm{o}} x_{\mathrm{o}}(\phi(k-l)).$$

Consequently, under the assumption that $V(k+s) \leq \eta V(k)$, $s \in I[-2R, 0]$, for a constant $\eta > 1$, the forward difference $\Delta V(k)$ can be computed as

$$\Delta V(k) \leq \left(-\frac{\gamma}{\lambda_{\max}(Q)} + 4\eta(1 + 2\sigma)\vartheta_{\mathrm{o}} R^2 (\varpi_{\mathrm{o}} + 10\varrho_{\mathrm{o}}) \right.$$
$$\left. + 8\eta R^2 (1 + \sigma)\vartheta_{\mathrm{o}} (\gamma\nu + 12\varrho_{\mathrm{o}}) \right) V(x_{\mathrm{o}}(k), e(k)),$$

where ϑ_{o}, ϖ_{o}, and ϱ_{o} are as defined in the proof of Theorem 6.2. With all the eigenvalues of A_{o} at $z = 1$, we have $\mathrm{tr}(A_{\mathrm{o}}) = n$ and $\det(A_{\mathrm{o}}) = 1$, which imply that ϑ_{o}, ϖ_{o} and ϱ_{o} all approach zero as γ does. This further implies that there exists a $\gamma^* \leq \gamma_1$ such that, for each $\gamma \in (0, \gamma^*)$,

$$4(1 + 2\sigma)\vartheta_{\mathrm{o}} R^2 (\varpi_{\mathrm{o}} + 10\varrho_{\mathrm{o}}) + 8R^2(1 + \sigma)\vartheta_{\mathrm{o}}(\gamma\nu + 12\varrho_{\mathrm{o}}) < \frac{\gamma}{\lambda_{\max}(Q)}.$$

The asymptotic stability of the closed-loop system (6.35) then follows from the Razumihkin Stability Theorem (Theorem 1.4). □

Theorem 6.3 suggests that, given a time-varying delay with an arbitrarily large upper bound, the delay independent output feedback TPF law (6.24) asymptotically stabilizes a discrete-time linear system with all its open loop poles at $z = 1$ or inside the unit circle. No restriction on the variation of the time-varying delay is imposed on the system. This is particularly remarkable even when compared to the predictor feedback for a general discrete-time linear system. Recall that the general predictor feedback, as described in Chap. 3, manages to stabilize the system by predicting the future state of the open loop system at the future time step $\phi^{-1}(k)$, which inevitably requires the inverse of $\phi(k) = k - r(k)$ to exist and be known. Such a requirement limits the ability of the predictor feedback to handle time-varying delays. Therefore, our proposed output feedback law manifests its advantage over the predictor feedback in the stabilization of discrete-time linear systems with time-varying delays.

6.3.3 Numerical Examples

Example 6.4 (A System with All Poles at $z = 1$) Consider system (6.21) with

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. \tag{6.42}$$

Clearly, the open loop system has all its poles at $z = 1$ and the triple (A, B, C) are both controllable and observable. We design the matrix L such that

$$\lambda(A + LC) = \left\{ \pm\frac{1}{2}, \pm\frac{1}{2} \right\}.$$

Theorem 6.3 suggests that the delay independent truncated predictor output feedback law (6.24) asymptotically stabilizes the system with a time-varying delay whose upper bound can be arbitrarily large. Consider a fast-varying delay

$$r(k) = \left\lceil 1 + \sin^2(100k) \right\rceil, \quad k \in \mathbb{N}$$

with an upper bound $R = 2$, where $\lceil \cdot \rceil$ denotes the ceiling function. Let $\gamma = 0.1$ and the initial condition be

$$x(k) = \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix}^{\mathrm{T}}, \quad \hat{x}(k) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad k \in I[0, R].$$

The evolutions of the closed-loop state, the state estimate error, and the corresponding input are shown in Figs. 6.4 and 6.5. □

Example 6.5 (A System with All Poles on the Unit Circle) Consider system (6.21) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}. \tag{6.43}$$

The open loop system has a pair of poles on the unit circle at $z = \pm j$ and another pole at $z = 1$. It can be easily verified that (A, B, C) are both controllable and observable. The matrix L is designed such that

$$\lambda(A + LC) = \left\{0, \pm\frac{1}{2}\right\},$$

and the feedback parameter is chosen as $\gamma = 0.1$. For simulation purpose, let a time-varying delay be

$$r(k) = \frac{1 + (-1)^k}{2}, \quad k \in \mathbb{N},$$

and the initial condition be

$$x(k) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^{\mathrm{T}}, \; \hat{x}(k) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \; k \in I[0, 1].$$

The state, the state estimation, the estimation error, and the control input are shown in Figs. 6.6 and 6.7. □

Example 6.6 (An Exponentially Unstable System) Consider system (6.21) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}. \tag{6.44}$$

The open loop system has an exponentially unstable poles at $z = 2$ and a pair of imaginary poles at $z = \pm j$. It can be readily verified that (A, B, C) are both controllable and observable. The matrix L is designed such that

$$\lambda(A + LC) = \left\{0, \pm\frac{1}{2}\right\},$$

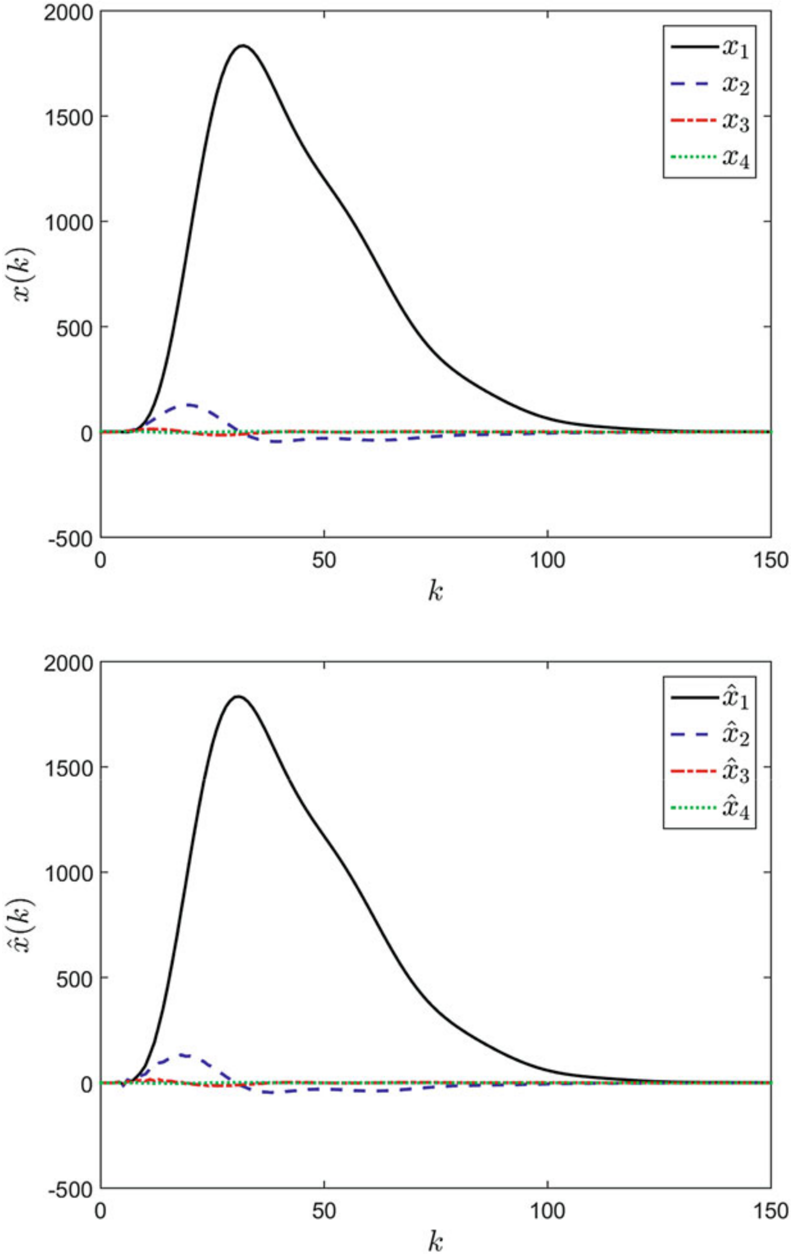


Fig. 6.4 Example 6.4: State and state estimation under the delay independent output feedback TPF law (6.24) with $\gamma = 0.1$

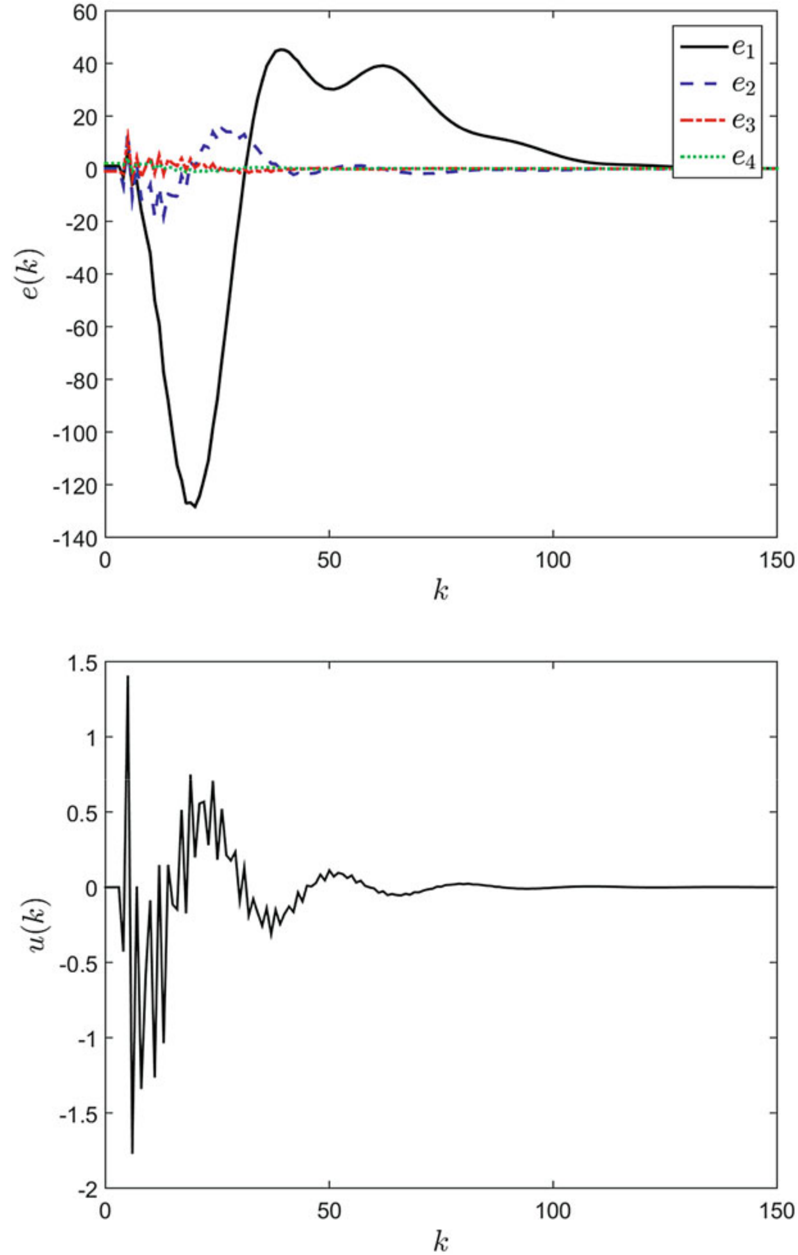


Fig. 6.5 Example 6.4: State estimation error and control input under the delay independent output feedback TPF law (6.24) with $\gamma = 0.1$

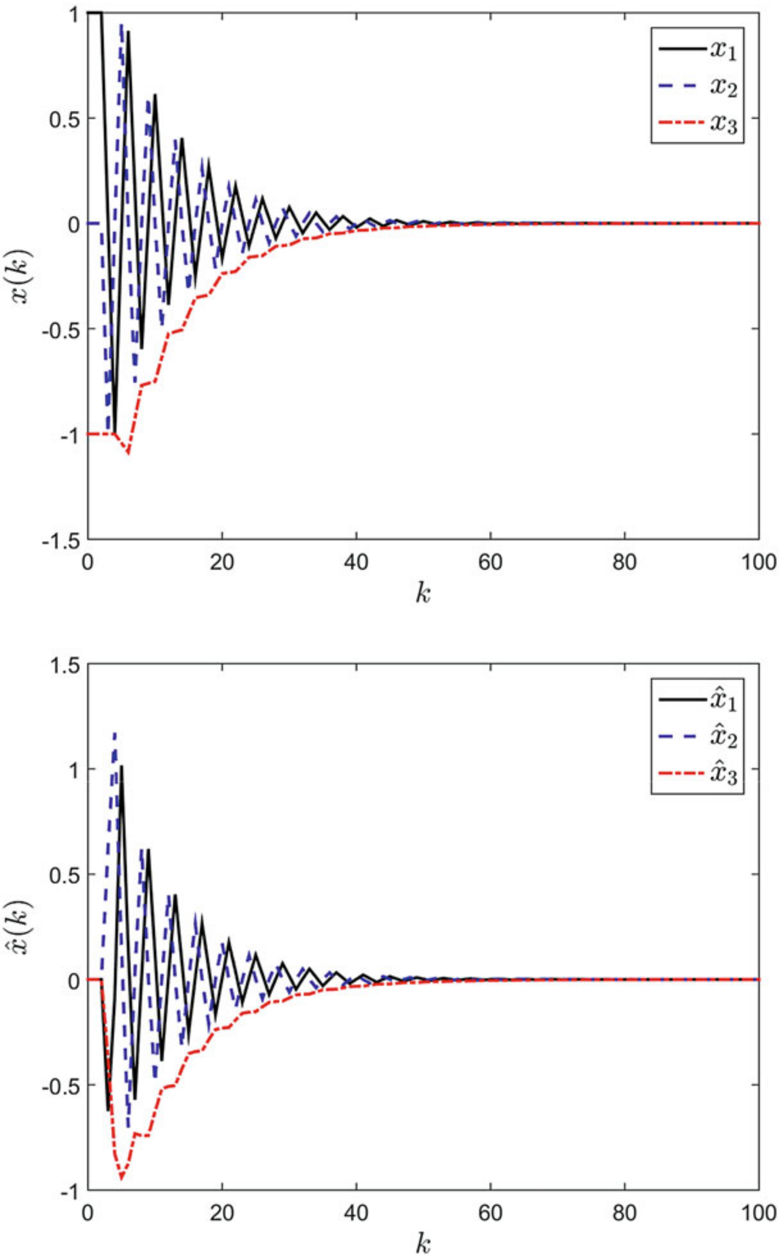
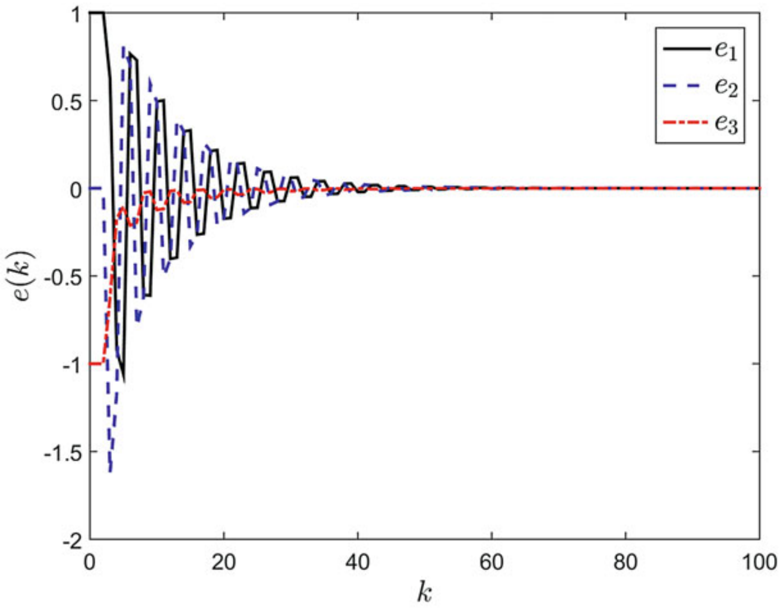


Fig. 6.6 Example 6.5: State and state estimation under the delay independent output feedback TPF law (6.24) with $\gamma = 0.1$



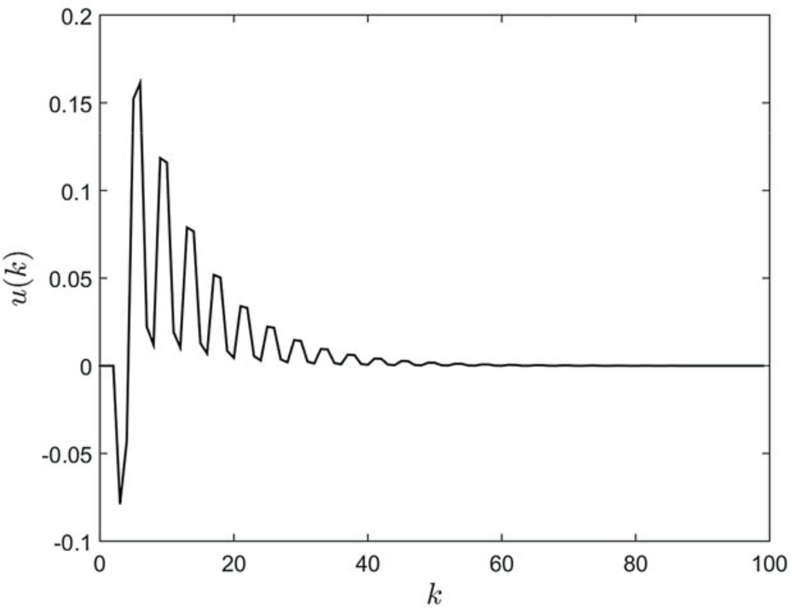


Fig. 6.7 Example 6.5: Estimation error and control input under the delay independent output feedback TPF law (6.24) with $\gamma = 0.1$

and the feedback parameter is chosen as $\gamma = 0.1$. Consider a constant delay

$$r(k) = 1, \quad k \in \mathbb{N}.$$

Let the initial condition be

$$x(k) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^{\mathrm{T}}, \ \hat{x}(k) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad k \in I[0, 1].$$

The convergent state, state estimation, estimation error, and control input are shown in Figs. 6.8 and 6.9. □

6.4 Conclusions

A general discrete-time linear system that is possibly exponentially unstable can be stabilized by delay independent TPF, either of state feedback type or of output feedback type, as long as the input delay does not exceed a certain bound. Since the construction of the delay independent TPF does not rely on the explicit knowledge of a time-varying delay, the feedback has the ability to handle time-varying delays that cannot be handled by the general predictor feedback of Chap. 3. Moreover, when all the open loop poles are at $z = 1$ or inside the unit circle, the delay independent predictor feedback laws achieve asymptotic stabilization no matter how large the bound on the time-varying delay is as long as the feedback parameter is chosen small enough.

6.5 Notes and References

The presentation of the results on the delay independent truncated predictor state feedback follows [106], whereas the results on its output feedback counterparts are from [109].

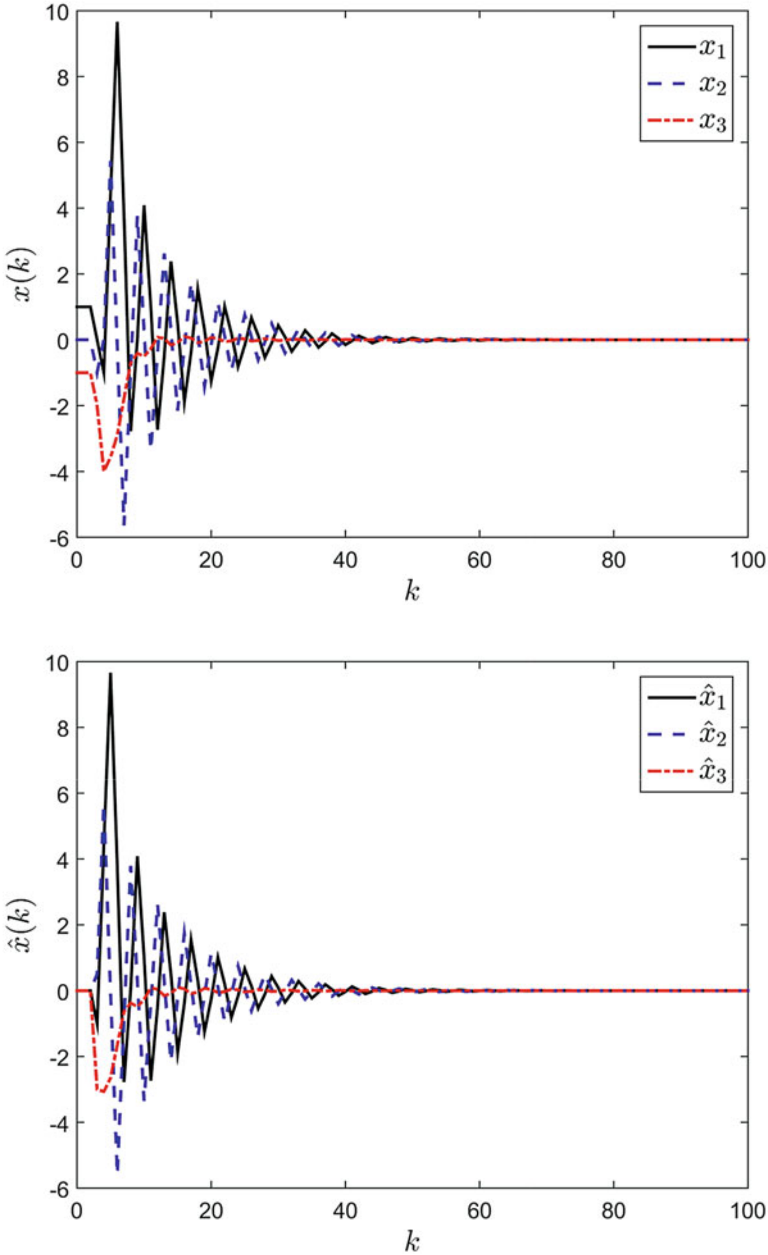


Fig. 6.8 Example 6.6: State and state estimation under the delay independent output feedback TPF law (6.24) with $\gamma = 0.1$

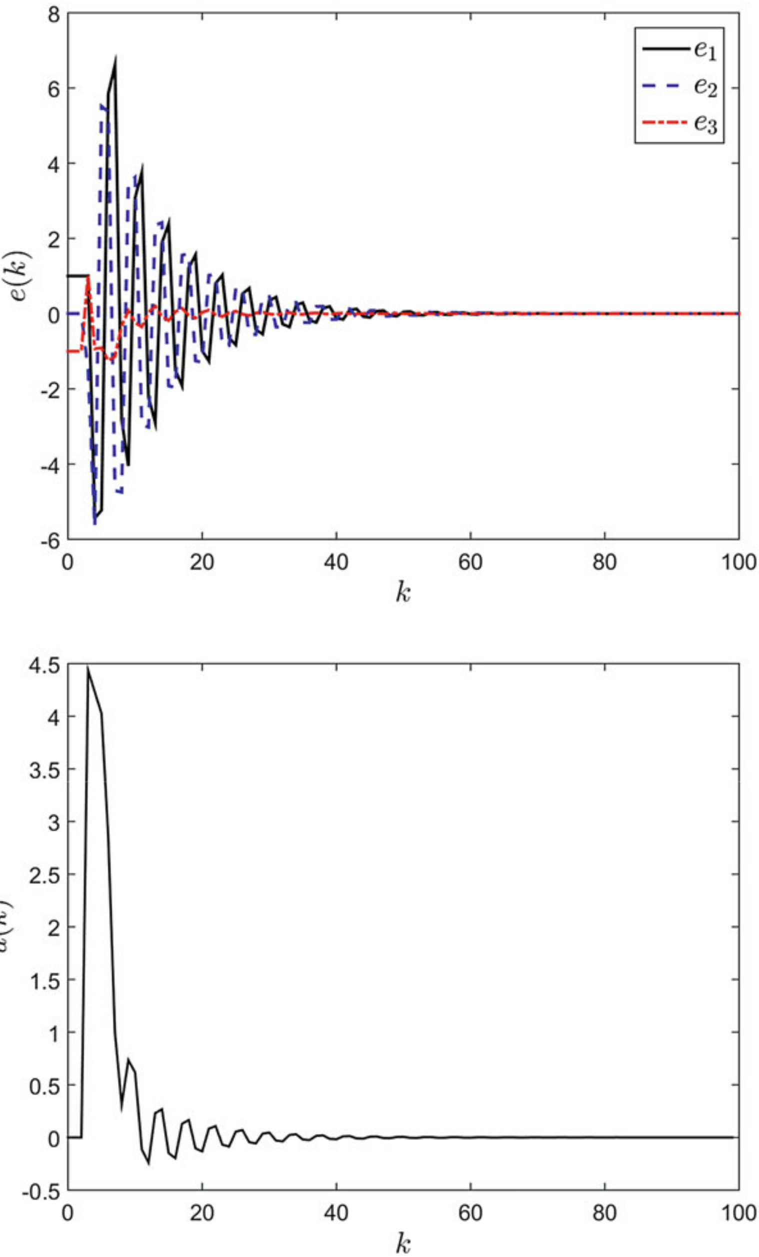


Fig. 6.9 Example 6.6: Estimation error and control input under the delay independent output feedback TPF law (6.24) with $\gamma = 0.1$

Chapter 7
Regulation of Continuous-Time Linear Input Delayed Systems Without Delay Knowledge

7.1 Introduction

In this chapter, we propose a control scheme that, in the absence of any knowledge of the delay, regulates to zero the state and the control input of a linear input delayed system whose open loop poles are at the origin or in the open left-half plane. Two main features of our control scheme are its non-distributed nature in the sense that only the current state is used in the feedback and its delay independence in the sense that no knowledge of the delay, neither its exact value nor its upper bound, is required. The main ingredients of our control scheme and the regulation proof include the design of a delay independent truncated predictor feedback law with a time-varying feedback parameter, Lyapunov function based adaptation of the time-varying parameter, a mechanism for switching between two update laws of the time-varying parameter, and the partial differential equation based analysis of the closed-loop system.

Both the truncated predictor feedback law and the delay independent truncated predictor feedback law, as presented in Chaps. 2 and 5, respectively, require knowledge of the input delay. The delay appears in the exponential factor of the truncated predictor feedback law. An upper bound of the delay is necessary for determining the value of a low gain feedback parameter of the delay independent truncated predictor feedback laws. In Sect. 5.2, a delay independent truncated predictor feedback law was proposed for a general linear system and an admissible delay bound for the closed-loop stability was established. A delay independent output feedback law was proposed in Sect. 5.4 for a general linear system, and an admissible delay bound that assures closed-loop stability was given. Also, an upper bound of the delay is required to be known for the stability guarantee. To study the potential of the delay independent truncated predictor feedback law to achieving faster convergence of the closed-loop system, Sect. 5.3 presented a time-varying feedback parameter that guarantees closed-loop stability of linear systems with all open loop poles at the origin or in the open left-half plane. An upper bound

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5_7

of the delay is still required for the design of the time-varying feedback parameter therein. For systems with open loop poles in the closed left-half plane, reference [103] developed adaptation of the truncated predictor feedback law to accommodate unknown delay, but the delay is required to be in a sufficiently small range whose upper and lower bounds are known. A control scheme that employs the full actuator states $U(t+\theta)$, $\theta \in [-\tau, 0]$, regulates a general linear system with input delay, but requires the upper bound of the delay to be known [58]. Typically, a delay-adaptive control scheme for a general linear system utilizes a lower bound and an upper bound of the delay (see [14] and [15]). Regardless of different types of systems and various control schemes, regulation of linear systems by using a feedback law independent of any knowledge of the delay still remains an open problem.

The delay independent truncated predictor feedback law as a non-distributed controller allows easy implementation and requires relatively less knowledge of the delay. To achieve regulation, in the absence of any knowledge of the delay, of linear systems with all open loop poles at the origin or in the open left-half plane, in this chapter, we consider the delay independent feedback law as the nominal controller for the design of a regulation scheme. The Lyapunov equation based low gain feedback design results in a feedback gain matrix with a single feedback parameter. An update algorithm for the feedback parameter consists of two update laws with a switching mechanism. This update algorithm guarantees the regulation of the state and the control input of the system to zero and the regulation of the feedback parameter to a positive constant. The regulation proof is carried out by the use of the partial differential equation (PDE) based system representation and a Lyapunov analysis of the closed-loop system. The development and the applications of the PDE approach to the representation and analysis of delayed systems can be found in great detail in [59].

7.2 A Feedback Law with a Time-Varying Parameter

We consider the design of a delay compensation scheme that regulates to zero the state and the control input of the following linear system with input delay,

$$\begin{cases} \dot{X}(t) = AX(t) + BU(t-\tau), & t \geq 0, \\ X(\theta) = \psi(\theta), & \theta \in [-\tau, 0], \end{cases} \tag{7.1}$$

where $X \in \mathbb{R}^n$ and $U \in \mathbb{R}^m$ are the state vector and the input vector, respectively, $\tau \geq 0$ is an arbitrarily large unknown delay, and the initial condition

$$\psi(\theta) \in PC[-\tau, 0].$$

It is assumed that all eigenvalues of A are at the origin and (A, B) is controllable.

In Chap. 5, the delay independent truncated predictor feedback law with the Lyapunov equation based low gain feedback design takes the form of

$$U(t) = -B^{\mathrm{T}} P(\gamma) X(t), \quad t \geq -\tau, \tag{7.2}$$

where γ is the low gain feedback parameter, $-B^{\mathrm{T}} P(\gamma)$ is the feedback gain matrix, and $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation,

$$A^{\mathrm{T}} P(\gamma) + P(\gamma) A - P(\gamma) B B^{\mathrm{T}} P(\gamma) = -\gamma P(\gamma), \quad \gamma > 0. \tag{7.3}$$

With the knowledge of an upper bound of the delay, an upper bound of γ that ensures the asymptotic stability of the closed-loop system was established.

In the absence of any knowledge of the delay value, an upper bound of the low gain parameter with stability guarantee cannot be determined. Inspired by the time-varying low gain feedback design in Sect. 5.3, we adopt

$$U(t) = -B^{\mathrm{T}} P(\gamma(t)) X(t), \quad t \geq -\tau, \tag{7.4}$$

with a time-varying feedback parameter

$$\gamma(t) > 0.$$

The initial condition of $\gamma(t)$ is given as

$$\gamma(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0].$$

It is assumed that

$$\phi(\theta) \in PC[-\tau, 0],$$

and satisfies

$$\phi(\theta) > 0, \quad \theta \in [-\tau, 0].$$

For notational brevity, $\gamma(0)$ is denoted as γ_0 in the rest of the chapter.

Remark 7.1 We have defined the initial condition of the closed-loop system consisting of (7.1) and (7.4) as $\psi(\theta)$ and $\phi(\theta)$ on $\theta \in [-\tau, 0]$. The initial condition for $U(t)$ is determined by the initial conditions $\psi(\theta)$ and $\phi(\theta)$. If we consider $t = 0$ as the starting time instant of the feedback law (7.4), then the initial condition of the closed-loop system can be defined by $X(0)$, γ_0 and

$$U(\theta) = \upsilon(\theta) \in PC[-\tau, 0).$$

□

The bottleneck to achieving the regulation of closed-loop signals including $X(t)$, $\gamma(t)$, and $U(t)$ under (7.4), without resorting to any knowledge of the delay, is the design of an appropriate delay independent update algorithm for $\gamma(t)$. Such an algorithm and its main features are presented in the next section.

7.3 An Update Algorithm for the Feedback Parameter

An update algorithm for $\gamma(t)$, which constructs a continuous $\gamma(t)$ on $t \geq 0$, consists of two update laws and a mechanism that governs the switching between them. Update law I takes the form of

$$\dot{\gamma}(t) = -\alpha \frac{V^p(t)}{V^p(t) + \beta} \gamma^q(t), \quad t \geq 0, \tag{7.5}$$

where

$$\alpha, \beta > 0, p \geq 1, q \geq 2,$$

and

$$V(X(t), t) = X^{\mathrm{T}}(t) P(\gamma(t)) X(t)$$

is a Lyapunov function commonly used for stability analysis of systems with or without delays. For notational convenience, we will denote

$$V(t) = V(X(t), t)$$

in the rest of the chapter.

Update law II takes the following form:

$$\dot{\gamma}(t) = -\zeta \gamma^r(t), \quad t \geq 0, \tag{7.6}$$

where

$$\zeta > 0, \quad r \geq 2.$$

The switching between the two update laws is based on an event-triggered switching condition according to the value of the Lyapunov function $V(t)$. The update law is switched from I to II whenever

$$V(t) \geq \epsilon,$$

where $\epsilon > 0$ is a threshold. Denote the time instant of a switch from update law I to update law II as $T_{\mathrm{I,II}}$. We consider isolated time instants

$$T_{\mathrm{I,II}} + \delta_i\,, \quad i \in \mathbb{N},$$

such that

$$\gamma\left(T_{\mathrm{I,II}} + \delta_i\right) = \frac{\gamma\left(T_{\mathrm{I,II}}\right)}{\xi^i}, \tag{7.7}$$

where

$$\xi > 1$$

and δ_i is computed by the use of (7.6) and (7.7) as

$$\delta_i = \frac{\gamma^{1-r}\left(T_{\mathrm{I,II}}\right)}{(r-1)\zeta}\left(\xi^{(r-1)i} - 1\right). \tag{7.8}$$

Then, the time instant of the next switch from update law II back to update law I is

$$T_{\mathrm{II,I}} = T_{\mathrm{I,II}} + \min_{i \in \mathbb{N}}\left\{\delta_i : V\left(T_{\mathrm{I,II}} + \delta_i\right) < \epsilon\right\}. \tag{7.9}$$

Note from (7.8) that δ_i is strictly increasing with respect to i. Thus, (7.9) indicates that after a switch from update law I to update law II, we check the value of V at the isolated time instants

$$T_{\mathrm{I,II}} + \delta_i\,, \quad i \in \mathbb{N},$$

and switch from update law II back to update law I at the first time instant

$$T_{\mathrm{I,II}} + \delta_i$$

at which

$$V\left(T_{\mathrm{I,II}} + \delta_i\right) < \epsilon.$$

Two main features of the update algorithm for $\gamma(t)$ are its non-distributed nature and delay independence. In particular, the update algorithm utilizes exclusively the current state as the feedback, rather than resorting to the past values of any closed-loop signal. On the other hand, no knowledge of the delay is required in the update algorithm. As a result, the integrated control scheme consisting of the feedback law (7.4) and the update algorithm is also non-distributed and delay independent.

Remark 7.2 According to the switching mechanism, the value of $V(0)$ uniquely determines which update law is to implement at $t = 0$. If

$$V(0) < \epsilon,$$

then the closed-loop system starts to evolve under update law I. Otherwise, it starts to evolve under update law II. By the definition of $V(t)$, the selection of the update law for $\gamma(t)$ at $t = 0$ solely depends on $\psi(0)$ and $\phi(0)$. □

Remark 7.3 Both $T_{\text{I,II}}$ and $T_{\text{II,I}}$ can be infinite, provided that the switching condition from one update law to another is never satisfied. In particular,

$$T_{\text{l,I,II}} = \infty$$

if

$$V(t) < \epsilon, \quad t \geq T_{\text{l,II,I}},$$

where $T_{\text{l,I,II}}$ and $T_{\text{l,II,I}}$ are the time instants of the last time of switch from update law I to update law II and from II to I, respectively. Similarly,

$$T_{\text{l,II,I}} = \infty$$

if

$$V\left(T_{\text{l,I,II}} + \delta_i\right) \geq \epsilon, \quad i \in \mathbb{N},$$

where δ_i is defined in (7.8) with $T_{\text{I,II}}$ replaced by $T_{\text{II,I}}$. □

Remark 7.4 On every time interval $\left[T_{\text{I,II}}, T_{\text{II,I}}\right]$, where $T_{\text{I,II}}$ is the time instant of a switch from update law I to update law II and $T_{\text{II,I}}$ is the time instant of the next switch from update law II back to update law I, $\gamma(t)$ evolves according to update law II. Equation (7.7) shows that the values of $\gamma(t)$ at the time instants of the sequence

$$\left\{T_{\text{I,II}} + \delta_i\right\}_{i=0}^{i=J},$$

where

$$J = \min\left\{j \in \mathbb{N} : V\left(T_{\text{I,II}} + \delta_j\right) < \epsilon\right\},$$

form a decreasing geometric sequence of a ratio

$$\xi^{-1} < 1.$$

From this perspective, update law II decreases the value of $\gamma(t)$ geometrically.

Another feature of update law II is that the switching condition from update law II to update law I is only required to be examined at the isolated time instants of the sequence

$$\{T_{\text{I,II}} + \delta_i\}_{i=0}^{i=J},$$

while for the rest of the time on the interval $\left[T_{\mathrm{I,II}}, T_{\mathrm{II,I}}\right]$, there is no feedback from other closed-loop signals to $\gamma(t)$. □

Remark 7.5 The frequency of switching between the two update laws for $\gamma(t)$ can be partially described by an examination of (7.9). Note that

$$T_{\mathrm{II,I}} - T_{\mathrm{I,II}} \geq \delta_1 = \frac{\gamma^{1-r}\left(T_{\mathrm{I,II}}\right)}{(r-1)\zeta}\left(\xi^{r-1} - 1\right) \geq \frac{\gamma_0^{1-r}}{(r-1)\zeta}\left(\xi^{r-1} - 1\right),$$

because of the nonincreasing monotonicity of $\gamma(t)$ on $t \in [0, \infty)$ and $r \geq 2$. Therefore, every time update law I switches to update law II, the time it takes to switch from update law II back to update law I is bounded from below by a positive constant. This implies that on any bounded time interval, the number of switches between the two update laws must be finite. Thus, the time-varying function $\dot{\gamma}(t)$ is piecewise continuous on every bounded time interval. □

In the rest of the chapter, we aim to prove the following two theorems on the well definedness and regulation to zero of the closed-loop signals under the proposed control scheme.

Theorem 7.1 *Under the update algorithm for $\gamma(t)$ and for any given initial conditions $\psi(\theta), \phi(\theta) \in PC[-\tau, 0]$, there exist unique solutions*

$$X(t), \gamma(t), U(t) \in C[0, \infty).$$

Moreover,

$$\gamma(t) \in (0, \gamma_0], \quad t \in [0, \infty).$$

Theorem 7.2 *The feedback law (7.4) with $\gamma(t)$ updated by the update algorithm achieves*

$$\lim_{t\to\infty} X(t) = 0, \quad \lim_{t\to\infty} U(t) = 0.$$

Moreover,

$$\lim_{t\to\infty} \gamma(t)$$

exists and is positive. In particular, on $t \in [0, \infty)$, the number of switches between the two update laws for $\gamma(t)$ is finite, and the last switch happens from update law II to update law I.

The proof of Theorem 7.1 is given in Sect. 7.4. Preliminary results for the proof of Theorem 7.2 are presented in Sects. 7.5 and 7.6, and the proof of Theorem 7.2 is given at the end of Sect. 7.6.

7.4 Proof of the Properties of the Closed-Loop Signals

Proof of Theorem 7.1 The proof is inspired by the continuation progress employed in the existence and the uniqueness proof of the solution of a functional differential equation (see Theorem 3.1 in [12]).

In view of the facts that

$$\psi(\theta), \phi(\theta) \in PC[-\tau, 0],$$

the open loop system (7.1) and the feedback law (7.4), there exists a unique solution $X(t)$ on $t \in [0, \tau]$ expressed as

$$X(t) = \mathrm{e}^{At} X(0) - \int_0^t BB^{\mathrm{T}} P(\phi(s-\tau))\psi(s-\tau)\mathrm{d}s,$$

which leads to

$$X(t) \in C[0, \tau].$$

With the solution $X(t)$, $t \in [0, \tau]$, the evolution of $\gamma(t)$ on $t \in [0, \tau]$ follows the update algorithm presented in Sect. 7.3. Consider an auxiliary signal

$$s(t) = \gamma^{-1}(t)$$

on $t \in [0, \tau]$. Then, $s(t)$ either satisfies

$$\dot{s}(t) = \alpha \frac{V^p(t)}{V^p(t) + \beta} s^{2-q}(t), \tag{7.10}$$

or

$$\dot{s}(t) = \zeta s^{2-r}(t),$$

corresponding to update laws I and update law II, respectively.

The existence of the signal $s(t)$ on $t \in [0, \tau]$ is shown by using the technique of proof by contradiction. Suppose that the solution $s(t)$ only exists on $t \in [0, t_{\mathrm{f}})$, where $0 < t_{\mathrm{f}} \leq \tau$. Note from Remark 7.5 that the number of switches between the two update laws on $t \in [0, t_{\mathrm{f}})$ is finite. Denote the time instant of the last time of switch as t_{s}. On $t \in [t_{\mathrm{s}}, t_{\mathrm{f}})$, either update law I or update law II is implemented. Suppose that it is the first case. According to (7.10),

$$\dot{s}(t) \leq \alpha s^{2-q}(t), \quad t \in [t_{\mathrm{s}}, t_{\mathrm{f}}),$$

which leads to

$$s(t) \le \left(\alpha(q-1)(t-t_s) + s^{q-1}(t_s)\right)^{\frac{1}{q-1}} < \infty, \quad t \in [t_s, t_f).$$

This contradicts with

$$\lim_{t \to t_f^-} s(t) = \infty.$$

Applying a similar argument to the second case, where update law II is implemented on $t \in [t_s, t_f)$, also results in a contradiction. Therefore, $s(t)$, and thus $\gamma(t)$, exist on $t \in [0, \tau]$. Note that

$$\dot{s}(t) \ge 0.$$

Thus,

$$s(t) \ge \gamma_0^{-1} > 0,$$

and

$$\gamma(t) > 0, \quad t \in [0, \tau].$$

The uniqueness of the solution $\gamma(t)$ on $t \in [0, \tau]$ again follows from proof by contradiction. Suppose that the time instant of the first time the solution $\gamma(t)$ becomes nonunique is t', where $t' \in [0, \tau)$. Again, recall from Remark 7.5 that the number of switches between the two update laws is finite on any bounded time interval. Therefore, there exists a sufficiently small

$$\varepsilon > 0$$

such that either update law I or update law II, is implemented on $t \in [t', t' + \varepsilon]$. Consider the first case where update law I is implemented. The right-hand side of (7.5) is continuous with respect to both t and γ at the point $(t', \gamma(t'))$ because

$$X(t) \in C[0, \tau]$$

and $P(\gamma)$ is infinitely differentiable with respect to γ (see Lemma 2.4). Furthermore,

$$\frac{\partial}{\partial \gamma}\left(-\alpha \frac{V^p(t)}{V^p(t) + \beta}\gamma^q\right) = -\alpha \frac{V^{p-1}(t)\gamma^{q-1}}{V^p(t) + \beta}\left(\frac{p\gamma\beta X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}X(t)}{V^p(t) + \beta} + qV(t)\right),$$

and is continuous with respect to t and γ at $(t', \gamma(t'))$. By the existence and uniqueness theorem on the solution of an ordinary differential equation, there exists an

$$\eta \in (0, \varepsilon]$$

such that the solution $\gamma(t)$ on $[t', t'+\eta]$ is unique. This contradicts the fact that $\gamma(t)$ starts to become nonunique from t'. Applying a similar argument to the second case where update law II is implemented also leads to a contradiction. Therefore, $\gamma(t)$ has a unique solution on $t \in [0, \tau]$.

With

$$X(t), \gamma(t) \in C[0, \tau],$$

the existence and the uniqueness of $X(t)$ and $\gamma(t)$ on $t \in [\tau, 2\tau]$ can be obtained similarly. Repeatedly applying the analysis along the time axis leads to the conclusion that there exists unique solutions

$$X(t), \gamma(t) \in C[0, \infty).$$

In view of the feedback law (7.4), the existence, the uniqueness, and the continuity of $U(t)$ then directly follow from those of $X(t)$ and $\gamma(t)$. On the other hand, by the use of the continuation progress, the positiveness of $\gamma(t)$ on $t \in [0, \infty)$ follows readily from the fact that

$$\gamma(t) > 0$$

on $t \in [0, \tau]$. By the nonincreasing monotonicity of $\gamma(t)$, we get

$$\gamma(t) \in (0, \gamma_0], \quad t \in [0, \infty).$$

□

Remark 7.6 The positiveness of $\gamma(t)$ on $t \in [0, \infty)$ as a conclusion of Theorem 7.1 shows that the update algorithm for $\gamma(t)$ successfully avoids the singularity at

$$\gamma = 0.$$

According to Chap. 2, such a singularity destroys the existence of a positive definite solution to (7.3). □

Remark 7.7 Theorem 7.1 also holds when $\gamma(t)$ is updated according to either update law I or update law II on $t \in [0, \infty)$. The proof for the case of a single update law is simpler, without the consideration of the other update law and the switching mechanism. □

In the rest of this section, we establish a lower bound of $\gamma(t)$ on $t \in [0, \infty)$ as a motivation for the more refined nonzero lower bound to be established in the proof of Theorem 7.2 in Sect. 7.6.

Proposition 7.1 $\gamma(t) \geq 1/\omega(t)$ *under the update algorithm for* $\gamma(t)$, *where*

$$\begin{aligned}\dot{\omega}(t) &= \max\left\{\alpha\omega^{2-q}(t), \zeta\omega^{2-r}(t)\right\}\\ &\triangleq f(\omega), \quad t \geq 0,\end{aligned} \tag{7.11}$$

and

$$\omega(0) = \gamma_0^{-1}.$$

Proof Consider an auxiliary signal

$$s(t) = 1/\gamma(t), \quad t \in [0, \infty).$$

The upper Dini derivative of $s(t)$ satisfies

$$\begin{aligned}D^+s(t) &= \limsup_{a\to 0^+} \frac{s(t+a) - s(t)}{a}\\ &\leq \max\left\{\alpha \frac{V^p(t)}{V^p(t)+\beta} s^{2-q}(t), \zeta s^{2-r}(t)\right\}\\ &\leq \max\left\{\alpha s^{2-q}(t), \zeta s^{2-r}(t)\right\}, \quad t \geq 0.\end{aligned} \tag{7.12}$$

Regardless of the values of α, q, ζ, r, and γ_0 given in Sect. 7.3, $\omega(t)$ has a unique solution on $t \in [0, \infty)$. In fact, the explicit solution of $\omega(t)$ can be obtained. Take the case where

$$q > r$$

and

$$\gamma_0 > \left(\frac{\zeta}{\alpha}\right)^{\frac{1}{q-r}}$$

for example. We compute

$$\omega(t) = \begin{cases} \left((q-1)\alpha t + \gamma_0^{1-q}\right)^{\frac{1}{q-1}}, & t \in [0, t_\omega],\\ \left((r-1)\zeta(t-t_\omega) + \left(\frac{\alpha}{\zeta}\right)^{\frac{r-1}{q-r}}\right)^{\frac{1}{r-1}}, & t \in [t_\omega, \infty),\end{cases}$$

$$t_\omega = \frac{\left(\frac{\alpha}{\zeta}\right)^{\frac{q-1}{q-r}} - \gamma_0^{1-q}}{\alpha(q-1)}.$$

The solution of $\omega(t)$ for other cases regarding the values of α, q, ζ, r can be obtained similarly. For brevity, we omit the solution of $\omega(t)$ for other cases.

By its definition in (7.11), $f(\omega)$ is locally Lipschitz in $w \in \mathbb{R}^+ \subset \mathbb{R}$. On the other hand, the interval $[0, \infty)$ is the maximum time interval of existence of both $\omega(t)$ and $s(t)$. By the continuity and the positiveness of $s(t)$ on $t \in [0, \infty)$,

$$s(0) = w(0)$$

and (7.12), it follows from the comparison lemma (Lemma 3.4 in [51]) that

$$s(t) \leq \omega(t), \quad t \in [0, \infty).$$

□

7.5 The PDE Description of the Closed-Loop System

The modeling of an input delayed system as a cascade of an ordinary differential equation (ODE) with a PDE brings a wealth of tools in the PDE analysis to the control systems analysis, advancing control techniques for state regulation, trajectory tracking, and compensation for unknown delays (see [4, 11, 14, 15], and [59]). While the PDE analysis in those earlier works applies to general linear or certain nonlinear delayed systems under predictor-based feedback laws, Sect. 5.3 introduces the PDE method for the analysis of system (7.1) under the feedback law (7.4). We recall from Sect. 5.3 the PDE-based modeling of such a closed-loop system.

We first define a series of functions and some auxiliary signals,

$$u(x, t) = U(t + \tau(x - 1)), \tag{7.13}$$

$$\hat{u}(x, t) = U\left(t + \hat{\tau}(t)(x - 1)\right), \tag{7.14}$$

$$\hat{\tau}(t) = \frac{h}{\gamma(t)}, \tag{7.15}$$

$$\hat{w}(x, t) = \hat{u}(x, t) - U(t), \tag{7.16}$$

$$\tilde{u}(x, t) = u(x, t) - \hat{u}(x, t), \tag{7.17}$$

$$\tilde{\tau}(t) = \tau - \hat{\tau}(t), \tag{7.18}$$

where

$$x \in [0, 1], \quad t \geq 0,$$

and h is some positive constant. Note from Theorem 7.1 that

$$\gamma(t) > 0, \quad t \in [0, \infty).$$

Thus, $\hat{\tau}(t)$ is well defined on the interval $[0, \infty)$. With the actuator state $u(x, t)$, the open loop system (7.1) is represented as a cascade of an ODE with a PDE,

$$\dot{X}(t) = AX(t) + Bu(0, t), \tag{7.19}$$

$$\tau u_t(x, t) = u_x(x, t), \tag{7.20}$$

$$u(1, t) = U(t). \tag{7.21}$$

Then, we arrive at the following closed-loop system representation:

$$\dot{X}(t) = \big(A - BB^{\mathrm{T}} P(\gamma(t))\big) X(t) + B\tilde{u}(0, t) + B\hat{w}(0, t). \tag{7.22}$$

Next, we recall from Sect. 5.3 the governing PDEs for $\hat{u}(x, t)$, $\hat{w}(x, t)$, $\tilde{u}(x, t)$ and $\hat{w}_x(x, t)$ as

$$\begin{cases} \hat{\tau}\hat{u}_t(x, t) = \big(1 + \dot{\hat{\tau}}(x - 1)\big)\hat{u}_x(x, t), \\ \hat{u}(1, t) = U(t), \end{cases} \tag{7.23}$$

$$\begin{cases} \hat{\tau}\hat{w}_t(x, t) = \hat{w}_x(x, t)\left(1 + \dot{\hat{\tau}}(x - 1)\right) + \hat{\tau} B^{\mathrm{T}} \dfrac{\partial P}{\partial \gamma}\dot{\gamma}(t) X(t) + \hat{\tau} B^{\mathrm{T}} P(\gamma(t)) \\ \qquad \times\Big(\big(A - BB^{\mathrm{T}} P(\gamma(t))\big) X(t) + B\tilde{u}(0, t) + B\hat{w}(0, t)\Big), \\ \hat{w}(1, t) = 0, \end{cases} \tag{7.24}$$

$$\begin{cases} \tau\tilde{u}_t(x, t) = \tilde{u}_x(x, t) - \dfrac{\tilde{\tau} + \tau\dot{\hat{\tau}}(x - 1)}{\hat{\tau}}\hat{w}_x(x, t), \\ \tilde{u}(1, t) = 0, \end{cases} \tag{7.25}$$

and

$$\begin{cases} \hat{\tau}\hat{w}_{xt}(x,t) = \hat{w}_{xx}(x,t)\left(1+\dot{\hat{\tau}}(x-1)\right) + \dot{\hat{\tau}}\hat{w}_x(x,t), \\ \hat{w}_x(1,t) = -\hat{\tau}B^{\mathrm{T}}\dfrac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t) - \hat{\tau}B^{\mathrm{T}}P(\gamma(t)) \\ \qquad\qquad \times\left((A-BB^{\mathrm{T}}P(\gamma(t)))X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t)\right), \end{cases} \tag{7.26}$$

respectively.

We finally recall a lemma from Sect. 5.3 on some properties of the functions (7.13)–(7.18).

Lemma 7.1 *Consider the closed-loop system consisting of (7.1) and (7.4). The following properties hold:*

$$\begin{aligned} &-\int_0^1 (1+x)\left(\tilde{\tau} + \tau\dot{\hat{\tau}}(x-1)\right)\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \\ &\le \left(\left|\tilde{\tau}\right| + \frac{1}{2}\tau\left|\dot{\hat{\tau}}\right|\right)\left(\rho||\tilde{u}(t)||^2 + \frac{1}{\rho}||\hat{w}_x(t)||^2\right), \end{aligned}$$

where $\rho > 0$ *is any constant,*

$$\int_0^1 (1+x)\left(1+\dot{\hat{\tau}}(x-1)\right)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \le \frac{1}{2}\left(\left|\dot{\hat{\tau}}\right|-1\right)\left|\hat{w}(0,t)\right|^2 + \left(\left|\dot{\hat{\tau}}\right| - \frac{1}{2}\right)\left|\left|\hat{w}(t)\right|\right|^2,$$

$$\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}B^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t)\mathrm{d}x \le h^{\frac{1}{2}}\left|\left|\hat{w}(t)\right|\right|^2 + \left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 h^{\frac{3}{2}}X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial \gamma}X(t),$$

$$\begin{aligned} &\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}B^{\mathrm{T}}P(\gamma(t))\left(\left(A-BB^{\mathrm{T}}P(\gamma(t))\right)X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t)\right)\mathrm{d}x \\ &\le \hat{\tau}n\gamma(t)\left|\left|\hat{w}(t)\right|\right|^2 + \frac{3}{2}\hat{\tau}n(n+1)\gamma^2(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t) \\ &\quad +3\hat{\tau}n\gamma(t)\left(|\tilde{u}(0,t)|^2 + \left|\hat{w}(0,t)\right|^2\right), \end{aligned}$$

$$\begin{aligned} &\int_0^1 (1+x)\left(1+\dot{\hat{\tau}}(x-1)\right)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x \\ &\le \left|\hat{w}_x(1,t)\right|^2 + \frac{1}{2}\left(\left|\dot{\hat{\tau}}\right|-1\right)\left|\hat{w}_x(0,t)\right|^2 + \left(\left|\dot{\hat{\tau}}\right| - \frac{1}{2}\right)\left|\left|\hat{w}_x(t)\right|\right|^2, \end{aligned}$$

$$|\hat{w}_x(1,t)|^2 \leq 2\hat{\tau}^2\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial \gamma}X(t) + 6\hat{\tau}^2 n^2\gamma^2(t)$$
$$\times\Big(\frac{n+1}{2}\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t) + |\tilde{u}(0,t)|^2 + |\hat{w}(0,t)|^2\Big).$$

In our regulation analysis later in this chapter, we adopt the following Lyapunov functional:

$$\mathcal{V}(X_t,\gamma(t)) = V(t) + b_1\tau\int_0^1(1+x)|\tilde{u}(x,t)|^2\mathrm{d}x + b_2\hat{\tau}(t)\int_0^1(1+x)|\hat{w}(x,t)|^2\mathrm{d}x$$
$$+b_2\hat{\tau}(t)\int_0^1(1+x)|\hat{w}_x(x,t)|^2\mathrm{d}x, \tag{7.27}$$

where

$$b_1, b_2 > 0$$

and their values are to be determined.

In the following lemma, we establish the continuity of $\mathcal{V}(t)$ with respect to time under the update algorithm for $\gamma(t)$.

Lemma 7.2 *Under the update algorithm for $\gamma(t)$ and with the initial conditions*

$$\psi(\theta), \phi(\theta) \in PC[-\tau, 0],$$

there exists

$$t_{\mathrm{c}} > 0$$

such that

$$\mathcal{V}(t) \in C[t_{\mathrm{c}}, \infty)$$

if

$$h \leq \frac{1}{2\max\left\{\alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2}\right\}}. \tag{7.28}$$

Proof We study the continuity of $\mathcal{V}(X_t,\gamma(t))$ term by term. The first term in $\mathcal{V}(X_t,\gamma(t))$ is continuous on $t \in [0,\infty)$ because of the continuity of $X(t)$ and $\gamma(t)$ on the same interval. By a change of variables,

$$\tau\int_0^1(1+x)|\tilde{u}(x,t)|^2\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{t-\tau}^{t} \left(2 + \frac{s-t}{\tau}\right) |U(s)|^2 \mathrm{d}s + \frac{\tau}{\hat{\tau}(t)} \int_{t-\hat{\tau}(t)}^{t} \left(2 + \frac{s-t}{\hat{\tau}(t)}\right) |U(s)|^2 \mathrm{d}s \\
&\quad - \frac{2\tau}{\hat{\tau}(t)} \int_{t-\hat{\tau}(t)}^{t} \left(2 + \frac{s-t}{\hat{\tau}(t)}\right) U^{\mathrm{T}}(s) U\left(\frac{\tau}{\hat{\tau}(t)} s + t\left(1 - \frac{\tau}{\hat{\tau}(t)}\right)\right) \mathrm{d}s,
\end{aligned}
$$

when $\tau \neq 0$. The continuity of $U(t)$ on $t \in [0, \infty)$ implies that the second term of $\mathcal{V}(X_t, \gamma(t))$ is continuous with respect to t if

$$
t \geq \max\left\{\tau, \hat{\tau}(t)\right\}.
$$

The continuity of the second term of $\mathcal{V}(X_t, \gamma(t))$ is obvious when

$$
\tau = 0.
$$

Noting that

$$
\begin{aligned}
\hat{\tau}(t) \int_0^1 (1+x)|\hat{w}(x,t)|^2 \mathrm{d}x &= \int_{t-\hat{\tau}(t)}^{t} \left(2 + \frac{s-t}{\hat{\tau}(t)}\right) |U(s)|^2 \mathrm{d}s + \frac{3}{2}\hat{\tau}(t)|U(t)|^2 \\
&\quad -2 \int_{t-\hat{\tau}(t)}^{t} \left(2 + \frac{s-t}{\hat{\tau}(t)}\right) U(s) \mathrm{d}s U(t),
\end{aligned}
$$

we deduce that the third term of $\mathcal{V}(X_t, \gamma(t))$ is continuous if

$$
\begin{aligned}
t \geq &\ \max\left\{\hat{\tau}(t), 0\right\} \\
= &\ \hat{\tau}(t).
\end{aligned}
$$

The last term of $\mathcal{V}(X_t, \gamma(t))$ needs careful examination because it involves $\dot{\gamma}(t)$, which is not necessarily continuous on $t \in [0, \infty)$ because of possible switching between the two update laws for $\gamma(t)$. It follows from a change of variables that

$$
\begin{aligned}
&\int_0^1 (1+x)|\hat{w}_x(x,t)|^2 \mathrm{d}x \\
&= \int_{t-\hat{\tau}(t)}^{t} \left(2\hat{\tau}(t) + s - t\right) \left(\dot{\gamma}^2(s) \left|B^{\mathrm{T}} \frac{\partial P}{\partial \gamma} X(s)\right|^2 + |B^{\mathrm{T}} P(\gamma(s)) A X(s)|^2\right. \\
&\quad + \left|B^{\mathrm{T}} P(\gamma(s)) B B^{\mathrm{T}} P(\gamma(s-\tau)) X(s-\tau)\right|^2 + 3\dot{\gamma}(s) X^{\mathrm{T}}(s) \frac{\partial P}{\partial \gamma} B B^{\mathrm{T}} P(\gamma(s)) A X(s) \\
&\quad - 3\dot{\gamma}(s) X^{\mathrm{T}}(s) \frac{\partial P}{\partial \gamma} B B^{\mathrm{T}} P(\gamma(s)) B B^{\mathrm{T}} P(\gamma(s-\tau)) X(s-\tau)
\end{aligned}
$$

$$- 3X^{\mathrm{T}}(s)A^{\mathrm{T}}P(\gamma(s))BB^{\mathrm{T}}P(\gamma(s))BB^{\mathrm{T}}P(\gamma(s-\tau))X(s-\tau)\Big)\mathrm{d}s. \tag{7.29}$$

On the time interval $[t - \hat{\tau}(t), t]$, where

$$t \geq \hat{\tau}(t),$$

$\dot{\gamma}(t)$ is piecewise continuous according to Remark 7.5. Also,

$$\begin{aligned} 0 \geq\ & \dot{\gamma}(t) \\ \geq\ & \min\left\{-\alpha\gamma^q(t), -\zeta\gamma^r(t)\right\} \\ \geq\ & \min\left\{-\alpha\gamma_0^q, -\zeta\gamma_0^r\right\} \end{aligned} \tag{7.30}$$

implies that $\dot{\gamma}(t)$ is bounded. In view of the continuity and the boundedness of $X(t)$ and $\gamma(t)$ on $[t - \hat{\tau}(t), t]$, the integrand of the right-hand side of (7.29) is Riemann integrable on the same interval as long as

$$t \geq \tau + \hat{\tau}(t).$$

Combining the continuity analysis for all the terms of $\mathcal{V}(X_t, \gamma(t))$, we conclude that, if

$$t \geq \tau + \hat{\tau}(t),$$

$\mathcal{V}(X_t, \gamma(t))$ is continuous. Under the update algorithm for $\gamma(t)$, the term

$$\frac{\dot{\gamma}(t)}{\gamma^2(t)}$$

is bounded on $t \in [0, \infty)$ because

$$\begin{aligned} 0 \geq\ & \frac{\dot{\gamma}(t)}{\gamma^2(t)} \\ \geq\ & \min\left\{-\alpha\gamma^{q-2}(t), -\zeta\gamma^{r-2}(t)\right\} \\ \geq\ & \min\left\{-\alpha\gamma_0^{q-2}, -\zeta\gamma_0^{r-2}\right\}. \end{aligned}$$

By the use of the comparison lemma, we compute

$$\gamma(t) \geq \frac{1}{\frac{1}{\gamma_0} + \max\left\{\alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2}\right\}t}, \quad t \geq 0. \tag{7.31}$$

If h satisfies (7.28), we have

$$h < \frac{1}{\max\left\{\alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2}\right\}}.$$

Therefore, there exists

$$t_c > 0$$

such that, for each $t \geq t_c$,

$$t \geq \tau + h\left(\gamma_0^{-1} + \max\left\{\alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2}\right\} t\right), \tag{7.32}$$

which implies that

$$t \geq \tau + \hat{\tau}(t)$$

according to (7.31). Then, the continuity of $\mathcal{V}(X_t, \gamma(t))$ follows. □

Remark 7.8 If $\gamma(t)$ is updated by either update law I or update law II on $t \in [0, \infty)$ alone, Lemma 7.2 also holds with the denominator of the right-hand side of (7.28) becoming

$$2\alpha\gamma_0^{q-2}$$

or

$$2\zeta\gamma_0^{r-2},$$

respectively. The proof for the case of a single update law resembles the proof of Lemma 7.2, with a difference in the estimate of the lower bound of $\dot{\gamma}(t)$ in (7.30). □

We compute the time derivative of $\mathcal{V}(X_t, \gamma(t))$ along the trajectory of the closed-loop system between switchings for $\gamma(t)$ as follows:

$$\begin{aligned}\dot{\mathcal{V}}(X_t, \gamma(t)) = {} & X^{\mathrm{T}}(t)\Big(-\gamma(t)P(\gamma(t)) - P(\gamma(t))BB^{\mathrm{T}}P(\gamma(t))\Big)X(t) \\ & + 2X^{\mathrm{T}}(t)P(\gamma(t))B\tilde{u}(0,t) + 2X^{\mathrm{T}}(t)P(\gamma(t))B\hat{w}(0,t) \\ & + X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t) + 2b_1\int_0^1 (1+x)\tilde{u}^{\mathrm{T}}(x,t)\tilde{u}_x(x,t)\mathrm{d}x \\ & - \frac{2b_1}{\hat{\tau}}\int_0^1 (1+x)\left(\tilde{\tau} + \tau\dot{\hat{\tau}}(x-1)\right)\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+2b_2\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\left(1+\dot{\hat{\tau}}(x-1)\right)\mathrm{d}x\\
&+2b_2\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}B^{\mathrm{T}}P(\gamma(t))\Big(\left(A-BB^{\mathrm{T}}P(\gamma(t))\right)X(t)\\
&+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x+2b_2\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)\mathrm{d}x\\
&+2b_2\int_0^1(1+x)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\left(1+\dot{\hat{\tau}}(x-1)\right)\mathrm{d}x\\
&+2b_2\int_0^1(1+x)|\hat{w}_x(x,t)|^2\dot{\hat{\tau}}\mathrm{d}x+b_2\dot{\hat{\tau}}\int_0^1(1+x)\left(\left|\hat{w}(x,t)\right|^2\right.\\
&\left.+\left|\hat{w}_x(x,t)\right|^2\right)\mathrm{d}x,
\end{aligned}
\tag{7.33}
$$

where we have used the closed-loop system representation (7.22), the Riccati equation (7.3), and the governing PDEs (7.23)–(7.26). It then follows that

$$
\begin{aligned}
&\dot{\mathcal{V}}(X_t,\gamma(t))\\
\le\ &\gamma(t)V(t)\left(-1+6b_2n^2(n+1)h^2+3b_2n(n+1)h\right)+|\tilde{u}(0,t)|^2\Big(2-b_1\\
&+6b_2nh+12b_2n^2h^2\Big)+|\hat{w}(0,t)|^2\left(b_2(\dot{\hat{\tau}}-1)+2+6b_2nh+12b_2n^2h^2\right)\\
&+||\tilde{u}(t)||^2b_1\left(-1+2\rho\frac{|\tilde{\tau}|+\frac{1}{2}\tau\dot{\hat{\tau}}}{\hat{\tau}}\right)+X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}X(t)\Big(-\frac{h}{\hat{\tau}^2}\dot{\hat{\tau}}+2b_2n\Big(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\Big)^2h^{\frac{3}{2}}\\
&+4b_2n\Big(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\Big)^2h^2\Big)+||\hat{w}_x(t)||^2\Big(\frac{2b_1}{\rho}\frac{|\tilde{\tau}|+\frac{1}{2}\tau\dot{\hat{\tau}}}{\hat{\tau}}+8b_2\dot{\hat{\tau}}-b_2\Big)\\
&+||\hat{w}(t)||^2b_2\Big(-1+2nh+2h^{\frac{1}{2}}+4\dot{\hat{\tau}}\Big)+|\hat{w}_x(0,t)|^2b_2(\dot{\hat{\tau}}-1),
\end{aligned}
\tag{7.34}
$$

where we have employed

$$
\int_0^1(1+x)\tilde{u}^{\mathrm{T}}(x,t)\tilde{u}_x(x,t)\mathrm{d}x=-\frac{1}{2}\left(|\tilde{u}(0,t)|^2+||\tilde{u}(t)||^2\right)
$$

and

$$
\frac{\partial}{\partial\gamma}P(\gamma)BB^{\mathrm{T}}\frac{\partial}{\partial\gamma}P(\gamma)\le n\frac{\partial}{\partial\gamma}P(\gamma),
$$

from Sect. 5.3, Young's Inequality, Lemma 7.1, and the facts that

$$h = \gamma(t)\hat{\tau}(t)$$

and $\gamma(t)$ is nonincreasing.

7.6 Regulation Under the Update Algorithm

The regulation analysis of the closed-loop signals under the update algorithm for $\gamma(t)$ is not straightforward due to the intrinsic mechanism of switching between the two update laws. We first establish two propositions on the regulation effects of update law I on $\gamma(t)$.

Proposition 7.2 *There exists $\gamma_0^\star > 0$ such that, for each $\gamma_0 \in (0, \gamma_0^\star]$, the feedback law (7.4) with $\gamma(t)$ updated by update law I as given in (7.5) achieves*

$$\lim_{t\to\infty} X(t) = 0, \quad \lim_{t\to\infty} U(t) = 0. \tag{7.35}$$

Also,

$$\lim_{t\to\infty} \gamma(t)$$

exists and is positive.

Proof Let

$$h = \gamma_0$$

in the definition of $\hat{\tau}(t)$ and take $\mathcal{V}(X_t, \gamma(t))$ given by (7.27) as the Lyapunov functional. Choose

$$b_1 = 3 > 0, \quad b_2 = 50\left(\tau + 1 + \frac{1}{2}\tau\alpha\gamma_0^{q-1}\right)^2 > 0,$$

in (7.27), and

$$\rho = \frac{1}{4\left(\tau + 1 + \frac{1}{2}\tau\alpha\gamma_0^{q-1}\right)} > 0$$

in (7.34). Define a positive constant

$$\gamma_0^* = (2\alpha)^{\frac{1}{1-q}}.$$

In view of update law I,

$$0 \leq \dot{\hat{\tau}}(t) = \alpha\gamma_0 \frac{V^p(t)}{V^p(t)+\beta}\gamma^{q-2}(t) \\ \leq \alpha\gamma_0^{q-1}, \tag{7.36}$$

which implies that the inequality

$$2n\alpha b_2\left(\gamma_0^{q-\frac{1}{2}} + 2\gamma_0^q\right) \leq 1$$

suffices for

$$-\frac{\gamma_0}{\hat{\tau}^2}\dot{\hat{\tau}} + 2b_2\left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 n\gamma_0^{\frac{3}{2}} + 4b_2\left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 n\gamma_0^2 \leq 0$$

to hold. Replacing $\dot{\hat{\tau}}$ by its upper bound $\alpha\gamma_0^{q-1}$ on the right-hand side of (7.34), we see that the non-positiveness of $\dot{V}(X_t, \gamma(t))$ is guaranteed by the following inequalities:

$$\begin{cases} -1 + 6b_2 n^2(n+1)\gamma_0^2 + 3b_2 n(n+1)\gamma_0 < 0, \\ 2 - b_1 + 6b_2 n\gamma_0 + 12b_2 n^2\gamma_0^2 < 0, \\ 2n\alpha b_2\left(\gamma_0^{q-\frac{1}{2}} + 2\gamma_0^q\right) \leq 1, \\ -1 + 2n\gamma_0 + 2\gamma_0^{\frac{1}{2}} + 4\alpha\gamma_0^{q-1} < 0, \\ \alpha\gamma_0^{q-1} < 1, \\ 2 + b_2\left(\alpha\gamma_0^{q-1} - 1\right) + 6b_2 n\gamma_0 + 12b_2 n^2\gamma_0^2 < 0, \\ -1 + 2\rho\dfrac{|\tilde{\tau}| + \frac{1}{2}\tau\alpha\gamma_0^{q-1}}{\hat{\tau}} < 0, \\ \dfrac{2b_1}{\rho}\dfrac{|\tilde{\tau}| + \frac{1}{2}\tau\alpha\gamma_0^{q-1}}{\hat{\tau}} + 8b_2\alpha\gamma_0^{q-1} - b_2 < 0. \end{cases} \tag{7.37}$$

Note that the last two inequalities in (7.37) still contain the time-varying term $\hat{\tau}(t)$. By the nonincreasing monotonicity of $\gamma(t)$, we obtain

$$\hat{\tau}(t) \geq 1,$$

which implies that

$$\left(|\tilde{\tau}| + \frac{1}{2}\tau\alpha\gamma_0^{q-1}\right)/\hat{\tau} \leq \tau + 1 + \frac{1}{2}\tau\alpha\gamma_0^{q-1}.$$

Thus, the value of ρ makes the last but one inequality in (7.37) hold. On the other hand, the last inequality of (7.37) holds if

$$8\left(\tau+1+\frac{1}{2}\tau\alpha\gamma_0^{q-1}\right)^2 b_1 < b_2\left(1-8\alpha\gamma_0^{q-1}\right). \tag{7.38}$$

In view of (7.37), (7.38), and the values of b_1, b_2, and ρ, we have that if

$$\begin{cases} b_2\alpha\gamma_0^{q-1}+6b_2n\gamma_0+12b_2n^2\gamma_0^2 < b_2-2, \\ \max\Big\{3b_2n(n+1)\gamma_0+6b_2n^2(n+1)\gamma_0^2,\ \ 6b_2n\gamma_0+12b_2n^2\gamma_0^2, \\ 2n\alpha b_2\left(\gamma_0^{q-\frac{1}{2}}+2\gamma_0^q\right),\ \ 2n\gamma_0+2\gamma_0^{\frac{1}{2}}+4\alpha\gamma_0^{q-1},\ \ 16\alpha\gamma_0^{q-1}\Big\} < 1, \end{cases}$$

then (7.37) holds. Notice that b_2 approaches

$$50(\tau+1)^2$$

as

$$\gamma_0 \to 0^+.$$

This shows that there exists a sufficiently small $\gamma_0^\star \le \gamma_0^*$ such that, for each $\gamma_0 \in (0, \gamma_0^\star]$,

$$\begin{aligned} \dot{\mathcal{V}}(X_t,\gamma(t)) \le&\ -\mu\gamma(t)V(t) \\ \le&\ 0, \end{aligned} \tag{7.39}$$

where

$$\begin{aligned} \mu =&\ 1-3b_2n(n+1)\gamma_0-6b_2n^2(n+1)\gamma_0^2 \\ >&\ 0 \end{aligned}$$

and we have used (7.34).

Fix a $\gamma_0 \in (0, \gamma_0^\star]$. It follows from

$$h=\gamma_0,\ \ \gamma_0 \le \gamma_0^*$$

and Remark 7.8 that there exists $t_1 > 0$ such that

$$\mathcal{V}(X_t,\gamma(t)) \in C[t_1,\infty).$$

In fact, Remark 7.7 and the proof of Lemma 7.2 imply that

$$\dot{\mathcal{V}}(X_t,\gamma(t)) \in C[t_1,\infty).$$

Then, by the use of the mean value theorem and (7.39), we have

$$\mathcal{V}(X_t, \gamma(t)) \leq \mathcal{V}(X_{t_1}, \gamma(t_1)), \quad t \in [t_1, \infty).$$

Furthermore, (7.39) implies that

$$\begin{aligned}\int_{t_1}^{t} \mu\gamma(s)V(s)\mathrm{d}s \leq & -\int_{t_1}^{t} \dot{\mathcal{V}}(X_s, \gamma(s))\mathrm{d}s \\ = & \ \mathcal{V}(X_{t_1}, \gamma(t_1)) - \mathcal{V}(X_t, \gamma(t)) \\ \leq & \ \mathcal{V}(X_{t_1}, \gamma(t_1)), \quad t \geq t_1, \end{aligned} \tag{7.40}$$

where we have used the fundamental theorem of calculus based on the fact that

$$\dot{\mathcal{V}}(X_t, \gamma(t)) \in C[t_1, \infty).$$

By update law I, we obtain

$$\frac{\mathrm{d}\gamma(t)}{\gamma^{q-1}(t)} = -\alpha \frac{V^{p-1}(t)}{V^p(t) + \beta} V(t)\gamma(t)\mathrm{d}t. \tag{7.41}$$

When $q = 2$, integrating both sides of (7.41) from t_1 to t gives

$$\begin{aligned}\gamma(t) = & \ \gamma(t_1)\exp\left(-\alpha \int_{t_1}^{t} \gamma(s)V(s)\frac{V^{p-1}(s}{V^p(s) + \beta}\mathrm{d}s\right), \\ \geq & \ \gamma(t_1)\exp\left(-\frac{\alpha}{\beta}\int_{t_1}^{t} \gamma(s)V(s)\mathrm{d}s\mathcal{V}^{p-1}(X_{t_1}, \gamma(t_1))\right) \\ \geq & \ \gamma(t_1)\exp\left(-\frac{\alpha}{\beta\mu}\mathcal{V}^p(X_{t_1}, \gamma(t_1))\right) \\ > & \ 0, \quad t \geq t_1, \end{aligned} \tag{7.42}$$

where

$$p \geq 1, \quad V(t) \leq \mathcal{V}(X_t, \gamma(t)),$$

and (7.39) and (7.40) are used. When $q > 2$, following a similar procedure leads to

$$\begin{aligned}\gamma(t) \geq & \ \left(\gamma^{2-q}(t_1) + \frac{\alpha}{\mu\beta}(q-2)\mathcal{V}^p(X_{t_1}, \gamma(t_1))\right)^{\frac{1}{2-q}} \\ > & \ 0, \quad t \geq t_1. \end{aligned} \tag{7.43}$$

The lower bounds on $\gamma(t)$ in (7.42) and (7.43) and the nonincreasing monotonicity of $\gamma(t)$ imply that

$$\lim_{t\to\infty} \gamma(t)$$

exists and is positive.

It remains to prove the regulation to zero of $X(t)$ and $U(t)$. By (7.40), the positive lower bounds of $\gamma(t)$ in (7.42) and (7.43), and the boundedness of $X(t)$ on $t \in [0, t_1]$, we get the square integrability of $X(t)$ on $[0, \infty)$. Moreover, the positive lower bounds of $\gamma(t)$ in (7.42) and (7.43), the continuity of $X(t)$, and

$$V(t) \le \mathcal{V}(X_{t_1}, \gamma(t_1))$$

on $t \ge t_1$ imply the boundedness of $X(t)$ on $[0, \infty)$, which leads to the boundedness of $\dot{X}(t)$ in view of (7.1) and (7.4). Then,

$$\lim_{t\to\infty} X(t) = 0$$

follows from the Barbalat's lemma (Lemma 2.14 in [92]), and

$$\begin{aligned}\lim_{t\to\infty} U(t) =& - B^{\mathrm{T}} P\Big(\lim_{t\to\infty} \gamma(t)\Big) \lim_{t\to\infty} X(t) \\ =&0.\end{aligned}$$

This completes the proof. □

Proposition 7.2 reveals the regulation effect of update law I when γ_0 is small. To illustrate the global behavior of the closed-loop system under update law I with respect to γ_0, we present a regulation result concerning an arbitrarily large γ_0 in the following proposition.

Proposition 7.3 *Under the feedback law (7.4) with $\gamma(t)$ updated by update law I as given in (7.5), for any given $\gamma_0 > 0$,*

$$\liminf_{t\to\infty} |X(t)| = 0, \quad \liminf_{t\to\infty} |U(t)| = 0, \tag{7.44}$$

and

$$\lim_{t\to\infty} \gamma(t) \tag{7.45}$$

exists and is positive. Moreover, if $X(t)$ is bounded on $t \in [0, \infty)$, then

$$\lim_{t\to\infty} X(t) = 0, \quad \lim_{t\to\infty} U(t) = 0. \tag{7.46}$$

Proof We consider the following two cases.

a) There exists $t_0 > 0$ such that

$$\gamma(t_0) \leq \gamma_0^\star,$$

where $\gamma_0^\star$ is as described in Proposition 7.2. Reset the starting point of system evolution at $t = t_0$, and define the initial conditions as $X(\theta)$ and $\gamma(\theta)$, $\theta \in [t_0 - \tau, t_0]$. Then, the regulation of $X(t)$ and $U(t)$ to zero as time goes to infinity, and the fact that

$$\lim_{t \to \infty} \gamma(t)$$

exists and is positive, are straightforward from Proposition 7.2.

b) There does not exist $t_0 > 0$ such that

$$\gamma(t_0) \leq \gamma_0^\star.$$

This implies

$$\gamma(t) > \gamma_0^\star, \quad t \geq 0,$$

With the nonincreasing monotonicity of $\gamma(t)$, we have that

$$\lim_{t \to \infty} \gamma(t)$$

exists and is positive. Denote this limit as $\underline{\gamma}$, which satisfies

$$\underline{\gamma} \geq \gamma_0^\star.$$

We claim that

$$\liminf_{t \to \infty} |X(t)| = 0.$$

Suppose the opposite. There exist $\nu > 0$ and $t_1 > 0$ such that

$$|X(t)| \geq \nu, \quad t \geq t_1.$$

Then,

$$\begin{aligned} V(t) \geq\ & \lambda_{\min}(P(\gamma_0^\star))\nu^2 \\ \triangleq\ & M \end{aligned}$$

on $t \in [t_1, \infty)$. By update law I,

$$\dot{\gamma}(t) \le \frac{-\alpha}{1+\frac{\beta}{M^p}}\gamma^q(t),$$

from which we have

$$\gamma(t) \le \left(\gamma^{1-q}(t_1) + \frac{\alpha(q-1)}{1+\frac{\beta}{M^p}}(t-t_1)\right)^{\frac{1}{1-q}} \to 0 \ \text{ as } \ t \to \infty.$$

This contradicts the fact that

$$\gamma(t) > \gamma_0^\star > 0$$

on $t \in [0, \infty)$, and the claim follows.

It follows from

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) > 0$$

(see Lemma 2.4),

$$|U(t)| \le |B||P(\gamma_0)||X(t)|$$

and

$$\liminf_{t\to\infty}|X(t)| = 0$$

that

$$\liminf_{t\to\infty}|U(t)| = 0.$$

The remainder of the proof is to show the regulation to zero of $X(t)$ and $U(t)$ under the assumption that $X(t)$ is bounded on $t \in [0, \infty)$.

Consider the factor

$$\frac{V^p(t)}{V^p(t)+\beta}$$

on the right-hand side of (7.5). We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{V^p(t)}{V^p(t)+\beta}\right) = \frac{p\beta V^{p-1}(t)}{(V^p(t)+\beta)^2}\Big(2X^{\mathrm{T}}(t)P(\gamma(t))\Big(AX(t)$$
$$-BB^{\mathrm{T}}P(\gamma(t-\tau))X(t-\tau)\Big) - X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}X(t)\alpha$$
$$\times\frac{V^p(t)}{V^p(t)+\beta}\gamma^q(t)\Big),$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{V^p(t)}{V^p(t)+\beta}\right)\right| \leq \frac{p}{\beta}\lambda_{\max}^{p-1}(P(\gamma_0))|X(t)|^{2(p-1)}\Big(2|A||P(\gamma_0)||X(t)|^2$$
$$+2|B|^2|P(\gamma_0)|^2|X(t)||X(t-\tau)|$$
$$+\alpha\gamma_0^q \max_{\gamma\in[\underline{\gamma},\gamma_0]}\left\{\frac{\partial P}{\partial \gamma}\right\}|X(t)|^2\Big)$$
$$< \infty.$$

In view of its boundedness, we deduce that the factor

$$\frac{V^p(t)}{V^p(t)+\beta}$$

is uniformly continuous. On the other hand, (7.5) implies that

$$\int_0^t \frac{V^p(s)}{V^p(s)+\beta}\mathrm{d}s = \frac{-1}{\alpha}\int_0^t \frac{\mathrm{d}\gamma(s)}{\gamma^q(s)}$$
$$= \frac{\gamma^{1-q}(t)-\gamma_0^{1-q}}{\alpha(q-1)}$$
$$\leq \frac{\underline{\gamma}^{1-q}}{\alpha(q-1)},$$

from which it follows that

$$\frac{V^p(t)}{V^p(t)+\beta} \in L^1.$$

Therefore, by the Barbalat's lemma (Lemma 2.16 in [92]),

$$\lim_{t\to\infty}\frac{V^p(t)}{V^p(t)+\beta} = 0,$$

and thus

$$\lim_{t\to\infty} V(t) = 0.$$

By the fact that

$$\gamma(t) > \gamma_0^\star$$

on $t \in [0, \infty)$ and the use of the squeeze theorem of limit, we obtain (7.46). □

Remark 7.9 Proposition 7.3 shows that, given any γ_0, update law I regulates $\gamma(t)$ to a positive constant, but the regulation of $X(t)$ and $U(t)$ to zero requires the boundedness of $X(t)$. Under update law I alone, we are unable to exclude the case where $X(t)$ is not bounded while still concurring with (7.44) and (7.45) exists. To guarantee that $X(t)$ is bounded, we introduce update law II and the switching mechanism to ensure that after some finite time, update law I remains in effect with a bounded $X(t)$. □

Before proceeding to the proof of Theorem 7.2, we establish a result on the regulation of the Lyapunov function $V(t)$ to zero by the update algorithm that is assumed to achieve the regulation of $\gamma(t)$ to zero. This regulation result is the key to designing the event-triggered switching mechanism between the two update laws for $\gamma(t)$ in the sense that an update law switches only when the value of $V(t)$ crosses a certain threshold.

Proposition 7.4 *If the closed-loop system consisting of (7.1) and (7.4) with $\gamma(t)$ updated by the update algorithm achieved*

$$\lim_{t\to\infty} \gamma(t) = 0,$$

then

$$\lim_{t\to\infty} V(t) = 0.$$

Proof Consider the Lyapunov functional $\mathcal{V}(X_t, \gamma(t))$ in (7.27). Pick

$$b_1 = 3, \quad b_2 = 100$$

of $\mathcal{V}(X_t, \gamma(t))$ in (7.27),

$$\rho = 1/8$$

in (7.34), and a small $h > 0$ such that

$$\max\Big\{ 6b_2n^2(n+1)h^2 + 3b_2n(n+1)h, \;\; 6b_2nh + 12b_2n^2h^2, \\ b_2Dh + 6b_2nh + 12b_2n^2h^2, \;\; 8b_2Dh, \;\; 2nh + 2h^{\frac{1}{2}} + 4Dh, \\ 2b_2nDh^{\frac{3}{2}}\left(1 + 2h^{\frac{1}{2}}\right)\Big\} < 1, \tag{7.47}$$

and

$$\left(1 - 6b_2n^2(n+1)h^2 - 3b_2n(n+1)h\right)2b_2h \le \min\left\{\frac{b_1}{2}, b_2 - 32b_1 - 8b_2Dh,\right.$$
$$\left. 1 - 2nh, -2h^{\frac{1}{2}} - 4Dh\right\}, \tag{7.48}$$

where

$$D = \max\left\{\alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2}\right\},$$

which, according to the update algorithm in Sect. 7.3, is an upper bound of

$$\left|\frac{\dot{\gamma}(t)}{\gamma^2(t)}\right|.$$

Note from Lemma 7.2 that such a selection of h implies that there exists $t_1 > 0$ such that

$$\mathcal{V}(X_t, \gamma(t)) \in C[t_1, \infty).$$

In view of the boundedness of

$$\frac{\dot{\gamma}(t)}{\gamma^2(t)},$$

and the continuity of $\gamma(t)$ on $t \in [0, \infty)$, as given in Theorem 7.1, we compute by using the comparison lemma,

$$\gamma(t) \ge \frac{1}{\frac{1}{\gamma_0} + Dt}, \quad t \ge 0. \tag{7.49}$$

The definition of $\hat{\tau}(t)$ and the boundedness of

$$\frac{\dot{\gamma}(t)}{\gamma^2(t)}$$

imply that

$$\begin{aligned} 0 &\le \dot{\hat{\tau}}(t) \\ &= -\frac{h}{\gamma^2(t)}\dot{\gamma}(t) \\ &\le Dh. \end{aligned} \tag{7.50}$$

Therefore, the inequality

$$2b_2nDh^{\frac{3}{2}}\left(1+2h^{\frac{1}{2}}\right) \le 1,$$

which is implied by (7.47), leads to

$$-\frac{h}{\hat{\tau}^2}\dot{\hat{\tau}} + 2b_2n\left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 h^{\frac{3}{2}} + 4b_2n\left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 h^2 \le 0.$$

The boundedness of

$$\frac{\dot{\gamma}(t)}{\gamma^2(t)}$$

implies that

$$\left|\frac{\dot{\gamma}(t)}{\gamma(t)}\right| \le D\gamma(t).$$

By

$$\lim_{t\to\infty}\gamma(t) = 0$$

and the use of the squeeze theorem of limit, we obtain

$$\lim_{t\to\infty}\frac{\dot{\gamma}(t)}{\gamma(t)} = 0,$$

and thus

$$\lim_{t\to\infty}\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)} = 0.$$

Then,

$$\frac{|\tilde{\tau}| + \frac{1}{2}\tau\dot{\hat{\tau}}}{\hat{\tau}} \le \frac{\tau}{h}\gamma(t) + 1 + \frac{1}{2}\tau\frac{\dot{\hat{\tau}}}{\hat{\tau}} \to 1 \text{ as } t\to\infty,$$

indicating that there exists $t_2 \ge t_1$ such that, for each $t \ge t_2$,

$$\frac{|\tilde{\tau}| + \frac{1}{2}\tau\dot{\hat{\tau}}}{\hat{\tau}} \le 2.$$

In view of (7.34) and (7.50), we deduce that the non-positiveness of $\dot{\mathcal{V}}(X_t, \gamma(t))$ on $t \geq t_2$ is guaranteed by (7.47). Therefore,

$$\dot{\mathcal{V}}(X_t, \gamma(t)) \leq -\kappa\gamma(t)V(t) - \sigma\left(||\tilde{u}(t)||^2 + ||\hat{w}(t)||^2 + ||\hat{w}_x(t)||^2\right), \quad t \geq t_2,$$

where

$$\begin{aligned}\kappa &= 1 - 6b_2n^2(n+1)h^2 - 3b_2n(n+1)h \\ &> 0\end{aligned}$$

and

$$\begin{aligned}\sigma &= \min\left\{\frac{b_1}{2}, b_2 - 32b_1 - 8b_2Dh, 1 - 2nh - 2h^{\frac{1}{2}} - 4Dh\right\} \\ &> 0.\end{aligned}$$

The definition of $\mathcal{V}(X_t, \gamma(t))$ leads to

$$V(t) \geq \mathcal{V}(X_t, \gamma(t)) - W(t)\left(||\tilde{u}(t)||^2 + ||\hat{w}(t)||^2 + ||\hat{w}_x(t)||^2\right),$$

where

$$W(t) = \max\left\{2b_1\tau, 2b_2\hat{\tau}(t)\right\},$$

and thus

$$\begin{aligned}\dot{\mathcal{V}}(X_t, \gamma(t)) \leq &- \kappa\gamma(t)\mathcal{V}(X_t, \gamma(t)) \\ &+ (\kappa W(t)\gamma(t) - \sigma)\left(||\tilde{u}(t)||^2 + ||\hat{w}(t)||^2 + ||\hat{w}_x(t)||^2\right).\end{aligned}$$

Note from (7.48) and

$$\lim_{t\to\infty} \gamma(t) = 0$$

that there exists $t_3 \geq t_2$ such that, for $t \geq t_3$,

$$\kappa W(t)\gamma(t) - \sigma \leq 0.$$

This implies that

$$\dot{\mathcal{V}}(X_t, \gamma(t)) \leq -\kappa\gamma(t)\mathcal{V}(X_t, \gamma(t)), \quad t \geq t_3.$$

By the use of the definition and the continuity of $\mathcal{V}(X_t, \gamma(t))$, and (7.49), we obtain by using the comparison lemma that, for each $t \in [t_3, \infty)$,

$$\begin{aligned} V(t) &\leq \mathcal{V}(X_t, \gamma(t)) \\ &\leq \mathcal{V}(X_{t_3}, \gamma(t_3)) \exp\left(-\kappa \int_{t_3}^{t} \gamma(s)\mathrm{d}s\right) \\ &\leq \mathcal{V}(X_{t_3}, \gamma(t_3)) \left(\frac{1 + D\gamma_0 t}{1 + D\gamma_0 t_3}\right)^{-\frac{\kappa}{D}}. \end{aligned}$$

Proposition 7.4 then follows from the squeeze theorem of limit. □

We now present the proof of Theorem 7.2.

Proof of Theorem 7.2 We claim that the number of switches between update law I and update law II is finite on $t \in [0, \infty)$. Suppose the number of switches from update law I to update law II is infinite, then there are an infinite number of disjoint time intervals on which update law II is in effect. By Remarks 7.4 and 7.5, and the fact that

$$\dot{\gamma}(t) \leq 0, \quad t \in [0, \infty),$$

we get

$$\lim_{t \to \infty} \gamma(t) = 0.$$

Therefore,

$$\lim_{t \to \infty} V(t) = 0$$

follows from Proposition 7.4. On the other hand, an infinite number of switches from update law I to update law II and Remark 7.5 imply that there exists a sequence $\{t_k\}_{k=0}^{\infty}$ such that

$$\lim_{k \to \infty} t_k = \infty$$

and

$$V(t_k) = \epsilon, \quad k \in \mathbb{N}.$$

This contradicts with

$$\lim_{t \to \infty} V(t) = 0.$$

Therefore, the claim follows.

We next claim that the last switch between the two update laws happens from update law II to update law I. Suppose the opposite. Denote the time instant of the last switch as t_{l}. Then, $\gamma(t)$ evolves according to update law II on $t \geq t_{\text{l}}$ and the closed-loop system under update law II satisfies

$$V(t_{\text{l}} + \delta_i) \geq \epsilon, \quad i \in \mathbb{N},$$

where δ_i is as defined in (7.8) with T_{12} replaced by t_{l}. By (7.6), we obtain

$$\lim_{t \to \infty} \gamma(t) = 0.$$

It then follows from Proposition 7.4 that

$$\lim_{t \to \infty} V(t) = 0,$$

which again contradicts the fact that

$$V\,(t_{\text{l}} + \delta_i) \geq \epsilon, \quad i \in \mathbb{N}.$$

We conclude that the last switch happens from update law II to update law I. Note that a natural consequence of this conclusion is that

$$V(t) < \epsilon, \quad t \geq t_{\text{I}},$$

according to the switching condition from update law I to update law II, where t_{I} is the time instant of the last switch and is also the time instant of the system evolution from which update law I remains in effect all the time. Recall from Proposition 7.3 that $\gamma(t)$ is bounded below by a positive constant. Therefore, $X(t)$ is bounded on $t \geq t_{\text{I}}$. The regulation of $X(t)$ and $U(t)$ to zero then follows directly from Proposition 7.3. □

Remark 7.10 The assumption on system (7.1) that all its open loop poles are at the origin can be relaxed to that all its open loop poles are at the origin or in the open left-half plane. Without loss of generality, we assume that the pair (A, B) has the following stability structural decomposition:

$$A = \begin{bmatrix} A_{\text{L}} & 0 \\ 0 & A_{\text{O}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\text{L}} \\ B_{\text{O}} \end{bmatrix},$$

where $A_{\text{L}} \in \mathbb{R}^{n_{\text{L}} \times n_{\text{L}}}$ is Hurwitz, all eigenvalues of $A_{\text{O}} \in \mathbb{R}^{n_{\text{O}} \times n_{\text{O}}}$ are at the origin, and $n_{\text{L}} + n_{\text{O}} = n$. Accordingly, we decompose system (7.1) into the following two subsystems:

$$\begin{cases} \dot{X}_{\text{L}}(t) = A_{\text{L}} X_{\text{L}}(t) + B_{\text{L}} U(t - \tau), \\ \dot{X}_{\text{O}}(t) = A_{\text{O}} X_{\text{O}}(t) + B_{\text{O}} U(t - \tau), \end{cases}$$

where

$$X(t) = \left[X_{\mathrm{L}}^{\mathrm{T}}(t)\ X_{\mathrm{O}}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$$

is the corresponding decomposition of the state vector $X(t)$, $U(t)$ is constructed for the X_{o} subsystem by following the design of our proposed control scheme. By Theorem 7.2, the regulation of the X_{o} subsystem is achieved under the constructed $U(t)$. It is then clear that the regulation of the whole system is achieved. □

7.7 A Numerical Example

Consider a linear system (7.1) with

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The unknown input delay is

$$\tau = 1$$

and the initial condition of the state is given by

$$\psi(\theta) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}, \quad \theta \in [-\tau, 0].$$

Note that (A, B) is controllable with all eigenvalues of A located at the origin. The parameters of our control scheme are chosen as

$$\alpha = 1, \quad \beta = 1, \quad p = 1, \quad q = 2, \quad \zeta = 1, \quad r = 2, \quad \epsilon = 1, \quad \xi = 1.1,$$

and the initial condition of $\gamma(t)$ is set to be

$$\phi(\theta) = 0.3, \quad \theta \in [-\tau, 0].$$

The evolutions of the closed-loop signals including $X(t)$, $U(t)$, $V(t)$, $\gamma(t)$, $\dot{\gamma}(t)$, and $\dot{\gamma}(t)/\gamma^2(t)$ are shown in Figs. 7.1, 7.2, and 7.3.

Ample information on the mechanism of the update algorithm for $\gamma(t)$ can be observed from Figs. 7.1, 7.2, and 7.3. With

$$\begin{aligned} V(0) &= 2.7 \\ &> \epsilon \\ &= 1, \end{aligned}$$

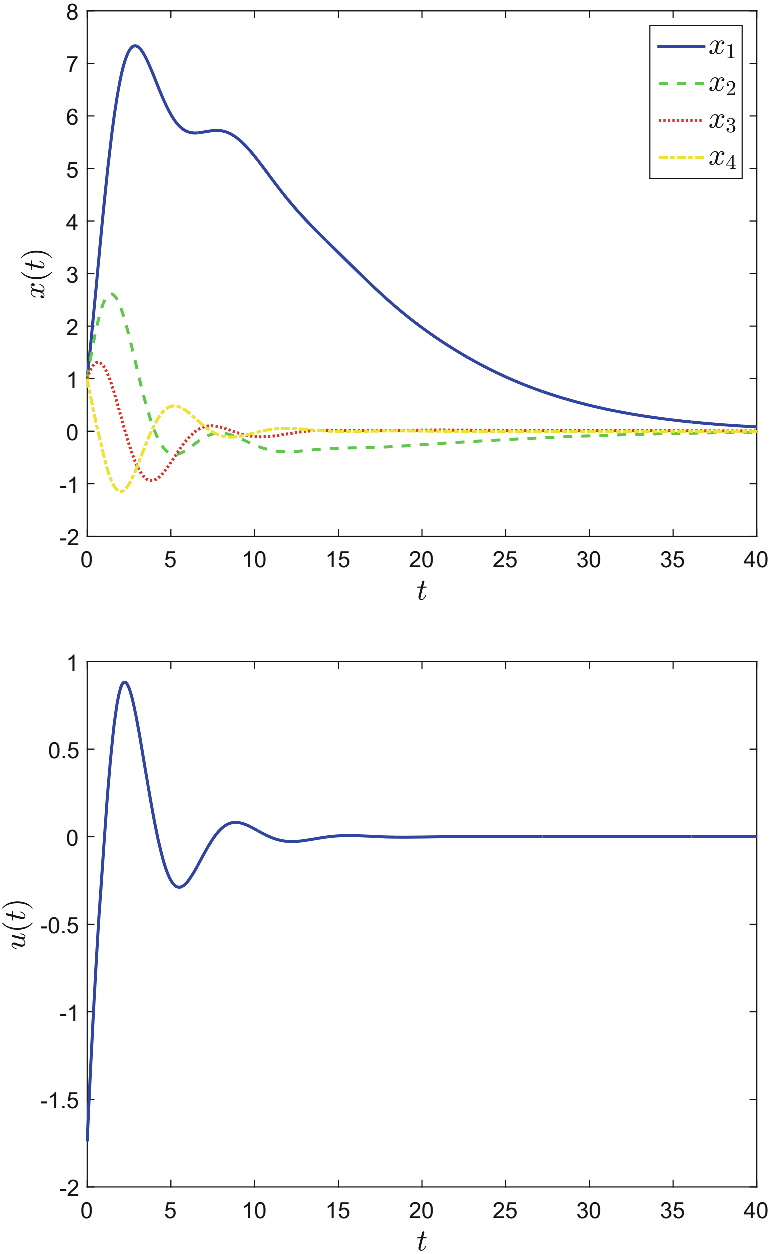


Fig. 7.1 Evolutions of the state and the input of the closed-loop system

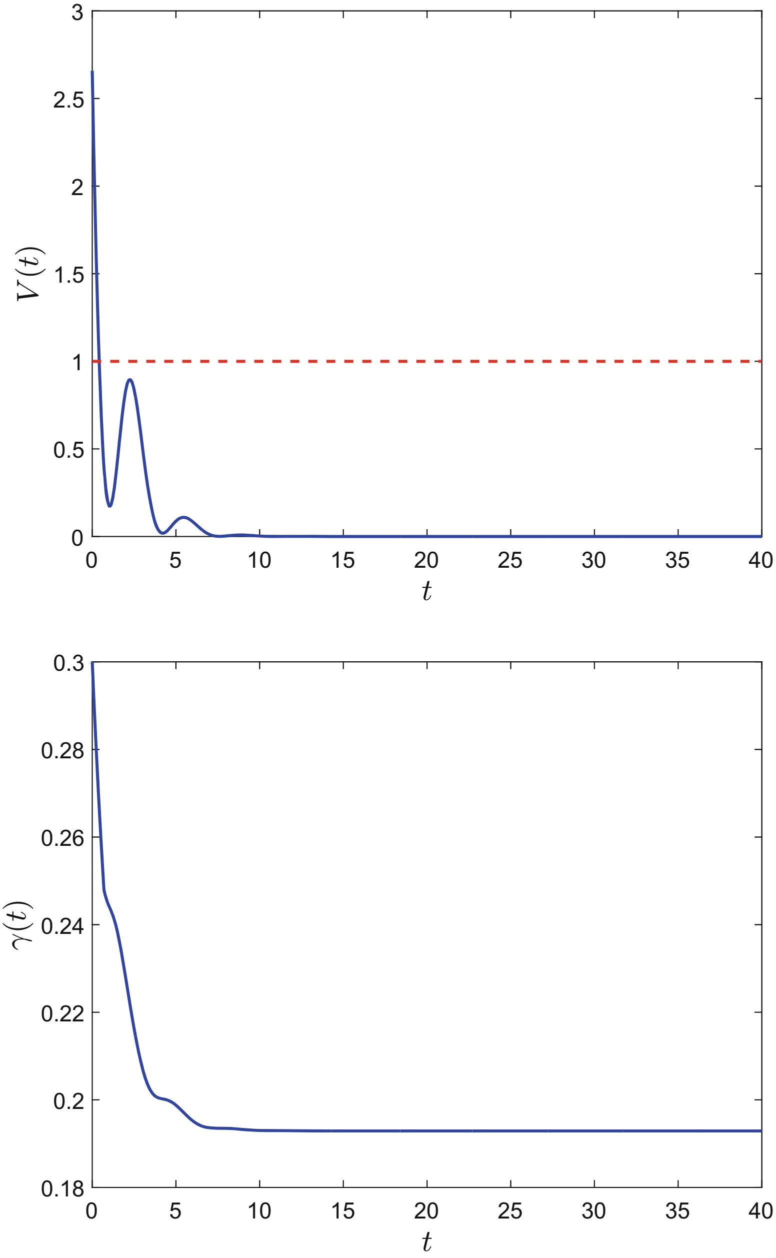


Fig. 7.2 Evolutions of the closed-loop signals $V(t)$ and $\gamma(t)$

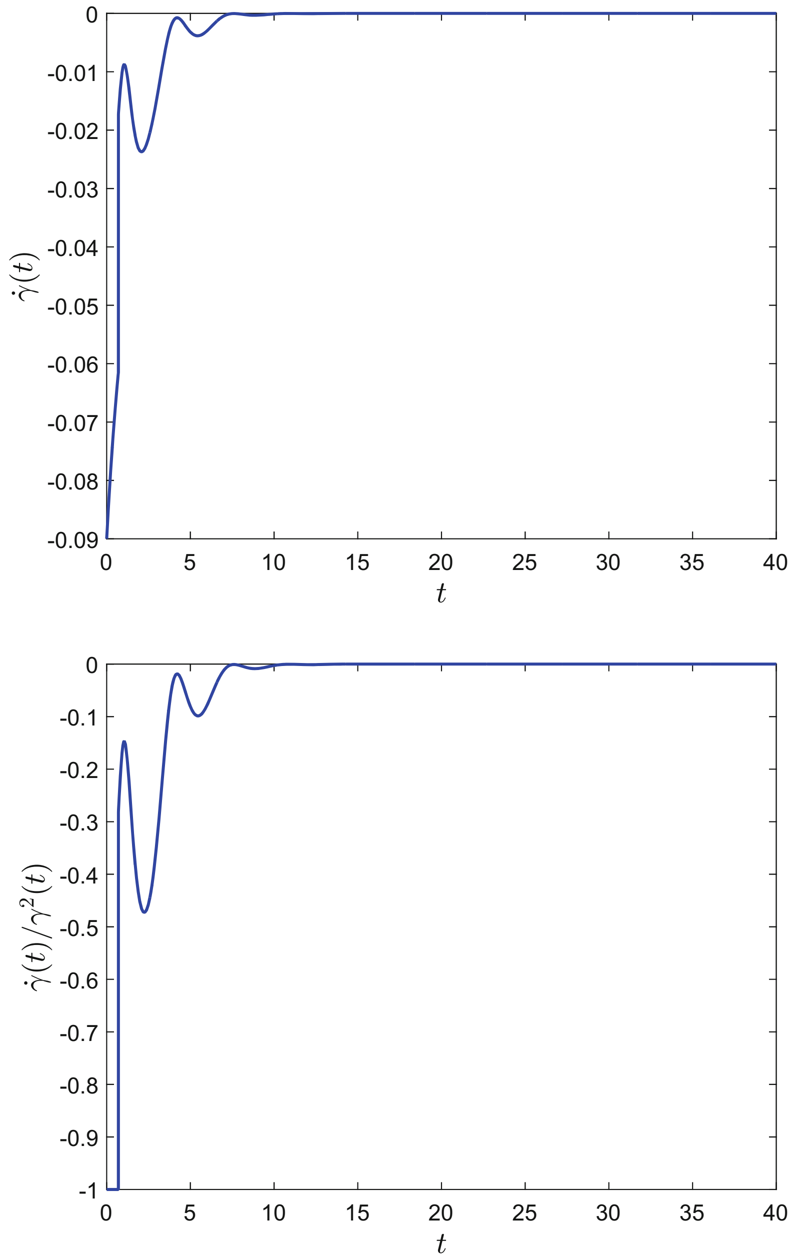


Fig. 7.3 Evolutions of the closed-loop signals $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$

update law II is implemented at the beginning phase of the system evolution. A switch from update law II to update law I does not happen at

$$t = \delta_1$$
$$= 0.3$$

because

$$V(\delta_1) > 1.$$

However, a switch from update law II to update law I happens at

$$t = \delta_2 = 0.7$$

because

$$V(\delta_2) < 1.$$

Note from the plot of $\dot{\gamma}(t)/\gamma^2(t)$ that the first switch, which in fact is the only switch between update laws I and II, happens at

$$t = \delta_2$$

by the discontinuity of $\dot{\gamma}(t)/\gamma^2(t)$ at

$$t = \delta_2.$$

After

$$t = \delta_2,$$

update law I remains in effect all the time because

$$V(t) < 1, \quad t \geq \delta_2.$$

It is interesting to mention here that a switch from update law II to update law I does not happen at $t = 0.4$ when $V(t)$ crosses the threshold $\epsilon = 1$, as marked in the plot of $V(t)$. This is because the switching condition from update law II to update law I is checked only at the isolated time instants

$$T_{\text{I,II}} + \delta_i, \quad i \in \mathbb{N},$$

where in this simulation,

$$T_{\text{I,II}} = 0.$$

To study the regulation performance of the control scheme, we carry out more simulation runs. In this simulation, we pick a larger

$$\gamma_0 = 1.$$

The rest of system parameters are the same as those in the previous simulation, except that

$$\epsilon = 50.$$

Figures 7.4, 7.5, and 7.6 illustrate the evolutions of the closed-loop signals with such a choice of γ_0. Note from the evolution of $X(t)$ that the overshoot and the convergence time of $X(t)$ are larger than those in the previous simulation. This is caused by the excessively large value of γ_0, which tends to destabilize the system at the starting phase of system evolution. The feedback law (7.4) starts to stabilize the system only after $\gamma(t)$ decreases to a small value. Also, we observe from the evolution of $\dot{\gamma}(t)/\gamma^2(t)$ that $\gamma(t)$ is updated by update law II only on the time interval $t \in [0.8135, 2.0154]$.

The regulation performance of the control scheme can also be examined under larger values of τ. Consider a closed-loop system whose system parameters are the same as those in the first simulation except that $\tau = 5$ and $\epsilon = 10$. Picking a larger ϵ is due to the expectation of a large overshoot of the closed-loop system in the presence of a large τ. The closed-loop evolution is presented in Figs. 7.7, 7.8, and 7.9. Consider an even larger $\tau = 10$ and an $\epsilon = 10$. The simulation results are presented in Figs. 7.10, 7.11, and 7.12. Note that, as the value of τ increases, the regulation effects of the control scheme becomes weaker, resulting in a larger overshoot and a slower convergence rate of the closed-loop system. This can be easily explained by the low gain nature of the feedback law (7.4). Smaller values of the feedback parameter increase the ability to achieve regulation at the cost of slower convergence rate of the closed-loop system. We also note that the computation of our control scheme is heavier in comparison with the delay independent truncated predictor feedback law (7.2) with a constant feedback parameter due to the adaptation of the time-varying feedback parameter. In particular, the real-time solution of the algebraic Riccati equation (7.3) contributes significantly to the computational burden.

Remark 7.11 As seen in Sect. 7.3, the update algorithm for $\gamma(t)$ provides a parameter space in which we can choose the values of $(\alpha, \beta, p, q, \zeta, r, \gamma_0)$ freely to achieve the regulation of the system. Given an open loop system, the analysis on the closed-loop performance with respect to the choice of the parameters in the space remains to be carried out. □

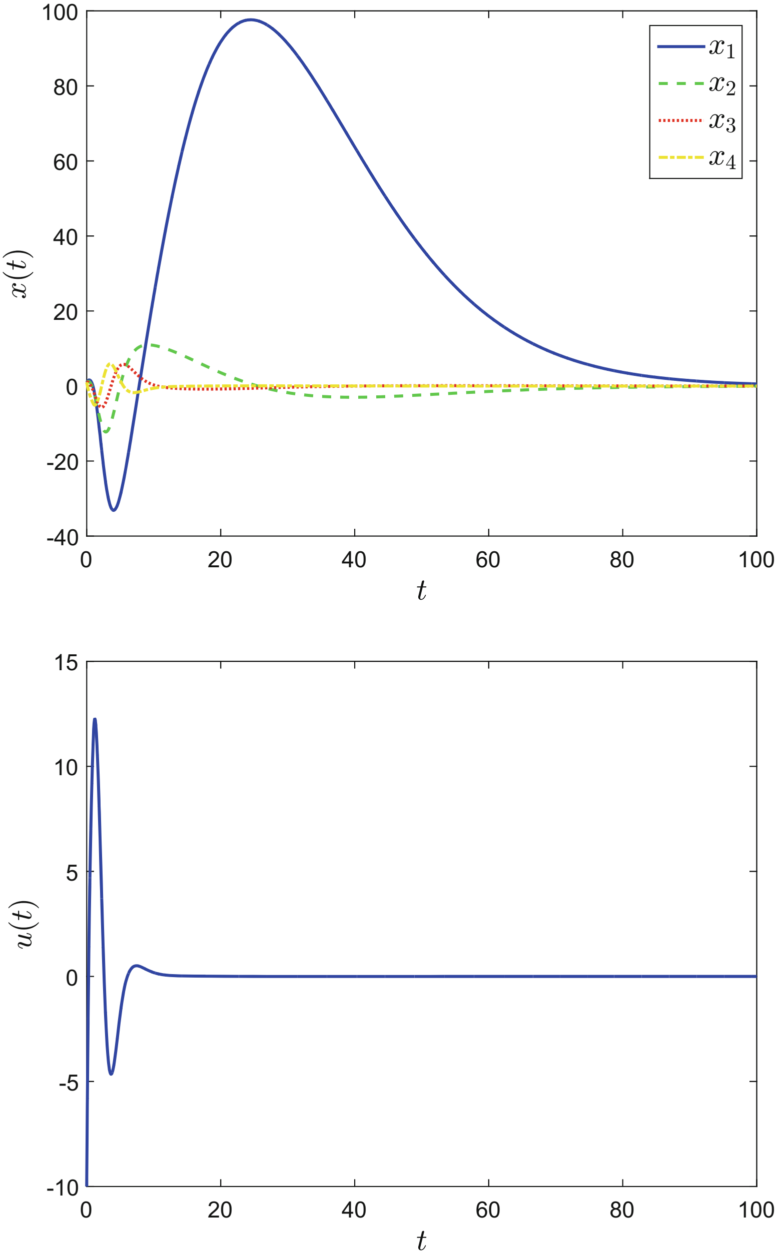


Fig. 7.4 Evolutions of the state and the input of the closed-loop system

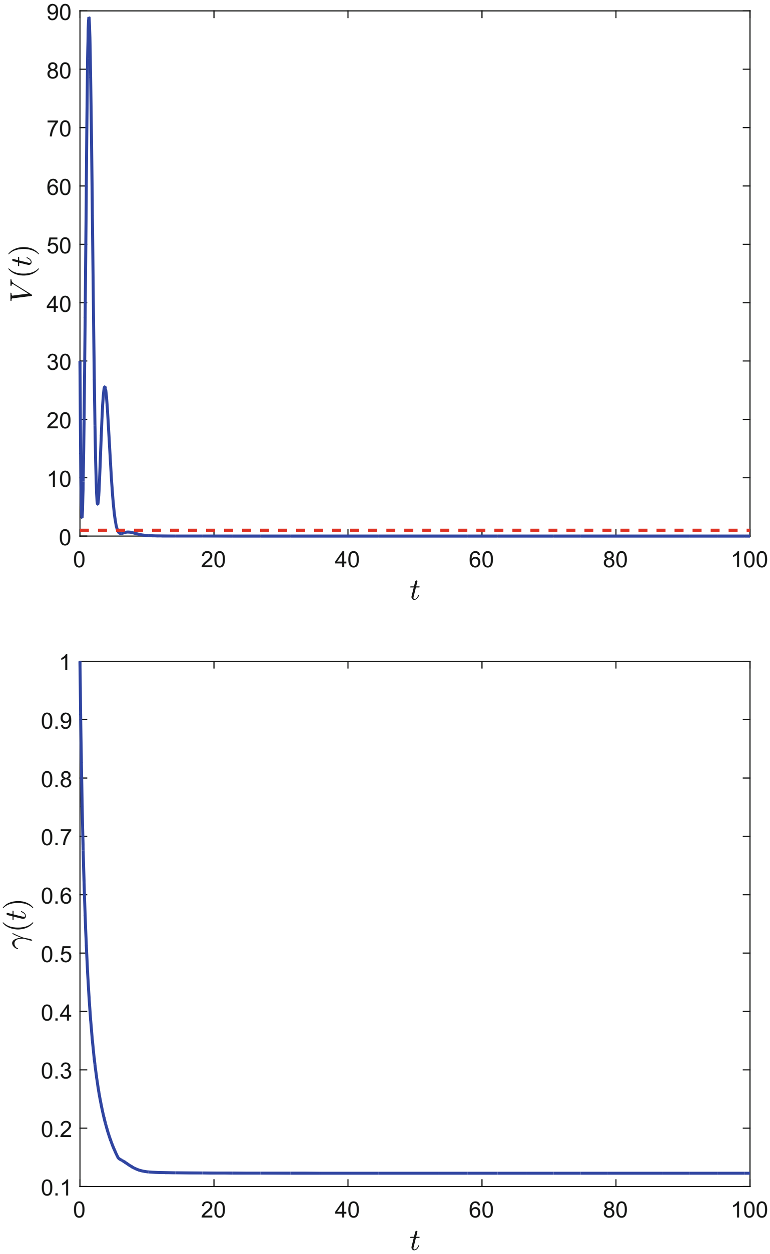
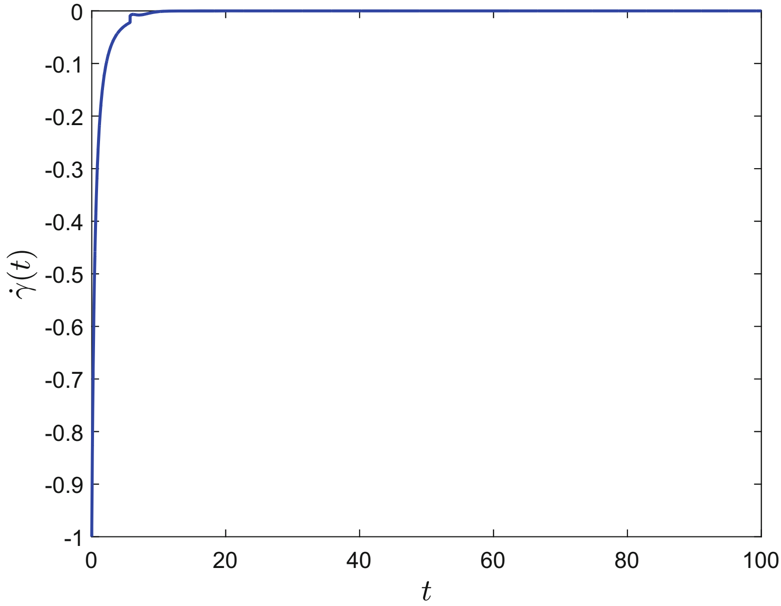


Fig. 7.5 Evolutions of the closed-loop signals $V(t)$ and $\gamma(t)$



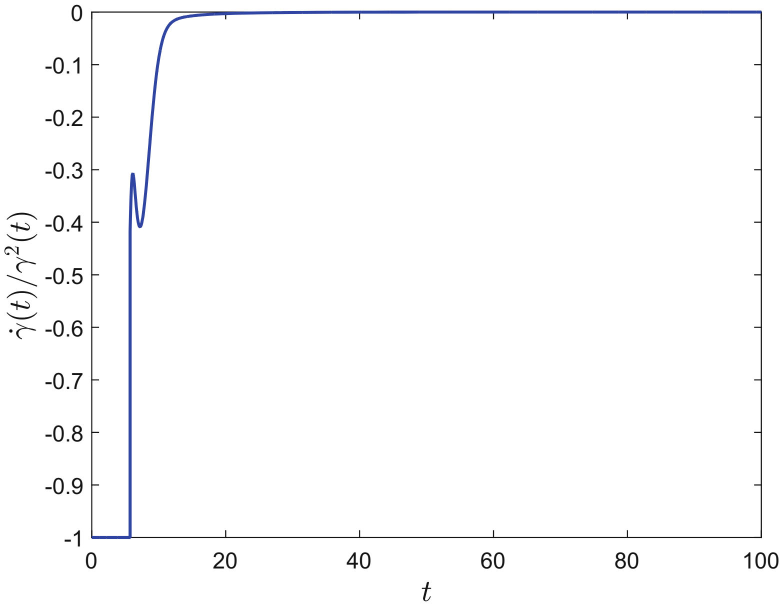


Fig. 7.6 Evolutions of the closed-loop signals $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$

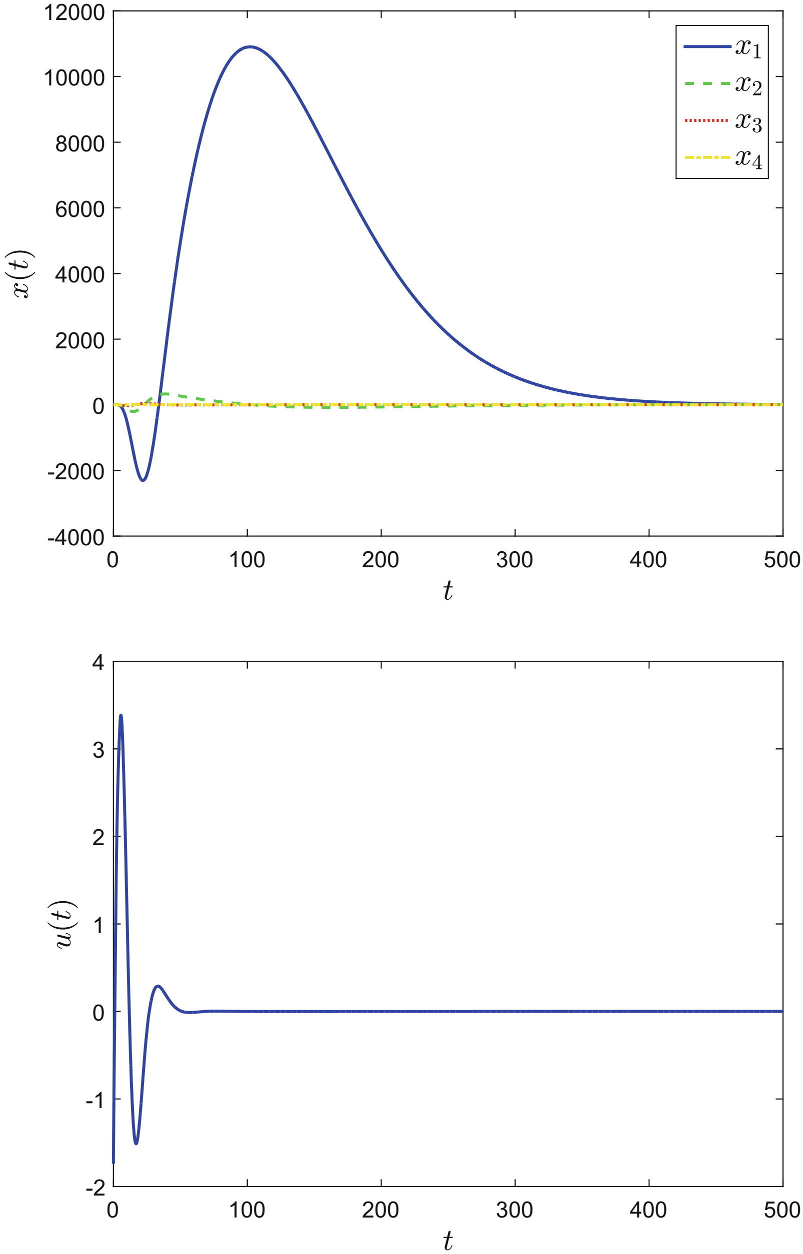


Fig. 7.7 Evolutions of the state and the input of the closed-loop system

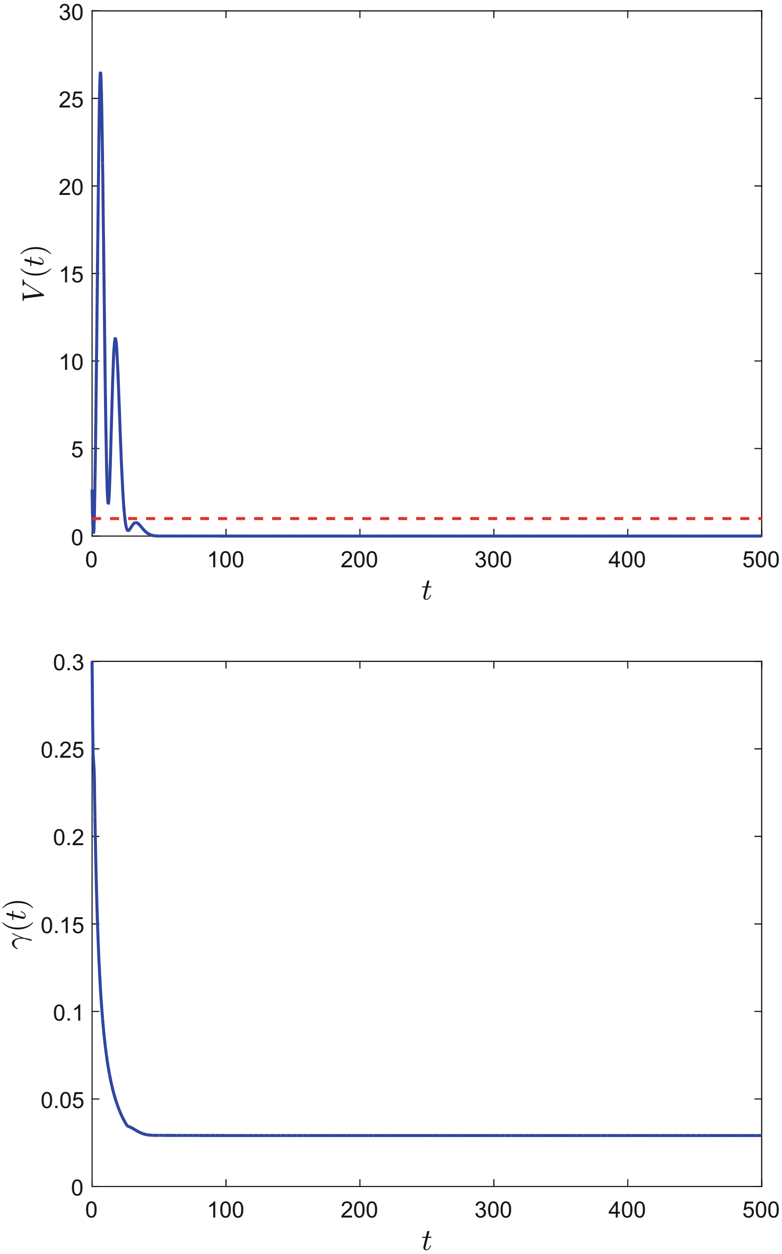


Fig. 7.8 Evolutions of the closed-loop signals $V(t)$ and $\gamma(t)$

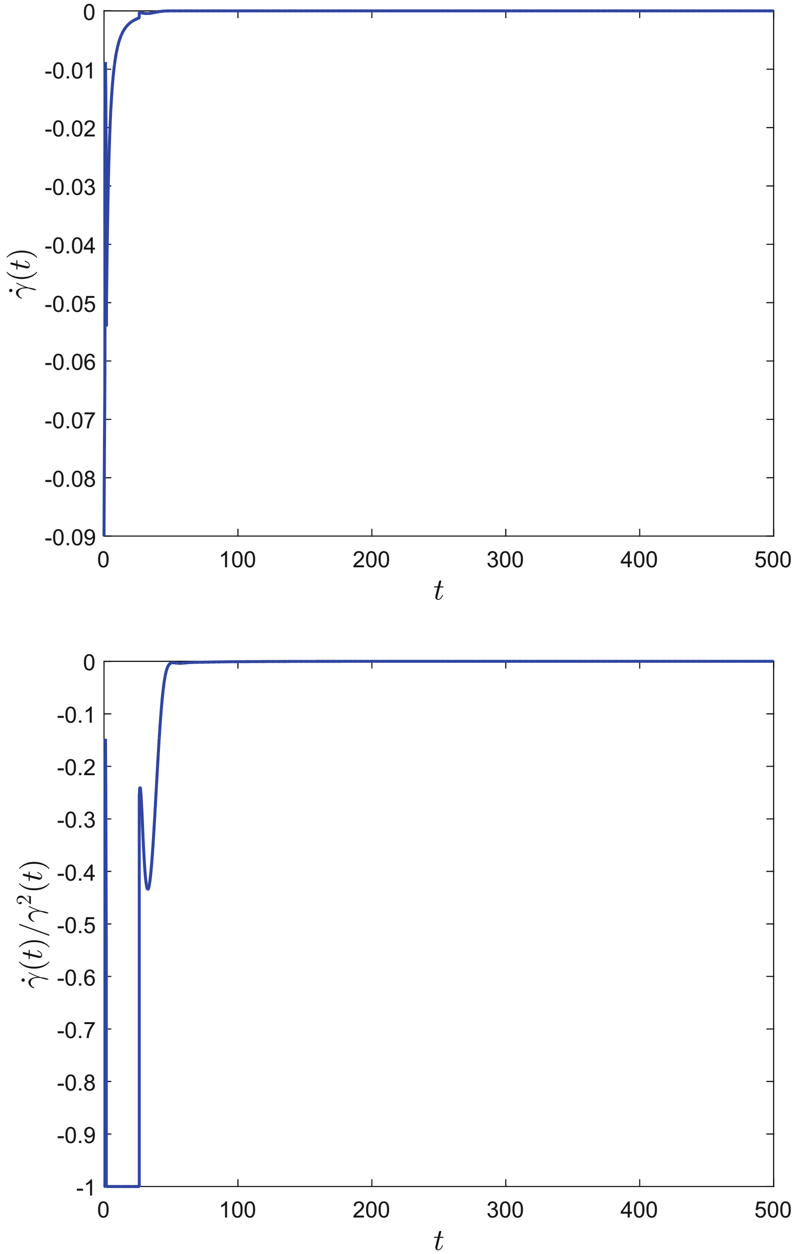


Fig. 7.9 Evolutions of the closed-loop signals $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$

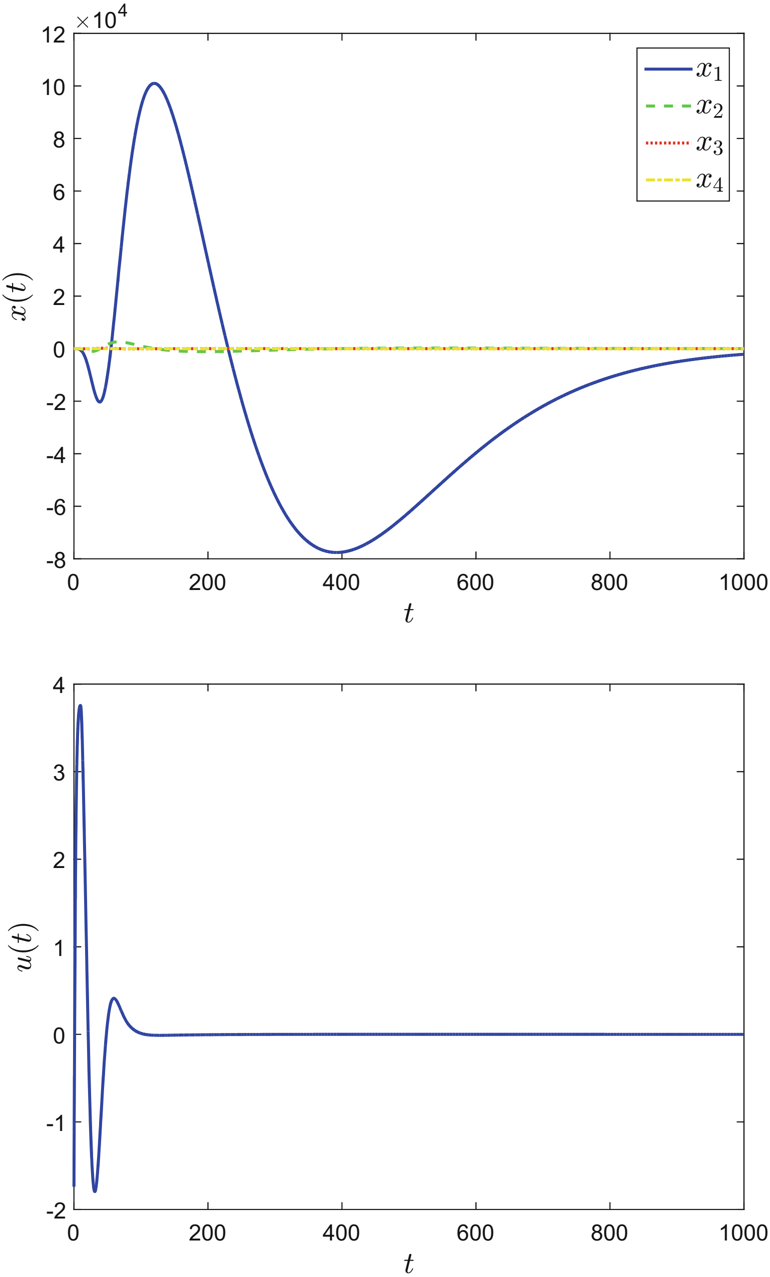


Fig. 7.10 Evolutions of the state and the input of the closed-loop system

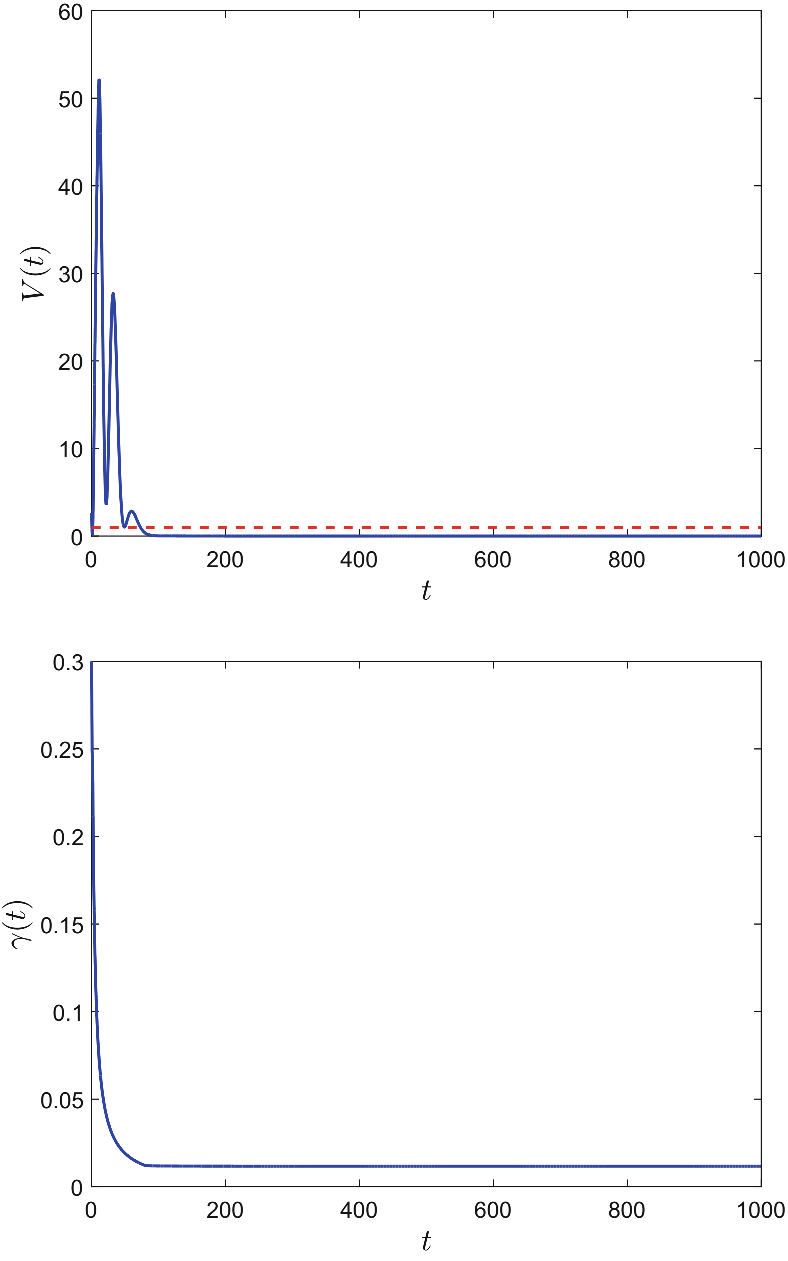


Fig. 7.11 Evolutions of the closed-loop signals $V(t)$ and $\gamma(t)$

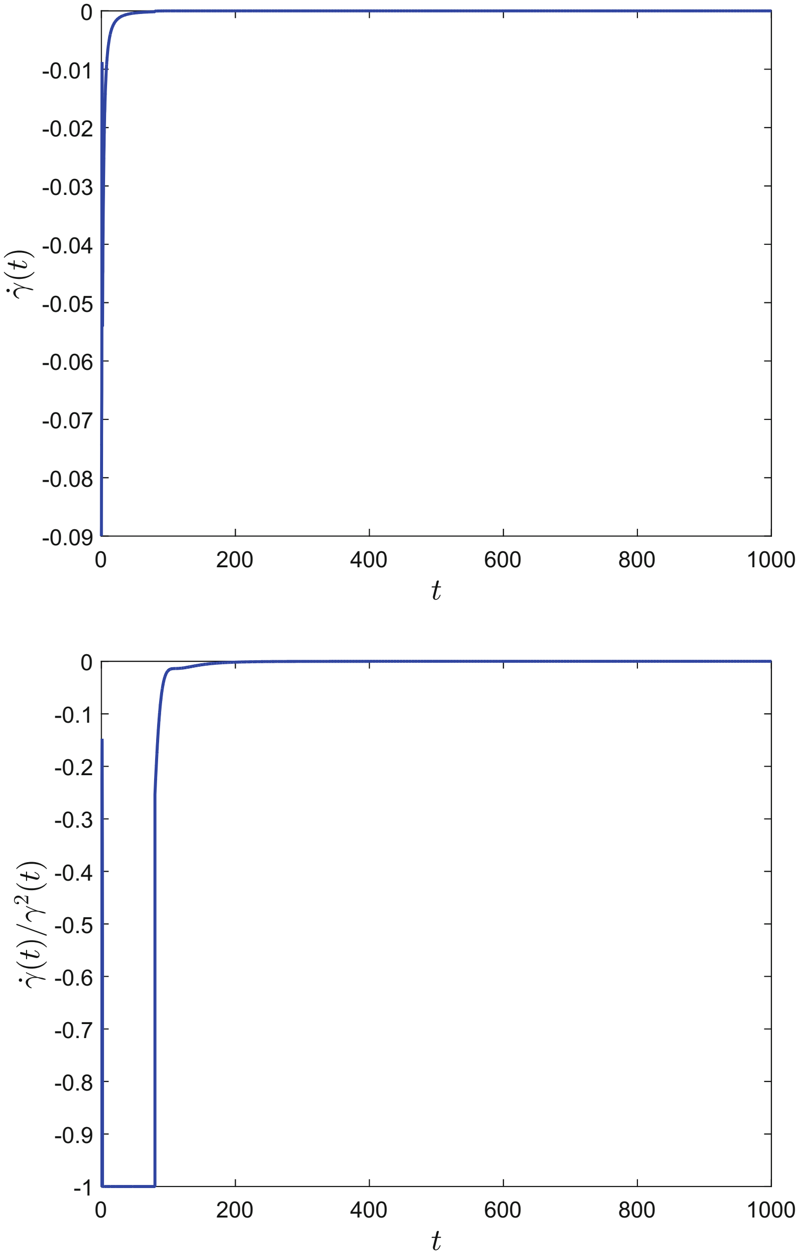


Fig. 7.12 Evolution of the closed-loop signals $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$

7.8 Conclusions

For an input delayed linear system with open loop poles at the origin or in the open left-half plane, the regulation of its state and control input is achieved without any knowledge of the delay. This is made possible by the adaptation of the delay independent truncated predictor feedback law with a time-varying feedback parameter. An update algorithm for the feedback parameter is proposed to compensate an arbitrarily large unknown delay. The use of only the current state as the feedback contributes to the non-distributed nature of the control scheme. A limitation of the results in this chapter is the restriction of the open loop systems to those with poles at the origin or in the open left-half plane. Generalization of the method for the regulation of more general systems entails further investigation.

7.9 Notes and References

This chapter is presented by following the presentation in [107].

Chapter 8
Regulation of Discrete-Time Linear Input Delayed Systems Without Delay Knowledge

8.1 Introduction

Regulation of linear systems without using any knowledge of the delay in the input is an open problem that is known to be challenging. This is the case both in the continuous-time setting and in the discrete-time setting. This chapter presents a solution to this problem for a discrete-time linear system with an arbitrarily large bounded input delay. When the system has all its open loop poles at $z = 1$ or inside the unit circle, an adaptive feedback law is proposed to regulate the state and the input of the system to zero as time tends to infinity. The main features of the feedback law are its accommodation to unknown delay and its memorylessness. No knowledge of the delay, not even the knowledge of its variation with time and its upper bound, is needed in the implementation of the feedback law. Moreover, only the current state is used for feedback. These two features of the feedback law contribute to the simplicity of its implementation. The simplicity of our adaptive control design in turn entails a delicate regulation analysis. A new paradigm of regulation analysis is developed that does not follow any Lyapunov type analysis for time delay systems. Numerical study demonstrates the analyzed regulation results and further provides an indication of the robustness of our control scheme to exponentially unstable open loop poles.

We briefly recall the design of the predictor feedback law for a discrete-time linear system with input delay. Consider a discrete-time linear system subject to input delay,

$$x(k+1) = Ax(k) + Bu(k-r),$$

where x and u represent the state and the input of the system, respectively, and r is the amount of input delay, the predictor feedback law takes the form of

$$u(k) = Fx(k+r),$$

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5_8

where F is the feedback gain matrix such that $A + BF$ is Schur stable, that is, all the eigenvalues of $A + BF$ are inside the unit circle. Under the predictor feedback, the closed-loop system

$$x(k+1) = (A + BF)x(k)$$

is asymptotically stable. Obviously, the predictor feedback law is not directly viable for implementation because it requires the future value of the state. Thanks to the linearity of the system, the future state of the system can be explicitly obtained as the solution of the system, which is the sum of the zero input solution and the zero state solution,

$$x(k+r) = A^r x(k) + \sum_{l=k-r}^{k-1} A^{k-1-l} Bu(l).$$

This leads to an explicit expression of the predictor feedback in an implementable form,

$$u(k) = FA^r x(k) + F \sum_{l=k-r}^{k-1} A^{k-1-l} Bu(l).$$

As we can see, the predictor feedback design does not place any restriction on the delay and manages to cancel the effect of the delay.

The predictor feedback law is clearly delay dependent. It requires the precise value of the delay to be known for its implementation. Chapter 3 proposed an easy-to-implement feedback law by truncating the finite summation term of the predictor feedback law,

$$u(k) = FA^r x(k),$$

where

$$F = F(\gamma)$$

is parameterized by the use of an eigenstructure assignment based low gain feedback design (see [61]). Such a parametrization enables the truncated predictor feedback law to compensate for an arbitrarily large delay in a discrete-time linear system with all open loop poles on or inside the unit circle. Still, the value of the delay appears in the state transition matrix A^r of the truncated predictor feedback law. In Chap. 3, an even simplified feedback law,

$$u(k) = F(\gamma)x(k),$$

was proposed that does not explicitly contain any information of the delay. It was pointed out that, by the same parametrization of the feedback gain matrix, such a delay independent truncated predictor feedback law also compensates for an arbitrarily large delay in a discrete-time linear system with all open loop poles at $z = 1$ or inside the unit circle. An alternative low gain feedback design technique, the Lyapunov equation based design, was adopted in Chap. 6, to reproduce the stabilization results in Chap. 3.

The delay independent truncated predictor feedback law in Chap. 6 is not completely independent of any knowledge of the delay. An upper bound of the delay is required to determine the value of the low gain feedback parameter γ involved in the parametrization of the feedback gain matrix $F = F(\gamma)$. We are thus still one step away from designing a feedback law that does not require any knowledge of the delay. An effort was made in [90] toward excluding any knowledge of the delay in a feedback design that is based on the delay independent truncated predictor feedback. However, the adaptive feedback law therein requires its low gain parameter to be switched and to check a switching condition on every step of the system evolution. Because the switching condition involves the past values of the state in a time window whose length is correlated with the amount of the delay, the adaptive feedback law in [90] is not memoryless.

In this chapter, we provide a much more simplified solution to this challenging problem of regulating, without any knowledge of the input delay, a discrete-time linear system whose open loop poles are at $z = 1$ or inside the unit circle. The simplicity of the solution lies in the complete delay independence of our proposed adaptive feedback law that accommodates any bounded input delay and its memorylessness property. To be specific, the proposed feedback law takes the form of the delay independent truncated predictor state feedback law with an online-updated low gain parameter, and only the current state is used as the feedback signal. The simplicity of our adaptive control scheme however incurs difficulty in the analysis of its regulation effects. A new paradigm for closed-loop analysis is developed that differentiates itself from any Lyapunov type analysis.

Inspired by the Lyapunov analysis method, we first define a quadratic-like function in terms of the state of the closed-loop system and its updated low gain parameter. By analyzing the forward difference of the quadratic-like function along the trajectory of the closed-loop system, we arrive at the boundedness of the quadratic-like function as time tends to infinity. Unlike any Lyapunov type analysis for time delay systems, the negativeness of the forward difference of the quadratic-like function is unnecessary in our analysis. Meanwhile, we establish the boundedness of the summation of the product of the low gain parameter and the quadratic-like function. These two boundedness properties, together with the update law for the low gain parameter, imply that the low gain parameter is bounded away from zero. Finally, by using the update law of the low gain parameter, we conclude the regulation of the quadratic-like function and that of the state and the input of the system. The contribution of this chapter is clear from the fact that not only does our adaptive law drastically simplify the existing adaptive law in [90], but also our regulation analysis adopts a completely new paradigm of proof, which is expected to be applicable to a wide range of regulation problems for time delay systems. Simulation study verifies the proven regulation results and further sheds light upon the robustness of our adaptive feedback design to the presence of open loop poles that are exponentially unstable.

This chapter solves the counterpart of the regulation problem for a continuous-time linear system without delay knowledge in the discrete-time setting. Unlike the adaptive control scheme in the continuous-time linear setting in Chap. 7, which involves two update laws for the feedback parameter and a switching mechanism between them, the adaptive control scheme in the discrete-time setting presented in this chapter is more elegant and easier to implement. Moreover, we adopt a completely new closed-loop analysis paradigm to prove regulation, while the regulation proof in Chap. 7 relies heavily on PDE analysis. Such simplification in terms of both the adaptive law and its regulation analysis is attributed partly to the reduced complexity in the analysis of discrete-time linear systems compared to their continuous-time counterparts. Because the regulation problems in both continuous-time and discrete-time settings have been solved by taking unparallel approaches, the development in this chapter is expected to inspire establishing PDE analysis for the regulation problem in the discrete-time setting, and vice versa, finding a simplified solution to the regulation problem in the continuous-time setting.

8.2 An Adaptive Feedback Law

Consider the regulation of the state and the input of a discrete-time linear system with a bounded input delay,

$$x(k+1) = Ax(k) + Bu(\phi(k)), \quad k \in \mathbb{N}, \tag{8.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the input, respectively, and (A, B) is stabilizable. It is assumed that all the eigenvalues of A are at $z = 1$ or inside the unit circle. The delayed effect of the input of the system is represented through $\phi(k)$, which is assumed to take the form of

$$\phi(k) = k - r(k).$$

Here, the time-varying delay $r(k)$ is bounded from above by an $R \in \mathbb{N}$, i.e.,

$$r(k) \leq R, \quad k \in \mathbb{N},$$

and the boundedness of $r(k)$ is the only assumption posed on the time-varying delay. This is a rather mild assumption on the input delay, considering that the compensation of an input delay in a discrete-time linear system by the predictor feedback law requires that the inverse function of $\phi(k)$ associated with $r(k)$ exists and is exactly known. In this chapter, not only is the widely adopted assumption on the invertibility of $\phi(k)$ not required, but also the upper bound of the time-varying delay R is not required to be known in the implementation of our adaptive feedback law. The initial condition of the system is given by

$$x(0) \in \mathbb{R}^n, \quad u(\theta) \in \mathbb{R}^m, \quad \theta \in I[-R, 0].$$

Without loss of generality, we assume that the pair (A, B) are in the form of

$$A = \begin{bmatrix} A_{\mathrm{I}} & 0 \\ 0 & A_{\mathrm{O}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{I}} \\ B_{\mathrm{O}} \end{bmatrix},$$

where all the eigenvalues of A_{I} are inside the unit circle and all the eigenvalues of A_{O} are at $z = 1$. Given such a structure, system (8.1) can be decomposed as

$$\begin{cases} x_{\mathrm{I}}(k+1) = A_{\mathrm{I}} x_{\mathrm{I}}(k) + B_{\mathrm{I}} u(\phi(k)), \\ x_{\mathrm{O}}(k+1) = A_{\mathrm{O}} x_{\mathrm{I}}(k) + B_{\mathrm{O}} u(\phi(k)), \end{cases}$$

where $x_{\mathrm{I}}(k)$ and $x_{\mathrm{O}}(k)$ are partitions of $x(k)$ corresponding to A_{I} and A_{O}, respectively. In view of the two subsystems of system (8.1), the regulationof the state and the input of the second subsystem, represented by the evolution of $x_{\mathrm{O}}(k)$, to zero, implies the regulation of the state and the input of the whole system. Since the overall objective of this chapter is to design a feedback law that regulates the state and the input of system (8.1), we, without loss of generality, make the assumption in the rest of the chapter that all the eigenvalues of A are at $z = 1$, and design a feedback law that regulates only the second subsystem of the system. By doing so, our overall control objective remains intact while the complexity of a feedback design and its corresponding regulation analysis will be reduced.

Our adaptive feedback law consists of two parts. The first part defines a state feedback law whose right-hand side is the product of a feedback gain matrix and the current state,

$$u(k) = F(\gamma(k))x(k), \quad k \in \mathbb{N}, \tag{8.2}$$

where, for a given value of $\gamma(k) = \gamma$, the feedback gain matrix $F(\gamma)$ is constructed by using the Lyapunov equation based low gain feedback design (see Chap. 3),

$$F(\gamma) = -\left(I + B^{\mathrm{T}} P(\gamma) B\right)^{-1} B^{\mathrm{T}} P(\gamma) A, \tag{8.3}$$

and $P(\gamma)$ is the unique positive definite solution to the following discrete-time parametric algebraic Riccati equation,

$$A^{\mathrm{T}} P(\gamma) A - P(\gamma) - A^{\mathrm{T}} P(\gamma) B \left(I + B^{\mathrm{T}} P(\gamma) B\right)^{-1} B^{\mathrm{T}} P(\gamma) A = -\gamma P(\gamma), \tag{8.4}$$

with $\gamma \in (0, 1)$. According to Chap. 3, $\gamma \in (0, 1)$ is necessary and sufficient to the existence and uniqueness of such a $P(\gamma)$. Two lemmas on the properties of the solution to the parametric algebraic Riccati equation (8.4) are presented at the end of this section, where a lemma known as Jensen's Inequality in the discrete setting is also recalled. All these three technical lemmas are extensively involved in the closed-loop analysis to be given in the next section.

Note that in (8.2), the feedback gain matrix is parametrized in a time-varying feedback parameter $\gamma(k)$, whose update as the second part of the adaptive feedback law is determined by

$$\gamma(k+1) = \gamma(k) - \alpha \frac{V^p(x(k), \gamma(k))}{V^p(x(k), \gamma(k)) + 1} \gamma^q(k), \quad k \in \mathbb{N}, \tag{8.5}$$

with any initial condition

$$\gamma(0) \in (0, 1).$$

In (8.5), $V(x(k), \gamma(k))$ is defined by

$$V(x(k), \gamma(k)) = x^{\mathrm{T}}(k) P(\gamma(k)) x(k),$$

$\alpha \in (0, 1]$, $p \geq 1$, and q is such that

$$q \geq 2, \quad q > \tilde{q}$$

Here, $\tilde{q}$ is the highest degree of the numerator polynomials in $P(\gamma)$ with respect to γ. Such a $\tilde{q}$ is well defined because by [125], all elements of $P(\gamma)$ are rational function of γ and

$$\lim_{\gamma \to 0^+} P(\gamma) = 0.$$

Clearly $\tilde{q} \geq 1$. Take a simple case where $A = 1$ and $B = 1$ for instance. We compute the solution $P(\gamma)$ to the parametric algebraic Riccati equation (8.4) as

$$P(\gamma) = \frac{\gamma}{1 - \gamma},$$

for which $\tilde{q} = 1$. Therefore, in this case, q in our feedback design is any positive constant that satisfies $q \geq 2$. It is worth mentioning that the knowledge of $\tilde{q}$ is obtained from A and B matrices, which are independent of R. Therefore, the update law (8.5) for $\gamma(k)$ can be implemented without using any knowledge of the time-varying delay $r(k)$.

The adaptive feedback law as a whole consists of the state feedback law (8.2) and the update law (8.5) for the time-varying feedback parameter, both of which are delay independent and utilizes only the current state as the feedback signal. The delay independence property and the memorylessness of our adaptive feedback law are both appealing features of a controller. The proposed feedback law allows easy implementation due to its simple form. Prior to its implementation, it only requires to determine three parameters α, p and q, the first and the second of which can be arbitrarily chosen based on the range of their predetermined values and the last of which can be determined by a simple examination of the matrix $P(\gamma)$. After the determination of these control parameters, the time-varying feedback gain matrix $F(\gamma)$ is updated online according to (8.5), and the regulation of the state and the input of the system is left to the state feedback law (8.2).

To examine the well posedness of our adaptive feedback law, the update law (8.5) for $\gamma(k)$ has to guarantee that

$$\gamma(k) \in (0, 1), \quad k \in \mathbb{N}.$$

The proof of this fact can be split into two parts. On one hand, since $\gamma(0) \in (0, 1)$, $q \geq 2$, and $\alpha \in (0, 1]$,

$$\begin{aligned} \gamma(1) &= \gamma(0)\left(1 - \alpha\frac{V^p(x(0), \gamma(0))}{V^p(x(0), \gamma(0)) + 1}\gamma^{q-1}(0)\right) \\ &> \gamma(0)(1 - \alpha\gamma(0)) \\ &> 0. \end{aligned}$$

Assume that $\gamma(k) > 0$ for some $k \in \mathbb{N} \setminus \{0\}$. Then, by following a similar argument, we have

$$\begin{aligned} \gamma(k+1) &= \gamma(k)\left(1 - \alpha\frac{V^p(x(k), \gamma(k))}{V^p(x(k), \gamma(k)) + 1}\gamma^{q-1}(k)\right) \\ &> \gamma(k)(1 - \alpha\gamma(k)) \\ &> 0. \end{aligned}$$

Thus, $\gamma(k) > 0$, $k \in \mathbb{N}$, is obvious by induction. On the other hand, $\gamma(k)$ is nonincreasing with respect to k according to the update law (8.5). Thus, $\gamma(k) < 1$ for all $k \in \mathbb{N}$. This completes the proof.

In the rest of this section, we present three technical lemmas as preparation for the regulation analysis to be carried out in the next section.

Lemma 8.1 *Let (A, B) be controllable with all eigenvalues of A at $z = 1$. For each $\gamma \in (0, 1)$, the unique positive definite solution $P(\gamma)$ to the parametric algebraic Riccati equation (8.4) satisfies*

$$\begin{aligned} F^{\mathrm{T}}(\gamma)B^{\mathrm{T}}P(\gamma)BF(\gamma) &\leq \varrho(\gamma)P(\gamma), \\ F^{\mathrm{T}}(\gamma)\left(I + B^{\mathrm{T}}P(\gamma)B\right)F(\gamma) &\leq \vartheta(\gamma)P(\gamma), \\ (A_{\mathrm{c}}(\gamma) - I)^{\mathrm{T}}P\left(A_{\mathrm{c}}(\gamma) - I\right) &\leq \varpi(\gamma)P(\gamma), \end{aligned}$$

where

$$\begin{aligned}
F(\gamma) &= -\left(I + B^{\mathrm{T}} P(\gamma) B\right)^{-1} B^{\mathrm{T}} P(\gamma) A, \\
A_{\mathrm{C}}(\gamma) &= A + BF(\gamma), \\
\varrho(\gamma) &= \frac{(1-(1-\gamma)^n)^2}{(1-\gamma)^{2n-1}}, \\
\vartheta(\gamma) &= \frac{1-(1-\gamma)^n}{(1-\gamma)^{n-1}}, \\
\varpi(\gamma) &= \frac{1}{(1-\gamma)^n} - 1 - n\gamma + \frac{(1-(1-\gamma)^n)^2}{(1-\gamma)^{n-1}}.
\end{aligned}$$

Lemma 8.1 can be readily obtained from Lemma 6.1 by letting $\mathrm{tr}(A) = n$ and $\det(A) = 1$. We omit the proof for brevity.

Lemma 8.2 *The polynomials $\varrho(\gamma)$ and $\varpi(\gamma)$ in Lemma 8.1 are infinitesimal quantities of at least second order with respect to γ, and they are strictly increasing with respect to γ. The polynomial $\vartheta(\gamma)$ in Lemma 8.1 is an infinitesimal quantity of at least first order with respect to γ and is strictly increasing with respect to γ. Moreover,*

$$\frac{\vartheta(\gamma)}{\gamma}$$

is nondecreasing with respect to γ.

Proof It is obvious that

$$\lim_{\gamma \to 0^+} \varrho(\gamma) = 0.$$

By

$$\frac{\mathrm{d}\varrho(\gamma)}{\mathrm{d}\gamma} = \frac{2n(1-(1-\gamma)^n)(1-\gamma)^n + (2n-1)(1-(1-\gamma)^n)^2}{(1-\gamma)^{2n}} > 0, \gamma > 0,$$

$\varrho(\gamma)$ is strictly increasing with respect to γ. Also, notice that $\lim_{\gamma \to 0} \frac{\mathrm{d}\varrho(\gamma)}{\mathrm{d}\gamma} = 0$. Thus, $\varrho(\gamma)$ is an infinitesmal quantity of at least second order with respect to γ.

As for $\varpi(\gamma)$, we have

$$\lim_{\gamma\to 0^+} \varpi(\gamma) = 0$$

and

$$\begin{aligned}
&\lim_{\gamma\to 0^+} \frac{\varpi(\gamma)}{\gamma} \\
&= \lim_{\gamma\to 0^+} \frac{\frac{1}{(1-\gamma)^n} - 1 - n\gamma}{\gamma} + \frac{(1-(1-\gamma)^n)^2}{\gamma(1-\gamma)^{n-1}} \\
&= \lim_{\gamma\to 0^+} \frac{1-(1-\gamma)^n}{\gamma(1-\gamma)^n} - n + \frac{(1-(1-\gamma)^n)^2}{\gamma(1-\gamma)^{n-1}} \\
&= \lim_{\gamma\to 0^+} \frac{n(1-\gamma)^{n-1}}{(1-\gamma)^n - \gamma n(1-\gamma)^{n-1}} - n + \frac{2n\,(1-(1-\gamma)^n(1-\gamma))}{(1-\gamma)-\gamma(n-1)} \\
&= 0.
\end{aligned}$$

Thus, $\varpi(\gamma)$ is an infinitesimal quantity of at least second order with respect to γ. Moreover,

$$\begin{aligned}
\frac{\mathrm{d}\varpi(\gamma)}{\mathrm{d}\gamma} &= \frac{n}{(1-\gamma)^{n+1}} - n + \frac{2n(1-(1-\gamma)^n)(1-\gamma)^n + (n-1)\,(1-(1-\gamma)^n)^2}{(1-\gamma)^n} \\
&> 0.
\end{aligned}$$

The rational function $\vartheta(\gamma)$ of γ is strictly increasing because

$$\begin{aligned}
\frac{\mathrm{d}\vartheta(\gamma)}{\mathrm{d}\gamma} &= \frac{n(1-\gamma)^n + (1-(1-\gamma)^n)\,(n-1)}{(1-\gamma)^{n+1}} \\
&> 0.
\end{aligned}$$

The monotonicity of

$$\frac{\vartheta(\gamma)}{\gamma}$$

can be examined as follows:

$$\begin{aligned}
\frac{\mathrm{d}\frac{\vartheta(\gamma)}{\gamma}}{\mathrm{d}\gamma} &= \frac{n\gamma(1-\gamma)^n - (1-(1-\gamma)^n)\,(2-n-\gamma)}{\gamma^2} \\
&= \frac{(n-1)\gamma(1-\gamma)^n + (n-2)\,(1-(1-\gamma)^n) + \gamma}{\gamma^2},
\end{aligned}$$

which is positive when $n \geq 2$. When $n = 1$,

$$\frac{\mathrm{d}\frac{\vartheta(\gamma)}{\gamma}}{\mathrm{d}\gamma} = 0.$$

This completes the proof. □

Lemma 8.3 *For any positive semi-definite matrix $M \geq 0$, two integers r_2 and r_1 with $r_2 \geq r_1$, and a vector valued function $\omega : I[r_1, r_2] \to \mathbb{R}^n$,*

$$\left(\sum_{i=r_1}^{r_2} \omega(i)\right)^{\mathrm{T}} M \left(\sum_{i=r_1}^{r_2} \omega(i)\right) \leq (r_2 - r_1 + 1) \sum_{i=r_1}^{r_2} \omega^{\mathrm{T}}(i) M \omega(i).$$

8.3 Closed-Loop Analysis

The closed-loop analysis is carried out in the following two steps. In the first step, we show that there exists a sufficiently small positive constant such that for any given initial condition of $\gamma(k)$ below such a constant, the regulation of the state and the input of the system (8.1) is achieved. In the second step, we further show that for any given initial condition $\gamma(0) \in (0, 1)$, such regulation can still be achieved. We formulate these two results in two theorems, respectively. To carry out the first step, we establish a new paradigm for the analysis of the closed-loop system. In doing so, we define a quadratic-like function in terms of the state of the system and the low gain parameter, and compute the forward difference of this quadratic-like function along the trajectory of the closed-loop system. We establish the boundedness of the quadratic-like function. The boundedness of the summation of the product of the low gain parameter and the quadratic-like function is also clear from this analysis. We next show that these two boundedness properties, together with the update law for the low gain parameter, imply that the low gain parameter is bounded away from zero. Then, the regulation of the state and the input of the system can be readily obtained by the use of the update law for the low gain parameter. The second step follows from the first step readily.

8.3.1 The Boundedness of $V(x(k), \gamma(k))$ and $\sum_{l=2R}^{\infty} \gamma(l) V(x(l), \gamma(l))$

Under the proposed adaptive feedback law (8.2), along with the update law for the feedback parameter $\gamma(k)$, system (8.1) takes a closed-loop form,

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(\phi(k)) \\ &= Ax(k) + BF(\gamma(\phi(k)))x(\phi(k)) \\ &= A_{\mathrm{C}}(\gamma(k))x(k) + B\left(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)\right), \end{aligned} \tag{8.6}$$

where $A_{\mathrm{C}}(\gamma(k))$ for each $k \in \mathbb{N}$ is defined by

$$A_{\mathrm{C}}(\gamma(k)) = A + BF(\gamma(k)).$$

Define a positive definite function along the trajectory of the closed-loop system,

$$V(x(k), \gamma(k)) = x^{\mathrm{T}}(k)P(\gamma(k))x(k). \tag{8.7}$$

The forward difference of the function (8.7) along the trajectory of the closed-loop system can be computed as

$$\begin{aligned} &\Delta V(x(k), \gamma(k)) \\ &= V(x(k+1), \gamma(k+1)) - V(x(k), \gamma(k)) \\ &= x^{\mathrm{T}}(k+1)P(\gamma(k+1))x(k+1) - x^{\mathrm{T}}(k)P(\gamma(k))x(k) \\ &= \left(A_{\mathrm{C}}(\gamma(k))x(k) + B\left(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)\right)\right)^{\mathrm{T}} P(\gamma(k+1)) \\ &\quad \times \left(A_{\mathrm{C}}(\gamma(k))x(k) + B\left(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)\right)\right) \\ &\quad - x^{\mathrm{T}}(k)P(\gamma(k))x(k) \\ &= x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k+1))A_{\mathrm{C}}(\gamma(k))x(k) + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k+1)) \\ &\quad \times B\left(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)\right) + (F(\gamma(\phi(k)))x(\phi(k)) \\ &\quad - F(\gamma(k))x(k))^{\mathrm{T}}B^{\mathrm{T}}P(\gamma(k+1))B(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\ &\quad - x^{\mathrm{T}}(k)P(\gamma(k))x(k) \\ &= x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k))A_{\mathrm{C}}(\gamma(k))x(k) + x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) \\ &\quad - P(\gamma(k)))A_{\mathrm{C}}(\gamma(k))x(k) + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k+1)) \\ &\quad \times B(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) + (F(\gamma(\phi(k)))x(\phi(k)) \\ &\quad - F(\gamma(k))x(k))^{\mathrm{T}}B^{\mathrm{T}}P(\gamma(k+1))B(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\ &\quad - x^{\mathrm{T}}(k)P(\gamma(k))x(k) \\ &= x^{\mathrm{T}}(k)(A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k))A_{\mathrm{C}}(\gamma(k)) - P(\gamma(k)))x(k) + x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k)) \\ &\quad \times (P(\gamma(k+1)) - P(\gamma(k)))A_{\mathrm{C}}(\gamma(k))x(k) + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k+1)) \\ &\quad \times B(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) + (F(\gamma(\phi(k)))x(\phi(k)) \\ &\quad - F(\gamma(k))x(k))^{\mathrm{T}}B^{\mathrm{T}}P(\gamma(k+1))B(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \end{aligned}$$

$$
\begin{aligned}
&= x^{\mathrm{T}}(k)(-\gamma(k)P(\gamma(k)) - F^{\mathrm{T}}(\gamma(k))F(\gamma(k)))x(k) \\
&\quad + x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))A_{\mathrm{C}}(\gamma(k))x(k) \\
&\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k+1))B(F(\gamma(\phi(k)))x(\phi(k)) \\
&\quad - F(\gamma(k))x(k)) + (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}} \\
&\quad \times B^{\mathrm{T}}P(\gamma(k+1))B(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&= -\gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) - x^{\mathrm{T}}(k)F^{\mathrm{T}}(\gamma(k))F(\gamma(k))x(k) \\
&\quad + x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))A_{\mathrm{C}}(\gamma(k))x(k) \\
&\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))B \\
&\quad \times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k))B(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&\quad + (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}}B^{\mathrm{T}}P(\gamma(k+1))B \\
&\quad \times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)),
\end{aligned}
$$

where we have used the parametric algebraic Riccati equation (8.4) to obtain

$$
A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k))A_{\mathrm{C}}(\gamma(k)) - P(\gamma(k)) = -\gamma(k)P(\gamma(k)) - F^{\mathrm{T}}(\gamma(k))F(\gamma(k)).
$$

By the construction of $F(\gamma)$ as given by (8.3), we get

$$
F(\gamma(k)) = -A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))P(\gamma(k))B,
$$

which implies that

$$
\begin{aligned}
&\Delta V(x(k), \gamma(k)) \\
&= -\gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) - x^{\mathrm{T}}(k)F^{\mathrm{T}}(\gamma(k))F(\gamma(k))x(k) \\
&\quad + x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))A_{\mathrm{C}}(\gamma(k))x(k) \\
&\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))B \\
&\quad \times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&\quad - 2x^{\mathrm{T}}(k)F^{\mathrm{T}}(\gamma(k))(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&\quad + (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}}B^{\mathrm{T}}P(\gamma(k+1))B \\
&\quad \times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&= -\gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) - x^{\mathrm{T}}(k)F^{\mathrm{T}}(\gamma(k))F(\gamma(k))x(k) \\
&\quad + x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))A_{\mathrm{C}}(\gamma(k))x(k) \\
&\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{C}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))B
\end{aligned}
$$

$$
\begin{aligned}
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&- 2x^{\mathrm{T}}(k)F^{\mathrm{T}}(\gamma(k))(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&+ (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}} B^{\mathrm{T}}(P(\gamma(k+1)) - P(\gamma(k)))B \\
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&+ (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}} B^{\mathrm{T}} P(\gamma(k))B \\
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)).
\end{aligned}
$$

By Young's Inequality, the evolution of the forward difference $\Delta V(x(k), \gamma(k))$ can be continued as follows:

$$
\begin{aligned}
&\Delta V(x(k), \gamma(k)) \\
\le& - \gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) - x^{\mathrm{T}}(k)F^{\mathrm{T}}(\gamma(k))F(\gamma(k))x(k) \\
&+ x^{\mathrm{T}}(k)A_{\mathrm{c}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))A_{\mathrm{c}}(\gamma(k))x(k) \\
&+ 2x^{\mathrm{T}}(k)A_{\mathrm{c}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))B \\
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&+ x^{\mathrm{T}}(k)F^{\mathrm{T}}(\gamma(k))F(\gamma(k))x(k) + (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}} \\
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&+ (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}} B^{\mathrm{T}}(P(\gamma(k+1)) - P(\gamma(k)))B \\
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&+ (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}} B^{\mathrm{T}} P(\gamma(k))B \\
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
=& - \gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) + x^{\mathrm{T}}(k)A_{\mathrm{c}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) \\
&- P(\gamma(k)))A_{\mathrm{c}}(\gamma(k))x(k) + 2x^{\mathrm{T}}(k)A_{\mathrm{c}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k)))B \\
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&+ (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}}(I + B^{\mathrm{T}} P(\gamma(k))B) \\
&\times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&+ (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}} B^{\mathrm{T}}(P(\gamma(k+1)) - P(\gamma(k))) \\
&\times B(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)).
\end{aligned}
$$

Because $\gamma(k)$ is updated nonincreasingly with respect to k and $P(\gamma)$ is strictly increasing with respect to γ (see Lemma 3.4),

$$
P(\gamma(k)) \ge P(\gamma(k+1)), \quad k \in \mathbb{N}, \tag{8.8}
$$

which, together with Young's Inequality, imply that

$$\begin{aligned}
&\Delta V(x(k), \gamma(k)) \\
&\leq -\gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) + x^{\mathrm{T}}(k)A_{\mathrm{c}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k+1)) - P(\gamma(k))) \\
&\quad \times A_{\mathrm{c}}(\gamma(k))x(k) + x^{\mathrm{T}}(k)A_{\mathrm{c}}^{\mathrm{T}}(\gamma(k))(P(\gamma(k)) - P(\gamma(k+1)))A_{\mathrm{c}}(\gamma(k))x(k) \\
&\quad + (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}}B^{\mathrm{T}}(P(\gamma(k)) - P(\gamma(k+1))) \\
&\quad \times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&\quad + (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}}(I + B^{\mathrm{T}}P(\gamma(k))B) \\
&\quad \times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&\quad + (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}}B^{\mathrm{T}}(P(\gamma(k+1)) - P(\gamma(k)))B \\
&\quad \times (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)) \\
&= -\gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) + (F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k))^{\mathrm{T}} \\
&\quad \times (I + B^{\mathrm{T}}P(\gamma(k))B)(F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)).
\end{aligned} \tag{8.9}$$

To examine the quadratic term associated with $F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k)$ after the equality sign in (8.9), we compute

$$\begin{aligned}
&F(\gamma(\phi(k)))x(\phi(k)) - F(\gamma(k))x(k) \\
&= (F(\gamma(\phi(k))) - F(\gamma(k)))x(\phi(k)) + F(\gamma(k))(x(\phi(k)) - x(k)),
\end{aligned}$$

which suggests that (8.9) can be continued as

$$\begin{aligned}
\Delta V(x(k), \gamma(k)) \leq &-\gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) + ((F(\gamma(\phi(k))) - F(\gamma(k)))x(\phi(k)) \\
&+ F(\gamma(k))(x(\phi(k)) - x(k)))^{\mathrm{T}}(I + B^{\mathrm{T}}P(\gamma(k))B)((F(\gamma(\phi(k))) \\
&- F(\gamma(k)))x(\phi(k)) + F(\gamma(k))(x(\phi(k)) - x(k))).
\end{aligned}$$

Again, by Young's Inequality, we obtain

$$\begin{aligned}
&\Delta V(x(k), \gamma(k)) \\
&\leq -\gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k) + 2x^{\mathrm{T}}(\phi(k))(F(\gamma(\phi(k))) - F(\gamma(k)))^{\mathrm{T}} \\
&\quad \times (I + B^{\mathrm{T}}P(\gamma(k))B)(F(\gamma(\phi(k))) - F(\gamma(k)))x(\phi(k)) + 2(x(\phi(k)) - x(k))^{\mathrm{T}} \\
&\quad \times F^{\mathrm{T}}(\gamma(k))(I + B^{\mathrm{T}}P(\gamma(k))B)F(\gamma(k))(x(\phi(k)) - x(k)).
\end{aligned}$$

By employing Lemma 8.1 and defining for each $k \in \mathbb{N}$,

$$\vartheta(\gamma(k)) = \frac{1 - (1 - \gamma(k))^n}{(1 - \gamma(k))^{n-1}},$$

we continue the evaluation as

$$
\begin{aligned}
&\Delta V(x(k),\gamma(k))\\
&\le -\gamma(k)x^{\mathrm{T}}(k)P(\gamma(k))x(k)+2x^{\mathrm{T}}(\phi(k))(F(\gamma(\phi(k)))-F(\gamma(k)))^{\mathrm{T}}\\
&\quad\times(I+B^{\mathrm{T}}P(\gamma(k))B)(F(\gamma(\phi(k)))-F(\gamma(k)))x(\phi(k))\\
&\quad+2\vartheta(\gamma(k))(x(\phi(k))-x(k))^{\mathrm{T}}P(\gamma(k))(x(\phi(k))-x(k)).
\end{aligned}
\tag{8.10}
$$

Further examination of $\Delta V(x(k),\gamma(k))$ relies on finding upper bounds of the quadratic terms in (8.10) that are associated with $F(\gamma(\phi(k)))-F(\gamma(k))$ and $x(\phi(k))-x(k)$. Based on Jensen's Inequality as given in Lemma 8.3, the nonincreasing property of $P(\gamma(k))$ as given by (8.8) and the fact that

$$
F(\gamma(\phi(k)))-F(\gamma(k))=\sum_{l=\phi(k)}^{l=k-1}(F(\gamma(l+1))-F(\gamma(l))),
$$

we derive

$$
\begin{aligned}
&(F(\gamma(\phi(k)))-F(\gamma(k)))^{\mathrm{T}}\left(I+B^{\mathrm{T}}P(\gamma(k))B\right)(F(\gamma(\phi(k)))-F(\gamma(k)))\\
&=\left(\sum_{l=\phi(k)}^{l=k-1}(F(\gamma(l+1))-F(\gamma(l)))\right)^{\mathrm{T}}(I+B^{\mathrm{T}}P(\gamma(k))B)\\
&\quad\times\left(\sum_{l=\phi(k)}^{l=k-1}(F(\gamma(l+1))-F(\gamma(l)))\right)\\
&\le R\sum_{l=\phi(k)}^{l=k-1}(F(\gamma(l+1))-F(\gamma(l)))^{\mathrm{T}}\left(I+B^{\mathrm{T}}P(\gamma(k))B\right)(F(\gamma(l+1))-F(\gamma(l)))\\
&\le R\sum_{l=\phi(k)}^{l=k-1}(F(\gamma(l+1))-F(\gamma(l)))^{\mathrm{T}}\left(I+B^{\mathrm{T}}P(\gamma(l))B\right)(F(\gamma(l+1))-F(\gamma(l)))\\
&=R\sum_{l=\phi(k)}^{l=k-1}\Delta F^{\mathrm{T}}(l)\left(I+B^{\mathrm{T}}P(\gamma(l))B\right)\Delta F(l),
\end{aligned}
$$

where $\Delta F(l)$ is defined as

$$
\Delta F(l)=F(\gamma(l+1))-F(\gamma(l)).
$$

By the mean value theorem and the nonincreasing property of $\gamma(k)$ with respective to k, we have

$$\begin{aligned}\Delta F(l) &= \left.\frac{\mathrm{d}F(\gamma)}{\mathrm{d}\gamma}\right|_{\gamma\in[\gamma(l+1),\gamma(l)]}(\gamma(l+1)-\gamma(l))\\ &= -\left.\frac{\mathrm{d}F(\gamma)}{\mathrm{d}\gamma}\right|_{\gamma\in[\gamma(l+1),\gamma(l)]}\alpha\frac{V^p(x(l),\gamma(l))}{V^p(x(l),\gamma(l))+1}\gamma^q(l),\end{aligned}$$

which implies that

$$\begin{aligned}&\Delta F^{\mathrm{T}}(l)\big(I+B^{\mathrm{T}}P(\gamma(l))B\big)\Delta F(l)\\ &\leq \alpha^2\gamma^{2q}(l)\left(\left.\frac{\mathrm{d}F(\gamma)}{\mathrm{d}\gamma}\right|_{\gamma\in[\gamma(l+1),\gamma(l)]}\right)^{\mathrm{T}}\big(I+B^{\mathrm{T}}P(\gamma(l))B\big)\\ &\times\left(\left.\frac{\mathrm{d}F(\gamma)}{\mathrm{d}\gamma}\right|_{\gamma\in[\gamma(l+1),\gamma(l)]}\right).\end{aligned}$$

According to Lemma 3.4, $P(\gamma)$ is a rational matrix in γ and

$$\lim_{\gamma\to 0^+} P(\gamma) = 0. \tag{8.11}$$

These two facts indicate that all the elements of $P(\gamma)$ are infinitesimal quantities of at least first order with respect to γ, which further implies that

$$\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma}$$

is bounded from above over $\gamma \in (0, \gamma_1(0))$, where $\gamma_1(0)$ is any positive constant within $(0, 1)$. Let $\gamma(0) \in (0, \gamma_1(0))$. We compute

$$\frac{\mathrm{d}F(\gamma)}{\mathrm{d}\gamma} = \big(I+B^{\mathrm{T}}P(\gamma)B\big)^{-2}B^{\mathrm{T}}\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma}BB^{\mathrm{T}}P(\gamma)A - \big(I+B^{\mathrm{T}}P(\gamma)B\big)^{-1}B^{\mathrm{T}}\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma}A,$$

from which and the property (8.11), the nonincreasing monotonocity of $\gamma(k)$ and $P(\gamma(k))$ with respect to k, we obtain

$$\left(\left.\frac{\mathrm{d}F(\gamma)}{\mathrm{d}\gamma}\right|_{\gamma\in[\gamma(l+1),\gamma(l)]}\right)^{\mathrm{T}}\big(I+B^{\mathrm{T}}P(\gamma(l))B\big)\left(\left.\frac{\mathrm{d}F(\gamma)}{\mathrm{d}\gamma}\right|_{\gamma\in[\gamma(l+1),\gamma(l)]}\right)\leq\beta I, \tag{8.12}$$

where β is a positive constant. The value of β solely depends on the value of $\gamma_1(0)$ and the pair (A, B).

Inequality (8.12) implies that

$$\begin{aligned} & x^{\mathrm{T}}(\phi(k))(F(\gamma(\phi(k))) - F(\gamma(k)))^{\mathrm{T}} \left(I + B^{\mathrm{T}} P(\gamma(k)) B\right) \\ & \times (F(\gamma(\phi(k))) - F(\gamma(k))) x(\phi(k)) \\ \leq & \ \beta\alpha^2 R \sum_{l=\phi(k)}^{k-1} \gamma^{2q}(l) x^{\mathrm{T}}(\phi(k)) x(\phi(k)) \\ \leq & \ \beta\alpha^2 R^2 \gamma^{2q}(\phi(k)) x^{\mathrm{T}}(\phi(k)) x(\phi(k)). \end{aligned}$$

By $q > \tilde{q}$, we have

$$\gamma^{2q}(\phi(k)) \leq \gamma^q(\phi(k)) P(\gamma(\phi(k))), \tag{8.13}$$

as long as the initial condition $\gamma(0)$ for $\gamma(k)$ is chosen sufficiently small. Then, there exists $\gamma_2(0) \in (0, \gamma_1(0))$ such that, for each $\gamma(0) \in (0, \gamma_2(0))$,

$$\begin{aligned} & x^{\mathrm{T}}(\phi(k))(F(\gamma(\phi(k))) - F(\gamma(k)))^{\mathrm{T}} \left(I + B^{\mathrm{T}} P(\gamma(k)) B\right) (F(\gamma(\phi(k))) - F(\gamma(k))) x(\phi(k)) \\ \leq & \ \beta\alpha^2 R^2 \gamma^q(\phi(k)) x^{\mathrm{T}}(\phi(k)) P(\gamma(\phi(k))) x(\phi(k)) \\ = & \ \beta\alpha^2 R^2 \gamma^q(\phi(k)) V(x(\phi(k)), \gamma(\phi(k))). \end{aligned}$$

On the other hand, we establish an upper bound of the quadratic term in (8.10) that is associated with $x(\phi(k)) - x(k)$. We compute

$$\begin{aligned} x(k) - x(\phi(k)) &= \sum_{l=\phi(k)}^{k-1} (x(l+1) - x(l)) \\ &= \sum_{l=\phi(k)}^{k-1} \left(A_{\mathrm{C}}(\gamma(l)) x(l) + B\left(F(\gamma(\phi(l))) x(\phi(l)) - F(\gamma(l)) x(l)\right)\right) - x(l) \\ &= \sum_{l=\phi(k)}^{k-1} \left(A_{\mathrm{C}}(\gamma(l)) - I\right) x(l) + B\left(F(\gamma(\phi(l))) x(\phi(l)) - F(\gamma(l)) x(l)\right), \end{aligned}$$

and then

$$\begin{aligned} & (x(k) - x(\phi(k)))^{\mathrm{T}} P(\gamma(k)) (x(k) - x(\phi(k))) \\ \leq & \ R \sum_{l=\phi(k)}^{k-1} \left(\left(A_{\mathrm{C}}(\gamma(l) - I) x(l) + B\left(F(\gamma(\phi(l))) x(\phi(l)) - F(\gamma(l)) x(l)\right)\right)^{\mathrm{T}} P(\gamma(k)) \right. \\ & \left. \times \left(\left(A_{\mathrm{C}}(\gamma(l) - I) x(l) + B\left(F(\gamma(\phi(l))) x(\phi(l)) - F(\gamma(l)) x(l)\right)\right)\right) \\ \leq & \ 2R \sum_{l=\phi(k)}^{k-1} x^{\mathrm{T}}(l) \left(A_{\mathrm{C}}(\gamma(l)) - I\right)^{\mathrm{T}} P(\gamma(k)) \left(A_{\mathrm{C}}(\gamma) - I\right) x(l) + \left(F(\gamma(\phi(l))) x(\phi(l))\right. \end{aligned}$$

$$
\begin{aligned}
&-F(\gamma(l))x(l)\big)^{\mathrm{T}}B^{\mathrm{T}}P(\gamma(k))B\big(F(\gamma(\phi(l)))x(\phi(l))-F(\gamma(l))x(l)\big)\\
&=2R\sum_{l=\phi(k)}^{k-1}x^{\mathrm{T}}(l)\big(A_{\mathrm{C}}(\gamma(l))-I\big)^{\mathrm{T}}\big(P(\gamma(k))-P(\gamma(l))\big)\big(A_{\mathrm{C}}(\gamma)-I\big)x(l)\\
&\quad+x^{\mathrm{T}}(l)\big(A_{\mathrm{C}}(\gamma(l))-I\big)^{\mathrm{T}}P(\gamma(l))\big(A_{\mathrm{C}}(\gamma)-I\big)x(l)+\big(F(\gamma(\phi(l)))x(\phi(l))-F(\gamma(l))x(l)\big)^{\mathrm{T}}\\
&\quad\times B^{\mathrm{T}}P(\gamma(k))B\big(F(\gamma(\phi(l)))x(\phi(l))-F(\gamma(l))x(l)\big)\\
&\le 2R\sum_{l=\phi(k)}^{k-1}\varpi(\gamma(l))x^{\mathrm{T}}(l)P(\gamma(l))x(l)+\big(F(\gamma(\phi(l)))x(\phi(l))-F(\gamma(l))x(l)\big)^{\mathrm{T}}\\
&\quad\times B^{\mathrm{T}}P(\gamma(k))B\big(F(\gamma(\phi(l)))x(\phi(l))-F(\gamma(l))x(l)\big)\\
&\le 2R\sum_{l=\phi(k)}^{k-1}\varpi(\gamma(l))x^{\mathrm{T}}(l)P(\gamma(l))x(l)\\
&\quad+2x^{\mathrm{T}}(\phi(l))F^{\mathrm{T}}(\gamma(\phi(l)))B^{\mathrm{T}}P(\gamma(k))BF(\gamma(\phi(l)))x(\phi(l))\\
&\quad+2x^{\mathrm{T}}(l)F(\gamma(l))B^{\mathrm{T}}P(\gamma(k))BF(\gamma(l))x(l)\\
&\le 2R\sum_{l=\phi(k)}^{k-1}\varpi(\gamma(l))x^{\mathrm{T}}(l)P(\gamma(l))x(l)\\
&\quad+2x^{\mathrm{T}}(\phi(l))F^{\mathrm{T}}(\gamma(\phi(l)))B^{\mathrm{T}}P(\gamma(\phi(l)))BF(\gamma(\phi(l)))x(\phi(l))\\
&\quad+2x^{\mathrm{T}}(l)F(\gamma(l))B^{\mathrm{T}}P(\gamma(l))BF(\gamma(l))x(l)\\
&\le 2R\sum_{l=\phi(k)}^{k-1}\varpi(\gamma(l))x^{\mathrm{T}}(l)P(\gamma(l))x(l)\\
&\quad+2\varrho(\gamma(\phi(l)))x^{\mathrm{T}}(\phi(l))P(\gamma(\phi(l)))x(\phi(l))+2\varrho(\gamma(l))x^{\mathrm{T}}(l)P(\gamma(l))x(l)\\
&=2R\sum_{l=\phi(k)}^{k-1}(\varpi(\gamma(l))+2\varrho(\gamma(l)))x^{\mathrm{T}}(l)P(\gamma(l))x(l)\\
&\quad+2\varrho(\gamma(\phi(l)))x^{\mathrm{T}}(\phi(l))P(\gamma(\phi(l)))x(\phi(l)),
\end{aligned}
$$

where we have employed the nonincreasing monotonocity of $P(\gamma(k))$ with respect to k and Lemma 8.1, and have defined for each $k \in \mathbb{N}$,

$$
\begin{aligned}
\varrho(\gamma(k)) &= \frac{\big(1-(1-\gamma(k))^n\big)^2}{(1-\gamma(k))^{2n-1}},\\
\varpi(\gamma(k)) &= \frac{1}{(1-\gamma(k))^n}-1-n\gamma(k)+\frac{\big(1-(1-\gamma(k))^n\big)^2}{(1-\gamma(k))^{n-1}}.
\end{aligned}
$$

Applying the derived upper bounds of the quadratic terms associated with $F(\gamma(\phi(k)))-F(\gamma(k))$ and $x(\phi(k))-x(k)$ in (8.9) then yields

$$
\begin{aligned}
&\Delta V(x(k),\gamma(k))) \\
\leq &-\gamma(k)V(x(k),\gamma(k)) + 2R^2\beta\alpha^2\gamma^q(\phi(k))V(x(\phi(k)),\gamma(\phi(k))) \\
&+ 4\vartheta(\gamma(k))R\sum_{l=\phi(k)}^{k-1}(\varpi(\gamma(l)) + 2\varrho(\gamma(l)))V(x(l),\gamma(l)) \\
&+ 2\varrho(\gamma(\phi(l)))V(x(\phi(l)),\gamma(\phi(l))). \qquad (8.14)
\end{aligned}
$$

Taking the summation of both sides of inequality (8.14) from $k = 2R$ to $k = N$, where $N \in \mathbb{N}$ and $N \geq 2R$, we obtain

$$
\begin{aligned}
&\sum_{k=2R}^{N}\Delta V(x(k),\gamma(k)) \\
\leq &-\sum_{k=2R}^{N}\gamma(k)V(x(k),\gamma(k)) + 2R^2\beta\alpha^2\sum_{k=2R}^{N}\gamma^q(\phi(k))V(x(\phi(k)),\gamma(\phi(k))) \\
&+ 4\sum_{k=2R}^{N}\vartheta(\gamma(k))R\sum_{l=\phi(k)}^{k-1}(\varpi(\gamma(l)) + 2\varrho(\gamma(l)))V(x(l),\gamma(l)) \\
&+ 2\varrho(\gamma(\phi(l)))V(x(\phi(l)),\gamma(\phi(l))) \\
\leq &-\sum_{k=2R}^{N}\gamma(k)V(x(k),\gamma(k)) + 2R^3\beta\alpha^2\sum_{l=R}^{N}\gamma^q(l)V(x(l),\gamma(l)) \\
&+ \sum_{k=2R}^{N}4\vartheta(\gamma(k))R^2\sum_{l=k-R}^{k-1}(\varpi(\gamma(l)) + 2\varrho(\gamma(l)))V(x(l),\gamma(l)) \\
&+ \sum_{k=2R}^{N}8\vartheta(\gamma(k))R^2\sum_{l=k-2R}^{k-1}\varrho(\gamma(l))V(x(l),\gamma(l)), \qquad (8.15)
\end{aligned}
$$

in which the second inequality sign holds for the following reasons. We examine the second summation after the first inequality sign in (8.15),

$$
\sum_{k=2R}^{N}\gamma^q(\phi(k))V(x(\phi(k)),\gamma(\phi(k))). \qquad (8.16)
$$

The index of this summation runs from $k = 2R$ to $k = N$, which implies that

$$
\phi(k) \in I[R, N].
$$

When the index of this summation is changed to $l = \phi(k)$, the range of l becomes $[R, N]$. Furthermore, for each $l \in I[R, N]$,

$$\gamma^q(l)V(x(l), \gamma(l))$$

is counted at most R times by the summation (8.16) because of the boundedness of $r(k)$. Thus, by a change of the index of this summation from k to $l = \phi(k)$, we get

$$\sum_{k=2R}^{N} \gamma^q(\phi(k))V(x(\phi(k)), \gamma(\phi(k))) \leq R\sum_{l=R}^{N} \gamma^q(l)V(x(l), \gamma(l)).$$

This, along with similar changes of the indices in the third and the fourth summations after the first inequality sign in (8.15), yield the second inequality in (8.15).

By the strictly increasing monotonicity of $\vartheta(\gamma)$ with respect to γ, as established in Lemma 8.2, we compute

$$\begin{aligned}
&\sum_{k=2R}^{N} \Delta V(x(k), \gamma(k)) \\
&\leq -\sum_{k=2R}^{N} \gamma(k)V(x(k), \gamma(k)) + 2R^3\beta\alpha^2\sum_{l=R}^{N} \gamma^q(l)V(x(l), \gamma(l)) \\
&\quad + 4R^2\sum_{k=2R}^{N}\sum_{l=k-R}^{k-1} \vartheta(\gamma(l))(\varpi(\gamma(l)) + 2\varrho(\gamma(l)))V(x(l), \gamma(l)) \\
&\quad + 8R^2\sum_{k=2R}^{N}\sum_{l=k-2R}^{k-1} \vartheta(\gamma(l))\varrho(\gamma(l))V(x(l), \gamma(l)) \\
&\leq -\sum_{k=2R}^{N} \gamma(k)V(x(k), \gamma(k)) + 2R^3\beta\alpha^2\sum_{l=R}^{N} \gamma^q(l)V(x(l), \gamma(l)) \\
&\quad + 4R^3\sum_{l=R}^{N-1} \vartheta(\gamma(l))(\varpi(\gamma(l)) + 2\varrho(\gamma(l)))V(x(l), \gamma(l)) \\
&\quad + 16R^3\sum_{l=0}^{N-1} \vartheta(\gamma(l))\varrho(\gamma(l))V(x(l), \gamma(l)) \\
&\leq -\sum_{k=2R}^{N} \gamma(k)V(x(k), \gamma(k)) + \sum_{l=0}^{N}(2R^3\beta\alpha^2\gamma^q(l)V(x(l), \gamma(l)) \\
&\quad + 4R^3\theta(\gamma(l))(\varpi(\gamma(l)) + 2\varrho(\gamma))V(x(l), \gamma(l))
\end{aligned}$$

$$
\begin{aligned}
&+ 16R^3\vartheta(\gamma(l))\varrho(\gamma(l))V(x(l),\gamma(l)))\\
\le &- \sum_{l=2R}^{N}\big(\gamma(l) - 2R^3(\beta\alpha^2\gamma^q(l) + 2\vartheta(\gamma)\big(\varpi(\gamma(l)) + 2\varrho(\gamma(l))\big)\\
&+ 8\vartheta(\gamma(l))\varrho(\gamma(l)))V(x(l),\gamma(l)) + 2R^3\sum_{l=0}^{2R-1}\big(\beta\alpha^2\gamma^q(l) + 2\theta(\gamma(l))\\
&\times(\varpi(\gamma(l)) + 2\varrho(\gamma(l))) + 8\vartheta(\gamma(l))\big)\varrho(\gamma(l))\big)V(x(l),\gamma(l)). \qquad (8.17)
\end{aligned}
$$

If

$$
\beta\alpha^2\gamma^{q-1}(l) + 2\frac{\vartheta(\gamma(l))}{\gamma(l)}\varpi(\gamma(l)) + 12\frac{\vartheta(\gamma(l))}{\gamma(l)}\varrho(\gamma(l)) < \frac{1}{2R^3} \qquad (8.18)
$$

for each $l \in I[2R, N]$, then the first summation after the last inequality sign in (8.17) is negative definite. By the nonincreasing monotonicity of $\gamma(k)$ and Lemma 8.2, there exists $\gamma_3(0) \in (0, \gamma_2(0))$ such that $\gamma(0) \in (0, \gamma_3(0))$ suffices for (8.18) to hold. Thus,

$$
\begin{aligned}
&\sum_{k=2R}^{N}\Delta V(x(k),\gamma(k))\\
\le\ & 2R^3\sum_{l=0}^{2R-1}(\beta\alpha^2\gamma^q(0) + 2\theta(\gamma(0))(\varpi(\gamma(0))\\
&+ 2\varrho(\gamma(0))) + 8\vartheta(\gamma(0)))\varrho(\gamma(0)))V(x(l),\gamma(l)). \qquad (8.19)
\end{aligned}
$$

This implies that, for each $\gamma(0) \in (0, \gamma_3(0))$,

$$
V(x(N),\gamma(N)) \le \sum_{l=0}^{2R} c_l V(x(l),\gamma(l)), \qquad (8.20)
$$

where

$$
c_l = 1 \qquad (8.21)
$$

for $l = 2R$ and

$$
c_l = 2R^3(\beta\alpha^2\gamma^q(0) + 2\theta(\gamma(0))(\varpi(\gamma(0)) + 6\varrho(\gamma(0))) \qquad (8.22)
$$

for $l \in I[0, 2R-1]$. Note that (8.20) holds for any $N \ge 2R$. Thus, $V(x(k),\gamma(k))$ is bounded over $k \in \mathbb{N}$.

From (8.17), we notice that, for each $\gamma(0) \in (0, \gamma_2(0))$,

$$\begin{aligned}
&\sum_{l=2R}^{N} \Delta V(x(k), \gamma(k)) \\
&\leq -\frac{1}{2}\sum_{l=2R}^{N} \gamma(l)V(x(l), \gamma(l)) - \frac{1}{2}\sum_{l=2R}^{N} \gamma(l)V(x(l), \gamma(l)) \\
&\quad + 2R^3 \sum_{l=0}^{N} \big(\beta\alpha^2\gamma^q(l) + 2\vartheta(\gamma(l))(\varpi(\gamma(l)) + 6\varrho(\gamma(l)))\big)V(x(l), \gamma(l)),
\end{aligned}$$

from which we get

$$\begin{aligned}
&\sum_{l=2R}^{N} \gamma(l)V(x(l), \gamma(l)) \\
&\leq -\sum_{l=2R}^{N} \gamma(l)V(x(l), \gamma(l)) + 4R^3 \sum_{l=0}^{N} \big(\beta\alpha^2\gamma^q(l) + 2\vartheta(\gamma(l))(\varpi(\gamma(l)) \\
&\quad + 6\varrho(\gamma(l)))\big)V(x(l), \gamma(l)) - 2\sum_{l=2R}^{N} \Delta V(x(k), \gamma(k)) \\
&= \sum_{l=2R}^{N} \Big(-\gamma(l) + 4R^3(\beta\alpha^2\gamma^q(l) + 2\vartheta(\gamma(l))(\varpi(\gamma(l)) + 6\varrho(\gamma(l))))\Big) V(x(l), \gamma(l)) \\
&\quad + 4R^3 \sum_{l=0}^{2R-1} \Big(\beta\alpha^2\gamma^q(l) + 2\vartheta(\gamma(l))\,(\varpi(\gamma(l)) + 6\varrho(\gamma(l)))\Big)\, V(x(l), \gamma(l)) \\
&\quad + 2(V(x(2R), \gamma(2R)) - V(x(N), \gamma(N)).
\end{aligned}$$

Using an argument similar to the one used in obtaining (8.19), we conclude that there exists $\gamma_4(0) \in (0, \gamma_3(0))$ such that, for each $\gamma(0) \in (0, \gamma_4(0))$,

$$\begin{aligned}
\sum_{l=2R}^{N} \gamma(l)V(x(l), \gamma(l)) \leq 4R^3 \sum_{l=0}^{2R-1} &\big(\beta\alpha^2\gamma^q(l) + 2\vartheta(\gamma(l))(\varpi(\gamma(l)) \\
&+ 6\varrho(\gamma(l)))\big)V(x(l), \gamma(l)) + 2V(x(2R), \gamma(2R)),
\end{aligned}$$

based on which we derive

$$\begin{aligned}\sum_{l=2R}^{N}\gamma(l)V(x(l),\gamma(l)) &\le 4R^3\sum_{l=0}^{2R-1}\big(\beta\alpha^2\gamma^q(l)+2\vartheta(\gamma(l))(\varpi(\gamma(l))\\ &\quad +6\varrho(\gamma(l)))\big)V(x(l),\gamma(l))+2V(x(2R),\gamma(2R))\\ &\le 2\sum_{l=0}^{2R}c_l V(x(l),\gamma(l)),\end{aligned} \tag{8.23}$$

where we have defined c_l as in (8.21) and (8.22).

8.3.2 The Boundedness of $\gamma(k)$ Away from Zero

By the update law (8.5) for $\gamma(k)$, we compute

$$\begin{aligned}&\sum_{l=2R}^{N}\gamma(k+1)-\gamma(k)\\ &=-\sum_{l=2R}^{N}\alpha\frac{V^p(x(k),\gamma(k))}{V^p(x(k),\gamma(k))+1}\gamma^q(k)\\ &=-\sum_{k=2R}^{N}\alpha\frac{V^{p-1}(x(k),\gamma(k))\gamma^{q-1}(k)}{V^p(x(k),\gamma(k))+1}V(x(k),\gamma(k))\gamma(k)\\ &\ge-\alpha\sum_{k=2R}^{N}V^{p-1}(x(k),\gamma(k))\gamma^{q-1}(k)V(x(k),\gamma(k))\gamma(k)\\ &\ge-\alpha\sum_{k=2R}^{N}V^{p-1}(x(k),\gamma(k))\gamma^{q-1}(0)V(x(k),\gamma(k))\gamma(k)\\ &\ge-\alpha\gamma^{q-1}(0)\left(\sum_{l=0}^{2R}c_lV(x(l),\gamma(l))\right)^{p-1}\sum_{l=2R}^{N}V(x(k),\gamma(k))\gamma(k)\\ &\ge-2\alpha\gamma^{q-1}(0)\left(\sum_{l=0}^{2R}c_lV(x(l),\gamma(l))\right)^{p},\end{aligned}$$

where we have applied an upper bound of $V(x(k),\gamma(k))$ for each of $k\ge 2R$, as given by (8.20), and an upper bound of $\sum_{l=2R}^{N}\gamma(l)V(x(l),\gamma(l))$, as given by (8.23). Then,

$$\gamma(N) \geq \gamma(2R) - 2\alpha\gamma^{q-1}(0)\left(\sum_{l=0}^{2R} c_l V(x(l), \gamma(l))\right)^p$$
$$\geq \gamma(0) - \alpha(2R+1)\gamma^q(0) - 2\alpha\gamma^{q-1}(0)\left(\sum_{l=0}^{2R} c_l V(x(l), \gamma(l))\right)^p.$$

From

$$\begin{aligned} V(x(l), \gamma(l)) &\leq \max_{l\in I[0,2R]}\{V(x(l), \gamma(l))\} \\ &\leq \sum_{l=0}^{2R} V(x(l), \gamma(l)) \\ &\leq \sum_{l=0}^{2R} x^{\mathrm{T}}(l) P(\gamma(l)) x(l) \\ &\leq |P(\gamma(0))| \sum_{l=0}^{2R} |x(l)|^2, \quad l \in I[0, 2R], \end{aligned} \tag{8.24}$$

we can see that as $\gamma(0)$ tends to zero, each $\gamma(l)$, $l \in I[0, 2R]$, tends to zero. This implies that over the time interval $k \in I[0, 2R]$, the state of the closed-loop system (8.6) tends to that of the open loop system

$$x(k+1) = Ax(k)$$

as $\gamma(0)$ tends to zero. Thus,

$$\sum_{l=0}^{2R} |x(l)|^2$$

in (8.24) approaches a positive constant that depends solely on A and R. Therefore, there exists a sufficiently small $\gamma_5(0) \in (0, \gamma_4(0))$ such that for each $\gamma(0) \in (0, \gamma_5(0))$,

$$\begin{aligned} \gamma(N) &\geq \gamma(0) - \alpha(2R+1)\gamma^q(0) - 2\alpha\gamma^{q-1}(0)\left(\sum_{l=0}^{2R} c_l V(x(l), \gamma(l))\right)^p \\ &\geq \gamma(0) - \alpha(2R+1)\gamma^q(0) - 2\alpha\gamma^{q-1}(0)\left(\sum_{l=0}^{2R} c_l |P(\gamma(0))| \sum_{l=0}^{2R} |x(l)|^2\right)^p \\ &> 0. \end{aligned} \tag{8.25}$$

Note that inequality (8.25) holds for each $N \geq 2R$. This implies that $\gamma(k)$ is bounded from below by a positive constant. In view of the nonincreasing monotonocity of $\gamma(k)$, we have that

$$\lim_{k\to\infty} \gamma(k)$$

exists and is positive as long as $\gamma(0) \in (0, \gamma_5(0))$.

8.3.3 The Regulation of the State and the Input Given a Sufficiently Small $\gamma(0)$

Based on the update law for $\gamma(k)$ (8.5), that is,

$$\gamma(k+1) = \gamma(k) - \alpha \frac{V^p(x(k), \gamma(k))}{V^p(x(k), \gamma(k)) + 1} \gamma^q(k),$$

and the fact that $\gamma(k)$ has a positive limit, which is denoted by γ_c, we derive

$$\begin{aligned}
\lim_{k\to\infty} (\gamma(k+1) - \gamma(k)) &= -\alpha \lim_{k\to\infty} \frac{V^p(x(k), \gamma(k))}{V^p(x(k), \gamma(k)) + 1} \gamma^q(k) \\
&= -\alpha \gamma_c^q \lim_{k\to\infty} \frac{V^p(x(k), \gamma(k))}{V^p(x(k), \gamma(k)) + 1} \\
&= 0,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} V(x(k), \gamma(k)) = 0.$$

By

$$V(x(k), \gamma(k)) \geq |P(\gamma_c)| \, |x(k)|^2,$$

we have

$$\lim_{k\to\infty} |x(k)| \leq \lim_{k\to\infty} \sqrt{\frac{V(x(k), \gamma(k))}{|P(\gamma_c)|}} = 0.$$

Thus,

$$\lim_{k\to\infty} x(k) = 0.$$

Obviously,

$$\lim_{k\to\infty} u(k) = \lim_{k\to\infty} F(\gamma(k)) \lim_{k\to\infty} x(k)$$
$$=0.$$

To summarize, we have the following result.

Theorem 8.1 *There exists* $\gamma^\star(0) \in (0,1)$ *such that for each* $\gamma(0) \in (0, \gamma^\star(0))$*, the adaptive feedback law (8.2) with* $\gamma(k)$ *updated by following (8.5) globally regulates system (8.1), that is,*

$$\lim_{k\to\infty} x(k) = 0, \quad \lim_{k\to\infty} u(k) = 0,$$

given any initial condition of the state x*. Moreover,*

$$\lim_{k\to\infty} \gamma(k)$$

exists and is positive.

8.3.4 The Regulation of the State and the Input Given Any $\gamma(0)$

Theorem 8.1 establishes global regulation of system (8.1) given a small enough initial condition for $\gamma(k)$. However, as seen in the proof of the theorem, the upper bound of this sufficiently small $\gamma(0)$ depends on the upper bound of the delay R. We now give a regulation analysis of system (8.1) under the adaptive feedback law with an arbitrarily initial $\gamma(0)$. It turns out that our adaptive feedback law achieves global regulation of the system for any given initial condition of $\gamma(k)$.

We consider two separate cases. The first case assumes that, given an initial condition of $\gamma(k)$, $\gamma(k)$ never decreases to the value of $\gamma^\star(0)$, where $\gamma^\star(0)$ is given in Theorem 8.1. This suggests that

$$\gamma(k) > \gamma^\star(0), \quad k \in \mathbb{N},$$

and thus,

$$\lim_{k\to\infty} \gamma(k)$$

exists and is a positive constant that is greater or equal to $\gamma^\star(0)$. Denote this limit by γ_d. Based on the update law for $\gamma(k)$, we obtain

$$\lim_{k\to\infty} \frac{V^p(x(k), \gamma(k))}{V^p(x(k), \gamma(k)) + 1} = 0,$$

and hence

$$\lim_{k\to\infty} V(x(k), \gamma(k)) = 0.$$

Thus,

$$\begin{aligned}\lim_{k\to\infty} |x(k)| &\le \lim_{k\to\infty} \sqrt{\frac{V(x(k), \gamma(k))}{|P(\gamma_{\mathrm{d}})|}} \\ &= 0,\end{aligned}$$

and

$$\lim_{k\to\infty} x(k) = 0.$$

In view of the boundedness of $\gamma(k)$ from below, we get

$$\lim_{k\to\infty} u(k) = 0.$$

Therefore, global regulation of the system (8.1) is achieved.

We now consider the second case where $\gamma(k)$ decreases to some $\gamma(0) \in (0, \gamma^\star(0))$ at some time instant $\tilde{k}$. Set this $\tilde{k}$ as the time instant at which the closed-loop system (8.6) starts evolution. Then, according to Theorem 8.1, the closed-loop system achieves

$$\lim_{k\to\infty} x(k) = 0, \quad \lim_{k\to\infty} u(k) = 0.$$

Also,

$$\lim_{k\to\infty} \gamma(k)$$

exists and is positive.

We now summarize the above conclusion in the following theorem.

Theorem 8.2 *The adaptive feedback law (8.2), with $\gamma(k)$ updated according to (8.5), globally regulates system (8.1), that is,*

$$\lim_{k\to\infty} x(k) = 0, \quad \lim_{k\to\infty} u(k) = 0.$$

Moreover,

$$\lim_{k\to\infty} \gamma(k)$$

exists and is positive.

To the best of our knowledge, no design achieves what the adaptive feedback law (8.2) has achieved. First, no knowledge of a time-varying bounded delay is required to implement our adaptive feedback law. Considering that the compensation of an arbitrarily large delay without using any delay knowledge is still open for a general linear system, the result in this chapter makes a step toward the solution to this long-standing problem. Second, the implementation of our adaptive feedback law is easy due to its memorylessness. Finally, any time-varying bounded delay can be compensated by our adaptive feedback law, which cannot be done by a traditional predictor based feedback. Because the predictor feedback handles only a group of time-varying delays whose associated $\phi(k)$ function has an inverse function, any adaptive feedback law that takes the predictor feedback law as the nominal controller will not be able to compensate for a time-varying delay whose associated $\phi(k)$ function does not have an inverse function.

8.4 A Numerical Example

The regulation and performance of the adaptive feedback law (8.2) are discussed in this section through numerical examples. Consider system (8.1) with

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{8.26}$$

All the open loop poles of this system are at $z = 1$, and the pair (A, B) is controllable. Let the time-varying delay be

$$r(k) = \mathrm{mod}(k, 5).$$

It can be readily seen that an upper bound of the delay is $R = 4$ and that the inverse function of

$$\phi(k) = k - r(k)$$

does not exist. Actually, any function $\phi(k)$ associated with a time-varying delay in the form of

$$r(k) = \mathrm{mod}(k, l),$$

where $l \in \mathbb{N} \setminus \{0\}$ does not have an inverse image. We pick this particular type of time-varying delay to numerically demonstrate the ability of our adaptive feedback law to regulate the system, which is not possessed by a traditional predictor based adaptive feedback law. Let the initial condition of the system be given by

$$x(k) = \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \quad k \in I[0, R].$$

Throughout the simulation study, we pick the control parameters in our design as follows:

$$\alpha = 1, \quad p = 1, \quad q = 2.$$

Here, $q = 2$ is chosen based on $\tilde{q} = 1$. In the simulation, we first let $\gamma(k)$ decrease from

$$\gamma(0) = 0.1.$$

Figures 8.1, 8.2, and 8.3 show the evolutions of the state, the input, and the feedback parameter of the closed-loop system, respectively, under our adaptive feedback law (8.1). It can be readily observed in Fig. 8.3 that $\gamma(k)$ has a positive limit as k tends to infinity.

Keeping all the parameters the same as in the first simulation, we change the initial condition of the feedback parameter to a larger value

$$\gamma(0) = 0.2.$$

Figures 8.4, 8.5, and 8.6 show respectively the evolutions of the state, the input, and the feedback parameter of the closed-loop system under our adaptive feedback law (8.1). We further increase the initial condition of $\gamma(k)$ to

$$\gamma(0) = 0.5,$$

under which the closed-loop evolution is shown in Figs. 8.7, 8.8, and 8.9.

These simulation results demonstrate that although the regulation of system (8.26) can be achieved no matter what the value of $\gamma(0)$ is, the closed-loop performance in terms of the overshoot and the convergence rate varies as the initial value of $\gamma(k)$ varies. Specifically, the value of $\gamma(0)$ cannot be chosen to be neither too large nor too small. An excessively large or small $\gamma(0)$ would induce large overshoot and slow convergence rate, while a modest value of $\gamma(0)$ tends to result in better closed-loop performance. A similar observation on how the value of the feedback parameter affects the closed-loop performance was made in Sect. 5.3.

More evidence of the ability of the adaptive feedback law in its regulation of system (8.26) can be seen by considering an even larger time-varying delay.

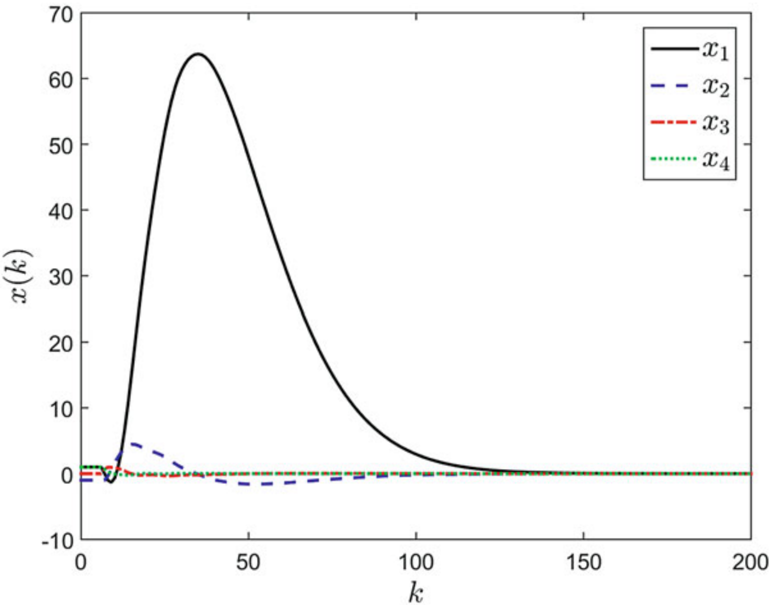


Fig. 8.1 State evolution of the closed-loop system with $\gamma(0) = 0.1$

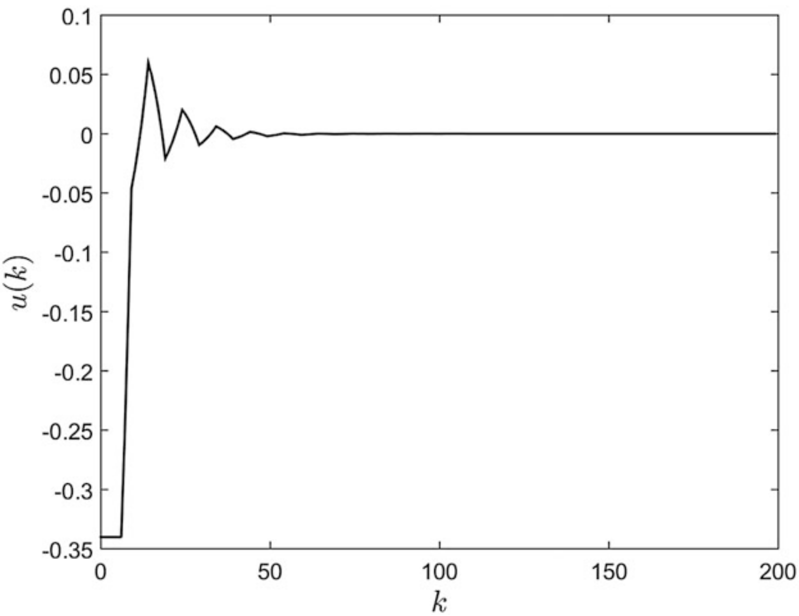


Fig. 8.2 Input evolution of the closed-loop system with $\gamma(0) = 0.1$

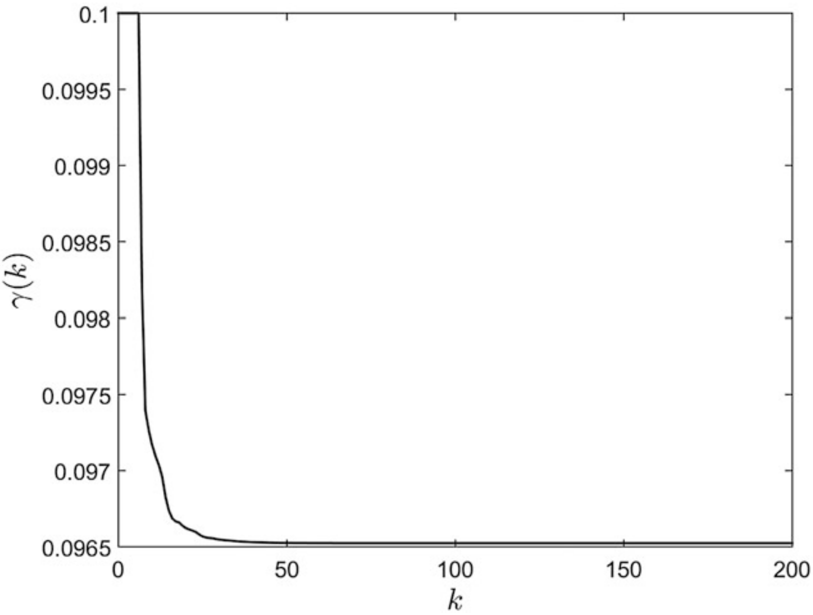


Fig. 8.3 Evolution of the feedback parameter of the closed-loop system with $\gamma(0) = 0.1$

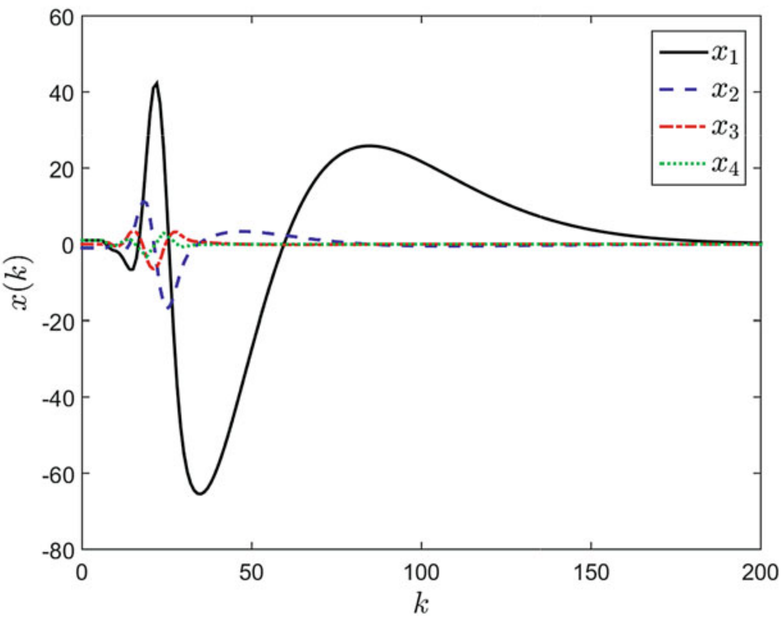


Fig. 8.4 State evolution of the closed-loop system with $\gamma(0) = 0.2$

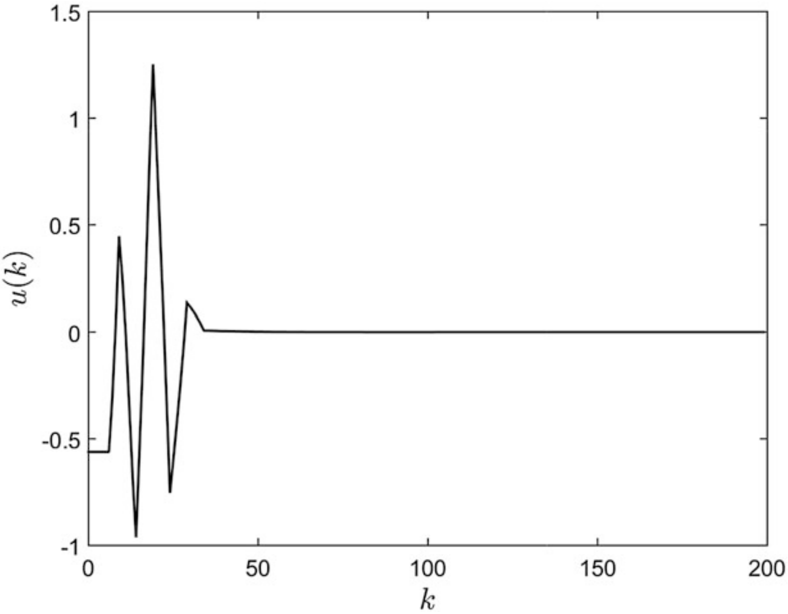


Fig. 8.5 Input evolution of the closed-loop system with $\gamma(0) = 0.2$

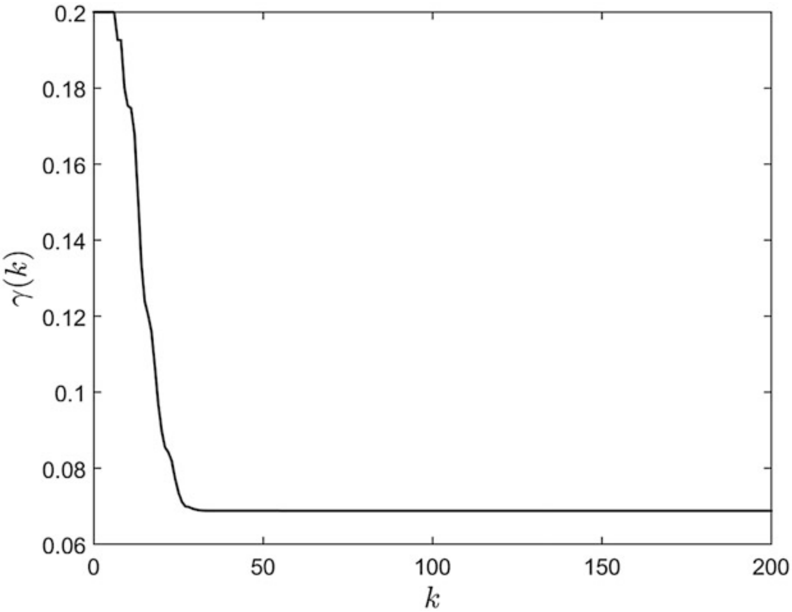


Fig. 8.6 Evolution of the feedback parameter of the closed-loop system with $\gamma(0) = 0.2$

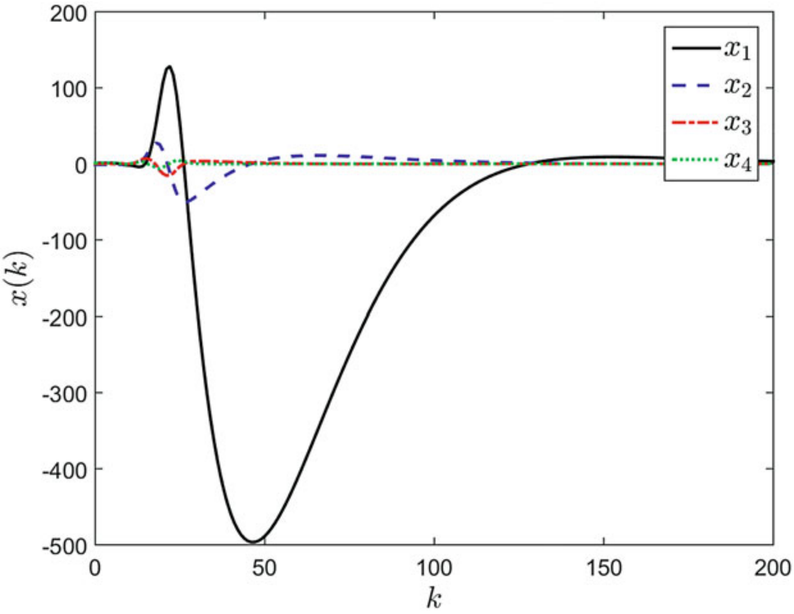


Fig. 8.7 State evolution of the closed-loop system with $\gamma(0) = 0.5$

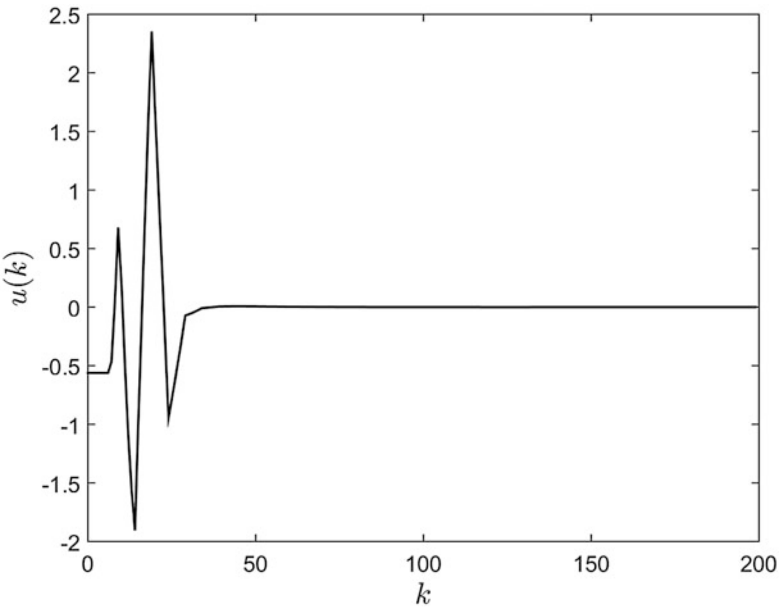


Fig. 8.8 Input evolution of the closed-loop system with $\gamma(0) = 0.5$

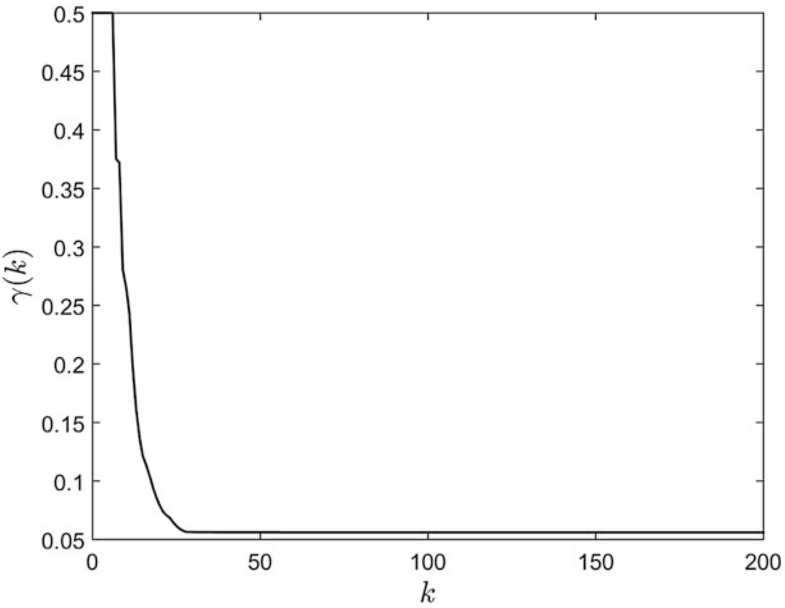


Fig. 8.9 Evolution of the feedback parameter of the closed-loop system with $\gamma(0) = 0.5$

Consider

$$r(k) = \text{mod}(k, 10),$$

whose peak value is $R = 9$. Set all the other parameters the same as those in the last simulation. Let the feedback parameter decrease its value from

$$\gamma(0) = 0.5.$$

Figures 8.10, 8.11, and 8.12 show the evolutions of the state, the input, and the feedback parameter of the closed-loop system. The performance of the closed-loop system is not as good as in the corresponding simulation with $R = 4$.

The following numerical study reveals that our adaptive feedback design is not sensitive to open loop poles that are exponentially unstable. Replace matrix A in system (8.26) with

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1.2 \end{bmatrix},$$

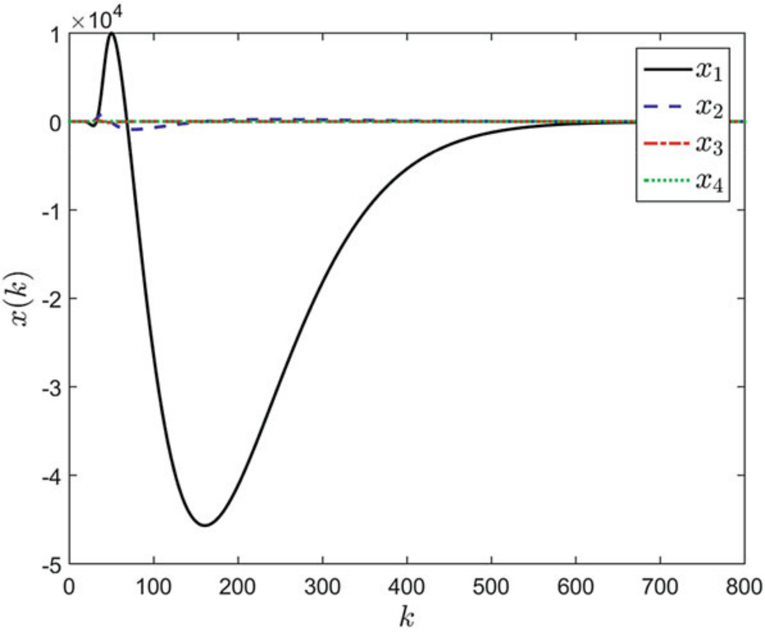


Fig. 8.10 State evolution of the closed-loop system with $R = 9$ and $\gamma(0) = 0.5$

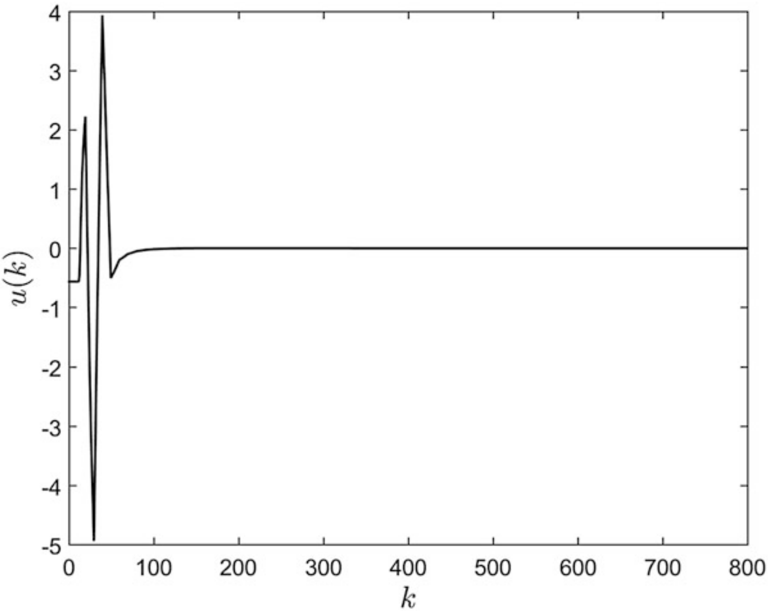


Fig. 8.11 Input evolution of the closed-loop system with $R = 9$ and $\gamma(0) = 0.5$

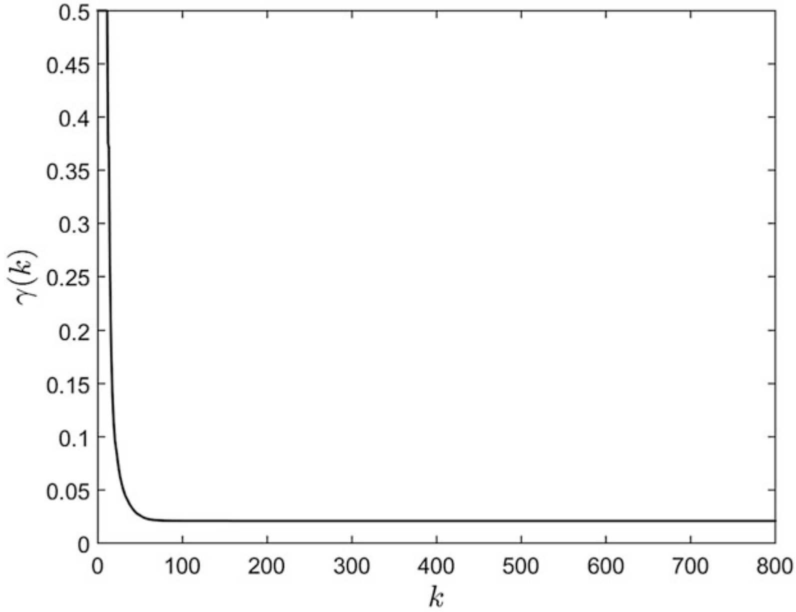


Fig. 8.12 Evolution of the feedback parameter of the closed-loop system with $R = 9$ and $\gamma(0) = 0.5$

which contains an exponentially unstable pole $z = 1.2$ of the open loop system. Choose the same set of system parameters as those in first simulation except the matrix A. Figures 8.13, 8.14, and 8.15 show the evolution of the closed-loop system with this unstable open loop pole. The simulation results show that the adaptive feedback design is robust to the presence of open loop poles that are exponentially unstable.

8.5 Conclusions

This chapter presented an adaptive feedback law that accommodates unknown input delay in discrete-time linear systems with all open loop poles at $z = 1$ or inside the unit circle. Two main features of the feedback law are its delay independence nature and its memorylessness. No knowledge of the delay is required for the implementation of the feedback law. Also, only the current state is used in the feedback. These two features contribute to the convenience in the implementation of the feedback law. A direct closed-loop analysis that does not base on Lyapunov type stability analysis was developed to establish regulation. Numerical examples demonstrated the regulation effects of the feedback design and its robustness to the presence of exponentially unstable open loop poles.

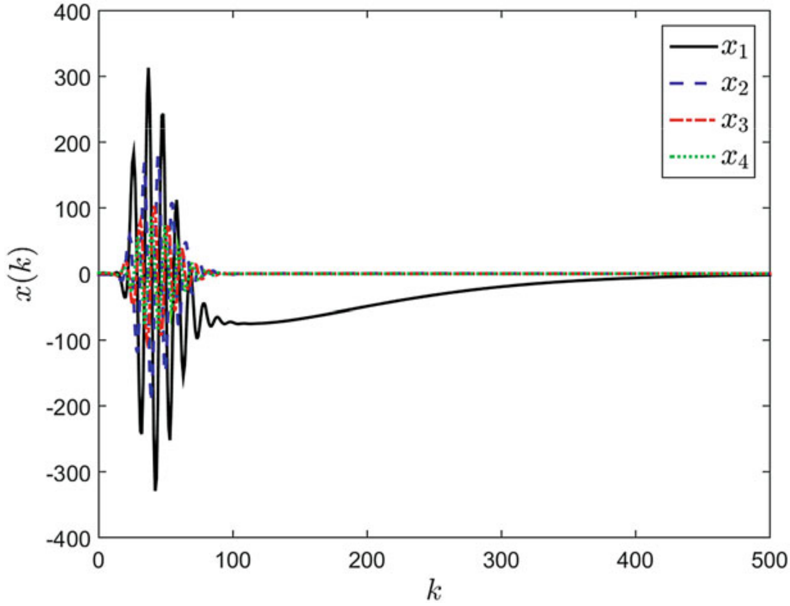


Fig. 8.13 State evolution of the closed-loop system with an exponentially unstable open loop pole

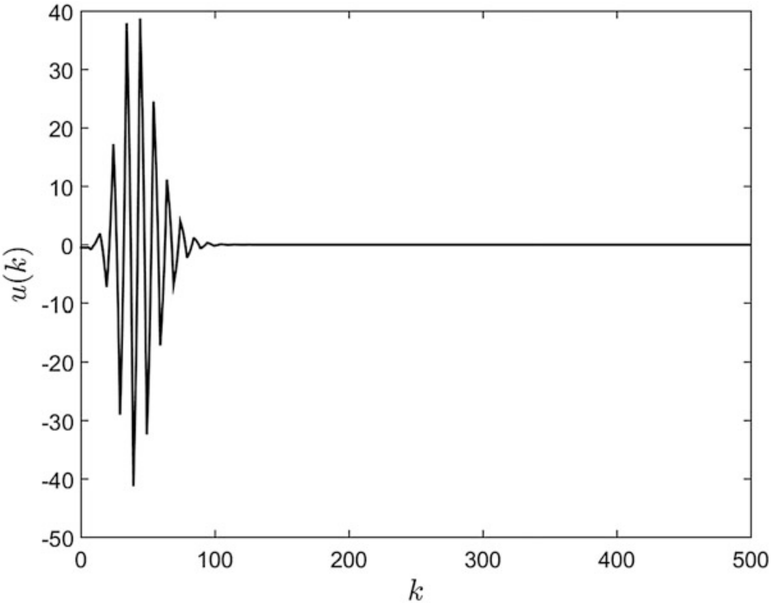


Fig. 8.14 Input evolution of the closed-loop system with an exponentially unstable open loop pole

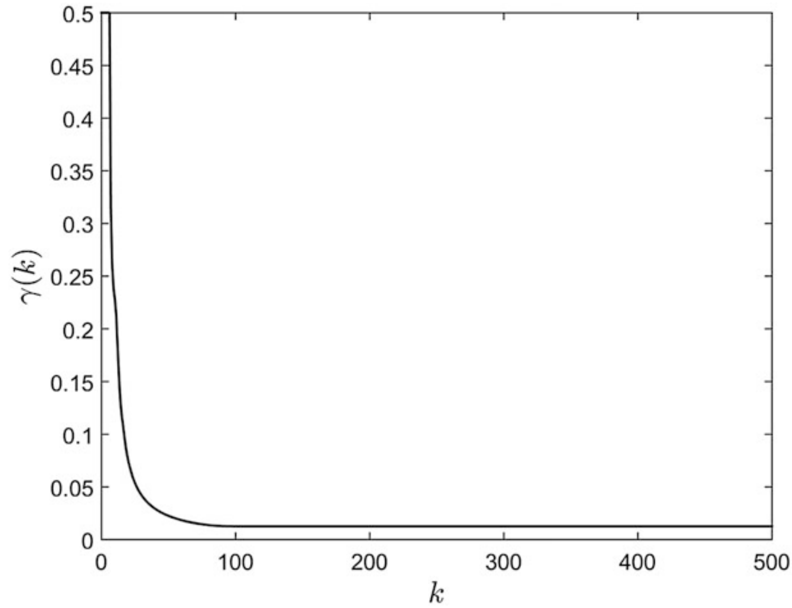


Fig. 8.15 Evolution of the feedback parameter of the closed-loop system with an exponentially unstable open loop pole

8.6 Notes and References

This chapter was presented to sole the regulation problem for a discrete-time linear system in the presence of any knowledge of the delay. The problem is the counterpart to the problem in the continuous-time setting studied in Chap. 7. The presentation follows from [108] submission.

References

1. K. Abidi, Y. Yildiz and B. E. Korpe, "Explicit time-delay compensation in teleoperation: An adaptive control approach," *International Journal of Robust and Nonlinear Control*, Vol. 26, No. 15, pp. 3388–3403, 2016.
2. K. Abidi and I. Postlethwaite, "Discrete-time adaptive control for systems with input time-delay and non-sector bounded nonlinear functions," *IEEE Access*, Vol. 7, pp. 4327–4337, 2019.
3. K. Abidi, I. Postlethwaite and T. T. Teo, "Discrete-time adaptive control of nonlinear systems with input delay," *The 38th Chinese Control Conference*, pp. 588–593, Guangzhou, China, 2019.
4. T. Ahmed-Ali, I. Karafyllis, F. Giri, M. Krstic and F. Lamnabhi-Lagarrigue, "Exponential stability analysis of sampled-data ODE-PDE systems and application to observer design," *IEEE Transactions on Automatic Control*, Vol. 62, No. 6, pp. 3091–3098, 2017.
5. H. I. Ansoff, "Stability for linear oscillating systems with constant time lag," *Journal of Applied Mechanics*, Vol. 16, pp. 158–164, 1949.
6. H. I. Ansoff and J. A. Krumhansl, "A general stability criterion for linear oscillating systems with constant time lag," *Quarterly of Applied Mathematics*, Vol. 6, pp. 337–341, 1948.
7. Z. Artstein, "Linear systems with delayed controls: a reduction," *IEEE Transactions on Automatic Control*, Vol. 27, No. 4, pp. 869–879, 1982.
8. K. J. Aström and R. M. Murray, *Feedback Systems: an Introduction for Scientists and Engineers*, Princeton University Press, 2010.
9. N. Bekiaris-Liberis and M. Krstic, *Nonlinear Control under Nonconstant Delays*, Vol. 25, SIAM, 2013.
10. N. Bekiaris-Liberis and M. Krstic, "Stability of predictor-based feedback for nonlinear systems with distributed input delay," *Automatica*, Vol. 70, pp. 195–203, 2016.
11. N. Bekiaris-Liberis and M. Krstic, "Predictor-feedback stabilization of multi-input nonlinear systems," *IEEE Transactions on Automatic Control*, Vol. 62, No. 2, pp. 516–531, 2017.
12. R. Bellman and K. L. Cooke, *Differential-Difference Equations*, New York: Academic, 1963.
13. G. Besancon, D. Georges and Z. Benayache, "Asymptotic state prediction for continuous-time systems with delayed input and application to control," *The 2007 European Control Conference*, pp. 1786–1791, Kos, Greece, 2007.
14. D. Bresch-Pietri and M. Krstic, "Adaptive trajectory tracking despite unknown input delay and plant parameters," *Automatica*, Vol. 45, No. 9, pp. 2074–2081, 2009.
15. D. Bresch-Pietri and M. Krstic, "Delay-adaptive predictor feedback for systems with unknown long actuator delay," *IEEE Transactions on Automatic Control*, Vol. 55, No. 9, pp. 2106–2112, 2010.

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5

16. F. Cacace and A. Germani, "Output feedback control of linear systems with input, state and output delays by chains of predictors," *Automatica*, Vol. 85, pp. 455–461, 2017.
17. B. Cahlon and D. Schmidt, "Stability criteria for certain high even order delay differential equations," *Journal of Mathematical Analysis and Applications*, Vol. 334, No. 2, pp. 859–875, 2007.
18. Y. Y. Cao, Z. Lin and T. Hu, "Stability analysis of linear time-delay systems subject to input saturation," *IEEE Transactions on Circuits and Systems*, Vol. 49, pp. 233–240, 2002.
19. B. M. Chen, Z. Lin and Y. Shamash, *Linear Systems Theory: A Structural Decomposition Approach*, Springer Science & Business Media, 2004.
20. B. S. Chen, S. S. Wang and H. C. Lu, "Stabilization of time-delay system containing saturating actuators," *International Journal of Control*, Vol. 47, pp. 867–881, 1988.
21. J. Chen and A. Latchman, "Frequency sweeping tests for stability independent of delay," *IEEE Transactions on Automatic Control*, Vol. 40, No. 9, pp. 1640–1645, 1995.
22. J. Chen, G. Gu and C. N. Nett, "A new method for computing delay margins for stability of linear delay systems," *Systems & Control Letters*, Vol. 26, No. 2, pp. 107–117, 1995.
23. J. H. Chou, I. R. Horng and B. S. Chen, "Dynamical feedback compensator for uncertain time-delay systems containing saturating actuators," *International Journal of Control*, Vol. 49, pp. 961–968, 1989.
24. K. L. Cooke and Z. Grossman, "Discrete delay, distributed delay and stability switches," *Journal of Mathematical Analysis and Applications*, Vol. 86, No. 2, pp. 592–627, 1982.
25. S. Elaydi and S. Zhang, "Stability and periodicity of difference equations with finite delay," *Funkcialaj Ekvacioj*, Vol. 37, No. 3, pp. 401–413, 1994.
26. K. Engelborghs, M. Dambrine and D. Roose, "Limitations of a class of stabilization methods for delay systems," *IEEE Transactions on Automatic Control*, Vol. 46, No. 2, pp. 336–339, 2001.
27. K. Engelborghs, T. Luzyanina and G. Samaey, "DDE-BIFTOOL v.2.00: A Matlab package for bifurcation analysis of delay differential equation," Dept. Comput. Sci., K.U. Leuven, T.W. Rep. 330, 2001.
28. H. Fang and Z. Lin, "A further result on global stabilization of oscillators with bounded delayed input," *IEEE Transactions on Automatic Control*, Vol. 51, No. 1, pp. 121–128, 2006.
29. Y. A. Fiagbedzi and A. E. Pearson, "Feedback stabilization of linear autonomous time lag systems," *IEEE Transactions on Automatic Control*, Vol. 31, No. 9, pp. 847–855, 1986.
30. Y. A. Fiagbedzi and A. E. Pearson, "A multistage reduction technique for feedback stabilizing distributed time-lag systems," *Automatica*, Vol. 23, No. 3, pp. 311–326, 1987.
31. E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems," *Systems & Control Letters*, Vol. 43, No. 4, pp. 309–319, 2001.
32. E. Fridman, *Introduction to Time-delay Systems: Analysis and Control*, Springer, 2014.
33. H. Gao and C. Wang, "A delay-dependent approach to robust H_∞ filtering for uncertain discrete-time state-delayed systems," *IEEE Transactions on Signal Processing*, Vol. 52, No. 6, pp. 1631–1640, 2004.
34. H. Gao, J. Lam, C. Wang and Y. Wang, "Delay-dependent output-feedback stabilisation of discrete-time systems with time-varying state delay," *IEE Proceedings-Control Theory and Applications*, Vol. 151, No. 6, pp. 691–698, 2004.
35. H. Gao and T. Chen, "New results on stability of discrete-time systems with time-varying state delay," *IEEE Transactions on Automatic Control*, Vol. 52, No. 2, pp. 328–334, 2007.
36. Q. Gao and N. Olgac, "Bounds of imaginary spectra of LTI systems in the domain of two of the multiple time delays," *Automatica*, Vol. 72. pp. 235–241, 2016.
37. H. Gorecki, S. Fuksa, P. Grabowski and A. Korytowski, *Analysis and Synthesis of Time Delay Systems*, New York: Wiley, 1989.
38. G. Gu and E. B. Lee, "Stability testing of time delay systems," *Automatica*, Vol. 25, No. 5, pp. 777–780, 1989.
39. G. Gu, P. P. Khargonekar and E. B. Lee, "Approximation of infinite-dimensional systems," *IEEE Transactions on Automatic Control*, Vol. 34, No. 6, pp. 610–618, 1989.

40. K. Gu, “An integral inequality in the stability problem of time-delay systems,” *Proc. 39th IEEE Conference on Decision and Control*, Sydney, Australia, pp. 2805–2810, 2000.
41. K. Gu, V. L. Kharitonov and J. Chen, *Stability of Time-Delay Systems*, Boston, MA: Birkhäuser, 2003.
42. A. Halanay, *Differential Equations: Stability, Oscillations, Time lags*, Vol. 23, Academic Press, 1966.
43. J. K. Hale, *Theory of Functional Differential Equations*, New York: Springer, 1977.
44. Q. L. Han and B. Ni, “Delay-dependent robust stabilization for uncertain constrained systems with pointwise and distributed time-varying delays," *Proc. 38th IEEE Conference on Decision and Control*, pp. 215–220, 1999.
45. F. Hoppensteadt, “Predator-prey model,” *Scholarpedia*, Vol. 1, No. 10, pp. 1563, 2006.
46. D. Israelsson and A. Johnsson, “A theory for circumnutations in Helianthus annuus,” *Physiologia Plantarum*, Vol. 20, No. 4, pp. 957–976, 1967.
47. M. Jankovic, “Control Lyapunov-Razumikhin functions and robust stabilization of time delay systems,” *IEEE Transactions on Automatic Control*, Vol. 546, No. 7, pp. 1048–1060, 2001.
48. M. Jankovic, “Cross-term forwarding for systems with time delay,” *IEEE Transactions on Automatic Control*, Vol. 54, No. 3, pp. 498–511, 2009.
49. M. Jankovic, “Forwarding, backstepping, and finite spectrum assignment for time delay systems,” *Automatica*, Vol. 45, No. 1, pp. 2–9, 2009.
50. M. Jankovic, “Recursive predictor design for state and output feedback controllers for linear time delay systems,” *Automatica*, Vol. 46, pp. 510–517, 2010.
51. H. K. Khalil, *Nonlinear Systems*, 3rd edition, Upper Saddle River, New Jersey: Prentice-Hall, 2000.
52. V. L. Kharitonov, *Time-delay Systems: Lyapunov Functionals and Matrices*, Springer Science & Business Media, 2012.
53. V. L. Kharitonov, “An extension of the prediction scheme to the case of systems with both input and state delay,” *Automatica*, Vol. 50, No. 1, pp. 211–217, 2014.
54. V. L. Kharitonov. “Predictor-based controls: the implementation problem,” *Differential Equations*, Vol. 51, No. 13, pp. 1675–1682, 2015.
55. V. L. Kharitonov, “Predictor based stabilization of neutral type systems with input delay,” *Automatica*, Vol. 52, pp. 125–134, 2015.
56. V. L. Kharitonov, “Prediction-based control for systems with state and several input delays,” *Automatica*, Vol. 79, pp. 11–16, 2017.
57. N. N. Krasovskii, J. McCord and J. Gudeman, *Stability of Motion*, Stanford University Press, 1963.
58. M. Krstic and D. Bresch-Pietri, “Delay-adaptive full-state predictor feedback for systems with unknown long actuator delay,” *The 2009 American Control Conference*, St. Louis, U.S.A., pp. 4500–4505, 2009.
59. M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Boston: Birkhäuser, 2009.
60. M. Krstic, “Lyapunov stability of linear predictor feedback for time-varying input delay,” *IEEE Transactions on Automatic Control*, Vol. 55, No. 2, pp. 554–559, 2010.
61. Z. Lin, *Low Gain Feedback*, London. U.K.: Springer-Verlag, 1998.
62. Z. Lin, B. M. Chen and X. Liu, *Linear Systems Toolkit*, Technical Report, Department of Electrical and Computer Engineering, University of Virginia, 2004.
63. Z. Lin and H. Fang, “On asymptotic stabilizability of linear systems with delayed input,” *IEEE Transactions on Automatic Control*, Vol. 52, No. 6, pp. 998–1013, 2007.
64. Z. Lin, “On asymptotic stabilizability of discrete-time linear systems with delayed input,” *Communications in Information and Systems*, Vol. 7, No. 3, pp. 227–264, 2007.
65. D. Ma and J. Chen, “Delay margin of low-order systems achievable by PID controllers,” *IEEE Transactions on Automatic Control*, Vol. 64, No. 5, pp. 1958–1973, 2018.
66. D. Ma, R. Tian, A. Zulfiqar, J. Chen and T. Chai, “Bounds on delay consensus margin of second-order multi-agent systems with robust position and velocity feedback protocol,” *IEEE Transactions on Automatic Control*, Vol. 64, No. 9, pp. 3780 - 3787, 2019.

67. M. S. Mahmoud, *Robust Control and Filtering for Time-delay Systems*, CRC Press, 2000.
68. A. Z. Manitius and A. W. Olbrot, “Finite spectrum assignment for systems with delays,” *IEEE Transactions on Automatic Control*, Vol. 24, No. 4, pp. 541–553, 1979.
69. D. Q. Mayne, “Control of linear systems with time delay,” *Electronics Letters*, Vol. 4, No. 20, pp. 439–440, 1968.
70. F. Mazenc, S. Mondie and S. I. Niculescu, “Global asymptotic stabilization for chain of integrators with a delay in the input,” *IEEE Transactions on Automatic Control*, Vol. 48, No. 1, pp. 57–63, 2003.
71. F. Mazenc, S. Mondie and S. I. Niculescu, “Global stabilization of oscillators with bounded delayed input,” *Systems and Control Letters*, Vol. 53, pp. 415–422, 2004.
72. F. Mazenc and M. Malisoff, “Local stabilization of nonlinear systems through the reduction model approach,” *IEEE Transactions on Automatic Control*, Vol. 59, No. 11, pp. 3033–3039, 2014.
73. F. Mazenc and M. Malisoff, “Stabilization and robustness analysis for time-varying systems with time-varying delays using a sequential subpredictors approach,” *Automatica*, Vol. 82, pp. 118–127, 2017.
74. F. Mazenc and M. Malisoff, “Stabilization of nonlinear time-varying systems through a new prediction based approach,” *IEEE Transactions on Automatic Control*, Vol. 62, No. 6, pp. 2908–2915, 2017.
75. W. Michiels and S. I. Niculescu, *Stability, Control, and Computation for Time-delay Systems: An Eigenvalue-based Approach*, SIAM, 2014.
76. N. Minorsky, “Self-excited oscillations in dynamical systems possessing retarded actions,” *Journal of Applied Mechanics*, Vol. 9, pp. A65-A71, 1942.
77. S. Mondié, R. Lozano and F. Mazenc, “Semiglobal stabilization of continuous systems with bounded delayed input,” *Proc. the 15th IFAC World Congress*, Barcelona, Spain, 2002.
78. S. Mondie and W. Michiels, “Finite spectrum assignment of unstable time-delay systems with a safe implementation,” *IEEE Transactions on Automatic Control*, Vol. 48, No. 12, pp. 2207–2212, 2003.
79. S. I. Niculescu, *Delay Effects on Stability: a Robust Control Approach*, Vol. 269. Springer Science & Business Media, 2001.
80. N. Olgac and R. Sipahi, “An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems,” *IEEE Transactions on Automatic Control*, Vol. 47, No. 5, pp. 793–797, 2002.
81. S. Oucheriah, “Global stabilization of a class of linear continuous time-delay systems containing saturating controls,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, Vol. 43, pp. 1012–1015, 1996.
82. P. Pepe, “The problem of the absolute continuity for Lyapunov–Krasovskii functionals,” *IEEE Transactions on Automatic Control*, Vol. 52, No. 5, pp. 953–957, 2007.
83. P. Pepe, “Input-to-state stabilization of stabilizable, time-delay, control-affine, nonlinear systems,” *IEEE Transactions on Automatic Control*, Vol. 54, No. 7, pp. 1688–1693, 2009.
84. P. Pepe, G. Pola and M. D. Di Benedetto, “On Lyapunov–Krasovskii characterizations of stability notions for discrete-time systems with uncertain time-varying time delays,” *IEEE Transactions on Automatic Control*, Vol. 63, No. 6, pp. 1603–1617, 2017.
85. P. Pepe and E. Fridman, “On global exponential stability preservation under sampling for globally Lipschitz time-delay systems,” *Automatica*, Vol. 82, pp. 295–300, 2017.
86. P. Pepe, “Converse Lyapunov theorems for discrete-time switching systems with given switches digraphs,” *IEEE Transactions on Automatic Control*, Vol. 64, No. 6, pp. 2502–2508, 2018.
87. J. P. Richard, “Time-delay systems: an overview of some recent advances and open problems,” *Automatica*, Vol. 39, No. 10, pp. 1667–1694, 2003.
88. S. Ruan and J. Wei, “On the zeros of transcendental functions with applications to stability of delay differential equations with two delays,” *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, Vol. 10, pp. 863–874, 2003.

89. O. M. Smith, "A controller to overcome deadtime", *ISA Journal*, Vol. 6, No. 2, pp. 28–33, 1959.
90. S. Su, Y. Wei and Z. Lin, "Stabilization of discrete-time linear systems with an unknown time-varying delay by switched low gain feedback," *IEEE Transactions on Automatic Control*, Vol. 64, No. 5, pp. 2069–2076, 2019.
91. G. Tao, "Model reference adaptive control of multivariable plants with delays," *International Journal of Control*, Vol. 55, No. 2, pp. 393–414, 1992.
92. G. Tao, *Adaptive Control Design and Analysis*, Vol. 37, John Wiley & Sons, 2003.
93. S. Tarbouriech and J. M. Gomes da Silva Jr., "Synthesis of controllers for continuous-time delay systems with saturating controls via LMI's," *IEEE Transactions on Automatic Control*, Vol. 45, pp. 105–111, 2000.
94. D. Tsubakino, M. Krstic and T. R. Oliveira, "Exact predictor feedbacks for multi-input LTI systems with distinct input delays," *Automatica*, Vol. 71, pp. 143–150, 2016.
95. V. Van Assche, M. Dambrine, J. F. Lafay and J. P. Richard, "Some problems arising in the implementation of distributed-delay control laws," *Proc. 38th IEEE Conference on Decision and Control*, Vol. 5, pp. 4668–4672, Phoenix, U.S.A., 1999.
96. G. H. D. Visme, "The density of prime numbers," *The Mathematical Gazette*, Vol. 45, No. 351, pp. 13–14, 1961.
97. C. Wang, Z. Zuo and Z. Ding, "Control scheme for LTI systems with Lipschitz non-linearity and unknown time-varying input delay," *IET Control Theory & Applications*, Vol. 11, No. 17, pp. 3191–3195, 2017.
98. P. J. Wangersky and W. J. Cunningham, "Time lag in prey-predator population models," *Ecology*, Vol. 38, No. 1, pp. 136–139, 1957.
99. Y. Wei and Z. Lin, "On the delay bounds of discrete-time linear systems under delay independent truncated predictor feedback," *Proc. the 2016 American Control Conference*, pp. 89–94, Boston, U.S.A., 2016.
100. Y. Wei and Z. Lin, "Stabilization of exponentially unstable discrete-time linear systems by truncated predictor feedback," *Systems & Control Letters*, Vol. 97, pp. 27–35, 2016.
101. Y. Wei and Z. Lin, "Maximum delay bounds of linear systems under delay independent truncated predictor feedback," *Automatica*, Vol. 83, pp. 65–72, 2017.
102. Y. Wei and Z. Lin, "Stability criteria of linear systems with multiple input delays under truncated predictor feedback," *Systems & Control Letters*, Vol. 111, pp. 9–17, 2018.
103. Y. Wei and Z. Lin, "Adaptation in truncated predictor feedback to overcome uncertainty in the delay," *International Journal of Robust and Nonlinear Control*, Vol. 28, No. 8, pp. 3127–3139, 2018.
104. Y. Wei and Z. Lin, "A delay-independent output feedback for linear systems with time-varying input delay," *International Journal of Robust and Nonlinear Control*, Vol. 28, No. 8, pp. 2950–2960, 2018.
105. Y. Wei and Z. Lin, "Time-varying low gain feedback for linear systems with unknown input delay," *Systems & Control Letters*, Vol. 123, pp. 98–107, 2019.
106. Y. Wei and Z. Lin, " Stabilization of discrete-time linear systems by delay independent truncated predictor feedback," *Control Theory and Technology*, Vol. 17, No. 1, pp. 112–118, 2019.
107. Y. Wei and Z. Lin, "Regulation of linear input delayed systems without delay knowledge," *SIAM Journal on Control and Optimization*, Vol. 57, No. 2, pp. 999–1022, 2019.
108. Y. Wei and Z. Lin, "Regulation of discrete-time linear systems in the absence of any knowledge of the input delay," *IEEE Transactions on Automatic Control*, submitted.
109. Y. Wei and Z. Lin, "A delay independent output feedback law for discrete-time linear systems with bounded input delay," *Automatica*, submitted.
110. W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, New York, 1979.
111. E. M. Wright, "A functional equation in the heuristic theory of primes," *The Mathematical Gazette*, Vol. 45, No. 351, pp. 15–16, 1961.

112. M. Wu, Y. He and J. H. She, "New delay-dependent stability criteria and stabilizing method for neutral systems," *IEEE Transactions on Automatic Control*, Vol. 49, pp. 2266–2271, 2004.
113. L. Xie, E. Fridman and U. Shaked, "Robust control of distributed delay systems with application to combustion control," *IEEE Transactions on Automatic Control*, Vol. 46, No. 12, pp. 1930–1935, 2001.
114. S. Xie and L. Xie, "Stabilization of a class of uncertain large-scale stochastic systems with time delays," *Automatica*, Vol. 36, pp. 161–167, 2000.
115. K. Yakoubi and Y. Chitour, "Linear systems subject to input saturation and time delay: stabilization and L_p-stability," *Proceedings of the 2nd Symposium on Structure, System and Control*, Oaxaca, Mexico, 2004.
116. S. Y. Yoon and Z. Lin, "Truncated predictor feedback control for exponentially unstable linear systems with time-varying input delay," *Systems & Control Letters*, Vol. 62, No. 10, pp. 312–317, 2013.
117. S.Y. Yoon and Z. Lin, "Predictor based control of linear systems with state, input and output delays," *Automatica*, Vol. 53. pp. 385–391, 2015.
118. H. Zhang, D. Zhang and L. Xie, "An innovation approach to H_∞ prediction with application to systems with delayed measurements," *Automatica*, Vol. 40, No. 7, pp. 1253–1261, 2004.
119. S. Zhang and M. P. Chen, "A new Razumikhin Theorem for delay difference equations," *Computers & Mathematics with Applications*, Vol. 36, No. 10–12, pp. 405–412, 1998.
120. X. Zhang, M. Wu, J. She and Y. He, "Delay-dependent stabilization of linear systems with time-varying state and input delays," *Automatica*, Vol. 41, No. 8, pp. 1405–1412, 2005.
121. B. Zhou, G. Duan and Z. Lin, "A parametric Lyapunov equation approach to the design of low gain feedback," *IEEE Transactions on Automatic Control*, Vol. 53, No. 6, pp. 1548–1554, 2008.
122. B. Zhou, Z. Lin and G. Duan, "Properties of the parametric Lyapunov equation based low gain design with application in stabilizing of time-delay systems," *IEEE Transactions on Automatic Control*, Vol. 54, No. 7, pp. 1698–1704, 2009.
123. B. Zhou, Z. Lin and G. R. Duan, "A parametric Lyapunov equation approach to low gain feedback design for discrete-time systems," *Automatica*, Vol. 45, No. 1, pp. 238–244, 2009.
124. B. Zhou, Z. Lin and G. Duan, "Truncated predictor feedback for linear systems with long time-varying input delays," *Automatica*, Vol. 48, No. 10, pp. 2387–2399, 2012.
125. B. Zhou and Z. Lin, "Parametric Lyapunov equation approach to stabilization of discrete-time systems with input delay and saturation," *IEEE Transactions on Circuits and Systems I: Regular Papers*, Vol. 58, No. 11, pp. 2741–2754, 2011.
126. B. Zhou, "Input delay compensation of linear systems with both state and input delays by nested prediction," *Automatica*, Vol. 50, No. 5, pp. 1434–1443, 2014.
127. J. Zhu, T. Qi, D. Ma and J. Chen, *Limits of Stability and Stabilization of Time-delay Systems: A Small-Gain Approach*, Springer, 2018.

Index

Y. Wei, Z. Lin, *Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay*, Control Engineering, https://doi.org/10.1007/978-3-030-53429-5