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FOUNDATIONS OF GEOMETRY AND INDUCTION

Containing
Geometry in the Sensible World
and
The Logical Problem of Induction

JEAN NICOD

With Prefaces by Bertrand Russell and André Lalande

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GEOMETRY IN THE SENSIBLE WORLD

TO

MY TEACHER

BERTRAND RUSSELL

AS A TOKEN OF MY APPRECIATION AND AFFECTION

J. N.

CONTENTS

PREFACE

THE premature death of Jean Nicod is more than a source of deep sadness for those who knew him; it means an irreparable loss in the realm of philosophical studies. Besides articles of considerable value, he had completed two theses for the doctoral degree at the University of Paris. The shortest of these is devoted to the logical problem of induction (pp. 193 ff.); the longer, whose text we have here, deals with a problem whose importance has appeared more and more clearly in the last few years: the relation between geometry and sense-perception.

The history of this problem in modern times is well known. Kant asserted that geometry is based on an *a priori* intuition of space and that experience could never contradict it because space constitutes a part of our manner of perceiving the world. Non-Euclidean geometry has led most thinkers to abandon this opinion; although from the logical point of view, it might be easy to maintain that Lobachevsky's work did not go counter to the Kantian philosophy. Another stronger but less known argument was employed against Kant; it is the argument derived from the attempt to reduce pure mathematics, at first to arithmetic, and then to logic. The implication was that an *a priori* intuition is no more necessary for abstract geometry than for the doctrine of the syllogism.

However, it was still possible to adopt a point of view which has certain affinities with that of Kant; for example, it was the view-point assumed by Henri Poincaré, who maintained that Euclidean geometry is neither true nor false, but that it is simply a convention. In a certain sense, this point of view may still be possible: in all experiment or physical observation, it is *the group* of applicable physical laws which constitutes the object of study, and if the results do not correspond to our expectation, we have a certain choice as to which of these laws should be modified. For example, Henri Poincaré would have maintained that if an astronomical observation seemed to prove that the sum of the angles of a triangle is not exactly equal to two right angles, this phenomenon would be more easily explained by assuming that light does not travel in a straight line than by giving up the system of Euclidean geometry. It is not surprising that Poincaré should have adopted this view; what is surprising, is that the progress of physics should have since shown in its own realm, that this point of view was ill taken. In fact, the eclipse observations undertaken to verify the Einsteinian theory of gravitation are explained usually by admitting both that space is non-Euclidean and that light is not propagated strictly in a straight line. Undoubtedly, it is still *possible* to hold to the view that space is Euclidean, as Dr. Whitehead does, but it is at least doubtful whether such a theory furnishes the most convenient explanation of the phenomenon.

In the following pages there will be found a different criticism, more fundamental than the theory of Henri Poincaré. When a logical or mathematical system is applied to the empirical world, we can distinguish, according to Jean Nicod's observation, two kinds of simplicity: simplicity intrinsic to the system and simplicity extrinsic to it. Intrinsic simplicity is the simplicity of the laws that establish the relations among the entities taken as primitive in the system. Extrinsic simplicity is the simplicity of the empirical interpretation of these entities. The points, lines, and planes of geometry give it the character of intrinsic simplicity, because they enable the axioms to be stated briefly; but they do not constitute what is empirically given in the sensible world. Consequently, if our geometry is to be applied to the perceived world, we shall have to define points, straight lines, and planes by means of terms which are at least similar to our sense-data. In fact, this definition is extremely complicated, and thus removes any character of *extrinsic* simplicity from our conventional geometry. To regain this extrinsic simplicity, we must start from data which are not in conformity with ordinary geometry, Euclidean or non-Euclidean; and we must formulate gradually, if we can, suitable logical constructs that enjoy the required properties. We cannot say in advance whether we shall obtain greater extrinsic simplicity by taking recourse to straight lines and Euclidean or non-Euclidean planes, although we admit that the possibility of one of the systems implies the possibility of the other and reciprocally.

Dr. Whitehead has examined, from the point of view of mathematical logic, how we can define in terms of empirical data the entities that traditional geometry considers as primitive. His method of "extensive abstraction" has great value and efficacy in this regard. But this method starts from the knowledge of the completed mathematical system which is the object to be attained, and goes back to entities more analogous to those of sense perception. The method adopted by Nicod follows the inverse order: starting from data of perception, it tries to attain the various geometries that can be built on them. This is a difficult and novel problem. To treat it logically, the author assumes as a starting point an entirely schematic simplicity of sensations, although it is easy to imagine some animals among whom it might exist. In his first example, he shows us an animal possessing only the sense of hearing and a perception of temporal succession, who produces notes of varying pitch as he proceeds up and down the keyboard of a piano. Now, such an animal, if we suppose him endowed with sufficient logical power, will be able to produce two geometries, both, naturally, in one dimension. The animals presented next come nearer to man in their perceptions; although they differ from most of us in that they are logicians and metaphysicians as penetrating as Nicod himself.

The distinction between pure geometry and physical geometry, which has gradually appeared of late, is presented as clearly as possible in Jean Nicod's work, the first part of which deals with pure geometry. This distinction and its consequences are not yet comprehended by philosophers as much as they deserve. In pure geometry we assume as a starting point the existence of a group of entities whose relations have definite logical properties and we deduce from them the propositions of the geometry under consideration. The existence of groups of entities having relations of this nature can in all usual cases be deduced from arithmetic. For example, all the possible triads of real numbers arranged in their natural order form the points of a tri-dimensional Euclidean space. The whole question belongs to the realm of pure logic and no longer raises philosophical problems. But in physical geometry, we are confronted with a much more interesting problem because it is far from having been completely solved. We know that experimental physics employs geometry; from this it follows that the geometry which it employs is applicable to the empirical world to the degree in which physics is exact. That is to say, it ought to be possible to find groups of sense-data and relations among these data such that the relations which are derived from these groups may approximately satisfy the axioms of the geometry employed in physics. Or, if the sense-data alone are not sufficient, they ought to be complemented in the same way as they are in physics, by means of inferences and inductions whose use is authorized by ordinary scientific method; for example, the inference which allows us to assume that the moon has another side which we do not see. This point of view is supported and faciliated by the absorption of geometry by physics as a result of the theory of relativity. However, the psychological aspect of this problem has been studied very little, probably because few psychologists possess a sufficient knowledge of modern physics or mathematical logic. We must build a bridge by beginning on both ends at the same time: that is to say, on one side, by bringing together the assumptions of physics and the data of psychology and, on the other, by manipulating the psychological data in such a way that we may build logical constructions that approximately satisfy the axioms of physical geometry. Jean Nicod has, in the last of these tasks, made progress of the highest importance. He has created a method much superior to that of his predecessors. We cannot say yet that the two sides of the bridge meet in his work, but the gap that remains to be filled today is smaller than it was before the writing of the following pages. That is why I recommend the study of them to all those who believe in the value of philosophical research and who are capable of appreciating in this work the rare clarity and beauty of its exposition, which reflects faithfully the equal beauty of the author's life and character.

BERTRAND RUSSELL.

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GEOMETRY IN THE PERCEIVED WORLD

INTRODUCTION

EXPERIENCE is the only test of the truth of particular propositions concerning objects around us. Let us call the group of such objects *physical*, taking the word in its widest sense. We are taking physics as an entirely empirical science: it attains certainty or probability only to the degree in which experience verifies its findings. Its sole claim upon our credence is the exactness with which it tells us what we shall see, hear, and touch in accordance with what we have seen, heard, and touched. If that is not its only task, such is anyway the end by which it wishes to be judged. No one would object to the feasibility of analyzing physics and its claims of verification in relation to what is given to the senses.

The analysis is still far from being achieved. Writers most occupied with the empirical meaning of propositions about the material world give us, in fact, only the most summary account of this meaning itself. They take any proposition from physics and say: "In experience it means something *like* this." But not *exactly* this. For on closer examination we would find that no physical fact is verified by sensations without ambiguity. Any perceived fact, we say, can spring from various physical causes, although not all these causes are equally probable; in other words, our senses can deceive us about objects. But, on the other hand, these same senses can also correct us by means of perceptions which are sometimes very different in kind from our first impressions. Thus the presentation of a physical fact in my experience goes beyond my present observation and extends into the group of my past and future observations. Because of this extension the sensory manifestations of various physical facts are not as distinct from one another as these facts themselves, but are, on the contrary, intimately fused. If one wishes to obtain the last word about the exact sensory meaning of any physical proposition whatsoever, he must seek it right in that realm of experience which is subject to the laws of physics. With respect to verification, as Duhem has well observed, all branches of physics form one whole. It is the form and structure of this whole that we wish to study in order to discern the concrete meaning of those simple and complicated, ordinary and sophisticated laws which make up our knowledge of nature.

Surely this content is already present in our mind. It furnishes us with special insights continually. But its wholeness escapes us. It stays in the shadow and yet guides us to the light; we know how to use it, but we do not know how to analyze it.

The reason for this strange fact is that the formation and growth of physics are pervasively dominated by the quest for simple laws, or rather, for the simple expression of laws. This expression can be obtained, indeed, only by cloaking the complexities of things with simple names. For nature is such that simple things do not enjoy simple laws, so that, in order to simplify laws, we must complicate the meaning of their terms. Energy, matter, object, space and time—all these physical terms and, generally, every word that physics employs outside of terms having simple designations, derive their meaning and utility from this tyrannical desire for simple and forceful embodiments of the laws of the sensible world.

The real complexity of these laws is hidden away in the very simplicity of the new terms. It emerges naturally again in the process of application; but then it ceases to overwhelm the mind. It even ceases to be distinct from mind. One might say that the mind remains attached to these new terms because of their æsthetic appeal. Thus the objective world becomes eclipsed by its representation; and in physics where we paint this picture, we must learn once more how to see the natural world.

Such an enterprise has not been undertaken until now. The result is that we believe in laws which are founded only on experience although we do not know exactly what they mean in terms of experience. It is true that the undertaking is difficult and, moreover, long. Besides, it would not fit those programs which philosophers have been accustomed to follow. For their sole interest in the sensory content of judgments about reality was determined exclusively by the desire to use it in arguments about the *general* nature of matter or of physics. These arguments proceeded from the existence of this sensory content, and not from its more determinate structures; and the existence of the sensible world being so indubitable that the most summary designation sufficed to render it obvious, philosophers went no further. The empirical analysis of nature, as soon as it was designated, no longer seemed to be actually worth while making.

We must think otherwise, however. The discernment of the sensory order around us, which forms the qualitative background of our life and of our science, and which is ever present however indistinctly, should certainly be a source of curiosity to any philosopher, even if his metaphysics should not obtain any aid from it. Such is the end at which we aim. We hope to approach it by the study of the objective aspect of geometry. It is impossible, in fact, to possess a proper idea of the order of our sensations if we are hampered by a false or confused idea of space.

This study might be a preface to the analysis of physics in terms of experience. It is also a beginning in it. For we shall find that the universal order of space to which every physical proposition seems to refer is, in truth, nothing but the very group of the laws of physics. The properties of space are already the most general schemata of physics and are nothing else. Thus—we shall be convinced of it as we proceed—the study of the spatial structure of a sensory universe is the study of the form and totality of all its laws.

We propose in this work to ascertain in what way geometry is an aid to physics; how its propositions are applied to the order of the perceived world; how knowledge of them helps us in the formulation of experiments and laws. For every statement in physics teems with geometry: every prediction of a perceptual fact is dependent on a certain disposition of the objects and observers, which is expressible in geometrical terms.

We are asking *how* geometry is exemplified in nature and not *why* it is. We are investigating the structure of the facts, not the reasons which render them possible or necessary. Analysis, indeed, should precede explanation; analysis is always possible, whereas explanation is not always possible.

In this problem, geometry appears as a form to which the objective world serves as matter. The natural order of this analysis is to study first what this form is, then what this matter is, and finally the particular way in which we find one in the other. Let us, in the first place, become acquainted with geometry in so far as it is a formal and wholly abstract science of the implications of certain principles involving terms and relations whose meanings are indeterminate. Let us next examine what terms and what relations are actually perceived by us in nature. Finally, let us investigate what meanings derived from these terms and relations are in agreement with the terms and relations of geometry, and comprise the laws of experience.

In the course of this work, I have been greatly aided by the advice and benevolent criticism of M. A. Lalande. To him I offer here the expression of my cordial appreciation.

I am equally indebted to M. E. Cartan for several valuable remarks.

PART ONE

GEOMETRIC ORDER

CHAPTER I

PURE GEOMETRY IS AN EXERCISE IN LOGIC

WHAT then is geometry considered as purely formal? It is whatever we can know about its structure without knowing its object; whatever we can understand in a treatise on geometry without being acquainted with the nature of the entities which it discusses.

In Kant's time, this point of view had not yet been reached. For geometry, which since Euclid was tending to liberate its proofs from the matter furnished by figures, for the purpose of basing them only on pure reason, had not yet succeeded in doing so. Deprived of concrete diagrams, its proofs seemed without force; the very concatenation of the propositions seemed to belong to these figures and not to the purely logical relations involved. All geometrical knowledge was in this way conceived as inseparable from the apprehension of space—a primitive matter, which, by imparting next its order to the sensible world, played with regard to the latter, the contrary rôle of a form. Thus the imperfect character and peculiar nature of the demonstrations of geometers supplied philosophers with the impression of a special mystery, and committed them to involved theories designed to account for the alleged existence of proofs which did not draw their force from common logic.

But the actual progress of geometrical science allows us to conceive the problem more simply. Indeed, while the philosophers were speculating over the extra-rational character of geometrical proof, the geometers succeeded in doing away with it altogether. They made it a principle that proof by figures is only the outline of a proof. They regarded the appeal to intuition as the index of a lacuna, the sign of the use of an assumed principle which they tried to make explicit; they would not accept any proof as regular unless it formed an entirely formal chain.

To obtain a proof, in this state of formal perfection, it is no longer *necessary* to illustrate it by a figure, to relate it to a matter, to attribute a determinate meaning to the geometrical expressions which it involves, for these concrete values add nothing to its force. It is *possible* to be convinced that the theorems flow from the axioms and postulates without knowing the meaning of *a point, a straight line* or *distance*; there is not a geometer today who would deny this. By becoming rigorous, that is to say explicit, geometrical proof has detached itself from all objects.

We do not have here any paradoxical development. Quite on the contrary, it puts an end to the paradox which opposes geometrical reasoning to all other reasoning. For a good demonstration, stated without anything implicit, is valuable for its form alone, independently of the truth and even of the meaning of its system of propositions. We may be astonished at this important fact, but we cannot doubt it. By freeing itself from all figures, by detaching itself from the meaning of the material terms which figure in it, geometrical demonstration has simply returned to common reason.

It is then possible today, as it was not a century ago, to take a completely abstract and fundamental view of geometrical science as independent of any object. It then appears as a chain of formal reasoning, which is in a certain sense blind, and which draws consequences from a group of premises formulated in terms of entities whose meanings, indifferent to the argument, remain quite indeterminate. Such is the universality of geometry. It is under this form, devoid still of any reality, that it may be fixed in our minds now. For, by conceiving it, at first, disengaged from any object, we are prepared to discern without any preconceived idea the objects of the universe to which the science is in fact applied.

Suppose then that we have not been taught geometry in school, and that we are acquainted with none of the particular terms of that science. Undoubtedly, the very things with which it is commonly supposed to deal, cannot fail to be familiar to us. But let us suppose that nobody has ever taught us their scientific names, and that, like the child Pascal, we call a straight line a bar and a circle a ring. Let us imagine that someone puts in our hands one of those treatises on geometry which aim only at rigour and which disdain all figures. What shall we get from it ? Let us try, however, to read it.

It is composed of a small number of initial statements entitled "axioms" or "postulates" and other propositions entitled "theorems," which appear to spring from the first by virtue of texts entitled "demonstrations." But if we understand the terms of current language only, and in particular the terms of ordinary logic, all the properly geometrical terms such as "point," "straight line," "distance," are entirely unknown to us; and these new terms seem to us at first very numerous. However we soon notice that they are for the most part introduced as simple abbreviations of complex expressions, in which we find only a small number of unknown terms. The latter are always identical, and must be only those contained in the initial propositions. There will be, for example, the class of "points," the relation of three points "in a straight line," and the relation of two couples of points "separated by the same distance"; thus, the term "sphere" will be defined as the abbreviation of the complex expression "class of points separated from a certain point by a constant distance."

We have taken inventory of the unknown expressions and we have reduced them to three. However, we have not eliminated them; since we are not aware of any subsistent expressions, we must admit that we do not understand what the " axioms," " postulates," " theorems " mean. But, to our surprise, we understand perfectly the intermediary steps called "demonstrations." The terms which embarrass us are still to be found there, but it is enough for us to understand the ordinary words which accompany them, and which belong to the logical sheathing of language, in order to follow the argument step by step, to grasp its march, to enjoy its ingenuity, and to discern its precision.

There is something surprising in this fact that the rigour or force of a demonstration can be apprehended without any knowledge of its matter. We are astonished to be able to proceed thus with our eyes closed. But this very force of form is found again in the most simple reasoning, and is valid in any given case because it holds for all possible and impossible cases. That is the constitutive fact of logic, remarkable, certainly, but common.

Then what do we learn from our reading? We may answer by saying: " I do not know what the author of this treatise calls a point, nor *a fortiori* what he calls three points in a straight line and two couples of points separated by a constant distance. But I know that if these three things really have, as he asserts, the properties that the axioms and postulates state, they cannot fail to have at the same time all the properties that the theorems state."

Reflecting on the fact that we have been able to establish the connection which links the various propositions in which these three terms with unassigned meanings figure, we can rise even to a more general view-point. Instead of assigning to these terms determinate but unknown meanings, we can take them as variables—a symbolic means of expressing this universal truth: "if a class π, a relation R having as terms three members of π, and a relation S having as terms two couples of members of π satisfy the axioms and postulates—in other words—if the assignment of three meanings π, R, S to the three expressions *point, in a straight line, separated by a constant distance*, transforms the axioms and postulates into true assertions—the meanings π; R, S also satisfy the theorems."

A geometrical proposition ceases then to be determinate and susceptible of being true or false by itself. It is no more than a formula with blanks to be filled by all kinds of different propositions, some false, others true, according to the meanings attributed to its variables: it is only a *propositional function*; and the systematic implication, *for all meanings*, of the propositional functions that are theorems as derived from the propositional functions that are axioms and postulates* forms all the instruction that we can obtain, in our ignorance, from the geometrical treatise which fell into our hands.

Let us close the treatise now and ask ourselves what motives could have impelled its author to write it. Perhaps it was the unique charm of the logical adventure, the singular pleasure of deducing the implications of a group of propositions chosen—like the rules of games of mental entertainment—for the sake of the diversity and harmony of their consequences. Perhaps, on the contrary, the author has tried to imitate nature by making axioms in accordance with natural objects. Has he not modelled his axioms on the demonstrated or conjectural properties of certain entities which are found in his universe, and perhaps also in ours? Let us try then to discover, or at least to conceive one or more systems of meanings satisfying the axioms of our author: we shall say that such a system of meanings is a *solution of this group of axioms*.

* The difference between a postulate and an axiom is only a matter of degree in regard to evidence, and does not exist for us because both, deprived of any fixed meaning, lack altogether any self-evidence. We shall then call *axioms* all the premises of a treatise on geometry in order to simplify language. Such is, besides, the usage of several modern geometers.

The domain of numbers furnishes an answer first. Let us in fact attribute these meanings:

(1) to the variable class of *points*, *the class of ordered triads of real numbers taken with their signs*;

(2) to the variable relation *in a straight line*, *the relation of three triads of real numbers* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) expressed by the equations

$$\frac{x_1 - x_2}{y_1 - y_2} = \frac{x_2 - x_3}{y_2 - y_3}, \quad \frac{x_1 - x_2}{z_1 - z_2} = \frac{x_2 - x_3}{z_2 - z_3}.$$

(3) To the variable relation *separated by a constant distance*, *the relation of two couples of trios of real numbers* (x_1, y_1, z_1), (x_2, y_2, z_2); (x_3, y_3, z_3), (x_4, y_4, z_4) expressed by the equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$
$$= (x_3 - x_4)^2 + (y_3 - y_4)^2 + (z_3 - z_4)^2.$$

It is known that the system of meanings (1), (2), (3), satisfies the axioms of Euclidean geometry.

This geometry admits then of one interpretation or purely arithmetic "solution." However, abstract geometry being more often confounded with its application to a particular interpretation called "space," the arithmetic interpretation is commonly introduced under the indirect form of a measure of space. But this detour is superfluous. As soon as geometry is conceived by itself as a form devoid of all application, it is seen that this scheme is applicable directly to numbers without any necessity of regarding them as measures or representations of a determinate subject matter. It is by thus substituting the purely arithmetic meanings (1), (2), and (3) for *points*, *rectilinearity*, and *congruence*, that the axioms, and consequently the theorems are resolved into arithmetical propositions. For example, the axiom which says that if the points *a*, *b*, *c* are in a straight line, and if the points *b*, *c*, *d* are also in a straight line, then the points *a*, *b*, *d* are in a straight line becomes in this interpretation: If the numbers x_a, y_a, z_a; x_b, y_b, z_b; x_c, y_c, z_c have the relation

$$\frac{x_a - x_b}{y_a - y_b} = \frac{x_b - x_c}{y_b - y_c}, \quad \frac{x_a - x_b}{z_a - z_b} = \frac{x_b - x_c}{z_b - z_c},$$

and if the numbers x_b, y_b, z_b; x_c, y_c, z_c; x_d, y_d, z_d have the relation

$$\frac{x_b - x_c}{y_b - y_c} = \frac{x_c - x_d}{y_c - y_d}, \quad \frac{x_b - x_c}{z_b - z_c} = \frac{x_c - x_d}{z_c - z_d},$$

then the numbers x_a, y_a, z_a; x_b, y_b, z_b; x_d, y_d, z_d will have the relation

$$\frac{x_a - x_b}{y_a - y_b} = \frac{x_b - x_d}{y_b - y_d}, \quad \frac{x_a - x_b}{z_a - z_b} = \frac{x_b - x_d}{z_b - z_d},$$

which is in fact an arithmetical theorem.

It is true that these arithmetical meanings of the primitive expressions in our treatise on geometry are not *simple*, and that being complex, they seem all the more artificial. Is it not strange that a point can be a class of three numbers? Not at all, since "point" is, prior to such a class definition, an empty word.

The discovery of one system of meanings satisfying a group of axioms is always logically very important: it constitutes the proof that these axioms do not contradict one another; and this is the only known proof of consistency. So arithmetic is one guarantee of the compatibility of the axioms of geometry. But there are perhaps, outside the domain of numbers, other "solutions" for this same group of axioms: it is the search for these that constitutes the object of this work.

CHAPTER II

FORMAL RELATIONSHIP OF VARIOUS SYSTEMS OF GEOMETRY

WE might conclude our introduction without stopping to consider the fact that geometry can be made to assume more than one form; for all such forms are equivalent. But we shall rise to a more general point of view by taking into account the plurality of these forms and by investigating the exact manner of their equivalence.

The geometry that we learn at school is the study of the consequences of a group of axioms that can be formulated in terms of three fundamental expressions: the point, the straight line and distance. We are accustomed to consider this group of axioms as the fixed and necessary seat of Euclidean geometry. It is nothing of the sort: that is only one of its possible foundations, for this geometry, like a polyhedron, can rest on a multitude of different bases.

Before abandoning primitive classical notions for others, it is possible to substitute for certain axioms other propositions which were theorems before, but which enable us to demonstrate inversely the axioms they replace. We can in this way form several equivalent sets of axioms for the Euclidean point, straight line, and distance. It is true that some of these postulate-sets are shorter than others, and that one of them can surpass them all in brevity and elegance; but all are equally correct. Hence, for the same system of primitive notions, the order of the propositions of Euclidean geometry and their distribution as axioms and theorems are not fixed. But it is still possible to change the primitive notions: the result of such a change will produce more far reaching consequences.

From the current system of the point, straight line and distance it is easy to eliminate the straight line as a primitive notion by defining it as the class of points equidistant from three given points. If geometry does not ordinarily proceed in this way, it is because it adheres to conventional classifications rather than to economy, and because it wishes to retain the value of the distinction between projective and metrical properties. But when geometry is considered as a single whole, it is often convenient to be able to take it as the science of the unique relation of congruence between couples of points.

Is this relation then its irreducible basis? No, for it is possible to replace it by many others.

Although in Euclid, only the notions of straight line and distance are to be found among his explicit axioms, Euclid has often recourse to the properties of the quite different idea—viz., *displacement*. As he was not concerned with formulating the use he made of it, some have tried to see in this loose sort of demonstration the proof of the persistence of a share of certain unformulable elements of intuition. But the present state of the science should make us think otherwise. On the one hand, provided with more complete sets of axioms, the geometry of the straight line and distance no longer borrows any assistance from the extraneous idea of displacement of figures. On the other hand, this idea can by itself be taken as the sole basis of geometry.* Thus, beside the geometry based on the straight line and distance, and conformable finally with the ideal of rigour pursued by Euclid, a geometry founded quite entirely on displacement has emerged from the same extraneous matter that had vitiated the proofs of the master.

But the point is not that congruence and displacement alone can serve as the unique primitive relation. A multitude of other relationships might be adopted, such as the relation of

* Cf. The "affine" geometry of Weyl: *Space, Time, and Matter*; also Hilbert, *Foundations of Geometry*.—Tr.

five points of a sphere: it is possible, in fact, to define all the entities of geometry in terms of this unique relationship of "sphericity"; this is only one random example. Geometry can be exhibited as the science of the logical properties of any one of various relations of points. The various deductive constructions thus obtained would differ undoubtedly in general elegance of expression; but there would not be one which would not have some particular advantage.

When one passes from one to the other of these equivalent geometries developed in terms of various primitive notions, the order of the propositions, particularly their division into axioms and theorems, is naturally modified. The geometry of congruence starts from the properties of congruence, the geometry of the sphere starts from the properties of the sphere, the geometry of displacement from the properties of displacement, and the order of propositions is thus found to vary with the point of departure.

Is it only the order itself which varies? Do those propositions which seem the same in all the systems, the hierarchy alone being changed, remain the same in truth? If we look more closely, their identity becomes merely apparent. In fact, the geometry of congruence defines all geometrical entities in terms of congruences, the geometry of the sphere defines them all in terms of sphericity, the geometry of displacement in terms of displacements: in each of these geometries, the word straight line assumes different expressions: in one case it is a function of congruences, in another a function of sphericity, in still another, a function of displacements. The same proposition then does not have the same meaning in every context. Take the proposition: *Any two points are included in a straight line.* This means in the first system: *For every point-couple x, y, there exist three points a, b, c such that x and y are each equidistant from a, b, and c.* In the second system, the same proposition means: *Every point-*

couple x, y belongs to two distinct classes M, N *of points which do not fall within any sphere containing three given points.* Finally in the third system, it has the meaning: *For every point-couple x, y, there exists a displacement which simultaneously transforms x into itself and y into itself.* Each system admits only its own primitive notions and cannot admit any other meanings. If one does succeed in finding the same statements in the different systems, it is due to the practice of using identical abbreviations to cover different concepts, a practice which is due only to a subtle play of words.

But since the various systems which result from the modification of primitive concepts contain nothing but different propositions, in what sense can we regard them as so many phases or aspects of the same geometry ?

Let us not answer too quickly that they all have the same subject-matter. For this sameness is not literally present and has to be shown. Moreover, it results only from a purely formal relationship: only by comparing two systems of Euclidean axioms, without assigning any meaning to their primitive propositions, can we be convinced that they apply to the same groups. To get at the exact nature of this identity, as well as to understand its ground, it is best to make explicit the formal relationship which determines it.

Let us compare the two systems which present Euclidean geometry as the science of an indeterminate relation named *congruence* between the two point-couples, and as the science of an indeterminate relation named *sphericity* among five points. The first deals only with congruences and it defines sphericity as a certain function of congruences. Likewise, the second treats only of sphericities, and employs the term congruence as a sign for a certain function of sphericities. These two functions are as follows:

In the geometry of congruence, *sphericity* is the name of the relation among five points x_1, x_2, x_3, x_4, x_5 including the existence of a point y (centre of sphere) such that the (radii) couples $x_1y, x_2y, x_3y, x_4y, x_5y$, are congruent.

In the geometry of sphericity, *sphere* is the name of the class of points x related to four given points by sphericity, when this class exists. In case it does not exist, we shall say that any one of the four given points is in the same *plane* with the other three. If two points x, y are the only points common to a certain sphere and to the (tangent) planes M and N respectively, and if z is a point common to M and N, we shall say that z is *equidistant* from x and y. Finally, we shall say that the point-couples ab, cd are congruent when there exist points $x_1, x_2, \ldots x_n$ such that in the series $abx_1x_2 \ldots x_ncd$, each interior term is equidistant from its two neighbouring ones.

Now we find the terms congruence and sphericity in both systems. But their meanings cannot be the same because the relations which define them are inverted. Each, within its own system, is more simple than the other. In the former sphericity is defined in terms of congruence; in the latter congruence is defined in terms of sphericity. Both cannot be taken simultaneously as the same literal meanings of the two terms. In order to unearth the fundamental relationship in both systems, we must dig under verbal identities. We must expose the verbal artifice in order to discover the ground which makes it advantageous.

We have already noticed that this application of the same words in different systems to things whose ordered relations are inverse produces the appearance of a complete identity among the *propositions* stated in all these systems. It is true that this identity is also verbal. But it does manifest a certain formal correspondence. It shows that it is *possible* to "translate" the simple expressions of one system by the complex expressions of the other in such a way that the axioms of the first system (and consequently all its propositions) are revealed as " translated " by the propositions of the second system. Such is the relationship which logically unites all conceivable systems of Euclidean geometry. It enables us thereby to establish a general definition of this geometry. Starting from any one of its forms, for example, from the science of congruences of point-couples, we can include all possible forms of Euclid's geometry under the concept of systems " translatable " into the former and *vice versa*.

This relation is familiar to the algebraist. It is nothing more than the transformation of a function into another by a change of variables. Just as the function $y^2+2xy+x^2+x+y$ is transformed into the function u^2+u if we substitute $u=x+y$, so we can transform our geometrical functions as follows:

E.g. *if there exists a point y such that* $x_1y, x_2y, x_3y, x_4y, x_5y$ *are congruent, and a point z such that* $x_2z, x_3z, x_4z, x_5z, x_6z$ *are congruent, then there exists a point u such that the couples* $x_1u, x_2u, x_3u, x_4u, x_6u$ *are congruent* may be transformed into the sphere-function if we substitute the equivalent of *sphericity = relation of five points such that there exists a sixth point forming with them five congruent couples*. The transformed function will be: *If the points* x_1, x_2, x_3, x_4, x_5 *and the points* x_2, x_3, x_4, x_5, x_6 *have the relation of sphericity, the points* x_1, x_2, x_3, x_4, x_6 *have it also.*

It is only necessary now to extend the formal relationship in question from numerical functions of numerical variables to any functions of any variables.

Assume two propositional functions $F_1\ (x_1 \ldots a_1 \ldots R_1 \ldots)$, $F_2\ (x_2 \ldots a_2 \ldots R_2 \ldots)$ in which the indeterminates can be either individuals $x_1, x_2, \ldots$ or classes $a_1, a_2, \ldots$ or relations $R_1, R_2, \ldots$ The number of such terms and their logical types are not restricted to being the same in both functions; let us remember that, on the contrary, a group of propositional functions can be replaced by a single function, the logical product of the former.

Now establish a correspondence between each one of the indeterminates of F_2 and a certain *logical* function of the indeterminates of F_1, that is to say, an expression containing outside these indeterminates only logical terms as follows:

$$x^1{}_2 = f(x_1 \ldots a_1 \ldots R_1 \ldots) \text{ etc.}$$
$$a^1{}_2 = g(x_1 \ldots a_1 \ldots R_1 \ldots) \text{ etc.}$$
$$R^1{}_2 = h(x_1 \ldots a_1 \ldots R_1 \ldots) \text{ etc.}$$

The simple expressions $x^1{}_2 \ldots$, $a^1{}_2 \ldots$, $R^1{}_2 \ldots$ being equivalent to the complex functions f, g, $h \ldots$, we may substitute the former in place of the latter. If the following equivalence results

$$F_1(x_1 \ldots a_1 \ldots R_1 \ldots)$$
$$\equiv F_2(x^1{}_2 \ldots a^1{}_2 \ldots R^1{}_2 \ldots)$$

we shall say that F_1 is *transformable* into F_2.

If we now denote by $F^1{}_1$, $F^{11}{}_1$, . . . the *logical consequences* of F_1, the theorems which are implied by the groups of axioms $F^1{}_1$ and which are naturally functions of the same indeterminates, it is possible that it is no longer the function F_1 which is transformable into F_2, but instead one of the functions $F^1{}_1$, $F^{11}{}_1$, . . . which are derived from F_1. A single letter can represent then the logical product of several theorems just as before it stood for the product of several axioms. In order to designate this more general formal relation, we shall say that the function F_1 *contains* the function F_2. Finally, when the functions F_1, F_2 are contained mutually in each other, we shall say that they are *inseparable*. Such is the definition of the general formal relationship of two systems of equivalent principles whether expressed or not in terms of the same primitive concepts. In fact this relationship *constitutes* their equivalence.*

* A fair statement in less condensed form of this relationship will be found in S. Buchanan's *Possibility* (this Library) under what the author aptly designates as "intervalence." Cf. also "analytical equivalence" and Prof. Sheffer's articles.—Tr.

CHAPTER III

MATERIAL CONSEQUENCES OF THIS RELATIONSHIP

OUR problem is now a material one, and not a formal one. We propose to investigate the objects which satisfy the axioms of Euclidean geometry. Now, these axioms are not unique. We are confronted with a choice of many possible systems. The problem then divides into many specific problems, each aiming at the particular objects capable of satisfying each separate system of axioms. But the formal relationship of all these systems re-establishes an *a priori* unity among the diverse objects to which they apply.

Let us consider the case of two inseparable systems of principles F(X), Φ (Y), formulated respectively in terms of two primitive propositions X and Y. The case would not be altered by increasing the number of undefined concepts.

The propositions X and Y do not in general have any common meaning. This incompatibility of the possible solutions of F (X) and Φ (Y) is evident in the case where X and Y are of different logical types, where, for example, X is the congruence relation of two couples of terms, and Y the sphericity relation of five terms. *Two systems of inseparable axioms are not then generally satisfied by the same values* : they are more often *incompatibles*.

But every solution of one logically furnishes a solution of the other. In fact, since F(X) contains Φ (Y) [by definition of " inseparable "] there exists a logical function

$$Y^1 = f(X)$$

such that $G(X) \equiv \Phi(Y^1)$ results from F(X). Now, suppose a certain meaning X_1 satisfies F(X), that is to say, that we have $F(X_1)$. Let us form the meaning:

$$Y^1{}_1 = f(X_1)$$

This second meaning, is established logically in terms of the first, since f is a logical function, and therefore satisfies Φ. For $\Phi(Y^1{}_1)$ results from $F(X_1)$, and $F(X_1)$ is true.

Thus the two equations $x^3+3x^2+4x+3=0$ and $y^3+y+1=0$, its transformation when we substitute $y=x+1$, have no common solution, but any solution of one furnishes a solution of the other. In the same way, no meaning can satisfy both the axioms of congruence and of sphericity simultaneously, that is, of $F(X)$ and of $\Phi(Y)$. But every meaning X_1 which satisfies the axioms of congruence furnishes a meaning $Y^1{}_1$ which satisfies the axioms of sphericity. And this is just the meaning $Y^1=f(X)$ or the definition of sphericity in the system of congruence, while congruence itself receives the meaning X_1.

Starting from the one "solution" of a system of axioms, we can thus form logically "solutions" of all the systems inseparable from the former system. Logically would mean without adding any matter. All these equivalent values have the same elements of meaning. They differ only as addition differs from subtraction. They are different concepts selected from the same facts; their different laws express the same state of things.

We can then say that two systems of inseparable (equivalent) axioms are true of the same realities. But it must be remembered that this proposition is still indeterminate, that what is called the same reality furnishes the content of a multitude of logically distinct entities. Yet these various forms must express the same order. Otherwise, this identity of the realities which verify inseparable systems of axioms would run counter in our minds to the incompatibility of their solutions. We should think that these systems which both imply and exclude each other can lay no claim to reality; they are only artifices. But that would be a confusion in thought. For inseparable systems of axioms do indeed apply to the same ideal things but not to the same concrete entities. Before the meanings of its primitive propositions are assigned, a system of axioms is neither true nor false, since it is then only an empty form. On the other hand, as soon as these meanings are concretely fixed (besides the realm which gives them content), they can satisfy only one system. The geometries of congruence and of sphericity are both true of a single ideal realm, that of numbers. But they are not true of the same entities in that domain. A numerical relation either does or does not possess the properties of congruence or of sphericity; it cannot possess both at the same time. The existence of formal systems that are inseparable (formally equivalent) and also (materially) incompatible is a commonplace fact. It does not diminish in any way the truth or falsity that arises in the application of these systems.

Henri Poincaré seems to forget this sometimes. He indeed notices that non-Euclidean systems can be translated into ordinary systems, a fact which leads him to draw the conclusion that the question of the truth and falsity of one or the other has no meaning, or at least, only a special meaning. Now, if non-Euclidean geometries are really translatable into any one of the Euclidean geometries, and if the inverse translation is equally possible, as can be easily verified, it follows from what we have just said that they do not differ from the Euclidean geometries any more than the latter differ among themselves. The main difference seems to arise from the fact that the same words straight line and congruence are used to designate different properties of the primitive propositions. By changing the meanings of the primitive propositions of any Euclidean system we can obtain a group of systems which will include the non-Euclidean. This general group then presents a remarkable unity, which does not however justify Poincare's conclusion. Undoubtedly, all these geometric systems are simultaneously true of a certain domain like the realm of number or the physical world, *because their interpretation in that way remains indeterminate.* But tell us which numerical relations and which physical relations you use for congruence and rectilinearity ; then the question of knowing whether these relations behave in a Euclidean or non-Euclidean way is perfectly definite and legitimate.

If it is true that in any domain there are terms and relations which satisfy the axioms of a given geometry, it is no less true that there are also *others* which satisfy any other system of geometrical axioms. We remain free to choose terms and relations in such a way that the order of this domain is expressed by the system that we prefer. But among all these groups of axioms certain ones are more simple in themselves: do they not therefore have an undeniable advantage over all the others, since it depends upon their intrinsic character ?

Poincaré thinks so. That is why he does not believe that any alteration of the universe or of consciousness can take away from traditional geometry its privileged intrinsic simplicity. But this opinion takes account of only one kind of simplicity and complexity, while there really are two kinds.

For one thing, one system of axioms can as a matter of fact be intrinsically simpler than another. It can contain fewer axioms or briefer ones; it may include fewer words altogether, or contain a smaller number of primitive expressions. This kind of simplicity has nothing to do with the meanings which these expressions may convey. It may be called *intrinsic simplicity* because it is inherent in the very form of the system. This is the type Poincaré has in mind.

But we must also consider the kind of simplicity inherent in the *meanings* of the fundamental concepts of the system. What relations can be taken for the meaning of *congruence*? The geometry of congruence does not give this meaning by itself; we have to look for it ourselves. Similarly, the geometry of sphericity does not tell us the meanings which its fundamental relation may allow. Any suitable meaning of congruence enables us to construct a suitable meaning for sphericity, and *vice versa*. But, which of these two expressions admits the simpler meaning in a given universe of discourse? That is the point which would confer a privilege upon one of the geometries over another in the given context, and that is exactly what we do not know at all *a priori*. For this second order of simplicity between two systems of axioms is *extrinsic*. It may be reversed whenever the objective context of interpretation is changed. It has nothing to do with the form in which the systems are presented.

We cannot infer the simplicity of a set of meanings from the simplicity of the formal laws of certain expressions. The fact is that these two orders of simplicity are independent. It is often necessary to complicate the entities that constitute a law in order to simplify it; the scientist knows this only too well. It is true that this complication annoys us, and that we tend to obliterate it from our dream about the world. We then assume that the complex meanings which obey simple laws refer to simple things, while we really have no clear ideas about them. Of such meanings we may list energy, matter, the notion of interval in relativity theory, and in general all the objects which enjoy the ideally simple property of invariance. But we need not be dupes of this metaphysic which imputes authoritatively to the world the simplicity of the laws of concepts. We know quite well that the applicable meanings of entities of this sort have in general an irreducibly complex nature.

As far as geometry is concerned is it not clear that experience alone determines, among the many groups of possible primitive notions, that one which in nature is the simplest? If rays of light travelled in circles, merely sighting an object would show that the relation of the objects is no longer in a straight

3

line, as it is with us, but along a circle. Circularity would then have a more simple empirical value than rectilinearity. If light rays formed Lobatchevskian lines, the geometry of Lobatchevski, although it might appear more complex in form than Euclid's, would be applied to more elementary natural objects. There is no geometrical system which is so complicated and so intrinsically cumbersome that it has no appropriate universe. In this universe it applies to things extrinsically in a more simple way than any of the other systems which are more simple in form, for we cannot impute to the order of nature these more elegant forms except by employing more complex conceptual devices. The appropriateness of one geometry rather than another to a given universe, or to certain regions, is indeed a matter of simplicity. But the latter does not refer to the formal simplicity of the system; it still belongs to the external simplicity of the interpretation which is given to it. Is there a form of geometry which is best for all portions of physics, and what is this form, if it exists? This is a positive empirical question. It cannot therefore be answered with certainty. But the probability of any particular solution depends on the state of physics and may vary with it.

We may be permitted to make the following criticism of Poincaré's discussion which is otherwise quite unassailable: although he constantly struggles against the idea of space as independent of things, he never quite frees himself from this very notion. He shows that space in itself has no determinate or intelligible character, and is therefore nothing at all; yet instead of rejecting this intermediary non-entity in the direct application of geometry to real things, he always ends by conceiving geometry as the science of space. That is why geometry does not seem true to him without convention; any spatial structure can be imputed to nature by compensatory changes in the statement of the laws of the world. This is an extremely indirect way of asserting that these laws, as soon as they contain values satisfying a given geometry, can conform to any other form of geometry, provided we translate properly the physical concepts involved in the first geometrical expression. But the fact that such and such meanings of the physical world satisfy the axioms of such and such a system does not depend upon any convention; on the contrary, it is the degree of the extrinsic simplicity of these meanings, and not the intrinsic simplicity of the system, which determines the appropriateness of that system in regard to its applicability.

A more complicated system of geometry may therefore be more appropriate than another to a given subject-matter. We cannot say that systems which are mathematically the most elegant admit the simplest interpretation of nature. To the pure geometer, the relation of the points of a straight line are simpler than the relation of the points of an ellipse, because its laws are simpler; but which of these two relations applies more simply to a case of geometrical structure in nature? We do not know *a priori*, and the intrinsic formal advantage of the first relation over the second does not seem to establish any probability in its favour.

CHAPTER IV

POINTS AND VOLUMES

It will be noticed that in all the systems of geometry which we have discussed so far, there is one expression that recurs in all. While primitive relations vary, the fundamental terms, namely, *points*, remain always the same. Might the point not be the indispensable element of geometry? Now, nature does not exactly present any simple objects having the properties assigned to points by the geometer. In order to obtain them it seems necessary to posit other terms not given in immediate apprehension and different in nature from the first. We ordinarily qualify the application of geometry in this way, because geometry, we say, is complete and rigorous when it posits points that are "simple and indivisible."

We already know that such a prejudice is not justified. In fact, even if it is true that geometry requires points as ultimate terms, it does not follow that the expression "point" should be taken in a simple and not in a complex sense. We have the proof of this in the arithmetical interpretation, where the point is a class of three numbers.* Geometry might accordingly dispense with the simple value of the point in nature, if some complex physical concept could be found to replace it.

But geometry itself leads us to the discovery of such a concept. For it is not true that it considers the point as a simple term. We can conceive systems which posit the point as complex, and which are composed of terms that are easier to interpret in nature.

* Three co-ordinates or distances from three perpendicular planes. —Tr.

Why is not the existence of these systems better known? It is because their discovery is recent; also because they do not interest the mathematician on account of their considerable intrinsic complexity. They do interest us, for they complete the demonstration of our thesis by showing the complexity and definability of the point, the notion that would be the last thing to appear as necessarily primitive in any rigorous exposition of geometry. Moreover, these more complex geometries are possibly closer to nature. It is well to devote a few moments to them here.

Instead of speaking of *points* and *relations among points*, these geometries speak of *volumes* and *relations among volumes*; so that where we otherwise call a volume a certain class of points, we here call a point a certain class of volumes. Now nature presents us undoubtedly with volumes rather than with points.

The idea of going from volume to surface, line, and point is not novel. There is something natural about it, and many geometries devote several preliminary lines to it. In fact, it is often said that a surface is the limit of a volume, that a line is the intersection of two surfaces, and that a point is the intersection of two lines. But that is not the same as defining them. For what sort of an entity is the limit of a volume? It is not a volume; it is then a new kind of entity whose existence is postulated. Likewise, what is the intersection of two surfaces and then of two lines if not some new entity that these words suggest, but do not analyze? These are not good geometrical definitions. A mathematical definition should be a synthesis of nothing but old terms and old relations to form the content of a new expression. In the geometry of points, a volume is a class of points having certain relations, and the relations of volumes are derived from the relations among points. Reciprocally, in a geometry which admits only the concept of volume, a point can only be a class of volumes having among themselves certain relations, and the relations of points can be nothing but the relations of their defining volumes. Does such a geometry then exist ? We actually have one as a result of the works of Dr. A. N. Whitehead.*

Dr. Whitehead adopts a point of view that is different from ours. He starts from an analysis of the terms and relations that nature presents, and looks for a combination of these entities which yields the properties of the geometrical point. But this combination itself turns into geometry, for the natural entities which it synthesizes already possess the properties of certain geometrical entities, namely, *volumes.* That is why Dr. Whitehead's construction is valuable as an investigation in pure geometry, the geometry of volume. It will be to our interest to consider it in this form.

Let us recall at first what volume is in the geometry of the point from which we start. Let us take a group of points A, and a point *p*. If any sphere with *p* as centre includes within its interior† at least one point of the group A, the point *p* is said to be a *limit-point* of A. If A is identical with its limiting points, the group A is called *perfect.*‡ Lastly, if between any two points of A and for any distance *d*, there exists a chain of points of A joining one of these two points to the other by a series of distances less than *d*, we shall say that the group A is *single-valued.*

A perfect single-valued group is a continuous domain: volume, surface or line. But the surface and line do not have interiority; each one of their points is a boundary point,

* B. Russell, *Our Knowledge of the External World*, ch. iv. Cf. A. N. Whitehead, *Principles of Natural Knowledge* and *The Concept of Nature*, where the method of "extensive abstraction" is applied to the four dimensional world of the theory of relativity.

† The interior of a sphere is in fact easy to define—it is the class of points nearer to the centre than the points on the sphere; for any particular volume, interiority can be defined this way. The difficulty arises in giving a *general* definition of volume.

‡ Any series which is both dense-in-itself and closed is called *perfect*. Cf. Huntington, *The Continuum*, pp. 51-52; Russell, *Principles of Mathematics*, p. 271.—Tr.

that is to say, a limit-point of the points outside the line or surface. On the other hand, the boundary of a volume everywhere limits points of the volume which are not boundary points of the volume. We shall then say that *a perfect, single-valued group of points* A *is a volume if any point of* A *which is a limit-point of points not in* A *is a limit-point of the points in* A *which are not themselves limit-points of points not in* A.*

That is what volume is in the geometry of points: a class of points connected by certain distance relations. We can now form relations among volumes by means of relations among their points. For example, we can agree to say that a volume *includes* another if it has all its points contained in the latter's. Two volume-couples AA , BB are *conjugate* if there exist single points in A and A , B and B, which are separated by the same distance;† it is the relation which corresponds to the experimental measurement of the distances between two bodies by placing one extremity of a ruler on the inside-surface of one of the two and the other extremity on the inside-surface of the other.

Now let us introduce the expressions just defined, wherever possible, into the theorems of the geometry of points. Certain propositions then will no longer refer to points nor to relations among points, for volumes and relations among volumes will take their stead. Propositions about volumes may be considered by themselves, if we write out the translated propositions and disregard propositions relating volumes to points. In other words, we take *volume* and the relations of *inclusion*‡

* If one wishes to avoid regarding two volumes joined by one point as forming a single volume, the condition must be added that any two points a, b belong to a perfect, single-valued group when all the other points of the group are interior points.

† We can also say that three volumes are *aligned* if the same straight line crosses them—such as one uses in sighting three stakes.

‡ It would have been more correct to employ instead of *inclusion*, which has a special meaning in logic, an entirely isolated term like *absorption*. It has seemed better to us however to use the current word *inclusion*, remembering that it here designates a variable relation, and consequently, a quite indeterminate one.

and of *conjugation* of volumes as primitive indeterminate meanings.

Let us call the whole group of these propositions (axioms and theorems) *properties of volumes*. Dr. Whitehead claims that these properties can be applied more directly to nature than traditional ones. Do the properties of volumes exhaust all geometry? If we know that they are formally contained in the properties of points, do we also know that they formally contain these? We can, in the system of points, form groups of points which obey the laws of volumes; that is how we have taken them. But can we, in the system of volumes, form groups of volumes which obey the laws of points?

We shall now examine Dr. Whitehead's solution.

For every point in the geometry of points we can substitute the limit of a unique aggregate of volumes. For every fundamental relation of congruence of point-couples, we can find a unique relation of corresponding volume-couples. This relation of the geometry of volumes translates every axiom referring to the congruence of points. In these volumes and their relation we have a "solution" of the geometry of points.

The group of volumes that serves to define a point is simply the group of all the volumes that contain the point. But this will not do as a definition, for we wish to mention nothing but volumes and relations among volumes. It will be necessary to make a long detour.

In order to obtain a class of volumes which have only one point in common, we think immediately of a series of volumes enclosed within one another and gradually reduced to a nucleus smaller than any given volume. That is to say, an *abstractive class* is defined by a class of volumes such that any one of two of its members is either included in the other or includes it, and no volume is included in all its members.

But these conditions are not sufficient to guarantee that all the volumes of an abstractive class would have only one point in common. They might have as their common nucleus not a volume, but a line or a surface. Thus an abstractive class formed from disks of a constant diameter and diminishing thickness converges to a circle; an abstractive class generated by a series of cylinders of constant height and decreasing diameter would reduce itself to a line segment (the altitude). We must therefore find the supplementary condition for an abstractive class to have only one point in common.

First we must introduce a limitation which does not occur in Mr. Russell's exposition,* but which is indispensable to the correctness of the following solution. Abstractive sets must be limited to those whose common points are *in the*

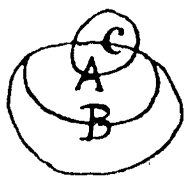

Fig 1

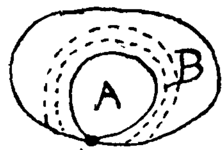

Fig 2

interior (and not at the surface) of all the volumes of the set. This preliminary condition is fulfilled if we postulate that of two volumes of the set, one must always be enclosed within the other *without touching*. But what is this new relation between two volumes ?

When the volume A, included within the volume B, touches the surface of B, not in a point or line, but in part of a surface, a volume C which is partly outside A can penetrate A without including any volume outside A and in B (Fig. 1). Such is the definition of the inclusion of A in B with their surfaces touching, in terms of relations among volumes.

We pass to the general case where they may touch in a line

* Neither is it to be found in the later works of Dr. Whitehead where he extends the method of extensive abstraction to the definition of a point-event in space-time, even though it also seems necessary there. [Cf. De Laguna's criticisms of Dr. Whitehead.—Tr.]

or in a point. The volume A, included within and touching B, forms the common nucleus of a class of volumes A′, A″, . . . included within one another, and all included in B with their surfaces touching B (Fig. 2). That gives us a general definition of inclusion with, and consequently, without the surfaces touching. Let us call the class of volumes included within one another without touching by the term *interior* abstractive set. We shall limit ourselves in what follows to interior abstractive sets.

Let us consider one of them (A) having part of a line or surface as its nucleus, and another (B) having part of the nucleus of the first as its nucleus, possibly a smaller part or a single point.

Every volume of A then includes a volume of B, or we can say that the class A *covers* the class B. Reciprocally, since the volumes of B surround their common nucleus quite closely, they leave outside themselves the greater part of the nucleus of A, and consequently, do not include any of the volumes of A. The class B then does not cover the class A. Any interior abstractive set whose common element contains more than one point must cover interior abstractive sets which do not cover it.

Suppose a class A has only one point p for its common element, and let B, a second class, be covered by A. Since the common element of B is contained in that of A, it is also reduced to the point p. Hence, every volume of B includes a volume of A which is nearer to the point p than it is, and the class B in its turn covers the class A.

Every interior abstractive set whose common element is formed by a unique point is then covered by *every interior abstractive set which it covers*. This is the condition required for an abstractive set of volumes to define a point. Let us give the name *point set* to those interior abstractive classes which fulfil this condition.

A point set of volumes determines a unique point which is its common element. But the converse does not hold true; for every point may form the nucleus of a point set of spheres, cubes, cylinders, and infinite others which all cover each other. They may be all designated as the aggregate of *volume members of point sets which cover a point set.* (The latter volumes are included by the fact that every class covers itself.)

This aggregate, which Dr. Whitehead calls *a point element,* constitutes the unique class of volumes that is associated with each point in the geometry of points. It is the class of volumes having this point in their interior, but defined solely by relations of inclusion of volumes.

Such is Dr. Whitehead's construction, or more exactly, what it becomes when taken not as an analysis of the real world but as pure geometry, to which realm it seems to belong.

We must now establish a correspondence between the fundamental relation of congruence of point-couples and a relation of couples of equivalent point elements. Thus relation consists in the conjugation of all couples of volumes that are respectively members of the two couples of point elements considered; we have here the *equivalence* of these two couples of elements.

In fact we have called " conjugation " of couples of volumes XY, X'Y' the existence in these volumes of two congruent couples of points xy, $x'y'$. It is clear that if the point elements A, B, A', B' have as common elements respectively, the points a, b, a', b' forming two congruent couples in this order, any pair of couples of volumes XY, X'Y' will be conjugate if they are members respectively of the elements A, B, A', B'. Conversely, if any pair of volume-couples XY, X'Y' members of the point elements A, B, A', B' contains two congruent point couples xy, $x'y'$, the points a b, a' b', nuclei of these elements, themselves form two congruent couples.

Every axiom of the congruence of points is then translated and expanded in the geometry of volumes, by means of a property of the equivalence of point elements. Let us now detach this geometry by regarding *volumes* only as a class of unknown terms, and by regarding *inclusion* and *conjugation* only as two relations, equally unknown, of these terms.

We can now compare this system of volumes with the initial system of points. Each of these systems is translated into the other if we make the following substitutions: in the system of points,

volume=a perfect single-valued set of points of which every boundary point is a limit-point of non-boundary points;
inclusion of volume A *by volume* B=the identity of every point of A with some point of B;
conjugation=the relation of two volume couples that contain respectively two congruent point couples;

and reciprocally in the system of volumes,
point=point element=an interior abstractive set covered by all the abstractive sets that it covers.
congruence=a relation of two point couples AA′, BB′ all of whose volumes form conjugate couples.

The geometry of volumes and the geometry of points are then *inseparable* (equivalent or translatable). Every meaning that satisfies the point and its relations supplies a more complex meaning that satisfies the volume and its relations, and vice versa. Thus geometry makes no demands on nature for volumes made of points rather than for points made of volumes. It gets along just as well with simple points as with simple volumes, in a world where there are nothing but points, as well as in a world where there are only volumes.

Imagine such a world as the latter, in which only volumes and their fundamental relations have simple meanings. The geometry of volumes alone fits it. But the geometry of points would appear to the scientists of that world as a refined form of the geometry of volumes; it would be considered as an extremely elegant invention. By introducing the complex concept of point element, it would simplify admirably the statement of the laws of nature. And if we assume that this world had any metaphysicians, they would not fail to imagine that the point element was the expression of a simple object, real or ideal; they would be concluding gratuitously from the intrinsic simplicity of laws to the extrinsic simplicity of the terms involved in the laws.

But let us notice how simplicity is introduced into laws by the example of the laws of volumes stated in terms of the abstractive set and the point element. Consider the relation between two couples of conjugate volumes. This relation does not enjoy the simple property of transitivity, but a more complicated one. If two couples of volumes *aa'*, *bb'* are conjugate to the same volume couple *cc'*, one of them—say *aa'*—will be conjugated, not necessarily with the couple *bb'* itself, but with a certain volume couple *xx'* intersecting *b* and *b'* respectively and smaller than *c* and *c'* respectively. (The intersection of two volumes being the inclusion of some volume, the volume *m* is smaller than the volume *n* if *n* includes a couple of volumes which is not conjugate with any couple of volumes included in *m*.) This is what is ordinarily and improperly expressed by saying that two volume couples conjugate to the same couple are conjugate to each other by a margin of error equal to the size of the volumes of the intermediary couple; the expression is improper, first, because the law does not say that these two couples are conjugate, and, secondly, " error " is not involved.

But if we take, instead of the relation of conjugation of volume couples, the relation of equivalence of couples of point abstractive sets, formed by the conjugation of all couples of their members, it is easy to show that the complex property of the simple relation of conjugation confers on the complex relation of equivalence the elementary property of transitivity. Is there a better example of the simplification of laws by the complication of concepts? In this respect, the point plays the same rôle in the geometry of volumes as that of energy in mechanics, of entropy in thermodynamics, and of any other complex entities whose laws elegantly formulate and simplify science.

There is one idea here which we cannot broach except by way of anticipation of certain fundamental problems. But since it arises now as a troublesome question, let us examine it for a moment.

Are volumes closer to nature than points? Nature seems to lend weak support to the affirmative, for its "volumes" are only perceivable when they are large enough. To claim that they nevertheless satisfy geometry, is then to predicate objects of a realm beyond the given. Do we, therefore, gain anything by forsaking points for volumes? Does not the application of any form of geometry to nature involve the same uncertain postulate?

No, for the geometry of volumes, as different from that of points, contains at least one feature that is illustrated in nature, and is dependent, moreover, on a really preferable assumption.

Indeed if we admit that there is a natural interpretation of volumes and their relations starting from a certain size of volume, all the propositions outside of those involving the existence of volumes of a smaller size are going to be translated by laws of nature. Thus the theorem cited before, relative to the "approximate" transitivity of the relation of conjugation of volume couples, would be a rule exactly verified in experience: for it is important to notice that the truth of an approximate property is something quite different from the approximate truth of an exact property. The second is only a confused perception of the first, which alone is something rigorous and satisfying. Of similar nature is the relation in experience of the geometry of points to that part of the geometry of volumes which does not posit the existence of volumes as small as we wish. That part of the geometry of volumes which are as large as we can see might be called its positive part.

But the very hypothesis of the presence of such volumes as are required in the point element, is quite different in character from the hypothesis of simple and indivisible points. In fact, instead of postulating entities which nature does not exemplify, we confine ourselves to positing new members of a known class, not different from the known individuals except as the latter differ among themselves. This is an intelligible and modest hypothesis. It emerges naturally from the positive part of the geometry of volumes, for we have seen that it brings into the laws of this geometry a simplicity which the mind cannot fail to appreciate.

But again, by imputing to nature a series of boxed volumes decreasing to infinity, are we not positing simple terms towards which these series converge? One of the merits of the study of geometry in its pure form is to invalidate such a conclusion. It is generally invalid to assume that simple terms define simple objects. Not knowing the concrete sense of volumes or of the inclusion of one volume in another, how can we say that the class of volumes that constitute a point element necessarily determine a new term, emerging suddenly out of the sky? Can the essence of such a term be other than that of its defining volumes? There is no logical necessity for it. If in nature the series of volumes of this type cannot exist without simple convergent terms, it can only be by virtue of a contingent property of natural volumes, a somewhat doubtful property whose absence would not affect the application of geometry to the world.

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We have tried to understand geometry in its abstract aspect. We have adopted the wholly formal viewpoint of a man who neither knows the meaning of the words proper to this science, nor has the slightest conception of what objects and what relations it claims to discuss. The reading of a geometrical treatise is nothing more, therefore, than an exercise of logic which consists in verifying the validity of the derivation of the theorems from the postulates. Is it, therefore, possible to follow a proof without knowing any material content? Undoubtedly, for all proof is absolutely independent of the significance of the terms involved. In order to know that every Frenchman is mortal, if every man is mortal and if every Frenchman is a man, it is not necessary to know the meanings of Frenchman, man, and mortal. Now, it is similarly unnecessary to know the meanings of point, straight line, and congruence in order to comprehend the force of a correct demonstration and even to appreciate its elegance. There is not a geometer today who will disagree.

But when we have turned over the last page of this treatise on an unknown science, called geometry, which by chance has come into our hands, when we have gone through the chain of its proofs, when we have sufficiently admired the necessity and subtlety of the ratiocination which develops to infinity the theme stated in the axioms, and weaves a few expressions, devoid of meaning to us, into a thousand patterns, our curiosity turns in a new direction. We wonder if there really are entities to which these "geometrical" expressions apply, or rather *might* apply correctly. The group of axioms with which our treatise opens then becomes the subject of a totally new problem concerning the geometrical expressions. *Are there any meanings, which attributed to the geometrical expressions appearing in the axioms will make the axioms true?* Any system of meanings answering this problem is an illustration, an instance, an interpretation, or better still, a *solution* of the group of axioms under consideration.

About these meanings the axioms tell us nothing by themselves. They leave us ignorant about their nature, about their degree of simplicity or complexity. Moreover, they may admit *diverse* independent solutions belonging to the most disparate realms. It is best then to be careful against postulating the existence of a purely geometric domain in some determinate region of the real or ideal.

But geometry can be founded just as well on the most dissimilar groups of axioms irrespective of the fact whether any concept common to all their primitive propositions can be found. We have defined the purely formal equivalence according to which all of these systems are diverse aspects of one single and whole geometry. And we have made precise the meanings in which their solutions, although different, are logically inseparable. Let us agree to call any system of meanings that renders the axioms of a geometry true by the term a *space*.

We ask whether there are any "spaces." There are, for we have discovered some of them in the domain of numbers. Hence, we know that geometry contains no contradiction because it lacks no concrete cases. These numerical solutions, being the only *a priori* ones, are even the only theoretically certain ones we have. However, they do not interest us, for it is not in the domain of pure ideas that we want to see the order of geometry reflected, but in nature as we get it through the senses. With this order in mind, let us then turn towards this material, in order to inquire whether it also does not offer one or several illustrations.

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PART TWO

TERMS AND RELATIONS

INTRODUCTION

WE must now take cognizance of the elementary terms or relations that nature presents to us in perception. These terms and these relations indeed form the very texture of its structure; the patterns which they offer will be the features of the object of geometry. But to speak of nature as a network of terms and relations seems to arouse all our philosophical scepticism. Are we not accepting a considerable postulate? What gives us the right to apply to the flux of sensation the categories of logic? There are many who would declare that reason is inadequate to the task.

So fundamental a suspicion cannot be met superficially. But if we advance to the details of analysis, we shall gradually define the issue more exactly, and perhaps remove the objection.

The elementary terms of nature are the entities called *sense data*. They are the *that* of anything immediately present to our senses; for examples, *that is a tree, that is a penny,** or again, *that is a falling star, that is the song of the nightingale.* For logic, which is independent of time, knows neither flux nor rest, and a sense datum does not need to be abstracted from change in order to become a "term." But how can the immediate flux of sensations in which everything is fused be resolved into distinct and simple terms? That is what we shall soon try to see. Let us proceed then to examine what elementary relations our mind can apprehend among those terms whose exact nature will become clearer as we advance.

* Cf. G. E. Moore, *Aristotelian Soc., suppl. vol. ii.* (1919), p. 180.

CHAPTER I

SPATIO-TEMPORAL RELATIONS INDEPENDENT OF THE DISTINCTION BETWEEN EXTENSION AND DURATION; THE NOTION OF A SENSE-DATUM

The relation of spatio-temporal interiority.—I follow with my eye the flight of an eagle crossing my field of vision with a slow and continuous swoop. The whole event forms a single visual term. In the middle of his flight, the eagle flapped his wings once. Between these two events as I saw them there stands out a very clear and very simple relation which I express by saying that the first of these two sensed terms is *interior* to the second.—Shutting my eyes, I slide a pencil over the fingers of my open left hand. Between the event of the passage of the pencil over my index finger and the larger event of the passage of the pencil over my whole hand, I apprehend again a very clear relation, which appears to be the same as the previous one, and which makes me say again that this term is *interior* to that term.—It is constantly the same relation that also appears to me between the sound of a word and the sound of the sentence in which it occurs, between the coloured spot of a painted figure and the greater spread of the whole of the picture in which it is contained. This relation of interiority is clear and manifests itself distinctly and prominently in every case.

Its relation to spatial inclusion and to temporal inclusion.—Without containing any distinction yet between *temporal* and *spatial relations*, the fundamental relation of *interiority* involves both the durations and the extensions of the sense data that it connects. For every sense datum that we perceive *interior* to another is evidently also included in that other with regard to both *extension and duration respectively.* But we must define these more special relations. I call *temporal inclusion* the relation of two of my sense data *a* and *b* when the duration of *a* is enclosed within the duration of *b*. So, during an August night, I can see one falling star emerge and die during the longer life of another one in a different part of the sky. The first flash of light is then included temporally (but not spatially) in the second. Moreover, I call *spatial inclusion* the relation of two of my sense data when the immediate extension of *a* is enclosed within the immediate extension of *b*. So, for the spectator of a fire, the little gray spot formed by the ashes left by the fire is included spatially (but not temporally) in the big flare made by the flame just before.

I have just defined temporal inclusion and spatial inclusion in terms of *durations and extensions*. We shall return to these definitions later, but at present, it is to be understood that when I speak of the *durations* and *extensions* of sense data, of the *field of sensations*, and more particularly, of the *field of such and such a sense*, and of the different *regions of that field*, I really mean to be discussing no terms other than my sense data, and no relations other than the relations I apprehend among these data.

Interiority entails both *spatial inclusion* and *temporal inclusion*: any sensed term interior to another has its extension and duration respectively enclosed in the extension and duration of the other term.

Might the converse not be true? Is it not sufficient for one term to be interior to another, that it should occur both within the duration and extension of the other? Then temporal inclusion and spatial inclusion would constitute the necessary and sufficient condition for interiority. Hence, the relation of interiority of sense data might not be the simple relation that we thought it to be. If it is equivalent in fact to the conjunction of spatial inclusion and temporal inclusion, interiority cannot be anything else than this conjunction.

Surely this analysis can be rejected, for two concepts can be inseparable in their instances and yet be different; for example, "equilateral triangle" and "equiangular triangle" are inseparable, but equilaterality is not the same as equiangularity. It is always direct inspection or intuition which in the last resort judges the reduction of one concept to another. Now it seems to me that I find in the relation of the flapping of wings to the flight, of the figure to the painting, of the word to the sentence, a simple relation which does not contain the distinction and the conjunction of one relation with respect to extension and another relation with respect to duration.

But is it even exact to say that interiority is inseparable from this conjunction ? Not at all, for when it comes to moving sense-data, temporal inclusion and spatial inclusion together do not involve interioritv.

Fixed and moving data.—The distinction between *fixed* and *moving* sense data is a qualitative contrast directly perceived. It cannot therefore be defined. The distinction can only be indicated by means of expressions which contain it. A datum is fixed if it retains a constant extension or the same position in the sense field during its whole duration. On the other hand, a datum is moving if its extension varies in the course of its duration, either by deformation or by displacement.

Reason obviously cannot transform a moving sensuous content into an analyzable entity. For logic knows nothing of change and persistence, motion and rest, just as it disregards colours and prefers no one of them. A spot may be said to be moving or fixed as it is said to be green or blue; with respect to reason sensible movement may be as simple a quality as rest.

Now interiority applies to fixed and moving data, no matter what else distinguishes them. No matter whether the perceived event is full of movement or rest, whether it is a battle or a landscape, interiority still relates a detail to the whole. The flapping of the wing is interior to the bird's flight, as the figure is interior to the picture, as the word is interior to the sentence, each one of the nouns here designating not the thing but the sense datum present to me on a certain unique occasion. Nor does temporal inclusion account for this distinction between moving and fixed terms: all terms endure in the same way; they either exclude or include each other according to their duration. Spatial inclusion, on the other hand, raises a difficulty in the case of moving terms; how can the extension of a moving datum be included in or include that of another when a moving datum is exactly one whose extension varies? Now either the question admits of no answer, from which we conclude that interiority cannot be reduced to the conjunction of two relations one of which has no meaning; or else, there is an answer which when we come to analyze it leads to the same conclusion that interiority is a clear and manifest relation while the other two (spatial and temporal inclusion) are not.

What is the fundamental meaning of spatial inclusion? It is one sense datum saying to another in my mind: "You have been nowheres where I have not been in my time; all your territory has been mine; you have never escaped from the shadow which you know as my extension." But from this point of view, the extension of a moving term is *the whole region of the sensible field "swept" by this term in the course of its existence.* For such is the domain that it has marked with its quality, the career that it has pursued, and over which its sovereignty reigns. Thus, the extension of a falling star is the whole line traced by it in my visual field.

Spatial inclusion would then apply to moving terms as well as to fixed terms. But *its conjunction with temporal inclusion no longer determines interiority.* Take, for example, a moving cloud which I see crossing the sky outlined by my window. Its extension is the whole band that it has swept while passing. But it has not covered its field in the same manner as a fixed datum. It is possible that a second datum, more restricted and briefer—the shining of a star—may occur in the cloud's extension and duration, and yet be exterior to it.

Facts, like intuition, seem to lead me to say that the interiority of sensed terms is a simple relation which entails spatial and temporal inclusion but is not implied by them, for it is antecedent to the separation of relations into extension and duration. The relation of interiority is so important for the concept of sensory terms that it is best to stop to consider its nature a little longer.

Interiority and logical wholes.—Does not the relation have a really rational validity? Does it not involve the logical relation of *component* and *composite*? The important consequence of this question is this: if the interiority of one sensory term in another implies the logical relation of part and whole, the complexity or simplicity of a sensory term will depend upon how many interior terms it contains. A metaphysical privilege ensues with regard to the smallest terms. In the eyes of reason, the reality of experience is precipitated, so to speak, into a multitude of distinct points and instants, for composites are only real through their elements. This view agrees, moreover, with the fundamental scientific maxim according to which physical reality is entirely given by the state of its elements at each point and at each instant. However, we must enter more fully into the question.

Criterion of the logical simplicity of a term.—We must first investigate more precisely what is meant logically by a simple term. Everybody will agree that when a term or content *x* is *part* of the term or content X, *it is impossible to conceive something about X without conceiving thereby something of x*. On the contrary, if I can predicate something of X without thereby predicating anything of any other term, it is clear that the term X is a simple subject.

This is not a mere matter of words. For the mind at times recognizes simple terms that have no names. Indeed every simple content can at first only be designated as *this*. Conversely, a single word often refers to a mass of details that do not form a whole. The verbal statement of judgments does not indicate with certainty to what simple subject they apply, for many words can cover either a simple content or a complex one, according to the fixating or shifting attitude of the mind that employs them. Thus, it is conceivable that a common part of our lives to which we refer by the same name, such as a walk taken together, may have been for one of us a succession of events, for the other a unique event. Perhaps even the one who remembers only a succession of events may discover, in a retrospective moment of particular freshness of perception, an æsthetic aspect of integral unity. And it may happen that the one who had at first apprehended the series of events as a totality loses its quality under the influence of some fatigue of his imagination. What was at first only a unified *this* becomes at certain times a discrete *this* and *that*, and words do not help us distinguish the simple from the complex contents envisaged by the mind.

The logically simple or complex character of objects is not, moreover, determined by the psychological conditions of their presence. Let us in fact conceive an object X whose quality is inseparable from qualities of objects x_1, x_2, x_3, . . . connected by the relations R_{12}, R_{13}, R_{23}, . . . The desire for economy imbedded in reason inclines us to say that the first object is nothing but the others "taken together," and that the quality X is resolved into the complex property derived from the qualities x_1, x_2, x_3, . . . and their mutual relations R_{12}, R_{13}, R_{23}, . . . However the principle of economy under the pretext of preventing us from seeing double should not blind us. It is direct inspection alone which decides in the last resort whether the quality X is *in itself or in its very meaning* a distribution of qualities and of relations belonging to a group of subjects, or if, on the contrary, being simple it is a simple subject.

Application of this criterion to macroscopic data.—Is this logical relation of component to composite necessarily involved between a sense-datum and those which are interior to it in extension and duration ? Take as examples the area covered by a checker-board and the spot covered by one of its boxes; the sound of chimes and one of the notes it contains. Evidently, there are some judgments about the checker-board which are not judgments about its boxes, judgments about the chimes that are not judgments about any one of its notes. For if I judge the appearance of the checker-board to be square, I attribute a certain property to the total spot that I see, and not to the smaller spots which are interior to it, although it happens that each one of them also possesses this property. Indeed if my judgment applies to each box as it does to the whole board, we may conclude that it does not distinguish between the boxes, considering them only as squares.

If we go from the spatial aspect of interiority to its temporal aspect, the same holds true. The chimes like the checker-board can be the simple subject of a judgment, and what better illustration is there than this, for we can find no quality which is common to the total sound of the chimes and the sound of one of its notes. Nevertheless, we admit the individuality of the total sound, for its melody may appear as a simple quality floating on the partial tones without attaching itself to any one. Now the possession of an irreducible quality has been posited as the criterion of logical individuality.

The relation of interiority does not imply the logical relation of component to composite, and the most extended and prolonged sense data of the richest internal diversity may be simple terms in the light of reason.

But how can we account for the mistake of inferring simplicity from interiority? Why does interiority which is a purely empirical and contingent relation between two terms seem to reduce one to a component of the other macroscopic terms?

Factual determination of macroscopic data by microscopic data.—Sense-data which contain other data are not constitutive of what we perceive in nature. They are distinct essences possessing simple natures of their own. But what is it that *determines* these characters? Does not the presence of the square appearance of the checker-board (held up before the face), as well as the melody of the chimes, admit any proximate cause? Yes, surely. The presence of these qualities in the total spot and in the total sound is determined and guaranteed by the presence of certain qualities and relations in the smaller spots and in the briefer tones that are interior to them. Thus, the checker-board appears square *because* this object contains eight square boxes in eight rows and eight columns. But it is not *identical* with them. It is sufficient for a coloured spot to fulfil these conditions in order to appear square. In the same way, the melody that I have recognized in the total sound of the chimes is present in every sound which contains in it other sounds having certain definite qualities and relations symbolized in the music sheet. We may generalize these two examples and lay down the principle that *all the properties of sense data which contain others are determined by the properties of the latter.*

Give me the arrangement of the note and their relations and you give me the melody; give me the colour of each visible point of the canvas and you give me the picture. It is then obvious that sense-data relatively large in extension and duration, such as the sound of a melody or the variegated scene of a picture come to be regarded as simple aggregations of more microscopic terms. It is imagined, as a result of giving the smaller terms as determinations of the macroscopic appearances, that the latter have no independent reality. But that is a confusion. The properties of microscopic terms *ideally determine* but do not *actually constitute* the others. Knowing the microscopic parts, I cannot infer the macroscopic whole unless I am already familiar with the whole. Let us imagine a mind incapable of apprehending as a whole unity certain aspects larger than those parts of which the whole is made. He may perceive each of the elements without seeing the whole, as one may see the brushes of paint on the canvas without seeing the picture, or each note of a song without hearing the song itself. Let us endow him with a more synthetic outlook, enabling him to survey larger regions of the same flux of sensations, and he will discover new and more comprehensive entities. These will be no less simple than the qualities and relations of the more restricted terms by which they are determined. Their simplicity does not reside in their elements but in themselves.

Moreover, is it reason that asserts the principle that the properties of a sense datum are constituted by the properties of the terms interior to it? No, it is only an empirical principle. Of course, we should be more astonished to find it false than to hear a tree speak. But no matter how strongly habit inclines us to accept it, reason remains indifferent. It cannot grant to the factual connection of macroscopic existences any essential microscopic connections. With respect to abstract reason, the conglomerations of macroscopic and microscopic data are all real and simple essences on the same plane.

The philosophical defence of macroscopic data.—We have thus removed the conflict between the philosopher, who defends the integral realities of life and art, and the scientific logician or physicist. In fact, data which are relatively comprehensive in extension and duration lend rhythm to experience and, so to speak, organize it. Common sense rebels at the disintegration resulting from the reduction of natural and vital terms into a mass of sensible minima. But it does not see where the error lies, and becomes dupe to it itself; for how can it refrain from believing that the perceived house is the same as the stones perceived as parts of the house ? The philosopher is friendly to the first protests of common sense and tries to satisfy them. The important sense terms, he says, although broad and full of diversity, are nevertheless individuals. If the ordinary man takes a checker-board as a single visual term, does not the painter also see a whole landscape as one ? Does not the musician enjoy a whole piece of music as a single audible object ? After our costly first experiences, do not certain days and even years have a peculiar quality which is unveiled in private moments ? Can I not even conceive my whole sensible past as constituting one and the same event, and my whole experience as forming one single object which is continually growing ?

There is an opposition between the technician's analytic attention directed towards sensible details that are difficult to get because of their microscopic nature, and the artist's synthetic attention directed, on the contrary, to larger vistas and richer terms that are difficult to embrace just because of the sweep and wealth of their extension and duration. At one extreme we have the discernment of point-instants, and at the other the perception of all experience as a single term.

But it is here that the confusion pointed out above makes itself felt. Only if we assume that a sensible term cannot be interior to another in extension and duration without forming one of its essential parts, and consequently, a more fundamental reality, do we have to choose between restricted and comprehensive data. We are artificially forced to find the reality of experience either in one or the other, either in the terms of the technician or in those of the artist, but no longer in both at the same time.

Once involved in this dilemma, one will choose, as Leibniz did, in favour of analysis, and the reality of sensible terms will be scattered into dust; or, as Bergson does, in favour of synthesis, and reality will belong, then, only to the totality of immediate experience. Or else, as Bergson does at other times, feeling how ill at ease the mind is while it is drawn to both extremes, one will accuse reason itself of imposing upon us a deceitful choice.

But it is only we who are causing the trouble by making reason intervene in a situation to which it is indifferent. This mystery of the sensible whole which is not the sum of its parts vanishes as soon as reason points out that these are not true parts, and that interiority in extension has no rational relation to duration.

Other relations of the same group.—Other relations of the same group as interiority are *penetration*, *exteriority* and its limiting case *continuity*. A row and a column of a checker-board are two sense terms which *interpenetrate*. They have a common interior term, but my perception of their inter-penetration seems something more simple and primitive than the discernment of the box which forms the nucleus of their intersection. Likewise the sound of the first three verses and the sound of the last three verses of a four versed stanza of poetry are two sense terms that *interpenetrate*.—I say, on the contrary, that two verses of one stanza, two rows in a checker-board, two actions separated by a rest, are two *exterior* terms in so far as they do not interpenetrate.—Finally, two exterior terms are said to be *continuous* if they touch each other without interpenetrating, like the motions of two relay messengers, one of whom starts as soon as the other arrives.

We can apply to all these relations the considerations which have led us to take interiority as a simple irreducible relation independent of extension and duration, and also as a factual relation without intrinsic rational value. In the same way as a sense datum can be a logically *simple* term although it comprehends many others, so a sense datum can also be logically *distinct* although it actually interpenetrates many others. Reason no more circumscribes sense-data in distinguishing them than it destroys their unity in doing so; it accepts their lack of sharp limits in extension and duration.

The indefiniteness of sense-data.—In what, as a matter of fact, does the indefiniteness of a sensible object like a cloud consist? It consists in the fact that there are other clouds whose exteriority to this one is doubtful. But the sensuous individuality of the first cloud remains above this doubt. For how else could we ask definitely whether any two terms which we apprehend have or have not such and such a particular relation? From the fact that we do not know or cannot say that two coloured spots belong to the same cloud, we cannot conclude that these two spots are not two distinct individual data. By what right then do we question the analogous individuality of anything because we are not sure of its relations of exteriority in extension and duration? What is really worthy of notice is that there does exist a limit of uncertainty. There is not a single sense term that is so variable that we cannot discern with respect to some other terms whether they are exterior or not to it. No matter how indefinite the object may be which I discern by remarking: "What a beautiful cloud!", I still know that this sea-shell is exterior to it. However dispersed the sunset may be, there certainly comes a moment when night has fallen. From the standpoint of the relations of interiority and exteriority, there are, then, for every sensible term other terms which can be put into definite relations with it. Now, while the existence of doubtful relations does not exclude the individual and distinct presence of the term considered, the existence of definite relations would seem to prove the presence of such.

Absolute reality of terms and of sensible relations.—There is no falsification involved in the procedure by which I discern in the bosom of the sense-flux terms related by the relations of interiority, exteriority, and penetration. Nothing compels me to attribute to these relations a rational value which they do not possess, and thus condemn myself to contradictions. When I say *this*, when I distinguish such and such a datum in order to predicate something of it, I do not cut it off from the continuous stuff of my experience, I do not stop it from changing, I do not elevate it above the flux of which it remains a passing wave. By considering it as a logical term, I do not assume that it has a more real unity than the microscopic data that it comprises, nor than the macroscopic data that comprise it, nor than those which interpenetrate and divide experience in other ways. There is in my immediate experience a multitude of realities, a surprising wealth of entities that interpenetrate without losing their original simple quality. But it does not require a kind of miraculous vision and abandonment of reason to see this. Of course, this interpenetration of simple realities would be a miracle if we had to take it in a purely logical sense; but we have seen that interpenetration is exactly not merely logical but empirical. Any contrary view rests on an inadequate idea of logical abstraction.

The contrary hypothesis may however be retained.—Suppose we admit that there is falsification in the discernment of distinct and related sense data. Our whole analysis would seem to vanish in smoke; but the fire would only be a painted one. For our opponent hurries to return what he had apparently taken away from us. There can be no doubt, he says, that there are *in a certain sense* distinct and related sense terms in perceived processes since the physicist, the astronomer, and the chemist observe them. And if there is any error or falsification in scientific analysis, it is not error in the ordinary sense, but in the metaphysical sense of spiritual deficiency rather than falsity. The adjectives applied to science are artificial, superficial, symbolic instead of false. These condemnatory words waver between error and sin in significance, between the truth and the value of ideas* and are quite current in what passes for metaphysics. They enable us to destroy everything or to save everything with the proper play on words, and we can do the same in this case. It will be sufficient to answer: "Since, according to what you say, it is not true that there are distinct and related sense-data, for it is not true that lightning precedes thunder but true *only in a certain sense*, understand whatever we say *in that sense whatever that may be*, for we mean to use words only in that sense."

But is such a special detachment of meanings among scientists founded on any philosophical basis? Yes, for the independence of two problems is a fact, not only relatively to our ignorance, but in itself. Of course, the philosopher refrains with difficulty from saying with Descartes: "All my beliefs hang together; they must be accepted or rejected all together." For are not our most precious beliefs always the least probable ones? But this excessive coherency is most often illusory. The idea is spreading today that philosophy will progress only by becoming more piecemeal like the

* Cf. H. Bergson, *Données immédiates*: "A definition of that sort contains *a vicious circle, or at least a very superficial idea of durée*" (p. 76).

sciences. The philosopher must learn to treat certain ideas abstractly without regard for his own doctrines as far as possible whenever he deals with a particular question. His solution should be presented in terms, and if possible in an attitude, as independent as possible, of all his other beliefs.

CHAPTER II

TEMPORAL RELATIONS AND THE HYPOTHESIS OF DURATIONS (DURÉES)*

LEAVING behind me the first group of relations, interiority, penetration, exteriority, and continuity, I pass to a second family, namely, the *temporal relations* of my sense terms.

We have already encountered *temporal inclusion*, which is the relation *during*. Let us add to it the following: *overlapping*, a relation between two data when one begins before the other ends; *separation* or non-interference; *prolongation*, a relation between two data when one begins just when the other ends.—This is not all; let *a* and *b* be two data related by separation, interference, or prolongation. I discern between them an asymmetric order that I state by saying that *b* (for example) is *after a* or *follows* it; that *a*, on the contrary, is *before b* or *precedes* it, either *completely* if *a* and *b* are separate (and *immediately* if *a* and *b* are in prolongation), or *partially* if *a* and *b* interfere. Let us name these ordering relations *complete succession*, *immediate succession*, and *partial succession*.—Let us finally point out the relation of *simultaneity* between two terms which begin and end together. This relation is formally and theoretically of very great importance.

These different relations clearly belong to the same family. In order to get some idea of their nature, it will not be amiss to ask in what their common relationship consists.

The hypothesis of durations (durées).—Common sense, or rather, the spirit inherent in language, offers this remarkable answer: all the immediate relations that I call temporal are related in that they directly connect, not two of my sense data,

* Nicod uses *durée* in Bergson's sense of absolute immediacy.—Tr.

but two terms of another sort, namely, two *durations* (*durées*). (These durations are themselves often decomposed into *instants*. But let us ignore for the time being this second analysis which posits a second type of terms even more removed from sense objects than durations.) These new terms, the durations, are in their turn related to sense objects by a relation *sui generis* which we shall call the *occupancy* of a durational whole by an object.* Each one of my sense-objects also has "its" durational context, and the temporal relations which I note among them are properly speaking relations among the durational wholes (*durées*) they *occupy*. Simultaneity assumes a particularly important rôle. It becomes in fact *the identity* (not *the equality*) of the *durées* of two data.

This conception may be called the doctrine of absolute sensible time. Between two of my sense data manifesting any one of the temporal relations described above, there have been introduced two new terms, two durational wholes, which by means of the secondary relation of occupancy join the temporal relation to the two sense data. Each one of the temporal relations between two *sense-data* a and b ceases then to be a *simple* relation going directly from one to the other, in order to become a *complex* of three distinct relations at first uniting the sense datum a to a duration α, then this duration α to another duration β, and finally this duration to the sense datum b. If we use R to symbolize a certain relation between two sense-data a and b, ρ the corresponding relation of their durations (*durées*) α, β, and O the relation of occupancy of a sense-datum to " its " *durée*, the relation R is analyzed† into $O \mid \rho \mid \breve{O}$.

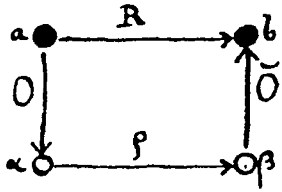
This view has two advantages and one disadvantage.

In the first place, it affords a striking explanation of the

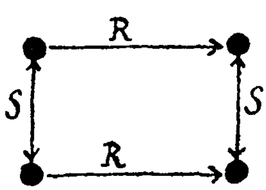
* Cf. Whitehead's *ingression*.—Tr.

† These logical symbols, as others we shall use, are borrowed from the *Principia Mathematica* of Whitehead and Russell.

community of the various temporal relations of my sense-data in so far as these relations admit durational wholes (*durées*) as immediate terms, and connect sense-data only by the *durées* they occupy.



Besides, the hypothesis of durational wholes accounts for the particularly remarkable formal law according to which two sensible objects that are *simultaneous* (that is to say, entirely contemporaneous) have themselves relations to all other objects: simultaneity, interference, precedence, and succession. That is because simultaneity *transmits* faithfully all temporal relations. With respect to each one of them, two simultaneous terms are interchangeable; a group of simultaneous terms constitutes an absolute community. No temporal relation can differentiate one from another among them. All the relations are attached not to any one term rather than another, but to all alike. If R represents any one of the temporal relations of my sense data, and S simultaneity, the "transmissibility" of R by S may be written $R = S \mid R \mid S$ or schematically



In the diagram we can imagine the relation R being transmitted from one couple of sense terms to the other by sliding in the direction S.

This remarkable property of simultaneity is made to appear as an analytic one in the hypothesis of durées. In fact, simultaneity is there taken to mean occupancy of the same durational whole. Moreover, every temporal relation of sense-data is a relation of durées occupied by them. Then, the property in question reduces itself down to this: "The relations that my sense-objects draw from the relations of the *durées* they occupy are the same for all those objects that occupy the same durée." The hypothesis of durées then takes into account an important formal law of temporal relations; but, however, only of that feature alone. For example, it does not explain the transitivity of succession or the symmetry of interference.

The disadvantage of the hypothesis of durées consists in its being lavish with entities. When I say that a datum *b*, for example, a roll of thunder, comes *after* a datum *a*, a flash of lightning, this hypothesis demands that I have no less than *four* types of entities: the two sense data *a* and *b*; the two corresponding *durées*; the relation of occupancy of each datum to its *durée*; and finally the temporal relation *after* between the two durées, *a* and *b*. Is it not unnecessarily complicating matters to replace the idea of "thunder following lightning" by the idea of "a durational whole occupied by thunder following a durational whole occupied by this flash of lightning." But even if I were certain that all these elements could be apprehended—the flash of lightning and the thunder, the *durées* of both, the relation of each to its *durée* and the relation of the *durées* to each other—the principle of economy as a maxim of method would prevent me from accepting this analysis of the temporal relations of my sensations. For it is possible to establish all the necessary constructions in the "geometries of sensations," as I do in the last part of this work, without taking the complications of the "*durée*" hypothesis into account. It is best to avoid superfluous assumptions, even of self-evident truths. But the hypothesis of *durées* is even lacking in self-evidence. While I am certain of really apprehending lightning, thunder and the perceived relation of time which orders them before and after, I remain in doubt about those intermediary terms, *the durées*, and the relation of occupancy by which they would be interpolated among the two primitive terms. Are these interpolated terms really simple natures or merely the shadows of words? The principle of economy, taken now as a maxim of probability, invites me to abstain, if I can, from so dispensable a hypothesis.—Perhaps, indeed, there are no durational wholes (*durées*) of my sense-objects, nor any relation, consequently, of these terms to their *durées*. Perhaps there is nothing more than these terms themselves and their direct temporal relations, simple and unanalyzable connections.

But if I assume that *durées* do not exist, what can I mean by the "*durée*" of one of my sense data? This question must be answered next.

The durée conceived as a class of sense-data.—A durational whole is not a *simple* entity. It is nothing more than the *class* of data simultaneous with the considered data, including this datum itself. (The result of this conception of *durées*, is, analytically, that a *durée* is always the *durée* of some datum, whereas this can be established only as a synthetic principle on the hypothesis that *durées* are independent entities.) A *durée* means "this, and everything that is simultaneous with this."

It cannot be objected that this is a vicious circle on the ground that *simultaneity* means nothing but *occupying the same durée*, for it is precisely this analysis that is denied. The difference between taking *durées* as entities and as classes consists exactly in the status of the temporal relations of the sense data which in the first case are complex and in the second case simple connections. To declare that the simultaneity of two data consists of the complex: *occupancy of the same durée,* is to posit the fundamental assumption of the theory of *durée-entities*; it postulates the very existence of what is in question. Far from the class-theory of *durées* being circular, the objection itself conceals a begging of the question.

Our treatment of *durées* as classes is only the first example of a logical method* to which we shall remain constantly faithful. We shall apply it again to the *extensions* and *qualities* of sense-objects. Since it will govern our treatment of the three pervasive domains of time, space and sense qualities, it will be worth while to examine briefly the nature and range of the method.

To reduce a duration, extension, or sense quality to the class of data which " have " or " fill " this duration, extension, or quality, is a kind of nominalism. But it is more limited and precise than traditional nominalism. First, it recognizes the necessity of retaining a universal under the form of *relation* which unites the members of these classes. Secondly, it does not appeal vaguely to *resemblances* as constitutive relations of classes or as substitutes for clear and distinct ideas. Of course, the relations which connect sense data " of the same duration," " of the same extension," " of the same quality " all belong to the general type of *similarity* which includes all symmetric and transitive relations; but we must evidently keep these three types of " similarity " distinct.

It is very important to understand the twofold import of this method of logical construction. In one sense, it serves as a particular hypothesis about the relative complexity of entities, as we have seen in the case of temporal relations. But in another sense, it avoids all hypotheses; therein lies its universal validity.

* Cf. B. Russell, *Principles of Mathematics*, " Principle of Abstraction "; *Our Knowledge of the External World,* passim.

For example, I wish to study the temporal relations of my sense-data. I am certain that the data are there. I am equally certain that the relations are there (without deciding whether they are simple or, on the contrary, admit further analysis). When I say: "This sound of thunder followed that flash of lightning," I am expressing a fact. Whether it is analyzable or not, it still remains perfectly definite. Now, about these facts I am asked to talk and think in terms of *durées*. But their whole utility is compatible with a hypothesis which reduces them to logical constructions whose sole elements are my sense-data and their temporal relations. Therefore, the wisest course is to understand them in this way. In fact, it is the only way of risking the introduction of superfluous entities. Even if *durées* do have more meaning than my sense-data and their relations, the latter meanings, no matter how artificial, still hold good as simple meanings. All the propositions in which I have employed durations as classes in speaking of the order of my sense-objects must be true *a fortiori* of those durations taken as more than classes.

Hence, we shall employ in our geometries of sensations only class-durations (and the same with class-extensions and class-qualities). In that way, we shall be doing no more than holding firmly to *sense-data* and to the relations we ascertain among them. By using these elements only to define classes which serve with regard to the facts all that is demanded of durations, extensions, and qualities, we do not exclude the hypothesis which treats temporal, spatial, and qualitative relations as complexes participating in non-sensible objects which would be the real durations, extensions, or qualities. This metaphysical theory is quite indifferent to our aim. We are neither adopting it nor excluding it in our inquiry into the fundamental relations of sensible objects. We are simply not cutting deeper in order to find complex relations of participation. The metaphysical complexity of non-sensible entities is beyond our scope. Our method amounts to treating relations *as if they were simple*, although we do not decide what their ultimate status in reality is.

Of the reduction of temporal relations to a single one.— Outside of the hypothesis of simple entities, *durées*, I can conceive of a second manner of analyzing the temporal relations of my sense-data. Without forcing them to submit to intermediary terms like durées, I can ask if all these relations cannot be reduced to some, or even, to a single one among them.

Let us take the relation of complete succession: all the other immediate temporal relations invariably coincide with logical complexes formed from it By giving to the sign = the sense of *equivalence* which it has in mathematical logic, I am able to write

a precedes $b = b$ follows a
a is during $b =$ any x which precedes b precedes a
and any x which follows b follows a

and so forth. Would not these equivalences yield all the temporal relations in terms of the single relation *precedes* ? Just because it is possible, the mathematician, the geometer of time, must accept it. With regard to any relation, he cares only for the structure that prevails among its terms. If two relations are inseparable, they are not for him distinct, since the first connection is merely reduplicated in the second equivalent relation without introducing any new arrangement in the universe. His rule is not only to refrain from multiplying entities, but also to reduce them as much as possible.

We shall presently do the same. But now, I am not trying to reconstruct the most economic statement of the order of the immediate flux I am simply inspecting the relations that I discover there, and among them it will be necessary for me to choose the connections which I shall need more particularly.

I try to apprehend each one of them just as it appears before my mind. Now it seems to me very doubtful whether my perception of the temporal relations of my sense-data could be reduced to the single perception of one of them like complete succession. For certain of these relations would then *consist of* or *be equivalent to* a complex relation *involving the totality of my data*, past, present, and future. Thus to state that a datum *a* is during a datum *b*, would be to state that *a* follows all the data that *b* follows, and precedes all the data that *b* precedes; for this is the very meaning of *during*. But can I ever ascertain a fact of this form? Undoubtedly, no. At most, I might be able to infer it. Now it seems to me that *a during b* is a relation which I ascertain directly very often between two sense objects. Hence, it is very probable that what I ascertain is the presence between *a* and *b* of a relation that entails no other sense object—it would be precisely the simple relation *during*—and that any second meaning of an all embracing relationship among all my other sense data is inferred only after the simple one. The general principle behind this procedure is that no definition of a relation expresses any real meaning if it entails *all* the members of an infinite class.

A second difficulty in reducing all temporal relations to a single one arises from a certain arbitrariness of choice. What is the ground for making the relation *precedes* the logical opposite of the relation *follows*? This is a difficulty that is easily solved by the geometer, but which is embarrassing to the philosopher. Perhaps this second difficulty, which attaches to all asymmetrical relations, is verbal. But the first is very real.—It seems then that the temporal relations of my sense-data consist of many original relations, and not only of one; intuition also supports this conclusion.

Definition of a natural group of sensible relations.—This brings us back to our initial question: Of what does the manifest unity of the temporal relations of my sense data consist? For I have just discovered a *plurality* of original relations. The hypothesis of real *durées* makes this unity reside in a common reference to *durées*. But in the hypothesis where, owing to the assumption that *durées* are complex realities, the constituent temporal relations are simple, must not the unity of simple relations be an ultimate unanalyzable similarity? That is, all these relations would have an indefinable *temporal aspect* which would be the irreducible residue of the general idea of sensible time.

This hypothesis of a simple temporal quality is not absolutely required. Certain formal laws, in fact, establish among these relations all the connections that exhaust their meaning. In the first place, the law of transmissibility through simultaneity, as we have seen, groups all the temporal relations involved in a *durée*. Again, we have the law that any two of my sense-data are related by one or the other of these relations. The result is that we no longer have a single regularity by virtue of which certain relations are inseparable. We see our relations separate, on the contrary, but only in order to share the mass of my sense-data, as several hunters divide the territory of a victim. Now have I not the right to say of relations that cover the sum total of the immediate flux that they hold true of the universe, that they form a kind of net thrown over the very face of experience?

While the current conception detaches from each one of my sense-data its *durée* (and decomposes this *durée* or event into *instants*), we have just seen that the only incontestable result is that I apprehend among these sense-data certain relations that I call temporal. It remains doubtful whether these relations imply *durées* (durational event-wholes), and *a fortiori* their elements, instants. It is even doubtful whether they all present one common quality.

The nature of the laws of sensible time and of analogous laws.—There are laws that state the general properties of the temporal relations of my sense data: Thus, simultaneity transmits each one of these relations; two of my sense data are always related; temporal inclusion is transitive, complete succession is transitive and asymmetrical. How do we know laws of this sort? What is the degree and nature of their evidence?

The answer depends in great measure upon the theory that I assume about the simplicity or complexity of each one of the considered relations. If I posit *durées* as real durations, the properties of simultaneity become analytical; for if I make it equivalent to a logical complex of relations, its properties flow logically from the latter. Generally speaking, any assumption of complexity in notions renders any one or more of their properties analytical. But there always subsist properties that are synthetic; and even if all relations could be reduced to a single one, that one could not be deduced from anything else and would hence be synthetic. According to what I take as the immediate content of these diverse relations, a more or less large part of the laws governing them are analytical; but there remains in all cases a nucleus of synthetic formal properties irreducible to identities.

Is there an *a priori* "chronology" involved here as Kant thought, or do the axioms of this science of sensible time have no more evidence than that of the incompatibility of black and white, of square and round? There is no doubt that such evidence is not solely inductive, and exercises a due function with respect to the imagination. But is it any other than the evidence that attaches pleasure or pain to such and such an object of sense? This is a delicate question, which we can only indicate.

CHAPTER III

GLOBAL RESEMBLANCE

On hearing a sound, scenting a perfume, enjoying a taste, I sometimes recognize the quality, and say that it is like such and such a sound, perfume, or taste I once experienced. This similarity between two sense terms has degrees like all resemblances, but in these cases is not limited expressly to only one of the various aspects of the terms which it compares. Since it takes each of the terms as single wholes, it may be called *global resemblance*. We shall soon see that there are two other types of more differentiated resemblance.

It is often thought that two given wholes never resemble each other in all their aspects, since they are apprehended in two different total perceptions. And even if all the circumstances could be reproduced, I would no longer experience them as different, since they would no longer be new to me. Moreover, when two data are similar, it is always with respect to some aspect. Thus, the global resemblances of my sense data would only form a chaos.

Two kinds of resemblance are confounded here. Resemblance may be either *direct* or *indirect*: it can pass immediately from one of its terms to the other, or else consist of a *relation common to some extraneous term*. Now only *direct* resemblances are under consideration here, whereas *indirect* resemblances can lead to similarities and differences between any two objects. The sense-data which accompany a certain datum *a*, even covering and surrounding it, the images which come forward in recognition, the feelings that it stirs, and all that taken as an individual mass, are so many logical terms extraneous to the datum *a* and consequently to the direct resemblances or differences that it may have to another datum.

But even if we should consent to allow as part of the quality of a sensible term everything that it happens to accompany and all that it arouses in my mind, it would still be necessary to admit that there are two parts to this quality, an indefinite reverberating and wavering nebulosity, and a nucleus of constitutive quality. The various odours, sights, and sounds that arouse such opposing states of mind at different times are yet incorporated each time in the olfactory, visual, and auditory data as inseparably as each quality in its object. Examine the contrast which makes me say: "Since yesterday nothing has changed, and yet, everything appears different." *Nothing* has changed: means that certain sensible terms present now are similar to those of yesterday because one essential quality stands out from all the aspects and relations that enter into the wider contexts. *Yet, everything appears different*: means that between today's and yesterday's experiences, there is a striking difference of aspects and relations outside of the central essential quality.

About this quality I may be uncertain and even mistaken. But it would be asking too much of the mind to expect it to function with no hesitations or mistakes. Difficulties and the dangers of error are everywhere present. No act of discernment is infallible; but neither is it proved futile because it allows error. To disengage the constitutive quality of a sense datum from the cloud of association and feeling that envelops it is a task difficult enough to allow the possibility of failure. But it is not an absurd task. The connoisseur of wines must indeed make an effort if he has to identify two tastes separated by two hours of his life and by a complete change of mental state. It is an effort of attention, of abstraction, if you please, but in any case a real effort, whose object is not a chimera.

The three logical forms of resemblance.—Global resemblance has an additional feature; it is a relation that has *degrees*. We must not confound this kind of more or less in the relation itself with the more and less in our certainty about its presence. This second intensive magnitude is universal and accompanies all relations indistinctly. Take the case of a relation R. It is always possible that there are three terms *a*, *b*, and *c* such that *a* R *b* is more certain to me than *a* R *c*, in so far as *a* has the relation R more certainly to *b* than to *c*. But that marks no degrees in the relation R itself; the degrees prevail between the two judgments *a* R *b* and *a* R *c*, each one of which has its meaning and degree of evidence independently of the other. On the contrary, the proposition " *a* resembles *b* more than *c* " is independent of the two propositions " *a* resembles *b* " and " *a* resembles *c*," because it directly connects three terms with a simple amount of evidence that does not consist in the comparison of two degrees of evidence.

This amounts to saying that a relation that admits degrees must be a three term relation; for example, " *a* resembles *b* more than *c*." It may be called *the order of global resemblance*. The reality of this relation cannot be doubted. Among three sense data that are very similar and yet manifestly different—three sounds, odours, shades—I apprehend the gradation sometimes very clearly. We must take care not to posit that among any three sense data one must be intermediate between the other two, for that is very doubtful.

We then have two relations, one of which, *global resemblance*, has two terms, the other, *the degree of global resemblance*, three terms. There is a possible additional relation between two terms viz. *perfect global resemblance* when I apprehend two data one of which seems to be the exact reproduction of the other. However, we can conceive the hypothesis that will reduce simple and perfect resemblance to the degree of resemblance. On such a hypothesis " *a* resembles *b* " would mean "there is a term *x* which *a* resembles more than *b*"; on the other hand, "*a* is exactly similar to *b*" would mean "there is no term *x* which *a* resembles more than *b*." There would be similarity between *a* and *b* so long as *a* resembled *b* more than *some* other term; there would be perfect resemblance or exact similarity between *a* and *b* when *a* would resemble *b* as much as or more than *any* other term.

This last formula conflicts, it is true, with the principle that no definable relation can call into play the totality of my sensible terms. But am I ever certain that two terms are exactly similar? Hence it is possible that perfect resemblance can never be ascertained as given, but can only be *inferred* from the fact that my imagination offers me no other term intermediate between the two terms in question. If that were the case the objection would be removed.

Since simple resemblance has a content as near to zero as we wish, and perfect resemblance remains uncertain, one losing itself in vagueness, the other in the ideal, degree of resemblance remains in all cases the most positive of the three relations.

CHAPTER IV

QUALITATIVE SIMILARITY AND LOCAL SIMILARITY

Over and above global resemblance are to be distinguished certain *partial similarities*. These involve new discriminations, for while I may discern by very acute hearing that *b* is or is not the exact reproduction of *a*, I may not know whether they differ, for example, in intensity, quality, or duration, and resemble each other in pitch. By global resemblance alone, I should be able to identify only those notes that are given in the same way by the same instrument. Likewise, I should not be able to recognize a colour, form, or change in my visual field unless they were apprehended all together; and the same for the sense of touch. My sense-data would have only one way of resembling each other (by a direct and simple relation) and not several ways.

But there are other modes of similarity, at least for certain senses. For impressions of touch, and especially of sight, global resemblance divides into two *partial* types of similarity which we shall call *local similarity* and *qualitative similarity*. This division is highly important.

While I am at rest, two identical sparks burst forth one after the other at the same point. The two data resemble each other as wholes (by global resemblance), like two odours or two tastes. Let one of the sparks be produced at my right, the other at my left. They still resemble each other, although less than before. Not only that, but they are extremely similar in one respect (globally), and not at all in another (locally). Finally, let there be two sparks that differ in colour, but not in place. As in the preceding case, they are similar to each other in one respect—in the very respect, place, in which the preceding case differed—and are not at all similar in respect of colour. We then name these two sorts of partial similarity of visual terms *local similarity* and *qualitative similarity* (taken in a narrow sense, for properly speaking, the quality of a sense-datum undoubtedly comprises also its immediate locality).

The same distinction holds true again of touch, and perhaps of all the senses. It is possible, in fact, that the immediate locality of a sensible object is always discernible from the rest of its quality. But then such a distinction would in actual experience hold true only of touch and sight. For while admitting that an odour, for example, has its immediate localization like visual or tactile data, this *localization* is the *same* for all odours. Every olfactory sense-datum fills the whole olfactory field. Locally, they all are similar; so that local similarity allows *no class distinctions* among odours, no serial arrangement, and consequently cannot enter into the expression of any law about my olfactory universe.

Let us take the kinesthetic sense. We must grant to each of its data a local character distinct from the whole of its quality, for we no longer have terms here that are all locally similar. Any local similarity doubles their global resemblance. In order for two kinesthetic sensations to have perfect local resemblance, they must, it seems, be perfectly identical and proceed from the same parts of bodies placed or displaced in the same way. If that is so, the perception of local similarity and diversity is again sterile. Before, it could introduce no order into olfactory data because it confounded them. Now, local similarity merely follows global resemblance, and is doubly dependent on it.

From the point of view of the order of my sense-data, the existence side by side of local and qualitative similarity is of no functional importance so long as they go on a par, or so long as neither of them can distinguish between terms. The framework of possible laws is by no means enriched thereby.

What is important is the existence of *divergent* local and qualitative similarities, so that *two overlapping structures or networks of similarities cross each other* and arrange the same data in two different ways. Touch and sight alone possess this rich source of structure. That is why they yield geometries that are particularly interesting. There are some however that are meagre; we shall see that it is possible to discern one and even several sorts of empirical geometries which utilize only global resemblance.

Are global, qualitative, and local resemblance elementary or complex relations ? Do they go straight from one sensible term to another, or do they consist, on the contrary, in the participation of these two terms in a common entity which would be their total (global) quality, their quality in the narrower sense, or their immediate localization ? With respect to temporal relations we have already indicated the general sense of our reply. On the one hand, it seems doubtful to us whether the similarities in question entail these special entities as well as the auxiliary relations by which sense particulars participate in them. On the other hand, this hypothesis is indifferent to the problem which occupies us; namely, to the question which concerns not the content of the relations of sense particulars, but the pattern or structural order that they outline. We must then remain neutral on the metaphysical issue. Now to remain neutral about the simplicity or complexity of a given relation amounts to treating it as if it were simple.

Qualitative and local similarity, like all resemblances, admit degrees and a maximum. Both then divide into three relations: a pure and simple resemblance, perfect resemblance, and an order of degrees of resemblance among three terms. We have already considered this trio of relations with respect to global resemblance and the possibility of deriving two of them from the third.

CHAPTER V

RELATIONS OF THE GROUP OF LOCAL SIMILARITIES

LOCAL resemblance may be considered the centre of several additional relations forming a family around it. These are relations which are *transmitted* by local similarity just as temporal relations are transmitted by simultaneity, that is to say, they may be symbolized by R=L | S | L, where L stands for perfect local resemblance.

They are (recalling the order of the corresponding temporal relations) local *inclusion*, *encroachment* and *separation*.

Relations of position.—But the group of local similarity contains, besides, relations which have no analogues among the temporal relations. I watch the lights of a city at night from the top of a mountain. Each lamp appears like a shining point. Do I not notice a strikingly definite similarity amongst the trios of shining points which come from three lamps in a straight line, no matter where they are situated in my visual field? It seems so, and it is this relation of three visible points, or if you wish, the special kind of similarity between two trios of visible points, that we wish to consider. Let us call the relation *alignment*. (We are not distinguishing the relation of three data proceeding from three objects in a *straight line* from the relation of three data whose physical objects are simply *in the same plane with my point of view*. Both hypotheses will be discussed later.) The relation of alignment is transmitted by local resemblance. Three visual data having perfect local resemblance to three other visual data in alignment are also in alignment. Three sparks, each one of which is locally perfectly similar to one of three preceding sparks which were in alignment, will also present the relation of alignment. It is easy to distinguish other relations in the same group. Thus, I may perceive a particular resemblance among all the couples of visible points coming from couples of small objects separated by equal angles of vision. This relation may be called *equality of divergence*.—These relations in the group of local similarity are *relations of position* and have no analogues in the group of temporal relations.

The first examples of these that I have taken are very small sense-data (sparks, distant lights) because we are accustomed to conceive relations of position as illustrative of the geometrical relations of points. But there is no doubt that these relations correspond rather to the relations of volumes on which, we have seen, geometry may be equally well founded. They undoubtedly hold among data of any magnitude. For instance, three large objects would be in the sensible relation of alignment if it could be said of them that " the same straight line passes through them."

Relations of position truly appear to be more illustrative of the abstract order of what we called geometry than all the other relations reviewed so far. If I seek other relations in the structure of my data and am suspected of looking for relations of position outside of their own field, it will be because I am conceiving geometry more abstractly.

We shall see later that these relations of position are not essential to the order by which sensible nature verifies the laws of science. They are, in fact, inseparable from complex relations formed from relations of other groups. They function, therefore, as formal equivalents of these other relations; they may possibly be reduced to the latter and thereby lose any original content of their own (cf. Part III., ch. viii.).

* * *

Such are the elementary terms and relations of sense immediacy and becoming. We have sought only to present them as they appear. If we have examined various views about their nature, it was only for the purpose of making it appear more plainly. Our only object was, thus, perception without system; for it is very important that these elements or constituents of all fact be themselves apprehended as facts apart from any theory.

Perhaps we have succeeded in showing that no hypothesis has been adopted concerning the representative content of the sensible relations which have just been enumerated. Simply distinguishing these relations from one another should be sufficient while our aim is simply to trace the order they introduce in nature. Similarly, we have assumed nothing about their conditions or historical origins. While I am considering the immediate reality designated in my thought as *this* relation, I have no direct acquaintance of such distant realities as bodies, sense organs, and nervous system. However, in order to designate it to you it may be necessary to add: whatever you apprehend in a situation involving objects and your body. But this detour serves only to make you think of *this* relation, and it is only this which I am really discussing.—In short, I do not know the history of those relations that appear to me as the elementary connections of nature. I do not claim that they have no history. I do not know what an animal, a savage, or a child apprehends. I do not believe that my perception has always been what it is now. But what I am considering is the observable universe of an adult, of a physicist. It is in this universe that we must discern the meaning and translation of physics.*

* There may be some astonishment at not finding any mention in the list of the ordering elements of observed nature of a primitive quality of *spatiality* (voluminosity, extensity)[1] which would be common to all

[1] William James held this theory. Cf. his *Principles of Psychology*, 2 vol.—Tr.

the data. But a property common to all the terms of a group does not order them. This very simple logical truth is forgotten when one sees in the spatiality of sense data the quality of geometricity (if we may be permitted to express nonsense in a barbarism) which would itself submit the data to the order of geometry. If, among the authors who have written on "space," some have thought that an analysis of the geometric order of experience should furnish an analysis of its original spatial quality, they surely were deceiving themselves, and we cannot follow this chimerical operation. Others were no less mistaken who thought that by simply indicating this indefinable quality of all sense experience one could found, construct, or replace the analysis of the geometric order that it presents. We cannot then be reproached in the last part of this work for untying with difficulty a knot that is too easily cut by intuition or words.

PART THREE

SOME GEOMETRIES OF SENSATION

INTRODUCTION

We have studied the formal structure of geometry. Then, we have drawn up lists of the primitive relations each of us perceives in sensible reality, simple connections of the order that it contains, elements of all the texts that we shall ever be able to read in it. Let us attempt to pass to the reading itself and see whether we can find geometry in the book of nature. But the task is too great to be accomplished with one effort. Geometry, in fact, does not come into nature before physics, but really through physics from whose more general canvas it abstracts its own perspective. It was long ago brought to light that experience has for its subject matter not "space" but bodies, or more generally, the sensible. However, geometry becomes realized in the expression of any experience by way of the situations of the objects and observers that constitute the conditions of any sensible fact. It exists only in these expressions; we cannot isolate it from them. Its sensible truth is no other than that of the group of the physical propositions that contain it.

But every physical proposition contains some geometry; the various branches of physics (astronomy and optics, for example) overlap in their empirical sensible data; and again to these data are joined those due to the physiology of the senses. The geometrical order of my experience, very far from leading back to a few simple and isolable facts, is ramified throughout my knowledge of the sensible world in an inextricable unity.

This generality is the essence of the problem. But it confuses the mind at first.

Let us then begin with problems of the same type (but proportionate with our mediocre power of analysis) by placing ourselves in a simplified type of experience. Since the geometrical order of the world is the group of its laws, let us begin by imagining worlds whose laws are simple enough to be apprehended in a glance. Thus we shall make our first contact, almost playfully, with the problem of the application of geometry to nature. We shall habituate our mind to some of its wider aspects.

We are going to control our experiment by choosing arbitrary conditions. Suppose we imagine a creature reduced to the sense of sight and placed in an immobile universe without his knowing anything about it except by changes in his field of vision. In this universe, we shall place only the objects we wish in any order we wish. The subject will have, if we so choose, a body invisible to himself. Or else, we shall consider a being who experiences only the contractions of his own body. We shall place him in a universe devoid of all visually perceptible differences in which we shall watch him wander about; but every time his movements bring him back to a certain position, we shall inform him by some particular impression (local sign).

We shall assume that these creatures possess the more subtle faculty of understanding the following artifices of science: homogeneous fields, material points, luminous points; and of constructing similar schemata of perception. We shall suppose they have senses that are infinitely delicate, a kinesthetic sense for which no two sensations are alike unless they represent two identical movements or attitudes, and a sense of vision which is able to distinguish two luminous points no matter how small their visual angle. Not only will its perception be perfect in intensive discrimination but also in extensivity or scope. Its vision will embrace in a single act the totality of space and all the material points in it. We shall cultivate these logical fictions in order to study the more serious problems before us.

Does not science itself, whose perceptual range we are investigating, invite us to proceed thus ? Physics does not hesitate to begin its approach to the real by assuming the most utopian fictions of simplicity: the material point, the luminous point—what magic wand could be waved more easily ? And having adopted these as objects, it gives us an exemplary study in painstaking analysis. While it is sure that these fictions are not to be found in nature, it sees in this no excuse for proceeding with less care and rigour. By mastering these schematic universes, it gains a vantage ground; then, it voluntarily relinquishes some part of its fictions in order to allow the entrance, one by one, of difficulties that separate it from nature. The physicist does not launch into the ocean of reality without precautions, for he would be taking the risk of drowning in it. Supported at first by a life-belt of ideal simplifications, he proceeds slowly, casting off part of his support only as he progresses in skill. We may imitate him, for the object of our study follows his which, fundamentally, is the object of our analysis.

The mind cannot do without these gradual approximations. Of the few authors who have considered the empirical meaning of geometry or, more generally, the sensory content of physical facts, there is not one who has not assumed extremely schematic conditions of experience. But since the mind leads naturally to this domain, they left their assumptions implicitly understood or indicated them rapidly with a word. To insist on these details might have seemed pedantic; is it not the privilege of the philosopher to transcend details in order to obtain a summary view in the shortest way possible ? Thus, leaving shadowy the simplifications introduced, and thereby giving up all rigorous construction, they aim only at producing before the mind an obscure image of a vague empirical order. Such teaching remains vague because it neglects to make explicit the hypotheses involved. One succeeds undoubtedly in veiling the interval that separates the image from reality, but the power of decreasing this interval is consequently lost.

Between this "natural" method of analyzing the order of sensations and the apparently artificial method we prefer, there is therefore exactly the difference between confusion and distinctness. Neatly postulating conditions that are in large part fictive, we shall study objectively their consequences. It is not that a spirit of unreality dominates us. On the contrary, the exact list of the assumptions that are at the basis of such schematisms enables us both to take them at their face value and to improve them later. For the spirit of approximation does not mean philosophically a method of negligence. It does not consist in multiplying inexact statements or modest guesses like "something near." It does not aim at disarming criticism, but at facilitating it. It says clearly with what it agrees and what the consequences are. Far from hiding anything artificial in its work, it aims to exercise the mind without encouraging illusion, that is, to prepare the mind for a better approximation.

When the light ray was no longer considered as a straight line without thickness so that light could be conceived as wave propagation, all optics had to be refashioned on this new model. Nothing remained of the preceding structure but a general set of features the memory of which guided the reconstruction. Likewise, the following sensible geometries are neither parts, nor models, nor originals of the infinitely more complex geometrical order of our world. Reared on fictions, these geometries can only be useful sketches, like the optics of the ray, or the mechanics of the material point.

As to the scientific turn of mind which we shall attribute to our imaginary observers at the same time that we impoverish their experience, our proposal should be regarded only as a device for exposition. Our investigation is not, in fact, psychological. In each one of the worlds that we posit, we are examining the order to be found there, not the reactions to it of a determined intelligence, memory, and curiosity. That is why our subjects enjoy an infinite power of intellect in order that they may discover all the order that is to be discovered. In relation to the structure of their sensible world their intelligence fulfils the function of a perfect *cicerone* (announcer) in a story whose real object is only descriptive.

Let us state the problem as accurately as possible.

Let us consider any geometry, for example: that in which the primitive terms are *points*, and where the only primitive relation is the *congruence* of two couples of points. Let $G(p, C)$ be the group of the axioms of this system, which is expressed in terms of a class p and a relation C between two couples of members of p. Let there be moreover the class s of my sense objects; let $R_1, R_2, \ldots R_n$ be the various elementary relations that I observe among them, and let $E(s, R_1, R_2 \ldots R_n)$ be the group of the laws that I would be led to regard as inductively probable. To discover an illustration, "a solution", of the formal system $G(p, C)$ in observed nature, is to form logically a relation C_o and a class p_o with the relations R and the class s such that the group $G(p_o, C_o)$ is implied in $E(s, R_1, R_2, \ldots R_n)$.

Let us illustrate this by a picture. As children we have all seen those picture puzzles which represent things that we cannot distinguish at the first glance; where it is a matter of discerning a giraffe or lion in the lines of a landscape deserted when first scanned. When we have " discovered " the picture hidden in them, we have seen nothing new. The contour of this little mountain is now the mane of the lion, and the knot in this tree-trunk is its eye. We had read in this network of lines a certain structure, the landscape, and now, we have just read a second structure, the lion. As to the lines themselves and the elementary relations which in the last analysis determined the whole design—angles, distances, intersections—we have in these the substance of the remainder: the very arabesque in which we can read a landscape by noticing that its elements grouped in a certain way reveals a certain border, and that a different grouping puts into vision a second structure, a lion.—The pattern that I have before me is sensible nature. The elementary relations that I know how to spell, so to speak, are the original relations of my sense-data. The figure that I tried to read is, for example, the geometry $G(p, C)$. What groups taken as elements, make this structure G appear in the relations which flow from their grouping? Would there be several modes of grouping answering this requirement; might one even find a lion in the landscape in more than one way?

In general, the relation C and the class p will be complex; the relation C will be a logical complex of $R_1, R_2, \ldots R_n$, and the class p will have for its members not members of the class s, but of the *classes* (if not of classes of classes) of these members, defined by means of the relations $R_2, R_1, \ldots R_n$ it might however happen that one of these relations R and the class s are themselves suitable meanings of C and p. This would take place if I saw every point in space and if I apprehended their equalities directly—as we shall suppose presently, in one of our cases. But, *even then*, it would be necessary for me to investigate whether the structure of the network of my experience $E(s, R_1, R_2, \ldots R_n)$ does not contain, besides, *other* solutions of the geometry G, other meanings of its relation C and its class p, meanings that are *complex* alongside of the simple meanings of the first system.

This logical formation of relations and terms possessing certain characters by means of relations and terms which do not possess them, the definition of points from entities that cannot be points, of "congruence" from relations no one of which has the properties of congruence, may seem very suspicious. One is tempted to think: if geometry is not present in the simple elements, it is not to be found suddenly in their complex synthesis.

But nothing is more commonplace than the creation of new formal properties by simple logical combination. The re-relation $>$ "greater than" is *asymmetrical* and *transitive*: it follows that its logical inverse $<$ "less than" is also asymmetrical and transitive. On the contrary, the logical sum of this relation and its inverse $\lessgtr$ "either greater than or less than" is a *symmetrical* and *nontransitive* relation. At the very beginning of this work, we have come across one interpretation of geometry in arithmetic: numbers when grouped by threes have filled the rôle of points, and congruence has been rendered by a group of two equations composed of various relations reducible to addition and multiplication. Thus, geometry which has already appeared once in an order of terms and relations was not something given in nature, but due solely to logical construction. Of course, nothing prevents such a geometry from being found in sensible nature.

In the sense-worlds that we are going to imagine, the network of order will, naturally, be composed of relations whose presence in immediate reality I have analyzed in Part II. But in these simplified experiences which are the only ones which can yield certainty in the present state of analysis, geometry is illustrated in structures which do not bring into play all the relations at once. It is *several groups* of relations, each one forming one or several geometrical structures. The most notable feature of a "geometry of sensations" is, therefore, the group of directly apprehended *relations* that it utilizes more than the meaning or meanings in which it takes its *terms*.

The spatio-temporal relations independent of the distinction between extension and duration do not play any rôle here, although they are very important, undoubtedly, in a less schematic plan of things. But temporal relations, global resemblance, local similarity, qualitative similarity, and the relations of position of sense data form in an ideally simple experience three sorts of geometric structure. One is in the combination of *succession* and *global resemblance*. The other resides directly in *relations of position*, if they are taken as original. The third is composed of *simultaneities* and *local* and *qualitative similarities*.

Let us examine in the first place the geometries of succession and of global resemblance among sense data.

These geometries divide into two kinds, according to whether we take the data of only one external sense, or whether we adjoin to them kinæsthetic sensations. It is true that the first kind include only rudimentary structures limited by the propositions of *analysis situs*. We shall begin with this kind.

CHAPTER I

SUCCESSION AND GLOBAL RESEMBLANCE

(Data of any external sense)

If we limit ourselves to the case of a geometry reduced to the properties of order in a line, we raise again, although from a different viewpoint, a little problem stated by Bergson in these terms: " . . . imagine an indefinite straight line and on this line a material point A which moves. If this point were to become self conscious, it would feel itself change since it is in motion; it would apprehend a quality of succession; but would this succession assume for him the form of a straight line ?" (*Données immédiates*, p. 78).

What is " the form of a line " ?—The question, in fact, is not the same for Bergson as it is for us. What does " the form of a line " mean to us ? It is as abstract a quality as it is complex. We call *line* a class of terms connected by a relation obeying the axioms of linear *Analysis Situs.* What terms, what relations are going to satisfy this definition, we do not know; or at least, if we do not wish to feign such complete ignorance, if we wish to recall a certain mental image evoked by the word, we must remember that this image is only one instance of what we are after, and that the other instances need not resemble it at all. We cannot then limit in any way the nature of the elements and relations which form a line in the analytic sense; we are seeking in observable nature cases obeying the axioms of linear order, and not a certain aspect serving alone as a line.

On the contrary, this is what Bergson is doing. He does not say what he means by a *line*, but his answer to the question that he asks shows sufficiently the meaning which he was giving to "line." The experience of motion would, he says, assume the form of a line "on the condition that it could in some manner rise above the line it traverses, and apprehend simultaneously several juxtaposed points." Whence this assertion? It arises from the fact that Bergson creates for himself a tyrannical image of the appearance a line should have. He represents to himself a group of simultaneous elements offering a certain original order that he calls juxtaposition; and seeing that in this way he has obtained a line satisfactory to geometry, he is unaware that there may be still others altogether different which are formed out of other elements and other relations. He limits the form of a line to one particular appearance. Having asked himself if his conscious point in motion would assume the form of a line, his answer is: undoubtedly yes, on the condition that it have or assume this form thanks to some illusion. But that is to abandon thought in order to yield to imagination; it is to fall into the arbitrary. For us, the form of a line will be only the laws of a line. If, as we shall see, these laws can be encountered in the intuitive order of an instantaneous apprehension, they can also be found in other aspects of experience.

It is true that Bergson believes that these laws, wherever they present themselves, can only be thought in terms of the particular image of simultaneous multiplicity which he calls the idea of space; and he appears to think that, inversely, a being equipped with this idea and using it in his thinking as a kind of blackboard cannot fail to impose its order on anything he thinks. Thus the application of geometry to nature would be contained entirely in this sole means of representation. But the first of these theses does not concern objective laws but only the mental laws governing our thinking of them. As to the second, according to which the use of these means would determine these laws themselves, we shall see that it is not maintainable (Part III., chap. v.).

Returning to our problem as just formulated, we examine the statement that in the moving body that describes the straight line, A " feels itself changing since it is moving." But this can occur in three different ways. Take the path of a horse and coach, the coachman sitting outside, and a traveller inside. The horse, because of his blinds, sees nothing but feels himself changing by experiencing the deformations of his limbs; the coachman, on the other hand, is aware of changes by noticing the continuous flight of the landscape; finally, the traveller who glances from time to time through the windows, knows he is changing in place only because each time he witnesses a new spectacle. Which of these three modes of experiencing displacement are we going to choose ? The experience of the horse is formed by internal sensations from which we are abstracting. On the other hand, the experience of the traveller, composed of impressions received in places which have among them no systematic connection of nextness, is too discontinuous to come under our analysis of relations of succession and of global resemblance. We are then left with an experience of the type had by the coachman, in which displacement is translated without a break by a series of external data; and since we are limiting ourselves now to the two relations of succession and global resemblance, we shall choose for clarity the relatively undifferentiated sense of hearing.

Open and simple line.—Imagine a creature having no other sense than hearing transported along a line divided into little segments such that with each passage over any segment A, a particular sound *a* is produced. (The existence of *indiscernible* segments is excluded by the qualities of the sounds which mark them; this simplification is convenient but it will be seen that it is not essential, as one might think.) This traced path is comparable to the length of a keyboard of an organ whose keys our subject would touch and play as he passed over them.

Concerning the sounds produced, which constitute his whole field of perception (we shall understand by *a sound*, an individual sound and not the abstract special quality of sound), our creature, let us say, wishes to know only two questions: *Was the sound y after the sound x? Was the sound y similar to the sound x?* Would the answers to these definite questions be the laws or physics of the objective world of our subject?

This physics can contain only two notions, namely, the two relations of succession and resemblance. (By resemblance, we shall understand perfect global similarity, and by succession, complete succession.) Their network is the whole structure of this universe. But prior to the laws which combine them, each of the two has its own laws.

The laws of succession are three. (1) '*The sound y follows the sound x*' excludes '*the sound x follows the sound y.*' (2) '*The sound y follows the sound x*' and '*the sound z follows the sound y*' implies '*the sound z follows the sound x.*' (3) '*Either the sound y follows the sound x* or *the sound x follows the sound y.*' Logic would express these laws by saying that the succession of the sounds is an asymmetrical, transitive and connected relation, or in one word, a series.*

Now a series is one of the two forms which define the order of an open and simple line: the serial succession of the sounds is then an open and simple line.

So far we have had only the science of the succession of sounds and not that of their resemblances yet; that is, of the flight of things, not of their return. Nothing yet translates the fact that the melody which forms this sensible universe constitutes what we call a displacement, and particularly, a

* Cf. B. Russell; *Introduction to Mathematical Philosophy*, ch. iv.

displacement along a certain kind of path. Does this translation exist ? That depends on the road we make our moving subject travel. So long as we do not reverse his direction, the sounds heard are all different. The observable order is reduced to the monotonous theme played by succession. The fact that the sounds which follow each other reflect what we call a progression across space is still unexpressed.

But as soon as we make the moving creature turn back on his journey, his science develops into two new branches. Beside succession, there now appears a second simple nature, resemblance. Like the other, it has its own laws, which make of it a symmetrical, transitive, and disconnected relation. To the science of succession is thus added the science of similarity. But in addition, a new complication arises.

Succession arranges all the sounds in a single series; on the other hand, similarity forms classes of sounds similar among themselves and different from all the others. Now these classes of similar sounds have their members dispersed in the order of time; *the double structure which is introduced by this fact is perhaps the most fundamental feature of objective nature.* This mingling of successions and resemblances forms all physics. In the universe of our subject, it admits a very simple formulation. Let us first see what it is for us, then how he would make the formulation.

Of any three sections of an open and simple line, there is one which is found *between* the two others and which one must cross each time in going from one to the other of these two. Now all the passages made by a moving particle across the same section are characterized by similar sounds. If then a, b and c designate three classes of similar sounds, one of these classes includes a sound in every series of sounds coming after a sound of the second and before a sound of the third or vice versa. This is the principle of the interweaving of successions and resemblances in the universe of our subject.

This principle is clearly quite empirically known to him without a shadow of intrinsic evidence or necessity. But let us see exactly how it manifests itself in experience and what complex notions its formulation reveals.

It is in the first place the relation of one sound to two other sounds one of which has preceded and the other followed it; that is, it *separates* them. It is also the class of all the sounds similar to a certain sound: let us call this class the *note* of this sound. Finally it concerns the relation of a note b to two other notes a and c which consists in the fact that every sound of a and every sound of c are separated by some sound of b; that is, the note b divides the notes a and c.

The principle then should be stated thus: *of any three notes, one of them divides the other two.* This principle exhausts the main content of the science of the combinations of succession and resemblance in the considered universe of sounds. All its other laws are deduced from this principle and from the properties of succession and resemblance, just as the theorems of mechanics or optics are deduced from the principles of these sciences and from the properties attributed to space and time.

Now these laws attribute to the relation of the division of notes the properties which define the relation of the *cut*, or the relation *between*, the second form of linear order. The two forms are moreover equivalent, and inseparable in the sense defined in Part I. In fact, beginning with a two-term (x, y) relation obeying the axioms of series, we can logically compose a three-term (x, y, z) relation obeying the axioms of the cut; that is,

$$A\ (xyz) = (xy) \text{ and } (yz) \text{ or } (zy) \text{ and } (yx).$$

Conversely, beginning with a relation of a cut (xyz) we can logically but with more difficulty compose a serial relation (xy). This structure of the open and simple line, in its two inseparable aspects of series and cut, is illustrated at first in the succession of individual sounds, and then in the division of classes of similar sounds or notes.

It must be noticed, however, that the serial aspect is simplest for sounds, and the cut, on the contrary, is simplest for notes. In fact, the succession of sounds, which is a simple relation, is a serial relation; in order to have a "cut" relation, we must employ the relation of the separation of two sounds by a third in accordance with formula A. On the other hand, the linear order of notes is expressed more simply by the relation of division, which is the same as the "cut" relation. Thus, of two inseparable systems of linear *analysis situs* one of which is founded on a binary relation of directed serial order, the other on a ternary relation of a cut, the first is more easily applied to the series of sounds, the second to the series of notes. Here we have already an illustration of the relative character of the extrinsic simplicity of systems of geometry.

Although geometry in this universe of sounds is illustrated twice, in the order of sounds and in the order of notes it is reduced really to the structure of an open and simple line. Nature very often weaves the same design several times. She likes to illustrate the same type of order in the sensible universe at first in a simple way and then in a more complex fashion; for example, in sounds and their immediately seized succession, then in classes of sounds and the relation of division among three of these classes laboriously analyzed by the mind.

These analogies, by which complexes are symbolically related to simples, are only with suspicion easily accepted as ultimate identities. How tempting it is to attribute their common root to nature or to the mind! However, if our subject took to philosophizing and thought he saw in the analogous orders of sounds and notes two expressions of only one fundamental fact, objective or subjective, would he not be mistaken? The order of sounds expresses a general property of sensible time, the order of notes a particular property of the path which it would be easy for us to modify or destroy. If notes like sounds form an open and simple line, that is purely accidental and not indicative of any unity. Perhaps it is the same way in our more complicated world; perhaps, the multiple aspects of its geometric order are several distinct facts whose unity is only apparent in the light of invalid speculation.

The simple series of the external sensations of a conscious moving point that is displaced along a straight line in an invariable universe therefore reflects *some* of the geometrical properties of the straight line, and its properties of linear order by a linear order of classes of similar sensations. But the more particular property of null curvature which fixes the form of the straight line and distinguishes it from the parabola or from the zig-zag has no place in this experience. In order to have before us a more distinct idea of the elementary geometrical properties which can be discovered there, and of the constitution of the terms and relations which illustrate them, let us briefly consider again what would happen if the trajectory our subject followed changed, not from a straight line which would alter nothing, but from a simple and open line.

Any line.—If we make our personage slide along a curve closed like an O or ramified like a Y, it will no longer be true that of any three sections, one will lie between two others. Consequently the principle of division disappears; the previous system of laws governing the intermingling of successions and resemblances of sounds is destroyed.

But another more complex law takes its place. For there is a principle which enables geometry to decompose methodically any line into a certain number of open and simple fragments, and at the same time to give the formula of its constitution. This principle consists in *cutting off sections* of this line. If

I cut off any section of a circle, there remains an open and simple line; this property defines the simple closed line or gives its formula. For ramified lines like the Y or 8 it is necessary to cut the curve in more than one point in order to obtain nothing but open and simple branches as remainders. (For the sake of simplicity we are excluding the existence of singular points situated on the boundary of a segment.) All these latter sections are distinctly characterized by being adjacent to more than two other segments and may be called *multiple sections*. Once these are cut off, the ordinary sections form a certain number of open and simple branches. Each one of these branches may comprise either only one or two sections adjacent to a multiple section, and these two sections may again be adjacent to different or the same multiple sections. In the first case, we have an isolated branch starting from an intersection or crossing, such as the bottom of the letter P; in the second, an interior branch, that is to say, one that unites two points of crossing, such as the middle bar of the letter H; in the third, a loop carrying back to the crossing from which it started such as the closed part of the letter P. A complex line is characterized by the formula which enumerates the branches and the multiple sections they touch. For instance, a Y or a T has the formula: three isolated branches touching a multiple section; a P or a 6: one isolated branch and one loop touching one multiple section; an A or an R: two interior branches and two isolated branches each of which touches one of two multiple sections; finally, an 8: two loops separated by a multiple section.

Instead of cutting off sections, our subject cuts off notes, that is to say, classes of similar sounds. He studies *the series of sounds that do not comprise any sound belonging to a certain note.* If we make him follow the road of a simple closed curve, the rule of the successions and resemblances of his experience is as follows: *In the series of sounds which do not include any* *sound of a certain note, whatever that note may be, there is one of three notes that divides the other two.* Let us now make him describe a ramified trajectory. He then distinguishes notes which have more than two neighbouring notes (*neighbours* would be notes whose sounds have succeeded each other immediately); he names them *multiple notes.* Calling the non-multiple notes *ordinary notes*, he classifies all those notes whose sounds have succeeded each other once without being separated by the sound of any multiple note; he calls these classes *sequences. In every succession of sounds belonging to the notes of the same sequence, one of any three notes divides the other two.* Thus within the limits of the notes of the same sequence there reappears the same structure which previously bound the notes as a whole. Moreover each sequence comprises either one or two notes neighbours to a multiple note, which may be different for each one of the two, or the same. That gives three kinds of sequences: *isolated sequences, interior sequences, closed sequences.* Suppose that his trajectory is an H curve. We say that it is composed of two multiple forkings united by an interior branch each one of which originates two isolated branches. Our subject is no less scientific, but he expresses himself differently and talks of other things. The universe, he says, is made of sounds; the sounds by their similarities form the classes that I call notes; and the group of notes is composed of two multiple notes connected by an interior sequence and neighbour of two isolated sequences each.

An external experience marking a displacement along any trajectory and containing only the two relations of succession and global similarity would thus offer a general principle of order and a formula of the constitution of the universe faithfully reflecting the general property of lines and the particular formula of its trajectory such as a treatise on *analysis situs* would state them.

Individuals, species and things.—There is, however, an important difference between our subject's way of seeing things and ours. For us the form of his trajectory is an individual fact; but for him, the formula which expresses it states a group of laws. For us, in fact, the *sections* of the line to which he is confined are individual facts; but for him the *notes* which translate them are *species* or *classes*. The relations of notes are then the class relations of the species of his universe, and the classification of these notes into multiple notes and sequences should be compared, not to a geographical map, but to a systematic table of natural classes of elements such as the chemists establish. We must not forget, of course, the extreme simplicity of these classes in the present case.

It is indeed worth while noticing that this universe contains only two entities: sensible individual objects and classes or species; that is, ephemeral *sounds* whose unity resides solely in their continuous existence (these sounds, naturally, have to vanish before they can be replaced by succeeding sounds); the classes are *notes* whose members, on the contrary, have no other relation among themselves than resemblance. In all this, nothing yet corresponds to the notion of a *physical object*. To the same physical thing there may belong two sensible terms separated in time; and, moreover, it is not because of that fact that these terms must be similar. The relation by virtue of which we place them among the " appearances " of the same physical thing is really very complex. This relation has not yet even begun to be introduced into the first schematism of our sensible world. We have deliberately referred it to the simple relation of similarity which characterizes the logical type of class species; we have accomplished this exactly by excluding the difficult case of two *indiscernible* " things " having exactly similar qualities. Thus, we have so far not distinguished a thing from a class.

Admitting indiscernibles.—Can we advance beyond the protection offered by the Leibnizian postulate? We must give up translating the diversity of sections crossed by the diversity of sounds heard. But we must not allow similar sounds to be associated with two sections which are so close that there is only one other between them: this is a *minimum of discernibility* below which all general order disappears in the considered universe. But let us admit that certain sections which are separated by two sections can at least produce similar sounds when they are traversed.

We shall only examine the fundamental case of an open and simple trajectory.

If during a walk, I see a telegraph post, and if a little later, I see a telegraph post again, the similarity of the two posts would not be sufficient to convince me that these two terms are two " appearances " of the same post: it would be necessary for me *to have retraced my steps* in the meanwhile. Now what does retracing one's steps mean from the summary point of view where we have removed by abstraction all kinesthetic sensations as well as observed changes and other complications? It means seeing the same landscape over again in reverse order.

This reversibility gives the complex relation which is to be substituted for similarity in the definition of the classes of sounds that mark passages over the same section. Briefly this relation is defined as follows: *Simple symmetric* sounds are any succession of sounds leaving in both directions from a central sound in such a manner that all those that precede it are similar to the same number following it. By neglecting all the sounds, save the first, of the simple symmetric sounds of a succession, we shall have a *general symmetric* series if the remainder forms a simple symmetric succession. Similar sounds separated by symmetric successions mark the successive passages of the observer on the same section. Such a class of sounds may be called a *unity*.

The principle of order which until now grouped sounds by notes now groups them by unities. It becomes: *Of any three unities, there is one which divides the other two.* Linear order persists then formally identical with itself, but attaching itself to more complex terms. This will happen to us often; each time we have caught the structure of an almost infinitely simplified universe, we shall approach a higher degree of reality by complicating the outline with a new feature which thus introduces a new order of which the first appears only as a particular case.

In fact, it is to be noticed that unless every note is common to several sections of the trajectory, certain notes will also be unities. That is what took place with all notes on the hypothesis that excluded indiscernibles, and that is why the more complex notion of unity was not free of the notion of a note. Not being necessary to any law, the notion of unity or self-identity had no occasion to appear behind the simple notion of a note.

Now again, it may be that on the concomitance of the two notions rests the rule that there is only one *note* which may participate in several *unities*. But that would not alter the fact that the whole value, and so to speak the whole weight, of the notion of a note, its whole importance as an articulation of the universe, has been irrevocably replaced by the more complex notion of *unity*. The laws of experience which previously held for notes have been degraded into common rules aware of the exception now that exact laws now depend on unity, to which the note is an insufficient approximation.

This notion of unity marks the first outlines of what we call a *thing*. It points out the fact that the simplest form of thinghood is incomparably more complex than the simplest form of a sensible class. It offers moreover the first indication of the way this complexity influences the objective order observable by each one of us. So long as this order refers to elementary sensible classes, the meeting of two data in the same ordering class is made by virtue of the single resemblance perceived between them, without there being any need of taking into account what happened in the interval of time that isolates them. On the other hand, when the ordering function passes from notes which are pure classes, to unities, which already have some analogy to things, the classifying of the two sounds in the same unity no longer results from their respective qualities alone, but also from the whole content of the sensible objective duration which separates them.

Surface or region.—So far, the sensible universe we have studied translates only the order of linear elements, not of surface or spatial ones. But let us make this same observer walk over a surface divided into squares, or in a space divided into cells, each one of which makes him experience a sensible quality different from those of his neighbours; the structure of this surface or space is reflected in his experience. For on a surface or in a space as along a line, although there are more possibilities, there are still classes of elements (squares or cells) which cut all the roads connecting two given elements. Now properties of this sort are translated into the experience we are considering by properties of the division of classes of similar impressions. Thus not only the order of a line, but also the order of a surface or space can be expressed solely by the relations of succession and global similarity of external data. Such an experiment is sufficient therefore to illustrate all of *analysis situs*. However, the more particular geometry of the straight line and distance has no place here. If the difference between a straight line and a circle, or a square and a cross is translatable by the method of this chapter, we still cannot translate the difference between a straight line and a helix, or an angle and an arc. This disproves incidentally the doubts of a certain philosophy as to whether our method of analysis is quite innocent. For our method is really incapable of introducing, by a kind of prestidigitation, a complete geometry into any manifold whatsoever.

CHAPTER II

SUCCESSION AND GLOBAL RESEMBLANCE

(*Kinesthetic data and data of any external sense*)

THE sensible universe which we are now going to study presents the same type of order as the preceding one in so far as it consists of *the same relations*: succession and global resemblance. But it is of another variety in so far as its terms are not only external data but also kinesthetic data. For our present purposes, the difference between these two sorts of data does not reside mainly in a qualitative difference, but in the difference of the physical causes which determine their similitudes. To an observer wandering in a motionless world, the resemblance of two external data signifies returning to the same place, whereas the resemblance of two kinesthetic data marks the repetition of the same change of place. The geometric order of the explored world will then be translated by various combinations of the successions and resemblances according to whether external data or " sensations of movement " form the resemblances.

It is to be noticed also that we are penetrating and accommodating experience not only for the propositions of *analysis situs*, but also for the whole content of geometrical treatises.*

* This chapter may be regarded as a development of the ideas of Henri Poincaré on the rôle of sensations of movement in the experimental aspect of geometry (*Science and Hypothesis*, ch. iv.). Poincaré claims, in fact, that geometry is illustrated in the alternation of external sensations and kinesthetic sensations; we are trying here to show more precisely how. But he assumes that without kinesthetic impressions there is no geometry of the sensible world; we have just seen that this exclusive dependence is not warranted. Besides, to the spatial order of sensations of movement combined with external sensations, he attributes a greater and more complex part than is necessary. For instance, the existence of changes of external impressions that are not accompanied by changes of internal impressions is according to him the essence of this order, whereas we shall see that it is nothing of the sort

Imagine a creature endowed with a perfect sense of kinesthesis, moving without external aid in a regular and motionless region where the effect of inertia is entirely absorbed; that is, as soon as the body of our subject interrupts the deformations which propel him, he stops on the spot, like a mole who stops digging. This assumption removes considerable complication. Two similar deformations, that is to say, two contractions starting from the same bodily attitude and passing through the same attitudes with the same rapidity bring about under these conditions two sensible displacements. The life of this creature is composed of deformations interrupted by rest, which produce in him kinesthetic sensations. Being infinitely precise no two of these sensations have perfect global resemblance unless they mark either two identical deformations (and consequently, two identical displacements), or two identical states of rest, that is, of the same attitude and duration.

Let us assume that he apprehends each one of his deformations and states of rest, no matter what their duration is, as an individual kinesthetic whole, provided that he does not distinguish more restricted terms. Thus, a series as long as one pleases of complex and varied contractions uninterrupted by any stops, is to his consciousness a unity that we shall leave undivided. He does not know it to be similar to another sensible term unless the latter is reproduced from beginning to end. Between two sensations of movement which have only one similar part (when one, for example, reproduces the other but endures or prolongs itself beyond the first), we assume that he will be aware only of a pure and simple difference. Each one of his movements, no matter how complex it is, can remain an individual effort within which he distinguishes nothing.—Likewise, we do not need to suppose that he apprehends a general similarity among all sensations which differ because of his attitude or because of their duration, or that he apprehends a general similarity among these sensations and those that correspond to movements. In other words, we are refraining from hypotheses that involve a certain selective interest.

Our subject has only kinesthetic sensations as yet. So long as the region he covers is devoid of perceivable differences, he will not have any other type of sensation. He wanders around therefore without meeting anyone or arriving anywheres. But the geometrical order of such a reduced experience cannot be translated. Its only observable laws are two rudimentary, they simply say that certain qualities of sensations never follow one another without an intermediary one; namely, those that signify rest in a certain attitude or movement ending in this attitude, and rest or movement starting from a different attitude. Thus a man who is walking cannot make one step twice in succession with the same foot, because this movement takes his body away in a certain attitude, and leaves it in another. But incompatibilities of this sort undoubtedly do not make a geometry.

We are obliged then to introduce external data marking returns to the same place. But whereas in the preceding experiment, it was necessary to attach to all the places traversed index qualities, we can here leave them all imperceptible, except one. Suppose then that in a single place, in a single attitude and in a single orientation our subject perceives a certain quality, for example, a sound. Let us call the whole quality of the place, attitude and orientation which determine the hearing of the sound *referential mark* or index position.

The laws of nature are going to consist of the recurrences of this referential sensation in the course of the series of movements. All the observations reduce themselves thereby to one single type: the ascertainment of which sensation or succession of sensations has separated two occurrences of the same referential sensation, or else (in our language) corresponds to a movement or succession of movements closed in its course. Is that sufficient to form a complete geometry? Granted that a sensible property serves as a distinguishing mark between closed and open courses of movement or successions of movement, does it follow that such sensible properties can serve also as marks that distinguish among the open course movements those that bring about the same displacement by different routes; among the various displacements those that are translations; and among the latter, translations in the same direction and of the same lengths? That is what we are going to show.

But of what use, it may be asked, is all this economical refinement? The real world, in fact, offers us in each one of its regions perceptions that have a qualitative and external point of reference; why show that only a single sort of these acts of reference is sufficient? In order to make the plurality of acts of this sort yield a plurality of objective spaces without any necessary apparent agreement of postulates, the fact of the independence of reference points constitutes a particular feature of the real world. It is thus that an analysis conducted with the greatest parsimony can alone produce all the rich diversity of the natural world.

Let us limit ourselves to the case of a plane geometry.

Plane movements.—If we restrict our observer to move only parallel to a certain plane—horizontally for instance—we can discover what laws he will discover.

We need a general term to designate the classes of similar sense data which translate either the same movement, or the same state of rest, or (for external data) the presence of an external point of reference. If we call them *sensations*, as we shall, a sensation will not be an individual datum, but the class of individual data similar to a certain datum (*definition* 1). We shall use *referential sounds* for all the external similar data of our subject, and call the class of such *a referential sensation* (*definition* 2).

In order for a movement to be possible immediately before or after resting in a referential position, it is necessary for it to take the body in the attitude of this position and to re-establish it in such a position. Such movements then bring about the same change of place as would be accomplished by displacing the body of the subject in a rigid attitude without deformation. These are the only kind of movements we shall study at first. Empirically they are translated by *sensations which can be preceded by an index sound and which can also be followed by an index sound.*

Let *rab . . . mnr'* be a chain of occurrences of the sensations **ab . . . mn** uniting two index sounds *r*, *r'*. The series **ab . . . mn** is to be called *closed* (*definition* 3).

If by inserting the sensation **x** between two terms of a closed series we still have a closed series, the sensation **x** is called a *null class* (*definition* 4). In fact, it then translates either a state of rest or a movement in a closed course equivalent to staying at rest.

In what follows we are to consider the property that certain series of sensations have because they are closed; it is really the only ascertainable property in this universe. The interpolation or omission of null sensations therefore changes nothing, and so we shall disregard them. Thus the series **xyz . . .** will be understood henceforth to be predicable to any succession of sensations **x**, **y**, **z . . .** which follow each other either in this order or are separated by any null sensations.

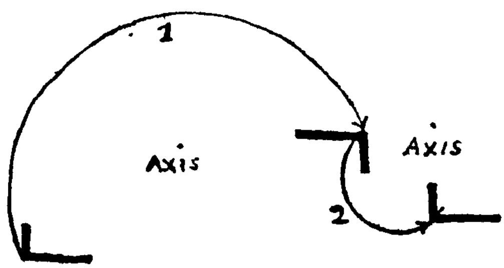
Let **x** and **y** be two sensations or series of sensations. If the series **xy** is closed, **x** and **y** are to be called *inverse* (*definition* 5). In fact, the movement corresponding to **y**, executed after the movement corresponding to **x** leads back to the position of departure.

Consider two movements B, C both inverse to the same movement A. They can differ in their courses and manner of motion, but they bring about the same displacement; namely, that which annuls the effect of the movement A. B and C are therefore equivalent. Our subject therefore defines as equivalent two sensations or series of sensations **b**, **c** which are inverse to the same sensation (*definition* 6).

We now see the experimental translation of the class of movements producing *the same displacement*. It is the class of all the kinesthetic data which are instances of one or the other of sensations *equivalent* to a certain sensation. We may call this class that is wider than a sensation *a unity* (*definition* 7).

Consider a movement A which, executed twice from the referential-position, compensates itself and leads back to this position. It is evident that it is equivalent to half a rotation of the body of the subject around a more or less distant vertical axis; that is actually the only horizontal displacement a repetition can compensate.—Our subject then defines as *an alternation* a unity which is its own inverse (*definition* 8).

Let us make the body of our subject experience two half-rotations around different vertical axes; the two half-turns compensating each other, the result is evidently a *horizontal translation*.



THE BLACK JOINTS REPRESENT THE POSITIONS OF THE BODY OF THE SUBJECT.

Conversely, we can find the parallel position that the body of the subject can assume on the same level as the referential-position by means of two half-rotations around vertical axes. Thus, the couples of half-rotations around a vertical axis is equivalent to the horizontal translations of his body starting from the referential position. He calls *double alternation* the unity formed by sensations equivalent to the succession of two alternations (*definition* 9). After what we have just said, the group of double alternations reflects in his experience the group of the horizontal translations of his body.

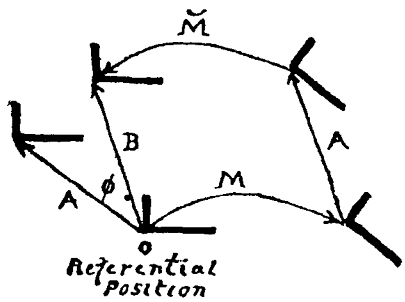
Consider all the parallel positions that this body can occupy in the series of these translations. These positions correspond to the points of the horizontal plane and present the same order as it does. For instance, the congruence of two couples of these positions (defined by the congruence of two couples formed by any four similar points), has the same laws as the congruence of points themselves. In order for geometry to be entirely expressed in the sensible universe we are now considering, it would then be sufficient to have the *congruences* of the positions of double alternation become reflected in some sensible relation among these translations from the index position.

Now this is exactly what does take place.

Let us see, first of all, how the equality of two translations is reflected; by equality is meant the congruence of the distances displaced by any point of the body of the subject.

What relation of the double alternations **a**, **b** which reflect empirically two translations A, B of equal length can express empirically the equality of these lengths? The difficulty of establishing among the movements A and B some experimental compensation arises in the case of the divergence of their directions. This difficulty in defining the congruence of displacements with directions other than translatory, can however be overcome.

Consider any translation A and a movement M which, not being a translation, changes the direction of the observer by an angle Φ. Starting from the initial index position *o*, let us execute the movement M, then the translation A, finally a movement M̆ which is the inverse of the movement M. It is evident that the result is equivalent to a translation B *of the same length* as A and *in a direction making with* A *the angle* Φ.



DETERMINATION OF CONGRUENCE BY ROTATORY DISPLACEMENT.

Inversely, every translation B equivalent to the succession of a movement M, of a translation A, and of the inverse of M is of the same length as A. That is, for any two translations whatsoever A, B, there exists a movement M such that the succession M-A-*inverse of* M is equivalent to B.

That gives a definition of the equality or congruence of two translations expressed in the relations of the corresponding double alternations. Our character defines two double alternations **a**, **b** as being *equal* if, for some unity **m** (**m̆** being the inverse unity of **m**) the succession **m a m̆** is equivalent to **b** (*definition* 10).

But what we are seeking to express is the relation of two couples of translations AB, CD whose displaced positions have between them the same distance *d*. This is easily expressed in terms of the *difference* of two translations; for it is clear that the *differences* (defined as the translations between two displacements) from A to B and from C to D have the same length.—Our subject having also named *the difference of two unities* **a**, **b** by a unity **x** such that the succession **ax** is equivalent to **b** (*definition* 11), finally ends with the following definition: when the differences of the double alternations **ab** and of the double alternations **cd** will be equal, I shall say that the two couples **ab**, **cd** are *connected* (*definition* 12).

Our object is thus attained. The connected couples of double alternations **ab**, **cd** answer in fact to the couples AB, CD of translations such that the couples **ab**, **cd** of the positions of these translations displaced from the initial position are congruent, and we know that *the congruences of parallel positions have the same laws as the congruence of points.* Plane geometry is then integrally reflected in the sensible universe of our subject. If he knows about this universe only by experience, and if he recalls a *Plane Geometry* which we suppose he has read, he notices that by taking *point* in the sense of a *double alternation* (*definition* 9) and the *congruence* of two couples of points in the sense of *a connection of two couples of double alternations* (*definition* 12), everything that the geometer says is verified in the flux of his own experience.

Movements in any plane.—Let us pass from the case of horizontal movements to movements in any plane, and let us see whether, under the conditions we have fixed, a geometric structure of sense data persists.

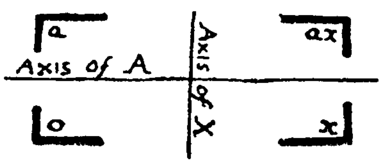
Certain of the preceding notions are not affected by our passing from plane movements to any movements. These are the *sensations* (*definition* 1), the class of *referential sounds* (*definition* 2), the *closed* series of successions (*definition* 3), the sensations or successions of *equivalent* sensations (*definition* 6), *unity* (*definition* 7), all of which still translate the same physical entities.

Likewise, *alternation* (*definition* 8) retains its value. For of all the movements that a body can execute, the only ones whose repetition brings it back to its starting position are the semi-rotations around an axis; only, however, the axes of two half-rotations are no longer necessarily parallel, as they were for movements in a horizontal plane. Consequently, the resultant of two successive half-rotations is no longer necessarily a translation, and these successive operations no longer compensate each other; that is, *double alternation* (*definition* 9) no longer furnishes the kinesthetic concretion of translatory motion. We must try again to discover one, if it still exists.

Let there be a half-rotation A. What kind of half-rotation X will make the displacement A followed by X the same as the displacement X followed by A? We state that the necessary and sufficient condition for this equivalence of AX and XA is that the axis of X cut the axis of A perpendicularly.*

Therefore, if A and B are two non-equivalent half-rotations, that is to say, around different axes, all the half-rotations X such that make AX and XA, BX and XB equivalent (if there exist such) have their axes perpendicular both to the axis of A and to the axis of X; they are therefore parallel to each other, and a succession MN of two of them is equivalent to a translation.

* Let o be the initial referential position, a the index position due to A, x the index position due to X, ax the index position resulting from AX (and by hypothesis from XA).



The half-rotation going from a to ax is the half-rotation X going from o to x. Therefore, x and ax are situated symmetrically on either side of the original position of the axis of A. But in order to go from x to ax, the subject must execute a half-rotation A after the rotation X, just as before it had to execute A before X. Because of that, x and ax are also situated symmetrically with respect to the axes of A and X. Now two volumes cannot be symmetrical with respect to more than one straight line. Consequently, the positions of the axis of A before X and after X should coincide. This can take place only when the axis of the half-rotation of X cuts the axis of the half-rotation A perpendicularly.

Our subject then defines the following notions. Two alternations **x**, **y** are *permutable* when the double alternation **xy** is equivalen to the double alternation **yx** (*definition* 13). If there exist two alternations **m**, **n** both permutable with the alternation **a** and with the alternation **b**, the double alternation **ab** is said to be *homogeneous* (*definition* 14).

Just as the positions obtained before by horizontal translations of the body of the observer starting from an index position were expressed in his experience by double alternations, so the positions obtained by *any* translations of his body starting from an index position correspond to the homogeneous double alternations which have been just defined. As to the congruence of two couples of these positions, which was expressed before by means of the *equality* (*definition* 10) and *difference* (*definition* 11) of two double alternations, that is, by the *connection* (*definition* 12) of two couples of double alternations, the same notion of congruence remains provided we add everywhere *homogeneous* double alternations.

The whole of geometry is expressed in this experience. *Homogeneous double alternations* and the *connection* of their couples offer to our character an interpretation of *points* and their *congruence* in everything that geometry predicates of them, and we already know that all geometry can be stated in propositions containing only these two expressions.

Let us notice, however, that the axioms which make up the simplest basis of geometry when it assumes the *point* and *congruence of points* as its primitive concepts, lose this privilege as soon as we substitute for the bare terms *point* and *congruence* the complex values *homogeneous double alternation* and *connection of two couples of homogeneous double alternations*. In fact, the simplest axiom takes on a most complex meaning. For instance, *two couples of points congruent to the same couple are congruent to each other* now signifies *two couples of homogeneous double alternations connected with the same couple are* *connected with each other.* Now any verification of this formula in the experiment under consideration puts into play at least a score of sense data. But on the other hand, this proposition has become demonstrable by starting from more simple propositions or the axioms of the new system; such as the one which postulates the symmetry of inversion: *if the sensation* **a** *is the inverse of the sensation* **b**, *the latter is in its turn the inverse of the sensation* **a** It would be interesting to investigate the axioms of this system in order to see the particular formal aspect that the structure of geometry would embody. But that would be a considerable task, and we shall not undertake it here.

Other spaces.—Does the logical personage whose experience we are studying find only *one* illustration of embodiment of geometry, only *one* possible interpretation of points and of their congruence ? Not at all, for he finds beside the one which we have just seen an extremely great multitude of solutions, all of them independent of each other. It happens to be the simplest of all.

It must be remembered that we have considered among the movements of the subject only those that begin in an attitude of a position of reference and end in one. The geometry which we have been able to disengage thus was founded entirely on kinesthetic data standing for particular motions.

Now let us consider a certain position p of the moving body, other than the referential position o and bearing a different attitude also. It is clear that the movements beginning and ending in this second attitude would be translated into the experience of the subject by means of the geometrical order we have seen, if the position p were a second referential position, marked by some recognizable impression. It is not anything of the sort, but the result is the same anyway. For if there does not exist any sensible index of the presence of the subject at the position *p*, there do exist determinate movements leading from this position to the index position *o*, and other movements, the inverse of the first ones, leading back from *o* to *p*. Let D and Ď be two of these movements inverse to each other, and let **d** and **ď** be the kinesthetic unities which translate them into experience. Let us take as "indices," not the class **r** of referential sounds any more, but the succession **drď**. A kinesthetic datum *x* which follows one of these new indices translates a movement which is a part, not of the position *o* any more, but of the position *p*; likewise a datum *y* which precedes an occurrence of the index succession *drď* translates a movement which has ended in the position *p*.

There now appears a new expression of geometry. Its notions will be defined as before; only the referential points **r** have been replaced by the complex indices **drď**, homogeneous double alternation and similitude remain as such. They illustrate in a new way the axioms of the geometry of the point and congruence. But this illustration, although similar to the preceding one in form is none the less independent of it. Its content is entirely new. Not a single one of the kinesthetic data that it puts into play entered the preceding system, for its material is the data that refer to the movements beginning and ending with* the attitude of the position *p*, different from the initial position, whereas the first interpretation was restricted exclusively to the kinesthetic translation of the movements beginning and ending with the attitude of the initial referential position.

To our subject, therefore, the "geometry" whose indices are **drď** is not deducible from the geometry whose indices are the class of sounds **r**. The fact that the same formal laws persist, after we replace **r** by **drď** by introducing an entirely

* Except the data **d** whose movements start from the attitude of the index position and the data **ď** whose movements end there. But these movements are themselves also outside of the first system, since they are conducted either at the beginning or at the end in a different attitude.

new sensible content made out of the notions of homogeneous double alternation and similitude, is an empirical discovery for our astonished subject.

But any other couple **mm̆** of inverse unities gives the same result as **dd̆**; that is, the adoption of the succession **mrm̆** as an index organizes geometrically a mass of sense data which is entirely new or entirely different from the two preceding "geometries" (provided that **m** itself is not included under them). All these interpretations are absolutely independent; their quantity is inexhaustible, since they are as numerous as the attitudes that the subject's body can assume.

Let us call *a space* a group of terms ordered by a relation of couples according to the Euclidean axioms of the congruence of couples of points. We shall say that the subject knows many analogous but distinct spaces constituted by groups of different sensations. We have just seen that their structures are established independently, and hence form so many primary facts. They do, however, present a certain unity which is itself another fact.

Formation of a total space.—Let E and E′ be two of these spaces. Their points are homogeneous double alternations that is to say, particular classes of kinesthetic data. These two spaces do not generally have in common any of these classes, nor even any of the data of which they are constructed. They are two sensible groups independent of each other. We know that they translate two groups of movements of different initial and final attitude. Let A be the movement leading from the origin of space E to the referential position. Let **a** be the corresponding unity and **ă** its inverse. The reference of space E is formed by the succession **ară** where **r** stands for the class of index sounds. Likewise, if A′ is the movement leading from the origin of space E′ to the index position, **a′** the corresponding unity and **ă′** its inverse, the succession **a'ră'** is the index of the space E'. The movements that are translated by the kinesthetic space E are the translatory motions resulting from the displacement A starting from the index position; likewise, the movements which are translated by the kinesthetic space E' are the translatory motions resulting from A' starting from an index position. Let T be any movement in space E, T' any movement in space E', and M a motion leading from the end position of T to that of T'. This motion M enjoys or is defined by the following property: Executed after *any* movement X in space E, it terminates in the end position of a certain movement X' in space E'. Moreover, the correspondence thus produced between the movements X and X' in the two spaces E and E' retains the relations of congruence: if X_1, X_2, X_3, X_4, correspond (as a result of the addition of M) to X'_1, X'_2, X'_3, X'_4 and if the couples X_1 X_2 and X_3 X_4 terminate in congruent couples of positions, so will the couples X'_1 X'_2 and X'_3 X'_4.

This property is reflected empirically in the following laws; the small letters designating the "unities" translating kinesthetically the movements designated by the corresponding capital letters. Both accented and unaccented letters designate homogeneous double alternations or "points" of the spaces E and E' whose origins are separated from the index position by A and A' respectively. If

axm is equivalent to **a'x'**,

then *any* succession **aym** is equivalent to a succession **a'y'** and vice versa; also, if the couples **st**, **uv** in space E are congruent, the same will be true of the couples **s't'**, **u'v'** in space E' which correspond to them in this way by means of the unity **m.**

Thus the same unity **m** added to the points of **E** transforms them into points of E' by respecting the congruences, and consequently all the geometric properties, of the groups of points. Space E′ may be said to be *applied* to space E by means of **m**. Likewise, space E″ is applied to space E′ by means of a unity **m′**, and so forth. All these kinesthetic spaces are applied to each other in such a way that their structural orders coincide. We may then call a *point* the group of all the points which can be applied to each other, and we may call *congruence* the relation of two couples of these new points whose members in each space are two congruent couples of old points. This new space units in itself all the preceding ones; it fuses them into a unique order. By means of the transformations that he discovers among all his kinesthetic spaces, our subject sees them organized in a manner to form one total space.

But the manner of organizing them is always arbitrary in so far as this total space can be formed in a great many incompatible ways no one of which is more obligatory than any other. In fact, consider two of the spaces it unites, E and E′. Let **t** be a point of space E and let τ be the point of total space to which **t** belongs. Which is the point of space of E′ belonging to this same point ? Is the point **t′** transformed from **t** by means of a certain unity **m**? Now this unity can be chosen in such a manner to " apply " the given point **t** of space E to *no matter what point* **t′** of space E′. Thus, the various kinesthetic spaces E, E′, E″, . . . can really be applied to each other, and constitute a totality of space, if one likes; but nothing tells us which of the points of these spaces should form together the same point of a total space. The total space of movements remains, therefore, here as an indeterminate form. Such is the least imperfect expression in this kinesthetic universe of the space we regard as embracing all nature in a single geometrical structure.

To our subject, a text on *Geometry* would in truth be an astonishing book. Enigmatic in itself, since the unknown expressions *point* and *congruence* (supposing only these) recur in it incessantly, this geometry is admitted to have not *one* objective interpretation—which would indeed be remarkable —but a host of such independent interpretations. The objective universe comprehends an infinite diversity of kinesthetic data outside of similar index sounds. In the bosom of this chaos, first one mass appears which illustrates or embodies geometry, then a second, then others still, each one formed by data which stand for movements that take the body in a certain position and leave it in the same attitude. The laws of all these different structures, differing by a change of "referential index" stipulated in each, as also by the diversity of the very data to which they apply, have the same form. They give so many different meanings—all of them true—to the group of axioms and theorems. Each one offers its integral and independent interpretation of points and their congruences. Each one constitutes what we have called *a space*, and all the spaces are in their turn bound together by invariant transformations.

If we were in the place of our subject, knowing what we know, or what we think we know, we should undoubtedly think we could *explain* this indefinite repetition of the same type of order in nature, and reduce it to a single fact. But in this study, we are limiting ourselves to *analysis*. It is meet then for us to stop in the complexity of geometric structures that our study has revealed so far. Even if this same complexity should arise from some simple root visible to the eyes of the philosopher, it would remain existing as complex in the universe of experience.

CHAPTER III

INTRODUCTION OF LOCAL DIVERSITY IN SENSE DATA

WE have just studied two primary types of geometries of sense data. Both have as elementary connections the same relations: *succession* and *perfect global resemblance.* But they interlace differently, because the resemblance of two terms marks in one case the presence of the observer in the same auditory place, and in the other case, the same propulsion of his organs. If experience is restricted to these two relations, the first case has illustrated geometry in the data of any external sense, and the second, in the data of the internal sense of movement and attitude.

But the first of these geometries, the external type, is not complete; the second, moreover is not a purely internal type. The succession of external data (sounds) studied at first expressed only that amorphous part of geometry that comes under *analysis situs.* Besides, if the mass of kinesthetic sensations examined next has formed a complete geometry, it is only by the aid of a minimum of external indices. We might be tempted to conclude from all this that the coöperation of several senses, and especially the conjunction of external and internal data, is essential to the expression of geometry in experience: that is the opinion reached by Henri Poincaré. But we shall see that this conclusion was inferred too hastily.

In the attempt to make these first two geometries operate conceptually in nature, did not the reader feel the resistance of his imagination? The melody that we at first posited and the flux of kinesthetic data which we next considered produce in the mind the awareness of a reality far too meagre and fleeting, embodied by too simple a succession to be fully an object of reflection and analysis. We are disconcerted by the absence of a certain structure to which we have been so accustomed that it is mingled with our very idea of an objective universe.

In fact, of the two fundamental types of order in our experience, only one was found to be present: the structure that the pure and simple recurrences of things form, by means of the relations of global resemblance running across succession. But there is another structure of which we have not yet spoken. In the case of vision there is a type of order possible in the *two diverse resemblances* of colour and immediate place. Global resemblance is repeated in qualitative similarity and local similarity. Whereas in the preceding universes a sonorous or kinesthetic term belongs to only *a single* class of terms grouped by similarity—a class that we have named in the first case a *note* and in the second case a *sensation*—a visual term appears to belong to both of the classes of similar terms. It is, on the one hand, the class of terms that resemble it locally—let us call this first class its *locality* or sensible place—and on the other hand, the class of terms that resemble it qualitatively—let us call this second class its *quality* (in the narrow sense).

The structure of similarity and succession is now complicated by the crossings of these two species of resemblance. A sound is only the return of its "note," a kinesthetic datum is only the return of its "sensation." But a visual term is an intersection of its "locality" and "quality." A "note" is or is not present in a given moment, but a colour is or is not present *here* or *there*, in such or such a place in the visual field. Sense immediacy, instead of having for its elements of order only pure and simple recurrents, becomes a spectacle in the course of which two sorts of entities, qualities and sensible places, are joined and separated in a thousand ways.

In addition we notice that if qualities are capricious in their occurrence, letting themselves drop into oblivion often, every sensible place in the visual field is continually presented to me by some given object: a stone, the leaf of a tree, part of a shadow, a piece of the sky, and so forth, from morning to night. Qualities—red, blue, green—visit and leave me, but sensible places are always represented in their complete totality. That gives them a certain privilege. Being constantly present as an aggregate, they constitute a sort of background in which qualities detach themselves, like a canvas on which the latter appear, or like a stage on which they perform as a company. A visual term does not disappear completely like a sound; it leaves behind itself an heir in one of its characters. Its sensible locality survives it in a new term. That removes something from the absoluteness of the flux. The appearance or disappearance of a datum comes only to realize or undo a certain possible arrangement, one of whose two factors, sensible place or the *here* and *there*, remains always present to me.

This play of two diverse resemblances across vast groups of simultaneous objects, this combination of two groups of characters one of which is constantly represented in its entirety, characterizes the universe of sight to an extent that is lacking in the more rudimentary sensations. That is what makes the latter appear so meagre, and so contrary to thought. Our imagination cannot hold on to them. As to the physical displacements by which I have defined the superficial successions of sensible objects that I wanted to consider, have I not represented them to myself under a quasivisual aspect? Have I not imagined myself seeing the subject here, then there, in some field analogous to that of vision—that is to say, a field in which a certain sensible *quality* changes *place*?

In this way we really see in the apprehension by the mind of some field, by means of which it can dispose various terms in various immediate places, the indispensable instrument of all intellectual construction, the ideal blackboard which alone permits ordered thinking. It would then be necessary to confer on our preceding subjects an imagination heterogeneous to their auditory or kinesthetic experience, or at least infinitely more subtle, since the latter does not realize any differentiation of resemblance into local resemblance and qualitative resemblance. It would be impossible to analyze the types of ordered succession without projecting them upon a simultaneous spectacle, in the middle of an imagination analogous to sight with respect to the diversity of the *heres* and *theres* the latter offers; the intuitive order of this simultaneous vision would be the fundamental expression of geometry.

But let us see, at first, what this new type of sensible order is in which succession no longer operates.

CHAPTER IV

RELATIONS OF POSITION

(*Visual Data*)

THE succession and return of global sensible qualities within the narrow limits of a type of experience in which certain given qualities follow each other in Indian file, already form, we have seen, two sorts of geometry. However, we do not naturally imagine the geometric order of nature in this way. To the imagination a space is not a group scattered in time, but rather a simultaneous multiplicity. And if we are considering physical realities, it is even true that in this realm of changing objects their geometric order cannot be understood except in terms of their order at a given instant. Nobody really denies this; even the relativists who make simultaneity, and consequently the geometric order of nature, depend upon a system of reference. In physics, spatial order is in its very essence an order of simultaneity.

But the simultaneity discussed there is not the immediate sensible relation that I call by the same name. In physics we imagine actually a great number of events being produced at the same instant in all regions in the universe. These *physical events* are simultaneous, we say; but I cannot perceive them simultaneously. The sensible events which manifest them are produced some earlier, others later, in my sensible time. What the physicist calls the order of space is inseparable from what he calls simultaneity; but the latter is not the simultaneity of immediate experience.

However, I have the idea, perhaps a chimera, of a space whose terms, immediately simultaneous to me, would offer me an intuitive order free from all succession. The group of the *relations of position* of sense data furnishes some stuff to this dream; for these relations, if they are really simple natures, realize in a greater or lesser degree the idea of a geometric order that is intuitive and instantaneous.

I scrutinize with one gaze the heaven full of stars. Among the trios that the stars form, I can discern those whose three terms appears in a straight line. The triangles formed by three stars not in a straight line may be classified into scalene, isosceles, etc. I can recognize equal angular distances between some stars, unequal distances between others. Of course, I would be more certain of these classifications if I used instruments. But even with the naked eye I apprehend grossly the similarities and differences of figures formed by the stars of various regions of the sky. Do I not have a geometry put together in an instant ?

Let us admit provisionally that these relations of position among the shining points that one look shows me scattered in the sky are in fact relations entirely contained in the instant, without intrinsic reference to past or future perceptions. That is tantamount to positing the existence of an intuitively objective geometry. To complete the idea, let us imagine this geometry as simple, as complete, and as perfect as possible.

Imagine each point of space inhabited by a material point, and a motionless spectator who would see all these points distinctly with one look. We are supposing a sort of extremely "idealized" vision, which embraces in one glance (without motion) the whole content of the universe. No object is opaque to it, so that every point is clearly distinguished. The material points which occupy the space filled by the body of observer himself are no exception; they are "seen" as distinctly as the others. As to this body itself, it would be convenient to imagine it made of an imperceptible and penetrable matter which is non-existent so far as being seen is concerned. The mind that we are supposing has, accordingly, before it an infinity of distinct and simultaneous sense data corresponding term for term to the points of our physical space. That is the maximum of an *intuitive* perception of space.

Besides, he apprehends immediately the relations of position of these data. We can consider only congruence now; among all the couples *a a′*, *b b′* of sensible terms corresponding to couples *αα′*, *ββ′* of luminous points themselves occupying two congruent couples AA′, BB′ of points in space, and among these couples only, he sees a certain indefinable relation that we shall call a connection. That is the maximum of an intuitive knowledge of space, since all geometry can be stated without introducing any other relation than congruence. We cannot therefore conceive a more perfect immediate knowledge of the spatial order of this sensible world.

Every proposition in geometry is then translated by a sensible law. Points beecome the *minima visibilia* (for we are supposing an infinitely fine vision), and congruence becomes the intuitive connection of their couples. For example, the axiom that congruence is transitive now says that connection is transitive, and so with the rest. Geometry, for this subject as for the preceding ones, then expresses the laws of nature, not more truly but more simply. For the present geometry is translated in his experience in one instant by means of a single order, and not bv means of various orders of an astonishing abundance (the auditory and kinesthetic worlds). Moreover, the fundamental geometrical relations—let us say, for simplicity, the congruence of point couples—admitted before only very complex meanings: recall the connection of couples of homogeneous double alternations! In the present case, congruence admits as its "value" a very simple nature, an original relation that is no longer composite. Finally, the sensible terms that it connects instead of being scattered in a duration, are present all at once, forming by themselves a single spectacle. Thus is explained fully why this type of the expression of geometry appears the more natural. Thought adopts it so involuntarily that we may well wonder whether such an arrangement of simultaneous terms in a field does not constitute for our mind, chained down by images, the base of all geometric order, or even of any order whatsoever.

We must stop a moment to examine the influence of this point of view on the analysis that we are pursuing. It means we must interrupt the exposition of this analysis. But is it not necessary to defend the principle of spatialized thought against doubts directed against it nowadays ?*

* Nicod is referring to the Bergsonian critique of spatialized thought. Cf. the following chapter.—Tr.

CHAPTER V

LIMITATIONS OF THE HYPOTHESIS OF A NATURAL SPATIAL SYMBOLISM

While we have endowed our subjects with perfect intelligence, we have said nothing specifically about the conditions governing its exercise. Now it may be admitted that the mind functions only by the aid of some sort of instinctive or deliberative representation of its objects and their relations on something like an ideal blackboard.

It is possible to conceive this auxiliary background as the projection plane of a tactile or visual field (these two senses offering a *here* or a *there*, a sharp empirical distinction between quality and immediate place), or else as the picture of any sensible field, by attributing to the mind the power of distinguishing quality from locality, even when they are always united. This means the mind will be able to represent to itself, even beyond experience, any qualitative term as situated *here* or *there* in the auditory, olfactory, and kinesthetic fields. We might then conceive the domain of the *heres* and *theres* serving intelligence as a kind of a blackboard, or as a sort of innate projection which is the real object of an *a priori* intuition. All these modes of regarding the mind's way of seeing amount to the same, for what follows. But in order to label our ideas, let us say we are dealing with a visual schematism.

Suppose then that the analysis of a sonorous or kinesthetic reality, such as we have just sketched, cannot take place directly but only by the projection of this reality upon the plane of a visual imagination which presents an intuitive order of immediate places. Does it follow, accordingly, that the analysis which has twice shown us illustrations of geometry in the succession of data being produced one at a time, was really carried out only when the field was used as a blackboard? Was our analysis merely investigating the laws of this imposing medium of representation?

It seems that we have already removed all means of proving the contrary. Such is not the case, however.

The conception whose limits we are studying rests entirely on the notion of *representation* or *symbolism.*

If imagining any fact by means of a symbol were the same as thinking in the presence of a certain ideal spatial background, a question arises of the following nature. After assuming a thing unthinkable in itself and a relation between the thing symbolized and the symbol such that if in this whole relation we substituted the symbol in its whole ideal nature for the thing in its whole unthinkable nature, would we not find the consideration of things by means of symbols replaced by the consideration of symbols themselves?

But this is hardly the case. Every kind of symbolism is partial, or again, abstract. In every symbol, that aspect of the symbol which symbolizes something is accompanied by other aspects which symbolize nothing and which have to be disregarded. In the letters of the alphabet, the form alone is symbolic of the sound: the colour of the ink, the dimension of the marks do not mean anything. The order of the words inside a line symbolize an analogous order of sounds. But the order of words placed vertically under one another is an accident due to the formation, and does not symbolize anything.

Notice that in China, the contrary is true. This shows plainly that from the same images arranged in the same way, the mind *selects* the qualities and relations to which it assigns the value of a symbol. In the presence of an unknown script, one cannot begin to investigate the sense of the signs found before having made some conjecture about what qualities determine the form of a sign. For it is not known—what should have to be known—whether, for example, the colour and the thickness of the marks are accidental or essential to the symbol.

Of course, we are often tempted to *extend* a symbolic system. We ask whether any new aspects of the models employed as symbols might not also be of service. It is in this manner that having pictured the chemical composition of bodies by groups of atoms, we have as a result made the relative arrangement of the atoms enter into the symbolic system. This kind of development is frequent in the history of the sciences. It is a regular procedure of investigation. But to go further and forget that every symbol remains to the end loaded with indifferent aspects, and to postulate that all significance resides in the symbol alone, is to leap from reason to nonsense. The mystic state of mind is perhaps the origin of thinking by symbols, perhaps also its weakness. But outside of some primitive tribes it is not the common mode.

A symbol is then only a symbol in such properties as the mind distinguishes. And likewise, the thing symbolized is not altogether symbolized, but only in certain of its properties. It is true that, according to the hypothesis of this discussion, I cannot think anything about the thing except by means of its symbol. But without leaving the ground of this assumption, I can say that one property A of a symbol represents nothing to me of its object and that, on the contrary, the properties B and C do represent something of it. Further, I can see that these two significant properties *do not represent the same thing*. For instance, in a word in italics, the form of the letters represents the general sound of the word, and the type represents an emphatic intonation. The tint of the ink indicates nothing; the pitch or rapidity of sound is not indicated by anything. Thus, each one of the symbolic aspects of the symbol is incontestably attached for the mind to a determinate aspect of the symbolized thing.

There is as much necessity in the agreement of symbolic properties with symbolized properties as there is contingency in the part of a symbol which has no meaning. I have just heard the clock strike four. The sounds are projected, that day, in my visual schematism under the form of four points in a line. I notice that the number of these points can symbolize a certain aspect of what I have heard, what I shall call a manifold. Their order from left to right can stand for another aspect of the sounds which I shall call their succession, provided that the symbols of the different sounds are arranged in a certain manner and not in any other. Finally, if the sounds were each of a different note, the order of the points along the straight line does not represent the order of the pitches except in the particular cases when the pitches of successive sounds are augmented or diminished continuously. In other cases, we need two different alignments to symbolize the order in time and the order of pitches. Or else, if a point is as brilliant as the pitch of the note it represents, we may let the order of brilliance stand for the order of pitch. Once I choose some such representative order, I can change nothing in it. For, as far as the possibility or impossibility of any system is concerned, that is independent of my choice or mental decrees.

Accordingly, my symbolic representations, even if I cannot dispense with them, instruct me nevertheless without blinding me, for they convey the recognition of the symbolic properties and the perception of the agreement of these properties with certain aspects of the thing represented.

What then is this agreement ? What relation between two properties makes one a possible symbol of the other ? Or else, to remain in the realm of symbolism itself: *what relation is there among all the possible systems of the same real property ?*

I can conceive the plurality of the clock-sounds by an infinity of various visual schematisms. What have these systems in common ? Nothing, except the number of their terms. I can conceive the general relation of sensible succession by the order of a straight line, a parabola, a sinusoid, a helix, and still many other lines, disposed in any way whatsoever as images in my mind. What have these figures in common ? Nothing, except the fact of being open and simple lines. We can generalize these examples. Among all the relations susceptible of correctly symbolizing the same aspect of the reality studied, there exists an abstract relation of *formal analogy*. In order for the relation R′ to symbolize the same thing as the relation R, it will be found in all cases necessary and sufficient that these two relations be *equivalent*, that is to say, that they have the same formal properties, or else again, that one can be replaced by the other and the field of one by the field of the other, *salva veritate*, in every proposition containing nothing besides logical or mathematical expressions.*

When I recognize that a certain aspect of reality which I am thinking can be correctly symbolized by the relation R, it is then the formal type of R whose analogy I recognize. Thus only the asymmetrical, transitive, and connected type can properly stand for immediate succession. But the properties of nature which we must think in our objective geometry are themselves formal properties. Therefore, the introduction of any schematism, visual symbolism for instance, only produces the following changes: instead of thinking: " There is in my experience a relation which I name succession and which *is* an asymmetrical, transitive, and connected relation," we shall think: " There is in my experience something that I name succession and which *may be correctly symbolized by* an asymmetrical, transitive, and connected relation "; and so forth for all the rest.

* Cf. L. Wittgenstein, *Tractatus logico-philosophicus* (London, 1923).

The analysis of the notion of symbolic representation then limits the use that the philosopher can make of it in his criticism of the truth of ideas. This limitation is characterized by three facts. In the first place, it is not the concrete image which symbolizes, but one or another of its properties. In the second place, it is not the concrete thing which is symbolized but one or another of its properties. In the third place, the appropriateness or inappropriateness of an intuitive relation to symbolize something, and something determinate, is an intelligible fact which depends only on the formal structure of this relation. These three characteristics constitute the abstract nature of all symbolism, thanks to which the mind, by the instrumentality of the symbol, apprehends the *thing*, and not merely the symbol instead of the thing.

Even though our characters might not have been able to think the relations of their kinesthetic or auditory data except by projecting them against the field of a visual imagination in which geometry is expressed by an intuitive order of position, the geometric order that they have discovered in these data need not have been the effect of this projection. For, confined to the alien sense properties of this symbolic field, geometrical order would not have been able to symbolize anything. That is, moreover, what takes place, in part, in the first geometry. If the notes marking the displacement of the observer along a straight line were projected for him in sections of visual straight lines, that would be purely accidental. For only the open and simple order of this image would symbolize to him a fact of nature. Its rectilinearity would mean nothing to him at all, and the difference in aspect of a straight line sign and a sinuous sign traced on his ideal blackboard, no matter how familiar it is to his imagination, would *represent* to him no character of *his* experience. Likewise, if his trajectory is ramified like a Y, he might imagine it indifferently by a Y, a T, an E, an F, since the structure common to all these forms and to many others is alone symbolic to him of the fact that is intelligible to him.

In our study of objective nature, we do not then have to be preoccupied with any hypothesis about the rôle of any schematism offering an intuitive geometrical order. The preceding analyses are not affected by any such hypothesis; for the schematism employed there is only an instrument of reason. It remains the servant and not the master of reason. It in no way determines the objective facts which it perceives by its means and which are our only concern here.

CHAPTER VI

RELATIONS OF POSITION, SIMULTANEITY, QUALITATIVE SIMILARITY, LOCAL SIMILARITY

(*Visual Data*)

We have met two types of objective geometry: the geometry of succession and global resemblance, and the geometry of relations of position. There exists still a third: the geometry of simultaneity, of local similarity and of qualitative similarity. It is most interesting because it best recalls the methods of physicists. However, let us approach it indirectly. Let us first see what nature would look like were the network of simultaneities, local similarities, and qualitative similarities not isolated, but interwoven with the web of the relations of position which we have just constructed. Relations of position, in fact, actualize the maximum of an intuitive geometric order, the very ideal of an immediate apprehension and science of space. It is worth while observing all that the other adds to it, however; and later, we shall show that relations of position are self sufficient.

Suppose, then, the same physical world and the same spectator as before. But instead of leaving this spectator motionless, as we have done until now, let us place him successively in different places. Suppose, for the sake of greater simplicity, that his power of perception disappears altogether during displacements, like that of a man who is jostled around while his eyes are bandaged. The single scene which formed his total universe then gives place to a second, then to a third—in short, we have a series of ephemeral scenes which follow and replace each other.

Experience becomes open and growing. The world divides into fragments and a pluriverse; what new order does it contain ?

Reality presents itself in massive wholes which succeed one another like so many transitory worlds. Let us call each one of these groups of perceived terms a *view*.

Each view is a space.—The initial scene presented the structure formed by the immediate relations of position of the perceived elements (reduced, for simplicity, to the connection of couples). This structure can be expressed by the formula G ($\mathbf{v}_o$, N) where the function G stands for the group of axioms about points and their congruence; $\mathbf{v}_o$ stands for the initial view, which plays the formal rôle of a class of points; and N stands for the intuitive relationship of connectedness between two couples of sensible elements which performs the rôle of congruence. Now, *within each one of the views* which come after the first, this same structure is found. The formula G ($\mathbf{v}$, N) is true for all values of $\mathbf{v}$. It means that every view furnishes a complete interpretation, a new illustration, an original solution or embodiment of the axioms. It means that every view is a space.

Similarities among elements of diverse views.—We are then confronted with a plurality of sensible totalities having the same nature as views, each one of which embodies the same structure of formal laws by the aid of the same primitive relationship, the connection of couples. But these successive wholes might have nothing in common. The sense particulars which constitute them might *not resemble each other in any respect*. Each view would then be entirely new and original by virtue of the quality of its elements.

However, let us exclude this radical sort of novelty by granting to our subject the perception of *local similarity* and *qualitative similarity* among data of different views (limiting ourselves to the perfect type of resemblance in all this). He then brings together in two ways the elements of any two views. In the one case, such an element *a* of a view **a** resembles an element *b* of a view **b** in quality: both are, we say, exactly of the same shade. In the other case, this same element *a* of the view **a** resembles locally a second element *b'* of the same view **b**—this is quite another type of similarity. Both occupy, we say, the same place in the perceived field. These two kinds of similarities, extending from the elements of past views to those of a present view, temper the novelty of views, making of their sequence various scenes of a play in which the same actors perform on the same stage.

Finally, we may exclude indiscernibles, as in the preceding cases, in order to simplify our problem as much as possible. Let us imagine that no two of the material points which fill his space are of the same shade. Thus, the appearances of the same luminous point in all the views will be united by the simple relationship of perfect qualitative similarity, so that we remove the particular complication which arises from the fact that, in our world, two appearances of the same object are not always similar, and two similar appearances do not always belong to the same object.

Views, perceived places, objects.—Neglecting for a moment the relationship N connecting two couples of terms of the same view, we shall consider *local similarity* (L), *qualitative similarity* (Q), and *simultaneity* (which may be regarded as a *temporal similarity* T).

Perceived elements may be then classified in three ways. Any element *x* belongs to the class of elements which are *temporally* similar, to the class of elements which are *locally* similar, and to the class of elements which are *qualitatively* similar. We have already called the first of these classes the *view of x*. We may call the second the *perceived place of x* and the third *the object of x*. (We are not prejudging by any means the existence or non existence of simple entities having a more natural right to these names, simply because we do not need to do so.—Cf. Part II, chap. ii). Again, we shall call the members of an object its *appearances*; and when a perceived place, an object, and a view have one element in common, we shall say that *this object appears at this perceived place in this view.*

Views, perceived places, objects are the three modes of classifying the similarities of the sense data of our imaginary spectator. The universe is for him the contents of all views. Again it is the contents of all perceived places. Finally, it is also the contents of all objects; for each of these is only one of three ways of analyzing the fundamental group of all perceived elements according to three distinct types of resemblance, respectively.

Fundamental laws.—These three relations which are added to the original relationship of connectedness when nature passes from rest to motion, are going to make for a new development of our science of geometry. The properties that they offer, either in isolation or in combination with each other and with connectedness actually furnish new laws.

Taken one by one, the three relations of local similarity (L), qualitative similarity (Q), and simultaneity or temporal similarity (T) have the same characteristics of *transitivity* and *symmetry* which ranks them in the formal class of perfect *similarities.* It is by virtue of these two characteristics that the relations L, Q, T distribute their terms into homogeneous classes which we have named perceived places, objects, and views.

If we pass to the combinations of the similarities L, Q, T, that is to say, to the conjunctions of views, perceived places, and objects, we obtain the following double law as an immediate consequence of our hypotheses: *Every view has one* *element, and only one, common to every perceived place, and to every object.* Or again: *In every view, every perceived place is present through one and only one datum, and every object possesses one and only one appearance.*

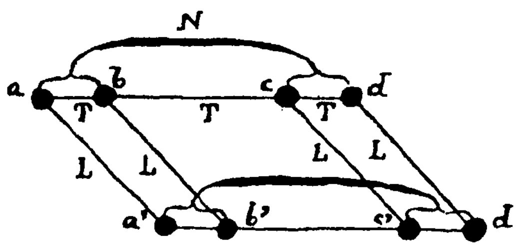
We proceed finally to the laws which join connectedness N with the similarities L, T, Q.

The combination of N with the single simultaneity T has already yielded the law: *for every view* **v**, *we have* G (**v**, N). This asserted that in each view the connections of couples of elements obey the same formal laws as the congruence of points, that is, they form a space. The combination of connectedness N with simultaneity T *and* local similarity L or qualitative similarity Q is going to express connections of perceived place or quality among these successive spaces.

We take local similarity first. Let $a, b, c, d, a', b', c', d'$, stand for any perceived elements.

$$(ab)\ \mathrm{N}\ (cd) \,.\, a'\ \mathrm{T}\ b' \,.\, a'\ \mathrm{T}\ c' \,.\, a'\ \mathrm{T}\ d'.$$
$$a\ \mathrm{L}\ a' \,.\, b\ \mathrm{L}\ b' \,.\, c\ \mathrm{L}\ c' \,.\, d\ \mathrm{L}\ d' \text{ entails } (a'b')\ \mathrm{N}\ (c'd').$$

Or else, in more intuitive terms: *Connectedness is transmitted from one view to others by local similarity.* Represent the perceived elements by points, simultaneity T by a horizontal line, local similarity L by a line descending from left to right

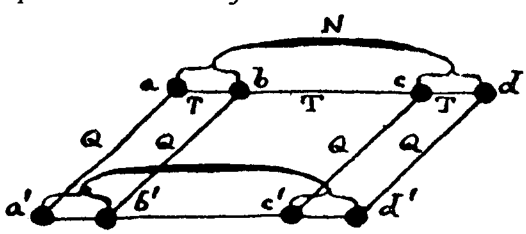


and the connectedness N of two couples of elements of one view by a double bracket. A view is then represented by all the points on one horizontal line; a perceived place and an object are designated by points on oblique lines in different directions. The transaction of connectedness by local similarity is then illustrated by the sliding of the bracket N towards the direction of L.

But all that the spectator has just noticed regarding local similarity, holds equally well concerning qualitative similarity: like couples of perceived places, couples of objects retain their connections in all views. (We must remember that we are dealing with a tri-dimensional field of vision, in which the connection of two couples of perceived elements translates, not the equality of the angular divergence of the corresponding material points, which depends on one's point of view, but their equality of distance, which is invariable, since no motion has been assumed.) We then have a new law:

$$(ab)\ \mathrm{N}\ (cd)\ .\ a'\ \mathrm{T}\ b'\ .\ a'\ \mathrm{T}\ c'\ .\ a'\ \mathrm{T}\ d$$
$$a\ \mathrm{Q}\ a'\ .\ b\ \mathrm{Q}\ b'\ .\ c\ \mathrm{Q}\ c'\ .\ d\ \mathrm{Q}\ d' \text{ entails } (a'b')\ \mathrm{N}\ (c'd').$$

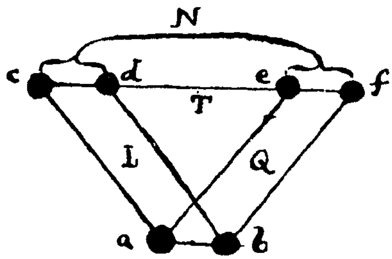
In other words: *Connectedness is transmitted from one view to others by qualitative similarity.*



The transmission of connectedness by local similarity is here translating the fact that the perceived field undergoes no deformation. The transmission of connectedness of qualitative similarity translates the other fact that the group of perceived material points also remains invariant in its internal proportions. But it might also be possible for the whole group of material points to contract or expand *uniformly* with respect to the perceived field. He would then note the following: two couples of objects $\mathbf{o}_1\mathbf{o}_2$, $\mathbf{o}_3\mathbf{o}_4$ which in one view $\mathbf{v}$ have two connected couples of appearances $e'_1e'_2$, $e'_3e'_4$ which are also connected. But the couple of elements $e'_1 e'_2$, appearances of the objects o_1 and o_2 in the view v', *is not necessarily connected* with the couple e''_1 e''_2 of the elements of this view v' which have the same perceived places that the appearances e_1 e_2 of the same objects o_1 o_2 had in the first view v.

We exclude from our hypotheses any contraction or expansion of this sort by assuming an absolutely motionless world. The variation just mentioned then never takes place, and we note the law.

a T b . a L c . b L d . a Q e . b Q f . entails (cd) N (ef).



TRANSMISSION OF CONNECTION BY LOCAL AND QUALITATIVE SIMILARITY.

This law differs from the foregoing laws in one important feature, whereas they can only state how a connection in one view transmits connectedness to another view (either by local similarity or by qualitative similarity), this law goes further. It claims that a certain complex of three similarities, temporal, local, and qualitative is sufficient to determine a connection. It terminates therefore in a connection without having started from a connection. That will enable us later to give up the hypothesis which makes the connection of two couples of perceived elements a primitive relation, and *to define it* instead as precisely this complex of three similarities T, L, Q.

All natural science is contained completely under the principles that we have just seen. Let us summarize them:

Law I. Within each view, perceived elements and their connections possess the properties of points and their congruences.

Law II. The elements of one view have one to one correspondences with another view, either by local similarity, or by qualitative similarity.

Law III. The relation of connectedness between two couples a b, c d, of elements of one view is transmitted to two couples a′ b′, c′ d′ of *elements which correspond locally in any other view.*

Law IV. Connectedness between two couples is also transmitted by qualitative similarity.

Law V. Finally, two couples a′ b′, a″ b″ of elements of a view which correspond, one qualitatively and the other locally, to the same couple ab of elements of any other view are connected.

Let us see now what kind of structure or form results for the world from these laws.

The world reduces to a single view, which we at first assumed had for its only constituents particular sense-data; then we found *the view* which envisaged them altogether and was the whole universe. Now, on the contrary, sense data compose different classes—perceived places, objects, and views—which are with respect to the sum total of experience, elements of a secondary order, more complex and more abstract than the elements which they comprise. Consider the group of these places, the group of these objects, the group of these views each of which comprises in its own manner the totality of the perceived world. What kind of structure does each of these three classes of classes offer to the mind ?

The space of perceived places.—We shall first examine the group of perceived places; by which we understand, it is to be remembered, the classes of sense data formed by local similarity. Take a certain view *v*. Each perceived place has one element in this view; conversely, each element of this view has its perceived place. The subject can then think any perceived place as *the perceived place of this element,* and their group as *the group of the places of elements perceived here.* He can regard the group of the elements of the view **v** as a picture of the group of perceived places, each element representing the perceived place of which it is a member.

Now these same elements form also, by the connections of their couples, groups possessing the formal properties of equal distances, and starting with these, they also possess the properties of straight lines, circles, spheres, and all the aggregates of points with which geometry is concerned. The mind might conceive the fancy of introducing this classification of the elements of the view **v** into the perceived places that they represent, by classifying as "*connected*" the couples of places represented by any connected couples of elements. To the straight lines, circles, spheres formed from the elements that the view **v** unfolds before the mind would then answer other "straight lines," "circles," "spheres" formed from the perceived places that these elements represent. The geometry of the elements of the view **v** would be reflected in a geometry with more complex terms, in which *points* are translated no longer by *elements of the view* **v**, but by *perceived places* of which these elements are a part. *Congruence* would be translated, no longer by *the connection of couples of elements of the view* **v**, but by the *relation of couples of perceived places* which is defined as occupancy in the view **v** by connected couples of elements.

Fancy, we were saying. In fact, the order of the connections of the elements of the view *v*, taken now as representatives of their perceived places, does not really belong to these places themselves. In an instant, the view **v** will have made way for the view **v**′: will the couples of perceived places embodied in **v** by connected couples of elements remain still connected in **v**′? Varying with the view chosen, these "connections" of perceived places will not characterize these places. The connections would remain accidental in relation to these places, if, as fixed sets of relations, they were deprived of that independence of its various members which defines the essential property of a class.

What would be necessary to make the "connections" of perceived places, defined by the connections of their members in a view chosen at random, hold characteristically of the order of these places themselves? We have hinted at it: it would be necessary for these relations to become *independent* of the choice of view. Now *this independence is exactly what the law of transmission of connection by local similarity affirms.* The relation thus defined between two couples of places really characterizes these couples themselves. But it reflects faithfully the connection of the couples of the elements of any view: and connection in its turn reflects the congruence of couples of points.

Let l stand for the class of perceived places, and N_l for the relation of two couples of these whose members in any view form two connected couples. We then have

$$G(l, N_l)$$

that is to say, *perceived places and their relation N_l form the points and congruences of a space.*

The space of objects.—The same manifestly holds for *objects*, or classes of sense particulars grouped by qualitative similarity. First, every object has, like every perceived place, a member or appearance in each view, and secondly, the connection of couples of particulars is transmitted from one view to others by qualitative similarity, by means of law IV. Therefore, among couples of objects there exists also a relationship of "connection." N_o, consisting of the connectedness of the couples of appearances of these objects *in any view*, which reproduces exactly the formal characteristics of the connection of the data of a view, which are equivalent to the formal characteristics of the congruence of points. Designating the class of objects by o, we have

$$G(o, N_o)$$

That is to say, *objects and their relation N_o form the points and congruences of a space.* Thus, geometry is illustrated or embodied not only by the sense particulars interior to each view, and not only by perceived places, but also by objects in their own way.

The space of a family of parallel views.—There remains for examination the order of *views*. But here, things are not as simple, and they change their aspect a little.

If we place ourselves again, with our spectator, in contemplation of a particular view **v**, can we, first of all, have any intuitive representation within this same view of preceding views ? Let us take three objects $\mathbf{o}_1$, $\mathbf{o}_2$, $\mathbf{o}_3$ which are actually presented to us through the three elements e_1, e_2, e_3 (not in a straight line), and let us ask at what other perceived places in any preceding view **v**′ we would find the appearances of these same objects $\mathbf{o}_1$, $\mathbf{o}_2$, $\mathbf{o}_3$; then let us note the elements e'_1, e'_2, e'_3 which have these places in the view **v**. In fact, it follows, as a result of law V, that if, in two views, three objects not in a straight line appear in the same places, *all* the objects appear then in the same places, and we consider two views of this sort only as a single one.* Each preceding view **v**′ is then exactly representable in the view **v** by three elements e'_1, e'_2, e'_3 which have in the view v the perceived places that the appearances of the three selected objects $\mathbf{o}_1$, $\mathbf{o}_2$, $\mathbf{o}_3$ had in the view **v**′.

But we can simplify the situation again, by supposing that

* This notion of indiscernible views alters the meaning of the term *view*: instead of designating a class of simultaneous data, it designates henceforth only the *content* of this class, that is to say, the way in which qualities are distributed among perceived places.

all the displacements of the subject are *translatory*. An object then cannot appear in the same perceived place in two views without the two views being identical; each view is therefore sufficiently characterized by the perceived place of the appearance, no longer of three objects, but only of one. Take an object **o** in the view **v**. Every other view **v′** (in this simplified case) will be represented in **v** by an element *e′* marking the place of the appearance of the object **o** in **v′**, *the view in which this object will appear* in that place. (As to the view **v** itself, it is clear that it is represented by the selected object's own appearance.) On the other hand, the preceding views always being finite in number, there would be far too many elements in the view **v** to represent each one of them.

But our observer now realizes an essential difference separating the view of perceived places from that of objects. From the outset of his experience he has been aware of all perceived places and of all objects, which accordingly can be increased only by the introduction of new members. On the contrary, views come forward each one as an individual and original whole. He should therefore consider future views if he wishes to consider the whole group of possible views. He cannot predict them, since we have not submitted his translatory movements to any rule. But he can, by falling back on the analogy of views already experienced, reject certain conceivable views as contrary to the laws of nature. Such would be the case for a view in which an object would appear at the same place as in another view, whereas another object would appear at a different place. The views which remain, not excluded by any law, constitute all *possible views*. They may be represented in the present view **v** in the same way as past views, by the elements of this view **v** whose places would be occupied by the appearance of the object **o**. Now we find that there are exactly as many views possible as there are elements in any view, so that each element of the view **v** thus represents a view.

We have now brought views to the same level of analysis as places and objects. We have only to repeat considerations that are already familiar to us. We can make the elements of the view **v** bear the same classification into connected couples (and consequently, into "straight lines," "circles," "spheres" etc. . . .) on the basis of the various possible views which these elements represent, by agreeing to say that two couples of views are "connected" when the elements of the view **v** which represent them form two connected couples.

But this relation would characterize two couples of views only if it were revealed independently of both the view **v** and the object **o** selected as the basis of the representation of all the views.

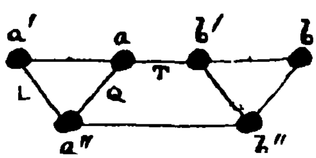
Now this is exactly what does take place.* Let us then form the relationship N_v of two couples of views which consists of the fact that for any object, two couples of the perceived places of its appearances in these views have the relationship N_l (for that is what the thing amounts tc); or in other terms, in the fact that two couples of views displace the appearances of all objects by the same distance. If we call v_1 the group of views, we have

$$G(v_1, N_v)$$

* This is possible, with respect to the view, through the transmission of connectedness by local similarity (law III), and, with respect to the object, through the new law:

$$aTb\ .\ aTa'\ .\ aTb'\ .\ a'La''\ .\ b'Lb''\ .\ aQa''\ .\ bQb''\ .\ a''Tb'' \text{ entails } (aa')\ N\ (bb')$$

which expresses the restriction of the movements of the observer to translations, by positing law VI: The appearances of all objects are displaced by the same distance in going from one view to any other view.



that is to say, *views and their relationship N_v form the points and congruences of a space.*

Formation of a total space of views.—But we have considered until now nothing but views oriented in the same way. We may raise this restriction by introducing into the experience of the spectator views of any sort of orientations caused by displacements of his body that are no longer confined to translations. The views that he considers as possible then form a multitude of families v_1, v_2, . . . of views which are oriented in the same way. Each one of these families may be easily defined by the persistence of law VI within each (equal displacements of the appearances of all objects). Each one of them constitutes a space, since we have $G(v_1, N_v)$, $G(v_2, N_v)$ etc.

All these spaces of views offer a unity which enables us to synthesize them into a single space. For it is possible to divide the totality of views into such classes that each view comprises one and only one view of each one of the spaces, and that in all the spaces the geometric relations of the views of the same classes are the same. In intuitive terms, it is possible to apply all the spaces of views one to the other in such a manner that their networks of order coincide. In fact, the spectator, selecting an object **o**, can class together the views in which this object appears at the same perceived place: the classes of views thus constituted have the required property. If we form the relationship N_t of two couples of these classes which consist of the fact that two couples of views which belong to them in any space of parallel views have the relation N_v, these classes constitute in their turn the points of a space whose relationship N_t is equivalent to congruence, and in which any possible view is situated. Let t be the group of these classes; we have again $G(t, N_t)$.

But this total space of views suffers from one defect that we have already noticed in the total space of the movements of a previous chapter: it is formed very arbitrarily. It rests on the selection of an object **o**, which is not determined by reason. If we change our choice, the classes of views which are its points are undone and have to be reconstructed differently. Thus, this total space of views has no natural existence for such an observer.*

We shall perceive a different state of things when we bring his vision down to only two dimensions.

* This same type of indetermination of total space will be found again in mechanics, where it corresponds to the restricted principle of relativity.

CHAPTER VII

REFLECTIONS ON THE PRECEDING UNIVERSE

In a universe like the foregoing visual one, geometry is completely illustrated a great number of times. It is integrally embodied in the first view met: its *points* are the *elementary data of this view*, and *congruence* is the original relation N that we have named *connection of couples of these data*. Then, it is embodied again in each one of the following views, successively taking for points and congruence the elementary data of each view and the connection of their couples. We may place all these interpretations, which are as numerous as there are views, in one class: for they have the same relation N for congruence, and terms of the same sort for points.—Next, geometry is illustrated in the manifold l of the *classes* of sense data grouped by local similarity, namely perceived places. These classes are its points, and the relation N_l of two couples of perceived places, whose members form two connected couples in every view, is the meaning of congruence for this geometry.—Again, geometry is illustrated in the manifold o of classes of sense data grouped by qualitative similarity, namely objects, taking these new classes for its points, and the relation N_o of two couples of objects whose members form two connected couples in every view for congruence.—Geometry is illustrated also in each group v_1 of possible views having the same orientation (that is to say, coming under law VI): it takes these views for points, and for congruence the relation N_v of two of their couples in which the appearances of any object have the perceived fields whose couples have the relation N_l.—Finally, geometry is illustrated in each total group t_1 of the classes of views in which the object o_1 has its members in the same perceived field: points, then, are these classes, and congruence is the relation N_t of two of their couples which "cut" every space v of parallel views into two couples of views having the relation N_v.

This visual universe gives, therefore, to the system G of the axioms of point and congruence the following solutions: *for every view* **v**, G(**v**, N); G(l, N_l); G(o, N_o); for every group v, G(v, N_v); for every *group* t, G(t, N_t). The order G appears first as the order of the simple relation N of connection within a view. Then, it is extended to the totality of the universe, considered as a group of perceived fields, as a group of objects, and finally as a group of possible views. The plurality and diversity of these perceivable spaces would be to the mind of the fictitious creature we have imagined, the "matter" of philosophy.

Irreducibility of these diverse geometrical orders to one single fact.—He would observe the various orders differ in their epistemological nature. The space interior to each view offers an intuitive order without succession. On the other hand, the space of objects and the spaces of views involve the comparison of successive views within the relations N_o and N which order them. They are grounded on laws IV and V which state that perceived material points remain immobile: they are laws of physical nature without any intrinsic self-evidence. If the order: *for every view* **v**, G(**v**, N) can be to our subject a geometry in the old sense of self evidence, the orders G(o, N_o) and G(v, N_v) are to him only a physics that is purely inductive.*

* As to the order G(l, N_l) of perceived fields, it will be successive and inductive, or on the contrary, timeless and intuitive, according to whether one rejects or adopts the hypothesis which analyzes the local similarity of two successive sense-data into an identity of "*ingression*" by the same simple entity, invariably present as their individual perceived field. In this hypothesis, in fact, the relation N of the connection of couples of sense data is referred to a relation n of individual occupied perceived places, and the transmission of the relation N of one view to successive views by local similarity, the basis of the order G (l, N_l), ceases to be an hereditary principle and becomes a logical identity.

It is often believed that in the geometric structure of nature, there is something both intuitive and elaborate; for geometric order is found both in the arrangement of the parts of an instantaneous perception and in the comparison of successive perceptions. But this use of intuition being very imperfect in us, it is natural to seek in its very imperfection the reason why the geometrical structure of the universe involves for us a share of both succession and assemblage. However, we have just defined and posited a perfect spatial intuition, and as a result other spaces, based on laws of succession, have appeared only more clearly. By eliminating the source of the accessory complications due to the senses, we have only shown more clearly the fundamental plurality of the spaces which order the particulars of one view, the perceived places, the objects, and the views themselves in the simplest visual universe possibly conceivable.

But why not suppose that all these spaces still derive from the simplest one among them that would intuitively contain the whole geometry of nature? Eager for unity, the mind tends to reason in the following way: all nature is reduced to sense-data; now every sense-datum enters immediately into the intuitive geometry within perception; the same is true of any collection of sense-data; consequently, it is certain that objects and views will in their turn be ordered geometrically.

Such a tendency of thought is highly confused. For the class of particulars which form any object or view spreads over the whole perceived field and fills it completely, so that objects or views, being everywhere present, derive no principle of order from the mere fact that they are given.

Or else we might say again, by means of a more indirect argument, if I try to analyze the group of objects or possible views, I can dispense with the intuitive group of the *heres* and *theres* of the perceived field only with great difficulty;

consequently, the order of this field cannot fail to accompany objects and views in my analysis.

But we know that even if the hypothesis of this argument were true, its conclusion would not follow. It presupposes both that every order of symbols has reference to things, and also, that this order cannot represent anything to the mind. If in the universe studied, objects and views possess in fact a geometrical structure analogous to that of the parts of one view, and symbolizable by the latter to this extent, the reason is not in the tyranny of this first intuitive order on the imagination, but only in the particular laws of the succession of views. The latter are inductive and not self evident.

So even when perceived nature is limited to vision and extremely simplified, the plurality of spaces already claims the last word. The imagined subject cannot in any noteworthy way discover a sovereign space, but only intermingled spaces whose terms, ordering relations, connection with time and with intuition are all distinct.

The space of views.—Of the four types of space that our spectator knows, the space of one view and even again the space of perceived places are forms that may be called intuitive and personal; the space of objects and the spaces of views are, on the contrary, empirical and impersonal forms, provided we do not press the quite vague meanings of personal and impersonal. The spaces of views, which are perhaps less familiar than the others (for a reason that we shall see presently), are particularly instructive. For the opposition between the order of views and the order of one view dispels the error that arises from reasoning on the nature of the order of space as if that nature were unique.

What really is a point in a space of views ? It is an entire view, that is to say, the entire content of the perceived field. It is a spectacle which is already in itself a complete space such as geometry presupposes. However, these successive spaces, through the similarities and differences of their contents, are in their turn ordered like so many points according to geometrical form. Thus the intuition which comprehends the points of a view with all their *here-there* order is at each instant enclosed within a single point of the space of views: the order of these points cannot appear to him more perfectly.

The geometry of nature brings to intuition two contrary relations. One is the classification of the internal details of an instantaneous perception. But the other is found—independently of the first—in the reflective classification of each perception, taken as an individual whole, in the group of possible perceptions or incompatible aspects of the world. Each one of these perceptions can only fill the whole of its spreading sensorial extent, and thereby the imagination, without leaving anything over. Geometrical order extends beyond the confines of a moment's total perception and applies to it in its turn a group, which can no longer be intuitive, of all total perceptions, in order to find a place for the moment's total perception in the series of all total perceptions. Thus, any universe in which there exists a space of views rejects the philosophy which sees "space" only as a structure internal to each view.

Likewise, the consideration of the spaces of views appears fatal to the thesis of the subjectivity of "space," ordinarily associated with its intuitive character. In fact, a space of views does not order only the successive perceptions of the same subject, but also the simultaneous perceptions of different subjects.

Let us imagine our character able to penetrate directly into the consciousness of other beings like himself, perceiving the same universe. He would take cognizance of the views present to these other minds, views different from the one he actually perceives, but similar to certain views he regards as possible. He would then locate the perceptions of his friends in the geometry of possible views just as if they were his own. For instance, three of these perceptions might be on a straight line, or else on a circumference with a fourth for a centre. This geometry of possible views has therefore formed the architecture of a common world in which the various visual worlds of individuals are coördinated by virtue of their differences according to rule.

It may be noted here that Kant, by maintaining the subjectivity of space (undoubtedly having before his mind some ideal intuitive and personal space analogous to the perfectly geometrical perceived field of our example), realized the danger of the conclusion that there were as many spaces as there are subjects. He was keenly, although obscurely, taking into account the fact that there is only one space for all of us. But he did not dream of distinguishing several spaces of opposing characters. He carries over the feeling that he has of the existence of the same space for all men to that space immediately intuited by everyone, whose subjectivity he thinks he clearly sees. What a strange and almost mystical use he makes of the notion of an impersonal subject! It seems he wants us to think: "Space is formed by a pure intuition of the subject, but this intuition does not reflect any of the particularities of the empirical ego: its subject is not Peter or Paul, but only man in himself. It is therefore a unity, and consequently, space is one and the same for all men." The conclusion is fallacious for all that is proven is the *perfect resemblance* of your intuitive space and mine, but not their *numerical identity*; since space is attached to the activity of the subject, there are two spaces as we are two subjects, and there is no way of recovering unity on these grounds.

Now, what establishes geometrical order among the views apprehended by various subjects (each view completely filling its personal space), is not the perfect resemblance of these spaces, but actually the systematic *differences* of the perceptions they have as contents. In order to coördinate all the views into a single space, it is not sufficient that they have similar *a priori* frameworks. They must have in addition different perceived contents; and different according to the complex and exact rules of perspective. It is these differences in your perception and mine, each one in his own private space, that constitute the single space in which we are both situated. Is it not manifestly an error to say that the existence of these intelligible divergences among the perceived *contents* present at the same moment to diverse subjects results from the fact that their minds are *similar* and that the *form* of space is the same for all ?

Leibniz, father of the idea of a geometry of the perspective views of the universe, has seen better than anyone else the double nature of the order of space, personal and subjective under one form, impersonal and universal under another. Spatial relations appear to him to connect the simultaneous objects of each monad's perception, and these same relations do not prevail among the monads. But the monads have nevertheless an analogous order, each one having at a given moment and by virtue of its complete perception a characteristic *point of view*. The quality of the space of one view and the space of views is clearly shown here.

Independence of the order of views and the order of objects.— There is still another question which is instructive to ask. Of the two empirical and impersonal forms of the geometry of nature in this case, the order of objects and the order of views, why is the second less familiar to us than the first ? That is because our vision in practice distinguishes only two dimensions. Indeed, the result of this limitation is to attach to every possible view of an object a particular relation that is expressed by our habit of saying that this view is *what is seen of this object*—of the top of this tower, of the edge of this lens. We shall examine presently of what this relation of a view to an object consists. But it is so natural that we always conceive the geometrical order of views identical with the order of objects in which they are situated. If we ask a physicist what he understands by the point of view of a certain observation of the universe, he will probably have no other reply than the following: "I understand by the point of view of an observation the place of the body of the observer in the group of other bodies."

But in our universe, such an answer is impossible. For none of the views has any distinct relation to any of the objects. Our subject cannot therefore designate any view as "the view tied to this object." In spite of that, a geometry of views has been discovered, thanks to which each view is placed in the group of the world considered not as the group of objects, but purely and directly as the group of views. Thus, this first visual universe shows that there can exist a geometrical order of the views of the world without these views being localized here or there in the group of bodies. This possibility is important, for even in a world where views are narrowly tied to bodies, it would be possible for the order of views to predominate over bodies as the more fundamental of the two.

CHAPTER VIII

ELIMINATION OF THE RELATIONS OF POSITION

In the preceding universe, we started with an intuitive order of data within each view; then the same type of order was found in the relations of perceived places, in those of objects, and in those of views. But each one of the relationships N_l, N_o, N_v, ordering these diverse groups in the manner of geometrical congruence contains in its definition the intuitive relationship N of the data of a view. What we have done therefore is made an *extension* of the intuitive order of each view, which we employed as a central element, to places, objects, and views. This central element was necessary but not sufficient in the construction of the other orders.

We may remove this initial order or neglect it without affecting the other orders, so that their dependence on it is only apparent and even possibly illusory.

Let us go back to the instances we gave of this initial intuitive order: the alignment of three stars, the equal divergences of two couples of stars. We seemed to see in these a simple original detail of the visible heaven. But on the other hand, all the relations of this sort among diverse parts of one view might be complexes and not simples; they might be composed of relations really due to comparing the present view with past views.

Imagine two spectators, one perceiving the relations of position of the data of each one of his views, the other not perceiving them. For the first, such couples or trios of data within one view as equal divergences or the property of alignment present a direct similarity. The second spectator on the contrary, does not discern among these same groups of data any particular relationship. The similarity or difference of two couples or trios is indifferent to him. He cannot understand the classification of couples into equal and unequal, of trios into aligned and not aligned, such as his analytical companion establishes. He is like a man who would hear individual sounds perfectly, and recognize them with accuracy, but would be tone-deaf, and would not catch any difference among the groups of three sounds forming a perfect chord and the groups of any other three sounds.

But this man who is tone-deaf might discover a relation among the notes which the musician tells him form a perfect chord. He might even anticipate with certainty his judgment about the three notes heard the first time, by counting the number of their vibrations and observing their relation. In the same way, our spectator without perception of the similarities of shapes can learn, even more easily, to discern the shapes which his companion would declare analogous.

It is enough for him to observe the operations that the other executes with a view to *verify* the accuracy of his own immediate judgments. To make sure that three stars appear well in line, he finds out whether they are disposed evenly along the edge of a pencil, and to verify the suitability of his standard, he finds out whether the edge of the pencil when seen from one end reduces to a point. That amounts to saying that three particulars a, b, c of a view are aligned if their perceived places A, B, C can be occupied by the simultaneous appearances of three objects α, β, γ such that there exists a view in which the appearances of two of these objects are both hidden by the appearance of the third. Now that is a complex relation composed of local similarities, qualitative similarities, simultaneities, and successions which the second observer has to analyze as such in order to apprehend it at all, for he cannot, by hypothesis, have it directly.

His companion similarly verifies the equal divergences of two couples of stars by successively covering both with the same segment-points of the pencil held out at arm's length. If his visual memory were perfect, it might even be enough for him to turn his head so that the second couple of particulars would be in the perceived places occupied formerly by the first couple; if the operation succeeded, the equality of the divergences is verified. But there again, we have an experimental relation which the second observer can apprehend as well as he.

It is not even necessary to execute any movements: the experiment may have been done once for all. In fact, if three data of one view are in line, the three data of any other view occupying the same perceived places are also in line; equality of divergences enjoy the same property. Hence, the places of the visual field may be classified experimentally once for all according to their alignments and the equality of their divergences. Our second spectator might then learn to judge, in the presence of a new view, the similarity of the shapes which are found there, as distinctly and as immediately as the other.

If we meet these two spectators after the end of this apprenticeship, we shall not be able to discern any difference between them: for by seeing them identify the same shapes with the same words, how shall we guess that they are judging relations in very different ways? Perhaps, they themselves do not suspect these differences, at least until they begin to explain to each other in very precise terms what and how they are judging.

" By alignment or equality of divergence," one would say, " I mean relations which are simple natures and do not refer to anything beyond the present."

The other would answer: " I do not know what to think of what you say, for I myself mean by these words relations which are very complex, formed from qualitative and local similarities, simultaneities and successions, relating present data by the mediation of past events whose scene they occupy. Far from the relation of two trios in line or two equal couples being a direct and self evident resemblance, to me it is only a relationship consisting of certain past facts, just like human ancestry. But are you not deluding yourself? Only your actions have taught me the experimental meanings of these two terms. Are you sure you yourself have not learnt it in the same way, but that you have forgotten it, now that all the necessary experiments are performed, and you know by heart what groups of your perceived places they characterize?"

"I really seem," the first observer would say, "to apprehend among these groups direct relations which have no historical content. But they are indeed invariably duplicated by the indirect relations of which you speak. I myself use these only when I am in need of a very precise statement (when for instance, I wish to construct a map of the sky) because the observations which constitute them are more exact than my simple perception of alignment or equality."

"Agree then," the other would reply, "that this simple perception which I do not share with you does not serve any important use. It adds to the picturesqueness of the world without adding to ts structure, since other relations which you yourself say are more precise, yield the same classification of appearances."

Which of these two observers do we most resemble? We appear to have some primitive feeling of the similarity of shapes, but we trust only in the indirect comparisons of real or apparent displacements of objects. The intuitive relations of position, if they exist, are not therefore necessary to the scientific order of nature; that is what we have already observed in the simplified nature just analyzed.

In the foregoing case, relations of position are represented by the relationship N of intuitive connection among couples of data caused by congruent couples of material points. Now this immediate relation is found duplicated by a complex relationship formed from qualitative, local, and temporal similarities. Indeed, law V says that two couples *ab*, *a'b'* of data of a view are always connected when *a* and *b* have the same perceived places, and *a'* and *b'* have the same objects as a couple *a"b"* of data of another view. Conversely, no law excludes the existence of a view in which the places of *ab* are taken by appearances *a"b"* of the objects of *a'b'*, provided exactly that the couples *ab*, *a'b'* are connected (Figure, p. 150). In the group of possible views, this complex relationship *N* is then equivalent to the simple relation N, since it characterizes the same couples of data.

Let us deprive our subject of the relation N and any other relation of position among diverse groups of data: there will remain nevertheless the multiple geometrical structure that we have analyzed, in which *N* takes the place of N. It is true that time enters more than before; for instead of characterizing these same couples of data by a self evident relation, it is necessary for him to await the experiences favourable towards comparing them. It is also true that he no longer knows any simple version of geometry in nature. All the meanings of congruence—and of any other geometrical relation—are now complex relations formed from local, qualitative, and temporal similarities which put into play several successive perceptions.

CHAPTER IX

THE GEOMETRY OF PERSPECTIVES*

LEAVING to one side all relations of position, let us limit ourselves to local, qualitative, and temporal resemblances. We may approximate one step further towards our own nature by studying the universe of a visual being for whom the diversity of places perceived no longer answers to the diversity of the places of the bodies perceived, but only to the differences of their directions. Such in fact is our so called visual distance which alone is correct enough for science. For it is true that we have a feeling of the difference of two visual data which are appearances of bodies lying in the same direction but at different distances. When these bodies are near, this feeling is much too vague to be utilized in observations.†

We shall show then what idea of the geometrical structure of the world an observer would have, were he reduced to a visual sense of two dimensions; to him life would be like a moving-picture show, or even like a magic lantern show since we are suppressing all perception during displacements. This finer differentiation of the order of the data of each view engenders new features, essential to our own universe, and visible here in a very simple setting.

We cannot posit a plenum of visible matter, for a two-dimensional vision would be blind in it. It requires transparent spaces. Suppose then six material points, like six

* Cf. B. Russell, *Our Knowledge of the External World*, ch. iii.

† The thesis according to which visual data would be all "at a distance," without further *differentiation*, appears to be confused. In any case, allowing the subsistence of identity among data proceeding from bodies lying in the same direction at varying distances, this hypothesis does not annul the bidimensional character of the visual manifold.

stars of different hues in a dark sky. These six stars are not in the same plane, and three of them are in a straight line; this last assumption, which simplifies matters very much, may be dispensable.

The observer to whom we show the spectacle of these six stars by successively and irregularly placing the material point which is his body in all sorts of positions, perceives, as in previous cases, the local, qualitative, and temporal resemblances of his perceived data, which are classified as before into *perceived places*, *objects*, and *views*.

The three objects **a**, **b**, **c**, which answer to the stars in a straight line offer this particular relation: namely, the appearances of two of them are both *absent* from certain views which we shall call (**a b c**) views. We would say that two of these three objects are hidden by the third, or else that these views are sighted along the straight line **a b c**.—The three objects thus related, when they appear all three together can only be found in certain trios of perceived places. We shall call these trios *aligned*, and *alignment* the class of the places aligned with two given places or with two of the places aligned with the latter.

An alignment is an appearance possible of perception by an observer placed in this plane and endowed with a vision that surveys all directions at once: it is the circle of the horizon. In fact, if we compare the visual field to a sphere, the alignments on it are great circles.

This symbol of a sphere is convenient; we shall employ it in our figures. But it goes without saying that a perceived field, not being an object, cannot possess a form that can be traced or modelled.

Consider two views V, V_1 separated by a displacement without translation of the observer. When we pass from one to the other, the appearances of the objects slide along alignments converging in two perceived fields a, a' (one of which is the point on the horizon towards which we are walking, and the other the opposite point from which we start out). We shall say that two views that are related thus are *transformable into each other according to a, a'*.

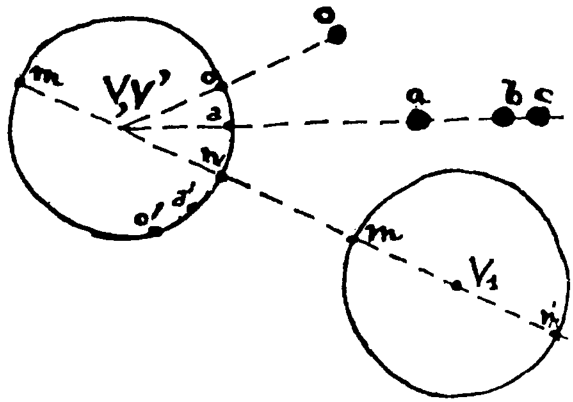
However, leaving the observer at the point of view of V, let us make him turn about in place one half-rotation around the straight line joining this point of view to that of V_1: the new view is transformable into V_1 according to the same places *a, a'*. Indeed, all the appearances have simply turned in a semi-circle around *aa'*: they therefore appear again on the same alignments intersecting in these two places.—Thus, two transformable views are either identical orientations, or separated by a semi-rotation around the straight line joining their viewpoints.

In this last case, any view transformable simultaneously into one and into the other, should be found on this same straight line *aa'*, since the transformation is made according to these places. The identical orientation of three views of which at least one and at most two are (**a b c**) views—this to assure their not being all three in a straight line from the same point of view, a case in which the following would not apply—is then expressed by the condition that they are transformable two by two according to three different couples of perceived places: we shall thus define *a family of parallel views.*

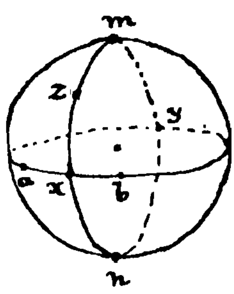
But before we proceed further, this double relative orientation possible among transformable views is going to serve in the definition of symmetry, of equidistance, and finally, of perceived places.

Let V be an (**a b c**) view transformable into a view V_1 which is not an (**a b c**) view, according to the couple of places, *m, n*. There is another view V', and only one, which equally satisfies this double condition: it is the view obtained from the same point of view as V and separated from V by a semi-rotation around the direction V V_1.

The two (**a b c**) views V and V′ thus tied by their common transformability into some view V_1 which is not an (**a b c**) view according to the same couple of places *m*, *n* are then symmetrical around *m*, *n*: the places *o*, *o′* occupied in these views by the appearance of the same object **o** will be said to be *symmetrical with respect to m and n.*



All the couples of places *x y* such that *x* and *y* are symmetrical with respect to *m* and *n*, *m* and *n* being inversely symmetrical with respect to *x* and *y*, form part of the same alignment which we shall call *the equator of m and n.* (See Figure below.)



Assume an alignment through *m* and *n* cutting their equator in *x*: every place *z* on this alignment will be said to be *equidistant* from any couple *a*, *b*, of places on this equator that are symmetrical with respect to x. In general, if any intermediate term of a series of places $p\ q\ x_1\ x_2\ .\ .\ .\ x_n\ r\ s$ is equidistant from its two neighbours, the extreme couples pq, rs will be called *congruent*.

The properties of these alignments and of these congruences determine completely the order of the group of the perceived places of our observer, or in other terms the structure of his field of perception. But this structure has ceased to be that of Euclidean space: the alignments of the places perceived possess only the order of the great circles of a sphere; the relation which we have just called their congruence illustrates only the congruences particular to the points of a spherical surface. If we continue to call a group of terms and relations satisfying the set of Euclidean axioms, "*space*," then the group of perceived places no longer forms a space.

The group of objects will have also lost this property. Whereas the preceding universe offered as many objects as geometry requires points, the present one includes only six, and a group of six terms does not evidently form a space.

What then is a space in this universe? What terms have the profusion of the points of geometry, what connecting ties have the properties of its relationships? Must we conclude that the data are reduced too much, and that a two-dimensioned visual experience is not sufficient to illustrate the complete structure of a space? No, for if the space of perceived places and the space of objects have disappeared, the spaces of views have persisted.

Consider the family of the views parallel to a given view, according to the definition which has been formed before. The views correspond exactly to what we call the points of space; for to each point corresponds that one of the views which represents the six stars seen from that point as lying in a common direction. We shall show that certain relations of these views reflect in the experience of our observer the relations of rectilinearity or of distance from their points of view.

If a view X is transformable into two views A and B according to the same places *m*, *n*, let us say that it is part of the straight line A B.

If a view X common to three straight lines *d*, *d'*, *d"*, is transformable into views of *d* by means of *m*, *n* and by means of *m'*, *n'* into views of *d'*, and into views of *d"* by means of *m"*, *n"*, one of the places *m*, *n* being symmetrical to one of the places *m"*, *n"* with respect to one of the places *m'*, *n'*, we shall say that the straight line *d* is *the bisector* of the two others.

If, in this definition, we identify *d"* with *d*, *m"* and *n* with *m* and *n*, we shall say that the two straight lines *d* and *d'* are *perpendicular*.

If the four views A, B, C, X are such that the straight line AX is the bisector of the straight lines AB and AC and is perpendicular to the straight line BC, we shall say that the view A is *equidistant* from the views B and C. Finally, if any intermediary view of the sequence P Q X_1 X_2 . . . X_n R S is equidistant from its two neighbours, we shall say that the extreme couples of views PQ, RS are *congruent*.

It is evident that the relation of two couples of views thus defined possess all the properties of Euclidean congruence, since it translates the equality of distance of corresponding view-points in the universe under consideration. It is remarkable that the geometrical structure of views survives the disappearance of the geometrical orders of perceived places and objects; for, it is not necessary, in order to make a space explicit in experience, to attach to each point a perceived locality or a thing, but only an aspect of the world.

But we have only considered until now a single family of parallel views; there are others, and each one of them forms a space of which these views are points. The group of the views therefore constitutes a multitude of spaces, without our seeing yet whether all these spaces are composed of only one "Space." The universe of the preceding chapter, made of three-dimensional views, already presented this state of things. However, a new fact arises.

The space of "points of views."—Before, we were able to superimpose the diverse spaces of parallel views, by selecting a certain object, then by grouping views according to the perceived place that its appearance occupies there, putting into the same class all the views where this place is the same. The classes thus formed comprise all by themselves separate views. Each family of parallel views is distributed among the classes as a class view, and the geometrical relations within these families cover each other exactly: if, in a family of parallel views, the views AB, CD being to the classes α, β, γ, δ are congruent, the same holds for all the other families. We have then been able to regard these classes of views as themselves forming a total space deriving its structure from the confused orders of all families of parallel views.

However, this total space was constructed very arbitrarily. In fact, the classes of views that it has for points are to be defined by means of a certain object, and if the object is changed, these classes are not the same any more. For two views which fall in the same class when we consider the place of the appearance of an object **a**, enter different classes when we adopt as a basis of classification the place of the appearance of an object **b**. Thus, the total space that might be formed by superposition of the spaces of parallel views had no natural existence in that it depended on an arbitrary selection.

It is exactly this sort of limitation which disappears in a universe having only two dimensions. Here classes reuniting the views of all orientations in order to furnish the points of total space, are formed in a unique manner by a relation free of any arbitrary reference. A tri-dimensional view has no more of a *viewpoint* than a statue: on the contrary, a bi-dimensional view has its point of view like a picture, and the views of all those orientations having the same point of view are united by a general relationship among their contents. In all these views, the angles of vision which separate objects remain the same. To each one of the viewpoints in which we can place the body of our observer in what we call empty space, something corresponds in his universe: a class of views or the appearances of any two objects occupy congruent couples of perceived places. We shall call a class of views defined thus a *point of views*—if we may be permitted to extend our terminology in this way.

As in the previous case, these classes comprehend all views; each space of parallel views is distributed among them, and the geometric relations of all these spaces are superposed, engendering a total space. But whereas, before, the classes in question were not well determined because the arbitrary choice of an individual object rendered their formation artificial, the grouping of views into " points of views " is on the contrary unique and based on a general relationship.

Our observer for whom views form only spaces will see in the order of " points of views," the most comprehensive space which contains and connects all the others. It will be to him the geometrical structure of the world, the " Space " which orders all phenomena.

But besides the formation of a single all embracing Space that includes the group of views, the hypothesis of a visual field reduced to two dimensions introduces a second characteristic feature into our world; it attaches each visible object to a particular " point of views."

Attachment of objects to " points of views."—In the case of tridimensional views, we have noticed the absence of any fixed connection between the group of objects and the group of views: no view or class of views was linked to one object rather than to another by any distinctive relation. Objects formed a space, and so did views, but these two sorts of space were not fixed to each other. Now, on the contrary, objects no longer form a space—there are no longer enough of them for that—but they are localized in the space formed by views.

In a general way, any view is fixed by the places of the appearances of four objects: the place that the appearance of a fifth object occupies is then determined, and it is possible for our observer to calculate it by starting from other views. Besides, this determination is *continuous*: if the first four places undergo a slight variation, the variation of the fifth place is also small. Thus, when the appearances of four objects in several series of views approach from given places by different avenues, the appearance of the fifth object tends towards the same place in all these series.

However, certain objects are exceptions in certain possible views. We cannot calculate the place which their appearance is to take. All processes of construction fail because of a particular disposition of the places which constitute the given: the alignments whose intersection should fix the desired place all become indeterminate. Moreover, the most complete sort of discontinuity is brought to light. According to whether we consider such and such a series in which the appearances of four objects tend by different roads towards the same places, we see the appearance of the fifth object tend with each series towards all the places in the perceived field. The view determined by the places of the appearances of the four normal objects is therefore a *singular view* with regard to the fifth object: it happens that all views belonging to the same "point of views," and only these views, share this singularity. Thus the observer is led to regard each "point of views" as singular with respect to such an object. In addition, each one of his objects has its singular "point of views."

What then is the special relation which attaches to every object in this way a certain collection of possible views? We express it by saying that these views form *the spectacle of the* universe seen from that *object.* Indeed, when we bring the material point which stands for the body of our spectator near to the material point A through different avenues, the appearances of the five distant material points tend towards the same group of places, while on the contrary the appearance of the point A remains fixed in different places.

It is to be noticed that we succeed in attaching to an object the class of views perceptible from that object even though our universe is purely visual and quite immobile. For it was not evident that the localization of a certain possible perception in a body might admit a meaning independent of all causality. Might it not seem that the content of a perception was situated in a certain body—as for example, a certain view of the sky in a certain observatory—only in relation to a particular body called the body of an observer, itself determining by concomitant variations the views which present themselves to the owner of the body? But in our universe, the observer is in his own eyes a pure spirit. He is not aware of any body; he sees only contingency in the succession of views. Despite that, he attaches to each of the objects seen a class of singular views. If he could penetrate into another mind like his, and if he found there one of these views, he could say: " this mind is situated in this object "—not of course with all the significance which we should give these words in our richer world, but yet with a meaning already determinate.

Just the substitution of bi-dimensional views for tri-dimensional views studied at first, with the assumption that it allows empty spaces in visible matter, has then important consequences. The space of perceived places and the space of objects disappear. On the other hand, the spaces of views subsist in a complex way, and come to be fused in the total space of "points of views." Finally, the objects which remain and no longer form a space are themselves inserted in this space of "points of views."

The order of views thus becomes the only fundamental space of nature. It seems that we have reversed matters. For we are accustomed to regard the spatial order of objects as fundamental, and to represent the spatial order of views as derived from the order of the objects which are their physical seat. That appears to us like a positivistic way of thinking. Nevertheless, the reversal which makes the order of views of contents of total perception appear autonomous, and analogous to the very structural framework of the bodies of this simplified world, is logically necessary. It furnishes in fact, in the light of rigorous analysis, the only possible application of geometry to the spatial content of such an experience.

Introduction of new objects.—The perceptual geometry which has just been presented operates with only six objects, three of which are situated in a straight line. But let us introduce in the universe of our observer a new group of six stars fulfilling the same condition. This new group is going to give matter to a geometry independent of the first. In fact, it supplies, *as though the first group were non-existent*, its own definition of alignment and of all the other entities which enter into the formation of the space of views and of a total space of "points of views." The coincidence of the two constructions thus obtained is, therefore, not a necessity, but a simple fact of existence. When he discovers the six new objects, the observer can know nothing of the rules according to which their appearances are going to be displaced from one view to the other. Will these rules be the same as for the first group of objects? The hypothesis is a natural one, but it remains an hypothesis. Far from being necessary, it is easy conceivable as false.

Perhaps the new objects, by the displacements of their appearances, communicate to views an order which is not at all geometrical, or else a two-dimensional or one-dimensional order: that is conceivable, for we know that the points of an entire space are capable of being arranged in a series. Perhaps, they are the basis of an entirely different geometry of views, non-Euclidean, for instance. But let us even suppose that the appearances of the objects of the new group are displaced according to the laws already known. The spaces of views, the total space defined by considering only these appearances have then the same structure as the analogous structures in terms of the initial group. However, it does not follow yet that the order of views of the second group of new objects will be the same. In fact, these formally similar structures are ignored: three perceived places aligned in the first group might not be aligned in the second; two views parallel in one might not be so in the other; two views might be of the same " point of views " with respect to the old objects but of different " points of views " with respect to the new objects. The views of the new objects would then form spaces intermingled in a complex structure, or even, without order.—For this spectator, the identity of the spaces of views based on the observation of the appearances of diverse groups of objects can then only be a fortunate simplicity in nature.

SUMMARY AND CONCLUSION

STARTING from the general idea of a space as a group satisfying a geometry,* after an inventory of the elementary terms and relations that nature offers was made, we began to investigate the different ways these relations and terms form spaces, either directly or by their combinations.

We found four distinct orders of geometry in nature. All have the particulars of sense naturally, for their last terms. Two of these orders have as elementary relations *global resemblance* and *succession*: they are concerned with the pure and simple recurrences of the contents of perception according to the order of perceived time. The first orders the particulars of any external sense; it only furnishes an illustration, it is true, of the rudimentary geometry of *analysis situs*. The second orders sensations of movement and attitude mingled with the recurrences of any external sensation of an invariable quality; it forms a complete geometric space, and even a host of such spaces.

The last two orders are composed of new relations, and bring us back to the order of the lone particulars of one external sense (vision, in the third case, where we had hearing in the first case). But this time we have a displacement of visual data in a field: in other words, the global resemblance of two particulars divides into local and qualitative similarities. These two species of similarity diverge and intersect.

Taking total perception in the sense of instantaneity, we have at first admitted that the various *groups* formed from the particulars contained within the total percept present new

* Cf. our article *Mathematics (Foundations of)*, *Encyclopedia Britannica*, 11th ed., supp. vol.

and simple similarities. Thus the trios of particulars would be subdivided into two classes of aligned and non-aligned trios; or again, each couple would divide all other couples into two classes according to the absence or presence among them of a certain sort of equality. We have classified the similarities of this third kind under the name *relations of position:* these relations might by themselves endow the content of each total perception with the order of a whole space or at least of a part of space.

But relations of position depend only on the perceived places of the particulars they relate, and not on the rest of their qualitative essence. By local similarity, relations of position can be transmitted from one instant's perception to that of another. Hence a variation of the preceding hypothesis offers itself to us. If we are willing to admit that a place in a perceived field is something more than a class of data resembling each other locally—that it is, for example, a quality common to these data, or even more, a simple subject directly apprehended in them, it becomes natural to take relations of position as immediately relating perceived places. These, relations of position make and order the perceived field into an absolutely motionless space which is the content of each total perception.

Relations of position supply the only simple translation of which geometry is susceptible in nature: in the other orders of space, all the geometrical relations receive meanings derived from combinations of relations. They also supply the only translation of the geometry which is contained in an instant; for no other perceived space is present all at once. That is why the intuitive relations of position occupy the foreground in the picture we make of a perceptual structure which embodies a geometry. However, we have seen that it is conceivable that relations of position do not really exist except as a figment of the imagination; besides, they can yield a space only from a three dimensional field without any limits, for on any other hypothesis the geometry of relations of position becomes incomplete. Whatever the case may be, even if there does exist an intuitive space of this type, no matter how it may impose itself on the imagination, we have shown conclusively that inductively perceived spaces involving temporal relations do not derive their order from relations of position. The latter may be instrumental in thinking them, but it can in no way give birth to them, even for the sake of the mind's comprehension.

Then, leaving relations of position aside, we have studied a last type of order, the richest and most ingenious of all. It requires *local similarity* and *qualitative similarity*. However, it neglects the order of succession of perceived particulars and takes into consideration only the presence or absence of *simultaneity*: it needs a more differentiated order of similarity than the first two constructions, but it is content with a more summary temporal relation. We have analyzed it for a vision with three, and then, with two dimensions: we have shown that it has its own way of ordering the perceived field, doubling the order of relations of position if this order exists, thus doing its work over again, and carrying geometry further into the order of objects and into the order of total perceptions.

We may draw a summary picture of these various sorts of natural geometries by imagining them gathered in one experience, which it will suffice to suppose is visual and kinesthetic. Let us review the visual universe of the last chapter. Attaching himself to the three relations of local, qualitative, and temporal similarity, the observer sees perceived places ordered like the points of a sphere, views in parallel families forming as many spaces, and these spaces in a total space of "points of views."—This framework of geometrical order requires only a group of six objects. It is repeated for each group and the resulting spaces are found to be the same.

Let us give to this spectator the intuition of simple relations among those of his perceived places which the preceding framework relates as aligned or unaligned. An intuitive order can only duplicate the experimental order in this first part of our construction.

But if we remove the bandage from the eyes of our spectator so that he can see in the interval during his displacements, a new science appears: the science of the succession of views. So long as he was transported from one view to another without seeing anything on the way, views followed each other without any rule of order. We must recall here that all order was derived up to that point by means of the comparison of the contents of diverse views, without being concerned with their distribution in perceived time. Now, this distribution presents in its turn certain laws. These laws are no other than those of *analysis situs*, whose perceptual form we studied in the first chapter of Part III. They divide views into classes such that a view of class *a* cannot follow a vision of class *c* without some view of class *b* appearing in the interval;* and they state the properties of this division.

It must be noticed that on account of this, each view is taken as a whole. If two total perceptions resemble each other globally, they are regarded as the same view. If they differ in the perceived place of some appearance, they are regarded as views which are purely and simply different. Whereas, before, the analysis of these differences was all there was to the science, now, they no longer play any rôle in these new laws which concern only the order of the pure and simple recurrences of the contents once experienced. The preceding framework determined the content of unknown views, but not the

* We are here neglecting the deformation due to relativity, which would introduce a new wealth of structure into this perceptual world.

times of their occurrences; the present one, on the other hand, connects the occurrences of known views.

These two orders are therefore independent and their coincidence constitutes a remarkable fact. Our spectator might sum it up by noticing that the views encountered *in the time* between a view A and a view B derive their *contents* from a class of "points of views" which possesses the formal characteristics of a continuous line from A to B.

Let us finally cease transporting our observer from place to place. Let him move himself and endow him with the kind of kinesthetic sensations under the conditions described in chapter ii of Part III. By means of any view whatsoever which is allowed to enjoy the rôle of an external referential datum, the world of these new sensations is ordered in its turn and in its manner into a multitude of spaces. Besides, for each view taken as a point of reference, there emerges an analogous and independent framework; and all these frameworks coincide.

This new order of the recurrence of views among the series of kinesthetic sensations has itself no necessary connection with the non-temporal order that views derive from their contents, nor with their order of succession. In truth, the coincidence of the spaces of the sensations of movement, taking different views as references, is reduced to the fact that the same kinesthetic sensation, starting from the same view, leads constantly to the same view. It does not follow therefore that the couples of views related by the same kinesthetic sensation ought to be found congruent in the space of "points of views." In the same way, the fact that the views A and C are always separated in perceived time by some view of a class *b* indicates nothing about the relation of the kinesthetic sensations connecting these diverse views, if we hold to the hypotheses which we found sufficient to establish a kinesthetic geometry. For we have only endowed our character with the discernment of impressions of total movements, without granting him the power of recognizing, in the midst of an uninterrupted movement leading from a view A to a view C by crossing a view B without a stop, the movements which would lead from A to B or from B to A. But two views separated by the same kinesthetic sensation are found to be separated by the same distance in the geometry of views, and elsewhere, views succeeding each other in perceived time are found separated by sensations of movement forming a continuous line in kinesthetic geometry. These are factual agreements between orders whose independence we have shown and whose diverse structures we have analyzed.

Such is the geometrical structure of one of the simplest perceptible worlds that can be imagined.

This world is still almost incomparable with ours. It is an image of it rendered summary by distance; it is a dream about it in which some of its features appear. Indeed, how much does it wholly neglect ? We have overlooked the inaccuracy of our senses, the narrowness of their scope, the deforming action of fields. We have excluded the existence of indiscernible objects. We have eliminated any change that goes further than a change of view-point. No perceived object is either modified or displaced by itself, the field remains invariable, the sensory mechanism of the observer suffers no alteration: there do not yet exist any other events than apparent events, and any other time than apparent or perceived time.

It is true that limitations of this sort give the impression of naïveté and obvious inevitability. The mind of anyone who wishes to form some idea of the order of sense particulars immediately accepts conditions of this kind. Take any passage at all that attempts to give an empirical expression of a proposition in geometry and you will find that it supposes the exclusion of all sorts of embarrassing circumstances, more tacitly perhaps, but no less boldly than we have done.

This simplification of the problem is a very legitimate means of approach. But we must not forget that the solution obtained by such a short cut cannot be more than a particular solution. It would be ridiculous to believe that nature manifests a geometric structure only to the extent that it is in conformity with our childlike ideal, that the spatial order of experience concerns fixed objects, with distinct and invariable aspects, perceived "normally" under no perturbing influences, to the exclusion of objects in movement and in change, of indiscernible objects, of perceptions reporting the effects of heterogeneous and moving fields.

This might be true of nature, but it actually is not. It might be possible for certain objects and certain perceptions of objects not to reside in the geometrical structure of nature that we call space: but it is not so, for all of the perceived world is ordered by this structure. Two objects having identical aspects, an object which moves and displaces itself, a view that is deformed by unequal refractions, a motion deviated by a current are not outside of space.

That amounts to saying that geometry does not apply to the perceived world in only a limited domain of physics, such, for example, as that of the displacements of rigid solids, but really applies to all physics in each one of its branches. Does not every physical proposition contain places, directions, and distances? The geometrical structure of the world is the structure of all its laws embodied in a few formal characters.

The application of geometry to nature is free of the limitations which we have imposed on ourselves. We have shown how certain bodies perceived in a certain way would present a perceptible spatial order. But in reality, this order embraces all bodies and all perceptions. We must guard against saying: Geometry does not apply to the universe except to the extent particular hypotheses of the solutions that have been exposed are satisfied. We must affirm on the contrary: These solutions are still only particular solutions of a general problem, which is no other than that of the empirical meaning of all physics. As specific solutions they have only the value of indications and experimental exercises. They are made not in order to stop the mind, but in order to aid it in the clarification of more complete solutions.

In this work, we have avoided following the road which descends from schematic abstractions towards reality. We have been contented with positing, first, a picture of the world: for we must start from something. Three times however, we have progressed one degree, giving up one of the simplifications admitted at first, and thereby drawing so much nearer to the real. The framework of the initial order was thus broken down to make place for a more general framework which would reproduce the structural form without obeying the same restrictions. We have, in fact, passed from a tridimensional vision of a space full of matter to a bidimensional vision of a scattered matter; we have noticed the elimination of the initial order, its resurrection under a more complex form, as well as the birth of new characters. At the very beginning, it is true, when dealing with the rudimentary structure of the succession of external data, we raised the restriction excluding the existence of identical realities in different places: the framework of the laws ordering "notes" was found to break down in certain exceptional cases, but only in order to make place for a more general framework of the same structure among "unities," concepts which are much less simple than notes and of which notes are particular cases. Finally, in Part I, we have provided a way in which voluminous data may be established in the place of fictive punctual data in perceptual geometries.

Of course, such reconstructions are easy alongside of those which the existence of heterogeneous fields imposes, and which, in addition to the profound changes the latter brings about in all things, postulates objects as well as sense-organs. However, our work already contains the idea of the method to be followed: it shows that the road is arduous, but also that it is practicable.

In summary, our hypotheses have not gone beyond the most rudimentary idea of a perceived world geometrically ordered. However we have studied it in a systematic spirit of rigour and consistency. On the one hand, the result of such rigorous analysis is to keep our primary conception of the perceptible content of geometry free from those accidental impressions which may lead to false generalizations. That is how Henri Poincaré, having perceived the existence of a geometrical type of order containing sensations of movement, and not realizing that a more attentive analysis reveals a purely visual illustration of this same order, concluded that the relationship of geometry to experience involves action primarily. The whole plausibility of his view derives from this lacuna in his analysis. Coming from a mind like his, such an illusion shows that we cannot afford to neglect any care in these first studies. On the other hand, just because our analyses are schematic, it is important that they should be exact. Because they are to bear considerable corrections in their application to what is known in nature, they must be precise; for we cannot correct what is not definitely stated. That is why we have depicted these first analyses of a perceptible order more arduously than is usual, and yet very simply. Perhaps they may already suggest an exact sketch, or render less plausible some error.

THE

LOGICAL PROBLEM OF INDUCTION

WITH A BIOGRAPHICAL PREFACE BY

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PREFACE

It is with deep sadness that I am now writing what I should have been glad to say for Jean Nicod on the day of the defence of his thesis concerning a book so full of promise as the present one, in which he was testing the foundations and setting the cornerstone of a structure that he was not to complete.

With him there disappears in full youth an unforgettable figure. His friends speak with profound feeling of his heart and character; his professors have been able to appreciate this free and noble spirit, ardent in the pursuit of ideas, in whose nature were compounded the rarest and the most varied qualities. His was primarily, as one of the friends who knew him best* wrote me, a sensibility that vibrated to the things of life and of art with the freshness of a childlike soul. He had an extraordinary faculty of enjoying a line, a colour, a sunbeam playing on the leaves of a tree; his was an imagination of a rare and charming fancy which the slightest stimulus aroused to delightful expression. But he retained all these qualities without ostentatious exuberance; if he laughed frankly at what amused him, he himself always spoke with tranquillity; enthusiasm or humour were only accompanied then by an instantaneous light in his eye, or by a hardly perceptible smile around the corners of his lips. He was endowed with an unremitting intellectual curiosity, aided by a facility of understanding and learning such as I do not recall ever having met before. I do not mean that banal ease which comes from

* Mr. I. Meyerson who was closely attached to him, and to whom I owe a great part of the biographical notice mentioned later.

mere memory and passive acceptance of the views of others, but rather, that quick grasp which comes from a solid apprehension of ideas, and which serves only as an instrument in the service of the higher power of personal judgment and creative reflection. The rigour of his reasoning was as great as the breadth of his imagination; there was something like genius in the rapidity and scope of his intellect, his friends used to say. The unanimous consensus of his comrades and teachers placed him in the front rank of the new philosophical generation.

Born in 1893 of a family of great intellectual culture, he had at first turned towards the sciences, and he had acquired by the age of eighteen, after two years of special mathematical studies, that solid fund of knowledge and technical habits which are obtained only with difficulty in later education. But philosophy appealed to him and absorbed his interests. He came to the Sorbonne, where in three years he obtained his degree, diploma of graduate studies, and his fellowship. He started his studies as a Fellow first in 1914, in the session interrupted by the declaration of war with Germany. Meanwhile, he had pursued graduate course in the École des Hautes-Études, and in the Faculty of Sciences; he had learned both Greek and English so well that he was accorded on his diploma-examination a grade of sixteen for an explication of Plato, and he also carried off first prize at Cambridge University in competition with British students.

Too frail in constitution to be drafted, he spent the greater part of the war period at Cambridge, working diligently on the most varied subjects (he even went so far as to learn Persian in a few months of his leisure time), taking the English degrees, studying particularly, under the invaluable direction of Bertrand Russell, problems of logic and logistics which had already awakened his curiosity during his studies at the Sorbonne. In this realm, to which few French mathematicians or philosophers devote themselves to-day, he brought the resources of a knowledge and ingenuity which promised an eminent successor to the work which was unfortunately interrupted by the tragic death of Louis Couturat—and which Nicod's departure leaves again in suspense. For what was at first with him only a certain lack of physiological resistance did not take long in assuming the form of a too well known illness against which medicine is almost disarmed. On his return from England he married one of his student comrades, Miss Jouanest, who brought him not only the tenderness of a warm devotion, but also the support of an intelligence capable of understanding his. At first, he followed the usual career of young Fellows: he taught philosophy at the lycees of Toulon, Cahors, and Laon; but the fatigue of lecturing made itself felt and he had to give up secondary teaching. With his extraordinary faculty of learning, and as a result of a competitive examination in which law and political economy played the principal part, he acquired a post, in 1921, with the International Bureau of Labour of the League of Nations. He became noticed quickly for the rapidity and accuracy of his work and the clarity of his mind; his linguistic knowledge made him an invaluable interpreter in international meetings. An improvement in health allowed him to come to Paris for some time where he was able to give a course on the history of Greek philosophy, and where he worked at the same time on his theses. But in the winter of 1922-1923, a rest at Leysin became necessary, and after that, in spite of periods of relative improvement in health, he was no longer destined to resume work. He had just returned to his functions at the International Bureau of Labour at Geneva, his doctoral theses were printed and handed in, and he was to defend them soon after at the Sorbonne, when abrupt complications set in; on February 16, 1924, he was removed from the affection of his family and friends.

Outside of an important article in the *Encyclopedia Brittanica* (New Volumes, Suppl. 1) entitled *Mathematical Logic and the Foundations of Mathematics*, he had presented in 1916 a very remarkable paper to the *Cambridge Philosophical Society*: *A reduction in the number of the primitive propositions of Logic* (Proceedings, vol. XIX., and separate extract by the *Cambridge University Press*). He had also published three reviews in the *Revue de Metaphysique*, one on Goblot's *Logique*, in which certain passages already anticipate the present work; another on the *Geometry of the Sensations of Movement*, in which are sketches of his future work on *Geometry in the Sensible World*; lastly, *The philosophical tendencies of Bertrand Russell*, in which it is seen to what degree he had been attracted to this new conception of logic extending far beyond its traditional limits, fused with mathematics, or more exactly (for we are not concerned here with geometry or even the theory of numbers) with the most general forms of order utilized by mathematics. He was content to deduce the conclusion that such a general logic becomes applicable to the most diverse realms, even to those which would lie outside the forms of space and time, or within the category of empirical determinism; it is so widely applicable that the knowledge of what exists in fact no longer implies a radical empiristic theory of *all* knowledge. It is around a special point of this conception that *The Logical Problem of Induction* is developed: granted that we are constantly making inductions, what are the logical principles that our experimental reasoning presupposes? Nothing is more opposed to the method of Lachelier, in his famous work on the same or related subject, than Nicod's own method, as Nicod himself admits. He frankly gives up, from the very beginning, a definitively fixed theory on which to build a philosophical system. All that he wants, is to make science advance one step further on a difficult ground whose pitfalls, he believes, have not been sufficiently heeded by philosophers. He pursues the method recommended so often by Rauh: he "takes up the line." He revises and finds doubtful the formulas which have been current until now; he analyzes and discusses the most recent work which has approached the problem in a really technical way: *A Treatise On Probability*, published in 1921 by John Maynard Keynes, the third part of which is devoted to the relations of probability with induction and analogy. Now accepting the latter's conclusions, then refuting them, but always scrupulously referring to them he tries to show: (1) that induction by simple enumeration is a fundamental mode of proof and that all those who have thought they can do without it have done so only by the aid of sophisms:

(2) that this style of reasoning would still retain its value, even if determinism were not postulated in advance;

(3) that it can increase the probability of a hypothesis, even when the new facts observed should do nothing but repeat without variation facts already known;

(4) that induction by the elimination of causes can never exceed mediocre probability;

(5) lastly, that is it not actually demonstrated by any procedure whatsoever, how inductive reasoning can raise the probability of a law to the point of indefinite proximity to a certainty.—This is not the place to investigate what might be opposed or added to these conclusions, which partake naturally, except perhaps the first, of the difficulties involved in the indispensable but quite obscure notions of probability. There is little doubt that if the creative and penetrating mind to whom we owe this analysis had been conserved a longer time, he would have pursued the study of this question which is central in the logic of the sciences; and without doubt, the critical part of his investigation would have resulted in a more positive construction. But it was a great step forward to

separate distinct problems, to seek the limits and conditions of each, and to show how illusory is the ease with which they were thought to be solved. For such a task nothing less was needed than the fine intelligence whose premature disappearance leaves those who have known him in such deep regret.

ANDRÉ LALANDE.

INTRODUCTION

Methodology and Logic

Logic is the study of proofs. The best proofs are encountered in the sciences, so that it is natural for the logician to stand near the scientist and watch his reasonings and his methods. But merely describing methods of proof is not enough: it is necessary to analyze them also. Thus the logician by this critical method of analysis proper to his discipline, can in his turn teach something to the scientist. He can make explicit to him the elements and premises used confidently by the scientist as means of proof. For the scientist may employ them while he is quite unaware of the conditions and causes of their power; he may take them as simples, whereas the logician finds them unexpectedly complex in structure. Thus, methodology is only one half of logic. Logic may sin by insufficient attention to the proofs of science; it then works with poor and inferior material. But it can sin equally by insufficient rigour in the analysis of these proofs: that is what has happened in the logic of induction.

We perceive at first glance two types of induction. One proceeds by simple enumeration of instances. It is founded on the number of such alone, and it claims to draw conclusions that are never more than probable. The other type, on the contrary, proceeds by analyzing the conditions or circumstances. Certain of itself, it leaves everything to care, and nothing to repetition in its search for certainty. Of these two theories, the last alone seems to answer to the practice and very spirit of science. A single experiment, the scientist thinks, if improvised very carefully, can in one successful result bring all the certainty accessible; and to wish to build anything whatsoever on repetition is unworthy of intelligence.

The logician generally accepts this proud thought too lightly. He closes his mind to the possibility that things are not what they seem, that the appearance of a certainty obtained by the analysis of conditions covers a probability founded on pure and simple repetitions, that scientific induction is resolved into a complex of enumerative inductions, and that, consequently, induction by the enumeration of instances may be the one which he has to justify in order to justify science. Dominated by the impression of power that scientific induction produces, he does not wish to know anything less promising. He says with Bacon: Induction by simple enumeration is a precarious process and is exposed to the danger of contradictory instances (*Inductio per enumerationem simplicem precario concludit et periculo exponitur ab instantia contradictoria*). He then turns toward the pursuit of a theory of induction that would count the repetition of instances for nothing and which, by more noble means, would rise to certainty.

This is letting the substance go for the sake of its shadow. For the doctrine thus constructed does not render an exact account of any sort of actual induction. The conditions that it assumes are never fulfilled; and besides, they are already unreal from the standpoint of pure logic. By placing itself on the grounds of certainty, by wilfully ignoring the influence of repetition, the theory of induction terminates in complete check.

It might be considered surprising that some doubts did not appear sooner. But these doubts, arriving too late and lacking in force against a prejudice decked in the prestige of science, are often replaced by an additional error. It is felt clearly that the theory is not applicable, that it is necessary to diminish some of the power it claims to confer on induction. It is then said: certain in theory, induction is only probable in practice;

and enough is thought to have been conceded. But in truth, if real inductions do not fulfil the conditions that would make them certain in theory, it follows that they are not certain at all; but by no means does it follow that they remain somewhat probable, or very probable, or extremely probable. If certainty is lacking altogether, the very possibility of probability remains to be established, and the whole theory of induction must be done over.

We propose in this work to confirm this principle and to study its consequences. We shall try to establish the logical problem of induction on its intended ground, that of probability. We shall pursue the solution of the problem, but we cannot admit having reached it. All that we can hope is to contribute to it, if only by showing how obscure the problem remains: obscure to the point that nobody has ever succeeded in proving or even stating principles capable of fully justifying induction under the conditions by which it operates.

There will be some discussion of the recent work of Mr. John Maynard Keynes, *A Treatise on Probability*. We shall attack his fundamental conception of the mechanism of induction and the proof of one or two of his essential theorems. But the value of the theorem which he has really and properly proved should only appear all the more forcibly. We regard it as the most important result yet possessed; and since we are to criticize Mr. Keynes several times, let us say here that in our opinion, no author since Mill has advanced the theory of induction as much as he.

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PRELIMINARY NOTIONS

It is useful, first of all, to determine certain notions and certain principles exactly.

Certain and probable inference.—What is induction? Induction is a species of inference: but it must be stated that an inference has no need of being certain, in order to be legitimate and rigorous in its way. We first regard inference as the perception of a connection between the premises and conclusion which asserts that the conclusion is true if the premises are true. This connection is implication, and we shall say that an inference grounded in it is a *certain inference*. But there are weaker connections which are also the basis of inferences. They have not until recently received any universal name. Let us call them with Mr. Keynes *relations of probability* (*A Treatise on Probability*, London, 1921, ch. i.). The presence of one of these relations among the group A of propositions and the proposition B indicates that in the absence of any other information, if A is true, B is probable to a degree p. A is still a group of premises, B is still a conclusion, and the perception of such a relation between A and B is still an inference: let us call this second kind of inference *probable inference*.

Certain and probable premises; general definition of inference.—These terms "certain inference" and "probable inference" undoubtedly lend themselves to equivocation. But we have no better terms and it will suffice to explain them.

First of all, probable inference says that in the absence of any other information, the truth of its premises renders its conclusion probable to a degree p. It then says *less* than what is claimed by certain inference, as we have just called it; but it says it with as complete certainty. It does not conclude *in a probable manner*, but it reaches *a probable conclusion* in as certain a manner as the other reaches its conclusion.

We have so far considered only premises that are certain. But any inference which yields something, starting from premises taken as certain, still yields something starting from premises taken only as probable, and this holds for both types of inferences, certain and probable. We can even assert that starting from premises which, taken together have a probability p, a certain inference will confer on its conclusion the same probability p; and a *probable inference*, which would confer on its conclusion the probability q if its premises were certain, will confer on its conclusion the probability pq.

It can therefore be said that certain inference transfers to its conclusion the totality of the certainty or probability of its premises taken together, and that probable inference transfers to the conclusion a part of its certainty. Inference in general will be defined as an operation which transports to its conclusion either the totality or part of the certainty or probability of its premises; it is only in this wide sense that it is legitimate to postulate that induction is an inference. Notice that the conclusion of an inference extracts at most no more certainty or probability than is equal to the conjunction of its premises, consequently, no more than any one of them, and in particular, no more than *the most uncertain* among them. This very evident truth will be very useful to us.

Definition of induction.—What sort of inference is induction ? It is defined in current times by the logical form of its premises and its conclusion by saying that is a passage from the individual to the universal.

At the beginning of an induction, then, *beside premises having any contents and forms whatsoever* not all of which are known, there are propositions about a certain class A of individuals or species, without assuming that the members under consideration exhaust the totality of the class A (for perfect induction does not concern us here). And in addition to this, the induction terminates in a proposition *about all the members* or else only *about any member* of this class. We shall return to this distinction in a moment.

Inductions about the relations of characters.—The extremely general term "proposition" includes mainly those propositions which refer to those secondary characters arising from the relations of characters—such as the fact, regarding a man, of having two eyes of different colours, or else of having a weight in kilogrammes equal to the number of centimetres his height exceeds a metre.

Let A be the character hypothecated by an inductive law. The character found in the conclusion often occurs in the following form: Let *b* and *c* be the classes of characters B_1, B_2, . . . B_n, . . . and C_1, C_2, . . . C_n . . .—for example, *b* and *c* may stand for temperature and density respectively, the "values" B, C being the different temperatures and different densities. Lastly, let R be a relation joining a C to each B. The character that induction attaches to the character A consists often of a certain relation R between the character of the class *b* and the character of the class *c* which accompanies the character A: thus the chemical composition of a body determines, not its density or its temperature, but only a relation R between the two, one of which remains a certain *function* of the other.

These functional laws are the objects of scientific inquiry, and are distributed elsewhere. Does not their particular form have any influence on the mechanism and logical principles of the inductions which they establish ? This is a problem we would not dare to probe.*

But no such form appears up to the point where this work stops. Therefore, everything which follows should be understood indifferently as induction about characters or about the relations of characters.

Inductions about classes of individuals and inductions about classes of classes.—We must notice in the first place that every functional law assumes the appearance of a law bearing on the classes of a class, rather than on the individuals of a class. In fact, let B_1 and C_1, B_2 and C_2, . . . B_n and C_n, . . . be the characters *b* and *c* associated by the relation R. The law linking to the character S the relation R of the characters *b* and *c* links C_1 to AB_1, C_2 to AB_2, . . . C_n to AB_n, . . . It is therefore a bundle of laws fixing the characters *c* of the classes AB_1, AB_2, . . . AB_n . . . of the class A.

But it has been thought that we should, besides, reserve in the previous definition the possibility of inductions operating directly with classes of classes just like other inductions dealing with classes of individuals. Indeed, a number—the number two, for instance—is a class—the class of couples—such that any arithmetical induction by verification of a formula for various numbers has as its immediate domain, not the individuals of a class, but in truth the classes of a class.

* It is in this way that induction about the relations of characters raises directly the problem of the connection of *simplicity* with *probability*. In fact, the relations connecting the members, taken two at a time, of even two series of given characters (e.g. the curves passing through a given collection of points) are multitudinous. Any induction in this domain supposes the choice of the simplest relationships. This choice is often explained by the psychological reason that the simplest is the most convenient. But to make induction valid, the choice must be justified. Psychology does not do so; for who can assure us that the most convenient formula of what we are familiar with, will be found to be the most probable formula of what is unknown to us ? This is a problem of the greatest logical difficulty. For we do not know perhaps with enough clarity what probability is, nor what the simplicity of a formula is: we can define simplicity differently from several points of view. But none of these assuredly must reduce itself to what is most flattering to the laziness of our minds.

The distinction between classes of classes and classes of individuals might very well make a difference to the logical theory of induction. But we shall not reach the point where this distinction makes itself felt, so that the substance of this study appears to us applicable indifferently to classes of particulars or to classes of classes.

Inductions from THESE to ALL and from THESE to ANY.—We must distinguish two forms that the conclusion of an induction may take with respect to *all* or *any* of the A's. Despite Stuart Mill's inference from particular to particular, these two forms have not been carefully discerned. This is because, remaining confounded in the domain of certainty, they are not separated until they appear in the domain of probability. In fact, if it is *certain* that any of the A's are B, it is certain that *all* the A's are B, and vice versa, so that these two forms *all* and *any* differ only verbally here. But when we have to do with a conclusion that is only probable, a hierarchy and even independence between *all* and *any* appear. If it is probable that *all* the A's are B, it is even more probable that *any* A is B, for then we have only one risk instead of several. On the other hand, when it is probable that *any* A is B, it may at the same time be improbable and even impossible that *all* the A's are B. This is what happens when it is certain or probable that there is, for instance, only one A out of a thousand that is not B. We must then, in the general study of induction, distinguish conclusions about *all* from conclusions about *any*, these last always being the more probable, sometimes the only probable ones.

Primary and secondary inductions.—We have said little about the premises of an induction. The conclusion referring to all the members or to any of the members of a class A, the premises should be comprised of particular propositions having as subjects members of A which are not all its members, so far as our knowledge goes. But nothing is said here about the other premises which must be taken together with these in order to obtain a valid inference. According to the form of these other premises, the force of induction arises possibly in several ways. Hence it is not necessary to assume as a principle that induction is a linear process and is analyzable in only one way.

In particular, suppose that an induction has among its premises the conclusion of another induction. We shall call it then a *secondary induction*. *Primary inductions* are those whose premises do not derive their certainty or probability from any induction.

It is possible that there are general modes of induction which, because of the premises they require, can only operate in secondary inductions. It may be that these modes are in themselves the most certain; that is to say, that they transmit to their conclusions a greater share of the certainty or probability of their premises than the modes of primary induction. But we cannot repeat too often that this intrinsic superiority is vain. For as a matter of fact, the probability conferred by an inference, of any sort, upon its conclusion is at most equal to that of the least probable of its premises. The probability supplied by any induction whatsoever cannot exceed the highest probability that a primary induction can yield. That is why primary induction should be analyzed before any other. For it is not only the logical foundation of induction, but it also marks the limit of all inductive assurance.

Mill's doctrine offers an excellent illustration of this same hierarchy of primary and secondary induction and of his misunderstanding of it. Mill thought he was presenting a very powerful method of induction. But this method is of the secondary mode because it requires as a premise the law of causality, which according to Mill is only an inductive generalization obtained by the lower and primary method of simple enumeration. If Mill's object were not merely the description of scientific inductions, but their analysis and the discovery of the principles which underlie them, why did he not make a profounder study of simple enumeration which in the end is the basis of his whole structure? Strange to note, he himself turned away from it. Simple enumeration appeared to him as primitive, prehistoric reasoning. But when he has to build on it, he assumes not only that it yields probability, but besides, a probability which approaches certainty indefinitely in the degree to which instances are multiplied; and this considerable postulate inspires hardly any doubt or curiosity in the most noted logicians of induction. And Mill has no sooner based his doctrine on simple enumeration, than he forgets it. He then seems to consider induction by simple enumeration as lacking in force, and to put all his faith in scientific induction. The method of induction, which is the basis of his logical theory, is seen only as an historical tradition, and he thinks himself free to distrust it as Bacon had done.

This place in Mill's doctrine has often been criticized. But, strangely enough, critics have not seen the real point of Mill's error. Mill might have been reproached for not realizing the mode of the primary induction which is required as the premise of the secondary induction with which he is so much concerned because of its praiseworthy results. But instead of that, he is most often reproached for making the law of causality itself based on induction. This is the objection which has become classically attached to his theory. And thus, discussion of this particular opinion of Mill has not made visible the exact nature of the lacuna in his system; moreover it is simply a lacuna, and not a contradiction or a vicious circle.

That is why logicians who came after him and rejected Mill's theory did not guard against another form of the same insufficiency in their inductive demonstration. This consists of requiring of the primary mode of induction which is disparaged and neglected, no longer, it is true, the law of causality, but some other premise which is less universal and yet necessary for the application of the mode of induction that is admitted. In fact, it is often estimated that it is not enough, in the practice of scientific induction to start with the principle that the effect studied has a cause, but that it is also necessary to have already some idea of the nature of this cause in order to eliminate from the very start a multitude of circumstances, observed as well as unperceived, so that we can be limited to the systematic examination of a few hypotheses. This prior limiting determinable, however, can only be viewed as an analogy derived from causes already known of effects of the same sort. We thus come back by a detour to Mill's general position which requires for the premise of scientific induction the mass of previous common knowledge, itself due to some pre-scientific mode of induction. And, like Mill again, we do not wish to see that the priority thus conceded to this primary mode is a very nice case of irrevocable logic. At present, the tendency is to present primary induction as solely historical and not concerned with pure logic. As a result, the elementary methodological distinction between the primary and secondary modes of induction is not at all a current fashion. Is not this a sign that thinking on this question has not been sufficiently clear ?

On the other hand, once we have recognized this simple distinction, we can no longer afford to neglect it. Every analysis and theory of the principles of induction becomes governed by the rule which makes the source and limit of the probability or certainty of induction in general depend on the probability or certainty of primary induction.

Probability and certainty.—Probability is different from certainty not only in degree, but in nature. For certainty is absolute and probability is relative. If I have judged a proposition to be certain—certain and not only infinitely probable—no new information can ever make it doubtful again, unless my judgment of its certainty was false in the first place. On the contrary, if I have judged a proposition to have some given degree of probability, some new information may make it more or less probable than it was; but I was not mistaken in the first place because of that. The probability of any proposition whatsoever is then relative to such and such a group of evidence, or else to employ the precise language of Mr. Keynes, it is a relation of this proposition to this group.*

Common sense has never ignored the distinction; for it believes that the actual realization of an improbable prediction cannot justify it as more than an improbable event, and that chance cannot contradict rational certainty. But the logicians of induction have not always remembered this.

And has not this forgotten point been the source of the suspicion which, since Bacon, has almost invariably been attached to induction by simple enumeration ? It was not satisfactory enough to say that this form of induction without analysis yields only a probability. The very reality of this modest result has been subjected to doubt. It seems that there is nothing convincing about simple enumeration and that it dissipates in absurdities. If induction by simple enumeration were valid, would it not be necessary to believe that the more one has lived, the less chances one has of dying ? (Cf. Keynes, *ibid.*, ch. xxi.) And so this mode of induction is traditionally accused, not of concluding in a probable way, which would be

* *A Treatise on Probability*, ch. I. Perhaps the question is, however, a little more complex: is there actually no intrinsic probability by which a proposition recommends itself more or less to the mind without being related to any of the other opinions which happen to be given ? This probability would then be as direct and immediate as certainty; there would only be a difference of degree. We do not see any reason for not admitting it, and it might be given the name of *plausibility* in order to distinguish it from the probability-relation. But the latter alone is produced by reasoning and more particularly by induction.

true, but of concluding in a very doubtful way, which would bé tantamount to not concluding at all.

Now all the paradoxes that this kind of induction seems to give birth to, disappear or lose their astonishing character as soon as we remember that probability is relative. In the first place, as a matter of fact, the occurrence of a single contrary event does not prove that a prediction founded on numerous instances was not very probable, or even infinitely probable: it proves only that this prediction was not *certain*. In the second place, if a prediction is probable, according to the principle of simple enumeration, when it is brought to a situation composed uniquely of numerous favourable instances, it is not longer probable by relation to a situation which comprises additional external facts making this prediction impossible or very improbable. The two sources of paradox thus disappear; for it is no longer astonishing that a probable prediction is later invalidated, and secondly, that a prediction which is known to be erroneous is not rendered probable by arguments which would make it such in the absence of such knowledge.

The perception of this principle that probability is a relation, not a quality of propositions, takes away from probability what appeared to be its fleeting and provisional character. It makes probability a fact as rigorous as implication, for instance. The propositions that a given group of propositions renders probable to a degree p are as determinate as the propositions that this same group renders certain, and they are sometimes as difficult to discover.

But the relative character of probability, while it solidly assures its existence against the doubts suggested by the first view, introduces a profound difference between probability and certainty, and makes it more difficult to compare them. Thus, it is commonly said that probability by increasing tends to approach certainty as a limit. But that is not true, rigorously speaking. In fact, it would then be necessary for infinite probability to be certainty. This identity is accepted in current discussions. But it is not exact; for there is nothing in the increase of probability, even carried to infinity, which renders this probability less relative to given information, less alterable by some new information: a relativity which separates it infinitely from certainty. The probability that an unknown number is not 1324 is infinite and we cannot conceive of a greater probability. Nevertheless, it is enormously different from certainty. For it is relative to a state of information in which the unknown number can have one value as well as any other. If we learn that there is a probability p, however small, that the true value of the number is under 10,000, the value 1324 in the light of this fact immediately acquires the finite probability of one-tenthousandth of p; and if we are informed that the first three ciphers are 1, 3, and 2, this probability becomes relatively to the new information equal to 1/10. A probability can be infinite but that will by no means make it more absolute. In short, probability is never identical with certainty. Are they at any time equivalent? This is a difficult question, for a probability, even when infinite, allows some chance: the unknown number may be all the while 1324. This chance is undoubtedly negligible; it is very tiny or as small as possible; but it is not nothing, since the chance does exist. As to saying that it is infinitely small, that has no meaning. In fact, the expression can only be applied to a function, and it means that for any value a, there exists a value of the variable or variables making the function smaller than a. But applied to a particular value such as that of the chance subsisting under determinate conditions, the expression infinitely small is a piece of nonsense. The difference between infinite probability and certainty is then a troublesome concept. We shall, in the course of this work, have occasion to say in conformity with usage that a probability approaches or tends towards certainty: let it be quite specific that we shall thereby understand simply that it approaches the highest probability conceivable.

Limitation of this work to inductions from THESE to ALL.— Inference then is the transition to a conclusion of all or part of the certainty of the premises, and induction is an inference whose conclusion refers either to *all* or to *any* of the members of a class A. Also, certain premises refer to such members of A which (since we are not concerned with perfect induction, so called) are not totally known; we propose to investigate the logical principles of induction, that is to say, the other premises which reason declares to be necessary for all inductions.

In this investigation, we shall seek especially those principles of induction which do not logically presuppose any other induction and which we have called primary: for their principles comprehend all inductive reasoning, just as their power limits it.

We shall limit this study to inductions which conclude about *all*. It may be doubted whether they are simpler than inductions about *any*. But, because the former have alone been studied until now, they appear to be more easily analyzable. Anyway, in our opinion, the difficulties in primary induction have not been appreciated, and perhaps it would be better after all not to begin with these mistakes (cf. Keynes, *ibid.*, p. 259). It would then be necessary to stay still further away from known theories and to proceed in an entirely new spirit. But we are not going to start in that way before we have been convinced that there remains no other.

HYPOTHESIS CONCERNING THE TWO ELEMENTARY RELATIONS OF A FACT TO A LAW

Confirmation, Invalidation

CONSIDER the formula or the law: *A entails* B.* How can a particular proposition, or more briefly, a fact, affect its probability? If this fact consists of the presence of B in a case of A, it is favourable to the law "*A entails B*"; on the contrary, if it consists of the absence of B in a case of A, it is unfavourable to this law. It is conceivable that we have here the only two direct modes in which a fact can influence the probability of a law. Given the hypothesis, either a fact realizes the conclusion and lends support to the law, or else it does not realize the conclusion and refuses to support the law: such would be the ultimate effects of the inductive process. A fact which consists of anything but the presence or absence of B in a case of A cannot then act *directly* on the probability of the law *A entails B*. But, once it consists of the presence or absence of N in an instance of M, it would act on the probability of the law *M entails N* either to strengthen or to weaken it. Where a fact does have an effect, it would influence the probability of the law *A entails* B, thanks to the relation of the probabilities of two laws, in the case where such a relation, favourable or contrary, is posited by some premise. Thus, the entire influence of particular truths or facts on the probability of universal propositions or laws would operate by means of these two elementary relations which we shall call *confirmation* and *invalidation*.

This hypothesis cannot claim the force of an axiom. But

* "A *entraîne* B" is translated hereafter as "*A entails B*," for the relation between the causal character A and the effect B is indeterminate. "A involves B" might also convey the sense.—Tr.

it offers itself so naturally and introduces such great simplicity, that reason welcomes it without feeling any imposition. We have not seen it stated in any explicit manner. However, we do not think that anything ever written on induction is incompatible with it.*

We may take this principle, therefore, for our guide.

Theoretical advantage of invalidation over confirmation.—The confirmation which a favourable case lends to a law and the invalidation which a contrary instance produces do not have the same value. A favourable case increases more or less the probability of a law, whereas a contrary case annihilates it entirely. Confirmation supplies only a probability; invalidation on the contrary, creates a certainty. Confirmation is only favourable, while invalidation is fatal.

Of the two elementary operations of facts on laws, the negative effect is therefore alone certain. By that very consideration, it becomes also the more accurate and clearer operation. Indeed, confirmation through a favourable case presents two difficulties which do not exist for invalidation by a contrary case. In the first place, the very reality of this confirmation is doubted when the case which is to bring it about reproduces identically a case already used; for it is a widely accepted opinion that two verifications which are identical in all respects count only as one. In the second place, one wonders how it is possible to measure this confirmation when it does exist, and one does not know what answer to make. The corroborative action of a favourable instance therefore appears enveloped in a kind of mist, whereas the effect of a contrary instance seems to be as limpid and intelligible as it is fatal.

* One might think of induction by *concomitant variations* as not being induction by *confirmation* or *invalidation*. But that cannot be maintained. In fact, what is called thus consists of an ordinary induction which renders certain or probable the law: "A *variation* of A entails a *variation* of B," and also of a deductive transition of this law to the following: "The *elimination* of B entails the *elimination* of A," that is to say, *A entails B*, by the aid of a so-called rational principle (which no mathematician, however, would regard as serious).

That is why, by virtue of its love of lucidity and certainty, the mind inclines without deliberation to a theory of induction based uniquely on the infirmative action of experience. In fact, an induction can conclude with certainty only on the condition that it does not utilize anything but the elementary operations of invalidation. Experience, in the matter of laws, having only the prerogative of denying, can attain as much assurance as it likes only when it affirms by negation. Secondly, this negative action of facts on the probability of laws is the only one which the mind understands clearly from the very first. To base itself solely on negation, is therefore to preserve the hope of conceiving a demonstrative induction, and this also means to satisfy reason.

This propensity of the mind appears to be linked to two opinions which are, so to speak, universal. According to one, induction must be certain in principle in order to be probable in practice. According to the other, the favourable instances or verifications of a law do not corroborate it by reason of their *number*, but only by virtue of their *variety*; the latter alone can appeal to reason. For in order to have an induction certain in principle, it must rest on operations of negation. And if the variety alone of favourable cases has an effect, and not their great number, is it not because these cases themselves corroborate only by excluding mere repetition? Thus the confirmation that the instances of a law appear to lend it directly would itself be indirect and negative in essence. The outcome of induction would reduce itself to the invalidation of possible laws by contrary cases.

Such seems to be the spirit of nearly all that has been written on induction. Sometimes this principle is openly avowed, often it is tacitly assumed, but in every case it directs thinking in the subject, and there is no denying that reason favours it. Let us then postulate it expressly and see where it leads.

INDUCTION BY INVALIDATION

The mechanism of induction by invalidation : elimination.— When we ask facts only to invalidate laws, it is necessary for a certain law to be confirmed. Several possible laws must then be found related in such a manner that the rejection of some of them favours those which remain. This mechanism is called in logic "*elimination.*"

But elimination may be either *partial* or *complete.*

If at least one of a group of propositions is true, elimination is complete when all these propositions except *one* are eliminated. The one that remains is then certain, without any need of knowing what the initial probabilities of the propositions at first were, nor the manner in which the rejection of the first, of the second, of the third, . . . has increased the probability of each of the remaining ones, until finally, the rejection of the next to the last had made the last certain. This final result of complete elimination does not depend on such a calculation.

On the other hand, elimination is partial so long as there remain *several* propositions not invalidated. In order to determine the value to which the probability of one of them amounts to, we must then know the initial probabilities, and we must besides suppose that the relations of the probabilities of the propositions that are not invalidated remain what they were at the beginning. If we make this assumption, the initial probability of the invalidated propositions is distributed among the remaining ones in proportion to the initial probabilities of the latter.

In order for induction by invalidation to operate, the conditions necessary for elimination must be present. The first of these conditions is a premise positing as true at least one of the possible laws of a certain group. If the facts furnish means of complete elimination in this group, this condition is sufficient. On the contrary, if elimination remains partial, it is indispensable, for the evaluation of the favour enjoyed by one of the remaining possible laws, to know the relation of the probabilities of these laws.

The assumption of determinism for any given character.— We must first obtain the general premise of all induction by elimination. How is it possible to form a group of laws, at least one of which is true? We can see only one way, and that is by assuming determinism. Given a character A, we postulate that *any case of A is a case of some other character X every case of which is a case of A*, or more briefly, that the character A cannot be produced without a cause. (By a cause, we mean any sufficient condition. By character we mean any property, whether it is a relation or an attribute, consisting of the existence of an antecedent or of a consequent of a specific kind. Besides, it would not be sufficient to postulate merely that there is some character X every case of which is a case of A, or that there is some cause producing A; for A might as well be produced without a cause, so that one would never be sure of finding in every cause of A some case of A. Now, that is just what is required, as we shall see.) We express this assumption by saying that the character A is *determined*.

Henceforth, every case of the character A furnishes a group of possible laws at least one of which is true. For if we designate by a the class of characters other than A belonging to the considered case, there must be at least one character of this class which *entails* A. The primary condition of induction by elimination used in favour of a law *X entails A* is therefore fulfilled if we suppose that the character A is determined.

This presupposition may, moreover, be made more restricted. We can, in fact, limit the nature of the characters capable of entailing A. Thus, we can suppose that when A is a character of events it is determined in each one of its occurrences by past circumstances, or more particularly, by the immediate past and by what is immediately adjacent in space. Such is undoubtedly the limit of what we can think of postulating *a priori*, that is to say, in a primary induction. But in inductions which are relative to effects of a type already known, agreement is more common. In any case, these ulterior limitations furnished by our present knowledge are founded on analogy and hence cannot be certain.

Induction by elimination requires a deterministic assumption.—The determinism of the consequent character or effect is an indispensable premise of all induction by elimination operating in favour of a law joining two characters. This determinism constitutes the very nerve of reasoning, the lever by which the rejection of certain possibilities redounds to those which remain. Here we have a proposition which can hardly be contested.*

Range of this assumption.—There is, however, room for making three observations.

Although the majority of authors are ready to admit that induction in general rests on a deterministic principle, we have just established determinism only in connection with induction by elimination, whose mainspring is the invalidation of laws by contrary cases. It may be that the same thing is true of all

* Mr. Keynes appears to be among the authors who do not admit this deterministic postulate. In fact, Mr. Keynes thinks he can demonstrate that induction by the accumulation of instances can confer on a law a probability higher than the initial probability of determinism itself. Hence, determinism cannot be a premise of this induction. However, Mr. Keynes maintains that this induction is based on the principle of elimination. Therefore, he simply does not recognize that elimination presupposes determinism as a postulate, and cannot consequently confer a probability exceeding that of its premise.

induction. But what we have just said cannot in any way be an argument in favour of this extension of the principle of determinism.

In addition, since induction by elimination is concerned with establishing a law of the form *X entails A*, it is not universal determinism, but only the determinism of the character A. For by furnishing for each case of A a class of characters, one or the other of which entails A, the determinism of A yields all that is required for an inquiry by elimination. And were A the only determined character in the world, the establishment of *X entails A* by the elimination of the rest of a group of possible laws would not be affected.

Lastly, if the determinism of the character A is the principle by virtue of which the invalidation of the law *Y entails A* favours the law *X entails A*, where X and Y are two characters observed in a same case of A; if, in other terms, the determinism of the character A is the backbone of the establishment of a law about the production of A, that should not be understood to imply that this procedure by elimination demands the *certainty* of this determinism.

For if any reasoning whatsoever confers on its conclusion the degree r of probability or certainty by supposing as certain the premise A, this same reasoning confers on its conclusion a degree r' of probability, weaker but not null, when we suppose this same premise A to be only probable to a degree s. Any argument which is favourable to a conclusion when its premise is certain is still favourable to it, although with less strength, when this premise is only probable. This is an incontestable axiom.

In fact, if it is no longer certain, but only probable to a degree s that at least one of the propositions of a given class is true, elimination operates again as before: it collects at each step for the benefit of the subsisting alternatives the initial capital of certainty or probability, until at last this capital falls entirely to the last one.

Suppose then that it is no longer certain but only probable to a degree s that the character A cannot be produced without a cause (or without a cause of a certain sort) and let α be the class of characters which accompany A in a given individual case. The observation of one of these characters in the absence of A increases again the probability that some one of the remaining ones entails A; and the elimination of all save one makes equal to s the probability that the last, X, entails A. For we can say at this point: either the character A has been produced in the given case without a cause (or without a cause of the supposed sort), or else X causes A; now there is a probability s that the first side of this alternative is false, consequently that the second is true.

It is then inaccurate to say that the certainty of the determinism of the character whose cause is sought, is a necessary prerequisite of induction by elimination. *The probability of this determinism is sufficient.* As slight as it may be, induction by elimination has some force and renders more probable the law in favour of which it operates. The general nature of inference demands this. For it posits that the degradation of a certain premise to a probable premise lessens the force of an argument without destroying it.

However, this same nature of inference postulates as the limit of the probability that induction by elimination can confer on a law of the form X *entails* A, the probability of the determinism of A. For an argument can convey to its conclusion only a probability at most equal to that of its least certain premise.* It then remains true that induction by

* Anyway, the probability to consider is not the one that admits a cause for A in *all its occurrences*, but the one that admits a cause in *any one of its occurrences*: for what is important is the probability of the presence of a single cause in the individual occurrence of A which furnishes the list of possible causes on which elimination is operati.

elimination cannot reach certainty or approach it indefinitely unless the determinism of the character whose cause is sought is certain or infinitely probable.

Such is the meaning, which is, moreover, simply in conformity with logic, of determinism as a premise of induction by elimination.

The other conditions of induction by elimination.—Let A be a character that is determined (or that is determined by some characters of a certain kind). What more is necessary for induction by elimination to operate in favour of a law of the form *X entails A* ? It is still necessary to form the list of characters (or of the characters of that kind) accompanying A in a given individual case. Finally it is necessary that the facts eliminate all the characters of this list except the character X. The law *X entails A* is rendered certain when these three super-premises are certain; and consequently when one or the other of these premises is only probable, the law is rendered probable to the same degree as their group. Such are the conditions and the power of complete induction by elimination. We see that this sort of induction is an inference that is certain, in the sense defined at the beginning.

When the second or third condition is not fulfilled, there subsist alongside of X other characters that are not eliminated, and elimination remains partial or incomplete. By coming under the general hypothesis according to which, with each elimination of a character, the chances of the remaining characters retain their relationship, the probability conferred on the law *X entails A* by the elimination of only some of the concurrent laws *depends on the initial probabilities* of this law and of the remaining laws. So long as these probabilities are not known, we do not know which one is conferred by partial elimination upon the law *X entails A*; every conjecture about this law is a conjecture about the others. The use of partial elimination does not require the knowledge of all the characters which can be the cause of A in any given particular case. On the contrary, it requires the knowledge of the initial probabilities of the characters that are being considered. This condition ought not to be forgotten.

We now know the conditions of induction by elimination, no matter whether it is complete or partial. Let us see if the universe fulfils them.

A.—Instances Completely Known

Let us first examine instances of induction by complete elimination. We are given a character A that is determined, and the list of the characters accompanying A in a given case that are capable of being its cause. It seems that the elimination of all but one of these characters is henceforth merely a question of skill or success. But on the other hand, it may have to face an impossibility that is altogether rational in nature.

Plurality of causes.—Suppose, in fact, that among the characters present with A in the given case and capable of being its cause, we find *more than one* which invariably accompanies A. This is possible because the postulate of the determinism of A says simply that there is *at least* one of these characters entailing A. Let X and Z be two among these, *each* entailing A. It is then impossible to establish by complete elimination *any one of these two laws*. That is evident, for no one of these two can be invalidated while they are both true. But when their truth is not known, induction by elimination stops at the incomplete result that at least one of the two must be true, without being able to say which one, nor if both are true.

But is this plurality of causes in the same case of an effect a special and very rare occurrence? Quite on the contrary, it is the general rule and is absent only in two particular cases.

The possibility of a complex cause renders complete elimination impossible.—As a matter of fact, it may be that the list of the possible causes of A in the case under consideration comprises, beside the characters L, M, N some characters compounded of several of these, such as LM, or LMN: this is what takes place every time that the information which is revealed about the determinism of A does not exclude with certainty causes that are composite or complex in nature.

Let X be the least complex character which entails A. Any other character of which X is a factor, such as LX, then also entails A. We now have the complexity of possible causes at the crux of the problem of elimination. This complexity moreover, is not exceptional, but rather normal, since it exists in all the cases where, on the list of the characters of the instance taken as a basis, the least complex and real cause is not at the same time the most complex of the possible causes.

Induction by elimination will then leave us with the possible laws *X entails A*, *LX entails A*, *MX entails A*, *LM . . . X entails A*, while we have not yet succeeded in demonstrating the first of these laws. It is true that the last is established, since it is presupposed by each one of the preceding ones. But it must be noticed that this last law already follows from the same premise which posits the determinism of A. For the character LM . . . X, the most complex of all, is simply the conjunction of all the possible causes of A in the considered case, and this conjunction, since A is supposed determined, cannot fail to entail A.

But when we are granted a determined character and the list of possible causes which are present in one of its occurrences, the establishment of a law of the causation of this character by the method of induction by complete elimination then meets an insurmountable theoretical obstacle in the possibility of a composite or complex cause.

Now this possibility surely exists, for many effects admit causes more complex than they are themselves. Thus, the colour of a mixture is determined by the diverse colours of its elements. We must conclude that elimination cannot be complete except for the case of the total character uniting all the causes, which has no need of the method of elimination since it entails the effect as a result of what has been postulated in the principle of determinism. And it is to be remembered that induction by complete elimination is the only type of induction that reaches conclusions with certainty.

Partial elimination: a principle directed against the complexity of causes.—It remains to be seen what probability can be gotten from an elimination which is condemned to incompleteness because of the possibility of a complex cause. Let X be the simplest character (of the list) which entails A. All the characters simpler than X or as simple as X have been eliminated; but, there remain the characters which cannot be eliminated without removing A also. These include X of course, and all the characters including X as a factor such as LX, MX, LMX, etc. The laws *X entails A*, *LX entails A*, etc., remain before us. We know that they share their total probability in the ratio of their initial probabilities.

Thereafter, we might think of appealing to an *a priori* principle of simplicity, saying that there is, for every chance p of complexity a certain degree π for a given effect to admit a cause of complexity lower or equal to p, and that this chance π, a function of p, increases in proportion as p increases and finally approaches certainty. Such a principle would be plausible, for it only puts into exact language the accepted opinion that an infinitely complex cause is infinitely less probable.

Insufficiency of such a principle.—It would give to X, the simplest cause possible, a certain advantage over LX, MX, LMX, etc. . . . However, this advantage would remain finite. Its measure would be in fact the value of π corresponding to the degree of complexity of the character X, a terminate finite value. This principle would not then permit induction by elimination to surmount the obstacle which proceeds from the possible complexity of causes except in a very imperfect manner. For it would not put induction by partial elimination, the sole remaining possibility, in a position to confer upon a given true conclusion either certainty (of course, henceforth impossible) or even a probability approaching certainty indefinitely.

Now, that is all that can be expected from induction. That induction does not yield certainty is admitted without difficulty. That practice, by limiting the facts at one's disposal, limits the probability of inductions, will be admitted still more easily. But that a purely theoretical reason, essentially indomitable, condemns induction to stop at a finite probability, is a conclusion that is only accepted as a last resort.

Another principle directed against the plurality of causes.—We may conceive another *a priori* principle of simplicity capable of bringing to induction by elimination an infinitely more efficacious aid against the obstacle of the complexity of causes —an aid that is, however, indirect. This principle would posit as improbable, no longer the *complexity* of causes but their *plurality*.

Suppose, in the first place, that this plurality is excluded. Assume not only that a character A is produced by some cause, but also that it is the effect of one identical cause, so that there exists a character X (of a specified kind or not) which is inseparable from A. Induction by elimination can then establish with certainty what this character is. It ought to be, in fact, one of those which accompany A in any particular case. But we now have the right to eliminate any character which is absent in the presence of A, as well as any character present while A is absent. In this manner we are delivered from superfluous characters such as LX by showing that if they may be sufficient to entail A, X alone is necessary. In order to establish the inseparability of A and X, it is then enough to have two cases of A having only X in common (by neglecting the characters which are not of the specified sort if there is one sort).

But to postulate in advance that a certain character A satisfies the unique condition of being both necessary and sufficient is a very bold assumption. It can be accepted by reason and yet not appear sure. Hence the following principle of probability seems to contain everything that is plausible and acceptable: *For any number n, there is a probability* n *that A involves a necessary condition formed by the alternative of less than n sufficient conditions, and* n *approaches certainty when n increases to infinity.*

It is necessary to speak in this way in order to avoid the following complication. If we said *that the number of causes of A should be less than n,* which would be the more natural way of expressing one's self, we would come up against the fact that when X is the cause of A, any character such as LX is also its cause; so that, in order to state what we mean, it would be necessary to carefully exclude from the account superfluous causes of this type. Thus the expression which appears at first to be most simple would involve in the end a much greater complexity of statement.

However, the principle is not yet quite satisfactorily stated. The preceding formula is good when we do not know anything yet about the number of the causes of the effect A. But suppose that we already know that these causes are pluralistic, that they are *at least* as many as *m*: this knowledge would result from the observation of *m* cases of A not having, taken two at a time, anything in common, i.e. they do not have anything in common that is known to be capable of causing A. Either this knowledge eliminates the operation of the above principle, or it lets it subsist. In the first case, it is impossible to render it infinitely probable that X entails A when we already know that Y entails A, which seems quite absurd. In the second case, what follows is the quite dubious consequence that the more diverse causes of the same effect we know, the less probable is it that there are still more to be admitted. That is because the aforestated principle, if it is posited as applicable to the effects whose diversity of causes is already admitted, goes beyond what is really necessary. All that is required, is that if m is the minimum number of diverse causes that are known to produce an effect A, it is infinitely improbable that an infinitely greater number of causes than m are admissible. We may express this in precise terms as follows: *Knowing that the character A does not admit any necessary condition formed from the alternative of less than m sufficient conditions, if we designate by* n *the probability that A will admit a necessary condition formed from the alternative of less than n sufficient conditions, the value of* $\mathbf{n}$ *approaches unity when* $\mathbf{n}$ *increases to infinity*. Such is the principle which seems to me most easily acceptable. When we know nothing about the plurality of the causes of A, the minimum m takes on the value l and the principle reduces itself to the preceding one.*

The indefinite increase of probability by multiplying instances.—Let there be two cases of A which have nothing in common but X. They no longer suffice, as before, to show that X is both a necessary and sufficient condition of A. But we can say: either X is a sufficient condition of A or it is not. If it is not, A involves in the two cases (since these cases have nothing in common outside of A and of X) two diverse causes. Any necessary and sufficient condition of A is then formed from the alternative of at least two characters. Therefore,

* Instead of making n a function of n, we make n a function of m and n. For a fixed value of m, whatever it is, n approaches 1 when n increases to infinity; but the value of n for a given n may diminish when m increases.

X entails A is as probable as the probability that A involves a necessary and sufficient condition formed from the alternative of less than two characters, namely one character. This probability has the value **n** when $n=2$. This probability **n** has undoubtedly a minimum value in this case, but it is never less than a finite number.

But the same holds true for more than two cases. If we have a hundred cases of A, any two of which have nothing in common but X, we shall demonstrate in the same way that if *X does not entail A*, any necessary and sufficient condition of A may derive from the alternative of at least 100 characters. Consequently, *X entails A* is as probable as the falsity of this consequence. Now this probability will have the value of **n** when $n=100$. It is then clear that the indefinite accumulation of cases of A, any two of which have nothing in common with X, conferring upon the law *X entails A* a probability equal to the successive values of **n** when n increases without limit, makes this law as probable as one wishes. This result is satisfactory, for certainty then seems accessible.

It calls for certain remarks.

In the first place, the kind of induction to which it is applicable is an induction by elimination, i.e. by infirmative invalidation. The multitude of the causes of AX on which it is founded do not serve in any to confirm the law *X entails A*, by reason of favourable instances, but really serve to invalidate other laws by reason of contrary instances. And it is from this invalidation alone that the corroboration of *X entails A* is established by means of our *a priori* principle. In fact, among the necessary and sufficient conditions of A which are initially possible, the observation of n cases of A having nothing but X in common, taken two at a time, eliminates all those possible causes which are formed from the alternative of less than n characters, with the exception of those causes which contain X as one of their characters. But to say that X is one of the characters whose alternative forms a necessary and sufficient condition of A is to say that X is a sufficient condition of A, or that X *entails A*. Such is the mechanism of this inductive inference: it really depends in the last resort only upon the infirmative action of facts on propositions of law.

It is hence remarkable that this infirmative process of elimination brings out the most striking feature of induction by simple enumeration, which is opposite to elimination in principle depending, as it seems, on the corroborative action of its instances. This feature is the rôle that the multiplication of instances plays in the establishment of the law. This numerical factor enters here without owing anything to the uncertainties of practice, since complete knowledge of the characters of each instance has been granted. Despite that, in order to make the law *X entails A* infinitely probable, an infinite number of cases of XA is required. It may appear that their direct confirmative action is being utilized. But in reality, we are utilizing only the infirmative action that they exercise on concurrent laws.

Idea of a theory of induction by repetition.—The appearance of corroboration and the reality of invalidation suggest a theory of induction by repetition. Perhaps, in fact, such induction is never anything more than what has just been described, or is always at least some analogous form of reasoning. The favourable strength of instances which verify a law would not be, as it seems, a direct confirmative influence. Their strength would itself be nothing more than the fatal virtue of those instances which invalidate laws by going counter to them. This conception, as we have just seen, is the basis of the opinion that a new instance does not fortify a law unless it satisfies the condition of being different from all former instances. It is even necessary, according to the preceding theory, where we suppose each instance completely known, that the new instance differs from each former instance in all its characters except the two which the law wishes to bind. Thus we regain through the operation of a number of varied instances that law which engenders and conditions such a variety: the conception of induction by enumeration thus qualified is satisfactory to reason. We shall come across it again in this study.

Summary.—We postulated at first a determined character **A** and the complete list of the characters which accompany it in one of its instances and are capable of entailing it. Unless we are sure that these characters, one or the other of which perhaps entails A, cannot entail several effects—and of that we are rarely sure—or unless A is on the contrary entailed only by the total character uniting all the others—and that is an exceptional state of things fortunately—elimination cannot be completed; so that we cannot establish by induction a law of the form *X entails A* with certainty. Incomplete elimination does not confer, moreover, on this law a definite probability except with the aid of a special principle of probability. If this principle is directed against the complexity of causes, the origin of the difficulty, the principle affords us only an inadequate basis. For it does not permit the law *X entails A*, supposed to be true, to exceed a mediocre probability. On the contrary, if this principle is directed against the plurality of causes, it furnishes a satisfactory solution by means of a detour. It permits us, in fact, to make the law *X entails A* as probable as we please on the condition that we have at our disposal as large a number as we wish of cases of XA not having anything in common but X, when taken two at a time. It thus places induction by elimination under the dependence of number, requiring, besides, variety. Such are the results of the criticism of elimination itself. These results followed from assuming that the characters of each instance of XA, at least those which are to be considered, were completely known, and secondly, that the determinism of the character A is certain or at least infinitely probable. It is these last conditions which we must now examine.

B.—Instances Incompletely Known

The individual samples of nature are known only incompletely.—Retaining determinism, let us ask whether the characters of each instance are in fact so well known in nature, where induction operates surely in so important a way. To ask the question is to answer it. The circumstances involved in any fact of nature, whether physical or mental, are never known except partially. If we do not restrict ourselves, these circumstances, in fact, embrace the totality of the universe in time and in space, a totality which escapes us infinitely. But even if we limit ourselves to the immediate neighbourhood and past, a more profound reason makes the complete knowledge of this limited realm no less inaccessible. It is a matter of fact that the total or partial cause of a comprehensive effect is sometimes hidden. It becomes comprehensible only as the outcome of a test which consists of an experiment properly speaking, or of the application of an instrument such as a microscope, which is also really an experiment. And that is true of mental effects as well as of physical effects. A mental phenomenon may have its total or partial cause in a state which the reaction to a certain test or the answer to skilful questions alone makes manifest to the very consciousness of the subject. This is the whole problem of personality and temperament in psychology.

In the domain of the facts of nature, the result of a test may then be an important circumstance. Hence, to be sure that we have not omitted any circumstance, it would be necessary to have applied all the tests possible. In order to determine surely and completely the state of a piece of matter, it would be necessary to test its behaviour by means of all the substances which will ever be discovered and to examine it by means of all the instruments which will ever be invented. Likewise, to determine with certainty the state of a mind, even for its own consciousness, it would be necessary to apply to it all the tests that the ingenuity of psychologists will ever imagine: a task that is perfectly impossible.

Undoubtedly, it may seem to us that certain tests are sufficient to show the complete manifestation of a given physical or mental state. But we cannot be sure of this, for it remains possible that these tests let differences escape whose action has not been noticed yet, but which really have effects. Such was the electrical state of bodies before their first effects had been discovered. And above all, this more or less strong assurance and presumption of knowing everything necessary to reveal the circumstances which should be taken into consideration, are founded on experience, that is to say, on previous inductions.

In fact, how do we know that in the investigation of the cause of a certain effect, we have made an inspection of the circumstances to be taken into consideration when we have noted certain characters and certain results ? It can only be by means of an analogy with the already known causes of other effects.

Observation can inform us only indirectly that such a character has no part in the production of such an effect. So long as one is in complete ignorance about what is involved, one is just as unaware of what is not involved; and it is only by indicating the characters which should be noted that experience excludes by past selection all the others. That is particularly clear for characters that are not revealed, such as was the state of electricity before it was conceived. For if we judge it improbable that a character of this still unknown sort enters the list of the possible causes of an effect that is being examined, it cannot be by a direct establishment of its lack of causal influence, since it has never been perceived and its very existence is not known. It is therefore really the analogy of laws already demonstrated or probable, and only this analogy, which limits the probable causes of a certain effect to a known part of the characters in the immediate spatio-temporal neighbourhood.

The force of this analogy between the unknown causes of a new effect and the known causes of similar effects is not everywhere the same. Its invariability makes induction more or less powerful according to the novelty of the effect and according to the knowledge acquired about effects of the same kind. This analogy is exact and rigorous in sciences already possessed of certainties; it is obscure and loose when its only foundation is the mass of common experience. Lastly, it is nearest failure when it deals with phenomena detached from both known science and practice, such as so-called " psychical " phenomena. It is then that we see the weakness of induction in its first steps, when it does not know yet where to turn.

Thus, induction gains autonomously a kind of momentum which increases in strength as it progresses. Its power has a " snow-ball " or cumulative effect. The limitation of " important " circumstances can finally have no more effect than a single type of character which must be taken into consideration in the production of a given effect. It suffices to discover the character of this type which is present in order to judge immediately whether it is the cause that is sought. Then that is why scientific induction says proudly: a single experiment is enough, provided that it is done by a man who knows how to direct his attention.

Conditions of primary induction.—But logic, which is sovereign, should lead back to modesty. Induction which depends on an induction for support is a secondary induction. Now no secondary induction, as certain as it may be, taken by itself, yields a result more probable than primary induction can, because no reasoning, even when certain, can render its conclusion more probable than the most doubtful of its premises. If the limitation of circumstances to be considered in the production of a certain effect is the conclusion of an induction—and we have just shown that it is—it must be because induction can be validly exercised without the principle of limited variety. And the probability it then reaches marks the definite limit of its power.

This question concerns logic and not history. In the group of laws that an empirical science contains, the problem is not that of discerning the first fruits of induction which are still devoid of analogy. The points of application of primary induction matter little. It is sufficient to have seen clearly that primary induction is at the basis of the whole induction. If care is not taken, the consideration that any law at all may be established after a few experiments by scientific induction —just as it is done in class room demonstrations and lectures —might lead one to the illusion that the same holds for all laws. But that would be to forget that any law is corroborated so easily only by being aided by the analogy of all the others already known. It would be as if one were to say that a table can stand without legs because any one of its four legs may be removed without making the table fall.

For a science of nature to be established by induction, it is then necessary that induction should know how to accommodate itself to phenomena which are partly unknown, without having any assurance that the unknown part is negligible. Pursuing the study of induction by invalidation, let us investigate how these new conditions of uncertainty alter the power that we have attributed to it.

Primary induction by elimination when applied to nature is not satisfactory.—Suppose in the first place that the complexity of the cause is excluded. Starting from one case of the effect A, it is then possible to eliminate all the characters of this case that have been observed, except the character X, since we have assumed the right to neglect characters more complex than X and containing X as an element. So long as we admitted that this complete elimination in favour of X bore on the totality of the characters present with A in the given case and capable of being its cause, the law *X entails A* was established with the same degree of probability or certainty given to the assumption of A's determinism. But here we admit that certain characters have escaped us. The complete elimination that we thought we could make has therefore affected only an incomplete list. It is really a partial elimination.

It is to be remembered that the result of a partial elimination depends on the values of certain initial probabilities. The initial probability of the eliminated characters is divided, according to the more general hypothesis, among the remaining characters in proportion to their initial probabilities. The probability that the law *X entails A* derives from the elimination of all the concurrent laws which have not escaped us, is not measured by the good will or care we have exhibited in the task of eliminating what is beyond our powers. We cannot say, as we should like, that the law is rendered as probable as is possible at the moment we have done everything in our power. No; the probability of a law depends on rules that are more firmly established than this rule of good will. The probability of the law *X entails A* depends exclusively upon the intrinsic probabilities of this law itself and upon other concurrent laws.

Of these intrinsic probabilities we have no idea. It would be necessary, in order to know them, to compare the chances that each of the observed characters has of being the cause of A, not only with those of the other characters, but again with those chances of the characters which have escaped observation, all of which are unknown, including their number. These probabilities are unknown. They are undoubtedly mediocre, because of their multitude. In any case, they are finite. Elimination can then operate to confer on the law *X entails A* only an insufficient probability and not anything near certainty.

Recourse to repetition.—We cannot be content with this result. For, once again, the power of this first indication which cannot yet depend on any analogy, limits the whole power of induction applied to nature. We should then try to improve the result. The means are evident; repeat experiments. It is commonly admitted, in fact, that in such a condition of ignorance, it is necessary to consent to depend on the number of instances.

But we are not looking simply for what is to be done. We are seeking the logical mechanism and the principle of these counsels of common sense. It is here suggested that we strengthen the obscure and mediocre probability that elimination has just reached by a procedure which is very similar to induction by simple enumeration. But we have already met the same necessity of accumulating instances of the law that we wished to establish. We recall, however, that this accumulation did not operate in a simple manner and according to appearance, but operated indirectly. It might be the same here. Under the multiplication of instances which is imposed upon us for the second time, the principle of elimination might be found again. It may thus suffice, against appearance, to render infinitely probable a law of nature. Such is the possibility which remains to be examined.

Attempt to found the influence of repetition on a principle of elimination: preliminary assumption.—This possibility requires us to posit anew an assumption on which it rested up to this point, namely, that the plurality of causes is improbable. In fact, so long as nothing is assumed in this respect, the elimination of the possible causes of A can be done only by adding to a case of A cases of *non*-A in which certain circumstances are found again and are thus eliminated. But the addition of several cases of A can eliminate nothing, so long as no weight is given to the conjecture that A proceeds in all cases from the same cause. With a view to explaining the influence of the multiplication of instances by the mechanism of elimination, it is necessary in a general way, it seems, to posit some principle directed against the plurality of causes.

However, this principle does not operate here as before. When we supposed each instance perfectly known, the probability conferred upon the law by a great number of its instances, having nothing in common when taken two at a time, was the very probability posited by the principle viz. that the effect did not proceed in so many cases from diverse causes. If we excluded the plurality oı causes, instead of taking it as merely improbable, two instances might suffice to demonstrate the law, and accumulation would no longer play any part. But we are now faced with imperfectly known instances: this imperfection, even excluding any plurality of causes, by itself makes necessary an infinite repetition.

Suppose, for simplicity, that the effect A admits a cause in all its occurrences. The encountering of two cases of A having nothing in common but X does not suffice *according to our knowledge* to give to the law *X entails A* a satisfactory probability. For the probability that they yield is the one obtained from assuming that their unknown parts have also nothing in common. This probability is finite, undoubtedly mediocre, and quite obscure. But the continuous accumulation of such cases of XA having nothing known in common appears capable of gradually raising the probability of the law *X entails A* and to make it as close as one wishes to certainty. Either this is necessarily true or the natural sciences ought to give up trying even to approach certainty. For in the presence of partially unknown instances—and such are all natural phenomena—the number of instances alone affords any hope of compensating the imperfection of analysis. Can its action be explained by a principle of elimination ?

Theory of the probability of elimination.—Yes, answers Mr. Keynes, and it has no other source. If a second case of XA increases the probability of the law *X entails A*, it is because it differs from the first according to our knowledge, or at least it has some chance of differing from it in our ignorance. This elimination, certain or probable, of some character from the initial concurrent case of X as a possible cause of A constitutes the whole favourable influence of a second case of XA on the probability of the law X *entails A*. Likewise, a third, a fourth, an *n*th case of XA operate only because they eliminate or have some chance of eliminating a character common to all the preceding cases. It is by this cumulative tendency of the cases of A to reduce their common part that it increases the chances of the persistent character X to be the cause of A: such would be the real source of induction by repetition.

Let us quote Mr. Keynes:

> "The whole process of strengthening the argument in favour of the generalization ϕ *entails* f* by the accumulation of further experience appears to me to consist in making the argument approximate as nearly as possible to the conditions of a perfect analogy, by steadily reducing the comprehensiveness of those resemblances ϕ_1, between the instances which our generalization disregards. Thus

* Designated in the text by $g(\phi f)$.

> the advantage of additional instances, derived from experience, arises not out of their number as such, but out of their tendency to limit and reduce the comprehensiveness of the class ϕ_1. . . ." (*A Treatise on Probability*, p. 227-228.)
>
> And again: "I hold then that our object is always to increase the Negative Analogy, or, which is the same thing, to diminish the characteristics common to all the examined instances and yet not taken account of by our generalization. Our method, however, may be one which certainly achieves this object, or it may be one which possibly achieves it. The former of these, which is obviously the more satisfactory, may consist either in increasing our definite knowledge respecting instances examined already, or in finding additional instances respecting which definite knowledge is obtainable. The second of them consists in finding additional instances of the generalization, about which, however, our definite knowledge may be meagre; such further instances, if our knowledge about them were more complete, would either increase or leave unchanged the Negative Analogy; in the former case they would strengthen the argument, and in the latter case they would weaken it; and they must, therefore, be allowed some weight." (*Ibid.*, p. 234.)

The theory of induction by repetition which these two passages summarize very clearly is most surely deductive. It justifies the opinion common among philosophers that several instances not possessing, according to one's sure knowledge, any difference, would not have more weight than a single instance. It encourages the idea that induction by multiplying instances is not really a valid principle, but is efficacious only to the extent that it imitates induction by analysis. It brings all induction back to elementary infirmative operations. So this doctrine has something distinctive about it which pleases the mind. The theory that Mr. Keynes proposes is hence incontestably the natural theory. But it is no less necessary to examine it.

Development of the theory of determinism.—We have shown at the beginning of this study that all induction by infirmative action in favour of the law *X entails A* requires as a premise the determinism of the character A, that is to say, the proposition that any sample of A is also an instance of at least a character entailing A.

But it is possible to have at one's disposal more definite knowledge. Let us take a certain sample of A. We have just posited that *the class* α *of all the characters of this sample (other than A)* contains at least one character which entails A. It is not inconceivable that we are in a position to say as much about a *more restricted* class. It is possible that we know that the class a of characters, which is only one part of the class α, contains even by itself at least one character which entails A. It is possible that we know this to be true of *several* partial classes a_1, a_2 . . . formed of characters of the considered sample of A. For it may be that A admits in each one of its occurrences *several* sufficient conditions. Finally, it is possible that for some of these classes—a_1 and a_2 for instance—it is certain that they contain a character which entails A, and that for others—a_3 and a_4 for instance—the same thing may be only *probable to degrees* p_1 *and* p_2 respectively.

This is what happens in the assumption commonly made about the determinism of the characters of natural phenomena. Indeed, if we were limited to the proposition that any one of the characters of a phenomenon is entailed by some other, that would amount to scarcely anything; for the characters of a phenomenon, if they are not limited, include its relationships to all other phenomena past, present, and future. On this point people are ordinarily agreed.

It is taken as certain that any character present in a phenomenon is entailed by at least one of its other characters which involve neither the future, nor even the present, but only *the past*. Furthermore, that any such character is entailed by at least one of the characters which involve only such a date, or more exactly *such a section of the past as we wish,* however short it may be; for we believe that the state of nature in any duration, however brief it is, determines its state at all subsequent times. Again, we assume sometimes that any character of a phenomenon is entailed, if we refer to a section of the past sufficiently proximate, by at least one of the characters which involve a *region of sufficiently restricted space* around the phenomenon studied.*

We may then represent the determinism of the characters of a natural phenomenon by the familiar image of the concentric waves produced by the fall of a stone into a lake. Except that we must imagine the process backwards: the waves starting from the periphery enclose each other and run back towards the place of perturbation, where upon their arrival they reach the cause. Thus, the conditions capable of determining an event occupy at a given previous date a region becoming vaster in proportion as this date recedes further back. Running in from the outer limits of the past, so to speak, they lock themselves around the event and converge towards the very space that the event fills.

In the heart of the class a of characters of a sample of A, those characters which involve any section of the past (and which are also restricted to a finite region of space, if we admit the last assumption) form then a partial class a about which it is *certain* that it contains at least one character which entails A. There is then an infinity of these classes a culminating in the total class a.

This is not all. Consider the class of the circumstances of

* It is clear that this postulate is necessary to eliminate the influence of probable causes escaping us because of their remoteness. This is very clearly stated by M. Painlevé in the article *Mécanique* in the volume *De la Méthode dans les Sciences* (On the Method of the Sciences). We may notice that this postulate, directed against action at a distance, is on the contrary a consequence of the principle according to which any influence is transmitted from the proximate to the proximate.

the given sample of A which are contained in the present like A itself—in other terms, the circumstances which are *contemporaneous* with the effect, and no longer *temporally prior* to it. Is there not some probability that the circumstance A is also entailed by at least one character among its concomitant characters ?

We shall first show that this probability cannot reach certainty as in the preceding case. The thing is evident; it may be demonstrated.

In fact, any character is just as well a conjunction of characters. Thus, all the characters of a phenomenon which are simultaneous with any character M make up all by themselves a character, and if it were *certain* that any present circumstance, hence also this integral character, is entailed by some present circumstance, it would be certain that any one at all of the present circumstances entails all the others. It would be certain that its recurrence would assure their recurrence. Now that is manifestly contrary to fact. In our universe many characters meet without entailing each other. There are then characters, at least complex ones, which are not entailed by any simultaneous character. From which fact it follows that it cannot be certain *a priori* that any character whatsoever is entailed by some simultaneous character.

It will even be admitted without difficulty that the *a priori* probability that such is the case, is very mediocre and very far from being practically equivalent to certainty, although it is not negligible completely. Let us designate it by Π. Likewise, there is a probability π for any character to entail some simultaneous character; and this probability π is neither extremely large nor negligible.

Application to induction by elimination.—In a general way, if the character A is entailed (with a certainty or probability **p**) by at least one member of a partial class *a* of the characters which accompany it in any given instance, the search for a character entailing A may operate by an elimination bearing on all the characters which are members of *a*. But once all these characters less one are rejected, it is known with a degree **p** of certainty or probability, and only to this degree, that the character left entails A.

These results are going to be useful to us in a moment.

Can the probability of elimination be the principle of induction by repetition in its application to nature?—It is not without some hesitation that we are presenting the following arguments. They are longer and more complex than might be desired; they demand an effort of attention. But believing them correct, it is best to offer them. We must follow the argument whither it leads: ὅπῃ ἂν ὁ λόγος ὥσπερ πνεῦμα φέρῃ, ταύτῃ ἰτέον.

If the only resource of induction is infirmative; if its only mode of operation is the rejection of possible causes in favour of the remaining ones by elimination; if the favourable influence of a new instance of the law *X entails A* consists altogether of the certain or probable elimination of some new member of a class *a* of circumstances present with X and A in the initial instance, at least one member of which it is certain or probable (to the degree **p**) entails A, the following results ensue:

We have already shown that this view requires some principle directed against the plurality of the causes of the characters A whose production is to be rendered in some probable law. Let us grant the maximum probability by excluding any plurality of causes on the assumption that there exists at least one character (simple or complex) entailing A, and conversely, entailed by A. Assume as certain that a partial class *a* of the circumstances of any instance of A contains such a character. According to what has just been said, this class cannot be the class of the circumstances contemporaneous with the effect A. It must be the class of the circumstances relative to some *antecedent* duration, that will be imagined, most naturally, very close to the very appearance of A. We shall call the members of this class (for we shall consider only one of them) the *antecedents* of A in the instance in question. We shall call the characters contemporaneous with A the *concomitants* of A.

Thus, we posit it as *certain* that at least one of the antecedents of any character is inseparable from it. On the other hand, it remains *probable to the degree* Π that any character whatsoever is entailed by at least one of its concomitants, and *probable to the degree* π that it entails at least one of its concomitants. The values Π and π are neither negligible nor practically equivalent to certainty. This group of assumptions is surely the maximum of what can be thought to agree *a priori* with the determinism of natural phenomena. If we succeed in showing that under these very favourable conditions the theory under consideration is not satisfactory, we shall have really proved *a fortiori* that it is not satisfactory under any conditions.

The probability conferred upon the law *X entails A* (X being an antecedent of A in a given case) by a number n of its instances is, in the concept under examination, a probability of the second order. It is the probability that we shall find among these n instances the realization of a certain possibility of elimination of the antecedents of A, which would itself give to the law *X entails A* a certain probability. It is in this way that induction by the multiplication of instances, instead of being a true argument, would be only the shadow of one.

This idea demands more exact statement. For we do not know with certainty just to what point our n instances of XA force the elimination of the antecedents of A. Perhaps they keep only X; perhaps they leave another subsist, or two others, or x others. But these different hypotheses may be unequally probable. In each one of them, the probability of the law *X entails A* is whatever the hypothesis would yield, if it were realized, multiplied by the probability of this realization. The round total of the probability conferred upon the law is then some average value among all these products, lower than the greatest probability among them.

In order for this probability to approach certainty when n increases, it is necessary for one of these products to do as much. And for that to happen, it is necessary that the multiplication to infinity of the instances of XA renders infinitely probable the realization of a possibility that elimination itself will render the law *X entails A* either certain or infinitely probable.

This possibility of elimination can be only one of the following two conditions: X *is the sole antecedent of A which is common to all the instances considered*; in this case, *X entails A* is certain. In the second case, antecedents of A other than X remain common to all these instances. But *it is infinitely probable that these other antecedents either entail A or else are entailed by* X. In both cases, *X entails A* is an infinitely probable effect.

We can even neglect here the first alternative.* For the antecedents other than those in question are characters which have escaped the means of observation employed. We know nothing of their nature, and it cannot be less probable, and more particularly, *infinitely* less probable that A is entailed by some one of them than by X.

A possibility of elimination making *X entails A* certain or infinitely probable is then a state in which X remains *alone* or else *in the presence of other antecedents of A which are with infinite probability entailed by* X.

* It is the hypothesis we met while studying the mechanism of repetition for completely known instances and for a plurality of causes which increases in improbability with an increase in their number.

Does the multiplication to infinity of the instances of XA render infinitely probable the realization of one or the other of these possible states ?

It cannot render the realization of the first possibility infinitely probable. In fact, it does not diminish the initial probability for X to entail some concomitant among the antecedents of A which escape observation; and this probability **p**, without being extremely small or large, is surely not negligible. In the presence of an infinite number of cases of XA not having, so far as our knowledge goes, any common character, a finite probability *at least* equal to **p** subsists then for all those cases having in common one or more antecedents of A unknown to us. We shall designate the group of such cases Y. The probability of a state of things is at least equal to the probability of any hypothesis which implies it.

But we must carefully notice that if the existence of an unobservable concomitant entailed by X implies the presence of the same unobservable character in all the examined cases of X, the latter, on the contrary, does not imply it. For it is evidently possible for a character Y to accompany X in all the cases under consideration and yet abandon it in others. Surely, in proportion as the number of examined cases increases, this hypothesis becomes infinitely probable. But let us take care to remember that it is exactly the very principle of induction by repetition which is behind the whole process, a principle which it is exactly our office to justify in the theory we are examining.

On this theory, is it then infinitely probable that any Y concomitant with X in an infinite number of cases is entailed by X ?

That is what the theory of the probability of elimination should prove. But it does not prove it; it cannot prove it, for only the falsity of the theory makes it true. Let us try to make this contradiction manifest.

Suppose X and Y are the only antecedents of A present in an infinite number of cases of A, and it is asked whether this supposition renders *X entails Y* infinitely probable.

Now, X is not an antecedent of Y, but a concomitant of Y. It is not *certain* that Y is entailed by some one of its concomitants; it is only probable to the degree Π. Hence, in the theory with which we are now concerned, the hypothesis of the elimination of all the concomitants of Y except X does not render *X entails Y* certain or infinitely probable, but only probable to the degree Π.

We might attempt to get around this conclusion through a detour by objecting that the concomitance of X and Y in an infinite number of cases makes it infinitely probable that the antecedent which entails X also entails Y. But this is to fall into a circle. For we have proved that it is not infinitely probable that all these cases will have only one antecedent in common, even if only one is observed. And it is clear that if there are several, there is a finite probability that one of them entails X and the other Y, unless it is infinitely probable that they both entail each other, in which case we find ourselves back at our start.

The whole argument may be summarized in the following way: In the theory which uses the principle of the probability of elimination to support induction by repetition, it is not infinitely probable that an antecedent followed by the effect in an infinite number of cases is the cause of it. For, in the first place, it is not infinitely probable that this antecedent does not entail some other unknown cause simultaneous with it. Secondly, it cannot be infinitely probable that if another antecedent actually accompanies the first in an infinite number of cases, it is then entailed by the first. For it is not infinitely probable *a priori* that a character is entailed by its concomitants.

Such is the chain of reasoning which seems to us to establish the untenability of both Mr. Keynes' theory and philosophical common-sense about the mechanism of induction by simple enumeration. Such induction does not really confer on the laws of nature a probability higher than the *a priori* probability that some character, simple or complex, is entailed by some concomitant character; and this probability Π is very far from certainty.

But we showed before that primary induction, which bears on all our empirical knowledge of nature, cannot yield a probability approaching certainty except by drawing on infinite repetition. It would be thus demonstrated that the idea of founding any induction on a principle of infirmative elimination leads in the end to an impasse, no matter how agreeable the idea is to the mind dominated so easily by this facile process of elimination. For to deliver to physics as a result of this principle only a mediocre probability separated from certainty by an irreducible interval, is an impasse or frustration to the physicist. It would be somewhat pessimistic to attribute such a cruel limitation to the very nature of things, and Mr. Keynes does not intend to do so.

It is interesting, if we wish to make clearer the exact scope of our results, to compare them with the position taken by M. Lachelier in his *Fondement de l'Induction* (*Foundations of Induction*).

M. Lachelier's ideas.—This writer seeks to formulate, but especially, to prove principles apparently capable of justifying induction, without delaying to determine the exact manner by which these principles apply in fact to inductions that confer a determinate probability. The analysis and verification of his principles are in our opinion fundamental to his theses.

He first presents the classical thesis according to which the essential premise of induction is the determination of phenomena by their antecedents. But—and this is his special thesis—this first principle is not sufficient; because among the multitude of the antecedents necessary for the production of a certain effect, there may be found, we even know that there will be found, unknowable factors. When we assert that the recurrence of such unknowable antecedents should entail the recurrence of the effect, we suppose evidently, by virtue of some other principle, that all the antecedents required are in fact reunited, at least in most cases. Lachelier gives as an illustration the biological law according to which similar reproduces similar. He then observes that the intervention of imperceptible conditions is no less present in physical or chemical phenomena than in the phenomena of biology. He shows that his principle of the coherency of simultaneous characters in groups is equally necessary for all the inductions of the natural sciences. He conceives this mutual determination of concomitant circumstances as "a principle of order which guards the preservation of kinds." He perceives a teleological necessity about it which he summarizes as follows: "We can then say in a word that the possibility of induction rests on the double principle of efficient and final causes." (Lachelier: *Fondement de l'Induction*, p. 12.)

He realizes that his second principle cannot be, like the first, a principle of certainty, but that it is only a principle of probability. In fact, the coherency of concomitant characters is limited: only certain groups form "kinds" which persist. We cannot, hence, be certain that the recurrence of observable characters by the means at one's disposal assures the recurrence of the imperceptible characters which are perhaps necessary to engender the effect.

Lachelier stops there. He thinks he has solved the properly logical problem of induction by having formulated principles which evidently justify induction. For him, as for most others, the important thing is not to see what the principles of induction are—that seems too easy to them—but indeed to prove them: "They abandon things, and run to causes."* In his haste to pass to this metaphysical task, he does not perceive that the principles whose proof he pursues are not sufficient in any way to justify inductions.

In fact, it is not *certain* that the recurrence of the knowable antecedents of an effect assures the recurrence of the imperceptible antecedents necessary for the reproduction of this effect. It is, therefore, only probable; and in the second place, we can recognize in Lachelier's principle of final causes the assumption which posits the *a priori* probability Π that any character is entailed by its concomitants. Thus, all that results from the two principles of Lachelier is a mediocre probability that the recurrence of the antecedents observed in a certain case assures the recurrence of the effect observed in this same case.

This result cannot be sufficient. We should be able to improve it. It can be done as a matter of fact. How? By multiplying instances. For it is admitted that the initial probability that perceived antecedents entail the perceived effect—and this is all that Lachelier's principles tell us—is capable of being annihilated by one contrary instance, and on the other hand, is also capable of being increased by favourable instances until it approaches certainty.

Lachelier does not stop to consider this ultimate bearing of the facts, nor do his principles take them into account. According to his theory, the probability of the connection of a consequent with an antecedent depends on the probability of the antecedents among themselves, that is to say, as concomitants. His second principle furnishes a certain *a priori* probability for the connection of any two concomitants. But who can assure us that the concomitants *observed the greatest number of times* together have the greatest chances of being

* Montaigne, III, ii.—In this way Lachelier devotes twelve pages to the formulation of the principles of induction and eighty to their proof.

connected; and that the probability of their being connected can even rise above the *a priori* probability that one or the other of them is connected with some concomitant, and approach certainty? This question is neither solved, nor even stated by him. Yet it is the most important as well as the most difficult question of induction.

Can the probability of elimination be the principle of induction by repetition in its application to numbers?—First of all, does induction by multiplying instances apply to numbers? Does the multiplication of the verifications of a formula or law uncertain by itself confer on the law a probability that increases towards certainty, in the domain of numbers as well as that of nature?

The possibility, in this domain, of certain demonstrations, and the exclusive value that is attached to them result in making induction by instances unnecessary in principle. It is not officially admitted in mathematics, which is content with nothing less than the certainty it has already once tasted. The very validity of induction by instances is doubted. It is thought, not only that the conclusions of induction in arithmetic state only a probability, but also that this probability is in itself precarious and unsubstantial. Illustrations may serve as suggestions and guides in discovery, but not in the establishment of theorems. If we employ them in the process of discovery, it is at our own risk and peril. No degree of certainty can be founded on experimental illustrations.

This view seems on reflection to be hardly rational. The precariousness of the probability founded on instances is not more real, in fact, in arithmetic than in nature. It proceeds in both cases from the fact that it is forgotten that probability is *relative to the information at our disposal*. Thus the discovery of a demonstration of the truth or falsity of a law about which we knew only numerous verifications cannot weaken the fact that the information which we had was sufficient then to render the law very probable, and only very probable.

But above all, if instances do not support any legitimate probability, mathematicians who still follow the guidance of instances in the investigation of theorems—and the best mathematicians have not failed to do so—are not acting rationally. Mr. Keynes expresses this point very well: "Generalizations have been suggested nearly as often, perhaps, in the logical and mathematical sciences, as in the physical, by the recognition of particular instances, even when formal proof has been forthcoming subsequently. Yet if the suggestions of analogy have no appreciable probability in the formal sciences, and should be permitted only in the material, it must be unreasonable for us to pursue them. If no finite probability exists that a formula for which we have empirical verification, is in fact universally true, Newton was acting fortunately, but not reasonably, when he hit on the Binomial Theorem by methods of empiricism. (See Keynes' reference to Jevons, *Principles of Science*, 1874, p. 231.)

"I am inclined to believe, therefore, that, if we trust the promptings of common sense, we have the same kind of ground for trusting analogy in mathematics that we have in physics, and that we ought to be able to apply any justification of the method, which suits the latter case, to the former also." (*A Treatise on Probability*, p. 243-244.)

Is not Mr. Keynes' judgment reasonable? It is inviting, and in any case authorizes us, to examine this application to numbers of his theory that the probability of elimination is the principle of induction by simple enumeration.

The insufficiency of this theory in this domain of numbers appears quite clearly. There is no need here to mention temporal distinctions in the determinism of arithmetical characters.

Is it *certain* or *infinitely probable* that any general character of the number *n*, formed from one or several properties—such as that of being the first, or perfect,* or square—is entailed by some other general character of this same number *n* ?

No, that is neither certain nor infinitely probable, but only probable to a finite degree **p**. For we know that the general characters of numbers form several groups and that it is not sufficient to fix one of the characters of a number for all the others to be equally fixed. Consequently, there is a finite probability not to be neglected, whose value is 1 - **p** that any general character A is not entailed by any other character. If it were entailed, it would form an independent group, complex or even simple; this would be a fundamental property which does not depend on any other property.

According to the theory which we are now discussing, " The whole process of strengthening the argument in favour of the generalization ϕ *entails* f by the accumulation of further experience appears to me to consist in making the argument approximate as nearly as possible to the conditions of a perfect analogy, by steadily reducing the comprehensiveness of those resemblances between the instances which our generalization disregards." (*A Treatise on Probability*, p. 227.) And by perfect analogy, Mr. Keynes understands the union of two or several causes of XA which eliminates all the rest, that is to say, which does not have any other character in common.

Any collection of numbers presenting the two properties X and A always has, undoubtedly, other general common properties. We should then have to say, according to Mr. Keynes, that this collection, no matter how numerous it is, does not exemplify a perfect analogy, and cannot but constitute an argument *inferior* to perfect analogy which is the ideal and limit. That is the whole thesis which Mr. Keynes defends.

* A perfect number is a number equal to the sum of its factors, e.g. $6=3\times 2\times 1=3+2+1$. For theorems concerning such rare numbers, cf. Dickson, *History of Theory of Numbers*.—Transl.

But what probability would perfect analogy itself confer on the law *X entails A* ? If we knew that two numbers *m* and *n* have no other general property in common but X and A (which is impossible), what degree of probability would result for the law *X entails A* ? Would it be certainty ? Would it be an infinite probability ? No, it would be only the finite probability **p**. In fact, there is a probality $1 - \mathbf{p}$ that the general property X depends on no other, and consequently does not depend on A. All that a perfect analogy can prove is that X is the only general property which *may* entail A. But *X entails A* does not result except to the degree in which it is probable that A is not an independent group of general properties of numbers or a fundamental general property not entailed by any other of its properties.

According to Mr. Keynes, the probability that a perfect analogy would establish, limits ideally the probability that the multiplication of instances can give. *The latter does not approach certainty, but remains lower than the finite probability* **p**: a result that is hardly satisfactory.

Conclusion of the study of induction by invalidation.—Let us summarize the preceding analysis.

We first stated the postulate that the whole influence of facts on the probability of laws resolves itself into these two primitive operations: confirmation and invalidation by means of favourable or opposing instances. We have analyzed the theoretical advantage there would be in conceiving infirmative invalidation as the only source of all inductive inference. We have noticed that philosophers and reason itself had such a propensity. We have tried to grasp the principle of this doctrine and pursue it rigorously, in order to decide whether it is tenable to the end.

In the first place we had to determine the essential and necessary form of induction by invalidation. It is the transference to one of the laws of a given group, by the rejection of all or part of the others, of the certainty or probability of the existence of at least one true law in the group. Such groups of possible laws containing certainly or probably a true law are furnished to induction by special deterministic assumptions. We must postulate that it is certain or probable to a degree **p** that in any one of the instances of the character A whose rule of generation we are trying to establish, there is, in the heart of a certain class α at least one character which entails A. The class α may comprise all the characters of the case or only certain ones selected from them. Starting then from some instance of the character A and from the class α which is in relations with it, induction seeks to make infirmative or to negate by means of contrary facts the connection of A with the greatest possible number of characters of α, thus transferring their initial chances of entailing A to those that remain. Such is *induction by elimination*. It is the only kind of induction possible on the basis of our assumptions.

We have examined the function of this type of induction in the ideal condition where individual cases, completely known to us, do not conceal from us any of their causal circumstances. But even this theory breaks down with the possibility of the complexity of causes. It cannot produce an inductive inference that is certain except by the indirect aid of some assumption directed against the plurality of causes. If we proceed to exclude the possibility of plural causes, certainty becomes accessible again. On the other hand, if we limit ourselves to the postulate that this plurality becomes more improbable as it becomes greater and greater, elimination again furnishes an infinitely probable inference, but on the condition of depending on an infinity of different favourable instances.

We next passed to nature. There no instance is known in all its circumstances for the reason mainly that a causal circumstance is not merely what one actually perceives, but also what one would perceive as a consequence of some experimental test, and we cannot make all the possibly relevant experiments in any one case, or rather we cannot be sure that we have made them. In the situation of primary induction, where we must give up expecting any assistance from empirical knowledge so long as we are aiming at universal connections, induction by elimination can obtain nothing sufficient from phenomena known incompletely to a degree also unknown.

It seems then that the doctrine which seeks in the infirmative rejection of laws by contrary instances the only source of induction should at this point be abandoned. The probability that approximates certainty, and which appears accessible to the natural sciences, must then be demanded principally from the confirmative action of favourable instances, that is to say, from induction by the multiplication of instances.

But in confirmative induction, itself, the doctrine we are studying is met again. That is, here again everything reduces to elimination; but it is an elimination which is only probable, and which would operate in a manner unknown to us on the unknown portion of the instances. It is only the probability of this hidden elimination which would produce the apparently direct confirmative force of collections of favourable instances. It is only by knowledge of the probability of their diversity that two cases indiscernible to us would count as more than one. Reason favours this theory which Mr. Keynes has so well expressed.

First of all, we have shown that it requires the aid of some principle directed against the plurality of causes. We have postulated that any character admits a unique cause, in order to remove this preliminary difficulty and examine in its very principle the theory of the probability of elimination.

Following this theory, all induction by repetition admits as its ideal and limit a certain induction by elimination, below which it always remains. Consequently, wherever the determination of a character by some other is *a priori* probable only to the degree **p**, the accumulation of instances cannot give to the connection of two characters anything but a probability lower than **p**; for this value is the maximum result that elimination itself can give. This we thought to be the case in the domain of numbers. On the other hand, wherever determinism distinguishes several partial classes in the heart of the total class of the characters of an instance, by serially ordering the circumstances of any instance into several sections $a_1, a_2, a_3, \ldots$ and postulating as *certain* that every character of one of these sections is entailed by some character of each preceding section; and also by assuming that it is *probable only to a finite degree* that it is entailed by some other character of its own section (and consequently that it entails some other character of its own section)—this is what Lachelier's second principle amounts to—we have tried to prove that the accumulation of instances, as far as they are pushed, cannot give to the proposition that one character entails another any but a finite probability. Now, such is the case with the phenomena of nature.

The conclusion we are led to is that neither in the domain of numbers, nor in the realm of nature, does the probability of elimination, postulated as the only source of induction by the repetition of instances, confer on this induction a force sufficient to approximate certainty. But the repetition of instances contained the only hope that remained to raise to practical certainty the mediocre probability furnished by deliberate elimination, employed on natural phenomena without any acquired knowledge to direct or certify its operation. Thus the theory which sees in invalidation either the overt or hidden principle of all induction found itself incapable of conferring on any law of nature an infinite probability. It does not allow physics to exceed, no matter how carefully and perseveringly it operates, a mediocre probability which is determined *a priori*.

We cannot absolutely reject this result, in our present great ignorance of the nature of induction, and consequently of its power and limits. On the other hand, we can only admit it if there exists no plausible theory allowing us to avoid it. Now, we have examined no more than one theory, which despite its advantages did not show itself reliable. Of the two elementary operations that the mind thinks it discerns in the relationship of facts to laws (invalidation and confirmation) only one of the two is admissible. For the mind tries to reduce the confirmation of a law by the proper instances to the invalidation of other concurrent laws. This doctrine, embraced by reason at first, willy-nilly, leads to nothing but a mediocre result, unfavourable to the natural sciences.

It is time to free ourselves from its prestige by observing that if we had examined it before following it, we would have recognized it as an extreme, and nearly a gamble.

There is then left the other road to be tried. It is possible and natural to conjecture that the corroboration of a law by its instances, viz. induction by repetition, possesses a force which comes elsewhere than from the probability of elimination. This is what we must do if we wish, to speak with Plato, to try to "save" (*Timeus*) our knowledge of nature and to open to it the approach, at least, to certainty.

But this idea brings us to a road where everything is still unknown. It means, in fact, to give up the doctrine which more or less distinctly, has dominated the thought of logicians. It means that we must turn away altogether from the direction in which the mind pursues first the explanation of the confirmative force of instances, and ends finally in the analysis of Mr. Keynes. We must seek elsewhere for an explanation and an analysis; for we cannot claim that this force is explained all alone. We remarked in the beginning: the invalidation of a law by a contrary instance is intelligible by itself. The mind does not ask any question about its foundation or measure.

It sees in invalidation a simple and decisive operation. On the other side, the corroboration of a law by a favourable instance does not have this clarity. Its value remains obscure. The principle according to which the probability produced by a number of instances increasing to infinity would indefinitely approximate certainty lacks evidence. It would require some proof. Now, it does not seem that we can obtain from classic works the slightest aid in the discovery of such a proof. For they all end with the theory of the probability of elimination; and we have seen that this condemns the possibility of certainty.

But a recent work offers us exactly what we are seeking: a justification of induction by repetition and a proof that it tends towards certainty when the number of instances increases to infinity. This work is already known: it is this same *Treatise on Probability* of Mr. Keynes, in which he supports the theory which we have presented and shown to be fatal to this very principle.

We shall present the elegant theorems of Mr. Keynes, detach them from a traditional philosophy definitely refuted by them, and finally examine the proofs given by their author. We shall see why one of them appears to us to be incorrect.

INDUCTION BY CONFIRMATION

MR. KEYNES' theory rests on the fundamental axiom concerning the probability of the conjunction of two propositions. The simplicity of this foundation is remarkable.

Probability of the conjunction of two propositions.—What is the probability that two propositions p and q are both true? The answer often given is that it is the *product* of their two probabilities. Now, it is not so in general, but only in the particular case where these two probabilities are independent of each other, that is to say, where the information that one of the propositions is true would not increase or diminish the probability of the other. On any other hypothesis it is clear that the probability of p and q together is no longer equal to the product of their separate probabilities. In fact, if p has q for its certain consequence, the probability of pq is that of p alone. And if p has q for its probable consequence, the joint probability of pq is even greater than the product. Inversely, if p has *non-q* for a certain consequence, the probability of pq is null; and if p has *non-q* for a probable consequence, the probability of pq is still less than the product.

The probability of pq is therefore not a function of the initial probabilities of p and of q, but a function of the initial probability of p and of the probability of q *if* p. Or else, for reasons of symmetry—for even if p and q refer to events, q may be known if we are given p—it is a function of the initial probability of q and of the probability of p, given q. Again by symmetry this is the same function. This function is, as before, the *product* of these two probabilities.

Designate by x/y the probability of x being concluded from

y. Let h be the initially given premises or information known. We shall then postulate

$$pq/h = p/h \times q/hp = q/h \times p/hq.$$

Such is the principle; it is at least infinitely plausible. We shall notice that it is universal and independent of any assumption or hypothesis whatsoever.

Justification of induction by repetition.—Let p be a general proposition or law, p/h the initial probability at the moment when it is considered to be indifferent whether no instance or on the contrary any number of instances of the law is known. Let q be the proposition that the law is going to be verified in the new instance E.

If the law p is true, q is certainly true also. We then have

$$q/hp = 1.$$

The principle

$$p/h \times q/hp = q/h \times p/hq$$

then yields the equation

$$\frac{p/h}{p/hq} = \frac{q/h}{1}$$

That is to say, the probability of the law before a verification is to the probability of the law after a verification $p/h : p/hq$ as the probability of this verification itself q/h is to certainty.

In order for the verification q to render the law more probable, we see then that it is necessary and sufficient

(*a*) *that p/h is not null*, that is to say, *that the law possesses independently of this verification some probability, no matter how weak it is;*

(*b*) *that q/h is less than unity*, that is to say, *that the verification q does not follow with certainty from what is already known.*

This theorem justifies induction by repetition.* It estab-

* Again we must not exaggerate the import of this purely theoretical proposition. For it says that the accumulation of the cases verifying the law renders more probable its verification in *all cases* (and conse-

lishes, besides, the dispensability of determinism as a premise. Its strength does not come from a probability of elimination, and even the probable variety of instances is not necessary. The result then is to overthrow the philosophy which we criticized before, and to which Mr. Keynes himself still remains attached. Let us stop a moment to study these important consequences.

Induction by repetition does not have determinism for a premise.—We have, in fact, just proved that any verification which was not certain in advance renders the law more probable only on the condition that the law already possesses some chance, however slight, of being true. Let X and A be the characters joined by the law. For the discovery of A in a given case of X, where its presence could not have been predicted with certainty from what was already known, i.e. to render *X entails A* more probable, the only assumption that must be made is, then, that *X entails A* has already some probability **p** which is not null but as small as one wishes. This assumption implies undoubtedly that the presence in any case of XA of some character entailing A has a probability of at least **p**. For it is probable to this degree that X itself is such a character. But that is *all* that it implies. It does not imply that the presence of such a character is *certain*.

Such induction by repetition requires only, in order to increase the probability of the law *X entails A*, that X should be determined with some degree of probability.

But in order to show that this kind of induction does not have determinism as a premise, we must show again that the law *X entails A* may attain by repetition a probability *higher* than the initial probability of the determinism of A.

quently, in *any case*); but not yet *in all the cases* STILL UNKNOWN, or *in any one of these cases*, a point which is necessary to justify real inductions. In fact, the proof given supposes that the cases recognized as favourable remain part of the sum of the cases. It is no longer valid if we consider only the sum of the cases still unknown.

That is easy. In fact, the initial probability **d** of the presence in any case of XA of some character entailing A should be equal to or higher than the initial probability p/h of the law *X entails A*. We have the right to postulate it as simply equal. It is sufficient to place one's self in the hypothesis where we would be sure that *X alone can entail A*. This particular hypothesis does not prevent the application of the theorem: the probability p/hq of the law *X entails A* after the new verification q is therefore higher than its probability p/h before this verification; and higher in respect of certainty than the probability q/h of the verification q in the prior state of information h. But it is certain that X alone can entail A: the initial probability **d** of the determinism of A is then precisely equal to the initial probability p/h of the law *X entails A*. The verification q then confers on this law a probability higher than the initial probability of the determinism of A. That proves, as we had proposed to show, that determinism is not a premise of induction by repetition.

The force of induction by repetition does not arise from a probability of elimination.—This results immediately from the preceding proposition. For all that perfect elimination can establish is that X alone *muy* entail A. The law *X entails A* would then be found to be heir to all of the initial probability of A's determinism, but *only* of this probability, that being the maximum result attainable by elimination. Now we have supposed it attained; and we have shown that the first new piece of information enabled the probability to increase. That can no longer arise because the new instance has a chance to eliminate some concurrent character of X on the ground of its being a sufficient condition of A, since this elimination has already been completed. The logical mechanism of confirmation by instances does not therefore reduce itself to a probability of elimination.

Let us take, in particular, the case where the existence of some other character inseparably joined to A would be probable to the degree p, and where we would have two instances of XA not having any other character in common. These two instances would embody what Mr. Keynes calls a perfect analogy. Letting himself be guided by the doctrine of elimination, he adds that no new instance could any longer add anything to the probability of the connection of X and A.* But his own theorem demonstrates just the contrary; for it shows that the verification of this connection in a third instance, provided only that we could not have predicted it with certainty, would make the law more probable than it was. We can never be sure, it is true, that two instances of XA differ in all other respects, and we are surely inclined to think that if we looked carefully, we should find other similarities in them. But the fact remains that the probable elimination of these resemblances is not the sole origin of the favourable operation of additional instances, since, even if we suppose this elimination completed, new instances may yet continue to fortify the law.

A new instance identical with an acquired or known instance may render the law more probable.—Let us conceive a universe where two instances might be numerically two without differing in any of their characters. This supposition is unreal, and even absurd. However, it may serve to illustrate a thesis. Mr. Keynes himself employs it to this end when he asserts: ' If the new instances were identical with one of the former instances, a knowledge of the latter would enable us to predict it." (Ibid., p. 236.) Consequently, the second would tell us nothing, and hence, as a result of the theorem, would not increase the probability of the law. It will then be permissible for us also to have recourse to the fiction of two identical

* *A Treatise on Probability*, p. 226.

instances although we do it in order to deny precisely what Mr. Keynes affirms from these examples.

Let us, first of all, determine exactly what makes two identical instances mutually inferrible. Given that the second instance reproduces, with X, all the characters of the first instance other than A, we should be certain that it also reproduces A, even before we have ascertained it.

But it is evident that this certainty is nothing but *the very certainty itself of the determinism of A*. For what we should be certain of, is that the total character, formed by the union of all the characters which accompany A in one of its instances, entails A and cannot be reproduced without A.

Undoubtedly, such is the case in our universe and it is not merely an assumption, simply and mainly because this total character cannot in fact be reproduced. On the other hand, in the fictive universe in which both Mr. Keynes and I discourse, he to assert that two identical instances would be inferable one from the other, and I to doubt the assertion, the objection that might be raised about the identity of indiscernibles no longer is relevant, and our assumption becomes quite real.

Now, this assumption does not operate effectively on the hypothesis of the theorem concerning us at present. This hypothesis, it will be remembered, is only that *X entails A* must possess some initial probability, however slight it may be. We then remain free to suppose the case where the determinism of the character A would not be certain. It would then not be certain that an instance of X presenting all the characters other than the A of an instance of XA already known, must also present A. By virtue of this theorem, the establishment of the presence of A in this second instance would then increase the probability of *X entails A*, since this would be a new verification which could not have been predicted with certainty.

That is not all. Let us even postulate as certain that A is strictly determined. The discovery of an instance of XA identical with an instance already acquired might again increase the probability of *X entails A*, and that by virtue of Mr. Keynes' same theorem which at first seems to imply the contrary.

Symbolize by L, M, N . . . the characters *other than X and A* of these two instances. The first instance being known, it is certain that XLMN . . . entails A; the establishment of *A with XLMN* . . . in the second instance does not add anything to our knowledge. But the establishment of *LMN with X* in this same case does teach us something; namely, it makes the law *X entails LMN* . . . more probable by being an instance of it. In fact the verification of this law in the second instance does not depend with certainty on its verification in the first. Otherwise it would be certain that *all* the instances of XA were identical, and only one of these instances would be sufficient to render *X entails A* certain, a hypothesis which would make it futile to ever investigate any new instances.

Outside of this hypothesis, too unnatural surely for any one to be long detained by it, an instance of AXLMN . . . identical with a preceding case increases then the probability of the law X *entails LMN* . . . when the law is recognized as possible. Now, we have postulated as certain that any X, if it is LMN . . ., entails A. The second instance of AXLMN . . . makes it more probable that any X is LMN . . . so long as the contrary is not rendered certain by the discovery of an X which is not LMN. . . . A second instance of AX identical with the first, would therefore make the law X *entails* A more probable *even if the determinism of A were strictly certain*.

Mr. Keynes' theorem has, therefore, a consequence precisely opposite to what he himself thinks he draws, deceived by the traditional conception of the mechanism of induction. Far from implying that two instances known to be identical can count for no more than a single one, his theorem implies just the contrary.

Most certainly, there are no identical cases and perhaps there cannot be any. Mr. Keynes' assertion was made only to illustrate the doctrine that in a number of cases, it is *only* their variety, certain or probable, which operates. Likewise, we have just illustrated by means of the same fiction the contrary thesis that in a number of instances, it is *not only* their variety, certain or probable, which operates. And we have shown this to be a direct consequence of Mr. Keynes' own theorem.

State of the question.—In the first part of this chapter, we convinced ourselves that the corroborative influence of collections of instances of a law did not have to draw all its force from a probability of elimination in order to approach certainty with regard to our inductive knowledge of the laws of nature, including as a subsidiary the laws of number. The preceding theorem establishes the fact that this condition of being independent of elimination is actually satisfied. It makes certain that induction by simple enumeration is not subjected to the conditions of induction by elimination, and that it is capable in principle of elevating the maximum result of the latter. The question of approaching certainty through the accumulation of instances is now reopened, under the very conditions in which the doctrine of possibility of elimination made this approach impossible. But it is hardly solved satisfactorily. It is not yet proved that the multiplication of the instances of a law confers a probability susceptible of attaining and exceeding any fixed value.

Mr. Keynes thinks he has also proved this, but he does it with the aid of a special postulate.

Two necessary and sufficient conditions for the probability of a law to approach certainty by the multiplication of its instances to infinity.—*It is at first necessary that the law possess, from the very start, a probability that is not null no matter how small it may be.* This condition is recognized as the one we already know necessary for any increase in probability through instances. But in order for this probability to be carried beyond any limit by an infinite number of instances, *it is necessary, besides, that on the hypothesis that the law is false, its successive verification in an infinite number of cases is infinitely improbable; or in more precise terms, that its improbability exceeds any limit for a sufficiently large number of cases.*

In fact, suppose that the law is verified in all the instances known and that these instances are infinite in number. Either the law is true or it is really false. Its truth would render certain the fact of its verification in all these instances. If we admit, in conformity with the above conditions, that the falsity of the law rendered this same fact infinitely improbable and that, besides, this falsity is not infinitely improbable by itself, it follows that this fact, once established, renders the initial law infinitely improbable. And this condition which is sufficient is also necessary.

All that results results directly from the axioms of probability. Let $\frac{p}{h}$ and $\frac{\bar{p}}{h}$ be the respective probabilities of the truth and of the falsity of the law p in the state h of knowledge from which we start. Let V be the fact that the law is found to be verified in an infinite number of cases not included in the given information h. Then V/hp and $V/h\bar{p}$ are the respective probabilities that the law will be certainly verified an infinite number of times relative to our present knowledge (h), on the two hypotheses respectively of the truth and falsity of the law p. But V/hp is given as certain, and is hence equal to unity. We are trying to find the probability p/hV or the degree of probability conferred on the law p relative to our knowledge h and the establishment of the fact V.

We have

$$(1)\ V/h = p/h \times V/hp + \bar{p}/h \times V/h\bar{p}$$

In fact, this means that the probability of V in our present state of knowledge h is divided into the respective probabilities of V on the two mutually exclusive and exhaustive alternatives p and $\bar{p}$, multiplied by the probabilities of these alternatives themselves. This is a fundamental proposition of the logic of probabilities.

Now, the principle postulated at the beginning of this whole development yields

$$p/h \times V/hp = pV/h = V/h \times p/hV$$

By substitution (1) becomes

$$V/h = V/h \times p/hV + \bar{p}/h \times V/h\bar{p}$$

And by transposition, this becomes

$$p/hV = 1 - \frac{\bar{p}/h \times V/h\bar{p}}{V/h}$$

or

$$= 1 - \frac{\bar{p}/h \times V/h\bar{p}}{p/h + \bar{p}/h \times V/h\bar{p}}$$

The upshot of this is that for the probability of p/hV to increase towards unity or certainty when the number of verifications constituting V increases to infinity, we see then that it is necessary and sufficient that p/h is not null and that $V/h\bar{p}$ tends to become null.

Replacing the second condition by a condition that is only sufficient.—Mr. Keynes substitutes for the condition that the probability of verifications of the falsity of p in the above discussion is not null i.e.

$$V/h\bar{p} > O$$

a more onerous condition which implies it, but is not implied by it, and which he thinks can be satisfied with certainty. Let us show wherein the substituted condition is onerous.

The rule of the composition of two probabilities, applied repeatedly with more and more verifications, gives for the joint probability V/hp of n verifications $x_1, x_2, x_3, \ldots x_n$ relative to the state of knowledge h and to the hypothesis of the falsity of the law p the value

$$V/h\bar{p}=x_1 x_2 \ldots x_n/h\bar{p}=x_1/h\bar{p} \times x_2/h\bar{p}x_1 \times \ldots \times x_n/h\bar{p}x_1x_2 \ldots x_{n-1}$$

that is to say: the probability of n successive verifications is equal to the product of their probabilities being given the probabilities of the preceding verifications.

The factors of this product are all less than unity. For the product to approach zero as their number increases, it suffices evidently for the factors not to approach unity but to remain less than a fraction f, itself less than unity. That is, that there exists a finite quantity ε such that we have, no matter what n is,

$$x_n/h\bar{p}\, x_1 x_2 \ldots x_{n-1} < 1-\varepsilon$$

Such is the condition Mr. Keynes tries to satisfy.

It is to be noticed that this condition is sufficient but no longer necessary. For a product of an increasing number of fractions may tend towards zero, whereas its factors tend towards unity; for instance the product

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1}$$

whose last term $\frac{n}{n+1}$ tends towards 1, and whose value $\frac{1}{n+1}$ (by cancellation) tends no less towards zero (as n approaches infinity).

We can then establish the fact, not only that on the hypothesis ($\bar{p}$) of the falsity of the law, is its verification in an infinite number of cases infinitely improbable, but in addition, that on this hypothesis ($\bar{p}$) not even an infinite number of positive verifications *will make it infinitely probable that the next instance will again verify the law p.*

We have just seen that the condition is no longer a necessary one. From the very first, it is doubtful whether our universe satisfies such a condition. It amounts to saying that if we *know* that a rule admits some exception, the observation of as many millions or billions of successive verifications as you please cannot reduce the chance below a fixed limit that the next instance has of just being an exception. It asserts that if we had once seen a man ten feet tall, the observation of as large a number of men as you please less than ten feet in height could not then render as probable as you please that men more than ten feet tall are as rare as you please; or else again, that if we had demonstrated that two properties of numbers are not always found together, the observation of their being in connection in as large as multitude of numbers as one wants to try, could not then make it as probable as one wants that they will be found together in the next number that will be tried. Such is the condition Mr. Keynes asks in order that the accumulation of instances in the absence of exceptions or contrary proof, may make it as probable as one wants that any man is less than ten feet tall or that two arithmetic properties are to be found together. We shall agree that it ought to be quite difficult to satisfy such a condition. However, he thinks he can fulfil the condition, as well as the fundamental condition of the existence of an initial probability that is not null in favour of the law p, by the aid of a very plausible postulate which he calls the postulate of the limitation of independent variety.

The postulate of the limitation of independent variety.—This postulate consists in assuming that the characters of the universe selected for consideration arrange themselves in a *finite* number of groups, a certain member of which entails the others. Mr. Keynes' work shows a very interesting development of the character and range of this postulate.

It satisfies the first condition.—The upshot of this postulate is that any character X taken at random possesses *a priori* a finite chance of entailing the character A, also taken at random. In fact, the character A possesses a finite chance of being a part of one or of several groups taken at random, since the number of these groups is finite. Hence, it possesses a finite chance of being a member of the group or groups of which X is a part, that is to say, A has a finite chance (not null) of being present in all the cases of X. The first condition would then be actually satisfied.*

But does it also satisfy the second condition? Mr. Keynes' reasoning.—It is the second condition which produces a difficulty. Let us analyze Mr. Keynes' reasoning since he exhibits it in his Treatise in a more condensed form (p. 254).

The number of the individuals or instances in the domain of natural phenomena or numbers may be conceived as infinite—and it even ought to be so—since we are considering what the probability of a law becomes at the limit when the series of its instances is indefinitely prolonged. *The number of the characters* of these instances, and even of any one of them, may also be infinite. What is finite, however, is only the number of the *groups of characters* entailed by a certain member of the group, or in other terms, *the number of the characters sufficient to determine all the others.*

Consequently, *the number of non-identical or distinct cases*

* To speak in all rigour, it would be necessary to assume not only that the number of groups of connected characters is some finite number x, but also that there is a finite probability that x is less than a given number—than a billion, for instance. For if all the finite numbers have the same chances of being x, it is infinitely more probable that *x is higher or lower than any assigned number*, and hence not finite.

is finite. For it is in fact limited by the number of the combinations of these determinant characters.

It is on this basis that Mr. Keynes works:

If the law p is false, he says, it is false in at least one case. But the number of distinct cases, say N, is finite. On the other hand, it is natural to admit by virtue of the principle of indifference, that it is not more probable at any moment whatsoever for the new instance which is going to appear to be one rather than another of the existing cases. Hence, the case or cases invalidating the law p have, no matter at what moment, a chance of appearing equal to $\varepsilon = \frac{1}{N}$, and we have for all the values of n,

$$x_n/h\bar{p}x_1x_2 \ldots x_{n-1} < 1 - \varepsilon$$

This reasoning rests on an unacceptable hypothesis.—The nerve of the argument is evidently the finitude of the number of cases. From this finitude should in fact follow the existence of a finite lower limit of the probability that the next case, supposing it to be taken at random, is one of the exceptions to the law. Now, what is given as finite, is *not the number of the individual cases* but only *the number of the non-identical or distinct cases,* which we may call *the number of the species,* including *infimæ species.* Mr. Keynes' reasoning then takes for granted that the cases which are not distinct, no matter how large their number is, constitute no more than a single instance.

Now, this is such a strange assumption that we should hesitate to attribute it to him, if we could doubt that his reasoning requires it.

In fact, it comes down to this; the proportion of the individuals encountered in different species cannot give any indication of the frequency or rarity of these species in the group of the individuals of a genus. If we have met with only a single exception among as large a number as we please of individuals of species belonging to a genus, we cannot say *a priori* that it was more or highly probable that these species are frequent in this genus, nor that any new individual of this genus, taken at random, will be found to belong to these species. Experience should then not be able to modify our initial ignorance about the relative importance of the existing species, about the chances that there are that an unknown sample of a genus belongs to one rather than to the other of its species. Observation should not be able to teach us anything *de multis et paucis* (about what is frequent and what is rare). Such an assumption is in truth unacceptable: and yet it is indispensable to Mr. Keynes' argument.

In fact, if we do not make this assumption his reasoning falls asunder. For we have assumed that the law *p*, "*X entails A*," is false. That is because there is at least one combination of characters, one species, where X is found without A; and the number of species is finite. It is then quite true that the probability, by drawing a *species* of the genus X under conditions where all are equally probable, that we shall find a species without A, is at any moment whatsoever higher than a finite value ϵ. *But we do not by any manner or means actually draw a species, but always an individual.* And for the reasoning to remain applicable, it would be necessary that it should be equally probable at any moment, *not that any one of the individuals of the genus* X *should be drawn*, but really *an individual which is a member of any one whatsoever of the species of the genus* X. Now, it seems to make good sense to say, if we have always encountered among the individuals of the genus X members of species containing A, and if that has happened during as long a series of events as one pleases, that it is thereby very probable (if not as probable as one pleases), that in the heart of the genus X, species lacking A are rarer than species containing A (if not as rare as one pleases). Consequently, the individual of the genus X which is to appear, will be also of a species containing A, *just because* our ignorance allows all individuals still unknown and belonging to the genus A the same chances of appearing.

The demonstration which Mr. Keynes based on the postulated finitude of the number of species rests then on the assumption, evidently contrary to the facts, that experience changes nothing of the initial ignorance which makes us regard an unknown individual of a genus as not having more chances of belonging to certain species of this genus than to others. That Mr. Keynes has let himself be misled by so ill grounded a construction, seems to follow from the effect of his general doctrine about the necessary diversity of fruitful cases. Although his assumption, while false in general, does apply to the probability of laws concerning the existence or non-existence of certain species, it is fully absurd to apply it to the frequency or rarity of existing species. We saw, when we studied his first theorem, that Mr. Keynes remained attached to a philosophy of induction incompatible with his positive theory. But with the second theorem, this philosophy has unfortunately introduced itself into his very argument and vitiated it.

Present state of the problem.—It seems to us that we have shown that if elimination is the only source of induction, as logicians and good sense itself incline to believe, no induction in favour of a law can exceed a mediocre probability. We also think we have shown that elimination is not the sole source of such inductions, and that the instances of a law have a corroborative force which is independent of elimination and of determinism. Finally, we have tried to show that nobody has been able to prove that these instances, by being multiplied to infinity, can raise the probability of the law above any limit. Such appears to us to be the present state of the logical problem of induction.

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