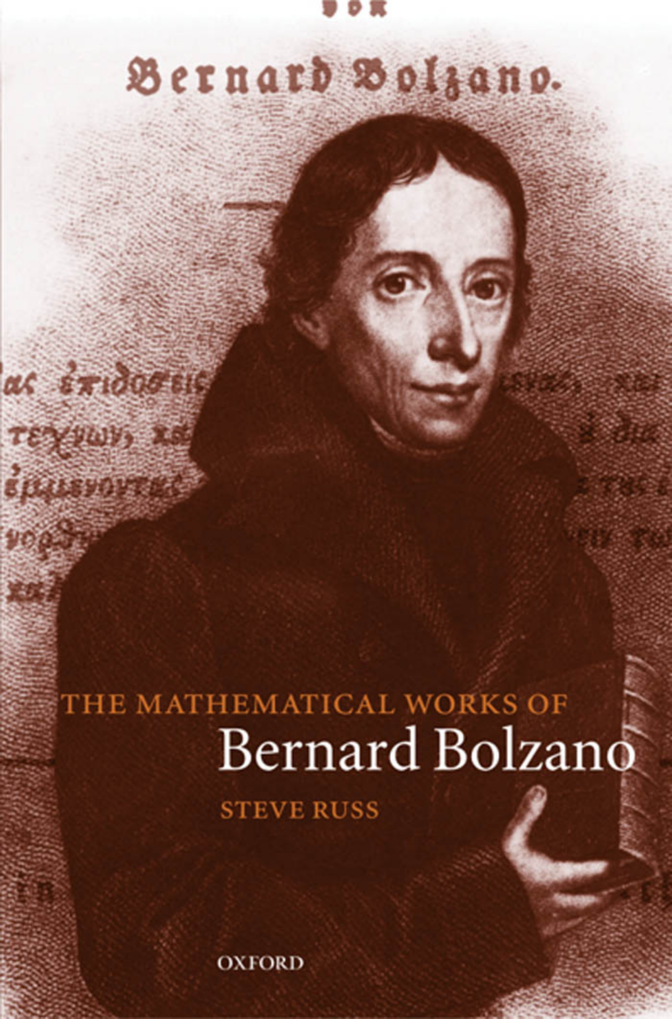

THE MATHEMATICAL WORKS OF

Bernard Bolzano

STEVE RUSS

OXFORD

Bernard Bolzano: oil painting by Heinrich Hollpein (1839–1840)

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The Mathematical Works of Bernard Bolzano

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In Memoriam

John Fauvel

David Fowler

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Abbreviations for the Works

Places, dates and original paginations are given for the first publication of those works that were published in, or close to, Bolzano's lifetime. For the works unpublished until recently (*RZ*, *F* and *F+*) the details are given here of the relevant volume in the *Bernard Bolzano Gesamtausgabe* (*BGA*). For each of the works these were the primary sources for the translation. Most of the works have also had other German editions published and these have always been consulted in the course of preparing the translations. Further details of these editions and other translations of some of the works are to be found in the *Selected Works* on p. 679.

BG Betrachtungen über einige Gegenstände der Elementargeometrie
Considerations on Some Objects of Elementary Geometry
Prague, 1804, X + 63pp.

BD Beyträge zu einer begründeteren Darstellung der Mathematik
Erste Lieferung
Contributions to a Better-Grounded Presentation of Mathematics
First Issue
Prague, 1810, XVI + 152pp.

BL Der binomische Lehrsatz, und als Folgerung aus ihm der polynomische, und die Reihen, die zur Berechnung der Logarithmen und Exponentialgrößen dienen, genauer als bisher erwiesen.
The Binomial Theorem, and as a Consequence from it the Polynomial Theorem, and the Series which serve for the Calculation of Logarithmic and Exponential Quantities, proved more strictly than before.
Prague, 1816, XVI + 144pp.

RB Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege.
Purely Analytic Proof of the Theorem, that between any two Values which give Results of Opposite Sign, there lies at least one real Root of the Equation
Prague, 1817, 60pp.

DP Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahmen des Archimedes, und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst: zugleich als Probe einer gänzlichen Umstaltung der Raumwissenschaft, allen Mathematikern zur Prüfung vorgelegt.

The Three Problems of Rectification, Complanation and Cubature solved without consideration of the infinitely small, without the hypotheses of Archimedes, and without any assumption which is not strictly provable. This is also being presented for the scrutiny of all mathematicians as a sample of a complete reorganisation of the science of space.
Leipzig, 1817, XXIV + 80pp.

RZ Reine Zahlenlehre
Siebenter Abschnitt. Unendliche Größenbegriffe.
Pure Theory of Numbers
Seventh Section: Infinite Quantity Concepts
BGA 2A8 (ed. Jan Berg) Stuttgart, 1976, pp. 100–168.

F Functionenlehre
Theory of Functions
BGA 2A10/1 (ed. Bob van Rootselaar) Stuttgart, 2000, pp. 25–164.

F+ Verbesserungen und Zusätze zur Functionenlehre
Improvements and Additions to the Theory of Functions
BGA 2A10/1 (ed. Bob van Rootselaar) Stuttgart, 2000, pp. 169–190.

PU Paradoxien des Unendlichen
Paradoxes of the Infinite
Leipzig, 1851, 134pp.

Other abbreviations used several times:

OED Oxford English Dictionary Ed. J.A. Simpson and E.S.C. Weiner. 2nd ed. Oxford: Clarendon Press 1989 and OED Online. Oxford University Press. Various dates of access. <http://dictionary.oed.com/>

DSB Dictionary of Scientific Biography Ed. C.C. Gillispie New York: Scribner 1970–

LSJ Greek-English Lexicon Ed. H.G. Liddell, R. Scott, H.S. Jones Oxford: Clarendon Press 1940

All references to German works, whether by Bolzano or by the translator, usually include the original German abbreviations. Some of the most common of these are:

B. = Bd. = Band = volume
S. = Seite = page
Th. = Thl. = Theil (old spelling for Teil) = part
Abth. = Abtheil = section
Aufl. = Auflage = edition

Note that a full point following a numeral indicates an ordinal in German so that 2.B 3.Thl has been rendered Vol. 2 Part 3.

Preface

Bernard Bolzano has not been well-served in the English language. It was almost 150 years after his first publication (a work on geometry) that any substantial appreciation of his mathematical work appeared in English. This was in the first edition of Coolidge's *The Mathematics of Great Amateurs* in 1949. A year later Steele published his translation of the *Paradoxes of the Infinite* with his own, still useful, historical introduction. Over the subsequent half century perhaps about a score of articles or books have been published in English, in all subject areas, that are either about Bolzano's work, or are translations of Bolzano. Some recent works from Paul Rusnock (see *Bibliography*) have gone some way to remedy this neglect in the area of mathematics. The present collection of translations is a contribution with the same purpose.

The main goal of this volume is to present a representative selection of the mathematical work and thought of Bolzano to those who read English much better than they could read the original German sources. It is my hope that the publication of these translations may encourage potential research students, and supervisors, to see that there are numerous significant and interesting research problems, issues, and themes in the work of Bolzano and his contemporaries that would reward further study. Such research would be no small undertaking. Bolzano's thought was all of a piece and to understand his mathematical achievements properly it is necessary to study his work on logic and philosophy, as well as, to some extent, on theology and ethics. Of course, it would also be necessary to acquire the linguistic, historical, and technical skills fit for the purpose. But the period of Bolzano's work is one of the most exciting periods in the history of Europe, from intellectual, political, and cultural points of view. And with over half of the projected 120 volumes of Bolzano's complete works (*BGA*) available, the resources for such research have never been better. The work on mathematics and logic has been particularly well-served through the volumes already published.

It was originally anticipated that each work translated here would be accompanied by some detailed critical commentary on its context and the mathematical achievements it contains. However, to do this properly a substantial proportion of the research described above would have to be completed. In particular, this would involve study of Bolzano's other mathematical and logical writings including his extensive mathematical diaries. To have prolonged the project even further in this way would hardly have been acceptable to any of the parties involved. Though there is, in fact, a certain amount of commentary in footnotes, particularly in *BG* and *RB*; the footnotes are confined, in the main, to translation issues or matters of clarification. Another relevant factor is that in the German *BGA* volumes there is detailed editorial comment on the mathematics as well as textual matters. This is invaluable and must be taken into account in any serious study of Bolzano's mathematics. But, of the nine works translated here, the *BGA* editions have only appeared for *RZ*, *F* and *F+*. In the case of these works, material from some of their most essential editorial footnotes has been included in this volume. The six other works will be published in the *BGA* series in due course. So whatever level of commentary had been given in these translations the coverage across the different works would inevitably be uneven.

Thus this volume contains little substantive assessment of Bolzano's mathematical context or his achievements. But the translations contained here, in so far as they are clear, accessible and mathematically faithful to their sources, will, I hope, prove to be a useful step in that direction. They are presented with a view to drawing greater, and wider, attention to the original works so that such an assessment, or at least work towards it, might more likely be made by other people. To help the reader gain some context and orientation on the works there is a general introduction together with short, more technical introductions to each of the three main parts into which the works are grouped: *Geometry and Foundations*, *Early Analysis* and *Later Analysis and the Infinite*.

I have learned a great deal through the preparation of this volume. First, and foremost, the original motivation for the translations—that Bolzano's mathematical work and thinking is still of sufficient interest that it deserves an English version—has been confirmed and reinforced in innumerable ways. Some of Bolzano's mathematics is very good by any standards; some is rather amateurish and long-winded, some is plain eccentric. But in each of the works included here, whether it be to do with notions of proof, concepts of number, function, geometry or infinite collections, his thinking is fundamental, pioneering, original, far-reaching, and fruitful. In each case his key contributions were taken up later by others, usually independently, and of course improved upon, but they all entered into the mainstream of mathematics. I know of no other mathematician, working in isolation, with such a consistent record of independent, far-sighted, and eventually successful initiatives.

A second lesson for me is that translation is a profoundly interesting process. Translation is often viewed in the English-speaking world as essentially trivial. It is seen as a surface activity, like a change of clothes or a re-wrapping of a parcel: merely a matter of changing the form while preserving the content. Yes, there will be difficulties, so this line of thinking goes, but they are on the 'puzzle' level, a bit of fiddling about and compromise, and a 'good enough' solution will emerge. The translator is seen as subservient to the author and the source text, performing an activity—ideally invisible—of replicating the meaning of the source text in the target language which, if it is English, is tacitly assumed always to be fully fit for the task. Translation is not itself seen as an intellectually interesting, or significant, process. This is not, I think, a caricature, but a widespread attitude, and it must be admitted that translation can be, and often is, done in just such a functional fashion. But such an attitude does little justice to the miracle that is language: the extraordinary richness, colour, mystery, nuance, and unconscious self-expression present in every communication. I wish I had learned earlier than I did of the exalted vision of Walter Benjamin for the translation process (Benjamin, 1999). Then there is also the glorious, detailed, celebration of translation from George Steiner (Steiner, 1998). I would be out of my depth in attempting any summary here of those authors' magisterial works. But I wish to put on record my debt to them for the inspiration and insights that I have gained from these works in particular, among the many others in the rapidly growing field of Translation Studies. It should be noted that Benjamin and Steiner are primarily discussing literary translation. Bolzano's works are neither literary nor technical though they contain technical parts. I have suggested in the *Note on the Translations* that these categories may not be particularly helpful in the context of translation.

Natural languages are not codes. Unlike the situation with a logical calculus, or with formal languages, there is just no possibility of an equivalence of meaning, or equivalence of effect, between natural languages. By their nature they represent vastly complex networks of meanings, associations and usages that are the incommensurable, and constantly changing, products of historical, social, and cultural forces. In practical terms, contemplating a source text, this means there will generally be a huge variety of choices for the exact form of the target text, and a negotiation must ensue between these choices and the source text within a context that includes the purposes of the texts, knowledge of the languages and the subject matter, and the prejudices and inclinations of the translator. The translation process therefore essentially involves an interplay between two kinds of meanings—those that can be accommodated in the source text, and those that can be accommodated in the target language.

A final, and quite unexpected, insight to emerge from this work is that the main strands that inevitably commingle throughout, namely the disciplines of history and of mathematics, and the process of translation, share a common characteristic mode of procedure: namely, what we have just concluded about translation, that at the heart of the process is an interplay between two kinds of meanings.

It is not hard to see the importance of this interplay of kinds of meaning within the discipline of history. To describe the way in which some events, or ideas, were understood at a certain time and place, or how that understanding changed, there are two, closely interacting, stories to tell. There is the view from the time itself, a view seen with the light available at the time, and the sensitivities of the time and the community concerned. But we who are telling the story, trying to take ourselves back into that earlier time, we are ineluctably of our own day, having our own hindsight, education and prejudice; it is a deliberate, imaginative, more or less informed, more or less sympathetic, effort to act as mediator, re-presenting events and ideas of the past in a theatre of the present. The history of mathematics shares completely in this dual aspect of history in general and is perhaps better thought of, at least when the emphasis is on the history, as part of the history of science or, better still, the history of ideas.

Since at least the time of the great Greek contributions to mathematics, there have been those who have explored mathematics for its own sake. Euclid's *Elements* is an outstanding example of assembling knowledge about geometry and arithmetic so as to display and emphasize its deductive structure rather than its practical use. In terms of knowledge about numbers it was *arithmetica* (the science of numbers) rather than *logistica* (or practical calculation). This distinction has been preserved and hallowed ever since. But the vast majority of mathematics throughout history has been motivated by, inspired by, or in the service of, other studies such as astronomy, navigation, surveying, physics, or engineering. This is what came to be called 'mixed mathematics', or by the nineteenth century, 'applied mathematics'. By contrast, the mathematics done for its own sake, became 'pure mathematics', or more commonly in the eighteenth and early nineteenth centuries, 'higher mathematics', or 'general mathematics', which would apply—as a science of quantity—to all quantities in general, with subjects like geometry, arising by specialization to spatial quantities. The abstract entities of pure mathematics, such as numbers, functions, ideal geometric extensions, have a universal, necessary, even—following Kant—an *a priori*, quality. It becomes a challenge for any philosophy of mathematics not only to explain the nature of such entities—seemingly pure and untouchable—but most importantly how they can become embodied and partake in the messy particularity of physical things such as chunks of steel and earth. In other words, how is applied mathematics possible? Here, the interplay of meanings arises in an especially interesting way. The meanings of the very general primitive concepts involved in arithmetic, analysis, geometry, set theory and so on, have typically, since the end of the nineteenth century, been given in a self-contained, implicit fashion through axiom systems. The choice of primitive concepts, and the framing of the axioms have, of course, been guided, or governed, by the mass of informal, intuitive meanings associated with the relevant domain. But when axiomatic theories are 'applied', it is those detached, 'sanitized' meanings that must return to their origins and interact with the meanings arising from particular, experienced, physical things, or patterns of observations. Such issues arising from the existence and success of applied mathematics are closely connected with lively debates in recent decades within the philosophy of mathematics. See, for example, Kitcher (1983) and Corfield (2003).

The work on these translations over many years has been accompanied, and delayed, by the everyday demands of research and teaching in a leading Department of Computer Science. That research has involved thinking about issues that are fundamental in computing. A particular interest of my own is how we might achieve a much closer integration of human and computer processes than has so far been exhibited by conventional computer systems. Typical questions that arise in pursuing this concern are the following. How is it possible to represent parts of the world? (Humans seem very good at it, but in order to have computers solve problems, or assist in doing so, we have to represent those problems somehow in the computer.) How far can human experience be represented on computers? How is it that we 'make sense' of our experience? How does my 'sense', or view, of some part of the world relate to yours? How does the private become public? A common feature of these questions is that they all have to do with semantics—the formation and communication of meanings. There are two kinds of semantics for a computer program. One is the process in machine memory that the program, and its inputs, evokes. That process may be hugely complex, but it is constrained within the memory and may be studied abstractly and mathematically. This is what the computer scientist usually means by the semantics of a program. The end-user of a program however, is apt to merge the program and its process, and the semantics of the program (and process) is then what the results mean for some application in the world (e.g. the projection of a business budget). The two kinds of semantics are closely related (e.g. via the display or other devices). It is this interplay between meanings of different kinds, that is crucial to the usefulness and the progress of a great deal of computing.[a]

So the fledgling discipline of computer science exhibits a similar interplay of two kinds of meanings in the important area of the semantics of programming. This observation was in fact the origin of the identification of the same commonality between the process of translation, and the disciplines of history and applied mathematics. No doubt there are other disciplines, or practices, that also exhibit this phenomenon. But it articulates, with hindsight, a common theme among my own interests, which was apprehended and felt at an early stage. The fact that this common theme has to do with meaning could hardly be more appropriate for a collection of works by Bolzano. The identification and understanding of the concept of meaning was at the heart of Bolzano's thinking. The attempt at a full-scale philosophical account of meaning was the substance of Bolzano's major logical work, *Theory of Science*. In the words of Coffa '[Bolzano] was engaged in the most far-reaching, and successful effort to date to take semantics out of the swamp into which it had been sinking since the days of Descartes' [Coffa, 1991, p. 23].

It seems to me that the separation of pure and applied mathematics in the later nineteenth century was attended by a kind of discontinuity, a tearing in the fabric of mathematical knowledge. Perhaps it was the degree of abstraction, or the reliance on axiom systems, but after this change there was no longer the semblance of an organic wholeness in knowledge. There was a severance of the connectivity, the mutual exchange and interaction of meanings, between experience and theory, and between sensation and thought. For much of applied mathematics, and engineering, this may not have had an adverse effect. Perhaps the 'experiences' required could usually be correlated with patterns of behaviour, or observation statements, which in turn could be successfully associated with mathematical variables. But the technology of computing now allows for, and requires, a deeper level of engagement between a 'formal' machine and experience. This

[a] I am indebted to the work of Brian Cantwell Smith for this analysis of the semantics of a program, for further details see [Smith, 1996, p. 32 ff.].

is an engagement at a cognitive level, more primitive than the recognition and use of standardized behaviours, one in which observations and their interpretation are context dependent, open and negotiable. It is the very nature of modern formal systems that make them, despite any amount of ingenious elaboration, unsuitable to act as foundation, or 'grounding', for the ways in which we are now using, and thinking about, computing. That nature, I believe, can be traced back in many ways to the nineteenth century. In order to develop a broader theory of computation, one that would include the existing theory but better support the ways computers are being used, and the closer integration of human and computing processes, we need, arguably, a breadth of outlook that would embrace both the formal and the informal. We need to restore the connectivity and wholeness that, I am suggesting, was broken during the nineteenth century. Of course, any such restoration, or enrichment of the formal sciences, is now a twenty-first century matter. Indeed a broader theory of computation, possibly based on just such an enrichment, is one of the current goals of the Empirical Modelling research group at the University of Warwick with which I have been actively involved for the past ten years.

History can, however, give us insight that would be foolish to ignore, into the weaknesses, or the fault-lines, the stresses and strains, the needs and motives, which attended the ways that pure mathematics and logic developed in their most formative decades. It would be a major historical, and philosophical, project to examine the suggestion that this development involved some kind of lasting discontinuity, or qualitative change, in the nature of mathematical knowledge. Bolzano's work is not a bad starting place for this study. He was thoroughly imbued with the integrated thinking about mathematics of the eighteenth century, yet he was also, ironically, to have some of the ideas that would contribute, later in the nineteenth century, to the very separation I have decried. But Bolzano's work would only be a starting point. The centre of gravity of this project would doubtless lie later in the nineteenth century.

I have always looked back with gratitude to the late G. T. Kneebone (of the former Bedford College of London University) as the one who first suggested Bolzano to me as 'an interesting person . . . only mentioned in footnotes' in relation to logic and the foundations of mathematics. I am very pleased to thank Clive Kilmister, formerly of King's College, London, who encouraged me in this area of study, Dan Isaacson of Oxford who has enthusiastically supported the work from the earliest days, and Graham Flegg, a founding member of the Open University, who very effectively managed my PhD thesis on Bolzano's early mathematical works (Russ, 1980*a*). That thesis had a long appendix consisting of (rather crude) translations of the five early works. It was the seed from which this volume has grown.

Over the years in which these translations have been prepared I have worked closely with two particular colleagues, Meurig Beynon and Martin Campbell-Kelly, who, in their very different ways, have been an indirect, but profound, influence to broaden and deepen my thinking about the nature of mathematics,

computing, and history. That has helped to shape and improve this work and I am grateful to them both. It is a personal sadness that John Fauvel, a great friend and collaborator, who did so much to support the completion of this volume by way of encouragement, vision, and advice has not lived to see its completion. His faith in his friends was a powerful source of energy for many people and many projects. It is a particular pleasure to record the debt I owe to David Fowler for his wonderful warmth and wisdom, his inspiring care and enthusiasm for the history of mathematics and for his unremitting prodding, teasing, and scholarly advice to me, which is now finally bearing fruit. Again, alas, the pleasure is overshadowed, at the time of writing, by sorrow at the loss of David after a long illness.

Both in the early stages and the more recent stages of preparation I have been very fortunate to have been able to draw on the expertise, and vast knowledge of Bolzano's mathematics, possessed by both Jan Berg and Bob van Rootselaar—through their published editorial commentary in the *Gesamtausgabe* volumes and through personal contact. Their unfailing detailed advice and support has been truly invaluable; I only regret that time has not allowed the inclusion of more of their technical insights from the *BGA* volumes. It has also been an enormous benefit in the final stages of preparation of the translations to be able to incorporate a substantial number of detailed and careful corrections, revisions, and suggestions from Annette Imhausen and Birte Feix. It was a great pleasure to be able to visit the Bolzano-Winter Archive in Salzburg in July 2003 and benefit from the resources there and the deep knowledge, and wise advice, of Edgar Morscher. Also in the later stages of this work it has been invaluable to benefit as I have from the encouragement, advice, and expert knowledge of both Peter Simons and Paul Rusnock. I am also glad to acknowledge the role of a former research student, David Clark. His thesis, Clark (2003), concerning the meaning of computing and the relationship of language to programming, is strongly connected with some of the issues touched upon earlier in this *Preface*. Our discussions were undoubtedly helpful to my own thinking about issues important for Bolzano's work and for computing. A great deal of the complicated typesetting of these works is due to Howard Goodman to whom I remain grateful for introducing me to the workings of TEX, the world of font families, and the subtle aesthetics and benefits of good typesetting. I am pleased also to thank Xiaoran Mo and Vincent Ng for help with some of the typesetting and for re-drawing all Bolzano's original figures so they could appear in convenient positions within the text in *BG* and *DP* for the first time. I am grateful to Ashley Chaloner for a postscript program that allows for the flexible display of the Bolzano function illustrated on p. 352.

A great number of other people have helped very significantly in the preparation of these translations. Help in terms of moral support is as valuable in a major project as technical advice. I have enjoyed both kinds of support for the project as a whole, or for specific parts of it, from the following: Joanna Brook, Tony Crilly, William Ewald, Jaroslav Folta, Ivor Grattan-Guinness, Jeremy Gray, the late Detlef Laugwitz, Dunja Mahmoud-Sharif, David Miller, Peter Neumann, Graham Nudd, Karen Parshall, Hans Röck, Jeff Smith, Jackie Stedall, the late Frank Smithies,

Julia Tompson, Claudia Wegener and Amanda E. M. Wright. My apologies, and thanks, go to any others whose names have inadvertently been omitted here and who have contributed in one way or another to this volume.

I am grateful to the University of Warwick, and in particular the Department of Computer Science, for their patient and generous support of this work, over the years, in innumerable ways. It is also a pleasure to acknowledge the support of a History of Science Grant from the Royal Society in 1992 which helped to fund visits to libraries and collections in both Prague and Vienna. I am pleased to acknowledge assistance from the Austrian National Library in Vienna for supplying digitized images of the title pages for *BG*, *BD*, *BL*, *RB*, and *PU* as well as the three images of manuscript pages from *RZ* and *F*.

Finally, I am very pleased to express my great thanks to Oxford University Press. All long relationships have times of strain, when character is tested, and patience and faith are stretched. I was gratefully amazed at the patience and understanding of Elizabeth Johnston during a long period when circumstances prevented my making progress with the work. I count myself fortunate that the new, vigorous management style of Alison Jones came at a time when I was, finally, able to complete the work; it was just what was needed. I am also very grateful to Anita Petrie who has helped calmly, and professionally, to smooth the way through a complex production process.

In spite of so much assistance I am all too conscious that the work will inevitably still contain errors, omissions, and defects of many kinds for which I alone am responsible. I should be very grateful to receive details of these as they are identified by readers. Corrections, comments, and suggestions for improvement may be sent to me at sbr@dcs.warwick.ac.uk. I anticipate maintaining a website with the original German texts translated here and with all known corrections as these are discovered. This website will be at http://www.dcs.warwick.ac.uk/bolzano/

The referencing system adopted in the volume is very simple. All references to works by Bolzano are made by a group of one or more upper-case characters and identified in the *Selected Works* section. The translated works have various textual forms further identified by a number. For example, the version of *PU* edited by van Rootselaar is referenced as *PU*(5). The selection of Bolzano's work is a very small fraction of what is available, being only what is needed for this volume. There is extensive bibliographic detail in the sources mentioned at the beginning of the *Selected Works*. All references to works by other authors are in a standard Harvard Style and included in the *Bibliography*.

Note on the Texts

The German source texts of the nine works translated in this volume are very varied in their origin and status. The first five works were published between 1804 and 1817, each by different publishers in Prague or Leipzig, and no manuscript copies are known to remain. None of these works has yet appeared in the *BGA*. But important traces and precursors of the ideas for these works are to be found in the manuscript material of the mathematical diaries appearing in the *BGA* Series II B. These diaries have already been published for material written up to the year 1820.

The next two works translated in this volume, *RZ* and *F*, date from the 1830s and until their appearance in the *BGA* (as volumes 2A8 in 1976, and 2A10/1 in 2000, respectively) they had only been published in partial forms. The works were part of Bolzano's mathematical legacy bequeathed to his former student Robert Zimmermann who took up a post as professor of philosophy at Vienna in 1850. The manuscripts came with Zimmermann and remain to this day in the Austrian National Library. They were 'discovered' (among very many other mathematical manuscripts) about 1920 by M. Jašek who realized their significance and began to publish material (mainly in Czech) about them. For further bibliographic details see Jarník (1981). The discovery led to a version of *F* being published by the Royal Bohemian Society for Sciences under the editorship of K. Rychlík in 1930. There are three versions of the manuscript for *RZ* and two for *F*, some are in Bolzano's own hand, others are in the hand of a copyist with corrections and additions by Bolzano. It should be noted that what is referred to here as *RZ* is only the final section of the whole work published in *BGA* 2A8. It is this same section that was given a partial publication in Rychlík (1962). Bolzano's handwriting can be notoriously hard to decipher and Rychlík included only the more easily readable parts and omitted parts he thought were not relevant. In fact some of the parts he omitted were significant improvements to Bolzano's theory. And he included material crossed out by Bolzano. The present author claims no proficiency in reading Bolzano's handwriting and has relied entirely on the expert editors of the *BGA* volumes. The 1930 edition of *F* is, according to van Rootselaar the editor of the *BGA* edition, 'still a valid edition and has always been taken into account in preparation of the present edition' [*BGA* 2A10/1, *Editionsbericht*]. However, it did not include the important improvements and corrections, found later within another separate manuscript and reported on in van Rootselaar (1964). These are included in *BGA* 2A10/1 and in the present translation as *F+*.

In writing his manuscripts Bolzano frequently referred to previous paragraphs by means of the symbol § without any specific numbers. In many cases the *BGA* editors have identified the appropriate paragraph number but in many other cases, especially in *F* and *F+*, the specific paragraph could not clearly be identified so only the section symbol § appears.

The final work *PU* was published posthumously in 1851 following Bolzano's request to his friend Příhonský to act as editor in its publication. Until recently it was believed that only substantial parts of the manuscripts that Příhonský used are extant, but not the final version used by the publishers. However, in the summer of 2003 Edgar Morscher discovered the printer's copy among the papers of Příhonský in Bautzen.[b] For further details of *PU* see Berg (1962, p. 25).

For the early mathematical works there is a facsimile edition *EMW* but both this and the original copies of these five works are now quite rare. Uniquely among these works *RB* also appeared in the Proceedings of the Royal Bohemian Society of Sciences and so enjoyed a significant European circulation. Schubring (1993) is a useful report on a number of reviews of several of the early works, especially in Germany, showing there was significant distribution to some important centres. With the exception of *BL* they have all had later editions as indicated in the *Selected Works*. Each of the first editions of these works has a significant number of errors, either deriving from Bolzano's manuscripts or from transcription errors by the printers. In each case the subsequent editions have corrected some first edition errors but introduced further errors themselves. The work *BL* had its own list of misprints included at the end but there are misprints and numerous omissions even in this list.

The original publications of these early works were made in the German Fraktur font. The convention for giving emphasis in this font was to adopt an extra 'spacing' of the characters of a word. This has been reproduced in these translations by the use of a slanting font although this has in many cases led to a more frequent use of emphasis than would now be usual, and in some cases hardly seems appropriate. It may, of course, have occurred sometimes in error, or merely to help the printer in justifying a line. We have sought to retain it in all cases.

For the works *RZ*, *F* and *F+* the emphasis is given by italics in the *BGA* volumes which corresponds to underlining in the manuscript, and such cases have been rendered with a slanting font in the translations. There is also excellent commentary by way of footnotes in those volumes on variants and corrections in the different manuscript versions. *PU* is the first publication of Bolzano's work appearing in a Roman font and having the diagrams *in situ* within the text.

Since the present volume is primarily for those who cannot easily read the original German it is not appropriate to give detailed accounts of the variations and errors in those texts. After taking account of later editors' comments and corrections such errors have mostly been corrected silently in making

[b] Personal communication, February 2004.

the translations. Occasionally attention is drawn to uncertain, or particularly significant, cases. Fuller details of all known variations and corrections do appear, however, in the electronic version of the German texts on the associated website (see p. xix).

Where footnotes by the editor of *BGA* 2A8, Jan Berg, have been translated and used here in *RZ* this has been indicated by '(JB)' following the footnote. Bob van Rootselaar, the editor of *BGA* 2A10/1, has authorized a general use of his footnotes in the translations *F* and *F+*. For all these works *RZ*, *F* and *F+* it is the *BG*A editions that have been the source, not the original manuscripts transcribed in those editions. Thus when there appear to be errors in the source this may be due to original error in the manuscript or to transcription error. When categorical statements are made about errors these have been checked with the editor concerned.

Note on the Translations

The first five of these translations began life as the appendix to an unpublished PhD thesis (Russ 1980*a*). A version of *RB* very similar to the thesis version was published as Russ (1980*b*). Revised versions of *BD* and *RB*, and the *Preface* of *BG*, appeared in Ewald (1996). In the main these revisions are the translations by the present writer in their versions of the time (1994). They have significant differences from the earlier thesis versions, which in some important cases are due to improvements made by William Ewald. The versions of all five early works appearing in the present volume are so different from the thesis versions, and from those in the Ewald collection, that it is hard to find a single sentence in them in common with the earlier versions. This is partly because of errors or omissions being corrected, but it is also witness to the flexibility and richness of language, the wider experience of the translator and the changing context in which the translations are presented. A text that deals significantly with concepts and meanings that are open to interpretation, rather than being tightly constrained or closed, thereby has a complexity and dynamic, a purposefulness, and yet an autonomy, that fully merits the metaphor of having a 'life' of its own. The present versions not only reflect better knowledge of the source and target languages but also of the purposes and contexts, both cultural and technical, in which the texts were written and in which they are now being re-presented.

The works *BL* and *DP* are published here in English in their entirety for the first time. There have been short extracts of the former work that appear in Rusnock (2000, pp. 64–69) and of the latter work contained in Johnson (1977). The major works *RZ* and *F* went unrecognized and unpublished in any form before the twentieth century; they have also not appeared in English before apart from some short extracts of each of them to be found in Rusnock (2000) and some extracts of *F* in Jarník (1981).

The translation of *PU* in Steele (1950) was the first of any of Bolzano's works to be available in English. It contains an historical introduction and a detailed summary of the contents of the sections laid out in parallel to the similar summary given by the posthumous editor, Příhonský, but reflecting 'a more modern analysis'. Steele has in many ways taken greater liberties with both text and mathematics than I have done and yet his translation still reads for the most part like an older style of English than its date would suggest. For both these reasons, as well as the intrinsic merit of the work, its influence in the nineteenth century, and the difficulty of obtaining Steele's version, it has seemed appropriate to include a new translation in this volume.

It is likely that a translation is generally read by people who are not in a position to evaluate the quality of the translation in its relation to the original source text. Nevertheless, it may not be without interest, and relevance, to the reader to know something of the translator's perception of both the process and the product.

The explicit principle governing these translations, for much of their life, has been to preserve everything to do with Bolzano's mathematics as faithfully and accurately as possible. This has led to the retention of his original notations and layout (unless these were obviously a restriction or convention of the time, such as the placing of collected diagrams on a foldout page). It has also led to the attempt to reflect closely the original terminology and thereby it has occasioned a somewhat literal rendering of the text, sometimes at the expense of a more natural English. The latter is probably inevitable, and even desirable, if a priority is to be made of the thought over the language in which it is expressed in so far as this is possible. But the basic principle of 'preserving the mathematics', while perhaps sounding innocent enough is, I now believe, naïve and flawed. The mathematics cannot, in general, be sharply separated from the insights and the attitudes to concepts and proofs, or beliefs about the status of mathematical objects, or even the motives for developing new theories. And even to the extent that the mathematics *can* sometimes be considered apart from these informal surroundings, the 'meanings' of either the mathematics, or of its informal framework, cannot be 'preserved'. We experience thought as almost inseparable from language; it is commonplace to find we do not know our thoughts until they are articulated, by ourselves or others. Thus it is not to be expected that we can translate Bolzano's language into the form he would have used if he had been expressing his thoughts in English. They would, in English, have been different thoughts. It is, in general, just not possible to separate content sharply from language. The challenge, therefore, is to translate both language and thought together. It is perhaps more useful to think in terms of transformation. The work of translation becomes that of transforming Bolzano's German thoughts into English thoughts in a way that respects their meanings—bringing them as close together as language and our understanding make possible.

It is a pleasing, albeit somewhat misleading, pair of images that Benjamin conjures in *The Task of the Translator*: 'While content and language form a certain unity in the original, like a fruit and its skin, the language of the translation envelops its content like a royal robe with ample folds' (Benjamin 1999, p. 76). But, of course, this is not always so. There is often not such unity in an original text and to be sure about the 'folds' of the translation assumes a direct access to the content of the original text. It is all too easy, in the first instance, to 'read into' the translated text a content that again fits skin-tight to the target language. Nevertheless Benjamin's imagery makes a serious point vividly: the same content will not 'fit' its expression in different languages equally well. The translator is thus not subservient to the preservation of an author's content but may at times need to re-create ideas afresh in the target language. It is this exalted, creative vision of translation that Benjamin extols in his essay. He suggests that translation is not so much a transmission as a re-creation. It is only information, he says, that can be transmitted and that is the inessential part of a text: attending primarily to the transmission function of a translation 'is the hallmark of bad translations'. With that, all the translations of this volume might seem to be condemned; it has been my chief aim to convey Bolzano's way of thinking about mathematical domains, about proofs and concepts, his particular insights into, among other things, numbers, functions and multitudes. Is this 'information'? For Benjamin the essential substance of a text is what it contains in addition to information, 'the unfathomable, the mysterious, the "poetic" '. A consequence of this, again calling for the re-creative function, is that a translation can be seen as part of the 'afterlife' of the original. The initial shock at Benjamin's apparent disparagement of the information content of a text is partly relieved by the fact that he explicitly refers to 'literary' texts.

It still appears to be common in writings on translation to make a broad distinction between 'literary' and 'non-literary' texts, or between 'literary' and 'technical' texts. This is in spite of some extensive studies of text types (such as in Reiss 2000). Although the meaning of 'literary' here is obviously wide, such binary divisions are clearly inadequate and unsuited for their purpose. The *OED* suggests that a primary meaning of 'literary' is 'that kind of written composition which has value on account of its qualities of form'. But on such a criterion a great deal of good mathematics would be literary. The widely admired qualities in mathematical writing of succinctness and clear structure, of economy and precision, and of appropriate notations are pre-eminently qualities of form. Such a classification is presumably not intended. A good deal of philosophy is surely both literary and technical and most texts are neither. As a counterpart to the wide spectrum of text types is the huge range of contents, or meanings, that lie in between Benjamin's 'information' and 'the unfathomable'. In this space lie assumptions, motives, contexts, viewpoints, interpretations and the use of metaphor, among many other components essential to texts of all sorts. All Bolzano's texts in this volume reside in large part in this space. This is for the simple reason that he is, in each of his works, exploring uncharted territory—he is doing original, radical, conceptual analysis of the abstract objects of mathematics, of meanings, of logical relationships, and of the nature of the infinite. Perhaps the spectrum of texts might be characterized with respect to translation in terms of their degree of openness to imaginative and varied interpretation. Thus literary or poetic texts would be deemed highly 'open', while more factual texts such as instruction manuals with their schematic diagrams would be relatively 'closed'. The place of scientific, or technical, texts would depend very largely, I suggest, on the place of those texts within the discipline at the time of their composition. The more they are presenting material that is original, fundamental, and tentative, the more open they are in the sense employed here. It is worth reflecting on the fact that the idea of a 'technical term' with which we are so familiar today, would mean little more at the time Bolzano was writing than the much weaker idea of a 'term of art'. Many of the very words Bolzano was using for major philosophical ideas had only been introduced as German (rather than Latin) philosophical terms less than a century before he began writing. For example, Christian Wolff introduced early in the eighteenth century *Vorstellung* for 'representation' (*representatio*) in general, but it shows how fluid matters were that for representations of *things* Wolff introduced *Begriffe* (concepts), while decades later Kant, for the same purpose was using *Erkenntnisse* (cognitions). To the extent that the translations of this volume engage with, and re-create the exploratory philosophical aspects of Bolzano's work they are open to interpretation, and call for interaction with the reader. In this respect they enter, I hope, at least part of the way into Benjamin's vision for translation.

I wish I had known earlier about the range of approaches and valuable insights already gained in the burgeoning and important subject of Translation Studies. A useful short introduction to the central issues and history of the subject may be found in Bassnett (2002). One of the major strands of theoretical thought in recent decades, which embraces both linguistic and cultural perspectives on translation, is the so-called 'functionalist approach'. The main argument of functionalist approaches is the apparently innocuous idea 'that texts are produced and received with a specific purpose, or function, in mind' Schäffner (2001, p. 14). That this might offer a governing principle for translation can be traced back to the *skopos* theory put forward by Vermeer in 1978 in German but with an English exposition in Vermeer (1996). Schäffner goes on to suggest that since the purpose of the target text may be different from the purpose of the source text, arguments about literal versus free translations, and similar contrasts, become superfluous: the style of translation should match the purposes of the texts. In this context the traditional dimensions of faithfulness and freedom in translation clearly become less significant.

The issue of purpose has been a useful consideration for the present volume. In the case of these texts of Bolzano there were undoubtedly multiple purposes. Some of them, such as making his ideas known to mathematicians of the time, and of gaining feedback on the value of the approach he was adopting, are clearly ones which cannot pertain to this translation. The time has gone. One purpose of these translations is that English-reading scholars of the early twenty-first century might appreciate in detail the context, insight, novelty and substance of his ideas and contributions to mathematics. A practical consequence of this purpose includes the demands mentioned above of preserving Bolzano's notations and paying close attention to his terminology. So although the principle enunciated above, of 'preserving the mathematics', may be flawed theoretically the outcome in practice has, I hope, not suffered unduly.

Translation has played an important, but neglected, role in the long histories of several scientific subjects. For an example of a rare study of translation in science see Montgomery (2000). Now that systematic and rigorous studies are being made to understand the way translation processes operate it will be important that historical scientific works also become part of Translation Studies, and that the theory and practice of translation maintain close connection over the whole spectrum of text types. It is possible that some standard and long-accepted translations could usefully both be informing current translation studies and themselves be reconsidered in the light of such studies.

In many translations of scientific and philosophical texts, including some works of, or on, Bolzano (e.g. George 1972, and Berg 1962), it has been common to give lists of the principal 'equivalences' between key German and English words. Although such lists may sometimes have their place, for example with certain technical terms, it has not seemed appropriate here. It will be clear from the whole tenor of this *Note* that I am sceptical about any attempts to mechanise the translation process. I hesitate to proclaim the consistency that the idea of equivalence suggests, and in any case I am unconvinced of the wisdom of aiming at a strict consistency. It has, however, often been convenient to give the original German word or phrase in square brackets following its translation. For example, it may be useful for the reader to know that it is the same German word (*Grund*) that has been variously rendered 'basis', 'foundation', 'ground' or 'reason' (among others). And conversely, that several German verbs *beweisen, erweisen, nachweisen, dartun* have, on occasion, all been rendered by the appropriate form of 'prove'. While *beweisen* has almost always been translated 'prove', the other terms mentioned have also given rise to suitable forms of 'establish', 'demonstrate' or 'show'.

Bolzano uses the verbs *bezeichnen* and *bedeuten* very frequently and as an experiment in consistency they were, for some time during revision of these translations, systematically rendered 'designate', or 'denote', respectively in such contexts as 'Let x designate (denote) a whole number'. It now seems unlikely there is any systematic distinction of meaning intended by the choice of these terms, but this explains the frequency of occurrences of the slightly cumbersome 'designate'.

The important term *Wissenschaft* is used on several occasions in *BG* and *BD*. It means a body of knowledge, especially knowledge rendered coherent or systematic by its subject matter, or the way in which it was acquired. There is no equivalent modern English word. In spite of the large range of alien connotations it is usually rendered by 'science' and after some experiment with 'subject' and 'discipline' I have fallen in with the conventional term. But the reader should strive to think only of the eighteenth-century meaning of 'science'. Dr Johnson's *A Dictionary of the English Language* (4th edition, 1773) gives as the first meaning for the entry *science* simply 'knowledge'. The *OED* offers 'knowledge acquired by study' and 'a particular branch of knowledge', each illustrated by quotations up to the early nineteenth century. Fichte (twenty years Bolzano's senior) used *Wissenschaftslehre* to mean an overall system of thought, and indeed one of his titles (in translation by Daniel Breazeale) runs *Concerning the Concept of the Wissenschaftslehre or of So-Called Philosophy* (in the collection Fichte (1988)). Breazeale decides not to translate *Wissenschaftslehre* at all in his own work while remarking that "Science of Knowledge' which has long been the accepted English translation of *Wissenschaftslehre*, is simply wrong' (Fichte 1994, p.xxxi). Bolzano's major philosophical work, also entitled *Wissenschaftslehre*, has been translated in each of the editions George (1972), and Berg (1973), with the title *Theory of Science*. In the opening section Bolzano explains that *Wissenschaft* has no generally accepted meaning (this is in 1837) and he declares his own meaning: a collection of truths of a certain kind, provided enough of them are known to deserve to be set forth in a textbook. It is clear from the title page of the work that he conflates his *Wissenschaftslehre* with his own broad understanding of logic. For further details the interested reader should consult the editors' introductions in the two translations just mentioned. Returning to the early works translated in this volume, the adjective *wissenschaftlich* has been translated, uncomfortably, but now for obvious reasons, as 'scientific'.

The range, references and connotations of *Größe* in German are different from those of 'quantity' in English which is nevertheless often the best translation. Each term has a complex pattern of usages and meanings overlapping in English with those of 'magnitude' and 'size'. When those latter terms are used in these translations they are always translating *Größe*, so we shall not usually indicate the German. But just as 'quantity' in early nineteenth century English was sometimes synonymous with 'number', so *Größe* was sometimes close to, but not the same as, *Zahl*. 'Number' in this volume is usually translating *Zahl* or *Anzahl*, so the occasions where the source has been *Größe* are indicated with square brackets or a footnote. (For example, both devices are used for this purpose on p. 87.) Quantity is the more general term embracing number, space, time, and multitude. So the work *RZ* (*Pure Theory of Numbers*) is a part of the overall unfinished enterprise *Größenlehre* (*Theory of Quantity*). Thus the late alteration by Bolzano of *Zahlen* into *Größen* at the opening of *RZ* on p. 357 is of interest and some surprise.

For a long time I have been convinced that 'set' is not the appropriate translation of Bolzano's use of *Menge* although it has been rendered this way in all previous translations of his mathematical work (including my own). In Bolzano's time, and still today, it is a word with a very wide everyday usage roughly meaning 'a lot of', or 'a number of'. It has also, since Cantor in 1895, been appropriated by mathematicians to take on a well-known technical meaning later enshrined in various axiomatisations, such as that of Zermelo and Fraenkel. Philip Jourdain, in translating Cantor's defining work, initially used 'aggregate' for *Menge*, while sometimes also needing to use 'number' (e.g. in the title of Cantor (1955)). But 'set' quickly became the standard English mathematical term for *Menge*, whether it be the axiomatized concept or Cantor's informal 'gathering into a whole of definite and distinct objects of intuition or thought'. For Bolzano working eighty years earlier, for example in *RB* with reference to collections of terms in a series, it would be misleading to use the term 'set' for his use of *Menge*. Later in *WL* §84 and *PU* §§3–8 where Bolzano is making more careful distinctions among collections, even if we are generous over his ambiguous use of 'part', his definition of *Menge* fails to correspond to the informal notion introduced by Cantor. In a thorough recent examination made by Simons (1997) of Bolzano's distinctions among various concepts of collection the proposal is made to translate his *Menge* with 'multitude'. This has been adopted in these translations with, I hope, quite satisfactory results. There are still some occurrences where 'number' must be used. For example, in *RB*§§1–3 are some good examples of *Menge* being used interchangeably with *Anzahl* in precisely the same context.

The term *Vielheit* is even more vexed than *Menge* and has been variously translated by others in the past as 'multiplicity', 'set', 'multitude', 'plurality', and 'manifold'. From what has already been said, plurality and multiplicity are the most obvious candidates. It occurs about 25 times in *PU*. Literally *Vielheit* means a many-ness and Bolzano describes it in *WL* §86, as well as *PU* §9, as a collection, the parts of which are 'units of kind A'. Many things of a specific kind suggest a grammatical plural although the grammatical connotations are not particularly suited to the context of use in *PU*. It is adopted in Simons (1997) with qualification as 'concrete plurality'; we have used simply 'plurality'.

I am grateful to Edgar Morscher for pointing out the difference between *gegenstandslos*, a fairly recent word meaning to become void, or irrelevant, and *gegenstandlos*, an older word—the one Bolzano uses—to mean, of an idea, that it is without an associated object (or, it is as it has usually been translated, 'empty'). The latter word is used in contrast to *gegenständlich* meaning, of an idea, that it does have objects associated with it. Bolzano often cites the cases of 0 or $\sqrt{-1}$ as number ideas that are empty (e.g. *PU* §34). See also the footnote on p. 594.

The translator of a text must also act to some extent as an editor. There have been, in this volume of translations numerous decisions of whether mathematical material should be displayed or left in-line with consequent problems of small font size and 'gappiness' in the lines of text. The manuscript works of *RZ*, *F* and *F*+ have closely preserved many of the human inconsistencies of the original manuscripts, for example, the number of dots used in a sequence or equation to indicate subsequent continuation. An editor faces an almost irrestible urge to 'tidy up' and to render 'consistent' the variations introduced by normal human production. We have not always managed to resist these urges. Continuation dots in the published early works, as in the later works rendered faithful to the manuscripts, have been reproduced as the conventional three dots (whether or not the original had one or two or more dots). The conventional German practice of continuation dots in arithmetic expressions being 'on the line' has been replaced by the English form of dots centred vertically at the operator level. The few cases of equation labels on the left-hand side of an equation have been replaced consistently by right-hand side labels. The breaking of very long expressions from one line to the next has not always followed the original form. Apart from these matters we have intended to follow Bolzano's notations exactly with only two exceptions: the matter of decimal commas on p. 268, and the subscripts inside an omega symbol on p. 574.

Some might regard it as undue pedantry to imitate such matters as Bolzano's astrological symbols labelling equations (e.g. in *BL*), or his wavering inconsistently in *F* between centred and right-hand superscripts. But it is hard to exaggerate the significance of writing and notation for our thinking. Bolzano knew this and took seriously the choice of good mathematical notation (see *BD* II §6 on p. 106). Usually he (or his printer) maintained a style of superscripting fairly consistently within one work. It seems surprising therefore, and should not be suppressed, that he was using, to any extent, superscripts so easily confused (we might suppose) with powers. It is hard enough to take ourselves back into the thinking of earlier times, surely it would be folly to deliberately erase anything that might possibly serve, alone or cumulatively, as a clue to the thinking we are working to recover?

Introduction

Bernard Bolzano was a Christian humanist who devoted a lifetime of thought and writing to a far-reaching and wide-ranging reform of the representation, organization, and discovery of knowledge. He was ordained as a Catholic priest in 1805 and thereafter, for almost fifteen years, he held a university post in Prague lecturing in theology and giving regular 'edifying discourses' to both the students and the public. He was eventually dismissed from his post by the Hapsburg Emperor, Franz I, because his public views, on social and political issues were deemed dangerously liberal, and his theological views even heretical. Nevertheless, he was popular as an educator and unflagging in his zeal as a reformer, driven in these roles, as in all parts of his life, by an ethical principle that he called the 'highest moral law', namely, 'always to behave in a way which will best promote the common good'. The well-being and progress—in the broadest sense—of humanity was his lifelong and overriding concern.

His working life occupied, almost exactly, the first half of the nineteenth century and was spent in the midst of a strong resurgence of Czech culture within a society dominated by German and Austrian influences. Born on 5 October 1781 of an Italian father and German mother, Bolzano clearly identified himself with the people and culture of his adopted country; he chose to describe himself as a 'Bohemian of the German tongue'. In spite of numerous obstacles, recurring illnesses, and persecution, his extraordinary energy, determination and hard work resulted in a prodigious output embracing philosophy, logic, mathematics, physics, politics, education, theology, and ethics. Evidence for the sheer scale of his writings lies in the monumental and meticulous complete edition [*Bolzano Gesamtausgabe*] (*BGA*) that is being prepared by Frommann-Holzboog of Stuttgart with a planned total of 120 volumes.

One result of his project for the reform of knowledge, and one which had a central place in Bolzano's thinking, was the four-volume work *Wissenschaftslehre* (*WL*) published in 1837, which was essentially his own novel reformulation and development of logic. It provoked Edmund Husserl to declare in 1900, 'we must count him one of the greatest logicians of all time, . . . logic as a science must be based upon Bolzano's work' (Husserl, 2001, i, p. 142). Substantial parts of *WL* have already been given independent English translations, one by Berg (1973) and one by George (1972).

Another result of his reform programme was a large amount of mathematics. Written throughout his lifetime and of varying quality, this is now generally recognized as containing within it substantial contributions that were original, profound, prescient of what would become fruitful and familiar to much later generations of mathematicians. His mathematics is closely allied to, and sometimes even promoted by, his own distinctive philosophical views about concepts and proofs. The material is in three main forms and complicated by the censorship imposed on Bolzano as a result of his dismissal in 1821, which prevented him publishing even mathematics within the Habsburg Empire, at least until the death of Franz I in 1835. First there are the mathematical works published in his lifetime (i.e. the works included here as *BG*, *BD*, *BL*, *RB*, *DP*); they were published in Prague or Leipzig and manuscripts for them do not survive. Then there is the *Nachlass*, the material remaining after his death, and this is in two very different forms. There is writing intended for publication and in a reasonable state of preparation but which was not published until after his death, either very soon as *PU*, or much later and in different forms as *RZ* and *F*. Finally, there is the material recorded throughout his lifetime in diaries. This is in the *BGA* Series II B *Miscellanea Mathematica* of which eighteen volumes have already appeared. The huge amount of diary material has been transcribed by Bob van Rootselaar and Anna van der Lugt from a handwriting that is difficult to read and full of personal abbreviations and corrections. They date from 1803 and provide a rich source for future scholars seeking to trace the origins and development of Bolzano's ideas.

We summarize here the main technical and mathematical contributions of Bolzano that are easily identified and recognized today. He gave the first topological definitions of line, surface, and solid (in 1817), and stated the Jordan curve theorem as a result requiring proof. Banishing the usual ill-defined infinitesimals of his time and skilfully employing an arithmetic limit concept, he defined and used concepts of convergence, continuity, and derivatives in a way very similar in detail to that found in modern textbooks. In the 1830s, he began an elaborate and original construction of a form of real numbers—his so-called 'measurable numbers'. Much of the work associated with definitions or constructions of real numbers (such as those by Weierstrass, Méray, Dedekind, and Cantor) dates from several decades later. He formulated and proved (1817) the greatest lower bound property of real numbers which is equivalent to what was to be called the Bolzano–Weierstrass theorem. He later gave a superior proof with the aid of his measurable numbers. In the course of an extensive treatment of functions of real variable he also constructed, in the early 1830s, a function everywhere continuous and nowhere differentiable. He proved the function had these properties on a dense subset of argument values. The discovery of such surprising functions is usually attributed to Weierstrass whose examples again date from at least thirty years later. He developed the first elementary theory of infinite collections, putting great emphasis on the concept of a 1-1 correspondence and clearly understanding that it is characteristic of infinite multitudes that they always have a 1-1 correspondence with some proper subcollection of themselves. Such fruitful, new mathematical concepts and theories were for Bolzano simply the consequences to be expected of what he regarded as his main contribution—namely, a fundamental reorganization of the principles and foundations of mathematics that placed practical demands on the concepts and proofs that were acceptable or appropriate as a theory was developed. Some details of these foundational views occur in all his works, but most particularly in the work *BD* of 1810.

It is one of the fascinations of Bolzano's thought that while he was well-read in his predecessors' work he nevertheless espoused a highly individual general philosophy that appeared to have an intimate and unusually formative effect on his own specific mathematical ideas. His philosophical views about mathematics acted as a productive driving force for new mathematics. In this way, even if, as seems to have been the case, he had relatively little influence on later mathematicians the reasons for his innovations, and the reasons for his reformulations of definitions and proofs may continue to be worth our while seeking to understand. Not only may this be valuable for understanding how mathematics has changed in the past, but such lessons may even yet not be irrelevant for suggesting fruitful directions in the future.

His mathematical publications began with the geometrical work *BG* in 1804. This was a bold effort to reorganize elementary geometry. His subsequent works *BD*, *BL*, *RB*, and *DP*, published while he was Professor of *Religionswissenschaft* were, on his own account, instalments intended to gain attention, advice, and criticism (see *RB Preface*). He may have seen these works foremost as an illustration and accompaniment to his developing views on knowledge and logic rather than part of a deliberate contribution to mathematics for its own sake. Later, his extensive mathematical research and achievements appear to have taken on a life and purpose of their own in the unfinished *Theory of Quantity* [*Größenlehre*], encompassing the works *RZ*, *F*, and *PU*, that he began planning and writing in the 1830s. It is only a small proportion of his mathematical work that is presented in translation in the present volume. Any such selection is likely to be somewhat arbitrary. It is not simply the 'highlights'; indeed, it contains some material that is very sketchy, or long-winded and amateurish. But it also contains some of his best insights and most far-reaching contributions. It is hoped that it is reasonably representative of both his published material, and his *Nachlass*. The eight works translated here have been put into three groups according to subject and chronology with a short introduction about the mathematical significance of each group.

We turn now to a brief account of Bolzano's life and work, and his own broad intellectual and moral concerns. His work is remarkable not only for its mathematical content but also for the circumstances, the context, and the significance of its creation.

There is an enigma at the heart of Bolzano's work that immediately strikes the modern reader as soon as she strays only slightly beyond the 'strictly' mathematical writings. But it is an enigma that demands resolution if we are to read, interpret, and judge his mathematics in a way that does justice to his perceptions and values as well as to ours. The chief motive in Bolzano's life and work was a moral one: to promote most effectively the 'common good' [*das allgemeine Wohl*].

Yet throughout his life he was predominantly occupied with philosophy, logic, and mathematics, and not the parts of these subjects that were associated with potential usefulness, but instead his preference was for topics—such as the foundations of mathematics and the nature of proof—that we would probably regard today as the very furthest from possible practical application. The society of early nineteenth century Bohemia had only recently emerged from feudal practices such as widespread serfdom and robot (i.e. forced labour for a landowner); it had suffered decades of unjustly heavy taxation and the frequent threats, and realities, of famine; there was poverty and social injustice on a huge scale. So we need to understand, for example, how the pursuit of the 'correct' or 'objective' proof of the intermediate value theorem, or the 'proper' definition of a line, could, without hypocrisy or blindness, be regarded as best promoting the common good. The entire drive and direction of these two goals—the moral and the mathematical—appear, while not directly opposed to one another, yet to be in such tension that it is hard for us to imagine them coalescing and forming part of the mainspring of the life and work of a single individual. But in one so single-minded of purpose, and so productive as was Bolzano, they must not only have coexisted but must surely have complemented, combined, and reinforced each other in some very constructive kind of way. In order to understand this better we shall seek a broader perspective by briefly surveying some the most memorable and significant of the turning points in the history of Bohemia that would be the natural 'landscape of the mind' for Bolzano as he grew up in late eighteenth-century Prague.

Of the numerous histories of Bohemia, the Czech peoples, and the Habsburg Empire the following works, in particular, have been consulted for this brief summary of the history and background of the society within which Bolzano grew up and was educated: Betts (1969), Kerner (1932) and Padover (1967). While origins so remote are inevitably uncertain, it was possibly a Roman tribe, the *boii*, that gave rise to the name Bohemia. It is likely that 'Czech' was simply the name of an early Slav leader. But without any doubt Bohemia was the scene, from medieval times to the Enlightenment, of some of the most dramatic and violent struggles in Europe over matters of government, religion, and language.

The Victorian Christmas carol 'Good King Wenceslas' is well-known in the English-speaking world for commemorating the Bohemian Christian Prince, Wenceslas, for bringing justice to 'the poor man gathering wood'. Wenceslas himself was violently killed by his brother after a political intrigue in AD 929. He was soon deemed a martyr and his memory seared into the consciousness of the Czech peoples as a veritable patron saint. In 1348, Charles IV as both King of Bohemia and Holy Roman Emperor founded the University of Prague. It was distinctive in having all four faculties for the arts, medicine, law, and theology, and soon becoming, along with Paris, Oxford, and Bologna one of the key centres of learning in the fourteenth century. Through the influence of John Wycliffe and his realist philosophy and theology there grew a strong relationship between Oxford and Prague. Numerous Czech theology students visited both Oxford and Paris. In 1390, the young Jan Hus, who was inspired and influenced by Wycliffe, matriculated in Prague. It was a very low point in the history of the Church when, at the Council of Constance in 1415, Jan Hus was condemned to be burned at the stake for heresy. This was taken as a national insult in Bohemia and the subsequent Hussite reform movement, and associated wars, was not an isolated outbreak but a local form of profound revolt that was manifesting itself across the whole of Europe.

The Habsburg dynasty first came to power in Bohemia in 1526 through Ferdinand I. By the late sixteeth century two-thirds of Bohemia was still Protestant and Czech speaking, the other third Catholic and German speaking. In 1620, there occurred perhaps the single most important event in the history of the Czech people, the Battle of the White Mountain. The Protestant Estates army faced a much larger imperial force. The Czechs were routed and the victory of the Catholic forces was devastating. This marked the end of independence for Bohemia. Three quarters of the manors of Bohemia were confiscated, 30,000 families including many estate owners and nobility fled from Bohemia immediately after the Battle and after the Thirty Years war there were less than a third of the original population of three million remaining. The recovery, over the next two centuries, of a sense of nationhood and identity was slow and arduous. The spirit of reform, alongside the face of absolutism and imperial power, had a wide, complex, national presence manifesting itself in political, social, linguistic, and religious ways. Maria Theresa, in power from 1740–80, saw the beginning of some reform and enlightenment, including the suppression of the Jesuits in 1774. Joseph II continued the reforming measures in an extraordinary burst of benevolence and social improvement (not always appreciated by his people) from 1780–90. Perhaps not surprisingly in a Europe convulsed by the French revolution and its aftermath, after Leopold's short reign, the Emperor Franz (from 1792 to 1835) was a conservative and a reactionary, and reversed much of the reforming progress made by his predecessors. In the final decades of the eighteenth century when German became the exclusive language for governmental affairs in Bohemia, and the official language of the university in Prague, the very preservation of the Czech language and culture was fragile and uncertain. What in the end was the mainspring of the recovery of national identity that actually occurred was crucially dependent on the revival and re-telling of Czech national history. The themes of repression and reform, outlined in very brief strokes above, would likely have made a vivid impression and influence on the youthful Bolzano.

The state of almost uninterrupted war which existed in most of Europe between 1789 and 1815 formed the backdrop for Bolzano's childhood and early career. Bohemia's dominant intellectual movement was the so-called 'Catholic Enlightenment', which emphasized the themes of rationality and usefulness in all things and did much to promote education at all levels. Bolzano's father, an Italian art-dealer, had emigrated to Prague in the 1760s and there married a German woman, Cecilia Maurer. Of their twelve children only two survived to adulthood. Bolzano himself was not a strong child, but, in spite of headaches and a weak heart, he wrote, 'I was a very lively boy who never rested for a moment'

(Winter, 1976, p. 56). This disposition to incessant activity in the face of frequent illness did not abate as he grew older. There are, for instance, over 8000 sheets of manuscript in his mathematical diaries in addition to similar diaries in logic, philosophy, and ethics. In 1796, Bolzano entered the Philosophy Faculty in the University of Prague, and for four years he followed courses mainly in philosophy and mathematics. Although he found both subjects rather difficult, he soon discovered in pure mathematics ample scope for the foundational and conceptual investigations, which appealed to him so strongly. In his autobiography Bolzano recollected, 'My special pleasure in mathematics rested therefore particularly on its purely theoretical parts, in other words I prized only that part of mathematics which was at the same time philosophy' (Winter, 1976, p. 64).

In the autumn of 1800, Bolzano began three years of theological study. Although he was basically an orthodox Catholic, he found that his rationalist inclinations did not fit as comfortably as he had hoped with his theological studies. He came to realize that teaching and not ministering defined his true vocation. Educational value no doubt influenced his constant concern for the clarity and correct ordering of concepts in any exposition. While pursuing his theological studies, Bolzano also prepared his doctoral thesis on geometry, which was published in 1804 and is the first translation in this volume. Unable to obtain a mathematics post and torn over his choice of career, Bolzano seemed initially to face an unsure future. However, after he decided in favour of a theological post, events moved swiftly. On 5 April 1804, Bolzano was awarded his doctorate; on 7 April, he was ordained; and on 19 April, he was appointed to the newly formed professorship in religious studies at the University of Prague. Such a post had been created at all universities with a view to curtailing the then current wave of liberalism and free-thinking. In addition to courses of lectures, Bolzano was required to give weekly sermons twice to the students and citizens of Prague. He performed these duties with seriousness and enthusiasm and soon became highly respected and popular in Prague, with over a thousand people regularly attending his sermons. The population of Prague was approximately 80,000 at this time. Despite his successes in the pulpit, Bolzano was never politically suited to such a post, and his appointment was viewed from the start with suspicion by the authorities in Vienna. He would only use the authorized textbook in order to criticize it, and he held distinctly pacifist and egalitarian views. After a long process (which he resisted strongly), Bolzano was dismissed in 1819 for heresy, put under police supervision, and forbidden to publish. This enforced early retirement probably greatly lengthened his life—he suffered from tuberculosis—for he was subsequently able to spend much of his time recuperating, and writing, as a guest on the estate of his friends, Joseph and Anna Hoffmann, at Těchobuz in southern Bohemia. After this crucial turning point in his career, Bolzano began to work on his two major projects: the *WL* on logic, and the *Größenlehre* on mathematics. Although the restrictions on him were gradually lifted after Franz I died in 1835, Bolzano took no further active part in politics, or in the revolution of 1848, the year of his death.

During the 1820's Bolzano worked on *WL*. As we have seen this was regarded both by Bolzano and his readers, for example Husserl, as primarily a work on logic. But from a modern perspective this requires at least as much explanation and empathy as we need in order to understand 'science' in its eighteenth century sense (see the *Note on the Translations*). Logic for Bolzano was a very much broader and richer subject than either the narrow Aristotelian focus on conceptual analysis and on syllogistic kinds of arguments, still popular in the eighteenth century, or the formal mathematical logic that has dominated scientific thinking since the seminal work of Frege at the end of the nineteenth century. We may distinguish two senses of logic in *WL*. There is a very wide sense, what Berg (1973, p. 1) calls 'a kind of metatheory', in which the objects are the various scientific theories themselves. This sense includes a theoretical part that resembles what we might now refer to as a 'philosophy of meaning'. Here Bolzano introduces the notion of a proposition in itself [*Satz an sich*], or the objective content of a proposition,

> ... by *proposition in itself* I mean any assertion that something is or is not the case, regardless whether or not somebody has put it into words, and regardless even whether or not it has been thought. (*WL* §19 as in George 1972, p. 20)

Bolzano distinguishes this abstract notion of proposition from the concrete expression of a proposition in mental or linguistic ways. Not surprisingly this provokes extensive discussion of the distinction between whether *there are* such propositions (which Bolzano defended) and whether such propositions *exist* (which Bolzano disputed). An excellent source for further material on this from Bolzano in translation is Rusnock and George (2004). This wide sense of logic also contains important practical parts, it is what Rusnock (2000, p. 90) calls 'a methodology or theory of science, concerned with the organization and presentation of truths'. It includes heuristics for the discovery of truths and rules for how we should go about composing textbooks.

By way of contrast it is with the narrow sense of Bolzano's logic that we are likely to feel more at home today. Here he treats relations of deducibility [*Ableitbarkeit*] and of ground-consequence [*Abfolge*]. The former is actually what we would now call logical consequence. It is defined in terms of Bolzano's 'logic of variation'. Any part of a proposition which is not itself a proposition is called an idea in itself. When an idea is considered 'variable' by Bolzano this means we should consider the class of propositions that arise by successive substitution for that idea from among a class of ideas of the same kind. This is something short of the notion of a propositional function but it is a simple, original, and powerful notion. It allows Bolzano to define such concepts as compatibility, deducibility, validity and analyticity. For example,

> Propositions *M*, *N*, *O*, . . . are *deducible* from propositions *A*, *B*, *C*, *D*, . . . with respect to variable parts *i*, *j*, . . . , if every class of ideas whose substitution for *i*, *j*, . . . makes all of *A*, *B*, *C*, *D*, . . . true also makes all of *M*, *N*, *O*, . . . true. (*WL* §155 as in George 1972, p. 209)

Bolzano's exposition at *WL* §155 is very detailed and thorough, extending to thirty-six numbered paragraphs and two explanatory notes. Several commentators have pointed out the similarity between Bolzano's definition of deducibility here and that of logical consequence given in Tarski (1983) originally appearing almost exactly one hundred years after the publication of *WL*. For example, see Etchemendy (1990). It should be noted that what we (and Rusnock and George) have rendered as 'deducibility' [*Ableitbarkeit*] is translated in Berg's works (and others) by 'derivability'. We take up the meaning of the ground-consequence relationship in the context of the early work *BD* where it undoubtedly had its roots (see pp. 18–20). What we have called the narrow sense of logic for Bolzano is almost entirely contained in the major section of *WL* entitled *Theory of Elements* that is about the nature and properties of ideas and propositions in themselves. The remaining parts of *WL* are concerned with logic in the wide sense.

This has been the merest sketch of Bolzano's logic and the contents of his *WL*. Space does not allow for further elaboration here of the remarkable and original treatment of logic in that work. Nor is it necessary because his logic is already much better served in English translation than his mathematics. The interested reader is referred to the editions of *WL* already mentioned, in addition to the detailed commentaries in Berg (1962) and Rusnock (2000, Ch. 4). On the logic of variation see also the chapters by Siebel and by Morscher in Künne (1997). For readers of French there is also an excellent further resource in the extensive and detailed study of both Bolzano's logic and his mathematics in Sebestik (1992). Another important French study is the collection of papers in Lapointe (2003).

The distinguished Bolzano scholar, Eduard Winter in *BGA* E1, has emphasized that for Bolzano logic and religion were inextricably related. On the one hand religion was the starting point for his logic. And at the same time Bolzano said himself that logic was the key to understanding his writings. The most fundamental role of logic becomes for him a moral, or ethical matter,

> The division of the totality of truths in disciplines and their presentation in individual treatises should be undertaken throughout in accordance with the laws of morality, and as a consequence also so that the greatest possible good (the greatest possible promotion of the general well-being) is thereby produced. (*WL* §395 as in Rusnock 2000, p. 91)

Bolzano understood religion in a rather special way. The supernatural was somewhat secondary and its agency, for example in miracles, was often to be understood metaphorically. Religion was primarily the wisdom by which people can live together more tolerantly. It is in this vision of logic and religion serving one and the same end that we might hope to find resolution of the enigma referred to earlier. Rusnock has described the position of logic for Bolzano as follows:

> The development of Bolzano's logic is thus guided by two strong principles: a commitment to make logic serve human ends, and an insistence on rigour in his characteristic sense. Neither asserts complete dominance: his logic might therefore be described as formalism with a human face. (Rusnock 2000, p. 92)

The provisional conclusion we draw here on the extraordinary blending in his life of energy and piety, of philosophy and mathematics, and of religion and logic, is that Bolzano was following in, and living out with great dedication, the historic and admirable tradition of Bohemian reform. He grew up at a time when this tradition was coming to public awareness in a new way through a revival of education in the Czech language in the schools and a vigorous movement at every level to recover the values and identity of the Czech peoples. Bolzano was a reformer in every sphere in which he worked. He sought social and political reform. He called for reform in the Church and in theology. He heralded and worked for reform in education at all levels in Bohemia. But principally he saw himself as having the vision and gifts to reform knowledge on a grand scale. Especially that fundamental preliminary to knowledge which was logic, and especially that part of knowledge for which he had obvious talent which was mathematics. This was his mission and his best way of promoting the common well-being.

Geometry and Foundations

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The first of Bolzano's publications (*BG*), on geometry, has several passages treating foundational or philosophical matters. His next publication (*BD*) on the methodology of mathematics was explicitly directed at addressing particularly the problems of geometry; it seems natural therefore to group these two together. The later work *DP* includes, curiously, several groups of geometrical definitions that are not actually used in the work itself, but offer a striking proposal for the problem of giving satisfactory definitions of geometric extensions (such as line, surface, and solid). A common theme connecting these three works is what Johnson calls Bolzano's 'definitional problem' (the problem is that of defining the geometric extensions just mentioned, see Johnson (1977)). A definition in this context is a report of the analysis of a concept, so this common problem is also that of how to analyse, and understand, geometrical concepts or entities. This has always been a pivotal issue in science and mathematics because it has implications and interest for theoretical issues (e.g. as the subject matter of proofs) and for applications (from astronomy to engineering) and for philosophical questions. We shall give here some background and overview of the material in *BG, BD*, and the geometrical parts of *DP*.

The modern reader, perusing the pages of *BG* for the first time could be forgiven for a certain sense of bewilderment. What kind of a work is this? Is it elementary, as the title and subject matter suggest, or is it advanced, as suggested by the frequent philosophical asides and the fact that it was submitted successfully for the award of a higher degree? Is it conventional, as the presentation and results suggest, or is it highly eccentric as suggested by the denial of angle as a quantity (*BG* I §6), and the wholesale rejection of concepts of the plane and of motion (see *BG Preface*), and even of congruence (*BG* I §49)? Is it for the instruction of beginners or to elicit criticism from those already very familiar with the material? Further study soon removes these paradoxes. The first alternative in each of these questions has only weak, or superficial grounds for support. The material is indeed difficult, unconventional, and proposed primarily to invite comment and criticism from experienced geometers. It is best understood in the context of Bolzano's overall programme of reform of the sciences where, as we have seen, he regards mathematics as one of the best examples of *wissenschaftlich* organization but even here there is much room for improvement and geometry is regarded as in the worst condition (*BD Preface*). After a summary of the contents of *BG*, we shall review briefly the situation of elementary geometry around 1800, especially through some contemporary German texts.

The work *BG* consists of three parts: a *Preface* outlines Bolzano's motivation and what he regards as original in the work, Part I contains a theory of triangles and parallel lines, and Part II is a sketch of ideas for a theory of the straight line. Part I is explicitly based on Part II although it is acknowledged that Part II is far from complete yet. We shall return to Bolzano's motivation and aims later. Part I, the most substantial part, contains the following material:

§§1–6. Definition of angle, principle that things that are determined in the same way are equal. Adjacent and vertically opposite angles. Errors of Euclid and discussion of angle as quantity.

§§7–15. Definition of triangle, equality of triangles. Equality and identity. Two sides and their included angle determines a triangle.

§§16–24. Similarity of triangles. Things that are determined by similar parts are similar. Axiom that there is no absolute distance. Two proportional sides with equal included angles determine similar triangles. Discussion of Wolff and Kant on similarity and motivation for the axiom.

§§25–49. Isosceles triangles. Possibility of right angles. Existence and construction of various lines and angles. Perpendiculars in isosceles triangles. One side and associated angles determine a triangle. Equal angles imply similar triangles. Pythagoras' theorem. Three sides determine a triangle, discussion of coincidence and congruence.

§§50–67. Intercepts, rectangles, parallels, corresponding and alternate angles. The parallel postulate, parallelograms, discussion of various geometers.

Part II is incomplete, tentative, and somewhat jumbled. The quality of ideas is mixed but there are some that are seeds for valuable insights in *DP*. The material can be organized as follows:

§§1–5. Identity and equality. Determination and possibility. Definitions and the nature of geometric objects.

§§6–24. The system of two points, distance and direction. Various properties, concept of opposite direction.

§§25–43. Concept of betweenness and definition of straight line. Relations of order. Midpoint and definition of plane.

During the eighteenth century in the German-speaking world, as elsewhere in many parts of Europe, there was a great growth in literacy, publishing, and education. Perhaps because of the political nature of the many independent German principalities there was a corresponding growth in universities so that by the end of the century there were at least twenty universities where German was the official language. Of course, this meant there were numerous textbooks and textbook writers—earlier mostly in Latin, but later in the century more often in German. But for mathematical education it is fair to say that there were two authors, Wolff and Kästner, whose work, between them, dominated the century in the German-speaking regions. Neither men were noted for being creative mathematicians but they were both committed to education and wrote highly systematic and comprehensive multivolume textbooks on mathematics that went through many editions and were very influential. Not surprisingly, they were both authors to whom Bolzano makes frequent reference in his early works.

Christian Wolff (1679–1754) studied at Jena and Leipzig, taught at Marburg and Halle and began publishing with his *Elementa Matheseos Universae* in five volumes that went through nine editions between 1713 and 1742 (with an English translation of parts of it in 1739). He also wrote an *Anfangsgründe der aller Mathematischen Wissenschaften* (Basic principles of all the mathematical sciences) as early as 1717. Wolff had regular correspondence with Leibniz over the period 1704–16 and his extensive philosophical work, some of which Bolzano refers to in *BG* I §24, was much influenced by Leibniz. A.G. Kästner (1719–1800) was Professor of Mathematics at Göttingen for most of his life and probably exerted considerable influence—especially in geometry. For example, he supervised an important thesis, Klügel (1763), on the history of the parallel postulate and several mathematicians attended his lectures some of whose students were later to be famous for their work on non-Euclidean geometry (such as Bolyai and Lobachevskii). Kästner wrote a great deal but among the most influential of his textbooks must have been his *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspektiv*. This ten volume work went through six editions from 1758 to 1800. Gauss attended some of Kästner's lectures at Göttingen but he regarded them as too elementary (*DSB, Kästner*). One of Bolzano's teachers at Prague, Stanislav Wydra to whom *BG* is dedicated, used Kästner's textbooks and Bolzano himself owned and annotated the fifth edition of the *Anfangsgründe*. Kästner was much admired by Bolzano because, 'he proved what is generally completely passed over, because everyone already knows it, that is, he sought to make very clear to the reader the basis [*Grund*] on which each of his judgements rests. That was what I liked most of all' (Winter, 1976, p. 62).

Folta (1966), identifies three different 'trends' in geometry around 1800. First there was a Euclidean tradition, then an empirical tradition illustrated by D'Alembert in his (1757) article *Géométrie* in the *Encyclopédie* that regards seeking foundations for geometry in axioms and definitions as 'chimerical', and finally a German tradition in which each author went his own way in seeking to improve on the logical conception of geometry. In fact these three trends seem to have developed in rather clear national boundaries since the Euclidean tradition was by far the strongest in Britain. V. Bobynin made an analysis of elementary geometry textbooks in the period 1759–99 and concluded that, 'the most important fact from the analysis of textbooks is that Euclid's *Elements* retained their original position as the single textbook of elementary geometry only in England in the second half of the eighteenth century' (Cantor, 1894–1901, iv, XXII). The Euclidean tradition was late developing in Germany with the first attempt to translate the *Elements* verbatim not appearing until 1773. Earlier works seem to have been mere summaries or selections. There is a survey of such translations in Heath (1956), p. 107. Of course, Bolzano was familiar with Euclid; here is a famous anecdote from his autobiography (Winter, 1976, p. 62)

> I should like to relate how the famous mathematician Euclid once became my doctor. It was during a holiday period which as usual, I, was spending in Prague, when I was struck by an illness I had never had before that

caused a shivering and shaking of all my limbs. In order to keep up my spirits as much as possible I took the *Elements* of Euclid and read the theory of proportion which I found treated here in a way completely new to me. The ingenuity of the Euclidean presentation gave me such vivid pleasure that I soon felt better again.

To gain some glimpse into the ways in which, as Folta suggests, German authors developed their own interpretations of geometry we quote from the opening sections of work by Wolff and Kästner. The following extract has been translated from the opening pages of the section *Principles of Geometry* [*Anfangsgründe der Geometrie*] in Wolff (1717):

Principles of Geometry

First Definition

Geometry is the science of space which is concerned with solid objects in respect of their length, breadth and thickness.

Second Definition

If length is considered without breadth and thickness, it is called a line. Its beginning and end are [each] called a point, which must therefore be thought of as without any parts. . . . . Now if a point moves from one place to another, a line is described.

[Two Notes omitted]

Third Definition

Similarity is the agreement [*Übereinstimmung*] of those things whereby objects are distinguished from one another.

[Note omitted]

Note

Similar objects can therefore not be distinguished unless they are brought together either actually, or in thought, with a third thing, for example, a ruler.

....

....

....

Fifth Axiom

If straight lines and angles cover one another then they are equal; and if they are equal they cover one another.

Sixth Axiom

Figures which cover one another are equal to one another; and those which are equal and similar cover one another.

The phrase 'equal and similar' appearing here was commonly used in the sense of 'congruent' at this time (see the footnote on p. 37). Some of Wolff's work was translated by John Hanna at Cambridge who writes in a note entitled *The Translator to the Reader* at the beginning of Wolff (1739):

> In his Geometry he [Wolff] sometimes uses Leibniz's method of demonstration from things being the same way determined, or rather, as he expresses it, from their similitude . . . because he counts it a more easy way than that of congruity, by which the mind is oft-times long led in winding paths.

And at the end of Chapter 1 of the *Elements of Geometry* of the same work we read:

> Corollary
>
> 120. In those things that are determined the same way, the things, by which they are to be distinguished, coincide, and so they are similar.

It is clear from these few quotations that Bolzano is by no means being original in giving a fundamental role to similarity and to the idea that some parts of a geometric object may determine the whole object. It is also clear that he has refined these concepts as applied in geometry—at least in relation to Wolff's work. For Wolff it appears that there is little difference between being determined in the 'same way' and being similar. Bolzano's definition and theorem in *BG* I §§16 and 17 are more discriminating. He goes on in much of his work to exploit the concepts of determination and similarity at least as a fruitful heuristic method in both geometry and analysis.

The *Geometry* section of Kästner (1792) begins with the following definitions:

> 1. The *boundary* of a thing is its exterior, or where it ceases.
> 2. A *continuous quantity* (continuum) is that [quantity] whose parts are all connected together in such a way that where one ceases, another immediately begins, and between the end of one and the beginning of another there is nothing that does not belong to this quantity.
> 3. A geometric extension is a space which is occupied with a continuous quantity.
>
> [Note omitted]
>
> 4. A solid extension, or a geometrical solid (solidum, corpus), is an extension which occurs within boundaries on all sides. The extension of a solid at its boundaries is called a surface (superficies), and the extension of a surface at its boundaries is a line (linea).
> 5. A point is the boundary of a line and therefore of all extension.
>
> Consequently it [a point] has neither extension nor parts, and no line consists of a multitude [*Menge*] of points next to one another.
>
> ....
> ....
> ....
>
> 6. A *straight* line is one all of whose points lie in one direction.

The boundary definitions 4 and 5 here (probably going back to William of Ockham) are explicitly rejected by Bolzano and in their place he offers the impressive series of definitions and examples in terms of neighbourhoods given in *DP* §11 ff. (lines), §35 ff. (surfaces), and §52 ff. (solids). It is explained in a footnote to *DP* §11 that 'A *spatial object* is in general every *system* (every collection) *of points* (which may form a finite or infinite multitude).' Then the first of a series of eight definitions is given (with associated diagrams) as follows:

> A spatial object, at every point of which, beginning at a certain distance and for all smaller distances, there is at least *one* and at most only a *finite* set of points as neighbours, is called a *line in general* (Figs. 1–7).

An excellent analysis and a thought-provoking assessment of Bolzano's achievements appears in Johnson (1977), together with numerous references to associated literature both by Bolzano and others. Johnson manages to take thorough account of the philosophical background to Bolzano's thinking while also looking ahead to the place his ideas played (whether explicitly or not) in the later development of dimension theory. It is essential reading for an appreciation of Bolzano's geometrical work. An interesting issue requiring further investigation is the origin of these definitions. Johnson clearly regards them as original with Bolzano. But van Rootselaar in the first volume of the mathematical diaries (*BGA* 2B2/1) refers to something probably written by Bolzano in 1803 and seeming to be a predecessor definition: 'a continuous line is a line with the property that for every point for which there is [another] point at distance *r*, there is also a point at every distance smaller than *r*'. He continues, 'most authors consider this definition an original achievement of Bolzano, but I have reasons to suppose the definition was already fairly common at that time'. In so far as this enigmatic remark is right, there is a gap in the standard histories and there is clearly more work to be done in bringing to light the full story of the emergence of concepts of continuity in both spatial extensions and their numerical counterparts.

From further reading of their works it becomes clear that both Wolff and Kästner adopt the Euclidean definition of angle as the 'inclination' of one line to another, and they assume the concept of the plane early on in their elementary geometry. They also concur with the conventional acceptance in various ways of the concept of motion in the study of geometric objects. On each of these matters Bolzano's negative position is unusual but not without precedent. He claims that angles and triangles can be conceived without assuming the plane, which is a more complex object. Angles should not, he says, be treated as quantities since they can only be added arithmetically if they are in the same plane. He ends up in *BG* II §12 with a rather abstract definition of angle simply as a pair of directions. See the footnote referenced from *BG* I §3. Notions of angle have a substantial history: the study by Schotten, 1890–93, ii, pp. 94–183, extends to nearly one hundred pages. However, Heath, 1956, p. 179 offers a convenient short summary. Curiously, although Schotten actually has several references to *BG* for the distinction of distance and direction made in *BG* Part II, he makes no mention of the unusual angle definition in Part I.

Proposition I, 4 of Euclid's *Elements* states that if two triangles have two sides and their included angles equal then the other sides and corresponding angles will be equal. It is, famously, proved by reference to the 'superpositioning' of the triangles. As Heath translates Euclid, 'if the triangle ABC be applied to the triangle DEF . . .'. This looks like an appeal to motion, in some sense, and has frequently been the source of controversy. Bolzano regarded any notion of the 'covering' of one triangle or figure by another (as Wolff uses in his axioms quoted above) as an essential part of the concept of congruence and implied a motion of some sort which he rejected completely for the reasons he explains in *BG Preface*. There are many other occasions when motion may be invoked in geometrical arguments. As Gray, 1989, pp. 43–7 points out, the concept of motion in geometry had been rejected in various ways by important Islamic mathematicians (such as Thabit ibn Qurra and Omar Kayyam) many centuries earlier, though this was unlikely to have been known by Bolzano. In place of congruence, or superposition, Bolzano employed his refinement of determination to do similar conceptual work. His proof of Euclid I, 4 appears in BG I §§12 and 14.

In *BG* I §59 there is the theorem that, 'Through the same point *o* . . . outside the straight line *xy*, there is only one straight line parallel to *xy*.' This has the appearance of being the Playfair equivalent of Euclid's fifth postulate, the parallel postulate, and has led some authors to claim that Bolzano had erroneously proved this postulate while not realizing (like Wallis) that it was equivalent to his earlier assumption of the existence of unequal similar figures (or, indeed, to the axiom at *BG* I §19). This is a mistake. It is certain that Bolzano never intended to prove the parallel postulate in the usual sense (i.e. from the other Euclidean axioms). He so thoroughly rejected Euclidean geometry that this would make no sense for him. There is no mention in *BG* of Euclid's form of the postulate. When Bolzano does refer to it in *BD* II §28 he only comments that 'it only holds under the condition that both lines lie in the same plane'. His final remark on his own work in *BG* I §67 is 'These are perhaps the most important propositions in the theory of parallels expressed here without the concept of the plane.' Thus, Bolzano did not intend here to solve this notorious problem of Euclidean geometry although it was still very topical in some quarters at this time. Nevertheless, it is clear in many references that he is pleased to have proved the result within his own system. He did not have the awkwardness that 'the Euclideans' had of a complex result having to be taken as an axiom. The answer to the question, 'Why did Bolzano not try to get acquainted with Klügel's thesis?', in Folta, 1966, p. 96, is that he was probably just not interested in investigating further a system in which he believed he had diagnosed far deeper errors than this mere symptom of the unproved parallel postulate. Bolzano was busy developing his own kind of 'non-Euclidean' geometry for which he could deduce the theory of parallels appearing in *BG* I §§50–66.

In another sense, however, Bolzano's *BG* was more 'Euclidean' than Euclid's *Elements*. And this was because of the assumptions made in *BG* Part II on the 'theory of the straight line' on which Part I is based. In Part I Bolzano assumes several results that are equivalent, in the plane, to the parallel postulate. Some of them we have already mentioned. Another is that all points equidistant from a straight line themselves form a straight line (*BG* I §34). It would be Bolzano's claim that these assumptions can all be proved from his definition and theory of the straight line in Part II. But there is a deeper reason why *BG* cannot be compared with the *Elements*. Euclid did not incorporate the intuitive idea of the 'straightness' of straight lines into his first four postulates. Nor is his attempted definition of the straight line used anywhere in actual proofs. Bolzano's definition of straight line in *BG* II §26 is stronger than Euclid's postulates 1 and 2 and could, if developed further, have been axiomatized to provide a rigorous foundation for *BG* Part I, if not, with suitable modification, for most of Euclidean geometry. In fact, van Rootselaar has shown in some detail how this might have been done (*BGA* 2B2/2 *Einleitung*). He also gives detailed references there to the various revisions and further efforts to complete work started in *BG* that are contained in this particular volume of Bolzano's mathematical diaries.

The titles of Folta (1966) and Folta (1968) rightly refer to Bolzano's early work on geometry as being primarily to do with the 'foundations of geometry'. But we can go further—they really have to do with the foundations of mathematics and use geometry as a kind of pilot study, an experiment with a new methodology. It was not the geometrical results that mattered so much as the correctness and efficacy, or otherwise, of the methods and proofs being used.

There was much concern at this time with the proper classification of mathematical topics. For example, Montucla (1799–1802), lists about twenty topics under the general division of 'physico-mathematics' in his wide-ranging history of mathematics. His was largely a cataloguing exercise in which theorems and results were grouped according to their 'object', for example, optics, astronomy, hydrodynamics, etc. In pure mathematics this was not so easy. Bolzano was concerned with the disorder and confusion both between and within the conventional divisions. He believed such a classification should not be arbitrary but should be a representation of 'hypothetical necessity'. Every theorem should be presented with its correct *ground*, which may, itself, consist of a finite sequence of ground-consequence relations. Disorder and confusion resulted from concepts and methods from one theory being employed in another. A far-reaching example of this was the use of geometrical ideas in arithmetic and analysis. Ever since Euclid, many 'algebraic' results had been interpreted, proved, and developed in geometrical terms. For Bolzano the starting points of the proof sequences in a theory are the propositions containing simple, but meaningful, concepts. To claim, as he did, that the disorder of proofs was not merely a subjective or aesthetic desire for 'purity' in proofs it was necessary to assume that there are genuine conceptual divisions of knowledge, or of truths, like the sharp divisions into species that were believed to exist in the organic world. The immediate consequence for mathematics was summed up in *BG Preface*, as follows:

> I could never be satisfied with a completely strict proof *if it were not derived from the same concepts* which the *thesis* to be proved contained, but rather

> made use of some fortuitous, alien, *intermediate concept* [*Mittelbegriff*], which is always an erroneous $\mu\varepsilon\tau\alpha\beta\alpha\sigma\iota\varsigma$ $\grave{\varepsilon}\iota\varsigma$ $\grave{\alpha}\lambda\lambda o$ $\gamma\varepsilon\nu o\varsigma$ [transition to another genus].[a]

There are really two ideas conflated here: that of a correct proof and that of a correct concept. In each case the correctness is relative to a given conclusion or theory. According to Bolzano logical or formal correctness is not the sole criterion of an adequate or correct proof; the concepts involved in the deduction are to appropriate, in some sense, to the conclusion. For example, with respect to the elementary theory of the triangle and parallel lines the concepts of straight line and direction are appropriate, whilst those of motion and the plane are deemed inappropriate. From these last two examples we can distinguish several ways in which concepts can be inappropriate. First, the concept of motion belongs to a different subject from geometry; it requires the empirical concept of an object occupying different positions in space and this is alien to a science which only studies space. To employ the idea of motion in a geometry proof is thus a 'transition to a different genus'. Second, the idea of motion is conceptually premature with reference to some hierarchical ordering. This seems to be based on the idea that if concept A 'contains' concept B, then B is prior to A in the hierarchy. And this priority or containment can itself occur in two ways. One way produces a logical circularity and motion illustrates this because, Bolzano says, to prove a geometrical result by means of motion one must first prove the possibility of the motion which, in turn, will require a proof of the result.

Another form of hierarchy that needs to be respected in proofs is illustrated by the rejection by Bolzano of the plane for his theory of triangles and parallels. Now it is not, of course, that the plane is not a geometric concept, and it is not that its use will conceal a logical circularity. It is simply a more 'advanced' concept. We can think of the system of two intersecting straight lines without any conceptual necessity to think of the plane that they happen to determine. Thus, there is a principle of what we might call 'conceptual correctness' at work in Bolzano's idea of an objective organization of knowledge. The other major principle is that of the 'ground-consequence relation'. This first explicitly appears in the work *BD* to which we must now turn.

The work *BD* must be one of the first books devoted to what we would now call foundations of mathematics, or philosophy of mathematics. (Indeed, this latter phrase was the title given to it by Fels for the second edition.) After a short *Preface* the first main part is devoted to the nature of mathematics and its proper classification. The second part deals with definitions, axioms, proofs, and theorems. It is here that the ground-consequence relation is introduced:

> in the realm of truth . . . a certain *objective connection* prevails . . . . some of these judgements are the grounds of others and the latter are the consequences of the former. (*BD* II §2)

[a] See the footnote on p. 32.

He goes on to explain that the proper purpose to pursue in a scientific exposition is to arrange the judgements so as to reflect this objective connection. Much later in the *Wissenschaftslehre* of 1837 there is a standard example to explain the ground-consequence relation (*WL* §162). Consider the propositions:

1. It is warmer at X than at Y.
2. The thermometer is higher at X than at Y.

If we know either 1 or 2 then we also know the other, at least it is a basis for knowing [*Erkenntnisgrund*] the other. But objectively 1 is the ground for the consequence 2 and not conversely. Bolzano explains in *BD* I §15 that he regards a cause as a ground that acts in time. He is also very clear that it should be distinguished from the semantic relationship of deducibility that has already been described (p. 7).

In *BD* Bolzano says little more about the ground-consequence relationship but he had already begun writing the 'second issue' (see *BD* title page) where he declared the terms 'ground' and 'consequence' incapable of definition although he did state some axioms that they should satisfy. (This writing has been published in *BGA* 2A5.) In his later philosophy it emerges that the ground-consequence relation holds in the *an sich* realm, that is, the collection of objective propositions, truths, and ideas in themselves. It is possible that the very idea of such a realm of objective contents emerged originally from sustained consideration of mathematics, and in particular, geometry. It is revealing for Bolzano's way of thought that in *BG* II §5 he acknowledges a point, and also lines and surfaces, as 'imaginary' objects for which nothing can be given in intuition (in contrast to solids). He explains in the same place that 'the definitions attempted in this paper of the straight line §26, and the plane §43, are made on the assumption that both are simply objects of thought [*Gedankendinge*]'. This does not imply that they are abstractions constructed by the mind, but rather, more consistent with Bolzano's outlook (and his Catholic intellectual heritage), that they are objective entities apprehended by the mind.

The ground-consequence relation is closely related to the idea of a correct proof, or as Bolzano often puts it, a scientific [*wissenschaftlich*] proof. The nature of such a proof is one of the main concerns of *BD*:

> We must therefore take the word in a *narrower* sense and understand by the *scientific proof* of a truth the representation of the *objective dependence* of it on *other truths*,' i.e. the derivation of it from those truths which must be considered as *the ground for it*—not fortuitously, but *actually and necessarily*— while the truth itself, in contrast, must be considered as their consequence [*Folge*]. (*BD* II §12)

The way this dependence can arise, and the nature of definitions, are subjects treated in the remainder of this major part of *BD*. An important contribution Bolzano makes here is a solution to the problem of how to define, or come to agreement, on the basic or simple concepts of a system. He says we should do so in the same way as we first learn terms in our mother tongue—by considering several propositions containing the term (see *BD* II §8). This is akin to what we would call implicit definition and, of course, is closely related to axiomatic systems. However, this is hindsight; there is no suggestion that Bolzano did, or could, consider axiom systems with the degree of formality familiar since the work of Hilbert.

Any such book as *BD* could hardly ignore Kant and his thinking about mathematics and accordingly there is an *Appendix* devoted to a criticism of Kant's theory of the construction of mathematical concepts through pure intuition. The central point of Bolzano's criticism was that the very notion of pure intuition was incoherent containing, he believed, internal contradictions. An excellent account of this, and indeed the whole methodology outlined in *BD*, is contained in Rusnock (2000), Ch. 2. For a more extended and philosophical discussion of Bolzano's views of proof and their relationship to Kant see Lapointe (forthcoming).

While in some important issues he was willing to change his mind, in many respects there is an impressive degree of cohesion and integrity in Bolzano's thought over his whole working life. The ideas and innovations in methodology that are signalled in his first work *BG*, and elaborated in *BD*, were not mere youthful enthusiasms. He continued in his later work to draw attention to *BG*, and even in 1844 he mentions in a letter to Příhonský the possibility of a second edition of the work (Winter, 1956, p. 245). These ideas, for example, of what we have called 'conceptual correctness' and the ground-consequence relation, as they are developed and explained in the short works that follow, are but part of a whole that represents a vast amount of thinking and working and to which his mathematical diaries are testimony. This latter material is of variable quality and value but the continuing diary publications offer an unprecedented opportunity for research into the context, the origins, and the significance of fundamental ideas that have proved surprisingly fruitful in ways, and for reasons, that are not yet well understood.

Betrachtungen

über

einige Gegenstände

der

Elementargeometrie

von

Bernard Bolzano.

Τας ἐπιδοσεις ὁρωμεν γιγνομενας, και των τεχνων, και των ἀλλων ἁπαντων, οὐ δια τους ἐμμενοντας τοις καθεστωσιν, ἀλλα δια τους ἐπανορθουντας, και τολμωντας ἀει τι κινειν των μη καλως ἐχοντων. Isocr. Evag.

Prag, 1804.
in Commission bey Karl Barth.

Considerations
on
Some Objects
of
Elementary Geometry

by

Bernard Bolzano

τὰς ἐπιδόσεις ἴσμεν γιγνομένας καὶ τῶν τεχνῶν καὶ τῶν ἄλλων ἁπάντων οὐ διὰ τοὺς ἐμμένοντας τοῖς καθεστῶσιν, ἀλλὰ διὰ τοὺς ἐπανορθοῦντας καὶ τολμῶντας ἀεί τι κινεῖν τῶν μὴ καλῶς ἐχόντων.

—*Isocrates*, Evagoras.

Prague, 1804
In Commission with Karl Barth

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Title page quotation:

Progress is made, not only in the arts, but in all other activities, not through the agency of those who are satisfied with things as they are, but through those who correct, and have the courage constantly to change, anything which is not as it should be.

Isocrates, Evagoras, Vol. III, p.9, Loeb Classical Library,

Tr. L.van Hook, Heinemann, 1945.

Dem

Hochwürdigsten, Hochgelehrten und Wohlgebornen

Herrn, Herrn

Stanislaus Wydra,

Director und Professor der Mathematik, emeritirten Rector Magnificus, Domherrn bey Aller Heiligen etc. etc.

zum Beweise

einer unbegränzten Hochachtung und Dankbarkeit

gewidmet

von seinem ehemaligen Schüler
dem Verfasser.

Dedicated to the
most worthy, learned and noble Gentleman

Herr

Stanislaus Wydra[a]

Director and Professor of Mathematics,
Emeritus Rector Magnificus, Canon of
All Saints, etc., etc.

as Proof
of an unbounded Respect and Gratitude

by his former Pupil,
the Author

[a] Stanislaus Wydra (1741–1804) was a Czech Professor of Mathematics at Prague University from 1772 until 1803 when he became blind. He published works (in Latin) on differential and integral calculus and a history of mathematics in Bohemia and Moravia. He also published a work on arithmetic in Czech.

Preface

It is well known that in addition to the widespread usefulness provided by its *application* to practical life, mathematics also offers a second use which, while not so obvious, is no less beneficial. This is the use which derives from the exercise and sharpening of the mind, from the beneficial promotion of a *thorough way of thinking*. It is this use that is chiefly intended when the state requires every student to study this science [*Wissenschaft*].[b] As I could no longer restrain the ambition to contribute something to the constant progress of this splendid science, in my spare time—and following my personal preferences—I have been considering, on the whole, only the improvement of theoretical [*spekulative*] mathematics, i.e. mathematics in so far as it will bring about the second benefit mentioned above.

It is necessary here to mention some of the rules which, among others, in my opinion apply to this matter.

Firstly, I propose for myself the rule that the *obviousness of a proposition* does not free me from the obligation to continue searching for a proof of it, at least until I clearly realize that absolutely no proof could ever be required, and why. If it is true that ideas are easier to grasp when they are everywhere clear, correct and connected in the most perfect order than when they are to some extent confused and incorrect, then we must regard the effort involved in tracing back all truths of mathematics to their ultimate foundations, and thereby endowing all concepts of this science with the greatest possible clarity, correctness and order, as an effort which will not only promote the *thoroughness* of education but will also make it *easier*. Furthermore, if it is true that if the first ideas are clearly and correctly grasped then much more can be deduced from them than if they remain confused, then this effort can be credited with a *third* possible use—the *enlargement* of the science. The whole of mathematics offers the clearest examples of this. At one time something might have seemed superfluous, as when Thales (or whoever discovered the first geometric proofs) took much trouble to prove that the angles at the base of an isosceles triangle are equal, for this is obvious to common sense. But Thales did not doubt *that* it was so, he only wanted to know *why* the mind makes this necessary judgement. And notice, by drawing out the elements of a hidden argument and making us clearly aware of them, he thereby obtained the key to new truths which were not so clear to common sense. The application is easy.

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[b] On the translation of *Wissenschaft* see the remarks in the *Note on the Translations*.

Secondly, I must point out that I believed I could never be satisfied with a completely strict proof *if it were not derived from the same concepts* which the *thesis* to be proved contained, but rather made use of some fortuitous, alien, *intermediate concept* [*Mittelbegriff*], which is always an erroneous *μεταβασιςἐις ἀλλο γενος*.[c] In this respect I considered it an error in geometry that all propositions about angles and ratios [*Verhältnissen*] of straight lines to one another (in triangles) are proved by means of *considerations of the plane* for which there is no cause in the *theses* to be proved. I also include here the concept of *motion* which some mathematicians have used to prove purely geometrical truths. Even *Kästner* is one of these mathematicians (e.g. *Geometrie, II. Thl., Grundsatz von der Ebne*).[d] *Nicolaus Mercator*, who tried to introduce a particularly systematic geometry, included the concept of motion in it as essential. Finally, even *Kant* claimed that motion as the *describing* of a space belonged to geometry. His distinction (*Kritik der reinen Vernunft*, S. 155)[e] in no way removes my doubt about the necessity, or even merely the admissability of this concept in pure geometry, for the following reasons.

Firstly, I at least cannot see how the idea of motion is to be possible without the idea of a *movable object* in space (albeit only imagined) which is to be distinguished from space itself. Because, in order to obtain the idea of motion we must imagine not only infinitely many *equal* spaces next to one another, but we must assume *one and the same thing* being successively in different spaces [*Räume*] as its *locations* [*Orte*]. Now if even *Kant* regards the concept of an object as an *empirical* concept, or if it is admitted that the concept of a thing *distinguished* from space is alien to a science which deals *merely* with space itself, then the concept of motion should not be allowed in geometry.

On the other hand, I think the theory of motion already presupposes that of space, i.e. if we had to prove the *possibility* of a certain motion which had been assumed for the sake of a geometrical theorem, then we would have to have recourse

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[c] *Translation*: crossing to another kind. Bolzano uses this phrase in *BD* II §29 and in *RB Preface* (see p. 126 and p. 254 respectively of this volume). It is a near quotation of a phrase used by Aristotle in the *Posterior Analytics* at 75[a]38. Bolzano has ἐις (to), where the text in Barnes has εξ (from). The complete sentence reads: 'One cannot, therefore, prove by crossing from another kind—e.g. something geometrical by arithmetic.' (Aristotle–Barnes, 1975, p. 13).

[d] The axiom of the plane [*Grundsatz von der Ebne*] referred to here is as follows. 'A straight line, of which two points are in a plane, lies completely within this plane (Theorem 1, Definition 7, and Axiom 1). But since the plane, in which this straight line is, can rotate around it as an axis, three points determine the position of a plane, and therefore every plane angle, and every triangle is in a plane.' Kästner (1792), I (iv), p. 350.

[e] The distinction referred to occurs in a footnote on p. 155 of the second German edition (Kant, 1787). 'Motion of an object in space does not belong to pure science and consequently not to geometry. For the fact that something is movable cannot be known *a priori*, but only through experience. Motion, however, considered as the describing of a space, is a pure act of the successive synthesis of the manifold in outer intuition, in general by means of the productive imagination, and belongs not only to geometry, but even to transcendental philosophy.' (Kant, 1929, p. 167)

to precisely this geometrical proposition. An example is the above-mentioned axiom of the plane (due to *Kästner*). Now because the assumption of any motion presupposes for the proof of its possibility (which one has a duty to give), particular theorems of space, there must be a science of the latter which precedes all concepts of the former. This is now called pure geometry.

In favour of my opinion there is also *Schultz* who, in his highly-regarded *Anfangsgründe der reinen Mathesis*, Königsberg, 1790, did not assume any idea of motion.[f]

In the present pages I am not providing any *œuvre achevé* but only a small sample of my investigations to date, which concerns only the *very first propositions of pure geometry*.

If the reception of this work is not wholly unfavourable then a second might follow it shortly on the first principles of mechanics.[g] I would especially like to have the judgement of those well-informed about contemporary geometrical ideas. That is the reason, as a more specific motive, that I have chosen to put something into print on this difficult material straight away rather than on another subject (as would certainly have been possible). Now something more about this material.

It is obvious that for a proper theory [*richtige Theorie*] of the straight line—I am thinking of the proofs of propositions such as: the possibility of a straight line, its determination by two points, the possibility of being infinitely extended, and some others—no considerations of *triangles or planes* can be used. On the contrary, the latter theory [*Lehren*] must only be based on the former. So I have set out in the *first part* an attempt to prove the *first propositions of the theory of triangles and parallels* only on the assumption of *the theory of the straight line*. As far as I am aware this has not been done before, because in all other places various *axioms of the plane* have been assumed, axioms which, if they had to be proved,[h] would require precisely that theory of triangles. Therefore in my view the first theorems of geometry have been proved only *per petitionem principii*;[i] and even if this were not so, a *probatio per aliena et remota*[j] has still been given which (as already mentioned) is absolutely not permissible.

[f] Johann Schultz (1739–1805) was a Professor of Mathematics at Königsberg and a friend of Kant. The work Schultz (1790) expresses several of the methodological principles, which were to be espoused by Bolzano, but it does not contain Bolzano's repudiation of the plane for the theory of the straight line and triangles.

[g] This almost certainly refers to a paper on the composition of forces not published until 1842. This is the work *ZK* listed in the *Selected Works of Bernard Bolzano* on p. 679. Bolzano explains in the *Preface* of that paper that he had been working on the material it contains forty years earlier.

[h] It was common at this time to regard an axiom simply as a self-evident statement. Bolzano did not think this was generally true and often sought for other kinds of justification. See, for example, *BD* II §21.

[i] *Translation:* by 'begging the question'.

[j] *Translation:* proof by alien and remote [ideas].

I regard the theory of the straight line itself, although *provable* independently of the theory of triangles and planes, yet still so little *proved*, that in my view it is at present the most difficult matter in geometry. In the *second part* I shall present extracts from my own considerations on the matter which seem to me to be the most fundamental, although they still do not reach the foundation. I only do this to find out whether I should continue on this path that I have taken.

I Attempt to Prove the First Theorems Concerning Triangles and Parallel Lines Assuming the Theory of the Straight Line

§ 1

Definition. Angle is that predicate of two straight lines *ca, cb* (Fig. 1), having one of their extreme points *c* in common, which is shared by every other system of two lines $c\alpha$, $c\beta$, which are *parts* of the former with the same initial point *c*. *c* is called the *vertex* of the angle, and the lines *ca, cb,* in so far as their length is disregarded so that the lines $c\alpha$, $c\beta$ could be taken instead, are called its *arms* [*Schenkel*].[k]

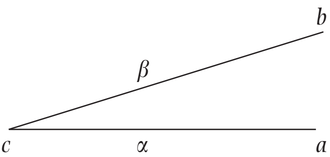

Fig. 1.

§ 2

Note. The phrasing which makes this a long definition is caused by the usage by which the angle *acb* is called *equal* to the angle $\alpha c\beta$, and accordingly *angle* is really a property of two *directions* (as I define the word in Part II[l]) and not two *lines*.[m] Others say: 'the angle is independent *of the magnitude* [*Größe*] of the arms,' which must, however, be understood with the qualification that the magnitude of an arm is never *negative*. Moreover, with some thought it becomes clear that the expression I have chosen is complete. For example, in order to prove that the angle $acb = \alpha c\beta$ (Fig. 2) we conclude directly *ex definitione*, $acb = ac\beta$, $ac\beta = \alpha c\beta$, therefore $acb = \alpha c\beta$.

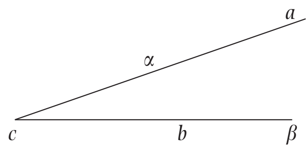

Fig. 2.

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k *Schenkel* is consistently used for the lines of indeterminate length bounding an angle, in contrast to *Seite* (side) for the determinate lines bounding a figure such as a triangle. See, for example, the proof in I §12. *Schenkel* is literally 'thigh' or 'leg', and although some authors have used the phrase 'the legs of an angle', it is perhaps in deference to a Victorian sense of propriety that 'arm' has prevailed in English as the most common term for this purpose and has been adopted here.

l See II §6.

m In II §12 angle is simply identified with 'the system of two directions proceeding from a point'. Compare Hilbert (1971), p. 11: 'Let α be a plane and *h, k* any two distinct rays emanating from *O* in α and lying on two distinct lines. The pair of rays *h, k* is called an *angle* . . .'.

§ 3

Theorem. Every angle determines [*bestimmt*] its adjacent angle.

Proof. From the definition (§1) an angle is determined if its arms are determined. Now the arms of the given angle determine at the same time the arms of the adjacent angle. For these are: one, an arm of the given angle itself, the other, an extension of the other arm of the given angle beyond its vertex. Now we know from the theory of the straight line that this extension (viewed apart from its length in the above sense (§1)) is determined [*gegeben*].[n]

§ 4

Corollary. Therefore if two angles are equal, their adjacent angles are equal. For *things which are determined in the same way are equal.*[o]

§ 5

Theorem. Vertically opposite angles[p] are equal, $ac\beta = bc\alpha$ (Fig. 3).

Proof. Their determining pieces are equal. The angle $ac\beta$ is an adjacent angle of acb, the angle $bc\alpha$ (in that order) is an adjacent angle of bca. Therefore, if $acb = bca$,[q] then also $ac\beta = bc\alpha$ (§§ 3, 4).

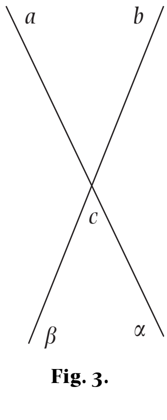

Fig. 3.

[n] Perhaps Bolzano has in mind II §15 where the concept of 'opposite direction' is introduced.

[o] The words 'same' and 'equal' here both translate *gleich*.

[p] The German *Scheitelwinkel*, is literally 'vertex angles'.

[q] In II §14 Bolzano says he still has 'no satisfactory proof' of this.

§ 6

Note. This is no different from the ordinary way of demonstrating the equality of two things: we conclude *ex datis* that their determining pieces are equal (§4). The Euclidean proof of the present theorem does not follow this method. But it has, in my view, two further *defects*. Firstly, the alien consideration of a *plane* has already been introduced here; for angles are added, which is only possible on the condition (albeit tacit) that the angles are in the same plane. Secondly, it assumes that angles are *quantities*, and on this assumption they are added and subtracted subject to the purely arithmetic axiom: 'equals taken from equals gives equal remainders'. A thing is called a *quantity* [*Größe*] in so far as it is regarded as consisting of a *number* [*Anzahl*] (plurality [*Vielheit*]) of things which are equal to the *unit* (or the measure). Therefore if I were to consider an angle as a *quantity* I must as a consequence imagine it as composed of several individual equal angles in one plane, which—it can be said explicitly or not—is really nothing but the idea of the area contained within the arms. So Schultz would be justified when he considers this infinite area as an essential property of angle. The author of *Bemerkungen über die Theorien der Parallelen des H. Hofpr. Schultz etc.* (Libau 1796) opposes this assumption of Schultz at length, yet does no better because he still considers angles as quantities and even brings in the concept of motion in defining (S. 55) angle as the concept of the ratio of the uniform motion of a straight line about one of its points to a complete rotation. But he thereby shows us clearly the true origin of all ideas of angles as quantities, which in my opinion is nothing but the empirical concept of motion. Now it is obvious that I can think of two lines with a common endpoint, therefore an *angle* (§1), without *having* to think of a surface, or of other lines drawn between them (component angles), or of a motion by which one of these lines comes from the other's position into its own position. Consequently, the angle in its essence is not a quantity. This was something the thorough [scholar] *Tacquet* surely already realised (*Elementa Geometriae I, Prop.* 3, *Schol.* 16). I am only surprised that he explains the usual, and contrary, kind of presentation as merely an abbreviation which, though improper, is a harmless *way of speaking* and may therefore be retained. If angle is a mere *quality* then one can only speak of *equality* or *inequality* of angles (or as *Tacquet* would have it: similarity or dissimilarity), but not of their being greater or smaller.[r] These denote two special *kinds* of inequality, which are really only valid for *quantities*, or at least one must first be agreed on their meaning. I shall therefore nowhere treat angles as *quantities*, and I shall reject as unusable all proofs in Euclid in which they are so regarded.[s] Nevertheless this makes no difference to the whole algebraic part of

[r] Tacquet does indeed begin his work as Bolzano describes but soon systematically adopts a quantitative treatment which is hard to explain as merely a 'way of speaking'.

[s] In other words, Bolzano is rejecting the Euclidean development of elementary (plane) geometry almost entirely.

geometry because here (as is well known) what we have in mind are *arcs* and not angles.

§ 7

Definition. Let us suppose the two points *a*, *b* in the arms *ca*, *cb* of an angle (Fig. 4) are different from the vertex *c*, and that through them there is a straight line *ab*, then the system of straight lines *ca*, *cb*, *ab* is called a *triangle*.

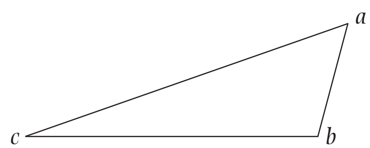

Fig. 4.

§ 8

Note. There is therefore no mention of any area.

§ 9

Corollary. In every triangle there are three angles. Each of these is *included* by two sides (i.e. it has them as arms) and stands *opposite* the third (i.e. it does not have it as an arm). Every side *lies on* two angles (i.e. it supplies one arm of the two angles).

These are in fact proper theorems,[t] but they are so easy to prove that I may save space by stating them in this way.

§ 10

Theorem. In two equal triangles, I. the *sides* are equal which stand across two equal sides, or which are opposite an equal angle, or which lie on two equal angles; II. the *angles* are equal which stand across two equal angles, or which are included by two equal sides, or which are opposite an equal side.

Proof. These sides (I), and angles (II), are determined in their triangles by the data [*Angabe*] given. For it follows from §9 that there is only one side or only one angle which belongs to these data. Now since the triangles themselves and their data are equal, the determining pieces of these sides and angles are equal.

[t] Bolzano distinguishes between theorems and corollaries in *BD* II §§24, 25 on pp. 120–121.

§ 11

Note. I call triangles which are otherwise called *equal and similar* simply *equal*.[u] According to usage the word *equal* says more than the word *similar*, so that if two objects are called equal they must already be *similar*. But *one property* of these objects (which does not determine them), e.g. the magnitude of two areas, can be equal without the objects, the areas themselves, being equal. This one property should not be given the name of the object itself; therefore one should not say, 'two triangles are equal' if actually one only wants to say that the magnitudes of their areas are equal. If one avoids this rather unmathematical metonymy then the addition of *similar* to the word *equal* is superfluous. But if some people want the word *equal* to be used of nothing but the property of *quantity*, then I ask them for another word which could be used generally to denote this concept? This word is not *identity*, for one and the same thing is only called identical in so far as it is compared with itself.

§ 12

Theorem. Two sides and the angle included by them determine[v] the triangle to which they belong.

Proof. From the definition (§1) it follows directly that the angle and the definite lengths of the pieces *ca*, *cb* of its arms together contain all predicates of the system of two lines *ca*, *cb*. For the angle alone contains that which is independent of the definite lengths of the lines *ca*, *cb*; therefore if this is included everything in the system is determined. Now we understand by sides [*Seiten*] (of a triangle)—in contrast to arms—determinate lines. Therefore, everything in the system of two lines *ca*, *cb* is determined; consequently also the points *a*, *b* and the straight line *ab* which is drawn through them (§7) are determined, as well as the angles which the line *ab* forms with the two other sides.

§ 13

Note. From this proposition two corresponding theorems will now follow, one about the equality, and the other about the similarity, of triangles.

§ 14

Theorem. Two triangles in which two sides and the included angle are equal, are themselves equal.

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[u] According to Vojtěch (1948), p. 190, Note 14 the phrase *gleich und ähnlich* (equal and similar) was used up to the end of the eighteenth century in German writings, and 'congruere' in Latin writings, to describe structures with equal boundaries. The word *congruiren* was used in German from around the end of the eighteenth century. Bolzano objects to it as empirical and superfluous (see I §49).

[v] Each occurrence of the words 'determine', 'definite', 'determined', and 'determinate' in this paragraph translates forms of the German *bestimmen*.

Proof. For their determining pieces are equal (§12).

§ 15

Corollary. This gives rise through mere negation of the conclusion (*conclusio hypothetica in modo tollente*)[w] to several propositions. For example, if two sides are equal but the third side is unequal, then the included angle must also be unequal. And so on.

§ 16

Definition. Two spatial objects are called *similar* if *all the characteristics* which arise from the comparison of the parts *of each one among themselves*, are *equal* in both; or if through every possible comparison of the parts of each one among themselves, no *unequal* characteristics can be perceived.

§ 17

Theorem. Objects whose determining pieces are similar are themselves similar.

Proof. Suppose they are not similar, then by making comparisons among the parts of one of them, an unequal characteristic must be perceived (i.e. one which is not present in the other). This inequality requires a basis [*Erkenntnisgrund*] in the objects themselves, and so in their determining pieces (for from these everything which is in the object itself must be perceived). There would therefore have to exist a difference in the determining pieces recognizable from a comparison among themselves, consequently they would not be similar (§16).

§ 18

Note. This proposition lies at the basis of the theorems of similarity, in the same way as the proposition 'objects whose determining pieces are equal, are themselves equal' (§4) lies at the basis of the theorems of equality (§6).

§ 19

Axiom. There is no special idea given to us *a priori* of any *determinate distance* (or absolute length of a line), i.e. of a determinate kind of separation of two points.[x]

[w] This medieval term occurs several times in this work and refers to an argument of the form: If *A* then *B*, but not-*B*; therefore not-*A*. The exact relationship of this principle to *reductio ad absurdum* (which is not mentioned here by Bolzano) is subtle, see Coburn and Miller (1977).

[x] It is pointed out in Gray (1989), p. 72 that Lambert had noticed the asymmetry between length and angle in that the latter has a natural absolute value of one revolution while the former has no such corresponding value (in Euclidean geometry). It appears that Lambert would have been aware, as Bolzano evidently was not, that the axiom of this section is equivalent to the parallel postulate. It is discussed further in I §24, but its origin in Bolzano's thinking seems to be a complicated matter.

§ 20

Theorem. All straight lines are similar.

Proof. Straight lines are determined by their two end-points. Now we have (§19) no special idea of any determinate separation of two points. Therefore, every separation of two points is similar to every other one. So too, all straight lines are themselves similar (§17).

§ 21

Theorem. Two triangles in which two sides enclosing an equal angle are in proportion, are themselves similar.

Proof. The determining pieces of these triangles are similar. These are (§12) an angle with the sides which include it, or (because the ratio of one line to another determines the former from the latter), an angle, a side and the ratio of the other side to the first. Now the angle and the ratio in both triangles are *equal* (and consequently also similar), but one side is similar, therefore the determining pieces are similar.

§ 22

Theorem. Similar angles are equal.

Proof. The word *angle* designates (§1) that which determines everything perceivable in the system of two directions *am*, *an* (Fig. 5). This is the distance *mn*, which every two points *m*, *n* have from one another, where *m*, *n* are determined by arbitrary distances *am*, *an* in the two directions; and the distance *pr*, which every point *p* in the one direction *am* has from a point *r* in *mn*; etc. If then in the two given angles all these perceivable pieces are *equal*, this means nothing other than that the angles themselves are equal. Now let *a* and α be *similar* angles (Fig. 6), then if $am : an = \alpha\mu : \alpha\nu$ we must also find (§16) $am : mn = \alpha\mu : \mu\nu$, otherwise the comparison of the parts of the angle α among themselves would not give rise to exactly the same idea as that of the parts of the angle *a*. Now consider αo (in direction $\alpha\mu$) = *am* and αu (in direction $\alpha\nu$) = *an*; then $\alpha\mu : \alpha\nu = \alpha o : \alpha u$. Therefore,

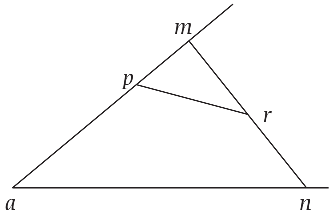

Fig. 5.

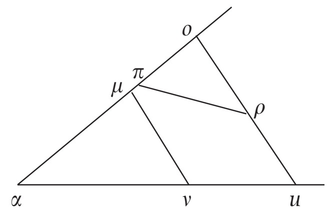

Fig. 6.

(if *ou* is drawn) (§21) $\triangle o\alpha\mu \backsim \triangle\mu\alpha\nu$,[y] whence (§16)

$$ou = \frac{\mu\nu . \alpha o}{a\mu} = \frac{mn . \alpha o}{am} = mn.$$

In the same way it can be shown that if π, ρ are taken so that $o\pi = mp$, $o\rho = mr$, then $\pi\rho = pr$. And so on. So therefore the condition mentioned above, of the equality of all characteristics, holds for angles a and α; accordingly these angles are equal.

§ 23

Corollary. Therefore in similar triangles, the angles opposite proportional sides are equal (§§ 16, 10, 22).

§ 24

Note. This theory of similarity, like its subsequent application, is a result of my own reflection, although *Wolff* has already put forward the same theory in detail in his *Philosophia prima seu Ontologia, Sect. III., Cap. I, de Identitate et Similitudine,* and also in the *Elementis Matheseos universae,* and thus it has been known to the academic world for a long time.[z] I myself have briefly read through the first work only recently, but I read the other several years ago, not in order to learn from them about the elementary theory but only with a view to finding in them some unknown problem. In doing this I skimmed the *Arithmetic* and *Geometry* so carelessly that I was not aware of this important change which is

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[y] Such a symbol for similarity, (or else the symbol $\sim$ as used in *BG*(2)) seems to have been first used by Leibniz, and then by Wolff. See Cajori (1929), i, §372 for further details.

[z] The concept of similarity introduced by Wolff in these works is very general but vague. For example, in the section of the *Philosophia* to which he refers, at §195, appears the definition 'Those things are similar in which the things by which they ought to be distinguished are the same . . . Similitude is the sameness [*identitas*] of those things by which entities ought to be distinguished from each other.' What Bolzano's theory of similarity has in common with that of Wolff is the central role played by the idea that a mathematical object may be 'determined' in a certain way by its component pieces. Both the pieces and the manner of being determined are important.

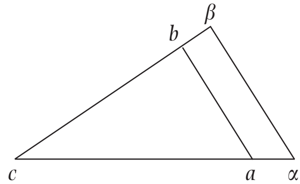

Fig. 7.

introduced in small scholia immediately after the definitions. The first form which I gave to my proof of the proposition §21, before I had read *Wolff's Ontologia*, is briefly as follows: we have no *a priori* idea of any determinate separation of two points, or more generally of any determinate spatial object. If therefore, an *a priori* knowledge of spatial objects is to be possible it must be valid for every unit of measurement adopted. For example, if in the $\triangle acb$ (Fig. 7), $cb = n.ca$, $ab = m.ca$, and in $\triangle \alpha c\beta$ with equal angles in the same way, $c\beta = n.c\alpha$, then $\alpha\beta$ must $= m.c\alpha$ because otherwise we would have to have had an *a priori* idea of the determinate line *ca* for which only the number *m* is valid. My intention on the discovery of this proof was to complete the well-known gap in the theory of parallels by means of the theory of the similarity of triangles. In fact, even if one is not completely satisfied with *Wolff's* proof (or mine) of the theory of the similarity of triangles, it still seems to me that the effort of proving this theory (from a basis independent of parallel lines and considerations of the plane) deserves more attention from geometers.[a] *Kant* has noted (*Von dem ersten Grunde des Unterschiedes der Gegenden im Raume*, 1768, to be found in the collection of some of his writings by *Rink*),[b] and he has repeated these thoughts elsewhere (*Prolegomena* p. 57ff),[c] that there are *differences* in spatial objects (therefore also the properties on which they are based) which cannot be perceived from any comparison among the parts of each one. For there are spatial objects which are completely equal and similar to each other and yet cannot be brought into the same space: therefore they must possess a difference. Such for example are two equal spherical triangles on opposite hemispheres. *Kant* called the basis of this difference the *direction*[d] in which the parts of the one and the other spatial object lie. This Kantian observation is

[a] Bolzano's outline of his 'first form' of proof for I §21 attempts to derive the result only from his axiom in I §19. The need to be independent of 'parallel lines and considerations of the plane' follows from Bolzano's requirement for the correct ordering of concepts, and is also a reference, by way of contrast, to the methods adopted in Book VI of Euclid (1956).

[b] The work referred to is in Kant (1968), ii, pp. 375–84.

[c] The passage referred to is now most easily found in Kant (1968), x, §13, and in English in Kant (1953), §13.

[d] The German here is *Gegend* as in the title of the work by Kant referred to a few lines above in this subsection. Dictionaries from the eighteenth century (such as Wolff (1747) and even the general dictionary Adelung (1793–1801)) make clear that in addition to its meaning as 'region', *Gegend* was commonly used as a synonym for *Richtung* (direction), which is the meaning intended here.

indeed quite correct; however, it is not only the *direction* but also secondly *the determinate kind of separation* (*the distance*) which is a property which cannot be perceived by any comparison of the parts of an object with each other. Indeed, if in two objects all properties which can be observed from the comparison of the parts of each one among themselves are equal, then it only follows that the two things are *similar*. They can still be *unequal*. If two objects are to be recognized as *equal*, then the one must be compared with a part of the other, or generally both must be compared with one and the same third thing (a common unit of measurement).* The reason for this is that we have no *a priori* idea of any determinate separation of two points (distance); so there is nothing we can do but to note the ratios of different distances to one another. This is the truth which I set up as an axiom in §19. As a motivation either for the acceptance of this axiom or the invention of some other way of proving the theory of similarity I may be permitted to recall the following. If a correct methodology requires of every systematic proof that it demonstrates the connection of the subject with the predicate, without the interference of fortuitous intermediate concepts, then our previous proofs of all theorems of similarity cannot stand up to any criticism. Let any expert cast a glance at our textbooks (at *Euclid*) in this respect and I hope to be cleared of any suspicion of slander. There therefore remains the obligation to look for error-free proofs for these theorems. These proofs would have to demonstrate—among other requirements just mentioned here—what is true of the *genus* without using an induction from the individual *species*. 'Volumes of similar solids vary as the cubes of similar sides, or more generally, *as* any other *solid* determined from them in a similar way. Surfaces vary as surfaces; lines (curved) vary as lines.' Where could Euclidean geometry demonstrate these propositions in this generality without resorting to the consideration of individual kinds (such as triangles)? But from §17 these propositions follow in complete generality and quite directly. For let A, a be two similar solids and let the solid B be determined from A in the same way as b is determined from a; consequently B, b are also similar and it is required to prove $A:B = a:b$. The system of solids A and B has the determining pieces: the solid A and the way that B comes from A. These determining pieces are *ex hypothesi* similar to the determining pieces in the system a and b. Therefore (§17) both systems are similar. Consequently everything which can be observed in the one system by the comparison of its parts is also equal in the other. Therefore if the volumes of the solids, A, B and a, b are compared, then it must be that $A:B = a:b$. The same proof can be applied to surfaces and lines. Finally, I may remark that I also use exactly this axiom (§19) in proving the first essential theorems in mechanics and that I believe that it can be usefully applied in all areas of mathematics (except arithmetic and

* *Kant* could therefore have cited in his paper not only the concept of direction but also that of separation as counter-examples to those philosophers who regard space as a pure relationship of coexisting objects.

algebra—because they have no particular thing for their object, but abstract plurality itself).

§ 25

Theorem. In an isosceles triangle the angles on the base are equal: $a = b$ (Fig. 8).

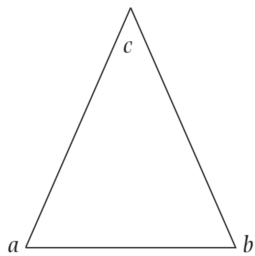

Fig. 8.

Proof. They are determined in the same way. From §14 it follows that $\triangle acb = \triangle bca$ (in the order of the letters). For ca in $\triangle acb = cb$ in $\triangle bca$; cb in $\triangle acb = ca$ in $\triangle bca$; $\angle acb$ in $\triangle acb = \angle bca$ in $\triangle bca$. Consequently (§10) $\angle b$ opposite ac in $\triangle acb = \angle a$ opposite bc in $\triangle bca$.

§ 26

Theorem. It is possible to erect from one point of a straight line another straight line in such a way that the two adjacent angles made with the segments of the former are *equal*.

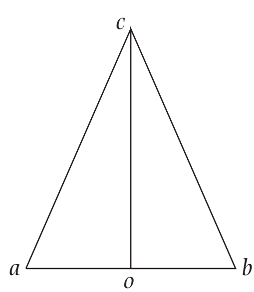

Fig. 9.

Proof. It is possible to think of two equal straight lines $ca = cb$ (Fig. 9) meeting at some angle c. Now if we draw ab and suppose o to be the *centre* [*Mitte*] of ab

(a concept defined in the theory of the straight line),[e] and finally we draw *co*, then (§25) $a = b$ and *per constructionem* $ac = bc$, $ao = bo$. Therefore (§14) $\triangle cao = \triangle cbo$. Hence (§10) $\angle coa = \angle cob$.

§ 27

Note. Euclid presents this proposition in the form of a *problem*. Now it is well known that theoretical geometry (e.g. that of *Euclid*) intends, by means of its problems, only to show the *possibility* of this or that spatial object. In contrast, the aim of giving a method whereby various spatial objects can be empirically constructed using a few simple instruments (e.g. straight edge and compass), is a *practical* aim. So the problems of *theoretical* geometry are really *theorems*, which is therefore the appropriate form for them. On the other hand, the practical part of geometry could contain the problems separately; and this opinion was also that of the Jesuit, *I. Gaston Pardies*. For this reason, the theoretician must also be allowed (and this is more important) to assume certain spatial objects without explaining the method of their actual construction, provided he has proved their *possibility*. With this in mind, I assumed the centre of *ab* in the present proposition without showing how it is to be found. And in what follows I shall assume the fourth proportional line for three given lines without showing how it would be constructed: it suffices that it is clear directly from §20 that we can think of a certain line *d* which has exactly the same ratio to the line *c* as that which the line *b* has to *a*.

§ 28

Theorem. It is possible to construct from one point of a straight line another straight line such that the adjacent angles formed are unequal.

Proof. As in §26, just let *ca*, *cb* be unequal, and instead of §14 apply §15.

§ 29

Corollary. From §§ 26, 28 there now follows the possibility of drawing from *any* point of any straight line another straight line so that the adjacent angles formed are either equal (§26) or unequal (§28).

Because any straight line, through being shortened or lengthened can become equal to *ab* (§§ 26, 28), and the given point in it can become its mid-point *o*, then what is possible for *ab* (§§ 26, 28) must be possible for any straight line and point.

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[e] See II §30.

§ 30

Theorem. Every system of a straight line produced indefinitely in both directions [*zu beiden Seiten*] and a point outside it, is similar[f] to every other such system (Fig. 10): $o, xy \backsim \omega, \xi\eta$.

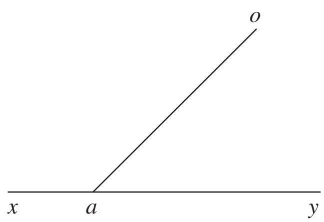

Fig. 10.

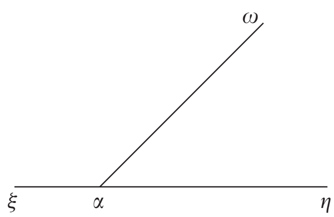

Fig. 10*.

Proof. For no difference can be observed in the two systems from the comparison of their parts (§16). Since the straight lines xy, $\xi\eta$ are produced indefinitely in both directions no point on these lines can be determined by the position which it has on them. Now since the points o, ω lie outside these straight lines, then it is essential that every line drawn from o, ω to a point a, α of the infinite line forms adjacent angles with the latter. The angles so formed in both systems can now either be unequal or equal. If they are *unequal*, I cannot conclude from this any difference in the two systems because the points a, α, on which these angles depend, are indeterminate. But if these angles are equal, $oax = \omega\alpha\xi$, $oay = \omega\alpha\eta$, then because the arms ax, ay; $\alpha\xi$, $\alpha\eta$ are indeterminate there can be no comparison of these with the lines oa, $\omega\alpha$. But in themselves these lines are similar, therefore o, xy; ω, $\xi\eta$ are similar systems in which oa, $\omega\alpha$ are similarly situated lines.

§ 31

Theorem. From any point o (Fig. 10) outside a straight line xy, produced indefinitely in both directions, it is possible to draw another straight line so that one angle which it forms with the line xy is *equal* to some given angle $\omega\alpha\xi$.

Proof. Consider a point ω in one of the arms of the given angle $\omega\alpha\xi$, then this is a point *outside* the other arm $\alpha\xi$. Now if the latter is produced indefinitely in both directions then one has a system ω, $\xi\eta$ of a point and an indeterminate

f The 1804 text has *gleich* (equal) here, which is clearly an error for *ähnlich* (similar). This is pointed out in *BG*(2).

straight line outside it, which is therefore *similar* (§30) to the given system o, xy. Consequently, because in the former system a line can be drawn from ω to $\xi\eta$ so that it forms the angle $\omega\alpha\xi$, then in the given system it must also be possible to draw a line from o to xy which forms an angle $oax = \omega\alpha\xi$.

§ 32

Theorem. From every point o (Fig. 11) outside a straight line xy, one and only one straight line can be drawn to the latter so that it forms *equal* adjacent angles on it.

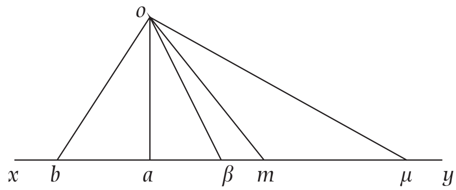

Fig. 11.

Proof. That one line can be drawn follows from §26 together with §31. Therefore let $oax = oay$. Likewise it follows from §28 together with §31 that a line om can be drawn from o to xy which forms *unequal* adjacent angles, omx not $= omy$. Now suppose there were another line ob which made $obx = oby$, then one may take $a\beta = ab$ (in the opposite direction to ab) and draw $o\beta$. Therefore (§14) $\triangle oab = \triangle oa\beta$, and (§10) $ob = o\beta$, $\angle oba = \angle o\beta a$; consequently (§4) $\angle obx = \angle o\beta y$. But *ex hypothesi* $\angle obx = \angle oba$, therefore $\angle oba = \angle o\beta y$. If one now assumes $\beta\mu = bm$ and draws $o\mu$, then (§14) $\triangle obm = \triangle o\beta\mu$. Therefore (§10) $om = o\mu$; $\angle omb = \angle o\mu\beta$. In $\triangle mo\mu$ (§25) $\angle om\mu = \angle o\mu m = \angle o\mu\beta$. Consequently $\angle omb = \angle om\mu$, i.e. $\angle omx = \angle omy$, *contra hypothesim*.

§ 33

Corollary. Therefore, the point o determines the line oa with the property of forming equal adjacent angles with xy. Consequently it also determines the nature of the angles oax, oay themselves.

§ 34

Theorem. All angles which are equal to their adjacent angles are also equal to each other.

Proof. In Fig. 12 let $\angle oax = \angle oay$, $\angle\omega\alpha\xi = \angle\omega\alpha\eta$; and if $\angle oax$ is not $= \angle\omega\alpha\xi$, then a line could be drawn from o which forms with xy an angle $= \omega\alpha\xi$ (§31). This forms equal adjacent angles (§4), therefore it cannot be different from oa (§32).

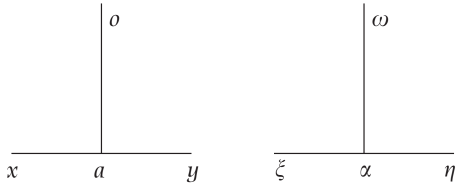

Fig. 12.

§ 35

Note. Since such angles are all equal to each other they may therefore be designated by the common name of *right angles*.

§ 36

Theorem. If from the point o (Fig. 13) oa is perpendicular to xy and a is the centre of mn, then I. the lines $om = on$, II. the angles $aom = aon$, III. the angles $amo = ano$.

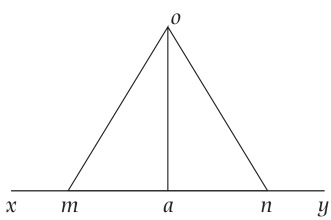

Fig. 13.

Proof. These follow directly from §14 and §10.

§ 37

Theorem. Conversely, if for the perpendicular oa either I. the lines $om = on$, or II. the angles $aom = aon$, or III. the angles $amo = ano$; then a is the centre of mn.

Proof. I. If (Fig. 14) $om = on$ and one takes p as the centre of mn, then it follows, as in §26, that op is perpendicular to mn, therefore (§32) p must be the same as [*einerlei*] a.[g] II. If (Fig. 15) $\angle aom = \angle aon$ and if one supposes $om : oa = on : o\alpha$ (the latter taken from o on oa) then (§21) $\triangle moa \backsim \triangle no\alpha$; consequently (§23) $\angle mao = \angle n\alpha o$, therefore = R.[h] Therefore α is the same as a (§32). Thus $oa = o\alpha$, therefore on account of the proportionality, also $om = on$ and (§10) $am = an$. III. If (Fig. 16)

[g] The word *einerlei* is translated by either 'identical' or 'the same as', but never by 'equal'. See II §1.

[h] It is not mentioned in Cajori (1929) but the letter R was in common use to denote a right angle, for example, in works by Schultz and Klügel.

$\angle amo = \angle ano$ and if one supposes $mo : ma = no : n\alpha$ (the latter taken from n on na) then (§21) $\triangle oma \backsim \triangle on\alpha$; consequently (§23) $\angle oam = \angle o\alpha n$ therefore = R. Therefore α is the same as a (§32). Thus, since (§21) $ma : ao = n\alpha : ao$, and $ao = \alpha o$, also $ma = n\alpha = na$.

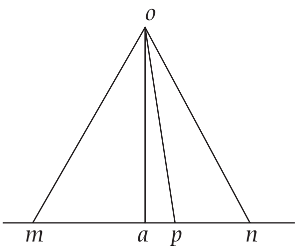

Fig. 14.

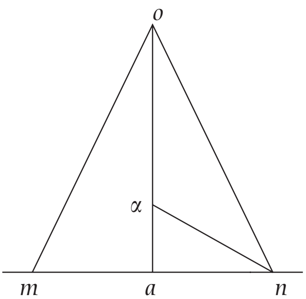

Fig. 15.

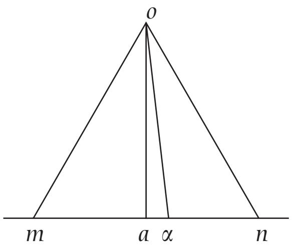

Fig. 16.

§ 38

Corollary. There are therefore no more than two lines om, on from o (Fig. 13) to xy for which either I. $om = on$, or II. $\angle aom = \angle aon$, or III. $\angle amo = \angle ano$.

For whichever of these occurs, $am = an$ (§37); now there are only two points in the straight line xy which are at an equal distance from the same point a.

§ 39

Theorem. From the same point o (Fig. 17) only one line oa can be drawn to the unbounded line xy so that it makes a given angle oax with the indefinitely extended part ax of this line.

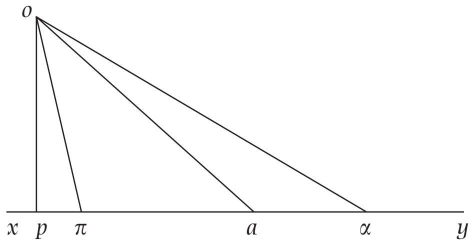

Fig. 17.

Proof. That one such line can be drawn follows from §31. That *only* one can be drawn is made clear as follows. Put $\angle o\alpha x = \angle oax$. Now consider the perpendicular op from o to xy. If p is at a or α then $\angle o\alpha x = \angle oax = \mathrm{R}$, therefore a, α are the same point (§32). If p is not at a or α, then consider $ao : ap = \alpha o : \alpha\pi$ (the latter taken in that arm of the two adjacent angles at α which is $= oap$ (§4)). Then (§21) $\triangle oap \backsim \triangle o\alpha\pi$. Therefore (§23) $\angle opa = \angle o\pi\alpha = \mathrm{R}$. Consequently, (§32) π is the same as p therefore $op = o\pi$ and by the ratio $op : pa = o\pi : \pi\alpha$ also $pa = \pi\alpha = p\alpha$. Therefore, (from the theory of the straight line) either p is the centre of $a\alpha$ or a is the same as α. The first alternative cannot hold because then (from the theory of the straight line) ap, αp would not be contained in one and the same indefinitely extended arm ax. Therefore the second alternative holds.

§ 40

Corollary. Conclusio in modo tollente.[i] If therefore two lines (Fig. 18) ac, bd form equal angles $cax = dbx$ with arms which are contained in the same indefinitely produced part of xy, then these lines at c, d cannot intersect anywhere. Likewise their extensions beyond a, b cannot intersect with one another, $a\gamma$ with $b\delta$. (But ac could indeed intersect $b\delta$ as in Fig. 18*.) If the angles $cax = dbx = \mathrm{R}$ (Fig. 18**) then $c\gamma$, $d\delta$ definitely cannot intersect; also ac cannot intersect with $b\delta$ towards c, δ because likewise $\angle cax = \angle\delta bx$ (§39).

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[i] See the footnote about this on p. 40.

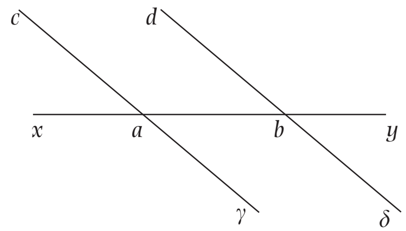

Fig. 18.

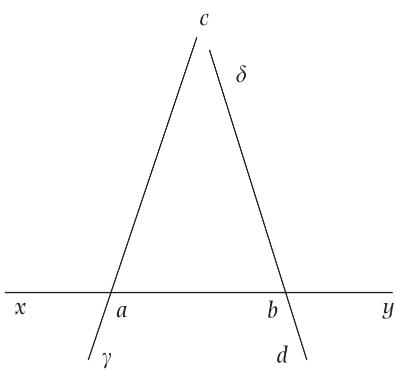

Fig. 18*.

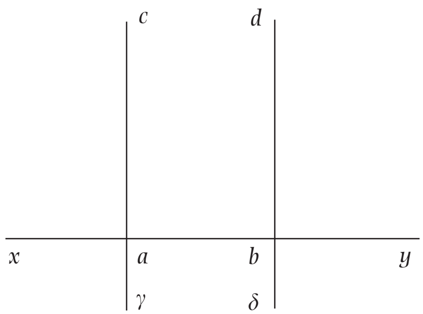

Fig. 18.**

§ 41

Theorem. One side and the two angles lying on it determine the triangle to which they belong.

Proof. If they did not determine it, there would have to be able to be another one. Therefore let (Fig. 19) *bac*, *bam* be two different triangles on the same angle *bac* and with the same side ab in which the second angle $abc = abm$. Suppose $ac : ab = am : an$ (the latter taken in ab from a) then (§21) $\triangle cab \backsim \triangle man$; consequently (§23) $\angle mna = \angle cba =$ (*ex hyp.*) $\angle mba$. Now since an lies in ab, the lines na, ba, produced beyond a, contain the same infinite part of the line ab (from the theory of the straight line). Hence (§39) n must be the same as b and consequently, because $ac : ab = am : an$, $ac = am$, therefore (§14) $\triangle mab = \triangle cab$.

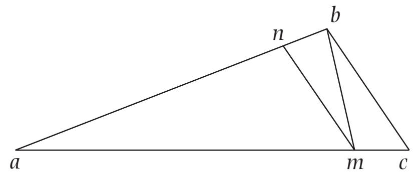

Fig. 19.

§ 42

Theorem. If in two triangles one side with the two angles lying on it are equal, then the two triangles themselves are equal.

Proof. Their determining pieces are equal (§41).

§ 43

Theorem. If in two triangles two angles are equal, then the two triangles themselves are similar.

Proof. Their determining pieces (§41): the side on which those angles lie (§20) and these angles themselves, are similar (§17).

§ 44

Theorem. In any triangle the sum of two sides is never equal to the third.

Proof. This sum is represented by extending ac beyond c so that $c\beta = cb$ (Fig. 20). Now if $a\beta = ab$ then (§25) it would follow in $\triangle \beta ab$ that $\angle \beta = \angle ab\beta$, and in $\triangle \beta cb$, that $\angle \beta = \angle cb\beta$. Therefore (§42) $\triangle c\beta b = \triangle a\beta b$. Consequently (§10) $a\beta = c\beta$, which is contradictory.

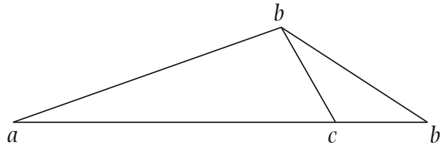

Fig. 20.

§ 45

Theorem. In a right-angled triangle, and only in such a triangle, the square (in the *arithmetic* sense) of the hypotenuse equals the sum of the squares of the other two sides[j] (Fig. 21): $ab^2 = ac^2 + bc^2$.

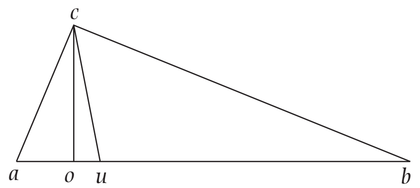

Fig. 21.

Proof.[k] I. To prove that (as *lines*)

$$ab = ac.\frac{ac}{ab} + bc.\frac{bc}{ab}$$

we construct lines of magnitude

$$ac.\frac{ac}{ab} \text{ and } bc.\frac{bc}{ab}.$$

Take $ab : ac = ac : ao$ (the latter in ab from a) and $ba : bc = bc : bu$ (the latter in ba from b) in order to obtain (§21) similar triangles, namely $\triangle bac \backsim \triangle cao$, $\triangle abc \backsim \triangle cbu$ (in the order of the letters). Consequently (§23) $\angle acb = \angle aoc$, $\angle bca = \angle buc$, and since $\angle c = \mathrm{R}$, the points o, u must be the same.[l] Now since *ex constructione* ao lies in the arm ab from a, $bu = bo$ in the arm ba from b, then o is in between a and b (from the theory of the straight line) and

$$ab = ao + ob = ac.\frac{ac}{ab} + bc.\frac{bc}{ab}.$$

II. If acb is *not* a right angle then o cannot be the same as u, otherwise aoc, buc would be equal adjacent angles; consequently it cannot also be that

$$ab = ao + bu = \frac{ac^2}{ab} + \frac{bc^2}{ab}.$$

[j] The word ‘sides’ here is translating *Katheten*, which corresponds to the now obsolete term ‘cathetus’ meaning ‘perpendicular’ (*OED*).

[k] According to Simon (1906), p. 109, this is a ‘completely original’ proof of Pythagoras theorem. It is certainly a muddled proof with the theorem being explicitly arithmetic, and the proof being explicitly to do with lines. Bolzano begins to address the underlying problem in II §33 ff. For a recent useful discussion of this important issue as it arises in Euclid, see Grattan-Guinness (1996).

[l] By I §32.

§ 46

Theorem. Three sides determine the triangle to which they belong.

Proof. If I just prove that three sides determine an angle, then it follows (by §12) that they also determine the triangle itself. I can only *infer* the determination of an angle (which is not given) in a triangle from the previous paragraphs (§§ 12, 41) if either (§12) two sides with the included angle, or (§41) one side and the two angles lying on it, are given. Therefore one given *angle* is always required. Therefore I form an angle (Fig. 22) in the $\triangle acb$, whose sides are given to me. Now since I can calculate the sides in right-angled triangles (§45), I will form a *right* angle and because it must be in a *triangle*, I will form it by dropping a perpendicular from a vertex c of the triangle on to the opposite side ab, thereby producing two right-angled $\triangle\triangle$ adc, bdc which (§45) give the equations $b^2 = x^2 + y^2$, $a^2 = x^2 + (\pm c \mp y)^2 = x^2 + (c - y)^2$ (according to whether d lies inside or outside ab). From these equations, x and y can be determined unambiguously. But these are two sides of $\triangle cda$ whose included angle $cda = \mathrm{R}$ is given. Hence (§12) the angle a is determined and thereby also the $\triangle abc$.

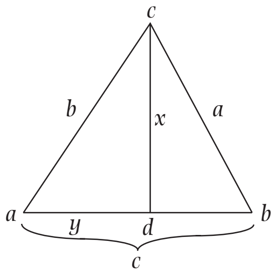

Fig. 22.

§ 47

Theorem. If in two triangles the three sides are equal then the triangles themselves are equal.

Proof. For their determining pieces are equal (§46).

§ 48

Theorem. If in two triangles the three sides are in proportion then the triangles themselves are similar.

Proof. For their determining pieces are similar (§§ 46, 17).

§ 49

Note. Why have I abandoned the usual proofs of the three propositions on the equality of triangles? I have not essentially altered the proof of the first theorem (§14) except for its presentation. As far as this *presentation* is concerned I have indeed omitted the concept of covering [*Decken*][m] which is usually used here and for several other theorems. I do not want to criticize here the inappropriateness of the choice of the German *word, Decken*, which easily misleads the beginner into thinking of a *lying-on-top-of-one-another* [*Übereinanderliegen*], and not of the identity of boundaries; instead of this it would be more appropriate to use 'coinciding' [*Ineinanderfallen*], or *congruence* [*Congruiren*] (συμπιπτειν).[n] But the concept of congruence is itself both empirical and superfluous. It is *empirical*: for if I say *A* is congruent to *B*, I imagine *A* as an *object* which I *distinguish* from the *space* which it occupies (likewise for *B*). It is *superfluous*: one uses the concept of covering to deduce the equality of two things if they are shown to cover each other in a certain position, according to the axiom 'spatial things which cover each other are *equal* to each other'. (In this way, one actually proves identity when only equality had to be shown.) Now one could never conclude that two things are congruent, i.e. that their boundaries are identical, until one had shown that all determining pieces are identical. But if this is proved one can also deduce without covering that these determining things themselves are identical. Therefore *Schultz* omitted the concept of covering throughout his *Anfangsgründe* without needing to alter much on this account. As for the proofs of the second and third theorems[o] these (even as more recent geometries have rearranged them) are completely based on theorems of the plane. Any expert will see this for himself. Therefore, I could not be satisfied with them because of the principles stated in the *Preface*. But do my own proofs indeed meet the requirement made in the *Preface*—to avoid *all fortuitous intermediate concepts*? I believe so. However, to avoid extensive detours in this small work I have not been able to give the detailed deduction of the necessity of every intermediate concept introduced. I therefore ask the learned reader to provide this by some thinking of his own.

§ 50

Theorem. If in the angle xcy (Fig. 23), $ca : cb = cd : ce$ then ab, de never intersect, however far they are produced.

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[m] An example of the 'usual' use of the concept of covering to which Bolzano refers is the following axiom: 'Figures which cover one another are equal to one another, and figures which are equal and similar cover one another' (Wolff, 1717, §50).

[n] On the word *Congruiren* see the footnote to I §11 on p. 39. The main meanings of the Greek term are 'to fall together', 'to coincide', 'to agree' (*LSJ*).

[o] That is, I §42 and I §47.

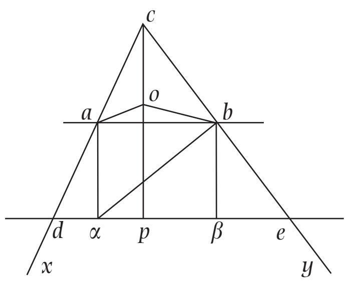

Fig. 23.

Proof. (§21) $\triangle acb \backsim \triangle dce$; therefore

$$ab = de.\frac{ac}{dc}.$$

Consider cp perpendicular to de and $cd : cp = ca : co$ (taken in cp from c); therefore (§21)

$$ao = dp.\frac{ca}{cd} \quad \text{and} \quad bo = ep.\frac{cb}{ce} = ep.\frac{ca}{cd}.$$

Hence

$$ao \pm ob = (dp \pm pe).\frac{ca}{cd} = de.\frac{ca}{cd},$$

therefore $ao \pm ob = ab$. From which (by §44 *in modo tollente*), it may be concluded that o is in the straight line ab. Also (§23) $\angle aoc = \angle dpc =$ R. Consequently, ab, de never intersect (§40).

§ 51

Theorem. Also (Fig. 23) the perpendiculars from a and b to de, $a\alpha = b\beta$, as also their distances $ab = \alpha\beta$. And the angles at a, b are also right angles.

Proof. I. By §43 $\triangle ad\alpha \backsim \triangle cdp$; $\triangle be\beta \backsim \triangle cep$. Consequently

$$a\alpha = cp.\frac{da}{dc} \quad \text{and} \quad b\beta = cp.\frac{eb}{ec} = cp.\frac{da}{dc}.$$

Therefore $a\alpha = b\beta$.

II. Furthermore

$$d\alpha = dp.\frac{da}{dc} \quad \text{and} \quad e\beta = ep.\frac{eb}{ec} = ep.\frac{da}{dc}.$$

Therefore

$$\alpha\beta = de - d\alpha - e\beta = de - (dp + pe).\frac{da}{dc}$$

$$= de.\left(1 - \frac{da}{dc}\right) = de.\frac{ac}{dc} = ab \quad \text{(as proved in §50).}$$

III. If $b\alpha$ is drawn, then (§47) $\triangle ba\alpha = \triangle\alpha\beta b$, therefore (§30) $\angle ba\alpha = \angle a\beta b = \mathrm{R}$. Similarly $\angle ab\beta = \mathrm{R}$.

§ 52

Theorem. If at four points, a, b, c, d (Fig. 24) the four angles $abc = bcd = cda = dab = \mathrm{R}$, then the line between any two of these four points equals the line between the other two: $ab = cd$, $ac = bd$, $ad = bc$.

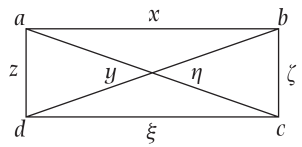

Fig. 24.

Proof. Let $ab = x$, $cd = \xi$, $ac = y$, $bd = \eta$, $ad = z$, $bc = \zeta$, then (§45)

$$y^2 = z^2 + \xi^2 = x^2 + \zeta^2$$
$$\eta^2 = z^2 + x^2 = \xi^2 + \zeta^2 .$$

Therefore $\xi^2 - x^2 = x^2 - \xi^2$, $\xi^2 = x^2$, or (in respect of length) $\xi = x$. Therefore also $y = \eta$, $z = \zeta$.

§ 53

Theorem. If (Fig. 25) $\angle a = \angle b = \angle\alpha = \angle\beta = \mathrm{R}$; then of the three conditions: I. $\angle m$ or $\angle\mu = \mathrm{R}$, II. $am = \alpha\mu$ and $bm = \beta\mu$, III. $m\mu = a\alpha$, each one has the others as consequences.

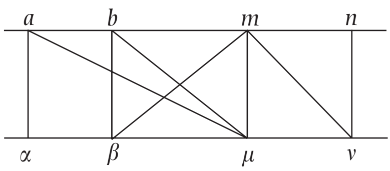

Fig. 25.

Proof. I. Let $m = \mathrm{R}$ and call $a\alpha = b\beta = a$, $ab = \alpha\beta = b$, $bm = x$, $\beta\mu = y$, $m\mu = z$. Now (§45), $a\mu^2 = a^2 + b^2 + 2by + y^2 = b^2 + 2bx + x^2 + z^2$ and $b\mu^2 = a^2 + y^2 = x^2 + z^2$, whence it follows that $2by = 2bx$, $y = x$ and $am = b + x = b + y = \alpha\mu$. Then from the right-angled $\triangle bm\mu$, $z^2 = a^2 + x^2 - x^2 = a^2$, therefore $z = a$.

Finally from (§47) $\triangle\beta bm = \triangle m\mu\beta$, therefore $\angle m\mu\beta = $ R. II. Let $am = \alpha\mu$ and $bm = \beta\mu$. Now if m were not a right angle there would still be a perpendicular from μ to ab, and for this (*ex dem.* I.) $am = \alpha\mu$, $bm = \beta\mu$; now there is only one point m in ab which has these two definite distances from a and b. Therefore m is a right angle. III. Let $m\mu = a\alpha$. Now if m were not a right angle there would still be a perpendicular from μ to ab whose length would be (*ex dem.* I.)$=a\alpha$. Now the hypotenuse μm cannot equal this side (§45); consequently this perpendicular is μm itself.

§ 54

Note. Therefore because all perpendiculars from points on one of the two lines ab, $\alpha\beta$ to the other are *equal*, they are called parallel lines. *Wolff* assumed this property for the definition of parallels without considering the obligation to prove the possibility of this property. This was a very unphilosophical error for that wise man.

§ 55

Corollary. The distances between any two perpendiculars $m\mu$, $n\nu$, are equal, $mn = \mu\nu$. This follows from the theory of the straight line, because (§53) it must also be that $am = \alpha\mu$, $bm = \beta\mu$, and these double distances from a, b determine the points m, n.

§ 56

Theorem. Every line $m\nu$, which intersects both parallels ab, $\alpha\beta$ (Fig. 25), forms equal adjacent angles with them.

Proof. Draw perpendiculars $m\mu$, νn from m, ν to the opposite parallel. Then (§47) $\triangle m\mu\nu = \triangle\nu nm$. Therefore (§10) the angles $nm\nu = \mu\nu m$.

§ 57

Corollary. The equal angles $m\nu\mu$, νmn are called *alternate angles*.[p] I would like to call the infinite parts mx, $\nu\eta$ (Fig. 25*) of the parallels which form the arms of the two associated alternate angles, *alternate arms*. It will be seen that the perpendicular from the initial point m of the one arm mx, or from any point r *outside this arm* meets the other alternate arm $\nu\eta$. For the perpendicular from m this is obvious from §56; for the perpendicular from r, I show this as follows. Because r is outside mx, and therefore (*ex demonstratione*) outside mn, then (by virtue of the properties of the straight line) nr is $> nm$, and $> mr$. Therefore also (by §55) $\nu\rho$ $(=nr)$ $>$ $\nu\mu$ $(=nm)$ and $>$ $\mu\rho$ $(=mr)$. Consequently (by the

[p] The term translated here as 'alternate angles' [*Wechselwinkel*] was also applied to the (unequal) angles arising from an intercept on non-parallel lines (see I §65).

theory of the straight line) μ, ρ lie in the identical direction from ν, or $\nu\mu$, $\nu\rho$ are identical arms.

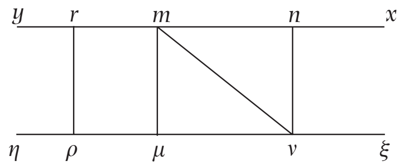

Fig. 25*.

§ 58

Theorem. The diagonals (Fig. 26) $a\beta$, $b\alpha$ in a rectangle intersect in their mid-point.

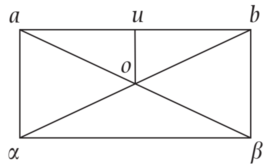

Fig. 26.

Proof. Take o, u the mid-points of $a\beta$, ab; then $\triangle oau \backsim \triangle \beta ab$ and $ou = \frac{1}{2}\beta b = \frac{1}{2}a\alpha$, and $\angle auo = \angle ab\beta = \mathrm{R} = \angle buo$. Therefore (§21) $\triangle buo \backsim \triangle ba\alpha$, therefore $bo = \frac{1}{2}b\alpha$. In the same way, $\alpha o = \frac{1}{2}\alpha b$. Consequently (§44 *in modo tollente*) o is in αb, and therefore $a\beta$ and $b\alpha$ bisect each other.

§ 59

Theorem. Through the same point o (Fig. 27), outside the straight line xy, there is only one straight line parallel to xy.[q]

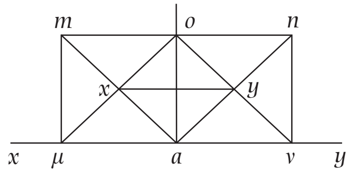

Fig. 27.

[q] On this apparent statement and proof of Euclid's parallel postulate, see the remarks on p. 19.

Proof. Let om, on be two parallels to xy. Take (for brevity) $om = on$ and drop perpendiculars to xy from m, n, then (§55) $om = a\mu$, $on = a\nu$. If am, $o\mu$ are drawn, these equal lines must intersect in their mid-point x (§58); similarly an, $o\nu$ in y. If the straight lines xy, mn are now drawn, then (§21) $\triangle \mu o \nu \backsim \triangle xoy$, from which $yx = \frac{1}{2}\nu\mu$. Furthermore (§47) $\triangle xoy = \triangle xay$. Also $\triangle man \backsim \triangle xay$, from which it follows that $mn = 2xy = \nu\mu$. Now since ν, a, μ are in the same straight line [*Gerade*] and $a\mu = a\nu$ then *either* ν, μ are the identical point and therefore also m, n are the identical point; hence om, on are the same lines, *or* ν, μ lie on opposite sides of a and therefore also n, m lie on opposite sides of o; consequently om, on are again in the same straight line [*geraden Linie*].[r]

§ 60

Theorem. If in each of the parallels ab, $\alpha\beta$ (Fig. 28) two points are given at equal distances $mn = \mu\nu$, then the lines $m\mu$, $n\nu$ which join these points in a particular way, are parallel and equal.

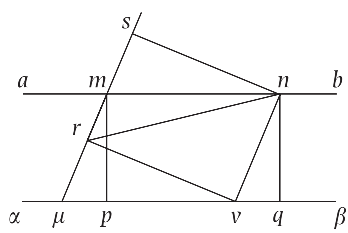

Fig. 28.

Proof. If the perpendiculars mp, nq are drawn, then (§55) $pq = mn$. But $mn = \mu\nu$, so $mn = pq$. Now it follows from the theory of the straight line that there is another combination of the four points μ, ν, p, q which occur in the straight line $\alpha\beta$ for which two distances are equal to each other. If this is $\mu p = \nu q$ then m, μ; n, ν are the points that must be joined to obtain the parallels. Now because also $mp = nq$ (§53), by (§14) $\triangle mp\mu = \triangle nq\nu$. Consequently $m\mu = n\nu$. If one draws perpendiculars to μm from n, ν, then it follows easily (from §21) that because one of the adjacent angles nms, $nm\mu$, must be $= \nu\mu r$ (§56), this is the angle nms in whose arm the perpendicular ns falls. Therefore (§43[s]) $\triangle nms \backsim \triangle \nu\mu r$. And since $nm = \nu\mu$, then $ns = \nu r$, $ms = \mu r$, consequently $\mu m = rs$, and since $\mu m = \nu n$, $rs = \nu n$. From this (§47) $\triangle rsn = \triangle n\nu r$, therefore $\angle n\nu r = \angle s = \mathrm{R}$. Hence $m\mu$, $n\nu$ are parallels (§53).

[r] See the footnote on this on p. 64.

[s] Both German editions have §44 by mistake.

§ 61

Theorem. If (Fig. 28*) *ab* is parallel to[t] *cd*, *ac* is parallel to *bd*, then also $ab = cd$ and $ac = bd$.

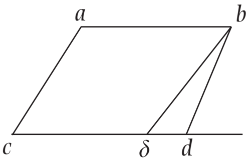

Fig. 28*.

Proof. If *ab* were not $= cd$ and one takes $c\delta = ab$ in the unbounded part of *cd* which is not an alternate arm of *ab*, then $b\delta$ would be parallel to *ac* (§60). Therefore (§59) *d* is identical with δ. (Because *bd* can have only one point in common with *cd*.)

§ 62

Theorem. If (Fig. 28*) *ab* is parallel to *cd* and $\angle acx = \angle bdx$ then also *ac* is parallel to *bd*.

Proof. For if we suppose $b\delta$ (drawn from *b*) is parallel to *ac*, then $\angle acx = \angle b\delta x$ (§60). Therefore $\angle bdx = \angle b\delta x$; hence (§39) *d* is identical with δ.

§ 63

Theorem. If the lines (Fig. 29) *ab*, *de* intersect the arms of the angle *xcy* in such a way that either $ca : cb$ not $= cd : ce$, or the angle *cab* not $=$ *cde* (where *a*, *d* are taken in one arm, *b*, *e* in the other) then *ab*, *de* intersect somewhere.

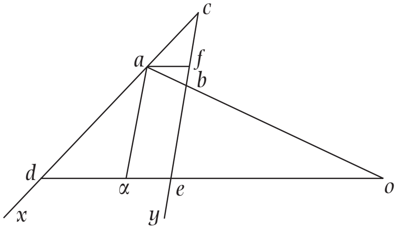

Fig. 29.

[t] Bolzano often abbreviated words, even in publications, so the original *par.* here, and in numerous other places, is better regarded as an abbreviation for German *parallel zu* than a piece of notation. Hence it has been translated.

Proof. Of these two conditions each one has the other as consequence as is clear from §§ 23, 24 *in modo tollente*. Now let

$$\frac{ca}{cb} < \frac{cd}{ce}$$

and $ca < cd$ (without loss of generality). Consider $cd : ce = ca : cf$ (the latter taken in ce from c). Therefore $cf < ce$, as well as $< cb$. Consequently f is in cb as well as in ce; hence $\angle afe = \angle afb$. Now (§23) $\angle cfa = \angle ced$, therefore also (§4) $\angle afe (= \angle afb) = \angle beo$. Consider $fb : fa = eb : eo$ (the latter taken from e in the arm of the last-named angle beo which $= \angle afb$). Therefore (§21) $\triangle afb \backsim \triangle oeb$. And

$$bo = ab.\frac{eb}{fb}.$$

Consider $dc : de = da : d\alpha$ (the latter drawn in de) then *subtrahendo* $dc : de = ac : \alpha e$. But $dc : de = ac : af$ therefore $\alpha e = af$. Similarly $fe = a\alpha$. Since $\triangle ad\alpha \backsim \triangle caf$ (§23), $\angle a\alpha d = \angle cfa$; therefore (§4) $\angle a\alpha o = \angle afb$. Furthermore

$$\alpha a : \alpha o = ef : \alpha e \pm eo = ef : af \pm af.\frac{eb}{fb}$$

$$= ef : af.\left(\frac{fb \pm eb}{fb}\right) = ef : \frac{af.fe}{fb} = fb : af.$$

Therefore (§21) $\triangle a\alpha o \backsim \triangle bfa$. Hence

$$ao = ab.\frac{a\alpha}{fb} = ab.\frac{ef}{fb}.$$

Therefore (in Fig. 29)

$$ab + bo = ab + \frac{ab.eb}{fb} = ab.\frac{ef}{fb} = ao.$$

Hence o is in the straight line ab (§44). Or (in Fig. 29*)

$$ao + ob = ab.\frac{ef}{fb} + \frac{ab.eb}{fb} = ab.\left(\frac{be + ef}{fb}\right) = ab.$$

Consequently o is again in the straight line ab.

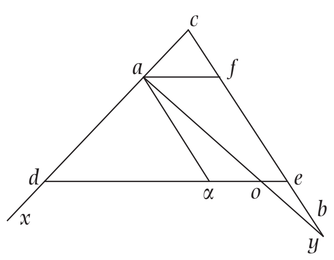

Fig. 29*.

§ 64

Theorem. If (Fig. 30) *ab*, *cd* are parallel, and the pieces *ab*, *cd* are unequal or the angles *cax*, *dbx* are unequal, then *ac*, *bd* intersect.[u]

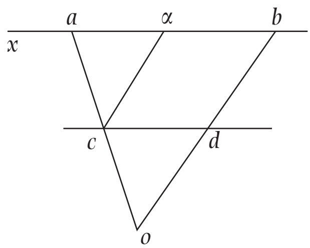

Fig. 30.

Proof. The first condition has the other as its consequence. For if $b\alpha$ is taken $= dc$ in the arm *bx* which is not an alternate arm to *dc*, then $c\alpha$, *db* are parallels (§60), and $\angle dbx = \angle c\alpha x$. Now $c\alpha x$, *cax* cannot be equal angles (§39). Therefore $\angle cax$ not $= \angle dbx$. Now suppose $a\alpha : ac = ab : ao$ (the latter drawn in that part of *ac* which forms an angle with *ab* equal to αac). Draw *bo*, *do*. Now $\angle dco$ is always $= \angle \alpha ac$. For if, for example, in one case the direction $a\alpha$ is identical to that of *ab*, then (*per constructionem*) *ac* also has the identical direction to that of *ao*. Now since (*ex hypothesi*) *bx*, *dc* are not alternate arms, neither are *ab*, *cd*; hence, since these parallels are cut by *ac*, $\angle bao (= \angle \alpha ac) = \angle dco$. Similarly in the other case. Furthermore, $a\alpha : ac = ab : ao = \alpha b : co$ (adding or subtracting) $= cd : co$. Therefore (§21) $\triangle dco \backsim \triangle \alpha ac$. Whence

$$do = c\alpha . \frac{cd}{a\alpha} = bd . \frac{\alpha b}{a\alpha}.$$

But from $\triangle bao \backsim \triangle \alpha ac$, it follows that

$$bo = c\alpha . \frac{ab}{a\alpha} = bd . \frac{ab}{a\alpha}.$$

From these equations, by comparison with *bd* it follows that *bd*, *bo* are identical straight lines.[v]

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[u] Other cases of what is described in the theorem are illustrated in Fig. 30* and Fig. 30** on the next page.

[v] They are identical when viewed as unbounded straight lines. Bolzano uses the word *Gerade* here for straight line. The first such use is in I §59. Previously he has usually used *gerade Linie* when referring to a bounded line segment, unless otherwise qualified. For example, in I §30, 'a straight line [*gerade Linie*] produced indefinitely in both directions'. However, the distinction of usage between *Gerade* and *gerade Linie* is not preserved consistently.

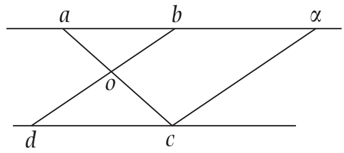

Fig. 30*.

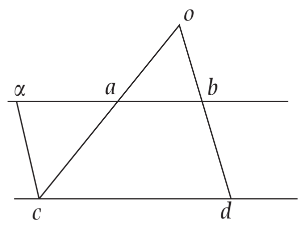

Fig. 30.**

§ 65

Theorem. If the two lines (Fig. 31) *ab*, *de* cut two other lines *ad*, *be* which are either parallel or intersecting, then the first are either parallel or intersecting depending on whether the fifth line *mn* by which they are cut forms equal or unequal alternate angles.

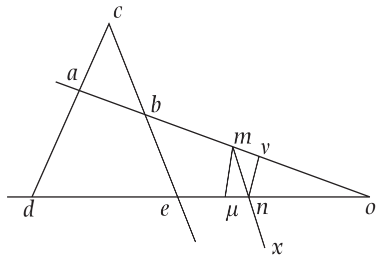

Fig. 31.

Proof. I. If these angles are *equal*, then *ab*, *de* cannot intersect. For if this happened in *o* and the perpendicular from *n* to *mo* fell inside *om*, then the perpendicular from *m* to *no* would fall outside *on*, because the angles $nm\nu = mn\mu$ are supposed to be *alternate angles* (§57). Therefore $\angle omx = \angle mn\mu =$ (§5) $\angle onx$ (if *nx* is an extension of *mn* beyond *n*), contradicting (§39). But if $\angle bac$, $\angle edc$ were unequal, then *ab*, *de* would have to intersect (§§ 63, 64). Therefore these angles are equal; therefore (§§ 50–54) *ab*, *de* are parallel. II. If the alternate angles are *unequal* then $\angle bac$, $\angle edc$

must also be unequal because otherwise these alternate angles would be equal (§§ 50, 57, 62). But if $\angle bac$, $\angle edc$ are unequal then *ab*, *de* intersect (§§ 63, 64).

§ 66

Theorem. If two lines (Fig. 32) *ab*, *cd* which are either parallel or intersecting have a third line *ac* cutting both, and the fourth line *bo*, which is either parallel to the third or intersects it, cuts the first *ab*, then it also cuts the other *cd* or is parallel to it.

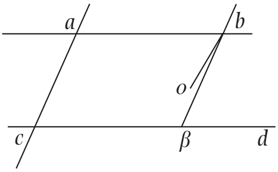

Fig. 32.

Proof. I. If *ab* is parallel to *cd* and *bo* is parallel to *ac* then *bo*, which cuts *ab*, must necessarily also cut *cd*. For if we take $c\beta = ab$, then $b\beta$ is parallel to *ac*, therefore *bo* is the same as $b\beta$ (§59). II. If *ab*, *cd* (Fig. 32*) intersect in *x*, but *ac* is parallel to *bo*, then take $xa : xb = xc : x\beta$; consequently (§§ 50, 54) $b\beta$ is parallel to *ac*. Therefore (§59) $b\beta$ is the same as *bo*. III. If *ab*, *cd* intersect as well as *ac*, *bo*, then *bo*, *cd* do not necessarily have to cut each other.[w] But one of the following two cases must hold: either *bo* is parallel to *cd* or they intersect. For here it is the case that in the angle *bac* two lines *cy*, *bx* cut both arms, and §62 can easily be applied.

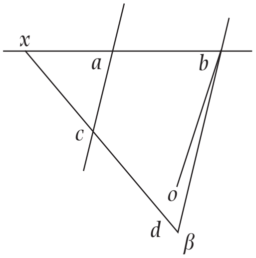

Fig. 32*.

§ 67

Note. These are perhaps the most important propositions of the theory of parallels expressed here without the concept of the plane (which is equivalent to the

[w] Case III in fact refers to Fig. 32**.

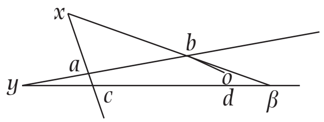

Fig. 32.**

condition: *that two lines either intersect or are parallel*), and from which, several other propositions, particularly trigonometric ones, can now be derived in the usual way. It will have been noticed here and there that I have assumed certain propositions from the theory of the straight line which are not explicitly mentioned in the usual textbooks of geometry. Yet they are indispensable to making geometrical language precise, and I would have wished to have been able to make stricter use of them, but I could not do so from fear that it would be interpreted as being over-pedantic. It is generally acknowledged that the efforts of geometers in the theory of parallels, up to the most recent ones of *Schultz*, *Gensich*, *Bendavid*, *Langsdorf*, have still all been inadequate. Now other people have already made the objection to the proof of *Schultz* (and the proof by the Frenchman *Bertrand* is essentially the same as that of *Schultz*) that it is based on a consideration of a different kind,[x] namely on the *infinite surface of the angle*, as well as on axioms of infinity, which have not yet gained general approval. *Gensich* aims, very ingeniously, to remove only the difficulties of infinity, but nothing is altered concerning the first matter. The fact that I cannot therefore be satisfied with this proof of the theory of parallels arises from the principles already stated (*Preface* and §6). *Bendavid's* proof is so hasty (somewhat unexpected for the author of *Die Auseinandersetzung des mathem. Unendlichen*) that it is rendered completely invalid. The proof offered by *Langsdorf* (*Anfangsgründe der reinen Elementar- und höheren Mathematik*, Erlangen, 1802) cannot satisfy me, nor all those who have not yet been convinced of the possibility and necessity of his *spatial points* (which are supposed to be *simple elements* in space, and from their accumulation together in finite numbers lines, surfaces and solids are formed). However, even if this truth-seeking scholar has not changed his conviction about spatial points, this does not necessarily prevent him, since he also accepts geometrical points and lines, from giving some approval to the methods of proof (if nothing else) in my present writing. Other *more recent* attempts concerning parallels are not known to me. Since I have now ventured on a goal which has so often proved unsuccessful in the past, it would be immodest if I claimed the discovery for myself after ἑυρηκα has so often been uttered too hastily. I prefer to leave it to the *judgement* of the reader and of the future.

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[x] Here, and in the last sentence of II §18, Bolzano uses the word *heterogen* to refer to the idea of one concept being of a different kind to another. More often he uses *fremdartig* (alien) for this purpose (e.g. in *Preface* on p. 32).

II

Thoughts Concerning a Prospective Theory of the Straight Line

§ 1

First of all let me say something about the concepts of identity [*Einerleiheit*] and equality [*Gleichheit*]. In a geometrical investigation it is not my job to look for perfectly correct definitions of these two words. But I must state the *specific meaning* which I attach to them (because otherwise it would be ambiguous). Therefore, I understand by *identity* (*identitas*) the concept which arises from the comparison of a thing (solely) *with itself*. Identity I put as contradictory to *difference*. Difference I divide again into the two contradictory *species*: *equality* and *inequality*. Consequently equality presupposes difference and it is correct to say, 'everything is identical with itself' (but not *equal* to itself), 'two *different* things are either *equal* or *unequal*' (never *identical*). Yet if one says, 'The thing A is identical with the thing B' this really means: A and B have been assumed, hypothetically, to be two different things and it has been proved that in fact they are not different but an *identical* thing. *Properties* of things can be called identical or different. But in so far as they are hypostasized and considered as *things* themselves, they are *eo ipso* different and can now be called equal or unequal.

§ 2

These theorems can now be proposed: things whose determining pieces are identical are themselves an identical thing, and conversely (*conversio simplex*). Things whose determining pieces are equal are themselves equal things, and conversely. I also take this opportunity to submit for criticism the following two propositions which I have often been inclined to assume in mathematical proofs. If, among the determining pieces of two things one is different (but the rest are identical), then there must also be a difference in the pieces determined. If, among the determining pieces of two things one is unequal (but the rest are equal), then there must also be an inequality in the pieces determined.

§ 3

Before the geometer applies the concept of equality to spatial things he should first demonstrate the *possibility of equal spatial things*. Perhaps the axiom proposed in I §19 may be used here, which may be expressed generally: we do not have an *a priori* idea of any determinate spatial thing (not even of a *point*). Therefore *several* completely *equal* spatial things must be possible for which all equal predicates hold. Therefore if some spatial thing A is possible at a point a, then also an equal spatial thing $B = A$, must be possible at the different point b.

§ 4

A genuine definition must contain only those characteristics of the concept to be defined which constitute its *essence* [*Wesen*], and without which we could not even conceive of it. Therefore we should regard as very artificial the definitions of the scholastic *Occam*, for solid, surface, line and point, according to which, solid is that kind of extension which cannot be the boundary of anything else, but surface is the boundary of a solid etc., because in order to conceive of only a point or a line they require in each case the idea of a *solid*. (*Langsdorf* also remarks on this, see the *Preface* of his *Anfangsgründe der Mathematik*.) However, it is obvious that we can perfectly well conceive of a surface, a line or a point, and that we do so without a solid which they bound. In my opinion, it would not be so very objectionable if someone were to turn this round and put forward definitions which required the idea of points for lines, and of lines for surfaces, etc.

§ 5

This much is, I hope, unobjectionable: that the *concept of point*—as a mere *characteristic of space* (*σημειον*)[y] *that is itself no part of space—cannot be dispensed with* in geometry. This point is indeed a merely imaginary object as I gladly concede to *Langsdorf*. Lines and surfaces are also like this, and indeed all three are so, in yet a different sense from the geometrical *solid*. Something adequate can be given for the latter in the intuition, but not for the former, (indeed everything that is given in the intuition is solid). And perhaps for this very reason any attempted pure *intuition* of *lines* and *surfaces* (say by the motion of a point) must be impossible. The definitions attempted in this paper of the straight line §26, and the plane §43, are made on the assumption that both are simply *objects of thought* [*Gedankendinge*].

§ 6

Since a point, considered in itself alone, offers nothing distinguishable, as we have no determinate *a priori* idea of it, it follows that the simplest object of geometrical consideration is a *system of two points*. From such a *simultaneous conception of two points* there arise certain predicates for these (concepts) which were not present with the consideration of a single point. Everything which can be perceived in the relation of these two points to one another, and indeed in the relation of *b* to *a* (Fig. 1), I divide into two component concepts [*Theilbegriffe*]: I. That which belongs to point *b* in relation to *a* in such a way that it is *independent* of the *specific point a* (*qua praecise hoc est et non aliud*),[z] and which can consequently be present *equally* in relation to *another* point, e.g. α, is called the *distance of point b from a*. II. That which belongs to point *b* in relation to *a* in such a way that it is *dependent simply on*

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[y] The main meanings given in *LSJ* are 'mark' (by which something is known), or 'sign', but also 'mathematical point', 'instant of time'.

[z] *Translation:* which is precisely this one and not another one.

the specific point a, where we have now separated off what is already present in the concept of distance, i.e. what can belong to point *b* also with respect to another point. This is called the *direction in which b lies from a*.[a]

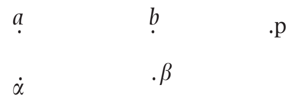

Fig. 1.

§ 7

Now to show the *possibility* of both concepts. I. *Distance*. The mere concept of *being different* [*Verschiedenseyns*] of the point *b* from the point *a* (of their being separate) is no part of the concept of the *relation of the point b to a* (of the *totius dividendi*[b] §6), but is necessarily presupposed by it. If *b* is to be related to *a*, then the idea that *b* is different from *a* must already be assumed. Therefore in order to demonstrate the reality [*Realität*] of the concept of distance as a component of the whole concept mentioned above, one must prove that it contains more than the mere fact that *b* is different from *a*. This I do as follows. If the concept of distance contained nothing else, then the other concept of direction would have to encompass completely the *totum divisum*, i.e. the whole concept of the relation of *b* to *a* would have to contain nothing other than what is dependent simply on the specific point *a*. In other words, the system *ab* could be completely determined by *that* which belongs to point *b* and is purely dependent on *a*, and so can belong to no *other* system. Therefore we would have a special *a priori* idea of *a*, which we would have of no other point, and this is contrary to our axiom. II. The concept of *direction* cannot be completely empty, because otherwise the concept of distance would again have to exhaust completely the concept being analysed. But *ex definitione* the concept of distance contains only what belongs to *b* independently of the *special* [*besonderer*] point *a*, so that it can also belong to the system *bα*. But the whole [concept] being analysed contains only as much as belongs to the system *ab* alone.

§ 8

Therefore since both concepts of distance and direction have a content, each one contains less than the whole concept being analysed. Hence neither distance alone, nor direction alone, determines the point *b*. In other words, there are several

[a] The context and use of the German *die Richtung in welcher b zu a liegt*, here and in subsequent paragraphs, shows it should be understood literally as 'the direction in which *b* lies at *a*', that is, the direction in which *b* lies for an observer at *a*.

[b] *Translation:* the whole [concept] being analysed.

points *b*, *β* which are at an *equal distance* from *a*, and similarly several points *b*, *p* which lie in the *same* [*einerlei*] *direction* from *a*.

§ 9

The assumption of the point *a*, and the distance and direction of the point *b* determine the latter *ex definitione*. And conversely the point *b* determines the distance from *a* and the direction from *a*. Therefore two different directions from the same point *a* can have no single point *in common*, i.e. there is no point to which they both belong.

§ 10

Theorem. For a given point *a* (Fig. 2) and in a given direction *aR* there is one and only one point *m*, whose distance from *a* equals the given distance of the point *y* from *x*.

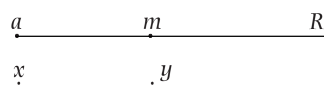

Fig. 2.

Proof. It follows that there is a point at the *given distance* from *a*, because otherwise there would be a distinction between the unrelated points *a*, *x*, which is not permitted. It follows that there is also such a point in the *given direction* from *a*, because otherwise we would have to have a special idea of a specific direction *aR*. Finally, it follows that there is only *one*, from §9.

§ 11

Up to now I have looked in vain for a satisfactory proof of the theorem that the distance of *b* from *a* is equal to the distance of *a* from *b*. Meanwhile the following reasoning may be proposed, although it is unsatisfactory to me, in order perhaps to inspire something better. If two things *A* and *B* are *equal*, then it must be *possible* to combine them in such a way that the relation of *A* to *B* is *equal* to the relation of *B* to *A*. Since the reason for its not being possible would have to lie in the things being unequal. Now since all points are *equal* things, it must be possible for two points to be combined in such a way that the relation of *b* to *a* equals the relation of *a* to *b*. But if such a combination is simply possible then it is also *actual*, for the combination of two points at a definite distance is a single thing. Now if the relation of *b* to *a* is to be equal that of *a* to *b*, then the *distance* of *b* from *a* must be equal to that of *a* from *b*, for the *directions* cannot be compared.

§ 12

The system of two directions proceeding from one point is called an *angle*. I have already indicated in I §2 why the concept of angle really belongs to *directions* and not to lines.

§ 13

However, I also have another definition of angle to offer which is completely analogous to the development of the concepts of direction and distance. Consider two directions R, S from the same point a (Fig. 3), and divide the whole concept of the relation of the direction S to R into the following two component concepts: I. that which belongs to the direction S independently of the specific direction R (and only this direction)—called the *angle which S makes with R;* II. that which belongs to the direction S only with respect to the direction R and to no other, where we have now separated from it what can belong to it equally with respect to another direction—called the *plane* in which S lies with R. (In this sense of the word '*plane*' it would include only that *half* of the *usual* plane through R and S which lies on *that* side of R in which S is.) But for the present I shall keep to the definition §12.

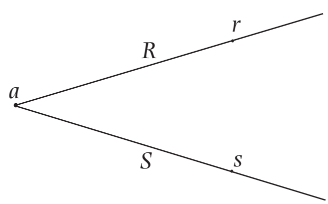

Fig. 3.

§ 14

In each case it needs to be shown that the idea which arises if S is related to R is equal to *that* which arises if R is related to S, i.e. the *angle* $sar = ras$. (Similar to the proposition of §11.) Here also I still have no satisfactory proof.

§ 15

If the angle between the directions R, S (Fig. 4) is such that it determines the direction S by the direction R, i.e. for the identical direction R there can be no direction different from S which forms an equal system with R, then the angle between S and R is called an *angle of two right-angles* (or as *Schultz* calls it) a *straight angle*.[c] The direction S is called *opposite* to the direction R. Therefore according

[c] This is Definition 19 in Schultz (1790), p. 270.

to §14, the direction *R* is also opposite to that of *S*. It also follows *ex definitione* that if the directions *R*, *S* (Fig. 3) are not opposite to one another then there are always directions different from *S*, which form an equal angle with *R*.

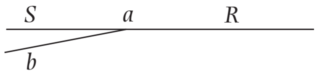

Fig. 4.

§ 16

However, it is something different, which does not follow *ex definitione*, that for any direction *R* there is only a *single* direction *S* opposite to it; for there could perhaps be *different* angles which belong to each *single* direction *S*.

§ 17

Theorem. If in the directions (Fig. 5) aC, $a\Gamma$ the points c, γ are taken at equal distances from a, and in the directions ca, γa the points m, μ are again taken at equal distances from c, γ, then the angles $ca\mu = \gamma am$.

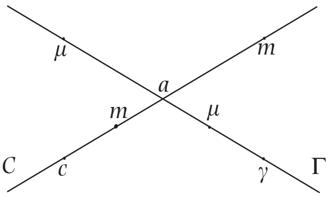

Fig. 5.

Proof. For since the angles $Ca\Gamma = \Gamma aC$, it can easily be shown that the determining pieces of the angle $ca\mu$ are *equal* to the determining pieces of the angle γam.[d]

§ 18

If the directions *ab*, *ac* from *a* (Fig. 6), and the distances *ab*, *ac* of the points *b*, *c* in these directions are given, then the points *b*, *c* are themselves also determined (§8); consequently the system of three points *a*, *b*, *c* is given which is called a *triangle*. The distance *bc* and the angles at *b*, *c* are also determined. Therefore one has a triangle here which consists only of three points and the three angles of the directions of every two points to the third—not of three lines. It may also be

[d] See I §5.

seen how several theorems about triangles, which in Part I are mixed up with the concept of the line, can already be presented here (with small changes). But I retained the concept there, although fundamentally heterogeneous, so that such abstractions should not become a burden.

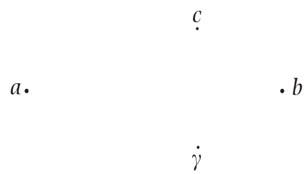

Fig. 6.

§ 19

Theorem. If the direction *ac* of the point *c* from *a* is not determined by its angle with the direction *ab* of the point *b* from *a*, then the direction *bc* is also not determined by the angle which it makes with *ba*.

Proof. By assumption there is at least one other direction different from *ac* which forms an equal angle with *ab*. Now if one takes the point γ in it at the distance $a\gamma = ac$, then the systems γab, *cab* (triangles) are *equal* because their determining pieces are equal. Consequently the angles $abc = ab\gamma$ and the distances $bc = b\gamma$. Hence the direction *bc* is not identical to that of $b\gamma$, for otherwise (§8) c, γ would be an identical point. Therefore there are different directions *bc*, $b\gamma$ which form an equal angle with *ba*.

§ 20

Corollary. The same holds of the angle *acb*. And this is actually the theorem that in every triangle there are three angles (I §9).

§ 21

But if the directions *ac*, *ab* (Fig. 7) are either identical [*einerlei*] or opposite, then the directions at *b*, *c* must also be identical or opposite. For if in one case neither of the two held, then neither of the two could also hold at *a* (§§ 19, 20).

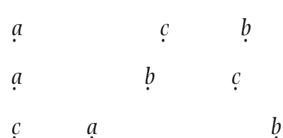

Fig. 7.

§ 22

It is also easy to prove the general proposition: If in a system of any number of points the rule holds that every single point, with another second point, lies in the identical direction, or the opposite direction, from a certain third point, then exactly the same holds of every two points from every third point.

§ 23

Theorem. If the two directions (Fig. 8) *oa*, *ob* are neither identical nor opposite, then the two directions *ao*, *bo* have only the single point *o* in common.

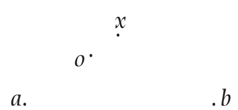

Fig. 8.

Proof. Assuming they had another point *x* in common, then it would follow (§21) that *oa*, *ob* were identical or opposite to *ox* and were consequently also identical or opposite to each other, *contra hypothesim*.

§ 24

As yet I am still not in a position to demonstrate the *possibility* of the concept of opposite direction. In general what I have to put forward as still unproved, as well as what has gone before (§§ 11, 14, 21), can be summed up in the following proposition. 'In a system *of three points* consider the relationship of the directions in which every two lie from the third: if these directions are *identical or opposite at one point* then they are *identical* at *two* points and *opposite at one* point.' A theory of the straight line can be based on these assumptions which must all be proved without the concept of the straight line so that they can be accepted for my purpose *absque petitio principii*. The chief propositions of this theory are as follows.

§ 25

Definition. The point *m* (Fig. 9) may be called (for the sake of brevity) *within* or *between a and b* if the directions *ma*, *mb* are opposite.[e]

---

[e] The need for an analysis of the concept of 'betweenness' was to be pointed out by Gauss in a letter to Bolyai in 1832, cited in Kline, 1972, p. 1006. Such an analysis in terms of axiomatic systems was carried out much later by, for example, Pasch, Hilbert, and Huntington.

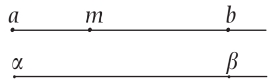

Fig. 9.

§ 26

Definition. An object which contains all and only those points which lie between the two points *a* and *b* is called a *straight line between a and b*.

§ 27

Note. The *possibility* of this object follows from what is assumed in §24. From the following it also appears that this object contains an infinite number of points, therefore it must be something qualitatively different from a mere *system of points*.

§ 28

Theorem. Two given points determine the straight line which lies between them.

Proof. For the straight line between *a* and *b* should contain all points which lie *between a* and *b* and no others. Therefore there is only a *single* thing which is called the straight line between *a* and *b*.

§ 29

Theorem. If the distances $ab = \alpha\beta$ (Fig. 9) then also the straight lines $ab = \alpha\beta$.

Proof. For their determining pieces (§28) are equal.

§ 30

Theorem. For every two given points (Fig. 10) *a*, *b* there is one and only one *mid-point* [*Mittelpunkt*], i.e. a point that is determined from both of them in the same way.

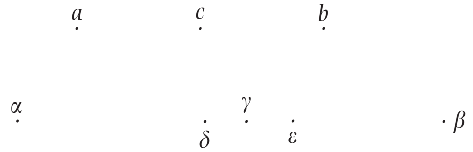

Fig. 10.

Proof. In the opposite directions $\gamma\alpha$, $\gamma\beta$ take the points α, β at arbitrary equal distances $\gamma\alpha = \gamma\beta$; then γ is determined from α, as it is from β. Now, if possible, let δ be another point that is determined from α as it is determined from β. Consequently it must be the case that the distances $\delta\alpha = \delta\beta$. Hence the directions $\delta\alpha$, $\delta\beta$ cannot be identical or α, β would be the identical point. Now if they were different but not opposite, then the directions $\alpha\beta$, $\alpha\delta$ would also be different and not opposite (§20); therefore the direction $\alpha\delta$ would not be determined by $\alpha\beta$, and also the point δ would consequently not be determined by α, β. Accordingly $\delta\alpha$, $\delta\beta$ must be opposite, and so (§24) $\alpha\beta$, $\alpha\delta$ are identical. But also the directions $\alpha\gamma$, $\alpha\beta$ are identical, therefore directions $\alpha\gamma$, $\alpha\delta$ are identical. Consequently (§24) the directions are opposite either at γ or at δ. I assume the first. (The deduction is similar in the other case.) Consider ε in the direction $\gamma\alpha$, which is opposite to that of $\gamma\delta$ or $\gamma\beta$, at the distance $\gamma\varepsilon = \gamma\delta$, then it follows because $\gamma\alpha = \gamma\beta$, that also the distances $\alpha\varepsilon = \beta\delta$, since they are determined in the same way. So as $\beta\delta$, $\beta\alpha$ are identical directions, then also $\alpha\varepsilon$, $\alpha\beta$ must be identical directions. Therefore (*per demonstrationem*) $\alpha\varepsilon$, $\alpha\delta$ are identical directions and also the distances $\alpha\varepsilon = \beta\delta = \alpha\delta$, therefore ε, δ are the identical point, which is contradictory. Thus the point γ is the only one which is determined in the same way from α, β. Now if α and β have a mid-point, then every other two points *a*, *b* must also have one (I §19).

§ 31

Theorem. If the point *c* (Fig. 11) lies *within* the points *a*, *b* then the straight lines between *a*, *c* and between *b*, *c* are *parts* of which the *whole* is the straight line between *a*, *b*.

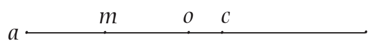

Fig. 11.

Proof. We need to prove that every point of the straight lines *ac* or *bc* is at the same time a point of *ab*, and that every point of *ab* is a point of either *ac* or *bc*. I. Let *m* be a point of *ac*, therefore (§26) the directions *ma*, *mc* are opposite, but those of *cm*, *ca* are identical (§24). And since *ex hypothesi ca*, *cb* are opposite, then so also are *cm*, *cb*. Hence (§24) *bc*, *bm* are identical. But likewise *bc*, *ba* are identical, therefore *bm*, *ba* are identical. Likewise it follows that *am*, *ab* are identical, therefore (§24) *ma*, *mb* are opposite and consequently *m* is a point in the straight line *ab*. II. Let *o* be a point in the straight line *ab*, consequently the directions *ao*, *ab* and *bo*, *ba* are identical. But *ex hypothesi ca*, *cb* are opposite, so *ac*, *ab* are identical. Hence *ac*, *ao* are identical. Consequently (§24) the directions of *ca*, *co* are either identical or opposite. If the former, then *o* therefore lies in the straight line *ac*. If the latter, then it is easy to see that *o* lies in the straight line *bc*.

§ 32

Theorem. If the points *m*, *n* (Fig. 12) both lie within *a*, *b* then the straight line *mn* is a *part* of the straight line *ab*.

Fig. 12.

Proof. In a similar way.

§ 33

Theorem. If the distances (Fig. 13) $ab = bc = cd =$ etc. and the directions *ba*, *bc*; and *cb*, *cd*; etc. are *opposite*, then the straight line between *a* and *d* can be considered as a *magnitude* which, if $n + 1$ is the number of points from *a* to *d*, represents the *number n* if its *unit* is the straight line *ab*.

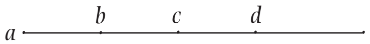

Fig. 13.

Proof. From (§31) it follows that the straight line *ac* can be viewed as a whole of which the component parts are *ab*, *bc*, and the straight line *ad* can again be viewed as a whole of which the parts are *ac*, *cd*, consequently also as a whole of which the component parts are *ab*, *bc*, *cd*, etc. But these parts *ab*, *bc*, *cd*, etc. are equal to each other because the distances $ab = bc = cd =$ etc. (§29). But it is easy to show that they are one fewer in number than the number of points. Consequently etc.

§ 34

Theorem. Every straight line *ab* (Fig. 14) can be divided into a given number of equal parts which together give the whole *ab* again.

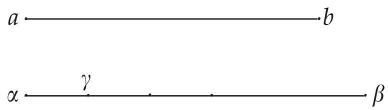

Fig. 14.

Proof. For with the assumption of some straight line $\alpha\gamma$, an $\alpha\beta$ can be conceived which consists of n parts $= \alpha\gamma$ (§33). Hence *ab* must also be capable of being divided in such a way (I §19).

§ 35

Theorem. The *unit* and the number (therefore the magnitude) determine the straight line to which they belong.

Proof. If two unequal lines were possible to which the same magnitude belonged, then we may conceive of them being from the same point *a* (Fig. 15) and in the same direction. Their two endpoints b, β must be different (§20). Consequently the directions *ba*, $b\beta$ are either identical or opposite. If, for example, the latter is the case, then (§31) the straight line $a\beta$ is a whole whose component parts are *ab*, $b\beta$. Therefore *ab* alone is not a component part of $a\beta$, consequently $a\beta$ does not have a magnitude which would be equal to that of *ab*.

Fig. 15.

§ 36

Corollary. All straight lines which have equal magnitude are therefore equal to one another.

§ 37

Theorem. If the directions *ca*, *cb* (Fig. 11) are opposite and the magnitudes of the lines *ac*, *cb* are represented, using a common unit, by the numbers m, n, then the magnitude of the straight line *ab*, using the same unit, is represented by the number $m + n$.

Proof. For according to §31 the line *ab* is a whole of which the integral [*integrierende*] parts are *ac*, *cb*; etc.

§ 38

Theorem. If the directions *ab*, *ac* (Fig. 11) are identical and the magnitudes of the lines *ab*, *ac* (Fig. 10) are represented using a common unit by the numbers $m + n$, m, then the magnitude of the straight line *bc*, using the same unit, is represented by the number n.

Proof. For *ac*, *cb* are the integral parts of the straight line *ab*, therefore $ac + cb = ab$, or in numbers, $m + cb = m + n$, hence $bc = n$.

§ 39

Theorem. For every three distances (Fig. 16) *ab*, *cd* and $\alpha\beta$ there is a fourth $\gamma\delta$ with the property that all predicates which arise from the comparison of the two distances *ab*, *cd* are equal to the predicates provided by the comparison of the two distances $\alpha\beta$, $\gamma\delta$.

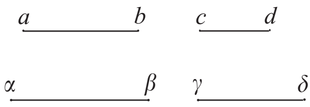

Fig. 16.

Proof. Otherwise we would have to have a special idea *a priori* of the *specific* distance *ab* according to which something would be true of it that is not true of $\alpha\beta$.

§ 40

Corollary. It is easy to demonstrate that the comparison between *ab*, *cd* can be carried on so far that the characteristics resulting from it determine *cd* from *ab*. And if *ab* determines *cd* then $\alpha\beta$ also determines $\gamma\delta$.

§ 41

Theorem. The same (as §39) holds also for straight lines.

Proof. Because these are determined by the distances (§28; I §17).

§ 42

Corollary. Therefore for every three given lines there is one and only one (§40) fourth proportional line.

§ 43

Concluding note. These few propositions are probably sufficient to show how I would think of building up a complete theory of the straight line on the axioms already stated. At the conclusion of this essay I want to add a *definition of the plane* according to which I have already sketched out a large part of a new theory of the plane. The *plane of the angle ras* (Fig. 3) is that object which contains all and only those points which can be determined by their relationship (their angles and distances) to the two directions *R*, *S*.[f]

[f] This definition is taken up again in *DP* §37.

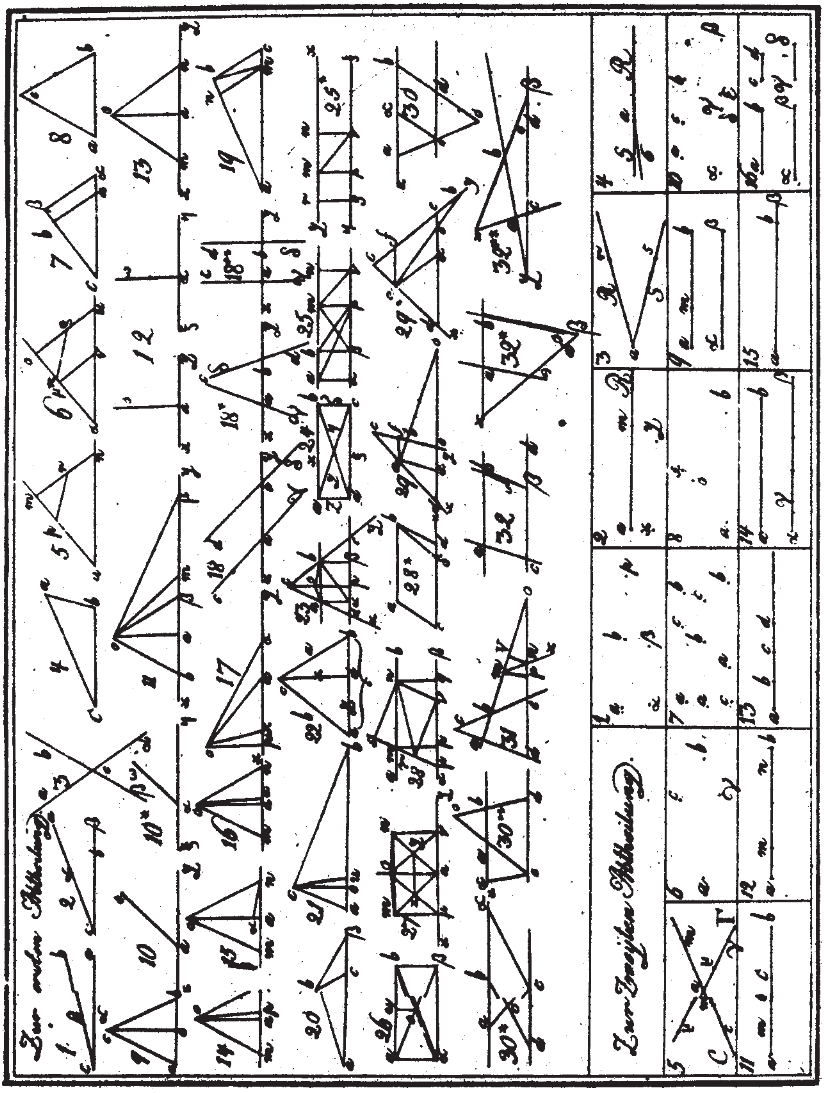

Plate of Figures as it appeared at the end of the first edition.

Beyträge zu einer begründeteren Darstellung der Mathematik.

Von

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Erste Lieferung.

Prag, 1810.
Im Verlage bey Caspar Widtmann.

Contributions
to a
Better-Grounded Presentation
of
Mathematics

by

Bernard Bolzano
Priest, Doctor of Philosophy, and Professor of Theology
at the Karl Ferdinand University

First Issue

Prague, 1810
Published by Kaspar Widtmann

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Multum adhuc restat operis, multumque restabit,
nec ulli nato post mille saecula praecludetur occasio
aliquid adhuc adjuciendi. . . . Multum egerunt, qui ante
nos fuerunt, sed non peregerunt. Suspiciendi tamen sunt,
et ritu deorum colendi. *Seneca*, Epistolae, *64*

Much still remains to do, and much will always remain,
and he who shall be born a thousand years hence will
not be barred from his opportunity of adding something further . . .

Our predecessors have worked much improvement, but have not
worked out the problem. They deserve respect, however,
and should be worshipped with a divine ritual.[a]

[a] *Seneca*, Epistle 64, Tr. R.M. Grummere, Heinemann (1953).

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Preface

Every impartial critic must admit that of all the sciences it is mathematics that comes nearest to the ideal of perfection. In the most ordinary textbook of mathematics there is more precision and clarity in the concepts, and more certainty and conviction in the judgements, than will be found at present in the most perfect textbook of metaphysics. But undeniable as that is, the mathematician should never forget that what is quoted above about all human knowledge is also true of his science, '*that it is only an incomplete work.*' However, the greatest experts in this science have in fact always maintained not only that the structure of their science is still unfinished and incomplete in itself, but also that even the first foundations of this (otherwise so magnificent) structure are still not completely secure and in order; or, to speak plainly, that some gaps and imperfections are still to be found even in the most elementary theories of all mathematical disciplines.

Now we give *some* reasons for this opinion. Have not the greatest mathematicians of modern times recognized that in *arithmetic* the *theory of negative numbers*,[b] together with all that depends on it, is still not clear? Is there not a different presentation of this theory in almost every arithmetic textbook? The chapter on *irrational* and *imaginary* numbers [*Größen*] is even more ambiguous, and in places full of self-contradictions. I do not want to mention anything here about the defects in *higher algebra* and the *differential* and *integral calculus*. It is well known that up till now there has not even been any agreement on the concept of a differential. Only at the end of last year the *Royal Jablonovsky Society of Sciences* at Leipzig gave as their prize-question *the analysis of different theories of the infinitesimal calculus and the decision as to which of these is preferable.*

None the less, it seems to me that arithmetic is still by far the most perfect of the mathematical disciplines, while *geometry* has much more important defects which are more difficult to remove. At present a precise definition for the important concepts of *line, surface, and solid* is still lacking. It has not even been possible to agree on the definition of a *straight* line (which could perhaps be given *before* the concept of *a line in general*). A few years ago, *Grashof* (*Theses sphaereologicae, quae ex sphaerae notione veram rectae lineae sistunt definitionem, omnisque geometriae firmum jaciunt firmamentum*, Berol 1806) presented us with a completely new definition which, however, is hardly satisfactory. But the most striking defect concerns the *theory of parallels*. As far as we know, people have been concerned with the

[b] The German *entgegengesetzten Größen* is literally 'opposite quantities'.

improvement of this since the time of *Proclus* and very probably even long *before Euclid.* Although so many attempts have been made in the past, no one has been successful enough as to enjoy general acclaim.

In *mechanics* the concepts of *speed* and *force* are almost as much of a stumbling-block as the concept of the straight line in geometry. It has also been agreed for a long time that the two most important theorems of this science, namely that of the *parallelogram of forces* and that of the *lever* have still not been rigorously proved. For this reason the *Royal Society of Sciences at Copenhagen* made a better-grounded *theory of the parallelogram of forces* a prize-problem in 1807. Since I have not yet seen the paper of Professor *de Mello* which gained the prize, I cannot be sure whether the attempt which I intend to give in *these* pages will be something new.[c] As for the theory of the *lever*, it is usually believed that *Kästner's* proof removes all the difficulties, but I believe that in the present work I shall show the contrary.

Finally, in *all* parts of mathematics, but especially in geometry, the *disorder* [*der Mangel der Ordnung*] has been criticized since the time of *Ramus*. In fact, what sorts of dissimilar objects are not dealt with in the individual theorems in *Euclid*? Firstly there are *triangles*, that are already accompanied by *circles* which intersect in certain points, then *angles*, adjacent and vertically opposite angles, then the *equality* of triangles and only much later their *similarity*, which is however, derived by an appalling detour, starting from the consideration of *parallel lines*, and even of the *area* of triangles etc.! But if we consider, *Ταῦθ᾽ ὅπως μὲν γέγραπται τοῖς καιροῖς καὶ ταῖς ἀκριβείαις*;[d] and if we reflect how every successive proposition, with *that* proof with which *Euclid* understands it, necessarily requires that which precedes it, then we must surely come to the conclusion that the reason for that disorder [*Unordnung*] must be more fundamental: the entire method of proof [*Beweisart*] which *Euclid* uses must be incorrect.

Now the purpose of the present pages is not only to make some contributions to the removal of what has just been criticized, but also to remove some other defects of mathematics whose presence can only be proved in what follows. I may reasonably be asked, *what qualifies me to do this*? I want to say here quite frankly what I know there is to say for or against me in this respect.

For about fifteen years—I have not known mathematics for longer—this science has always been one of my favourite studies, the theoretical part [*spekulative Teil*] in particular, as a branch of philosophy and as a way of practising correct thinking. As soon as this became known to me through *Kästner's* splendid textbook, I was struck by [first] one and then another defect and I occupied myself in my leisure

[c] The references here, and in the following sentence, are to later intended issues of the current work. See Bolzano's account in the final paragraph of this *Preface*. Some of the material for the second issue is published in BGA 2A5.

[d] 'How well this discourse has been composed with respect to appropriateness and finish of style'. From the concluding sentence of Isocrates' letter to Philip of Macedon. *Isocrates*, I, 339, Tr. G. Norlin, Heinemann (1928).

time with removing them, certainly not from vanity, but out of an intrinsic interest that I found in such theoretical problems [*Speculationen*]. With further reflection the number of defects which I believed I had discovered increased yet again. Indeed I gradually succeeded in removing some of them but I did not immediately trust the solution from fear of deceiving myself, because I loved the truth more than the pleasure of an imagined discovery. Only if I had examined my opinion from all sides and had always found it confirmed, would I have more confidence in it. Meanwhile, as far as I have been allowed by my other studies, and for the past five years my teaching post, as well as other circumstances, I also examined *those books* which have been written with the intention of perfecting the scientific system [*wissenschaftliche System*] of mathematics. In these, I found that some of the things to which I had been led by my own thinking had already been presented. On the other hand, there are some things I have still not found anywhere. However, since I could not acquire a complete knowledge of the mathematical literature, it is possible that some things which I regard as new have already been said somewhere, but this will certainly not be the case with everything.

Moreover, I am *well aware* that it is surely a *daring* enterprise to want to change and improve something in the first foundations of mathematics. Kästner says somewhere, and it is a historical truth, '*hitherto all those who wanted to surpass Euclid have come to grief themselves.*' Surely a similar fate also awaits me, especially as prejudice and obstinacy will oppose me even if I should have truth on *my* side? However, from the failure of *several* attempts it certainly does not follow that all the *others* must fail. Also, the method [*Weg*] which I adopt here is very different from the methods attempted previously. I therefore regarded it as my duty to submit it to the judgement of the experts.

As early as 1804 I published a small sample of my alterations under the title *Betrachtungen über einige Gegenstände der Elementargeometrie*.[e] But the small scope of the booklet, its uninformative title, the over-laconic style, the obscurity of the author, and many other factors, were certainly not favourable for attracting attention to it. Therefore nothing further followed except that it was announced in some scholarly journals (e.g. in the *Leipziger Jahrg.*, 1805 Jul., St 95, in the *Jen.*, 1806 Febr., No. 29) without an *obvious* mistake[f] being pointed out in the theory of parallels presented in it. Now I have naturally made progress in my ideas since that time and therefore I believe many things are now presented better and more correctly than they were then. Therefore in these *contributions* I intend to go through the individual *a priori* disciplines of mathematics one at a time, according to the order set out in the present work in Part I, §20. The contributions are to appear in small issues, like the present one, at indefinite intervals, and I cannot determine in advance how many there will be. Most of the alterations, and the most important ones, will concern *geometry*. I shall therefore present these as

[e] This is the work *BG* appearing on p. 24 of this volume.

[f] Possibly Bolzano had become aware that some of his early 'results' in *BG* were logically equivalent in the plane, to his parallel postulate 'theorem' of *BG* I §59.

soon as possible so that through the criticism of experts either my views will be confirmed, or my error made clear, and I shall not lose any more time on a wrong course.

—εἴ τίς μοι ἀνὴρ ἅμ' ἕποιτο καὶ ἄλλος,
μᾶλλον θαλπωρὴ καὶ θαρσαλεώτερον ἔσται
—μοῦνος δ' εἴ πέρ τε νοήσῃ,
ἀλλά τέ οἱ βράσσων τε νόος, λεπτὴ δέ τε μῆτις.[g]

Iliad X, 222.

[g] '. . . I wish another man would go with me. I should feel more comfortable, and also more inclined to take a risk. [Two men together seize advantages that one would miss;] whereas a man on his own is liable to hesitate, if he does see a chance, and make stupid mistakes.' (Diomedes speaking to Nestor in reference to a proposed visit to a Trojan camp. The bracketed words are omitted from Bolzano's text. *Homer*, Iliad, X, 222. Tr. E.V. Rieu, Heinemann 1950.)

I On the Concept of Mathematics and its Classification

§ 1

It is well known that the oldest mathematical textbook, *Euclid's Elements*—which in some ways is still unsurpassed—*contains no definition* of the science with which it is concerned. Whether its immortal author did this out of a kind of wilfulness, or because he thought it was not worthwhile, or because he did not know any valid definition to give us, I shall not venture to decide. By contrast, in all *modern* textbooks of mathematics this definition is put forward: '*mathematics is the science of quantity*'.[h] *Kant* has already found fault with this definition in his *Kritik der reinen Vernunft* (see the 2nd edition, S. 742) because in it, as he says, '*no essential characteristic* of mathematics is stated, and *the effect* is also mistaken *for the cause.*'

§ 2

Naturally everything here depends on what is understood by the word '*quantity*'. The anonymous author of the book *Versuch, das Studium der Mathematik durch Erläuterung einiger Grundbegriffe und durch zweckmässigere Methoden zu erleichtern. Bamberg and Würzburg,* 1805 (S. 4), puts forward the following definition of quantity, '*A quantity is something that exists and can be perceived by some sense.*' This definition is always one of two things, either too wide or too narrow, according to whether the author takes the words '*exists*' and '*can be perceived*' in their widest sense when they mean a purely *ideal existence* and a *possibility of being thought*, or in their narrower and proper sense in which they hold only for a *sensible object*[i] *which actually exists*. In the first case, quantity would be *every conceivable thing without exception* and if we then defined mathematics as the science of quantity we would basically bring all sciences into the domain of this one science. On the other hand, in the second case, only *sensible objects* would be quantities, and the domain of mathematics would obviously then be excessively restricted—because immaterial things, e.g. spirits and spiritual forces, can also become an object of mathematics, and particularly of arithmetic.

§ 3

However, this definition of quantity (§2), interpreted as above or otherwise, is in fact quite contrary to the use of language. I have only mentioned it here to show subsequently that *this* author might also have had in his mind (although only dimly) a certain idea which seems to me true. If we do not wish to move too far

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[h] The German *Wissenschaft der Größen*, literally 'science of quantities' betrays a more 'distributive' view of quantity than in the standard, singular, form in English used here.

[i] The German *sinnliche Gegenstand*, an object open to the senses; while slightly archaic, the *OED* supports this meaning of 'sensible'.

away from the use of language (something which we should surely never do even in the sciences without necessity), then we must understand by quantity, a *whole in so far as it consists of several equal parts*, or even more generally, *something which can be determined by numbers*. Assuming this meaning of the word 'quantity', the usual definition of mathematics as a science of quantity is certainly defective and indeed *too narrow*. For quantity is considered *alone and in abstracto* only in pure *general mathesis*,[j] i.e. in *logistic* or *arithmetic*, but it does not exhaust the content even of *this* science. The concept of quantity, or of number, does not even appear in many problems of the *theory of combinations* (this very important part of general mathesis). For example, if the question is raised: *which permutations*—not *how many*—*of the given things a, b, c, . . . are admissible?* In the *particular*[k] parts of mathematics, *chronometry*, *geometry* etc., as the names suggest, some object *other* than the concept of *quantity* (e.g. time, space, etc.) appears everywhere, and the concept of quantity is just *frequently applied* to it. So that in all these disciplines there are several axioms and theorems which do not even contain the concept of quantity. Thus, for example, in chronometry the proposition that *all moments are similar to each other*, and in geometry that *all points are similar to each other*, must be established. Such propositions, which do not contain the concept of a quantity or number at all, could never be established in mathematics if it were merely a *science of quantity*.

§ 4

It will not be as easy to put a better definition in the place of the usual one as it has been for us to criticize and reject it. We have already observed that those *special objects* which appear in the individual parts of mathematics, alongside the concept of quantity, are of such a nature that the latter can easily be applied to them. This could perhaps lead to the idea of *defining* mathematics as *a science of those objects to which the concept of quantity is especially applicable*. And it really seems that those who adopted the definition quoted in §1 basically intended nothing but this to be understood. However, a more careful consideration shows that even this definition is objectionable. The concept of quantity is *applicable* to *all* objects, even to *objects of thought*. Therefore if one wanted to consider the mere *applicability of the concept of quantity* to an object a sufficient reason for counting the theory of that object among the mathematical disciplines, all sciences would in fact have to count as mathematics, e.g. even the science in which the proposition is proved that there are only *four* (or as *Platner* more correctly states, only *two*) syllogistic figures; or the science which states that there are *no more and no less than four sets of three* pure simple concepts of the understanding (categories), etc. Therefore in

[j] Bolzano uses the Latin term, as here, and German *Mathematik*, interchangeably—see, for example, his usage in §§11, 12 of this Part. Mathesis is used in translation wherever Bolzano uses it.

[k] The German *besondere*, 'particular' in the sense of specialised; throughout this Part it is contrasted with 'general' [*allgemeine*].

order to salvage this definition one would have to take into account the *difference between rarer and more frequent applicability*, i.e. count only those objects to be in mathematics to which the concept of quantity can be applied *often and in many ways*. But anyone can see that this would be an extremely vague, and not at all scientific, determination of the boundaries of the domain of mathematics. We must therefore look for a *better* definition.

§ 5

The *critical philosophy* seems to promise us one. It thinks it has discovered a definite and characteristic difference between the two main classes of all human *a priori* knowledge, philosophy and mathematics. Namely that *mathematical knowledge* is *capable* of adequately presenting, i.e. *constructing*, all its concepts in a *pure intuition* [*reine Anschauung*], and on account of this is also *able to demonstrate* its theorems. On the other hand, *philosophical knowledge*, devoid of all intuition, must be content with purely *discursive concepts*. Consequently the essence of mathematics may be expressed most appropriately by the definition *that it is a science* [*Vernunftwissenschaft*] *of the construction of concepts* (cf. *Kant's Kritik der reinen Vernunft*, S. 712). Several mathematicians who adhere to the critical philosophy have actually adopted this definition. Among others there is *Schultz* who deserves much credit for the grounding [*Begründung*] of pure mathematics in his *Anfangsgründe der reinen Mathesis*, Königsberg, 1791.

§ 6

For my part I wish to admit openly that I have not yet been able to convince myself of the truth of many doctrines in the critical philosophy, and especially of the correctness of the Kantian claims about *pure intuitions* and about the *construction of concepts using them*. I still believe that in the *concept of a pure* (i.e. *a priori*) *intuition* there already lies an intrinsic contradiction. Much less can I persuade myself that the concept of *number* must necessarily be constructed in *time* and that consequently the intuition of time belongs essentially to arithmetic. Since I say more about this in the appendix to this paper, I am content here only to add that among the independent thinkers in Germany there are some, and probably many, who agree just as little as I do with these claims of Kant. There are even some who had been inclined to the Kantian definition at first, but subsequently found they had to abandon it. Such, for example, was *Michelsen* in his *Beiträge zur Beförderung des Studiums der Mathematik*, Berlin, 1790, see 1.B., 5. Stück.

§ 7

But more instructive to me than what *Michelsen* says in his paper was what I came across in the general *Leipz. Literatur Zeitung* (1808 Jul., St. 81). The learned reviewer condemns the usual definition of mathematics as a *science of quantity* and then says, 'Quantity is only an object of mathematics because it is the most *general form, to be finite*, but in its nature *mathematics* is a *general theory of forms*. Thus, for example, is *arithmetic*, in so far as it considers *quantity* as the general *form of finite things*; *geometry*, in so far as it considers *space* as the general *form of Nature*; the *theory of time*, in so far as it considers the general *form of forces*; the *theory of motion*, in so far as it considers the general form of *forces acting in space.*' I do not know whether I understand these definitions quite in the sense of their author, but I must admit that they helped me to develop and further improve the following definition and classification of pure mathematics which I had already sketched in its main outline.

§ 8

I therefore think that mathematics could best be defined as a *science which deals with the general laws (forms) to which things must conform* [*sich richten nach*] *in their existence* [*Dasein*]. By the word '*things*' I understand here not merely *those* which possess an *objective existence* independent of our consciousness, but also those which simply exist in our *imagination*, either as *individuals* (i.e. *intuitions*), or simply as *general concepts*, in other words, *everything which can in general be an object of our capacity for representation* [*Vorstellungsvermögens*]. Furthermore, if I say that mathematics deals with *the laws to which these things conform in their existence*, this indicates that our science is concerned not with the proof of the *existence* of these things but only with the *conditions of their possibility*. In calling these laws *general*, I mean it to be understood that mathematics never deals with a single thing as an *individual* but always with whole *genera* [*Gattungen*].[1] These genera can of course sometimes be higher and sometimes lower, and the classification of mathematics into *individual disciplines* will be based on this.

§ 9

The definition given here will certainly not be found to be *too narrow* for it clearly covers everything that has previously been counted in the domain of mathematics. But I am all the more afraid that it might be found rather *too wide* and the objection might be made that it leaves too little for *philosophy* (metaphysics). The latter will be limited by my definition to the single concern of proving, from *a priori* concepts, the *actual existence* of certain objects. Mathematics and metaphysics, the two main parts of our *a priori* knowledge would, by this definition, be contrasted with each other so that the *former* would deal with the general conditions under which the existence of things is *possible*; the latter, on the other hand, would seek to prove *a priori* the *reality* of certain objects (such as the freedom of God and the immortality of the soul). Or, in other words, the *former* concerns itself with the question, *how must things be made in order that they should be possible?* The *latter* raises the question, *which things are real*—and indeed (because it is to be

[1] 'Class' might be used for *Gattung* but Bolzano refers to the classical terminology of 'species' and 'genus' (see I §11).

answered *a priori*)—*necessarily real?* Or still more briefly, *mathematics* would *deal* with *hypothetical** *necessity*, *metaphysics* with *absolute necessity*.

§ 10

If I come upon some ideas which are *new* to me I am accustomed to asking myself the question, whether anyone before me has held the same view. If I find this is the case, then naturally I gain *conviction*. Now as far as the above definition is concerned, I need hardly say how closely what the astute reviewer has said (§7) coincides with my presentation, that is, if it does not amount to exactly the same thing. This idea also seems to have been in the mind of the author of the book mentioned in §2, although only dimly. For in defining quantity, or the object of mathematics, as *that which is*, he seems to have felt that mathematics is concerned with *all* the forms of things, not merely with their *capacity to be compounded out of equal parts* (their *countability*). *Kant* defines *pure natural science* [*reine Naturwissenschaft*] (which has always been regarded, under the name of *mechanics*, as a part of mathematics) as *a science of the laws which govern the existence of things* (of phenomena). This definition can lead very easily to our definition as given above. *Time* and *space* are also *two conditions* which govern the existence of appearances, therefore *chronometry* and *geometry* (which consider the properties of these two forms *in abstracto*) deal likewise, though only *indirectly*, with the laws which govern the existence of things (i.e. things open to the senses [*sinnliche Dinge*][m]). Finally *arithmetic*, which deals with the laws of *countability*, thereby develops the *most general* laws according to which things must be regulated in their existence, even in their *ideal* existence.

§ 11

Now from this definition of mathematics, let us try to derive a *logical classification* of this science into several individual disciplines. If we succeed fairly naturally with this classification, then this may even be another new confirmation of the validity of that definition. According to it, mathematics is to be a *science of the laws to which things must conform in their existence*. Now these laws are either so general that they are applicable to *all things completely without exception*, or not. The former laws, put together and ordered scientifically, will accordingly constitute the *first* main part of mathematics. It can be called *general mathesis*; everything else is then *particular mathesis*.

* Although not all its propositions have this hypothetical *form* because it is tacitly assumed, especially in chronometry and geometry, that the condition is the same for all propositions.

[m] It is sometimes also convenient to translate this phrase by the term, ‘sensible things’, for example, in the table in §20. We have already used the term ‘sensible object’ in I §2.

Note. To this *general* mathesis belong, as we shall see below, *arithmetic*, *the theory of combinations*, and several other parts. These parts of mathematics must therefore not be considered as *coordinate with the rest* (chronometry, geometry, etc.); it is rather that the latter are *subordinate* to the general mathesis as a whole, as species [*Art*] of the genus. And because the concept of *number* is one of those of the *general* mathesis it will also appear *frequently* in all these particular parts, but it will not exhaust their content.

§ 12

Now in order to obtain the *particular* or *special* parts of mathematics we must put the *things themselves* with whose general forms mathematics is concerned into certain *classes*. But before we do this, let us draw attention to a certain concept of our understanding, which (as far as I can see) is *not completely* applicable *to all things* and therefore, in all strictness, should not be included in *general mathesis*. On the other hand, it is applicable to things *of such different kinds* that it would hardly be suitable for a classification of mathematics into individual disciplines. This is the *concept of being opposite* [*Entgegensetzung*]. I do not believe that to *every* thing there is an *opposite*; but the *past* and *future* in *time*, *being on this side* and *that side* in *space*, *forces* which act in opposite directions in *mechanics*, *credit and debit* in the calculation of *accounts*, *pleasant* and *unpleasant* in *feelings*, *good* and *evil* in *free choices*, and similar things, are clear examples of *being opposite* which sufficiently prove how widely applicable is this remarkable concept. At the same time, we see from just these examples how little suited this concept is, as a basis for a main division of the mathematical disciplines into those to which it would be applicable, and those to which it would not be applicable. On the contrary, the classifications which actually exist (and which we cannot completely over-ride) are made according to a quite different basis of classification which we can only dimly imagine. Included in every individual discipline is whatever particularly arises from the application of the concept [of being opposite] to the object of the discipline. The general part, however, which deals without exception with all things capable of being opposite, certainly deserves separate consideration. It can be presented (as has already been done to a certain extent) in a special *appendix to the general mathesis*.

§ 13

Everything which we may ever think of as *existing* [*existierend*] we must think of as being one or the other: either *necessary or free* (i.e. not necessary) in its existence.* That which we think of as something *free* is subject to no conditions and laws in its

* An example of the first kind is the *speed* of a moving body; an example of the second kind, every *human decision*.

becoming (or *existence***), and is therefore not an object of mathematics.*** That which we think of *as necessary* in its existence is so, either *simply* (i.e. *in itself*) or only *conditionally* (i.e. on the presupposition of something else). The *necessary in itself* is called *God* and is considered in *metaphysics* not as a merely *possible* object but as an *actual* object. Therefore there remains only the *hypothetically necessary* [object] which we consider as *produced through some ground* [*Grund*].[n] Now there are certain *general conditions* according to which everything which is produced through a ground (*in* or *out* of time) must be *regulated* in its *becoming* or *existence*. These conditions taken together and ordered scientifically will therefore constitute the *first* main part of mathesis, which I call, for want of a better name, *the theory of grounds* [*Grundlehre*] or *aetiology*.

Note. This part of mathematics contains the theorems of ground and consequence [*Grund und Folge*], some of which also used to be presented in *ontology*, e.g. that *similar grounds* have *similar consequences*. It also contains the theories known by the name of the *calculus of probability*. I only mention this here so that it will not be thought that perhaps we have completely ignored this important part of mathematics. Moreover, in a scientific exposition aetiology must precede chronometry and geometry because the latter appeal to certain theorems of the former, as we shall see more clearly in due course.

§ 14

Everything which we not only *think of* as real but which has to be *perceived* as real, must be perceived in *time* and—if in addition we are to recognize it as a thing *outside ourselves*—also in *space*. In other words, *time* and *space* are the two *conditions* which must govern all things *open to the senses*, i.e. all things which *appear* to us as real. Therefore if we develop the properties of *time* and *space in abstracto* and order them scientifically, these sciences must also be counted as mathematics in that they also *deal*, albeit only indirectly, with the *conditions to which things must conform in their existence*. We therefore have the *second* and *third* components of particular mathesis, the *theory of time* (chronometry), and the *theory of space* (geometry).

Note. It really does not matter which of these two sciences we put before the other in the system, as the properties of time and space are completely independent of each other. However, because the concept of time is applicable to *more* objects than that of space it seems more appropriate to allow chronometry to precede geometry.

** In case it does not *become*, but only *is*, as e.g. the *free activity of the Deity* as far as we are not considering it in *time*.

*** But it is certainly the object of *morality* which investigates the question: *how that which happens* (or *is*) *freely, should happen* (or *be*).

n See Bolzano's footnote on p. 98.

§ 15

If, finally, time and space are not to be considered merely *in abstracto* but as occupied with *actual things* and indeed with such things as are not free in their existence but are subject to the laws of causality, then two *new sciences* appear, which are, as it were, composed from the one mentioned in §14 with that in §13. Namely:

(a) The general laws to which *unfree things*, which are in *time*, must conform in their existence (and in their changes), form the content of a particular science which I call, for want of a more suitable name, *the theory of causes* [*Ursachenlehre*] or *temporal aetiology*.*
(b) The general laws which govern *unfree things* which are *in both time and space* at once form the content of that mathematical discipline which is called *pure natural science*, and otherwise *the theory of motion* or *mechanics*.

Note. To the domain of *temporal aetiology* belong, for example, the theorems: 'every effect is simultaneous with its cause; the size of the effect originating from a constant cause varies as the product of the degree of the cause and the time for which it acts', and similar ones. These theorems are in fact so general that they hold not only for spatial material things but also for spiritual forces, our ideas, and generally for all things which appear in time and are subject to the law of causality. *Pure natural science* is already familiar.

§ 16

A science has often been spoken of in which there should appear the general laws *of the possibility of motion without regard to a force producing the movement*—therefore the concepts of *time*, *space*, and *matter* without that of a *cause*. *Hermann*, *Lambert*, and *Kant* have called this science *phoronomy*[o] [*Phoronomie*] and *Kant* considered it as part of pure natural science, although it would *not* belong there according to our definition above. *E. G. Fischer*, in his *Untersuchung über den eigentlichen Sinn der höhern Analysis nebst einer idealen Übersicht der Mathematik und Naturkunde nach ihrem ganzen Umfange*, Berlin, 1808, likewise puts forward this science under the name *phorometry* and has it following *geometry* as the *second* main part of *spatial mathematics*. However, if the views which I intend to give subsequently are not entirely incorrect, such a science cannot even exist. For all propositions which have so far been put forward in it are in fact only provable with the aid of the concept of cause.

* Thus I distinguish the words '*ground*' [*Grund*] and '*cause*' [*Ursache*]. The latter means for me a *ground* which acts in *time*.

[o] Phoronomy is the purely geometrical theory of motion, or kinematics (*OED*).

§ 17

The individual disciplines of mathematics are usually divided into *elementary*[p] and *higher*. However, I do not know up until now of any *genuinely scientific basis for this classification*. It will be more appropriate to discuss to what extent such a basis for classification applies only to a single discipline (e.g. geometry), in the particular treatment of this discipline. Therefore we deal here only with those bases for classification which are to apply throughout the whole domain of mathematics. This is the case for all those which have been suggested for *general* mathesis. Classifications which are made for this must therefore pervade all the particular parts of mathematics. In *Michelsen's Gedanken über den gegenwärtigen Zustand der Mathematik, usw.*, Berlin, 1789, elementary and higher mathesis are so divided that the one has for its object *constant quantities*, and the other *variable quantities* (or in general, *things*). I do not believe I can accept this classification because the *assumption* on which it is tacitly based, which many have adopted even in the definitions of mathematics, is that it is the sole business of mathematics *to find quantities which are not given, from others which are given*. If this were correct, then all the propositions of mathematics should have the form of *problems*. *Axioms* and *theorems*, etc., could not actually appear in it at all. However, before it can be asked, what *follows* from given things, one must first have *proved*, or have accepted as a *postulate*, that these things *can* be given, i.e. are *possible*. A completely different classification was suggested by *Michelsen* in his above-mentioned *Beyträge* in 1.B., 2. St. (*Über den Begriff der Mathematik und ihre Teile*). Here he assumes three main parts of general mathesis:

1. the *lower* [*niedere*], which has quantities consisting of *equal* components;
2. the *higher*, in which quantities are considered to be composed partly of *equal* components and partly of *unequal* components (the theory of differences and sums);
3. the *transcendental*, in which the components of quantities are proper *elements* or *units of magnitude* [*Größeneinheiten*] in the strictest sense (the differential and integral calculus).

I do not know if I understand this classification correctly. For it seems to me that in the sense in which calculations of sums and differences can be said to regard quantities as composed partly from equal and partly from unequal quantities, elementary arithmetic already does this, for example, in viewing $2 + \frac{1}{2}$ as a whole. Still less do I see how differentials can be considered to be *units of magnitude* in the strictest sense of the word, since we also attribute a magnitude to them, at least in the sense of an intensive magnitude.[q]

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[p] The German *gemeine* is more usually translated as 'common', but in this context where it is contrasted with 'higher', 'elementary' has been used.

[q] Here, and elsewhere, both 'magnitude' and 'quantity' translate the same German word *Größe*. See remarks on this in the *Note on the Translations*.

The best procedure might well be to count as *higher* mathesis only that in which the concept of an *infinity* (whether infinitely great or small), or of a *differential*, appears. At the present time this concept has not yet been sufficiently explained. If, in the future, it should be decided that the *infinite* or the *differential* are nothing but *symbolic expressions* just like $\sqrt{-1}$ and similar expressions, and if it also turns out that the method of proving truths using purely symbolic inventions is a method of proof which is indeed *quite special*, but is always correct and logically admissible, then I believe it would be most appropriate to continue to refer the concept of *infinity*, and any other equally symbolic concept, to the domain of higher mathematics. *Elementary* mathesis would then be that which accepts only *real concepts or expressions* in its exposition—*higher* mathesis that which also accepts purely *symbolic* ones.

§ 18

We also have something to say about the classification of mathematics into *pure* and *applied*. If, as commonly happens, one understands by *applied mathematics* the same as *empirical mathematics* then we cannot, without contradicting ourselves, even admit the *existence* of such a thing, because in our above definition we counted the *whole* of mathematics among the pure *a priori* sciences. But it need not be feared that in this way we shall lose a substantial part of the mathematical disciplines. The history of mathematics shows increasingly that whatever has been accepted at first merely from experience is subsequently derived from concepts and therefore comes to be treated as a part of pure *a priori* mathesis. And this may be sufficient ground for making no scientific distinction between pure and empirical mathematics. For example, because *we* do not know* how to derive *a priori* the existence of an attractive force, and the law that it acts in inverse proportion to the square of the distance, does it follow from this that our descendants will never know it either and that it is absolutely not derivable *a priori*? Furthermore, what one accepts from *experience* in the so-called *applied* parts of mathematics does not fundamentally make these disciplines *empirical*. For mathematics does not deal at all with what actually *takes place* but with the *conditions* or *forms* which something must have *if it is to take place*. Therefore it is only necessary to present those propositions offered by experience, purely hypothetically, and then to derive by *a priori* deductions, on the one hand, the *possibility* of these hypotheses, and on the other hand, the *consequences* which follow from them. Hence no empirical judgement appears in the whole exposition, and the science is therefore *a priori*. So, for example, in the exposition of optics there is no need to borrow from *experience* the law that light bends in going from air to glass in the ratio 3 : 2, and similar

* *Kant* nevertheless attempted it.

things; it is sufficient to make comprehensible *a priori* the mere *possibility* of a material like light and the fact that it bends in passing through different media. On this basis the statement is established in hypothetical form, '*if there is a material which etc. . . . then this and that consequence must follow from it*'. But that *possibility* can never be difficult to prove in that everything which is to be perceptible as real in experience, must already be recognised as possible.

§ 19

Therefore in so far as one wishes to understand by *applied* mathematics something *which is essentially based on some propositions borrowed from experience*, I do not believe its existence can be justified. But one can also understand something completely different by the name *applied* mathematics, something which I prefer to call *practical*—or using a term borrowed from the critical philosophy and surely more definite—*technical mathematics*. This is *an exposition of the mathematical disciplines established particularly for useful application in everyday life*. Such an exposition is distinguished in a specific way from the *purely scientific* by the difference in *purpose*; that of the *latter* is the greatest possible *perfection of scientific form* and thereby also the *best possible exercise in correct thinking*, while that of the *former*, in contrast, is *direct usefulness for the needs of life*. Therefore in the practical exposition all the *excessively general views* which are not absolutely essential to the application are put aside, while many *examples* and special *references to actual cases* are inserted. It is not compulsory to cite these actual cases as mere *possibilities* (as must happen in a purely scientific exposition) rather they are put forward directly as *realities* proved by experience. Moreover, I need hardly say that most existing textbooks of mathematics are based on a certain *mixed* approach which aims at combining those two purposes, the purely scientific and the practical. It is certainly not my opinion that this is generally a *fault* in those textbooks. A wholly appropriate textbook composed using this *mixed* method would in fact be a far more widely useful work than a purely scientific one. But I believe the first cannot be achieved until the *purely scientific* system has been completed. Whoever works for the perfection of the latter can be allowed, for the time being, to put the *second* purpose from his mind completely, so as to turn his attention solely to the *first*, scientific perfection.

§ 20

From all this I believe it is now clear that in a scientifically ordered mathematics there are only the *main parts* mentioned in §§ 11–15. The following table sets these out in a convenient summary. The bracketed words designate the object of each discipline.

A
General mathesis
(things in general)

B
Particular mathematical disciplines
(particular things)

I
Aetiology
(things which are not free)

II
(sensible things which are not free)

a
(form of these things *in abstracto*)

α	β
Theory of time	Theory of space
(time)	(space)

b
(sensible things *in concreto*)

α	β
Temporal aetiology	Pure natural science
(sensible things in time)	(sensible things in time and space)

II On Mathematical Method

§ 1

The method employed by mathematicians for the exposition of their science has always been praised on account of its high degree of perfection, and up to the time of *Kant* it was also believed* that its *essential* features could be applied to *every* scientific subject. I personally still firmly adhere to this opinion, and maintain that the so called *methodus mathematica* is, in its *essence*, not in the least different *from any scientific exposition*. On this assumption a work on mathematical method would be basically nothing but logic and would not even belong to mathematics at all. Meanwhile I may be permitted to make some remarks, as briefly as possible, on individual parts of this method—especially since everything which we shall say here refers only to mathematics, and primarily to the removal of certain of its imperfections.

§ 2

Although I do not know whether it will cause a certain group [*Klasse*] of my readers to lose all faith in me, but out of love for the truth, I must admit at the outset that I am not completely clear myself on the true nature of scientific exposition. The reason for this will be understood better subsequently. But this much seems to me certain: in the realm of truth, i.e. in the collection of all true judgements, a certain *objective connection* prevails which is independent of our accidental and *subjective recognition* of it. As a consequence of this some of these judgements are the grounds of others and the latter are the consequences of the former. Presenting this objective connection of judgements, i.e. choosing a set of judgements and placing them one after another so that a consequence is represented as such and conversely, seems to me to be the real *purpose* to pursue in a scientific exposition. Instead of this, the purpose of a scientific exposition is *usually* imagined to be the greatest possible *certainty* and *strength of conviction*. It therefore happens that the obligation to prove propositions which, in themselves, are already completely certain, is discounted. This is a procedure which, where we are concerned with the practical purpose of certainty, is quite correct and praiseworthy; but it cannot possibly be tolerated in a scientific exposition because it contradicts its essential aim. However, I believe that *Euclid* and his predecessors were in agreement with me and they did not regard the mere *increase in certainty* as any part of the purpose of their method. This can be seen clearly enough from the trouble which these men took to provide many a proposition (which in itself had complete certainty) with a proper [*eigener*] *proof*, although it did not thereby become any more certain. Whoever became more *certain* after reading *Elements* Bk. I Prop. 5 that in an isosceles triangle the angles at the base are equal?

* Especially in the *Leibniz–Wolff* school.

No, the most immediate and direct purpose which all genuinely philosophical thinkers had in their scientific investigations was none other than the search for the ultimate grounds of their judgements. And this search then had the *further purpose, on the one hand,* of putting themselves in the position of deriving from these clearly recognized grounds *some* of our judgements, perhaps also some *new* judgements and truths;[r] and *on the other hand,* of providing an *exercise* in correct and orderly thinking which should then *indirectly* contribute to greater *certainty* and strength in *all* our convictions. This has been a preliminary remark on the purpose of mathematical method in general. Now to its individual parts.

A. On Descriptions,[s] Definitions and Classifications

§ 3

It is usually said, that '*the mathematician must always begin with definitions.*' Let us see if this does not require rather *too much,* for it is easy to see that *excessive* demands often do just as much harm in the sciences as too much *tolerance.* Logicians understand by a *definition* (*definitio*), in the truest sense of this word, the *statement of the most immediate components* (two or more) *out of which a given concept is composed.*

Note. The *general form* which comprehends all definitions is the following, where the letters a, α, A designate concepts: a, *which is* α, is A; or (a *cum* α) $= A$. If a concept is determined by a negative characteristic *non* α, then instead of (a *cum non* α), we can write even more briefly (a *sine* α). But in each case neither a, nor α, *on their own* may be $= A$.

§ 4

From this it is already clear that there are true definitions only for concepts which are *composite,* and therefore also *decomposable* again; but for these they do always exist. *Simple concepts,* i.e. those that cannot be decomposed into two or more components different from each other and from the original concept itself, provided there are such others, *cannot be defined.* However, in my opinion, the existence of such simple concepts cannot be derived from anything but our own *consciousness.* Since a concept is called *simple* only if *we ourselves* can distinguish no more *plurality* in it, then it would follow from the opposite assertion (that there are *no* simple concepts *at all*), that each of our concepts could be analysed *indefinitely,* and yet we are not actually conscious of this.

r The phrase 'perhaps also some *new* judgements' [*vielleicht noch manche* neue *Urteile*] occurring in the first edition here is omitted in the second edition.

s The German *Umschreibungen,* refers to a means of reaching agreement on what a new term shall mean by giving several statements containing the term. This is explained in §8. It is description in a technical sense and works like an implicit definition. Rusnock (2000) uses 'circumscription'.

§ 5

Accordingly, to decide whether a given concept is composite or simple ultimately depends on our consciousness, on our ability or inability to decompose it. We note the following as a *few rules* which could make this decision easier in actual cases:

(a) *If we think of an object as composite, then precisely for this reason the concept of it is not a simple one.* For the *concept* of an object is nothing but what we think when we *think* of the object.*

If this remark is correct it follows immediately that the concepts of the *straight line*, the *plane*, and several others, which have so often been taken as simple concepts, are not such at all, and that one therefore can never discount the obligation to define them. For obviously the straight line and the plane are *objects of a composite kind*, in which we imagine, for instance, innumerably many points as well as particular relationships which these points must have to some given ones.

(b) *Not every concept which is subordinate to a more general one therefore ceases to be simple*. That is to say, a concept only ceases to be simple if it is decomposable. But in a *decomposition*, at least *two* components must be given each of which are *conceivable in themselves*. Now in fact we consider the *more general* concept actually as a *component* of the narrower one which is subordinate to it. But it could well be that for this *first* component (*genus proximum*) no *second* one (*differentia specifica*) is to be found, i.e. that that which must *be added to* the general concept, in order to produce from it the narrower one, is not in itself *representable*. So, for example, the concept of a point is certainly narrower than that of a *spatial object*, the latter is narrower than that of an *object in general* and the *most general* of all human concepts is, as is well known, only one, namely that of an *idea in general*. But it certainly does not follow from this that all preceding concepts are decomposable and definable. For example, if one tries to go from the concept of a spatial object, as *genus proximum* $= a$, to the concept of a point $= A$ in the form of a definition then it will be seen that the characteristic that must be added to a so as to obtain A is none other than the *concept of a point itself* $= A$, which is what was to be defined. Therefore in *general* if we want to convince ourselves whether a certain concept is *simple* or *decomposable* then we assume a *genus proximum* for it and we try to think

* That which one thinks when one *thinks* of an object (i.e. as already *contained* in it) is something quite different from that which one *can* (if desired) *add to it mentally*, or *think of as connected with it*. The clearer this difference seems to me the more I suspect that I should accept that the great *Lambert* has made an error about it. In his *deutscher gelehter Briefwechsel*, I.B., S. 348, he writes to *Kant*: '*The simple concepts are individual concepts. For genera and species contain the fundamenta divisionum et subdivisionum in themselves and are therefore the more composite the more abstract and general they are. The concept ens is the most composite of all concepts.*' *Lambert* often returns in his writings to this assertion which is so completely opposed to our definition above, and he derives from it the peculiar consequence, that in metaphysics one must not *begin* with the concept *ens* but rather *end* with it. The whole thing seems to me an error in which the *composite nature* of a concept has been confused with its *capacity to be composed*.

of some *differentia specifica* to add to it which is not itself already identical with the concept to be defined. If this cannot be done in any way, the concept concerned is a simple one.

Note. The question also arises here of *whether one and the same concept may admit of several definitions*. We believe this must be denied in the same way as we deny below (§30) the similar question of *whether there are several proofs for one truth*. One and the same concept consists only of the same simple parts. If it has *more than two* simple component concepts and one does not go back in the definition to the *ultimate* components, then it is of course possible to divide it in different ways into two integral parts and to this extent there are several definitions of the same concept. But the difference between these definitions lies only in the *words*, and so it is merely subjective, not objective and not *scientific*. The usual distinction between *nominal*, *real*, *generic*, and other definitions therefore seems unacceptable to us. What was falsely regarded as a *definition* was very often really a *theorem*. For example, the proper *definition of beauty* might not really be difficult to find, but what is sought is not the *concept* but a *theorem* which shows us what must be the nature of what produces the feeling of beauty in us.

§ 6

Propositions in which it is stated that, in future, one intends to *assign* this or that particular *symbol* to a certain concept, are called *conventions*.[t] Therefore *definitions*, in so far as they are expressed in *words* and assign a particular *word* to the composite concept, are also a *kind of convention*. But there are also conventions which are nothing but definitions, e.g. the proposition: *the symbol for addition is* +.

Note. We must not be misled by the *name* 'conventions' and believe that the kind of symbol chosen to denote this or that concept is *totally arbitrary*. Semiotics[u] prescribes certain rules here. The symbol must be easily recognized, possess the greatest possible similarity to the concept denoted, be convenient to represent, and most importantly, it must not be in contradiction with any symbols already used or cause any ambiguity. Several mathematical symbols could be more appropriately chosen in this respect. For example, the symbol for the separation of decimal fractions, a stroke to the *right* after the units is clearly defective. Why to the right? It would be just as good standing to the *left*. It should rather stand *above* or *below* the position of the *units* so that numerals which are equally far from the *symbol*, to left or right, would refer to an equal positive or negative power of ten. The difficulty which this mistake in notation causes can be clearly perceived when instructing a beginner. So before a symbol is put forward, one should first prove in a preceding proposition that it has the properties mentioned above and is therefore suitable. In

[t] The German *willkürliche Sätze*, is literally 'arbitrary propositions'. See, for example, *RB* §1 on p.264.

[u] In German *Semiotik* has had a long philosophical usage. In 1764 Lambert uses the term to denote, 'a characteristic language of symbols to avoid the ambiguities of everyday language' (*DSB, Lambert*).

the theory of exponential quantities we shall make particular use of this rule. Now if conventions are not completely arbitrary, it is possible to judge in what sense the proposition, *definitions are arbitrary*, must be understood. Indeed nothing is really arbitrary in definitions but the *word* which is chosen to denote the new composite concept, though it is obvious here that one should not do unnecessary violence to the use of language. On the other hand, it is not arbitrary which *concepts* are combined into a single one. *Firstly* these combinations must be made according to the law of *possibility*, and *secondly* one must select, from the *possible* combinations, only those which it can be *useful* to consider.

§ 7

From what has just been said the *place* can now be determined where definitions in a scientific exposition should be presented. They obviously *cannot* be *the first thing*, with which we begin. We must first have the insight [*eingesehen haben*] that a certain combination of two or more *words* (and the concepts they denote) produces a new and real concept; only then is it worthwhile to give this combination a *name* of its own. This process should also enable us to realize the *purpose* for which this combination is made and considered, if not perfectly clearly, then at least to anticipate it dimly. Therefore it is an error, contrary to good method, when *Euclid* gathers all his definitions at the beginning; *Ramus* has already justifiably criticized him for this.

§ 8

After all this, and certainly after the example quoted in §5(b), there is now no more question as to whether the mathematician can be required to let a definition precede all the concepts which he puts forward. He *cannot* do so, for among the concepts which concern him there are several which are completely simple. '*But how does he begin to reach an understanding* [*verständigen*] *with his readers about such simple concepts and the word that he chooses for their designation?'* This is not a great difficulty. For either his readers already use certain words or expressions to denote this concept and then he need only indicate these to them, e.g. 'I call *possible* that of which you say that it *could be*', or else they have no particular symbol for the concept he is introducing, in which case he assists them by stating several propositions in which the concept to be introduced appears *in different combinations* and is designated by its own word. From the comparison of these propositions the reader himself then abstracts which particular concept the unknown word designates. So, for example, from the propositions: *the point* is the *simple* [*object*] in space, it is the *boundary* of a line and itself no *part* of the line, it has neither extension in length, breadth, nor depth, etc., anyone can gather which concept is designated by the word 'point'. This is well known as the means by which we each came to know the first meanings of words in our mother tongue. Moreover, since concepts which are *completely simple* tend to be used only rarely in social life and thus either have *no* designation at all, or only a very *ambiguous* one,* then in a scientific exposition which begins with simple concepts it is anything but superfluous first to explain the particular designation for these concepts in one of those two ways. To distinguish such explications [*Verständigungen*] from a real definition we could call them *designations* or *descriptions* [*Umschreibungen*]. They also belong to the class of *conventions*, in so far as they are intended only to supply a certain concept with its particular symbol. They would thus be the *first* thing with which any scientific exposition must begin in so far as it has simple concepts.

§ 9

The *classifications* also belong in a scientific exposition, to give it *order*, and to make it easy to *have an overview*. But I believe that every genuine classification can only be a *dichotomy*.** Indeed a genuinely scientific classification only arises if for a certain concept *A* (*the one to be classified*), there is a certain second concept *B* (*the basis of classification* [*Einteilungsgrund*]) which must be consistent with *A*, and which can either be adjoined to *A* or excluded from it. The general form of all classifications would then be: *All things which are contained in the concept A are either contained in the concept* (*A cum B*) *or in the concept* (*A sine B*). From this it may be seen immediately that the concepts obtained through classification [(*A cum B*), (*A sine B*)] are always *composite*, and therefore *definable*, concepts. Thus it may be seen that definitions with *negative* characteristics cannot be rejected absolutely, in that there certainly are, and must be, concepts with negative characteristics [(*A sine B*)] (§3 *Note.*).

Note. An objection made to mathematics, often not incorrectly, is that it makes hardly any use of *classifications* and from this arises the striking *disorder* found in the mathematical disciplines. But in fact nothing could be more difficult than to remove this disorder and to introduce a true and natural order, rather than a merely *apparent* order. For this, of course, one must be clear about all the simple concepts and axioms of these disciplines, and know already exactly which premisses each axiom needs, or does not need, for its logically correct proof. Until this has been done, all efforts to remove that deficiency depend only on good luck and it is not surprising if they fail. For example, if the concepts which we intend to put forward below in geometry are correct, then all the ways of dividing up this science, attempted so far by *Schultz* and others, are unusable.

* An example, among several others, is the concept just mentioned of the mathematical point. It surprises me that the astute Locke could maintain just the opposite on this.

** Kant mentions it as noteworthy that trichotomy appears in his table of categories. However, in *my* view there is no true *classification* here because otherwise the categories could not be *simple, fundamental concepts* [*Stammbegriffe*].

B. On Axioms and Postulates

§ 10

In the usual mathematical textbooks and even in many logic books, it is said of axioms that 'they are propositions which on account of their intuitiveness (obviousness) require no proof, or whose truth is recognized as soon as their meaning is understood.' According to this, therefore, the distinguishing feature of an axiom would lie in its intuitiveness [*Anschaulichkeit*]. However, with some reflection it will easily be seen that this property is very little suited for providing a *firm basis for the classification* of all truths into two classes, that is, into axioms and theorems. For *firstly*, *intuitiveness* is one of those properties which allows of innumerable differences in its *degree*. It will therefore never be possible to determine precisely what degree of it should really be sufficient for an axiom. *Furthermore*, the intuitiveness of a truth depends on all sorts of *very fortuitous circumstances*, for instance, whether, by education or our own experience and the like, we have been led to recognize it often or only rarely. *Finally*, for this reason the degree of intuitiveness is also *very different* for different people; what for one person is perfectly obvious, often appears obscure to another. The greatest mathematicians, as we have already noted (§2), seem to have always dimly felt this, in that they also put the clear truths, provided they knew how to find a proof for them, into the class of theorems. *Euclid* and his predecessors proved what they *could* prove, and the notorious *parallel postulate*, together with some other propositions, were certainly only put among the so-called *κοινας ἐννοιας*[V] because they still did not know how to prove them.*

§ 11

Simply *not knowing how to prove* a truth does, indeed, make it into a *κοινη ἐννοια*, i.e. *common and naïve* [*ungelehrten*] *knowledge* (for thus knowledge is called which does not rest on clearly recognized grounds). It would also be good if such truths were set out, separated from all the rest at the beginning of the textbook so as to separate the proved from the unproved and to draw the attention of the student particularly to the latter. But this would then be only a purely *subjective* classification of propositions, made not for the *science in itself* but only for the benefit of *those working on it*. What one person puts forward today as a *κοινη ἐννοια*, another

* *Michelsen* puts forward the conjecture in his *Gedanken über den gegenwärtigen Zustand der Mathematik, usw.* that *Euclid's postulates* and *axioms* were originally nothing but certain aids to the memory, for beginners discovering solutions and proofs. It is indeed *possible* that *originally* this was their purpose, because they are certainly *very useful* for that goal. Yet in *Euclid's* time this suggested purpose was surely no longer in mind, or many other propositions would have been added to them, e.g. the one about the square on the hypotenuse, which appears in proofs at least as frequently as the 11th axiom. However that may be, this purpose of axioms would be anything but scientific and we do [not][W] imitate it.

[V] Common notions or axioms.

[W] The omitted 'not' must be a mistake from the whole sense of Bolzano's footnote.

would find a proof for tomorrow, and it would therefore be deleted from that *list of liabilities*. If therefore the word '*axiom*' is to be taken in an *objective* sense we must understand by it a *truth* which we *not only do not know how to prove but which is in itself unprovable*.

Note. Common notions can also be used in a systematic exposition for the proofs of other propositions, only if one is convinced that the as yet unknown proof of the first propositions may be provided without the assumption of just those propositions for whose proof one wants to use them. For example, in the theory of *triangles* all theorems *of the straight line* can be presupposed as *κοινας ἐννοιας*.

§ 12

Now another question arises: what should really be understood by the *proof* of a truth? One often calls any sequence of judgements and deductions [*Schlüsse*] by which the truth of a certain proposition is made generally *recognizable and clear*, a *proof of the proposition*. In this *widest* sense, *all* true propositions, of whatever kind they may be, can be proved. We must therefore take the word in a *narrower* sense and understand by the *scientific proof* of a truth the representation of the *objective dependence* of it on *other truths*, i.e. the derivation of it from those truths which must be considered as *the ground for it*—not fortuitously, but *actually and necessarily*—while the truth itself, in contrast, must be considered as their consequence [*Folge*]. Axioms are therefore *propositions* which in an *objective* respect can only ever be considered as *ground* and never as *consequence*. It is now high time to discuss here *how many simple, and essentially different kinds of deduction* [*Schlußarten*] *there are*, i.e. how many ways there are, in which one truth can be dependent on another truth. It is not without hesitation that I proceed to put forward my opinion, which differs so greatly from the usual one. Firstly concerning the *syllogism*, I believe there is only a *single*, simple form of this, namely *Barbara* or *Γραμματα* in the *first* figure. But I would like to modify it by putting the *minor* [premiss] before the *major* so that in this way the three concepts, S, M, P proceed in steps from the particular [*Speziellen*] to the general. I find it more natural to argue: *Caius is a man, All men are mortal, therefore Caius is mortal*, than the more usual order: *All men are mortal, Caius is a man, therefore Caius is mortal*. However these are small matters. Every other figure and form of the syllogism seems to me to be either not essentially different from *Barbara*, or not completely simple. But on the other hand, I believe that there are some *simple kinds of inference* [*Schlußarten*] apart from the syllogism. I want to point out briefly those which have occurred to me so far.

(a) If one has the two propositions:

A is (or *contains*) *B*, and

A is (or *contains*) *C*

then by a special kind of inference there follows from these the *third* [proposition]:

A is (or *contains*) [*B et C*].

This proposition is obviously different from the first two, each considered in itself, for it contains a different predicate. It is also not the same as their *sum*, for the latter is not a *single proposition* but a collection of *two*. Finally, it is also obvious that according to the necessary law [*Gesetz*] of our thinking the first two propositions can be considered as *ground* for the third and not, indeed, conversely.

(b) In the same way it can also be shown that from the two propositions,

A is (or *contains*) *M*, and

B is (or *contains*) *M*

the third proposition:

[*A et B*] *is* (or *contains*) *M*

follows by a simple inference.

(c) Again, it is another simple inference, which, from the two propositions

A is (or *contains*) *M*, and

(*A cum B*) *is possible* or *A can contain B*

derives the *third*:

(*A cum B*) *is* (or *contains*) *M*.

This inference has a great deal of similarity to the syllogism but is nevertheless to be distinguished from it. The syllogistic form would really arrange the second of the two premisses thus: (*A cum B*) *is* (or *contains*) *A*. But *this* proposition first needs, for its verification [*Bewährung*], the proposition, (*A cum B*) *is possible*. But if this is assumed, one can dispense with the first as being merely analytic, or take it in *such* a sense that, in fact, both mean the same thing and only differ verbally. All these kinds of inference, including the syllogism, have the common property that from *two* premisses they derive only *one* conclusion [*Folgerung*]. On the other hand, the following might look like an example of a kind of inference whereby *two* conclusions come from *one* premiss:

A is (or *contains*) [*B cum C*]

therefore

A is (or *contains*) *B*, and

A is (or *contains*) *C*.

But I do not believe that this is an *inference* in that sense of the word which we established at the beginning of this section. Having recognized the truth of the *first* of these three propositions I can, indeed, *recognize* subjectively the truth of the two others, but I cannot view the first *objectively* as the *ground* of the two others. I cannot let myself go into a detailed discussion of all these assertions here.

§ 13

Now the two questions arise: whether in general there *are* truths which are in themselves unprovable? and further, whether there are definite *characteristics* of this unprovability? Both must be answered affirmatively if there are to be axioms in the sense of the word given above (§11). Since some people still doubt whether there are any judgements in the realm of truth which absolutely cannot be proved, it seems to me worth the trouble of attempting a short proof of this claim here. Every provable judgement is to be viewed, according to the definition given in §12, as a *consequence* and its premisses taken together as its *ground*. Therefore to claim that all judgements are provable means to accept a *series of consequences* in which no *first* ground appears, i.e. no *ground* such that it is *not in turn* a consequence. But this is absurd. On the contrary, therefore, one must necessarily accept some judgements—at least two (§12)—which are themselves not consequences but basic judgements [*Grundurteile*] in the strictest sense of the word, i.e. *axioms*.

Note. As the contradictory nature of a *series of consequences without a first ground* is quite obvious with a *finite number* of terms, one could try to make it less noticeable by extending the series into *infinity*. However, it can easily be shown that this does not remove the contradiction at all. For this rests, not on the number of terms, but only *on the fact* that the denial of a *first ground* (according to our definition of this term given above) is the supposition of a *consequence* which has no *ground*. If it is assumed that the series stretches backwards to infinity, then it only follows from this, that whoever starts to count backwards from a given term never reaches that contradiction. However, although he does not immediately find it in *this* way, he must none the less think of it as *existing*. Refuting that notorious objection to the possibility of motion called the *argumentum Achilleum* is almost exactly the same. In the way which the authors of this objection deliberately make their attack, one can of course never arrive at the moment at which Achilles reaches the tortoise. But from this does not follow that this moment does not exist: it can actually be found very easily in another way. One may compare on this the well-known paper by *Cochius*: *Ob jede Folge einen Anfang habe* in *Hißmans Magazin*, Bd. 4.

§ 14

But how does one recognize that a proposition is unprovable? To answer this question properly we shall have to go back somewhat further, namely to the concept of a judgement and its different kinds. The usual definition of a *judgement*, as a *combination of two concepts*, is obviously too wide, because every *composite concept* is also a *combination of two* (or more) *concepts*. Of course, it is *one* combination by which we join two concepts into a new composite *concept* and *another one* through which we join two concepts into a *judgement*, but *both* combinations are in my opinion *simple, indefinable* performances of our mind. *Kant* certainly claims to have given us a *precise, distinct definition* of judgement (*Metaph. Anfangsgründe der Naturwissenschaft*, 3. Aufl., Leipz., S. XVIII, Vorr.; likewise in his *Logik*) as an *action through* *which given ideas first become cognitions of an object*. But provided I understand this definition correctly (by the *given ideas* I understand the *predicate*, and by the *object* the so-called *subject* of the judgement), then the whole concept of judgement here lies in the word '*cognition*'. And the expressions 'to consider an idea (namely the predicate) *as the cognition of an object* (otherwise called the subject)' or, 'to view an idea as the *criterion or characteristic of another one*' etc., are only different *descriptions*, but not proper *definitions*, i.e. not *decompositions* of the concept of judgement. This assertion is important to me because in the opposite case, if the concept of judgement were a composite one, then the concepts of the different *kinds* of judgement would also have to be composite concepts and we should therefore not simply *enumerate* them, as we shall now be doing, but would have to *classify* them according to a logical basis of classification.

§ 15

As far as I know logicians have hitherto always assumed that all judgements could be traced back to the form *A is B*, where *A* and *B* represent the two connected *concepts* and the little word '*is*' (called the *copula*) expresses the *way* in which the understanding [*Verstand*] *connects A* and *B* in the judgement. Now it seems to me that this *way of connecting* [*Verbindungsart*] the two concepts is not the same for all judgements and therefore it should not be denoted by the identical *word*. The most *essential difference between judgements* seems to me to lie in the variety of the kind of connection [*Verbindungsart*] and accordingly the following kinds of judgements have occurred to me so far.

1. Judgements which can be traced back to the form: *S is a kind of P*, or what amounts to the same, *S contains the concept P*, or, *the concept P belongs to the thing S*. The connecting concept in these judgements is the concept of the *belonging* of a certain *property*, or what is just the same, of the *inclusion* of a certain thing, as *individual* or *kind*, under a certain *genus*. This concept, although it is expressed here using *several words*, seems to me nevertheless to be a *simple one*, and if not identical to the concept of *necessity*, it is *preferably* included under it. Therefore so as to give them their own name I would like to call *this* class of judgements, *judgements of necessity*. An example of such a judgement of necessity is the proposition: *two lines which cut the arms of an angle in disproportional parts meet when sufficiently extended*. It is properly expressed thus: *the concept of two lines which cut the arms of an angle in two disproportional parts (= S), is a kind, of the concept of two lines which have a point in common (= P)*. Moreover, these judgements can be affirmative or negative, which also holds for the subsequent classes.
2. Judgements which *state a possibility* and are included in the form: *A can be a kind of B*. Their connecting concept is the concept of *possibility* so I call them *judgements of possibility*. An example is the proposition: *There are equilateral triangles*. It is properly expressed thus: *The concept of a triangle (= A), can be a kind, of the concept of an equal-sided figure (= B)*.

3. Judgements which express a *duty* and are included in the form: *You*, or generally *N*, *should do X*. The connecting concept here is that of *obligation* or *duty*, and the *subject N* is essentially a *free rational being*. These are called *practical judgements*.
4. Judgements which express some mere *existence*, without necessity, and can be included in the form: *I perceive X*. Here *perception* is taken in its widest sense in which one can perceive not only ideas through the senses [*sinnliche Vorstellungen*] but generally *all* one's ideas. Its essential subject is *I*. We call them *empirical judgements, judgements of perception* or *judgements of reality*.
5. Finally (it seems to me) *judgements of probability* also form a proper class of judgements whose connecting concept is that of *probability*. Yet I am still not clear about their real nature.

§ 16

The *most important matter* on which I differ from others with this enumeration (§15) consists in my putting certain concepts into the *copula* of the judgement which otherwise are put into the predicate or subject. I must therefore briefly indicate what caused me to make this change. It was principally the *judgements of possibility* and *duty*. I believe I have found that all judgements whose subject or predicate are composite concepts must be *provable* judgements (see below, §20). Now if *judgements of possibility* are expressed (by the usual method) so that the concept of possibility seems to form the *predicate*, then their subject would essentially be a composite concept, for it is well known to be superfluous to assert the possibility of a *simple* concept. (*A cum B*) *is possible*, would then be the general form of all judgements of this kind where (*A cum B*) would represent the subject, and *possible* the predicate. From the remark just made, therefore, all these judgements should be *provable*. Nevertheless it is easy to see that there must be some absolutely unprovable judgements of *this* kind, because every judgement of possibility, if it is to be proved, presupposes a premiss in which the concept of possibility is already present, i.e. *another* judgement of possibility. But if we take the concept of possibility into the copula there can be judgements of possibility whose subject and predicate are both completely simple concepts and which we can, therefore, without objection, allow as unprovable judgements. That is, if *A* and *B* are simple concepts then it is not unreasonable to assume that the *judgement* '*A can be B*' is unprovable, because its subject and predicate are simple. It is the same also with *practical judgements*, which, if one transfers the concept of *obligation* or *duty* to the predicate or subject they must always be composite judgements. And nevertheless there must be a first practical judgement (namely the highest moral law), which is absolutely unprovable.

Note. This much at least may be seen, that it is often not so easy to determine what really belongs to the subject and what to the predicate of a judgement. The appearance is deceptive. Thus, for example, in the proposition, '*In every isosceles* *triangle are the angles on the base-line equal*',[x] it would clearly be quite wrong to make the words before '*are*' the subject, and those following, the predicate. For here the subject would be repeated in the predicate, in that the concept *of the angles on the base-line* tacitly includes that *of the isosceles triangle*, because only those angles *which are opposite equal sides* and of which equality can be asserted *are called the angles on the base-line*. The proposition must therefore be: '*The concept of the relation of the two angles on the base-line in an isosceles triangle (= S), is a kind, of the concept of the equality of two angles (= P).*'

§ 17

A classification of judgements quite different from those considered so far, which has, since *Kant*, become particularly important, is the classification into *analytic* and *synthetic* judgements. In our so-called *necessity judgements*, §15 no. 1, the subject appears as a *species* [*Art*] whose genus is the predicate. But this relation of species to genus can be of two kinds: either there is a characteristic which can be thought of and stated in itself, which is added in thought as a *differentia specifica* to the *genus* (predicate *P*) to produce the *species* (subject *S*), or not. In the first case the judgement is called *analytic*; in every other case, which may be any of the classes mentioned in §15, it is called *synthetic*. In other words an *analytic* judgement is such that the predicate is contained directly, or indirectly, in the definition of the subject, and every other one is *synthetic*.

§ 18

From this definition it now follows immediately that analytic judgements can never be considered as *axioms*. Indeed, in my opinion they do not even deserve the name of *judgements*, but only that of *propositions*,[y] they teach us *something new* only as *propositions*, i.e. in so far as they are expressed in words, but not as *judgements*. In other words the *new* [fact] which *one can learn from them* never concerns *concepts and things in themselves* but at most only their *designations*. Therefore, they do not even deserve a place in a scientific system, and if they are used, it is only to recall the concept designated by a certain word, just as with conventions. In any case, it is decided even according to the usual views, that analytic judgements are not *axioms*, for their truth is not recognized from them themselves, but from the definition of the subject.

[x] The word order of the German has been artificially preserved in order to make sense of what follows.

[y] The German *Satz* could, appropriately, have been translated 'sentence' in this paragraph (among others) because of the emphasis here on the verbal expression of the proposition. However, throughout this work *Satz* has been translated 'proposition' because it is used variously and ambiguously to refer both to the meaning or content of a sentence and the syntactic sequence of words that constitutes the sentence. In the later *Wissenschaftslehre* Bolzano systematically uses *Satz an sich* and *Satz* respectively for these two meanings. For example, see *WL* §19.

§ 19

Therefore, if all our judgements were analytic there could also be no unprovable judgements, i.e. axioms, at all. And because there is still actually some support for this opinion we want to try and demonstrate, in a way independent of §15, that there actually are *synthetic judgements*.

1. *All judgements whose subject is a simple concept are thereby already also synthetic*. This is clear without further proof from the definition in §17. And no one will doubt that there actually are such judgements with simple subjects, unless they deny the existence of simple concepts themselves (§4). For it must be possible to form judgements about every concept, of whatever kind it may be, because for each concept every other [concept] must either belong, or not belong, to it as predicate.
2. *All negative* [*negative*] *judgements, if their subject is a positive concept, are likewise synthetic*. For if the subject is a *positive* concept, i.e. either completely simple or composed by pure *affirmation* from several simple [concepts], in which no negation appears, then it can never be proved from its mere definition that this predicate could not belong to it as subject. Now there really are such negative judgements, e.g. '*A point has no magnitude.*'

Note. There is an *essential* difference between positive and negative concepts which does not rest merely on the arbitrary *choice of words*. We have already defined *positive* [concepts]; *negative* concepts are those which contain some *negation* (i.e. an *exclusion*, not merely a *non-being*). But, of course it is not always immediately apparent from the *word* which designates a concept, whether it is affirming or negating.

Only if one attempts to define the concept and proceeds with this definition to the simple characteristics oneself, will it always be discovered whether or not it contains a negation. For example, the concept of a *right angle* is a positive one for it is the concept of *an angle which is equal to its adjacent angle*. On the other hand, the concept of an *oblique* [*schiefen*] *angle*, i.e. *a non-right angle*, is obviously negating, but the concepts of an *acute* and an *obtuse* angle are again positive etc. Among the negative concepts the *simplest* are those which arise from a mere negation or exclusion of a certain positive concept *A*, without thereby putting forward anything definite. They are of the form: '*Everything which is not A*'. I denote them (Π *sine A*) and call them *indeterminate or infinite concepts* (*terminos indefinitos*). They appear when the *inverse*[z] [*Umkehrung*] *of an affirmative proposition* is formed. For if one has the proposition, '*M is A*', for example, then it follows that: *Everything which is not A* = (Π *sine A*) *is also not M*. From this example it is seen immediately what is *really* intended by the expression, '*everything which is not A*'. Namely, the word '*everything*' is not taken *collectively* so as to designate the *totality of the*

[z] Bolzano clearly intends the contrapositive of a proposition, he sometimes makes use of the Latin form *propositio inversa* (e.g. in II §32). Confusingly *umgekehrt* is often used for the very different concept 'conversely'.

things which are not A, but *distributively*, i.e. so that one understands by it *this or that object which is undetermined except that it may not be A*. In so far as every concept must *put forward* something, the infinite concepts put forward something quite indeterminate, except that it may not be *A*, hence their name, *indeterminate concepts*. The judgements in which they appear as *predicates* are called *limiting* [*limitierende*] or *infinite judgements*. They are not different from *negating ones* in so far as the two are *equipollent*; but the former essentially appear as the *minor* in *negating* syllogisms. Only in *negating* judgements can such concepts occur as the *subject*. For nothing which is both affirmative and general can be predicated of *something which is not A*, except that it is *something that is not A*, which is a *purely identical* proposition. But one can certainly form negating propositions in which (Π *sine A*) is the subject. If one knows, for example, that *a is always A*, then one can *say*: (Π *sine A*) *is never a*. On the other hand, the other negative concepts which have some positive way of being determined (*M sine A*), can certainly also lead to affirmative judgements. For example, *if two numbers are not equal to each other then one of them is greater than the other*.

§ 20

After this preliminary discussion we can now answer the question we put in §14 in the following way:

(a) *Judgements whose subject is a composite concept, if they can be recognized as true a priori, are always provable propositions*. For it is clear that a composite [thing] and its properties must be dependent on those simple things of which it is composed, as well as on their properties. Therefore if any subject is a composite concept then its properties, i.e. the *predicates* which can be attributed to it, must depend on those individual concepts of which it is composed and on their properties, i.e. on those *judgements* which can be formed about these simple concepts. Thus every proposition whose subject is a composite concept is a proposition *dependent* on several other propositions and thus (in so far as it can be known *a priori*) it is actually also a *derivable*, i.e. a *provable proposition* and therefore can in no way be regarded as an *axiom*.

(b) *Judgements whose predicate is a composite concept, if they can be recognized as true a priori, are always provable propositions*. That a certain predicate belongs to a certain subject depends as much on the subject, as on the predicate and its properties. Now if the latter is a composite concept, then its properties depend on those individual concepts of which it is composed and on their properties, i.e. on those judgements which can be formed about these concepts. Therefore the truth of a judgement whose predicate is a composite concept depends on several other judgements and so, as before, it is clear that it cannot be an *axiom*.

(c) Hence it now follows that the really unprovable propositions, or axioms, are only to be sought in the class of those judgements in which both subject and predicate are completely simple concepts. And because in general, there *are* axioms (§13), they *must* be found there. The following, *third* theorem can provide us with a more precise determination. *For every simple concept there is at least one unprovable* *judgement in which it appears as subject*. For there are, in general, *judgements* about it. Therefore let '*A is B*' be such a judgement. If we now suppose it is *provable*, then *on account of its simple subject and predicate it can only be proved by a syllogism whose premisses are of the form: 'A is X' and 'X is B'*. If the premiss '*A is X*' should again be provable, then in the same way this presupposes another of the form '*X is Y*' etc. Therefore if one did not wish to concede that there was at least one axiom in the form '*A is M*', one would have to accept an infinite series of consequences without a first ground.

Note. In the proof of this last proposition the claim which is printed with emphasis [*durchschossen Lettern*] seems to need further elucidation. For the question could be raised as to why I restrict this claim only to propositions with a simple subject and predicate. For if it is *generally* valid, then *too much* follows from my proof, namely, that for *every* concept, even composite ones, there is an unprovable judgement in which it appears as predicate, which would contradict the *first theorem*. Therefore for the complete verification of this first claim *three things* will really be needed: (1) that propositions with simple concepts are provable only through syllogisms, (2) that these syllogisms always presuppose a premiss whose subject is one and the same simple concept *A*, and (3) that at least one of these two conditions holds exclusively of propositions with *simple* concepts—and not also of those with *composite concepts*. Now I have already mentioned in §12 that I do not regard the syllogism as the only simple kind of inference. Moreover from the example quoted there it may be seen that for propositions with composite concepts, but only for these, there are at least *three other* simple forms of inference. On the other hand, how propositions with *simple concepts* could be proved other than through a syllogism, I really do not know. I therefore assume that (1) and (3) are correct and this only leaves (2). It is quite clear to me that (2) holds at least for propositions with simple concepts, but whether it is a different matter for propositions with composite concepts, I shall not at present venture to decide. Of course the question arises here (one which is also interesting in another respect) whether every judgement can only be expressed in *one and the same* way, i.e. so that it retains the same subject and predicate. This is obvious for judgements with simple concepts; but for other kinds it might be thought that one and the same judgement could be transformed into a *new one* containing a quite different subject and predicate, without changing its sense, merely by taking some characteristics of the subject over to the predicate (or conversely). Now if this should be the case,* then a composite judgement could be proved by a series of syllogisms, without any of these syllogisms containing a premiss whose subject was the same unaltered subject of the judgement. This would then be a *second* reason why my above claim could also be extended, not against my will, to *composite* judgements.

§ 21

If by the foregoing we have now proved that every judgement which is to be an axiom must consist only of simple concepts, it is nevertheless not proved,

* But I readily admit that the *contrary* seems much more probable.

conversely, that every judgement which consists of simple concepts is an axiom. Therefore in order to prove that a given proposition '*A* is *B*' is an axiom, it is not enough to show that the concepts *A* and *B* are both completely simple; one must show further that there are no two propositions of the form '*A is X*' and '*X is B*' from which ['*A* is *B*'] could be inferred. In most cases this will require a *special consideration* [*eigene Betrachtung*], which, to distinguish it from a real *proof* (or a *demonstration*), I give the particular name of a *derivation* (or *deduction*).[a] Axioms are therefore not *proved*, but they are *deduced*, and these *deductions are an essential part of a scientific exposition* because without them one could never be certain whether those propositions which are used as axioms actually are axioms.

Note. If we consider that what we said about simple *concepts* in §8 also holds for *axioms*, then it will not be expected that all axioms should appear in our minds with perfect vividness. On the contrary, our clearest and most vivid judgements are obviously *inferred*. The proposition that *a curved line between two points is longer than the straight line between the same points* is far clearer and more intuitive than some of those from which it must laboriously be derived. The proposition (to give, for once, an example from another science), '*You should not lie*', is far clearer and more obvious than that principle from which it follows, '*You should further the common good*'. Indeed, it could even be that an axiom may appear *suspicious* and *doubtful*, particularly from a misunderstanding of its words, or because we do not immediately see that the things we recognize at once as true can be derived from it. (Thus some people find the merely identical proposition, '*Follow reason*', suspicious, because they understand it as though it nullified the obligation to obey divine commands or those of a legitimate authority.) In such cases the deduction of the axiom must first instil in us a confidence in its truth and this will happen if it proceeds from some generally accepted and unmistakably clear propositions which are however basically nothing but *consequences*, and even judgements *inferred* from that axiom which we wish to deduce. By making this connection apparent we will become convinced of the truth of the axiom itself.

§ 22

If the foregoing is correct, the question can now be answered, '*whether mathematics also has axioms?*' Of course, if all mathematical concepts were *definable* concepts, then there could be no axioms in the mathematical disciplines. But since there are *simple* concepts which belong properly to mathematics (§8), one certainly has to acknowledge actual *axioms* in it. The domain of the axioms stretches as far as that of the purely simple concepts: where the latter end and the *definitions* begin, there also the axioms come to an end and the *theorems* begin.*

* From this one sees how wrong it is to say, as the usual textbooks of mathematics do, '*the axioms follow the definitions.*'

[a] For the alternatives in parentheses in this sentence Bolzano used *Demonstration* and *Deduction*, with the verb form *deduciret* in the following sentence.

§ 23

Mathematicians are in the habit of introducing, as a special kind of axiom, *postulates*, by which they understand those axioms which assert the *possibility* of a certain object. According to §16 there are, and there must be, postulates (unprovable judgements of possibility). However, they occur, at most, only with concepts which are composed of *two* simple ones. The possibility of a concept composed of three or more simple components is a *provable* proposition. On the other hand, the possibility of *completely simple* concepts is really not even a judgement, for the predicate which the word 'possible', in the verbal expression of the proposition, only seems to represent, is missing. Possibility, just like impossibility, only occurs with composite concepts.

C. On Theorems, Corollaries, Consequences[b] and their Proofs

§ 24

It is well known that provable propositions are sometimes called *theorems*, and again sometimes *corollaries* or *consequences*. It seems that up till now mathematicians have not regarded it as worth their while to distinguish the former from the latter. What one of them puts forward as a *theorem*, another is to be seen treating as a *corollary* or as a *consequence*. However, I should think it would not be a disadvantage to the science if we were made clearly aware of what difference these various names were actually intended to indicate. Most mathematicians up till now, following some vague feeling, seem to have accorded the character of theorem to the *more noteworthy* propositions. But it is clear that this property is not very suitable for providing a reliable means of distinguishing propositions, since it is purely relative and extremely vague [*schwankend*]. I would therefore propose the following basis for the distinction. All theorems and corollaries have in common, that they are judgements inferred from previous propositions. But in order to be able to derive this consequence from the previous proposition, one may either require the assistance of *a proper axiom in this science* (or what amounts to the same thing, a proposition already inferred from such an axiom), or not. In the *first* case the inferred proposition may be called a *theorem*, in the *second*, i.e. if it arises from the previous proposition simply by means of an axiom of *another* science or simply by reference to an existing definition or something similar, then it may be called a *corollary* or *consequence*.

Note. According to this rule, for example, if it has first been proved that *the arithmetic square on the hypotenuse is equal to the sum of the squares on the other two sides*, then the proposition that *the side of a square is in the proportion* $= 1 : \sqrt{2}$ *to its diagonal* would be a mere corollary. The same is true for the proposition that *all irrational ratios of the form* $\sqrt{n} : \sqrt{m}$ *can be represented by the ratio of two*

[b] 'Consequence' here and in §§24,25 translates *Folgerung*, later (e.g. in §30) in the context of the ground-consequence relation it translates *Folge*.

straight lines, and similar propositions. For all that these propositions require, to be derived from that first one, are only particular applications of arithmetic, but no new geometrical axiom. In contrast, the proposition that *the area of the square on the hypotenuse equals the areas of the squares on the other two sides* would be a new *theorem*, because it can only be proved with the aid of a new geometrical axiom, or what amounts to the same thing, a new theorem proceeding from such an axiom, namely, the proposition that *the areas of similar figures are proportional to the arithmetic squares of their corresponding sides*.

§ 25

If one also wanted to adopt a difference between *corollaries* and *consequences*, then one could certainly retain the difference already actually made in the use of language. Namely, that it is usual to call *corollaries* (likewise, *additional propositions*) those judgements which are preceded by some true *proposition*, i.e. a *theorem*. By contrast, other propositions, e.g. those which follow directly from a definition, cannot, on this account, appropriately be called corollaries: they are therefore called mere *consequences*. We can thus combine this definition with the previous one: *consequences are those provable judgements which follow from mere definitions, but corollaries are those which follow from a preceding theorem without the help of any new axiom of this science.*

§ 26

From the above discussions several consequences can be derived about the *organization and succession of propositions* in a scientific system, and about the nature of their proofs. We still wish to mention some of these, even if they will contain nothing that is fundamentally new or that has not been said before, *because* they are still not being *followed* and it seems they are not sufficiently acknowledged as being *indispensable*.

1. *If several propositions appearing in a scientific system have the same predicate, then the proposition with the narrower subject must follow that with the wider subject, and not conversely*. For if the two judgements, *S is P*, and *Σ is P*, are valid where Σ is *narrower* than *S*, then either $\Sigma = (S \textit{ cum } s)$ or the proposition, *Σ contains S*, is valid. In the first case one knows that the proposition, *Σ is P*, is to be considered as a consequence of the two: *S is P*, and (*S cum s*) *is possible*, according to the third kind of inference in §12. In the second case the proposition, *Σ is P*, is to be viewed as a consequence, by a syllogism, of the two propositions, *Σ contains S* and *S is P*. Moreover, the truth of this assertion has always been felt and is expressed in the following phrase, '*in a scientific exposition one must always proceed from the general to the particular.*' For this means nothing else but that the proposition with a *narrower* subject must always follow the proposition with a wider subject.

Note. However, a proposition can sometimes be so expressed that it seems from its words to have a greater generality than a certain other one, without this in fact being the case. One must therefore not allow oneself to be mistaken on this account. The following proposition provides an example: *In every triangle bac*, $ab^2 = ac^2 + bc^2 \pm 2ac.cd$, *according as the perpendicular ad falls outside or inside cb*. This proposition seems to have a *wider subject* than the *well known theorem of Pythagoras* from which nevertheless it is only derived. Yet in fact it is otherwise. Notice first of all that there are basically *two* propositions present here which only seem to be put together as a single one. There are really *two* subjects as well as *two* predicates according as the perpendicular mentioned, falls outside or inside the side *cb*. Euclid was therefore quite correct in forming two *separate* theorems from it. Furthermore it must not be supposed that the *theorem of Pythagoras* can be considered as being contained within one of these two propositions. Because the *case* in which the perpendicular meets the vertex of the included angle is a quite *special* case, in which nothing can be said about the rectangle *ac.cd*. Thus the subjects of these three propositions are not subordinate to one another, but are co-ordinate concepts.

§ 27

2. *If several propositions appearing in a scientific system have the same subject then the proposition with the more composite predicate must follow that with the simpler predicate and not conversely*. For in the proposition, *S contains* (*P cum* Π) the proposition, *S contains P*, is presupposed in such a way that one definitely has to think of the latter before the former and not conversely (§12). This truth has forced itself particularly clearly upon those who have reflected upon the nature of scientific exposition, '*One must*' they said, '*always teach more, not less, in the later propositions than in the preceding ones.*' Moreover, it is obvious here that we cannot extend our assertion *further* and replace the expression, '*the proposition with the more composite predicate*', with the more general one, '*the proposition with the narrower predicate*'. For whenever we make an inference using a syllogism, one of the premisses (namely the so-called *major*), having just the same subject as the conclusion [*Conclusion*], has a predicate (namely the *terminus medius*) which is narrower than that of the conclusion. *S contains M*, *M contains P*, therefore *S contains P*. Where the concept *M* must obviously be narrower than *P* because otherwise the proposition, *M contains P*, could not be true, and yet the judgement, *S contains M*, must be considered as one which precedes the judgement, *S contains P*.

§ 28

As regards the *proofs* which must be supplied for all provable propositions in a scientific system, we shall content ourselves with mentioning just *two properties* which are required for their correctness as *conditio sine qua non*.

1. *If the subject* (or the hypothesis) *of a proposition is as wide as it can be so that the predicate* (or the thesis) *can be applied to it, then in any correct proof of this proposition all characteristics of the subject must be used, i.e. they must be applied in the derivation*

of the predicate, and if this does not happen the proof is incorrect. For in such a proposition the *whole subject* (not just one of its component concepts) is seen as the sufficient basis [*Grund*] for the presence of the predicate. But if one were not to *use* some characteristic of the subject in the proof at all, i.e. not to derive any consequences from it, then the predicate would appear independent of this one characteristic. Thus no longer the whole subject but only a part of it would be considered as the ground for the predicate. Therefore if some characteristic of the subject is not used in the proof this is a sure sign that either the theorem itself must be expressed too narrowly and contain superfluous restrictions, or if this is not the case, that the proof itself contains some hidden false conclusion. As the well-known maxim of the logicians says: *quod numium probat, nihil probat*.[c]

Note. Though simple and clear in itself, this observation has been often overlooked *in praxi*. Sometimes futile effort has been made to find a proof for a certain theorem without seeing how all the conditions present in the *hypothesis* will be used in this proof. At other times, proofs have been put forward, which, because they did not use all the conditions of the *hypothesis*, must obviously have been defective. An example of the *former* is the notorious *parallel postulate*, in which the intersection of the two lines clearly only holds under the condition that both lines lie in the same plane. However, very few people concerned with discovering a proof of this proposition have considered how this condition could be used in the proof. This would actually have caused them to seek it in quite a different way. An example of the *second* case is offered by *Kästner's theory of the lever* which is usually considered the best. After *Kästner* in his *Mathematischen Anfangsgründe*, Part II, 1st Sec. in 4th ed. in (16) has very correctly demonstrated the *theorem*: '*that equal lever arms and weights produce equilibrium*', he concludes in the corollary (18): '*Loads, which are not to fall, must be supported. There is nothing which could act as the support for A. Therefore this takes the complete load* $2P = 2Q$', i.e. (according to a definition of this phrase which he gives in (29)), '*if one wanted to hold this weighted lever on a thread AZ then one would have to pull along AZ with a force* $F = 2P$.' In this argument the condition does not appear that the two forces must act perpendicularly to the lever, or at least in *parallel directions*, a condition which is nevertheless necessary for the above conclusion. Therefore this proof is obviously false for it proves too much.

§ 29

2. As well as the characteristics of the subject several other *intermediate* concepts can also appear in the proof. However, if the proof is to contain nothing superfluous, then *for an affirmative proposition there should only appear intermediate concepts which are not narrower than the subject and not wider than the predicate. But for a negative proposition they should only be wider than the subject or wider than the predicate*. From §17 it may be seen that in all *a priori* judgements, at least in all judgements of necessity and possibility (such as all mathematical

[c] *Translation*: what proves too much, proves nothing.

judgements are), if they are affirmative, the predicate is a concept which, if not *wider*, is at least *as wide** as the subject. Now if such an affirmative judgement, e.g. *A contains B*, is to be proved then in addition two other similarly affirmative judgements are required as premisses. And by whichever of the four *simple forms of inference* mentioned above (§12) it can be proved, it is apparent that the intermediate concepts used are always narrower than A and wider than B. This will be seen without any trouble from the following table, in which the signs $>$ and $<$ mean *wider* and *narrower*, if one always makes use of the relationship which has been asserted between the extents of the two concepts of an affirmative judgement.

First Form

A is a kind of M
A is a kind of N

A is a kind of $(M \textit{ et } N) = B$

$$\left| \begin{array}{l} M > A \text{ and } < B \\ N > A \text{ and } < B \end{array} \right.$$

Second Form

M is a kind of B
N is a kind of B

$(M \textit{ et } N) = A$ is a kind of B

$$\left| \begin{array}{l} M > A \text{ and } < B \\ N > A \text{ and } < B \end{array} \right.$$

* This seems to contradict what *Selle* (*De la realité et de l'idéalité des objets de nos connaissances*, in the Mémoires de l'Academie de Berlin 1787, p. 601) claims to have discovered, that the real difference between analytic and synthetic judgements does not lie in what we said above in agreement with *Kant*, but only in the fact that with analytic judgements the predicate is contained in the subject, while in synthetic judgements the latter is contained in the former. Now the expression, 'a concept A is contained *in* another one B' is *ambiguous* in French just as it is in German and can mean either that A or B is the *narrower* of the two. But in each case this means that there are judgements in which the predicate is *narrower* than the subject, which to me at least seems absurd in the two classes of judgement mentioned. First, for *negative* judgements, the predicate is obviously neither narrower nor wider than the subject but each completely excludes the other. However, for *positive* judgements, if they are *particular* and *disjunctive* it might be possible for it to seem at first as if their subjects were wider than their predicates. Thus one could say that in the *particular* judgement: '*some quadrilaterals are squares*' the subject '*some quadrilaterals*' is obviously wider than the predicate '*squares*'. However, with more careful consideration one surely sees that the form of particular judgements is not even purely *a priori* but is *empirical*. For the fact that some quadrilaterals *are really* squares, is, when expressed *thus*, an empirical assertion. The only assertion which can be called purely *a priori* is: *the concept quadrilateral can contain the concept of a figure with only equal sides and angles*. This judgement belongs to the class of purely *a priori* judgements of possibility (§15) in which the predicate is obviously a wider concept than the subject. The *disjunctive judgements* are really all of the form: *A is either B or not B*. This judgement really amounts to: *the concept of the sum of* (*A cum B*) *and* (*A sine B*) *contains the concept of all A*. Now here the subject, '*the sum of* (*A cum B*) *and* (*A sine B*)' is again obviously a narrower concept, only a kind of the concept of the predicate, '*the totality of things which are A*'.

Third Form

M is a kind of B
M can be a kind of N

$(M \textit{ cum } N) = A$ is a kind of B
$|M > A$ and $< B$
$|N > A$ and $< B$

Fourth Form

A is a kind of M
M is a kind of B

A is a kind of B
$|M > A$ and $< B$

In a *negative* judgement subject and predicate are mutually exclusive, so one cannot say of either of the two that it is the *wider* or the *narrower*. But it is clear, as previously, that the *intermediate concepts* which are required in the proof of such negative judgements *A is not a kind of B*, must always be *wider* than the *subject A or wider than the predicate B*:

First Form

A is not a kind of M
A is not a kind of N

A is not a kind of $M \textit{ et } N = B$
$|M > B$
$|N > B$

Second Form

M is not a kind of B
N is not a kind of B

$(M \textit{ et } N) = A$ is not a kind of B
$|M > A$
$|N > A$

Third Form

Whatever is a kind of M is not a kind of B
M can be a kind of N

$(M \textit{ cum } N) = A$ is not a kind of B
$|M > A$
$|N > A$

Fourth Form

Whatever is a kind of M is not a kind of B
A is a kind of M

A is not a kind of B
$|M > A$

Note. Therefore if in a proof there appear *intermediate concepts* which are, for example, *narrower* than the subject, then the proof is obviously defective; it is what is otherwise usually called a $\mu\varepsilon\tau\alpha\beta\alpha\sigma\iota\varsigma$ $\dot{\varepsilon}\iota\varsigma$ $\dot{\alpha}\lambda\lambda o$ $\gamma\varepsilon\nu o\varsigma$. Sometimes, of course, it is clear right at the first glance that a certain concept, being completely alien, does not belong to the proof of a proposition. For example, in *Theorie des fonctions analytiques No.* 14,[d] the important assertion that the function

$$f(x+i)=f(x)+ip+i^2q+i^3r+\cdots$$

in general varies continuously with i, is derived from a *geometrical* consideration: namely from the *fact* that a continuous [*continuierlich*] curved line which cuts the abscissae-line[e] has no smallest ordinate. Moreover, we are in a real *circulus vitiosus* here, because only by assuming the purely arithmetic assertion about to be proved, can it be shown that every equation of the form $y=fx$ gives a continuous curved line. But in other cases it can only be decided whether or not some superfluous concept is involved, through a detailed analysis of the whole proof into its simple inferences, and by decomposing all its concepts into their simple components.

§ 30

We must give our opinion here about the question of *whether there can possibly be several proofs for one proposition*. It depends on what one counts as the *essence* of a proof. If one counts the *order of the propositions* (*which make use of certain premisses which may, or may not, be expressed in words*) as part of the essence of a proof, then a proof will be called a different one if the propositions in it merely follow each other in a different order, and some intermediate propositions introduced explicitly in one are omitted in the other. Then there is no doubt at all that there could be *several* proofs for the one proposition. On the other hand, if one regards what is essential in a proof as lying in those *judgements* upon which the [conclusion] to be proved rests, as a consequence does on its ground, (irrespective of whether these judgements are all explicitly stated, or whether some of them are merely tacitly assumed, or indeed whether they follow one another in this or that order) then for every true judgement there is only one *single* proof. For although in general not every consequence determines its ground and equal consequences can sometimes proceed from unequal grounds, yet it is a different matter with the *grounds of knowledge*. It is clear from the foregoing that the one or two intermediate concepts which are required for every simple inference are always *determined* and cannot arbitrarily be taken in *different* ways. To save space we shall show this, only for one of those kinds of inference, e.g. the syllogistic kind, in such a way that it will be seen that the same also holds for the others. Let $M, N, O, \ldots$ be *intermediate concepts* between A and B, i.e. they are $>A$ and $<B$, then one can use each of them for a syllogism from which the judgement, *A is a kind of B*, is to be inferred.

[d] Bolzano refers to Lagrange (1797).

[e] On this literal translation of *Abscissenlinie* see the footnote on p. 254.

That is, one can put forward the syllogisms, *A is a kind of M*, *M is a kind of B*, *therefore A is a kind of B*. In the same way, *A is a kind of N*, *N is a kind of B*, *therefore A is a kind of B*, etc. So far it may seem now as though there really were several proofs of the judgement, *A is a kind of B*. However, if between *A* and *M* there is some other intermediate concept *L*, then the judgement, *A is a kind of M*, is itself provable, consequently in a complete proof in which nothing is to be omitted, the syllogism, *A is a kind of L*, *L is a kind of M*, therefore *A is a kind of M*, must be put first. So in other words there must be as many syllogisms put forward as there are intermediate concepts between *A* and *B*. And since there is obviously only a single *definite* number of these intermediate concepts, then also the number and form of these syllogisms, and therefore the whole proof, is determined.

§ 31

From this it is obvious how one must judge the classification of proofs into *analytic* and *synthetic*. The whole difference between these two kinds of proof rests merely on the *order and sequence* of the propositions in the exposition. This is, according to the remark made by the admirable *Platner* (in his *Philosoph. Aphorismen*, 1.B., §554, 2. Aufl.) exactly like the distinction between the fourth and first syllogistic figures, in which the two premisses are simply interchanged. Now such distinctions certainly cannot establish an objectively valid or scientific classification, but nevertheless they are not to be dismissed completely. Unless I am mistaken, the real difference between theorems and problems rests on this distinction, and we shall say more about it presently.

§ 32

Finally we should give our opinion briefly about the *apagogic kind of proof*[f] since it is used so frequently in mathematics. First of all, though, we must put in a remark on the so called *inverse* [*Umkehrung*] *of a proposition*. If the affirmative judgement, *A is B* (or more definitely, *A is a kind of B*) holds then it is well known that there also holds a *propositio inversa*, the judgement, '*what is not B is also not A*' (or more definitely, *what is not a kind of B is also not a kind of A*). Usually the latter is considered as a kind of *consequence* from the *former*. But should the inverse proposition be derived from its affirmative counterpart by a *proper inference*, i.e. should it be so derived that the *former* would have to be considered as *ground*, and the *latter* as a *consequence*? Can the former not be derived from the latter just as well as the latter from the former, so that we are therefore equally justified in viewing the former as consequence and the latter as ground? I therefore think that *the inverse of a proposition* should in no way be counted in the class of *kinds of inference* of which we spoke above (§ 12).

[f] An apagogic proof is one by *reductio ad absurdum* (OED).

§ 33

Having said this, two particular kinds of apagogic proofs have been distinguished. The first kind is as follows. The proposition, *A is B*, is proved by assuming the contrary, *A is not B*, and deriving from it an impossibility, i.e. some contradiction with a proposition, *A is C*, which has already been proved. Now in this kind of proof the *essential thing* (I believe) does not lie in the false hypothesis, *A is not B*, but solely in the fact that premisses of the form, '*what is not M is also not N*' appear, so that in my opinion *only negating propositions essentially require an indirect proof—affirmative* propositions can always be proved *directly*. In its usual form, the apagogic proof is completed thus: '*A is B*, for *suppose A were not B then likewise A would not be C, which is absurd.*' This reasoning could also have been arranged in the following form: *whatever is C is always B, A is C, and so therefore A is also B*. Here no negating proposition appears at all. On the other hand, if the proposition to be proved were negating, e.g. *A is not B*, then one would proceed by the usual method thus: *suppose A were B, then it would follow that A is also C, which is absurd*. However, this can also be presented: *what is not C is also not B, A is not C, A is therefore also not B*.

§ 34

The second kind of indirect proof derives, from the *false* hypothesis, *A is not B*, the *true* proposition *A is B*. Now this was justifiably found objectionable until *Wolff* and *Lambert*, amongst others, showed that it is not really the false premiss, *A is not B*, on which one builds, but that at the beginning of the proof it is *just left undetermined* as to whether *A is a kind of B* or not. Therefore unless any other inverse proposition appears as an essential premiss in it, this kind of proof is not really apagogic.

D. On Problems, Solutions, Notes, etc.

§ 35

Problems with their *solutions* are a particular kind of mathematical proposition. Mathematicians' definitions of their real nature are still not fully in agreement. According to most *modern definitions* problems are those '*provable propositions which state the possibility of a concept*'. It is in their favour that *theorems* and *problems* are thus distinguished just like *axioms* and *postulates*. However, if we look at the *use* which the mathematicians *actually* always make of their problems, we find it does not correspond to this concept at all. For it is not only *propositions of possibility* that are presented under the title of problems, but also many *propositions of necessity*. For example, '*To find the third side of a triangle from two sides and the included angle*', in which there is no mention of any *possibility* at all. Similar propositions which do not state any possibility, also appear in *Euclid* under the title of *problems*, e.g. *Elements* Book II Prop. 14, Book III Prop. 1, Book VII Prop. 2, and several others. Conversely, on the other hand, several *propositions of possibility* are put forward under the title of *theorems*. Such is *Elements* Book I Prop. 7 Theor. 4: *It is impossible, on the same straight line, in the same plane and on one side, to draw two equal straight lines to more than one point*—a negative proposition of possibility. I am not against making the distinction, already established among *unprovable propositions*, between axioms and postulates and also in conformity with this, distinguishing the *provable propositions* into those which express a *necessity* and those which express a *possibility*, and devising suitable terminology for these. Indeed, there is a great deal to *recommend* this because the propositions of possibility are also to be proved from quite different grounds from the propositions of necessity: namely, the former from postulates, but the latter from axioms. However, I believe the distinction between *problems and theorems* to be yet *another distinction* and it is worth preserving, although it is not *objectively scientific* and only concerns the mere *manner of exposition*. Of course, provable propositions can be presented in a scientific system in *two* ways: either they are first just *stated* (i.e. first only their *sense* is made known) and the persuasion as to their truth and the representation of their objective connection [*Zusammenhang*] with other propositions is made to *follow*, or this is not done. The *first* way gives the form of the *theorem*, the *second* way that of the *problem*. The *theorem* therefore has *two components*: the *proposition* (*thesis*), in which the new judgement is merely *expressed*, and the *proof*, in which its objective connection with other truths is shown. However, the *problem* also has *two components*: the *question* (otherwise also called the *problem in the narrower* sense) in which one determines only the object about which one now intends to state something new, and the *solution*, in which the new truth which is sought is reached by proceeding from unprovable, or already proved, truths. This therefore shows that it is a misuse to provide the solution with its own *proof*, for thereby the problem reverts back again to the form of a *theorem*. Rather solution and proof should, through the use of the synthetic method, be amalgamated into one.

§ 36

Assuming these concepts, it can now be more precisely determined for which truths the form of a theorem, and for which that of a problem, is more suitable. Namely for those propositions whose statement *is unsurprising* and whose truth one can grasp immediately, even though dimly, from what went before, the form of the *theorem* is suitable. On the other hand, *for those which one would never have thought of*, the form of a *problem* is more appropriate. So for example, the proposition *that factors taken in a different order give one and the same product* is suited, by its nature, to being a *theorem*. On the other hand, the proposition about *how the highest common factor of two numbers is to be found* is always more appropriately presented as a *problem*.

Note. With regard to the place that is given to the problems, it should be noted that care must be taken to see the *possibility of solving them* with the preceding propositions. For example, before one puts forward the problem: '*to calculate* *the area of a triangle from its three sides*', one must have proved that this area is *determined* by those three sides.

§ 37

This also seems to be the right place to denounce a certain arbitrary *restriction* which mathematicians themselves have introduced, especially in geometry. Namely, *never* to accept an object as *real*, before having shown the *method* by which it can be constructed with certain instruments. It is well known that in *Euclidean* geometry no spatial object is accepted as real unless its construction has first been demonstrated by means of *plane, circle, and straight line*. This restriction betrays its *empirical* origin clearly enough. For *board, compass, and ruler* are the simplest instruments which were initially used for drawing. However, considered in themselves the *straight line*, the *circle* and finally also the *plane* are such composite objects that their possibility cannot be accepted in any way as a *postulate*. On the contrary it must first be proved from the possibility of just *those* things which *Euclid* teaches us to construct using those three. For example, the proposition that between every two points lies a *mid-point* is far simpler than the proposition that between every two points a *straight line* can be drawn. Nevertheless *Euclid* proves the former from the latter, amongst several others. For the *theoretical* exposition of mathematics (Part I, §18) it is sufficient to prove the *possibility* of every conceptual connection which is put forward. How, and in what way, an *object* analogous to the concept can be produced *in reality* belongs to *practical mathematics*. For example, it is enough to prove *that to every three straight lines there must be a fourth proportional line*; we do not need to show the way in which it can be *found*.

§ 38

Finally there is one more heading which is common in the *a priori* disciplines of mathematics, namely the *note*. This covers remarks which do not belong to the science in an objective respect, but have only a subjective purpose: e.g. historical remarks, explanations, other proofs, examples, applications, warnings of misunderstandings and similar things. But why have *reminders*, *introductions*, *transitions*, amongst other similar terms, not also been accepted into the mathematical apparatus since they are already common in other scientific expositions? Would mathematical exposition not thereby lose some of its rigidity and be able to be more flexible, illuminating and accessible? If everything that has been said so far is now gathered together, then the *complete mathematical apparatus* (to give a brief summary of it) would consist of the following parts:

1. *Designations* (for simple concepts).
2. *Definitions* (decompositions of composite concepts into their simple components).
3. *Conventions* (terms for simple as well as composite concepts).
4. *Classifications* (which always give composite concepts).

5. *Postulates* (unprovable propositions which state a possibility).
6. *Axioms* (unprovable propositions which state a *necessity*).
7. *Theorems* (provable propositions inferred from axioms of the same science which are either (a) *propositions of possibility* or (b) *propositions of necessity*).
8. *Consequences* (provable propositions which are consequences of definitions with the help of extraneous axioms).
9. *Corollaries* (provable propositions which are consequences of theorems with the help of extraneous axioms).
10. *Problems with their solutions* (provable propositions which are presented according to the synthetic method).
11. *Introductions, transitions, notes.*
12. *Common knowledge* (provable propositions of whose truth one is convinced, but whose systematic proof is not yet known).

Appendix On the Kantian Theory of the Construction of Concepts through Intuitions

§ 1

To have first correctly drawn attention to the important difference that exists between the analytic and synthetic parts of our knowledge remains an *achievement* which has to be attributed to *Kant*, even if we cannot justify, and agree with, everything that this philosopher has claimed about the intrinsic nature of our synthetic judgements. It is certain that the truth of analytic judgements rests on a quite *different* basis from that of the synthetic. If, in fact the *former* deserve the name of genuine *judgements* (which I do not admit without hesitation*), then they are all based on that one general proposition which is expressed by the formula: (*A cum B*) *is a kind of A*. If this is called the *law of identity* or of *contradiction* then it can always be said that the *law of contradiction is the common source of all analytic judgements*. However, it is entirely different with the *synthetic judgements*: these obviously cannot be derived from that axiom. *Kant* therefore raised the question, 'what is the basis *here* which makes us decide [*unsern Verstand bestimmt*] to attribute to a certain subject, a predicate which is certainly not contained in the *concept* (in the *definition*) of the former?' And he believed he had found that this basis could be nothing but an *intuition* which we connect with the concept of the subject and which at the same time contains the predicate. Accordingly, to all concepts about which we can form synthetic judgements there must correspond *intuitions*. But if these intuitions are always merely *empirical* then the judgements conveyed by means of them are also always *empirical*. Now since there are also *a priori* synthetic judgements (such as mathematics and pure natural science undeniably contain) there must also be, strange as it may sound, *a priori intuitions*. Once it has been decided that there can be such things, it is easy to be convinced, that with respect to mathematics and pure natural science, these are *time* and *space*.

§ 2

We may reasonably ask here what Kant understands by an *intuition*. From his *Logik* (ed. *Jäsche*), for example, and from many other places in his writings (e.g. *Kritik der reinen Vernunft*, S. 47 etc.) we obtain the answer: All ideas are either *intuitions*, i.e. ideas of an individual, or *concepts*, i.e. ideas of something general. Finally, if we ask what a *pure a priori intuition* is meant to be, then it seems to *me* at least, that no other answer is possible here than: *an intuition which is combined with the awareness of the necessity that it must be so and not otherwise*. For only if this awareness of the necessity is contained in the intuition can it also lie in

* See Part II, §18.

the connection, made by means of the intuition, between the subject and the predicate, i.e. in the *judgement*.

§ 3

It is well known that several people have already taken exception to these *a priori intuitions* of the *critical* philosophy. For my part, I readily admit that there has to be a certain *basis*, quite different from the law of contradiction, by which the understanding connects the predicate of a synthetic judgement with the concept of the subject. But how this basis can be, and be called, *intuition* (and even, with *a priori* judgements, *pure* intuition) I do not find clear. Indeed, if I am to be really honest, all this seems to me to rest on a distinction, which is not clearly enough thought out, between that which is called *empirical*, and that which is called *a priori* in our cognitions. The *Kritik der reinen Vernunft* indeed begins with this distinction, but it gives no proper *definition* of these things, and I already found this unsatisfactory on my first acquaintance with this book. Now how can this deficiency be remedied? Since the two concepts *empirical* and *a priori* are mutually contradictory, it would be enough to determine only one of them properly, e.g. that of the *empirical*, then the determination of the other would simply be given by the contrary propositions. What therefore do we properly call *empirical*? It is not desirable to give the answer, '*the empirical is what we obtain through the five senses—or through an external object.*' As philosophers we certainly cannot presuppose what the *five senses* are, nor that there are *external objects*.

§ 4

In my view, the distinction between the empirical and *a priori* in our cognition extends originally only to our *judgements*, and it is only through these that it can also be indirectly extended to our *concepts* or *ideas*. That is, I am conscious of making judgements of the form, '*I perceive X*'; I call these judgements *empirical judgements, judgements of perception* or *judgements of reality*, and the *X* in them I call an *intuition* or, if preferred, an *empirical idea*. The essential *copula* of all these judgements is the concept of *perceiving* which I consider to be a *simple*, and therefore *indefinable, concept*. But in order to *describe* it and to guard against misunderstanding one could perhaps say it is the concept of an *existence* [*Sein*], (a) of a simple, *pure* existence without necessity, (b) of an existence not of an *external object* as such, but only a simple *idea in me* (namely the *intuition**). Now the *rest* of my judgements, namely those which express (a) a *necessity*, (b) a *possibility*, or (c) an obligation (cf. §15 of Part II), I call *a priori*, and the concepts which appear in them as subject or predicate, I call *a priori concepts*.

* For it must first be *inferred* that an external object corresponds to the idea as its basis [*Grund*].

§ 5

By the *principle of sufficient reason* [*Satz vom Grunde*] I am, of course, bound to look for a certain *basis* [*Grund*] for all my judgements. This is a quite different one for the empirical judgements than for the *a priori* ones. The former, the so-called *judgements of reality*, have the characteristic, that I seek their basis in *that which is*, (in something real, in *things*), and indeed, according to circumstances, partly in that which I call, '*the particular nature of my perceptive faculty*', and partly in certain '*things different from me, i.e. external things*', which (as the phrase goes), '*affect my perceptive faculty*'. This is not the case with my *a priori judgements*, for which I can assume that the basis on which I attribute the predicate to the subject, cannot possibly lie anywhere other than in the *subject itself* (and in the characteristic nature of the predicate).* We have already done this above in Part II, §20. *Intuitions* are not useful and in my view they cannot be used for anything here: this will perhaps be made clearer with several examples by the following paragraphs.

§ 6

There is actually one kind of judgement, the so-called *judgements of experience* or *probability* (see Part II, §15), in which the connection of the predicate with the subject is in fact brought about *through intuitions*. For if I only have the judgements of perception, '*I perceive the intuitions X and Y, and indeed never X without Y*', then I derive from these, by means of the principle of sufficient reason, the *judgement of probability*, '*the thing which is the basis of the intuition X, is probably connected with the thing which is the basis of Y, like a cause with its effect.*' In my view all our so-called *judgements of experience* are of this form. If we say for example, *the sun warms the stone*, then basically this means nothing but: *the object* (sun), *which is the cause of the intuition X* (namely the shining disc of the sun), *is also the basis of the intuition Y* (namely that of a warm stone). But each of these judgements has, by its nature, only a *probability*.

§ 7

But how could judgements which are *absolutely certain*, such as all *a priori* judgements, result from the connection with intuitions?** It seems that *Kant* wants to say: 'If I connect the general concept, of a *point* or of a *direction*, or *distance*, for example, with an *intuition*, i.e. I imagine a *single* point, a *single* direction or

* If it is sometimes said that this basis lies in the *absolute necessity of the thing* or in the *special nature of our understanding*, then these are, I believe, *empty phrases* which, in the end, say no more than *it is so because it is so*.

** All *a priori* judgements are absolutely certain, but this is also true of empirical judgements. It is not only judgements of *necessity* which have absolute certainty, as one usually imagines, but also *judgements of possibility* and of *duty*—in short, all our judgements apart from those of which we have spoken in the previous paragraph. These accordingly deserve the characteristic name of *judgements of probability*.

distance, then I find that this or that predicate belongs to these single objects, and feel at the same time that this *is likewise the case* for *all other* objects which belong to this concept.' If this is the opinion held by *Kant* and his followers, then I now ask: but *how* do we come, from the intuition of that *single object*, to the *feeling, that what we observe in it also applies to every other one*? Is it through that which is *single* and individual in this object, or through that which is *general*? Obviously only through the latter, i.e. through the *concept*, not through the *intuition* (§2).

§ 8

How dubious the Kantian theory of intuition is, becomes particularly clear if it is extended to other propositions outside *geometry*. The principle of sufficient reason, and the majority of propositions of arithmetic are, according to *Kant*'s correct observation, *synthetic propositions*. But who cannot feel how contrived it is, that *Kant*, in order to carry through his theory of intuitions generally, has to assert that even *these* propositions are based on intuition, indeed (for what else should it be?) the *intuition of time*. Yet the *principle of sufficient reason* also holds where there is no time, and (according to a remark that has already often been made) it was only as a result of *this* proposition that *Kant* himself accepted the existence of the *noumena* which are not in time. The propositions of *arithmetic* do not require the intuition of time in any way. We shall only analyse a single example. *Kant* gave the proposition, $7 + 5 = 12$, instead of which, to make it easier, we shall take the shorter, $7 + 2 = 9$. The proof of this proposition is not difficult as soon as we assume the general proposition, $a + (b + c) = (a + b) + c$, i.e. that with an *arithmetic* sum one only looks at the *number* of terms not their *order* (certainly a wider concept than *sequence in time*). This proposition excludes the concept of *time* rather than presupposing it. But having accepted it, the proof of the above proposition can be carried out in the following way: the statements $1 + 1 = 2$, $7 + 1 = 8$, $8 + 1 = 9$ are mere *definitions* and *conventions*. Therefore, $7 + 2 = 7 + (1 + 1)$, *per def*.) $= (7 + 1) + 1$, (*per propos. praeced*.) $= 8 + 1$(*per def*.) $= 9$, (*per def*.).

§ 9

'However,' it will be said, 'it is true at least in *geometry*, that there are certain underlying intuitions. For in fact, however much we like to *think of* only the concept *point*, we already also have the *intuition* of a point in mind.' But this *image* accompanying our pure *concept* of the point, is of course not connected with it *essentially*, but only through the association of ideas, because we have often thought both of them together. Therefore also the nature of this image is certainly very different for different people and is determined by thousands of fortuitous circumstances. For example, someone who had always seen only rough and thickly drawn *lines*, or to whom straight lines had always been represented by chains or sticks, would have in mind, for the idea of a line, the image of a chain or a stick. With the word '*triangle*' one person always has in mind an *equilateral* triangle, another a *right-angled* triangle, a third perhaps an *obtuse-angled* triangle. I therefore do not understand at all how *Kant* has been able to find such a great difference between the intuition produced by some triangle which is actually *sketched* in front of us, and that produced by a triangle *constructed* only in the *imagination*, that he declares the former to be altogether superfluous and insufficient for the proof of an *a priori*, synthetic proposition, but the latter to be necessary and sufficient. According to my ideas it is of course *unavoidable* that when we think of some frequently seen spatial object, our imagination paints us a *picture* of the same thing. It is also *useful and convenient* that this image appears in our minds, as it makes the assessment of the object easier. But I do not regard it as being absolutely *necessary* for this assessment. There are actually even theorems in geometry for which we have no intuitions at all. The proposition that every straight line can be extended to infinity has no intuition behind it: the lines which our imagination can picture are not infinitely long. In *stereometry* we are often concerned with such complicated spatial objects, that even the most lively imagination is no longer able to imagine them clearly; but we none the less continue to calculate with our *concepts* and find truth.

§ 10

But if it is not *intuitions* which make the essential difference between mathematics and the other subjects, from where does the former derive its great *certainty and obviousness*? I answer that it is because one can very easily *test* the results [*Resultate*] of mathematics by *intuitions* and *experiences*. For instance, everyone tests by innumerable experiments, that the straight line really is the shortest one between two points, long before he can prove it with arguments. Also the renowned obviousness of mathematics gradually disappears where we lack the experience. In the same way, propositions which are inferred often have a far higher degree of intuitiveness than genuine axioms (cf. Part II, §21 Note).

§ 11

Would this therefore allow no distinction at all between those intuitions which *Kant* called *a priori* and the empirical ones? All objects must have a *form*, but they need not possess *colour*, *smell*, and the like. I would answer that not *all* objects which may *appear* to us must possess a *form*, but only those which appear to us as *external*, i.e. in *space*. But even these must then also have something which *occupies* [*erfüllt*] this form, and this, due to the particular nature of our perceptive faculties, can only be one of the following five things, either a *colour* or a *smell* etc. Therefore, colour, smell etc., are also *a priori* forms in the same sense of the word as space and time, except that the *range* to which the former relate is narrower than that of the latter, just as the form of *space* has a narrower *range* than that of *time*. Our conclusion is that among *concepts* there is no justifiable distinction according to which they could be divided into *empirical* and *a priori*: instead they are all *a priori*.

—Tu, si quae nosti rectius istis,
Candidus imperti! si non: his utere mecum.[g]

[g] '. . . If you know of a more correct solution, share it with me openly: otherwise go along with me in this one.' *Horace*, Epistles 1, 6, 67–8, Tr. H.R. Fairclough, Loeb Classical Library, Heinemann (1969).

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Early Analysis

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By the end of the eighteenth century the algebraic and infinitesimal methods that had been used to solve the geometry problems of the seventeenth century had proved so successful in a wide range of physical problems that, under the general heading 'analysis', they had become the dominant part of mathematics. The intuitive concepts of space, time, and motion, long associated with geometry and mechanics, were losing ground to the more formal manipulation of functions and equations by the methods of calculus. It was perhaps inevitable after the advent of analytic geometry that these developments should lead, during the eighteenth century, to analytical or 'rational' mechanics. In the *Preface* to his *Méchanique analytique* (1788) Lagrange wrote:

> No diagrams will be found in this work. The methods which I expound in it demand neither constructions nor geometrical or mechanical reasonings, but solely algebraic operations subject to a uniform and regular procedure. Those who like analysis will be pleased to see mechanics become a new branch of it . . . . (Translation as in Kline, 1972, p. 615)

The purely algebraic, or analytic, treatment of mechanics described in this quotation was evidence of the huge, wide-ranging transition taking place in the development of mathematics. This was not only the rise of analysis and its subsequent arithmetization. It was the wholesale replacement of empirical and intuitive elements in mathematics by more formal symbolic and arithmetic procedures. More fundamentally, arithmetic and its operations with symbols, was being freed from the need to be interpreted geometrically. The truth of Euclidean geometry was giving place to the truth of arithmetic. Geometry, however, did not decline, it developed in great varieties of ways (and in its turn was finally freed from the need to be empirically interpreted) over the period of the 'arithmetization' of analysis. The latter was a huge and haphazard process, occurring to no one suddenly and being advanced and retarded over many generations until there was some kind of stable consensus.

After the work of Euler and the Bernoullis it was the French mathematicians of the late eighteenth century who contributed most to the prominence of analysis. By around 1800, analysis was not simply a new and fruitful branch of mathematics, it had displaced geometry as the paradigm of mathematics. And this was understood, for example, by Laplace, in terms of its superior generality and degree of abstraction. In his *Exposition du système du monde* he writes:

> The algebraic analysis soon makes us forget the main object [of our researches] by focussing our attention on abstract combinations and it is only at the end that we return to the original objective. But in abandoning oneself to the operations of analysis, one is led by the generality of the method and the inestimable advantage of transforming the reasoning by mechanical procedures to results often inaccessible to geometry. (Translation as in Kline, 1972, p. 615)

Confidence in the future of analysis was reflected in the great number of new and widely read works appearing soon after the French revolution. These included Lagrange's *Théorie des Fonctions analytiques* (1797, German 1798, 2nd ed. 1810), and *Leçons sur le calcul des fonctions* (1806), *Traité élémentaire* (1802, 2nd ed. 1806, English 1816, German 1817), Carnot's *Réflexions sur la métaphysique du calcul* (1797, German and English 1800, 2nd ed. 1813).

The main purpose of these works, and others like them, was to spread and teach the methods and achievements of the calculus. In most standard textbooks one could expect to find all the elementary processes and rules of differentiation and integration for a wide range of functions, applications to finding maxima, minima, and singularities of curves and surfaces (also their rectification and quadrature), the solutions to many ordinary and partial differential equations, and perhaps Lagrange's calculus of variations.

There were numerous ways of introducing the concepts of differential and of derivative. They either involved infinitesimals explicitly, the use of a limit without a proper arithmetic definition (though L'Huilier came close to this), or the use of infinite series without a clear definition of convergence. None of these methods were regarded as wholly satisfactory though Lagrange's technique of assuming a Taylor series expansion for all functions enjoyed considerable, if short-lived, enthusiasm in the first decade or so of the nineteenth century. The works of Lacroix and Carnot mentioned above adopt a kind of amalgam of several of these methods. In spite of being a matter for concern, and numerous attempts to improve the relevant definitions, there was really little sense of 'crisis'. The foundations of analysis were not as significant to the mathematicians of the early nineteenth century as they sometimes seem to appear to modern eyes. Logical structure was secondary to truth. For although various peculiarities and paradoxes were known to arise from using infinitesimals and infinite series these could cast no doubt at all on the truth of the main body of analysis. That was guaranteed both by its overall coherence and its overwhelming success in applications. The foundations were desirable not so much for the sake of truth but in order to conform the subject to the newly developing ideal of mathematics as being independent of any intuitive appeal to such things as vanishing quantities, motion, or ideas borrowed from geometry.

Bolzano was fairly well acquainted with the mathematical literature of his time. In his papers of 1816 and 1817 there are over twenty references to important works on analysis which had been published within the previous twenty years. These included works by Lagrange, Lacroix, Gauss, and Crelle. He also knew at least some of the works of Newton, Euler, and D'Alembert. To a large extent then it was natural that he inherited the general views of the time on analysis that we have outlined so far. There are a number of specific remarks which show that this was true. In *BL Preface* the differential and integral calculus, are classified as higher analysis. Bolzano says he regards this subject as containing 'the most important discoveries in mathematics' (*DP Preface*). At the same place he explains the term 'purely analytic' as being equivalent to 'purely arithmetic, or algebraic' and says that a 'purely analytic procedure' is,

> . . . one by which a certain function is derived from one or more other functions through certain changes and combinations which are expressed by a rule completely independent of the nature of the quantities designated.

The example is given of forming $(1 + x)^n$ from $(1 + x)$. Thus far this seems quite a straightforward interpretation of 'analytic' in terms of algebraic operations. On closer inspection, however, there are some significant differences in Bolzano's view of analysis not only from a modern understanding but also from the views of his contemporaries. From a modern viewpoint what comes first logically in developing analysis—defining a domain of values such as the real number system—came about last of all historically. It is only after a proper, analytic definition of the number concept, as well as the continuity of a function, that we now regard the main result of *RB*, the intermediate value theorem, as a theorem of analysis. Bolzano regards the result, as was usual at the time, as part of 'the theory of equations' (*RB Preface*). Doubtless he would have regarded this theory of equations as part of analysis (the *Analysis der endlichen Größen*, the analysis of finite quantities in typical German classifications), but more because it is algebraic than because of an underlying limit concept. So in spite of the careful continuity definition (applied in *BL* and *RB*), the outlook and priorities are rather different from the viewpoint which emerged later in the century. It was regarded as more significant to find a way of solving an equation than to prove a property of continuous functions.

As for the way in which Bolzano's understanding of analysis differed from that of his contemporaries there are two main issues: the categorical rejection of the infinite in all its forms, and the very sharp separation, and removal, of geometric concepts from those of number. With regard to the infinite there was the infinitely great (e.g. value of a rational function whose denominator becomes zero, or perhaps just infinitely small), the infinitely small—notoriously something which was smaller than every conceivable quantity but was not zero—and there was the infinitely numerous, such as the collection of summands in an expression representing the sum of all natural numbers. Bolzano says that the usual definition of infinitesimal quantities as 'actually smaller than every . . . conceivable quantity' is 'contradictory' (*BL, DP Preface*). With specific reference to calculus Bolzano assumes 'it must be known to everyone' that the rules 'can be expressed in such a way that the concept of the *infinitely small* (which *otherwise* would surely be associated with the expressions dx, dy, dz, . . .) may be completely avoided (*DP Preface*). By this time this was probably not even a minority view. It was common to reject both infinitesimals and limit concepts. The full title of Lagrange's 1797 work was, *Théorie des fonctions analytiques contenant les principes du calcul différentiel, dégagés de toute considération d'infiniment petits, d'évanouissans, de limites et de fluxions, et réduits à l'analyse algébrique des quantités finies*. And a substantial work by Dubourguet from 1810, referred to in *DP*, is entitled, *Traités élémentaire du calcul* *différentiel et du calcul intégral, indépendans de toutes notions de quantités infinitésimal et de limites*.

Bolzano himself was not in the tradition of these works; he preserved a clear and essential limit concept (without, however, using the term), while rejecting the infinitely small. A glance at the opening pages of *BL* is enough to show how cautious and careful Bolzano was with regard to references to the infinite—at this stage he is at pains to reject them all. His treatment of the cases of the binomial theorem for negative and rational exponents are entirely in terms of strictly finite initial segments of the 'infinite series'—a term which he rejected and avoided in all his early analysis works.

The other distinctive feature of Bolzano's thinking on analysis is the extent to which he dissociated the subject from geometry. For someone like Isaac Barrow in the seventeenth century the early problems of calculus were geometrical problems and they were most appropriately solved by purely geometrical means. On the Continent during the eighteenth century the algebraic formulation of problems, aided by Leibniz's notation and the development of the function concept, changed from being merely a convenient description of a geometrical or mechanical situation to being an independent body of theory capable of geometrical or mechanical interpretation. A symbolism that began as a servant to geometry became, not its master, but independent and superior (in its generality) to geometry. To guarantee this independence it was therefore essential that analysis should borrow nothing from geometry unless it could be reformulated completely in arithmetic terms. Bolzano saw this very clearly, especially in terms of the generality of arithmetic and analysis. The quantities of geometry were 'spatial quantities' and accordingly theorems about them were only a special case of more general theorems.

The attitudes of late eighteenth century mathematicians to the relationship of analysis and geometry were various and often muddled. The intermediate value theorem of *RB* provides a good example. The theorem states that a continuous function of one real variable that is negative for one value of its argument and positive for another must be zero for some argument value in between these two. It was widely accepted and used. It was clearly 'true', but such clarity and truth was a product of the geometrical interpretation and did not extend to the general functional formulation of the theorem. Kästner and Gauss saw the need to give a properly analytic proof but many others did not and seem to have been quite content with the appeal to geometric intuition. He is therefore far from being conventional when Bolzano emphasizes repeatedly in *RB Preface* that the proof of the intermediate value theorem must not make use of concepts or methods borrowed from geometry. The very title of *RB* (*Purely Analytic Proof* . . .) reinforces this understanding of 'analytic', and in the first paragraph of the *Preface*, 'a purely analytic truth' is contrasted with 'a geometric consideration'. The implication is that it is necessary to the proper sense of 'analytic' that it implies 'non-geometric'. The arguments for this in *RB* are the matter of generality already mentioned (leading to logical circularity) and also the 'genus argument' against using concepts from one kind of theory in another. In fact, in consequence of Bolzano's regard for 'kinds' one might be left in doubt from reading *RB* alone whether he would actually endorse the application of algebra to geometry (or anything else), that is, whether he would allow analytical geometry as valid mathematics. Such doubt is dispelled by, 'the most general way of determining the nature of a spatial thing is to state certain equations between co-ordinates' (*DP Preface*). Nevertheless the tension in these early works between the principles of genus on the one hand and generality on the other, is never resolved. The genus principle invoked on several occasions with the phrase *μεταβασις ἐις ἀλλο γενος* [transition to another genus][a] seems to imply mutual exclusion of theories and their respective concepts. The generality principle implies a one-way relationship of inclusion, or of the application of the more general to the less general. With *DP* the generality principle is clearly the dominant one; the rectification problems are geometrical but they are being solved by the powerful, more general methods of analysis. However, the main conclusion to which we wish to draw attention here is that in both *RB* and *DP* various considerations led Bolzano to emphasize more than many of his contemporaries that analysis derived its meaning partly in contrast to geometry: being analytic implied being non-geometric.

One of Bolzano's continuing motivations in his mathematical work was the improvement of the foundational aspects of theories: the clarifying of concepts, and the provision of rigorous, appropriate proofs for the important theorems. It would naturally have been the foundational problems in analysis that most interested him and there was plenty of scope for his contributions. That he was well aware in 1810 of the continuing confusion in this area is clear from the following:

> I do not want to mention anything here about the defects in the higher algebra and the differential and integral calculus. It is well known that up till now there has not been any agreement on the concept of a differential. Only at the end of last year the Royal Jablanosvky Society of Sciences at Leipzig gave as their prize-question the ... discussion of different theories of the infinitesimal calculus and the decision as to which of these is preferable. (*BD Preface*).

That he says no more at this stage about the foundations of calculus is perhaps because there were already more problems than he could solve to his satisfaction in the more elementary parts of mathematics.

The programme started in *BD* of re-organizing mathematical theories (including the simplest ones of arithmetic and geometry) did not progress far. A part of the *zweyte Lieferung* (second issue) of *BD* was written but not published (until 1974 in *BGA* 2A5). There was no lack of enthusiasm on Bolzano's part, but to continue the work he clearly needed such enthusiasm or interest to be shown on the part of other mathematicians. Since this was not forthcoming in reviews or correspondence he decided to postpone the major work of *BD* and, as he candidly acknowledges, 'make myself better known to the learned world by publishing some papers which, by their titles, would be ... more suited to arouse attention'

[a] see footnote c on p. 32.

(*RB Preface*). He explains in the same passage that this also applies to *BL* and *DP*. There may have been some difficulty in finding publishers for these works—each had a different publisher—so in addition to obtaining criticism and interest in his work there may also have been a simple financial motive. At all events the topics of these analysis works were chosen for publicity purposes—to gain attention. In this respect they were not conspicuously more successful than his earlier efforts. There are few new results proved in these papers; their main purpose in each case is to give new proofs of well-known theorems that were essential to analysis. Bolzano regarded them each as being the first truly rigorous proof. This attention to the proofs of basic theorems was the result of fundamental conceptual requirements: the removal from analysis of ideas of infinity and infinitesimals, as well as the remnants of geometrical intuitions. These papers represent just a few examples of how Bolzano would like to reorganize analysis. He describes them as, 'a sample of a new way of developing analysis' (*BL Preface*). The aim of attracting some attention by means of these analysis works was eventually achieved, but long after Bolzano's death. The first important recognition of Bolzano's work was in connection with *BL* in Hankel's article *Grenze* (Limit) in Ersch and Grüber's *Allgemeine Encyklopädie* in 1871. Thereafter there are regular references to Bolzano's early work on analysis in the literature—often, however, confined to footnotes (e.g. there are more than a dozen such footnote references to *BL* and *RB* in the *Encyklopädie der mathematischen Wissenschaften* between 1898 and 1916).

However, the modern recognition of Bolzano's work raises a historical problem. From Hankel's article in 1871 to the extracts in Birkhoff (1973) commentators have been inclined to give particular credit to Bolzano for matters which at the time he saw in a very different light from these later critics. We are thinking here of the arithmetic concept of limit and the concept of the convergence of infinite series that are commonly adopted today. These concepts had been used in some form for a long time and judging from other examples in his writings, Bolzano would not have been too modest to claim them as new and original if he had regarded them as such. He does not do so. Undoubtedly he had great confidence in these definitions; they satisfied his conceptual requirements, he knew they would be fruitful and effective in the development of analysis, but never does he claim them to be his own. Nor could this be explained by his trying to avoid disapproval at a time when, at least in some quarters, the limit concept was simply unfashionable—he could hardly disguise the fact that its *use* was fundamental to his approach in all three of these works. Thus in judging Bolzano's work it is worth distinguishing carefully between insight into a concept and the definition or symbolic formulation of that concept that allows effective use in a theory. The effective use of a new concept requires the vision of an overall context or theory within which the new concept has clear connections with other already well-established concepts, and consequently clear connections with the existing problems of the theory. As we shall see Bolzano saw how he would put together the concepts of convergence and continuity from *BL* with the main result in *RB* (an early form of Bolzano–Weierstrass theorem) to prove the intermediate value theorem.

It is commonly assumed (Rusnock, 2000, p. 63 and Russ, 1980, p. 197) that following the introduction of quantities symbolized by ω, or Ω, possibly with subscripts, there is outlined in *BL* §14 ff. a fairly standard theory of limits. The irony is that Bolzano, along with most of his contemporaries, would have associated limits with infinite processes (or infinitely small quantities). And so he would, at this time, have been horrified to be associated with such a theory. Similar remarks apply to his work on the convergence of series. He believed he was treating the binomial series for negative and rational exponents in a purely finite manner. The way in which he used his ω quantities—variable quantities that can become smaller than any given quantity, or that can become as small as we please, naturally appealed implicitly to an infinite range of values. We might call them 'arbitrarily small quantities'. Rusnock suggests that such a concept of a variable quantity that can become as small as desired was common at the time. It is some sort of counterpart to a physical variable quantity. He suggests that Bolzano's ω's might be interpreted as *ranges* of values containing zero. This is in line with his later 'logic of variation' but the idea raises as many questions as it answers. The very concept of variable and the transition from physical variable quantities to logical variables or mathematical variables has been given surprisingly little attention by historians of mathematics. It is a subject ripe for further research and the way that Bolzano and his contemporaries were working with variables in analysis offers a useful way into the topic.

Although the work *RB* is often referred to as the highlight of Bolzano's early mathematics it is *BL* in which the concepts and methods essential to *RB* are first introduced, defined, and used. Two major problems facing any strict treatment, or proof, of the binomial theorem in 1816 were how to deal rigorously with infinite series and how to give a meaning to an irrational exponent. For much of the eighteenth century when the main use of the binomial series was the calculation of approximate numerical values for practical purposes, neither of these issues were recognized as problems at all. Bolzano's work identified and addressed both problems. He had two purposes in mind with *BL* apart from that of attracting some attention to his work. It was 'a sample of a new way of developing analysis' (*Preface*) and at the same time it was meant to be a new, substantial contribution to analysis in being the first strict proof of the binomial theorem and associated results. Consequently it had to cater for two kinds of readers. As a way of developing analysis it was to be accessible to beginners and suitable as a textbook, but as a thoroughly strict proof it was to be complete and rigorous. To cope with this Bolzano indicates near the end of the *Preface* many paragraphs that can be omitted on a first reading. He also points out that the main proof is not nearly so long as might be supposed from the total number of pages (it occupies only 20 of the 144 pages in the original). Apart from the style, which is far from terse, there are several reasons for the work's length. It includes proofs of the polynomial (or multinomial) theorem and the exponential and logarithmic series. There is explanation and motivation suitable for beginners. The concepts of convergence and differentiation which Bolzano regards either as novel, or as unsatisfactorily explained in earlier texts, are treated on each occasion by his elaborate method of working explicitly with his arbitrarily small quantities.

There are three main sections according to the topics mentioned in the title. The binomial theorem occupies §§1–51, the polynomial theorem is dealt with in §§52–59, and exponential and logarithmic series are the subject of §§60–74. Before the general binomial theorem is considered there is a long section (§§11–29) of general purpose preliminaries. These contain an example of his treatment of convergence of an infinite series (§12), the continuity of a function (§29), and a limit definition of derivative (§23). With his usual reticence about mentioning terms associated with infinitesimals he makes no mention of the term derivative or differentiation in *BL* (unless to cast aspersions on their validity or foundations). The important lemma §29 is intended to justify the term by term differentiation of convergent series. It is helpful to realize that by $\overset{r}{F}x$ Bolzano has in mind (in modern terminology) the difference between the partial sum of a convergent series and the limit function of that series. The lemma states that if the value of $\overset{r}{F}x$ becomes arbitrarily small with increasing r, and a given value of x, then so does its derivative $\overset{r}{f}x$. An important gap in the proof, acknowledged in a footnote (on p. 183), is the assumption of the intermediate value theorem which is the subject of the work *RB*.

The proof falls into two parts corresponding to uniqueness and existence. The first part for which key sections are §§30, 32, 33, shows that if there is a power series in x, the value of which for given x and n, becomes arbitrarily close to $(1 + x)^n$, then it must be the binomial series:

$$1 + nx + n.\frac{n-1}{2}x^2 + \cdots + n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}x^r.$$

For Bolzano the 'binomial series' is this finite series of arbitrary length. Then the 'binomial equation' holds if the value, for given x and n, of the binomial series is arbitrarily close to the value of $(1 + x)^n$. The necessary conditions found in this part of the proof serve to delimit the possible range of validity. The second part of the proof, for which key sections are §§38, 40, 41 shows that the binomial equation actually does hold for $|x| < 1$ and for positive or negative, and rational or irrational n. In spite of numerous small errors and the general long-windedness, the logical structure of this proof, and the treatment of convergence and continuity, put it above most of what had gone before as rigorous proofs of the theorem. More detailed commentary on the work of *BL* can be found in Russ (1980) and Rusnock (2000).

The structure of the work *RB* is simple. There are two key concepts and two main theorems. The concepts are those of the continuity of a function and that of the convergence of an infinite series. These are formulated more clearly and more generally than they appeared in *BL*. The continuity definition appears in the

Preface as:

> According to a *correct definition*, the expression *that a function fx varies according to the law of continuity for all values of x inside or outside certain limits* means only that, *if x is any such value the difference $f(x + \omega) - fx$ can be made smaller than any given quantity, provided ω can be taken as small as we please* or (in the notation we introduced in §14 of *Der binomische Lehrsatz* etc., Prague, 1816) $f(x + \omega) = fx + \Omega$.

The convergence criterion is given in terms of a series of partial sums:

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \ldots, \overset{n}{F}x, \ldots, \overset{n+r}{F}x, \ldots$$

The value of x here is constant: there is no concept of uniform convergence. The notation is simply following on from the context of §29 of *BL*. Then the crucial result on convergence is stated in §7:

> If a series of quantities
>
> $$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \ldots, \overset{n}{F}x, \ldots, \overset{n+r}{F}x, \ldots$$
>
> has the property that the difference between its nth term $\overset{n}{F}x$ and every later one $\overset{n+r}{F}x$, however far this latter term may be from the former, remains smaller than any given quantity if n has been taken large enough, then there is always a certain *constant quantity*, and indeed only *one*, which the terms of this series approach and to which they can come as near as we please if the series is continued far enough.

A sequence with the property described here usually bears the name of Cauchy. Four years after Bolzano published *RB* in Prague, Cauchy described the same property in his widely read *Cours d'Analyse*. There has been discussion in the literature on the possibility that Cauchy might have plagiarized from Bolzano. See Grattan-Guinness (1970), Freudenthal (1971) and Sinaceur (1973).

Bolzano then continued with an interesting proof of the result quoted. We may, following the analysis in Kitcher (1975), identify four steps:

(i) if such a constant quantity (X) exists, it can be determined as accurately as desired;
(ii) therefore, the assumption of such a quantity X 'contains no impossibility';
(iii) therefore, there is such a real quantity X; and
(iv) this quantity X is unique.

Bolzano provided clear, rigorous proofs of (i) and (iv). In, or between, steps (ii) and (iii), however, the proof seems doubtful to the the modern reader. It is worth reflecting on the account Bolzano himself gives in the proof of §7. Kitcher suggests that the thinking behind the claim (ii) may have been that if any contradiction could be derived from the possibility of X, the same contradiction could be derived from a sufficiently close approximation to X. Thus, he might have seen (i) as a kind of relative consistency proof of (ii).[b]

Many commentators (e.g. Steele 1950) say that, of course, there is a logical error in the whole proof due to the lack of a proper existence proof, and that Bolzano was doing the best he could in the absence of a definition of real numbers. We could argue almost the opposite, however. Rather than committing a logical error, Bolzano both argued for, and then appealed to, a profound insight into the nature of number, namely that a correct concept of number is one in which every sequence with the 'bunching' property of §7 has a limit. Using such a perception as an essential part of a proof bears comparison both with adopting the existence of such a limit as an axiom, and with introducing a definition through certain infinite sets (cuts or equivalence classes). Because we cannot hope to give *formal* proof of the consistency of such axioms or definitions, the consistency of their introduction is a belief grounded in the common intuition, or insight, which they capture. Bolzano may not have been the first to have such an insight into number, but he was probably the first to express it so usefully in terms of the property of §7. For further discussion of this proof see Kitcher (1975), pp. 247–51, Russ (1992), pp. 50–51, and Rusnock (2000), pp. 69–84.

The main theorem of *RB* is in §12 and could be called the 'greatest lower bound' property:

> If a property M does not apply to *all* values of a variable quantity x but does apply to *all* values *smaller* than a certain u, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.

If A is a non-empty sequence of real numbers and the property M signifies 'not being a member of A', then the theorem asserts that if A has a lower bound, it must have a greatest lower bound. The proof Bolzano gave is well organized, clear, and rigorous. Starting from the interval $[L, R]$ (where $L = u$ and $R = u + D$ for some positive quantity D) with the property that M applies to all $x < L$ but not to all $x < R$, it proceeds by repeated bisection to produce a sequence of nested intervals for which the endpoints maintain this property and, therefore, enclose the value U as claimed. Either U coincides with one of the points of bisection or there are infinitely many intervals. After comparison with a geometric progression with common ratio $\frac{1}{2}$, the convergence criterion is used to show that their endpoints converge to the value U. The greatest lower bound property is equivalent to what is often referred to as the Bolzano-Weierstrass theorem in a form such as 'a bounded sequence always contains convergent subsequences'. Many introductory textbooks on analysis ask students to accept some further (equivalent) result as an axiom about the real numbers, for example, 'An increasing sequence, bounded above, must be convergent.' For a discussion of these particular forms, and proofs of their equivalence, see, for example, Scott & Tims (1966).

[b] I am indebted to Dan Isaacson for first suggesting this idea.

An argument such as in the above summary must have been well known for a finite number of bisections in the context of solving equations numerically. It also resembles certain geometrical arguments in Euclid Books X and XII which hinge on an indefinite number of bisections. But in so far as Bolzano's application of the bisection argument is in the context of a process which may never end it appears to have been original.

Bolzano next (§15) proved a general case of the intermediate value theorem:

> If two functions of x, fx and ϕx, vary *according to the law of continuity* either for *all* values of x or for all those lying between α and β, and furthermore if $f\alpha < \phi\alpha$ and $f\beta > \phi\beta$, then there is always a certain value of x between α and β for which $fx = \phi x$.

The proof Bolzano gave of this result is an elegant application of his continuity definition and the earlier completeness result from §12. He summarized the whole procedure in the *Preface* in a succinct example of how to convey the 'bones' of a proof, before moving on to give the formal details. It could almost be given verbatim as an instructional model to undergraduate students today.

The material of *DP* is of a different character from the other two works of this section. The main themes are those of determination and similarity but both concepts, which in *BG* were prominent in a purely geometric guise, are now extended to analytic functions. In the *Preface* Bolzano explains that he rejects the usual derivations of the length of curved lines. In the first place he tries to improve on the usual presentation of deriving curved lengths from a limiting process from small straight lengths:

> For the *rectification* of lines the proposition: *The relation of the length of an arc curved according to the law of continuity* (whether simple or double) *to its chord comes as close as we please to the relation of equality if the arc itself is taken as small as we please.* (p. 286)

But he rejects this (and the corresponding result for complanation of surfaces) as much too compound to be adopted as an axiom. He rejects the 'method of limits' in which the area of a circle is approximated by inscribed and circumscribed polygons because the latter, rectilinear objects are 'alien' to the curved area required.

A long initial section on the determination of functions sees Bolzano for the first time entertaining infinite collections (see *DP* §3). This section is followed by a remarkable sequence of geometrical definitions to which we have already referred on p. 18. These definitions and commentary are interspersed with the rectification proofs in a way that leaves the connection between the two unclear. The material can be summarized as follows:

Preface. Review and criticism of previous attempts at the problem. Summary of method adopted here.

§§ 1–10. Various properties of continuous functions. The determination of functions from given sequences of functions using a process analagous to geometrical similarity.

§§11–12. Definitions of kinds of line and commentary.

§§13–31. Definitions of determinable spatial object, straight line, length, and distance. Application of similarity to general lines.

§§32–34. Solution of rectification problem.

§§35–37. Definitions of kinds of surface and the plane.

§§38–49. Properties of surfaces and the determination of areas.

§§50–51. Solution of the complanation[c] problem.

§§52–60. Definitions of kinds of solids, their properties, and the determination of their volumes.

§§61–62. Solution of the cubature problem.

Appendix. Critical comment on the contemporary work Crelle (1816).

The main steps of his approach to rectification are outlined in the *Preface* (pp. 329–31). The crucial idea is, '*that the lengths of similar lines are in proportion to the lengths of other lines determined from them in a similar way*'. So if Fx and Φx are lengths of lines with equations $y = fx$ and $y = \phi x$ respectively, we simply take $\phi x = \alpha + \beta x$ which is a straight line because we then know the form of Φx. By the principle mentioned Fx must have the same relationship to fx as that holding between Φx and ϕx. The principle used here, which Bolzano describes in §10 as a kind of 'higher rule of three' is discussed in greater detail in that section and also applied to the kinematics of a particle.

This whole work is of great importance to understanding better how Bolzano regarded the concept of determination between mathematical objects and also of how he regarded similarity and the potential of these two concepts to work together in definitions and proofs. As far as this author is aware, no proper study has yet been undertaken of these significant issues.

[c] Note the distinction between complanation and quadrature. *Complanation* comes from the verb *complanate*, meaning to make plane or level, to flatten (*OED* from 1643). Thus complanation is the reduction of a curved surface to an equivalent plane area. The *Philosophical Transactions* of 1695 speaks of *The Rectification and Complanation of Curve Lines and Surfaces*. This usage was still current in the early nineteenth century: the Edinburgh Review of 1816 speaks of *The cubature and complanation of solids*. *Quadrature*, by contrast, is strictly the reduction of a plane surface or figure with curved or polygonal boundary to a square; in Edmund Stone's *Mathematical Dictionary* of 1743, for example, *Quadrature of any Figure in the Mathematics, is finding a square equal to the area of it*. I am indebted to the late John Fauvel for this clarification.

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Der binomische Lehrsatz,

und als

Folgerung

aus ihm der

polynomische, und die Reihen,

die zur Berechnung der

Logarithmen und Exponentialgrößen

dienen,

genauer als bisher erwiesen

von

Bernard Bolzano,

Doctor der Philosophie, k. k. Professor der Religionswissenschaft an der Carl-Ferdinandischen Universität, und ordentl. Mitglied der k. Gesellschaft der Wissenschaften zu Prag.

Prag, 1816.

In der C. W. Enderschen Buchhandlung.

The

Binomial Theorem

and as a Consequence from it the

Polynomial Theorem and the Series

which serve for the Calculation of
Logarithmic and Exponential Quantities
proved more strictly than before

by

Bernard Bolzano

Doctor of Philosophy, Professor of Theology at the Karl Ferdinand University
and Ordinary Member of the Royal Society of Sciences at Prague

Prague, 1816
Published by C. W. Enders

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Preface

The *binomial theorem* is usually quite rightly considered as one of the most important theorems in the whole of analysis. For not only can the *polynomial theorem* and all the formulae used for the calculation of *logarithmic* and *exponential quantities* be derived from it by easy arguments, as will be shown here, but upon it also rests, as a further consequence, the widely applicable *Taylor's theorem* which can in no way be proved strictly, unless the binomial theorem is presupposed. It would hardly be an exaggeration to say that almost the whole of the so-called differential and integral calculus (higher analysis) rests on this theorem. However, the main theorems in every science have the property that it is usually their presentation in particular that is most beset by difficulties, and this is also the case with the binomial theorem. Therefore it is not surprising that the list of mathematicians who have tried to discover a completely strict proof of the theorem is already so long. *Newton's* immortal name stands at the top of the list. Following him—to mention only those known to *me*—*Colson*, *Horsley*, *Th. Simpson*, *Robertson*, *Sewell*, *Landen*, *Clairaut*, *Aepinus*, *Castillon*, *L'Huillier*, *Lagrange*, *Kästner*, *Euler*, *Segner*, *Scherfer*, *Klügel*, *Karsten*, *Busse*, *Pfaff*, *Rothe*, *Hindenburg*, *Kaussler*, *Schultz*, *Pasquich*, *Rösling*, *Jungius*, *Fischer* and *Krause*, *Crelle*, *Nordmann*, amongst many others.

However, grateful as we must be that so much really excellent work on our theorem has already been achieved through the efforts of all these men, there are certainly still a few things that they have left to be gleaned by later scholars [*Bearbeiter*]. I may be permitted to state here generally, and without mentioning anyone by name, what I find missing in previous presentations of this theorem.

I. First of all I think that the *meaning* of the theorem itself has been, I will not say incorrectly understood, but at least not very clearly presented.

1. Generally, if the exponent was not a whole positive number, the terms of the binomial series have been allowed to continue to *infinity*. Now I must remind you that every assumption of an infinite series, as far as I can see, is the assumption of a sum of infinitely many quantities, and every attempt to calculate its value is an attempt to calculate the infinite, a true *calculus infinitesimalis*. Therefore if one does not want to get involved in such things (and it seems as though most contemporary mathematicians actually have this very commendable intention, even if for no better reason than the many difficulties which this calculus contains) then one must also refrain altogether from the assumption and calculation of infinite series. I for one have kept to this rule, not only in this work, but also everywhere else.

Likewise, instead of the so-called *infinitely small quantities* I have always made use, with equal success, of the concept of those quantities, *which can become smaller than any given quantity*, or (as I sometimes call them to avoid monotony but less precisely) quantities, *which can become as small as desired*.[a] I hope no one will mistake the difference between quantities of this kind, and what is otherwise thought of under the name 'infinitely small'. The requirement of imagining a quantity (I am thinking of a variable quantity) which can always become smaller than it has already been taken, and generally can become smaller than any given quantity, really contains nothing that anyone could find objectionable. On the contrary, surely anyone must be able to see that there are very often such quantities, in space as well as in time? On the other hand, the idea of a quantity which cannot only be *assumed* to be smaller, but is really to *be* smaller than every quantity, not merely every *given* quantity but even every *alleged*, i.e. conceivable, quantity, is this not contradictory? Nevertheless this is the usual definition of the infinitely small.

2. It is equally common to present the binomial equation as an equation valid for *every value* of the exponent and for every kind of quantity consisting of two parts.[b] Nevertheless it is certain that this equation really only holds precisely if the exponent is a whole positive number. In every other case, if one does not want to accept any infinite series, its validity can only be maintained in the sense that the difference between the two sides,[c] i.e.

$$(1+x)^n - 1 - nx - n.\frac{n-1}{2}x^2 - \cdots - n.\frac{n-1}{2}\cdots\frac{n-r+1}{r}x^r$$

can be made smaller than any given quantity, if the number of terms in the series is taken large enough, but even this is only true if $x < \pm 1$.[d] What am I to say if the equation is applied, without the least scruple, to the case where the two-part quantity is negative and the exponent is of the form $\frac{2n+1}{2m}$, e.g. if it was supposed that

$$\sqrt{-1} = (1-2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}.2 - \tfrac{1}{2}.\tfrac{1}{4}.4 - \tfrac{1}{2}.\tfrac{1}{4}.\tfrac{3}{6}.8 - \tfrac{1}{2}.\tfrac{1}{4}.\tfrac{3}{6}.\tfrac{5}{8}.16 - \cdots ?$$

---

[a] The frequently occurring phrase *so klein [nähe], als man nur immer will* is literally 'as small [close], as ever one wishes'. It has been rendered 'as small [close] as desired' or 'as small [close] as we please'. For discussion on the nature of these quantities see p. 147. An example using *nähe* [close] occurs just before section II of this *Preface* on the following page.

[b] The German *für jede Beschaffenheit der zweiteiligen Größe*, literally, for every nature of the two-part quantity.

[c] The German *Glied* is used to mean both a term in an equation (or series) and also, as here, the complete side of an equation.

[d] It is clear from his usage that, for example, by $x < \pm 1$ Bolzano meant what we would write as $|x| < 1$. There was no standard symbol at this time for absolute value or modulus. According to Cajori (1929), §492, the symbol of vertical bars around the argument was introduced in 1841 by Weierstrass. It was not unusual, however, to refer in words to 'absolute value' although Bolzano does not do so until around 1830 (e.g. *RZ*, §42). Bolzano interprets an inequality $x > c$, where c is a numerical constant, as follows: x takes values with the same sign as c, and in the range such that $|x| > |c|$; similarly for $x < c$. This can be confusing: $x > -1$ in Bolzano's usage is what we mean today by $x < -1$.

3. In fact the inadmissability of the equation for values of $x > \pm 1$ has been agreed to be much too clear for it not to be acknowledged by everyone, at least afterwards and half-heartedly. But why only afterwards and not right at the start? And even then, why not always in the appropriate way? For it is usually said only that 'the equation does not hold' (or using an even vaguer expression, 'it is not suitable *for calculation*'), 'if the terms of the series *do not converge*'. Are we not tacitly given to understand by this, that if the terms converge then it will always be correct? However, this is also not always so, for if $x = 1$ and n is positive, then the terms do always decrease starting from a certain term: but however far this may be continued the value of the series never comes as close as desired to the value of $(1 + x)^n = 2^n$.

II. If the meaning of a proposition has not been presented with complete precision, then it may be anticipated that the *proofs* of it will also be more or less unsuccessful. And all previous proofs of the binomial theorem must actually be defective, *because they prove too much*. For, as stated, this proposition holds, for a fractional or negative exponent only if $x < \pm 1$. But in which of the previous proofs is even the slightest regard given to this indispensable condition? What good is it to prohibit the application of the proposition in cases where $x =$ or $> \pm 1$ only afterwards, if it cannot be seen from the proof itself why it does not also hold in these cases? Can proofs possibly be correct if they do not use all the conditions on which the truth of a proposition rests? Thus it is already decided at the outset that there must be some defect to be found in each of the previous proofs. But we want to point out briefly only the commonest and most important [defects].

1. The *most general [fault]* (which occurs not only in the proof of the binomial theorem but also elsewhere) is to allow the correctness of a given equation to be regarded as sufficiently proved if it has been shown that its first, second and generally the rth terms (where r can become as large as desired), are equal on both sides. [This is done] without investigating whether there may perhaps be, beyond this rth term, another one or more terms which are unequal to each other, and whose difference can perhaps never be reduced as much as desired. This latter [condition] must necessarily be proved if one is to be able to say of the two expressions concerned, that they are, in some sense of the word, *equal* to each other. Only from the idea that there are *infinite* series, in which consequently there is no *last* term, is it understandable how such a clear obligation could have been renounced. But the consequence was, that to one's own great astonishment, one found oneself entangled in various completely absurd assertions. For example, the binomial formula stated that

$$\frac{1}{11} = (1 + 10)^{-1} = 1 - 10 + 100 - 1000 + \cdots \textit{in inf.},$$

an assertion which is an affront to common sense. If the series had always been assumed only to be finite, as it should have been, then it would have been immediately apparent that the binomial series:

$$1 - x + x^2 - x^3 + x^4 - \cdots \pm x^r$$

can be put *equal* only in a certain sense, to the true value of

$$(1 + x)^{-1} = \frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \cdots \pm \frac{x^r}{1 + x},$$

if $x^r - \frac{x^r}{1+x}$ can become as small as desired by increasing r, i.e. if $x < \pm 1$.

2. All those who wanted to prove the binomial theorem by proceeding from the general form,

$$(1 + x)^n = A + Bx^\beta + Cx^\gamma + \cdots$$

sought to determine by inferences [*Schlüsse*] the nature of the quantities $\beta, \gamma, \ldots, A, B, C, \ldots$.

(a) A common mistake of which they were guilty is contained in the following kind of inference:

> 'If
>
> $$(1 + x)^n = A + Bx^\beta + Cx^\gamma + \cdots \qquad \odot$$
>
> then also
>
> $$n(A + Bx^\beta + Cx^\gamma + \cdots) = (1 + x)(\beta Bx^{\beta-1} + \gamma Cx^{\gamma-1} + \cdots) \qquad ♁$$
>
> Also conversely therefore, if one determines $\beta, \gamma, \ldots, A, B, C, \ldots$ in such a way that the equation ♁ holds, then the equation ⊙ must also hold.'

Clearly this is a wrong conclusion since no logic justifies us deducing the affirmative and general converse of an affirmative proposition without proof. But if this kind of proof is analysed more carefully it is basically found to contain *two* such unauthorized—here actually *false*—converses. The *first* occurs in the proposition:

> 'If
>
> $$(1 + x)^n = A + Bx^\beta + Cx^\gamma + \cdots \qquad \odot$$
>
> then also
>
> $$n(1 + x)^{n-1} = \beta Bx^{\beta-1} + \gamma Cx^{\gamma-1} + \cdots \qquad ☾\text{'}$$

From which it is inferred by the converse,

> 'Therefore if with a certain determination of the quantities $\beta, \gamma, \ldots, A, B, C, \ldots$ the equation ☾ holds, then ⊙ also holds.'

This is clearly a false assertion, since from ☾ it is known only that

$$(1 + x)^n = \text{Const.} + Bx^\beta + Cx^\gamma + \cdots$$

follows, where Const. is arbitrary. However, then the *second* proposition is formulated:

> 'If
>
> $$n(1 + x)^{n-1} = \beta Bx^{\beta-1} + \gamma Cx^{\gamma-1} + \cdots \qquad ☾$$

and at the same time,

$$(1 + x)^n = A + Bx^\beta + Cx^\gamma + \cdots \qquad \odot$$

then it must also be that

$$n(A + Bx^\beta + Cx^\gamma + \cdots) = (1 + x)(\beta Bx^{\beta-1} + \gamma Cx^{\gamma-1} + \cdots) \qquad \text{♁.'}$$

The converse of this proposition is also formed and it is concluded, 'Therefore if equation ♁ holds, then ⊙ also holds.' Now this obviously makes the proof all the more uncertain. Nevertheless I do not know of even a single mathematician whose proof of the binomial theorem (if he did it this way) has gone further than the determination of the quantities $\beta, \gamma, \ldots, A, B, C, \ldots$ from equation ♁. They all believe that here their task is completed although it has only been proved that a series which is to correspond to that of ⊙ must have the form

$$A + nAx + n.\frac{n-1}{2}.Ax^2 + \cdots$$

But whether every series of this form, or whether even only one of them, and for what kinds of quantities A, n, and x themselves the equation ⊙ is actually satisfied—these things must still be decided on completely different grounds.

(b) Furthermore, the way the equation ♁ is usually derived from the equation ⊙ is never satisfactory.

(α) Most people make use of the differential calculus for this, i.e. a calculus which, as it has so far been presented, is still based on the shakiest foundations; for example, on the self-contradictory concepts of infinitely small quantities, and on the assumption that even zeros can have a ratio to one another. The logician justifiably remarks: *non entis nullae sunt affectiones*.[e]

(β) Other people artificially imitate the procedure of the differential calculus without mentioning it by name, i.e. they avoid the word without removing the difficulties from the method. Quantities which are at first used as *divisors* are eventually allowed to denote *zero*; in my opinion this can never be allowed since it is certainly possible to divide by every finite (i.e. actual) quantity, but never by a zero (i.e. by nothing). If dividing is in fact nothing but seeking that quantity which, when multiplied by the divisor, gives the dividend, then obviously no quantity can be conceived of which, when multiplied by zero, i.e. taken no times, gives an actual something.

(γ) Yet others want to introduce into mathematics a special axiom for the derivation of the equation ♁ or else a condition equivalent to it; namely, '*that which is consistent with all truths, is itself also true*'. Although one may be inclined to readily admit this proposition, I very much doubt the possibility of using it as a sound

[e] *Translation:* nothingness [or, non-being] has no properties. I am indebted to Mariano Artigas and Ignatius Angelelli for expert advice on this translation. The phrase is used more than once by Bolzano in mathematical contexts and in numerous forms it has a rich history. See I. Angelelli, 'The interpretations of "nihili nullae sunt proprietates"' in Imaguire (2004).

criterion for the discovery of new truths in any science. Not everything true is known to us, so simply from the fact that a given proposition does not contradict the truths known to us, how are we ever supposed to be able to conclude with certainty that it does not also contradict any others? At the least one could never produce in this way the *objective ground* of a truth, which must be done for its proper representation in the science.

(c) Finally there is something else reprehensible in this method. The exponents of x in the series $A + Bx^\beta + Cx^\gamma + \cdots$, which is supposed to be $= (1 + x)^n$, are assumed, without sufficient proof [*Beweisgrund*], to be $= 1, 2, 3, \ldots$. Indeed the analogy of the powers $(1 + x)^2, (1 + x)^3, \ldots$ is suggested as a justification for this assumption. But what if someone had concluded from the very same analogy with $(1 + x)^{-2}, (1 + x)^{-3}, \ldots$ that the exponents had to be *negative*?

3. A *second* method of proof for the binomial theorem is based on Taylor's theorem. This procedure has,

(a) already against it, the fact that Taylor's theorem is a much more difficult theorem and, at least as far as I understand, can only be strictly proved by assuming the binomial theorem.

(b) Further on this point, to apply Taylor's theorem to the binomial theorem, the first term of the development of $(1+x)^n - 1$ must already be known. Now in order to prove that this term is nx, even the strictest mathematicians assume, without proof, that it is of the form $x.f(n)$. I say 'without proof' because this cannot be deduced from Taylor's theorem which states that the development of $f(y + \omega)$ must, in general, have the form $fy + \omega f'y + \cdots$. It must never be forgotten that that development only holds in general, i.e. indeed for infinitely many, but not for *all* values of y. Now since y must be given a *definite* value here, namely 1, the question arises whether this may not be just one of those values for which that development does not apply? I am of course, aware of the clever method by which one scholar has recently sought to remove this difficulty. But he works from the proposition, '*that no term of a series can contain the quantity x if the value of this series does not itself depend on x*'. This assertion is shown to be incorrect by the single example that for every integer value of x it is well known that,

$$\begin{aligned}\sin(2x\pi + \phi) &= (2x\pi + \phi) - \frac{(2x\pi + \phi)^3}{1.2.3} + \frac{(2x\pi + \phi)^5}{1.2.3.4.5} - \cdots \\ &= \sin\phi\end{aligned}$$

therefore it is certainly not dependent on x although this x appears in every term!

4. The attempt to determine the form of the binomial series when the exponent is fractional or negative, by considering that $(a + b)^{\frac{n}{m}}$ is some term interpolated in the geometric series,

$$\ldots, (a + b)^{-2}, (a + b)^{-1}, (a + b)^0, (a + b)^1, (a + b)^2, \ldots$$

already betrays its incompleteness, because in this way it is not shown at all how far the series for $(a+b)^{\frac{n}{m}}$ must be continued, and whether it holds generally, or only if $\frac{a}{b} < \pm 1$.

5. The most thorough procedure is surely that adopted by those who proved the validity of the binomial equation, first for whole positive exponents, and then sought to demonstrate, by considering the nature of those products whose factors are of the form,

$$1 + nx + n.\frac{n-1}{2}.x^2 + \cdots = {}^{n}S,$$

that that equation also holds for fractional and negative exponents. In fact after one had learned to avoid here the hastiness which would deduce the equal *form* of two series directly from their equally great *values*, proofs arose on this basis, of which (as well as their somewhat irksome length) nothing could be criticized if they were not based on the inadmissable idea of infinite series. If one wants to avoid this idea, as is proper, then several of the arguments employed there no longer hold. Series are obtained which are indeed equal to one another from their first term up to arbitrarily many terms, but then they have just as many unequal terms, so that in order to claim the equality of their values it becomes necessary to show that the sum of the unequal terms can become smaller than any given quantity, when the number of equal terms is made large enough.

Similar mistakes to those we have just condemned in the previous proofs of the binomial theorem are also made in the derivation of those formulae which serve for the calculation of the *exponential quantities* and *logarithms*. In general the series which are supposed to express the values of a^x and log y are assumed to continue to infinity; moreover, use is made of the idea of infinitely small quantities, or quantities which are initially considered as real [*wirklich*] and have been used as divisors, are allowed in the course of the calculation to take the value zero, etc.

In the present work—as a sample of a *new way of developing analysis*—the theorems mentioned in the title are presented in such a way that, I hope, not only have the mistakes just condemned been avoided, but also several other blunders contrary to good method which are frequently allowed. As far as *method* is concerned I have generally kept to the principles set out in the *Beyträge zu einer begründeteren Darstellung der Mathematik, First Issue,* 1810, *Second Part,* because I am still convinced of their correctness. None the less (and this I must say to avoid any misunderstanding) the fragmentary form of the present work does not allow those principles to be followed in *all* its parts. Otherwise the exposition would have to have begun straight away with the first concepts and theorems of arithmetic, since in a complete system even these would have to be presented considerably differently from how it has been until now; understandably this also has some influence on the presentation of the subsequent material. All references in this work to concepts and theories not in current use have been avoided, and for this reason I hope it is sufficiently comprehensible that it might be useful not only as an introduction for public instruction but also be understandable to beginners, who are not completely inexperienced, even without the assistance of a teacher. In this respect it will nevertheless not be superfluous to note which sections of this work can be passed over—according to circumstances—either completely or at first reading. If one merely wishes to be convinced of the *validity* of the *binomial formula* without at the same time requiring the proof that there are no other formulae for the representation of the developed value of $(1 + x)^n$, then one can go directly from §12 to §38. If one is satisfied with the same thing for the formulae which serve for the calculation of *logarithms* then it is sufficient to read only no.1 of §66. It is equally reasonable to omit §§ 49–51 initially because they simply investigate the nature of a formula, which would give the value of $(a + x)^n$ for as many x and n as possible. In general the §§ 6, 9, 11, 12, 31, 35, 36, 63, 67, 68 can also be omitted initially. Thus it will be seen that the kind of proof supplied in this work is not so very long as might perhaps be supposed merely from the number of pages. Every expert will realize the importance of the proposition in §29, for it will be clear to him that the application of all the operations of the differential and integral calculus to all equations with variable quantities is based entirely on this proposition (together with §27).

Raditsch, 5th Sep. 1815

Bolzano

§ 1

Preamble. According to the known rules of multiplication, the second, third, fourth, ... power of every quantity consisting of two or more parts, like $a + b$, or $a + b + c + d + \cdots$, can be expanded in a series whose individual terms contain nothing but powers of the individual parts $a, b, c, d, \ldots$ or products of such powers, possibly also multiplied by a specific number. Thus, simply by multiplication we find:

$$(a + b)^2 = a^2 + 2ab + b^2,$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$\text{etc.}$$

And similarly,

$$(a + b + c + d + \cdots)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 + 2ad + 2bd + 2cd + d^2 + \cdots,$$
$$(a + b + c + d + \cdots)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 + \cdots,$$
$$\text{etc.}$$

This observation gives rise to the idea of whether perhaps in *general* every function consisting of two or more parts of the form $(a + b)^n$, $(a + b + c + d + \cdots)^n$, where n may denote any kind of *whole* number, indeed also a *fractional*, *irrational* or *negative* quantity, can be expanded in a series which, like the above, contains nothing but powers of the individual parts $a, b, c, d, \ldots$ or products of such powers, possibly also multiplied by a quantity dependent merely on n. This is the question with which we shall now occupy ourselves. Its affirmative answer in respect of a *two-part* quantity of the form $(a+b)^n$ is called the *binomial theorem*, and in respect of a quantity consisting of *several parts*, of the form $(a + b + c + d + \cdots)^n$, it is called the *polynomial theorem*.

§ 2

Definition. A sum of terms which, like that described in §1, contains nothing but powers of the individual quantities $a, b, c, d, \ldots$ or products of such powers, possibly also multiplied by a quantity independent of $a, b, c, d, \ldots$, is called a *developed* function of the quantities $a, b, c, d, \ldots$. On the other hand, functions which, like $(a + b)^n$, contain powers of a sum of two or more of the quantities $a, b, c, d, \ldots$ and possibly even another quantity, are called *complex* [*komplexe*] functions of $a, b, c, d, \ldots$. A sum of terms which are formed according to a specific law is called a *series*, particularly if their number may be increased arbitrarily.

§ 3

Definition. The series

$$1 + \frac{n}{1}.x + \frac{n}{1}.\frac{n-1}{2}.x^2 + \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.x^3 + \cdots$$
$$+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\ldots\frac{n-(r-1)}{r}.x^r$$

whose first term $= 1 = x^0$, and of those following—the number of which can be increased arbitrarily—each one is formed from the previous one by increasing the exponent of the quantity x by 1, and multiplying the coefficient by $\frac{n-(r-1)}{r}$ where r is the exponent already found, is called the *binomial series belonging to the nth power of* $(1 + x)$, or the *binomial formula* for $(1 + x)^n$. If it should be possible to prove that the value of this series, under certain circumstances, is equal to the corresponding power of $(1 + x)$, then we call the expression of this equality, for short, the *binomial equation.*

§ 4

Corollary. If n is a whole (positive) number, and x is likewise positive, then all the terms in this series, up to the $(n + 1)$th, which is

$$\frac{n}{1}.\frac{n-1}{2}\ldots\frac{n-(n-1)}{n}.x^n = x^n,$$

are positive, but each of the following ones are equal to zero.

§ 5

Theorem. The binomial equation holds for the value of the exponent $n = 0, = 1, = 2, = 3, \ldots$, whatever x denotes, provided the series is not broken off while r is still $< n$.

Proof. Under this condition, the value of the binomial series for $n = 0$, however far its terms may be continued, is obviously only $= 1 + 0 + 0 + 0 + \cdots = 1$, and this is just the value of the corresponding power of $(1 + x)$, namely the value of $(1 + x)^0$, whatever x denotes. For $n = 1$, the value of the binomial series $= 1 + x + 0 + 0 + \cdots = 1 + x$, and this is just the value of the corresponding power, i.e. the *first* power of $(1 + x)$. For $n = 2$, the binomial series gives the value $1 + 2x + x^2 + 0 + \cdots$, which is exactly the value of its corresponding, i.e. *second*, power of $(1 + x)$. For $n = 3$, the binomial series is $= 1 + 3x + 3x^2 + x^3 + 0 + \cdots$, and this is also the value of the corresponding power, i.e. the *third* power of $(1+x)$. And so on.

§ 6

Note. If anyone should ask how the form of this strange series should ever be thought of, or could ever have been thought of, we would give him the following explanation. From the particular way in which multiplication proceeds it soon becomes clear that a product of n factors of the form $(1+a)(1+b)(1+c)(1+d)\dots$ equals the following sum: $1 + (a + b + c + d + \cdots) + (ab + ac + ad + bc + bd + cd + \cdots) + (abc + abd + acd + bcd + \cdots) + (abcd + \cdots) + \cdots$, i.e. that it consists of the following parts: (1) unity, (2) the sum of all the quantities $a, b, c, d, \dots$ taken singly, (3) the sum of all products which can be formed by combining every two of these quantities, (4) the sum of all products which arise by combining every three of these quantities, etc. Finally the $(n + 1)$th part is the product which results from combining all the quantities. If one then investigated the number of the terms from which each of the sums just described is formed, it would first of all be clear that with the *first*, i.e. with the sum of the quantities *taken singly*, $a, b, c, d, \dots$ this number is $= n$. It would not be so obvious how large the number of terms [*Gliederzahl*] is in each of the subsequent sums. But this much would be clear, that each one is as large as the number [*Menge*] of combinations which is possible from among n quantities $a, b, c, d, \dots$ first, two at a time, then three at a time, etc., if each of them appears in each combination only once, and none of these combinations has the same components as another. Now if we called the number [*Anzahl*] of combinations taking r terms at a time (where r must denote a number $< n) = R$, then it could soon be seen that the number of combinations taking $(r + 1)$ terms at a time, S, must be $= \frac{n-r}{r+1}.R$. In every combination of the quantities $a, b, c, d, \dots$ for the number of those r at a time there appear only r of these quantities; $(n - r)$ of them are omitted. Therefore if to each of these combinations one of the others is added, in turn, from the $(n - r)$ omitted quantities, then from each of the previous combinations $(n - r)$ new ones are formed each of which has $(r + 1)$ terms. Therefore one obtains $(n - r).R$ combinations in all. Amongst these combinations, each consisting of $(r + 1)$ terms, several are completely equal to one another in respect of their components. Indeed, a little thought shows that, of each kind $(r + 1)$ are equal. For it is obvious that with the rule of formation that has been followed, each of the new combinations must arise in $(r + 1)$ different ways, since each of the $(r + 1)$ quantities of which it consists, was at one time the omitted one, which was added. Therefore the number of combinations which can be distinguished in respect of their components is $S = \frac{n-r}{r+1}.R$. Once this was recognized, then the quantities $a, b, c, d, \dots$ in the product $(1 + a)(1 + b)(1 + c)(1 + d)\dots$ need only all be put equal to one another, e.g. $= x$, then it becomes the nth power of $(1+x)$. On this assumption the *first* term of this product still remains $= 1$, but the *second* would now be the sum of as many xs as before there were quantities $a, b, c, d, \dots$, i.e. it would be $= n.x$. The *third* term would be the sum of as many x^2s as previously there were combinations of the quantities $a, b, c, d, \dots$ two at a time. By the formula $S = \frac{n-r}{r+1}.R$, if $R = n$ and $r = 1$, this number is found to be $= \frac{n.n-1}{2}$. Therefore the third term of the product $= \frac{n.n-1}{1.2}.x^2$. And so on. It may be seen how one can continue to argue in this way and thus be able to produce the series of §3. Hence it is also immediately clear that this series must give correct results for the value of $(1+x)^n$, not only for the values of n mentioned in §4 but for every whole number. None the less we still want to prove this truth specially in the following sections, because the present consideration does not seem to us to be a genuinely scientific proof, since it derives the truth to be proved from an extraneous concept.

§7

Theorem. If the binomial equation holds for any *specific* values of x and n then it also holds for x and $n+1$, if r is taken large enough.

Proof. From the assumption, for certain specific values of x and n the equation

$$(1+x)^n = 1 + n.x + n.\frac{n-1}{2}.x^2 + \cdots + n.\frac{n-1}{2}.\frac{n-2}{3}\ldots\frac{n-(r-1)}{r}.x^r \quad \odot$$

is valid, however large r is taken to be, provided it is not assumed to be smaller than a certain quantity. Therefore if both sides of this equation are multiplied by $(1+x)$ we have:

$$(1+x)^{n+1} = 1 + nx + \cdots + n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}.x^r \qquad \text{(A)}$$

$$+ x + \cdots + n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}.x^{r+1}. \qquad \text{(B)}$$

If one combines the equal powers of x in the two series (A) and (B) which are to be added together, then a series appears in which the first, second, third, ... terms are obviously of the same nature as the first terms of the binomial formula for $(1+x)^{n+1}$ should be, namely,

$$1 + (n+1)x + (n+1).\frac{n}{2}.x^2 + \cdots.$$

But in order to see that this is also the case for each subsequent term, let p designate a whole number which is not $>r$. Then every term in the series (A) which is identical with the series $\odot$, is of the form

$$n.\frac{n-1}{2}\ldots\frac{n-(p-1)}{p}.x^p$$

and every term in the series (B) is of the form

$$n.\frac{n-1}{2}\ldots\frac{n-(p-2)}{p-1}.x^p$$

so that all the terms of these two series can be obtained if one successively puts $0, 1, 2, \ldots, r$ for p in these two forms. For the same value of p, these two forms represent the two terms in the series (A) and (B) which contain the same power of x, whose sum we have therefore found. It gives the quantity

$$n.\frac{n-1}{2}\ldots\frac{n-(p-2)}{p-1}.\left(1+\frac{n-(p-1)}{p}\right).x^p$$
$$=n.\frac{n-1}{2}\ldots\frac{n-(p-2)}{p-1}.\frac{(n+1)}{p}.x^p$$
$$=(n+1).\frac{n}{2}.\frac{n-1}{3}\ldots\frac{n-(p-2)}{p}.x^p$$
$$=(n+1).\frac{n}{2}.\frac{n-1}{3}\ldots\frac{n+1-(p-1)}{p}.x^p.$$

Now this is exactly the form which each term should have in the binomial series belonging to $(1+x)^{n+1}$, from which we see that the sum of all its terms up to the last,

$$n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}.x^{r+1}$$

has the form of the binomial series for $(1+x)^{n+1}$. But if, as assumed, the series in ⊙, therefore the series in (A) may also be continued arbitrarily beyond the term x^r without affecting the equation, then one can also put in (A) after x^r the term

$$n.\frac{n-1}{2}\ldots\frac{n-r}{r+1}.x^{r+1}.$$

But thereby the last term in the sum to be found will be

$$=n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}.\left(1+\frac{n-r}{r+1}\right)x^{r+1}$$
$$=(n+1).\frac{n}{2}.\frac{n-1}{3}\ldots\frac{n+1-r}{r+1}.x^{r+1}$$

and is therefore likewise of the form which it should be in the binomial series for $(1+x)^{n+1}$. Finally, since in the equation ⊙, r may be taken as large as desired, provided it is not taken below a certain limit, then it will also be the same in the series now found which expresses the value of $(1+x)^{n+1}$, so $(r+1)$ can be taken as large as desired. Accordingly the series found has, throughout, all the properties which are required of a binomial series for $(1+x)^{n+1}$.

§ 8

Theorem. The binomial equation holds for every *whole-numbered, positive* value of n, whatever x is, provided r is not taken smaller than n.

Proof. As a consequence of §5 this equation holds whatever x is, for $n=1$, $n=2$, etc. Therefore the same equation, by §6, also holds for $n=2+1=3$. And therefore, as a consequence of the same section, also for $n=3+1=4$, etc. This kind of argument can always be continued further, and by repeatedly increasing a whole number by 1, every higher number can be reached. Therefore the binomial equation holds for every whole number, provided r is taken appropriately. That it is sufficient that r should not be $< n$ is now clear as follows. Every value of r, which is not $< n$ must either be $= n$ or be $> n$. Now if r is taken to be $= n$, the last term of the series

$$= n.\frac{n-1}{2}.\frac{n-2}{3}\ldots\frac{n-(n-1)}{n}.x^n = x^n.$$

But the one immediately following, if it were continued, would be

$$= n.\frac{n-1}{2}\ldots\frac{n-n}{n-1}.x^{n+1} = 0$$

and consequently every subsequent term would be $= 0$ because they are formed from previous terms merely by multiplication (§3). Therefore since all later terms are $= 0$ the equation cannot be affected: one may include them or omit them.

§ 9

Note. It has been proved here that the binomial equation is *valid* provided r is *not* taken to be $< n$. But the converse proposition, that if r is taken to be $< n$ it does *not* hold, still does not follow from this. It cannot be deduced simply from the fact that the terms which are then left out are not each $= 0$, for several of them taken together could be $= 0$. In fact, for each particular n, a value of x can be given for which the sum of several terms is of the form

$$n.\frac{n-1}{2}\ldots\frac{n-p+1}{p}.x^p + n.\frac{n-1}{2}\ldots\frac{n-p}{p+1}.x^{p+1} + \ldots + x^n = 0.$$

Hence it may therefore be seen that it would even be a *false proposition* to assert that the binomial series must always be continued up to the term in x^n.

§ 10

Theorem. The binomial theorem holds for every *whole positive number* as exponent.

Proof. If one puts $x = \frac{b}{a}$ in the equation of §7, then a, b, can denote any kind of quantities and it is always the case that

$$(1+x)^n = \left(1+\frac{b}{a}\right)^n = 1 + n.\frac{b}{a} + n.\frac{n-1}{2}.\frac{b^2}{a^2} + \cdots$$
$$+ n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}.\frac{b^r}{a^r}$$

provided n is a whole positive number. Therefore also,

$$a^n\left(1+\frac{b}{a}\right)^n = (a+b)^n = a^n + n.a^{n-1}.b + n.\frac{n-1}{2}.a^{n-2}.b^2 + \cdots$$
$$+ n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}.a^{n-r}.b^r.$$

Therefore in this way the two-part quantity $(a + b)^n$ can be expanded, if n is a whole positive number, into a series with nothing but the powers of the individual parts and of products of such powers, possibly multiplied by a quantity depending only on n. Now this is the binomial theorem for a whole positive number as exponent (§1).

§ 11

Corollary. Now in order to find out whether the binomial equation may perhaps hold *in general*, i.e. also for every negative, fractional and irrational exponent, suppose n is taken $= -1$ in the series of §3, then one obtains the following: $1 - x + x^2 - x^3 + \cdots \pm x^r$, where in the last term the upper or lower sign holds according to whether r is even or odd. Now if the binomial equation also held for the value $n = -1$, this series would have to be $= (1 + x)^{-1}$. However, by the rules of division

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \pm \frac{x^r}{1+x},$$

an expression which is obviously only equal in value to the first one in the single case where $x = 0$, since $\frac{x^r}{1+x}$ can never be $= x^r$ unless $x = 0$. We therefore see from this example that the binomial equation certainly does not hold for every value of n and x.

§ 12

Corollary 2. But if x is a *proper fraction*, then the remarkable situation occurs that the binomial series $1 - x + x^2 - x^3 + \cdots \pm x^r$ can be brought as close to the value $(1 + x)^{-1}$ as desired, merely by sufficiently increasing its [number of] terms. For the difference between the two is

$$x^r - \frac{x^r}{1+x} = \frac{x^{r+1}}{1+x},$$

which, if x is a proper fraction, can be made smaller than any given quantity if r is taken large enough. For if $x < +1$, then $\frac{1}{x}$ will be $= (1 + u)$, where u designates a positive quantity. Now if $\frac{x^{r+1}}{1+x}$ is to be $< D$, for instance, then one just takes a value for

$$r > \frac{\frac{1}{D(1+x)} - 1}{u} - 1, \quad \text{so that} \quad r + 1 > \frac{\frac{1}{D(1+x)} - 1}{u}$$

and

$$(r+1)u > \frac{1}{D(1+x)} - 1, \quad \text{or} \quad 1 + (r+1)u > \frac{1}{D(1+x)}.$$

But according to §7,

$$(1+u)^{r+1} = 1 + (r+1)u + (r+1).\frac{r}{2}.u^2 + \cdots > 1 + (r+1)u.$$

Therefore *a fortiori*

$$(1+u)^{r+1} > \frac{1}{D(1+x)}.$$

Therefore

$$\frac{1}{(1+u)^{r+1}} = x^{r+1} < D(1+x),$$

and $\frac{x^{r+1}}{1+x} < D$. The proof proceeds in almost the same way if x is negative. Therefore whenever x designates a quantity which is $< \pm 1$, the binomial equation also holds for the value $n = -1$, not in fact precisely, but the difference can be made smaller than any given quantity if r, i.e. the number of terms, is taken large enough.

§ 13

Transition. Now it is all the same, as far as *practical calculation* is concerned, whether an expression found represents the value of a quantity sought *completely precisely* or only expresses the value so closely that the difference can be arbitrarily reduced. Furthermore, those equations which are not completely exact, but are nevertheless valid, if we imagine a quantity being added to one side which can become smaller than any given quantity, can also be used *in science* as an aid for the discovery of important truths. For these reasons we must also pay them more careful attention here. Since in the previous section we have an example of one case of the binomial equation where it does not hold exactly, but it does hold in the sense that we think of being added to one side of the equation, a quantity which can be smaller than any given quantity. This then gives rise to the conjecture that there may be several other such cases. We therefore set ourselves the task of investigating, 'What are all the values which n and x can take, for which the binomial equation is either completely exact, or else holds if to one side a quantity is added which can be smaller than any given quantity?' But somebody might think that the binomial formula we gave above (§3) is perhaps not the expression which suits our present intention best, i.e. that perhaps another function developed in powers of x could be discovered which would give the value of $(1+x)^n$ for *several* kinds of values of n and x. So we shall first discuss the following question: 'What is the most general form of a series developed in powers of x which would either be completely equal, or come as close as required, to the value of the complex function $(1+x)^n$ for as many values of n and x as possible?' For the present we shall understand by as *many values of* n *and* x *as possible* only those which lie between *zero* and another limit (positive or negative) which is as large as possible; so that the equation always holds starting from a certain value of x and *for all smaller* *values*.[f] For the usefulness of an equation of this kind rests chiefly on the values of x becoming smaller and smaller. However, we must first of all start with some lemmas.

§ 14

Convention. To designate a quantity which can become smaller than any given quantity, we choose the symbols ω, Ω or something similar.

§ 15

Lemma. If each of the quantities $\omega, \overset{(1)}{\omega}, \overset{(2)}{\omega}, \ldots, \overset{(m)}{\omega}$ can become as small as desired while the (finite) number of them does not alter, then their algebraic *sum* or *difference* is also a quantity which can become as small as desired, i.e.

$$\omega \pm \overset{(1)}{\omega} \pm \overset{(2)}{\omega} \pm \cdots \pm \overset{(m)}{\omega} = \Omega.$$

Proof. For if the *sum* of these quantities is to be $<D$, where D designates some finite quantity, then if there is a constant number n of them, each of them may be taken $< \frac{D}{n}$, which is possible as a consequence of the assumption. Then certainly $\omega \pm \overset{(1)}{\omega} \pm \overset{(2)}{\omega} \pm \cdots \pm \overset{(m)}{\omega} < D$, even if the terms of this sum should all be positive, and all the more so in any other case.

§ 16

Corollary. Therefore also $(A + \omega) \pm (B + \overset{(1)}{\omega}) \pm (C + \overset{(2)}{\omega}) \pm \cdots \pm (R + \overset{(r)}{\omega}) = A \pm B \pm C \pm \cdots \pm R + \Omega$, if the number of these terms does not change, while ω, $\overset{(1)}{\omega}, \ldots, \overset{(r)}{\omega}$ can become as small as desired. For in fact,

$$\begin{aligned}(A + \omega) \pm (B + \overset{(1)}{\omega}) \pm (C + \overset{(2)}{\omega}) \pm \cdots \pm (R + \overset{(r)}{\omega}) \\ = A \pm B \pm C \pm \cdots \pm R + (\omega \pm \overset{(1)}{\omega} \pm \overset{(2)}{\omega} \pm \cdots \pm \overset{(r)}{\omega}) \\ = A \pm B \pm C \pm \cdots \pm R + \Omega \quad (\S 15).\end{aligned}$$

§ 17

Lemma. Every *product* of a quantity which remains constant, and another which can become smaller than any given quantity, can also itself become smaller than any given quantity. That is, $A.\omega = \Omega$.

Proof. For if $A.\omega$ is to be $< D$, then just take $\omega < \frac{D}{A}$.

[f] For Bolzano this means smaller in absolute value, i.e. for a certain c then for all x such that $|x| < c$.

§ 18

Lemma. $(A+\omega)(B+\overset{(1)}{\omega}) = A.B + \Omega$.

Proof. For $(A+\omega)(B+\overset{(1)}{\omega}) = A.B + \omega.B + \overset{(1)}{\omega}.A + \overset{(1)}{\omega}.\omega = A.B + \Omega$ (§§ 17, 15).

§ 19

Lemma.

$$\frac{A+\omega}{B+\overset{(1)}{\omega}} = \frac{A}{B} + \Omega.$$

Proof. For by the rules of division,

$$\frac{A+\omega}{B+\overset{(1)}{\omega}} = \frac{A}{B} + \frac{B.\omega - A.\overset{(1)}{\omega}}{B^2 + B.\overset{(1)}{\omega}}.$$

But if $B.\overset{(1)}{\omega}$ is *positive* then obviously

$$\frac{B.\omega - A.\overset{(1)}{\omega}}{B^2 + B.\overset{(1)}{\omega}} < \frac{B.\omega - A.\overset{(1)}{\omega}}{B^2}$$

which by § 17 and § 15 can become as small as desired, and so this is all the more certain for $\frac{B.\omega - A.\overset{(1)}{\omega}}{B^2+B.\overset{(1)}{\omega}}$. But if $B.\overset{(1)}{\omega}$ is *negative*, then in order to make $\frac{B.\omega - A.\overset{(1)}{\omega}}{B^2+B.\overset{(1)}{\omega}} < D$, one first takes a value for ω such that

$$\omega < \frac{D(B^2 + B.\overset{(1)}{\omega})}{2B} = \frac{D(B + \overset{(1)}{\omega})}{2},$$

then also $\frac{B.\omega}{B^2+B.\overset{(1)}{\omega}} < \frac{D}{2}$. If $\overset{(1)}{\omega}$ decreases[g] further then $B^2 + B.\overset{(1)}{\omega}$ increases, therefore $\frac{B.\omega}{B^2+B.\overset{(1)}{\omega}}$ decreases and certainly remains $< \frac{D}{2}$. Now take $\overset{(2)}{\omega} < \overset{(1)}{\omega}$ and at the same time $< \frac{B^2D}{2A-BD}$, then provided D is so small that $BD < 2A$ then also $\frac{A.\overset{(2)}{\omega}}{B^2+B.\overset{(2)}{\omega}} < \frac{D}{2}$, consequently it is certain that $\frac{B.\omega - A.\overset{(2)}{\omega}}{B^2+B.\overset{(2)}{\omega}} < D$.

§ 20

Lemma. Every *positive* (whole or fractional) power of a quantity which can become as small as desired can itself become as small as desired.

[g] When Bolzano says of a negative quantity that it decreases, he means it decreases in absolute value (so it actually increases).

Proof. Let p and q be any whole positive numbers (unity itself is not excluded), then $\omega^{\frac{p}{q}}$ represents any kind of positive (whole or fractional) power of ω. Now if $\omega^{\frac{p}{q}}$ is to become $< D$ then ω may be taken smaller than the number $D^{\frac{q}{p}}$, then certainly also (if something smaller is multiplied by something smaller), $\omega^2 < D^{\frac{2q}{p}}$, also consequently, for just the same reason again, $\omega^3 < D^{\frac{3q}{p}}$ etc. Hence it may now be seen (because p is a whole positive number) that it must be that $\omega^p < D^q$. But from this it follows further that also $\omega^{\frac{p}{q}} < D$. For $\omega^{\frac{p}{q}}$ cannot ever $= D$ because otherwise ω^p would also be found $= D^q$. Still less can $\omega^{\frac{p}{q}} > D$, because otherwise one would obtain (by repeated multiplication as before) $\omega^p > D^q$. Therefore $\omega^{\frac{p}{q}}$ must be $<D$.

§ 21

Corollary. Therefore $A\overset{(1)}{\omega}{}^{\alpha} + B\overset{(2)}{\omega}{}^{\beta} + \cdots + R\overset{(r)}{\omega}{}^{\rho}$ can also become smaller than any given quantity, if $\overset{(1)}{\omega}, \overset{(1)}{\omega}, \ldots, \overset{(r)}{\omega}$ can, in themselves, each become as small as desired, and provided the exponents $\alpha, \beta, \ldots, \rho$ are all positive, and they, as well as the coefficients $A, B, \ldots, R$, and the number of terms itself do not change (§§ 20, 17, 15).

§ 22

Lemma. Every product of an arbitrarily large number of proper fractions which all remain smaller than a given [quantity] can, by the increase of its factors, be made smaller than any given quantity.

Proof. For if all these fractions, whose number can be called r, are smaller than the given quantity x, then their product is also smaller than the result of replacing each of them with the fraction x, i.e. it is $< x^r$. But since x is a proper fraction, x must be $= \frac{1}{1+u}$, where u represents a positive quantity. Therefore $x^r = \left(\frac{1}{1+u}\right)^r$, and this becomes $< D$ as soon as one takes

$$r > \frac{\frac{1}{D} - 1}{u}.$$

For then $r.u > \frac{1}{D} - 1$, and $1 + r.u > \frac{1}{D}$. However,

$$(1+u)^r = 1 + r.u + r.\frac{r-1}{2}.u^2 + \cdots + u^r \quad (\S 8)$$

is certainly $> 1 + r.u$ (§4). Therefore all the more certainly $(1+u)^r > \frac{1}{D}$, and consequently $\frac{1}{(1+u)^r} = x^r < D$. All the more certainly therefore, that product of the r fractions is $< D$.

§ 23

Lemma. The quantity

$$\frac{(x+\omega)^n - x^n}{\omega}$$

can be brought as close to the value nx^{n-1} as desired, if ω is taken small enough; n and x may denote whatever desired, provided x is not $= 0$. That is,

$$\frac{(x+\omega)^n - x^n}{\omega} = nx^{n-1} + \Omega.$$

Proof. We assume for the sake of brevity that ω is *positive*, because the proof in the opposite case proceeds in almost exactly the same way and we need the proposition here only for the one case.

1. For $n = 0$ it is self-evident.
2. *Secondly* therefore let n be *positive* and indeed $= \frac{p}{q}$, where p and q represent two whole numbers (not excluding unity). Now

$$(x+\omega)^n = (x+\omega)^{\frac{p}{q}} = x^{\frac{p}{q}}\left(1+\frac{\omega}{x}\right)^{\frac{p}{q}}$$

provided x is not $= 0$. The quantity $(1+\frac{\omega}{x})^{\frac{p}{q}}$ must necessarily be >1. For, first of all, it cannot be $= 1$ because otherwise

$$\left(1+\frac{\omega}{x}\right)^p = 1 + p.\frac{\omega}{x} + \cdots = 1^q = 1,$$

would also have to hold, contrary to §4. Still less can $\left(1+\frac{\omega}{x}\right)^{\frac{p}{q}} < 1$, for instance, $= \frac{1}{1+u}$, where u represents a positive quantity. For then it would have to be that

$$\left(1+\frac{\omega}{x}\right)^p = \left(\frac{1}{1+u}\right)^q = \frac{1}{(1+u)^q}.$$

However, according to §4, $\left(1+\frac{\omega}{x}\right)^p > 1$, but for the same reason, $\frac{1}{(1+u)^q} < 1$. Therefore it is certain that $\left(1+\frac{\omega}{x}\right)^{\frac{p}{q}} = 1+u$ where u represents a positive quantity. I now claim that the latter can become as small as desired, if ω is taken small enough. For if, for example, $u = \left(1+\frac{\omega}{x}\right)^{\frac{p}{q}} - 1$ is to become $< D$ then it is only necessary that

$$\left(1+\frac{\omega}{x}\right)^{\frac{p}{q}} < 1 + D, \quad \text{or} \quad \left(1+\frac{\omega}{x}\right)^p < (1+D)^q,$$

i.e. (by §8),

$$1 + p.\frac{\omega}{x} + p.\frac{p-1}{2}\left(\frac{\omega}{x}\right)^2 + \cdots + \left(\frac{\omega}{x}\right)^p < (1+D)^q,$$

or

$$\frac{\omega}{x}\left(p + p.\frac{p-1}{2}\left(\frac{\omega}{x}\right) + \cdots + \left(\frac{\omega}{x}\right)^{p-1}\right) < (1+D)^q - 1.$$

Now if this relationship does not occur for a certain value of ω then one just takes a new value $\overset{1}{\omega}$ which is $<\omega$, and that is

$$< \frac{(1+D)^q - 1}{\frac{1}{x}\left(p + p.\frac{p-1}{2}.\left(\frac{\omega}{x}\right) + \cdots + \left(\frac{\omega}{x}\right)^{p-1}\right)}.$$

Then it follows that

$$\frac{\overset{1}{\omega}}{x}\left(p + p.\frac{p-1}{2}.\left(\frac{\omega}{x}\right) + \cdots + \left(\frac{\omega}{x}\right)^{p-1}\right) < (1+D)^q - 1;$$

therefore all the more certainly, if one puts, instead of ω, the smaller $\overset{1}{\omega}$,

$$\frac{\overset{1}{\omega}}{x}\left(p + p.\frac{p-1}{2}.\left(\frac{\overset{1}{\omega}}{x}\right) + \cdots + \left(\frac{\overset{1}{\omega}}{x}\right)^{p-1}\right) < (1+D)^q - 1,$$

and therefore

$$\left(1 + \frac{\overset{1}{\omega}}{x}\right)^{\frac{p}{q}} - 1 < D.$$

Consequently in the equation

$$\left(1 + \frac{\omega}{x}\right)^{\frac{p}{q}} = 1 + u,$$

v represents a quantity which can become smaller than every given quantity. Now,

$$\left(1 + \frac{\omega}{x}\right)^p = (1+u)^q \qquad \text{or}$$

$$1 + p.\frac{\omega}{x} + p.\frac{p-1}{2}\left(\frac{\omega}{x}\right)^2 + \cdots + \left(\frac{\omega}{x}\right)^p = 1 + qu + q.\frac{q-1}{2}.u^2 + \cdots + u^q.$$

Therefore,

$$\frac{\omega}{x}\left(p + p.\frac{p-1}{2}\left(\frac{\omega}{x}\right) + \cdots + \left(\frac{\omega}{x}\right)^{p-1}\right)$$
$$= u\left(q + q.\frac{q-1}{2}.u + \cdots + u^{q-1}\right).$$

But since $(x+\omega)^{\frac{p}{q}} - x^{\frac{p}{q}} = x^{\frac{p}{q}}(1+u-1) = x^{\frac{p}{q}}u$, then one obtains,

$$\frac{(x+\omega)^{\frac{p}{q}} - x^{\frac{p}{q}}}{\omega} = x^{\frac{p}{q}}.\frac{u}{\omega} = x^{\frac{p}{q}-1}\left(\frac{p + p.\frac{p-1}{2}\left(\frac{\omega}{x}\right) + \cdots + \left(\frac{\omega}{x}\right)^{p-1}}{q + q.\frac{q-1}{2}.u + \cdots + u^{q-1}}\right).$$

According to §21 and §19, the quantity inside the brackets can come as close to the value $\frac{p}{q}$ as desired, if ω, and therefore also u, are taken to be small enough. Therefore by §17, the value of the whole expression, also comes as close to the value $\frac{p}{q}.x^{\frac{p}{q}-1}$ as desired.

3. *Thirdly* if n denotes a *negative* quantity then using the same notation as before,

$$(x+\omega)^n = (x+\omega)^{-\frac{p}{q}} = x^{-\frac{p}{q}}(1+\omega)^{-\frac{p}{q}} = x^{-\frac{p}{q}}(1+u)^{-1}.$$

Therefore,

$$\frac{(x+\omega)^{-\frac{p}{q}} - x^{-\frac{p}{q}}}{\omega} = -x^{-\frac{p}{q}-1}\left(\frac{p+p.\frac{p-1}{2}\left(\frac{\omega}{x}\right)+\cdots+\left(\frac{\omega}{x}\right)^{p-1}}{q+q.\frac{q-1}{2}.u+\cdots+u^{q-1}}\right)\frac{1}{1+u},$$

which by §§ 19, 21, comes as close as desired to the value $-\frac{p}{q}.x^{-\frac{p}{q}-1}$.
4. Finally, if n is *irrational* then there is always a fraction $\frac{p}{q}$ (positive or negative) which comes as close as required to n. But then it follows from the definition of the *concept* of an irrational power that the quantity $a^{\frac{p}{q}}$ gives a value as close to that of a^n as desired, if $\frac{p}{q}$ comes as close to the value n as desired. Therefore $(x+\omega)^n = (x+\omega)^{\frac{p}{q}} + \Omega$ and $x^n = x^{\frac{p}{q}} + \overset{1}{\Omega}$ where Ω, and $\overset{1}{\Omega}$ can become as small as desired for the same x and ω (merely by changing $\frac{p}{q}$). Therefore also in

$$\frac{(x+\omega)^n - x^n}{\omega} = \frac{(x+\omega)^{\frac{p}{q}} - x^{\frac{p}{q}}}{\omega} + \frac{\Omega - \overset{1}{\Omega}}{\omega}$$

the term $\frac{\Omega-\overset{1}{\Omega}}{\omega}$ will, by §15 and §17, be able to become as small as desired. But by what has just been proved,

$$\frac{(x+\omega)^{\frac{p}{q}} - x^{\frac{p}{q}}}{\omega}$$

comes as close to the value $\frac{p}{q}.x^{\frac{p}{q}-1}$ as desired, therefore also, by §18, to the value $nx^{\frac{p}{q}-1}$ or nx^{n-1}. Consequently (§15) $\frac{(x+\omega)^n-x^n}{\omega} = nx^{n-1} + \overset{2}{\Omega}$.

§ 24

Corollary. Therefore if x is not $= 0$ and ω can become as small as desired, then the difference $(x+\omega)^n - x^n$ can likewise become as small as desired. For it is $= nx^{n-1}.\omega + \omega.\Omega$ (§17, §15).

§ 25

Lemma. If, in the series $Ax^\alpha + Bx^\beta + Cx^\gamma + \cdots + Rx^\rho + S$, only x changes and α is the *greatest positive* exponent, then there is always a value of x for which, as well as for all greater values, the term Ax^α is greater than the sum of all the others $(Bx^\beta + Cx^\gamma + \cdots + Rx^\rho + S)$.

Proof. Since α is to be the greatest positive exponent, the series

$$\frac{Bx^{\beta-\alpha}}{A} + \frac{Cx^{\gamma-\alpha}}{A} + \cdots + \frac{Rx^{\rho-\alpha}}{A} + \frac{Sx^{-\alpha}}{A}$$

obtained by dividing $(Bx^\beta + Cx^\gamma + \cdots + Rx^\rho + S)$ by Ax^α contains only negative exponents. Now if the *smallest* of these $= -\mu$ then for every value of x which is > 1, $x^{-\mu} = \frac{1}{x^\mu}$ is equal to or greater than each of the quantities $x^{\beta-\alpha}$, $x^{\gamma-\alpha}, \ldots, x^{\rho-\alpha}, x^{-\alpha}$. It is *equal*, if its exponent is the same as μ and *greater* if it is greater. Let one of these greater ones $= -\mu - \pi$ and $x = 1 + u$, where u represents a positive quantity; then it is clear, as in §23 no. 2., that as well as $(1+u)^\mu$ also $(1+u)^\pi > 1$, therefore certainly $(1+u)^{\mu+\pi} > (1+u)^\mu$, therefore

$$\frac{1}{(1+u)^{\mu+\pi}} = x^{-\mu-\pi} < \frac{1}{(1+u)^\mu} = x^{-\mu}.$$

Furthermore, if the greatest of the coefficients $\frac{B}{A}, \frac{C}{A}, \ldots, \frac{R}{A}, \frac{S}{A}$ is $= M$ then $M.x^{-\mu}$ is a quantity which is equal to or greater than each of

$$\frac{Bx^{\beta-\alpha}}{A}, \frac{Cx^{\gamma-\alpha}}{A}, \ldots, \frac{Rx^{\rho-\alpha}}{A}, \frac{Sx^{-\alpha}}{A},$$

and if n is the number of these quantities, then certainly

$$nMx^{-\mu} > \left(\frac{Bx^{\beta-\alpha}}{A} + \frac{Cx^{\gamma-\alpha}}{A} + \cdots + \frac{Rx^{\rho-\alpha}}{A} + \frac{Sx^{-\alpha}}{A}\right).$$

Therefore it is only necessary to choose an $x > 1$, and $=$ or $> \sqrt[\mu]{nM}$, then $x^\mu =$ or $> nM$, and therefore $1 =$ or $> nMx^{-\mu}$, consequently

$$1 > \left(\frac{Bx^{\beta-\alpha}}{A} + \frac{Cx^{\gamma-\alpha}}{A} + \cdots + \frac{Rx^{\rho-\alpha}}{A} + \frac{Sx^{-\alpha}}{A}\right)$$

and

$$Ax^\alpha > (Bx^\beta + Cx^\gamma + \cdots + Rx^\rho + S).$$

§ 26

Lemma. If a function $F(r)$, whose variable quantity r is only capable of taking *whole-numbered* values (positive or negative), takes, for each of these values only a finite number of values, none of which is zero, and it is to be possible for this function to become smaller than any given quantity, then this can only happen by increasing the value of r beyond any given limit.

Proof. Because the value of r is always supposed to be a whole number, then within any two given limits there are only a finite number of these, and therefore, because $F(r)$ also takes only a definite number of values for each value of r, there are only a finite number of values of $F(r)$. Therefore one of them must be the smallest, or else be so small that there is none smaller among them. But since, nevertheless, this value is not to be zero, it is also not smaller than every given quantity, namely not smaller than itself. Therefore if the values of $F(r)$ are to be capable of becoming smaller than any given quantity, then there must be even smaller values than those which this function takes within the fixed limits of r, i.e. r must be taken outside these limits.

§ 27

Lemma. If the quantities ω, $\overset{\mathrm{I}}{\omega}$ in the equation $A+\omega = B+\overset{\mathrm{I}}{\omega}$ can become as small as desired, while A and B remain unchanged, then it must be that $A = B$ exactly.

Proof. For if A and B were unequal there would have to be one of them, e.g. A, which is the greater one. Therefore $A = B + D$ where D would be a constant quantity, because A and B themselves are constant. Then also $B+D+\omega = B+\overset{\mathrm{I}}{\omega}$, whence $D = \overset{\mathrm{I}}{\omega} - \omega$. And accordingly $\overset{\mathrm{I}}{\omega} - \omega$ could not become smaller than every given quantity (namely not smaller than D), contrary to §15.

§ 28

Lemma. If an equation is of the form $A + Bx^{\beta} + Cx^{\gamma} + \cdots + Rx^{\rho} = \mathfrak{A} + \mathfrak{B}x^{\mathfrak{b}} + \mathfrak{C}x^{\mathfrak{c}} + \cdots + \mathfrak{R}x^{\mathfrak{r}}$ in which no *complex*[h] function of x appears, and among the exponents none is equal to another one on the same side, either for all values of x, or for all values smaller than a certain one, then for each term on the one side there must be another *completely equal* one (i.e. one that has an equal exponent and coefficient) on the other side, i.e. the equation must be *identical*.

Proof. 1. First of all let it be supposed that on neither side is there a *negative* exponent, but that on one side there may occur a constant term $\mathfrak{A}$, then by §21 the quantity on the right hand side of the equality sign comes as close to the value $\mathfrak{A}$ as desired, if x is taken small enough. But if there were no constant term here, the quantity on the left-hand side could become smaller than any given quantity. Therefore, by §27, the equation could not hold. We must therefore accept that if there is a constant term $\mathfrak{A}$ on one side, there is also one on the other side and indeed an equal one, therefore $\mathfrak{A} = A$. Now if the two terms found equal to each other are subtracted from both sides of the given equation, which is then divided by that power of x which now appears in the lowest of the terms on the right or left (and that there is such, follows, because all exponents on one side are different from one another), then one obtains a new equation in which again no negative exponents can exist, but on one side a constant appears. (One could have done the same thing with the same result, if in the initial equation there had been no constant quantity present). Now suppose $\mathfrak{b}$ is that least exponent, then the new equation is

$$Bx^{\beta-\mathfrak{b}} + Cx^{\gamma-\mathfrak{b}} + \cdots + Rx^{\rho-\mathfrak{b}} = \mathfrak{B} + \mathfrak{C}x^{\mathfrak{c}-\mathfrak{b}} + \cdots + \mathfrak{R}x^{\mathfrak{r}-\mathfrak{b}}$$

in which the term $\mathfrak{B}$ is necessarily a constant. We therefore conclude, as before, that there must be another constant term equal to this one on the other side. Therefore one of the exponents $\beta - \mathfrak{b}, \gamma - \mathfrak{b}, \ldots, \rho - \mathfrak{b}, = 0$ and then its corresponding coefficient $= \mathfrak{B}$. For example, let $\beta - \mathfrak{b} = 0$, then $\beta = \mathfrak{b}$ and $B = \mathfrak{B}$. Therefore the term $\mathfrak{B}x^{\mathfrak{b}}$ on one side is also completely equal to Bx^{β} on the other

[h] See §2 for the definition of complex.

side. Since these arguments can always be continued as long as there is a term on one side, the correctness of the claim may be seen.
2. But if some terms of the equation have *negative* exponents, then there is one among them, which has none greater than it, because otherwise the number of terms could not be finite. Now if one multiplies both sides of the equation by a positive power of x whose exponent is as large as this negative one, then a new equation is obtained in which a negative exponent no longer appears. It therefore holds of this equation, by no. 1, that every term on one side corresponds to a completely equal one on the other side. But since the coefficients of the individual parts are not changed at all by the multiplication, and their exponents are merely increased by the same quantity, it follows that the terms which are now equal were also equal before multiplication.

§ 29

Lemma. Suppose a function of x, $\overset{r}{F}x$, of arbitrarily many terms, formed according to a particular rule, has the property that either for all x, or for all x within certain limits a and b, it can become as small as desired, merely by increasing its number of terms r. Suppose furthermore that $\overset{r}{f}x$ denotes a second function of the same arbitrary number of terms, which depends on the former in such a way that for every value of x within a and b, the equation

$$\frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} = \overset{r}{f}x + \Omega$$

holds, in which Ω can become as small as desired if the same holds for ω. Then I claim that the function $\overset{r}{f}x$ also has the property, that it can become as small as desired for the same values of x as for $\overset{r}{F}x$, if its number of terms r is taken large enough.

Proof. 1. That the concept of a function with an arbitrarily large number of terms, which can thereby become as small as desired, is not an impossibility, is shown for example, by the following function: $(1+x)^{-1} - 1 + x - x^2 + x^3 - \cdots \pm x^r$. For, as is clear from §12, for every x which lies between $+1$ and -1 this can become smaller than every given quantity, merely by the accumulation of its terms. Also, that the assumption of such a relationship between two functions as is required by the equation,

$$\frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} = \overset{r}{f}x + \Omega$$

is not impossible in general is proved by §23, for the x^n and nx^{n-1} given there are such a pair of functions, since

$$\frac{(x+\omega)^n - x^n}{\omega} = nx^{n-1} + \Omega.$$

Now one may suppose that among all the values which the function $\overset{r}{f}x$ takes, for the same r, when x is given all values between a and b, the greatest positive (or if none are positive, the smallest negative) $=\overset{r}{f}p$; and the smallest positive (or if there are no positive ones, the greatest negative) $=\overset{r}{f}q$. For every value of x, with the exception of p and q, the two expressions $\overset{r}{f}x - \overset{r}{f}p$ and $\overset{r}{f}q - \overset{r}{f}x$ will then designate two positive quantities.[i] But by assumption, provided x is taken within a and b, the equation holds:

$$\frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} = \overset{r}{f}x + \Omega.$$

Therefore the following two also certainly hold:

$$\frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} - \overset{r}{f}p = \overset{r}{f}x - \overset{r}{f}p + \Omega,$$

and

$$\overset{r}{f}q - \frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} = \overset{r}{f}q - \overset{r}{f}x - \Omega.$$

Now since, with the two exceptions mentioned for $x = p$ or q, the quantities $\overset{r}{f}x - \overset{r}{f}p$ and $\overset{r}{f}q - \overset{r}{f}x$ are not zero, then an ω can always be taken small enough (without regard to its sign) that the value of Ω is $< \overset{r}{f}x - \overset{r}{f}p$ and at the same time also $< \overset{r}{f}q - \overset{r}{f}x$, since for the same r and x the quantities $\overset{r}{f}x, \overset{r}{f}p, \overset{r}{f}q$ are given. Therefore with this value of ω, both the quantities $\overset{r}{f}x - \overset{r}{f}p + \Omega$ and $\overset{r}{f}q - \overset{r}{f}x - \Omega$ are positive, and therefore the expressions which equal them,

$$\frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} - \overset{r}{f}p \quad \text{and} \quad \overset{r}{f}q - \frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega},$$

are positive. According to the definition, $\overset{r}{f}p, \overset{r}{f}q$ are necessarily either both positive or both negative. In the first case it must be, according to mere value, that

$$\overset{r}{f}p < \frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} \quad \text{and} \quad \overset{r}{f}q > \frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega};$$

and in the second case just the converse,

$$\overset{r}{f}p > \frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} \quad \text{and} \quad \overset{r}{f}q < \frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega}.$$

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[i] Clearly the differences are in error in this sentence and should be, respectively, $\overset{r}{f}p - \overset{r}{f}x$ and $\overset{r}{f}x - \overset{r}{f}q$. The consequences of the error have been followed through consistently. It can easily be remedied and makes no difference to the argument of this part.

Therefore in each case $\overset{r}{f}x$ is a function of such a nature that at one time it is $>$, and at another time it is $< \frac{\overset{r}{F}(x+\omega)-\overset{r}{F}x}{\omega}$, whence it follows by a well-known theorem* that there must also exist a value of x, indeed one lying between p and q (therefore also between a and b), $= \xi$ for which $\overset{r}{f}\xi = \frac{\overset{r}{F}(x+\omega)-\overset{r}{F}x}{\omega}$.

2. If for the same x and ω, r is taken ever greater, then by the assumption[k] it should be possible to make the value of $\overset{r}{F}x$ as well as $\overset{r}{F}(x+\omega)$, as small as desired. Hence it follows that by this increase in r the function $\overset{r}{f}\xi$ can also be made as small as desired. For if, for example, it is to be made $<D$ then r is just taken so large that (without respect to sign) $\overset{r}{F}x$ and $\overset{r}{F}(x+\omega)$ become $<\frac{\omega D}{2}$, then certainly it must turn out that

$$\overset{r}{f}\xi = \frac{\overset{r}{F}(x+\omega) - \overset{r}{F}x}{\omega} < D.$$

3. Therefore by increasing r the function $\overset{r}{f}x$, at least for the value $x = \xi$, can become smaller than any given quantity. But it is now very easy to prove that this is also possible for every other value of x, provided it lies between a and b. For example, let α be such a value of x between a and b, and take a second one β which is likewise between a and b and to which α comes as near to as desired. Since the property of the function $\overset{r}{F}x$, that merely by increasing r it can become smaller than any given quantity, holds for all values between a and b, then it must also hold for all values between α and β (since what is between α and β must also be between a and b). Therefore, as in no.1 and no.2, merely from the fact that for all values of x lying between a and b the function $\overset{r}{F}x$ can become smaller than any given quantity it follows that between a and b there must also be a value ξ for which $\overset{r}{f}\xi$ can become smaller than any given quantity. Then for the same reason there must be a value $x = \zeta$ between α and β for which $\overset{r}{f}\zeta$ becomes smaller than any given quantity if r is sufficiently increased. But if $\overset{r}{f}\zeta$ can become smaller than any given quantity then $\overset{r}{f}\alpha$ must also be able to do so. For since the difference between and α and ζ can be made as small as desired (because it must always be smaller than that between α and β), then the values of $\overset{r}{f}\zeta$ and $\overset{r}{f}\alpha$ must also be able to come as close to one another as desired, since the

* Indeed in my own opinion this theorem, which belongs to the theory of equations, has never before been correctly proved. However, I believe I have been so fortunate as to have found a completely sound proof of it. It is already sketched out in a special paper and should soon be printed.[j]

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[j] The theorem is the intermediate value theorem, and the paper forms the work *RB* that appears next in this volume.

[k] Namely, the assumption in the statement of this lemma.

function $\overset{r}{f}x$ is continuous. For a function is called continuous if the change which occurs for a certain change in its argument, can become smaller than any given quantity, provided that the change in the argument is taken small enough. Now first of all the equation

$$\frac{\overset{r}{F}(x+\omega)-\overset{r}{F}x}{\omega}=\overset{r}{f}x+\Omega$$

indicates that $\overset{r}{F}x$ is continuous, because $\overset{r}{F}(x+\omega)-\overset{r}{F}x=\omega(\overset{r}{f}x+\Omega)$ can become smaller than any given quantity if (for the same r and x) ω is taken small enough. But from this, it follows further that $\overset{r}{f}x$ must also vary continuously. For if x changes to $x+i$, then if $x+i+\omega$ still lies within a and b,

$$\frac{\overset{r}{F}(x+i+\omega)-\overset{r}{F}(x+i)}{\omega}=\overset{r}{f}(x+i)+\overset{\mathrm{I}}{\Omega}.$$

Therefore

$$\overset{r}{f}(x+i)-\overset{r}{f}x=\frac{\overset{r}{F}(x+i+\omega)-\overset{r}{F}x}{\omega}-\overset{\mathrm{I}}{\Omega}+\Omega,$$

which for the same x and r, can become as small as desired merely by diminishing ω and i. That is, merely by diminishing ω, Ω and $\overset{\mathrm{I}}{\Omega}$ can be made as small as desired. Then again for the same ω, i can be so taken such that $i+\omega$, and consequently also $\overset{r}{F}(x+i+\omega)-\overset{r}{F}x$ become as small as desired, therefore the same also applies to $\frac{\overset{r}{F}(x+i+\omega)-\overset{r}{F}x}{\omega}$, and also to the whole quantity

$$\frac{\overset{r}{F}(x+i+\omega)-\overset{r}{F}x}{\omega}-\overset{\mathrm{I}}{\Omega}+\Omega.$$

§ 30

Problem. A series developed in powers of x is either to be completely equal, or at least come as near as desired if the number of its terms is large enough, to the value of the complex function $(1+x)^n$, for as many values of n as possible, and for all x which lie between zero and a limit (positive or negative), which is as large as possible. The problem is to find certain conditions which the series must necessarily satisfy.

Solution. The assumption that a series developed in powers of x, i.e. a series of the form, $Ax^\alpha+Bx^\beta+Cx^\gamma+\cdots+Rx^\rho$, is either completely equal, or else comes as close as desired, to the value of the function $(1+x)^n$, is, by the argument of §§ 8 and 12, possible at least in certain cases. The series occurring there are included under the form of the present one if $\alpha=0$, $\beta=1$, $\gamma=2$, etc., and $A=1$, $B=n$,

$C = n.\frac{n-1}{2}$, etc. We must therefore be allowed to proceed on the assumption that an equation of the form

$$(1+x)^n = Ax^\alpha + Bx^\beta + Cx^\gamma + \cdots + Rx^\rho + \Omega \qquad \odot$$

holds *in an indeterminate number of cases*, and therefore to investigate now what conditions the quantities $\alpha, \beta, \gamma, \ldots, \rho, A, B, C, \ldots, R$, must satisfy whenever this equation holds.

1. Now if the equation $\odot$ holds either for every value of x or for every value smaller than a certain value b, then if, instead of x, we put $x+\omega$, and take ω so that $x+\omega < b$, then the following equation must also hold:

$$(1+x+\omega)^n = A(x+\omega)^\alpha + B(x+\omega)^\beta + C(x+\omega)^\gamma + \cdots + R(x+\omega)^\rho + \overset{1}{\Omega}.$$

Subtraction of the two and division by ω produces the third equation,

$$\frac{(1+x+\omega)^n-(1+x)^n}{\omega} = \frac{A(x+\omega)^\alpha - x^\alpha}{\omega} + \frac{B(x+\omega)^\beta - x^\beta}{\omega} + \frac{C(x+\omega)^\gamma - x^\gamma}{\omega} + \cdots + \frac{R(x+\omega)^\rho - x^\rho}{\omega} + \frac{\overset{1}{\Omega} - \Omega}{\omega}.$$

Now let ω denote a quantity which can be as small as desired (which is perfectly compatible with the previous condition), then it follows from §23 that, if one substitutes $1+x$ for x in it, the quantity

$$\frac{(1+x+\omega)^n-(1+x)^n}{\omega}$$

for every value of x which is not -1 and for every value of n, can come as close as desired to the quantity $n(1+x)^{n-1}$ if ω is taken small enough. But also under this condition, whatever $\alpha, \beta, \gamma, \ldots$ denote, and provided x is not $= 0$, the quantities

$$\frac{(x+\omega)^\alpha - x^\alpha}{\omega}, \frac{(x+\omega)^\beta - x^\beta}{\omega}, \ldots, \frac{(x+\omega)^\rho - x^\rho}{\omega}$$

come as near as desired to the following: $\alpha x^{\alpha-1}$, $\beta x^{\beta-1}, \ldots, \rho x^{\rho-1}$. Therefore if we write,

$$\frac{(1+x+\omega)^n-(1+x)^n}{\omega}=n(1+x)^{n-1}+\overset{(n)}{\omega},$$
$$\frac{(x+\omega)^\alpha-x^\alpha}{\omega}=\alpha x^{\alpha-1}+\overset{(\alpha)}{\omega},$$
$$\frac{(x+\omega)^\beta-x^\beta}{\omega}=\beta x^{\beta-1}+\overset{(\beta)}{\omega},$$
$$\ldots$$
$$\frac{(x+\omega)^\rho-x^\rho}{\omega}=\rho x^{\rho-1}+\overset{(\rho)}{\omega},$$

then $\overset{(n)}{\omega}, \overset{(\alpha)}{\omega}, \overset{(\beta)}{\omega}, \ldots, \overset{(\rho)}{\omega}$ denote values which can all become as small as desired if ω is taken small enough and we obtain by substitution the equation,

$$\frac{\overset{1}{\Omega}-\Omega}{\omega}=n(1+x)^{n-1}-\alpha Ax^{\alpha-1}-\beta Bx^{\beta-1}-\cdots$$
$$-\rho Rx^{\rho-1}+\overset{(n)}{\omega}-A\overset{(\alpha)}{\omega}-B\overset{(\beta)}{\omega}-\cdots-R\overset{(\rho)}{\omega}.$$

Now as a consequence of the assumption, $\Omega=(1+x)^n-Ax^\alpha-Bx^\beta-\cdots-Rx^\rho$ is a function of x which for all values of x which lies inside 0 and b can be made as small as desired merely by increasing its number of terms, therefore it is comparable with the $\overset{r}{F}x$ of §29.[1] The quantity $\overset{1}{\Omega}$ is $=(1+x+\omega)^n-A(x+\omega)^\alpha-B(x+\omega)^\beta-\cdots-R(x+\omega)^\rho$ and is therefore $=\overset{r}{F}(x+\omega)$ if one had put $\Omega=\overset{r}{F}x$. Consequently

$$\frac{\overset{1}{\Omega}-\Omega}{\omega}=\frac{\overset{r}{F}(x+\omega)-\overset{r}{F}x}{\omega}.$$

The quantity $\overset{(n)}{\omega}-A\overset{(\alpha)}{\omega}-B\overset{(\beta)}{\omega}-\cdots-R\overset{(\rho)}{\omega}$ consists of a number of terms which $=r+1$, but which, merely by diminishing ω with the same r, can become smaller than any given quantity. For example, if it is to be $<D$ then one just takes ω so small that

$$\overset{(n)}{\omega}<\frac{D}{r+1}, \overset{(\alpha)}{\omega}<\frac{D}{(r+1)A}, \overset{(\beta)}{\omega}<\frac{D}{(r+1)B}, \ldots, \overset{(\rho)}{\omega}<\frac{D}{(r+1)R},$$

[1] Original has §23 by mistake.

which by §23 must always be possible and then certainly $\overset{(n)}{\omega} - A\overset{(\alpha)}{\omega} - B\overset{(\beta)}{\omega} - \cdots - R\overset{(\rho)}{\omega} < D$. Therefore this quantity can be compared with the Ω of §29 and consequently the quantity, $n(1+x)^{n-1} - \alpha Ax^{\alpha-1} - \beta Bx^{\beta-1} - \cdots - \rho Rx^{\rho-1}$, considered as a mere function of x can be compared with the $\overset{r}{f}x$ of that paragraph. Accordingly it also holds of this quantity that for every value of x which lies between 0 and b it can become as small as desired if its number of terms is taken sufficiently large. We can therefore write,

$$n(1+x)^{n-1} - \alpha Ax^{\alpha-1} - \beta Bx^{\beta-1} - \cdots - \rho Rx^{\rho-1} = \overset{2}{\Omega}.$$

If one now multiplies this equation again by $(1+x)$, and puts for $(1+x)^n$ the series which this function should equal by $\odot$, then after simplifying those terms which obviously contain the same power of x, and noting that $n\Omega - (1+x)\overset{2}{\Omega} = \overset{3}{\Omega}$, the following equation appears which must hold as a *condition* for the validity of the equation $\odot$:

$$\begin{aligned}(n-\alpha)Ax^{\alpha} + (n-\beta)Bx^{\beta} + \cdots + (n-\rho)Rx^{\rho} \\ = \alpha Ax^{\alpha-1} + \beta Bx^{\beta-1} + \cdots + \rho Rx^{\rho-1} + \overset{3}{\Omega}.\end{aligned} \tag*{♁}$$

§ 31

Corollary. The last but one equation of the previous paragraph was

$$n(1+x)^{n-1} = \alpha Ax^{\alpha-1} + \beta Bx^{\beta-1} + \cdots + \rho Rx^{\rho-1} + \overset{(1)}{\Omega}.$$

If one applies the arguments by which this was derived from equation $\odot$, also to itself, then one gets the equation:

$$\begin{aligned}n(n-1)(1+x)^{n-2} = \alpha(\alpha-1)Ax^{\alpha-2} + \beta(\beta-1)Bx^{\beta-2} + \cdots \\ + \rho(\rho-1)Rx^{\rho-2} + \overset{(2)}{\Omega}.\end{aligned}$$

If this is treated in the same way, it gives:

$$\begin{aligned}&n(n-1)(n-2)(1+x)^{n-3} \\ &\quad = \alpha(\alpha-1)(\alpha-2)Ax^{\alpha-3} + \beta(\beta-1)(\beta-2)Bx^{\beta-3} + \cdots \\ &\qquad + \rho(\rho-1)(\rho-2)Rx^{\rho-3} + \overset{(3)}{\Omega}.\end{aligned}$$

And so on. Hence it may be seen that the possibility of the equation ⊙ generally requires the possibility of the following:

$$\begin{aligned}
n(n-1)(n-2)\dots(n-p)(1+x)^{n-p}&\\
=\alpha(\alpha-1)(\alpha-2)\dots(\alpha-p)Ax^{\alpha-p}&\\
+\beta(\beta-1)(\beta-2)\dots(\beta-p)Bx^{\beta-p}&\\
+\cdots&\\
+\rho(\rho-1)(\rho-2)\dots(\rho-p)Bx^{\rho-p}&\\
+\overset{(p)}{\Omega}&
\end{aligned}$$

where p can denote any whole and positive number. If one now multiplies by $(1+x)^p$ and puts for $(1+x)^n$ the equivalent series, then one obtains by use of §§ 15, 17 this *generally valid condition*:

$$n(n-1)(n-2)\dots(n-p)(Ax^{\alpha}+Bx^{\beta}+\cdots+Rx^{\rho})$$

$$=(1+x)^p\left\{\begin{array}{ll} & \alpha(\alpha-1)(\alpha-2)\dots(\alpha-p)Ax^{\alpha-p}\\ + & \beta(\beta-1)(\beta-2)\dots(\beta-p)Bx^{\beta-p}\\ + & \dots\dots\dots\dots\\ + & \rho(\rho-1)(\rho-2)\dots(\rho-p)Rx^{\rho-p}\end{array}\right\}+\overset{(p)}{\Omega} \qquad ♂$$

§ 32

Problem. To determine the form of the series $Ax^{\alpha}+Bx^{\beta}+\cdots+Rx^{\rho}$ which is to $=(1+x)^n-\Omega$ for as many n and x as possible, as far as this can be done merely from the *condition* ♁ given in §30.

Solution. The equation ♁, whose validity is a condition of the validity of ⊙, already suffices for determining the nature of the exponents $\alpha, \beta, \gamma, \dots, \rho$ as well as determining the relationship that must hold between the coefficients A, B, $C, \dots, R$, for the equation ⊙ to hold for as many n and x as possible.

1. Suppose that in the equation

$$(n-\alpha)Ax^{\alpha}+(n-\beta)Bx^{\beta}+\cdots+(n-\rho)Rx^{\rho}$$
$$=\alpha Ax^{\alpha-1}+\beta Bx^{\beta-1}+\cdots+\rho Rx^{\rho-1}+\overset{3}{\Omega},$$

the exponents $\alpha, \beta, \gamma, \dots, \rho$ are all *different* (which is possible because the terms in the series $Ax^{\alpha}+Bx^{\beta}+\cdots+Rx^{\rho}$ which contain the same power of x could be combined into one term that always remains in the form Ax^{α} or Bx^{β}). Furthermore suppose that these terms are all arranged according to the size of their exponents starting from the greatest negative (in case there is such—this must be possible since the number of terms is only finite). Then this equation has the form of the one described in §28 and therefore if it is to hold for all values of x which are smaller than a certain value, then every power of x on one side of the equality sign corresponds to a completely equal one on the other side with at most the exception of one or more parts which can be as small as desired. Hence, first of all, the exponents $\alpha, \beta, \gamma, \ldots, \rho$ of the series are $= 0, 1, 2, 3, \ldots, r$, i.e. they must proceed from zero in the order of the natural numbers. This is because all the exponents of the terms on the right-hand side of the equality sign are one lower than those of the equally many terms on the left hand side, thus it would be impossible for the equation to hold unless the lowest term among them, i.e. $\alpha Ax^{\alpha-1}$ vanished, since there is nothing equal to this on the other side. But the coefficient A itself cannot be $= 0$. For if one were to assume that, then the term Ax^{α} in the series ⊙ would really not be present at all. But now we understand by Ax^{α} the *lowest* term in this series and it must necessarily have such a term since the number of terms is assumed to be only finite, therefore A cannot be zero. Therefore $\alpha Ax^{\alpha-1}$ can only vanish if $\alpha = 0$. Hence it follows that the term $(n-\alpha)Ax^{\alpha}$ on the left-hand side $= nA$, a constant. Therefore also on the right-hand side there must be a constant term which can only be the lowest, therefore $\beta Bx^{\beta-1}$. Consequently $\beta - 1 = 0$, $\beta = 1$. Hence further, if n is not $= 1$ then on the left-hand side the term $(n-1)Bx$ exists. If this is to correspond to an equal one on the right-hand side it can only be the term $\gamma Cx^{\gamma-1}$, because all the following ones are still higher. Therefore $\gamma - 1 = 1$, i.e. $\gamma = 2$. It may be seen how these arguments can always be continued further. However, to understand even more clearly that none of the exponents $\alpha, \beta, \gamma, \ldots, \rho$ can be *fractional* or *negative*, one may consider the following. If there were fractional or negative exponents in the series ⊙ then one of them would have to come *first* in the arrangement established above, i.e. would be the smallest positive or largest negative. If we call this one μ then if μ were fractional then $\mu - 1$ would also represent a fractional exponent which is nevertheless smaller and if μ is negative, denotes an even greater negative exponent. Now since on the right-hand side a term with exponent $\mu - 1$ actually appears, there must also be on the left-hand side a term with a fractional or negative exponent which is less than μ. Therefore this contradicts the assumption and one must assume that all exponents of the series (with the exception of one which can be *zero*) are positive and whole-numbered. Hence it now appears that one could not be in error if the exponents are taken according to the order of the natural numbers. For then none of the terms which should actually appear can be omitted and if one term too many is assumed this will show itself in that its coefficient will be found to be $= 0$. But a little thought also shows that the exponents of the series do not increase to a larger number without first having gone through all smaller ones. For example, if μ is some exponent which actually occurs then there also appears in the series a term of the form $\mu Lx^{\mu-1}$, i.e. if μ is not *zero*, a term with exponent which is one smaller.

2. Now if one puts these values just found for the exponents $\alpha, \beta, \gamma, \ldots,$ and calls the *last* of them r (because one does not know how many there are), then the

equation obtained above takes the following form:

$$nA + (n-1)Bx + (n-2)Cx^2 + \cdots + (n-r)Rx^r$$
$$= B + 2Cx + 3Dx^2 + \cdots + rRx^{r-1} + \Omega.$$

From this the application of §28 obviously yields:

$$B = nA$$

$$2C = (n-1)B \quad \text{or} \quad C = \frac{n-1}{2}B$$
$$= n.\frac{n-1}{2}.A$$
$$3D = (n-2)C \quad \text{or} \quad D = \frac{n-2}{3}C$$
$$= n.\frac{n-1}{2}.\frac{-2}{3}.A, \quad \text{etc.}$$
$$rR = (n-(r-1))Q \quad \text{or} \quad R = \frac{n-(r-1)}{r}Q$$
$$= n.\frac{n-1}{2}.\frac{n-2}{3}\ldots\frac{n-(r-1)}{r}.A.$$

3. This determination takes care of all terms on both sides of the equation with the exception of the highest $(n-r)Rx^r$ which is

$$= n.\frac{n-1}{2}.\frac{n-2}{3}\ldots\frac{n-(r-1)}{r}.(n-r)A.x^r.$$

Therefore if the equation is to be possible there must be some value of the number r for which this term can either be made exactly zero or else smaller than any given quantity. Accordingly the form of the series which is to express the value of $(1+x)^n$ for as many values of n and x as possible—as far as this arises from the condition of §30—is as follows:

$$A + A.n.x + A.n.\frac{n-1}{2}.x^2 + A.n.\frac{n-1}{2}.\frac{n-2}{3}.x^3 + \cdots$$
$$+ A.n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}.x^r.$$

§ 33

Theorem. A series developed in powers of x which is either to give exactly the value of the complex function $(1+x)^n$ or is to be so close that the difference can become smaller than any given value for as many values of n as possible and all x which lie between zero and the greatest possible limit (positive or negative) can have no other form than that of the binomial series belonging to the power $(1 + x)^n$ if the nature of its terms is to be independent of their number, i.e. it must be

$$1 + nx + n.\frac{n-1}{2}x^2 + \cdots + n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}x^r.$$

Proof. As a consequence of the previous section this series must have the form $A + A.nx + A.n.\frac{n-1}{2}x^2 + \cdots + A.n.\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}x^r$ just in order to satisfy the condition of §30. But if the value of it, for all x between zero and some positive or negative limit, is either to be exactly equal to $(1 + x)^n$, or else to come as close to it as desired, then A must necessarily be $= 1$. For if x is taken as small as desired then the value of the function $(1 + x)^n$, whatever n denotes, approaches the value 1 as much as desired (§§ 17, 15). But with this assumption the value of the above series comes as close as desired to the value A. Therefore A must be $= 1$ (§27).

§ 34

Note. Hence we see that the equation of §30 still does not cover all the conditions which a series must satisfy which is to be $= (1 + x)^n$ for as many values of n and x as possible. For it is satisfied by every series of the form of §32, whatever A may be. But in order to be $= (1 + x)^n$, A may not have many values but only the definite value 1. We cannot therefore generally deduce that a certain substitution of the values x and n which satisfies equation ♁ will also hold for ⊙, but only conversely that if for certain assumed values of n and x equation ♁, or even one of the equations mentioned in §31, cannot be satisfied, then we certainly know that also ⊙ cannot hold. Furthermore, since in the series $1 + nx + n.\frac{n-1}{2}.x^2 + \cdots$, n and x everywhere only occur in whole-numbered powers then the series obviously has only a single value for every determinate n and x. On the other hand, the function $(1 + x)^n$ sometimes possesses *several values*—namely, if n is a fraction with an even denominator and what is intended to be expressed by $(1 + x)^n$ is a quantity capable of taking opposite signs, i.e. two equal and opposite values. Therefore at most one equation can be expected between $(1 + x)^n$ and that series if by $(1 + x)^n$ is to be understood as merely abstract quantity or if only one of its value is required. Finally $(1 + x)^{\frac{2p+1}{2q}}$ is well known to be *imaginary* for $x > -1$[m] and therefore again the binomial equation cannot be valid for all such values of x and n.

§ 35

Problem. To determine, merely from consideration of the *conditions* of §§ 30, 31 in which cases it is definitely impossible to express the value of $(1 + x)^n$ by the corresponding binomial series, where it may now either be completely exact or such that the difference can become smaller than every given quantity.

[m] In modern terms this means $x < -1$. See footnote on p. 158.

Solution. It can be decided merely from consideration of the equations ♁ and ♂ (§§ 30, 31) that the binomial series can never give the value of $(1+x)^n$ if x is > or even $= \pm 1$, unless at the same time n is either a whole positive number or zero.

1. For, *firstly*, let $x > +1$, then it is apparent that the equation ♁ cannot hold. For it is part of its nature that

$$n\frac{n-1}{2}\cdots\frac{n-(r-1)}{r}(n-r)Ax^r$$

can become as small as desired (§32, no.4.). Now if n is not $= 0$, nor any whole positive number, then it is first of all clear that this quantity can never completely vanish whatever may be put for r since none of the factors of which it consists:

$$n, \frac{n-1}{2}, \ldots, \frac{n-(r-1)}{r}, n-r, A, x^r$$

can ever become zero, since r only ever denotes a whole positive number. Therefore

$$n\frac{n-1}{2}\cdots\frac{n-(r-1)}{r}(n-r)Ax^r,$$

for the same A, n, and x, is a function of r which can never become zero. Therefore, however small its value may be for some definite r, it is, nevertheless, not smaller than any given quantity; if it is to be such, then, according to §26, it must only be by assuming ever greater and greater values of r. But if this is tried here, it will soon be apparent that, starting from a certain r, and for all greater ones, the values of this function, instead of decreasing, grow. For

$$n\frac{n-1}{2}\frac{n-2}{3}\cdots\frac{n-(r-1)}{r}(n-r)Ax^r$$

is a product of the two factors

$$n\frac{n-1}{2}\frac{n-2}{3}\cdots\frac{n-(r-1)}{r}Ax^r, \quad \text{and} \quad (n-r),$$

each of which, starting from a certain r, will only get larger, the larger r is taken. This is obvious with the factor $n-r=-(r-n)$, from where $r>n$ it grows with r even if n is positive, and if n is negative it always grows with r. But the factor

$$n\frac{n-1}{2}\frac{n-2}{3}\cdots\frac{n-(r-1)}{r}x^rA$$

also grows with r. For every later value of it, i.e. the one belonging to $(r+1)$, arises from the one directly before it, i.e. from the one belonging to r, since it is multiplied by $\frac{n-r}{r+1}x = -\frac{(r-n)}{r+1}x$. Now if, (α) n is *positive*, then from the point where $r>n$, and > the positive quantity $\frac{nx+1}{x-1}$, and for all greater values, $r(x-1) > nx+1$, or $rx - r > nx + 1$; therefore also, because $r>1$ and $rx > nx$, $rx - nx > r+1$, and consequently $\frac{(r-n)}{r+1}x > 1$. Therefore every subsequent value of the factor

$$n\frac{n-1}{2}\frac{n-2}{3}\dots\frac{n-(r-1)}{r}Ax^r$$

is greater than the preceding one.
(β) But if n is *negative* and for example $= -m$ where now m represents a positive quantity, then it is all the more obvious that with the same value of r as before it must be that, $\frac{(r+m)}{r+1}x > 1$, if $\frac{(r-n)}{r+1}x > 1$.
2. Now *secondly* let $x = +1$, then the expression

$$n\frac{n-1}{2}\dots\frac{n-(r-1)}{r}(n-r)Ax^r = n\frac{n-1}{2}\dots\frac{n-(r-1)}{r}(n-r)A.$$

Furthermore, let (α) n be *negative*, e.g. $= -m$ where now m represents a positive quantity, then the above expression

$$= (-m)\frac{-m-1}{2}\frac{-m-2}{3}\dots\frac{-m-(r-1)}{r}(-m-r)A.$$

The value of it for $r+1$ is

$$(-m)\frac{-m-1}{2}\frac{-m-2}{3}\dots\frac{-m-(r-1)}{r}\frac{-m-r}{r+1}(-m-r-1)A.$$

Therefore the quotient of the latter by the former, or that by which one would multiply the former to obtain the latter

$$= \frac{-m-r-1}{r+1} = -\frac{r+1+m}{r+1} > 1.$$

Therefore every subsequent value will be greater than the preceding value, the equation ♁ can therefore not hold.
(β) But in order to understand that it could not also hold in the case that n is *positive* it needs a rather more extensive investigation. To this end notice from §31 that the possibility of the equation ♁ also requires the possibility of equation ♂. If one imagines the multiplication indicated there to be actually done then one may understand that the highest power of x on one side of the equality sign is $n(n-1)(n-2)\dots(n-p)Rx^\rho$ and on the other side is $\rho(\rho-1)(\rho-2)\dots(\rho-p)Rx^\rho$. Therefore if the equation is to hold then

$$(n(n-1)(n-2)\dots(n-p) - \rho(\rho-1)(\rho-2)\dots(\rho-p))\,Rx^\rho$$

must either be zero or else can be made as small as desired. But in this expression, as we know from §31, $\rho =$ some whole positive number r and

$$R = n\frac{n-1}{2}\dots\frac{n-(r-1)}{r}A$$

it is therefore the same as

$$(n(n-1)(n-2)\dots(n-p)-r(r-1)(r-2)\dots(r-p))$$
$$\times\, n.\frac{n-1}{2}\dots\frac{n-(r-1)}{r}.Ax^r$$

and its subsequent value for $r+1$ is $=$

$$(n(n-1)(n-2)\dots(n-p)-(r+1)r(r-1)\dots(r+1-p))$$
$$\times\, n.\frac{n-1}{2}\dots\frac{n-(r-1)}{r}.\frac{n-r}{r+1}.Ax^{r+1}.$$

Now if n is not a *whole positive* number and not *zero*, then it is obvious that no factor of these two expressions $=0$; one can therefore divide the former into the latter in order to find what the former must be multiplied by to obtain the latter. It is

$$\left(\frac{n(n-1)(n-2)\dots(n-p)-(r+1)r(r-1)\dots(r+1-p)}{n(n-1)(n-2)\dots(n-p)-r(r-1)\dots(r-p)}\right)\frac{n-r}{r+1}x.$$

For $x=1$ therefore this factor

$$=\left(\frac{(r+1)r(r-1)\dots(r+1-p)-n(n-1)(n-2)\dots(n-p)}{r(r-1)\dots(r-p)-n(n-1)(n-2)\dots(n-p)}\right)\frac{n-r}{r+1}.$$

If one actually works out the multiplication indicated of the factors which contain r, in the numerator as well as the denominator of this fraction, and arranges everything by powers of r, then one obtains an expression of the following form:

$$\frac{r^{p+2}+(-n+1+0-1-2-\cdots-(p-1))\,r^{p+1}+\cdots}{r^{p+2}+(+1+0-1-2-\cdots-p)\,r^{p+1}+\cdots}.$$

The terms which are indicated here merely by dots contain only lower powers of r, but the coefficients of r^{p+1} are determined by the known rule, as a consequence of which the coefficient of the second term in the development of a product of the form $(x+a)(x+b)(x+c)\dots$ must be $=(a+b+c+\cdots)$ (§6). Therefore if one actually divides the denominator into the numerator, the quotient,

$$=1+\frac{(-n+1-1-2-\cdots-(p-1)-1+1+2+\cdots+p)r^{p+2}+\cdots}{r^{p+1}+(+1-1-2-\cdots-p)r^{p+1}}$$

$$=1+\frac{(p-n)r^{p+1}+\cdots}{r^{p+2}+(+1-1-2-\cdots-p)r^{p+1}+\cdots}.$$

Now in this expression p can denote any whole positive number, therefore also one which is larger than the positive n. But then in the last fraction the coefficient of the first term in the numerator is positive, just like the first term in the denominator. Since r can be taken as large as desired and all terms following the first, in the numerator as well as the denominator, clearly contain only lower powers of r, then r can always be taken so great that the first term in the numerator and denominator is greater than all the following ones taken together (§25). For such an r, and for all greater ones, the value of that fraction is certainly positive, and consequently,

$$1 + \frac{(p-n)r^{p+1} + \cdots}{r^{p+2} + \cdots} > 1.$$

Therefore, every subsequent value of the expression

$$(n(n-1)(n-2)\ldots(n-p) - r(r-1)(r-2)\ldots(r-p))\,Rx^r$$

will be greater than the one before it and therefore the equation ♂, and therefore ♁ and ⊙, cannot possibly hold.

3. Now if the quantities which we have considered hitherto cannot become as small as desired whenever $x >$ or $= +1$, then they could also not do this for the values of $x >$ or $= -1$. For they contain no other function of x than the factor x^r which at most changes only its sign and not its magnitude if x changes its sign. Therefore the binomial equation holds for no value of x which is $>$ or even only $= \pm 1$, unless at the same time n is a whole positive number or zero.

§ 36

Corollary. On the other hand, whenever $x < \pm 1$ then the equation ♁ holds without exception, n may denote whatever desired, because the term

$$n\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}(n-r)Ax^r$$

can then always be made as small as desired if r is taken large enough. For the term belonging to $r+1$, i.e. the next value of this quantity is

$$n\frac{n-1}{2}\ldots\frac{n-(r-1)}{r}\frac{n-r}{r+1}(n-r-1)Ax^{r+1}.$$

Therefore the quotient of the latter by the former, or that by which one must multiply the former in order to obtain the latter, is

$$= \frac{n-r-1}{r+1}x = \frac{-r+1-n}{r+1}x.$$

Now if, (α) firstly n is *positive*, then as soon as $(r+1) > n$ and for all following values of r, $\frac{r+1-n}{r+1}$ is obviously a proper fraction which indeed increases if r increases, but always remains <1; therefore the product $\frac{r+1-n}{r+1}.x$ is always somewhat smaller than the proper fraction x. Therefore every subsequent value of the quantity

$$n\frac{n-1}{2}\ldots\frac{n+r-1}{r}(n-r)Ax^r$$

arises from the previous one by multiplication with a proper fraction which always remains smaller than x. Therefore this quantity can be considered as a product of which one factor (namely the value which it has for the greatest r which is still not $>(n-1)$) remains unchanged, while the other forms a product of an

arbitrarily large number of fractions each of which is $<x$. By §22 and §17 such a quantity can become smaller than any given quantity.
(β) But *secondly* if n is *negative*, e.g. $= -m$, then $\frac{r+1-n}{r+1} = \frac{r+1+m}{r+1}$ which is always >1. But a value of r can always be given, for which and for all greater values, the product of this improper fraction with the proper fraction x is <1. For to do this one just takes $(r+1) > \frac{mx}{1-x}$, then also $(r+1)(1-x) > mx$, consequently, since $1 > x$, $(r+1) > (r+1)x + mx = (r+1+m)x$. Therefore $\frac{(r+1+m)}{r+1}x$ is a proper fraction. But the larger one takes r from this value onwards, the smaller will be this proper fraction since one factor of it, the improper fraction $\frac{r+1+m}{r+1}$, always approaches closer to unity. Therefore the above arguments also hold here, or the quantity

$$n\frac{n-1}{2}\ldots\frac{n-r+1}{r}(n-r)Ax^r$$

is a product of a constant quantity with an arbitrarily large number of proper fractions which are all smaller than $\frac{r+1+m}{r+1}x$.

§ 37

Transition. Now it is already decided from the foregoing [sections] that the binomial equation does not hold for any value of $x > \pm 1$ unless the exponent is zero or a whole positive number. Indeed this is decided simply from the condition ♁. But as we have just seen, this equation does not say that the binomial equation itself would be invalid for all values of x which are $<\pm 1$. And thereby it has shown all that it can show us. For in no case can we expect of it such an *affirmative* decision that the binomial equation *actually holds* for those values. Therefore we must now turn to completely different grounds to decide whether the binomial equation holds in all cases, and if not, in which cases, when $x < \pm 1$.

§ 38

Theorem. If the two binomial series,

$$1 + px + p\frac{p-1}{2}x^2 + \cdots + p\frac{p-1}{2}\ldots\frac{p-(r-1)}{r}x^r$$

and

$$1 + qx + q\frac{q-1}{2}x^2 + \cdots + q\frac{q-1}{2}\ldots\frac{q-(s-1)}{s}x^s$$

in which p and q denote any kind of quantity, are multiplied together and the product is arranged by powers of x, then all terms of the product starting from the first up to the term x^r or x^s according to whether r or s is the smaller number, are identical with the equally many terms of the binomial series belonging to $(1+x)^{(p+q)}$.

Proof. The terms of the binomial series belonging to $(1+x)^{(p+q)}$ starting from the first up to the term x^m are, by §3:

$$1 + (p+q)x + (p+q)\left(\frac{p+q-1}{2}\right)x^2 + \cdots$$
$$+ (p+q)\left(\frac{p+q-1}{2}\right)\cdots\left(\frac{p+q-m+1}{m}\right)x^m.$$

Therefore the equally many terms of the former product, if it is developed appropriately, will be identical with these. Now it is very easily made clear that the *first, second, third,* etc., terms, do in fact coincide by actually doing the multiplication. But that this also holds for the subsequent terms up to the term with the power x^r or x^s, is made clear in the following way.

1. If p and q denote two *whole and positive* numbers, then by §8:

$$1 + px + p\frac{p-1}{2}x^2 + \cdots + p\frac{p-1}{2}\cdots\frac{p-r+1}{r}x^r = (1+x)^p$$

and $$1 + qx + q\frac{q-1}{2}x^2 + \cdots + q\frac{q-1}{2}\cdots\frac{q-s+1}{s}x^s = (1+x)^q,$$

provided r and s are not taken smaller than p and q. Therefore the product of these two series, which we shall call M, necessarily has the value

$$= (1+x)^p(1+x)^q = (1+x)^{p+q}$$
$$= 1 + (p+q)x + (p+q)\left(\frac{p+q-1}{2}\right)x^2 + \cdots$$
$$+ (p+q)\left(\frac{p+q-1}{2}\right)\cdots\left(\frac{p+q-t+1}{t}\right)x^t,$$

because $(p+q)$ also represents a whole positive number. This equality holds for *every* value of x, therefore also for those that are smaller than a certain limit. Therefore if M is arranged appropriately by powers of x, then §28 may be applied to this equation, i.e. to every term of it on the one side there must be an exactly equal term on the other side. Now if m represents some whole positive number which is not $>r$ or $>s$ and the coefficient in the product M belonging to the power x^m is denoted by $F(p,q)$ (because for the same m it can only be a function of p and q) then at least in value, it must be that,

$$F(p,q) = (p+q)\left(\frac{p+q-1}{2}\right)\cdots\left(\frac{p+q-m+1}{m}\right).$$

Now if m is left constant, while p and q are arbitrarily *increased*, then obviously the *form* of $F(p,q)$ cannot change at all (i.e. the rule indicated in this expression for deriving the value of $F(p,q)$ from p and q). For the increase in the numbers p and q changes nothing in the series $1 + px + \cdots$, and $1 + qx + \cdots$, which constitute the factors in the product M, unless the number of their terms has to be increased beyond r and s. But since m is to be neither $>r$, nor $>s$, then none of the new terms which are introduced to the factors by the increase of p and q, have an influence on the formation of the coefficient of x^m. Naturally only those terms of the two factors whose exponents are equal to, or smaller than m have an influence on it. Therefore there are innumerably many values which can be put for p and q without disturbing the equation,

$$F(p,q) = (p+q)\left(\frac{p+q-1}{2}\right)\dots\left(\frac{p+q-m+1}{m}\right),$$

and it follows from this that they must be *identical* in respect of the quantities p and q. In order to understand this consequence very clearly let us consider the form of the two terms of this equation somewhat more precisely. The binomial coefficient

$$(p+q)\left(\frac{p+q-1}{2}\right)\dots\left(\frac{p+q-m+1}{m}\right),$$

if it is developed by powers of one of the two quantities p or q, e.g. p, gives an expression of the form $Ap^m + Bp^{m-1} + Cp^{m-2} + \cdots + Lp$, where $A, B, C, \dots, L$ represent partly constant quantities and partly functions of q alone. But also $F(p,q)$, if it is arranged by powers of p, can only be of the form[n] $A^1p^m + B^1p^{m-1} + C^1p^{m-2} + \cdots + L^1p$, since this function can obviously contain no higher power of p than the mth, because no higher power appears in those terms of the factors of M from the combination of which $F(p,q)x^m$ has been formed. Therefore the equation holds:

$$Ap^m + Bp^{m-1} + Cp^{m-2} + \cdots + Lp = A^1p^m + B^1p^{m-1} + C^1p^{m-2} + \cdots + L^1p$$

or

$$(A-A^1)p^m + (B-B^1)p^{m-1} + (C-C^1)p^{m-2} + \cdots + (L-L^1)p = 0,$$

in which one must be able to put innumerably many values for p. Hence it is now clear that every coefficient of a particular power of p must, in itself, $= 0$ because otherwise, if some of these coefficients were real quantities then by a well-known property of equations only a finite number of values would be possible for p (at most m). Accordingly,

$$A = A^1, B = B^1, C = C^1, \dots, L = L^1.$$

These quantities themselves, which are partly still functions of q can clearly only be of the form

$$aq^m + bq^{m-1} + \cdots + lq; \quad \text{and}$$
$$a^1q^m + b^1q^{m-1} + \cdots + l^1q$$

[n] The printer of Bolzano's text uses numeral '1' here, and below, as a distinguishing mark, or prime, not as a power.

where now $a, b, \ldots, l$ and $a^{\mathrm{I}}, b^{\mathrm{I}}, \ldots, l^{\mathrm{I}}$ denote only constant quantities. Now, for example, let $aq^m + bq^{m-1} + \cdots + lq$ represent one of the equal quantities $A = A^{\mathrm{I}}$, or $B = B^{\mathrm{I}}$, etc., and $a^{\mathrm{I}}q^m + b^{\mathrm{I}}q^{m-1} + \cdots + l^{\mathrm{I}}q$ the other, then always the equation $aq^m + bq^{m-1} + \cdots + lq = a^{\mathrm{I}}q^m + b^{\mathrm{I}}q^{m-1} + \cdots + l^{\mathrm{I}}q$ holds for innumerably many values of q, from which it necessarily follows that $a = a^{\mathrm{I}}$, $b = b^{\mathrm{I}}, \ldots, l = l^{\mathrm{I}}$. Therefore the function $F(p, q)$ is composed from p and q and certain constant quantities $a, b, \ldots, l$, etc. in exactly the same way as

$$(p+q)\left(\frac{p+q-1}{2}\right)\cdots\left(\frac{p+q-m+1}{m}\right),$$

therefore it is identical with it. Now since m can denote every whole positive number which is not $> r$ or $> s$, it is clear that all terms in the product M of which the power is not $> r$ or $> s$, are identical to the equally many terms in the binomial series belonging to $(1 + x)^{p+q}$ if p and q denote whole positive numbers.
2. But the same must also be the case if p and q denote any other kind of quantity (fractional, negative or even irrational). For the way one proceeds with the multiplication of the two series $1 + px + \cdots$, $1 + qx + \cdots$, remains the same whatever the letters p and q may denote. If we therefore know (from 1.) that in the case in which p and q are whole positive numbers one obtains the coefficients of the binomial series belonging to $(1 + x)^{p+q}$, then we can conclude from this with certainty that this must also happen in every other case.

§ 39

Note. An exact equality between that product and the binomial series still does not follow from what has just been proved, but only that both series are formed in the same way [*gleichartig*] up to the term which is multiplied by the smaller of the two powers x^r, x^s. Now r and s can indeed be taken as large as desired and therefore one may extend the equality [*Gleichförmigkeit*] of the two series as far as desired. But if one continues this further and for this purpose one takes r and s larger, then there are always a number of other terms after x^r (or x^s) whose exponents are higher (namely which increase up to $(r + s)$), and these are not at all identical with the equally many powers of the binomial series belonging to $(1 + x)^{p+q}$. The number of these terms will be greater as r and s are taken greater because it will always equal the greater of these two numbers. Therefore, however many terms in the product M conform to the binomial series, at least as many also deviate from it. And if $x > \pm 1$, and n is not a whole positive number then the deviating terms are always much greater, in value, than the conforming terms. Therefore in such a case the two series certainly cannot be said to be equal. Only in the case when x is a proper fraction does the special circumstance occur that the unequal terms always become smaller and that one can actually make the value of their sum as small as desired by the increase in r and s. This is proved in the following paragraph.

§ 40

Theorem. The value of the product of the two series

$$1 + px + p\frac{p-1}{2}x^2 + \cdots + p\frac{p-1}{2}\cdots\frac{p-r+1}{r}x^r$$

and $$1 + qx + q\frac{q-1}{2}x^2 + \cdots + q\frac{q-1}{2}\cdots\frac{q-s+1}{s}x^s$$

differs from the value of the series,

$$1 + (p+q)x + (p+q)\left(\frac{p+q-1}{2}\right)x^2 + \cdots + (p+q)\left(\frac{p+q-1}{2}\right)\cdots\left(\frac{p+q-t+1}{t}\right)x^t,$$

by a quantity which can be made smaller than any given quantity if r, s, t are taken large enough and x is a proper fraction.

Proof. According to §38 the product of those two series up to the term x^r (if r denotes the smaller of the two numbers r and s) is identical with the binomial series last mentioned. But then in the product other terms follow which may be found if one of the series, e.g. the first, is multiplied term by term with those terms of the second which when combined with the multiplier produce a power higher than x^r. To save space we shall denote the coefficients of the first series in order by, $1, p, \overset{(2)}{p}, \overset{(3)}{p}, \ldots, \overset{(r)}{p}$, those of the second series by, $1, q, \overset{(2)}{q}, \overset{(3)}{q}, \ldots, \overset{(s)}{q}$. Accordingly the terms which appear later on in the product are the following:

$$\begin{aligned}&\left(\overset{(r+1)}{q}x^{r+1} + \overset{(r+2)}{q}x^{r+2} + \cdots + \overset{(s)}{q}x^s\right)\\ &+ \left(\overset{(r)}{q}x^r + \overset{(r+1)}{q}x^{r+1} + \cdots + \overset{(s)}{q}x^s\right)px\\ &+ \left(\overset{(r-1)}{q}x^{r-1} + \overset{(r)}{q}x^r + \cdots + \overset{(s)}{q}x^s\right)\overset{(2)}{p}x^2\\ &+ \cdots\\ &+ \left(qx + \overset{(2)}{q}x^2 + \cdots + \overset{(s)}{q}x^s\right)\overset{(r)}{p}x^r.\end{aligned}$$

The sum of these terms is the difference of the product M and the binomial series belonging to $(1+x)^{p+q}$, if one takes in it $t = r$. Therefore if it can be shown that this sum can be made smaller than any given quantity by the increase in r, then the theorem is proved. But obviously the value of this sum is smaller than that which appears if all terms of it are viewed as of the same kind (e.g. as positive), and in the place of the different coefficients $p, \overset{(2)}{p}, \overset{(3)}{p}, \ldots, \overset{(r)}{p}; q, \overset{(2)}{q}, \overset{(3)}{q}, \ldots, \overset{(s)}{q}$ one always puts the greatest of its kind which we shall denote by P and Q. This gives the sum,

$$
\begin{aligned}
&= Qx^{r+1}(1 + x + x^2 + \cdots + x^{s-r-1}) \\
&\quad + PQx^{r+1}(1 + x + x^2 + \cdots + x^{s-r}) \\
&\quad + PQx^{r+1}(1 + x + x^2 + \cdots + x^{s-r+1}) \\
&\quad + \cdots \\
&\quad + PQx^{r+1}(1 + x + x^2 + \cdots + x^{s-1}) \\
&= Qx^{r+1}\frac{1 - x^{s-r}}{1 - x} \\
&\quad + PQx^{r+1}\left(\frac{1 - x^{s-r+1}}{1 - x} + \frac{1 - x^{s-r+2}}{1 - x} + \cdots + \frac{1 - x^{s}}{1 - x}\right) \\
&= Q\frac{x^{r+1} - x^{s+1}}{1 - x} + PQ\frac{x^{r+1}}{1 - x}(r - x^{s-r+1}(1 + x + x^2 + \cdots + x^{r-1})) \\
&= Q\frac{x^{r+1} - x^{s+1}}{1 - x} + PQ\frac{x^{r+1}}{1 - x}\left(r - x^{s-r+1}\frac{1 - x^{r}}{1 - x}\right) \\
&= Q\frac{x^{r+1} - x^{s+1}}{1 - x} + PQ\frac{rx^{r+1}}{1 - x} - PQ\frac{x^{s+2} - x^{s+r+2}}{(1 - x)^2}.
\end{aligned}
$$

Now if the exponent is *positive*, then the binomial coefficients, starting from a certain value of r (namely from that for which $r > \frac{1}{2}n$) always become ever smaller, the larger r is taken. But if the exponent is *negative*, they grow steadily. If we can therefore prove that the expression itself in the latter case can become as small as desired, then it is clear that it can all the more certainly become as small as desired in every other case. Now in this case, the greatest coefficient is always the last, therefore

$$P = p\frac{p-1}{2}\cdots\frac{p-r+1}{r} \quad \text{and} \quad Q = q\frac{q-1}{2}\cdots\frac{q-s+1}{s}.$$

If we substitute these values then it appears that each of the above three terms can become as small as desired through the increase in r and s, consequently by §15, this also applies to their sum. The term $Q\frac{x^{r+1}-x^{s+1}}{1-x}$ can, of course, become as small as desired if the two parts Qx^{r+1}, Qx^{s+1} of its numerator can become so since the denominator $1 - x$ is a constant quantity (§17). Now

$$Qx^{r+1} = q\frac{q-1}{2}\cdots\frac{q-s+1}{s}x^{r+1}.$$

And if one arbitrarily increases r and s, e.g. each of them by unity, then the new value which this term takes arises from the previous one by multiplication by $\frac{q-s}{s+1}x = -\frac{s-q}{s+1}x$, a quantity which, starting from a certain s and for all greater ones, is a proper fraction which steadily approaches the value x even if q is *negative*. Therefore the term considered, by the arbitrary increase of r and s will be multiplied by an arbitrarily large number of proper fractions, which become ever smaller and can therefore become smaller than any given quantity (§22). The same thing can be shown in the same way of the term,

$$Qx^{s+1} = q\frac{q-1}{2}\dots\frac{q-s+1}{s}x^{s+1},$$

and also of the terms PQx^{s+2}, PQx^{s+r+2}, which form the third part of the expression to be calculated. Finally also the part $\frac{PQrx^{r+1}}{1-x}$ can become as small as desired if r, s are taken large enough. For it is

$$= p\frac{p-1}{2}\dots\frac{p-r+1}{r}.q\frac{q-1}{2}\dots\frac{q-s+1}{s}.r\frac{x^{r+1}}{1-x}.$$

Now if one arbitrarily increases r and s, e.g. each of them by 1, then the new value which this quantity takes arises from the previous one by multiplication by

$$\frac{p-r}{r+1}\frac{q-s}{s+1}\frac{r+1}{r}x = \frac{r-p}{r}\frac{s-q}{s+1}x.$$

Now even if p and q are negative, i.e. if $\frac{r-p}{r}$, $\frac{s-q}{s+1}$ are improper fractions, their values can come as close to unity as desired if r and s are taken large enough, there is a value of r and s, for which, and for all subsequent values, the product $\frac{r-p}{r}\frac{s-q}{s+1}x$ is a proper fraction which becomes ever smaller and always approaches the value x. Therefore also $\frac{PQrx^{r+1}}{1-x}$ must be able to become as small as desired if r and s are taken large enough.

§ 41

Theorem. If the binomial equation holds for two specific values of the exponent, p and q, either exactly or so that the difference can become smaller than any given quantity, then it holds in just the same sense for the exponent $(p+q)$ provided x always remains the same and is a proper fraction.

Proof. As a consequence of the assumption,

$$(1+x)^p = 1 + px + p\frac{p-1}{2}x^2 + \dots + p\frac{p-1}{2}\dots\frac{p-r+1}{r}x^r + \overset{(p)}{\Omega},$$

and

$$(1+x)^q = 1 + qx + q\frac{q-1}{2}x^2 + \dots + q\frac{q-1}{2}\dots\frac{q-s+1}{s}x^s + \overset{(q)}{\Omega}$$

and $\overset{(p)}{\Omega}$ and $\overset{(q)}{\Omega}$ in these equations either denote zero or a quantity which can be made smaller than any given quantity if r and s are taken large enough.

Therefore $(1+x)^p(1+x)^q = (1+x)^{p+q}$

$$= \left(1 + px + \cdots + p\frac{p-1}{2}\ldots\frac{p-r+1}{r}x^r + \overset{(p)}{\Omega}\right)$$
$$\times\left(1 + qx + \cdots + q\frac{q-1}{2}\ldots\frac{q-s+1}{s}x^s + \overset{(q)}{\Omega}\right)$$

$$= \left(1 + px + \cdots + p\frac{p-1}{2}\ldots\frac{p-r+1}{r}x^r\right)$$
$$\times\left(1 + qx + \cdots + q\frac{q-1}{2}\ldots\frac{q-s+1}{s}x^s\right)$$
$$+\left((1+x)^p - \overset{(p)}{\Omega}\right)\overset{(q)}{\Omega}$$
$$+\left((1+x)^q - \overset{(q)}{\Omega}\right)\overset{(p)}{\Omega}.$$

The latter two quantities can, by §17, become as small as desired if r, s are taken large enough. But the product of those series can, because x is to be a proper fraction, by §40, be brought as close as desired, if r and s are taken large enough, to the value of the series:

$$1 + (p+q)x + (p+q)\left(\frac{p+q-1}{2}\right)x^2 + \cdots$$
$$+ (p+q)\left(\frac{p+q-1}{2}\right)\ldots\left(\frac{p+q-r+1}{r}\right)x^r,$$

(where r denotes the smaller of the two numbers r and s). Therefore, by §15, we can write,

$$(1+x)^{p+q} = 1 + (p+q)x + (p+q)\left(\frac{p+q-1}{2}\right)x^2 + \cdots$$
$$+ (p+q)\left(\frac{p+q-1}{2}\right)\ldots\left(\frac{p+q-r+1}{r}\right)x^r + \overset{2}{\Omega}$$

where $\overset{2}{\Omega}$ can become smaller than any given quantity if r is taken large enough. With this meaning of the letter 'r' the last series is what is called the binomial series belonging to the power $(1+x)^{p+q}$. Therefore the binomial equation also holds for the exponent $(p+q)$ in the sense that the difference can be made either zero or smaller than any given quantity if r is taken large enough.

§ 42

Theorem. The binomial equation also holds for every *whole negative* number as exponent, in the sense that the difference can become smaller than any given quantity if x is a proper fraction and the number of terms is taken large enough.

Proof. This equation holds, as we have already seen in §12, for the value $n = -1$ in the sense that the difference can become smaller than any given quantity if r is taken large enough and x is a proper fraction. Hence if one puts in §41 $p = -1$, $q = -1$, the binomial equation also holds in the same sense for the exponent $= -2$. And again if one now puts $p = -2$, $q = -1$, it also holds for the exponent $= -3$, etc. Since one can always continue to argue in this way and can reach every whole negative number by repeated addition of -1 to -1, the truth of the theorem is clear.

§ 43

Theorem. The binomial equation also holds for every *fractional positive* value of the exponent of the form $\frac{1}{m}$ where m denotes a whole number, in the sense that the difference can be made smaller than any given quantity if one increases the number of terms sufficiently and x designates a *proper fraction*; it is always assumed that by $(1+x)^{\frac{1}{m}}$ is to be understood only the *real* and *positive* value which this expression can denote.

Proof. The value of the series,

$$1 + \frac{1}{m}x + \frac{1}{m}\frac{\frac{1}{m}-1}{2}x^2 + \cdots + \frac{1}{m}\frac{\frac{1}{m}-1}{2}\cdots\frac{\frac{1}{m}-r+1}{r}x^r$$

which is the binomial series belonging to the power $(1+x)^{\frac{1}{m}}$, always remains, however large r may be taken in it, smaller than a certain constant quantity, provided $x < \pm 1$. For the coefficients of this series,

$$\frac{1}{m}, \frac{1}{m}\frac{\frac{1}{m}-1}{2}, \ldots, \frac{1}{m}\frac{\frac{1}{m}-1}{2}\cdots\frac{\frac{1}{m}-r+1}{r}$$

are all proper fractions since the first one is a proper fraction and each one arises from the preceding one by multiplication by a proper fraction of the form,

$$\frac{\frac{1}{m}-p+1}{p} = -\frac{p-1-\frac{1}{m}}{p}.$$

Therefore suppose all terms of the series were of the same kind with respect to their sign, then their sum would certainly always be smaller than,

$$1 + x + x^2 + \cdots + x^r = \frac{1-x^{r+1}}{1-x}.$$

Now if $x < \pm 1$, then x^{r+1} is all the smaller, the larger r is taken. If at the same time x is *positive*, the value of $\frac{1-x^{r+1}}{1-x}$ is obviously always $< \frac{1}{1-x}$. But if x is negative then $-x^{r+1}$ is positive for every even value of r and therefore $\frac{1-x^{r+1}}{1-x}$ is larger than $\frac{1}{1-x}$ but nevertheless always smaller than the value of the same quantity for some smaller even r, therefore always smaller than (if one puts $r=0$), $\frac{1-x}{1-x}=1$. Therefore $\frac{1}{1-x}$ or 1 are the limits which the value of that series never reaches however large r may be taken in it. If we therefore call a particular value of that series U, then U is a quantity which always remains smaller than $\frac{1}{1-x}$ or 1. If the series is multiplied by itself, then the value of this product, by §41, if one puts $p=\frac{1}{m}$, $q=\frac{1}{m}$ there, differs from the value of the series,

$$1+\frac{2}{m}x+\frac{2}{m}\frac{\frac{2}{m}-1}{2}x^2+\cdots+\frac{2}{m}\frac{\frac{2}{m}-1}{2}\cdots\frac{\frac{2}{m}-r+1}{r}x^r$$

by a quantity which can be made smaller than any given quantity if r is taken large enough. Therefore one can write,

$$U^2=1+\frac{2}{m}x+\frac{2}{m}\frac{\frac{2}{m}-1}{2}x^2+\cdots+\frac{2}{m}\frac{\frac{2}{m}-1}{2}\cdots\frac{\frac{2}{m}-r+1}{r}x^r+\Omega.$$

If again this equation is multiplied by

$$U=1+\frac{1}{m}x+\frac{1}{m}\frac{\frac{1}{m}-1}{2}x^2+\cdots+\frac{1}{m}\frac{\frac{1}{m}-1}{2}\cdots\frac{\frac{1}{m}-r+1}{r}x^r$$

one obtains

$$U^3=\left(1+\frac{2}{m}x+\cdots+\frac{2}{m}\frac{\frac{2}{m}-1}{2}\cdots\frac{\frac{2}{m}-r+1}{r}x^r\right)$$
$$\times\left(1+\frac{1}{m}x+\cdots+\frac{1}{m}\frac{\frac{1}{m}-1}{2}\cdots\frac{\frac{1}{m}-r+1}{r}x^r\right)+U\Omega.$$

By §17 the last term can become as small as desired if by increasing r, Ω is diminished sufficiently. But in the same circumstances that product of the two series approaches as close as desired, by §41, to the value of the series,

$$1+\frac{3}{m}x+\frac{3}{m}\frac{\frac{3}{m}-1}{2}x^2+\cdots+\frac{3}{m}\frac{\frac{3}{m}-1}{2}\cdots\frac{\frac{3}{m}-r+1}{r}x^r.$$

Therefore by §15 one has,

$$U^3=1+\frac{3}{m}x+\frac{3}{m}\frac{\frac{3}{m}-1}{2}x^2+\cdots+\frac{3}{m}\frac{\frac{3}{m}-1}{2}\cdots\frac{\frac{3}{m}-r+1}{r}x^r+\overset{3}{\Omega}.$$

Since these arguments can be continually repeated, one must, because m is a whole positive number, be able finally to reach the following equation,

$$U^m=1+\frac{m}{m}x+\frac{m}{m}\frac{\frac{m}{m}-1}{2}x^2+\cdots+\frac{m}{m}\frac{\frac{m}{m}-1}{2}\cdots\frac{\frac{m}{m}-r+1}{r}x^r+\overset{m}{\Omega}$$
$$=1+x+\overset{m}{\Omega}.$$

If one therefore extracts the mth power root from both sides of this equation and only the positive result is taken on both sides,

$$U = (1 + x + \overset{m}{\Omega})^{\frac{1}{m}}.$$

However, by §24,

$$(1 + x + \overset{m}{\Omega})^{\frac{1}{m}} = (1 + x)^{\frac{1}{m}} + \omega.$$

Therefore,

$$(1 + x)^{\frac{1}{m}} = 1 + \frac{1}{m}x + \frac{1}{m}\frac{\frac{1}{m} - 1}{2}x^2 + \cdots + \frac{1}{m}\frac{\frac{1}{m} - 1}{2}\ldots\frac{\frac{1}{m} - r + 1}{r}x^r - \omega.$$

§ 44

Theorem. The binomial equation holds for any kind of *fractional, but positive* value of the exponent, in the sense that the difference may be made smaller than any given quantity if the number of terms is increased sufficiently, if x designates a *proper fraction*, and if by $(1+x)^{\frac{n}{m}}$ one always means only the *real* and *positive* value of this expression.

Proof. Every fractional positive value can be represented in the form $\frac{n}{m}$, if n and m denote any kind of whole positive numbers. Now in just the same way as in §43, if one repeats the arguments n times instead of m times, one obtains,

$$U^n = 1 + \frac{n}{m}x + \frac{n}{m}\frac{\frac{n}{m} - 1}{2}x^2 + \cdots + \frac{n}{m}\frac{\frac{n}{m} - 1}{2}\ldots\frac{\frac{n}{m} - r + 1}{r}x^r + \overset{n}{\Omega}.$$

However, U is, as shown in §43 $= (1+x)^{\frac{1}{m}} + \omega$, if one understands by $(1+x)^{\frac{1}{m}}$ only the real and positive value of the expression. Therefore $U^n = \left((1 + x)^{\frac{1}{m}} + \omega\right)^n = (1 + x)^{\frac{n}{m}} + \overset{1}{\omega}$ (§24). Therefore by §15,

$$(1 + x)^{\frac{n}{m}} = 1 + \frac{n}{m}x + \frac{n}{m}\frac{\frac{n}{m} - 1}{2}x^2 + \cdots + \frac{n}{m}\frac{\frac{n}{m} - 1}{2}\ldots\frac{\frac{n}{m} - r + 1}{r}x^r + \Omega.$$

§ 45

Theorem. The binomial equation also holds for all *fractional*, and at the same time, *negative* values of the exponent, in the sense that the difference may become smaller than any given quantity if one increases the number of terms sufficiently, if x designates a proper fraction, and if by $(1 + x)^{-\frac{n}{m}}$ is understood only the *real* and *positive* value which this expression has.

Proof. Every fractional negative quantity can be produced by the algebraic sum of a whole negative quantity and a fractional positive quantity. For let $-\frac{n}{m}$ be any kind of fractional negative quantity of the form that n, m are whole numbers. Then there is certainly a positive whole-numbered quantity a with the property that $am > n$, where consequently $\frac{am-n}{m}$ represents a positive fraction. But then the algebraic sum of the negative whole-numbered quantity $-a$ and this positive fraction, i.e. $-a + \frac{am-n}{m} = -\frac{n}{m}$. Now as a consequence of §42 the binomial equation holds for every negative whole number as exponent, therefore for $-a$, and as a consequence of §44 also for every fractional positive quantity, therefore for $\frac{am-n}{m}$. Consequently by §41, if one puts there $p = -a$, $q = \frac{am-n}{m}$, also for $-\frac{n}{m}$.

§ 46

Theorem. The binomial equation also holds for every *irrational* value of the exponent $= i$, in the sense that the difference may become smaller than any given quantity if the number of terms is increased sufficiently, if x designates a proper fraction and if by $(1 + x)^i$ one understands only the *real* and *positive* value of this expression.

Proof. Two whole (positive or negative) numbers m and n can always be given with the property that the fraction $\frac{n}{m}$ comes as close as desired to the value of the irrational quantity i. But then as a consequence of the concept which is connected with the expression $(1 + x)^i$, also $(1 + x)^{\frac{n}{m}}$ comes as close to the value $(1 + x)^i$ as desired. However $(1 + x)^{\frac{n}{m}}$ is, if $x < \pm 1$, by §44 or §45,

$$= 1 + \frac{n}{m}x + \frac{n}{m}\frac{\frac{n}{m} - 1}{2}x^2 + \cdots + \frac{n}{m}\frac{\frac{n}{m} - 1}{2}\cdots\frac{\frac{n}{m} - r + 1}{r}x^r + \Omega.$$

Therefore also (§15),

$$(1 + x)^i = 1 + \frac{n}{m}x + \frac{n}{m}\frac{\frac{n}{m} - 1}{2}x^2 + \cdots + \frac{n}{m}\frac{\frac{n}{m} - 1}{2}\cdots\frac{\frac{n}{m} - r + 1}{r}x^r + \overset{1}{\Omega}.$$

If one puts in this series, with constant r, i everywhere instead of $\frac{n}{m}$, then the value of each of its terms, by §17, will be changed by a quantity which can be made as small as desired merely by suitable increase in n and m. Therefore also the sum of these changes, whose number is r, can by §15, be made as small as desired. Therefore also,

$$(1 + x)^i = 1 + ix + i\frac{i - 1}{2}x^2 + \cdots + i\frac{i - 1}{2}\cdots\frac{i - r + 1}{r}x^r + \overset{2}{\Omega}.$$

§ 47

Theorem. If the two-part quantity, $a+b$, has $a > b$ in mere value, and a^n is possible, then the following equation holds for *every real* value of n:

$$(a + b)^n = a^n + na^{n-1}b + n\frac{n - 1}{2}a^{n-2}b^2 + \cdots$$
$$+ n\frac{n - 1}{2}\cdots\frac{n - r + 1}{r}a^{n-r}b^r + \Omega$$

where Ω can be made as small as desired if r is taken large enough and by $(a + b)^n$ is to be understood the *real* and *positive* value of this expression.

Proof. Whenever n is a *whole positive* number the assertion of this theorem holds as a consequence of §10 even without the restriction that a should be $>b$. But in every other case this restriction is certainly necessary. For $(1+x)^n$ is

$$= 1 + nx + n\frac{n-1}{2}x^2 + \cdots + n\frac{n-1}{2}\cdots\frac{n-r+1}{r}x^r + \Omega$$

only on the condition that $x < \pm 1$. Therefore if one wants to put $\frac{b}{a}$ instead of x then $\frac{b}{a}$ must be $< \pm 1$, therefore a, in value, must be $> b$. But then one obtains,

$$\left(1+\frac{b}{a}\right)^n = 1 + n\frac{b}{a} + n\frac{n-1}{2}\frac{b^2}{a^2} + \cdots + n\frac{n-1}{2}\cdots\frac{n-r+1}{r}\frac{b^r}{a^r} + \Omega$$

if by $\left(1+\frac{b}{a}\right)^n$ is understood only the real and positive value of this expression. Now if also a^n is *possible* (i.e. it is not the case that a is negative and at the same time n is of the form $\frac{2p+1}{2q}$ where p and q denote whole numbers), then multiplication of this equation by a^n gives, by §17 and §15:

$$a^n\left(1+\frac{b}{a}\right)^n = a^n + na^{n-1}b + n\frac{n-1}{2}a^{n-2}b^2 + \cdots$$
$$+ n\frac{n-1}{2}\cdots\frac{n-r+1}{r}a^{n-r}b^r + \overset{1}{\Omega},$$

in which instead of $a^n\left(1+\frac{b}{a}\right)^n$ one can write $(a+b)^n$ if one understands by this only the real and positive value of this expression.

§ 48

Corollary. The equation

$$(1+x)^n = 1 + nx + n\frac{n-1}{2}x^2 + \cdots + n\frac{n-1}{2}\cdots\frac{n-r+1}{r}x^r + \Omega,$$

whose validity we have now proved for every value of n and for every $x < \pm 1$, takes the following form for a negative x and n:

$$(1-x)^{-n} = 1 + nx + n\frac{n+1}{2}\frac{n+2}{3}x^3 + \cdots$$
$$+ n\frac{n+1}{2}\frac{n+2}{3}\cdots\frac{n+r-1}{r}x^r + \Omega.$$

By means of these two formulae it can be proved that the *real* value of the expression y^n, in so far as it has one, may always be represented by a series which proceeds in increasing powers of a quantity easily derived from y, namely $y-1$ or $\frac{y-1}{y}$. For *firstly*, if y is *positive* but still $<+2$, one just puts $y^n = (1+(y-1))^n$ and then $(y-1) < \pm 1$ and it can therefore replace the quantity x in the first of those two formulae. From this, if one understands by y^n only the one real and positive value of this expression, the equation gives:

$$y^n = 1 + n(y-1) + n\frac{n-1}{2}(y-1)^2 + n\frac{n-1}{2}\frac{n-2}{3}(y-1)^3 + \cdots$$
$$+ n\frac{n-1}{2}\frac{n-2}{3}\cdots\frac{n-r+1}{r}(y-1)^r + \Omega.$$

If y^n possesses several values then there is only one more, an equally large negative one, that is real. It can therefore also be expressed by the same series if one changes the sign of all the terms. Therefore, if desired, both values can be represented in one expression:

$$y^n = \pm 1 \pm n(y-1) \pm n\frac{n-1}{2}(y-1)^2 \pm n\frac{n-1}{2}\frac{n-2}{3}(y-1)^3 \pm \cdots$$
$$\pm n\frac{n-1}{2}\frac{n-2}{3}\cdots\frac{n-r+1}{r}(y-1)^r \pm \Omega.$$

But if $y > +2$, indeed even if it is only $> +\frac{1}{2}$, then $\frac{y-1}{y}$ is a proper fraction and $\left(1 - \frac{y-1}{y}\right)^{-n} = y^n$, therefore if one puts $\frac{y-1}{y}$ instead of x in the second formula the following equation appears:

$$y^n = 1 + n\frac{(y-1)}{y} + n\frac{n+1}{2}\frac{(y-1)^2}{y^2} + n\frac{n+1}{2}\frac{n+2}{3}\frac{(y-1)^3}{y^3} + \cdots$$
$$+ n\frac{n+1}{2}\frac{n+2}{3}\cdots\frac{n+r-1}{r}\frac{(y-1)^r}{y^r} + \Omega.$$

If y^n has a *double* real value this can also be represented here as before. But if, *secondly*, y is *negative*, e.g. $= -z$, where z again indicates a positive quantity then one has $(-z)^n = (-1)^n z^n$ and z^n can be expanded, by what has just been shown, in a series of increasing powers of $(z-1)$ or $\frac{z-1}{z}$, according as $z < +2$ or $> +\frac{1}{2}$. Now if $(-z)^n$ denotes something real, then also $(-1)^n z^n$, and consequently $(-1)^n$, must be real. If we therefore multiply all the terms of the series found for z^n by the real value of $(-1)^n$ (which is well known to be either $+1$ or -1), then one has also represented the value of $(-z)^n$ in a series of increasing powers of $(z-1)$ or $\left(\frac{z-1}{z}\right)$.

§ 49

Transition. We have seen in §33 that the most general form of a series developed in powers of x which is to express the value of the function $(1+x)^n$ for as many values of n and x as possible is none other than the *binomial series*. By *as many values of x as possible* is to be understood as all those which lie between zero and a positive or negative limit which is as large as possible. It has finally been shown here that this second limit may not be greater than ± 1. We have also already mentioned in §13 the basis on which we established that one of those limits should be zero. However, there might also be cases where one wanted to calculate the value of the function

$(1+x)^n$ for *greater values* of x. We shall therefore investigate the supplementary question, 'of what form a series developed in powers of x would have to be if it is to give the value of $(1+x)^n$ for as many values of n as possible and for all values of x which lie between the given limit a and another whose difference from the former is as great as possible.'

§ 50

Problem. To find the form of a series developed in powers of x which gives the value of $(1+x)^n$ for as many values of n as possible, and for all values of x which lie between a given limit a and another which differs as much as possible from the former, either exactly or so nearly that the difference can become smaller than any given quantity if one takes the number of its terms suitably.

Solution. 1. The series should be of the form

$$Ax^\alpha + Bx^\beta + Cx^\gamma + \cdots + Rx^\rho$$

where it may be assumed that the exponents $\alpha, \beta, \gamma, \ldots, \rho$ all differ from one another and are arranged according to their magnitude from the greatest negative (if there is such) up to the greatest positive. If one now puts $a+y$ instead of x then the equation for y which one obtains in this way:

$$(1+a+y)^n = A(a+y)^\alpha + B(a+y)^\beta + C(a+y)^\gamma + \cdots + R(a+y)^\rho + \Omega \quad \odot$$

should hold starting from a certain value of y and for all smaller ones. However, if $y < (1+a)$ and this is *positive*, $(1+a+y)^n$ is, for every value of n, expandable by §47 in a series of increasing powers of y. The same holds generally of the functions $(a+y)^\alpha$, $(a+y)^\beta, \ldots, (a+y)^\rho$, if $\alpha, \beta, \ldots, \rho$ are whole positive numbers, but in every other case only if one puts $y < a$. Therefore one obtains:

$$(1+a)^n + n(1+a)^{n-1}y + n\frac{n-1}{2}(1+a)^{n-2}y^2 + \cdots$$
$$+ n\frac{n-1}{2}\cdots\frac{n-r+1}{r}(1+a)^{n-r}y^r$$

$$= Aa^\alpha + A\alpha a^{\alpha-1}y + A\alpha\frac{\alpha-1}{2}a^{\alpha-2}y^2 + \cdots$$
$$+ A\alpha\frac{\alpha-1}{2}\cdots\frac{\alpha-r+1}{r}a^{\alpha-r}y^r$$
$$+ Ba^\beta + B\beta a^{\beta-1}y + B\beta\frac{\beta-1}{2}a^{\beta-2}y^2 + \cdots$$
$$+ B\beta\frac{\beta-1}{2}\cdots\frac{\beta-r+1}{r}a^{\beta-r}y^r$$
$$+ \ldots.$$

$$+ Ra^\rho + R\rho a^{\rho-1}y + R\rho\frac{\rho-1}{2}a^{\rho-2}y^2 + \cdots$$
$$+ R\rho\frac{\rho-1}{2}\cdots\frac{\rho-r+1}{r}a^{\rho-r}y^r + \Omega$$ ☾

where Ω can become as small as desired if one takes r large enough. From this, by application of §28, the following conditions arise for the determination of the quantities, $\alpha, \beta, \gamma, \ldots, \rho; A, B, C, \ldots, R$:

$$\begin{aligned}
(1+a)^n &= Aa^\alpha + Ba^\beta + Ca^\gamma + \cdots + Ra^\rho \\
n(1+a)^{n-1} &= A\alpha a^{\alpha-1} + B\beta a^{\beta-1} + C\gamma a^{\gamma-1} + \cdots \\
&\quad + R\rho a^{\rho-1} \\
n(n-1)(1+a)^{n-2} &= A\alpha(\alpha-1)a^{\alpha-2} + B\beta(\beta-1)a^{\beta-2} \\
&\quad + C\gamma(\gamma-1)a^{\gamma-2} + \cdots \\
&\quad + R\rho(\rho-1)a^{\rho-2} \\
&\cdots\cdots \\
n(n-1)\ldots(n-r+1)(1+a)^{n-r} &= A\alpha(\alpha-1)\ldots(\alpha-r+1)a^{\alpha-r} \\
&\quad + \cdots \\
&\quad + R\rho(\rho-1)\ldots(\rho-r+1)a^{\rho-r}.
\end{aligned}$$

The number of these equations is $(r+1)$, and of the quantities to be determined $\alpha, \beta, \gamma, \ldots, \rho; A, B, C, \ldots, R$ there are $2(r+1)$. Therefore half of them still remain undetermined and it is left to our choice to add a new condition to the previous ones whereby the remaining quantities would be determined.

2. If for example we wanted to put forward, in this respect, the condition that *the form of the series to be found is independent of the quantity a*, then the first of these equations would immediately determine all the quantities $\alpha, \beta, \ldots, \rho$; $A, B, \ldots, R$. For if the equation,

$$\begin{aligned}
(a+1)^n &= a^n + na^{n-1} + n\frac{n-1}{2}a^{n-2} + \cdots \\
&\quad + n\frac{n-1}{2}\cdots\frac{n-r+1}{r}a^{n-r} \\
&= Aa^\alpha + Ba^\beta + Ca^\gamma + \cdots + Ra^\rho
\end{aligned}$$

is to hold for indefinitely many values of a, then (as in §38, 1.) every power of a on one side must correspond to an equal power with an equal coefficient on the other side. If ρ is to be the greatest exponent this would give $\rho = n$, $R = 1$, and if Qa^q is the penultimate term, $q = n-1$, $Q = n$, etc. One would obtain in this way the binomial series belonging to the power $(1+x)^n$, only limited to n terms, therefore if the equation is to hold n would have to be a whole positive number.

Hence the series obtained would only be valid for those cases where the exponent is a whole positive number. If it is to hold more generally we must therefore drop the requirement that its form is independent of a.

3. Let us therefore now try to establish another condition, namely *that the exponents $\alpha, \beta, \gamma, \ldots, \rho$ are all only whole numbers*, because the use of the series is not at all convenient with fractional exponents. But because this condition alone is not sufficient to determine $(r+1)$ quantities we stipulate straight away that $\alpha, \beta, \gamma, \ldots$ should proceed according to the order of the natural numbers from 0 onwards, since in this way the calculation of the coefficients $A, B, C, \ldots, R$ works out most simply. The above equations then take the following form:

$$\begin{aligned}
(1+a)^n &= A + Ba + Ca^2 + \cdots + Ra^r \\
n(1+a)^{n-1} &= B + 2Ca + 3Da^2 + \cdots \\
&\quad + rRa^{r-1} \\
n(n-1)(1+a)^{n-2} &= 1.2.C + 2.3.Da + 3.4.Ea^2 \\
&\quad + \cdots + (r-1)rRa^{r-2} \\
n(n-1)(n-2)(1+a)^{n-3} &= 1.2.3D + 2.3.4Ea + \cdots \\
&\quad + (r-2)(r-1)rRa^{r-3} \\
&\ldots\ldots \\
n(n-1)(n-2)\ldots(n-r)(1+a)^{n-r} &= 1.2.3\ldots rR.
\end{aligned}$$

The last of these gives

$$R = \frac{n.n-1.n-2\ldots n-r}{1.2.3\ldots r}.(1+a)^{n-r}.$$

And this value put in the penultimate one gives Q, etc.

4. The series which is obtained in this way for the value of $(1+x)^n$ has the *great inconvenience* that the form of each of its coefficients taken individually depends on the multitude of *all* of them, so that one may not *increase* (or *decrease*) the number of terms in it by a single one without at the same time changing them all.

5. Concerning the limits which the value of y may not exceed it is obvious from the above procedure (1.) that they are $= \pm(1+a)$. For it is obvious that $y < \pm(1+a)$ is the only condition which is required (together with the positive value of $(1+a)$) for the possibility of the expansion of $(1+a+y)^n$ in the series: $(1+a)^n + n(1+a)^{n-1}y + \cdots$ for every value of n. The expansion of the functions $(a+y)^\alpha$, $(a+y)^\beta, \ldots$ in developed series is possible without restriction after one has put $\alpha, \beta, \gamma, \ldots = 0, 1, 2, \ldots$. Now if these expansions hold then so does the equation ☾, and consequently also ⊙, because it is identical with it. But it follows from this that the equation ⊙ also does not hold for any greater values of y. For if this were

so, then for these same values ☾ would also hold, since if for some value of y,

$$(1 + a + y)^n = A + B(a + y) + C(a + y)^2 + \cdots + R(a + y)^r + \Omega,$$

then also (by development of the values of $(a + y)^2$, etc.)

$$\begin{aligned}(1 + a + y)^n = A &+ Ba + Ca^2 + \cdots + Ra^r \\ &+ (B + 2Ca + 3Da^2 + \cdots + rRa^{r-1})y \\ &+ \frac{1}{2}(2C + 2.3Da + \cdots + (r-1)rRa^{r-2})y^2 \\ &+ \ldots\ldots \\ &+ \frac{1.2.3\ldots r}{1.2.3\ldots r} Ra^0 y^r.\end{aligned}$$

But since by virtue of the determination of the coefficients $A, B, C, \ldots, R$,

$$\begin{aligned}A + Ba + Ca^2 + \cdots + Ra^r &= (1 + a)^n \\ B + 2Ca + \cdots + rA^{r-1} &= n(1 + a)^{n-1}\end{aligned}$$

etc., then one obtains by substitution:

$$\begin{aligned}(1 + a + y)^n = (1 + a)^n + n(1 + a)^{n-1}y + \frac{n.n-1}{1.2}(1 + a)^{n-2}y^2 + \cdots \\ + \frac{n.n-1\ldots n-r+1}{1.2\ldots r}(1 + a)^{n-r}y^r + \Omega\end{aligned}$$

which would be the binomial equation, which nevertheless cannot hold unless y is $< \pm(1 + a)$, or at the same time n were a whole positive number. Therefore the limits for $x = a + y$ are the two quantities $a - (1 + a) = -1$ and $a + 1 + a = 2a + 1$, i.e. x must always be $> \pm 1$ and $< \pm(1 + 2a)$.

§ 51

Example. Let $a = 1$ and the number of terms of which the series is to consist be only *four*, then for the determination of its four coefficients one has the equations:

$$\begin{aligned}2^n &= A + B + C + D \\ n.2^{n-1} &= B + 2C + 3D \\ n.n - 1.2^{n-2} &= 1.2C + 2.3D \\ n.n - 1.n - 2.2^{n-3} &= 1.2.3D.\end{aligned}$$

From which

$$D = \frac{n.n-1.n-2}{1.2.3}.2^{n-3}$$
$$C = \frac{n.n-1}{1.2}.2^{n-3}(4-n)$$
$$B = n.2^{n-3}\left(\frac{4+(n-1)(n-6)}{2}\right)$$
$$A = 2^{n-3}\left(8 - \frac{n^3-9n^2+32n}{6}\right).$$

Therefore

$$(1+x)^n = 2^{n-3}\left(8 - \frac{n^3-9n^2+32n}{6}\right) + n.2^{n-3}\left(\frac{4+(n-1)(n-6)}{2}\right)x + \frac{n.n-1}{1.2}2^{n-3}x^2 + \frac{n.n-1.n-2}{1.2.3}2^{n-3}x^3.$$

This equation is actually exact for the values $n = 0, = 1, = 2, = 3$ whatever x may denote. For higher values it cannot be exact because one has only four terms. For $x = 1$ it holds for all values of n; on the other hand, for other values of x which lie between ± 1 and $\pm(1+2a) = \pm 3$, it only holds approximately, and indeed not very precisely, because it has been restricted to only four terms. Thus for $x = 2$ and $n = -1$, it gives $(1+2)^{-1} = \frac{5}{16}$, which comes fairly near to the true value $= \frac{1}{3}$; for $x = 2$ and $n = -2$ one obtains $(1+2)^{-2} = \frac{1}{16}$ which now deviates significantly from the true value $= \frac{1}{9}$, etc.

§ 52

Transition. We have now therefore finished the first part of the investigation we undertook in §1: whether, and in what way, a power of a quantity *consisting of two parts* could be expanded in a series developed in powers of its simple parts. There still remains the investigation of whether and how this could happen with every quantity consisting of *several parts*. This can be dealt with briefly.

§ 53

Theorem. Every quantity consisting of many parts, of the form $(a+b+c+d+\cdots)^n$, can be expanded exactly in a series developed in powers of its simple parts a, b, c, d, ..., in the case where n denotes a *whole positive* number, but in every other case only in the sense that the difference between the complex function $(a+b+c+d+\cdots)^n$ and the developed series can become smaller than any given quantity if one increases the number of terms sufficiently and one of the quantities a, b, c, d, ..., e.g. a, is greater than the sum of the others and at the same time a^n is possible.

Proof. For if n is a *whole positive* number then whatever a and $(b+c+d+\cdots)$ may be, by §10, $(a+b+c+d+\cdots)^n$ can be expressed exactly in a series developed in powers of a and $(b+c+d+\cdots)$. Now in this series $(b+c+d+\cdots)$ always appears with only purely whole-numbered and positive powers, it is therefore generally possible to put instead of this complex function, certain series developed in powers of b and $(c+d+\cdots)$ etc. Now since each of the quantities $(a+b+c+d+\cdots)$, $(b+c+d+\cdots)$, $(c+d+\cdots)$, ... contains one term less than its predecessor, it follows that by continuing this procedure one must finally come to a quantity consisting of two parts, in the expression for the development of which all complex functions vanish completely. For each of the developed series is merely multiplied by one or more powers of simple parts (namely the preceding ones) whereby it does not cease to be a developed series. But if the exponent n is not a whole number then the first of the developments made here only proceeds in the case when a^n is possible, and $a > (b+c+d+\cdots)$, and even then the developed series $a^n + na^{n-1}(b+c+d+\cdots) + n\frac{n-1}{2}a^{n-2}(b+c+d+\cdots)^2 + \cdots$ is not exactly equal to the function $(a+b+c+d+\cdots)^n$ but only in the sense that the difference can become smaller than any given quantity if one takes the number of terms in the series large enough. Now concerning the succeeding developments, namely those of $(b+c+d+\cdots)^2$, $(b+c+d+\cdots)^3, \ldots$, these are possible by what has just been said and because in every case they only involve whole-numbered and positive powers, whatever the single quantities $b, c, d, \ldots$, might be. But the value which the developed functions, put in the place of the complex functions $(b+c+d+\cdots)^2$, $(b+c+d+\cdots)^3, \ldots$, take, is exactly equal to the latter, therefore the difference of the equation remains the same as before, i.e. it is Ω.

§ 54

Transition. Therefore it only remains to determine the *general form* which is taken by a series originating from such successive developments. We intend to seek this in the paragraphs immediately following. The generality of this investigation which we are now going to undertake will not be affected in the least (rather it may even be advantageous for the cases where the terms of the given quantity of many parts proceed in powers of a single one of them), if we take the form of the series as: $1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \overset{(3)}{a}x^3 + \cdots + \overset{(m)}{a}x^m$, where $\overset{(1)}{a}, \overset{(2)}{a}, \overset{(3)}{a}, \ldots, \overset{(m)}{a}$, denote any quantities which we designate by the numerical arrangement (1), (2), (3), ..., (m), in order to note conveniently to which power of x each of them belongs as coefficient. If the given quantity consisting of many parts has no such increasing powers of x, one need only put $x = 1$ and if its first term is not, as in our formula, $= 1$ then one can put $(a+b+c+d+\cdots)^n = a^n\left(1+\frac{b}{a}+\frac{c}{a}+\frac{d}{a}+\cdots\right)^n$

provided a^n is not impossible. If one then develops $\left(1+\frac{b}{a}+\frac{c}{a}+\frac{d}{a}+\cdots\right)^n$ in the form $\left(1+\overset{(1)}{a}x+\overset{(2)}{a}x^2+\cdots+\overset{(m)}{a}x^m\right)^n$, then one obtains the development of $(a+b+c+d+\cdots)^n$ if one just multiplies all the terms of the series by a^n.

§ 55

Problem. A series developed in powers of x is either to be exactly equal to the value of the complex function $\left(1+\overset{(1)}{a}x+\overset{(2)}{a}x^2+\cdots+\overset{(m)}{a}x^m\right)^n$ for as many values of n as possible and for all values of x which lie between zero and some limit (positive or negative) as far from zero as possible, or else to come so close to this value that the difference becomes smaller than any given quantity if the number of terms of the series is sufficiently increased. The problem is to find certain conditions which the series must satisfy.

Solution. Let the series be $A+Bx^\beta+Cx^\gamma+\cdots+Rx^\rho$, then for every value of n and x for which the equation

$$\left(1+\overset{(1)}{a}x+\overset{(2)}{a}x^2+\cdots+\overset{(m)}{a}x^m\right)^n$$

$$=A+Bx^\beta+Cx^\gamma+\cdots+Rx^\rho+\Omega \qquad \odot$$

holds, also the following equation must hold:

$$\left(1+\overset{(1)}{a}(x+\omega)+\overset{(2)}{a}(x+\omega)^2+\cdots+\overset{(m)}{a}(x+\omega)^m\right)^n$$

$$=A+B(x+\omega)^\beta+C(x+\omega)^\gamma+\cdots+R(x+\omega)^\rho+\overset{1}{\Omega}$$

provided ω is so small that not only x, but also $x+\omega$, lies inside the limits established. If for the sake of abbreviation one writes, $1+\overset{(1)}{a}x+\overset{(2)}{a}x^2+\cdots+\overset{(m)}{a}x^m=Y$, then by §10:

$$1+\overset{(1)}{a}(x+\omega)+\overset{(2)}{a}(x+\omega)^2+\cdots+\overset{(m)}{a}(x+\omega)^m$$

$$=Y+\omega\left(\overset{(1)}{a}+2\overset{(2)}{a}x+3\overset{(3)}{a}x^2+\cdots+m\overset{(m)}{a}x^{m-1}\right)+\cdots$$

where the terms omitted at the end only contain higher powers of x which are multiplied by certain constant quantities, i.e. not dependent on ω, and the number of terms does not vary with ω. Therefore

$$\omega\left(\overset{(1)}{a}+2\overset{(2)}{a}x+3\overset{(3)}{a}x^2+\cdots+m\overset{(m)}{a}x^{m-1}\right)+\cdots$$

is a quantity which, by §§ 17, 15, can become as small as desired just by diminishing ω. Let us designate it by $\overset{2}{\Omega}$, then

$$\frac{\overset{2}{\Omega}}{\omega} = \left(\overset{(1)}{a} + 2\overset{(2)}{a}x + 3\overset{(3)}{a}x^2 + \cdots + m\overset{(m)}{a}x^{m-1}\right) + \cdots$$

and the terms omitted can, for just the same reason as mentioned before, become as small as desired. However if one subtracts the first of the two equations from the other and divides by ω one obtains, with the notation just introduced,

$$\frac{(Y + \overset{2}{\Omega})^n - Y^n}{\omega} = \frac{B((x+\omega)^\beta - x^\beta)}{\omega} + \frac{C((x+\omega)^\gamma - x^\gamma)}{\omega} + \cdots + \frac{R((x+\omega)^\rho - x^\rho)}{\omega} + \frac{\overset{1}{\Omega} - \Omega}{\omega}.$$

Now by §23,

$$\frac{(Y + \overset{2}{\Omega})^n - Y^n}{\overset{2}{\Omega}} = nY^{n-1} + \overset{(1)}{\omega},$$

therefore,

$$\frac{(Y + \overset{2}{\Omega})^n - Y^n}{\omega} = nY^{n-1}\frac{\overset{2}{\Omega}}{\omega} + \frac{\overset{(1)}{\omega}\overset{2}{\Omega}}{\omega},$$

which by §§ 17 and 18

$$= nY^{n-1}\left(\overset{(1)}{a} + 2\overset{(2)}{a}x + 3\overset{(3)}{a}x^2 + \cdots + m\overset{(m)}{a}x^{m-1}\right) + \overset{(n)}{\omega},$$

provided ω is taken small enough. But with the same condition the quantities:

$$\frac{(x+\omega)^\beta - x^\beta}{\omega} = \beta x^{\beta-1} + \overset{(\beta)}{\omega}$$

$$\frac{(x+\omega)^\gamma - x^\gamma}{\omega} = \gamma x^{\gamma-1} + \overset{(\gamma)}{\omega}$$

$$\cdots\cdots\cdots$$

$$\frac{(x+\omega)^\rho - x^\rho}{\omega} = \rho x^{\rho-1} + \overset{(\rho)}{\omega}.$$

Therefore one obtains the equation:

$$nY^{n-1}\left(\overset{(1)}{a}+2\overset{(2)}{a}x+3\overset{(3)}{a}x^2+\cdots+m\overset{(m)}{a}x^{m-1}\right)$$
$$=\beta Bx^{\beta-1}+\gamma Cx^{\gamma-1}+\cdots+\rho Rx^{\rho-1}+B\overset{(\beta)}{\omega}+C\overset{(\gamma)}{\omega}+\cdots$$
$$+R\overset{(\rho)}{\omega}+\frac{\overset{(1)}{\Omega}-\Omega}{\omega}.$$

Now it can be proved of the expression $\frac{\overset{(1)}{\Omega}-\Omega}{\omega}$, as it was shown of the similar expression in §30, that for every value of ω it can be made as small as desired merely by increasing the number of terms in $A+Bx^\beta+Cx^\gamma+\cdots+Rx^\rho$. But with the same number of terms, $B\overset{(\beta)}{\omega}+C\overset{(\gamma)}{\omega}+\cdots+R\overset{(\rho)}{\omega}$ can become smaller than any given quantity merely by diminishing ω. Therefore,

$$nY^{n-1}\left(\overset{(1)}{a}+2\overset{(2)}{a}x+3\overset{(3)}{a}x^2+\cdots+m\overset{(m)}{a}x^{m-1}\right)$$
$$=\beta Bx^{\beta-1}+\gamma Cx^{\gamma-1}+\cdots+\rho Rx^{\rho-1}+\overset{3}{\Omega}.$$

If one multiplies this equation by Y and puts instead of Y^n the series which is to be of equal value, then one finds, by application of §§15, 17, the following *condition* which the quantities $\beta, \gamma, \ldots, \rho, A, B, C, \ldots, R$ must satisfy if ⊙ is to hold:

$$n\left(\overset{(1)}{a}+2\overset{(2)}{a}x+\cdots+m\overset{(m)}{a}x^{m-1}\right)\times(A+Bx^\beta+Cx^\gamma+\cdots+Rx^\rho)$$
$$=\left(1+\overset{(1)}{a}x+\overset{(2)}{a}x^2+\cdots+\overset{(m)}{a}x^m\right)$$
$$\times(\beta Bx^{\beta-1}+\gamma Cx^{\gamma-1}+\cdots+\rho Rx^{\rho-1})+\Omega. \qquad ♁$$

§ 56

Problem. To determine the form of the series $A+Bx^\beta+Cx^\gamma+\cdots+Rx^\rho$ in so far as it is possible to do so only from the *condition* of §55.

Solution. 1. If we imagine, as in the similar problem of §32, that all exponents in $A+Bx^\beta+Cx^\gamma+\cdots+Rx^\rho$ are different from one another and are arranged according to magnitude then the equation ♁ gives, if one actually carries out the multiplication indicated in it and then arranges all terms by powers of x, the form of the equation of §28. So if this is to be valid for all x which are smaller than a certain value, every term on one side must correspond to an exactly equal one on the other side with at most the exception of one or more parts which can become as small as desired.

2. Hence it is clear, firstly that A cannot be zero, and that the exponents $\beta, \gamma, \dots, \rho$ must progress in the order of the natural numbers 1, 2, 3, ... For on the right-hand side of the equality sign there is obviously no lower term than $\beta B x^{\beta-1}$ which cannot be zero since neither B nor β may be zero because in the first case the term would not be present at all, and in the second case it would not be of the form Bx^{β} but of the form A. Therefore an equal term must correspond to this on the left-hand side, and it can only be $n\overset{(1)}{a}A$. Therefore A may not be zero. But it is quite clear, as in §32, that none of the exponents $\beta, \gamma, \dots, \rho$ could be fractional or negative.

3. Therefore if one puts for $\beta, \gamma, \dots, \rho$ the numbers $1, 2, 3, \dots, r$ one obtains the equation ♁ in the following form:

$$n\left(\overset{(1)}{a} + 2\overset{(2)}{a}x + 3\overset{(3)}{a}x^2 + \cdots + m\overset{(m)}{a}x^{m-1}\right) \times (A + Bx + Cx^2 + Dx^3 + \cdots + Rx^r)$$
$$= \left(1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \overset{(3)}{a}x^3 + \cdots + \overset{(m)}{a}x^m\right) \times (B + 2Cx + 3Dx^2 + \cdots + rRx^{r-1}) + \Omega$$

from which the values of the coefficients $A, B, C, D, \dots, R$ can be determined by $\overset{(1)}{a}, \overset{(2)}{a}, \dots, \overset{(m)}{a}$ and A. But in order to understand clearly the general rule by which each of them is formed from the preceding ones, let their rule of arrangement be expressed in the same way as for the quantities $\overset{(1)}{a}, \overset{(2)}{a}, \dots, \overset{(m)}{a}$, i.e. by their symbols. Therefore let the coefficient of x^0, or A, be designated by $\overset{(0)}{A}$, that of x^1, or B, by $\overset{(1)}{A}$, that of x^2, or C, by $\overset{(2)}{A}, \dots,$ that of x^r by $\overset{(r)}{A}$, now the above equation has the following appearance:

$$n\left(\overset{(1)}{a} + 2\overset{(2)}{a}x + 3\overset{(3)}{a}x^2 + \cdots + m\overset{(m)}{a}x^{m-1}\right) \times \left(\overset{(0)}{A} + \overset{(1)}{A}x + \overset{(2)}{A}x^2 + \overset{(3)}{A}x^3 + \cdots + \overset{(r)}{A}x^r\right)$$
$$= \left(1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \overset{(3)}{a}x^3 + \cdots + \overset{(m)}{a}x^m\right) \times \left(\overset{(1)}{A} + 2\overset{(2)}{A}x + 3\overset{(3)}{A}x^2 + \cdots + r\overset{(r)}{A}x^{r-1}\right) + \Omega.$$

Now in order to determine the coefficient of some term, e.g. that which belongs to x^{p+1}, one may calculate all terms on both sides of the equality sign which give the power x^p. On the left-hand side there is the following:

$$n\overset{(1)}{a}\overset{(p)}{A}x^p + 2n\overset{(2)}{a}\overset{(p-1)}{A}x^p + 3n\overset{(3)}{a}\overset{(p-2)}{A}x^p + \cdots + (p+1)n\overset{(p+1)}{a}\overset{(0)}{A}x^p,$$

but on the right-hand side,

$$(p+1)\overset{(p+1)}{A}x^p + p\overset{(1)}{a}\overset{(p)}{A}x^p + (p-1)\overset{(2)}{a}\overset{(p-1)}{A}x^p + \cdots + \overset{(p)}{a}\overset{(1)}{A}x^p.$$

Therefore by §28 the former sum must be exactly equal to the latter, whence,

$$\begin{aligned}(p+1)\overset{(p+1)}{A} &= n\overset{(1)}{a}\overset{(p)}{A} + 2n\overset{(2)}{a}\overset{(p-1)}{A} + 3n\overset{(3)}{a}\overset{(p-2)}{A}\\ &\quad + \cdots + (p+1)n\overset{(p+1)}{a}\overset{(0)}{A} - p\overset{(1)}{a}\overset{(p)}{A} - (p-1)\overset{(2)}{a}\overset{(p-1)}{A} - \cdots - \overset{(p)}{a}\overset{(1)}{A}\end{aligned}$$

or

$$\begin{aligned}(p+1)\overset{(p+1)}{A} &= (n-p)\overset{(1)}{a}\overset{(p)}{A} + (2n-p+1)\overset{(2)}{a}\overset{(p-1)}{A} + (3n-p+2)\overset{(3)}{a}\overset{(p-2)}{A}\\ &\quad + \cdots + (pn-1)\overset{(p)}{a}\overset{(1)}{A} + (p+1)n\overset{(p+1)}{a}\overset{(0)}{A}.\end{aligned}$$

This is an equation whose rule of formation is sufficiently clear by inspection and by means of this rule every subsequent coefficient can be calculated from all the preceding ones. But the first $\overset{(0)}{A}$ remains undetermined.

§ 57

Note. *Hindenburg*, *Schultz* and others have indicated methods by which one can calculate each of these coefficients *directly*, i.e. without first having found the preceding ones. It is outside our original purpose to repeat these methods here.

§ 58

Theorem. If the series found in §56 is to express the value of the complex function $\left(1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)^n$ correctly for as many values of n and x as possible, then $\overset{(0)}{A}$ must be $= 1$.

Proof. For if x is taken ever smaller then the true value of

$$\left(1 + \overset{(1)}{a}x + \cdots + \overset{(m)}{a}x^m\right)^n$$

approaches the value 1, whatever n may be. But the value of the series approaches the value of its first term $\overset{(0)}{A}$. Therefore by §27 it must be that, $\overset{(0)}{A} = 1$.

§ 59

Theorem. The series found in §56, through the more precise determination which §58 gives to it, yields the value of the complex function $\left(1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)^n$ precisely, whenever n denotes a *whole positive* number provided r is not allowed to be $<mn$. On the other hand, for every other value of n it only gives the value of that function if $\left(\overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right) < \pm 1$, in the sense that the difference can become smaller than any given quantity if the number of its terms is taken large enough.

Proof. 1. By virtue of §55 every series developed in powers of x which is to express the value of the function

$$\left(1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)^n$$

for all x which are smaller than a certain one, must be of the form given in §56, and moreover by §58 the quantity A in it must be $= 1$. But after this two-fold determination there is nothing left undetermined in it, in fact for every particular value of the quantities x, $\overset{(1)}{a}$, $\overset{(2)}{a}, \ldots, \overset{(m)}{a}$ and n it only takes one single value. Therefore if the value of the function $\left(1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)^n$ cannot be represented exactly by this series in any way then it must generally be impossible to express its value exactly. But by §50 this must always be possible if n represents a *whole positive* number. Therefore the series of §53 must have the required property. But it is clear that if x is to remain undetermined in it, this is only possible if it is continued at least up to the power x^{mn} because $\left(1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)^n$, if it is expanded by multiplication, would certainly go up to a term of the form x^{mn} (but not higher). Now if the series of §58 is to be able to be put equal to this expansion for all x which are smaller than a certain one, then both expressions must be identical, and consequently also there must be a term in the former with the power x^{mn}. But it is also clear, for the same reason, that it could never be necessary to continue the series beyond the term x^{mn}. Because the expansion by multiplication gives no higher powers, then also they may not appear in the series; the coefficient of every such term, if it was sought, would be found $= 0$.

2. But if n is not a whole positive number then, by §53, it is possible, under the condition that $\left(\overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right) < \pm 1$, to transform that complex function into a series expanded in powers of x which comes as close to its value as desired if the number of its terms is taken large enough. Therefore the series of §58 must achieve this because it would not be possible otherwise. However, this can of course only be expected of it if the number of its terms is taken large enough. Because there is only one series that can do this it must be identical with

that which would be found by the method of §53, i.e. by successive expansion. Now the first expansion gives:

$$1 + n\left(\overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)$$
$$+ n.\frac{n-1}{2}.\left(\overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)^2$$
$$+ \ldots\ldots$$
$$+ n.\frac{n-1}{2}\ldots\frac{n-p+1}{p}.\left(\overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)^p,$$

in which, if the difference between the value of this series and of the given function is to be made as small as desired, p must be able to be taken as large as desired. But if one can take p here as large as desired, then the highest power which appears in the expansion of this series, namely x^{mp}, can exceed every given limit. Consequently in the series of §58 which is identical with the present series one must also be able to take the number of terms as large as desired.

§ 60

Definition. A quantity y which depends on two other quantities a and x, so that $y = a^x$, is called an *exponential quantity* in so far as one regards x, at least, as variable, if not also a. The x in this relation is called the *logarithm of y with base number* (or basis) a. In order to indicate that x is the logarithm of y one usually writes $x = \mathrm{l}\, y$ or $\log y$ or something similar. If, in the same calculation, one has logarithms with different base numbers it is necessary to choose a particular method of notation for the logarithms referring to each particular base number so that they are not confused with one another. For if a and α are different and $a^x = \alpha^\xi = y$, then also x and ξ are different, i.e. the same quantity has different logarithms with respect to different base numbers. Therefore in such cases it might perhaps be most appropriate to indicate the base number which the logarithm refers to, in its designation itself, e.g. to represent the logarithm of y with respect to the basis number a by $\overset{(a)}{\mathrm{l}}\, y$.

§ 61

Transition. Having already achieved the expansion, under certain conditions, of every function of the form $(1 + x)^n$, and also for every function of the form $\left(1 + \overset{(1)}{a}x + \overset{(2)}{a}x^2 + \cdots + \overset{(m)}{a}x^m\right)^n$, in a series proceeding by powers of x, this leads to the idea of whether the functions a^x and $\log y$ may not also be expanded in certain series in powers of their variable quantity, the former x, the latter y, or even in powers of another quantity easily derivable from x and y. We now investigate this in the following.

§ 62

Lemma. Suppose that by y is to be understood either a merely *abstract* quantity (i.e. one which cannot take opposite signs) or just a *positive* quantity, but by y^ω is understood only the *real* and *positive* value of this expression, then for all y which are $> +\frac{1}{2}$, the equation holds:

$$\frac{y^\omega - 1}{\omega} = \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r} + \Omega;$$

but for all y which are $< +2$, the following holds:

$$\frac{y^\omega - 1}{\omega} = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \cdots \pm \frac{1}{r}(y-1)^r + \overset{1}{\Omega}.$$

In these equations Ω and $\overset{1}{\Omega}$ can become as small as desired if ω is taken sufficiently small but r, the number of terms in the series, sufficiently large.

Proof. 1. First let $y > +\frac{1}{2}$, then by §48 one has, for every value of ω, the equation:

$$y^\omega = 1 + \frac{\omega(y-1)}{y} + \omega\frac{\omega+1}{2}\frac{(y-1)^2}{y^2} + \omega\frac{\omega+1}{2}\frac{\omega+2}{3}\frac{(y-1)^3}{y^3}$$
$$+ \cdots + \omega\frac{\omega+1}{2}\frac{\omega+2}{3}\cdots\frac{\omega+r-1}{r}\frac{(y-1)^r}{y^r} + \Omega.$$

If one actually carries out the multiplications merely indicated here by the factors,

$$\frac{\omega+1}{2}, \frac{\omega+2}{3}, \ldots, \frac{\omega+(r-1)}{r}$$

and one arranges everything according to powers of ω, then one arrives at an equation of the following form:

$$y^\omega = 1 + \omega\left(\frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r}\right)$$
$$+ A\omega^2 + B\omega^3 + \cdots + R\omega^r + \Omega,$$

where $A, B, \ldots, R$ represent certain functions of the quantity $\frac{y-1}{y}$. Hence,

$$\frac{y^\omega - 1}{\omega} = \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r}$$
$$+ A\omega + B\omega^2 + \cdots + R\omega^{r-1} + \frac{\Omega}{\omega}.$$

I now claim that the quantity

$$A\omega + B\omega^2 + \cdots + R\omega^{r-1} + \frac{\Omega}{\omega}$$

can be made as small as desired merely by diminishing ω and increasing r.
(α) In order to see this let us first consider only the case where ω, as well as $\frac{y-1}{y}$, is a positive quantity and therefore $y > +1$. In this case, if one considers the quantities $(y-1)$ and $(r-1)$ as unanalysable then throughout the series:

$$1 + \omega\frac{(y-1)}{y} + \omega\frac{\omega+1}{2}\frac{(y-1)^2}{y^2} + \cdots$$
$$+ \omega\frac{\omega+1}{2}\frac{\omega+2}{3}\cdots\frac{\omega+(r-1)}{r}\frac{(y-1)^r}{y^r}$$

there appear no subtraction signs and no negative quantities. Therefore also in the expansion of this series in powers of ω all the terms must be positive. Therefore $A, B, \ldots, R$ are all positive. Hence, furthermore, the value of Ω must always be positive. For if Ω were negative, i.e. if for some r the value of the above series were greater than y^ω, then by increasing (and not diminishing) the number of terms in this series, i.e. adding new positive terms, the difference could not be made smaller than every given quantity, as it should be by §48. But if all the terms in $A\omega^2 + B\omega^3 + \cdots + R\omega^r + \Omega$ are positive, then it follows that if one omitted this quantity (still divided by the positive ω), then it must be that:

$$\frac{y^\omega - 1}{\omega} > \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r}.$$

Now in order to find a quantity from the other side to which $\frac{y^\omega-1}{\omega}$ stands in the relationship of being the smaller, consider the following. Each of the factors,

$$\frac{\omega+1}{2}\frac{\omega+2}{3}, \frac{\omega+1}{2}\frac{\omega+2}{3}\frac{\omega+3}{4}, \ldots, \frac{\omega+1}{2}\frac{\omega+2}{3}\frac{\omega+3}{4}\cdots\frac{\omega+r-1}{r}$$

which appear in the above series is contained in the general form:

$$\frac{\omega+1}{2}\frac{\omega+2}{3}\cdots\frac{\omega+p-1}{p}$$
$$= \frac{\omega+1}{1}\frac{\omega+2}{2}\cdots\frac{\omega+p-1}{p-1}\frac{1}{p}$$
$$= (1+\omega)\left(1+\frac{\omega}{2}\right)\left(1+\frac{\omega}{3}\right)\cdots\left(1+\frac{\omega}{p-1}\right)\frac{1}{p}$$

if p is allowed to denote $2, 3, 4, \ldots, r$ successively. Now certainly,

$$(1+\omega)\left(1+\frac{\omega}{2}\right)\left(1+\frac{\omega}{3}\right)\cdots\left(1+\frac{\omega}{p-1}\right) < (1+\omega)^{p-1},$$

since the $(p-1)$ factors of this last product are all equal to the greatest in the first product. But since $(p-1)$ denotes a whole positive number, $(1+\omega)^{p-1}$ is, by §8,

$$\begin{aligned}
&= 1 + (p-1)\omega + (p-1)\left(\frac{p-2}{2}\right)\omega^2 \\
&\qquad + (p-1)\left(\frac{p-2}{2}\right)\left(\frac{p-3}{3}\right)\omega^3 + \cdots \\
&= 1 + (p-1)\omega\left(1 + \frac{p-2}{2}\omega + \frac{p-2}{2}\frac{p-3}{3}\omega^2 + \cdots\right)
\end{aligned}$$

a series in which all terms are positive or zero (§4). Therefore its value is certainly

$$\begin{aligned}
&< 1 + (p-1)\omega\left(1 + (p-2)\omega + \frac{(p-2)(p-3)}{2}\omega^2 + \cdots\right) \\
&= 1 + (p-1)\omega(1+\omega)^{p-2}.
\end{aligned}$$

Thus all the more certainly,

$$(1+\omega)\left(1+\frac{\omega}{2}\right)\left(1+\frac{\omega}{3}\right)\cdots\left(1+\frac{\omega}{p-1}\right) < 1 + (p-1)\omega(1+\omega)^{p-2}.$$

Therefore if, in the series which expresses the value of y^ω, instead of each factor of the form,

$$\frac{\omega+1}{2}\frac{\omega+2}{3}\cdots\frac{\omega+p-1}{p} = (1+\omega)\left(1+\frac{\omega}{2}\right)\cdots\left(1+\frac{\omega}{p-1}\right)\frac{1}{p}$$

a quantity of the form $\frac{1+(p-1)\omega(1+\omega)^{p-2}}{p}$ is substituted, then the value of the series must necessarily be $> y^\omega$ even if the Ω there is omitted. For this latter quantity can, by §48 for constant y and ω, become smaller than any given quantity merely by increasing r, therefore also smaller than one of the positive quantities added here which has a definite value for the same y and ω. Therefore

$$\begin{aligned}
y^\omega &< 1 + \omega\frac{(y-1)}{y} + \frac{\omega}{2}\frac{(y-1)^2}{y^2}(1+\omega) \\
&\quad + \frac{\omega}{3}\frac{(y-1)^3}{y^3}(1 + 2\omega(1+\omega)) \\
&\quad + \frac{\omega}{4}\frac{(y-1)^4}{y^4}(1 + 3\omega(1+\omega)^2) \\
&\quad + \ldots .
\end{aligned}$$

$$+\frac{\omega}{r}\frac{(y-1)^r}{y^r}(1+(r-1)\omega(1+\omega)^{r-2})$$

$$=1+\omega\frac{(y-1)}{y}+\frac{\omega}{2}\frac{(y-1)^2}{y^2}+\frac{\omega}{3}\frac{(y-1)^3}{y^3}+\cdots+\frac{\omega}{r}\frac{(y-1)^r}{y^r}$$
$$+\frac{\omega^2}{2}\frac{(y-1)^2}{y^2}+\frac{2\omega^2}{3}\frac{(y-1)^3}{y^3}(1+\omega)+\frac{3\omega^4}{4}\frac{(y-1)^4}{y^4}(1+\omega)^2$$
$$+\cdots+\frac{r-1}{r}\omega^2\frac{(y-1)^r}{y^r}(1+\omega)^{r-2}.$$

The last series here, which contains the factor ω^2 is certainly smaller than the following one which arises from it if instead of the fractions $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{r-1}{r}$ one puts unity throughout,

$$+\omega^2\frac{(y-1)^2}{y^2}+\omega^2\frac{(y-1)^3}{y^3}(1+\omega)+\omega^2\frac{(y-1)^4}{y^4}(1+\omega)^2+\cdots$$
$$+\omega^2\frac{(y-1)^r}{y^r}(1+\omega)^{r-2}$$
$$=\omega^2\frac{(y-1)^2}{y^2}\left(1+\frac{(y-1)}{y}(1+\omega)+\frac{(y-1)^2}{y^2}(1+\omega)^2+\cdots\right.$$
$$\left.+\frac{(y-1)^{r-2}}{y^{r-2}}(1+\omega)^{r-2}\right).$$

Now the series inside the brackets is a geometric progression whose ratio[o] is $\frac{(y-1)}{y}(1+\omega)$. Therefore if ω is taken so small that the product of $(1+\omega)$ with the proper fraction $\frac{(y-1)}{y}$ still remains a proper fraction, then for this value and for all smaller values the series will be *decreasing* and its value

$$=\frac{1-(y-1)^{r-1}(1+\omega)^{r-1}}{1-\frac{(y-1)}{y}(1+\omega)}$$

always remains, *however large r may be taken*,

$$<\frac{1}{1-\frac{(y-1)}{y}(1+\omega)}=\frac{y}{1-\omega(y-1)}.$$

Therefore always,

$$\frac{y^\omega-1}{\omega}<\frac{(y-1)}{y}+\frac{1}{2}\frac{(y-1)^2}{y^2}+\frac{1}{3}\frac{(y-1)^3}{y^3}+\cdots+\frac{1}{r}\frac{(y-1)^r}{y^r}$$
$$+\frac{\omega(y-1)^2}{y-\omega y(y-1)}.$$

[o] The German *Exponent* was used both for exponent in a modern sense and also for the common ratio of a geometric progression. See Klügel (1803–36).

And previously,

$$\frac{y^\omega - 1}{\omega} > \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r}.$$

Here therefore we have two limits between which the value of the function $\frac{y^\omega - 1}{\omega}$ always remains enclosed. Their difference

$$\frac{\omega(y-1)^2}{y - \omega y(y-1)},$$

can, as may be seen from the proof of §19, become smaller than any given quantity if ω is taken small enough. Therefore the true value is

$$\frac{y^\omega - 1}{\omega} = \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r} + \overset{1}{\Omega}.$$

(β) Now let ω always be positive but $y < +1$ although $> +\frac{1}{2}$. Accordingly $\frac{y-1}{y}$ now represents a negative quantity, but because $y > \frac{1}{2}$, the following equation still holds:

$$\frac{y^\omega - 1}{\omega} = \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r}$$
$$+ \overset{1}{A}\omega + \overset{1}{B}\omega^2 + \cdots + \overset{1}{R}\omega^{r-1} + \frac{\overset{1}{\Omega}}{\omega}.$$

However, for every y which is $> \frac{1}{2}$ and at the same time < 1, there is a quantity z which satisfies the equation $\frac{y-1}{y} = -\frac{z-1}{z}$ and at the same time is $> +1$. For this it is necessary only to take $z = \frac{y}{2y-1}$ then z is positive and > 1. Therefore, also the following equation holds:

$$\frac{z^\omega - 1}{\omega} = \frac{(z-1)}{z} + \frac{1}{2}\frac{(z-1)^2}{z^2} + \frac{1}{3}\frac{(z-1)^3}{z^3} + \cdots + \frac{1}{r}\frac{(z-1)^r}{z^r}$$
$$+ A\omega + B\omega^2 + \cdots + R\omega^{r-1} + \frac{\Omega}{\omega},$$

in which, by what has already been proved, the quantity $A\omega + B\omega^2 + \cdots + R\omega^{r-1} + \frac{\Omega}{\omega}$ can become as small as desired merely by increasing r and diminishing ω. In particular this also holds of the part $A\omega + B\omega^2 + \cdots + R\omega^{r-1}$. But if we compare this with $\overset{1}{A}\omega + \overset{1}{B}\omega^2 + \cdots + \overset{1}{R}\omega^{r-1}$ then it is soon clear that the coefficients $\overset{1}{A}, \overset{1}{B}, \ldots, \overset{1}{R}$ are composed in exactly the same way from the quantity $\frac{z-1}{z}$ as the coefficients $A, B, \ldots, R$ are composed from the quantity $\frac{y-1}{y}$. Now since $\frac{z-1}{z}$ and $\frac{y-1}{y}$ are equal to one another in value, and since, as noted in (α), the individual parts

$$\frac{z-1}{z}, \frac{(z-1)^2}{z^2}, \frac{(z-1)^3}{z^3}, \ldots, \frac{(z-1)^r}{z^r}$$

from which $A, B, \ldots, R$ are composed, are all positive and are combined only by additions, while in $\overset{1}{A}, \overset{1}{B}, \ldots, \overset{1}{R}$ there appear some negative terms like

$$\frac{(y-1)}{y}, \frac{(y-1)^3}{y^3}, \frac{(y-1)^5}{y^5}, \ldots$$

and some positive terms like

$$\frac{(y-1)^2}{y^2}, \frac{(y-1)^4}{y^4}, \frac{(y-1)^6}{y^6}, \ldots,$$

then it follows undeniably that the total value of $\overset{1}{A}\omega + \overset{1}{B}\omega^2 + \cdots + \overset{1}{R}\omega^{r-1}$ must be smaller than that of $A\omega + B\omega^2 + \cdots + R\omega^{r-1}$. Now if the latter, however large r may be taken, always remains

$$< \frac{\omega(z-1)^2}{z - \omega z(z-1)} = \frac{\omega(y-1)^2}{2y^2 - y + \omega y(y-1)}$$

then all the more certainly, $\overset{1}{A}\omega + \overset{1}{B}\omega^2 + \cdots + \overset{1}{R}\omega^{r-1}$ must always remain

$$< \frac{\omega(y-1)^2}{2y^2 - y + \omega y(y-1)}$$

however large r is taken. Therefore if $\overset{1}{A}\omega + \overset{1}{B}\omega^2 + \cdots + \overset{1}{R}\omega^{r-1} + \frac{\overset{1}{\Omega}}{\omega}$ is to become smaller than some given quantity D then ω is first taken so small that

$$\frac{\omega(y-1)^2}{2y^2 - y + \omega y(y-1)} < \frac{D}{2}$$

and for every value of r, $\overset{1}{A}\omega + \overset{1}{B}\omega^2 + \cdots + \overset{1}{R}\omega^{r-1}$ remains $< \frac{D}{2}$. Therefore if one now takes r so large that $\overset{1}{\Omega} < \frac{\omega D}{2}$, then $\frac{\overset{1}{\Omega}}{\omega} < \frac{D}{2}$, and $\overset{1}{A}\omega + \overset{1}{B}\omega^2 + \cdots + \overset{1}{R}\omega^{r-1} + \frac{\overset{1}{\Omega}}{\omega}$ is certainly $<D$, as was required. Therefore not only for every $y > +1$, but also for every $y > +\frac{1}{2}$ the following equation is correct:

$$\frac{y^\omega - 1}{\omega} = \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r} + \Omega$$

provided ω is positive.

(γ) Finally it is easily shown that it is also correct if ω is *negative*. For since

$$\frac{y^{-\omega} - 1}{-\omega} = \frac{y^\omega - 1}{\omega}\frac{1}{y^\omega},$$

and since the factor $\frac{1}{y^\omega}$, if ω is taken as small as desired, approaches as close to unity as desired, then

$$\frac{y^{-\omega} - 1}{-\omega} = \frac{y^\omega - 1}{\omega}(1 + \Omega).$$

But from the previous work it may be seen that the value of $\frac{y^\omega - 1}{\omega}$, however small ω may be taken, remains within certain finite limits. For the value of the series,

$$\frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r},$$

which comes as near to it as desired if one takes r sufficiently large, is obviously smaller than the value of the geometric progression,

$$\pm\frac{(y-1)}{y} + \frac{(y-1)^2}{y^2} \pm \frac{(y-1)^3}{y^3} + \cdots \pm \frac{(y-1)^r}{y^r}$$

in which all terms are taken to be positive. Now the ratio of this progression, $\frac{y-1}{y}$, is a proper fraction, therefore its value is always smaller than a certain finite quantity; therefore by §17,

$$\frac{y^\omega - 1}{\omega}\Omega = \overset{1}{\Omega},$$

and consequently by §16:

$$\frac{y^{-\omega} - 1}{-\omega} = \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r} + \overset{(2)}{\Omega}.$$

2. Now the proof of the *second* formula has no further difficulties. Because for all values of y which are $< +2$, $\frac{1}{y}$ denotes a quantity which is $> +\frac{1}{2}$. Now since, by 1. (γ),

$$\frac{y^\omega - 1}{\omega} = \frac{y^{-\omega} - 1}{-\omega} - \overset{1}{\Omega} = -\frac{\frac{1}{y^\omega} - 1}{\omega} - \overset{1}{\Omega},$$

then $\frac{\frac{1}{y^\omega} - 1}{\omega}$ can be expanded by the formula of 1. if one puts $\frac{1}{y}$ instead of y. But by this $\left(\frac{y-1}{y}\right)$ is changed into $(1-y)$ and one obtains

$$\frac{y^\omega - 1}{\omega} = -\left((1-y) + \frac{1}{2}(1-y)^2 + \frac{1}{3}(1-y)^3 + \cdots + \frac{1}{r}(1-y)^r\right) - \overset{1}{\Omega}$$

or, what amounts to the same,

$$\frac{y^\omega - 1}{\omega} = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \cdots \pm \frac{1}{r}(y-1)^r - \overset{1}{\Omega}.$$

§ 63

Corollary. For a *negative* value of the quantity y^ω the numerator of the fraction $\frac{y^\omega - 1}{\omega}$ approaches the value -2 as much as desired, while its denominator ω becomes as small as desired, from which it is clear that $\frac{y^\omega - 1}{\omega}$ becomes greater than any given quantity if ω is continually diminished; therefore it is not representable by any formula which would be similar to that of §62. But if y itself is negative then it depends on the nature of ω whether $\frac{y^\omega - 1}{\omega}$ can be expanded in a way similar to that of §62. If it is of the form $\frac{2n}{m}$, i.e. a fraction with even numerator, then $(-y)^\omega = (+y)^\omega$ and therefore everything holds of this case which has just been shown true of $\frac{(+y)^\omega - 1}{\omega}$. But if ω is of the form $\frac{2n+1}{m}$, then it depends on the nature of the denominator m. If m is also odd, then $(-y)^\omega$ is negative and consequently $\frac{(-y)^\omega - 1}{\omega}$ grows without limit. Finally, if m is even, then $(-y)^\omega$, and consequently also $\frac{(-y)^\omega - 1}{\omega}$, is imaginary.

§ 64

Problem. To investigate whether, and in what way, the *exponential quantity a^x* can be expanded in a series of powers of x as long as a is positive and by a^x is understood only the real and positive value of this expression.

Solution. 1. By §48 one of the following two equations holds for every value of x according as $a < +2$ or $a > +\frac{1}{2}$: either

$$a^x = 1 + x(a-1) + x\frac{x-1}{2}(a-1)^2 + x\frac{x-1}{2}\frac{x-2}{3}(a-1)^3 + \cdots + x\frac{x-1}{2}\frac{x-2}{3}\cdots\frac{x-r+1}{r}(a-1)^r + \Omega,$$

or

$$a^x = 1 + x\frac{(a-1)}{a} + x\frac{x+1}{2}\frac{(a-1)^2}{a^2} + x\frac{x+1}{2}\frac{x+2}{3}\frac{(a-1)^3}{a^3} + \cdots + x\frac{x+1}{2}\frac{x+2}{3}\cdots\frac{x+r-1}{r}\frac{(a-1)^r}{a^r} + \overset{1}{\Omega}.$$

Now if one actually carried out the multiplication that is merely indicated here by the factors

$$\frac{x-1}{2}, \frac{x-2}{3}, \ldots, \frac{x-r+1}{r}, \frac{x+1}{2}, \frac{x+2}{3}, \ldots, \frac{x+r-1}{r},$$

and then one combined all quantities which contain the same power of x into one term, one would obtain from the two equations what was desired, a series proceeding by powers of x, which comes as near to the value of the function a^x as desired if r, i.e. the number of its terms, is taken large enough. Therefore from this the *possibility* of such a series for every value of x is made clear, only it would be difficult to discover in this way the rule of formation for the coefficients of its terms. We therefore look for this in another way.

2. The series which is obtained from the expansion of one of the two equations above would obviously have the first term $= 1$, and the subsequent terms would only have purely positive and whole-numbered powers of x. But someone could consider that the form derivable from those two equations is perhaps not the only one which a series proceeding by powers of x must have in order to be either completely equal, or to come as close as desired, to the value of a^x for as many values (viz., as we now know, for all values) of x. So we shall assume that

$Ax^\alpha + Bx^\beta + Cx^\gamma + \cdots + Rx^\rho$ is the most general form which such a series must possess, i.e. for all values of x,

$$a^x = Ax^\alpha + Bx^\beta + Cx^\gamma + \cdots + Rx^\rho + \Omega \qquad \odot$$

if one increases the number of terms in this series sufficiently. Therefore also it must be that:

$$a^{x+\omega} = A(x+\omega)^\alpha + B(x+\omega)^\beta + C(x+\omega)^\gamma + \cdots + R(x+\omega)^\rho + \overset{\text{I}}{\Omega}.$$

Consequently, by subtraction and division by ω:

$$a^x \frac{a^\omega - \text{I}}{\omega} = A\left(\frac{(x+\omega)^\alpha - x^\alpha}{\omega}\right) + B\left(\frac{(x+\omega)^\beta - x^\beta}{\omega}\right) + \cdots$$
$$+ R\left(\frac{(x+\omega)^\rho - x^\rho}{\omega}\right) + \frac{\overset{\text{I}}{\Omega} - \Omega}{\omega}.$$

The quantity $\frac{a^\omega - \text{I}}{\omega}$, by §63, approaches as close as desired, if ω is taken small enough, a certain constant value which, according as a is $<+2$ or $>+\frac{\text{I}}{2}$, is given as accurately as desired, if r is taken large enough, by either the series:

$$(a-\text{I}) - \frac{\text{I}}{2}(a-\text{I})^2 + \frac{\text{I}}{3}(a-\text{I})^3 - \cdots \pm \frac{\text{I}}{r}(a-\text{I})^r,$$

or the series:

$$\frac{(a-\text{I})}{a} + \frac{\text{I}}{2}\frac{(a-\text{I})^2}{a^2} + \frac{\text{I}}{3}\frac{(a-\text{I})^3}{a^3} + \cdots + \frac{\text{I}}{r}\frac{(a-\text{I})^r}{a^r}.$$

We shall designate this value briefly, by $\mathfrak{A}$, because it depends on a, and put $\frac{a^\omega - \text{I}}{\omega} = \mathfrak{A} + \overset{(a)}{\omega}$. Under the same condition that ω can be taken as small as desired, by §23 we also have:

$$\frac{(x+\omega)^\alpha - x^\alpha}{\omega} = \alpha x^{\alpha-\text{I}} + \overset{(\alpha)}{\omega},$$
$$\frac{(x+\omega)^\beta - x^\beta}{\omega} = \beta x^{\beta-\text{I}} + \overset{(\beta)}{\omega},$$
$$\cdots\cdots\cdots$$
$$\frac{(x+\omega)^\rho - x^\rho}{\omega} = \rho x^{\rho-\text{I}} + \overset{(\rho)}{\omega}.$$

Therefore we obtain the equation:

$$a^x.\mathfrak{A} = A\alpha x^{\alpha-\text{I}} + B\beta x^{\beta-\text{I}} + \cdots + R\rho x^{\rho-\text{I}}$$
$$- a^x\overset{(a)}{\omega} + A\overset{\alpha}{\omega} + B\overset{(\beta)}{\omega} + \cdots + R\overset{(\rho)}{\omega} + \frac{\overset{\text{I}}{\Omega} - \Omega}{\omega}.$$

As for the similar expression in §30 it can be shown here that $\frac{\overset{1}{\Omega}-\Omega}{\omega}$ can be made as small as desired merely by increasing the number of terms in the series $Ax^{\alpha} + Bx^{\beta} + \cdots + Rx^{\rho}$. But then with the same number of terms, merely by diminishing ω, the sum $A\overset{(\alpha)}{\omega} + B\overset{(\beta)}{\omega} + \cdots + R\overset{(\rho)}{\omega} - a^x\overset{(a)}{\omega}$ can become smaller than any given quantity (§§ 17, 15). Therefore,

$$a^x.\mathfrak{A} = A\alpha x^{\alpha-1} + B\beta x^{\beta-1} + \cdots + R\rho x^{\rho-1} + \overset{(2)}{\Omega};$$

and if one puts instead of a^x the series which is to equal this value, then by applying §§ 17, 15 again:

$$\mathfrak{A}Ax^{\alpha} + \mathfrak{A}Bx^{\beta} + \mathfrak{A}Cx^{\gamma} + \cdots + \mathfrak{A}Rx^{\rho}$$
$$= \alpha Ax^{\alpha-1} + \beta Bx^{\beta-1} + \gamma Cx^{\gamma-1} + \cdots + \rho Rx^{\rho-1} + \overset{(3)}{\Omega}. \qquad ♁$$

Since this equation must hold as a necessary condition for the validity of the equation ⊙, §28 can be applied to it and it may be found, by similar arguments as were used in §32, that $\alpha = 0$, $\beta = 1$, $\gamma = 2, \ldots, \rho =$ some whole number r which is one smaller than the number of terms in the series. Furthermore, $B = \mathfrak{A}A$, $C = \frac{\mathfrak{A}^2 A}{2}$, $D = \frac{\mathfrak{A}^3 A}{2.3}, \ldots, R = \frac{\mathfrak{A}Q}{r} = \frac{\mathfrak{A}^r A}{2.3\ldots r}$. And with these determinations, all terms of the equation are accounted for up to the highest $\mathfrak{A}Rx^{\rho} = \frac{\mathfrak{A}^{r+1}Ax^r}{2.3\ldots r}$. But it is easy to see that this difference, which if the equation is to hold must be able to become as small as desired, does actually have this property. For every subsequent value of it, $\frac{\mathfrak{A}^{r+2}Ax^{r+1}}{2.3\ldots(r+1)}$ arises from the value immediately preceding, $\frac{\mathfrak{A}^{r+1}Ax^r}{2.3\ldots r}$, by multiplication by $\frac{\mathfrak{A}x}{r+1}$. Now since for the same $\mathfrak{A}$ and x, r can be as large as desired, then from the value when $(r+1) > \mathfrak{A}x$ and for all greater values, $\frac{\mathfrak{A}x}{r+1}$ is a proper fraction which becomes ever smaller. Therefore every subsequent value of that difference is produced from the preceding value by multiplication with a proper fraction which becomes ever smaller, therefore this difference itself will certainly become as small as desired if r is taken large enough. Consequently the form of the series $Ax^{\alpha} + Bx^{\beta} + \cdots + Rx^{\rho}$, which is required by the condition ♁, is the following:

$$A + A\mathfrak{A}x + \frac{A\mathfrak{A}^2x^2}{1.2} + \frac{A\mathfrak{A}^3x^3}{1.2.3} + \cdots + \frac{A\mathfrak{A}^rx^r}{1.2.3\ldots r}.$$

Here A still remains completely undetermined. But the equation ⊙ requires that $A = 1$, for otherwise it would be impossible for every value of x that:

$$a^x = A + A\mathfrak{A}x + \frac{A\mathfrak{A}^2x^2}{1.2} + \cdots + \frac{A\mathfrak{A}^rx^r}{1.2.3\ldots r} + \Omega.$$

For if one takes $x = \omega$, a^x approaches unity; but the series

$$A + A\mathfrak{A}x + \cdots + \frac{A\mathfrak{A}^rx^r}{1.2.3\ldots r} + \Omega,$$

for the same r, approaches the value A as close as desired. Therefore by §27 it must be that $A = 1$. From this determination our series now appears as follows:

$$1 + \mathfrak{A}x + \frac{\mathfrak{A}^2x^2}{1.2} + \frac{\mathfrak{A}^3x^3}{1.2.3} + \frac{\mathfrak{A}^4x^4}{1.2.3.4} + \cdots + \frac{\mathfrak{A}^rx^r}{1.2.3\ldots r}.$$

Therefore nothing remains undetermined in its form, so we find that there is only a *single* form for a series proceeding by powers of x, if it is to be exactly equal, or as close as desired, to the value of a^x for every x. Now since by 1. such a series is in fact possible, it must be the one found and therefore for every value of x:

$$a^x = 1 + \mathfrak{A}x + \frac{\mathfrak{A}^2x^2}{1.2} + \frac{\mathfrak{A}^3x^3}{1.2.3} + \cdots + \frac{\mathfrak{A}^rx^r}{1.2.3\ldots r} + \Omega$$

provided one takes r sufficiently large.

§ 65

Corollary. If ever a^x is to designate a quantity capable of taking opposite signs, then the two real values which it may have sometimes (provided, of course, x is of the form $\frac{p}{2n}$) are, as is well-known, equal in magnitude and differ only in sign. Hence both may be represented by the formula:

$$a^x = \pm 1 \pm \mathfrak{A}x \pm \frac{\mathfrak{A}^2x^2}{1.2} \pm \frac{\mathfrak{A}^3x^3}{1.2.3} \pm \cdots \pm \frac{\mathfrak{A}^rx^r}{1.2.3\ldots r} \pm \Omega.$$

But if a is a quantity capable of taking opposite signs and is *negative*, then it is known that a^x can only be something real if x is not of the form $\frac{2n+1}{2m}$ but is either $= \frac{2n}{2m}$ or $\frac{2n+1}{2m+1}$. However, $(-a)^{\frac{2n}{2m}}$ is, of course, $= (+a)^{\frac{2n}{2m}}$ and $(-a)^{\frac{2n+1}{2m+1}} = -(+a)^{\frac{2n+1}{2m+1}}$, therefore the value of a^x even in these cases, i.e. *all the real values* of a^x can be expressed in the expansion of a series proceeding by powers of x.

§ 66

Problem. To investigate whether, and in what way, the function $\overset{(a)}{\mathrm{l}}\, y$ is expandable in a series of powers of y or else of a quantity easily derivable from y, if by a is understood a quantity which is either merely abstract or only positive, and by every power of it is understood only the real and positive value.

Solution. 1. If we write for abbreviation $\overset{(a)}{\mathrm{l}}\, y = x$, then $a^x = y$ and consequently whatever ω denotes,

$$a^{\omega x} = y^{\omega}, \quad \text{and} \quad \frac{(a^{\omega x} - 1)}{\omega x} x = \frac{y^{\omega} - 1}{\omega}.$$

Now if ω denotes a quantity which can be become as small as desired, then for the same x, ωx is also such a quantity. Therefore merely by diminishing ω, $\frac{a^{\omega x}-1}{\omega x}$ can be brought as near as desired to a certain constant quantity, which we have already denoted by $\mathfrak{A}$ in §64; it is therefore $= \mathfrak{A} + \Omega$. But under the same condition, also $\frac{y^\omega - 1}{\omega}$, according as $y < +2$ or $> +\frac{1}{2}$, is either

$$= (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \cdots \pm \frac{1}{r}(y-1)^r + \overset{1}{\Omega},$$

$$\text{or} \quad = \left(\frac{y-1}{y}\right) + \frac{1}{2}\left(\frac{y-1}{y}\right)^2 + \frac{1}{3}\left(\frac{y-1}{y}\right)^3 + \cdots + \frac{1}{r}\left(\frac{y-1}{y}\right)^r + \overset{2}{\Omega}.$$

We therefore obtain the equation,

$$x(\mathfrak{A} + \Omega) = \begin{cases} (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \cdots \pm \frac{1}{r}(y-1)^r + \overset{1}{\Omega}, \\ \text{or} \\ \left(\frac{y-1}{y}\right) + \frac{1}{2}\left(\frac{y-1}{y}\right)^2 + \frac{1}{3}\left(\frac{y-1}{y}\right)^3 + \cdots + \frac{1}{r}\left(\frac{y-1}{y}\right)^r + \overset{4}{\Omega}. \end{cases}$$

Therefore by application of §§ 17, 15:

$$x = \frac{1}{\mathfrak{A}}\left((y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \cdots \pm \frac{1}{r}(y-1)^r\right) + \overset{3}{\Omega}$$

if $y < +2$, or

$$x = \frac{1}{\mathfrak{A}}\left(\left(\frac{y-1}{y}\right) + \frac{1}{2}\left(\frac{y-1}{y}\right)^2 + \frac{1}{3}\left(\frac{y-1}{y}\right)^3 + \cdots + \frac{1}{r}\left(\frac{y-1}{y}\right)^r\right) + \overset{4}{\Omega}$$

if $y > +\frac{1}{2}$.

Here therefore we already have what was required, $x = \overset{(a)}{\mathrm{l}}\, y$ expressed by a series of increasing powers, not indeed of y itself, but of a quantity easily derivable from it, namely either $(y-1)$ or $\left(\frac{y-1}{y}\right)$.

2. Now there is only the question of whether it may not be possible to represent $\overset{(a)}{\mathrm{l}}\, y$ in another way, namely by a series of increasing powers of the quantity y itself, or of an expression derived from it which is even simpler than $(y-1)$, $\left(\frac{y-1}{y}\right)$. For this purpose let z be such a quantity that $x = \overset{(a)}{\mathrm{l}}\, y$ can be expressed in increasing

powers of z, and suppose,

$$x = Az^{\alpha} + Bz^{\beta} + \cdots + Rz^{\rho} + \Omega. \qquad \odot$$

If this equation is to hold for all z which are smaller than a certain one, then it must also hold if one puts $z + \omega$ instead of z, whereby x may be transformed to $x + \overset{\text{I}}{\omega}$. Therefore,

$$x + \overset{\text{I}}{\omega} = A(z+\omega)^{\alpha} + B(z+\omega)^{\beta} + \cdots + R(z+\omega)^{\rho} + \overset{\text{I}}{\Omega},$$

or by subtraction and division of ω, if the notation already well known from §§ 30, 55 and 64 is introduced,

$$\begin{aligned}\frac{\overset{\text{I}}{\omega}}{\omega} &= \alpha A z^{\alpha-\text{I}} + \beta B z^{\beta-\text{I}} + \cdots + \rho R z^{\rho-\text{I}} \\ &\quad + A\overset{(\alpha)}{\omega} + B\overset{(\beta)}{\omega} + \cdots + R\overset{(\rho)}{\omega} + \frac{\overset{(\text{I})}{\Omega} - \Omega}{\omega}.\end{aligned}$$

It can also be shown here in exactly the same way as in the paragraphs just mentioned that the quantity,

$$A\overset{(\alpha)}{\omega} + B\overset{(\beta)}{\omega} + \cdots + R\overset{(\rho)}{\omega} + \frac{\overset{(\text{I})}{\Omega} - \Omega}{\omega}$$

can become as small as desired merely by increasing the number of terms and diminishing ω. We therefore have,

$$\frac{\overset{\text{I}}{\omega}}{\omega} = \alpha A z^{\alpha-\text{I}} + \beta B z^{\beta-\text{I}} + \cdots + \rho R z^{\rho-\text{I}} + \overset{(2)}{\Omega}.$$

Now if z is to be some quantity derived from y, then conversely y can be considered as a function of z and therefore we can write $y = fz$. The simpler the nature of the function fz that is assumed the simpler also will be the way z is derived from y. But if $a^x = y = fz$ then also, it must be that:

$$a^{x+\overset{\text{I}}{\omega}} = f(z+\omega).$$

This gives $a^{x+\overset{\text{I}}{\omega}} - a^x = f(z+\omega) - fz$, and

$$\frac{a^{x+\overset{\text{I}}{\omega}} - a^x}{a^x} = a^{\overset{\text{I}}{\omega}} - \text{I} = \frac{f(z+\omega) - fz}{fz},$$

therefore

$$\frac{a^{\overset{\text{I}}{\omega}} - \text{I}}{\overset{\text{I}}{\omega}} = \mathfrak{A} + \overset{2}{\omega} = \frac{f(z+\omega) - fz}{\overset{\text{I}}{\omega} fz},$$

from which by application of §§ 17, 15:

$$\frac{\overset{1}{\omega}}{\omega} = \frac{f(z+\omega) - fz}{\omega} \frac{1}{\mathfrak{A}fz} + \overset{3}{\omega}.$$

Therefore the following equation must hold:

$$\frac{f(z+\omega) - fz}{\omega} \frac{1}{\mathfrak{A}fz} = \alpha A z^{\alpha-1} + \beta B z^{\beta-1} + \cdots + \rho R z^{\rho-1} + \overset{3}{\Omega}$$ ♁

as an indispensable *condition* for the validity of the equation ⊙. It cannot be decided from this condition how fz must be constituted but it can, nevertheless, be decided how it may *not* be constituted, since for every form of the function fz for which even ♁ does not hold, then also ⊙ does not hold, but not conversely.

(α) We see from this, first of all, that $y = fz$ is not $= z$ itself, also it is not $=$ a mere multiple of z, e.g. nz. For if we take $fz = nz$ then

$$\frac{f(z+\omega) - fz}{\omega} \frac{1}{\mathfrak{A}fz} = \frac{1}{\mathfrak{A}z}.$$

Therefore it would have to be that,

$$\frac{1}{\mathfrak{A}z} = \alpha A z^{\alpha-1} + \beta B z^{\beta-1} + \cdots + \rho R z^{\rho-1} + \overset{3}{\Omega},$$

$$\text{or} \quad 1 = \alpha A \mathfrak{A} z^{\alpha} + \beta B \mathfrak{A} z^{\beta} + \cdots + \rho R \mathfrak{A} z^{\rho} + \overset{4}{\Omega},$$

which by §28 is impossible because it would presuppose that $\alpha = 0$, $A = 1$, $B = 0$, $C = 0, \ldots, R = 0$, i.e. that one has in mind no function of z at all but merely a constant quantity 1.

(β) We shall now take for fz the form $m + nz$, as being the simplest of those functions which are more complicated than nz. Then one finds

$$\frac{f(z+\omega) - fz}{\omega} \frac{1}{\mathfrak{A}fz} = \frac{n}{\mathfrak{A}(m+nz)}$$

and therefore it must be that,

$$\frac{n}{\mathfrak{A}(m+nz)} = \alpha A z^{\alpha-1} + \beta B z^{\beta-1} + \cdots + \rho R z^{\rho-1} + \overset{3}{\Omega},$$

or

$$n = \alpha A m \mathfrak{A} z^{\alpha-1} + \beta B m \mathfrak{A} z^{\beta-1} + \cdots + \rho R m \mathfrak{A} z^{\rho-1}$$
$$+ \alpha A n \mathfrak{A} z^{\alpha} + \beta B n \mathfrak{A} x^{\beta} + \cdots + \rho R n \mathfrak{A} z^{\rho} + \overset{4}{\Omega}.$$

This condition gives, by arguments like those of §32 the following determinations: $\alpha = 1, \beta = 2, \gamma = 3, \ldots, \rho =$ some whole number r, then

$$\begin{aligned} A &= \frac{n}{m\mathfrak{A}}, \\ B &= -\frac{nA}{2m} = -\frac{n^2}{2m^2\mathfrak{A}}, \\ C &= -\frac{2nB}{3m} = +\frac{n^3}{3m^3\mathfrak{A}}, \\ &\ldots\ldots\ldots \\ R &= -\frac{(r-1)Qn}{rm} = \pm\frac{n^r}{rm^r\mathfrak{A}}. \end{aligned}$$

Therefore one obtains the form,

$$\frac{nz}{m\mathfrak{A}} - \frac{n^2z^2}{2m^2\mathfrak{A}} + \frac{n^3z^3}{3m^3\mathfrak{A}} - \cdots \pm \frac{n^rz^r}{rm^r\mathfrak{A}},$$

or since $z = \frac{y-m}{n}$, the following:

$$\frac{1}{\mathfrak{A}}\left(\left(\frac{y-m}{m}\right) - \frac{1}{2}\left(\frac{y-m}{m}\right)^2 + \frac{1}{3}\left(\frac{y-m}{m}\right)^3 - \cdots \pm \frac{1}{r}\left(\frac{y-m}{m}\right)^r\right).$$

Now here the quantity n itself has been completely cancelled out but the equation ♁ leaves it completely undetermined how m is to be taken. It also cannot be decided from the equation alone, whether and for what values of y the above series in fact yields the value of $\overset{(a)}{\mathrm{l}}\, y$. Only for other reasons, e.g. from the fact that for $y = m$ the series gives the value of $\overset{(a)}{\mathrm{l}}\, m$ equal to zero, can it be concluded that m must be $= 1$, because only the logarithm of 1 can be $= 0$. But it follows simply from the proof which we gave in 1. that the series holds only for values of y which are $<+2$, and it holds for these without exception.

(γ) Finally, if we take as the simplest of *fractional* functions $fz = \frac{m}{n+pz}$, or, because this form can always be brought to $\frac{1}{\frac{n}{m}+\frac{p}{m}z}$, if we take $fz = \frac{1}{m+nz}$, then we obtain,

$$\frac{f(z+\omega) - fz}{\omega}\frac{1}{\mathfrak{A}fz} = \frac{-n}{\mathfrak{A}(m+nz)} + \overset{2}{\omega}.$$

Therefore it must be that:

$$\frac{-n}{\mathfrak{A}(m+nz)} = \alpha Az^{\alpha-1} + \beta Bz^{\beta-1} + \cdots + \rho Rz^{\rho-1} + \overset{3}{\Omega},$$

or

$$\begin{aligned} -n = {} & \alpha Am\mathfrak{A}z^{\alpha-1} + \beta Bm\mathfrak{A}z^{\beta-1} + \cdots + \rho Rm\mathfrak{A}z^{\rho-1} \\ & + \alpha An\mathfrak{A}z^{\alpha} + \beta Bn\mathfrak{A}z^{\beta} + \cdots + \rho Rn\mathfrak{A}z^{\rho} + \overset{4}{\Omega}. \end{aligned}$$

This gives the determinations: $\alpha = 1$, $\beta = 2$, $\gamma = 3, \ldots, \rho = r$, then

$$A = -\frac{n}{m\mathfrak{A}},$$
$$B = -\frac{An}{2m} = +\frac{n^2}{2m^2\mathfrak{A}},$$
$$C = -\frac{2Bn}{3m} = -\frac{n^3}{3m^3\mathfrak{A}},$$
$$\ldots\ldots\ldots$$
$$R = -\frac{(r-1)Qn}{rm} = \pm\frac{n^r}{rm^r\mathfrak{A}}.$$

Therefore the form of the series which should express the value of $\overset{(a)}{\mathrm{l}}\,y$ is:

$$-\frac{nz}{m\mathfrak{A}} + \frac{n^2z^2}{2m^2\mathfrak{A}} - \frac{n^3z^3}{3m^3\mathfrak{A}} + \cdots \pm \frac{n^rz^r}{rm^r\mathfrak{A}},$$

or because $z = \frac{1-my}{ny}$, it is:

$$\frac{1}{\mathfrak{A}}\left(\left(\frac{my-1}{my}\right) + \frac{1}{2}\left(\frac{my-1}{my}\right)^2 + \frac{1}{3}\left(\frac{my-1}{my}\right)^3 + \cdots + \frac{1}{r}\left(\frac{my-1}{my}\right)^r\right).$$

How m is to be taken cannot be learned from ♁ but certainly from the fact that for $y = \frac{1}{m}$ the value of $\overset{(a)}{\mathrm{l}}\,\frac{1}{m}$ found from the series $= 0$, hence it may be seen that $\frac{1}{m}$ must $= 1$, and $m = 1$. But it is proved in 1. that this series only holds for values of $y > +\frac{1}{2}$, and for these values it holds without exception.

3. Therefore $(y-1)$ and $\left(\frac{y-1}{y}\right)$ are the two *simplest* functions of y which have the property that the value of the function $\overset{(a)}{\mathrm{l}}\,y$ can be expressed by a series of their increasing powers. But it is in no way said here that there may not be some other functions among the *more complicated* functions which also possess this property. For example, if one puts, $y = fz = \frac{m+nz}{p+qz}$ or $= \frac{1+mz}{n+pz}$ then one would again obtain a formula for $\overset{(a)}{\mathrm{l}}\,y$ which proceeds in powers of $z = \frac{1-ny}{py-m}$ and by arguments like those in (β) and (γ) it would be found that $m = 1$, $n = 1$, $p = -1$, etc.

§ 67

Corollary. It can now also be decided whether, and in which cases, the function $\overset{(a)}{\mathrm{l}}\,y$ is representable by a series of increasing powers of a quantity derivable from y, even *if y is negative*. For as is well known, $(-a)^x$, whenever it is real and negative, $= -(+a)^x$, if one now understands by $(+a)^x$, when this expression possesses a double real value, only the positive one. Now since §66 shows how the quantity x can be represented by a series of increasing powers of a quantity derived from the positive y, then because every derivation from $+y$ is also a derivation from $-y$, one has, in the very same formula given by §66, expressed the value of x by y, even if the latter is negative. But, as already said, this is only true if $y = a^x$ is real, i.e. in the case of a negative a and a rational x provided the latter has the form $\frac{2p+1}{2q+1}$.

§ 68

Note. It is clear from §66 that for any two abstract or positive values of the quantities a and y there is an x for which the equation $y = a^x$ holds. That is, every abstract or positive quantity has a possible logarithm in every system of which the base number is itself an abstract or positive quantity. And likewise it is clear from §64 that to every real logarithm in such a system there is also a corresponding real quantity. On the other hand, if one wanted to take a *negative* quantity, $-a$, for the base [*Basis*] of the system then first of all it is certain that neither for all positive nor for all negative quantities would there be a corresponding logarithm. Not for all *positive* quantities, because for all values of y, which are of the form $\frac{1}{(+a)^{2n+1}}$, obviously no x is possible for which $(-a)^x = y = \frac{1}{(+a)^{2n+1}}$. Also not for all *negative* quantities, since for all y which are of the form $-\frac{1}{(+a)^{2n}}$, no value of x can be given which makes $(-a)^x = -\frac{1}{(+a)^{2n}}$. Another inconvenience would be that there would not be a real y corresponding to every real x. For whenever x were of the form $\frac{2n+1}{2m}$, y would be imaginary. This would give rise to a third inconvenience in all cases where one did not know the value of the logarithm x exactly, but only *approximately* and one had to deduce from it whether there was a y corresponding to it and what kind of y it was. No certain information could be found out about it if the value of x is represented sometimes by the form $\frac{2n+1}{2m}$, and sometimes by $\frac{2n+1}{2m+1}$ according to whether the approximation is broken off earlier or later. From the former one would suppose that y was imaginary and from the latter that it was real. From all this it may now be seen that the quantity which is adopted for the base number of a system of logarithms must only be a *positive* or rather a merely *abstract* number. Then to every conceivable abstract or positive quantity there will be a real logarithm, and conversely to every logarithm a real quantity. For *negative* quantities there can, of course, be no logarithm. But they are not needed since the use of logarithms chiefly rests on using them to transform multiplications and divisions into additions and subtractions, and raising to powers and extracting roots into multiplications and divisions. All these things are possible when the quantities which have been taken for these operations are negative since this circumstance does not change the value but at most only the sign of the result—assuming the latter is real.

§ 69

Transition. The uses just mentioned for a list of precisely calculated logarithms make desirable the discovery of certain formulae by means of which one could work out the logarithm corresponding to each number, for some base-number, more conveniently than by the formulae of §66. Since the form of these formulae turns out most simply if the quantity $\mathfrak{A}$ could be put equal to *unity* this gives rise to the conjecture that a logarithmic system for which a base number is adopted so that $\mathfrak{A}$ would be found $= 1$, would generally be the easiest to calculate with. We must first of all investigate whether such a base-number is in fact possible.

§ 70

Problem. To investigate whether there is a quantity a with the property that in the equation $\frac{a^\omega - 1}{\omega} = \mathfrak{A} + \overset{(a)}{\omega}$, the quantity $\mathfrak{A}$ is $= 1$, i.e. that the value of the expression $\frac{a^\omega - 1}{\omega}$ approaches as close to the value 1 as desired if ω is taken as small as desired.

Solution. Suppose there were such a quantity and it was a, then because generally,

$$a^x = 1 + \mathfrak{A}x + \frac{\mathfrak{A}^2 x^2}{1.2} + \frac{\mathfrak{A}^3 x^3}{1.2.3} + \cdots + \frac{\mathfrak{A}^r x^r}{1.2.3\ldots r} + \Omega$$

(§64), for this special case where $\mathfrak{A}$ is to be $=\ 1$,

$$a^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \cdots + \frac{x^r}{1.2.3\ldots r} + \Omega$$

and consequently, if one puts $x = 1$ the value of a itself is

$$= 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots + \frac{1}{1.2.3\ldots r} + \Omega.$$

Now first of all it is certain that this series actually expresses a *real finite* quantity since its value not only always remains finite, but even approaches a certain constant quantity as close as desired provided it is continued far enough. The former is easily made clear because

$$1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots + \frac{1}{1.2.3\ldots r}$$

obviously always remains $< 1+1+1 = 3$. But in order to understand the latter as well, let us put the value of the series, if it is continued only up to a certain definite term $= \frac{1}{1.2.3\ldots r}$, $= R$, then $R < 3$ and if the series is now continued further, as far as desired, then its increase

$$= \frac{1}{1.2.3\ldots r(r+1)} + \frac{1}{1.2.3\ldots r(r+1)(r+2)} + \cdots + \frac{1}{1.2.3\ldots r(r+1)\ldots(r+s)}$$

is undeniably smaller than the value of the *geometric* series:

$$\frac{1}{1.2.3\ldots r(r+1)} + \frac{1}{1.2.3\ldots r(r+1)^2} + \cdots + \frac{1}{1.2.3\ldots r(r+1)^s}$$

which arises from the former if instead of the factors $(r+2), (r+3), \ldots, (r+s)$ in the denominators, one puts the smaller $(r+1)$ throughout. But the value of this latter series is, however far it is continued, always

$$< \frac{1}{1.2.3\ldots r(r+1)} . \frac{1}{1-\frac{1}{r+1}} = \frac{1}{1.2.3\ldots r.r},$$

a quantity which by increasing r can be made smaller than any given D. So all the more certainly the series

$$1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots + \frac{1}{1.2.3\ldots r}$$

can be continued so far that its increase for every further continuation remains smaller than any given quantity D. But it follows from this that there would have to exist a certain constant quantity which this series steadily approaches and to which it can come so near that the difference is smaller than any given quantity. And from any particular value of the series $= R$, one also incidentally learns, what this quantity is (though not exactly), since it can only differ from R by something that is $<D$. We shall designate the value of this constant quantity once and for all by e. Therefore the following equation holds:

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots + \frac{1}{1.2.3\ldots r} + \Omega.$$

Now since for every real quantity the following equation holds (§64):

$$a^x = 1 + \mathfrak{A}x + \frac{\mathfrak{A}^2x^2}{1.2} + \frac{\mathfrak{A}^3x^3}{1.2.3} + \cdots + \frac{\mathfrak{A}^rx^r}{1.2.3\ldots r} + \overset{1}{\Omega}$$

then the same holds also for e and therefore, if we designate by $\mathfrak{C}$ the value that $\mathfrak{A}$ takes when e is put for a,

$$e^x = 1 + \mathfrak{C}x + \frac{\mathfrak{C}^2x^2}{1.2} + \frac{\mathfrak{C}^3x^3}{1.2.3} + \cdots + \frac{\mathfrak{C}^rx^r}{1.2.3\ldots r} + \overset{2}{\Omega}$$

therefore also if one puts $x = 1$,

$$e = 1 + \mathfrak{C} + \frac{\mathfrak{C}^2}{1.2} + \frac{\mathfrak{C}^3}{1.2.3} + \cdots + \frac{\mathfrak{C}^r}{1.2.3\ldots r} + \overset{2}{\Omega}.$$

If we now compare with this expression the one adopted above,

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots + \frac{1}{1.2.3\ldots r} + \Omega,$$

then it is clear that the constant quantity $\mathfrak{C}$ has to be equal to *unity*. For if it were >1 then also

$$1+\mathfrak{C}+\frac{\mathfrak{C}^2}{1.2}+\frac{\mathfrak{C}^3}{1.2.3}+\cdots+\frac{\mathfrak{C}^r}{1.2.3\ldots r}+\overset{2}{\Omega}$$

must necessarily be

$$>1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\cdots+\frac{1}{1.2.3\ldots r}+\Omega$$

and in the opposite case it will necessarily be found that the former is smaller than the latter. Therefore it is now proved that there must actually exist a quantity, i.e.

$$e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\cdots+\frac{1}{1.2.3\ldots r}+\Omega,$$

for which the function $\frac{e^\omega-1}{\omega}=\mathfrak{C}+\overset{(e)}{\omega}$ comes as near to the value 1 as desired if ω is taken as small as desired.

§ 71

Definition. A logarithmic system for which the quantity e of §70 is taken as base number is called a *natural* system, the logarithms themselves are called *natural* logarithms, otherwise also called *hyperbolic* logarithms. We shall designate them by $\overset{(e)}{\mathrm{l}}$.

§ 72

Corollary. Therefore with reference to *natural* logarithms, if one takes $a=e$ in the formula of §64,

$$e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\cdots+\frac{x^r}{1.2.3\ldots r}+\Omega,$$

is a formula by means of which, for every given *natural logarithm* $=x$, the number corresponding to it $=e^x$ can be calculated conveniently enough. For if one has calculated the value of the series only up to the term $\frac{x^r}{1.2.3\ldots r}$ (where r denotes a specific number), then the increase which can occur for every further continuation

$$=\frac{x^{r+1}}{1.2.3\ldots r(r+1)}+\frac{x^{r+2}}{1.2.3\ldots r(r+1)(r+2)}+\cdots$$
$$+\frac{x^{r+s}}{1.2.3\ldots r(r+1)\ldots(r+s)}$$

$$< \frac{x^r}{1.2.3\ldots r(r+1)} \cdot \frac{x}{1 - \frac{1}{r+1}}$$
$$= \frac{x^r}{1.2.3\ldots r} \cdot \frac{x}{r}.$$

Therefore if r is so large that $\frac{x^r}{1.2.3\ldots r} \cdot \frac{x}{r}$, i.e. the value of the term with which one breaks off the calculation multiplied by $\frac{x}{r}$, turns out $<D$, then one also knows that the value of a^x calculated from this piece of the series can deviate from its true value only by something that is $<D$. Now since the terms of the series from where r is $>x$ onwards, steadily decrease, and since the fraction, by multiplication with which every subsequent term arises from the preceding term, becomes constantly smaller, then however large x should become, one will soon reach a value of $\frac{x^r}{1.2.3\ldots r} \cdot \frac{x}{r} < D$. But conversely if the natural logarithm $= x$ corresponding to the given number $= y$ is to be calculated, then §66 offers us, if we put $\mathfrak{A} = 1$ there, the formulae:

$$\overset{(e)}{\mathrm{l}}\, y = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \cdots \pm \frac{1}{r}(y-1)^r + \Omega,$$

if $y < +2$, and

$$\overset{(e)}{\mathrm{l}}\, y = \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r} + \overset{1}{\Omega},$$

if $y > +\frac{1}{2}$. These two formulae are very inconvenient to use in several cases—the former whenever y is very small or very near to $+2$, the latter whenever y is very large or very near to $+\frac{1}{2}$—since then their terms decrease only very slowly. This makes the following problem necessary.

§ 73

Problem. To give convenient formulae by means of which the natural logarithms of every whole, fractional or even irrational, non-negative number can be calculated.

Solution. 1. It is merely necessary to find some formulae which could be conveniently used for the calculation of the natural logarithms of *all prime numbers*. For once these have been calculated the logarithms of all other quantities which are not negative can be derived from them with little effort.

(a) The logarithms of the *whole numbers* which are not prime numbers but are products of such are found by simple addition of the logarithms of their factors.

(b) *Fractional quantities* which are nevertheless rational can be brought to the form $\frac{m}{n}$ in which m, n denote two whole numbers, therefore their logarithms can be found from (a) by a simple subtraction of the logarithm of n from that of the number m.

(c) The logarithms of *irrational quantities* which are of the form $(\frac{m}{n})^p$ where m, n denote whole numbers and p is any kind of real quantity, are given by a simple multiplication of the logarithm of $\frac{m}{n}$ (found by (b)) by p.

(d) Finally if the irrational quantity $= i$ is of any kind, then there are, provided it is positive, two positive numbers m, n such that the fraction $\frac{m}{n}$ comes as close to i as desired. Therefore also the logarithm of $\frac{m}{n}$, determined by (b), comes as near to the logarithm of i as desired.

2. Since the logarithm of 1 is $= 0$ then the smallest prime number whose logarithm is to be looked for $= 2$. This could always be calculated conveniently enough from the formula:

$$\overset{(e)}{\mathrm{l}}\, y = \frac{(y-1)}{y} + \frac{1}{2}\frac{(y-1)^2}{y^2} + \frac{1}{3}\frac{(y-1)^3}{y^3} + \cdots + \frac{1}{r}\frac{(y-1)^r}{y^r} + \Omega$$

since in this case it becomes the following:

$$\overset{(e)}{\mathrm{l}}\, 2 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{4}\left(\frac{1}{2}\right)^4 + \cdots + \frac{1}{r}\left(\frac{1}{2}\right)^r + \Omega.$$

Since one has here a series which decreases more quickly than a geometric series with ratio $\frac{1}{2}$, it can be decided in just the same way as in §72 how large the difference can be, at most, between the true value of the logarithm to be found and the value which is obtained if the series is broken off at its rth term. For the calculation of the natural logarithm of 3 the formula would give the following series:

$$\overset{(e)}{\mathrm{l}}\, 3 = \frac{2}{3} + \frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{4}\left(\frac{2}{3}\right)^4 + \cdots + \frac{1}{r}\left(\frac{2}{3}\right)^r + \Omega,$$

which decreases still more slowly, etc.

3. On the other hand, from the formula

$$\overset{(e)}{\mathrm{l}}\, y = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \cdots \pm \frac{1}{r}(y-1)^r + \Omega$$

which holds for $y < +2$, if one takes $y = 1 + z$, the following is obtained:

$$\overset{(e)}{\mathrm{l}}\,(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \cdots \pm \frac{1}{r}z^r + \Omega,$$

and this must hold for every $z < \pm 1$. Therefore also for just the same value of z the following formula holds:

$$\overset{(e)}{\mathrm{l}}\,(1-z) = -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4 - \cdots - \frac{1}{r}z^r + \overset{1}{\Omega}.$$

Therefore by subtraction also the following holds:

$$\overset{(e)}{\mathrm{l}}\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots + \frac{1}{2n+1}z^{2n+1} + \overset{2}{\Omega}\right)$$

where $2n + 1$ denotes an odd number. Now let u denote any conceivable quantity, provided it is *positive*, so $\frac{u-1}{u+1}$ is always $<\pm 1$, and can therefore be put in the place of z. But then $\frac{1+z}{1-z}$ will $= u$ and one obtains:

$$\overset{(e)}{\mathrm{l}}\, u = 2\left(\left(\frac{u-1}{u+1}\right) + \frac{1}{3}\left(\frac{u-1}{u+1}\right)^3 + \frac{1}{5}\left(\frac{u-1}{u+1}\right)^5 + \cdots + \frac{1}{2n+1}\left(\frac{u-1}{u+1}\right)^{2n+1} + \Omega\right)$$

a formula which already decreases notably quicker than the previous one if $\frac{u-1}{u+1}$ is a sufficiently small fraction. At least for the calculation of the logarithms of 2 and 3 it may be found convenient enough. For if $u = 2$, $\frac{u-1}{u+1} = \frac{1}{3}$, for $u = 3$, $\frac{u-1}{u+1} = \frac{1}{2}$, therefore the former series decreases more quickly than a geometric series with ratio $\frac{1}{9}$, the latter decreases more quickly than a geometric series with ratio $\frac{1}{4}$. But for larger numbers the fraction $\frac{u-1}{u+1}$ will be too large.

4. Now in order to arrive at a formula which would also be applicable to *larger* numbers, at least prime numbers, one may note that whenever u denotes a prime number, $u - 1$ and $u + 1$ are certainly not such numbers. For if u is a prime number which is larger than 2, u must be odd and accordingly $u - 1$, $u + 1$ are certainly even and therefore are divisible, at least, by the number 2. It is equally clear that the factors into which the two numbers $u - 1$, $u + 1$ are divisible must be $< u$. Therefore if one proceeds with the calculation of the logarithms of the prime numbers in their natural order, then the logarithms of the factors of $u - 1$, $u + 1$ would already have been calculated before one came to the calculation of the logarithm of u. Therefore by 1. (a) the logarithms $\overset{(e)}{\mathrm{l}}\,(u-1)$ and $\overset{(e)}{\mathrm{l}}\,(u+1)$ can be regarded as already known for the calculation of $\overset{(e)}{\mathrm{l}}\, u$; therefore also $\overset{(e)}{\mathrm{l}}\,(u-1) + \overset{(e)}{\mathrm{l}}\,(u+1) = \overset{(e)}{\mathrm{l}}\,(u^2-1)$. But by the formula of 3. the logarithm of every quantity which is not much > 1, can fairly easily be calculated. For example, such a quantity is $\frac{u^2}{u^2-1}$ provided u is considerably > 1 and it is all the more so the larger u is. Therefore also $\overset{(e)}{\mathrm{l}}\left(\frac{u^2}{u^2-1}\right)$ is to be considered as a quantity which can be calculated. But if $\overset{(e)}{\mathrm{l}}\,(u^2-1)$ and $\overset{(e)}{\mathrm{l}}\left(\frac{u^2}{u^2-1}\right)$ are known then so also is $\overset{(e)}{\mathrm{l}}\,(u^2)$ and therefore $\overset{(e)}{\mathrm{l}}\, u$. For,

$$\overset{(e)}{\mathrm{l}}\left(\frac{u^2}{u^2-1}\right) = \overset{(e)}{\mathrm{l}}\,(u^2) - \overset{(e)}{\mathrm{l}}\,(u^2-1)$$
$$= 2\,\overset{(e)}{\mathrm{l}}\, u - \overset{(e)}{\mathrm{l}}\,(u^2-1).$$

Therefore

$$\overset{(e)}{\mathrm{l}}\,u = \frac{1}{2}\left(\overset{(e)}{\mathrm{l}}\left(\frac{u^2}{u^2-1}\right) + \overset{(e)}{\mathrm{l}}\,(u^2-1)\right).$$

In fact if we put $\frac{u^2}{u^2-1}$ into the formula of 3. in the place of the u that is there, or $\frac{1+z}{1-z}$, then $z = \frac{1}{2u^2-1}$ and we therefore obtain the formula:

$$\begin{aligned}\overset{(e)}{\mathrm{l}}\,u = {}& \frac{\overset{(e)}{\mathrm{l}}\,(u-1) + \overset{(e)}{\mathrm{l}}\,(u+1)}{2} \\ & + \left(\frac{1}{2u^2-1}\right) + \frac{1}{3}\left(\frac{1}{2u^2-1}\right)^3 + \frac{1}{5}\left(\frac{1}{2u^2-1}\right)^5 + \cdots \\ & + \frac{1}{r}\left(\frac{1}{2u^2-1}\right)^r + \Omega.\end{aligned}$$

This series obviously decreases all the quicker the larger the prime number u which is to be calculated. Even for $u = 5$, i.e. for the next prime number which has to be calculated after 2 and 3, $\frac{1}{2u^2-1} = \frac{1}{49}$ and therefore the series decreases more rapidly than a geometric series whose ratio $= (\frac{1}{49})^2 = \frac{1}{2401}$. With larger numbers the rate of decrease is even greater.

§ 74

Problem. If the natural logarithms are already calculated, to calculate the logarithms of every other system that has a positive or abstract base number and also to calculate the number in this system corresponding to every given logarithm.

Solution. 1. Let the base number of the system be a, let y be some number and x the logarithm corresponding to it, therefore $y = a^x$. Now if the natural logarithms of all positive or abstract quantities are already calculated then one knows $\overset{(e)}{\mathrm{l}}\,a$ as well as $\overset{(e)}{\mathrm{l}}\,y$. Let the former $\overset{(e)}{\mathrm{l}}\,a = \alpha$, then $a = e^\alpha$ and therefore $y = a^x = e^{\alpha x}$ and so αx is the natural logarithm of y, $\alpha x = \overset{(e)}{\mathrm{l}}\,y$, therefore

$$x = \frac{\overset{(e)}{\mathrm{l}}\,y}{\alpha} = \frac{\overset{(e)}{\mathrm{l}}\,y}{\overset{(e)}{\mathrm{l}}\,a},$$

i.e. one obtains the logarithms of every other system from the natural logarithms just by dividing the latter by the natural logarithm of the base number of that system.

2. In order to find the quantity corresponding to a given logarithm one may remember from §64 that generally,

$$y = a^x = 1 + \mathfrak{A}x + \frac{\mathfrak{A}^2x^2}{1.2} + \frac{\mathfrak{A}^3x^3}{1.2.3} + \cdots + \frac{\mathfrak{A}^r x^r}{1.2.3\ldots r} + \Omega$$

in which the quantity $\mathfrak{A} = \frac{a^\omega - 1}{\omega} + \overset{(a)}{\omega}$, either

$$= (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \cdots \pm \frac{1}{r}(a-1)^r + \overset{1}{\Omega}$$

$$\text{or} \quad = \left(\frac{a-1}{a}\right) + \frac{1}{2}\left(\frac{a-1}{a}\right)^2 + \frac{1}{3}\left(\frac{a-1}{a}\right)^3 + \cdots + \frac{1}{r}\left(\frac{a-1}{a}\right)^r + \overset{2}{\Omega},$$

according to whether a is $< +2$ or $> +\frac{1}{2}$. Hence by comparison with §72, $\mathfrak{A}$ must be $= \overset{(e)}{\mathrm{l}}\, a$ (§27). Therefore if the natural logarithms of every non-negative quantity are known, then also $\overset{(e)}{\mathrm{l}}\, a$ is known, and therefore the number y corresponding to the given logarithm x can be calculated by means of the formula:

$$y = a^x = 1 + (\overset{(e)}{\mathrm{l}}\, a)x + \frac{(\overset{(e)}{\mathrm{l}}\, a)^2x^2}{1.2} + \frac{(\overset{(e)}{\mathrm{l}}\, a)^3x^3}{1.2.3} + \cdots + \frac{(\overset{(e)}{\mathrm{l}}\, a)^r x^r}{1.2.3\ldots r} + \Omega.$$

However large x and $\overset{(e)}{\mathrm{l}}\, a$ may be, from the term where r has become $> x(\overset{(e)}{\mathrm{l}}\, a)$ onwards, this series decreases more rapidly than a geometric series whose ratio is the proper fraction $\frac{x(\overset{(e)}{\mathrm{l}}\, a)}{r}$. From this fact it can be estimated, just as in §72, how close one is to the true value of y if one took the value of the series up to the term $\frac{(\overset{(e)}{\mathrm{l}}\, a)^r x^r}{1.2.3\ldots r}$ for y itself.

§ 75

Concluding note. In the whole of this work we have deliberately said nothing about the case where one of the quantities in the calculation, e.g. the quantity which is to be raised to a power, or the exponent itself, has become imaginary. Just as in §67 we have also not mentioned at all the case when a *negative* quantity is to be raised to a power with an *irrational* exponent, e.g. $(-2)^{\sqrt{3}}$. These are subjects which we could not deal with thoroughly until various concepts had first been clarified more carefully than hitherto. First the concept of *imaginary expressions* itself, then that of the *irrationality* of a quantity, that of the *mathematical opposite*[p] and finally also that of the *raising of a quantity to a power*, must have been clearly

[p] That is, in this context, negative numbers.

developed if one wants to decide the above questions thoroughly. Now since in the present work we have decided to proceed everywhere only from concepts which are already known and common, and to avoid all the more difficult innovations, those investigations could have no place here. But we look forward to publishing our opinion on the subjects just mentioned at a later time if this present work is received favourably.

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Rein analytischer
Beweis des Lehrsatzes,

daß

zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege;

von

Bernard Bolzano,

Weltpriester, Doctor der Philosophie, k. k. Professor der Religionswissenschaft, und ordentlichem Mitgliede der k. Gesellschaft der Wissenschaften zu Prag.

Für die Abhandlungen der k. Gesellschaft der Wissenschaften.

Prag. 1817,
gedruckt bei Gottlieb Haase.

Purely Analytic
Proof of the Theorem

that

between any two Values,
which give Results of Opposite Sign,
there lies at least one real Root of the Equation

by

Bernard Bolzano

Priest, Doctor of Philosophy, Professor of Theology and
Ordinary Member of the Royal Society of Sciences at Prague

For the Proceedings of the Royal Society of Sciences

Prague, 1817
Printed by Gottlieb Haase

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Preface

There are two propositions in the theory of equations for which, up until recently, it could still be said that a perfectly correct proof was unknown. One is the proposition: *between every two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation*. The other is: *every algebraic rational integral function of one variable quantity can be decomposed into real factors of first or second degree*. After several unsuccessful attempts by *D'Alembert, Euler, de Foncenex, Lagrange, Laplace, Klügel* and others at proving the latter proposition, *Gauss* finally provided us, last year, with several proofs which leave hardly anything to be desired. This outstanding scholar had in fact already presented us with a proof of this proposition in 1799,[*] but it still had the defect, which he himself admitted, that it based a purely analytic truth on a *geometrical consideration*. But his two *most recent proofs*[**] are quite free of this defect, since the *trigonometric functions* which appear in them can, and should, be understood in a purely analytical sense.

The other proposition which we mentioned above is just not one which has concerned scholars so far to any great extent. Nevertheless, we do find very distinguished mathematicians concerned with this proposition and *various* kinds of proof for it have already been attempted. Anyone wishing to be convinced of this need only compare the various treatments of this proposition given, for example, by *Kästner*,[†] *Clairaut*,[††] *Lacroix*,[‡] *Metternich*,[‡‡] *Klügel*,[§] *Lagrange*,[§§] *Rösling*,[¶] as well as by several others.

However, a more careful examination very soon shows that none of these kinds of proof can be regarded as satisfactory.

[*] *Demonstratio nova Theorematis, omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse.* Helmstadii, 4°, 1799.

[**] *Demonstratio nova altera* etc., and *Demonstratio nova tertia*; both from the year 1816.

[†] *Anfangsgründe der Analysis endlicher Größen*, 3te Aufl., §316.

[††] *Elémens d'Algèbre*, 5eme Edit., Supplemens, Chap. I, no. 16.

[‡] *Elémens d'Algèbre*, 7eme Edit.

[‡‡] In his *translation* of the above work of *Lacroix*, Mainz, 1811, §211.

[§] In his *Mathematisches Wörterbuch*, 2.Band, S. 447 ff.

[§§] *Traité de la résolution des équations numériques de tous les degrés*, Paris, 1808.

[¶] *Grundlehren von den Formen, Differenzen, Differentialien und Integralien der Functionen*, I.Thl, §49.

I. The *most common* kind of proof depends on a truth borrowed from *geometry, namely: that every continuous* [*continuirlich*][a] *line of simple curvature of which the ordinates are first positive and then negative* (or conversely), *must necessarily intersect the abscissae-line*[b] *somewhere at a point lying between those ordinates*. There is certainly nothing to be said against the *correctness*, nor against the *obviousness* of this geometrical proposition. But it is also equally clear that it is an unacceptable breach of *good method* to try to derive truths of *pure* (or general) mathematics (i.e. arithmetic, algebra, analysis) from considerations which belong to a merely *applied* (or special) part of it, namely *geometry*. Indeed, have we not long felt, and acknowledged, the impropriety of such a $\mu\varepsilon\tau\alpha\beta\alpha\sigma\iota\varsigma$ $\grave{\varepsilon}\iota\varsigma$ $\grave{\alpha}\lambda\lambda o$ $\gamma\varepsilon\nu o\varsigma$?[c] Are there not a hundred other cases where a method of avoiding this [transition] has been discovered, and where the avoidance was considered a virtue?* So if we wish to be consistent must we not strive to do the same here? In fact, anyone who considers that scientific proofs should not merely be *confirmations* [*Gewißmachungen*], but rather *groundings* [*Begründungen*], i.e. presentations of the objective reason for the truth to be proved, realizes at once that the strictly scientific proof, or the objective reason of a truth, which holds equally for *all* quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities which are in *space*. If we adhere to this view we see instead that such a *geometrical* proof is, in this as in most cases, really circular. For while the geometrical truth to which we refer here is (as we have already admitted) extremely *obvious* and therefore needs no *proof* in the sense of *confirmation*, it none the less does need a *grounding*. For it is apparent that the concepts of which it consists are so combined that we cannot hesitate for a moment to say that it cannot possibly be one of those *simple* truths, which are called *axioms*, or *basic truths* [*Grundwahrheiten*], because they are the *basis* [*Grund*] for other truths and are not themselves consequences. On the contrary, it is a *theorem* or *consequent truth* [*Folgewahrheit*], i.e. a kind of truth that has its basis in certain other truths and therefore, in science, must be proved by a derivation

* The papers of *Gauss* quoted earlier provide an example.

[a] Throughout his works Bolzano makes a distinction between the continuity [*continuirlich*] of a line or curve, and the continuity [*stetig*] of a function. There is no explicit statement of the connection between these concepts except to say that a curve being *continuirlich* cannot justify the claim that the corresponding function is *stetig* (e.g. see the end of *BL* §29 on p. 184). It is not clear, in the light of his example of a continuous, non-differentiable function (*F* §135 on p. 507) whether Bolzano believed that a function could be continuous in an analytic sense and yet the corresponding curve not be continuous in some geometric or topological sense.

[b] This literal translation of *Abscissenlinie* as ‘abscissae-line’ suits the context and reflects the fact that Bolzano does not use the phrase *die Axe von x* which was beginning to be used in German at this time. The older term carries connotations of a measuring line, rather than the more abstract number line of the modern term ‘*x*-axis’.

[c] Crossing to another kind. See footnote c on p. 32.

from these other truths.* Now consider, if you will, the objective reason why a line, as described above, intersects its abscissae-line. Surely everyone will soon see that this reason lies in nothing other than that general truth, as a result of which every continuous function of x which is positive for one value of x, and negative for another, must be zero for some intermediate value of x. And this is precisely the truth which is to be proved here. It is therefore quite wrong to have allowed the latter to be derived from the former (as happens in the kind of proof we are now examining). Rather, conversely, the former must be derived from the latter if we intend to represent the truths in science exactly as they are related to each other in their objective connection.

II. The proof which some people have produced from the concept of the *continuity* of a function mixed up with the concepts of *time* and *motion*, is no less objectionable. 'If two functions fx and ϕx', they say, 'vary according to the law of continuity and if for $x = \alpha$, $f\alpha < \phi\alpha$, but for $x = \beta$, $f\beta > \phi\beta$, then there must be some value u, lying between α and β, for which $fu = \phi u$. For if we imagine that the variable quantity x in both these functions successively takes all values between α and β, and in both always takes the same value at the same moment, then at the *beginning* of this continuous change in the value of x, $fx < \phi x$, and at the *end*, $fx > \phi x$. But since both functions, by virtue of their continuity, must first go through all intermediate values before they can reach a higher value, there must be some *intermediate moment* at which they were both equal to one another.' This is further illustrated by the example of the *motion* of two bodies, of which one is initially *behind* the other and later *ahead* of the other. It necessarily follows that at one time it must have *passed* the other.

No one will want to deny that the concept of *time*, as well as that of *motion*, is just as alien to general mathematics as the concept of *space*.[e] Nevertheless, we would have no objection if these two concepts were only introduced here for the sake of *clarification*. For we are in no way party to a *purism* so exaggerated, that it demands, in order to keep the science free from everything alien, that in its exposition one cannot even use an *expression* borrowed from another field, even if only in a figurative sense and with the purpose of describing a fact more briefly and clearly than could be done in a description involving purely specialist terms. Nor [do we object to such use] if it is just to avoid the monotony of the constant repetition of

* Compare on all this my *Beyträge zu einer begründeteren Darstellung der Mathematik*, First issue, Prague, 1810, II. Abthl. §§ 2, 10, 20, 21, where the logical concepts which I have assumed here as known are developed further.[d]

[d] The sections of *BD* cited here appear in translation beginning on p. 103.

[e] For the modern reader the emphasis here should be on '*general* mathematics'. The concepts of time, motion, and space were all important in mathematics; they were, respectively, central to the three areas of applied mathematics to which Bolzano refers in §13, namely: chronometry, mechanics, and geometry.

the same word, or to remind us, by the mere name given to a thing, of an example which could serve to confirm a claim. It follows immediately that we do not regard *examples* and *applications* as detracting in the least from the perfection of a scientific exposition. There is only one thing that we do strictly require: that examples never be put forward instead of *proofs*, and that the essence [*Wesenheit*] of a deduction never be based on the merely figurative use of phrases or on associated ideas, so that the deduction itself becomes void as soon as these are changed.

In accordance with these views, the inclusion of the concept of *time* in the above proof may still perhaps be excused, because no conclusion is based on phrases containing it, which would not also hold without it. But the last *illustration* using the *motion* of a body really cannot be regarded as anything more than a mere *example* which does not prove the proposition itself, but instead must first be proved by it.

(a) Therefore let us leave this example and concentrate on the rest of the reasoning. *First of all*, let us notice that this is based on an incorrect concept of *continuity*. According to a *correct definition*, the expression *that a function* fx *varies according to the law of continuity for all values of* x *inside or outside certain limits** means only that, *if* x *is any such value the difference* $f(x+\omega)-fx$ *can be made smaller than any given quantity, provided* ω *can be taken as small as we please*,[g] or (in the notation we introduced in §14 of *Der binomische Lehrsatz* etc., Prague, 1816),[h] $f(x+\omega)=fx+\Omega$. But, as assumed in this proof, the continuous function is one which never reaches a higher value without having first gone through all lower values, i.e. $f(x+n\Delta x)$ can take every value between fx and $f(x+\Delta x)$ if n is taken arbitrarily between 0 and +1. That is indeed a very *true* assertion, but

* There are functions which are continuously variable for *all* values of their argument, e.g. $\alpha+\beta x$.[f] However, there are also others which vary according to the law of continuity only for values of their argument inside or outside certain limits. Thus $x+\sqrt{(1-x)(2-x)}$ varies continuously only for all values of x which are $<+1$ or $>+2$, but not for the values which lie between +1 and +2.

[f] Three matters arise from this sentence. First, the standard linear function given corrects the misprint of '$\alpha x+\beta x$' in the first edition which has often been preserved in later editions and translations. Second, the word translated here as 'argument' is *Wurzel* which means 'root', both in the sense of a value of an unknown which satisfies an equation (e.g. as in the title of the current work *RB*), and in the sense used in the phrase 'square root'. Bolzano also uses *Wurzel* to refer to the independent variable of a function, for which the appropriate English word is now 'argument'. The intended meaning of *Wurzel* is always clear from the context. Finally, the phrase '... are continuously variable' sounds a little odd but reflects the German phrasing *stetig veränderlich sind*, and is significantly distinct from the two more common ways of saying the 'same' thing which occur respectively in the following two sentences of Bolzano's footnote, i.e. 'vary according to the law of continuity' [*nach dem Gesetze der Stetigkeit ändern*], and 'varies continuously' [*ändert sich ... stetig*].

[g] This important, recurring phrase is first used in *BL Preface*, see the footnote on p. 158.

[h] This section of *BL* appears in translation on p. 173. The definition of continuity just given already appears clearly, but informally, in this earlier work. 'For a function is called continuous if the change which occurs for a certain change in its argument, can become smaller than any given quantity, provided that the change in the argument is taken small enough.' *BL* §29.

it cannot be regarded as a *definition* of the concept of continuity: it is rather a *theorem* about continuity. In fact it is a theorem which can only itself be proved on the assumption of the proposition to whose proof one wishes to apply it here. For if M denotes any quantity between fx and $f(x+\Delta x)$, then the assertion that there is some value of n between 0 and +1 for which $f(x+n\Delta x) = M$ is only a special case of the general truth that if $fx < \phi x$ and $f(x+\Delta x) > \phi(x+\Delta x)$, then there must be some intermediate value $x + n\Delta x$ for which $f(x+n\Delta x) = \phi(x+n\Delta x)$. The first assertion follows from this general truth in the special case when the function ϕx becomes a constant quantity M.

(b) But even supposing one could prove this proposition in another way, the proof which we are examining would have yet another defect. That is, from the fact that $f\alpha > \phi\alpha$ and $f\beta < \phi\beta$ it would only follow that, if u is any value lying between α and β for which $\phi u > \phi\alpha$ but $<\phi\beta$, then fx becomes equal to ϕu in going from $f\alpha$ to $f\beta$, i.e. for *some* x lying between α and $\beta, fx = \phi u$. But whether this happens for *exactly the same* value of x which $=u$, that is (since u can be any arbitrary value lying between α and β which makes $\phi u > \phi\alpha$ and $< \phi\beta$) whether there is some value of x lying between α and β for which *both* functions fx and ϕx are equal to one another would still not follow.

(c) The deceptive nature of the whole proof really rests on the fact that the concept of *time* has been involved in it. For if this were omitted it would soon be seen that the proof is nothing but a re-statement in different words of the proposition to be proved. For to say that the function fx, before it passes from the state of being smaller than ϕx to that of being greater, must first go through the state of being equal to ϕx, is to say, without the concept of time, that among the values that fx takes if x is given every arbitrary value between α and β, there is one that makes $fx = \phi x$, which is exactly the proposition to be proved.

III. Others prove our proposition on the basis of the following, which is either given completely without proof, or just supported by a few examples borrowed from geometry: *Every variable quantity can only make a transition from a positive state to a negative state through the state of being zero or that of being infinite.* Now since the value of an equation cannot be *infinitely large* for any finite value of the argument, they conclude that the transition that occurs here must be through *zero*.

(a) If we wish to separate from the above proposition the figurative idea of a *transition*, which contains the concept of a change in *time* and *space*, thereby also ridding ourselves of the nonsensical expression '*a state of non-existence*',[i] then we eventually obtain the following proposition: *If a variable quantity, depending on another quantity x, is found to be positive for $x = \alpha$ and negative for $x = \beta$, then there is always a value of x lying between α and β for which the quantity is zero, or one for which it is infinite.* Now everyone surely sees that such a complex assertion is not a

[i] A similar verbal shift occurs in the German from *den Zustand des Nullseyns* ('the state of being zero' in the paragraph above) to *eines Zustandes des Nichtvorhandenseyns* ('the state of non-existence'—in the sense of being lacking or absent, in this 'quotation').

basic truth but would have to be proved, and that its proof could hardly be easier than that of the very proposition which we wish to establish.
(b) Indeed, a closer examination shows that the assertion is fundamentally *identical* with the proposition. For it must not be forgotten that this assertion is really only true if it refers to purely *continuously variable quantities*. Thus for example, the function $x+\sqrt{(x-2)(x+1)}$ certainly has a *positive* value for $x = +2$ and a *negative* value for $x = -1$, yet because it does not vary within these limits according to the law of continuity there is no value of x between $+2$ and -1 for which the function is zero or infinite. However, if we restrict the assertion only to those quantities that vary continuously, we must also exclude those functions which become *infinite* for a certain value of their argument. For such a function as, for example $\frac{a}{b-x}$, does not actually vary continuously for *all* values of x, but only for all values which are $>$ or $<$ b. For the function does not have any *determinate* value when $x = b$, but becomes what is called *infinitely large*. Therefore it cannot be said that the values which it takes for $x = b + \omega$, all of which are *determinate*, can come as close as we please to the value it has for $x = b$. But this is part of the concept of continuity (II.(a)). If the concept of continuity is now added to the above assertion and the case of the function *becoming infinite* is omitted, then it becomes, word for word, the proposition we had to prove, namely: that every continuously variable function of x which is positive for $x = \alpha$, and negative for $x = \beta$, must be zero for some value lying between α and β.

IV. In some works we read the following argument: *Because fx is positive for $x = \alpha$ and negative for $x = \beta$, there must be, between α and β, two quantities a and b at which the transition from the positive values to the negative values of fx takes place, in such a way that between a and b no more values of x occur for which fx would still be positive or negative.* This error scarcely needs a refutation and would not be introduced here at all if it did not serve to prove how unclear the concepts of even some distinguished mathematicians still are on this subject. It is, after all, sufficiently well known that between any two values, however close to one another, of a *quantity that varies independently*, such as the argument x of a function, there are always infinitely many *intermediate* values; likewise that for any continuous function there is no *last* x which makes it positive, and no *first* x which makes it negative, so there are no such a and b as described here!
V. The failure of these attempts to prove *directly* the proposition with which we are concerned led to the idea of deriving it from the *second* proposition we mentioned at the beginning, namely from the fact that *every function can be decomposed into certain factors*. There is also no doubt that, if the latter is allowed, the former can be deduced from it. But the fact remains that such a derivation could not be called a genuinely scientific *grounding*, in that the second proposition clearly expresses a much *more complex* truth than our present one. The second can therefore certainly be based on the first, but not, conversely, the first on the second. In fact no one has really yet succeeded in proving the second without presupposing the first. With regard to the proofs which *Gauss* has already shown to be inadmissible

in his paper of 1799, it is precisely because they have already been proved to be unacceptable that it is unnecessary to investigate whether or not they are based on our present proposition. *Laplace's* proof* likewise has its faults, which however we need not consider here because it is explicitly based on our present proposition. And in the same way, we need also pay no regard to *Gauss*'s *first* proof because it relies on *geometrical* considerations. Moreover, it would be easy to show that even in that proof our proposition is implicitly accepted, in that the geometrical considerations employed in it are very similar to those mentioned in I. So now it all comes down to *Gauss*'s *Demonstratio nova altera* and *tertia*. The former refers explicitly to our proposition, in that it presupposes on S. 30: *aequationem ordinis imparis certo solubilem esse*,[j] an assertion which is well known to be nothing but an easy consequence of our proposition. It is not so obvious that the *Demonstratio nova tertia* depends on our proposition. It is based among other things on the following theorem: *If a function always remains positive for all values of its variable quantity x which lie between α and β, then its integral taken from $x = \alpha$ to $x = \beta$*[k] *also has a positive value*. Now in the proof provided by *Lagrange*** for this theorem no explicit reference to our proposition is to be found. But there is still a gap in *Lagrange*'s proof. Namely, it requires that the quantity i be taken sufficiently small that

$$\frac{f(x+i)-fx}{i} - f'x < \frac{f'x + f'(x+i) + f'(x+2i) + \cdots + f'(x+(n-1)i)}{n},$$

where the product $i.n$ is to remain equal to a *given* quantity and the familiar notation $f'x$ represents the first derived function of fx. Now the question arises here of whether it is even possible to satisfy this requirement. The smaller i is taken in order to reduce the difference

$$\frac{f(x+i)-fx}{i} - f'x,$$

the greater n, the *divisor* of the right-hand side, must be taken if $i.n$ is to remain constant. Now also the number [*Menge*][l] of terms in the numerator increases, but whether this increase in the numerator grows in proportion to the denominator, or whether the value of the whole fraction decreases as i decreases, perhaps by

* In the *Journal de l'école normale*, or also in the *Traité du calcul différéntiel et intégral* of *Lacroix*, T.I., no. 162, 163.

** *Leçons sur le Calcul des Fonctions*, Nouvelle Edition, Paris, 1806, Leç. 9, p. 89.

[j] *Translation*: An equation of odd degree is certainly soluble.

[k] Literally, 'its integral, so taken that it vanishes at $x = \alpha$ and [calculated] up to $x = \beta$'.

[l] Usually *Menge* has been rendered 'multitude' but here and in the opening sections of this work Bolzano uses *Menge* interchangeably with *Anzahl*, 'number', which is more appropriate here. See the *Note on the Translations*.

less than, or by more than, the expression

$$\frac{f(x+i)-fx}{i}-f'x,$$

remains to be shown. Now if this gap is to be filled, it can really only be done by calling on our present proposition, since we already had to refer to the latter for the proof of a theorem,* which although much simpler, is related to this one due to *Lagrange*.[m]

Thus all the proofs so far of the proposition which forms the title of this paper are inadequate. Now the one which I submit here for the judgement of scholars contains, I flatter myself, not a mere *confirmation*, but the objective *grounding* of the truth to be proved, i.e. it is strictly scientific.**

The following is a short survey of the proof. The truth to be proved, that between the two values α and β which give results of opposite sign there always lies at least one real root, clearly rests on the *more general truth* that, if two continuous functions of x, fx and ϕx, have the property that for $x = \alpha$, $f\alpha < \phi\alpha$, but for $x = \beta$, $f\beta > \phi\beta$, there must always exist some value of x lying between α and β for which $fx = \phi x$. If $f\alpha < \phi\alpha$, then by virtue of the law of continuity $f(\alpha + i) < \phi(\alpha + i)$, if i is taken small enough. *The property of being smaller* therefore belongs to the function of i represented by the expression $f(\alpha + i)$ for all values of i smaller than a certain value. Nevertheless this property does not hold for *all* values of i without restriction, namely not for an i which is $=\beta - \alpha$, since $f\beta$ is already $>\phi\beta$. Now the *theorem* holds that whenever a certain property M applies to all values of a variable quantity i which are smaller than a given value, and yet not for *all values in general*, then there is always some *greatest* value u, of which it can be asserted that all i which are $<u$ possess the property M. For this value of i itself, $f(\alpha + u)$ cannot be $<\phi(\alpha + u)$ because then by the law of continuity $f(\alpha + u + \omega) < \phi(\alpha + u + \omega)$ if ω is taken small enough. And consequently it would not be true that u is the greatest of the values for which the assertion holds that all lower values of i make $f(\alpha + i) < \phi(\alpha + i)$, for $u + \omega$ would be a still greater value for which this holds. But still less can $f(\alpha + u) > \phi(\alpha + u)$, for then $f(\alpha + u - \omega) > \phi(\alpha + u - \omega)$ would also be true if ω is sufficiently small, and consequently it would not be true that for all values of i which are $<u$, $f(\alpha + i) < \phi(\alpha + i)$. So therefore it must be that $f(\alpha + u) = \phi(\alpha + u)$, i.e. there is a value of x lying between α and β,

* Namely the proposition §29 in the paper *Der binomische Lehrsatz etc.*[m]

** All the same do not expect me to comply here with *all* the rules which I myself set out in the *Beyträge zu einer begründeteren etc.* II. Abth. for the construction of a *strictly scientific* exposition. For although I am still completely convinced of the correctness of these rules, it is only possible to follow them precisely in a situation when one begins the exposition of a science from its *first* propositions and concepts, but not where one is only dealing with some of its theories taken out of the context of the whole, as is done here. This observation clearly also applies to the paper *Der binomische Lehrsatz etc.*

[m] Translation begins on p. 181.

namely $\alpha + u$, for which the functions fx and ϕx are equal to one another. It is now only a question of proving the *theorem* mentioned. We prove this by showing that those values of *i* of which it can be asserted that all smaller values possess the property M, and those of which this can no longer be asserted, can be brought as near one another as we please. From which it follows, for anyone having a correct concept of *quantity*, that the idea of an *i* which is the greatest of those [quantities] of which it can be said that all below them possess the property M, is the idea of a real, i.e. *actual*, quantity [*einer reellen, d.h. wirklichen Größe*].[n]

Before concluding this preface allow me to make a confession and a request which apply not only to this *present work*, but to all my writings, also, God willing, to my *future* ones.

A careful reader could already have gathered from the few writings which have appeared *up until now*, but particularly from that outline of a *new logic* given in the first issue of *Beyträge zu einer begründeteren Darstellung der Mathematik* in its *second* part, headed *Über die mathematische Methode*,[o] that I hold certain views which, if they are not found to be altogether wrong, must lead to a *complete reorganization of all purely a priori sciences.* I have already examined the majority of these ideas and their most important aspects for such a long time, and with so much impartiality, that it is really not premature for me to venture to speak more openly about them here. There are two ways in which views encompassing the entire field of one or more sciences can be made known, either by stating them all at once in a connected form or in parts in individual papers. The *first* way has been, until now, by far the most common, and it is also without doubt the way anyone must adopt if all he wants is to gain a reputation in the shortest possible time among contemporary scholars. But for the improvement of the sciences, it seems to me the *second* procedure is much more beneficial for the following reasons.

Firstly, because in this way the discoverer of new ideas runs much less risk of being rushed, in that the piecemeal presentation of his opinions allows him to postpone to a later time his explanation of points on which he himself is initially in doubt. He can learn from the criticisms made of work already published, and still correct some things which had been incorrect.

Secondly, with such an exposition of his views appearing in instalments, he can also expect a far stricter examination on the part of the reader. For whoever presents an already completed system asks us to attend to a larger number of new assertions all in one go than we could hope anyone to examine with as much care as if they had been presented individually. Whoever supplies a complete theory shows, or at least should show, how those truths which are recognized by common

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[n] At the end of *BD* I §17 (p. 100 of this volume) Bolzano draws a contrast between real [*reelle*] concepts or expressions, as used in elementary mathematics, and the 'purely symbolic' ones such as $\sqrt{-1}$ and infinity used in higher mathematics.

[o] This section of *BD* appears in translation on p. 103.

sense with unshakeable certainty can be derived from his *unfamiliar* premisses. This reconciles us to those premisses straight away and results in our acknowledging them much more rapidly than if he had presented them individually and left us in doubt as to whether, and to what extent, they were consistent with all the rest of what we hold to be true. Finally, it is surely not to be denied that the mere sight of a *large, thick book* promising a complete system of this or that science, instils in us a kind of reverence before we have even read it. Now if we discover on reading it a certain coherence in its assertions, if the structure of human knowledge outlined for us here has a pleasing form, if everything is laid out according to measure and number and symmetry, then our judgement will be swayed; then we even begin to wish that here, at last, might be *that single correct system* which we have already sought for so long! And the least that happens is that because of the coherence we observe, we imagine that we are only free either to accept or to reject the entire system, whereas in fact neither the one nor the other should happen!

These were the main reasons for which I decided as early as 1804 not to begin in any science with the publication of a *complete treatise*, but first only to present these unorthodox concepts of mine in individual papers. And if, after much correction, these have met with approval from a proportion of their readership, only then should the preparation of an entire system be considered—that is, unless death should oblige us to leave this latter task to others.

I began my publications with a paper concerning *mathematics* under the title *Betrachtungen über einige Gegenstände der Elementargeometrie*, (Prague, C. Barth, 1804).[p] In this I put forward, along with several other views, *a new theory of parallels*.* Some years later I made up my mind to publish all my ideas in the field of *mathematics* in instalments under the title *Beyträge zu einer begründeteren Darstellung der Mathematik*. But then the *first* of these instalments (Prague, C. Widtmann, 1810)[r] had the misfortune, despite the importance of its contents, not even to be announced and reviewed in some scholarly journals, and in others only very superficially. This forced me to postpone the continuation of these contributions to a later time and meanwhile just to attempt, even though I may not succeed, to make myself somewhat better known to the academic world by publishing some works whose titles would be better suited to arouse attention. To this end there appeared in 1816 the work already mentioned, *Der binomische Lehrsatz* . . .

* This theory might deserve attention for at least two reasons: *firstly*, because it is the only one in which no obvious error has been detected; *then* because the greatest living French geometer *Legendre* came to just the same view of these things[q] quite independently of me in the *tenth* edition of his *Elémens de Géométrie*, Paris, 1813.

[p] This is *BG*, see p. 24.

[q] The phrase 'these things' is advisedly vague; apart from the fact that Legendre also sought to prove the parallel postulate as a theorem there is little similarity with Bolzano's work in results or methodology. The work Legendre (1813) is far more comprehensive, systematic, and conventional than Bolzano's contributions in his *BG* and *DP*.

[r] This is *BD*, see p. 82.

(Prague, Enders).[s] My wish is that the present paper should also serve this purpose; moreover, its publication was necessary in any case because in that earlier paper I had already made appeal to the proposition proved here.[t] There are also some *other* papers already prepared and ready for printing, e.g. one with the following title: *Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahmen des Archimedes und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst*.[u] These are still awaiting their publisher.

If I am to be able to proceed further in this direction, which seems to me the most beneficial, then the only *favour* for which I must *ask* the public is that this separate work be not overlooked on account of its limited scope, but rather examined with all possible rigour and the results of this examination made known publicly, so that what is perhaps unclearly expressed may be explained more clearly and what is quite incorrect may be retracted. The sooner truth and correctness become generally accepted, the better.

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[s] This is *BL*, see p. 154.

[t] See Bolzano's footnote in *BL*, §29 on p. 183.

[u] This is *DP*, see p. 278.

§ 1

Convention. Suppose that for a *series* [*Reihe*][v] *of quantities* the special case does not occur that from a certain term onwards all the terms are each *zero*, as happens for example after the $(n + 1)$th term in the *binomial series* for every positive whole-numbered exponent n. Then it is obvious that the *value of this series*, that is, the quantity resulting from the summation of its terms, cannot always remain the same if the number of terms is arbitrarily increased. In particular this value must certainly change every time the number of terms is increased, even by a *single one* which is not zero. Hence the value of a series depends not only on the *rule* determining the formation of the individual terms but also on their *number*. So this value represents a *variable* quantity even though the *form* and *magnitude* of the individual terms remain unchanged. With this in mind, we denote a *function* of x, which consists of an arbitrarily extendable *series* of terms and whose value therefore also depends not only on x but also on the *number of terms* r, by $\overset{(r)}{F}(x)$ or $\overset{r}{F}x$.[w] So, for example, $A + Bx + Cx^2 + \cdots + Rx^r = \overset{r}{F}x$, while $A + Bx + Cx^2 + \cdots + Rx^r + \cdots + Sx^{r+s} = \overset{(r+s)}{F}x$.

§ 2

Corollary 1. The *change in value*, i.e. the *increase* or *decrease* in value, of a series on increasing its number of terms by a *specific* number, e.g. by one, can, according to the nature of the circumstances, at one time be a *constant* quantity (namely, if the terms of the series are all equal to one another) but at another time may also be a *variable* quantity. In the latter case, the change may be either a quantity which sometimes increases and sometimes decreases, or one which increases steadily, or one which decreases steadily. Thus the change in the series

$$1 + 1 + 1 + 1 + \cdots,$$

if it is increased by *one* term, is a *constant* quantity; the change in the series

$$a + ae + ae^2 + ae^3 + \cdots$$

[v] The primary meaning for *Reihe* given in the mathematical dictionary Klügel (1823), iv is: 'a sequence [*Folge*] of quantities which are formed according to a common rule'. So what is translated here as 'series' should be understood as a rule-based sequence of indeterminate length. When the phrase the 'value of the series' appears in the following sentence the idea of summation is added, so then the concept corresponds more closely with what we now mean by 'series'.

[w] Although Bolzano introduces the notation $\overset{r}{F}x$ as denoting a function depending on x as well as r, in the following subsections, and in §7 in particular, the value of x is assumed fixed. The reason for the appearance of x at all is probably that the need for the proof contained in this paper first arose in the context of *BL* §29 where this notation was being used for certain functions of x.

by the increase of *one* term is a *variable* quantity, provided e is not $=1$. It becomes *larger and larger* if $e > \pm 1$, and *smaller and smaller* if $e < \pm 1$.[x]

§ 3

Corollary 2. If the *change in value* (*increase* or *decrease*) of a series due to the increase in its number of terms by a *specific* number (e.g. by one) always remains exactly the *same* or even always *increases*—and additionally if in both cases it retains the *same sign*—then it is clearly possible for the *value* of this series to become *greater than any given quantity* if it may be continued far enough. For suppose the growth in the series by each increase of n terms is $=$ or $>d$ and we want to make the series so large that it exceeds a given quantity D. Then we need only take a whole number r which is $=$ or $> \frac{D}{d}$ and extend the series by $r.n$ terms, thereby obtaining an increase which is $=$ or $>$ $(r.d = \text{or} > \frac{D}{d}d = D)$.

§ 4

Corollary 3. On the other hand there are also series whose value *never exceeds a certain quantity*, however far they may be continued. The series

$$a - a + a - a + \cdots$$

is just of this kind, whose value, however far the series is continued, is always either 0 or a and therefore never exceeds the quantity a.

§ 5

Corollary 4. Particularly remarkable among such series is the class of those series with the property that the *change* in value (*increase* or *decrease*), *however far the continuation* of its terms is taken, always remains *smaller* than a certain quantity, which itself can be taken *as small as we please* provided the series has been continued far enough beforehand. That there *are* such series is proved to us not only by the example of all those whose terms after a certain point are all *zero* (and which therefore really have no continuation *beyond* this term and are no more capable of changing value than the *binomial series* of §1), but also of this kind are all series in which the terms decrease at the same rate, or even faster than, the terms of a *geometric progression* whose ratio [*Exponent*][y] is a proper fraction. The value of the geometric series

$$a + ae + ae^2 + \cdots + ae^r$$

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[x] See the footnote d on p. 158.

[y] See footnote on p. 226.

is well known to be $= a.\frac{1-e^{r+1}}{1-e}$. And if this series is extended by s terms, then the increase is

$$ae^{r+1} + ae^{r+2} + ae^{r+3} + \cdots + ae^{r+s} = ae^{r+1}.\frac{1-e^s}{1-e}.$$

Now if $e < \pm 1$, and r has been taken sufficiently large, then this increase remains smaller than any given quantity, however large s may later become. For because e^s always remains $<\pm 1$, then $ae^{r+1}.\frac{1-e^s}{1-e}$ is obviously always *smaller* than $ae^{r+1}.\frac{2}{1-e}$. But this latter quantity can be made smaller than any given quantity by increasing r, because the value it takes for the next larger value of r is just the previous result multiplied by e, a constant proper fraction. (See *Der binomische Lehrsatz* . . ., §22.[z]) Therefore every geometric progression whose ratio is a proper fraction can be continued so far that the increase caused by every further continuation must remain smaller than some given quantity. This must hold all the more for series whose terms decrease even more rapidly than those of a decreasing geometric progression.

§ 6

Corollary 5. If the values of the sums of the first $n, n+1, n+2, \ldots, n+r$ terms of a series like those of §5 are denoted (§1) by $\overset{n}{F}x$, $\overset{n+1}{F}x$, $\overset{n+2}{F}x, \ldots, \overset{n+r}{F}x$, respectively, then the quantities

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \ldots, \overset{n}{F}x, \ldots, \overset{n+r}{F}x, \ldots$$

represent a *new* series (called the *series of sums* of the previous one). By assumption this has the special property that the difference between its nth term $\overset{n}{F}x$ and every later term $\overset{n+r}{F}x$ (no matter how far from that nth term) stays smaller than any given quantity, provided n has first been taken large enough. This difference is the increase produced in the *original* series by a continuation beyond its nth term, and by the assumption, provided n has been taken large enough, this increase should remain as small as we please.

§ 7

Theorem. If a series of quantities

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \ldots, \overset{n}{F}x, \ldots, \overset{n+r}{F}x, \ldots$$

has the property that the difference between its nth term $\overset{n}{F}x$ and every later one $\overset{n+r}{F}x$, however far this latter term may be from the former, remains smaller than any given quantity if n has been taken large enough, then there is always a certain *constant quantity*, and indeed only *one*, which the terms of this series approach

[z] On p. 175.

and to which they can come as near as we please if the series is continued far enough.[a]

Proof. It is clear from §6 that a series such as that described in the theorem is possible. But the hypothesis that there exists a quantity X which the terms of this series approach as closely as we please when it is continued ever further certainly contains nothing impossible, provided it is not assumed that this quantity be *unique* and *constant*.[b] For if it is to be a quantity which may vary then it can, of course, always be taken so that it is suitably near the term $\overset{n}{F}x$ with which it is just now being compared—even exactly the same as it. But also the assumption of a *constant* quantity with this property of proximity to the terms of our series contains nothing impossible because on this assumption it is possible to determine this quantity as accurately as we please. For suppose we want to determine X so accurately that the difference between the assumed value and the true value of X does not exceed a given quantity d, no matter how small. Then we simply look in the given series for a term $\overset{n}{F}x$ with the property that every succeeding term $\overset{n+r}{F}x$ differs from it by less than $\pm d$. By the assumption there must be such an $\overset{n}{F}x$. Now I say that the value of $\overset{n}{F}x$ differs from the true value of the quantity X by at most $\pm d$. For if r is increased arbitrarily, for the same n, the difference $X - \overset{n+r}{F}x = \pm\omega$ can become as small as we please. But the difference $\overset{n}{F}x - \overset{n+r}{F}x$ always remains $<\pm d$, however large r is taken. Therefore the difference

$$X - \overset{n}{F}x = (X - \overset{n+r}{F}x) - (\overset{n}{F}x - \overset{n+r}{F}x)$$

must also always remain $<\pm(d + \omega)$. But since for the same n this is a *constant* quantity, while ω can be made as small as we please by increasing r, then $X - \overset{n}{F}x$ must be $=$ or $<\pm d$. For if it were *greater* and $= \pm(d + e)$, for example, it would be impossible for the relation $d + e < d + \omega$, i.e. $e < \omega$, to hold if ω is reduced further. The true value of X therefore differs from the value of the term $\overset{n}{F}x$ by at most d, and can therefore be determined as accurately as we please since d can be taken arbitrarily small. There *is* therefore a *real quantity* [*eine reelle Größe*] to which the terms of the series under discussion approach as closely as we please if the series is continued far enough. But there is only *one* such quantity. For suppose that besides X there was another *constant* quantity Y which the terms of the series approach as much as we please if it is continued far enough, then the differences $X - \overset{n+r}{F}x = \omega$ and $Y - \overset{n+r}{F}x = \overset{\mathrm{I}}{\omega}$ can be made as small as we please if r is allowed to be large enough. Therefore this must also hold for their own difference, i.e. for

[a] This is the first statement of the convergence criterion, usually named after Cauchy; for a brief discussion of the following proof see pp. 149–50 where there are further references.

[b] The word 'constant' has been used to translate both *beständig* (occurring in the statement of the theorem) and *unveränderlich* (used at this point in the proof).

$X - Y = \omega - \overset{1}{\omega}$ which, if X and Y are to be *constant* quantities, is impossible unless one assumes $X = Y$.

§ 8

Note. If we try to determine the value of the quantity X in the way described in the previous section, namely by using one of the terms of which the given series is composed, then we shall never determine X entirely *accurately* unless all the terms of this series from a certain term onwards are equal to one another. But we must take care not to conclude from this that the quantity X must always be *irrational*. For if we consider the series

$$0.1, 0.11, 0.111, 0.1111, \ldots$$ [c]

(which is the series of sums of the geometrical progression

$$\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots)$$

then the quantity to which the terms approach as closely as we please is not irrational at all but is the fraction $\frac{1}{9}$. Thus it does not follow from the fact that a quantity cannot be determined accurately *in a certain way*, that it cannot be completely determined in any *other* way, and would therefore be *irrational*.

§ 9

Corollary. Therefore if some given series has the property that each individual term is *finite* but the change in it when it is continued, no matter how far, is smaller than any given quantity, provided only that the number of terms taken initially is large enough, then there is always one and only *one constant quantity* to which the value of this series comes as close as we please if the series is continued far enough. For such a series is of the kind described in §5, and hence the *values* which are the sums of its n, $n+1, \ldots$ terms form a series like those of §§ 6 and 7; therefore such a series also has the property proved in §7.

§ 10

Note. It should not be thought that in the above proposition (§9) the condition: *that the change* (increase or decrease) *in the series for every continuation must remain smaller than any given quantity provided it has been continued far enough initially*, is superfluous and that the proposition could perhaps even be expressed with greater generality: *If it is possible for the terms of a series, as it is continued, to become ever smaller and as small as we please, then there is always a constant quantity which the value of the series, when it is continued, approaches as closely as we please.* This

[c] Here we have not followed our usual practice of preserving Bolzano's mathematical notation. The original text uses commas (as in modern German) to denote what is rendered here as a decimal point.

assertion would immediately be contradicted by the following example. The terms of the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

can be made as small as we please and yet it is a truth, familiar from the properties of the *rectangular hyperbola* (but also derivable from purely arithmetic considerations), that the value of this series can become greater than any given quantity if it is continued far enough.

§ 11

Preamble. In investigations of applied mathematics it is often the case that we learn that a definite property M applies to *all* values of a *variable quantity* x which are *smaller* than a certain u, without at the same time learning that this property does not apply to values which are greater than u. In such cases there can still perhaps be some $\overset{1}{u}$ that is $> u$ for which, in the same way as it holds for u, all values of x lower [than $\overset{1}{u}$] possess property M. Indeed this property M may even belong to *all values of x without exception*. But if *this alone* is known, *that M does not belong to all x in general*, then by combining these two conditions we will now be justified in concluding: *there is a certain quantity U which is the greatest of those for which it is true that all smaller values of x possess property M.*[d] This is proved in the following theorem.

§ 12

Theorem. If a property M does not apply to *all* values of a variable quantity x but does apply to *all* values *smaller* than a certain u, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.

Proof. 1. Because the property M holds for all x *smaller* than u but nevertheless not for *all x*, there is certainly some quantity $V = u + D$ (where D represents something positive) of which it can be asserted that M does not apply to all x which are $< V = u + D$. If I then raise the question of whether *M in fact applies to all x which are* $< u + \frac{D}{2^m}$ where the exponent m is in turn first 0, then 1, then 2, then 3, etc., I am sure the *first* of my questions will *have to be answered 'no'*.[e] For the question of whether M applies to all x which are $< u + \frac{D}{2^0}$ is the same as that

[d] This is probably the first formulation of a result that is equivalent to the Bolzano–Weierstrass theorem. It may be brought into a more familiar form if we consider the set of 'non-*M*' values. This is bounded below by assumption, and the result described here, and proved in the following section, states that there exists a greatest lower bound, U, for that set.

[e] The German *verneinen* used here means to say 'no' as a response, as well as to deny a charge or person. It is not correct in English to 'deny' a question, and hence the somewhat response-ridden rendering of this section. There is a similar situation with regard to *bejahen* and 'affirm'.

of whether M applies to all x which are $<u+D$, which is ruled out by assumption. What matters is whether all the *succeeding* questions, which arise as m gradually gets larger, will also be ruled out. Should this be the case, it is evident that u *itself* is the greatest value for which the assertion holds that all smaller x have property M. For if there were an even greater value, for example $u+d$, i.e. if the assertion held that also all x which are $<u+d$ have the property M, then it is obvious that if I take m large enough, $u+\frac{D}{2^m}$ will at some time be $=$ or $<u+d$. Consequently if M applies to all x which are $<u+d$, it also applies to all x which are $<u+\frac{D}{2^m}$. We would therefore not have said 'no' to this question but would have had to say 'yes'. Thus it is proved that in this case (when we say 'no' to all the above questions) there is a certain quantity U (namely u itself) which is the greatest for which the assertion holds that all x below it possess the property M.

2. However, if one of the above questions is *answered 'yes'* and m is the particular value of the exponent for which this happens *first* (m can be 1 but, as we have seen, not 0), then I now know that the property M applies to all x which are $<u+\frac{D}{2^m}$ but not to all x which are $<u+\frac{D}{2^{m-1}}$. But the difference between $u+\frac{D}{2^{m-1}}$ and $u+\frac{D}{2^m}$ is $=\frac{D}{2^m}$. If I therefore deal with this as I did before with the difference D, i.e. if I raise the question of whether M applies to all x which are

$$<u+\frac{D}{2^m}+\frac{D}{2^{m+n}},$$

and here the exponent n denotes first 0, then 1, then 2, etc., then I am sure once again that at least the *first* of these questions will have to be answered 'no'. For to ask whether M applies to all x which are

$$<u+\frac{D}{2^m}+\frac{D}{2^{m+0}}$$

is just the same as asking whether M applies to all x which are $<u+\frac{D}{2^{m-1}}$, which had previously been denied. But if all my *succeeding* questions are also to be answered negatively as I gradually make n larger and larger, then it would appear, as before, that $u+\frac{D}{2^m}$ is that greatest value, or the U, for which the assertion holds that all x below it possess the property M.

3. However, if one of these questions is answered positively and this happens first for the particular value n, then I now know M applies to all x which are

$$<u+\frac{D}{2^m}+\frac{D}{2^{m+n}}$$

but not to all x which are

$$<u+\frac{D}{2^m}+\frac{D}{2^{m+n-1}}.$$

The difference between these two quantities is $=\frac{D}{2^{m+n}}$ and I deal with this again as before with $\frac{D}{2^m}$, etc.

4. If I continue this way as long as I please it may be seen that the result that I finally obtain must be one of two things.

(a) Either I find a value of the form

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

which appears to be the greatest for which the assertion holds that all x below it possess the property M. This happens in the case when the questions of whether M applies to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r+s}}$$

are answered with ‘no’ for every value of s.
(b) Or I at least find that M does indeed apply to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

but not to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}}.$$

Here I am always free to make the number of terms in these two quantities even greater through new questions.

5. Now if the *first* case occurs the truth of the theorem is already proved. In the *second* case we may remark that the quantity

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

represents a series whose number of terms I can increase arbitrarily and which belongs to the class described in §5. This is because, depending on whether $m, n, \ldots, r$ are all $=1$, or some of them are greater than 1, the series decreases at the same rate, or more rapidly than, a geometric progression whose ratio is the proper fraction $\frac{1}{2}$. From this it follows that it has the property of §9, i.e. there is a certain *constant quantity* to which it can come as close as we please if the number of its terms is increased sufficiently. Let this quantity be U; then I claim the property M holds for all x which are $< U$. For if it did not hold for some x which is $< U$, e.g. for $U - \delta$, then the quantity

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

must always keep at the distance δ from U because for all x that are smaller than it, the property M is to hold. Since every x that is

$$= u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}} - \omega,$$

however small ω is, possesses the property M, while on the other hand, M is not to apply to $x = U - \delta$, it must therefore be that

$$U - \delta > u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}} - \omega$$

or

$$U - \left[u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}\right] > \delta - \omega.$$

Hence the difference between U and the series cannot become as small as we please, since $\delta - \omega$ cannot become as small as we please because δ does not change, while ω can become smaller than any given quantity. But just as little can M hold for all x which are $<U + \varepsilon$. For the value of the series

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}}$$

can be brought as close to the value of the series

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

as we please because the difference between the two is only $\frac{D}{2^{m+n+\cdots+r}}$. Further, the value of the latter series can be brought as close as we please to the quantity U. Therefore the value of the first series can also come as close to U as we please. So

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}}$$

can certainly become $< U + \varepsilon$. But now by assumption M does not hold for all x which are

$$<u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}};$$

so much less therefore [does M hold] for all x which are $<U + \varepsilon$. Therefore U is the greatest value for which the assertion holds that all x below it possess the property M.

§ 13

Note 1. The above theorem is of the greatest importance and is used in all branches of mathematics, in analysis as well as in the applied areas, in geometry, chronometry and mechanics. In the past this false proposition has often been used instead: *If a property M holds, not for all x, but for all x smaller than a certain value, then there is always some greatest x to which the property M applies.* This I say is *false*, as a consequence of the theorem just proved. For if there is some quantity U which is the greatest of those of which it can be said that all x below them have property M, then for that reason there is certainly *no greatest x* to which this property applies, *provided x is either a freely or a continuously variable quantity.* For it is well known that for every quantity that varies *freely or according to the law of continuity* there is never a *greatest* value that is smaller than a certain limit U, because however close it may already be to this limit it can always be brought closer. In order to explain this by an example, consider a *rectangular hyperbola* and take one of its asymptotes as the abscissae-line and take the origin, not at the centre c, but at some other point a on this asymptote which is at a distance D from c. Now let us define the direction ac as the *positive direction of the abscissae*, and the direction ab which is the perpendicular ordinate of the point a, as the *positive direction of the ordinates*. Then every abscissa, x, which is *smaller* than a certain one, say smaller than $\frac{D}{2}$, has the property *that a positive ordinate corresponds to it*. However, this property (M) will not hold for *all* positive abscissae, namely not for those which are greater than D. Now is there in fact a *greatest* abscissa here, a greatest value of x, to which the property M applies? By no means, but there is certainly a U, i.e. an abscissa, which is the greatest among those of which it can be said that all smaller than it have positive ordinates, i.e. possess the property M. In fact, this abscissa is $+D$.

§ 14

Note 2. Perhaps someone might think that the proof of the theorem in §12 could have been expressed quite briefly in the following way: If there were no greatest U for which the assertion holds that all x below it possess the property M, one could take *u ever larger and larger*, and therefore as large as we please, and consequently M would have to hold for *all x without exception.* However, this would be a very mistaken conclusion because it would rest on the tacit assumption: *that a quantity that can be taken ever larger than it has already been taken can become as large as we please.* How false this is may be proved, for example, by the well-known series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, whose value can always be made greater than it already is and yet always remains <1! We would not even mention such an easily noticeable error if it did not happen from time to time that mathematicians were guilty of it—as one was only recently in his *Vollständige Theorie der Parallelen.*

§ 15

Theorem. If two functions of x, fx and ϕx, vary *according to the law of continuity* either for *all* values of x or for all those lying between α and β, and furthermore if $f\alpha < \phi\alpha$ and $f\beta > \phi\beta$, then there is always a certain value of x between α and β for which $fx = \phi x$.

Proof. We must remember that in this theorem the values of the functions fx and ϕx are to be compared with one another only as *absolute* quantities, i.e. without regard to signs, or as though they were quantities incapable of having opposite signs. Rather, what matters are the signs which α and β have.

I. 1. *Firstly* assume that α and β are both *positive* and that (because it does not matter) β is the *greater* of the two, so $\beta = \alpha + i$, where i denotes a positive quantity. Now because $f\alpha < \phi\alpha$, if ω denotes a positive quantity which can become as small as we please, then also $f(\alpha + \omega) < \phi(\alpha + \omega)$. For because fx and ϕx are to vary continuously for all x lying between α and β, and $\alpha + \omega$ lies between α and β whenever we take $\omega < i$, then it must be possible to make $f(\alpha + \omega) - f\alpha$ and $\phi(\alpha + \omega) - \phi\alpha$ as small as we please if ω is taken small enough. Hence if Ω and $\overset{\prime}{\Omega}$ denote quantities which can be made as small as we please, $f(\alpha + \omega) - f\alpha = \Omega$ and $\phi(\alpha + \omega) - \phi\alpha = \overset{\prime}{\Omega}$. Hence,

$$\phi(\alpha + \omega) - f(\alpha + \omega) = \phi\alpha - f\alpha + \overset{\prime}{\Omega} - \Omega.$$

However, $\phi\alpha - f\alpha$ equals, by assumption, some positive quantity of constant value A. Therefore

$$\phi(\alpha + \omega) - f(\alpha + \omega) = A + \overset{\prime}{\Omega} - \Omega,$$

which always remains positive if Ω and $\overset{\prime}{\Omega}$ are allowed to become small enough, i.e. if ω is given a very small value, and even more so for all smaller values of ω. Therefore it can be asserted that for *all* values of ω *smaller* than a certain value the two functions $f(\alpha + \omega)$ and $\phi(\alpha + \omega)$ stand in the relationship of smaller quantity to greater quantity. Let us denote this property of the variable quantity ω by M, then we can say that all ω that are smaller than a certain one possess the property M. But nevertheless it is clear that this property M does not apply to *all* values of ω, namely not to the value $\omega = i$, because $f(\alpha + i) = f\beta$ which, by assumption, is no longer $<$ but $>$ $\phi(\alpha + i) = \phi\beta$. As a consequence of the theorem of §12 there must therefore be a certain quantity U which is the greatest of those of which it can be asserted that all ω which are $<U$ have the property M.
2. And this U must lie *between* 0 and i. For *firstly* it cannot be $=i$ because this would mean that $f(\alpha + \omega) < \phi(\alpha + \omega)$, whenever $\omega < i$, and however near it came to the value of i. But in exactly the same way that we have just proved that the assumption $f\alpha < \phi\alpha$ has the consequence $f(\alpha + \omega) < \phi(\alpha + \omega)$, provided ω is taken small enough, so we can also prove that the assumption $f(\alpha + i) > \phi(\alpha + i)$ leads to the consequence $f(\alpha + i - \omega) > \phi(\alpha + i - \omega)$, provided ω is taken small enough. It is therefore not true that the two functions fx and ϕx stand in the relationship of smaller quantity to greater quantity for all values of x which are $< \alpha + i$. *Secondly*, still less can it be that $U > i$ because otherwise i would also be one of the values of ω which are $<U$, and hence also $f(\alpha + i) < \phi(\alpha + i)$ which directly contradicts the assumption of the theorem. Therefore, since it is *positive*, U certainly lies between 0 and i and consequently $\alpha + U$ lies between α and β.
3. It may now be asked what relation holds between fx and ϕx for the value $x = \alpha + U$? *First of all*, it cannot be that $f(\alpha + U) < \phi(\alpha + U)$, for this would also give $f(\alpha + U + \omega) < \phi(\alpha + U + \omega)$, if ω were taken small enough, and consequently $\alpha + U$ would not be the *greatest* value of which it can be asserted that all x below

it have the property M. *Secondly*, just as little can it be that $f(\alpha + U) > \phi(\alpha + U)$, because this would also give $f(\alpha + U - \omega) > \phi(\alpha + U - \omega)$ if ω were taken small enough and therefore, contrary to the assumption, the property M would not be true of all x less than $\alpha + U$. Nothing else therefore remains but that $f(\alpha + U) = \phi(\alpha + U)$, and so it is proved that there is a value of x lying between α and β, namely $\alpha + U$, for which $fx = \phi x$.

II. The same proof is also applicable to the case when α and β are *both negative* as long as ω, i and U are taken to be *negative* quantities, because then in the same way $\alpha + \omega$, $\alpha + i$, $\alpha + U$, $\alpha + U - \omega$ represent quantities between α and β.

III. If $\alpha = 0$ and for *positive* β then just by taking i $(= \beta)$, ω, U positive, and for *negative* β by taking these others negative, the proof I. can be applied word for word.

IV. Finally, if α and β are of *opposite sign* and (because it does not matter) α is negative and β positive, for instance, then the assumption of the theorem in respect of the continuity of the functions fx and ϕx states that this continuity applies to all values of x which, if *negative*, are $<\alpha$ and, if positive, are $<\beta$. Then the value $x = 0$ is also included among these. One therefore investigates the relationship which holds between fx and ϕx for $x = 0$. If $f(0) = \phi(0)$ then the theorem is automatically already proved. But if $f(0) > \phi(0)$, then since $f\alpha < \phi\alpha$ by III. we have a value between 0 and α, and finally if $f(0) < \phi(0)$, a value between 0 and β, for which $fx = \phi x$. Therefore in every case there is a value of x lying between α and β which makes $fx = \phi x$.

§ 16

Note. It is by no means claimed here that there is *only a single value* of x which makes $fx = \phi x$. Namely, if $f\alpha < \phi\alpha$ and $f(\alpha + U) = \phi(\alpha + U)$ we must indeed have $f(\alpha + U + \omega) > \phi(\alpha + U + \omega)$ if ω is taken small enough, i.e. the function fx which was previously *smaller* than ϕx must, soon after they have become *equal* to one another, become *greater* than ϕx. But as ω is increased *more and more* it is certainly possible that, before $\alpha + U + \omega$ has been made $=\beta$, there are values for which *once again* $fx < \phi x$. In such a case, it follows directly from our theorem that there must be *another two* values of x, besides U, between α and β which make $fx = \phi x$. For if $f(\alpha + U + \chi) < \phi(\alpha + U + \chi)$ then, because $f(\alpha + U + \omega)$ was already $>\phi(\alpha + U + \omega)$, there must be a value of x between $\alpha + U + \omega$ and $\alpha + U + \chi$, i.e. also lying between α and β, for which $fx = \phi x$. And in the same way, because $f(\alpha + i)$, or $f\beta$, is again $>\phi\beta$, there is also a value of x between $\alpha + U + \chi$ and β which makes $fx = \phi x$. In this way it becomes clear in general that there must always exist *an odd number* of values of x which make $fx = \phi x$.

§ 17

Theorem. Every function of the form $a + bx^m + cx^n + \cdots + px^r$, in which $m, n, \ldots, r$ denote positive integer exponents, is a quantity which varies *according to the law of continuity* for all values of x.

Proof. If x changes to $x + \omega$, the change in the function is obviously

$$= b[(x+\omega)^m - x^m] + c[(x+\omega)^n - x^n] + \cdots + p[(x+\omega)^r - x^r],$$

a quantity for which it can easily be shown that it can become as small as we please if ω is taken small enough. For by the *binomial theorem*, whose validity for positive integer exponents we have shown (§8 of *Der binomische Lehrsatz* ...[f]) to be independent of the investigations with which the present paper is concerned, this quantity is

$$= \omega \left\{ \begin{array}{l} mbx^{m-1} + m.\dfrac{m-1}{2}bx^{m-2}\omega + \cdots + \omega^{m-1} \\ +ncx^{n-1} + n.\dfrac{n-1}{2}cx^{n-2}\omega + \cdots + \omega^{n-1} \\ + \cdot \;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ +rpx^{r-1} + r.\dfrac{r-1}{2}px^{r-2}\omega + \cdots + \omega^{r-1} \end{array} \right\}.$$

As we know, the number of terms in the factor contained in the brackets is always *finite*, and *independent* of the values of the quantities x and ω. Since these appear everywhere only with *positive powers* the value of every individual term, and consequently of the whole expression, is always *finite* for every value of x and ω (also for $x = 0$). But if for the same x, the value of ω is made smaller, then the terms in which ω appears decrease, while the others remain unchanged. So if we denote by S the quantity arrived at by putting a definite value, say $\overset{1}{\omega}$, for ω in all the individual terms of the expression, and summing as if they all had the same sign, then the actual value of this expression for this particular $\overset{1}{\omega}$ is certainly not $>S$, but that [value] which it takes for every *smaller* ω is surely $<S$. Hence if it is required that the change in the function $a + bx^m + cx^n + \cdots + px^r$ should be $<D$, then we simply take an ω that is at the same time $<\overset{1}{\omega}$ and also $<\frac{D}{S}$. Then $\omega.S$ and, all the more, the product of ω with a quantity which is $<S$, must be $<D$.

§ 18

Theorem. If a function of the form

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + px + q,$$

in which n denotes a positive integer, takes a *positive* value for $x = \alpha$, but a *negative* value for $x = \beta$, then the equation

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + px + q = 0$$

has at least one *real root* lying between α and β.

[f] On p. 169.

Proof. 1. If α and β are both of the same sign (both are either positive or negative) then it is clear that exactly the same terms of the function which are positive or negative for $x = \alpha$ also keep this sign for $x = \beta$ and for all values of x lying between α and β. Now suppose the value of the function is positive for $x = \alpha$, but negative for $x = \beta$; this change can only occur because for $x = \alpha$ the sum of its positive terms turns out *greater* than that of the negative ones while for $x = \beta$ it is *smaller*. But the sum of the former, as well as of the latter, is of the form

$$a + bx^m + cx^n + \cdots + px^r$$

of §17, i.e. a continuous function. Let us therefore denote the one by ϕx and the other by fx, then because $f\alpha < \phi\alpha$ and $f\beta > \phi\beta$, there must, by §15, be some value of x lying between α and β for which $fx = \phi x$. But for this value $fx - \phi x$, i.e. the given function, becomes zero, therefore this value is a root of the equation

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + px + q = 0.$$

2. But if α and β are of *opposite sign*, consider the value taken by the given function for $x = 0$. If this value is zero, it follows at once that the given equation has a real root lying between α and β, namely $x = 0$. But if this value (the quantity q) is *positive* then it is now known that the given function is positive for $x = 0$ but becomes negative for $x = \beta$. Since the same terms which are positive or negative for $x = \beta$ also retain this sign for all values lying between 0 and β, one can prove, by the same arguments as in no. 1, that there must be a value of x lying between 0 and β which makes the function zero. Finally, if q is *negative*, then what we have just said holds if α is substituted for β. Now since a value lying between 0 and β or between 0 and α also lies between α and β, if they are of opposite sign, then our theorem is shown to be true for every case.

Die

drey Probleme

der

Rectification, der Complanation

und der

Cubirung,

ohne Betrachtung des unendlich Kleinen, ohne die Annahmen des Archimedes, und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst;

zugleich als

Probe einer gänzlichen Umstaltung der Raumwissenschaft,

allen Mathematikern zur Prüfung

von

Bernard Bolzan[illegible]

Weltpriester, Doctor der Philosophie, k. k. Profe[illegible] gionswissenschaft, und ordentlichem Mitgliede der k. [illegible] schaft der Wissenschaften zu Prag.

Leipzig, 1817.

bey Paul Gotthelf Kummer.

The Three Problems of Rectification, Complanation and Cubature,

solved without consideration of the infinitely small, without the hypotheses of Archimedes and without any assumption which is not strictly provable

This is also being presented for the scrutiny of all mathematicians as a sample of a complete reorganization of the science of space.

by

Bernard Bolzano

Priest, Doctor of Philosophy, and Professor of Theology and Ordinary Member of the Royal Society of Sciences at Prague

Leipzig, 1817
Paul Gotthelf Kummer

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Preface

The rectification of curved lines, the complanation of surfaces and the cubature of solids are three main problems with the solution of which elementary geometry is just as concerned as higher geometry. However, although the solutions for these three problems are easy and general, especially those which have been found more recently, there has been a great lack until now of strict and genuinely scientific proofs of their correctness, at least in respect to the first two.

In this connection it is striking that in the oldest textbook of geometry, in which we still admire the unsurpassed rigour, the *Elements of Euclid*, in spite of having such a great collection of propositions, there is not a single one which determines anything about the *length* of a curved *line* or the *area* of a curved *surface*. Can this silence be explained in any other way than that the profound author of this work realized that the few propositions which he allowed himself to include in his system without proof (under the name of *axioms*) are completely insufficient for working out the quantitative relationship which holds between straight and curved lines or surfaces?

Therefore *later* geometers (from whom, it could be said, the spirit of rigour has increasingly departed) solved *individual* problems in this area only by taking the liberty of increasing the number of geometrical axioms by adding some *new ones*. But it is worth noticing that the first person who ventured to do this, the excellent *Archimedes*, introduced his new axioms with only the weaker term *assumptions* [*Annahmen*].* Essentially these were the following four:

I. *Every curved line is longer than the straight line lying between the same end-points.*

II. *Of two curved lines which are both concave on one side the enclosing one is longer than the enclosed one.*

III. *If a curved and a plane surface have the same boundaries the former is greater than the latter.*

IV. *Of two curved surfaces which are both concave on one side the enclosing one is greater than the enclosed one.*

* *Λαμβἀνω δε τâυτα*,[a] he said. Also they are entitled simply ‘*Υποθεσεις*.[b]

[a] *Translation*: I make these postulates. The specific phrase occurs near the beginning of Archimedes, *On the Sphere and Cylinder*, in *Greek Mathematical Works*, II, 45, Tr. Ivor Thomas, Heinemann (1941).

[b] *Translation:* hypotheses.

Among the axioms of *Euclid* one had always been found objectionable and its acceptance had been resisted throughout the centuries because everyone felt that this proposition, although certain and clear, was not really an *axiom*, i.e. was not the sort of proposition which allows of no further ground for its truth.* It could not be much clearer concerning those *new* hypotheses of *Archimedes* that they certainly do not deserve the position at the top of the system as genuine axioms, but at best [they are] the sort of proposition to which one particularly wishes to draw attention because they have already been used although their proofs have not yet been found. In fact we can see from the history of geometry that people have always sought to do one of these two things: either first to justify those Ὑποθεσεις [hypotheses] of Archimedes by a separate proof, or to derive the theorems which had been based on them in another way. However, while there was no general overview [*Übersicht des Ganzen*], and only the magnitude of *some individual* lines and surfaces had been derived (more through certain fortunate techniques than by a generally applicable method), and while people had no idea of the *general relationships* which hold between the equations for the nature of a line, surface or solid and the expressions for their magnitude it was almost impossible to discover the unique, objective, or genuinely scientific, proofs, even for those solutions which were already actually known.

It was left to the age of the most important discoveries in the field of mathematics, namely the age of the invention of the *differential* and *integral calculus*, to make these remarkable relationships known. The length of any line is $\int\sqrt{dx^2+dy^2+dz^2}$, the area of any surface

$$=\iint dx\,dy\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2},$$

the volume of any solid $=\iiint dx\ dy\ dz$, where, of course, x, y and z refer to the three perpendicular co-ordinates of these spatial objects [*Raumdinge*]. Now since the most general way of determining the nature of a spatial object is to state certain equations between co-ordinates and since, as soon as these are given, the derivation of its magnitude by the formulae just quoted is a *purely analytic procedure*,** then the latter really express the three most general theorems that

* Concerning this concept of an axiom compare the *Beyträge zu einer begründeteren Darstellung der Mathematik. First issue*, Prague, 1810, S. 59–96.[c]

** A *purely analytic* (also purely arithmetic, or algebraic) *procedure* is one by which a certain function is simply derived [*ableitet*] from one or more other functions through making certain changes and combinations with them which are expressed *by a rule that is completely independent of the nature of the designated quantities*. Thus, for example, the way the function $(1+x)^n$ is derived from $(1+x)$ is a purely analytic procedure, for $(1+x)^n$ is obtained from $(1+x)$ by making certain changes and combinations in the latter which are given by a rule which is completely independent of the nature of the quantity designated by $(1+x)$. Equally, everyone who understands differential and integral

[c] On pp. 109–120.

can ever be stated about rectification, complanation and cubature. Therefore if these three formulae were to have been properly proved then all problems which could appear in this three-fold domain would already be solved at one stroke geometrically, and there would merely remain, in individual cases, a problem to solve in analysis concerning the integration of the expressions found.

Furthermore, if those three propositions really are the most general ones, then it follows that without first having properly derived them from their objective ground no other theorem about the measurement of individual lines, surfaces and solids can be correctly proved. This is because the nature of a genuinely scientific proof requires that the more particular proposition is always only proved from the more general.* Actually I claim that none of the proofs known so far of those three general propositions is genuinely scientific. However, since I intend soon (in a special work) to give detailed criticism of *all the defects of our previous systems of geometry*, I may be allowed for the proof of my present claim, to put forward now just enough to show it is not inconceivable; but for the rest to refer in advance to this forthcoming book.

I shall first distinguish the two problems of rectification and complanation from that of cubature.

All those geometers who, like the inventors of the infinitesimal calculus, or in more recent times like *Johann Schultz*, make use of the concept of the *infinitely small*, can never avoid the suspicion of contradiction in this concept itself,** nor can they adequately answer the following question. 'Why does the length of an infinitely small arc only coincide with the length of that straight line which goes through *the ordinates* which bound it if it has the direction of the *chord* or *tangent*, but not if it goes through the ordinates at some other kind of angle?' Of course, these geometers do accept a difference between the piece of arc and its chord but claim that in respect of *length* it is an infinitely small quantity of the *second* order, while the difference between the arc and *another* straight line will amount to an infinitely small quantity of the *first* order. Now where is the *proof* of these claims?

calculus knows that, for example, the expression $\int\sqrt{dx^2 + dy^2 + dz^2}$ designates a function which can be derived from the two given functions $y = fx$ and $z = \bar{f}x$ by rules which do not depend at all on the nature of the quantities designated by these expressions. Moreover, it must also be known to everyone that these rules can be expressed in such a way that the concept of the *infinitely small* (which *otherwise* would surely be associated with the expressions $dx, dy, dz, \ldots$) may be completely avoided. I therefore hope that no one will object that in this work we have tacitly assumed the concept of infinitely small quantities just because we have used expressions of the form $\frac{dy}{dx}$, $\int y\, dx$ etc.

* On this see the previously mentioned *Beyträge*, S. 102 ff.[d]

** See *Der binomische Lehrsatz* etc., Prague, 1816, Preface, S. IV.[e]

[d] On p. 121 ff.

[e] On p. 158.

Or can judgements which are so very composite really be viewed as genuine *basic truths* [*Grundwahrheiten*]?*

Those who define the infinitely small as an *absolute nothing* have first of all to deal with the objection of how *nothings* can have a ratio to one another. Suppose they replied to this that expressions like $\frac{dx}{dy}$ do not designate the *ratio* of the differences Δx, Δy which have become *nothing*, but only the relationship in which the *way* Δx vanishes stands to the *way* Δy vanishes. Then as before I ask again here, why do only arcs and chords have the relation of equality to one another in their vanishing, but not other lines, e.g. the arc and the increment of the abscissa? What can be more arbitrary than the following argument which is none the less to be found in the work of several very noteworthy mathematicians? (For example, *Langsdorf* in his *Einleitung in das Studium der Elementargeometrie, Algebra*, etc., Manheim and Heidelberg, 1814, *Elemente der höhern Geometrie*, §§10, 11; also with Dubourguet in *Traités élémentaires du calcul différentiel et du calcul intégral, indépendans de toutes notions de quantités infinitésimals et de limites*, Paris, 1810, P. II, §§529, 532, 534.)

> '*The arc Δs and its chord $\sqrt{\Delta x^2 + \Delta y^2}$ are different only as long as they have not both become zero, therefore in this latter case one has, in complete strictness,*
>
> $$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}.\text{'}$$

Could one not maintain in just the same way that also

$$ds = dx \quad \text{or} \quad \frac{ds}{dx} = 1?$$

From the fact that two quantities vanish *at the same time* it does not follow at all that the relationship of the way they vanish is equal in both, or to express this better, that the relationship of these quantities can come as near as desired to the relationship of equality if they are taken as small as we please.

Of course, those people remain free from such errors who have avoided all concepts of the infinitely small and all ratios of zero, and combined the *method of limits* and *Taylor's theorem* as it is done more or less distinctively by *Lagrange*, *Pasquich*, *Gruson*, *Buzengeiger*, *Bohnenberger*, *Crelle* and others. However, these all resort explicitly to the so-called *axioms of Archimedes*. See for example Lagrange, *Théorie des fonctions analytiques*, Art. 136, 137. Or by the same author, *Leçons sur le calcul des fonctions, Nouv. Edit.*, Leç. 9; or in *Pasquich Anfangsgründe einer neuen Exponentialrechnung* in *Hindenburg Archiv*, 8. Heft, 1798, p. 419; or *Crelle Versuch einer Darstellung der Rechnung mit veränderlichen Größen*, 1. Band, Göttingen, 1813; and there are others.

But perhaps these formerly unproven *assumptions* of Archimedes have already been sufficiently proved by the efforts of modern geometers? Actually the accolade

* Concerning these refer to the work already mentioned, *Beyträge*, S. 87 ff.[f]

[f] On p. 117.

of 'perfect strictness' has been conferred on first one, and then another, of their attempts; nevertheless this appears to have been only for the sake of being able to refer to them with all the more justification. The proverb applies here: *homines quod volunt, credunt.* [g]

Legendre, whose proof has been said to be the most thorough, proceeds in the *tenth* edition of his very admirable, *Éléments de Géométrie*, Paris, 1813, p. 115, 244 with just the same error to which E. G. Fischer has drawn some attention (in his *Untersuchung über den eigentlichen Sinn der höhern Analysis*, Berlin, 1808). '*If the enclosed line* (he says) *is not shorter than all the enclosing lines there must be, among the latter, one which is shorter than all the rest and either as long as, or shorter than, the enclosed line.*' Two claims are contained in this proposition. One of them, namely, '*if the enclosed line is not shorter than all the enclosing lines then among the latter there must exist one which is as long as, or still shorter than, the enclosed line*' is, of course, indubitable. It is an *identical*, and therefore (like all analytic propositions) also a *useless proposition*.* It is all the easier to pass over the second claim which here, maybe intentionally or by accident, is incorporated into the former, '*if the enclosed line is not shorter than all the enclosing lines there must be, among the latter, one which is shorter than all the rest.*' I absolutely do not see what connection holds between the premiss and the conclusion of this claim, or what kind of major premiss it must have been which the author has had in mind with this enthymeme.[i] If he thought, '*that among all quantities of a certain kind* (here, therefore, among all lines which enclose another) *one must be the smallest*,' this would obviously have been a wrong idea which he himself immediately contradicts in what follows; the premiss would also then have been added superfluously. However, the objectionable nature of the whole judgement is clearest as soon as it is remembered that its *subject* (the hypothesis, '*if the enclosed line is not shorter than all the enclosing lines*') is something impossible. Something true can never be predicated of an *impossibility* and therefore judgements whose subject contains an impossibility should never be put forward in science. For example, it would be just as foolish to make the judgements, *if all angles in a triangle are right angles, then it is equilateral, two sides of it are parallel* etc. In any case, it could already be seen that the author's proof must be mistaken from the fact that the condition, which is indispensable for the truth of the proposition, that *the enclosed line is concave on one side*, is indeed *mentioned* but not in fact *used*.** A significant number of other geometers assume, sometimes tacitly sometimes explicitly, the false proposition: '*If one of two lines* (also surfaces)

* See *Beyträge*, S. 81[h].

** Compare *Beyträge*, S. 106–110.[j]

[g] *Translation:* Men believe what they want to believe.

[h] On p. 115.

[i] A syllogism in which one premiss is suppressed. (*OED*)

[j] On pp. 122–123.

lies closer, with respect to all its points, to a certain third line then it also comes closer to it with respect to its length (for the surfaces, with respect to their area).' Thus *Lorenz* (in his *Elementen der Mathematik*, 2te Aufl., Leipzig, 1793, 1.Teil, §209), *Lacroix* (in his *Anfangsgründe der Geometrie* translated by Hahn, Berlin, 1806, §§152, 270), *Ide* (in his *Anfangsgründe der reinen Mathematik*, Berlin, 1803, 2.Theil, §§9, 158, 159, 261), *Thibaut* (in his *Grundrisse der reinen Mathematik*, 2te Aufl., Göttingen, 1809, S. 277, 387), *Pasquich* (in his *Anfangsgründe der gesammten theoretischen Mathematik*, Vienna, 1812, 2.B., §§306, 861) and many others. Nevertheless *Kästner* had already drawn attention to this clearly false claim (*Anfangsgründe der Geometrie*, 41.Satz, 1.Zus. X).

Others say, '*Two lines* (and equally two surfaces) *come closer to one another in length the more points they have in common with one another.*' *Klügel* says this in his *Mathematisches Wörterbuch*, 1.B., Art. *Bogen*, S. 343, Art. *Complanation*, S. 512; and others.

This, and the previous proposition, can at best only be defended as true if they are understood to be about lines or surfaces which are *concave only on one side*. But even with this restriction it can be seen that they cannot hold as *axioms* but instead they are rather remote *consequences*. For how *composite* is merely the concept referred to as, '*concave on one side*'! How many words are needed for the definition of just this single concept! What seems to have escaped the attention of geometers so far is that all these propositions still do not suffice if it is a matter of calculating the *curves of double curvature*. The axioms of Archimedes, as well as the propositions which have been suggested instead of them, are at best applicable to lines which lie completely in one and the same plane, because it is only to such lines that the concept of *curving on one side*, which is indispensable for their truth, can be applied. But, as is well known, there are also lines of double curvature and these have a length which it must be possible to calculate. I now ask anyone how they intend to modify the above propositions so that they will extend to lines of this sort?

Here it would no doubt have been better, instead of all those earlier propositions, to have set out only the following two:

I. For the *rectification* of lines the proposition: *The relation of the length of an arc curved according to the law of continuity* (whether simple or double) *to its chord comes as close as we please to the relation of equality if the arc itself is taken as small as we please.*

II. For the *complanation* of surfaces the similar proposition: *The relation of the area of a surface curved according to the law of continuity which is enclosed by a plane, to the area of this bounding plane, comes as close as we please to the relation of equality if the surface itself is taken as small as we please.**

* The condition '*according to the law of continuity*' is essential in these two propositions. For if a *cusp* occurs somewhere in the *line* then an arc in which this cusp lies has a ratio to its chord that never comes as close to the ratio 1 : 1 as desired if the arc is taken arbitrarily small. The same holds for a *surface* which is not curved according to the law of continuity but goes somewhere into a *cusp* or *sharpness*; the former, for example, would be the curved surface of a *cone* at its vertex; the latter, the case where the curved surface meets its base.

However, these two propositions could not possibly be valid as *axioms* because they obviously consist of composite concepts, and axioms can only be found among the class of those propositions of which the subject, predicate and copula are absolutely simple concepts. But in order to be able to prove one of the latter, or even one of the above propositions properly (i.e. from its objective ground), the method of calculating lines and surfaces in general must necessarily be presupposed as already known. Therefore exactly the converse is the case: instead of being allowed to use the Archimedean, or similar, propositions for the derivation of the formulae for rectification and complanation one must, in a genuinely scientific exposition, derive the correctness of the former from the latter. So the usual representation of these two theorems of geometry can quite correctly be rejected as being circular.

The problems of the *quadrature of plane surfaces* and the *cubature of solids* have been more fortunate. For the proofs which *Lagrange* and others have offered here are at least not circular. The reason why it was easier with these two problems was this. For every *plane surface*, even if bounded by a *curved line*, two other *rectilinear surfaces* can be specified of which one is a part of the surface bounded by a curved line, and therefore smaller than it, while the other represents a whole of which the one bounded by a curved line is only a part, so that the former is greater than the latter. Thus one devises some law of formation for these two rectilinear surfaces (which is always easy to arrange) by which their areas can be brought as close to one another as we please. (For example, one may take *inscribed* and *circumscribed polygons* whose sides can be made as small as desired.) Then one calculates the quantity which these areas are approaching so that the difference can be made as small as desired, then this quantity is the area of the surface bounded by curved lines that was sought. A similar thing can be done with *solids*. There is, however, no such relationship for curved *lines* or curved *surfaces* which can neither be considered as parts of straight lines or of plane surfaces, nor divided into such things. Here therefore the *method of limits* is not applicable, instead a completely different method must be adopted if one none the less wishes to obtain a quantitative comparison between curved lines or surfaces and straight lines or plane surfaces.

However, if we say that this method of quadrature and cubature *just mentioned* is *better* than that of rectification and complanation, we still cannot really describe them as free from error and *scientific*. For a little thought shows that the truths to be proved are not derived, by the method of limits, in the way they should be in a truly scientific proof—from the concepts of the thesis itself—but only through certain *associated concepts* [*Nebenbegriffen*] *that have been introduced in here quite fortuitously* (*per aliena et remota*). Anybody should realize that those infinitely many regular polygons circumscribed around a circle and inscribed within it are completely alien objects if one wishes to find not their area but that of the circle itself.

So in the end all three problems concerning the measurement of spatial objects are, in a scientific respect, still as good as unsolved.

The present work now offers a method which, I believe, meets the demands of science in the most complete way and is also short and easy. In order to understand it, nothing is presupposed but a knowledge of *Taylor's theorem*.

To give readers some idea of the nature of this method as quickly as possible we intend to explain it here briefly in application to a *particular case*. The following work will present it in its greatest generality and in combination with several other new concepts.

Therefore the *length of a line* is to be calculated which is of *simple curvature* and which lies in the same plane as its orthogonal co-ordinate system. The equation given for this curved line is $y = fx$ and the length to be found, of the piece that belongs to abscissa x, equals Fx. While x increases by Δx this length increases by a quantity $F(x + \Delta x) - Fx$, which by Taylor's theorem is

$$\Delta x \left[\frac{\mathrm{d}Fx}{\mathrm{d}x} + \frac{\Delta x}{2} . \frac{\mathrm{d}^2 Fx}{\mathrm{d}x^2} + \cdots \right].$$

This quantity obviously does not depend on the nature of the piece of curve up to x but only on that piece of arc which is over the piece of abscissa Δx. Now since this piece of arc is determined solely by the ordinates which belong to the abscissae which do not lie outside the limits x and $x + \Delta x$, it follows also that the function $F(x + \Delta x) - Fx$ depends only on the values which fx takes for all values of its argument which do not lie outside x and $x + \Delta x$. Or, what amounts to the same thing, $F(x + \Delta x) - Fx$ is determined merely by the values which $f(x + m\Delta x)$ takes for every conceivable proper fraction m including 0 and 1. Furthermore, if a new abscissae-line is taken which runs parallel with the first so all values of y are increased or decreased by an equally large piece d while x remains unchanged, then again nothing can change in the nature of the function Fx, nor in $F(x + \Delta x) - Fx$, because the same piece of arc still belongs to the same x. This shows that for the determination of the function $F(x + \Delta x) - Fx$ the absolute magnitude of the values which $f(x + m\Delta x)$ takes, for all conceivable proper fractions m including 0 and 1, are not necessary. Merely the statement of the values of the *differences* $f(x + m\Delta x) - d$ is sufficient. If we now consider for different values of Δx that the quantity x, and therefore the quantity y, is constant, then we can take the arbitrary constant $d = y = fx$, and it may be said that the function $F(x + \Delta x) - Fx$ must be determinable merely through all the values which $f(x + m\Delta x) - fx$ takes for all conceivable proper fractions m together with 0 and 1. Finally, if in two or more curves the increase in the abscissa Δx stands in one and the same ratio to that of the ordinate $= f(x + \Delta x) - fx$, i.e. if the quotient $\frac{f(x+\Delta x)-fx}{\Delta x}$ for these lines is equal in magnitude, and if further, the quotients $\frac{f(x+m\Delta x)-fx}{m\Delta x}$, where m is any kind of proper fraction, are of equal magnitude, then the arcs belonging to Δx in these lines must be *similar* to one another. From the theory of similarity it is provable that also the *lengths* of these arcs $= F(x+\Delta x)-Fx$ and Δx have an equal ratio throughout, i.e. also the quotient $\frac{F(x+\Delta x)-Fx}{\Delta x}$ is equal for all these lines. Therefore we finally find that the function

$\frac{F(x+\Delta x)-Fx}{\Delta x}$ is determinable merely through the values which $\frac{f(x+m\Delta x)-fx}{m\Delta x}$ yields for every conceivable proper fraction m together with 0 and 1. For as long as the latter quantities all remain unchanged then also the quantity $\frac{F(x+\Delta x)-Fx}{\Delta x}$ remains the same (by what has just been shown) however the absolute values of $\Delta x, fx, Fx$, etc. change. Since all these assertions hold however small Δx is taken, and since then the value of $\frac{F(x+\Delta x)-Fx}{\Delta x}$ approaches as close as desired to the value $\frac{dFx}{dx}$, and also the quantities expressed by $\frac{f(x+m\Delta x)-fx}{m\Delta x}$ approach as close as desired to the value $\frac{dfx}{dx}$, it is clear that also the quantity which $\frac{F(x+\Delta x)-Fx}{\Delta x}$ changes into for $\Delta x = 0$, i.e. $\frac{dFx}{dx}$, must be determinable from the quantity which the function $\frac{f(x+m\Delta x)-fx}{m\Delta x}$ changes into for $\Delta x = 0$, i.e. $\frac{dfx}{dx}$. Assuming this let $y = \phi x$ denote the equation for some other line and let Φx be its length. Then Fx and Φx designate things of the same kind, namely lengths of lines, and since it is known that the complete nature of a line, therefore also its *length*, is determined by its *equation* then there is, without doubt, some identical [*gleichlautend*] law by which for all lines the functions Fx and Φx can be derived from the functions fx and ϕx. But we have proved that the functions $\frac{dFx}{dx}$ and $\frac{d\Phi x}{dx}$ are determined merely by the values of the functions $\frac{dfx}{dx}$ and $\frac{d\phi x}{dx}$ so that they are completely independent of their intrinsic nature. Now since $\frac{dFx}{dx}$ and $\frac{d\Phi x}{dx}$ are derived from Fx and Φx by the same law, and $\frac{dfx}{dx}$ and $\frac{d\phi x}{dx}$ are derived from fx and ϕx by the same law, it follows that the determination of $\frac{dFx}{dx}$ from the value of $\frac{dfx}{dx}$ and the determination of $\frac{d\Phi x}{dx}$ from the value of $\frac{d\phi x}{dx}$ occurs by the same law. Therefore if for some definite value of x the quantity $\frac{dfx}{dx} = \frac{d\phi x}{dx}$ then the determining pieces of the functions $\frac{dFx}{dx}$ and $\frac{d\Phi x}{dx}$ are completely equal to one another and therefore they themselves certainly are, i.e. $\frac{dFx}{dx}$ must be constructed from $\frac{dfx}{dx}$ just as $\frac{d\Phi x}{dx}$ is constructed from $\frac{d\phi x}{dx}$. Now if we let $y = \phi x$ denote the equation of a *straight* line, then we know how to find Φx, and so also $\frac{d\Phi x}{dx}$, and thereby we also get to know $\frac{dFx}{dx}$. For a straight line the function ϕx is, of course, of the form $\alpha + \beta x$ and then Φx is $= x\sqrt{1+\beta^2}$, therefore $\frac{d\Phi x}{dx} = \sqrt{1+\beta^2}$. But if ϕx is $= \alpha + \beta x$, then $\frac{d\phi x}{dx} = \beta$, therefore $\frac{d\Phi x}{dx} = \sqrt{1+\left(\frac{d\phi x}{dx}\right)^2}$, and therefore also

$$\frac{dFx}{dx} = \sqrt{1+\left(\frac{dfx}{dx}\right)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2},$$

the well-known formula for the length of a line of simple curvature.

With this derivation we have obviously used no concept of the infinitely small, no calculation with zeros, nothing similar to the axioms of Archimedes and generally no assumption of a kind that would not certainly be provable earlier than the theorem which was to be proved. Only with one assumption could anyone take offence: *that the lengths of similar lines are in proportion to the lengths of other lines determined from them in a similar way.* It might be thought that this could only be strictly proved if one could already calculate the lengths of lines or that it could only be proved through the division of a line into infinitely small pieces which are regarded as straight. But this is not so. Instead this proposition is an easy consequence from the concept of similarity as this is shown (with several other consequences) in the work itself §30.

Thus the reader should now have obtained some idea of the method, which makes its appearance here for the first time, and which can be applied not only to the *three geometrical objects* mentioned in the title of this work but also to others. In §10 there appears an example of its application to *mechanics* and by this means two important theorems are proved which could never be properly proved up till now.

No one should fear the small amount of effort which must be made in order to be more thoroughly acquainted with this method and the other new concepts which are put forward in this present work so that they will be in a position to make a thorough judgement on its value or lack of value! Also everyone who is convinced, by appropriate consideration, of the correctness of our views should promote them (according to his means) so as to bring them to general recognition. If the merit of a discovery is lost through the double suspicion that it may only be the desire to say something new and it is perhaps mostly a pure accident, then in contrast the participation in the spread and defence of a discovery made by someone else has a more unambiguous merit. Because it is all the more certain that one is convinced of the correctness of the discovery through an impartial consideration, and that one has taken over its dissemination only out of a love for the truth.

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§ 1

Preamble. It often happens in geometry and mechanics that the magnitude and nature of an *unknown function* $F(x, y, \ldots)$, for all values of its arguments $x, y, \ldots$ however small, are known to depend only on the *values* taken by one or more *other* functions $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ if one puts for $m, n, \ldots$ in them all values not lying outside 0 and 1 (i.e. every conceivable proper fraction including 0 and 1). In order to explain this by an *example*, let Ft be the unknown function which expresses the *distance* which a particle moving in a straight line covers in time t; let ft be another function which expresses the *speed* of the particle at the end of each time t. If t increases by Δt then it is easy to see that the distance described in Δt depends only on the collection of all those speeds which occur inside the time Δt. Now the function $f(t + m\Delta t)$ gives these speeds if one puts for m every conceivable proper fraction together with 0 and 1. On the other hand, $F(t + \Delta t) - Ft$ expresses the distance covered. Therefore it can be asserted that $F(t + \Delta t) - Ft$, or if preferred $\frac{F(t+\Delta t)-Ft}{\Delta t}$, is a function which depends only on the values which the function $f(t + m\Delta t)$ takes if one puts for m in it all values lying not outside 0 and 1. This assertion holds however small Δt becomes, and therefore by imagining that with the same t only Δt changes, one can view $\frac{F(t+\Delta t)-Ft}{\Delta t}$ as a function of $\Delta t = x$, and equally $f(t + m\Delta t)$ as a function merely of $m\Delta t = mx$. From this it is clear that the relationship just mentioned of these two functions to one another is an example that confirms the correctness of our above claim in the simplest case where there is only one of the variable quantities $x, y, \ldots$. Other examples appear subsequently. But to consider this example further, if one wanted to put the increase $\Delta t = 0$, then if the function $\frac{F(t+\Delta t)-Ft}{\Delta t}$ had been expanded by Taylor's theorem as follows:

$$\frac{\mathrm{d}Ft}{\mathrm{d}t} + \frac{\Delta t}{2}\frac{\mathrm{d}^2 Ft}{\mathrm{d}t^2} + \cdots,$$

it would always retain a *real* value, namely it would be $\frac{\mathrm{d}Ft}{\mathrm{d}t}$, and $f(t + m\Delta t)$ would change into ft. But whether the relationship existing before still holds, i.e. whether $\frac{\mathrm{d}Ft}{\mathrm{d}t}$ depends purely on the *value* of ft could not be deduced directly, at least on *mechanical* grounds. For now the grounds from which one would have deduced this previously would no longer be applicable because if Δt is $= 0$ no *motion* occurs inside this Δt, and consequently also no *distance* is described and there is no *speed*, therefore the above arguments cannot now be used at all. Nevertheless, for another reason it is certain that the fact does still hold in this case, it even holds generally, '*that every (continuous) function $F(x, y, \ldots)$ which is determined for all values of its arguments $x, y, \ldots$, however small, merely by the values which certain other functions $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ take if one puts for $m, n, \ldots$ every conceivable proper fraction together with* 0 *and* 1, *is also still determinable from these functions alone when $x, y, \ldots$ have all become zero.*' This truth, which is of crucial importance for the determination of the nature of $F(x, y, \ldots)$ as we shall get to know in more detail, is an easy consequence on purely *analytic* grounds, as shown now in the following paragraphs.

§ 2

Theorem. Suppose that the function fx varies by *the law of continuity* either for all values of x or at least for all those values of x which approach as close as desired to a. Furthermore, suppose it is known about the quantity X that it is derivable from the *values* of fx for the values of x just mentioned in such a way that the *rule* of this derivation (that is, the intrinsic nature of the function of fx which $= X$), may perhaps vary for every different value of fx but always according to a certain *law of continuity so that* the variation of X can be made smaller than any given quantity if the variation in fx is made small enough. Then I claim that this relationship of being determinable [*Bestimmbarkeit*] of X and fx must also hold for $x = a$ itself, i.e. the value which X takes for $x = a$ is independent of the *nature* of the function fx and determinable merely from the *value* which it takes for $x = a$.

Proof. It will be proved that the value of the quantity X at $x = a$ depends not on the intrinsic nature of the function fx but only on its value at $x = a$ if we prove that it remains unchanged, however the nature of the function fx varies, provided its value for $x = a$, i.e. fa, remains the same. For this purpose let $f'x$ be another form of the function fx but such that $f'a = fa$. Let the value of X belonging to $f'x$ be X' and the values which X and X' take for $x = a$ be A and A'. By the given continuity of the functions fx and $f'x$ for all x which are as close to a as desired it may be asserted that the values fx and $f'x$ come as close as desired to the values fa and $f'a$ if x comes as close to a as desired. But by virtue of that kind of continuity which holds for the changes in the quantity X the values at fx and $f'x$, namely X and X', must, under the same condition, come as close as desired to A and A'. Therefore if quantities which can be smaller than any given ones are denoted by $\Omega, \Omega', \ldots$, one has, $X = A + \Omega$ and $X' = A' + \Omega'$. But since $fa = f'a$ the values of fx and $f'x$ come as near one another as desired, and since, as long as x has not become equal to a, the quantity X should depend merely on the values of the functions fx or $f'x$ without respect to their intrinsic nature, then also X and X' must come as close to one another as desired, or $X = X' + \Omega'', \ldots$. Therefore also $A + \Omega = A' + \Omega' + \Omega''$, from which since A and A' can be constant while $\Omega, \Omega', \Omega''$ become as small as desired (by §27 of *Der binomische Lehrsatz*[k]) it must be the case that $A = A'$.

§ 3

Theorem. Suppose the functions $fx, \bar{f}x, \bar{\bar{f}}x, \ldots$, which may form a finite or infinite multitude [*Menge*], vary according to the law of continuity either for all x or for those values of x which are as close as desired to a. Suppose furthermore that

[k] On p. 180.

the quantity X is known to be derivable, for the values of x just mentioned, from the values of the functions fx, $\bar{f}x$, $\bar{\bar{f}}x, \ldots$, in such a way that the *rule* of this derivation varies for each different value of $fx, \bar{f}x, \bar{\bar{f}}x, \ldots$ but always according to a *law of continuity in such a way* that the change in X can be made smaller than every given quantity if the changes in $fx, \bar{f}x, \bar{\bar{f}}x, \ldots$ are made small enough. Then I claim that the relationship of X being determinable from the values of $fx, \bar{f}x, \bar{\bar{f}}x, \ldots$ must also hold for $x = a$, i.e. the value of X is not dependent on the *intrinsic* nature of the functions $fx, \bar{f}x, \bar{\bar{f}}x, \ldots$, but only on the *values* which they take for $x = a$.

Proof. Suppose firstly that the functions $fx, \bar{f}x, \bar{\bar{f}}x, \ldots$ are completely independent of one another and now one of them, e.g. fx, is allowed to vary while all the rest remain unchanged. Then from §2 (which now applies) it is clear that the quantity X is determinable not only for values of x which are as close to a as desired, but also for the value $x = a$, independently of the nature of the function fx and merely from the value which it has in that case. Therefore this also holds for the functions $\bar{f}x, \bar{\bar{f}}x, \ldots$ whether they form a finite or infinite multitude; that is, the quantity X depends, for $x = a$, not on the intrinsic nature of any one of these functions but merely on the values which they take for $x = a$. But the same must also hold in the case when these functions are dependent on one another. For a quantity which can be determined from certain other ones if these are independent of one another, must also be able to be determined from them (indeed perhaps from even fewer) if some of them are already determined from the others.

§ 4

Note. Perhaps someone wonders whether it may not be contradictory to *determine* a quantity (like the X here) through certain other quantities (namely the values of $fx, \bar{f}x, \bar{\bar{f}}x, \ldots$) if it is admitted that this latter *multitude* may itself possibly become *infinite* and consequently *not determinable*. However, some thought will dispel this doubt. A quantity, or generally even some object, does not cease to be *determinate* or *determinable* because there are *infinitely many* quantities, or generally objects, necessary to determine it, provided there is some *rule* given, or even a finite multitude of rules, by means of which they can all be determined. There is nothing impossible in the idea that a rule may determine infinitely many objects. Each line, surface and solid provides an example of this. For it is well known that each of these spatial objects contains infinitely many points and these must all be determined if the object itself is to be said to be determined. Nevertheless no one doubts that such spatial objects can be determined and often the statement of only a few points, and one or more laws from which the others are to be found, is sufficient for their complete determination. Thus, for example, a straight line will be determined by the mere statement of its two end-points and by a single law (see below §15). Of course, for the determination of a fact one rightly requires the complete determination of its ground. Now if this consists of an infinite multitude of things it appears as though it were not determinable. But in fact only the *multitude* of its parts is not determinable, not it itself, provided there is some law that determines each individual from these parts.

§ 5

Theorem. Suppose the functions $fx, \bar{f}x, \bar{\bar{f}}x, \ldots,$ of which there may again be a finite or infinite multitude, vary by the law of continuity either for all x or for those values of x which come as close to zero as we please. Suppose also that the quantity X, for all values of x just mentioned, is known to be determinable from all those values which the functions $f(mx), \bar{f}(mx), \bar{\bar{f}}(mx), \ldots,$ take if one puts for m in them every conceivable proper fraction together with 0 and 1, and suppose moreover, the law of continuity as described in §3 holds. Then I claim that also the value of X belonging to $x = 0$ is determinable merely from those values which the functions $f(mx), \bar{f}(mx), \bar{\bar{f}}(mx), \ldots,$ take for $x = 0$, i.e. from $f(0), \bar{f}(0), \bar{\bar{f}}(0), \ldots.$

Proof. Each of the functions $f(mx), \bar{f}(mx), \bar{\bar{f}}(mx), \ldots$ gives, if one puts for m in it every conceivable proper fraction, an infinite multitude of functions of x which differ, as x changes, only in their constant quantity. And if the functions $fx, \bar{f}x, \bar{\bar{f}}x, \ldots$ vary continuously for all values of x which come as close as we please to zero, then also the functions represented by $f(mx), \bar{f}(mx), \bar{\bar{f}}(mx), \ldots$ in which m designates some proper fraction, 0 or 1, are continuously variable for the same values of x. Therefore the case of the previous paragraph holds here and X is a quantity which, for every x, however small that is not zero, is determined by the values which certain functions (actually infinitely many) take for these values of x. Therefore, etc.

§ 6

Theorem. Suppose the functions $f(x, y, \ldots), \bar{f}(x, y, \ldots), \ldots$ vary according to the law of continuity however small the arguments $x, y, \ldots$ may be taken. Suppose furthermore that the function $F(x, y, \ldots)$ is also known to vary continuously for the values of $x, y, \ldots$ mentioned but always depends merely on the values which the functions $f(mx, ny, \ldots)$, $\bar{f}(mx, ny, \ldots), \ldots$ take, if one lets $m, n, \ldots$ in them denote all conceivable proper fractions together with 0 and 1. Then I claim that also that quantity which the function $F(x, y, \ldots)$ changes into for $x = 0$, $y = 0, \ldots$ i.e. $F(0, 0, \ldots)$, depends only on the values taken by the functions $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ for $x = 0, y = 0, \ldots$ i.e. on $f(0, 0, \ldots), \bar{f}(0, 0, \ldots), \ldots.$

Proof. If we regard only one of the quantitites $x, y, \ldots,$ e.g. x, as variable then $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ appear as functions of x alone, and $F(x, y, \ldots)$ which is then likewise also a function of x alone, is a quantity like the X described in §5, and therefore the value which it takes for $x = 0$ depends only on the values which the functions $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ take for $x = 0$. Now since the same holds also for y, it is clear that if one regards the $x, y, \ldots$ all as variable at the same time, the value of the function $F(x, y, \ldots)$ for $x = 0$, $y = 0, \ldots$ i.e. $F(0, 0, \ldots)$, can depend on nothing but the values which the functions $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ take for $x = 0, y = 0, \ldots$ i.e. on $f(0, 0, \ldots), \bar{f}(0, 0, \ldots), \ldots$.

§ 7

Corollary. Now suppose one knows the following about two functions $F(x, y, \ldots)$ and $\Phi(x, y, \ldots)$. *Firstly*, both are derivable by one and the same rule (although unknown to us) merely from the values taken by certain other functions of the form $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ and $\phi(mx, ny, \ldots), \bar{\phi}(mx, ny, \ldots), \ldots$, if $m, n, \ldots$ are taken in them to denote all conceivable proper fractions together with 0 and 1. *Secondly*, the former, as well as the latter, vary according to the law of continuity however small $x, y, \ldots$ are. *Thirdly* and finally, the values changed into by the functions $f(mx, ny, \ldots)$ and $\phi(mx, ny, \ldots)$; and $\bar{f}(mx, ny, \ldots)$ and $\bar{\phi}(mx, ny, \ldots), \ldots$ for $x = 0, y = 0, \ldots$ are pairwise equal, namely $f(0, 0, \ldots) = \phi(0, 0, \ldots)$, $\bar{f}(0, 0, \ldots) = \bar{\phi}(0, 0, \ldots), \ldots$. Then I claim that the quantities $F(0, 0, \ldots)$ and $\Phi(0, 0, \ldots)$ are not only equal but both are composed in the same way from their respective functions, the former from $f(0, 0, \ldots), \bar{f}(0, 0, \ldots), \ldots$ the latter from $\phi(0, 0, \ldots), \bar{\phi}(0, 0, \ldots), \ldots$. For it is proved in §6 that $F(0, 0, \ldots)$ is derivable from the values, $f(0, 0, \ldots), \bar{f}(0, 0, \ldots), \ldots$, and $\Phi(0, 0, \ldots)$ is derivable from the values $\phi(0, 0, \ldots), \bar{\phi}(0, 0, \ldots), \ldots$. But in consequence of the assumption there is some identical law according to which $F(x, y, \ldots)$ can be derived from the values represented by $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ and $\Phi(x, y, \ldots)$ can be derived from the values of $\phi(mx, ny, \ldots), \bar{\phi}(mx, ny, \ldots)$, so also the law by which $F(0, 0, \ldots)$ is derivable from $f(0, 0, \ldots)$ and $\bar{f}(0, 0, \ldots), \ldots$ and the law by which $\Phi(0, 0, \ldots)$ is derivable from $\phi(0, 0, \ldots)$, $\bar{\phi}(0, 0, \ldots), \ldots$ must be expressible in an identical way. For these latter functions are derived from the former in an identical way, namely by taking $x = 0, y = 0, \ldots$ in both. Moreover since the quantities $f(0, 0, \ldots)$ and $\phi(0, 0, \ldots), \bar{f}(0, 0, \ldots)$ and $\bar{\phi}(0, 0, \ldots)$ etc. are equal in value to one another it may be seen that the quantities $F(0, 0, \ldots)$ and $\Phi(0, 0, \ldots)$ have completely identical determining pieces. Therefore not only must the *values* be equal $F(0, 0, \ldots) = \Phi(0, 0, \ldots)$ but also $F(0, 0, \ldots)$ must be composed from $f(0, 0, \ldots), \bar{f}(0, 0, \ldots), \ldots$ in just the same way as $\Phi(0, 0, \ldots)$ is composed from $\phi(0, 0, \ldots), \bar{\phi}(0, 0, \ldots), \ldots$.

§ 8

Corollary. Suppose therefore that of the functions $F(x, y, \ldots)$ and $\Phi(x, y, \ldots)$ one, e.g. $\Phi(x, y, \ldots)$ is known to us, and we also wish to know $F(0, 0, \ldots)$. Now if also $\phi(mx, ny, \ldots), \bar{\phi}(mx, ny, \ldots), \ldots, f(mx, ny, \ldots)$, $\bar{f}(mx, ny, \ldots), \ldots$ were given, then we could, from comparison of $\Phi(0, 0, \ldots)$ with $\phi(0, 0, \ldots), \bar{\phi}(0, 0, \ldots), \ldots$ get to know the manner in which $\Phi(0, 0, \ldots)$ is composed from $\phi(0, 0, \ldots)$, $\bar{\phi}(0, 0, \ldots), \ldots$ and since $F(0, 0, \ldots)$ is composed in the same way from $f(0, 0, \ldots)$, $\bar{f}(0, 0, \ldots), \ldots$, then also $F(0, 0, \ldots)$ could be found.

§ 9

Problem. To show how the previous propositions can often be used for the determination of an unknown function.

Solution. 1. If in a function $F(x, y, \ldots)$ of one or more variable quantities $x, y, \ldots$ these cannot be taken as small as desired, at least there will be *increases* of $x, y, \ldots$ i.e. Δx, $\Delta y, \ldots$ which can be taken as small as desired. If one now calculates with these, then by considering the $x, y, \ldots$ to be the same and only the Δx, $\Delta y, \ldots$ to vary arbitrarily, the function turns into a function of Δx, $\Delta y, \ldots$. Therefore it will be comparable with the $F(x, y, \ldots)$ of §6 where Δx, $\Delta y, \ldots$ here denote quantities like the $x, y, \ldots$ in §6, which can be taken as small as desired.
2. Now if it is to be possible to determine the unknown nature of the function $F(x, y, \ldots)$ then at least some one or more functions of $x, y, \ldots$ must be given by which it is determined in some way. If it depends merely on the *value* which these functions take for values of $x, y, \ldots$ which are equally great, or which stand in some other relationship, then $F(x, y, \ldots)$ is properly to be considered as a function of these values. The determination of its nature occurs in the usual ways. The second case, which is the only appropriate one here, is if $F(x, y, \ldots)$ is not determined by any *single value* that those functions have, but rather by *all* the values which they take if one puts in for $x, y, \ldots$ all values lying between 0 and $x, y, \ldots$.
3. In this case, by *Taylor's* well-known theorem* we have:

$$\begin{aligned} &F(x + \Delta x, y + \Delta y, \ldots) \\ &\quad = F(x, y, \ldots) + \Delta x \frac{\mathrm{d}^x F(x, y, \ldots)}{\mathrm{d}x} \\ &\qquad + \Delta y \frac{\mathrm{d}^y F(x, y, \ldots)}{\mathrm{d}y} + \cdots + \frac{\Delta x^2}{2} \frac{\mathrm{d}^{xx} F(x, y, \ldots)}{\mathrm{d}x^2} \\ &\qquad + \Delta x \Delta y \frac{\mathrm{d}^{xy} F(x, y, \ldots)}{\mathrm{d}x\,\mathrm{d}y} \\ &\qquad + \cdots + \frac{\Delta y^2}{2} \frac{\mathrm{d}^{yy} F(x, y, \ldots)}{\mathrm{d}y^2} + \cdots \end{aligned}$$

* I denote here, following *Dubourguet*, by $\frac{\mathrm{d}^x F(x,y,\ldots)}{\mathrm{d}x}$, $\frac{\mathrm{d}^y F(x,y,\ldots)}{\mathrm{d}y}$, $\frac{\mathrm{d}^{xy} F(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y}$, the functions derived from $F(x, y, \ldots)$ by the well-known rules of differentiation either only in respect of x or $y, \ldots$ alone, or in respect of x and $y, \ldots$ together and differentiating once or several times as often as divided by $\mathrm{d}x, \mathrm{d}y, \ldots$. It hardly needs mentioning that the derivation of these functions from $F(x, y, \ldots)$ is a purely algebraic procedure which can be achieved without any consideration of the infinitely small or anything similar. But with regard to *Taylor's* theorem mentioned here I do not wish to conceal that I do not allow it quite in the sense and generality in which it is usually presented. Since I cannot develop my ideas on this further at this point I have kept to the rule of using the proposition only under such restriction, and in such a way, as I believe I am able to justify by my own concepts—which I shall do in due course.

Therefore, that part of the increase of $F(x, y, \ldots)$ which contains all the Δx, $\Delta y, \ldots$ as factors is of the following form:

$$\Delta x \Delta y \ldots \left[\frac{\mathrm{d}^{xy} F(x, y, \ldots)}{\mathrm{d}x\, \mathrm{d}y} + \cdots \right] = P$$

where the terms merely indicated after the + sign all contain one or more of the quantities Δx, $\Delta y, \ldots$ as factors.

4. Now if $F(x, y, \ldots)$ depends only on the values taken by certain functions of $x, y, \ldots$ for all values of their arguments lying between 0 and $x, y, \ldots$, then it can always be shown that the quantity P depends only on the values which these functions and, according to circumstances even simpler ones, take if one puts for Δx, $\Delta y, \ldots$ in them all values lying between 0 and Δx, $\Delta y, \ldots$.

5. Again, if one imagines (as in no.1) that for the same $x, y, \ldots$ only Δx, $\Delta y, \ldots$ change, then these functions are really only functions of Δx, $\Delta y, \ldots$. We shall designate them by $f(\Delta x, \Delta y, \ldots)$, $\bar{f}(\Delta x, \Delta y, \ldots), \ldots$.

6. These functions stand in the same relationship to the quantity P as the $f(x, y, \ldots), \bar{f}(x, y, \ldots), \ldots$ of §6 stand to the function $F(x, y, \ldots)$ which occurs there. Namely as $F(x, y, \ldots)$ there depends only on the values which the functions $f(mx, ny, \ldots), \bar{f}(mx, ny, \ldots), \ldots$ take if for $m, n, \ldots$ one puts every conceivable proper fraction, so also P depends only on the values which the functions $f(m\Delta x, n\Delta y, \ldots)$, $\bar{f}(m\Delta x, n\Delta y, \ldots), \ldots$ take if one lets $m, n, \ldots$ denote every conceivable proper fraction.

7. The same therefore also holds of the quantity

$$\frac{P}{\Delta x \Delta y} = \frac{\mathrm{d}^{xy} F(x, y, \ldots)}{\mathrm{d}x\, \mathrm{d}y} + \cdots$$

and hence the value which it changes into for $\Delta x = 0$, $\Delta y = 0, \ldots$ i.e. the function $\frac{\mathrm{d}^{xy} F(x,y,\ldots)}{\mathrm{d}x\, \mathrm{d}y}$ must depend merely on the values of $f(m\Delta x, n\Delta y, \ldots)$, $\bar{f}(m\Delta x, n\Delta y, \ldots), \ldots$ for $\Delta x = 0$, $\Delta y = 0, \ldots$, i.e. on $f(0, 0, \ldots), \bar{f}(0, 0, \ldots), \ldots$.

8. The function $F(x, y, \ldots)$ has a certain meaning; it is to determine the magnitude of a certain object (which depends on the quantities $x, y, \ldots$). Now if we consider the whole class [*Gattung*] of those objects, whose magnitudes $F(x, y, \ldots)$ can be determined in the same way, i.e. by some common rule from certain appropriate functions $f(m\Delta x, n\Delta y, \ldots), \bar{f}(m\Delta x, n\Delta y, \ldots), \ldots$, then perhaps one of them can be found, possibly the simplest, whose magnitude $\Phi(x, y, \ldots)$ is a *known* function of $x, y, \ldots$. With this, one also knows the functions $\phi(m\Delta x, n\Delta y, \ldots)$, $\bar{\phi}(m\Delta x, n\Delta y, \ldots), \ldots$ from which the quantity

$$\frac{\Pi}{\mathrm{d}x\, \mathrm{d}y} = \frac{\mathrm{d}^{xy} \Phi(x, y, \ldots)}{\mathrm{d}x\, \mathrm{d}y} + \cdots$$

belonging to $\Phi(x, y, \ldots)$ is derivable in the same way as $\frac{P}{\mathrm{d}x\, \mathrm{d}y}$ can be derived from $f(m\Delta x, n\Delta y, \ldots), \bar{f}(m\Delta x, n\Delta y, \ldots), \ldots$.

9. If this is found one needs nothing but to determine arbitrarily the constant quantities in $\phi(\Delta x, \Delta y, \ldots), \bar{\phi}(\Delta x, \Delta y, \ldots), \ldots$ to make $\phi(0, 0, \ldots) = f(0, 0, \ldots)$; $\bar{\phi}(0, 0, \ldots) = \bar{f}(0, 0, \ldots)$, etc. Then also in §7 not only is $\frac{d^{xy}\Phi(x,y,\ldots)}{dx\,dy} = \frac{d^{xy}F(x,y,\ldots)}{dx\,dy}$ in value, but the latter function is composed in the same way from the quantities $f(0, 0, \ldots), \bar{f}(0, 0, \ldots), \ldots$ as $\frac{d^{xy}\Phi(x,y,\ldots)}{dx\,dy}$ is formed from the quantities $\phi(0, 0, \ldots), \bar{\phi}(0, 0, \ldots), \ldots$. Now since one knows $\Phi(x, y, \ldots)$ and $\phi(\Delta x, \Delta y, \ldots), \bar{\phi}(\Delta x, \Delta y, \ldots), \ldots$ and therefore also $\frac{d^{xy}\Phi(x,y,\ldots)}{dx\,dy}$ and $\phi(0, 0, \ldots), \bar{\phi}(0, 0, \ldots), \ldots$, then $\frac{d^{xy}F(x,y,\ldots)}{dx\,dy}$ will be found and from this by means of the known rules of integration, $F(x, y, \ldots)$ can itself be derived.

§ 10

Note. I ask the reader to give special attention to the few propositions which I have just put forward. Because they not only constitute the foundation of the whole theory that appears in this work but they are also of the greatest use in other ways. I believe I have discovered in them the right path which leads successfully from the truths of elementary mathematics to the magnificent results of the differential and integral calculus which have hitherto been found to be separated from the former by insurmountable gaps. Because of this it will not be superfluous to add some remarks which may clarify the correct understanding and assessment of this method, and to give an example of its application. If we require in §§7–9 as a condition that the functions $F(x, y, \ldots)$ and $\Phi(x, y, \ldots)$, or rather only P and Π, are derivable *by one and the same rule*, the former from the values of $f(m\Delta x, n\Delta y, \ldots), \bar{f}(m\Delta x, n\Delta y, \ldots), \ldots$, the latter from the values of $\phi(m\Delta x, n\Delta y, \ldots), \bar{\phi}(m\Delta x, n\Delta y, \ldots), \ldots$, then this is to be distinguished from the requirement that *every* rule by which P is derivable from $f(m\Delta x, n\Delta y, \ldots), \bar{f}(m\Delta x, n\Delta y, \ldots), \ldots$ must also hold for the derivation of Π from $\phi(m\Delta x, n\Delta y, \ldots)$, $\bar{\phi}(m\Delta x, n\Delta y, \ldots), \ldots$. No, provided the functions $f(\Delta x, \Delta y, \ldots)$ and $\phi(\Delta x, \Delta y, \ldots)$; $\bar{f}(\Delta x, \Delta y, \ldots)$ and $\bar{\phi}(\Delta x, \Delta y, \ldots), \ldots$ etc. are not pairwise identical, then a rule specially modified for the derivation of $F(x, y, \ldots)$ from the $f(\Delta x, \Delta y, \ldots), \bar{f}(\Delta x, \Delta y, \ldots), \ldots$ can be thought out which is not also applicable to $\Phi(x, y, \ldots)$. But it is enough for our requirement that there is *some one* general rule which is sufficient for the derivation of P from $f(\Delta x, \Delta y, \ldots)$, $\bar{f}(\Delta x, \Delta y, \ldots), \ldots$, as well as for Π from $\phi(\Delta x, \Delta y, \ldots), \bar{\phi}(\Delta x, \Delta y, \ldots), \ldots$. Thus, for example, there is a common rule by which the two functions $\sqrt{1 + 4x^2}$ and $\sqrt{1 + 49x^{12}}$ are derivable from the two functions x^2 and x^7. Namely, it is 'that one first finds the derivative of the given function with respect to x, then square this, add one and take the square root of the sum'. But taken individually there are several ways of deriving each of these two functions from its original function which do not work for the other. For example, $\sqrt{1 + 4x^2}$ results from x^2 by multiplying the given function by 4, adding one and taking the square root of the sum; but $\sqrt{1 + 49x^{12}}$ cannot be derived from x^7 in this way. Moreover, it is not necessary to *know* that rule of derivation of the functions F and Φ from their appropriate

$f, \bar{f}, \ldots$ and $\phi, \bar{\phi}, \ldots$ but it is sufficient to know that *one exists*. The great usefulness of the above propositions is based on this fact. For without knowing the nature of a certain function one can often deduce from other circumstances that it must be derivable from one or more *given* functions $f, \bar{f}, \ldots$ by the same rule by which another known function Φ is derivable from one or more likewise known functions $\phi, \bar{\phi}, \ldots$. Then by §9 F can be determined from $\phi, \bar{\phi}, \ldots, f, \bar{f}, \ldots$ and Φ. If anyone wants to compare this procedure with that by which one calculates the fourth proportional from three given quantities, we have nothing against that idea. If our method is perhaps called a kind of *higher rule of three* which happens incidentally to achieve for higher mathematics what holds everywhere for elementary mathematics, namely that a pair of functions F and Φ are derivable from one or more pairs of others f and $\phi, \bar{f}$ and $\bar{\phi}$, etc. by an *equal* rule, we are justified in accepting this, providing there is some common concept under which we can bring the relationship of dependency of F on $f, \bar{f}, \ldots$ as well as that of Φ on $\phi, \bar{\phi}, \ldots$. For example, it could be that Ft and Φt denote two functions of time t which, for every t, give the distance travelled in each time by a particle moving in a straight line, while ft and ϕt could denote the velocities corresponding to each t. Then it is certain that the nature of the functions Ft and Φt must be derivable from those of ft and ϕt by some identical rule. For there can always be given a special rule by which the distance can be calculated from the velocity if the latter is constant (when $ft = c$) and a particular rule if the velocity increases in equal ratio with the time (when $ft = ct$) and many other rules for other cases. Nevertheless, because $Ft, \Phi t, \ldots$ all have in common that they are expressions for the distance covered and all of $ft, \phi t, \ldots$ are expressions for the velocities achieved, and because the distance is determined by the velocities which have occurred during the motion, there must also necessarily be a general rule by which all the $Ft, \Phi t, \ldots$ are derivable from their corresponding $ft, \phi t, \ldots$. This however is only the *first* condition required for the application of our method to an object. *Secondly*, it is required that the quantities P and Π, derived in the same way from $F(x, y, \ldots)$ and $\Phi(x, y, \ldots)$ depend only on the values which certain functions $f(m\Delta x, n\Delta y, \ldots)$ and $\phi(m\Delta x, n\Delta y, \ldots)$; $\bar{f}(m\Delta x, n\Delta y, \ldots)$ and $\bar{\phi}(m\Delta x, n\Delta y, \ldots)$ etc. take if one puts for $m, n, \ldots$ all conceivable proper fractions together with 0 and 1. The meaning of this requirement is self-evident. But an example where it is satisfied is the one just given. For here P, or that part of the increase in the function $F(x, y, \ldots)$ which contains all $\Delta x, \Delta y, \ldots$ as factors, is the complete increase itself, namely

$$F(t + \Delta t) - Ft = \Delta t \frac{dFt}{dt} + \frac{\Delta t^2}{2} \frac{d^2 Ft}{dt^2} + \cdots ,$$

therefore it is the distance which the particle moves in Δt. Now it has already been noted in §1 that this distance depends only on the velocities which occur during the time of its movement, that is, only on the values taken by the function $f(t + m\Delta t)$ if one puts for m every conceivable proper fraction together with 0 and 1. The same therefore holds of the even simpler function

$$\frac{P}{\Delta t} = \frac{\mathrm{d}ft}{\mathrm{d}t} + \frac{\Delta t}{2}\frac{\mathrm{d}^2 ft}{\mathrm{d}t^2} + \cdots .$$

Since this relationship holds for every value of Δt, however small, it follows from §6 that it also holds for $\Delta t = 0$, i.e. that quantity which

$$\frac{\mathrm{d}ft}{\mathrm{d}t} + \frac{\Delta t}{2}\frac{\mathrm{d}^2 ft}{\mathrm{d}t^2} + \cdots$$

changes into for $\Delta t = 0$, namely $\frac{\mathrm{d}ft}{\mathrm{d}t}$, depends only on the value which $f(t + m\Delta t)$ takes for $\Delta t = 0$, i.e. on ft. But in just the same way $\frac{\mathrm{d}\Phi t}{\mathrm{d}t}$ also depends only on the value of ϕt. Therefore, if for some value of $t, ft = \phi t$ then $\frac{\mathrm{d}Ft}{\mathrm{d}t}$ is composed from ft in the same way as $\frac{\mathrm{d}\Phi t}{\mathrm{d}t}$ is composed from ϕt. Now there is at least one motion for which we know the velocity as well as the distance, i.e. ϕt as well as Φt, namely *uniform* motion in which the velocity $ft =$ some constant quantity c, and the distance described Φt is $=ct$. Therefore if we take c so that $ft = \phi t = c$, then $\frac{\mathrm{d}Ft}{\mathrm{d}t}$ can be derived from ft just as $\frac{\mathrm{d}\Phi t}{\mathrm{d}t}$ can be derived from $\phi t = c$. However, $\frac{\mathrm{d}\Phi t}{\mathrm{d}t} = \frac{\mathrm{d}(ct)}{\mathrm{d}t} = c$. Therefore $\frac{\mathrm{d}Ft}{\mathrm{d}t}$ must also be $= ft$ and consequently $Ft = \int(ft)$ which is the well-known theorem about the relationship of distance to velocity for every kind of motion. If, on the other hand, one wanted to understand by ft and ϕt the *forces* which are acting on the particle in the time t then the *first* condition that Ft and Φt are derivable from ft and ϕt by the same rule would still hold, but the *second* would no longer hold. For the distance P described in Δt is not only dependent on the forces acting on the particle in the interval Δt, but also on those which had acted earlier. But if we now designate by Ft and Φt the velocities, so that we are looking for the relationship in which the *velocity* and *force* stand to one another for any motion in a straight line, then again both conditions hold. For that *increase* in the velocity during Δt, i.e.

$$F(t + \Delta t) - Ft = \Delta t\frac{\mathrm{d}Ft}{\mathrm{d}t} + \frac{\Delta t^2}{2}\frac{\mathrm{d}^2 Ft}{\mathrm{d}t^2} + \cdots$$

certainly depends only on the forces which act during Δt. Our method can therefore be applied here. It may easily be proved that if the force acts uniformly the velocity must increase as the time. If we therefore put the force $\phi t =$ a constant quantity p then the function Φt is well known, i.e. it equals pt. We thus obtain, as before, $\frac{\mathrm{d}\Phi t}{\mathrm{d}t} = p$, and $\frac{\mathrm{d}Ft}{\mathrm{d}t} = ft$, or, $Ft = \int(ft)$ which is the well-known formula. Thus we see here *two main theorems of mechanics* proved without needing any consideration of the *infinitely small* which was still regarded only recently as indispensable! Just as little have we needed to resort to the artificial *axiom* which *Lagrange*, *Pasquich* and others bring in here, '*that of two causes which act through an equal time but of which one exceeds the other in magnitude at each individual moment, an unequal effect must be produced, in fact a greater effect from the one that is always greater.*' This is a proposition which has the greatest similarity with the *second Archimedean axiom* concerning the length of an enclosed line (*Preface*, p. IV),[l] especially in the form which the modern writers have given to it. But the latter can be viewed as an axiom just as little as the former. Its composite nature already shows that it is a *theorem* which must be *proved* and if this is to be done it cannot be otherwise than by assuming those two formulae by means of which one calculates every effect if its cause is a given function of time. These few things are perhaps sufficient to draw attention to the *fruitfulness* of our method. In conclusion therefore we just say a few words about what is *distinctive* in the method. For if a theorem should suffice to prove truths which one is not able to prove without it then it must contain some combination of concepts not attempted before, briefly, something distinctive. Wherein then does this lie with our method? I say it is this, that one derives the nature of a function (namely $\frac{\mathrm{d}^{xy\cdots} F(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$) not from the nature of *one* or *more* quantities, but from the nature of an *infinite multitude of quantities*, namely the values which $f(m\Delta x, n\Delta y, \ldots), \bar{f}(m\Delta x, n\Delta y, \ldots), \ldots$ take if one puts for $m, n, \ldots$ every conceivable proper fraction. But this does not happen here by attempting a calculation of that which is in itself *infinite* and consequently *incalculable*. Neither does it occur by viewing the expression to be found $\frac{\mathrm{d}^{xy\cdots} F(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$ as a *function of all those quantities* which form an infinite multitude. For this would also be an attempted calculation of the infinite because a function of infinitely many quantities would also necessarily have to be viewed as composed from infinitely many parts. Instead the whole situation is transformed merely by the observation that the nature of the function to be found $\frac{\mathrm{d}^{xy\cdots} F(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$, depends only on the values of those quantities which the functions $f(m\Delta x, n\Delta y, \ldots), \bar{f}(m\Delta x, n\Delta y, \ldots), \ldots$ change into for $\Delta x = 0$, $\Delta y = 0, \ldots$ and that therefore if these values are equal to one another for several functions of the same kind, $\frac{\mathrm{d}^{xy\cdots} F(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$, $\frac{\mathrm{d}^{xy\cdots} \Phi(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$, namely, $f(0,0,\ldots) = \phi(0,0,\ldots)$; $\bar{f}(0,0,\ldots) = \bar{\phi}(0,0,\ldots), \ldots$, the *determining pieces* of the functions $\frac{\mathrm{d}^{xy\cdots} F(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$, $\frac{\mathrm{d}^{xy\cdots} \Phi(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$ equal one another, therefore the functions themselves must also be *identical*. Therefore one only needs to know the manner of composition of one of them, e.g. $\frac{\mathrm{d}^{xy\cdots} \Phi(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$ from its corresponding $\phi(0,0,\ldots), \bar{\phi}(0,0,\ldots), \ldots$ in order to deduce the manner of composition of every other $\frac{\mathrm{d}^{xy\cdots} F(x,y,\ldots)}{\mathrm{d}x\,\mathrm{d}y\ldots}$ from its $f(0,0,\ldots), \bar{f}(0,0,\ldots), \ldots$ and thereby the nature of $F(x, y, \ldots)$ itself.

§ 11

Definition. A spatial object,* at every point of which, beginning at a certain distance and for all smaller distances, there is at least *one* and at most only a *finite* set of points as neighbours, is called a *line in general* (Figs. 1–7).

* A *spatial object* is in general every *system* (every collection) *of points* (which may form a finite or infinite multitude).

[l] On p. 281.

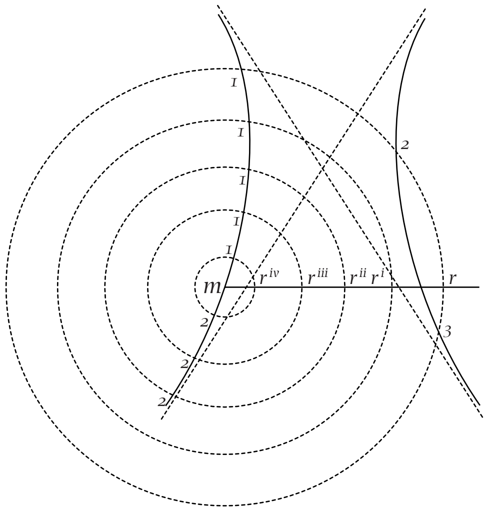

Fig. 1.

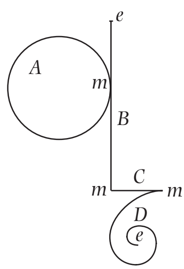

Fig. 2.

2. A spatial object of which every part which can be viewed as a line according to the definition just given, has at least *one point* in common with the remaining part which then likewise must be viewed as a line, is called an *absolutely connected line* (Figs. 2–7).

3. A spatial object of which every point, beginning from a certain distance and for all smaller distances, has at most *two* neighbours is called a *simple* line (Figs. 4, 6, 7).

4. A spatial object of which every point, beginning from a certain distance and for all smaller distances, has an *even number* of neighbours, and thereby no point whose distance from others is greater than a given distance, is called a *self-returning* or *closed line* (Figs. 3, 4, 7).

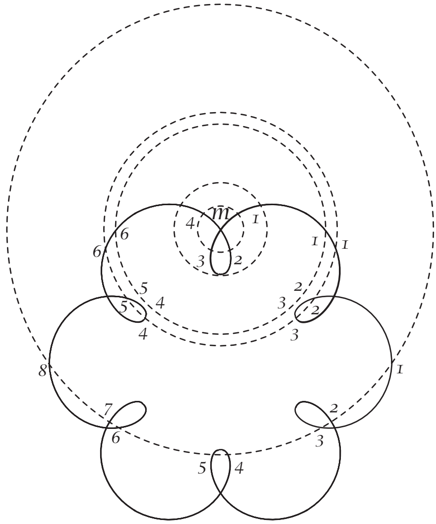

Fig. 3.

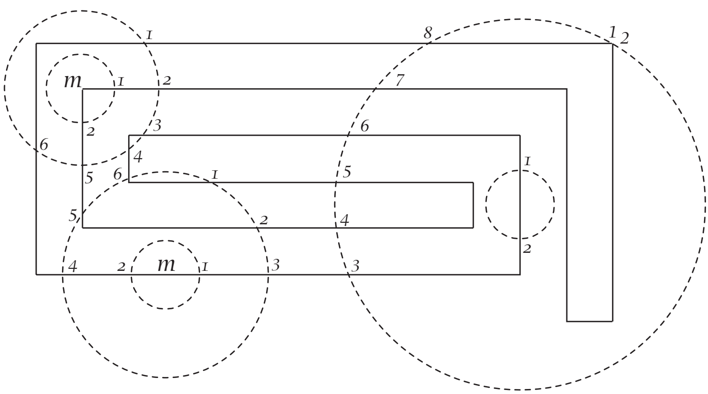

Fig. 4.

5. If this number is everywhere only two then it is a *simple self-returning* line (Figs. 4 and 7).
6. On the other hand, a line in which there are points which, beginning from a certain distance and for all smaller distances, have only *one* neighbour, is called a *bounded* line (Figs. 2, 5, 6).
7. Those points in it (e.g. *e*, *e*) are called *boundary-* (or *end-*) *points*, the others (e.g. *m*, *m*) are *internal* points.
8. If these latter [points] each have only *two* neighbours the line is called a *simple, bounded* line (Fig. 6).

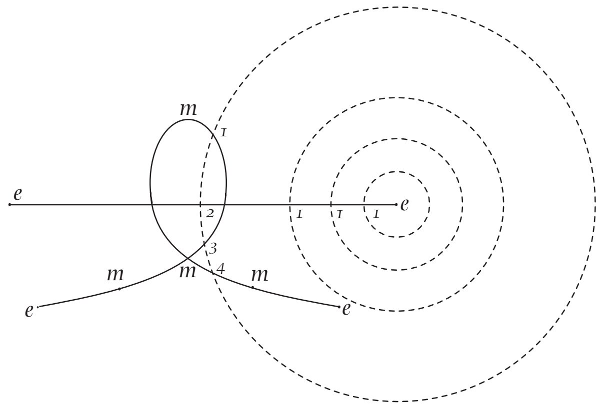

Fig. 5.

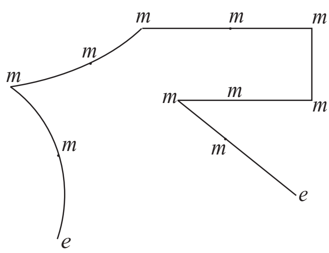

Fig. 6.

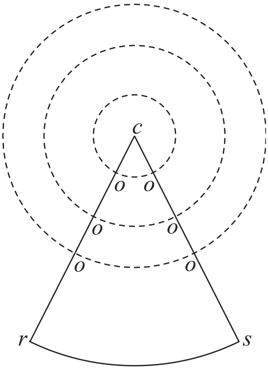

Fig. 7.

§ 12

Note. It is not necessary to have these definitions in mind (nor some other definitions which follow), in order to judge and understand the theory appearing here. Nevertheless it seems useful to me to give them a place here, even if it only gives a small preview of the nature of that complete *reorganization of geometry* at which I have worked for years but have so far communicated very few samples to the public. Having now mentioned these definitions I must add something by way of explanation and justification. Obviously, at each point, e.g. m, of every line (Fig. 1) there is no *next point*, i.e. no point that is so near to it that another could not be said to be even nearer to it. Rather, at every point m a certain distance can be found, e.g. mr, *for which and for all smaller distances*, e.g. mr', mr'', mr''', . . . it can be asserted that there are points in the line which have this distance from m. However, this property of *lines* also belongs to every *surface* and every *solid*, and is therefore as it were, the *higher concept* (*genus proximum*) which comprehends all these three kinds of extension. But what belongs *exclusively* to the line (the *differentia specifica*) is that for every single distance e.g. mr, mr', mr'', . . . only a *finite* set of points are found as neighbours, e.g. for mr three, for mr' one, for mr'' two. For with the *surface* and the *solid* this is different: in the former at each point there is a complete *line* of points and in the latter even a complete *surface* of points which have the same distance from the given point. Of course, in a *line* there can sometimes be single points like c (Fig. 7), which for a certain distance cr have a complete *line segment* [*Linienstück*] rs full of neighbouring points. But even then there will always be a small distance e.g. co, for which, and for all smaller distances, only a finite multitude of neighbours is to be found. Moreover, according to this definition a whole consisting of several disconnected branches, like Fig. 1, bears the name '*a* line' which also agrees with the geometric use of language. But if something is to be called just a *single absolutely connected line* then to the foregoing must be added the characteristic mentioned in the *second* definition above. Now with some thought, and by comparison with the figures, one will hopefully see the meaning of this definition and the following ones, as well as their agreement with the use of language. Someone might find it more objectionable that in consequence of these definitions I allow the line to arise from the mere *composition of points*. I remind such a person that such an idea would only be an error if either, (a) the line was viewed as a *sum* of points in the arithmetic meaning of this word, and it was therefore forgotten that for a line (as for every spatial object) one looks not only at a *multitude* of points but also at the way they are *put together*; or, (b) it were believed that a merely *finite multitude* of points may be sufficient for a line, since in fact every line requires infinitely many; or, (c) it were imagined that of the points of a line each one, I do not know in what way, must *border directly on the next one*, because otherwise there would remain an empty space between two of them. But these are all errors which our definition contradicts instead of sanctioning.* Another objection which is easily anticipated is that all the propositions which I have put

* The individual things of which another thing *consists* or *is composed* are usually called its *parts* or its *components*. According to our definition one may therefore assert that every *line* contains points as its *parts*, and similarly from the definitions that appear later that every *surface* contains points and lines, and every *solid* contains points, lines and surfaces as *components*. That might be a fresh offence to the ear of the geometer, but this is only because he is accustomed to using the word '*part*' in a *narrower*

forward here as *definitions* are basically only *theorems*. Because it is obvious that the terms, *line*, *self-returning line*, *boundary point* and so on, refer, according to their *characteristic* meaning, to completely different concepts from those which I attributed to them in §11. So e.g. the name '*line*' ($\Gamma\varrho\alpha\mu\mu\varepsilon$) undeniably refers to a spatial object that can be described by the motion of a *material point* or *particle*. A *boundary point* in a line is, according to the etymology of the word, a point by which the motion which is necessary for the describing of the line is *bounded*, i.e. from which it must be either *begun* or *ended*. *A self-returning line* is, from the original meaning of this expression, a line which is described by the motion of a particle which returns again to the point of its departure. And so on. I admit all this, but reply that in a *science* it is well known that one has to look not at the *original* meaning of a word but only at that which is meant by it, and *must* be meant if it is to belong to *this* science. Now for reasons which I have already indicated in 1804 in the *Preface* to *Betrachtungen über einige Gegenstände der Elementargeometrie*,[m] it is undeniable that no concepts belong to the *science of space* which include in themselves the concept of *motion*. Therefore, if the words '*line*', '*boundary point*', '*self-returning line*', etc. are to be used in the *science of space* they must not be taken in their *first* and *original* meaning any more than, for example, the words '*intersection*', '*centre of gravity*', etc., but in a *special* meaning from which the concept of motion (as well as all other alien concepts) is removed, or should be removed. The *purely geometric* and the *mechanical* concepts of one and the same word are therefore no less to be distinguished than the *arithmetic* and *geometric* meanings of the words '*square*', '*cube*', etc. These are concepts which usually have the same *range*, but not the same *content* (i.e. not the same *components*); they are indeed *interchangeable concepts*. For example, the *mechanical* concept of a *spatial object that can be described by the motion of a particle* and the *purely geometric* concept which we put forward above (§11, no. 2) under the name of an *absolutely connected line*, are actually two interchangeable concepts. In the *science of space* however, one pays no regard at all to the mechanical meaning of such words, it is only in *mechanics* that it is shown which *mechanical* properties can be predicated of these geometrical objects. And this shows the reason why they were given these names. For example, the concept of *centre of gravity* which several *geometers* have already accepted in their science is defined in geometry so that the concept of *weight*, contained in the original meaning of that word, is completely avoided. But

sense, i.e. the things $a, b, c, \ldots$ which constitute the thing M are only called *parts* of it if one considers in them a property which they possess *in common* with M, i.e. if they are of the *same kind* as M. But is it useful to employ this word in so narrow a sense? Should one not rather distinguish *similar* and *dissimilar* parts and require the condition just mentioned only of the *former*? Just as one makes no objection, for example, to saying *body* and *soul* constitute the two *components* of human beings, so one should not find fault if someone calls *points components* of a *line* in just the same sense of the word. Of course they are only *dissimilar* components of the line and not *similar* ones.

[m] On p. 32.

in *mechanics* it must be shown what mechanical property this geometrical point possesses which explains how it comes to be called a *centre of gravity*. Similarly it is only in *mechanics* that we must prove that that spatial object defined by us under the name of a *line* can actually be described by the motion of a particle, and that that other spatial object, which above we called a *self-returning line*, has the property that a particle describing it finally returns to the same place from which it began.* And so on. Perhaps it would not be superfluous if we give a small example so as to clarify to our readers to some extent how mechanics achieves all this. Let us therefore show how mechanics proves the *theorem* that *every particle, if the cause of its motion does not change, describes a straight line*. From this proposition, in combination with some others, can subsequently be derived the correct theorem that *every particle, however it moves, describes a line in general*. The spatial object that contains all points which the moved particle occupies in time t will be denoted by S, and that which includes all the places [*Orte*] occupied during the interval θ will be denoted by Σ. Then it is certain that S and Σ are things which are completely determined by the given position of the particle at the beginning of the times t and θ, by the time intervals themselves and by the given cause of the motion, i.e. the *speed* of the particle. Now if this speed does not change at any time then I say that the determining pieces of S and Σ are *similar* to one another. For the positions m and n in which the particle is at the beginning of the times t and θ are, like *all points*, similar to one another. The same thing also holds of the times t and θ because all *time intervals* are similar to one another. But also no difference can be alleged in the relationship in which the positions m and n stand to the times t and θ, or to the speeds of the particles, provided the latter remain unchanged. Accordingly, all determining pieces of S and Σ are similar to one another. Now we have proved in the *Betrachtungen* ... etc., I.Abth., §17,[n] that if the determining *pieces of two things* are similar to one another then also the things themselves must be similar to one another. Therefore S and Σ are similar to one another. This similarity holds in whatever relationship the times t and θ stand to one another. So for example the time θ can be part of t, in which case Σ must also be a part of S because the positions which the particle occupies during θ are then included among the positions which it occupies during t. From this it follows that S is a spatial object

* The usual definition of a line, *through the motion of a point* (or particle), could also have been seen to be *objectionable* from the fact that there are generally *several* ways in which one and the same line can be described (with a *simple* line at least *two* but with a *self-returning line, infinitely many* because one can start from each of its points). Now if the concept of a line were nothing other than the concept of that which arises from the motion of a particle then the line which is described by the motion of a particle from a to b would be different, *as a concept*, from that described when the particle goes through just the same points as before only in the reverse order from b to a. Then one really would have *two lines* in mind which completely contradicts the meaning which the geometer connects with the word 'line'. For he certainly recognizes in this case only *one single* line, and hence it clearly follows that the *describing* of the line in the sense of the geometer does not belong to the concept of it.

[n] On p. 40.

of such a nature that every part of it, Σ, is similar to the whole. From which again it follows that it is nothing other than a *straight line*, because as geometers know, there is no other spatial object of which every part is similar to the whole.

§ 13

Definition. A spatial object is called a *determinate* or *determinable* object if all the points of it are either *actually determined* or are *determinable* from a certain number of *given* points by means of a *finite multitude of rules*.

§ 14

Note. It is well known that if only *two* points are given in some way, e.g. *directly* or (as the saying goes) through *intuition*, an infinite set of other points can be determined (namely all those which lie with the former on a *straight line*) by a mere statement of the ratio of their distances from the given points to the distance of these two from one another, therefore by mere *rules* (or by general concepts). Moreover, if *three points* are given of which none is already determined by its relationship to the other two (i.e. three points which do not lie in the same line) then a still greater multitude of points can be determined (namely all which lie in the *plane* of the three given points) by a mere description of the ratio of the relevant distances, therefore by mere concepts. Finally, if *four* points are given, of which none is already determined by its ratio to the other three (i.e. four points which do not lie in the same plane) then *every possible fifth point*, wherever it lies, can be determined by a description of the ratio of its four distances to the distances which the four given points have to one another; therefore by mere *rules*. With these truths in mind, I hope everybody will understand the expression in our above definition, '*to be determinable from a certain number of given points by mere rules.*' The *points* which must be *given* so as to be able to determine all the other points of a spatial object, by what has just been said, need only be very few in number, at most *four*. But the *rules* which must be known in order to determine all the other points from the given points often need to be very numerous. In fact when the object contains an *infinite multitude* of points (as is the case with all lines, surfaces and solids) then if every point requires a *special* rule for its determination one would have to conceive an infinite multitude of rules, i.e. the object would be as good as *indeterminable*. If then it is to be determinable either *all* its points must be determined by one and the same rule, or it must at least be divisible into a finite multitude of parts of which each part can be determined by a single rule. It is assumed here that an infinite multitude of points can be determined by a single rule. This is indeed possible. For example, by the short rule, '*consider all points which have two given distances from two given points a* and *b*', an infinite set of points is determined which form a complete *circle* whose axis is *ab*. Moreover, it is self-evident that the *magnitude* of spatial objects which are not determinable cannot be calculated. They will therefore not be considered subsequently in this work. If an *example* of such a spatial object is required there is the concept applied by many geometers for the supposed purpose of calculating *curves*, i.e. the concept of a '*broken line which is composed of infinitely many, infinitely small, straight lines*'. Such a line is, as an *indeterminable object*, already clearly incapable of having its length calculated, much less could it be used for the calculation of the lengths of other lines.

§ 15

Definition. The spatial object which contains no other points but the two points *a* and *b*, and all the points which lie *between* them, i.e. all those which, from *a*, are in the same direction as *b*, and from *b*, are in the same direction as *a*, is called the *the straight line* [*Gerade*] *contained between the points a and b*.

§ 16

Theorem. The *straight line* is a *line* and in fact a *determinable line*.

Proof. It is easy to show that the characteristics required in §11 belong to this spatial object and consequently it is a *line*. The rule by which all its points can be determined is already stated in §15, therefore this spatial object is also a *determinable line*.

§ 17

Problem. To give the most appropriate method by which every determinable line can be determined from a sufficient number of given points.

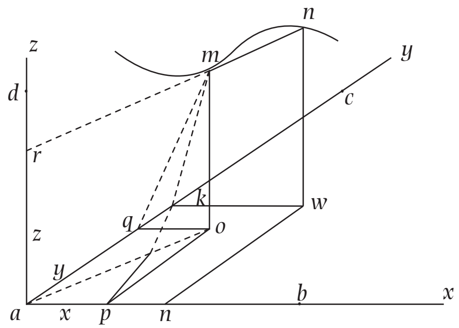

Fig. 8.

Solution. Every point *m* (Fig. 8) is determinable most conveniently from four given points if three of them e.g. *b*, *c*, *d* are so related to the fourth point *a* that the three directions *ab*, *ac*, *ad* are mutually perpendicular. Then the three *distances* from a to the feet of the perpendiculars from the point m to these directions, namely ap, aq $(= po)$, ar $(= om)$, are given. Suppose then that these perpendiculars are drawn to the three directions ab, ac, ad (which are called *axes*) from all the points of the line to be determined (their distances from a are called *abscissae* or *ordinates*). Now if the co-ordinates for m are represented by x, y, z; and for another point n by $x + \Delta x, y + \Delta y, z + \Delta z$, then the symbols Δx, Δy, Δz, cannot *all* denote *zero* otherwise n would be the same point as m. But the distance mn is $= \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = u$, a quantity which, by §11, for every m must be able to take all values less than a definite limit. From this it follows that for every x, y, z, there are Δx, Δy, Δz, as small as desired, i.e. that x, y, z in so far as they vary at all vary by the law of continuity. Now since, by what has just been said, one of these variables, e.g. z, necessarily varies and must take infinitely many values, then at most the following cases can arise: (a) infinitely many different z belong to one y and one x, (b) infinitely many z and y belong to one x, (c) only a finite number of values of z belong to one y and only a finite number of y belong to one x. In fact all three cases are possible with a line, as the mere example of the *straight line* shows. For with a line which is parallel to the z-axis one has the *first* case, with one that is not parallel to the z-axis but is perpendicular to the x-axis, the *second* case, and with yet another position, the *third* case. Finally with a *broken* line which is composed of three such straight lines all three cases hold at once. We must now consider each of these separately:

1. In the first case it is easy to see there cannot be infinitely many pairs of x and y to each of which there are infinitely many z. One of the following two cases must hold: by the infinitely many z which belong to the same x and y is to be understood either *all* z which lie between some definite limits or only *certain* z which have to be determined by a particular rule.

(a) The *first* case would give infinitely many neighbours, for a definite distance, at every point m, because from the infinitely many Δx, Δy, infinitely many corresponding pairs can be produced for each of which there is a Δz which with them makes $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} =$ a definite quantity u.
(b) In the *second case* it depends on whether by the infinitely many x or y is to be understood *all* values contained between definite limits or only *certain* values. The *first* would be the same case as (a) except that instead of z one now has x or y. Therefore there remains the *second* case. If the quantities Δx, Δy, Δz, in fact had infinitely many values but these were not *all* lying between certain limits they can give to the expression $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$, by their various combinations, infinitely many, but not *all* values less than a given quantity as it should be for a line. Therefore there can only be a *finite* set of x and y for every pair of which the quantity z must have infinitely many values, in fact *all* values between definite limits, e.g. a and $a + b$, then between $a + b + c$ and $a + b + c + d$, etc. If the line is to be a *determinate* line we must be *given* these limits. Hence it may be seen that the line which one had in mind in the *first* case is a *straight line* or a collection of *several straight lines* which all lie parallel to the z-axis.

2. In the *second* of the cases considered above, it can be proved, in a similar way as in the first case, that there may again only be a finite multitude of xs to each of which belong infinitely many y and z. Assuming this, and as at every m there can only be a finite multitude of neighbours, for a definite u, therefore only a finite multitude of $y + \Delta y$ and $z + \Delta z$, it follows that there must be such a relationship of dependence between the quantities of $y + \Delta y$ and $z + \Delta z$ that there is only a finite multitude of corresponding Δy and Δz for which $\sqrt{\Delta y^2 + \Delta z^2} = u$, i.e. for every $y + \Delta y$ there is only one value (or there are several values) of $z + \Delta z$ and therefore also for every y only one or several values of z. Therefore for every value of y there must be some *rule* that restricts the corresponding z co-ordinate so that it takes only one or several definite values. But since there are *infinitely many* of the values which y takes it is clear that the line would not be *determinable* if for every value of y one required a *particular* rule for the determination of the corresponding value z. Therefore for several, indeed for *all* values of y lying between given limits the corresponding value of z must be determinable by some single rule. In other words, we must be given a finite multitude of equations of the form $z = fy$, and moreover the limits of y must be stated within which each of them is to be used for determining z. It is also clear that the line which is now in mind lies in one or more *planes* perpendicular to the x-axis.
3. Finally in the *third* case it can be shown in a similar way that the line is only determinable if one or n pairs of equations of the form $y = fx$ and $z = \bar{f}x$ are given, and moreover, it is stated within which limits of x each of these pairs is to be used for determining y and z. The line which one has in mind in this case, is, or at least can be, one of those called *lines of double curvature* (*courbes à double courbure*).

§ 18

Note. It is normally said that every line can be determined by a pair of equations of the form $y = fx$ and $z = \bar{f}\,x$. This is not quite correctly stated. The proposition certainly does not hold for *indeterminable* lines. But even if a line is *determinable*, and determinable through a single pair of equations, one does not always want the *collection of all* points which are determinable through these two equations if every value is taken for x, but only to have some *piece* of the line. This happens, for example, if one thinks of the three sides of a triangle where one obviously does not imagine the three straight lines in all their possible indefinite extension. Therefore so as to be able to assert of such lines that they are determined by a pair of equations one must at least add that the equations are only meant for those values of x which lie between certain limits $x = a$ and $x = a + b$, etc. Furthermore, since there can also be lines whose individual parts proceed according to completely different laws, as in the case with the example just mentioned of the *perimeter of a triangle*, one must add that often more than *one* pair of equations are required. Thus one finally arrives at what was said in the previous paragraph.

§ 19

Definition. The *length* of a line is a quantity which is derivable from the nature of the line, with respect to a given distance E, according to a rule such that if the length for a certain piece of line $= l$ and for another piece $= \lambda$, then for the whole which consists of these two pieces together the length is $= l + \lambda$.

§ 20

Note. Everything *new* is apt to meet opposition just because it is still unfamiliar. So the present definition can hardly expect any better fate. I can already anticipate certain objections which will be brought against it, so I shall mention them and answer them briefly straight away. *Firstly*, it might surprise some people that I distinguish here the concept of *distance* from that of *length* since these are usually regarded as the same. That this distinction is well-founded and that the concept of distance is much *simpler* than that of length can be gathered from the fact that for every system of two points there is a *distance* without needing to consider them connected by a *line*, let alone a *straight line*. *Furthermore*, it might appear that my definition is not based on the *essence* of the concept but only on a *fortuitous* property, because in order to arrive at the concept of the *length* of a line it does not really seem necessary to consider it as a whole consisting of several parts. To this I reply that the concept of length certainly contains that of a *quantity*. But since all quantities have to be either *extensive* or *intensive* surely no one will deny that length is to be counted as a quantity of the first kind. Now quantities are called *extensive* when their parts are all located outside one another. Thus one never thinks of the length of a line without thinking of it as a whole consisting of parts. This therefore is the *first* characteristic (the *genus*) in the concept of length. Now could the *second* characteristic (the *differentia specifica*) be anything but this: that one has to consider *such* a quantity for each part that the quantity of the whole, if it is determined by exactly the same rule, is equal to the sum of the quantities of the individual parts? Of course, it might be *further* objected that such a definition of the concept of length, instead of producing the concept, already assumes it as *known*. Because it must already be known what length is if it is desired to *add* the lengths of two pieces and *compare* them with the whole. If this objection were valid, then for the same reason many other definitions which everyone acknowledges to be correct would have to be rejected. For example, the definition of *difference*, 'that is that quantity which added to the *subtrahendus* produces the *minuendus*'. For it could be said that in order to test, by this definition, whether a certain quantity is a difference one would already have to possess this concept. However, just as this objection would be false here (without already having the concept of difference one can find it with certainty by deductions straight from this definition), so also without already knowing the lengths of the lines *ab*, *bc*, *ac* one can determine these quantities merely from the definition that *length ab* + *length bc* = *length ac*, and knowing that these three quantities are all to be derivable from the nature of the line by *one and the same rule*. Now, of course, all this only proves that the features accepted in our definition of the concept are *characteristic*, i.e. that they belong *exclusively* to the length of a line. But from this it still does not follow that they are the actual *components* of the concept to be defined any more than the concept '*of a figure whose angle sum is* 180°' is the concept of a *triangle* although this characteristic belongs exclusively to the triangle. It is however undeniable that in the concept of *length* there is the concept of *quantity* and indeed that of an *extensive* quantity, i.e. a whole consisting of several parts. Now anyone would find it hard to mention a characteristic which would be simpler than the one which we have given, which could be added (as *differentia specifica*) to the concept of a whole (as *genus proximum*) to produce a concept of equal range with that of length. But logic teaches us that when there are several combinations of different characteristics of a kind that the concept which each produces belongs exclusively to one and the same object, the one which is the simplest of them is suited for the *definition* of the object. If, after all that has been said here (albeit briefly) someone is still not able to convince himself of the correctness of this definition then I remind him that it is enough for our theory if he will allow the present paragraph to hold only as a *theorem*. Now whoever could not do *this*—to be willing to admit that the length of every line is equal to the sum of the lengths of its individual parts?

§ 21

Theorem. Lines which are *geometrically equal* are also of equal length.

Proof. As a consequence of the definition of §19 the length of a line is a certain quantity which is derivable from the nature of the line, with respect to a given distance *E*, according to a law which is the same for all lines. Now if two lines are *geometrically equal*, i.e. are of such a nature that every two equally-positioned [*gleichnahmig*] points in the one have exactly the same distance from one another as in the other, then everything that can be perceived in these lines by comparison of them with a given distance *E* must be identical for both lines. Therefore also the *length*, which is something determined by one and the same law from the sum of these characteristics in both lines, must be found to be the same in the one as in the other.

§ 22

Note. I have here called *geometrically equal* what otherwise used to be called *equal and similar*, or according to an even older use of language, *congruent*. The unsuitability of the latter term is clear from the fact that spatial objects which have both *similarity* as well as *equality* nevertheless do not always have that property which the word '*congruence*' originally indicated and for the designation of which it would be reasonable to retain the word. I mean the property by which spatial objects are capable of becoming *places* [*Orte*] for one and the same material object (understood at different times). Thus, to repeat an example already given by *Kant*, the left glove can be completely equal and similar to the right one and yet will not fit on both hands. Of course, *the theory of congruence* in this meaning of the word has hitherto hardly been worked out at all. What *Robert Simson* (in his *Elementis Euclidis*, Glasgow, 1756) or *Legendre* (in his *Éléments de Géométrie*) provides for it is still very little. But in a complete system of geometry a section must be given to this theory and its results will be sufficiently interesting. At least the word '*congruence*' would thus be allowed the meaning established by ordinary use which it will necessarily have in that section of geometry. As for the expression '*similar and equal*', first of all it is unsatisfactory to designate a single concept by two words combined with 'and'. Then also there must be added (in thought) a restriction to the word '*equal*', i.e. '*merely with respect to quantity*' which is a rather forced ellipsis! As a consequence of the use of language prevailing in ordinary life (which, as far as it goes, should also be respected in the sciences), the word '*equality*' designates the agreement of all characteristics which are perceivable through *comparison* of two things in themselves. Therefore it seems to me necessary to add '*geometrical*' to this word because in geometry one understands it of the coincidence merely of all those characteristics which are perceivable through comparison with a certain *distance*. But the concept of *similarity* I define as *Wolff* and others have already done as the equality of the *intrinsic* characteristics of two things, i.e. those characteristics which can be recognized *in a thing in and for itself* without comparing it with something outside it (as for example a given *distance* would be).

§ 23

Theorem. Straight lines in which the end-points are as far apart from one another in the one as in the other are also of equal length.

Proof. The straight line is well known to be completely determined by its end-points. Therefore those straight lines whose end-points are at equal distances from one another have determining pieces which are geometrically equal. Therefore they themselves are geometrically equal and hence, as a consequence of §21, their lengths must also be equal.

§ 24

Corollary. On the other hand, straight lines of which the end-points in one are not as far apart as in the other are of unequal length. In fact, the one whose end-points are *closer* to one another is of less length, as it is usually said, it is *shorter*. For if the distance *cd* (Fig. 9) is *wider* than the distance *ab*, then inside the line *cd* a point β can be specified whose distance from *c* is equal to the distance *ab*. Consequently the line *cd* can be considered as a whole whose integral parts are the two lines $c\beta$ and βd. Now since each of these must have a certain length, but the length of $c\beta$ by §23 is equal to length *ab* so the length of *cd* must necessarily be greater than that of *ab*.

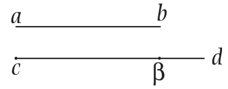

Fig. 9.

§ 25

Note. Distances in themselves possess no magnitude, they are either *equal* or *unequal* and in the latter case, either *wider* or *narrower*. In so far as the determinate distance between two points contains the ground for the possibility of the straight line between those two points, the length of which is a determinate magnitude, one also attributes a *magnitude* to distance itself. But accordingly this is only an *intensive* quantity and it must be expressed by the same numbers by which one expresses the length of the straight line. It can be seen that distance is not an *extensive* quantity from the fact that it does not consist of any *parts*. For it can certainly be asserted of the *line* between *a* and *b* that it is composed of the *two* lines *ac* and *cb*, but it cannot be said of the *distance* between *a* and *b* that it is composed of the *two* distances *ac* and *cb*.

§ 26

Theorem. If the directions *ba* and *bc*, *cb* and *cd*, *dc* and *de*, *ed* and *ef*, etc. (Fig. 10) are, in each pair, *opposite* to each other, and the distances $ab = bc = cd = de = ef$, etc., and finally if the number of points $a, b, c, d, e, f, \ldots = n + 1$, or the number of lines $ab, bc, cd, de, ef, \ldots = n$, then the length of the line $af = nx$, if the length of *ab* is expressed by the quantity x.

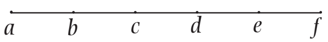

Fig. 10.

Proof. By virtue of the assumption that the directions *ba* and *bc*, *cb* and *cd*, etc. are opposite to one another in pairs, the straight lines ab, bc, $\ldots$ each lie completely outside each other but all in the straight line *af* of which they are integral parts. Therefore as a consequence of the definition of §19,

$$\textit{length } af = \textit{length } ab + \textit{length } bc + \cdots .$$

Because the distances $ab = bc = \cdots$, then the lengths of all these lines are, by §23, equal to one another. But the number of them is n, therefore the length of $af = nx$, if the length of $ab = x$.

§ 27

Corollary 1. If, conversely, the length of the straight line $af = l$ is given, then the length of the piece *ab* can be found since $x = \frac{l}{n}$.

§ 28

Corollary 2. If the integral parts of the line *ah* (Fig. 11) are the following:

1. *m* lines *ab*, *bc*, *cd*, ... which are equal in length to the line $\alpha\beta = x$;
2. then a piece *de* which contains *n* lines of which each is $\frac{1}{r}x$, i.e. *r* of them are as long as $\alpha\beta$ (§27);

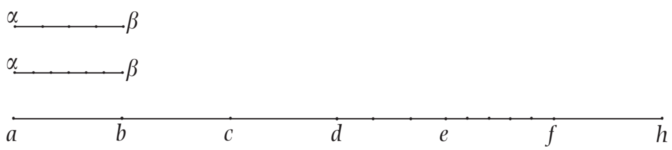

Fig. 11.

3. then a piece *ef* which contains *s* lines each of which is $\frac{1}{t}x$ etc.

then the length of *ah*, $l = \left(m + \frac{n}{r} + \frac{s}{t} + \cdots\right)x$.

§ 29

Note. From the foregoing it may also be seen that the length of a straight line can only be specified by comparison with the length of another line. But it does not matter which line is chosen for the measuring unit of this comparison. One can, for example, choose that straight line whose end-points have the distance *E*. If we let the length of this one straight line be designated by 1, then the lengths of all other lines can be expressed by *definite magnitudes* but only by means of reference to that length adopted as the unit.

§ 30

Theorem. Lengths of lines which are similar to one another are in proportion to the lengths of other lines determined from them in a similar way.

Proof. 1. In this proposition it is first assumed that *there are lines which are similar to one another* and in fact such that differ in their magnitudes. This truth, as generally the whole theory *of the possibility of similar spatial objects* which are geometrically unequal, is based on three simple propositions: that all *points*, all *directions* and all *distances are similar to one another*. These three propositions and several others whose proofs were not known to me in 1804 when I published the *Betrachtungen über einige Gegenstände der Elementargeometrie*, arise very easily, on my present view, from a correct definition of the *concept of space*. Namely, from the definition that *space is the possibility of position* and *position is that relationship among the things of appearance which contains the ground for why they act in this or that temporal relationship to one another*. Without being able to develop this further at present it is enough if it is agreed that the three propositions of the similarity of all *points*, *directions* and *distances* are certainly much *simpler* than those of the possibility of similar *lines*, *surfaces* and *solids*. For it will already follow from this that the former can justifiably be assumed for the proof of the latter. But with their assumption this proof is very easy. For let there exist some one of these spatial objects, e.g. a *line*, and let there also be given some point *A* in it through which there are three mutually perpendicular directions, and also some distance *E*. Then, provided the spatial object is not indeterminable, all the points of it, therefore the whole object, can be determined by merely specifying its relationships to the given five pieces (therefore by mere concepts). Now consider some other point *a* and through it likewise three mutually perpendicular directions together with a certain distance *e* different from *E*. Then because all points, directions and distances are similar to one another, there will be, for the point *a*, the three directions through it and the distance *e*, a spatial object which has exactly the same relations to the pieces mentioned as the previous one had to its pieces. These two spatial objects are therefore similar to one another, but on account of the different distances *E* and *e* they are geometrically unequal.

2. *Secondly*, it follows from the concept of similarity itself that in similar lines the lengths are in proportion to the lengths of other lines derived from them in a similar way. For lines which are derived from other similar ones in a similar way must themselves necessarily be *similar objects* (because their determining pieces are similar to one another). Therefore every intrinsic characteristic of one of these objects must also be found in the other. But among these intrinsic characteristics belong the relationship in which the length of these lines stands to the lengths of those from which they are derived because lengths can always be compared with lengths. Therefore this relationship must be the same in both cases.

§ 31

Note. The foregoing theorem is so necessarily fundamental to the theory of rectification which will be presented here that it would be a crucial defect if it could not be proved earlier and quite independently of it. Nevertheless, the usual proof of this theorem (by dividing the line into infinitely small parts which are then viewed as *straight lines*) assumes a truth only provable from the theory of rectification. (Namely that the ratio of the length of an arc to the length of its chord comes as close to 1 : 1 as desired by taking the arc itself as small as desired.) However, this usual proof is objectionable in any case and the proposition is not provable in any other way than in the way in which we have just done it, when it appears as an easy consequence of the concept of similarity. In this circumstance I may be allowed to recount how one of the greatest living geometers confirms these views of mine in a very striking way. In the often-mentioned *Betrachtungen* etc., and therefore already in 1804, I had put forward *the theory of similarity* as well as several new views in a way which corresponds exactly to the presentation here and I derived from them, among other consequences, *a proper theory of parallels*. This little work was reviewed, not unkindly, in the academic journals and was recommended to the public for further examination. Nevertheless, for very understandable reasons no general attention was aroused and the work was gradually completely forgotten. So it was gratifying for me to find the famous French geometer *Legendre* who, as is well known, had tried in vain to establish a sound theory of parallels in a different way, finally in his *tenth* edition of the *Élémens de Géométrie* of the year 1813 had come to the same views which I had put forward nine years earlier and had even derived quite similar consequences from them. (Compare his Note II, pp. 280–286, with §16 ff. and especially §27 of the *first* part of my *Betrachtungen*.) My small work certainly never came to the notice of the French scholar, so this coincidence of my ideas with his is all the more noteworthy and indeed the ideas of both of us with those of older ones of *Leibniz* and *Wolff* (see the latter's *Elem. Arithm.*, §27, his *Elem. Geom.*, his *Ontol.*, P. I, Sect. III, Cap. I, §201 ff.). If several people come to the same view, without one being caused to do so by any of the others, will it not be very probable that it is correct?

§ 32

Problem. To find the length of every determinable line if the sufficient number of equations between rectangular co-ordinates are given for its determination, as well as the distance E to which the expression for this length is to be referred.

Solution. 1. We shall take the *most general* case straight away and show how to calculate the length of those lines which, as considered in §17, no.3, can be of *double curvature*. Then everyone will know how to apply the method which we use here for the far simpler cases of no.1 and no.2. Also we need only show how to calculate the length of a piece of the line such that for all the points the *same pair* of equations $y = fx$ and $z = \bar{f}x$ holds. For if one knows the lengths of all such pieces then by §19 their sum gives the length of the whole line.
2. Therefore for all values of abscissa x which do not lie outside the limits a and $a + b$ let the pair of equations $y = fx$ and $z = \bar{f}x$ hold.
3. Now to find the length of the piece of line belonging to all these values of x we consider the length l of a *variable part* of it, in fact of that which contains all points of the line from abscissa $x = a$ up to a certain one which we shall not determine more precisely than that it should lie between a and $a + b$, and which we designate simply by x.
4. This length is obviously also a variable quantity, because if x increases, e.g. by the piece Δx, (keeping inside the limits mentioned) a certain part is also added to the line, namely that which contains all points from the abscissa x up to the abscissa $x + \Delta x$. Moreover, by virtue of the definition of §19, l depends only on the nature of the curved line itself, and in fact only on the nature of the part of it lying between the abscissa a and x, and on the distance E. Now since this nature of the line is determined by the two equations $y = fx$ and $z = \bar{f}x$ and the constants E and a, one can also say that l is a function of x which must be determinable from the nature of the functions fx and $\bar{f}x$ and the constants E and a. We shall designate this by Fx.

5. If x increases by Δx, then the increase in l or Fx, namely $F(x + \Delta x) - Fx$ is a quantity which, on geometrical grounds, is dependent on E as well as on the nature of just that piece of the line that belongs to the piece of abscissa Δx, therefore just on the values taken by the co-ordinates $y = f(x + \Delta x)$ and $z = \bar{f}(x + \Delta x)$ for all values of the abscissa which do not lie outside the boundaries x and $x + \Delta x$. Or, what amounts to the same thing, $F(x + \Delta x) - Fx$ depends only on the values which the functions $f(x + m\Delta x)$ and $\bar{f}(x + m\Delta x)$ take if one puts for m every conceivable proper fraction together with 0 and 1. Furthermore, if a *new* abscissae-line is taken which is parallel to the previous one all y and z are increased or decreased by equal amounts d and e while x remains unchanged. Therefore nothing may change in the nature of the function Fx, nor therefore in $F(x + \Delta x) - Fx$, because the same piece of arc still always belongs to the same x. From this it follows that for the determination of the function $F(x + \Delta x) - Fx$ it is not even necessary to have the absolute size of the values which $f(x + m\Delta x)$ and $\bar{f}(x + m\Delta x)$ take if one puts for m every conceivable proper fraction together with 0 and 1. Instead, the mere statement of the values of the *differences* $f(x + m\Delta x) - d$ and $\bar{f}(x + m\Delta x) - e$, suffices. If we now regard the quantity x, and therefore also y and z, as constant for different values of Δx, then it is permitted to put the arbitrary constants d and e equal to the values fx and $\bar{f}x$. Hence it follows that the function $F(x + \Delta x) - Fx$ is determined merely from all the values which the functions $f(x + m\Delta x) - fx$ and $\bar{f}(x + m\Delta x) - \bar{f}x$ take if one puts for m every conceivable proper fraction together with 0 and 1. Finally, consider two or more curves in which the increase in abscissa Δx stands in exactly the same ratio to the increase in the ordinates $= f(x + \Delta x) - fx$ and $\bar{f}(x + \Delta x) - \bar{f}x$, i.e. for which the quotients $\frac{f(x+\Delta x)-fx}{\Delta x}$ and $\frac{\bar{f}(x+\Delta x)-\bar{f}x}{\Delta x}$ have the same magnitude and further, that all quotients representable by $\frac{f(x+m\Delta x)-fx}{m\Delta x}$ and $\frac{\bar{f}(x+m\Delta x)-\bar{f}x}{m\Delta x}$ as m is every kind of proper fraction, are always of equal magnitude, then the pieces of arc belonging to Δx are *similar* to each other, and it follows from §30 that the *lengths* of these pieces of arc $= F(x + \Delta x) - Fx$, have the same ratio to the length of Δx which is similarly situated in each case, i.e. that the quotient $\frac{F(x+\Delta x)-Fx}{\Delta x}$ for all these lines is of equal size. Finally, we gather from this that the function $\frac{F(x+\Delta x)-Fx}{\Delta x}$ is determinable merely through the values which the functions $\frac{f(x+m\Delta x)-fx}{m\Delta x}$ and $\frac{\bar{f}(x+m\Delta x)-\bar{f}x}{m\Delta x}$ take, if one puts for m every conceivable proper fraction together with 0 and 1. As long as the latter quantities all remain unchanged, however the absolute values of Δx, fx, Fx, etc. vary, then so also (by what has just been said) the quantity $\frac{F(x+\Delta x)-Fx}{\Delta x}$ will remain unchanged.

6. Now let $\eta = \phi x$ and $\zeta = \bar{\phi}x$ denote the equations for another line for which the other quantities are also a and E. Let its length $= \Phi x$. Therefore Fx and Φx denote things of the same kind, namely lengths of lines, and since according to no.4 Fx and Φx must be determinable for the same a and E, merely from the nature of the functions fx, $\bar{f}x$ and ϕx, $\bar{\phi}x$, then without doubt there is some *general law* by which for *all* lines the functions Fx and Φx are derivable from the nature of fx, $\bar{f}x$

and ϕx, $\bar{\phi}x$. But according to no.5 the functions $\frac{F(x+\Delta x)-Fx}{\Delta x}$ and $\frac{\Phi(x+\Delta x)-\Phi x}{\Delta x}$ will be determined by the values which the functions

$$\frac{f(x+m\Delta x)-fx}{m\Delta x}, \frac{\bar{f}(x+m\Delta x)-\bar{f}x}{m\Delta x}$$

$$\text{and} \quad \frac{\phi(x+m\Delta x)-\phi x}{m\Delta x}, \frac{\bar{\phi}(x+m\Delta x)-\bar{\phi}x}{m\Delta x}$$

take if one puts for m every conceivable proper fraction, together with 0 and 1. Therefore this determination proceeds according to some law that is identical for all lines. For these functions just mentioned are themselves derived from the previous ones (i.e. the former from Fx, Φx and the latter from fx, ϕx, $\bar{f}x$, $\bar{\phi}x$) by a law which is identical for both cases.

7. Now since, as is clear from §17, the functions fx and $\bar{f}x$, ϕx and $\bar{\phi}x$ must be continuous, then the values of $\frac{f(x+m\Delta x)-fx}{m\Delta x}$ and $\frac{\bar{f}(x+m\Delta x)-\bar{f}x}{m\Delta x}$ can come as close as desired to the values $\frac{\mathrm{d}fx}{\mathrm{d}x}$ and $\frac{\mathrm{d}\bar{f}x}{\mathrm{d}x}$, and the values of $\frac{\phi(x+m\Delta x)-\phi x}{m\Delta x}$ and $\frac{\bar{\phi}(x+m\Delta x)-\bar{\phi}x}{m\Delta x}$ can come as close as desired to the values $\frac{\mathrm{d}\phi x}{\mathrm{d}x}$ and $\frac{\mathrm{d}\bar{\phi}x}{\mathrm{d}x}$ if Δx is taken small enough. Finally, since at least for certain lines, e.g. for *straight* lines, the function Φx is also continuous and therefore $\frac{\Phi(x+\Delta x)-\Phi x}{\Delta x}$ can come as close to the value of $\frac{\mathrm{d}\Phi x}{\mathrm{d}x}$ as desired, it follows that the same must also be the case with the function Fx. Therefore now all conditions of §9 are present. So if for some x one has the equations $\frac{\mathrm{d}fx}{\mathrm{d}x} = \frac{\mathrm{d}\phi x}{\mathrm{d}x}$ and $\frac{\mathrm{d}\bar{f}x}{\mathrm{d}x} = \frac{\mathrm{d}\bar{\phi}x}{\mathrm{d}x}$, then the functions $\frac{\mathrm{d}Fx}{\mathrm{d}x}$ and $\frac{\mathrm{d}\Phi x}{\mathrm{d}x}$ must both be composed in the same way, the former from the values of $\frac{\mathrm{d}fx}{\mathrm{d}x}$ and $\frac{\mathrm{d}\bar{f}x}{\mathrm{d}x}$, the latter from the values of $\frac{\mathrm{d}\phi x}{\mathrm{d}x}$ and $\frac{\mathrm{d}\bar{\phi}x}{\mathrm{d}x}$.

8. Now suppose ϕx and $\bar{\phi}x$ denote the equations for a *straight line* and are therefore of the form $\alpha+\beta x$ and $\gamma+\delta x$. For such a line Φx is also known to us and in fact $=\sqrt{(1+\beta^2+\delta^2)}(x-a)$. Therefore $\frac{\mathrm{d}\phi x}{\mathrm{d}x}=\beta$, $\frac{\mathrm{d}\bar{\phi}x}{\mathrm{d}x}=\delta$, and

$$\frac{\mathrm{d}\Phi x}{\mathrm{d}x} = \sqrt{(1+\beta^2+\delta^2)} = \sqrt{1+\left(\frac{\mathrm{d}\phi x}{\mathrm{d}x}\right)^2+\left(\frac{\mathrm{d}\bar{\phi}x}{\mathrm{d}x}\right)^2}.$$

9. Hence also the way in which $\frac{\mathrm{d}Fx}{\mathrm{d}x}$ must be composed from $\frac{\mathrm{d}fx}{\mathrm{d}x}$ and $\frac{\mathrm{d}\bar{f}x}{\mathrm{d}x}$ is made clear. Namely, it must be

$$\frac{\mathrm{d}Fx}{\mathrm{d}x} = \sqrt{1+\left(\frac{\mathrm{d}fx}{\mathrm{d}x}\right)^2+\left(\frac{\mathrm{d}\bar{f}x}{\mathrm{d}x}\right)^2}$$

from which, according to the known rules of integral calculus, the function Fx itself can be found, provided the constants to be added are determined by reference to the conditions of no. 3.

§ 33

Note. It may be seen that this solution is based on no other assumptions than the following:

1. that the length of every piece of a line depends on the nature merely of this one piece and on the given distance E by which the line is measured, and in no way on the nature of the bounding parts;
2. that there must be some *rule identical* for *all* lines, according to which their length can be derived from the nature of those functions which determine the nature of the line;
3. that this rule is certainly of a kind that if the length of a certain piece has been found by it and $= l$, and that of another piece $= \lambda$, the length of the whole must be found to be $= l + \lambda$;
4. that the lengths of *similar lines* are in proportion to the lengths of other lines determined from them in a similar way; and finally
5. that the method of §9 is correct.

The first three assumptions are much too obvious to be objectionable to anyone. Also the *fourth* assumption will be admitted by anyone who has carefully considered the content of §§30 and 31. Thus it all depends on the method of §9. So whoever is suitably convinced of the correctness of this method will, I hope, not withhold his approval of the *theory of rectification* just presented.

§ 34

Note 2. If through the point m of a line, which belongs to abscissa x a (straight) *tangent* is drawn to the line, then as a consequence of the theory of lines of contact the two equations

$$\eta = m + \frac{\mathrm{d}y}{\mathrm{d}x}\xi \quad \text{and} \quad \zeta = n + \frac{\mathrm{d}z}{\mathrm{d}x}\xi$$

hold for this straight line, where ξ, η, ζ are its three co-ordinates from the same origin and in the same directions as x, y, z; $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}x}$ are two constant coefficients for a fixed value of x, and m and n are also certain constant quantities; namely

$$m = y - \frac{\mathrm{d}y}{\mathrm{d}x}x \quad \text{and} \quad n = z - \frac{\mathrm{d}z}{\mathrm{d}x}x.$$

If we compare these expressions with those of §32, no.8, $\eta = \alpha + \beta x$, $\zeta = \gamma + \delta x$, where it is found $\beta = \frac{\mathrm{d}y}{\mathrm{d}x}$, $\delta = \frac{\mathrm{d}z}{\mathrm{d}x}$, then we discover that the straight line, whose length helps us for the calculation of the length of every other line is *parallel* with the tangent through m if it is taken on the same co-ordinates. Indeed this line, if α and γ are determined suitably (and this determination is arbitrary), goes through the point m itself and therefore is *identical* with this tangent. This fact seems to be the true cause of why the *tangent* and the so-called *characteristic triangle* have been used with such good effect for the calculation of curved lines. From the two equations for the tangent

$$\eta = m + \frac{\mathrm{d}y}{\mathrm{d}x}\xi \quad \text{and} \quad \zeta = n + \frac{\mathrm{d}z}{\mathrm{d}x}\xi$$

it follows that it approaches the curved line in the points *lying immediately around m* more closely than every other conceivable straight line through m. From this it used to be concluded that also the *length* of the tangent, for these smallest values of Δx, comes closer than that of every other straight line to the length of the curved line. Now since the increase in length of the tangent for $\xi + \Delta x$ is

$$= \Delta x \left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]^{1/2},$$

while the increase in length of the curved line, by Taylor's formula is $F(x + \Delta x) - Fx = \Delta x \frac{\mathrm{d}Fx}{\mathrm{d}x} + \cdots$, then it was inferred that it must be that

$$\frac{\mathrm{d}Fx}{\mathrm{d}x} = \left[1 + \left(\frac{dy}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]^{1/2}.$$

The conclusion was correct but the proper basis on which it rests is only given in §32. For merely from the fact that a certain line comes closer than another one in all its *points* to a given line, it does not follow in any way that its *length* comes closer to the length of the given one.

§ 35

Definition. 1. A spatial object at each point of which, beginning from a certain distance and for all smaller distances, there is at least *one* and at most only a *finite* set of *separate lines* full of points is called a *surface in general* (Figs. 13–19).[o]
2. A spatial object, of which every part which can be viewed by the previous definition as a surface has at least *one line* in common with the other part, which must likewise be viewed as a surface, is called a single absolutely *connected surface* (Figs. 14–19).
3. A spatial object every point of which, starting from a certain distance and for all smaller distances, has only *one simple* line full of points near to it is called a *simple surface* (Figs. 16, 18 and 19).
4. A spatial object at every point of which, starting from a certain distance and for all smaller distances, has completely *self-returning* lines as neighbours and thereby no points whose distance from others is greater than a given one is called a *self-returning* or *closed surface* (Figs. 15 and 16).
5. If these lines are *simple* then it is a *simple closed surface* (Fig. 16).
6. On the other hand, a surface in which there are points which, starting from a certain distance and for all smaller distances has only *bounded* lines as its neighbours is called a *bounded surface* (Figs. 17–19).

[o] Bolzano's Fig. 12 is out of order and appears on p. 329.

7. Those points themselves are called *boundary points* of the surface (e.g. *e*, *e*, . . .). The others are *inner points* (e.g. *m*, *m*, . . .).
8. If these latter each have, from a certain distance downwards, only *one simple* line [of points] as neighbours then it is a *simple bounded surface* (Figs. 18 and 19).

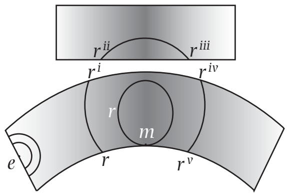

Fig. 13.

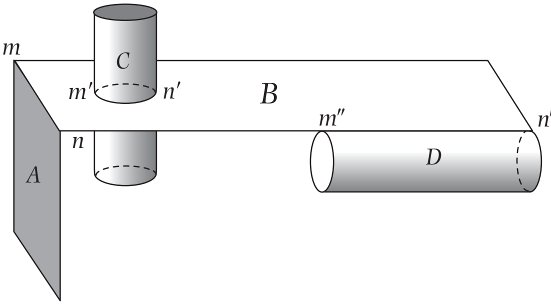

Fig. 14.

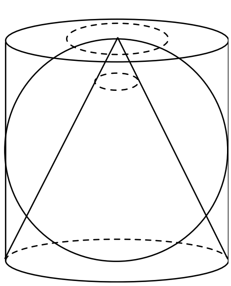

Fig. 15.

§ 36

Note. Whoever has understood the definitions of §11 should find hardly any difficulty in the present ones. At each of the points of a surface, e.g. *m* (Fig. 12),

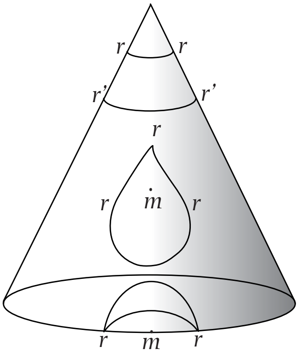

Fig. 16.

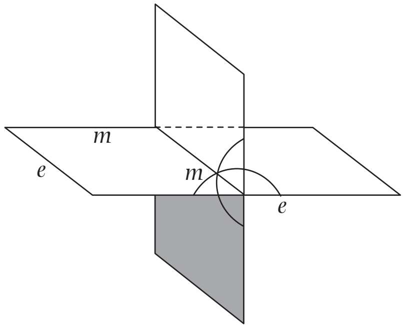

Fig. 17.

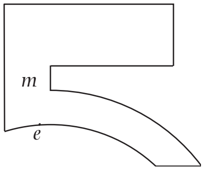

Fig. 18.

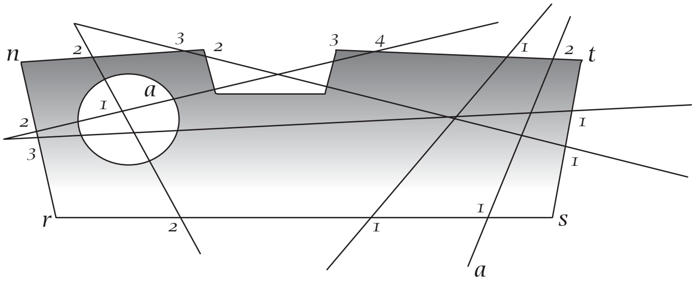

Fig. 19.

there is of course no *next one*. But if one considers all the points which have given a distance, e.g. *mr*, from *m*, then there is an infinite set of them so that in fact taken together they form one or even perhaps several complete *lines* e.g. rr', $r''r'''$, $r^{IV}r^{V}$, ... In fact in many surfaces (e.g. in a hemi-sphere enclosed by a plane) with certain of their points (here with the centre) and for a certain distance (that of the radius), there can even be a whole *surface* of points. But then a smaller distance can always be taken, of which it holds that starting from this one and for all smaller distances there are only one or several separate lines as neighbours. The other definitions of this paragraph can be elucidated further by the diagrams adjoined. Fig. 13 is a surface consisting of two disconnected pieces. In Fig. 14 *A* and *B* can be considered a pair of planes and *C* and *D* a pair of cylinders. The lines *mn*, $m'n'$, $m''n''$, are then that which combines these different surfaces into a single surface. Fig. 15 represents the three surfaces of a cylinder, a cone and a sphere. And so on.

§ 37

Definition. The *plane of the points a, b, c* is that spatial object which contains all those points (and no others) which can be determined by their *relationship* (i.e. by their distances) to the three points *a*, *b*, *c*, which do not lie on a straight line.

§ 38

Theorem. The plane is a surface.

Proof. It is easy to prove that every point of the spatial object described in §37 has, for every distance, a whole line of points (in fact a circle) as neighbours (§35).

§ 39

Definition. A *point* is said *to be enclosed by a line on a surface* if it lies in the surface so that it is impossible to imagine a connected line between it and a second point whose distance from it is as large as desired so that the line remains, throughout its whole course, in the given surface (or in another of which the former is a part) without having a single point in common with the given line (Fig. 19).

§ 40

Note. It can be shown that only a *self-returning* line can enclose a point on a surface, and also that a point *i* is enclosed if—with at most a finite number of exceptions—every plane containing it has a line of intersection with the surface in which both lie of such a kind that in the two pieces into which it is divided by *i* there are an *odd number* of points of intersection of it with the given line. If on the other hand this number is *even* or *zero* as is the case with the point *a* then it is not enclosed but lies *outside*.

§ 41

Definition. By the *surface figure nrstn* (Fig. 19) we understand the spatial object which contains, together with the *self-returning simple* line *nrstn* also all those points which are enclosed by it, where they are thought of as lying in a certain surface. The enclosing line *nrstn* is called the *boundary* of this surface figure.

§ 42

Note. It can be proved that the spatial object described in §41 is a *surface*, and likewise that all points of this surface and consequently the surface itself, are *determined* as soon as the surface in which the enclosing line lies and the line itself are given.

§ 43

Problem. To specify the most appropriate method by which every determinable surface can be determined from a sufficient number of given points.

Solution. Considerations, which are similar in all respects to those we put forward in §17, show here that to one of the three coordinates x, y, z there must belong infinitely many values of the two others or at least of one other. For otherwise there would not be, at every point m (Fig. 13), a complete line full of neighbours as the definition of §35 requires, because it is necessary for this that in the equation $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = u$, for the same u, infinitely many values if not of all, or least of two of the three quantities Δx, Δy, Δz, are possible. Therefore at most two cases can arise here, either (a) for one x there are infinitely many y and for one y infinitely many z, or (b) for one x there are infinitely many y but for one y only one or only a finite multitude of z. The example of the *plane surface* shows that these two cases are possible: the *first* case occurs if it is perpendicular to the *axis* of x, and the *second* case if it intersects the axes of x, y, z obliquely.

1. In the *first* case it can be shown in a similar way to §17, no.1 that the set of x may only be *finite*. Assuming this it is clear that for the same y there must be not only *certain* values of z but *all* values contained between specific limits. Of course, if one thinks of the point m as an *inner* point (§35), then it must have self-returning lines as neighbours for all values of u lying below a certain limit e. Now it is provable from the proposition of §40 that this is impossible if there are not also points of this line in the direction of the ordinate z. But these points are always at a distance Δz from m therefore $\Delta z = u$ must be appropriate to *all* values below e. For every y therefore the ordinate z has not only *infinitely* many values but *all* values contained between certain limits. But as may easily be conceived if the surface is to be *determinate* the limits mentioned must not only be finite in number but also given. We shall designate this boundary value of z belonging to each y by z', then for every y there are infinitely many z' and it is clear, as in §17, no.2, that for the determination of them one or more equations of the form

$z' = \psi y$ must be given and at the same time it must also be shown within which values of y each of them holds. The equation $z' = \psi y$ must, as is obvious, give an *even* number of values of z' and if one orders these from the greatest positive to the greatest negative (or conversely) then the first and second, the third and fourth, in general the $(2n + 1)$th and $(2n + 2)$th, are two *associated* limits inside which values of z are to be taken. On the other hand, inside the second and third, the fourth and fifth, in general the $2n$th and $(2n + 1)$th there should be no values of z. Moreover the surface which one has in mind in this *first* case is a mere plane surface and its equations which are of the form $z' = \psi y$ are at the same time the equations for its boundary lines.

2. In the *second* case, for every x there are to be infinitely many y but for every y only one or more z. Here it can first be shown in a similar way to §17, no.1, that x as well as y must possess not only infinitely many values but *all* values contained between specific limits. Thus for every value of the one, e.g. x, the infinitely many values of the other y must be contained inside certain *given* limits if the surface is to be determinable. If we designate this *boundary value* of y belonging to each x by y', then one or more equations of the form $y' = \psi x$ must be given, and it must be stated at the same time for what values of x each of them holds. Then, because for each specific x and y there are only one or more values of z there must be given one or more equations of the form $z = f(x, y)$ and it must be stated at the same time for what values of x and y each of them is applicable.

§ 44

Note. It is usually just said that every surface can be determined by an equation of the form $z = f(x, y)$. But from what has just been shown it may be seen that this is even less correctly stated than the statement to which we objected in §18. If one tries to determine the surface, e.g. of some *triangle*, that has an oblique position with respect to the three axes, then the insufficiency of the usual method will be seen and the necessity of our additions to it, especially the equation $y' = \psi x$.

§ 45

Definition. The *area* [*Inhalt*] *of a surface* is a quantity which is derivable from the nature of the surface, by means of reference to a given distance E, by a rule such that if the area for a certain piece of surface is represented by s, and for another piece by σ, then for the piece that contains both together it must be represented by $s + \sigma$.

§ 46

Theorem. Surfaces which are geometrically equal are also of equal area.

Proof. This proposition is very similar to that of §21 and is proved in the same way.

§ 47

Theorem. If the area of a parallelogram which has sides of length a and b is expressed by the quantity N, then the area of another parallelogram, with the same angles, which has sides of length x and y must be expressed by the quantity $N\frac{xy}{ab}$.

Proof. The quantity X, by which the area of the second parallelogram is to be expressed must double if one of the quantities x or y itself doubles. For it is easily proved that the area of a parallelogram one of whose sides$=2x$, the other$= y$, can be considered as a whole whose integral parts are two parallelograms with the same angles which have the sides x and y. But these are geometrically equal so by §46 also have equal areas. Therefore by the definition of §45 the area of the whole $= X + X = 2X$. Hence it now follows that the areas of equiangular parallelograms in which only the side x changes are in proportion to the length of this side. Since in this case the area always merely depends on the length of this side, one can say that it is a function of x, and in fact a function Fx such that $2Fx = F2x$. But analysis shows that such a function can only be of the form Ax. Now if both sides of the parallelogram vary its area must be a homogeneous function of x and y and therefore of the form Cxy. But if $x = a$ and $y = b$ then Cxy must be $= N$. Therefore $C = \frac{N}{ab}$ and the required area $=N\frac{xy}{ab}$.

§ 48

Note. It will be seen from these few examples how, using our concepts, we happen to reach exactly the same results as are obtained on the usual view—in particular how we come to the theorem that the area of a triangle whose three sides are x, y, z

$$= \frac{\sqrt{(x+y+z)(x+y-z)(x+z-y)(y+z-x)}}{4}.$$

We will refer to this subsequently.

§ 49

Theorem. Areas of surfaces which are similar to one another are in proportion to the areas of other surfaces derived from them in a similar way.

Proof. This proposition corresponds exactly with §30 and is proved in the same way.

§ 50

Problem. To find the area of any determinable surface if the sufficient number of equations between rectangular co-ordinates are given for its determination, together with the distance E to which the area is referred.

Solution. 1. It will be sufficient to show the calculation of surfaces like those considered in §43, no.2 because it is easy to apply the method occurring here to the simpler case of no.1 which contains only the *plane* surfaces. Also we need only show how to calculate the area of those pieces of surface for which the one equation $z = f(x, y)$ holds, and for which the determination of the boundary values of y at each x the same auxiliary equation $y' = \psi x$ holds. For if one knows the area of each such piece then by the definition of §45 the sum of all such pieces gives the area of the whole surface.
2. Accordingly, let the equation $z = f(x, y)$ hold for all values of x which are not outside the limits a and $a + b$, and for all y which are not outside the boundary values determined by the equation $y' = \psi x$.
3. Now in order to find the area of the piece of surface belonging to all these values of x, y, z we consider firstly the area s of a variable *part*, in fact of that which is obtained if one takes the following values for x and y: (a) for x, all from a to one which we do not determine any more closely than that it is $<a + b$ and which we call simply x; (b) for each of these x all y which, (α) are greater than the smaller of every pair of boundary values which the equation $y' = \psi x$ determines, and (β) smaller than a certain value which we call simply y and is only determined in that it is to be smaller than the greater boundary value belonging to the last x which the equation $y' = \psi x$ gives.
4. This area s is also obviously a variable quantity in that if x increases by Δx and y by Δy then a certain part is also added to that piece of surface, namely that which contains all points with abscissae from x to $x + \Delta x$ and from y to $y + \Delta y$. s therefore depends upon x and y, and on the nature of the surface itself in these regions, therefore on the equations $z = f(x, y)$, $y' = \psi x$ and the constant quantity a, as well as the distance E. We can therefore say s is a function of x and y which must be determinable from the nature of the functions ψx and $f(x, y)$ and from the constants a and E. We shall designate it by $F(x, y)$.

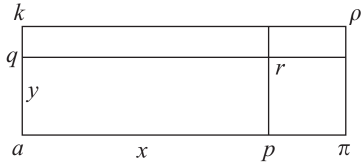

Fig. 12.

5. Let the abscissa $x = ap$ (Fig. 12) increase by $\Delta x = p\pi$ and the abscissa $y = aq$ increase by $\Delta y = qk$, then the total increase in the surface $= F(x + \Delta x, y + \Delta y) - F(x, y)$ is a piece of surface which consists of three parts which can be distinguished as follows. The points of one part are all those vertically above the rectangle πr, those of the other above the rectangle $r\rho$ and finally those of the third vertically above the rectangle rk. The middle piece, namely that which is above $r\rho$ contains merely those points which belong to the abscissa from x to $x + \Delta x$ and

from y to $y + \Delta y$. From this it follows that its area, which we shall designate by P, depends only on those values which the function $z = f(x + m\Delta x, y + n\Delta y)$ takes if one puts for m, n in it every conceivable proper fraction together with 0 and 1. Furthermore, suppose a new co-ordinate system is adopted with x- and y-axes parallel to the previous ones so that all z co-ordinates are reduced by an *equal* quantity d while the x and y co-ordinates remain unaltered. Then it is quite understandable that nothing alters in the function $F(x, y)$ nor, therefore, in the quantity P. From this it follows, as in §32, no.5, that P really depends only on the values which $f(x + m\Delta x, y + n\Delta y) - d$ takes, or if we put $d = f(x, y)$, on the values which $f(x + m\Delta x, y + n\Delta y) - f(x, y)$ takes, if m and n are put equal to every conceivable proper fraction together with 0 and 1. Finally if one considers two or more surfaces of such a nature that the quotient

$$\frac{f(x + m\Delta x, y + n\Delta y) - f(x, y)}{m\Delta x}$$

as well as

$$\frac{f(x + m\Delta x, y + n\Delta y) - f(x, y)}{n\Delta y}$$

is always of equal magnitude for the same m and n, however different the absolute sizes of Δx, Δy, etc. are in these surfaces, then it is known that the surfaces are *similar* to each other and it therefore follows from §49 that also the area of this surface, $= P$, always has the same ratio to the area of a surface determined from it in a similar way, for example $r\rho = \Delta x.\Delta y$, i.e. that $\frac{P}{\Delta x.\Delta y}$ is of equal magnitude in all these surfaces. Now this shows us that the function $\frac{P}{\Delta x\, \Delta y}$ depends only on the values which the two functions

$$\frac{f(x + m\Delta x, y + n\Delta y) - f(x, y)}{m\Delta x}$$

and

$$\frac{f(x + m\Delta x, y + n\Delta y) - f(x, y)}{n\Delta y}$$

take if one puts for m, n every conceivable proper fraction together with 0 and 1.
6. Now let $\zeta = \phi(x, y)$ denote the equation for *another* surface for which the auxiliary equation $y' = \psi x$ and the quantities a and E are the same as before. Let its area be $= \Phi(x, y)$, and let the quantity which is derived from $\Phi(x, y)$ in just the way that P was derived from $F(x, y)$ be represented by Π. Therefore $F(x, y)$ and $\Phi(x, y)$ designate things of the same kind, namely areas of surfaces, and since by 4. $F(x, y)$ and $\Phi(x, y)$ must be determinable for the same $\psi x, a$ and E merely from the nature of the functions $f(x, y)$ and $\phi(x, y)$ then there is certainly some *general rule* according to which for every kind of surface the functions $F(x, y)$, $\Phi(x, y)$ can be derived from the functions $f(x, y)$, $\phi(x, y)$. But by

5. the quantities $\frac{P}{\Delta x\,\Delta y}$, $\frac{\Pi}{\Delta x\,\Delta y}$ will be determined only by the values which

$$\frac{f(x+m\Delta x, y+n\Delta y)-f(x,y)}{m\Delta x}, \frac{f(x+m\Delta x, y+n\Delta y)-f(x,y)}{n\Delta y}$$

and

$$\frac{\phi(x+m\Delta x, y+n\Delta y)-\phi(x,y)}{m\Delta x}, \frac{\phi(x+m\Delta x, y+n\Delta y)-\phi(x,y)}{n\Delta y}$$

take if one puts for m and n every conceivable proper fraction together with 0 and 1. Therefore this determination is made from some rule which holds equally for all surfaces, because the quantities just mentioned are derived from the previous ones, namely from $F(x,y)$ and $\Phi(x,y)$, $f(x,y)$ and $\phi(x,y)$ themselves according to a rule which holds equally for both cases.

7. One can deduce from §43 that the functions $f(x,y)$, $\phi(x,y)$ must be continuous and consequently the values

$$\frac{f(x+m\Delta x, y+n\Delta y)-f(x,y)}{m\Delta x}$$

and

$$\frac{f(x+m\Delta x, y+n\Delta y)-f(x,y)}{n\Delta y}$$

approach the values

$$\frac{\mathrm{d}^x f(x,y)}{\mathrm{d}x}+\frac{n\Delta y}{m\Delta x}\frac{\mathrm{d}^y f(x,y)}{\mathrm{d}y} \quad \text{and} \quad \frac{\mathrm{d}^y f(x,y)}{\mathrm{d}y}+\frac{m\Delta x}{n\Delta y}\frac{\mathrm{d}^x f(x,y)}{\mathrm{d}x};$$

and the values

$$\frac{\phi(x+m\Delta x, y+n\Delta y)-\phi(x,y)}{m\Delta x}$$

and

$$\frac{\phi(x+m\Delta x, y+n\Delta y)-\phi(x,y)}{n\Delta y}$$

approach the values

$$\frac{\mathrm{d}^x \phi(x,y)}{\mathrm{d}x}+\frac{n\Delta y}{m\Delta x}\frac{\mathrm{d}^y \phi(x,y)}{\mathrm{d}y} \quad \text{and} \quad \frac{\mathrm{d}^y \phi(x,y)}{\mathrm{d}y}+\frac{m\Delta x}{n\Delta y}\frac{\mathrm{d}^x \phi(x,y)}{\mathrm{d}x}$$

as closely as desired if the quantities Δx, Δy are taken small enough. Finally, at least for certain surfaces, e.g. for plane surfaces, the function $\Phi(x,y)$ is also continuous and therefore by Taylor's well-known theorem:

$$\begin{aligned}\Phi(x+\Delta x, y+\Delta y) - \Phi(x,y) = \Delta x\frac{\mathrm{d}^x\Phi(x,y)}{\mathrm{d}x} + \Delta y\frac{\mathrm{d}^y\Phi(x,y)}{\mathrm{d}y} \\ + \frac{\Delta x^2}{2}\frac{\mathrm{d}^{xx}\Phi(x,y)}{\mathrm{d}x^2} \\ + \frac{\Delta y^2}{2}\frac{\mathrm{d}^{yy}\Phi(x,y)}{\mathrm{d}y^2} \\ + \Delta x\,\Delta y\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y} + \cdots .\end{aligned}$$

By the remark of no.5 this is the area of the three-fold piece of surface that is over the rectangles $\pi r, r\rho, rk$. If one puts $\Delta y = 0$ then of this piece of surface there remains only the part over πr. But in the formula just given, when $\Delta y = 0$ there remain only those terms which contain no Δy, i.e. only these,

$$\Delta x\frac{\mathrm{d}^x\Phi(x,y)}{\mathrm{d}x} + \frac{\Delta x^2}{2}\frac{\mathrm{d}^{xx}\Phi(x,y)}{\mathrm{d}x^2} + \cdots$$

which give the area of the piece of surface over πr. In the same way it can be proved that the terms containing no Δx:

$$\Delta y\frac{\mathrm{d}^y\Phi(x,y)}{\mathrm{d}y} + \frac{\Delta y^2}{2}\frac{\mathrm{d}^{yy}\Phi(x,y)}{\mathrm{d}y^2} + \cdots$$

express the area of the piece over rk. Hence it evidently follows that the remaining terms, namely those in which Δx and Δy are multiplied together give the area of the piece over $r\rho$, therefore that

$$\Pi = \Delta x\,\Delta y\left[\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y} + \cdots\right]$$

and therefore

$$\frac{\Pi}{\Delta x\,\Delta y} = \frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y} + \cdots .$$

Therefore $\frac{\Pi}{\Delta x\,\Delta y}$ is a function which approaches the value $\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y}$ as closely as desired if $\Delta x, \Delta y$ are taken small enough. It follows that the same relationship must also hold between the functions $\frac{P}{\Delta x\,\Delta y}$ and $\frac{\mathrm{d}^{xy}F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$ and hence all the conditions for the application of §9 are present. Consequently if one has, for some x and y, the equations $\frac{\mathrm{d}^x f(x,y)}{\mathrm{d}x} = \frac{\mathrm{d}^x\phi(x,y)}{\mathrm{d}x}$ and $\frac{\mathrm{d}^y f(x,y)}{\mathrm{d}y} = \frac{\mathrm{d}^y\phi(x,y)}{\mathrm{d}y}$ then the determining pieces of the functions $\frac{\mathrm{d}^{xy}F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$ and $\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y}$ into which $\frac{P}{\Delta x\,\Delta y}$ and $\frac{\Pi}{\Delta x\,\Delta y}$ go for $\Delta x = 0, \Delta y = 0$, namely the quantities

$$\frac{\mathrm{d}^x f(x,y)}{\mathrm{d}x} + \frac{n\Delta y}{m\Delta x}\frac{\mathrm{d}^y f(x,y)}{\mathrm{d}y} \quad \text{and} \quad \frac{\mathrm{d}^x\phi(x,y)}{\mathrm{d}x} + \frac{n\Delta y}{m\Delta x}\frac{\mathrm{d}^y\phi(x,y)}{\mathrm{d}y},$$

and

$$\frac{\mathrm{d}^y f(x,y)}{\mathrm{d}y} + \frac{m\Delta x}{n\Delta y}\frac{\mathrm{d}^x f(x,y)}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}^y \phi(x,y)}{\mathrm{d}y} + \frac{m\Delta x}{n\Delta y}\frac{\mathrm{d}^x \phi(x,y)}{\mathrm{d}x},$$

are obviously equal to one another. Therefore $\frac{\mathrm{d}^{xy} F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$ is composed from the quantities $\frac{\mathrm{d}^x f(x,y)}{\mathrm{d}x}$ and $\frac{\mathrm{d}^y f(x,y)}{\mathrm{d}y}$ in exactly the same way as $\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y}$ is composed from the quantities $\frac{\mathrm{d}^x \phi(x,y)}{\mathrm{d}x}$ and $\frac{\mathrm{d}^y \phi(x,y)}{\mathrm{d}y}$.

8. Now let $\zeta = \phi(x,y)$ denote the equation for a *plane surface* and therefore let it be of the form $\zeta = \alpha + \beta x + \gamma y$. For such a surface we know how to calculate $\Phi(x,y)$. But without entering into this calculation, and to derive first (by differentiation) $\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y}$, which is all we need, we can calculate the latter directly if we consider that

$$\Pi = \Delta x\,\Delta y\left[\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y} + \cdots\right]$$

is the area of that piece of the plane that is over $r\rho$. This piece of area is clearly a parallelogram* whose four vertices are at the four ordinates, $z, z + \beta.\Delta x$, $z + \gamma.\Delta y, z + \beta.\Delta x + \gamma.\Delta y$. From this circumstance the sides and a diagonal of it can easily be calculated and hence finally its area which will be found to be $= \Delta x\,\Delta y[1 + \beta^2 + \gamma^2]^{1/2}$. Therefore

$$\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y} = [1 + \beta^2 + \gamma^2]^{1/2}.$$

But from $\phi(x,y) = \alpha + \beta x + \gamma y$ it follows that $\frac{\mathrm{d}^x \phi(x,y)}{\mathrm{d}x} = \beta$ and $\frac{\mathrm{d}^y \phi(x,y)}{\mathrm{d}y} = \gamma$. Therefore

$$\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y} = \left[1 + \left(\frac{\mathrm{d}^x \phi(x,y)}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}^y \phi(x,y)}{\mathrm{d}y}\right)^2\right]^{1/2}.$$

Consequently, we must also have

$$\frac{\mathrm{d}^{xy}F(x,y)}{\mathrm{d}x\,\mathrm{d}y} = \left[1 + \left(\frac{\mathrm{d}^x f(x,y)}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}^y f(x,y)}{\mathrm{d}y}\right)^2\right]^{1/2}.$$

Whence $F(x,y)$ can be found through integration, if one suitably determines the constant quantities by means of the auxiliary equation $y' = \psi x$ and the given a and E.

* Due to an oversight *Dubourguet* in the work quoted before (*Preface*, p. IX[p]) describes this parallelogram (§534) as a *rectangle*.

[p] On p. 284.

§ 51

Note. The plane surface, the consideration of which has helped us here for the calculation of the area of every other surface is, if one takes its co-ordinates on the same axes as the latter, a plane of contact through the point m which belongs to x, y, z: either parallel to it or identical with it. This circumstance gives rise to similar remarks to those in §34.

§ 52

Definitions. 1. A spatial object at each point of which, starting from a certain distance and for all smaller distances, there exists at least *one* absolutely connected surface full of points, is called a *solid in general* (Figs. 20 and 21).
2. A spatial object every part of which can be viewed as a solid according to the previous definition and which has at least *one surface* in common with the other part which can be viewed as a solid, is called a *single absolutely connected solid* (Fig. 21).
3. A solid in which there are points which, starting from a certain distance and for all smaller distances, have only a *bounded surface* full of points for their neighbours is called a *bounded solid* (Fig. 21).
4. Those points themselves e.g. $e, e, \ldots$ are the *boundary points* of the solid; the others, e.g. m, n, (which therefore have a *closed* surface, namely a *sphere* around them) are called *inner points*.

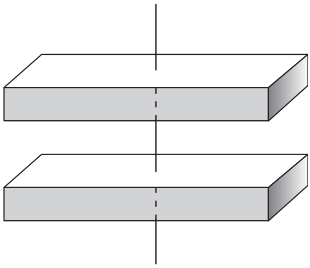

Fig. 20.

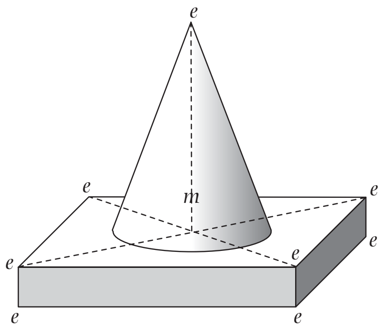

Fig. 21.

§ 53

Problem. To specify the most appropriate method of determining every determinable solid from a sufficient number of given points.

Solution. Considerations like those in §17 and §43, show that for every solid there must be infinitely many x, and at each of them (with at most only a finite set of exceptions) infinitely many y, and at each of these (with the same restriction) infinitely many z. This is because the equation $(\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2} = u$ for the same x, y, z and u must allow so many values of Δx, Δy, Δz that all the points belonging to $x + \Delta x$, $y + \Delta y$, $z + \Delta z$ form a complete surface. Furthermore, x must not only generally have infinitely many values but *all* which are between certain limits, and similarly y for each value of x, and z for each value of y. This is clear in exactly the same way as the similar assertion in §§17 and 43. Finally, if the solid is to be determined then all these (infinitely many) limits must be determined. Therefore firstly there must be given a *finite* set of *limits* within which all the x are contained, e.g. a and $a + b$, then $a + b + c$ and $a + b + c + d$ etc., then a finite set of *equations* of the form $y' = \psi x$ which determine the boundary values of all y belonging to each x. It must also be indicated within which values of x each of these equations is to be applied. Finally, a finite set of equations of the form $z' = f(x, y)$ which establish the boundary values of all z belonging to every x and y, with which again it must be noted within which values of x and y each of them holds.

§ 54

Note. Thus the equations which are required for the determination of a *solid* are exactly of the same form as those for a *surface* and this can easily be understood. It is also just as easy to show that the points which belong to the *boundary values* of z, i.e. to the ordinate $z' = f(x, y)$ are *boundary points* of the solid which also, together, form a *surface*, in fact a *closed* surface. Hence it follows that every determinable solid must be bounded by a self-returning surface and is determined by the specification of this surface.

§ 55

Definition. A *prism* is a spatial object which contains all the following points and no others:

1. those which lie in two geometrically equal plane surfaces A and B whose sides are parallel to one another and in the spatial object C that joins all equally-positioned boundary points of A and B by *straight* lines;

2. All the points which are *enclosed* by the spatial object consisting of the union of A, B, C.

§ 56

Theorem. Every prism is a solid and indeed, if A (or B) is determinable, a determinable solid.

Proof. It can be shown,

1. that the spatial object C is a surface;
2. that the surface consisting of the three surfaces A, B, C is a *self-returning* surface which therefore *encloses** points and in fact the following points:
3. points of a *solid*, i.e. points of a kind which have the property described in §52;
4. that also the points of the surfaces A, B, C are points of this solid and are in fact,
5. *boundary points*.

Hence it follows by §54 that if A (or B), consequently also C, is determinable then also this whole solid is determinable.

§ 57

Definition. The *volume of a solid* is a quantity which is derivable from the nature of the solid, by means of reference to a given distance E, by a rule such that if the volume for a certain solid $= s$ and for another $= \sigma$, then for the solid which contains the two as integral parts it will be found to be $= s + \sigma$.

§ 58

Theorem. Solids which are geometrically equal are also of equal volume.

Proof. As §21.

§ 59

Theorem. The volumes of equiangular parallelepipeds are in joint proportion to the lengths of their sides.

Proof. In a similar way to §47.

§ 60

Theorem. The volumes of similar solids are in proportion to the volumes of other solids derived from them in a similar way.

Proof. As that of the similar proposition §30.

* A point is said to be *enclosed* by a surface if every straight line passing through the point has, in both directions, an odd number of points in common with the surface.

§ 61

Problem. To find the volume of every determinable solid if a sufficient number of equations between rectangular co-ordinates are given together with the distance E to which this volume is to be referred.

Solution. 1. We need only show how to calculate the volume of a piece of solid for which the same equation $z' = f(x, y)$ holds for the determination of the boundary values of all z and the same auxiliary equation $y' = \psi x$ holds for the determination of the boundary values of y belonging to each x. For if one knows the volume of each such piece alone then the sum of all of them, by §57, gives the volume of the whole solid.
2. Accordingly let the equation $z' = f(x, y)$ hold for the boundary values of z and hold for all values of x which do not lie outside the limits a and $a + b$, and for all values of y which do not lie outside the boundary values of y determined by the equation $y' = \psi x$.
3. In order to find the volume of the piece of solid belonging to all these values of x, y, z, we firstly consider the volume s of a variable *part* of it. In fact of that part which one obtains if one takes for x, y, z the following values:

(a) for x, all from a to one which we determine no more closely than that it is to be $<a + b$ and which we call simply x;
(b) for each of these x, all y which: (α) are *greater* than the smaller of the two corresponding boundary values which the equation $y' = \psi x$ gives, and (β) are *smaller* than a certain value which we call simply y and only determine in that it is to be smaller than the greater boundary value belonging to the last x which the equation $y' = \psi x$ gives;
(c) finally, for each of these x and y, all z which are not outside the pairs of boundary values determined by the equation $z' = f(x, y)$, i.e. all z which for the x and y mentioned are possible in consequence of the equation $z' = f(x, y)$.

4. Obviously this volume is a variable quantity. If, for example, x increases by Δx, and y increases by Δy then also a part is added to the former piece of solid, namely that which contains all points with abscissae from x to $x + \Delta x$ and from y to $y + \Delta y$. s therefore depends on x and y but then also on the nature of the solid itself in these regions, i.e. on the equations $y' = \psi x$ and $z' = f(x, y)$ and on the constant quantity a as well as on the distance E. One can therefore say s is a function of x and y which must be determinable from the nature of the functions $\psi x, f(x, y)$ and the constants a and E. We shall designate it by $F(x, y)$.
5. If the abscissa x (Fig. 19) $= ap$ increases by $\Delta x = p\pi$, and the abscissa $y = aq$ increases by $\Delta y = qk$ then the total increase in the solid $= F(x + \Delta x, y + \Delta y) - F(x, y)$; this represents a piece of solid which consists of three parts which are distinguished in that the points of one of them are vertically above the rectangle πr, those of the other vertically above that of $r\rho$ and finally those of the third vertically above the rectangle rk. The middle piece, namely that which is over $r\rho$, contains only those points which belong to the abscissa from x to $x+\Delta x$ and from y

to $y+\Delta y$ whence it follows that its volume which we shall designate by P, depends only on those values which the function $z' = f(x+m\Delta x, y+n\Delta y)$ takes if one puts for m, n every conceivable proper fraction together with 0 and 1. On the other hand, the mere *increase* of z' or $f(x+m\Delta x, y+n\Delta y) - f(x,y)$ would obviously be insufficient for the determination of this piece of solid because the points which belong to each $x+m\Delta x$ and $y+n\Delta y$ are not simply determined by the magnitude of the difference $\Delta z'$ but by the different values which z' has for one and the same $x+m\Delta x$ and $y+n\Delta y$, and of course only the function $f(x+m\Delta x, y+n\Delta y)$ itself can give us these. But if the function P is determined merely from the values which $f(x+m\Delta x, y+n\Delta y)$ takes if one puts for m, n every conceivable proper fraction together with 0 and 1, then certainly the function $\frac{P}{\Delta x\,\Delta y}$ is also determined by Δx, Δy and the values just mentioned of $f(x+m\Delta x, y+n\Delta y)$.

6. Now let $\zeta' = \phi(x,y)$ be the equation of *another* kind of solid for which the auxiliary equation $y' = \psi x$ and the quantities a and E are the same as before. Let the volume of this solid be $= \Phi(x,y)$ and a quantity which is derived from $\Phi(x,y)$ in the same way as P was derived from $F(x,y)$ is to be represented by Π. Therefore $F(x,y)$ and $\Phi(x,y)$ designate things of the same kind, namely volumes of solids, and since by no.4 with the same $\psi x, a$ and E the functions $F(x,y)$ and $\Phi(x,y)$ depend only on the nature of the functions $f(x,y)$ and $\phi(x,y)$ then there is surely some *general* rule by which the functions $F(x,y)$, $\Phi(x,y)$ for every kind of solid, can be derived from $f(x,y)$, $\phi(x,y)$. But by no.5 the functions $\frac{P}{\Delta x\,\Delta y}$ and $\frac{\Pi}{\Delta x\,\Delta y}$ are determined merely by the values which $f(x+m\Delta x, y+n\Delta y)$ and $\phi(x+m\Delta x, y+n\Delta y)$ take if one puts for m, n every conceivable proper fraction together with 0 and 1, therefore *this* determination also certainly results from some rule which holds equally for all solids. For the quantities just mentioned are derived from the previous $F(x,y)$ and $\Phi(x,y)$, $f(x,y)$ and $\phi(x,y)$ themselves by the same rule.

7. It is to be concluded from §53 that the functions $f(x,y)$, $\phi(x,y)$ must be continuous, so that the values of $f(x+m\Delta x, y+n\Delta y)$ and $\phi(x+m\Delta x, y+n\Delta y)$ can come as close to the values $f(x,y)$ and $\phi(x,y)$ as desired if Δx, Δy are taken small enough. Finally, at least for *certain* solids, e.g. for those which are bounded by planes, $\Phi(x,y)$ is also a continuous function and therefore by Taylor's theorem,

$$\begin{aligned}\Phi(x+\Delta x, y+\Delta y) - \Phi(x,y) = \Delta x\frac{d^x\Phi(x,y)}{dx} &+ \Delta y\frac{d^y\Phi(x,y)}{dy}\\ &+ \frac{\Delta x^2}{2}\frac{d^{xx}\Phi(x,y)}{dx^2}\\ &+ \frac{\Delta y^2}{2}\frac{d^{yy}\Phi(x,y)}{dy^2}\\ &+ \Delta x\,\Delta y\frac{d^{xy}\Phi(x,y)}{dx\,dy} + \cdots\end{aligned}$$

which by the remark of no.5 is the volume of the three-fold piece of solid which is above the rectangles πr, $r\rho$ and rk. One may now be convinced in the same way as in §50, no.7 that the volume of the piece of solid above $r\rho$ is expressed merely by those terms of the formula just quoted in which Δx and Δy are multiplied together. Therefore

$$\Pi = \Delta x\, \Delta y \left[\frac{\mathrm{d}^{xy} \Phi(x, y)}{\mathrm{d}x\, \mathrm{d}y} + \cdots \right]$$

and consequently

$$\frac{\Pi}{\Delta x\, \Delta y} = \frac{\mathrm{d}^{xy} \Phi(x, y)}{\mathrm{d}x\, \mathrm{d}y} + \cdots .$$

The function $\frac{\Pi}{\Delta x\, \Delta y}$ therefore comes as close as desired to the value $\frac{\mathrm{d}^{xy} \Phi(x,y)}{\mathrm{d}x\, \mathrm{d}y}$ if $\Delta x, \Delta y$ are taken small enough. Consequently the same relation also holds between the functions $\frac{P}{\Delta x\, \Delta y}$ and $\frac{\mathrm{d}^{xy} F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$. Therefore here again all the conditions of §9 are fulfilled. The quantities into which the functions $\frac{P}{\Delta x\, \Delta y}$ and $\frac{\Pi}{\Delta x\, \Delta y}$ change for $\Delta x = 0$, $\Delta y = 0$, namely $\frac{\mathrm{d}^{xy} F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$ and $\frac{\mathrm{d}^{xy} \Phi(x,y)}{\mathrm{d}x\, \mathrm{d}y}$ therefore depend only on the values into which the quantities $f(x + m\Delta x, y + n\Delta y)$ and $\phi(x + m\Delta x, y + n\Delta y)$ change for $\Delta x = 0$, $\Delta y = 0$, i.e. on $f(x, y)$ and $\phi(x, y)$. Consequently if for some x and $y, f(x, y) = \phi(x, y)$ then the determining pieces of the functions $\frac{\mathrm{d}^{xy} F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$ and $\frac{\mathrm{d}^{xy} \Phi(x,y)}{\mathrm{d}x\, \mathrm{d}y}$ are equal to one another. Therefore $\frac{\mathrm{d}^{xy} F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$ is composed out of the quantity $f(x, y)$ in exactly the way that $\frac{\mathrm{d}^{xy} \Phi(x,y)}{\mathrm{d}x\, \mathrm{d}y}$ is composed out of the quantity $\phi(x, y)$.

8. Now if the surface in which the end-points of the ordinates $\zeta' = \phi(x, y)$ lie, consists of one or several *planes* then we shall very easily be able to calculate $\Phi(x, y)$ or $\frac{\mathrm{d}^{xy} \Phi(x,y)}{\mathrm{d}x\, \mathrm{d}y}$, if we consider that

$$\Pi = \Delta x\, \Delta y \left[\frac{\mathrm{d}^{xy} \Phi(x, y)}{\mathrm{d}x\, \mathrm{d}y} + \cdots \right]$$

is the volume of just that piece of solid which is over $r\rho$. Therefore let us assume (because this is the simplest) that the planes just mentioned, in which the points belonging to ζ' lie, are *parallel* to the plane of x and y so that the value of ζ' is constant for all x and y. But in order to be able to put $\zeta' = z' = f(x, y)$ one must assume as many pairs of parallel planes in this solid as pairs of values of the quantity z' in the equation $z' = f(x, y)$. Now if we designate these different values from the greatest positive to the smallest negative (or conversely) by $z', z'', z''', z^{\mathrm{IV}}, \ldots$ then the piece of solid above $r\rho$ is a sum of parallelepipeds whose base-areas $= r\rho = \Delta x\, \Delta y$ and whose heights are $= z' - z'', z''' - z^{\mathrm{IV}}, \ldots$. The volume of them is therefore, by §59,$= \Delta x\, \Delta y[(z' - z'') + (z''' - z^{\mathrm{IV}}) + \cdots] = \Pi$.

Therefore $\frac{\mathrm{d}^{xy}\Phi(x,y)}{\mathrm{d}x\,\mathrm{d}y}$, or the quantity into which $\frac{\Pi}{\Delta x\,\Delta y}$ changes for $\Delta x = 0$, $\Delta y = 0$, $= (z' - z'') + (z''' - z^{\mathrm{IV}}) + \cdots$. Therefore it must also be that

$$\frac{\mathrm{d}^{xy}F(x,y)}{\mathrm{d}x\,\mathrm{d}y} = (z' - z'') + (z''' - z^{\mathrm{IV}}) + \cdots$$

whence, since the values $z', z'', z''', z^{\mathrm{IV}}, \ldots$ are given by the equation $z' = f(x,y)$, one can find by integration the function $F(x,y)$ itself, provided the additional constants are suitably determined by means of the equation $y' = \psi x$ and the given a and E.

§ 62

Note. In general it is said that $\frac{\mathrm{d}^{xy}F(x,y)}{\mathrm{d}x\,\mathrm{d}y} = z'$; a formula which arises from ours in the special case when the solid to be calculated is not an *absolutely connected solid* of the kind that each of the ordinates z' cuts its surface in only two points of which one always lies in the plane of x and y. In other words, this is the special case when one seeks the volume only of such a piece of solid as is bounded on one side by the plane of x and y. In this case, of course $z'' = 0$ and the quantities $z''', z^{\mathrm{IV}}, \ldots$ do not even exist; therefore $(z' - z'') + (z''' - z^{\mathrm{IV}}) + \cdots$ simply becomes z'. Moreover, since the parallelepiped with which the comparison has helped us here for the calculation of the volume of all other solids, goes through the point m which belongs to x, y, z, then considerations can also be put forward here like those of §34.

§ 63

Concluding note. The obvious *analogy* which exists in the definitions of *line* (§11), *surface* (§35) and *solid* (§52), is met with just as much for the definitions of the *length* of a line (§19), the *area* of a *surface* (§45) and the *volume* of a *solid* (§57). Also notice the *ease* and *uniformity* with which the formulae for the measurement of these three kinds of geometrical extension can be derived in their greatest generality from these six definitions merely by the application of the most simple method of §9. All these things should arouse, I believe, the most favourable opinion for the theory expounded here. But the more strictly the reader examines it and the more consideration he gives to it the more completely will he be convinced of its correctness.

Appendix

The foregoing work had been ready for printing for a long time when the paper of *Dr A. L. Crelle* (Royal Prussian Chief Adviser) appeared, *Über die Anwendung der Rechnung mit veränderlichen Größen auf Geometrie und Mechanik*, Berlin, 1816, in which a new attempt is made at the three problems of *rectification, complanation* and *cubature*, together with those two *theorems of mechanics* which we presented as examples in §10 of our paper, as well as the *theory of contact* and *Taylor's theorem*. In fact the author, in his concern for the truth, discovered certain defects in his earlier presentation of these theories (*Versuch einer Darstellung der Rechnung mit veränderlichen Grössen*, 1.Band, Göttingen, 1813) which he endeavours now to rectify. Whether this is achieved in respect of the last two topics is a question which is not appropriate here, but it certainly is appropriate to investigate whether the first five sections are presented so satisfactorily in this latest treatment as to make any further effort superfluous.

I. Here again the *rectification of the curve in the plane* is based by *Crelle* on the *proposition*, '*that every line which encloses another one that is convex along its whole length on one and the same side, and with which it has two points in common, is longer than the enclosed line*' (S. 51). About his proof for this proposition he says (S. 51) that it is 'in substance the one which *Legendre* had in mind (*Élémens de Géométrie*, l.IV, Propos. IX, cinq.Edit., p. 116) but which he did not express completely clearly. *Crelle* begins it in the following way. '*If the enclosed line is not itself the shortest of all lines by which AMB is enclosed on the convex side, then among these enclosing lines there is some other which is the shortest.*'

Now it is shown in the well-known way that no such shortest line on the convex side of *AMB* can be given, because for each one which might be taken for it there can be another still shorter and it is then deduced, because this assumed line is not to lie on the convex side of *AMB*, that *AMB* itself is this shortest line. In this proof, the expression '*the shortest line among those which enclose AMB*' is ambiguous because it is left vague whether one is to understand by it a line which itself belongs to the class of enclosing lines, or not. If it should be the former, then it is absurd to say *AMB* itself is this line for *AMB* does not enclose *AMB*. But if by, '*the shortest line of all those which enclose AMB*' is to be understood only a line in general which has the property of being shorter than all lines enclosing *AMB*, then it is absurd to look for it in the class of those enclosing lines. For how can a line which encloses *AMB* be shorter than every line which encloses *AMB*, and therefore also be shorter than itself? Nevertheless these contradictions only lie in the way the proof is *expressed* and could easily be avoided with a different presentation. But the most crucial mistake is in the first proposition where the following is actually stated: '*If the enclosed line is not shorter than every line enclosing it then among the latter there must be one which is shorter than all the rest.*' This is exactly the same assumption which *Legendre* also made and we have already explained in detail why it is inadmissable in the *Preface*, p. XI ff.[q]

It was encouraging for us to find that in *Crelle's* opinion the proposition that has to be accepted in this proof as a *second* assumption, namely that *the straight line is the shortest between two points*, is not an axiom but is in need of a proof. But we could not approve of the kind of proof which he subsequently offers on S. 53 ff.

If some curved line *AMB* were shorter than the straight line *AB* then all kinds of broken lines *AQPB* could be formed which (by Euclid I, 20) are greater than *AB* and consequently would exceed *AMB* still more. '*Now,*' says Crelle, '*among these lines enclosing AMB which exceed AMB there will necessarily be one which exceeds AMB the most.*' But this cannot be the case as he correctly shows. '*Consequently it is impossible that the straight line AB is longer than some other convex line between A and B, therefore it is the shortest among all of them.*' This proof, as everyone may see, is even more mistaken than the previous one. Firstly, from the fact that the straight line *AB* is *not longer than AMB* it certainly does not follow that it must be shorter. Moreover, how arbitrary is the assumption, '*among the lines which enclose AMB and exceed it there must necessarily be one which exceeds it the most.*'! This proposition is just as absurd as the one rejected before which we have dealt with in our *Preface*. In fact its subject is something which is self-contradictory. Therefore it would be quite wrong to accept it without proof as an axiom, instead it must straight away be removed from those propositions which may be put forward in science because it does not express *any truth* at all.

II. *Crelle* derives the *rectification of curved lines of double curvature* from that of lines of simple curvature by considering (S. 60) surfaces of a cylindrical kind in which all the *z* ordinates lie and this is then stretched out into a plane. Such a stretching out is, as everyone can see, a kind of *motion* and all ideas of motion are to be avoided in *geometry* as something alien. It would also first have had to be proved that by the stretching of this cylindrical surface into a plane the curved line which is drawn on it undergoes no change in its *length*. To express this purely geometrically: *If in two lines, one of which is a straight abscissa line and the other a curved abscissa line (but of simple curvature) there always belong, for equal lengths of abscissa, equal ordinates, then these lines are also of equal length.* The condition that the curved abscissa line may only be of simple curvature is essential in this proposition. Now whoever would want us to admit this as an axiom without *proof*?

III. For the *complanation of surfaces Crelle* refers (S. 61) to an argument which he claims to be similar to that used for the rectification of lines but here space prevents him following it through. But it would doubtless have therefore been open to the same objections with which we rejected the former.

q On p. 285.

IV. The author proves the two *theorems of mechanics* in exactly the way which we have already indicated as unsatisfactory in our work §10, S. 49.[r]

V. The *quadrature of plane surfaces* and the *cubature of solids Crelle* achieves in a way rather different from the usual method (S. 42–48). However, that objection regarding the *introduction of fortuitous associated concepts* which we have already made to the *method of limits* (*Preface*, p. XVII[s]) also applies here. Anyone can see that that consideration of an ordinate lying between the smallest and largest, which is just so large that a rectangle formed from it and the increase in abscissa equals the increase in surface bounded by a curved line, is a very alien kind of consideration if one only wishes to calculate the latter. It is another mistake that the author permits an equation, whose validity is proved only for cases where the variable quantity has a value not equal to zero, to be applied exactly in the case when this value is equal to zero. However, this mistake would be easy to rectify in the well-known way which is followed in §28 of *Der binomische Lehrsatz*.[t]

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[r] On pp. 299–300.

[s] On p. 287.

[t] On p. 180.

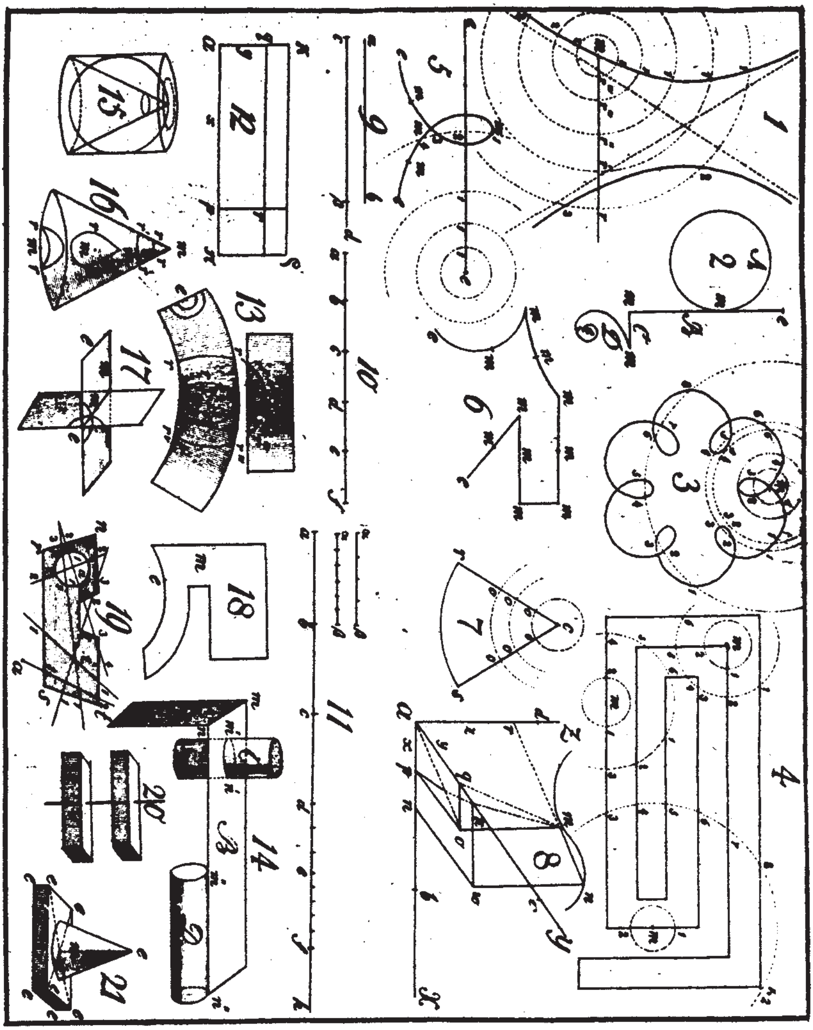

Plate of Figures as it appeared at the end of the first edition.

Later Analysis and the Infinite

The final group of works translated here, *RZ*, *F*, *F*+, and *PU* date from the last two decades of Bolzano's life and the earliest of them (*RZ* and *F*) were written about fifteen years after the publication of his early analysis works *BL*, *RB*, and *DP*. Those years had been a momentous middle period of Bolzano's life. He had been deposed from his university post in 1819 and forced to retreat from public life. He turned this to advantage by composing his four-volume *Wissenschaftslehre* where he develops the conceptual and philosophical framework that is necessary for his views and contributions in the fields of logic and the theory of knowledge.

It was around 1830 that he turned his full attention to the second major writing project of his logical and mathematical work, namely, the *Theory of Quantity* [*Größenlehre*]. This was conceived on a grand scale. In a letter to one of his former students Michael Josef Fesl dated 5 April 1835 Bolzano wrote that he had, 'one book near completion with the title *Pure Theory of Numbers* consisting of two volumes: an *Introduction to Mathematics*, the first concepts of the general theory of quantity, and then the *Theory of Numbers* itself. Another book should have the title the *Pure Theory of Quantity*, and a third should contain the *Theories of Time and Space*. (Quoted by Jan Berg in *BGA* 2A7 p. 10.) This was Bolzano's outline of the contents of his projected huge compendium of all of mathematics. Both the works *RZ* and *F* would have belonged to the volume on the *Theory of Numbers* (which would, judging from extant manuscripts, actually have comprised several volumes).

We can see in Bolzano's plan some influence of his work twenty years earlier in the classification of mathematics at the conclusion of *BD* I (see p. 102). But two major changes in Bolzano's views on mathematics over those years are as follows. In *BD*, I §11 he adopted a very general view of mathematics as 'the science of the laws to which things must conform in their existence'. This was deliberately in contrast to the more conventional view of mathematics as the science of quantity. His 1810 classification of mathematics is consequently in terms of types of 'things', and does not mention quantity. It is clear from the title *Theory of Quantity* for his mature comprehensive work on mathematics that he has moved closer to a 'science of quantity' view of the nature of mathematics. Second, there was the question of the infinite in mathematics. We have seen (p. 143) that Bolzano rejected the infinite in all forms in most of his early work. Although in *DP* he is beginning to be more accepting of infinite collections, he still rejects the possibility of a calculation involving an infinite multitude of terms. But the title of this seventh part of *RZ* is *Infinite Quantity Concepts*. And the second paragraph makes clear that such concepts correspond to expressions requiring an infinite multitude of operations and he gives examples such as $1 + 2 + 3 + \cdots$ *in inf*., and $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$ *in inf*. By the time of preparing *PU* Bolzano (or Příhonský on his behalf) felt so comfortable with the infinite as to quote, on the title page, Leibniz almost revelling in the concept of the actual infinite (see p. 591). And in *PU*, §2 we read 'it is precisely in *mathematics* ... where we speak most frequently of the infinite'. What brought about such a significant change in his thinking? It was a crucial preliminary to his many major insights in these later works. There are many lines along which to open up this enquiry. His careful treatment in *BL*, though often long-winded, was actually a fine demonstration of how the notion of infinite series could be legitimized by being based on purely finite entities and finite processes. Further reflection on the significance of this work might have been influential. In the *Theory of Science* he presented proofs that there were infinitely many propositions in themselves. His important neighbourhood definitions of geometric objects in *DP* in terms of certain point sets could hardly be supported in terms of merely finite sets. His notion of 'arbitrarily small quantities', those represented by ω, or Ω, as introduced in *BL* §14 and used in subsequent sections, is subtle and may already contain some concept of the infinite. These, and many other issues, would be part of any study into how he came to change his views on the infinite. It is an open question, but the status of the infinite, in its various forms, was a well-known, controversial question of the time that has great importance for the history of mathematics more broadly than for Bolzano's studies alone. The case of Bolzano's change of views simply epitomizes the issue vividly and offers a useful case study to be considered alongside the views of his contemporaries.

Another issue we can hardly ignore on reading the opening of this seventh section of *RZ* is that of Bolzano's alteration of 'Infinite Number Concepts' into 'Infinite Quantity Concepts'. (See the footnote on p. 357.) What is the significance of this change? Perhaps not very much, there are numerous unaltered occurrences of the term 'number concepts' in the succeeding sections. But the initial changes were probably intended to be representative. Both numbers and multitudes [*Mengen*] are examples for Bolzano, of quantities which is therefore the more general concept. On the philosophical issues arising from the alteration a good starting place for further study is Chapter 7 (Bolzano's Philosophy of Mathematics) in Berg (1962). On the related language issues see the *Note on the Translations*.

The material of the seventh section of *RZ* is a theory of what Bolzano called 'measurable numbers'. This has been edited, assessed, or given critical comment, in works such as Rychlík (1962), van Rootselaar (1964), Laugwitz (1962–66), Berg, *BGA* 2A8, Spalt (1991), Sebestik (1992), and Rusnock (2000). It is difficult to summarize the variety of views represented by these authors partly because we have learned more about the primary source over the period in which they are writing. The edition of Rychlík (1962) was only a partial edition having some important sections omitted and some portions included that were deleted by Bolzano. Major defects in the theory were diagnosed by van Rootselaar and some ways of repairing these were suggested by Laugwitz. With Berg's definitive edition in 1976 (*BGA* 2A8) it was discovered that Bolzano had himself made some of the revisions independently suggested by Laugwitz. There is a broad consensus that the work is an attempt at a theory of real numbers and that while having numerous technical defects and complications it represents important insights and a surprising achievement for its time. It is also probably repairable and feasible, but this is perhaps less important now than the conceptual progress that it represented then which has enduring value.

We have already referred to the problem of the existence of the limit of a sequence satisfying the (Cauchy) convergence criterion described in §7 of *RB*. This is a matter of the completeness of the real numbers and conventional wisdom now has it that such existence should be guaranteed by a suitable construction or axiom. Bolzano made an interesting remark in reference to this limit in *RB Preface*. With regard to a situation where values which are lower bounds (for some property) and those which are not, can be brought as near one another as we please, he said that 'for anyone having a correct concept of *quantity*' the idea of the greatest such lower bound is 'the idea of a real, i.e. *actual*, quantity'. Thus, as usual for Bolzano, he saw the problem of the existence of numbers as primarily a conceptual matter. This means it required analysis, definition, and a theory. This was the aim of *RZ*, and in particular its major seventh section. There can be no doubt that Bolzano saw the roots of this work in the earlier *RB*. The two main theorems of *RB* (in §§7 and 12) are re-formulated here in terms of measurable numbers and proved in a more detailed and rigorous manner towards the end of *RZ* in §§107 and 109, respectively.

Having introduced expressions that require an infinite multitude of operations in §2 he proceeds to define in §6 what he means by a number expression S being *measurable*. The intuition has much to do with determination again. Imagine a ruler with units divided into q equal divisions, call this a q-ruler. Then for every linear quantity, and every natural number q, placing the q-ruler against the quantity will 'measure' it so that it will either match up exactly against a division or else it will be in between two divisions. So for Bolzano S is *measurable* if for every natural number q there is an integer p such that

$$S = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$

where P^1 is either zero or positive and P^2 is positive. (As often the case with Bolzano these superscripts only distinguish, they are not powers.) This kind of determination Bolzano suggests will be useful when, in an infinite number concept, we cannot actually complete the infinite number of operations in its associated expression.

There is a consensus among the main commentators on Bolzano's work that it is helpful for the sake of clarity, though not explicitly supported by Bolzano, to interpret his infinite number concepts as sequences of rational numbers. Thus, $1 + 2 + 3 + \cdots$ *in inf*. corresponds to $\{\frac{1}{2}n(n+1)\}$ and $b/(1 + 1 + 1 + \cdots$ *in inf*.) corresponds to $\{\frac{b}{n}\}$.

The bulk of the work develops a theory of measurable numbers, including their closure and order properties preparatory to the new proofs of the convergence criterion (§107) and the greatest lower bound theorem (§109). A particularly interesting section is that of §§54, 55 where he extends notions of equality and order to measurable numbers. This effectively suggests identifying real numbers with equivalence classes of measurable number concepts.

Rusnock (2000), p. 179 ff. cogently summarizes some of the defects of Bolzano's theory as it stands, outlines the possibilities for their repair and assesses the improvements of this work over his earlier achievements in 1817.

By way of a short summary of the *Theory of Functions*, and its *Improvements*, see the 'Overview of Contents' given by van Rootselaar and translated here on p. 430 and p. 432.

The work is characterized at the outset by Bolzano's reflective approach. His motivation for studying functions is not one of problem-solving or applications. Instead he says that the question of the nature of the change in a dependent number if all, or some, of the numbers on which it depends vary, will 'be especially useful for improving our knowledge of the nature of functions themselves' (§1). He begins from a simple case of a function whose analytic form depends on the range within which the argument lies (§2) but soon moves on to reveal (§37) he is thinking of altogether more sophisticated functions such as:

$$W = \begin{cases} ax & \text{if } x \text{ is of the form } \frac{2m+1}{2^n} \\ ax + b & \text{otherwise} \end{cases}$$

Functions such as this, which is clearly discontinuous everywhere, are generally associated with Dirichlet whose work was probably slightly earlier than Bolzano's. His well-known example of a function taking distinct values on rational and irrational arguments appears in 1829 (Dirichlet, 1889–97, i, pp. 117–32).

The section §49 is given the heading 'Uniform continuity does not follow from continuity' by van Rootselaar. And here Bolzano was probably ahead of Dirichlet who is known to have lectured on the topic of uniform continuity in 1854 (see the historical note in Rusnock (2000), p. 172). For some years this had been a controversial issue. Rusnock cites Rychlík, and van Rootselaar in 1969, as denying that Bolzano had a clear idea of uniform continuity at all. From the introduction to *BGA* 2A10/1 it is clear that van Rootselaar has revised his view on this in favour of Bolzano having the concept of uniform continuity. In the *Improvements* §6 there is the theorem that a function continuous on the closed interval $[a, b]$ is also uniformly continuous there. There is a detailed study of the matter in Rusnock and Kerr-Lawson (2005).

There are many significant improvements here on the earlier work in analysis. The definitions and examples of continuity are more detailed and instructive. Left and right-sided continuity are introduced, as well as left and right-sided derivatives. A clear and detailed critical account of the achievements and limitations of Bolzano's work in *Theory of Functions* may be found in Rusnock (2000), p. 159 ff. where he emphasizes the importance of three theorems in particular. These are:

1. If f is continuous on a closed interval $[a, b]$ then f is bounded there, (§57).
2. If f is continuous on a closed interval $[a, b]$ then f takes a greatest and a least value on $[a.b]$, (§60).

3. If f is continuous on a closed interval $[a, b]$ and f takes two values L and M on $[a, b]$, then f also takes all values in between L and M on $[a, b]$, (§65, this is the intermediate value theorem).

Bolzano's strategy in the *Theory of Functions* seems to be to attack the problem of the nature of functional dependency from two directions. Not only is he at pains to build up the theory in terms of positive results such as the three just considered, but at the same time he is exploring all the ways in which functions may fail to be continuous and may fail to have derivatives. He considers in §38 the cases of functions that have *jumps* and *gaps*. At §85 he begins a series of results about the monotonicity of functions. At §101 he constructs a continuous function that can alternate its sign infinitely many times over a given interval. In §111 he defines a function, this time a limit of a convergent sequence of functions, that is continuous on a closed interval but not monotonic over any subinterval, however small. Then in §135 he proves that the function of §111 has no derivative at a dense subset of points. In fact it has no derivative at any point. It was an astonishing achievement even to conceive, and explore the possibility, of such a pathological function at this time, let alone to construct one and prove its properties. Bolzano acknowledges that others were beginning to think of functions in wider terms than the analytic expression view familiar from Euler (e.g. Fourier and Dirichlet). But a typical contemporary opinion on the properties of functions is shown when Bolzano goes on (in §136) to quote Galois offering an alleged proof that every function must have a derivative, except possibly at certain isolated points.

In his early work Bolzano had developed the technique of using arbitrarily small quantities in a way equivalent to limit arguments to prove standard results about derivatives. For example, in *BL* §29 he proves that differentiability of a function at a point implied its continuity at the point. It is an impressive testimony to his 'conceptualist' approach to mathematics that within about fifteen years he had so thoroughly gained insight into the trio of concepts function, continuity, and derivative, that he was able to prove that, contrary to all expectation from intuition, the converse of his earlier result, namely that continuity implies differentiability, was false.

We give here an outline of the construction of the 'Bolzano function'. It is the function F where the value $F(x)$ is the limit of the values $y_n(x)$ as n increases indefinitely. (Bolzano again uses the confusing superscript notation y^n.) He defines the functions $y_n(x)$ recursively as follows. The value of $y_1(x)$ is the linear function from (a, A) to (b, B) so:

$$y_1(x) = A + (x - a)\frac{B - A}{b - a};$$

then $y_2(x)$ replaces the single line segment of $y_1(x)$ by a graph consisting of four linear segments. It coincides with $y_1(x)$ at each of (a, A), (b, B), and their midpoint but has in addition the nodes $(a + \frac{3}{8}(b - a), A + \frac{5}{8}(B - A))$ and $(a + \frac{7}{8}(b - a), A + \frac{9}{8}(B - A))$. Then for all n, $y_{n+1}(x)$ is obtained by decomposing each

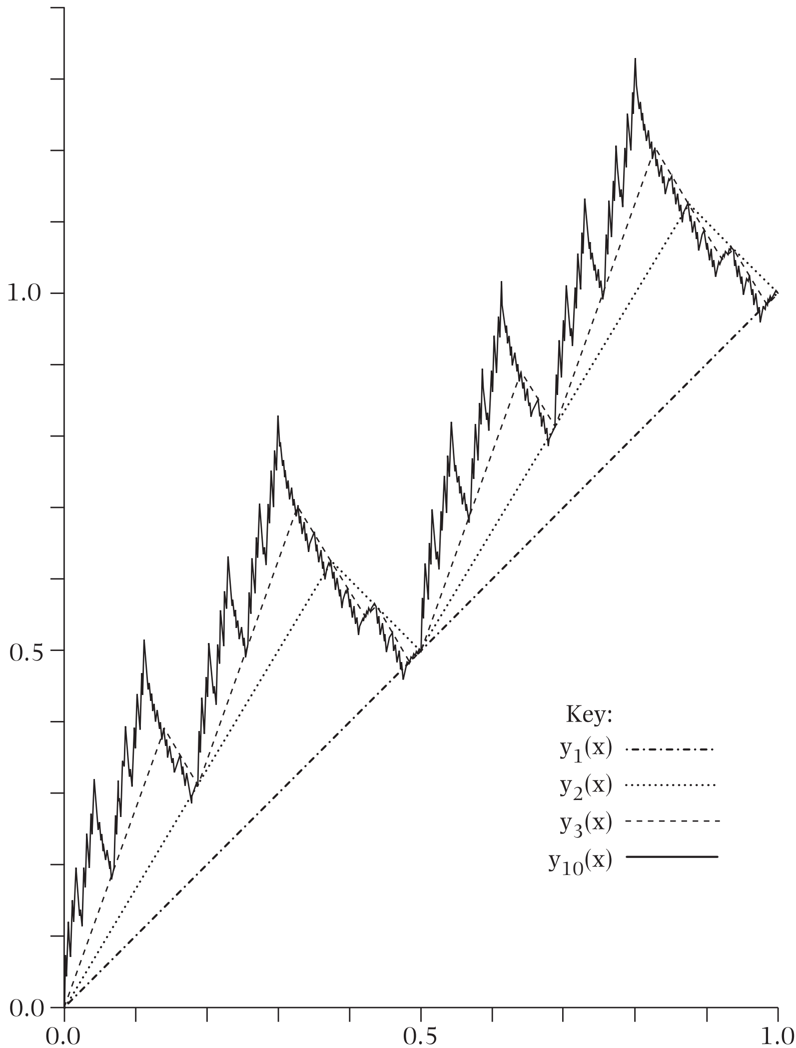

Fig. 1. The first few iterations of Bolzano's function.

segment of $y_n(x)$ into four new segments in the same way that $y_2(x)$ was obtained from $y_1(x)$. Taking the initial segment joining the origin to $(1, 1)$, the first few iterations are shown in Figure 1. The limit function $F(x)$ of the $y_n(x)$ is the Bolzano function and is, presumably, the first analytically defined fractal set.

The manuscript for *PU* was prepared in Bolzano's final years and the fact that he chose to do this as a priority suggests the value he put upon the ideas it contains. The work illustrates well the 'wholeness' of Bolzano's thinking. In focussing on the infinite he appears to move with ease between fields of knowledge where today we would be likely to hesitate self-consciously at the boundaries. Very broadly, the first half of the work (§§1–37) is concerned with the occurrence and use of infinities in mathematics in the sense of a theory of quantity, and so emphasizing the infinitely large and small. The next quarter §§38–49 is concerned with infinities implicitly involved in the continuous extensions of time and space. The final quarter (§§50–70) deals with the infinite in physics and metaphysics where ideas such as spiritual substance, the ether and 'dominant' substances play a significant part. But each of these broad sections are interspersed with ideas and comparisons that are philosophical and theological. More details of the division of the material are given in the table of contents compiled by the editor Příhonský, and given here at the beginning of the work. There is an even more detailed list of contents, compiled by Steele, in his translation Steele (1950). It should be noted that Příhonský himself states in his introduction that he found some of the manuscript illegible and some to be incorrect. Some details of earlier drafts of *PU* are to be found in Berg (1973), pp. 25–27, but see also the *Note on the Texts*.

In characteristic fashion Bolzano begins in §2 'we may hope that a more precise investigation of the question of the circumstances in which we define a multitude as finite or as infinite, will also give us information about the *infinite in general*.' To this end he says we must study the concept of conjunction and therefore of various kinds of collection (§3). In the following sections he defines increasingly specialized collections—multitudes, pluralities, sums, and series culminating in a definition of whole numbers (§8), and of *infinite plurality* (§9). For a careful and thought-provoking analysis of Bolzano's thinking on collections see Simons (1997). It has been explained in the *Note on the Translations* why *Menge* has been rendered by 'multitude' rather than 'set' on most of its many occurrences in *PU*.

In §§20–22 appears the insight that is probably most well known about *PU*. He says in §20 that two infinite multitudes may have a bijection between them and yet their multiplicities may have 'the most varied relationships'. In particular, they need not be equal. He gives examples showing that an infinite multitude may have a bijection with a proper part of itself. It is clear that he knows this property can be proved for any infinite multitude. It is characteristic only of finite multitudes, he says in §22, that the existence of a bijection between any two of them is equivalent to their having equal multiplicity. For the infinite case, he suggests further information might be possible which, together with the existence of a bijection, would guarantee equal multiplicity. He suggests this might be something like that the multitudes have the same 'determining grounds' or 'mode of specification'. Here, yet again, the interesting notion of the 'determination' of a mathematical object arises although Rusnock is clearly not impressed with the suggestion. See Rusnock (2000), pp. 188–96 for further commentary on the mathematical issues arising in *PU*. In the same passage there is also a translation of the suggestion by Berg, quoting a late letter of Bolzano to R. Zimmermann, that in fact Bolzano may have changed his mind and actually accepted the bijection alone as sufficient criterion for equal multiplicity. This was the step that Dedekind would take in 1888, thus opening the way for Cantor's theory of cardinal numbers (see §64 of

The Nature and Meaning of Numbers in Dedekind (1963). On this particular point Rusnock takes a more cautious position than Berg. But such matters remain controversial and await, as so many of the issues raised here, further research. We can be optimistic about progress being made on this, and many other questions, in the light of recent studies, more manuscript material continuing to be edited and published, and a burgeoning interest in Bolzano's work and times in the research community of the history and philosophy of mathematics.

Pure Theory of Numbers

Bernard Bolzano

Seventh Section: Infinite Quantity Concepts

Translated from the edition of Bolzano's manuscripts prepared

by

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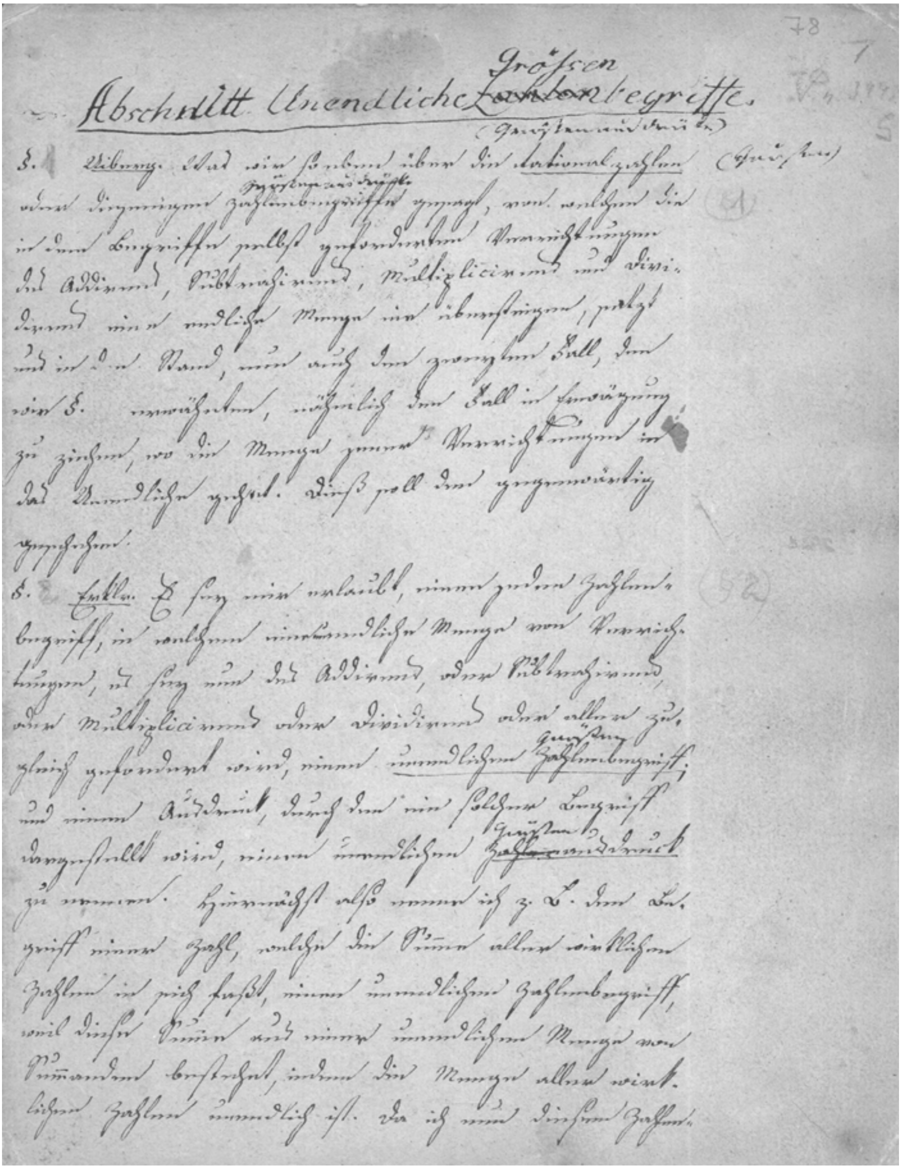
in the

Bernard Bolzano Gesamtausgabe Bd. 2A8



This mss in Series Nova 3469: 78 in the Manuscript Collection of the Austrian National Library. It shows the opening paragraphs of the seventh section of *RZ* with alterations in Bolzano's hand as described in footnotes on the facing page.

Seventh Section. Infinite Quantity Concepts[a] (Quantity expressions)[b]

§ 1

Introduction. What we have just said about *rational numbers* (quantities) or those quantity expressions[c] in which there are never more than a finite multitude [*Menge*][d] of the operations of addition, subtraction, multiplication and division required in the concept itself enables us now to the consider the second case which we mentioned (§1 of the 4th Section[e]), namely the case where the number of those operations increases indefinitely [*in das Unendliche gehet*]. This should now be done.

§ 2

Definition. Let me call every number concept in which an infinite number of operations is required—these may be additions, subtractions, multiplications or divisions, or all of them together—an *infinite quantity concept*, and an expression by which such a concept is represented, an *infinite quantity expression*. For example, I call the concept of a number which comprises the sum of all actual numbers, an infinite number concept because this sum consists of an infinite multitude of summands, since the multitude of all actual numbers is infinite. Now, since I designate this number concept by the following expression: $1 + 2 + 3 + \cdots$ *in inf.*, I shall also call this an infinite expression. But $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$ *in inf.* would also be such an expression. It contains the sum of all those fractions which arise if the unit is first divided by the number 2 and then every successive fraction is formed by multiplying the denominator of the previous one by -2. The product $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\left(1 - \frac{1}{16}\right) \cdots$ *in inf.*, where the rule by which the factors are to be continued indefinitely is self-evident, represents yet another infinite number concept.

§ 3

Note. It may be worth remarking, for some readers, that in this definition I am only saying that, in an infinite number concept, an infinite multitude of operations

[a] Bolzano first formulated the title as 'Infinite Number Concepts' [*Unendliche Zahlenbegriffe*]. This was crossed out and replaced by the present title 'Infinite Quantity Concepts' [*Unendliche Größenbegriffe*].

[b] Added by Bolzano, possibly at the time of the emendation described in the above footnote.

[c] In the manuscripts there appeared originally *Zahlenbegriffe* [number concepts] but this was crossed out and replaced by Bolzano with *Größenausdrücken* [quantity expressions]. (*JB*)

[d] For remarks on the translation of *Menge* see the *Note on the Translations*.

[e] Such references to 'Section' are to earlier sections of the work *RZ*—of which this is the seventh and last. All sections are published in *BGA* 2A8.

of addition, subtraction, multiplication or division is *required*. I am not saying that it contains the idea of each one of these operations individually, and thereby that it contains an infinite multitude of ideas as its characteristic components. In this latter case such a concept would be composed of infinitely many parts, and therefore it would be inconceivable for a finite mind such as ours. However, one must remember that the components from which the object of a concept is composed do not necessarily have to be represented in this concept—indeed that it may often be possible to form a concept of an object (I always have in mind a concept which refers to it exclusively) without thinking, when considering the concept, of a single one of the components of which the object consists. Thus, for example, the concept of a pocket-watch is only the concept of 'a watch, that is, an instrument suitable for the measurement of time, which is so made that it can be carried comfortably in the pocket'. In this concept there appears no idea at all of the wheels, springs and other parts from which such a watch is usually composed. In a similar way we can easily form the concept of a number which is composed from infinitely many parts without these parts themselves, indeed without any of them, having to be represented in the concept. Thus the concept of 'a number which equals the sum of all actual numbers' contains only a very small number of components, among which there does not occur a single idea of one of the individual parts (1, 2, 3, ...) from which that sum is to be composed. Therefore if I call such a concept *infinite*, we must not be misled by this *figurative* term. It was only chosen in order to indicate that those actual or only imagined numbers, which are the object of such a number concept, are always thought of as having arisen through an infinite multitude of operations of addition, subtraction, multiplication or division.

§ 4

Corollary. If an infinite number concept is to be composed in a comprehensible fashion (§3 of the 4th Section) then the number which it represents must also be thought of as composed from an infinite multitude of other numbers. For only a finite number of numbers can be combined in a comprehensible fashion by means of a merely finite number of additions, subtractions, multiplications and divisions (§4 of the 4th Section).

§ 5

Theorem. Among the infinite number concepts there are some which are of such a kind that, to every arbitrary actual number [*wirklichen Zahlen*][f] q which we want to consider as the denominator of a fraction, a numerator p can be found which is again a positive or negative actual number, or even sometimes zero, with the

[f] In this context Bolzano's expression 'actual number' means 'whole number'. (*JB*)

property that we obtain the two equations

$$S = \frac{p}{q} + P \quad \text{and} \quad S = \frac{p+1}{q} - P^1,$$

in which the symbol S denotes the infinite number expression but the P and P^1 denote a pair of strictly positive number expressions, or sometimes the former may also merely denote zero.

Proof. The following is such an infinite number expression

$$S = a + \frac{b}{1+1+1+\cdots \textit{in inf.}}$$

if a and b denote a pair of actual numbers. For we already know from §30 of the 3rd Section that for every arbitrary actual number q, there is a p such that the relationships $p = aq$, or $p < aq$ and $p+1 > aq$, hold, provided that p can represent an actual number or, in special cases, even zero. Now if the first case arises, i.e. if $p = aq$, then $\frac{p}{q} = a$ and we therefore obtain, on the one hand, the equation

$$S = a + \frac{b}{1+1+1+\cdots \textit{in inf.}} = \frac{p}{q} + P$$

because the part

$$\frac{b}{1+1+1+\cdots \textit{in inf.}}$$

is a strictly positive number expression. On the other hand,

$$S = a + \frac{b}{1+1+1+\cdots \textit{in inf.}} = \frac{p+1}{q} - \left(\frac{1}{q} - \frac{b}{1+1+1+\cdots \textit{in inf.}}\right).$$

But instead of

$$\frac{1}{q} - \frac{b}{1+1+1+\cdots \textit{in inf.}}$$

we can also write, according to § ,

$$\frac{(1+1+1+\cdots \textit{in inf.}) - qb}{q(1+1+1+\cdots \textit{in inf.})}.$$

Now since qb is an actual number, the difference $(1+1+1+\cdots \textit{in inf.}) - qb$ consists of a sum of infinitely many units because if from an infinite multitude $(1+1+1+\cdots \textit{in inf.})$ a merely finite multitude qb is subtracted, the remainder still always contains an infinite multitude of units (§135 of the *Einleitung*[g]). Therefore

$$\frac{(1+1+1+\cdots \textit{in inf.}) - qb}{q(1+1+1+\cdots \textit{in inf.})}$$

[g] This refers to *EG* III. (*JB*)

is to be considered as a strictly positive expression and we can therefore write

$$S = a + \frac{b}{1+1+1+\cdots \textit{in inf.}} = \frac{p+1}{q} - P^1.$$

In the second case, if $p < aq$ but $p + 1 > aq$, then it must be that $p = aq - r$ and $p + 1 = aq + s$, where r and s denote a pair of actual numbers. Therefore

$$a = \frac{p}{q} + \frac{r}{q} = \frac{p+1}{q} - \frac{s}{q}.$$

Consequently this gives, on the one hand

$$S = a + \frac{b}{1+1+1+\cdots \textit{in inf.}} = \frac{p}{q} + \left(\frac{r}{q} + \frac{b}{1+1+1+\cdots \textit{in inf.}}\right),$$

i.e. $S = \frac{p}{q} + P$, because

$$\left(\frac{r}{q} + \frac{b}{1+1+1+\cdots \textit{in inf.}}\right)$$

is obviously a strictly positive expression. On the other hand, we have

$$S = a + \frac{b}{1+1+1+\cdots \textit{in inf.}} = \frac{p+1}{q} - \left(\frac{s}{q} - \frac{b}{1+1+1+\cdots \textit{in inf.}}\right).$$

However, the expression

$$\left(\frac{s}{q} - \frac{b}{1+1+1+\cdots \textit{in inf.}}\right)$$

can be changed into the following:

$$\frac{s(1+1+1+\cdots \textit{in inf.}) - qb}{q(1+1+1+\cdots \textit{in inf.})}.$$

As before, it is clear that $s(1+1+1+\cdots \textit{in inf.}) - qb$, which is what remains if a merely finite multitude is subtracted from an infinite multitude of units, is strictly positive, therefore

$$S = \frac{p+1}{q} - P^1.$$

§ 6

Definition. It is easy to see that the relationship just proved to hold between some infinite number concepts on the one hand, and certain simple fractions with arbitrarily chosen denominators on the other hand, is significant and could be used to discover the characteristics of a given infinite number concept. For obviously we express a property relevant to the given number concepts if we determine what sort of p must be chosen for every arbitrary q so that the equations

$$S = \frac{p}{q} + P \quad \text{and} \quad S = \frac{p+1}{q} - P^1$$

arise. It will also be appreciated that this method of determining infinite number concepts must be all the more welcome to us the less we are in the position of determining the value of such a number concept by actually doing and completing the infinitely many operations which are indicated in it. I therefore say, and this in fact applies to every number expression S—not only infinite ones but also finite ones—that we *determine by approximation or measure* it, if we determine, for every arbitrary and positive number q, the number p that must be chosen (whether zero or else some kind of actual positive or negative number) so that the two equations

$$S = \frac{p}{q} + P^{1} \quad \text{and} \quad S = \frac{p+1}{q} - P^{2}$$

arise, in which P^{1} and P^{2} denote a pair of strictly positive number expressions (the former possibly denoting zero). I also say that we *measure* or *determine up to* $\frac{1}{q}$ the number S by this specification [*Angabe*] of the numerator p belonging to the denominator q. And if we are in the position of specifying the appropriate p for every arbitrarily large value of q, then I say that we determine the number S *by approximation as precisely as we please, or that the* determination *by approximation, or the measurement of S can be carried out as far as we please.* A number expression for which there is a p with the property described corresponding to every arbitrary value of q, such that the two equations

$$S = \frac{p}{q} + P^{1} = \frac{p+1}{q} - P^{2}$$

hold, is called by me a *measurable* [*meßbarer*] or *estimable* [*ermeßlicher*] expression. In contrast every other is *unmeasurable* or *inestimable*. The fraction $\frac{p}{q}$ I call the *measuring fraction* and the fraction $\frac{p+1}{q}$ the *next greater* fraction. Since $S = \frac{p}{q} + P^{1}$, I call P^{1} the *completion* of the measuring fraction. In the special case when $P^{1} = 0$, and therefore we have $S = \frac{p}{q}$, I call the measuring fraction *full* or *perfect*, or the *full* or *perfect measure* of S.

§ 7

Theorem. Every *rational number* is a measurable number and indeed a complete measure can be given for it. Conversely every number for which a complete measure can be given is a rational number.

Proof. 1. Every rational number S can be brought to a fraction of the form $\frac{m}{n}$, where n denotes an actual number and m denotes an actual number or zero. Now let q denote any kind of actual positive number, then the symbol p can always represent a positive or negative actual number, or just zero, with the result that we have the two relationships

$$\frac{p}{q} \leq \frac{m}{n}, \quad \text{but} \quad \frac{p+1}{q} > \frac{m}{n}.$$

To obtain this we only need to attempt the division of the product qm by n; if it turns out that the quotient is $+p$, then $\frac{+p}{q} = \frac{m}{n}$. Therefore $\frac{p}{q}$ is the required measuring fraction, in fact it is the complete measure. But if this division does not go exactly, then there is, according to §30 of the 3rd Section, a whole number, or zero, $=p$ such that

$$p < \frac{mq}{n} \quad \text{but} \quad p + 1 > \frac{mq}{n}.$$

Therefore

$$\frac{p}{q} < \frac{m}{n} \quad \text{and} \quad \frac{p+1}{q} > \frac{m}{n}.$$

Therefore $\frac{p}{q}$ is the required measuring fraction. But it is clear that for every rational number $= \frac{m}{n}$ there must be a denominator q by means of which a complete measure may be found for it: we take $q = n$, and obviously also $p = m$.
2. Conversely, whenever the number S has a complete measure, it must be rational, for it is $S = \frac{p}{q}$ which is the general form of any rational number.

§ 8

Theorem. If a rational number is positive then its measuring fraction is either likewise positive or zero, if it is negative then its measuring fraction is either likewise negative or zero.

Proof. A positive rational number is representable by the positive fraction $\frac{+m}{n}$. Now if the numerator of the measuring fraction of $\frac{+m}{n}$ were neither zero nor positive, then it would have to be negative (because there is no fourth possibility). Therefore we would have to have the equation

$$+\frac{m}{n} = \frac{-p+1}{q} - P^2 = -\left[\frac{p-1}{q} + P^2\right] \quad \text{or} \quad \frac{-m}{n} = \frac{p-1}{q} + P^2.$$

Now since p is not to be zero, $\frac{p-1}{q}$ represents either zero or something positive, thus $\left[\frac{p-1}{q} + P^2\right]$ is also a strictly positive expression which therefore could not possibly equal $-\frac{m}{n}$. The proof of the second part proceeds in a similar way.

§ 9

Theorem. Every measurable number of which the measuring fraction is either zero or positive is likewise one of the two things, either zero or positive, the former only in the case when the completion of the number is zero. But if the measuring fraction of a number is negative then the number is always to be considered as negative.

Proof. If the measuring fraction $\frac{p}{q}$ of the number S is positive, then the equation $S = \frac{p}{q} + P^1$ shows that S can be represented by a *strictly positive* expression, therefore without doubt it is to be regarded as positive. But if the measuring fraction $\frac{p}{q} = 0$, then $S = P^{1}$ and therefore will also be a strictly positive expression if P^{1} is not zero. But if $P^{1} = 0$ then, of course, S itself is also $=0$. Finally if the measuring fraction is negative, $=\frac{-p}{q}$, then we have

$$S = \frac{-p+1}{q} - P^2 = -\left[\frac{p-1}{q} + P^2\right]$$

where $\frac{p-1}{q}$ is either zero or has a positive value, therefore $\left[\frac{p-1}{q} + P^2\right]$ always represents a strictly positive expression and so S represents a negative expression.

§ 10

Theorem. If A is measurable then $-A$ is also measurable.

Proof. For if A is measurable then for every value of q there is some p that can be given such that we obtain the pair of equations

$$A = \frac{p}{q} + P^{1} = \frac{p+1}{q} - P^2.$$

But from this also follows

$$-A = -\frac{p}{q} - P^{1} = -\frac{(p+1)}{q} + P^2.$$

If we now designate the number $-(p+1)$ by π, then $-p$ is designated by $\pi + 1$, and we obtain

$$-A = \frac{\pi+1}{q} - P^{1} = \frac{\pi}{q} + P^2.$$

Now if P^{1} should be $=0$ then $-A$ would be $=\frac{\pi+1}{q}$ a rational number and therefore, of course, measurable (§7). But if P^{1} is not zero, then the equations

$$-A = \frac{\pi}{q} + P^2 = \frac{\pi+1}{q} - P^{1}$$

are obviously just as the definition requires, so that $-A$ can be called measurable.

§ 11

Theorem. For one and the same number expression A, which is measurable, and for one and the same number q which is to be taken as the denominator, there are not two different numbers p^{1} and p^2 which can be chosen as the numerators of the measuring fractions so that we have the equations

$$A = \frac{p^{1}}{q} + P^{1} = \frac{p^{1}+1}{q} - P^2$$

$$\text{and} \quad A = \frac{p^2}{q} + P^3 = \frac{p^2+1}{q} - P^4.$$

Proof. Since

$$\frac{p^1+1}{q} - P^2 = \frac{p^2}{q} + P^3$$

it must also be that

$$\frac{p^1 - p^2 + 1}{q} = P^3 + P^2.$$

Therefore obviously it cannot be that $p^2 > p^1$, because otherwise $\frac{p^1-p^2+1}{q}$ would either $= 0$ or would even be negative, consequently it could not be $= P^2 + P^3$. But since also

$$\frac{p^1}{q} + P^1 = \frac{p^2+1}{q} - P^4$$

then it must be that

$$\frac{p^2 - p^1 + 1}{q} = P^1 + P^4.$$

Therefore p^1 may not be $> p^2$ because otherwise $\frac{p^2-p^1+1}{q}$ would either be $= 0$ or be negative, and thus could not be $= P^1 + P^4$. Therefore it only remains that $p^1 = p^2$.

§ 12

Theorem. If a pair of rational numbers A and B are not equal, therefore one of them, e.g. B, is the *greater*, then there is some q large enough that the p belonging to this q for B is a larger one than for A.

Proof. If we denote the p belonging to A by p^1, and the one belonging to B by p^2, then

$$A = \frac{p^1}{q} + P^1 = \frac{p^1+1}{q} - P^2$$

$$\text{and} \quad B = \frac{p^2}{q} + P^3 = \frac{p^2+1}{q} - P^4.$$

But because $B > A$, then by §4 of the 5th Section it must be that $B = A + \frac{m}{n}$, where m and n designate a pair of actual numbers. Accordingly, if we put into this latter equation the value of A from the first equation,

$$B = \frac{p^1}{q} + \frac{m}{n} + P^1 = \frac{p^1+1}{q} + \frac{m}{n} - P^2.$$

If we therefore take $q = n$, then

$$B = \frac{p^1+m}{q} + P^1 = \frac{p^1+m+1}{q} - P^2.$$

But since also

$$B = \frac{p^2}{q} + P^3 = \frac{p^2+1}{q} - P^4,$$

it follows from the previous section that it must be that $p^2 = p^1 + m$ and therefore $p^2 > p^1$. It is therefore proved that there is a value of q, namely $= n$, to which a greater p corresponds with B than with A. Thus, for example, $\frac{5}{4} > \frac{7}{6}$, in fact $\frac{5}{4} - \frac{7}{6} = \frac{1}{12}$, therefore if we take 12 for the denominator of the measuring fraction, it is found that, $\frac{5}{4} = \frac{15}{12}$, $\frac{7}{6} = \frac{14}{12}$.

§ 13

Corollary. Therefore if, conversely, two number expressions which are both finite, appear to be identical in the process of measuring, so that to the same q, not only sometimes but always, the same p belongs to both, they must be equal to each other.

§ 14

Theorem. If for some q, taken as denominator in the process of measuring, a larger numerator p is required for the finite or infinite, but measurable, expression B than for the finite or infinite, but measurable, expression A, then the difference in this numerator for smaller values of q, can become smaller, in fact it can even vanish, but it can never become negative.

Proof. If we designate the value of p, for A, belonging to the definite value of q, by p, and for B by $p + \pi$, where π denotes an actual and positive number, but p can be positive, negative or even zero, then

$$\left.\begin{aligned} A &= \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2 \\ B &= \frac{p+\pi}{q} + P^3 = \frac{p+\pi+1}{q} - P^4. \end{aligned}\right\} \qquad \text{I}$$

For some other value of q, which we shall designate by q^1, let the equations

$$\left.\begin{aligned} A &= \frac{p^1}{q^1} + P^5 = \frac{p^1+1}{q^1} - P^6 \\ B &= \frac{p^1+\pi^1}{q^1} + P^7 = \frac{p^1+\pi^1+1}{q^1} - P^8 \end{aligned}\right\} \qquad \text{II}$$

hold where we are determining at all whether π^1 denotes a positive or negative actual number, or merely zero. The combination of the second equation of I, with the first equation from II, gives

$$\frac{p+1}{q} - P^2 = \frac{p^1}{q^1} + P^5$$

from which we obtain,

$$p^1 = q^1 \frac{(p+1)}{q} - q^1(P^2 + P^5). \qquad \text{(a)}$$

The combination of the third equation from I, with the fourth equation from II, similarly gives

$$\frac{p+\pi}{q} + P^3 = \frac{p^{\mathrm{I}} + \pi^{\mathrm{I}} + 1}{q^{\mathrm{I}}} - P^8$$

from which we obtain,

$$p^{\mathrm{I}} + \pi^{\mathrm{I}} + 1 = q^{\mathrm{I}}\frac{(p+\pi)}{q} + q^{\mathrm{I}}(P^3 + P^8). \qquad \text{(b)}$$

Therefore by subtracting (a) from (b)

$$\pi^{\mathrm{I}} + 1 = q^{\mathrm{I}}\frac{(\pi - 1)}{q} + q^{\mathrm{I}}(P^2 + P^3 + P^5 + P^8).$$

It is clear to see from this equation that π^{I} is at most $=0$, but could never be negative. For the smallest negative value of π^{I}, namely -1 would give

$$0 = q^{\mathrm{I}}\frac{(\pi - 1)}{q} + q^{\mathrm{I}}(P^2 + P^3 + P^5 + P^8)$$

which is absurd because the smallest value of π is 1, for which $q^{\mathrm{I}}.\frac{(\pi-1)}{q}$ but not $q^{\mathrm{I}}(P^2 + P^3 + P^5 + P^8)$, would be zero. But we can easily be convinced by an example that under certain circumstances π^{I} could also vanish, i.e. $=0$. For if $A = \frac{4}{3}$, $B = \frac{5}{3}$, then for $q = 5$ we would have

$$A = \frac{6}{5} + \frac{2}{15} = \frac{7}{5} - \frac{1}{15} = \left(\frac{4}{3}\right)$$
$$B = \frac{8}{5} + \frac{1}{15} = \frac{9}{5} - \frac{2}{15} = \left(\frac{5}{3}\right)$$

i.e. the p belonging to $q = 5$ for A would be $= 6$, but for $B = p + \pi = 8$ so that we would have $\pi = 2$. But for the value $q = 2$,

$$A\left(=\frac{4}{3}\right) = \frac{2}{2} + \frac{1}{3} = \frac{3}{2} - \frac{1}{6}$$
$$B\left(=\frac{5}{3}\right) = \frac{3}{2} + \frac{1}{6} = \frac{4}{2} - \frac{1}{3}$$

where therefore the difference π is only $= 1$. For $q = 1$ we would especially find,

$$A\left(=\frac{4}{3}\right) = \frac{1}{1} + \frac{1}{3} = \frac{2}{1} - \frac{2}{3}$$
$$B\left(=\frac{5}{3}\right) = \frac{1}{1} + \frac{2}{3} = \frac{2}{1} - \frac{1}{3}$$

where therefore $\pi = 0$.

§ 15

Theorem. If a finite or infinite number expression A is measurable and there is therefore a p, for every arbitrary q, such that $A = \frac{p}{q} + P^{1} = \frac{p+1}{q} - P^{2}$, then that p, namely p^{1}, which belongs to a greater q, namely q^{1}, is *never smaller* than p. And whenever q^{1} is some multiple of q, e.g. nq, then p^{1} is at least the same multiple, i.e. $= np$, or it has an even greater value than nq,[h] yet it is never greater than $np + n - 1$.

Proof. 1. The equations

$$A = \frac{p}{q} + P^{1} = \frac{p+1}{q} - P^{2}$$

which express the conditions for the measurability of the number A contain no other determination of the positive expressions designated by P^{1} and P^{2} (of which the first may also just be zero) than that their sum, $P^{1}+P^{2}$, must always $= \frac{1}{q}$. Now if the denominator q^{1} of a measuring fraction $\frac{p^{1}}{q^{1}}$ is greater than the denominator q of another measuring fraction $\frac{p}{q}$ of the same number A, then I claim that the numerator p^{1} can at least never be smaller than the numerator p of the other fraction. For if it were that $p^{1} < p$, then p would have to be either $=$ or $> p^{1} + 1$. But since $A = \frac{p}{q} + P^{1}$ then also $A = \frac{p^{1}+1}{q} + P^{3}$ and because $\frac{p^{1}+1}{q^{1}}$ is $< \frac{p^{1}+1}{q}$ all the more certainly $A = \frac{p^{1}+1}{q^{1}} + P^{4}$. Therefore it is impossible that it could also be that $A = \frac{p^{1}+1}{q^{1}} - P^{5}$ as the assumption that $\frac{p^{1}}{q^{1}}$ be a measuring fraction of the number A requires. Therefore always $p^{1} =$ or $> p$.

2. In particular if $q^{1} = nq$, then p^{1} can be at least not $< np$. For even the largest number which is smaller than np, namely $np - 1$, gives a fraction, $\frac{np-1}{nq}$ which cannot be the measuring fraction of A because

$$\frac{(np-1)+1}{nq} = \frac{p}{q},$$

therefore something must not be subtracted but rather something zero or something positive, in fact P, must be added if this fraction is to become $= A$. This is all the more certain with every smaller fraction, like $\frac{np-2}{nq}$, $\frac{np-3}{nq}$, ... etc. Therefore p^{1} must either be $=$ or $> np$.

3. Finally, there is still to be proved that p^{1} could in no case be *greater* than $np + n - 1$. The next greater value is $np + n$, which would give the fraction

$$\frac{np+n}{nq} = \frac{p+1}{q}.$$

However we already know about this that it is not the measuring fraction because something positive, namely P^{1},[i] must be subtracted from it for A to be obtained.

[h] Presumably np is intended here.

[i] Presumably P^{2} is intended here.

This is all the more certain of every fraction with even greater denominator[j] e.g. $\frac{np+n+1}{nq}$, $\frac{np+n+2}{nq}$, ... : that something positive must be subtracted to obtain A. Therefore none of these fractions is a measuring fraction.

§ 16

Corollary. If, for the denominator of the measuring fraction, we proceed from 1 through all terms of the natural number series then the numerator, if it does not happen to remain constantly $= 0$, gradually takes ever greater (never smaller) values. But the value of the *fraction itself* will not steadily rise but often go lower, namely if the denominator increased by one or more units without the numerator being increased. This happens for example, if $A = \frac{2}{3}$ and the following measuring fractions arise: $\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{4}{7}, \frac{5}{8}, \frac{6}{9}, \frac{6}{10}, \frac{7}{11}, \frac{7}{12}$[k] etc.

§ 17

Corollary. Generally, if $\frac{p}{q}$ and $\frac{p^1}{q^1}$ denote two measuring fractions of one and the same measurable number A, as a consequence of the definition in §6, either zero or something strictly positive must *be added to* these fractions to obtain the number A, but from the fractions $\frac{p+1}{q}$, $\frac{p^1+1}{q^1}$ something definitely positive must be *subtracted*, to obtain the number A. It follows from this that neither $\frac{p^1}{q^1} = \frac{p+1}{q}$, nor $\frac{p^1}{q^1} > \frac{p+1}{q}$, for both mean that we must still *add* zero, or something strictly positive, to $\frac{p+1}{q}$ to obtain A. Therefore, on the contrary, it must be that $\frac{p^1}{q^1} < \frac{p+1}{q}$, and for the same reason also, $\frac{p}{q} < \frac{p^1+1}{q^1}$. This gives the relationships for p^1,

$$p^1 > \frac{q^1 p}{q} - 1 \quad \text{and} \quad p^1 < \frac{q^1(p+1)}{q},$$

i.e. the numerator of the measuring fraction of which q^1 is the denominator must always be *greater* than the fraction $q^1.\frac{p}{q}$ reduced by one, and always *smaller* than the fraction $q^1.\frac{p+1}{q}$.

§ 18

Theorem. If

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$

$$\text{and} \quad B = \frac{p+\pi}{q} + P^3 = \frac{p+\pi+1}{q} - P^4$$

[j] Presumably 'numerator' was intended here.

[k] The last term should be $\frac{8}{12}$. (*JB*)

so that the numerators of the measuring fractions of the numbers A and B which arise for the same denominator q differ by π, then for every other denominator nq which is a multiple of q, the new difference π^{1} in the numerators is at least $=n(\pi - 1) + 1$ and at most $n(\pi + 1) - 1$.

Proof. If we designate the numerators, belonging to the denominator nq, for A and B by p^{1} and $p^{1} + \pi^{1}$, then it must be that:

$$A = \frac{p^{1}}{nq} + P^{5} = \frac{p^{1} + 1}{nq} - P^{6}$$
$$B = \frac{p^{1} + \pi^{1}}{nq} + P^{7} = \frac{p^{1} + \pi^{1} + 1}{nq} - P^{8}.$$

But we know from the previous theorem that p^{1} cannot be smaller than np and cannot be greater than $np + n - 1$, and that $p^{1} + \pi^{1}$ cannot be smaller than $np + n\pi$ and cannot be greater than $np + n\pi + n - 1$. From this it follows that the difference $(p^{1} + \pi^{1}) - p^{1} = \pi^{1}$ can also not be smaller than $n\pi$[1] and cannot be greater than $n\pi + n - 1$. For obviously π^{1} cannot be smaller than it will be in the case where $p^{1} + \pi^{1}$ takes its smallest possible value and p^{1} takes its greatest possible value. Now the smallest value of $p^{1} + \pi^{1} = np + n\pi$, and the greatest value of $p^{1} = np + n - 1$; therefore the smallest value of $\pi^{1} = n\pi - n + 1 = n(\pi - 1) + 1$. On the other hand, it is obvious that π^{1} can be no greater then it is in the case when $p^{1} + \pi^{1}$ takes its greatest value and p^{1} takes its smallest value, therefore when $p^{1} + \pi^{1} = np + n\pi + n - 1$ and $p^{1} = np$. Therefore this greatest value of $\pi^{1} = n\pi + n - 1 = n(\pi + 1) - 1$. For example, suppose

$$A = \frac{71}{30} = \frac{7}{3} + \frac{1}{30} = \frac{8}{3} - \frac{9}{30}$$
$$B = \frac{117}{30} = \frac{11}{3} + \frac{7}{30} = \frac{12}{3} - \frac{3}{30}$$

and now we take $n = 12$, then

$$A = \frac{71}{30} = \frac{85}{36} + \frac{6}{1080} = \frac{86}{36} - \frac{24}{1080}$$
$$B = \frac{117}{30} = \frac{140}{36} + \frac{12}{1080} = \frac{141}{36} - \frac{18}{1080}.$$

Therefore the difference between the numerators of the first two measuring fractions is $\pi = 4$, the difference between the last two measuring fractions is $\pi^{1} = 55 > 3.12 + 1\ (=37)$ and $< 5.12 - 1\ (=59)$.

[1] The manuscript omits the completion of the expression '$-n + 1$'. (*JB*)

§ 19

Corollary 1. Therefore if the difference in the numerators of the measuring fractions of two numbers, for some denominator, amounts to more than one, e.g. 2, then it can be made greater than every given number merely by increasing the denominator. For, if $\pi > 1$, then the smallest value of $\pi^{1} = n(\pi - 1) + 1$ is an expression which grows indefinitely with the increase of n, if $\pi > 1$.

§ 20

Corollary 2. But if, for a certain denominator q, the difference found in the numerators of the measuring fractions of two numbers A and B amounts to only *one*, then we cannot assert with certainty that this difference could be made greater by increasing q. Instead there are cases for which it remains constantly one. For example, if we put

$$A = 1 - \frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}}$$

but $B = 1$, then for every q, however large, the numerator of the measuring fraction for A, or p, will be found to be $= q - 1$, but for B, $p = q$, so the difference π will therefore remain constantly $=1$.

§ 21

Theorem. Among the infinite number concepts there are also some, of such a kind, that in the process of measuring, the numerator of the measuring fraction is always found $=0$, but without our being justified in calling the number concept concerned itself zero.

Proof. The infinite number concept which the expression

$$S = \frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}}$$

designates is of this kind. For if the equations

$$S = \frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}} = \frac{p}{q} + P^{1} = \frac{p+1}{q} - P^{2}$$

are to hold, then p must be put $= 0$ however large q may be taken. For if p were an actual number, then from the equation

$$\frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}} = \frac{p}{q} + P^{1}$$

the value of $-P^{1}$

$$= \frac{p}{q} - \frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}} = \frac{p(1 + 1 + 1 + \cdots \textit{in inf.}) - q}{q(1 + 1 + 1 + \cdots \textit{in inf.})}.$$

Now since the numerator in this last fraction comprises an infinite number of units whenever p is an actual number, then the value of the fraction itself cannot possibly be $= -P^1$ (§135, EG III). But if $p = 0$, then indeed the equation $S = \frac{0}{q} + P^1$ surely holds, as also $S = \frac{0+1}{q} - P^2$. The first one is self-evident because

$$S = \frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}}$$

therefore P^1 certainly denotes a strictly positive expression if we put $S = \frac{0}{q} + P^1$. The second one is apparent because

$$S = \frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}} = \frac{1}{q} - \left(\frac{1}{q} - \frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}}\right),$$

and

$$\left(\frac{1}{q} - \frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}}\right)$$

can be put

$$= \frac{(1 + 1 + 1 + \cdots \textit{in inf.}) - q}{q(1 + 1 + 1 + \cdots \textit{in inf.})}$$

which again is regarded as equal to a strictly positive expression, and thus can be expressed by P^2. Finally, it is clear that we are not justified, at least by the concepts so far, in considering the expression

$$\frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}}$$

as equivalent to zero. For according to the definition of §116, EG III, zero is only the idea of such a thing which has to be added to A, so that A itself again appears, or $A - A$.

§ 22

Definition. The number, which a number concept like that described in the previous theorem represents as its object, I allow myself to call an *infinitely small*, and indeed *absolute* [*absolute*], or also *positive* number. Therefore I always understand by such a number one which in the attempted process of measuring for any q, however large it is taken, the numerator of the measuring fraction is always $= 0$, or for which the two equations hold, $S = P^1 = \frac{1}{q} - P^2$ where q is taken as large as we please. If S denotes an infinitely small absolute number, then by contrast I call $-S$ an infinitely small *negative* number. Accordingly therefore, for every infinitely small negative number the equations $S = -P^1 = -\frac{1}{q} + P^2$ hold. A number which is measurable and yet *not* infinitely small should be called a *finite* number. Henceforth, therefore, finite *expressions* and finite *numbers* may not be confused with one another. For an *infinite expression* can also designate a merely *finite number* (§5).

§ 23

Corollary. According to this definition, the number I call *infinitely small*, is not an actual number—it is not even a fraction of which the denominator and numerator are actual numbers. For no actual number, and likewise also no fraction of which the numerator and denominator are actual numbers, satisfies the equations $S = \pm P^{1} = \pm\left(\frac{1}{q} - P^{2}\right)$, if q may become as large as we please. If we denote the absolute value of this actual number, or of this fraction, by A, then it would have to be that $A + P^{2} = \frac{1}{q}$, which is absurd if $\frac{1}{q}$ can decrease indefinitely.

§ 24

Corollary. If the two equations

$$S = \pm P^{1} = \pm\left(\frac{m}{n} - P^{2}\right)$$

hold, in which n, for the same m, may be taken as large as we please, then S is infinitely small. For if we put $n = mq$ then

$$\frac{m}{n} = \frac{1}{q}.$$

Therefore

$$S = \pm P^{1} = \pm\left(\frac{1}{q} - P^{2}\right)$$

from which it is clear that S behaves exactly like an infinitely small number.

§ 25

Theorem. If, in the process of measuring a given number A, to every multiple of q, $=nq$, as denominator of the measuring fraction, only the same multiple of the numerator $= np$ occurs, so that always

$$A = \frac{np}{nq} + P^{1} = \frac{np+1}{nq} - P^{2},$$

however large n is taken, then P^{1} must be *infinitely small*.

Proof. For $P^{1} = \frac{1}{nq} - P^{2}$ and $\frac{1}{nq}$ can decrease indefinitely (§4 of the 6th Section).

§ 26

Theorem. Among the infinite number concepts there are also some of such a kind that with the attempt at measuring them to every given q, a p can indeed be found that satisfies one of the two equations

$$S = \frac{p}{q} + P^{\mathrm{I}} = \frac{p^{\mathrm{I}} + 1}{q} - P^2,$$ [m]

but no p that satisfies both equations.

Proof. The infinite number concept $1 + 2 + 3 + 4 + \cdots$ *in inf.*, which contains the sum of all the natural numbers, is of such a kind. For it is self-evident that to every q, a p can be given that satisfies the first of the two equations above, namely $S = \frac{p}{q} + P^{\mathrm{I}}$. For example, if we take $p = q$, so $\frac{p}{q} = 1$ and

$$1 + 2 + 3 + 4 + \cdots \textit{in inf.} = \frac{p}{q} + (2 + 3 + 4 + \cdots \textit{in inf.})$$

where $(2 + 3 + 4 + \cdots \textit{in inf.})$ is obviously a strictly positive expression. But that no p can be given of such a kind that S would be $= \frac{p+1}{q} - P^2$ is also clear from the following. Certainly there is, to every q a p large enough so that $\frac{p+1}{q} = N + 1$, however large a number N denotes. For this only needs p to be taken $= qN + q - 1$. But if $\frac{p+1}{q} = N + 1$, then, because $1 + 2 + 3 + 4 + \cdots$ *in inf.* can obviously also be written $= 1 + 2 + \cdots + N + (N + 1) + (N + 2) + \cdots$ *in inf.* $= (N + 1) + [1 + 2 + \cdots + N + (N + 2) + (N + 3) + \cdots$ *in inf.*$]$,

$$S = N + 1 + [1 + 2 + \cdots + N + (N + 2) + (N + 3) + \cdots \textit{in inf.}]$$

where the expression in square brackets is strictly positive, and therefore can in no way $= - P^2$. If we had taken $S = -1 - 2 - 3 - \cdots$ *in inf.* it would have been shown in the same way that there is a p which satisfies the equation $S = \frac{p+1}{q} - P^2$, but none which satisfies the equation $S = \frac{p}{q} + P^{\mathrm{I}}$.

§ 27

Definition. The number which represents an infinite number concept of the kind just described I wish to give the name an *infinitely large number*. Therefore I call a number S *infinitely large*, if for every q there is a p such that one of the two equations

$$S = \frac{p}{q} + P^{\mathrm{I}} = \frac{p + 1}{q} - P^2$$

is satisfied but there is no p which satisfies them both at once. If the number S is of such a kind that the first equation, but not the second can be satisfied, then I call the infinitely large number S *positive*. On the other hand, in the case where the second equation can be satisfied but not the first, then I call the infinitely large number S *negative*. Therefore, for example, $1 + 2 + 3 + \cdots$ *in inf.* is a positive, but $-1 - 2 - 3 - \cdots$ *in inf.* is a negative, infinitely large number.

[m] Presumably the p^{I} in this equation should read p.

§ 28

Corollary. The number to which, by this definition, I give the name of an *infinitely large* number is therefore not unmeasurable (§6), and therefore is neither an actual number nor even a fraction of which the numerator and denominator are actual numbers (§7).

§ 29

Corollary. However, the converse does not hold: every number that is unmeasurable does not have to be infinitely large. For example, $1 - 1 + 1 - 1 + \cdots$ *in inf*. is indeed not measurable and yet is not infinitely large. I say the number which this expression represents is unmeasurable because no value can be given for p which satisfies even only one of the two equations

$$S = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2.$$

But just on account of this, the number represented by this expression is not infinitely large because the definition given would require as a consequence of this that at least one of those two equations could be satisfied.

§ 30

Corollary. There can also be number expressions whose number is neither finite, nor infinitely small, nor infinitely large. For example, zero is of this kind, and also the number expression mentioned before, $1 - 1 + 1 - 1 + \cdots$ *in inf*.

§ 31

Theorem. If a number S is infinitely large then for every actual number N, however large, the equation $S = \pm(N+P)$ holds, and conversely, whenever for every value of N, however large, the equation $S = \pm(N + P)$ holds, then S is infinitely large.

Proof. The first part is clear directly from the given definition. If for every p, however large, $S = \pm\left(\frac{p}{q} + P\right)$ then if we take $p = qN$, $S = \pm(N + P)$. If, conversely, the equation $S = \pm(N + P)$ holds for every value of N however large, then there is no p so large that $S = \pm\left(\frac{p+1}{q} - P\right)$. For if $S = \pm\left(\frac{p+1}{q} - P^1\right)$, then also if we take $N > \frac{p+1}{q}$ it would have to be that $\pm(N + P) = \pm\left(\frac{p+1}{q} - P^1\right)$, i.e. $N - \frac{p+1}{q} = -(P + P^1)$, which is absurd since $N - \frac{p+1}{q}$ has a positive value.

§ 32

Theorem. If A designates an infinitely large number, but $\frac{m}{n}$ is some rational number, then also $A \pm \frac{m}{n}$ is an infinitely large number.

Proof. Suppose, first, that A is positive, then (§31), $A = N + P$. Now if also $\frac{m}{n}$ is positive, then there is no doubt that

$$A + \frac{m}{n} = N + \left(P + \frac{m}{n}\right) = N + P^{\mathrm{I}}$$

is again infinitely large. But also $A - \frac{m}{n}$ is infinitely large. Because the number N can be taken as large as we please, if we put $N = N^{\mathrm{I}} + m$, then also N^{I} still denotes a number which can increase indefinitely. But

$$A - \frac{m}{n} = N - \frac{m}{n} + P = N^{\mathrm{I}} + m - \frac{m}{n} + P = N^{\mathrm{I}} + \left(\frac{m(n-1)}{n} + P\right) = N^{\mathrm{I}} + P^{\mathrm{I}}$$

and is therefore infinitely large. The proof is done in a similar way if A is negative.

§ 33

Theorem. If an infinite number expression A is of such a nature that for every arbitrary positive or negative value of the numbers m and n the expression $A - \frac{m}{n}$ is either positive or negative or zero, then A is either infinitely large or measurable.

Proof. If it turns out that the expression $A - \frac{m}{n}$ always has the same sign, however large the positive or negative numerator m is taken while we keep n unchanged, then it follows from §27 that A must be *infinitely large*, and indeed positive if the sign of $A - \frac{m}{n}$ is always positive, but negative if this sign is always negative. If, on the other hand, there are certain values of m and n, and a certain sign for m, for which the expression $A - \frac{m}{n}$ is zero, then $A = \frac{m}{n}$, is simply a rational number and therefore by §7 is, without doubt, measurable. Finally, if there should be no values at all of n and m for which $A - \frac{m}{n}$ is zero, but there are values for which this expression is positive, and others for which it is negative, then

$$A - \frac{m}{n} = P \quad \text{and} \quad A - \frac{m^{\mathrm{I}}}{n^{\mathrm{I}}} = -P^{\mathrm{I}}.$$

For brevity we shall assume $\frac{m}{n}$, as well as $\frac{m^{\mathrm{I}}}{n^{\mathrm{I}}}$, are positive and $\frac{m^{\mathrm{I}}}{n^{\mathrm{I}}} > \frac{m}{n}$, because in other cases the proof proceeds in a very similar way. Now let q designate an arbitrary number which we want to take as the denominator for the attempted measuring of the expression A, and let p be the actual, or next smaller, quotient of the attempted division of n into mq, and p^{I} the actual, or next smaller, quotient of the attempted division of n^{I} into $m^{\mathrm{I}}q$. So we therefore have

$$\frac{m}{n} = \text{ or } > \frac{p}{q} \quad \text{and} \quad \frac{m^{\mathrm{I}}}{n^{\mathrm{I}}} = \text{ or } > \frac{p^{\mathrm{I}}}{q},$$

and certainly

$$\frac{m^{\mathrm{I}}}{n^{\mathrm{I}}} < \frac{p^{\mathrm{I}} + 1}{q}.$$

If we now consider the series of fractions

$$\frac{p}{q}, \frac{p+1}{q}, \frac{p+2}{q}, \frac{p+3}{q}, \ldots, \frac{p^1}{q}, \frac{p^1+1}{q};$$

or rather the series of remainders

$$A-\frac{p}{q}, A-\frac{p+1}{q}, A-\frac{p+2}{q}, A-\frac{p+3}{q}, \ldots, A-\frac{p^1}{q}, A-\frac{p^1+1}{q},$$

then we know, by virtue of the assumption of our theorem, that the property of being either positive or negative must belong to each of these expressions. Moreover, we know that the first of these remainders is positive, but the last is negative. Because $A-\frac{m}{n}$ is positive, but $\frac{m}{n}=$ or $>\frac{p}{q}$ then all the more $A-\frac{p}{q}$ must certainly be positive because

$$A-\frac{p}{q}=A-\frac{m}{n}+\left(\frac{m}{n}-\frac{p}{q}\right).$$

And because $A-\frac{m^1}{n^1}$ is negative, then all the more, since $\frac{p^1+1}{q} > \frac{m^1}{n^1}$, then

$$A-\frac{p^1+1}{q}=A-\frac{m^1}{n^1}-\left(\frac{p^1+1}{q}-\frac{m^1}{n^1}\right)$$

must certainly be negative. It therefore follows from §188, EG III that among the positive terms occurring in this series there must be a highest one. Let us denote this by $\frac{p+\pi}{q}$, then we have

$$A-\frac{p+\pi}{q}=P^1 \quad \text{and} \quad A-\frac{p+\pi+1}{q}=-P^2$$

from which the two equations

$$A=\frac{p+\pi}{q}+P^1 \quad \text{and} \quad A=\frac{p+\pi+1}{q}-P^2$$

arise, which proves the measurability of the expression A.

§ 34

Theorem. The sum of two infinitely small and infinitely large numbers, which both have the same sign, again gives rise to an infinitely small or infinitely large number with the same sign.

Proof. Clearly it will be sufficient to prove both propositions only for the case where those signs are positive.

1. If S and T denote a pair of *infinitely small* numbers which are both positive, then it must be that

$$S=P^1=\frac{1}{q}-P^2 \quad \text{and} \quad T=P^3=\frac{1}{r}-P^4$$

where q and r can be as large as we please. Therefore the sum

$$S + T = P^1 + P^3 = \frac{1}{q} + \frac{1}{r} - (P^2 + P^4) = \frac{r + q}{qr} - (P^2 + P^4).$$

Now it is easy to see that q and r can always be taken so that $\frac{r+q}{qr}$ equals an arbitrary fraction of the form $\frac{1}{n}$, with n as large as we please. For example, if we just take $q = r = 2n$, then

$$\frac{r + q}{qr} = \frac{2n + 2n}{4nn} = \frac{1}{n}.$$

Then we obtain

$$S + T = P^5 = \frac{1}{n} - P^6,$$

from which we see that the sum $(S + T)$ behaves exactly like an infinitely small number (§22).

2. If S and T denote a pair of *infinitely large* numbers then it must be that $S = N^1 + P^1$, $T = N^2 + P^2$, where N^1 and N^2 denote a pair of actual numbers increasing indefinitely (§1 of the 6th Section). This yields the sum $S + T = (N^1 + N^2) + (P^1 + P^2)$. But if N^1 and N^2 can become as large as we please then $(N^1 + N^2)$ also denotes an actual number which can become as large as we please. Therefore the sum $(S + T)$ behaves exactly like an infinitely large positive number (§27).

§ 35

Corollary. Therefore every sum composed of a merely finite multitude of infinitely small numbers of the same sign, is infinitely small, and every such sum of infinitely large numbers, is infinitely large.

§ 36

Corollary. It is not therefore the same if the numbers combined in a sum have *different* signs or, what amounts to the same, if we ask about their *difference* instead of their sum. The difference between a pair of *infinitely large* numbers can be, according to circumstances, zero, infinitely small, finite or infinitely large. The first case can occur because the two numbers can be equal to one another, the other cases can arise because a number which is infinitely large remains so if an infinitely small number or a finite number or even an infinitely large number is added to it. But as for the difference between *infinitely small* numbers, we shall determine it later.

§ 37

Theorem. If an infinitely small or an infinitely large number is multiplied by a finite number, different from zero, then the resulting product is, in the first case infinitely small, in the second case infinitely large. And this is regardless of the order of the two factors.

Proof. We need only prove these propositions for the case when both factors are positive, for the remaining cases differ at most in the sign of the result.

1. Now if J denotes an infinitely small positive number, then

$$J = P^1 = \frac{1}{q} - P^2$$

for every value of q, however large. Furthermore, if A is a finite, positive number and different from zero, then

$$A = \frac{r}{s} + P^3 = \frac{r+1}{s} - P^4$$

and r is not $=0$. Therefore the product

$$J.A = P^1.\left(\frac{r}{s} + P^3\right) = \left(\frac{1}{q} - P^2\right)\left(\frac{r+1}{s} - P^3\right)$$
$$= \frac{r+1}{qs} - \left[\frac{1}{q}.P^3 + P^2\left(\frac{r+1}{s} - P^3\right)\right].$$

Now since $P^1.\left(\frac{r}{s} + P^3\right)$, and likewise the expression occurring in the square brackets, are obviously positive, and the fraction $\frac{r+1}{qs}$ is always equal to the fraction $\frac{1}{n}$, if, for example, we just take $s = n$ and $q = r + 1$, then it is clear that JA behaves like an infinitely small number. The same holds of the product

$$AJ = \left(\frac{r}{s} + P^3\right) P^1 = \left(\frac{r+1}{s} - P^3\right)\left(\frac{1}{q} - P^2\right)$$
$$= \frac{r+1}{s.q} - \left[\frac{r+1}{s}P^2 + P^3\left(\frac{1}{q} - P^2\right)\right].$$

2. If U designates an infinitely large positive number then it must be that $U = N + P$. Consequently

$$U.A = (N+P)\left(\frac{p}{q} + P^1\right) = N.\frac{p}{q} + \left[NP^1 + P\left(\frac{p}{q} + P^1\right)\right].$$

Now since $N.\frac{p}{q}$ can become equal to every actual number however large, and the expression in the square brackets is strictly positive, then it may be seen that $U.A$ behaves like an infinitely large number. The same holds of the product

$$A.U = \left(\frac{p}{q} + P^1\right)(N+P) = \frac{p}{q}.N + \left[\frac{p}{q}P + P^1(N+P)\right].$$

§ 38

Theorem. If a pair of infinitely small numbers, or a pair of infinitely large numbers are multiplied with one another, then the product in the first case is infinitely small, and in the second case is infinitely large.

Proof. Let us consider only the case when both factors are positive.

1. Now if S and T are a pair of infinitely small positive numbers, then it must be that

$$S = P^1 = \frac{1}{q} - P^2 \quad \text{and} \quad T = P = \frac{1}{r} - P^4$$

for every value of q and r. Therefore

$$S.T = P^1.P^3 = \left(\frac{1}{q} - P^2\right)\left(\frac{1}{r} - P^4\right) = \frac{1}{q}.\frac{1}{r} - \left[\frac{1}{q}.P^4 + \frac{1}{r}P^2 - P^2P^4\right].$$

Now since $\frac{1}{q}.\frac{1}{r}$ can become equal to any arbitrary fraction of the form $\frac{1}{n}$ if we take $r = n$, $q = 1$, then the product $S.T$ obviously behaves like an infinitely small number.

2. If S and T are a pair of infinitely large positive numbers then it must be that $S = N + P$ and $T = N^1 + P^1$. Therefore $S.T = (N + P)(N^1 + P^1) = NN^1 + [N.P^1 + P(N^1 + P^1)]$ which is obviously the form of an infinitely large number.

§ 39

Theorem. If every factor of a product is a finite number, and the number of factors is itself only finite, then the product is neither infinitely small nor infinitely large.

Proof. We need to prove the proposition for a product of *two* factors and, moreover, we can assume both are positive. Now if A and B are a pair of such factors then

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2,$$
$$B = \frac{r}{s} + P^3 = \frac{r+1}{s} - P^4.$$

Therefore

$$A.B = \left(\frac{p}{q} + P^1\right)\left(\frac{r}{s} + P^3\right)$$
$$= \frac{p}{q}.\frac{r}{s} + \left[\frac{p}{q}.P^3 + P^1\left(\frac{r}{s} + P^3\right)\right],$$

and from this equation it is clear that $A.B$ cannot be infinitely small. Furthermore

$$A.B = \left(\frac{p+1}{q} - P^2\right)\left(\frac{r+1}{s} - P^4\right)$$
$$= \frac{p+1}{q}.\frac{r+1}{s} - \left[\frac{p+1}{q}.P^4 + P^2\left(\frac{r+1}{s} - P^4\right)\right];$$

and from this equation it is clear that $A.B$ is not infinitely large (§27).

§ 40

Theorem. Even if a finite number is *divided* by another finite number, different from zero, the quotient is neither infinitely small nor infinitely large.

Proof. For in the first case, the product of a finite number with an infinitely small number would have to give a finite number, and in the second case the product of a finite number with an infinitely large number would have to give a finite number, neither of which is the case (§37).

§ 41

Theorem. If a finite, or even infinite, number expression A is measurable, then it can be determined for every arbitrary positive or negative rational number $\frac{m}{n}$ whether the difference $A - \frac{m}{n}$ is zero, or positive, or negative, and it is always only one of these three.

Proof. If A is measurable then by the definition of §6 for every arbitrary value of q, therefore also for $q = n$, it must be that a p can be found which satisfies the two equations,

$$A = \frac{p}{n} + P^{1} = \frac{p+1}{n} - P^{2}$$

where P^{1} is zero or positive, but P^{2} must always be positive.

1. Now if $P^{1} = 0$ then $A = \frac{p}{n}$ and now it merely depends on whether $m = p$, or is $>p$, or is $<p$. One and only of these three can occur. If $p = m$, then we have $A - \frac{m}{n} = 0$. If $m > p$ then we have $\frac{m}{n} > \frac{p}{n}$. If $m < p$ then we have $\frac{m}{n} < \frac{p}{n}$. Therefore for the case $P^{1} = 0$ one of these three relationships undoubtedly occurs.
2. But if P^{1} is not zero then the expression $A - \frac{p}{n} = P^{1}$ represents a positive number. All the more certainly the expressions

$$A - \frac{p-1}{n}, A - \frac{p-2}{n}, A - \frac{p-3}{n}, \ \ldots \textit{in inf.},$$

which arise from the previous expression if we add $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots$, respectively, are positive. Furthermore, $A - \frac{p+1}{n}$ represents a negative number because it has to be $= -P^{2}$; therefore all the more certainly the expressions

$$A - \frac{p+2}{n}, A - \frac{p+3}{n}, A - \frac{p+4}{n}, \ \ldots \textit{in inf.},$$

which arise from the former expression if we subtract $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots$ are also negative. Now since the value m must obviously appear in one of the two series

$$p, p-1, p-2, p-3, \ldots \textit{in inf.};$$

$$p+1, p+2, p+3, \ldots \textit{in inf.}$$

then it is already decided whether $A - \frac{m}{n}$ is positive or negative according to whether p appears in the first, or the second, series. Certainly only one of these two cases can occur, and therefore, altogether only one of the three cases $A - \frac{m}{n} = 0$, $A - \frac{m}{n} = P$, $A - \frac{m}{n} = -P$ can occur.

§ 42

Corollary. If, with the same positive value of n, we take the number m to be at one time positive, and at another time negative, and in either case we let its absolute value increase indefinitely, then the value of the expression $A + \frac{m}{n}$ must certainly sometime become positive, and that of the expression $A - \frac{m}{n}$ must certainly sometime become negative. For if $A + \frac{m}{n}$ were never positive, however large m was made, then by §27, A would have to be a negative infinitely large number, because for every value of p and q we would have $A = -\frac{p}{q} - P$. And if, on the other hand, $A - \frac{m}{n}$ were never negative however large m was made, then by §27, A would have to be a positive infinitely large number because we would always have $A = \frac{m}{n} + P$. In no case then would A be measurable.

§ 43

Theorem. If A represents a measurable number, and $\frac{m}{n}$ represents an arbitrary positive or negative rational number, then also $A \pm \frac{m}{n}$ represents a measurable number, i.e. the sum, and likewise the difference, of a measurable number and a rational number are themselves measurable numbers.

Proof. For every arbitrary number q which we take for the denominator, and for every arbitrary numerator p which we can put with this denominator, it can be determined whether the expression $\left(A \pm \frac{m}{n}\right) - \frac{p}{q}$ is zero, or positive, or negative, and furthermore it is always one and only one of these three. This results from the previous theorem because

$$\left(A \pm \frac{m}{n}\right) - \frac{p}{q} = A \pm \frac{mq \mp np}{nq}$$

and $\frac{mq \mp np}{nq}$ represents a positive or negative rational number like the $\frac{m}{n}$ in the theorem. If we now find

$$\left(A \pm \frac{m}{n}\right) - \frac{p}{q} = 0,$$

then

$$\left(A \pm \frac{m}{n}\right) = \frac{p}{q},$$

therefore $\frac{p}{q}$ is the perfect measure of the number $\left(A \pm \frac{m}{n}\right)$. But if we find $\left(A \pm \frac{m}{n}\right) - \frac{p}{q}$ is positive, then if we make the numerator p gradually greater, by continually adding 1 to it, an expression arises which is certainly no longer positive but is negative; and if $\left(A \pm \frac{m}{n}\right) - \frac{p}{q}$ is negative then by similar decrease of p this expression would certainly not become positive. This follows from the previous corollary because the indicated increase or decrease of p results in an indefinite increase or decrease of the rational number $\frac{mq \mp np}{nq}$. But if there is a value of p, e.g. $p+\pi^{1}$, which makes the expression $\left(A \pm \frac{m}{n}\right) - \frac{p}{q}$ negative and one, e.g. $p-\pi^{2}$, which makes this expression positive then there is no doubt that in the series

$$p-\pi^{2}, p-\pi^{2}+1, p-\pi^{2}+2, p-\pi^{2}+3, \ldots, p+\pi^{1}-2, p+\pi^{1}-1, p+\pi^{1}$$

there is a value of p (we will denote it simply by p) which is the greatest of those which make the expression negative. Therefore the one following it makes this expression either zero or positive. In the first case

$$\left(A \pm \frac{m}{n}\right) = \frac{p}{q}$$

and so $\frac{p}{q}$ is the perfect measure of $A \pm \frac{m}{n}$, but in the second case we have both the equations

$$\left(A \pm \frac{m}{n}\right) = \frac{p}{q} + P^{1} = \frac{p+1}{q} - P^{2},$$

which proves the measurability of the expression $\left(A \pm \frac{m}{n}\right)$.

§ 44

Corollary. The symbols P^{1} and P^{2} in the two equations which establish the measurability of a number, namely

$$A = \frac{p}{q} + P^{1} = \frac{p+1}{q} - P^{2},$$

therefore themselves always represent a pair of measurable numbers. For if A is measurable, then so also must be $A - \frac{p}{q}$ and $A - \frac{p+1}{q}$.

§ 45

Theorem. If a pair of numbers A and B are measurable then also their sum or difference represents a measurable number.

Proof. Understandably it will be enough to prove the proposition concerning the sum. Now if A and B are measurable then for every arbitrary value of q,

$$\left.\begin{aligned} A &= \frac{p^{1}}{q} + P^{1} = \frac{p^{1}+1}{q} - P^{2} \\ B &= \frac{p^{2}}{q} + P^{3} = \frac{p^{2}+1}{q} - P^{4}. \end{aligned}\right\} \quad \text{I}$$

But according to the previous corollary also P^{1}, P^{2}, P^{3} and P^{4} are measurable numbers and therefore for every arbitrary value of q, and therefore also the value nq, however large n is taken, we must have the following equations:

$$\left.\begin{aligned} P^{1} &= \frac{r^{1}}{nq} + P^{5} = \frac{r^{1}+1}{nq} - P^{6} \\ P^{3} &= \frac{r^{2}}{nq} + P^{9} = \frac{r^{2}+1}{nq} - P^{10} \end{aligned}\right\} \quad \text{II}$$

$$\left.\begin{aligned} P^2 &= \frac{s^1}{nq} + P^7 = \frac{s^1+1}{nq} - P^8 \\ P^4 &= \frac{s^2}{nq} + P^{11} = \frac{s^2+1}{nq} - P^{12}. \end{aligned}\right\} \qquad \text{III}$$

From the combination of I with II and III there results

$$A + B = \frac{p^1+p^2}{q} + \frac{r^1+r^2}{nq} + P^{13} = \frac{p^1+p^2+2}{q} - \frac{s^1+s^2}{nq} - P^{14}. \qquad \text{IV}$$

Now if for some value of n,

$$\frac{r^1+r^2}{nq} \geqq \frac{1}{q}, \quad \text{or} \quad \frac{s^1+s^2}{nq} \geqq \frac{1}{q},$$

the measurability of the number $A+B$ is beyond doubt. For in the first case

$$A + B = \frac{p^1+p^2+1}{q} + P^{15} = \frac{p^1+p^2+2}{q} - P^{16}$$

therefore the measuring fraction of $A + B = \frac{p^1+p^2+1}{q}$; but in the second case

$$A + B = \frac{p^1+p^2}{q} + P^{17} = \frac{p^1+p^2+1}{q} - P^{18},$$

therefore the measuring fraction is $\frac{p^1+p^2}{q}$. Therefore there only remains the case in which, however large the value of n, each of the two expressions

$$\frac{r^1+r^2}{nq} \quad \text{and} \quad \frac{s^1+s^2}{nq}$$

always remains smaller than $\frac{1}{q}$. The addition of the four initial equations in II and III gives

$$P^1 + P^2 + P^3 + P^4 = \frac{r^1+r^2+s^1+s^2}{nq} + P^{19}$$

and the addition of the four final equations gives

$$P^1 + P^2 + P^3 + P^4 = \frac{r^1+r^2+s^1+s^2+4}{nq} - P^{20}.$$

But it is known from the four equations in I that

$$P^1 + P^2 + P^3 + P^4 = \frac{2}{q}$$

must hold; we therefore obtain

$$\frac{2}{q} = \frac{r^1+r^2+s^1+s^2}{nq} + P^{19} = \frac{r^1+r^2+s^1+s^2+4}{nq} - P^{20}.$$

From this it is clear first of all that

$$P^{19} + P^{20} = \frac{4}{nq}$$

and consequently that each of the numbers P^{19} and P^{20} decrease indefinitely with the infinite increase in n, from which it further follows that

$$\frac{r^1 + r^2 + s^1 + s^2}{nq} = \frac{2}{q} - P^{19}$$

approaches the value $\frac{2}{q}$ indefinitely while n increases indefinitely. Therefore each of the two expressions

$$\frac{r^1 + r^2}{nq} \quad \text{and} \quad \frac{s^1 + s^2}{nq}$$

must approach indefinitely the value $\frac{1}{q}$, or we may write

$$\frac{r^1 + r^2}{nq} = \frac{1}{q} - \Omega^1 \quad \text{and} \quad \frac{s^1 + s^2}{nq} = \frac{1}{q} - \Omega^2,$$

if we understand by Ω^1 and Ω^2 a pair of numbers which decrease indefinitely with the infinite increase in n. Then from IV,

$$A + B = \frac{p^1 + p^2}{q} + \frac{1}{q} - \Omega^1 + P^{13} = \frac{p^1 + p^2 + 2}{q} - \frac{1}{q} + \Omega^2 - P^{14}.$$

Therefore

$$P^{13} + P^{14} = \Omega^1 + \Omega^2,$$

i.e. P^{13} and P^{14} also decrease indefinitely and accordingly (§8 of the 6th Section)

$$A + B = \frac{p^1 + q^1 + 1}{q}.$$ [n]

We therefore see that in each case $A + B$ behaves like a measurable number.

§ 46

Corollary. Therefore every algebraic sum of measurable numbers, provided there is not an infinite number of them, is again a measurable number.

§ 47

Theorem. If the summands $A, B, C, \ldots, L$, of which there are n, have respectively the measuring fractions $\frac{p^1}{q}, \frac{p^2}{q}, \frac{p^3}{q}, \ldots, \frac{p^n}{q}$, then the measuring fraction of the sum $A + B + C + \cdots + L$ can never be smaller than $\frac{p^1+p^2+p^3+\cdots+p^n}{q}$, and can never be greater than $\frac{p^1+p^2+p^3+\cdots+p^n+n-1}{q}$.

[n] Presumably the numerator of the right-hand side should be $p^1 + p^2 + 1$.

Proof. For if

$$A = \frac{p^1}{q} + P^1 = \frac{p^1 + 1}{q} - P_1$$
$$B = \frac{p^2}{q} + P^2 = \frac{p^2 + 1}{q} - P_2$$
$$C = \frac{p^3}{q} + P^3 = \frac{p^3 + 1}{q} - P_3$$
$$L = \frac{p^n}{q} + P^n = \frac{p^n + 1}{q} - P_n$$

then

$$\begin{aligned} A + B + C + \cdots + L &= \frac{p^1 + p^2 + p^3 + \cdots + p^n}{q} \\ &\quad + [P^1 + P^2 + P^3 + \cdots + P^n] \\ &= \frac{p^1 + p^2 + p^3 + \cdots + p^n + n}{q} \\ &\quad - [P_1 + P_2 + P_3 + \cdots + P_n]. \end{aligned}$$

From this it is now very clear that the measuring fraction could *not be smaller* than $\frac{p^1+p^2+p^3+\cdots+p^n}{q}$ because this fraction will in fact only be the measuring fraction if

$$\frac{p^1 + p^2 + p^3 + \cdots + p^n + 1}{q} = (A + B + C + \cdots + L) - P$$

i.e. if

$$P^1 + P^2 + P^3 + \cdots + P^n = \frac{1}{q} - P.$$

But it is also just as clear that the measuring fraction could *never be greater* than $\frac{p^1+p^2+p^3+\cdots+p^n+n-1}{q}$ because the next fraction $\frac{p^1+p^2+p^3+\cdots+p^n+n}{q}$ already amounts to $(A + B + C + \cdots + L) - P$.

§ 48

Theorem. If a number A is measurable then the two equations

$$A = \frac{\alpha}{a} + \frac{\beta}{ab} + \frac{\gamma}{abc} + \frac{\delta}{abcd} + \cdots + \frac{\mu}{abcd \ldots m} + P^1$$
$$A = \frac{\alpha}{a} + \frac{\beta}{ab} + \frac{\gamma}{abc} + \frac{\delta}{abcd} + \cdots + \frac{\mu + 1}{abcd \ldots m} - P^2$$

can always be formed, in which the symbols $a, b, c, d, \ldots, m$ denote certain arbitrary actual and absolute numbers which, starting from the second, are all > 1. The symbol α denotes a positive or negative actual number, or even just zero, but all the symbols $\beta, \gamma, \delta, \ldots, \mu$, denote only positive actual numbers or zeros so that, at least, $\beta < b$, $\gamma < c$, $\delta < d, \ldots, \mu < m$. Finally the symbols P^1 and P^2 have the usual meaning, namely the first should represent, if not zero, something positive, the second should represent something positive.

Proof. Because A is measurable, then for every arbitrary actual and positive number, therefore also for the number a which is chosen for the denominator of the measuring fraction, there must be a numerator α which is either positive or negative or even equal to zero, which satisfies the equations

$$A = \frac{\alpha}{a} + P_1 = \frac{\alpha + 1}{a} - P_2.$$

Now if $P_1 = 0$ then the two equations alleged in the theorem are satisfied if we let the symbols $\beta, \gamma, \delta, \ldots, \mu$ and P^1 all denote zero, but P^2 denote $\frac{1}{a}$. But if P_1 is not zero it is, according to §44, nevertheless a measurable number, and there must therefore be for each denominator, so also for ab, a numerator β which satisfies the two equations

$$P_1 = \frac{\beta}{ab} + P_3 = \frac{\beta + 1}{ab} - P_4.$$

But because P_1 must be $<\frac{1}{a}$, as is clear from the last two equations, then it is clear that β must be $<b$. If we now substitute these two values of P_1 in the first equation above, we obtain

$$A = \frac{\alpha}{a} + \frac{\beta}{ab} + P_3 = \frac{\alpha}{a} + \frac{\beta + 1}{ab} - P_4$$

two equations which are exactly as our theorem requires them. If the number designated by P_3 is not zero, then since it must nevertheless be measurable, the measuring fraction belonging to it can again be determined for any arbitrary denominator so also for the denominator abc. If we designate this fraction by $\frac{\gamma}{abc}$ then γ will be a number which in any case is $<c$, because P_3 itself must be $<\frac{1}{ab}$, and we will have

$$P_3 = \frac{\gamma}{abc} + P_5 = \frac{\gamma + 1}{abc} - P_6.$$

If we put these two values of P_3 in the previous equation then

$$A = \frac{\alpha}{a} + \frac{\beta}{ab} + \frac{\gamma}{abc} + P_5 = \frac{\alpha}{a} + \frac{\beta}{ab} + \frac{\gamma + 1}{abc} - P_6.$$

It is now evident how these steps can be continued and lead to the general formulae of the theorem.

Example. Suppose $A = \frac{1}{3}$ and we had taken $a = 1$, $b = c = d = e = 10$ then it would be found that $\alpha = 0$, $\beta = 3$, $\gamma = 3$, $\delta = 3$ etc. so that,

$$\frac{1}{3} = \frac{0}{1} + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10\,000} + P^1$$

$$\frac{1}{3} = \frac{0}{1} + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{4}{10\,000} - P^2.$$

§ 49

Theorem. If A and B are a pair of measurable numbers then their product $A.B$ is also a measurable number.

Proof. Let

$$\left.\begin{aligned} A &= \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2 \\ B &= \frac{r}{s} + P^3 = \frac{r+1}{s} - P^4 \end{aligned}\right\} \qquad \text{I}$$

and for some value of the number n, which we want to take once and for all >1 we find,

$$\left.\begin{aligned} A &= \frac{\pi}{nq} + P^5 = \frac{\pi+1}{nq} - P^6 \\ B &= \frac{\rho}{ns} + P^7 = \frac{\rho+1}{ns} - P^8. \end{aligned}\right\} \qquad \text{II}$$

By multiplication, these equations give

$$\left.\begin{aligned} A.B &= \frac{p}{q}.\frac{r}{s} + P^9 = \frac{rp+p+r+1}{qs} - P^{10} \\ \text{or also} \quad A.B &= \frac{\pi.\rho}{nnqs} + P^{11} = \frac{\pi\rho+\pi+\rho+1}{nnqs} - P^{12} \end{aligned}\right\} \qquad \text{III}$$

yet we know that $\frac{\pi}{nq}$ = or $>\frac{p}{q}$, and $\frac{\rho}{ns}$ = or $>\frac{r}{s}$, and that the differences $\frac{\pi}{nq} - \frac{p}{q}$, $\frac{\rho}{ns} - \frac{r}{s}$ cannot decrease with the increase of n, but can only increase. But if we consider the fraction $\frac{\pi\rho+\pi+\rho+1}{nnqs}$ then it appears merely from the comparison of the first and fourth equations in III that this can never be $<\frac{p}{q}\cdot\frac{r}{s}$. But whether also $\frac{\pi\rho+\pi+\rho+1}{nnqs} > \frac{pr+1}{qs}$, and generally what relationship this fraction has to the fractions of the following series

$$\frac{pr+1}{qs}, \frac{pr+2}{qs}, \frac{pr+3}{qs}, \ldots, \frac{pr+p+r}{qs}$$

depends on the particular nature of the expressions designated by P^1 and P^3 and, for the same q and s, on the value of n. Thus if P^1 and P^3 are both $=0$, then π is always $=np$ and $\rho = nr$, and so,

$$\frac{\pi\rho+\pi+\rho+1}{nnqs} = \frac{nnpr+np+nr+1}{nnqs} = \frac{pr+\frac{p}{n}+\frac{r}{n}+\frac{1}{nn}}{qs}.$$

From this it is clear that for all values of n, which are sufficiently large that $\frac{p}{n}+\frac{r}{n}+\frac{1}{nn} < 1$, the value of the fraction concerned always remains $<\frac{pr+1}{qs}$. But there can certainly also be values for P^1 and P^3 for which the fraction $\frac{\pi\rho+\pi+\rho+1}{nnqs}$, for large enough n, turns out to be $>\frac{pr+1}{qs}$, or even $>\frac{pr+2}{qs}$ etc. But since, according to §18, the greatest value which π can have $= np + n - 1$, and the greatest value which ρ can have $= nr + n - 1$, then the greatest value which the fraction $\frac{\pi\rho+\pi+\rho+1}{nnqs}$ can take is

$$= \frac{(np + n - 1)(nr + n - 1) + (np + n - 1) + (nr + n - 1) + 1}{nnqs}$$
$$= \frac{pr + p + r + 1}{qs}.$$

Now it is easy to see that with particular values of P^1 and P^2, and for particular q and s, the fraction $\frac{\pi\rho+\pi+\rho+1}{nnqs}$ can be made greater than every single fraction in the above series merely by the increase in n.

1. First of all now, if P^1 and P^3 are such that with given q and s the value of the fraction $\frac{\pi\rho+\pi+\rho+1}{nnqs}$, however large n is taken, always remains $=$ or $< \frac{pr+1}{qs}$ then, because $A.B = \frac{\pi\rho+\pi+\rho+1}{nnqs} - P^{12}$ it is certain also that $A.B = \frac{pr+1}{qs} - P^{13}$. Since, on the other hand, we have $A.B = \frac{pr}{qs} + P^5$, then we see that for these values of q and s the product $A.B$ behaves as the condition of measurability requires.

2. If instead there is a value of n so great that $\frac{\pi\rho+\pi+\rho+1}{nnqs} > \frac{pr+1}{qs}$, then we can further investigate whether there is also a value of n for which the fraction concerned is $> \frac{pr+2}{qs}$. If we were to say 'no' to this latter question, i.e. if for all values of n however large, $\frac{\pi\rho+\pi+\rho+1}{nnqs} =$ or $< \frac{pr+2}{qs}$ then $A.B$ must be $= \frac{pr+2}{qs} - P^{14}$. On the other hand, from the fact that $\frac{\pi\rho+\pi+\rho+1}{nnqs} > \frac{pr+1}{qs}$, it follows that by suitable increase in n, just the fraction $\frac{\pi\rho}{nnqs}$ can be made $> \frac{pr+1}{qs}$. For the part $\frac{\pi\rho}{nnqs}$ will never become smaller, with continual increase in n, but will always become greater, because it is $= \frac{\pi}{nq}.\frac{\rho}{ns}$ and each of these factors cannot decrease with the increase in n, but can only increase. But the part $\frac{\pi+\rho+1}{nnqs}$ decreases indefinitely with the indefinite increase of n because $\frac{\pi}{nq}$ must always remain $< \frac{p+1}{q}$ and $\frac{\rho}{ns}$ must always remain $< \frac{r+1}{s}$, therefore

$$\frac{\pi + \rho + 1}{nnqs} < \frac{p+1}{q}.\frac{1}{ns} + \frac{r+1}{s}.\frac{1}{nq} + \frac{1}{nnqs}$$

in which sum each of the three summands obviously decreases indefinitely. Therefore there is certainly a value of n so large, that for this value and for all greater values, $\frac{\pi\rho}{nnqs}$ becomes and remains $> \frac{pr+1}{qs}$. Now since we have (III) $A.B = \frac{\pi\rho}{nnqs} + P^{11}$ it is all the more certain that $A.B = \frac{pr+1}{qs} + P^{15}$. This, combined with the equation $A.B = \frac{pr+2}{qs} - P^{14}$ found earlier, shows that also in this case and with the values of q and s taken originally, the product $A.B$ behaves like a measurable number.

3. It may be seen how these arguments can be continued and so it is clear that for every value of the numbers q and s, there is one fraction in the above series of fractions of such a kind that if we denote it by $\frac{pr+m}{qs}$, we have the two equations

$$A.B = \frac{pr + m}{qs} + P^{15} = \frac{pr + m + 1}{qs} - P^{16}.$$

Now since, if q and s are arbitrary, qs can represent any arbitrary denominator, it follows that the product $A.B$ satisfies completely the concept of a measurable number.

§ 50

Corollary. Therefore a product of any arbitrary multitude of factors, which are all measurable, is again measurable.

§ 51

Theorem. If a pair of expressions A and B constantly gives one and the same measuring fraction then the difference $A - B$ can only be one of two things, either zero or infinitely small.

Proof. Since both expressions are measurable, and in the process of measuring they give one and the same measuring fraction, then if we designate this fraction by $\frac{p}{q}$, it must be that,

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$
$$\text{and} \quad B = \frac{p}{q} + P^3 = \frac{p+1}{q} - P^4,$$

so that the two expressions can differ at most in the parts P^1, P^2, P^3, P^4. Now if $P^1 = P^3$, or $P^2 = P^4$, then the two expressions A and B would obviously be completely equal, and the difference $A - B = 0$. But in any case this difference must, according to §45, be measurable, and therefore to every arbitrary n there must be an m of such a kind that the equations

$$A - B = \frac{m}{n} + P^5 = \frac{m+1}{n} - P^6$$

hold. Just by subtraction of the third from the second and the fourth from the first of the above equations we also have

$$A - B = \frac{1}{q} - (P^2 + P^3) = -\frac{1}{q} + (P^1 + P^4).$$

The combination of the two first equations gives

$$\frac{1}{q} - \frac{m}{n} = P^2 + P^3 + P^5.$$

Now since with the same A and B, therefore with the same n and m, q can increase indefinitely, it is clear that this latter equation can only hold if m is either $= 0$ or negative. If the former holds, i.e. if $m = 0$ for every value of n, then $A - B = P^5 = \frac{1}{n} - P^6$ is, according to the definition of §22, a positive infinitely small number. But if the latter holds and m is negative, then the combination of the two second equations for $A - B$, $\frac{m+1}{n} + \frac{1}{q} = P^1 + P^4 + P^6$ shows us that

m can have no other value than $m = -1$, because any other greater value would make the expression $\frac{m+1}{n} + \frac{1}{q}$ negative with the increase of q. But if $m = -1$ then we have

$$A - B = -\frac{1}{n} + P^5 = -P^6,$$

equations which prove that $A - B$ is a negative infinitely small number.

§ 52

Corollary. If we take A and B to be infinitely small then the previous theorem shows us that the algebraic difference of the two, and therefore of every finite multitude of infinitely small numbers, always yields zero or an infinitely small number.

§ 53

Transition. The foregoing already gives occasion for the remark that the comparison of infinite number expressions, either with one another or with other finite expressions, can introduce relationships of such a kind that we are justified, by our concepts so far, neither in defining the two expressions to be compared with one another as *equal*, nor that one of them is *greater* and the other is smaller. For example, if we compare the two expressions, A and $A + \frac{1}{1+1+1+\cdots in\ inf.}$, our previous concepts do not justify defining the two as *equal* for their difference $\frac{1}{1+1+1+\cdots in\ inf.}$ is not identical with zero. But also we are just as little justified in claiming one of these expressions, possibly the latter, to be *greater* than the former, not even if we follow the extended meaning of this word established at §4 in the 5th Section. For according to this $A > B$ only if the difference $A - B$ can be brought to a fraction of the form $+\frac{m}{n}$ where m and n denote actual numbers. But the difference $\frac{1}{1+1+1+\cdots in\ inf.}$ is not of this form because the denominator, as an infinite multitude of units, does not represent any actual number. If, in order to give a second example, we compare the infinite expression $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots in\ inf.$ with the finite expression $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$ and subtract the latter from the former, then the remainder is $= \frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} + \cdots in\ inf.$ Now if the multitude of terms in this remainder were merely finite, then they could be put together according to §5 of the 4th Section into a single fraction of the form $+\frac{m}{n}$ and it would then, of course, be permitted to define the former expression as *greater* than the latter. Just because the number of fractions here is infinite, we are not in a position directly to call the first expression greater, and the other one smaller. Just as little is it allowed for the second expression to be called greater, or to say the two [expressions] are *equal* to one another. It is nevertheless desirable that we obtain concepts which can be applied to the relationship between infinite number concepts in several of the cases (if not all) where our previous concepts have not been adequate. Therefore we wish to pose the following problem.

§ 54

Problem. To investigate whether there may not be an appropriate extension of our existing concepts, which would enable us to apply the relationships of *equality* and *order* (being greater and being smaller) also to infinite number expressions, if the existing concepts do not allow this.

This § is to be rewritten.[o]

Numbers which differ only by an infinitely small [amount] can behave differently in the process of measuring. For example, 1 and $1 - \frac{1}{1+1+1+\cdots in\ inf.}$, give, with every measuring fraction q, the numerators $p = q$ and $p = q - 1$, the numbers $\frac{3}{5}$ and $\frac{3}{5} - \frac{1}{1+1+1+\cdots in\ inf.}$ are different with the denominators 5, or 2.5, 3.5, generally $n.5$ etc. I therefore believe that the two definitions of this § should be revised as follows. If the pair of numbers A and B have a difference $A - B$ which, considered absolutely, has the same characteristics as zero itself in the process of measuring (i.e. it behaves like zero) in that for every denominator q, however large, the numerator of the measuring fraction is found to be $= 0$, and so it has only the two equations $A - B = \frac{0}{q} + P^1 = \frac{1}{q} - P^2$, then we say that $A = B$. But if the difference has the characteristics of a number different from zero, and its true value is positive, then $A > B$, if it is negative, then $A < B$. Also with this definition there is no difficulty in showing that it is not really the concept of equality but only its application which has been modified, that is, the viewpoint from which the objects concerned are being considered. A and B are here called equal to one another in the sense that both have the same properties, and that their difference $A - B = B - A$, considered absolutely, has equal characteristics in the process of measuring to those of zero.

NB. It must be proved in a theorem of its own that for the establishment of the concept of equality the case could not arise that the measuring fraction of two numbers, with any arbitrary denominator however large, may differ only by a 1.

Solution. 1. The first of the two examples of the previous section, or rather the general theorem of §51, shows us that among the infinite number concepts there are some which—although not *equivalent* in the meaning of this expression up until now—nevertheless have so much in common that they always produce the same measuring fraction in the attempted process of measuring, i.e. for every arbitrary q always the same positive or negative p may be found, for both expressions A and B, which satisfies the equations

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$

$$\text{and} \quad B = \frac{p}{q} + P^3 = \frac{p+1}{q} - P^4.$$

[o] That is this section §54. The passage from here to, but not including, 'Solution 1.' below, is added in Bolzano's handwriting. It is a supplement to, and an improvement upon, the definition of equality to be given here and in §55. (*JB*)

This naturally leads to the question whether we might not put just this equal behaviour in the process of measuring to be *equality* in the wider meaning we are seeking here? Then, therefore, we would define all expressions as *equal* or *equivalent* providing they are of such a nature that in the process of measuring they always yield one and the same measuring fraction. Now this would, of course, be a concept which could apply both to finite, as well as to infinite, number expressions. And in the application to the former it would not prove to be too wide, for only those finite number expressions which we would have viewed as equivalent to one another according to our previous concepts would also be so called with this new concept. For if A and B are a pair of finite number expressions, then we know from §7 that for every arbitrary q, for each of these expressions, there is some p with the property that the following equations arise

$$A = \frac{p^1}{q} + P^1 = \frac{p^1 + 1}{q} - P^2$$

$$\text{and} \quad B = \frac{p^2}{q} + P^3 = \frac{p^2 + 1}{q} - P^4.$$

And if $A = B$ here, then also $p^1 = p^2$, and if, conversely, for all values of q, $p^1 = p^2$, then $A = B$. However, the phrase 'that certain number expressions behave in one and the same way in the attempted process of measuring', could also be understood by someone in such a way that a pair of expressions are to be called *equal* to one another if, in the attempted process of measuring, they both prove to be unmeasurable. Whoever wanted to extend the concept of equality so far as this therefore wants to treat all expressions which are just not measurable as equivalent to one another. Indeed, whoever would define as equal all those number expressions for which the measuring attempt shows that to every arbitrary q we can find no p large enough for the equations $A = \frac{p+1}{q} - P^2$ and $B = \frac{p+1}{q} - P^4$ to hold, such a person would have set up a concept which is too wide and is actually unusable. For now we would have to view and treat as equivalent, all those numbers which we called in §27 infinitely large. But since we saw in §32 that an infinitely large number still remains infinitely large if an arbitrary rational number, or even another infinitely large number (of the same sign), is added to it, we would have to allow as valid, if U denotes an arbitrary infinitely large number, but x denotes some other number, the equations $U + x = U$ and so, if U was subtracted from both sides, also the equation $x = 0$, and, because x can even represent U itself, even the equation $U = 0$. Certainly, therefore, we must restrict our concept of equality so that we call equal to one another only *measurable* expressions if they show the same features in the process of measuring, i.e. to every arbitrary q for both of them always one and the same p may be found which yields the measuring fraction $\frac{p}{q}$ common to both. But if we wanted to ask for even more and require that also the two expressions P^1 and P^3 which have to be added to $\frac{p}{q}$ should be the same, and therefore also the two expressions P^2 and P^4 which are subtracted from $\frac{p+1}{q}$, should be the same, then we would not have extended our concept of equality at all, instead it would have the same scope as it had before. For that A is to be put $= B$ if $A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$, and B is again $= \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$, follows immediately from the earlier concept. So if we wish to obtain a concept that is actually wider we must make the equality refer only to the measuring fraction, but we must leave how P^1 and P^3, and likewise P^2 and P^4, are to be related to one another as an open question. But, of course, this extension of the concept [of equality] will have the consequence that we must treat all expressions which represent a so-called *infinitely small* number as being equivalent with zero itself. For if A represents an arbitrary measurable number while J represents an infinitely small number then according to §34, A and $A + J$ are measured by the same numbers, we must therefore have the equation $A = A + J$. But this has the consequence that J has to be regarded as equivalent to zero. This also follows immediately from the concept in question. For zero and an infinitely small number yield one and the same measuring fraction in the process of measuring, namely zero. Yet another consequence is that if an equation is to be preserved we may not be allowed to multiply its two sides by one and the same expression if it is infinitely large. For example, if we wanted to multiply the equation

$$\frac{1}{1 + 1 + 1 + \cdots \textit{in inf.}} = \frac{3}{1 + 1 + 1 + \cdots \textit{in inf.}},$$

which we must admit as correct according to this concept, on both sides by the infinitely large number $(1 + 1 + 1 + \cdots \textit{in inf.})$ then we would obtain $1 = 3$ which is absurd. This, and some similar restrictions, may indeed seem inconvenient. However, it is certain that the introduction of this new concept will open the way to the knowledge of so many new relationships between numbers that the inconvenience will be far outweighed by the advantage.

2. If we decide once and for all to adopt this concept, and therefore if the *equality* of two expressions refers only to the equality of the measuring fraction, then there is little doubt how we should determine the *second* concept, namely that of *order*, or of *being greater or smaller*. That is to say, the first idea which arises in order to extend our previous concept of being greater so that it can also apply to relationships such as that shown in the second example introduced in the previous section, would be to say that B is to be called greater than A if the difference $B - A$ is equal to some strictly positive expression. With this concept we would obviously find that the same finite expressions for which we could assume a relationship of being greater and being smaller (according to §4 of the 5th Section), would still be in the same relationship. For if $B > A$, in the sense of §4 of the 5th Section, then it must be that $B - A = +\frac{m}{n}$, where n and m denote actual numbers. Therefore $B - A$ is certainly positive. And conversely, whenever $B - A$ is positive it must be of the form $+\frac{m}{n}$ because every finite elementary expression[p] can be reduced to

[p] Bolzano defines an *elementary expression* in §2, 4th Section, as a number expression that contains no operations other than an arbitrary number of occurrences of the four operations addition, subtraction, multiplication, and division. (*BGA* 2A8, p. 74.)

the form $+\frac{m}{n}$. However, if we cannot determine the nature of the expression P in our concept any more precisely than that it should be a strictly positive expression then the new concept would, in fact, still be too wide. For as a consequence of this definition B would also have to be called $>A$ if the difference, $B - A = P$, is an infinitely small number. But in this case A and B can, by the definition already established, be called *equal*. The relationships of equality, and of being greater or smaller, would therefore not be exclusive of one another. In order to avoid this defect we must necessarily restrict our concept further. Now the most natural way to try this is to ask whether it cannot be said that B is only to be called $>A$ if the difference $B - A$ is positive and not infinitely small? In fact, it is apparent that this addition is enough to produce a contrast [*Gegensatz*] between the relationships of equality, on the one hand, and that of being greater or smaller, on the other hand. For now, certainly, in no case where it can be said $B > A$ could it also be said that $B = A$. If A and B are both not measurable, or if even one of these two expressions is not measurable, then already from the definition of equality suggested in no.1 we may not call them equal to one another! But if both are measurable, and their difference $B - A$ is positive and not infinitely small, then in the process of measuring, A and B do not always give the same measuring fraction, for as we saw from §51 the difference $B - A$ between two measurable expressions which both always yield the same measuring fraction, must be zero or infinitely small. Therefore it does not seem that we would have cause to think of an even narrower restriction of this concept. But with its introduction it will indeed be necessary to observe some precautionary rules in order to preserve the relationship $B > A$, such as 'both terms are never to be multiplied by one and the same infinitely small number'. Restrictions of this kind we can very willingly allow.

§ 55

Definition. So from now on I allow myself to call a pair of number expressions A and B (they may be finite or infinite) *equal* to one another or *equivalent*, and to write $A = B$, if only both are measurable, and in the process of measuring they give identical numbers in the sense that to every arbitrary q there always belongs the same positive or negative p for A as there does for B, which satisfies the four equations:

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$
$$B = \frac{p}{q} + P^3 = \frac{p+1}{q} - P^4,$$

in which the symbols P^1 and P^3 are either zero or strictly positive expressions, while the symbols P^2 and P^4 always denote only strictly positive expressions. On the other hand, I say that B *is greater than* A or A *is smaller than* B, and write $B > A$ or $A < B$, whenever the difference $B - A$ is positive and not infinitely small.

§ 56

Note. In order to prevent a possible misunderstanding, let me remark that it is not the concept of *equality* in itself which is extended by the previous definition but only that I have here revised to some extent the *object* to which equality is being applied. That is to say, by setting up my definition I am confirming that in the future, if it is a question of deciding the equality or inequality of numbers, I want to take into consideration *only their behaviour in the process of measuring*, or to speak more precisely, *only the nature of their measuring fraction*. This is rather like how the geometer also does not change the concept of equality, but only the object of it, when he explains that he wants to call two lines equal to each other provided they have the same length. Furthermore, I believe that the concept of equality, as well as that of order [*Höhe*], are defined here in a way which lets us already foresee the usefulness of both concepts for pure *number theory*, or (what amounts to the same) their suitability for the discovery of new relationships between numbers. Nevertheless I do not deny that it was chiefly for the sake of an application to be made first in the *theory of quantity* that this concept is interpreted here in this particular way. The number expressions which we here call equal or equivalent should be able to be used for the measurement of quantities which are considered as equal quantities. On the other hand, a number expression which we here call *greater* than another one should measure a quantity which is a whole, of which, that which the smaller expression measures, is a part. It is similar with several other concepts and terms which are introduced into pure number theory because they already have here some usefulness, although their chief application, and that which gave rise to their formation, belongs to another science, e.g. the theory of quantity or the theory of space. Thus it is with the terms *square number*, *cubic number*, *pyramidal* number etc. It is just the same with the so-called *trigonometric numbers*, which have their most important application in the science of space, as they were invented in the first place for the purpose of this science.

§ 57

Corollary. All infinitely small numbers must be regarded as equivalent to zero itself. For if J is infinitely small then $A + J = A$, therefore $J = 0$.

§ 58

Note. Since the infinitely small numbers are not equivalent to zero in every respect, but only in respect of their behaviour in the process of measuring, it might be expedient, if we call such numbers zero, to call them a *relative* zero or *respective* zero [*relative oder beziehungsweise Null*]. In contrast to that, the concept of zero which we already met in §116, *EG* III, might be called the absolute zero.

§ 59

Corollary. If the summands [*Addenden*] A, B, C, ... form a finite multitude and they are measurable numbers of which each one has only a single value, then also the sum is a measurable number which also has only a single value. For according to §45 we cannot regard two different, i.e. unequal, numbers as sums of the two numbers A and B because the measuring fraction of $A + B$ is determined by the measuring fractions of A and B. The same holds of a difference, and likewise of any product (§51).

§ 60

Corollary. If $A = B$ then it is not also the case at the same time that $A > B$ or $A < B$; and if $A > B$ then it is not also the case that at the same time that $A < B$ or $A = B$. This follows directly from what we have just seen. In particular, if $A > B$ then $A - B = +P$, and therefore it is not that $B - A = +P$ but $= - P$.

§ 61

Corollary. If A and B are a pair of rational numbers then one of the following three relationships must hold between them: either $A = B$, or $A > B$, or $A < B$ (§7 of the 5th Section).

§ 62

It is not the same if one of the two expressions A and B, or even both at the same time, are infinite number expressions. Then a relationship can hold of such a kind that we are not justified, *even according to the wider concepts now established*, in declaring the two expressions either as *equal*, nor one of them as being the *greater* and the other being the *smaller*. For example, if A were an arbitrary expression, but $B = A + 1 - 2 + 3 - 4 + 5 - \cdots$ *in inf*. then the difference $B - A = 1 - 2 + 3 - 4 + 5 - \cdots$ *in inf*., an expression of which we are neither justified in saying it is positive, nor that it is negative, nor that it is zero or infinitely small. Therefore we may neither put $B = A$, nor $B > A$, nor $B < A$.

§ 63

Corollary. If $A > B$ and $B > C$ then also $A > C$. Because $A > B$, it must be that $A = B + P^1$ and because $B > C$, it must be that $B = C + P^2$, where P^1 and P^2 are not infinitely small. Therefore $A = C + P^2 + P^1 = C + (P^2 + P^1)$ where $(P^2 + P^1)$ is again not infinitely small. Therefore $A > C$.

§ 64

Corollary. If $A = B - P$, where P represents a strictly positive expression, then B is at least *not* $<A$, because otherwise $A = B + P^1$ therefore it would have to be that

$B = A - P^{\mathrm{I}}$. But it cannot be deduced immediately that $B > A$, for if, for example, P were infinitely small, then B would be not $>A$, but $=A$.

§ 65

Corollary. If, in the process of measuring a given number A, we find that to every arbitrary multiple of the denominator q, $= nq$, there is no other numerator in the measuring fraction than just the same multiple of p, namely np, then it is already known from §25 that the P^{I} appearing in the equation $A = \frac{p}{q} + P^{\mathrm{I}}$ must be infinitely small. Therefore it follows from §57 that in such a case we may put $A = \frac{p}{q}$. In every other case, i.e. whenever we only know that P^{I} is *not infinitely small*, we have the relationships $A > \frac{p}{q}$ and $A < \frac{p+1}{q}$. This follows directly from the definition §55 because the differences $A - \frac{p}{q}$, $\frac{p+1}{q} - A$ are always positive and not infinitely small.

§ 66

Corollary. Since in the two equations

$$A = \frac{p}{q} + P^{\mathrm{I}} = \frac{p+1}{q} - P^2$$

the number designated by P^2 may, according to the definition in §6, never be zero, therefore (by §57) it may also never be infinitely small, then in every case P^{I} is $< \frac{1}{q}$, because it must be $= \frac{1}{q} - P^2$.

§ 67

Corollary. The propositions established in §§ 11, 12 of the 5th Section: that equal things added to, or subtracted from greater things, adding greater things to greater things, subtracting smaller things from greater things, all give greater things, or that whenever, $A > B$ then also $A \pm C > B \pm C$, and if additionally $D > E$, then also $A + D > B + E$, or $A - E > B - D$, are valid also for the extensions of these concepts, introduced here, and for infinite numbers. For the proofs indicated there can also be applied again here.

§ 68

Corollary. If $A > B$ then also $nA > nB$, where n denotes any actual number. For it follows by repeated addition.

§ 69

Corollary. Every so-called *infinitely large number* may, if it is positive, be called *greater*, and if it is negative be called *smaller*, than every actual number. For if A is infinitely large and positive then if $N + 1$ designates any actual number, by §31,

$A = (N + 1) + P$, therefore $A - N = 1 + P$, is a difference which is obviously positive and not infinitely small. Therefore $A > N$. On the other hand, if A is negative, then we have $A = -(N + 1) - P$, therefore $A + N = -(1 + P)$, or $A - (-N) = -(1 + P)$, from which it follows by the definition §55 that $A < -N$.

§ 70

Corollary. Every measurable number has a greater measurable number above it, and a smaller measurable number below it. For if A is measurable, then by §43 also $A + 1$ and $A - 1$ are measurable. And certainly by the definition of §55, $A + 1 > A$ and $A - 1 < A$.

§ 71

Corollary. We have seen in §37 that every product of which one factor is a measurable number and the other is an infinitely small number itself represents an infinitely small number. But infinitely small numbers are now equivalent to zero, so every product in which one factor A is measurable, and the other is zero, is to be put equal to zero. Therefore not only $0.A = 0$ (as already follows from earlier assumptions) but also $A.0 = 0$.

§ 72

Theorem. If, in the process of measuring two given numbers A and B we get a denominator q for which there belongs a *greater* p (in the sense of §31 of the 1st Section) for the measuring fraction of B than that for A, then $B > A$.

Proof. If we designate the p belonging to A by p, but that belonging to B by $p + \pi$, where therefore π denotes an actual and positive number, then we have,

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$

$$\text{and} \quad B = \frac{p+\pi}{q} + P^3 = \frac{p+\pi+1}{q} - P^4,$$

therefore by subtraction $B - A = \frac{\pi}{q} + P^3 - P^1$. At the same time $P^1 = \frac{1}{q} - P^2$. Therefore we obtain

$$B - A = \frac{\pi}{q} - \frac{1}{q} + P^3 + P^2.$$

According to §18 it is always possible to take q so large that π will be >1. For such a value

$$B - A = \frac{\pi - 1}{q} + P^3 + P^2$$

is obviously positive and not infinitely small. Therefore $B > A$.

§ 73

Theorem. If A and B are a pair of measurable numbers then always one and only one of these three cases holds, either $A = B$, or $A > B$, or $A < B$.[q]

Proof. If both numbers A and B are measurable then to every arbitrary q there is one and only one p which represents the numerator of the measuring fraction of A, and one and only one p which represents the numerator of the measuring fraction of B. If it is always the same p which holds for A as for B, then A and B are equivalent to one another as a consequence of the definition in §55. But if it happens that for some q there is a different p for B than there is for A, then (by §55) one of them must be greater and the other one smaller. If that of B is greater, then by §72, $B > A$; if that of A is greater, then $A > B$. Therefore there is always one and only one of these three, either $A = B$ or $A < B$ or $A > B$.

§ 74

Theorem. If A and B are a pair of measurable and finite numbers, both of which we also consider as positive or absolute, then there is always some multiple of one which is *greater* than the other, and some aliquot part[r] of the same one which is *smaller* than the other.

Proof. Because both numbers A and B may be considered as positive and absolute, and they are measurable, we have

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$

$$\text{and} \quad B = \frac{\pi}{q} + P^3 = \frac{\pi + 1}{q} - P^4.$$

And because neither of the two numbers are infinitely small, neither p nor π may be $=0$. It will now be sufficient to prove that if one of the two numbers e.g. $B < A$, a multiple of B, nB, can be found that makes $nB > A$, and conversely, if $B > A$, an aliquot part of B, $= \frac{B}{n}$, can be found which makes $\frac{B}{n} < A$.

1. For A to be $>B$ there must be some value of q large enough that $p > \pi$. But if $p > \pi$ there is always a number n large enough that $n.\pi > p + 1$. If we now multiply the last but one of the previous equations by this number n, then $nB = \frac{n\pi}{q} + nP^3$, therefore $nB =$ or $> \frac{n\pi}{q}$. But $\frac{n\pi}{q} > \frac{p+1}{q}$ and $\frac{p+1}{q} > A$. Therefore without doubt $nB > A$.

2. If $A > B$ and a divisor n is to be found which makes $\frac{A}{n} < B$ then we need only seek an n which makes $nB > A$. Then undeniably $\frac{A}{n} < B$.

[q] This theorem, like many others, is false with Bolzano's first definition of equality (§55), but is correct with his second definition (actually given in §54 and to the effect that $A = B$ if and only if $A - B$, in absolute value, is infinitely small). (*JB*)

[r] An aliquot part is a part, 'contained in another a certain number of times without leaving any remainder' (*OED*).

§ 75

Definition. It has already been stated in §18 of the 5th Section concerning only rational numbers, that one number, B, lies *between* two others A and C if it is greater than one and smaller than the other. I now extend this expression also to infinite number concepts and say that the number represented by the finite or infinite number concept B lies *within*, or between, the numbers represented by the finite or infinite number concepts A and C, if B is greater than one of them and smaller than the other, where both [these relationships] have that wider meaning established in §55.

§ 76

Corollary. Therefore if B is to lie between A and C then A, B and C must be unequal. This follows just as in §19 of the 5th Section.

§ 77

Corollary. If $\frac{p}{q}$ is the measuring fraction of a number A, and P^{1} is not zero, then $\frac{p}{q} < A$ and $A < \frac{p+1}{q}$. Therefore one can say, not of all measuring fractions, but of every measuring fraction which is not a perfect measure (§6), that the number which it measures lies between it and the *next greater* fraction.

§ 78

Theorem. Every *positive* measurable number lies between zero and a *positive* infinitely large number, and every *negative* measurable number lies between zero and a *negative* infinitely large number.

Proof. If P is positive, and S designates a positive infinitely large number, then $P > 0$ because $P - 0 = P$, and $P < S$ because $S - P = P^{1}$ (§31). On the other hand, $-P < 0$, and $-S < -P$, because $+P^{1}$ must still be added to $-P$ and to $-S$ to make 0 and $-P$.

§ 79

Theorem. If A and C are a pair of unequal measurable numbers then there is always a third measurable number which lies between the two.

Proof. Like that of the similar proposition, only that some strictly positive and measurable expression B is to be used instead of $\frac{m}{n}$, regardless whether it is finite or infinite.

§ 80

Theorem. If B lies between A and C then only one of two things can occur: either $B > A$ and then $B < C$, and all the more $A < C$, or $B < A$, and then $B > C$ and all the more $A > C$.

Proof. As in §21 of the 5th Section.

§ 81

Theorem. If the measurable numbers L and R both lie between the numbers A and Z, then the difference $R - L$, taken in its absolute value, is smaller than the difference $Z - A$ taken in its absolute value.

Proof. As in §23 of the 5th Section.

§ 82

Theorem. If the measurable numbers L and R both lie between the numbers A and Z, and the number M lies between L and R, then M also lies between A and Z.

Proof. As in §25 of the 5th Section.

§ 83

Theorem. If the number M lies between L and R then there is always a measurable number μ small enough that also $M + \mu$, and likewise $M - \mu$, lie within L and R.

Proof. As in §27 of the 5th Section only that instead of §9 of the 5th Section, §74 is to be used.

§ 84

Theorem. The multitude of measurable numbers which lie between every two different measurable numbers is infinite.

Proof. As in §28 of the 5th Section.

§ 85

Theorem. If the measurable number M lies between the measurable numbers L and R, and A is some finite and positive number, then the product $A.M$ also lies between the products $A.L$ and $A.R$.

Proof. Let R designate the greater of the two numbers L and R, then $M = L + P^1 = R - P^2$, where P^1 and P^2 designate a pair of positive number expressions which are not infinitely small. Therefore also (§)

$$A.M = A(L + P^1) = A(R - P^2).$$

However, by §, $A(L+P^{1}) = AL + AP^{1}$ and $A(R-P^{2}) = AR - AP^{2}$. Accordingly we obtain $A.M = A.L + A.P^{1} = AR - AP^{2}$. Now since A, P^{1} and P^{2} are finite, then also the products $A.P^{1}$ and $A.P^{2}$ are finite. Therefore by §55 $AM > AL$ and $<AR$.

§ 86

Definition. If a certain property B belongs to all measurable numbers which lie within the two unequal [numbers] L and R, but not to the numbers L and R themselves, then we say that *B belongs to all measurable numbers from L exclusively to R exclusively*. But if also L, or R, or both, have the same property then we say B belongs to all measurable numbers from *L inclusively to R exclusively*, or from *L exclusively to R inclusively* or finally from *L inclusively to R inclusively*. Finally, if a certain property B does belong to the number M, but there is no measurable number μ so small that it could be said that the property B belongs to all numbers between $M+\mu$ and $M-\mu$, then I say the property B belongs to the number *M in isolation*. Thus the property, of being greater than 4 and smaller than 5, belongs to all measurable numbers from 4 exclusively to 5 exclusively, but the property of being not smaller than 4 and not greater than 5 belongs to all measurable numbers from 4 inclusively to 5 inclusively. The property of being even belongs to the number 8 in isolation.

§ 87

Corollary. Therefore if a certain property B belongs to all measurable numbers from L to R, inclusively or exclusively, then the property B cannot belong in isolation to any number lying within L and R, and if L or R are included this also cannot be said of L or R. And conversely if it holds of some number lying within L and R that the property B belongs to it in isolation, then it can certainly not be said that the property B belongs to all numbers lying within L and R.

§ 88

Theorem. If the property B belongs to all measurable numbers from L to R, inclusively or exclusively, and M and Q are a pair of numbers lying between L and R, then the property B belongs to all measurable numbers from M inclusively to Q inclusively.

Proof. Because the property B belongs to all measurable numbers lying within L and R, and M and Q are a pair of such [numbers], the property B belongs to the numbers M and Q. Furthermore, because every measurable number which lies between M and Q also lies, by §82, between L and R, so the property B belongs to all measurable numbers lying within M and Q. Therefore as a consequence of the definition the property B belongs to all measurable numbers from M inclusively to Q inclusively.

§ 89

Theorem. If the two numbers L and R, of which we are told that the property B is to belong to all measurable numbers from L to R, inclusively or exclusively, are both infinitely large, and indeed one is positive and the other is negative, then the property B belongs to all measurable numbers in general. And if one of these numbers, e.g. $L = 0$, but the other R is an infinitely large positive number, then the property B belongs to all measurable numbers which are positive, and if R is an infinitely large negative number, then the property B belongs to all measurable numbers which are negative.

Proof. Follows from §88.

§ 90

Theorem. If L and R are a pair of measurable numbers and R represents the greater of the two, then among the measurable numbers from L *inclusively* to R inclusively or exclusively, L is the smallest and among the numbers from L inclusively or exclusively to R *inclusively*, R is the greatest, but among the numbers from L *exclusively* to R inclusively or exclusively there is none which is the *smallest* and among the numbers from L inclusively or exclusively to R *exclusively* there is none which is the *greatest*.

Proof. 1. Among the numbers from L *inclusively* to R, L is the *smallest*, for numbers which are smaller than L would, on that account, be neither L itself, nor lie within L and R.
2. Among the numbers from L to R *inclusively* R is the greatest, for a number which is greater than R would, on that account, be neither R itself nor lie within L and R.
3. Among the numbers from L *exclusively* to R there is no *smallest*. For if M is one of these numbers then $M > L$ and $<R$ and there is always a μ small enough that also $M - \mu > L$. Therefore $M - \mu$ is a smaller number than M which nevertheless lies between L and R.
4. Among the numbers from L to R *exclusively* there is no *greatest*. For if $M < R$ but $>L$ then there is always a μ small enough that $M + \mu < R$ and $>L$. Therefore $M + \mu$ is a greater number than M which nevertheless lies between L and R.

§ 91

Theorem. If the symbols Ω^1, $\Omega^2, \dots, \Omega^n$ denote variable measurable numbers which can decrease indefinitely in exactly the sense in which this expression is defined in §1 of the 6th Section,[s] and furthermore, if the multitude of these numbers is finite and constant then the algebraic sum $\Omega^1 + \Omega^2 + \cdots + \Omega^n$

[s] The definition is framed in terms of a 'number idea' [*Zahlenvorstellung*]. A number idea can *decrease indefinitely* if for any positive fraction however small, there is a value of the number idea that is an even smaller positive fraction.

again represents a number which can decrease indefinitely, assuming it is not constantly $= 0$.

Proof. As in §8 of the 6th Section, because it also holds of those measurable numbers which are not even rational numbers that they can be made $< \frac{1}{nN}$ as long as it is true that they can decrease indefinitely.

§ 92

Theorem. If A and B denote a pair of measurable numbers which remain unchanged, while the measurable numbers Ω^1 and Ω^2 decrease indefinitely and the equation $A \pm \Omega^1 = B \pm \Omega^2$ is always to hold, then it must be that $A = B$.

Proof. Because A and B are measurable and unchangeable, i.e. have only one value, then (§73) either $A = B$, or $A > B$, or $A < B$. Now if $A > B$ then we would have $A = B + P$, where P would be unchangeable and the equation $A \pm \Omega^1 = B \pm \Omega^2$ would give $B + P \pm \Omega^1 = B \pm \Omega^2$ or $P = \pm\Omega^2 \mp \Omega^1$, which is absurd, because $\pm\Omega^1 \mp \Omega^2$, by §8 of the 6th Section is either $= 0$ or again decreases indefinitely. The same absurdity would arise if we try $A < B$. Therefore it only remains that $A = B$.

§ 93

Theorem. The next greater fraction $\frac{p+1}{q}$ belonging to the measuring fraction $\frac{p}{q}$ of a number A can always be made even smaller than it already is by increasing q.

Proof. If we put instead of q an arbitrary multiple nq, then the numerator of the measuring fraction belonging to the denominator nq is, by §15, $=$ or $< np + n - 1$. Therefore the next greater fraction $=$ or $< \left(\frac{np+n}{nq} = \frac{p+1}{q}\right)$. However the case that the numerator of the measuring fraction belonging to the denominator nq takes the greatest value possible for it, namely $np + n - 1$, can certainly not *always* occur, however large the number n can become. For if the numerator, for the denominator nq, was always found to be $np + n - 1$, however large the value of n, then it would have to be that

$$A = \frac{np + n - 1}{nq} + P^1 = \frac{p + 1}{q} + P^1 - \frac{1}{nq}.$$

Now since the fraction $\frac{1}{nq}$ can decrease indefinitely then by the previous theorem one of the two [possibilities] would have to hold, either $A = \frac{p+1}{q}$ or $A > \frac{p+1}{q}$. But then the measuring fraction of A would not be $\frac{p}{q}$ but would be $\frac{p+1}{q}$.

§ 94

Theorem. If X denotes a number which is measurable and constant, or which changes but only so that its absolute value always remains smaller than a given rational number, while the measurable number Ω can decrease in absolute value indefinitely, then also the products $X.\Omega$ and $\Omega.X$ are numbers which can decrease indefinitely in their absolute value.

Proof. By the assumption made X can always be written $= +\frac{m}{n} - P^{1}$, where m, n designate a pair of actual and constant numbers, but P^{1} designates a measurable and strictly positive number which as long as X is constant must likewise be considered as constant, but if X changes it is constrained by the condition to remain always smaller than $\frac{m}{n}$. Furthermore, if Ω can decrease indefinitely then it can also become $<\frac{1}{\frac{m}{n}(N+1)}$, or (what amounts to the same) $= \frac{1}{\frac{m}{n}(N+1)} - P^{2}$. But then

$$X.\Omega = \left(\frac{m}{n} - P^{1}\right)\left(\frac{1}{\frac{m}{n}(N+1)} - P^{2}\right)$$
$$= \frac{1}{N+1} - \left[\frac{m}{n}.P^{2} + P^{1}.\Omega\right]$$

or because $\frac{1}{N+1} = \frac{1}{N} - \frac{1}{N(N+1)}$,

$$X.\Omega = \frac{1}{N} - \left[\frac{1}{N(N+1)} + \frac{m}{n}.P^{2} + P^{1}.\Omega\right].$$

Now since the expression in the square brackets is strictly positive and certainly not infinitely small, we have the product $X.\Omega < \frac{1}{N}$. Finally, if X and Ω are positive, then $X.\Omega$ is certainly also positive, so it is evident that this number could decrease indefinitely. In the same way the product $\Omega.X$ is

$$= \left(\frac{1}{\frac{m}{n}(N+1)} - P^{2}\right)\left(\frac{m}{n} - P^{1}\right)$$
$$= \frac{1}{N+1} - \left[\frac{1}{\frac{m}{n}(N+1)}.P^{1} + P^{2}.\Omega\right]$$
$$= \frac{1}{N} - \left[\frac{1}{N(N+1)} + \frac{1}{\frac{m}{n}(N+1)}.P^{1} + P^{2}.\Omega\right].$$

Therefore obviously also $\Omega.X < \frac{1}{N}$ and is decreasing indefinitely.

§ 95

Corollary. Therefore every product of a finite and constant multitude of measurable numbers, providing only one factor among them decreases indefinitely while the others are constant or remain in their absolute value smaller than a given rational number, can decrease indefinitely in its absolute value.

§ 96

And this also happens if several, or all, factors decrease indefinitely. For numbers which decrease indefinitely are, likewise, numbers which in their absolute value always remain smaller than a given rational number.

§ 97

Theorem. If A and B are a pair of measurable and constant numbers but Ω^1 and Ω^2 are a pair of variable numbers which can decrease indefinitely, then

$$(A \pm \Omega^1)(B \pm \Omega^2) = A.B \pm \Omega^3$$

where Ω^3 again denotes only a measurable number which can decrease indefinitely if it is not always $= 0$.

Proof. If A and B are to be measurable and constant numbers then it must be that

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$
$$\text{and} \quad B = \frac{r}{s} + P^3 = \frac{r+1}{s} - P^4.$$

And if the numerators of the measuring fractions belonging to the denominators nq, ns are π and ρ, so that we have,

$$A = \frac{\pi}{nq} + P^5 = \frac{\pi+1}{nq} - P^6$$
$$B = \frac{\rho}{ns} + P^7 = \frac{\rho+1}{ns} - P^8$$

then it remains that the fractions $\frac{\pi}{nq} < \frac{p+1}{q}$, $\frac{\rho}{ns} < \frac{r+1}{s}$, however large n may be taken. But the measurable numbers P^5 and P^7 decrease indefinitely, because they must always remain smaller respectively than $\frac{1}{nq}$ and $\frac{1}{ns}$. Therefore we want to represent them by Ω^5 and Ω^7. But $A \pm \Omega^1 = \frac{\pi}{nq} + \Omega^5 \pm \Omega^1$ is, by §91, $= \frac{\pi}{nq} \pm \Omega^9$; $B \pm \Omega^2 = \frac{\rho}{ns} + \Omega^7 \pm \Omega^2$ is, by §91, $= \frac{\rho}{ns} \pm \Omega^{10}$. Therefore

$$(A \pm \Omega^1)(B \pm \Omega^2) = \left(\frac{\pi}{nq} \pm \Omega^9\right)\left(\frac{\rho}{ns} \pm \Omega^{10}\right)$$
$$= \frac{\pi}{nq}.\frac{\rho}{ns} + \frac{\pi}{nq}.\Omega^{10} \pm \Omega^9\left(\frac{\rho}{ns} \pm \Omega^{10}\right).$$

By the previous theorem the two last terms are numbers which can decrease indefinitely because $\frac{\pi}{nq}$ and $\frac{\rho}{ns} \pm \Omega^{10}$ denote numbers which always remain smaller than a given rational number. We therefore obtain by §11 of the 6th Section

$$(A \pm \Omega^1)(B \pm \Omega^2) = \frac{\pi}{nq}.\frac{\rho}{ns} \pm \Omega^{11}.$$

But on the other hand we also have

$$A.B = \left(\frac{\pi}{nq} + \Omega^5\right)\left(\frac{\rho}{ns} + \Omega^7\right) = \frac{\pi}{nq}.\frac{\rho}{ns} + \Omega^{12}.$$

If we subtract the last equation from the previous one, and notice that $\pm\Omega^{11} - \Omega^{12}$ can again only be Ω^{13} or zero, then we obtain

$$(A \pm \Omega^1)(B \pm \Omega^2) = A.B \pm \Omega^{13}.$$

§ 98

Corollary. Therefore also for a product of a finite and constant number of factors

$$(A \pm \Omega^1)(B \pm \Omega^2)\dots(L \pm \Omega^n) = A.B\dots L \pm \Omega,$$

provided that the symbols $A, B, \dots, L, \Omega^1, \Omega^2, \dots, \Omega^n$ and Ω all denote measurable numbers, the $A, B, \dots, L$ are either constant or always remain smaller than certain given rational numbers, but the numbers $\Omega^1, \Omega^2, \dots, \Omega^n$, and Ω can decrease indefinitely.

§ 99

Theorem. The theorem about the permutation of factors which holds for all rational numbers, §17 of the 4th Section, also holds for all measurable numbers in general.

Proof. We need only prove that if A, B, C denote three measurable numbers, $A.(B.C) = (A.B).C$. Now if A, B and C are measurable there must be three equations of the form

$$A = \frac{p}{q} + \Omega^1, \quad B = \frac{r}{s} + \Omega^2, \quad C = \frac{t}{u} + \Omega^3$$

in which the numbers Ω^1, Ω^2 and Ω^3 can decrease indefinitely, while the fractions $\frac{p}{q}$, $\frac{r}{s}$ and $\frac{t}{u}$ always remain smaller than three given rational numbers. Therefore

$$A.(B.C) = \left(\frac{p}{q} + \Omega^1\right).\left[\left(\frac{r}{s} + \Omega^2\right)\left(\frac{t}{u} + \Omega^3\right)\right]$$
$$= \frac{p}{q}.\left(\frac{r}{s}.\frac{t}{u}\right) + \Omega^4.$$

In the same way

$$(A.B).C = \left[\left(\frac{p}{q} + \Omega^1\right)\left(\frac{r}{s} + \Omega^2\right)\right].\left(\frac{t}{u} + \Omega^3\right)$$
$$= \left(\frac{p}{q}.\frac{r}{s}\right)\frac{t}{u} + \Omega^5.$$

However by §17 of the 4th Section

$$\frac{p}{q}\left(\frac{r}{s}.\frac{t}{u}\right) = \left(\frac{p}{q}.\frac{r}{s}\right)\frac{t}{u}.$$

Therefore if we express the value of this product from the two foregoing equations,

$$A.(B.C) - \Omega^4 = (A.B)C - \Omega^5.$$

Now since the numbers $A(B.C)$ and $(A.B)C$ are, by §50, measurable and certainly constant it follows from §92 that it must be that $A(BC) = (AB)C$.

§ 100

Corollary. Therefore if A and B denote a pair of measurable numbers and moreover, B is not zero, then we always have $\frac{A.B}{B} = A$. For $\frac{A.B}{B}$ denotes, if B is not zero, by the definition of §24 of the 3rd Section, a number which when multiplied by B gives $A.B$. Now this is A because $B.A = A.B$.

§ 101

Theorem. The equation proved in § only for rational numbers, namely that $A(B \pm C \pm \cdots) = AB \pm AC \pm \cdots$, also holds generally for all measurable numbers provided the number of terms from which the multiplicand $(B \pm C \pm \cdots)$ is composed, is finite.

Proof. Under this assumption the sum $(B \pm C \pm \cdots)$ is itself a measurable number (§45). Therefore the product $A(B \pm C \pm \cdots) = (B \pm C \pm \cdots)A$. But the latter is, by the definition of § always equivalent to $BA \pm CA \pm \cdots$. Finally $BA = AB$, $CA = AC$, etc. Therefore $A(B \pm C \pm \cdots) = AB \pm AC \pm \cdots$.

§ 102

Definition. If the difference between the two measurable numbers X and Y, considered in its absolute value, can decrease indefinitely then I say (as §17 of the 6th Section) that *they can approach each other as closely as we please.*

§ 103

Theorem. Suppose that two measurable numbers X and Y can approach each other as closely as we please. Then at least one of them must be variable and can take infinitely many values. Among these there is *no greatest* if this number is the *smaller*, and there is no *smallest* if this number is the *greater*. If, in addition, there is a third constant number A which always lies between the first two, then both must be variable and have infinitely many values, among which there can be no greatest for the smaller number and no smallest for the greater number.

Proof. 1. It is obvious that the difference $Y - X$, considered in its absolute value, cannot decrease indefinitely if both numbers X and Y are constant because then this difference itself is also constant, i.e. it is only one [value] (§45).
2. But if one of these numbers, e.g. X, is taken as variable, then assume that it has only a finite multitude of different values. Then the difference $Y - X$ would also have only a finite multitude of values, and therefore one of them would certainly be the smallest. Hence it could not decrease indefinitely. Therefore X must have infinitely many values.
3. Furthermore, if X is the smaller of the two numbers X and Y, then I say that there can be no *greatest* among its infinitely many values. For if X^1 were the greatest, then the difference $Y - X$ could obviously never be smaller than $Y - X^1$, and therefore could not decrease indefinitely. I claim about Y, or the *greater* of the two numbers X and Y, that there may be *no smallest* among its infinitely many values. For if Y^1 were the smallest then the difference $Y - X$ could obviously never become smaller than $Y^1 - X$, therefore again it could not decrease indefinitely.
4. But if a third number A is always to lie between X and Y, and we take Y as the greater, and X as the smaller number, then it should always be the case that $Y - A = P^1$ and $A - X = P^2$. So $Y - X = P^1 + P^2$ decreases indefinitely. Now if one of the former two numbers, e.g. Y, were constant, then $Y - A = P^1$, as a difference of two constant numbers, would itself also be constant. Therefore $Y - X = P^1 + P^2$ could certainly never become $<P^1$ and thus could not decrease indefinitely. The same would be the case if we wished to take X as constant.
5. Finally [the fact] that the greater of the two numbers X and Y can have no *smallest* value, while the smaller can have no *greatest* value, and that therefore (this then follows directly[t]) both numbers must have an infinite multitude of values is clear as follows. Suppose the greater of the two numbers, namely Y, had a smallest value Y^1. Then the difference $Y - A$ could not become smaller than $Y^1 - A = P$, therefore all the less could the difference $Y - X$, which must always be $>Y - A$, decrease indefinitely. On the other hand, if the smaller number, i.e. X, had a greatest value X^1, then the smallest value of the difference $A - X$ would be $= A - X^1$. So $A - X$, and therefore also $Y - X$, could not become smaller than $A - X^1$, i.e. could not decrease indefinitely.

§ 104

Theorem. If a constant number A always lies between the two variable but measurable numbers X and Y, whose difference $Y - X$ can decrease indefinitely, then A is itself a measurable number.

Proof. We shall have proved the measurability of the number A if we prove that, for every arbitrary value of q taken as denominator, there is a numerator p which

[t] The German *von selbst*, is literally 'by itself'.

satisfies the equations

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2.$$

Now because the difference $Y - X$ can decrease indefinitely, let us for the present take the numbers X and Y so that this difference is at least $<\frac{1}{q}$. Now we look for those values of p for which the fraction $\frac{p}{q}$ is either $= X$ or indeed $<X$, yet so that $\frac{p+1}{q}$ will already be $>X$. There must be such a p in every case because all values of X are measurable numbers.

1. Now let p be such that $\frac{p}{q} = X$. Then, since we denote the greater of the two numbers X and Y by Y, and $\frac{1}{q} > Y - X$, by addition $\frac{p+1}{q} > (Y + X - X) = Y$. Now since A lies between X and Y, so $Y > A > X$, then obviously $A > \frac{p}{q}$ and $<\frac{p+1}{q}$. In this case we can therefore assert without doubt the equations $A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$.
2. But if the second case occurs and we have $\frac{p}{q} < X$, $\frac{p+1}{q} > X$, then it is a question of whether $\frac{p+1}{q}$ always remains $>X$ for all other values which X can take, or whether there is some value of X for which $\frac{p+1}{q} =$ or $<X$. One of these two things must necessarily be the case, because $\frac{p+1}{q}$ and X are measurable numbers (§73).

(a) Now if it is the first case, i.e. for all values which X can take, however great, $\frac{p+1}{q}$ remains $>X$, then I assert that A must be $=$ or $<\frac{p+1}{q}$.[u] For if $\frac{p+1}{q} > X$ for every value of X, then $\frac{p+1}{q} - X = P^3$, where P^3 designates a number, perhaps decreasing indefinitely, but always measuring and measurable. Furthermore, because A is to lie between X and Y, it must always be that $A - X = P^4$, where P^4 necessarily denotes a number decreasing indefinitely, because otherwise $Y - X$ could not also decrease indefinitely. But the subtraction of the last equation from the first gives $\frac{p+1}{q} - A = P^3 - P^4$. Now if P^3 always remains greater than a certain value, then, because P^4 decreases indefinitely, $P^3 - P^4$ is positive and not infinitely small. Therefore $\frac{p+1}{q} > A$. But if P^3 is also to decrease indefinitely, then $P^3 - P^4$ could also only represent a number which decreases indefinitely, or a number which is constantly $= 0$. Now since the former is absurd, because $\frac{p+1}{q} - A$ is a number expression constant with respect to X, Y, there only remains the latter, that is that $A = \frac{p+1}{q}$. Now if this latter holds then without doubt A is measurable. But if the former holds and $\frac{p+1}{q} > A$, then because we also have $\frac{p}{q} < X < A$, the two equations

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$

again hold undeniably.

[u] The manuscripts have '$>$' but '$<$' is certainly intended. (*JB*)

(b) If, on the other hand, there is some value of X—we shall designate it by X^{I}—for which $\frac{p+1}{q}$ = or $<X^{\mathrm{I}}$, then, because A must always be $>X^{\mathrm{I}}$, we have $A = \frac{p+1}{q} + P^5$. However, for a smaller value of X, we had $\frac{p+1}{q} > X$ and also for this X, $Y - X$ was $<\frac{1}{q}$. Therefore $\frac{p+1}{q} + \frac{1}{q} > X + (Y - X)$, i.e. $\frac{p+2}{q} > Y$. It is all the more certain that for every other *smaller* value of Y, which we shall designate by Y^{I}, $\frac{p+2}{q} > Y^{\mathrm{I}}$. Nevertheless for this value of Y it must also be that $A < Y^{\mathrm{I}}$, because otherwise it would not lie between X^{I} and Y^{I}. Therefore we may write $A = \frac{p+2}{q} - P^6$. Therefore two equations also hold for this case: $A = \frac{p+1}{q} + P^5 = \frac{p+2}{q} - P^6$, which correspond exactly to the concept of measurability if we just consider $\frac{p+1}{q}$ as the measuring fraction.

§ 105

Theorem. There do not exist two measurable and variable[v] numbers, which are different from one another, i.e. *unequal*, and which always lie within the same two variable but measurable limits, if these limits can approach one another as closely as we please.

Proof. If A and B are two measurable and constant numbers which are different from one another, i.e. unequal, then one of them, e.g. B, must (§73) be the greater and we must have $B = A + P^{\mathrm{I}}$, where P^{I} designates a constant measurable number which is not infinitely small. Now if the two numbers A and B are to lie between the same limits X and Y, then if we take Y as the greater of the two, it must be that $Y - B = P^2$ and $A - X = P^3$. By addition this would give $Y - B + A - X = P^2 + P^3$, or $Y - X = (B - A) + P^2 + P^3$. Now since $B - A = P^{\mathrm{I}}$ is constant, the sum $P^{\mathrm{I}} + P^2 + P^3$, or the difference $Y - X$, cannot decrease indefinitely.

§ 106

Theorem. If the measuring fraction $\frac{p}{q}$ of a number A always lies within the two limits X and Y whose positive difference $Y - X$ decreases indefinitely, then at most the greater of these two numbers, namely Y, can be constant, and this must then be equivalent to the number A itself. But if both numbers X and Y are variable then the number A itself always lies between the two.

Proof. 1. If one of the two limits X and Y is to be *constant*, then I claim that this must be the *greater* one Y, and then A itself must be equivalent to this one. That is, if Y designates the greater of the two numbers X and Y then, because $\frac{p}{q}$, the measuring fraction of A, is always to lie between X and Y, it must be that $Y - \frac{p}{q} = P^3$, $\frac{p}{q} - X = P^4$. In addition $A - \frac{p}{q} = P^{\mathrm{I}}$, and P^3, as

[v] The German is *veränderliche* here, but presumably 'constant' [*unveränderliche*] is intended.

well as P^4, here denote numbers which can decrease indefinitely, but P^1 can if need be denote merely zero. Now if we subtract the last of these three equations from the first, and add it to the second, then we obtain $Y - A = P^3 - P^1$ and $A - X = P^4 + P^1$. First of all it is obvious here that the constant limit cannot be X. Since there is no case in which A can be $<\frac{p}{q}$, and so A cannot be $<X$, this leaves only the two cases $A = X$ or $A > X$. However, by virtue of the last equation $A = X$ would give $P^4 + P^1 = 0$. This is absurd because, as long as pq is actually to lie *within* X and Y, possibly $P^1 = A - \frac{p}{q}$, but not $P^4 = \frac{p}{q} - X$, can represent zero.

But the assumption $A > X$ also contradicts the equation $A - X = P^4 + P^1$ if X is to be constant, for P^4 and also P^1 (if it is not $=0$) decrease indefinitely and therefore their sum $P^4 + P^1$ cannot equal the constant number which the difference $A - X$ represents.

There remains therefore only the possibility that Y is the constant limit. However, Y is $= A + P^3 - P^1$, and $P^3 - P^1$ can represent only one of two things, either a number which decreases indefinitely or a number which is always zero. The first contradicts the fact that $P^3 - P^1 = Y - A$ is to be constant. Therefore only the second can occur, i.e. it must be that $Y = A$.

2. But if both numbers X and Y are *variable* then I claim that, together with the measuring fraction, the number A itself also always lies between both. We have already seen that A is always $>X$, because $A - X = P^4 + P^1$. Therefore it only remains to show that A is always $<Y$. Because by §103 there can be no smallest value of Y, if we take one [value] which is $= A$, it would immediately follow that there are yet other values of Y which are even $<A$. Now if Y^1 were such a value, namely one which is $<A$, then $A - Y^1 = P^5$ would be positive. Now if we took a q large enough so that $\frac{1}{q} < P^5$, then, because we must have $A = \frac{p+1}{q} - P^2$ or $\frac{p+1}{q} > A$, i.e. $\frac{p}{q} > A - \frac{1}{q}$, it would be all the more certain that $\frac{p}{q} > A - P^5$. But $A - P^5 = Y^1$. Therefore $\frac{p}{q} > Y$, which contradicts the assumption that all values of $\frac{p}{q}$ lie within X and Y. Therefore, on the contrary, it must be that Y always remains $>A$, and therefore A always lies between X and Y.

§ 107

Theorem. Suppose the infinitely many measurable numbers $X^1, X^2, X^3, \ldots, X^n, \ldots, X^{n+r}, \ldots$, which we can consider as the terms of an infinitely continuing series distinguished by the indices $1, 2, 3, \ldots, n, \ldots, n+r, \ldots$, proceed according to such a rule that the difference between the nth term and the $(n+r)$th term of the series, i.e. $(X^{n+r} - X^n)$, considered in its absolute value, always remains, however large the number r is taken, smaller than a certain fraction $\frac{1}{N}$ which itself can become as small as we please, providing the number n has first been taken large enough. Then I claim that there is always one and only one single measurable number A, of which it can be said that the terms of our series approach it indefinitely, i.e. that the difference $A - X^n$ or $A - X^{n+r}$ decreases indefinitely in its absolute value merely through the increase of n or r.[w]

Proof. Naturally the proposition is only in need of a proof if the numbers X^1, X^2, X^3, ... do not all have one and the same value. But if they have different values they can either be always *increasing*, i.e. every successive term is always somewhat greater, or at least never smaller, than the previous term, or, on the contrary, they can always be *decreasing* i.e. becoming smaller, or finally they could be alternating, sometimes increasing, and sometimes decreasing.

1. Let us prove the proposition first of all only for the case when the numbers $X^1, X^2, X^3, \ldots$ are continually growing, because the proof for the other two cases will then follow automatically. That there are indeed numbers which proceed according to a rule, such as is assumed here, is shown to us by §48 where such numbers actually appear. Now it will be proved that wherever such numbers exist, there is always also a measurable number A which they approach indefinitely, as soon as we show that to every arbitrary value of q taken as denominator, a value of p can be found which satisfies the two equations $A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$. By the assumption of the theorem the numbers X^1, X^2, X^3, ... proceed according to a rule such that to every given fraction $\frac{1}{N}$, however small, and therefore also to the fraction $\frac{1}{2q}$, one of these numbers, e.g. X^n, can be found with the property that the difference between it and every successive X^{n+r}, i.e. $X^{n+r} - X^n$, remains $<\frac{1}{q}$ however large r may be taken. Now let us consider the smaller of these two numbers, X^n, and let us investigate whether there is indeed a whole number p of such a kind that the equation $X^n = \frac{p}{q}$ can be asserted.

2. If this is the case then there is no doubt that the equations $A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$ can also be asserted. For if we do not determine the values of P^1 and P^2 any more precisely than is done merely by these two equations, then certainly these establish nothing which would contradict the assumption that A is to be a number which the values of the numbers X^1, X^2, X^3, ... approach indefinitely. Because every number following X^n, such as X^{n+r}, is *greater* than X^n, and so certainly A itself must also be thought of as greater than X^n and hence $= X^n + P = \frac{p}{q} + P$, if it is to be true that every greater number which follows X^n can approach A as closely as we please. But on the other hand, since none of these numbers reaches the value $X^n + \frac{1}{q} = \frac{p+1}{q}$, it is therefore just as necessary to write $A = \frac{p+1}{q} - P^2$.

3. But if no value of p exists which satisfies the equation $X^n = \frac{p}{q}$, then, because X^n is to be measurable, there is a number π of such a kind that $\frac{\pi}{q} < X^n$, but $\frac{\pi+1}{q} > X^n$. And now one of the following two cases must occur: either all values which X^n can take by the increase of n, or which X^{n+r} can take by the increase of

[w] An earlier formulation of this convergence criterion occurs in *RB* §7 on p. 266.

n and r, i.e. all successive terms of the series, are smaller than $\frac{\pi+1}{q}$, or else there are also terms in our series which are $=$ or even $>\frac{\pi+1}{q}$.

(a) In the first case, if all numbers of the series $X^1, X^2, X^3, \ldots$ which follow after X^n remain $<\frac{\pi+1}{q}$, it is still a question of whether the difference which arises if each of these terms is subtracted from $\frac{\pi+1}{q}$, namely $\frac{\pi+1}{q} - X^{n+r}$, always remains greater than a certain number, or whether this difference can decrease indefinitely. In the first case it is again clear that then we have the equations $A = \frac{\pi}{q} + P^3 = \frac{\pi+1}{q} - P^4$. This is so because among the numbers $X^1, X^2, X^3, \ldots$ there are some which are $>\frac{\pi}{q}$. Thus A must be assumed $>\frac{\pi}{q} + P^3$ and, because each of these numbers remains *smaller* than $\frac{\pi+1}{q}$ by a certain given number, then A must be assumed $= \frac{\pi+1}{q} - P^4$. But in the second case, and when the difference $\frac{\pi+1}{q} - X^{n+r}$ decreases indefinitely if we only let r increase continually, then it is clear that the number $\frac{\pi+1}{q}$ itself has the same property as is described of the number A in our theorem. Therefore A is to be put $= \frac{\pi+1}{q}$.

(b) Finally if terms appear in the series $X^1, X^2, X^3, \ldots$ which are *equal* to the number $\frac{\pi+1}{q}$ then certainly terms also appear later which are $>\frac{\pi+1}{q}$, and therefore it is easy to see that we must have the equation $A = \frac{\pi+1}{q} + P^5$. But on the other hand we know that each of the numbers $X^1, X^2, X^3, \ldots$, however large it may be, remains $<X^n + \frac{1}{q}$, therefore since $X^n < \frac{\pi+1}{q}$, also $< \left(\frac{\pi+1}{q} + \frac{1}{q} = \frac{\pi+2}{q}\right)$. We can therefore also say $A = \frac{\pi+2}{q} - P^6$, and thus the measuring fraction of the number A is in this case $\frac{\pi+1}{q}$.

4. Finally it is very easy to prove that there can, in any case, only be a single measurable number like A, which the numbers $X^1, X^2, X^3, \ldots$ approach as closely as we please. For if B were a second measurable number, different from A, which the terms of our series also approached indefinitely, then $B - X^n$ would have to be able to decrease indefinitely providing we take n large enough. We would therefore have to have the two equations $A - X^n = \omega^1$ and $B - X^n = \omega^2$, from which we obtain $A - B = \omega^1 - \omega^2$, which is absurd if we do not assume $A = B$.

5. Now the two other cases which we mentioned at the beginning still remain to be considered. If the numbers $X^1, X^2, X^3, \ldots$ are continually decreasing instead of continually increasing, then it is self-evident that the proof just presented, with some very insignificant changes, can be retained. But if the series of numbers $X^1, X^2, X^3, \ldots$ is of such a kind that for every nth term of it, X^n, there are, among the succeeding terms, not only some that are greater but also some that are smaller, then we can remove from the whole collection of these numbers only that part of it which has the property that every succeeding term is greater or smaller than the preceding term. For the new series of numbers which we obtain in this way, the rule also holds: that the difference between the value of a certain term in it, e.g. the nth, and every successive $(n+r)$th term, however large r may be taken,

remains smaller than a given fraction $\frac{1}{N}$ which itself can become as small as we please provided n is taken large enough. It has been proved in the foregoing also for this series that there is one and only one measurable number A which the terms of the series approach indefinitely. But if it is also true that these single terms, taken out of the given series, $X^1, X^2, X^3, \ldots$, that they approach the number A indefinitely, then this must also be true of the remaining terms, i.e. of the whole given series. For if we designate by X^n some term of the given series which was also included in the new series, but by X_1^{n+r} a term which was not included in [the new series], then the difference $X^n - X_1^{n+r}$, considered in its absolute value, must decrease indefinitely providing we take n large enough, however large r may be taken afterwards. But under precisely this condition of the continual increase in n, the difference $A - X^n$ can also decrease indefinitely. From this it follows necessarily that the difference $A - X_1^{n+r}$ must also decrease indefinitely in its absolute value.

§ 108

Problem. If the rule is given according to which the two rational numbers $\frac{m}{n}$ and $\frac{r}{s}$ vary, and their difference $\frac{r}{s} - \frac{m}{n}$ can decrease indefinitely, then we have to determine the measuring fraction, belonging to any arbitrary denominator q, of a number A which always lies within the limits $\frac{m}{n}$ and $\frac{r}{s}$.

Solution. A glance back at the proof of §104, in which the measurability of the number A was proved, might provide us with the procedure which can be used for the present purpose. But since the expressions $\frac{m}{n}$ and $\frac{r}{s}$, which appear here in place of the X and Y appearing there, designate simply rational numbers which can always be reduced to simple fractions, we may assume that m, n, r, s denote actual numbers (m and r even zeros). Furthermore, if $\frac{m}{n}$ is the smaller of these two fractions then because A is always to lie within them, it must be that, $\frac{m}{n} < A < \frac{r}{s}$, for all values of the fractions. Finally, if we denote by x the numerator to be found of the measuring fraction, then it should be that $A = \frac{x}{q} + P^1 = \frac{x+1}{q} - P^2$, where P^1 can possibly represent merely zero. Now because the difference $\frac{r}{s} - \frac{m}{n}$ can decrease indefinitely, then let us first of all give these two fractions those values for which their difference is at least $<\frac{1}{q}$. After this let us investigate whether we can claim the existence of an actual number p which makes the fraction $\frac{p}{q} = \frac{m}{n}$ (the smaller of the two limits), i.e. whether mq is divisible by n.

1. If this is so, then it is already decided that the number $p = x$, and therefore the required measuring fraction of A is found to be $= \frac{p}{q}$. For if $\frac{p}{q} = \frac{m}{n}$, then as $\frac{m}{n}$ is $<A$ we also have $\frac{p}{q} < A$, therefore we can put $A = \frac{p}{q} + P^1$. And since, on the other hand, we have $\frac{r}{s} - \frac{m}{n} < \frac{1}{q}$, then it is all the more certain that $\left(A - \frac{m}{n} = A - \frac{p}{q}\right) < \frac{1}{q}$. Therefore $A < \frac{p+1}{q}$, and thus we may put $A = \frac{p+1}{q} - P^2$. From the combination of this with the equation for A found before it now follows clearly that $\frac{p}{q}$ is the required measuring fraction.

2. But if mq is not divisible by n, then let π denote the next smaller quotient, or we have $mq = n\pi + \upsilon$, where $\upsilon < n$. From this equation it follows that $\frac{\pi}{q} < \frac{m}{n}$ and $\frac{\pi+1}{q} > \frac{m}{n}$. Now let us try to discover by means of the known law according to which the values of the fraction $\frac{m}{n}$ vary, whether there are values of $\frac{m}{n}$ which are not *smaller* than $\frac{\pi+1}{q}$ like that just considered, but rather are equal to it or greater than it.

(a) If it should be shown that the limit of $\frac{m}{n}$ remains $<\frac{\pi+1}{q}$ in all its values, then in exactly the same way as in the proof of §104 (2 (a)) it is clear that A must be either $=$ or $>\frac{\pi+1}{q}$. The first case occurs if we find that the difference $\frac{\pi+1}{q} - \frac{m}{n}$ can be decreased indefinitely with the same q and p. The second case occurs if this difference does not decrease indefinitely. In the first case $\frac{\pi+1}{q}$ is a fraction which measures the number A *perfectly*. In the second case, since $\frac{\pi+1}{q} > A$ and $\frac{\pi}{q} < A$, it is obvious that the measuring fraction of A which we seek $= \frac{\pi}{q}$.

(b) If, on the contrary, some value of the limit $\frac{m}{n}$, I shall designate it by $\frac{m^1}{n^1}$, is $=$ or $>\frac{\pi+1}{q}$, then it can be seen from §104 as already mentioned, that $\frac{\pi+1}{q}$ and not $\frac{\pi}{q}$ represents the measuring fraction of the number A. And thus this can be determined in each case.

Example. Suppose $A = \frac{13}{4}$ and it is required to find the measuring fraction of this number for the denominator 7, and suppose $\frac{29}{9}$ and $\frac{30}{9}$ were a pair of limits of A given to us for this purpose, whose difference $\frac{1}{9}$ is $<\frac{1}{7}$. Here the attempted division of $n = 9$ into $mq = 29.7$ would not be possible, but gives the next smaller quotient 22. Now if, furthermore, we can infer from the law which governs the variation of the limits $\frac{29}{9}$ and $\frac{30}{9}$ that the smaller of these two fractions always remains smaller than $\frac{23}{7}$, indeed that the difference $\frac{23}{7} - \frac{m}{n}$ cannot decrease indefinitely, then it is decided that $\frac{22}{7}$ is the measuring fraction of A for the denominator 7.

§ 109

Theorem. If we know about a certain property B, that it belongs, not to all values of a variable measurable number X which are greater (or smaller) than a certain value U, but to all which are *smaller* (greater) than U, then we can definitely claim that there is a measurable number A which is the *greatest* (smallest) of those of which it can be said that all *smaller* (greater) X have the property B. It is left still undecided here whether the value $X = A$ itself also has this property.[x]

Proof. We need to prove the truth of this proposition only for the one case described outside the brackets, because the proof for the other case described inside the

[x] An earlier formulation of the Bolzano–Weierstrass theorem occurs at *RB* §12 on p. 269.

brackets follows of its own accord providing the words *greater* and *smaller* are exchanged for one another.

1. Now if the property B belongs, not to all X which are *greater*, but to all which are *smaller* than a certain U, then there is surely some number $U + P$ (where P represents a positive measurable number) of which it can be said that B does not belong to all X which are $<U + P$. For if no P of this kind could be found then B would have to belong to all X which are greater than U, because all measurable numbers which are greater than U are representable in the form $U + P$. If we therefore raise the question whether B belongs to all X which are $<U + \frac{P}{a}$, where the divisor a first of all denotes 1, then 2, 3 4, . . . , etc. in order. Then we are certain that we must say 'no' to the first of our questions, because $U + \frac{P}{1} = U + P$, and it has already been assumed that B does not belong to all X which are $<U + P$. It is only a matter of whether we shall also say 'no' to the *succeeding* questions which arise if we make a ever greater. If this should be the case, then it is clear that U itself is the greatest of the values of which the claim holds that all X which are smaller than it have the property B. For if there were a greater one, e.g. $U+d$, i.e. if the claim were to hold that also all X which are $<U+d$ have the property B, then it is obvious from §6 of the 6th Section that if we take a great enough, $U + \frac{P}{a}$ becomes = or $<U + d$, and consequently, if B belongs to all X which are $<U+d$, then B would also have to belong to all X which are $<U+\frac{P}{a}$. Thus we would not have said 'no' to this question, but 'yes'. It is therefore proved that in the case where we say 'no' to all the above questions, there is a certain measurable number A, namely $=U$, which is the greatest of those for which the claim holds that all values of X below it possess the property B.

2. If, on the contrary, we say '*yes*' to the above question for a certain value of a, and now let a designate that definite value for which it was *first* affirmed (as we have seen, a must be >1). Now we know that the property B belongs to all X which are $<U + \frac{P}{a}$, but not for all which are $<U + \frac{P}{a-1}$. But the difference between $U + \frac{P}{a-1}$ and $U + \frac{P}{a}$ is $= \frac{P}{a(a-1)}$. Therefore if we proceed with this, as we did before with the difference P, i.e. if we raise the question of whether B belongs to all X which are $<U + \frac{P}{a} + \frac{P}{a.b}$, and here b denotes first of all $a - 1$, then a, $a + 1$, $a + 2$ etc., then we are again certain that we must say 'no' at least to the first of these questions. For asking whether B belongs to all X which are $<U + \frac{P}{a} + \frac{P}{a(a-1)}$ means the same as asking whether B belongs to all X which are $<U+\frac{P}{(a-1)}$ (because $\frac{P}{a} + \frac{P}{a(a-1)} = \frac{P}{a-1}$), which has already been denied. But if we should say 'no' to all our succeeding questions, however large we gradually make b, then, as made clear before, $U + \frac{P}{a}$ would be that greatest value, or that A, for which the claim holds that all values of X below it have the property B.

3. If, on the contrary, we say 'yes' to one of these questions, and this happens first for the definite value b then we now know that B belongs to all X which are $<U+\frac{P}{a}+\frac{P}{ab}$, but not for all which are $<U+\frac{P}{a}+\frac{P}{a(b-1)}$. The difference between these two numbers is $\frac{P}{ab(b-1)}$ and we can proceed with this as we did before with $\frac{P}{a(a-1)}$.

4. If we imagine that we continue in this way, then eventually only one of two outcomes could arise:

(a) Either we would find a value of the form

$$U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m},$$

which is the *greatest* for which the claim holds, that all smaller values of X have the property B. This happens in the case when we say 'no' to the question whether B belongs to all X which are

$$< U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m} + \frac{P}{abc\ldots m.n}$$

for every value of n.

(b) Or we find, at least, that B does indeed belong to all X which are

$$< U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m}$$

but not to all which are

$$< U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m} + \frac{P}{abc\ldots m.n}$$

and in this latter case we are free to make the number of terms in the two expressions even greater through new questions.

5. Now if it is the first case that holds, then the measurable number

$$U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m}$$

is itself that A whose existence our theorem asserts.

6. In the *second* case, let us note that the infinite multitude of numbers of the form

$$U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m}$$

which would appear if we could continue the procedure described above indefinitely would form a series with the property of §107. That is, the series of measurable numbers in which every term is greater than the previous one but in such a way that the difference between the nth and $(n+r)$th terms, however much we may increase r, always remains smaller than a given fraction $\frac{1}{N}$, which itself can be taken as small as desired, provided n has first been taken sufficiently large. For if we consider as the nth term in the series mentioned above,

$$U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m}$$

but as the $(n+1)$th term,

$$U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m} + \frac{P}{abc\ldots mn}$$

etc., and finally as $(n+r)$th term,

$$U+\frac{P}{a}+\frac{P}{ab}+\frac{P}{abc}+\cdots+\frac{P}{abc\ldots m}+\frac{P}{abc\ldots mn}+\cdots+\frac{P}{abc\ldots mn\ldots r}$$

then the difference between the nth and $(n+r)$th terms

$$=\frac{P}{abc\ldots mn}+\cdots+\frac{P}{abc\ldots mn\ldots r}=\frac{P}{abc\ldots m}\left(\frac{1}{n}+\cdots+\frac{1}{n\ldots r}\right).$$

Since the numbers $a, b, c, \ldots, m, n, \ldots, r$ are all >1, it is clear that this difference always remains $<\frac{P}{abc\ldots m}$, however much the number r, i.e. the number of fractions contained in the brackets $\left(\frac{1}{n}+\cdots+\frac{1}{n\ldots r}\right)$, may be increased. Furthermore, since the number of factors $a.b.c\ldots m$ which appear in the denominator of the fraction $\frac{P}{abc\ldots m}$ is $=n$, and can therefore be increased indefinitely by increasing n, then it is clear that the difference spoken of can be decreased indefinitely. From the §107 just mentioned we therefore know that there is a certain number, in fact a unique, measurable number A, which the terms of our series, i.e. the numbers of the form

$$U+\frac{P}{a}+\frac{P}{ab}+\frac{P}{abc}+\cdots+\frac{P}{abc\ldots m},$$

approach indefinitely. Now I claim of this number A that it is the greatest of all those of which it can be said that all values of X below it have the property B.

(a) In the first place it is obvious that all values of X which are $<A$ still have the property B. If we were to imagine that some X which is still $<A$, e.g. $=A-\delta$, does not have the property B, then the number

$$U+\frac{P}{a}+\frac{P}{ab}+\frac{P}{abc}+\cdots+\frac{P}{abc\ldots r}$$

would always have to keep at distance δ from U because for all X which are smaller than it the property B should hold. Every X that is

$$=U+\frac{P}{a}+\frac{P}{ab}+\frac{P}{abc}+\cdots+\frac{P}{abc\ldots r}-\omega,$$

however small ω may be, has the characteristic B, on the other hand it should not belong to $X=A-\delta$. Therefore it must be that

$$A-\delta>U+\frac{P}{a}+\frac{P}{ab}+\frac{P}{abc}+\cdots+\frac{P}{abc\ldots r}-\omega$$

$$\text{or}\quad A-\left[U+\frac{P}{a}+\frac{P}{ab}+\frac{P}{abc}+\cdots+\frac{P}{abc\ldots r}\right]>\delta-\omega.$$

Thus the difference between A and the numbers appearing in the square brackets could not be made as small as desired by merely increasing r, since $\delta-\omega$ cannot be made as small as desired merely by the increase of r since δ does not change, while ω may become smaller than every given number through such increase in r.

(b) But it is clear that in no case can there be a number greater than A, e.g. $A+\delta$, of which it can be said that all X below it still have the property B. It was already said in 4 (b) that the property B does not belong to *all* X which are

$$< U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots(m-1)},$$

in which the number of factors in the last fraction $\frac{P}{abc\ldots(m-1)}$ can be taken as large as desired. The difference between this number and the number

$$U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m},$$

of which it still holds that all X below it have the property B, is obviously

$$= \frac{P}{abc\ldots(m-1)} - \frac{P}{abc\ldots m} = \frac{P}{abc\ldots m(m-1)},$$

a difference which can be decreased indefinitely, and can therefore also become $<\delta$, merely by the increase in the number of factors occurring in the denominator. Now since, in any case,

$$A > U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots m}$$

then also by addition

$$A + \delta > U + \frac{P}{a} + \frac{P}{ab} + \frac{P}{abc} + \cdots + \frac{P}{abc\ldots(m-1)}.$$

Therefore if the property B does not belong, by assumption, to all X which are smaller than the latter expression, then much less can this property belong to all X which are $<A+\delta$. Therefore it is not $A+\delta$, but only A, which is that greatest measurable number for which the claim holds that all values of X which are smaller than it have B.

7. Nevertheless whether this property belongs only to all values of X which are *smaller* than A, but not to the value $X = A$ itself, or, on the contrary, whether the value $X = A$ also has the property B, must remain undecided. For the one possibility, as much as the other, can be the case in particular circumstances. For example, if we ask about those values of X which have the property of making $A - X$ *positive*, the answer is obviously that this property belongs to all those X which are $<A$, but not to the value $X = A$ itself. On the other hand, if it were asked which X have the property of only making the expression $A-X$ not *negative*, then it would have to be answered that this property belongs not only to all those values of X which are $<A$, but also to the value $X = A$ itself.

§ 110

Theorem. If the variable but measurable number Y always remains *greater* than the variable but measurable number X, and if also there is no greatest value of the former and no smallest value of the latter, then there is always at least one measurable number A which always lies between the two limits X and Y. Furthermore, if the difference $Y - X$ cannot decrease indefinitely, then there are infinitely many such measurable numbers lying between X and Y. But if this difference does decrease indefinitely, then there is only a single [number]. Finally if the difference $Y - X$ decreases indefinitely and either X has a greatest value, or Y has a smallest value, then there is not a single measurable number which always lies between X and Y.

Proof. 1. Because the variable Y always remains greater than the variable X then there is some number U small enough that all numbers which are $<U$ have the property that they cannot become equal to Y itself. For this purpose it is only required to take U equal to one of the values that X can take. For then all values of Y are greater than U therefore certainly every number which is $<U$ has the property that no Y can be $=$ or $<$ it. On the other hand, it is nevertheless certain that what has just been said does not hold of *all* numbers however large they may be taken. For if we were to take U to be $=$ or $>$ a certain value of Y, e.g. Y^{I}, then obviously it could no more be asserted that the property mentioned still belongs to all numbers which are $<U$. For among these there is indeed one which is in fact equal to a value of Y. According to the previous theorem there must therefore be a measurable number A which is the greatest of those of which it can be said that all smaller ones possess the property mentioned, namely that no value of Y is equal to or smaller than it. I now claim of this number A that it always lies between the limits X and Y, or that always $Y > A > X$.

(a) Because no value of Y can become equal to a number which is smaller than A then all values of Y must be greater than A, or there can be at most only one which is equal to A. But if no value of Y can be a *smallest* one as is assumed in the first part of the theorem, then no value of Y can become equal to the number A, for this would certainly have to be the smallest value of Y, because every smaller number, which is therefore $<A$, never represents a value of Y. Therefore the only possibility remaining is that every value of $Y > A$.

(b) But it is equally necessary that every value of $X < A$. For if some value of X, e.g. X^{I}, were $>A$ then A would not be the *greatest* number of which it can be said that all below it have the property that no Y can be $=$ or $<$ than it. If we put $X^{\mathrm{I}} - A = \alpha$ then $A + \alpha$ is a number $>A$ of which it holds in the same way as of A that every smaller number has the property that no Y can be $=$ or $<$ than it. It is therefore absurd to assume that some value of X may be $>A$. But there cannot even be one value of X which $=A$ if the assumption holds that among all the values of X there is no greatest one. For if there were a value of $X = A$ then this would certainly be the greatest of all values X can take since there is no greater one, i.e. none which is $>A$. It is thus proved that X must always remain $<A$ and thus the two relations hold that $Y > A > X$, i.e. A always lies between the two limits X and Y.

2. Furthermore, if the difference $Y - X$ cannot decrease indefinitely, therefore it always remains $>P$ (for example), then I claim that there are infinitely many numbers which lie between the two limits X and Y. In fact I say that every number which lies between A and $A - P$, e.g. $A - P + d$, if $d < P$, also always lies between X and Y. For because Y always remains $>A$ then it is evident that also Y always remains $>B$, if this is $<A$. But because A is the greatest number of which it can be said that all smaller [numbers] have the property that no value of Y is $=$ or $<$ it, there must be values of Y which come as close to the number A as we please, in such a way that $Y - A$ can become $= \omega$. But because, on the other hand, the difference $Y - X$ should always remain $>P$, we have that always $X < Y - P$, i.e. if we put for Y the value $A + \omega = B + P - d + \omega$, always $X < (B + P - d + \omega - P) = B - d + \omega$. Thus B is always $>X + d - \omega$, therefore always $>X$. Therefore there is no doubt that the relationship $Y > B > X$ always holds, i.e. that P also lies between X and Y. Now since according to §84 there are infinitely many measurable numbers which lie between A and $A - P$, then it is proved that there are even infinitely many numbers which always lie between the limits X and Y, if the difference $Y - X$ cannot decrease indefinitely.
3. But if this difference can decrease indefinitely, then A is the single measurable number which always lies between X and Y. For a number which were to be $>A$ would not always be $<Y$, and a number which were to be $<A$ would not always remain $>X$ since the values of Y and X approach A as closely as we please.
4. Finally, if Y indeed always remains $>X$ but the difference $Y - X$ decreases indefinitely, and moreover, either Y has a smallest [value], or X has a greatest value, then there is no number at all which always lies between X and Y. For as we have just seen, as long as it is assumed that neither Y has a smallest value, nor X has a greatest value then there is only a single measurable number always lying between X and Y, namely A. But if Y has a *smallest* value, then this $=A$. For no value of Y can become smaller than A, and a value of Y which is $>A$ cannot be the smallest because the difference $Y - A$ decreases indefinitely. If X has a *greatest* value then this is $=A$. For no value of X can become greater than A, and a value of X which is $<A$ cannot be the greatest because the difference $A - X$ decreases indefinitely. Therefore in neither of these two cases can it be said that A always lies within X and Y, because in the first case a value of Y, and in the second case a value of X, is identical with A itself.

§ 111

Theorem. If A and B are a pair of measurable numbers, and moreover B is not infinitely small or zero, then the quotient $\frac{A}{B}$ again represents a measurable number and indeed only a single measurable number, as long as A and B are also one-valued [*einförmig*].

Proof. We need only prove the proposition for the case when A and B are both positive or absolute, for in other cases at most the sign of the quotient changes.

1. Now if, first of all, the dividend A is infinitely small or (what means the same here) is a mere zero, then there is no doubt that at least one of the numbers which $\frac{A}{B} = \frac{0}{B}$ represents, is zero. For by §71 $B.0 = 0$. Now since zero is a measurable number, and it is known from §39 that no other measurable number (i.e. finite number) gives a product with B which $= 0$, then the truth of the theorem for this case is proved.

2. But if not only B but also A is a finite number then there is always a q and an s large enough that the numerators p and r of the measuring fractions $\frac{p}{q}$ and $\frac{r}{s}$ are not zero, and because we then have:

$$A = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$$
$$B = \frac{r}{s} + P^3 = \frac{r+1}{s} - P^4$$

it is self-evident that also the numerators of the fractions $\frac{p+1}{q}$ and $\frac{r+1}{s}$ could never be zero because otherwise A and B could not be positive. Now if we imagine that the denominators q and s take all conceivable values (with the limitation already mentioned) then I claim that the two expressions $\frac{p+1}{q}.\frac{s}{r}$ and $\frac{p}{q}.\frac{s}{r+1}$ can be viewed as limits within which a certain measurable number, and indeed only a single one, always lies. By the previous § this assertion will be proved if we show that no single one of the infinitely many values which the expression $\frac{p+1}{q}.\frac{s}{r}$ can take, can be equal to a single one of the infinitely many values which the expression $\frac{p}{q}.\frac{s}{r+1}$ can take, but that the former are always greater than the latter, that the difference $\frac{p+1}{q}.\frac{s}{r} - \frac{p}{q}.\frac{s}{r+1}$ decreases indefinitely, and finally that $\frac{p+1}{q}.\frac{s}{r}$ has no *smallest* value and that $\frac{p}{q}.\frac{s}{r+1}$ has no greatest value.

(a) Now it is very easy to see that the number that the expression $\frac{p+1}{q}.\frac{s}{r}$ represents is greater than the number which the expression $\frac{p}{q}.\frac{s}{r+1}$ represents in the case where the variable numbers q and s, and therefore also p and r, have the same values in the two expressions. For obviously $\frac{p+1}{q} > \frac{p}{q}$ and $\frac{s}{r} > \frac{s}{r+1}$ (§4 of the 5th Section), therefore also (§16 of the 5th Section) $\frac{p+1}{q}.\frac{s}{r} > \frac{p}{q}.\frac{s}{r+1}$. However, it is to be shown that every value of the expression $\frac{p+1}{q}.\frac{s}{r}$ is greater than *every* [value] of the expression $\frac{p}{q}.\frac{s}{r+1}$, not only for such as have the same values for q and s but also for those in which other arbitrary values of q and s appear. Now if we denote the values of q and s in one of these expressions by q and s, and in the other by q^1 and s^1, and the numerators belonging to these denominators by p^1 and r^1, then it is to be proved that $\frac{p+1}{q}.\frac{s}{r} > \frac{p^1}{q^1}.\frac{s^1}{r^1+1}$, whatever relationship holds between the numbers q and q^1, and between s and s^1. Now it is known that $\frac{p+1}{q}$ is always $> \frac{p^1}{q^1}$, and likewise that $\frac{r^1+1}{s^1}$ is always $> \frac{r}{s}$, therefore by §15 of the 5th Section, $\frac{s}{r}$ is always $> \frac{s^1}{r^1+1}$. Thus by multiplication $\frac{p+1}{q}.\frac{s}{r}$ is always $> \frac{p^1}{q^1}.\frac{s^1}{r^1+1}$.

Therefore *every* value which $\frac{p+1}{q}.\frac{s}{r}$ can represent is certainly *greater* than *every* value which $\frac{p^1}{q^1}.\frac{s^1}{r^1+1}$ can represent.
(b) Furthermore, it is now to be proved that there are values for these expressions for which the difference $\frac{p+1}{q}.\frac{s}{r} - \frac{p^1}{q^1}.\frac{s^1}{r^1+1}$ becomes smaller than every fraction of the form $\frac{1}{N}$. I claim that such values, whose difference decreases indefinitely, arise if we allow the denominators q, s, q^1, s^1 to increase indefinitely. That is, because $\frac{p+1}{q}.\frac{s}{r} = \frac{ps}{qr} + \frac{s}{qr}$ and $\frac{p}{q}.\frac{s}{r+1} = \frac{ps}{qr} - \frac{ps}{qr(r+1)}$, it may be seen that at least for those values of the expressions $\frac{p+1}{q}.\frac{s}{r}$ and $\frac{p}{q}.\frac{s}{r+1}$ being compared, which arise from the same q and s, the difference decreases indefinitely as long as q and s grow indefinitely. For this difference is $= \frac{s}{qr} + \frac{ps}{qr(r+1)} = \frac{s}{qr}\left(1 + \frac{p}{r+1}\right)$. Now if q and s change into mq and ns [respectively] then the greatest value which the p corresponding to mq, can take $= mp + m - 1$, and the smallest value which r can take corresponding to ns, $=nr$. Therefore the difference to be estimated is certainly never greater than $\frac{ns}{mq.nr}\left(1 + \frac{mp+m-1}{nr+1}\right)$ which arises if we take the numerators in the given difference as large, and the denominators as small, as they can ever become. Now if for the same q, s, p, r the numbers m and n are taken equal and are increased indefinitely, then the value of the latter expression becomes $\frac{s}{n.qr}\left(1 + \frac{p+1-\frac{1}{n}}{r+\frac{1}{n}}\right)$ which obviously decreases indefinitely. Accordingly it is proved that the difference $\frac{p+1}{q}.\frac{s}{r} - \frac{p}{q}.\frac{s}{r+1}$ decreases indefinitely, at least if we multiply the numbers q and s indefinitely.
(c) It only remains to show that $\frac{p+1}{q}.\frac{s}{r}$ has no *smallest* value, and that $\frac{p}{q}.\frac{s}{r+1}$ has no *greatest* value. For this it is enough to show that for every given $\frac{p+1}{q}.\frac{s}{r}$ another can be produced which is smaller, and to every given $\frac{p}{q}.\frac{s}{r+1}$ another can be produced which is greater. Now it is known, from §15, that the value of the fraction $\frac{p+1}{q}$ can always become smaller than the [value] of the *next greater* measuring [fraction] merely by the increase of q. But if we reduce the value of the fraction $\frac{p+1}{q}$ while we leave that of the fraction $\frac{s}{r}$ unchanged, then we also reduce the value of the expression $\frac{p+1}{q}.\frac{s}{r}$. Exactly the opposite holds of the expression $\frac{p}{q}.\frac{s}{r+1}$, because merely by the increase of s, $\frac{r+1}{s}$ is always made smaller, so conversely $\frac{s}{r+1}$, and consequently also the product $\frac{p}{q}.\frac{s}{r+1}$, is always made greater. Therefore it is proved that both the expressions $\frac{p+1}{q}.\frac{s}{r}$ and $\frac{p}{q}.\frac{s}{r+1}$, if we give the numbers q and s all possible values, can be viewed as limits within which a certain measurable number lies, and indeed only a single one.

3. Now I claim further that this measurable number, which I shall designate by C, corresponds to the idea of $\frac{A}{B}$. This will be proved, by the definition of §24 of the 3rd Section, if I show that $B.C = A$. Now because C always lies between the two limits $\frac{p+1}{q}.\frac{s}{r}$ and $\frac{p}{q}.\frac{s}{r+1}$, and the first expression is the greater, then (by §80) we have the relationship, $\frac{p+1}{q}.\frac{s}{r} > C > \frac{p}{q}.\frac{s}{r+1}$. Therefore if we

multiply the terms of this relationship, which are all measurable numbers, by the measurable number B, we also have $B.\frac{p+1}{q}.\frac{s}{r} > B.C > B.\frac{p}{q}.\frac{s}{r+1}$ (§). But since $B = \frac{r}{s} + P^3 = \frac{r+1}{s} - P^4$, if we put the first value in the first term and the second value in the third term then we shall have,

$$\left(\frac{r}{s} + P^3\right)\frac{p+1}{q}.\frac{s}{r} > BC > \left(\frac{r+1}{s} - P^4\right)\frac{p}{q}.\frac{s}{r+1}$$

$$\text{i.e.}\quad \frac{p+1}{q} + P^3.\frac{p+1}{q}.\frac{s}{r} > BC > \frac{p}{q} - P^4.\frac{p}{q}.\frac{s}{r+1}.$$

Furthermore, $\frac{p+1}{q} = A + P^2$ and $\frac{p}{q} = A - P^1$, therefore

$$A + P^2 + P^3.\frac{p+1}{q}.\frac{s}{r} > BC > A - P^1 - P^4.\frac{p}{q}.\frac{s}{r+1}.$$

Therefore it must be that

$$A + P^2 + P^3.\frac{p+1}{q}.\frac{s}{r} - BC = P^5$$

$$\text{and}\quad BC - A + P^1 + P^4.\frac{p}{q}.\frac{s}{r+1} = P^6,$$

where P^5 and P^6 denote two positive number expressions yet to be determined. The numbers P^1 and P^2, and likewise P^3 and P^4, can be decreased indefinitely merely by the increase of q and s (§66). Therefore the same also holds for the products $P^3.\frac{p+1}{q}.\frac{s}{r}$ and $P^4.\frac{p}{q}.\frac{s}{r+1}$ because the fractions $\frac{p+1}{q}, \frac{s}{r}, \frac{p}{q}, \frac{s}{r+1}$ never exceed a given limit through the increase of q and s. Accordingly, if we designate by Ω^1 and Ω^2 a pair of positive numbers which can decrease indefinitely, we have

$$\left.\begin{aligned} A + \Omega^1 - BC &= P^5 \\ \text{and}\quad BC - A + \Omega^2 &= P^6 \end{aligned}\right\} \qquad \odot$$

which by addition gives $\Omega^1 + \Omega^2 = P^5 + P^6$, and shows that P^5 and P^6 both also decrease indefinitely. Now since A and BC are unchangeable while Ω^1, Ω^2, P^5 and P^6 decrease indefinitely then the equations $\odot$ show that it must be that $A = BC$. Thus it is proved that there is always a certain measurable number C which corresponds to the quotient $\frac{A}{B}$.

4. But it is also clear that C is the unique measurable number which corresponds to the idea $\frac{A}{B}$. Suppose C^1 were a second measurable number distinct from C which can be considered as the quotient of A by B, so therefore it must be that $BC^1 = A$. But if C and C^1 are unequal, one of them, e.g. C^1, is the greater and we have $C^1 = C + D$. Then $BC^1 = BC + BD$ therefore also $BC + BD = A$ and thus if we subtract $BC = A$, $BD = 0$ which by no. 1 is impossible unless we assume that $D = 0$.

§ 112

Corollary 1. Therefore if A and B are a pair of equal measurable numbers and C is a measurable number different from zero, then also $\frac{A}{C} = \frac{B}{C}$, i.e. equations of two measurable numbers are not affected if both sides are divided by a third measurable number which is not zero.

§ 113

Corollary 2. If A and B are a pair of measurable numbers and moreover B is not zero, then $\frac{A}{B}.B = A$. For under this assumption $\frac{A}{B}$ is measurable, therefore $\frac{A}{B}.B = B.\frac{A}{B}$ (§99) $= A$ (§100).

§ 114

Corollary 3. If A, B, C are three measurable numbers and moreover B is not zero, then it must be that $C.\frac{A}{B} = \frac{CA}{B}$. For both sides of this equation give one and the same product $= CA$ when multiplied by B. Because $\frac{A}{B}$ is measurable, then theorem §99 can be applied to the product $B\left(C.\frac{A}{B}\right)$ and $B.\left(C.\frac{A}{B}\right) = C.\left(B.\frac{A}{B}\right)$. But $B.\frac{A}{B}$, is, by the definition of §113 $= A$. Therefore $B\left(C.\frac{A}{B}\right) = C.A$. But by the same definition, also $B.\frac{CA}{B} = CA$. Therefore also it must be that $C.\frac{A}{B} = \frac{CA}{B}$.

§ 115

Corollary. If A, B, C are three measurable numbers and B, C are not zero then it must be that $\frac{A}{B} = \frac{C.A}{C.B} = \frac{A:C}{B:C}$. The correctness of the first equation, $\frac{A}{B} = \frac{C.A}{C.B}$ is clear because both sides multiplied by CB give the same product. That is, $(CB).\frac{A}{B}$ is, by §99 $= C.\left(B.\frac{A}{B}\right) = C.A$. And $CB.\left(\frac{CA}{CB}\right)$ is, by §113 $= C.A$. From this also follows the correctness of the second equation, $\frac{A}{B} = \frac{A:C}{B:C}$. Because $A : C$, $B : C$ are measurable numbers, and moreover the latter is not zero if B is not zero, then the fraction $\frac{A:C}{B:C}$ is of the form of the fraction $\frac{A}{B}$. Therefore, by what has just been proved, it is not affected if we multiply numerator and denominator by the same measurable number different from zero, namely C. But this gives the fraction $\frac{A}{B}$.

§ 116

Corollary. If A, B, C are three measurable numbers and moreover C is not zero, then we have $\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$. For on multiplication by C both expressions give the identical result, $A + B$. For $C\left(\frac{A+B}{C}\right)$ is, by the definition §113, $= A + B$. And because $\frac{A}{C}$ and $\frac{B}{C}$ are measurable numbers then by §101 $C\left(\frac{A}{C} + \frac{B}{C}\right) = C.\frac{A}{C} + C.\frac{B}{C}$ which by the definition just mentioned $= A + B$.

§ 117

Corollary. On the same assumption also $A + \frac{B}{C} = \frac{CA+B}{C}$. For instead of A we can write $\frac{CA}{C}$ (§113).

§ 118

Corollary. If A, B, C, D are four measurable numbers and moreover neither C nor the algebraic sum $C + D$ is zero, then we have

$$\frac{A+B}{C+D} = \frac{A}{C} - \frac{AD - CB}{C\,(C+D)}.$$

For on multiplication by $(C + D)$ both expressions give the identical result $A + B$. For the first one this is obvious. But in respect of the second expression we must first notice that $\frac{A}{C}$ as well as AD, CB, $C(C + D)$ and $\frac{AD-CB}{C(C+D)}$ designate measurable numbers. Therefore

$$\begin{aligned}(C+D)\left[\frac{A}{C} - \frac{AD-CB}{C(C+D)}\right] &= (C+D)\frac{A}{C} - (C+D)\frac{AD-CB}{C(C+D)}\\ &= C.\frac{A}{C} + D.\frac{A}{C} - \frac{(C+D)(AD-CB)}{C(C+D)}\\ &= A + \frac{AD}{C} - \frac{AD-CB}{C}\\ &= A + \frac{AD}{C} - \frac{AD}{C} + B\\ &= A + B.\end{aligned}$$

§ 119

Corollary. If A, B, C, D are four measurable numbers and B and D are not zero, then $\frac{A}{B}.\frac{C}{D} = \frac{AC}{BD}$. For $BD\left(\frac{A}{B}.\frac{C}{D}\right)$, by §99, $= \left(B.\frac{A}{B}\right)\left(D.\frac{C}{D}\right) = A.C$.

§ 120

Corollary. If A, B, C, D are four measurable numbers, and moreover the last three are not zero, then also $\frac{A}{B} : \frac{C}{D} = \frac{AD}{BC}$. For $\frac{C}{D}\left(\frac{A}{B} : \frac{C}{D}\right) = \frac{A}{B}$ (by §113). And $\frac{C}{D}\left(\frac{AD}{BC}\right) = \frac{CAD}{DBC}$ (by the previous §) $= \frac{A.C.D}{B.C.D} = \frac{A}{B}$.

§ 121

Corollary. If the two measurable numbers A and B are in the relationship $A > B$, and C designates a third measurable number, different from zero and positive, then it must also be that $\frac{A}{C} > \frac{B}{C}$. For because $\frac{A}{C}$, $\frac{B}{C}$ are measurable numbers, it must be that either $\frac{A}{C} = \frac{B}{C}$ or $\frac{B}{C} > \frac{A}{C}$ or $\frac{A}{C} > \frac{B}{C}$. The first case cannot be because it would also give, by §113, $A = B$, the second case cannot be because by § it would give $B > A$. Therefore only the third case can occur, $\frac{A}{C} > \frac{B}{C}$.

§ 122

Corollary. If A, B, C, D are four measurable numbers then $\frac{A}{B+\frac{C}{D}} =$ [y]

For the theory of measurable numbers

Perhaps the theory of measurable numbers could be simplified if we formulated the definition of them so that A is called measurable if we have two equations of the form $A = \frac{p}{q} + P = \frac{p+n}{q} - P$, where for the identical n, q can be increased indefinitely.

Theorem. If a, b, c, d denote four measurable numbers then the equation $\frac{a}{b+\frac{c}{d}} = 0$ cannot hold unless either a or $d = 0$.

Proof. Because a is not zero then neither can $b + \frac{c}{d}$, nor b, nor c be $= 0$. For none of these assumptions would give $\frac{a}{b+\frac{c}{d}} = 0$. And $b + \frac{c}{d} = 0$ would give $\frac{a}{0} = \infty$. Therefore $\frac{a}{b+\frac{c}{d}} = \frac{ad}{bd+c} = d$, $d = 0$.

[y] The manuscript of the copyist ends here. The rest has been added in Bolzano's hand.

Theory of Functions

with

Improvements and Additions to the
Theory of Functions

Bernard Bolzano

Translated from the edition of Bolzano's manuscripts prepared

by

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in the

Bernard Bolzano Gesamtausgabe Bd. 2A10/1

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[a] This is a translation of the contents as compiled by Bob van Rootselaar in *BGA* 2A10/1. The numeration in parentheses refers to the numbering by Rychlík given in the edition *F*(1). This numeration is used, for example, in Rusnock (2000).

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2. Improvements

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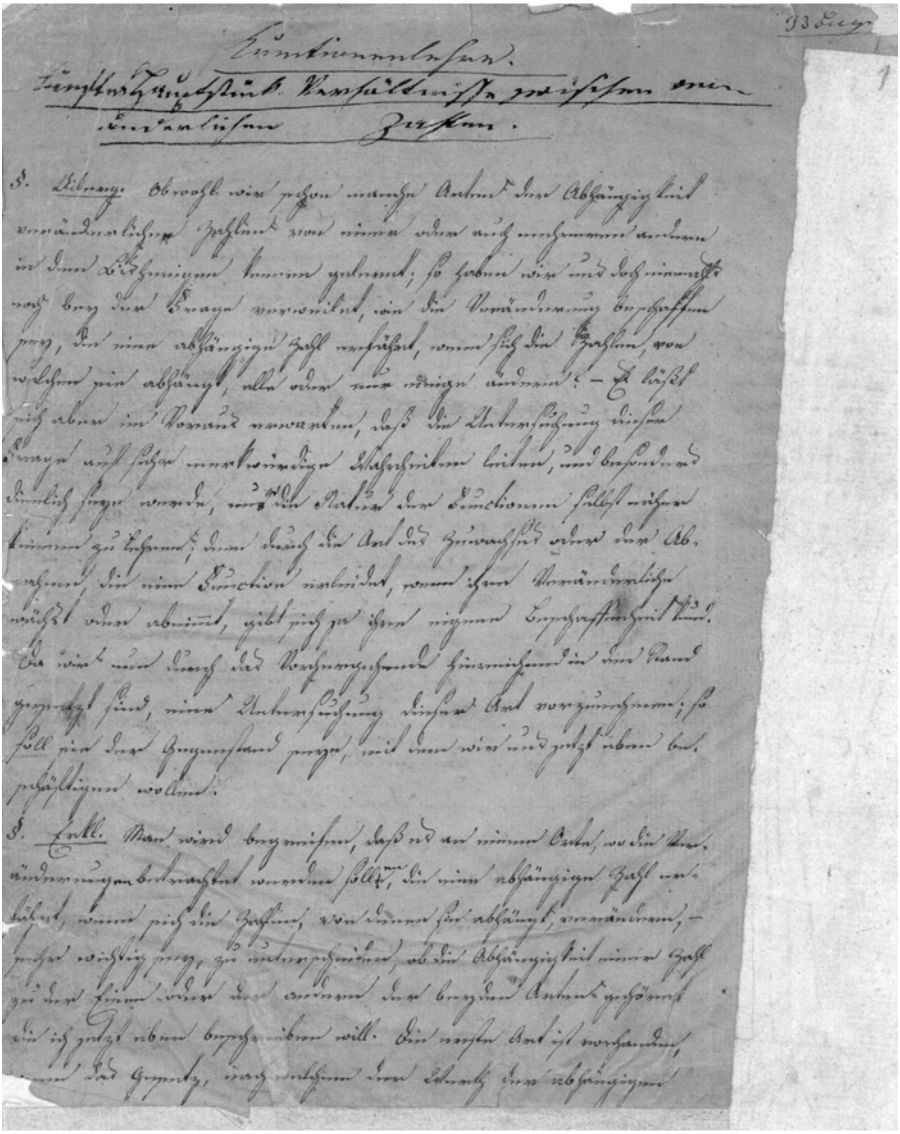

This is the opening sheet of *Functionenlehre* in the mss Series Nova 3471b from the Manuscript Collection of the Austrian National Library. It is in the hand of Bolzano's copyist.

Theory of Functions

Fifth Part.[b] Relationships between variable numbers

§ 1

Introduction. Although we have already met, in previous sections, with many kinds of dependence of variable numbers on one, or even several others, we have not yet paused to raise the question of the nature of the change in a dependent number if all, or some, of the numbers on which it depends vary. We can anticipate that the investigation of this question will lead to very significant truths and will be especially useful for improving our knowledge of the nature of functions themselves. For it is the way in which a function increases or decreases, when its variable increases or decreases, that reveals its particular nature. Since what we have said so far has been sufficient for us to take up this kind of investigation, this is the subject that we want to deal with now.

§ 2

Definition. It will be appreciated that where we are considering the changes in a dependent number, due to changes in the numbers on which it depends, it is very important to distinguish whether the dependence is of one or other of the two kinds that I shall now describe. The first kind occurs when the law, by which the value of the dependent number can be determined from the values of the freely variable numbers on which it depends, can be presented in a way which makes no mention of any *particular* value of the freely variable numbers; on the contrary, there is a rule by virtue of which the dependent number can be determined from the free variables whatever the value of the latter may be. When this does not apply, the second kind of dependency occurs. This is when there is no law for the derivability of the values of the variables from the values belonging to the free variables which would hold equally for all values of the latter. We have an example of the first if we define the number W so that the value belonging to it can be determined in each case by the values of the variables x, y and z, on which it depends, by multiplying x by 2, y by 3 and z by 4 and then combining these products into a sum, or letting $W = 2x + 3y + 4z$. Thus we give here a rule for the determination of W from x, y and z in the expression of which there obviously occurs no mention of any particular value of the numbers x, y and z, but rather the given rule holds generally for every one of these values. An example of the other kind is if W denotes the prize that shooters are rewarded with for skill at target practice if we decide that a shot in the bullseye should get 100 *Reichsthaler*, a shot which is a distance x inches from the bullseye, where x is not more than 2, gets $(100 - 25x)$ *Reichsthaler*, and a shot with distance from the bullseye >2 and <5 gets $(58 - 2x^2)$ *Reichsthaler* etc. Thus W depends here on x according to a law

[b] Being the fifth main part of the *Theory of Quantity* [*Größenlehre*].

which cannot be expressed without mentioning certain particular values of x, since from $x = 0$ to $x = 2$ the equation $W = 100 - 25x$ applies, but from $x = 2$ to $x = 5$ the equation $W = 58 - 2x^2$ applies, etc. We say about functions of the first kind that they can be determined *by a single identical law for all values of their variables*, but concerning functions of the second kind we say that they follow *several laws, different laws holding for different values of their variables*. In what follows we shall deal mainly, though not exclusively, with functions of the first kind.

§ 3

Note. It will be noticed that I have said in this definition that a function only belongs to the second kind if it is *not possible* to determine its values by a single law which is completely independent of any particular value of its variable and thus a generally valid law, but not if it is only possible to state certain particular laws for its determination which are valid for particular values of its variables. For this latter case is possible for every function (even those of the first kind) because whenever a generally valid law will suffice, then also several different laws can be devised valid for different values. For example if $W = 3x$ we could define W to be 3 for the value $x = 1$, but for every value which is <3 it should be determined by the equation $W = \frac{6x}{2}$, and for every value which is >3 by the equation $\frac{12x}{4}$.

§ 4

Definition. From among the many (perhaps infinitely many) values which can be given for a certain number idea x (the idea of a so-called *variable* number) let us distinguish one, e.g. $\overset{1}{x}$, and investigate the question of what sort of a number must be added to this $\overset{1}{x}$, or subtracted from this $\overset{1}{x}$, in order to obtain any other value of x, e.g. $\overset{2}{x}$. In this connection we call the value $\overset{1}{x}$ a *fundamental*, or *original*, or principal value, while the others which we compare with it, like $\overset{2}{x}$, the *modified* values. Finally, that actual, or only imagined, number which must be added to x[c] to obtain $\overset{2}{x}$, we call *the change*, or also *the increase*, or *the increment*; but most commonly we call it the *difference* of $\overset{1}{x}$ and we designate it by $\Delta\overset{1}{x}$. If the original value of x is designated simply by x, then the change, or that which must be added to x in order to obtain a modified value like x'', may also be designated simply by Δx. If the symbol for the number whose change we want to express is composed of several other symbols then for the sake of greater clarity we will put them in brackets. So, for example, the increase in the number xy is represented by $\Delta(xy)$. If we want to represent the increase, or to say it better, the *change*, in $W = F(x, y, z, \ldots)$ which depends on several variables $x, y, z, \ldots$ if only one of these variables, namely x, varies, then we write $\Delta_x W$ or $\Delta_x F(x, y, z, \ldots)$. Thus $\Delta_x F(x, y, z, \ldots)$ really designates nothing but the

[c] There should be $\overset{1}{x}$ instead of simply x here.

difference $F(x + \Delta x, y, z, \ldots) - F(x, y, z, \ldots)$, $\Delta_y F(x, y, z, \ldots)$ designates nothing but the difference $F(x, y + \Delta y, z, \ldots) - F(x, y, z, \ldots)$. The difference in W if x and y vary we designate by $\Delta_{xy} F(x, y, z, \ldots)$ etc.

§ 5

Note. Therefore the *difference* between two *given* numbers, e.g. $6 - 4$, is one thing, but the difference of a variable number $\Delta \overset{1}{x} = \overset{2}{x} - \overset{1}{x}$ is quite a different thing. For example, if x is a freely variable number, then $\overset{1}{x}$ just as much as $\overset{2}{x}$ can designate any arbitrary actual, or even merely imagined, value and therefore also $\Delta \overset{1}{x}$ can represent any arbitrary value; while the difference of two given numbers is always something definite and constant.

§ 6

Corollary 1. Depending on the nature of the two values which are being compared with one another, namely the original one and the modified one, the difference of a variable number x can sometimes be an actual number, and sometimes a merely imagined number, and, if the unit to which the variable x refers is a unit capable of an inverse[d] then the difference may sometimes be positive and sometimes negative. Thus, if the original value of $x = 6$, and the modified value $= 10$, the difference $\Delta x = 10 - 6 = 4$. If the original value is, as before $= 6$, but the modified value $= 3$ then by subtraction the difference is $3 - 6 = -3$. If, in a particular case, the value $\overset{2}{x}$, which we are thinking of as the modified one, is equal to the value $\overset{1}{x}$, which we are considering as the original one, then the idea of $\Delta x = \overset{2}{x} - \overset{1}{x}$ is just $= 0$, and therefore empty [*gegenstandlos*].

§ 7

Corollary 2. If the original value is x, and Δx is the change, then the modified value $= x + \Delta x$.

§ 8

Corollary 3. If the variable W depends on the variables $x, y, z, \ldots$ and is therefore $= F(x, y, z, \ldots)$, and the value W corresponds to the original values of x, y, z, $\ldots$ but the value $W + \Delta W$ corresponds to the modified values $x + \Delta x$, $y + \Delta y$, $z + \Delta z, \ldots$ then $W + \Delta W = F(x + \Delta x, y + \Delta y, z + \Delta z, \ldots)$, and therefore $\Delta W = F(x + \Delta x, y + \Delta y, z + \Delta z, \ldots) - F(x, y, z, \ldots)$.

[d] The German *ein des Gegensatzes fähige Einheit*, is literally, a unit capable of an opposite.

§ 9

Corollary 4. Also

$$\begin{aligned}\Delta_{xy}F(x,y) &= F(x+\Delta x, y+\Delta y) - F(x,y)\\ &= F(x+\Delta x, y) - F(x,y) + F(x+\Delta x, y+\Delta y)\\ &\quad - F(x+\Delta x, y)\\ &= \Delta_x F(x,y) + \Delta_y F(x+\Delta x, y).\end{aligned}$$

And likewise,

$$\begin{aligned}\Delta_{xyz}F(x,y,z) &= F(x+\Delta x, y+\Delta y, z+\Delta z) - F(x,y,z)\\ &= F(x+\Delta x, y, z) - F(x,y,z)\\ &\quad + F(x+\Delta x, y+\Delta y, z) - F(x+\Delta x, y, z)\\ &\quad + F(x+\Delta x, y+\Delta y, z+\Delta z)\\ &\quad - F(x+\Delta x, y+\Delta y, z)\\ &= \Delta_x F(x,y,z) + \Delta_y F(x+\Delta x, y, z)\\ &\qquad + \Delta_z F(x+\Delta x, y+\Delta y, z).\end{aligned}$$

And so on.

§ 10

Theorem. The increase in a function W of several variables $x, y, z, \ldots$ is, in general, again only a function of these same variables $x, y, z, \ldots$ and of their increases $\Delta x, \Delta y, \Delta z, \ldots$. But in individual cases this increase can be completely independent of one or other of these variables $x, y, z, \ldots$.

Proof. It is self-evident that $\Delta W = F(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) - F(x,y,z,\ldots)$ can depend on no *other* variable number than $x, y, z, \ldots$ and their increases $\Delta x, \Delta y, \Delta z, \ldots$. For only these variables appear in the concept of ΔW if in the concept of W or $F(x,y,z,\ldots)$ no other variable numbers than $x, y, z, \ldots$ appear, i.e. if W is, in fact, a function of only the variables $x, y, z, \ldots$ and no others. Examples can show us that in individual cases one or other of those variables $x, y, z, \ldots$ can disappear completely from this expression, in which case ΔW is completely independent of this variable. Suppose the function W or $F(x,y,z,\ldots)$ were of the form $x+\phi(y,z,\ldots)$, where $\phi(y,z,\ldots)$ designates a number dependent only on y and z. Then obviously $W+\Delta W = x+\Delta x+\phi(y+\Delta y, z+\Delta z, \ldots)$, and by subtraction $\Delta W = \Delta x+\phi(y+\Delta y, z+\Delta z, \ldots)-\phi(y,z,\ldots)$. In this expression the variable x does not appear any more, and therefore ΔW is certainly quite independent of it. In a similar way it is clear that the difference of the function $x+y+\phi(z)$ contains neither x nor y but, in addition to z and Δz, only Δx and Δy. And so on.

§ 11

Theorem. If a function $W = F(x, y, z, \ldots)$ is single-valued [*einförmig*] for all values of its variables or for all those values to whose concept the values x and $x + \Delta x$, y and $y + \Delta y$, z and $z + \Delta z, \ldots$ belong, then also ΔW is single-valued.

Proof. For $\Delta W = F(x + \Delta x, y + \Delta y, z + \Delta z, \ldots) - F(x, y, z, \ldots)$. Now if $F(x + \Delta x, y + \Delta y, z + \Delta z, \ldots)$ and $F(x, y, z, \ldots)$ are a pair of single-valued expressions, then certainly the difference of them, or ΔW, is also single-valued.

§ 12

Corollary. If $W = F(x, y, z, \ldots)$ is many-valued [*mehrförmig*], then ΔW need not, conversely, have equally many different values (§).

§ 13

Definition. Since the difference of a given function W of one or more variables $x, y, z, \ldots$ is in most cases again a dependent number, and indeed dependent on the same variables $x, y, z, \ldots$ and also on their changes $\Delta x, \Delta y, \Delta z, \ldots$ then, if we consider certain values of these variables $\overset{1}{x}, \overset{1}{y}, \overset{1}{z}, \ldots, \Delta\overset{1}{x}, \Delta\overset{1}{y}, \Delta\overset{1}{z}, \ldots$ as the originals, and certain others, $\overset{2}{x}, \overset{2}{y}, \overset{2}{z}, \ldots, \Delta\overset{2}{x}, \Delta\overset{2}{y}, \Delta\overset{2}{z}, \ldots$ as the modified ones, we can investigate the nature of the change in this new function ΔW. It is usual to call the change in the latter, i.e. the difference in the difference of W, the *second difference* of W, and in contrast with it the one which we considered earlier, i.e. the proper difference of W, is called the *first* difference. The concept of the *third*, *fourth*, and in general the *m*th difference, is formed in a similar way. We denote these differences respectively by $\Delta^2 W, \Delta^3 W, \Delta^4 W, \ldots, \Delta^m W$ etc.

§ 14

Theorem. For every value of n and m

$$\Delta^n(\Delta^m x) = \Delta^{n+m} x.$$

Proof. The proposition obviously holds for every value of m if $n = 1$. For it follows directly from the definition that $\Delta^1(\Delta^m x) = \Delta(\Delta^m x) = \Delta^{m+1} x$. But if the proposition $\Delta^n(\Delta^m x) = \Delta^{n+m} x$ holds for some value of n then it also holds for the next larger [value]. For if $\Delta^n(\Delta^m x) = \Delta^{m+n} x$, then also $\Delta^{n+1}(\Delta^m x) = \Delta^{m+n+1} x$, because $\Delta^{n+1}(\Delta^m x)$ is certainly $= \Delta(\Delta^n(\Delta^m x))$. But if $\Delta^n(\Delta^m x) = \Delta^{n+m} x$, then $\Delta^{n+1}(\Delta^m x) = \Delta(\Delta^{n+m} x) = \Delta^{n+m+1} x$, by the previous remark that $\Delta(\Delta^m x) = \Delta^{m+1} x$ for every value of m. Therefore $\Delta^n(\Delta^m x) = \Delta^{n+m} x$ also for $n = 2$, $n = 3$, and for every successive value.

§ 15

Theorem. If a function $W = F(x, y, z, \ldots)$ of one or more variables $x,\ y,\ z, \ldots$ is single-valued then in addition to its first *difference*, every successive *difference* is also single-valued.

Proof. For if W is indeed single-valued then so also is ΔW (§11), and if ΔW is single-valued then by the same proposition also $\Delta(\Delta W) = \Delta^2 W$ is single-valued etc.

§ 16

Definition. If we imagine W as a function of the variables $x,\ y,\ z, \ldots$ such that when the values $x,\ y,\ z, \ldots$ change into the values $x + \Delta x,\ y + \Delta y,\ z + \Delta z, \ldots$ the change in the function is equivalent [*gleichgeltend*] to the given function $\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$ of the variables $x,\ y,\ z, \ldots$ and $\Delta x,\ \Delta y,\ \Delta z, \ldots,$ then we call W, in this regard, the *sum* [*Summe*] belonging to the function $\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$, in fact the *first* sum. On the other hand, if we imagine W to be a function of the variables $x, y, z, \ldots$ such that not the first but the *second*, *third* or the *m*th difference which arises if $x,\ y,\ z, \ldots$ and then in addition to $x,\ y,\ z, \ldots$ also $\Delta x,\ \Delta y,\ \Delta z, \ldots$ etc. vary, is equivalent to the given function $\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots, \Delta^2 x, \ldots)$ then we call W the *second*, *third*, generally the *m*th *sum* of $\phi(x, y, z, \Delta x, \Delta y, \Delta z, \ldots, \Delta^2 x, \Delta^2 y)$. We designate these sums respectively by

$$\Sigma\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots),\ \Sigma\Sigma\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z,\ \ldots)$$

or

$$\Sigma^2\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots),\ \Sigma^3\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$$

and generally by $\Sigma^m\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$.

§ 17

Note. Therefore one must sharply distinguish between the sum of certain given numbers in the sense defined in §2, and the sum of a given number expression, which we regard as a difference, in the sense adopted here. The first is nothing but a collection of those given numbers in which no regard is paid to the order of the parts, and the parts of the parts are to be considered as parts of the whole. On the other hand, a sum in the sense established here is a function.

§ 18

Theorem. Generally $\Sigma^m \Sigma^n \phi = \Sigma^{m+n}\phi$.

Proof. In an exactly similar way to §14.

§ 19

Theorem. If a function W of the variables $x, y, z, \ldots$ is of such a nature that the change in it when $x, y, z, \ldots$ become $x + \Delta x, y + \Delta y, z + \Delta z, \ldots$ is in fact equal to the given [function] $\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$, then there is not merely a single such function but infinitely many, all differing from one and the same W only in that they have the form $W + C$, where we designate by C a number which is independent of $x, y, z, \ldots, \Delta x, \Delta y, \Delta z$, and moreover a number which is completely arbitrary.

Proof. If the change in W, or $\Delta W, = \phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z)$, then also the change in $W + C = \phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$ for all values of $C, x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots$ because always $\Delta(W + C) = (W + \Delta W + C) - (W + C) = \Delta W$. But it is clear that there can be no other function which satisfies this condition. Suppose that $F(x, y, z, \ldots)$ and $\Phi(x, y, z, \ldots)$ were two functions for which the changes are both $= \phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$. Then for all values of the variables $x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots$ the equation must hold

$$F(x + \Delta x, y + \Delta y, z + \Delta z) - F(x, y, z, \ldots)$$
$$= \Phi(x + \Delta x, y + \Delta y, z + \Delta z, \ldots) - \Phi(x, y, z, \ldots).$$

Therefore if, on both sides, we subtract $\Phi(x + \Delta x, y + \Delta y, z + \Delta z, \ldots)$ and add $F(x, y, z, \ldots)$:

$$F(x + \Delta x, y + \Delta y, z + \Delta z, \ldots) - \Phi(x + \Delta x, y + \Delta y, z + \Delta z)$$
$$= F(x, y, z, \ldots) - \Phi(x, y, z, \ldots).$$

Since this equation should hold for every value, not only of $x, y, z, \ldots$ but also of $\Delta x, \Delta y, \Delta z, \ldots$, it must also hold if we choose $\Delta x = a - x$, $\Delta y = b - y$, $\Delta z = c - z$ etc. But in this case the first side of the equation becomes $F(a, b, c, \ldots) - \Phi(a, b, c, \ldots)$ which is obviously a number idea quite independent of $x, y, z, \ldots$. If we represent this by C, then $C = F(x, y, z, \ldots) - \Phi(x, y, z, \ldots)$. Therefore $F(x, y, z, \ldots) = \Phi(x, y, z, \ldots) + C$ as stated in the theorem.

§ 20

Corollary 1. If the given function $\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$ is not actually of such a nature that there is a function of $x, y, z, \ldots$, $W = \Phi(x, y, z, \ldots)$ which, when $x, y, z, \ldots$, become $x + \Delta x$, $y + \Delta y$, $z + \Delta z, \ldots$, produces the difference $\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$ then the idea $\Sigma\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$ is empty. Therefore also $\Delta\Sigma\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$ is empty in approximately the same way that $(x - y) + y$ or $y(x/y)$ are empty if $x - y$ or x/y are empty (*RZ* 2, §18, *RZ* 3, §27). However, as we found reasons (*RZ* 2, §19, *RZ* 3, §28) to extend the original concepts of a difference and of a quotient, so that the equations $(x - y) + y = x$ and $y(x/y) = x$ might be established generally, so we can also establish here, by a similar extension of the concepts, that the equation $\Delta\Sigma\phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z) = \phi(x, y, z, \ldots, \Delta x, \Delta y, \Delta z, \ldots)$ holds generally.

§ 21

Corollary 2. Then it will also be the case in general, whatever ϕ is taken to be, $\Delta^m \Sigma^{m+n} \phi = \Sigma^n \phi$.

§ 22

Corollary 3. But it is not definitely the case that $\Sigma\Delta\phi$ is $= \phi$, but only one of the values which $\Sigma\Delta\phi$ can represent is $= \phi$. But all the values which $\Sigma\Delta\phi$ represents are of the form $\phi + C$.

§ 23

Theorem. The difference of a function which is an algebraic sum of several numbers, some of them variable and some of them constant, consists of the algebraic sum of the differences of the individual variable summands.

Proof. If we denote the algebraic sum which the constant numbers form on their own, by a, and the remaining variable numbers by $x, y, z, \ldots$ then the whole sum, i.e. the function whose change we are to determine, is $W = a \pm x \pm y \pm z \pm \cdots$. Furthermore, by §8, $W + \Delta W = a \pm (x + \Delta x) \pm (y + \Delta y) \pm (z + \Delta z) \pm \cdots$. Therefore by subtraction, $\Delta W = \pm\Delta x \pm \Delta y \pm \Delta z \pm \cdots$.

§ 24

Corollary 1. The difference of a completely constant number is therefore $= 0$.

§ 25

Corollary 2. Since the numbers $x, y, z, \ldots$ do not have to be *free* variables they could also represent functions of one or more other variables and thus the formula of the previous theorem shows us also how to find the difference of a function that is itself an algebraic sum of several others—if we know how to find the differences of the latter.

§ 26

Corollary 3. The validity of the formula of the previous theorem is not affected if the multitude of the terms that are combined by the signs $+$ or $-$ is infinite.

§ 27

Corollary 4. But if the multitude of variable terms $x, y, z, \ldots$ is only finite and the individual increases $\Delta x, \Delta y, \Delta z, \ldots$ can all decrease indefinitely while the number of them does not change, then it is clear from *RZ* 7, §§ 35, 52[e] that also the difference ΔW can decrease indefinitely.

§ 28

Theorem. The difference of a function, which is a product of a constant measurable number and a variable but always measurable number, consists of the product of the constant number and the difference of the variable number.

Proof. If we designate the constant number by a and the variable by x, then the given function W is either ax or xa. But if a and x are to be measurable numbers then we have $ax = xa$. Furthermore, $W + \Delta W = a(x + \Delta x)$. But if not only x but also $x + \Delta x$ is measurable, then also (by *RZ* 7, §45) Δx must be measurable so the last expression can be reduced to $ax + a\Delta x$. And we obtain by subtraction that $\Delta W = a\Delta x$.

§ 29

Corollary 1. Therefore if Δx decreases indefinitely, then ΔW also decreases indefinitely (*RZ* 7, §37).

§ 30

Corollary 2. In a similar way the difference of the quotient $\frac{x}{a}$, provided a is not zero, is $=\frac{\Delta x}{a}$. For by the assumption that a is not zero, $\frac{x}{a} = x.\frac{1}{a}$ and $\frac{1}{a}$ is measurable.

§ 31

Theorem. The difference of a function which is a product of *two* variables is obtained if we multiply each of the variables by the difference of the other one, then multiply the two differences by one another, and add all three products.

Proof. If there are only two factors then W is $=xy$, and $W + \Delta W = (x + \Delta x)(y + \Delta y)$ which, because x, y, Δx, Δy are measurable numbers can be expanded into $xy + y\Delta x + x\Delta y + \Delta x\Delta y$. Therefore by subtraction the required difference $\Delta W = \Delta(xy) = y\Delta x + x\Delta y + \Delta x\Delta y$.

§ 32

Corollary 1. If the given function is a product of *three* variable factors $W = xyz$ then if we view the product of two of them, e.g. yz, as a single variable factor u, we obtain by the previous formula, $\Delta W = \Delta(xu) = u\Delta x + x\Delta u + \Delta x\Delta u$. Now if we put for u its value yz and expand $\Delta u = \Delta(yz)$ again by the previous formula

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[e] Bolzano's several references to *RZ*7 correspond to the work *RZ* translated in this volume. For example, the paragraphs referred to here may be found on pp. 377 and 390 respectively.

into $y\Delta z + z\Delta y + \Delta y\Delta z$, then $\Delta W = \Delta(xyz) = yz\Delta x + x(y\Delta z + z\Delta y + \Delta y\Delta z) + \Delta x(y\Delta z + z\Delta y + \Delta y\Delta z)$ or $\Delta(xyz) = yz\Delta x + xz\Delta y + yz\Delta x + z\Delta x\Delta y + y\Delta x\Delta z + x\Delta y\Delta z + \Delta z\Delta y\Delta z$.

§ 33

Corollary 2. It is evident how this procedure could be extended to a product of any arbitrary number of variable factors.

§ 34

Corollary 3. If the increases Δx, Δy, $\Delta z, \ldots$ of the individual factors from which the product $xyz\ldots$ is composed, can all individually decrease indefinitely, while the number of them does not change, then also $\Delta(xyz\ldots)$ can decrease indefinitely. If there are only two factors then it is obvious that $\Delta(xy) = y\Delta x + x\Delta y + \Delta x\Delta y$ decreases indefinitely if Δx and Δy decrease indefinitely (*RZ* 7, §38). But if the proposition holds for a product of n factors then it also holds for $(n+1)$ factors. For if we denote one factor by x and the product of the other n factors by y then the whole product $W = xy$ decreases indefinitely, by what has just been said, providing Δx and Δy decrease indefinitely.

§ 35

Theorem. The difference of a *quotient*, if the numerator and denominator are both variable but always remain measurable numbers, and moreover the denominator never becomes zero, arises as follows. We multiply the denominator by the difference of the numerator, and the numerator by the difference of the denominator, subtract the latter product from the former, and divide the result by the product of the denominator and the denominator increased by its difference.

Proof. If the numerator and denominator are variable then we have $W = \frac{x}{y}$ and $W + \Delta W = \frac{x+\Delta x}{y+\Delta y}$ and $\Delta W = \frac{x+\Delta x}{y+\Delta y} - \frac{x}{y}$, which as long as y and $y + \Delta y$ are not zero can also be written $\Delta\left(\frac{x}{y}\right) = \frac{y\Delta x - x\Delta y}{y(y+\Delta y)}$.

§ 36

Corollary. If the increases in x and y, i.e. Δx and Δy, can decrease indefinitely then also $\Delta\left(\frac{x}{y}\right)$ can decrease indefinitely providing y is not $=0$. For on this assumption $\frac{y\Delta x - x\Delta y}{y(y+\Delta y)}$ can decrease indefinitely (*RZ* 7, §37).

This sheet from Bolzano's original version of *Functionenlehre* is Series Nova 3471a:9r from the Manuscript Collection of the Austrian National Library. It begins with the last six lines of §37 and illustrates the challenge of transcription for Bolzano's copyist as well as for modern editors.

First Section. Continuous and Discontinuous Functions

§ 37

Introduction. We saw from §§ 29, 34, 36 there are functions whose difference decreases indefinitely as long as the difference of their variables itself decreases indefinitely. In fact this case occurs with the functions mentioned for every arbitrary, but measurable, value of their variables. But the function of §36 is an exception to this rule in the special case when the denominator $y = 0$. For then $\Delta\left(\frac{x}{y}\right) = \Delta\left(\frac{x}{0}\right) = \frac{x+\Delta x}{\Delta y} - \frac{x}{0}$ is an expression which indicates a negative and infinitely large number. Furthermore, since we may think of the law of dependency of one number on another as we wish, it is evident that we can think of a number W whose value was determined by the value of another number x, in such a way that the difference ΔW fails to have the property of decreasing indefinitely when Δx decreases indefinitely not only for some, but for *all* values of x. [This could be] either because among the values that Δx takes as it decreases indefinitely there are some for which ΔW has no measurable value, or because this ΔW does not decrease indefinitely. For example, such a case would occur if we defined a certain number W to be $= ax$ for every value of x which is of the form $\frac{2m+1}{2^n}$, but to be $= ax + b$ for every other value. Since to every value of x which is not of the form $\frac{2m+1}{2^n}$, there are others which are of this form, and which are so close to x that the difference Δx can become smaller than every given fraction $\frac{1}{N}$, it is easy to see that for every x and for every Δx, however small it is taken, there is a yet smaller Δx for which the corresponding ΔW turns out $= a\Delta x + b$ or $= a\Delta x - b$, and therefore certainly could not decrease indefinitely. This difference in the behaviour of functions is certainly important enough to deserve being designated by special technical terms.[f]

§ 38

Definition. Suppose a single-valued function Fx of one, or several, variables has the property that the change in it when one of its variables x goes from the specific value x to the changed value $x + \Delta x$, decreases indefinitely if Δx decreases indefinitely. Therefore supposing the value Fx is measurable, as well as the value $F(x + \Delta x)$ (at least, for the latter starting from a certain value of the difference Δx and for all smaller values below), but the difference $F(x + \Delta x) - Fx$, in its absolute value, becomes and remains smaller than any given fraction $\frac{1}{N}$, providing only that Δx is taken small enough and then however much smaller it may become, then I say that the function Fx *varies continuously* for the value x, and indeed *with a positive increase* or in a positive direction, if what has just been said occurs with a positive value of Δx. On the other hand, I say that it *varies continuously*

[f] The German *Kunstworte*, is literally, made-up words.

with a *negative increase in x or in a negative direction* if what has been said occurs with *a negative value* of Δx. Finally, if what has been said holds with a positive increase as well as with a negative increase in x, then I simply say that Fx is *continuous* for the value of x, or if afraid of some misunderstanding I add that Fx is continuous for positive as well as negative increases. In the opposite case I say that the function Fx, for a positive or negative Δx, or both, *breaks the law of continuity*, or *is discontinuous* or *changes suddenly*. In the special case when the failure of continuity for the value $x = a$, is due to the fact that all differences that can be represented by $F(a + \Delta x) - Fa$ remain $> \frac{1}{N}$ if the positive or negative Δx decreases indefinitely, then let me say that with the value $x = a$ the function Fx makes a *jump* [*Sprung*]. But if the failure mentioned is due to the fact that for the value $x = a$ of its variable our function does not even have a *measurable* value then I say that for this value of its variable it has a *gap* [*Lücke*].

§ 39

Note. The concept of continuity has already been defined essentially as I do here by others, e.g. *Klügel*'s *Wörterbuch* (Article *stetig*), by *Cauchy* (*Cours d'Algèbre*,[g] Ch. 2, §2), *Ohm* (*Anal.*, B. 2, §456). And if some of these, like *Ohm* (who otherwise is so precise), use the expression that the change $F(x + \Delta x) - Fx$, in its absolute value, becomes smaller than any given quantity D and *must become all the smaller, the smaller Δx is taken*, then the latter might only have been said through oversight, since we really only require that $F(x + \Delta x) - Fx$ must become and *remain* $<D$, if Δx is made ever smaller. For that the difference $F(x + \Delta x) - Fx$ always becomes smaller if Δx is made smaller does not happen with every function to which, nevertheless, continuity is generally attributed, e.g. with $x^2 \sin\log x$ for $x = 0$, where the difference $F(x + \Delta x) - Fx$ becomes $(\Delta x)^2 \sin\log \Delta x$, and to each Δx however small, an even smaller one can be given with which this difference again becomes greater. Some very respected mathematicians like *Kästner* (*höhere Mechanik*, Auflage 2, §§ 183 ff.) and *Fries* (*Naturphilosophie*, §50) define the continuity of a function Fx as that property of it by virtue of which it does not go from a certain value Fa, to another value Fb, without first having taken all the values lying in between. However, it will be seen subsequently that this definition is too wide if in fact the concept intended is to be equivalent to the one above. The definition of *Eytelwein* (*Höhere Analysis*, Band 1, §16) is somewhat peculiar: that a function is to be called *continuous* if all the values which it takes inside certain limits of its variables, are real and finite; it is called *discontinuous* if one or more values become infinitely great or *impossible*. According to this definition we have to claim, contrary to all usage, that the function mentioned in the earlier §37, of which the values are always either $= ax$ or $= ax + b$ according to whether x is of the form $\frac{2m+1}{2^n}$ or not, is a continuous function. *Eytelwein* obviously derived his definition only by considering common functions which can be represented by our

[g] This should be *Cours d'Analyse*.

customary algebraic symbols. Of these it does indeed hold that inside those limits within which they become neither infinitely great nor imaginary (so remain measurable), they are also continuous. However, it is well to note that this is only the case with some of them, particularly with the so-called transcendental functions, because we (tacitly) establish in the definition of their concept that they should only vary according to the law of continuity—as I hope to show more clearly in its [proper] place. *Lacroix* has conceived of the concept of continuity much more narrowly when he (in his *Traité élémentaire de calcul differentiel et intégral*, §60) puts the nature of continuity in the general property (according to his idea) of functions of admitting a limit to the ratio $\frac{\Delta Fx}{\Delta x}$. This important property, not of all functions but of very many, does indeed deserve to be designated with a special term. However, the attribute which we described in the previous § also deserves a name just as much. Now since both are not only distinct in themselves but are also separable in that the continuity described above can occur without that of *Lacroix* having to be there, then it will be best to remain with the usage introduced by *Lagrange*, *Cauchy* and others, and to understand by the continuity of a function only the property described in the previous §. But we may say of functions for which the quotient $\frac{\Delta Fx}{\Delta x}$ approaches indefinitely a certain limit, independent of Δx, that *they have a derivative* (*une fonction dérivée*). Moreover, I must point out that the circumstance (admittedly rare) where a function is continuous only in one direction, that is only in respect of a positive increase or only of a negative increase, has not only been completely ignored so far in the *definition*, but also (as it seems to me) has had too little attention in application.

§ 40

Theorem. Suppose a number W is completely independent of x (which we are considering as variable), for example, because x does not appear in the concept of W at all, or it appears in it, but in such a way that the value of W does not change whatever value x may take. For example, if we had $W = \frac{ax-bx}{cx}$, where a, b and c denote constant numbers independent of x. Then W can also be considered as *continuous*, and in fact continuous in respect of every value of x and in both directions.

Proof. If we designate the values of W belonging to x and $x + \Delta x$ by Fx and $F(x + \Delta x)$, then we must put $Fx = F(x + \Delta x)$ and therefore $F(x + \Delta x) - Fx = 0$. Therefore it can always be said that the difference ΔFx becomes and remains $< \frac{1}{N}$, for 0 is certainly $< \frac{1}{N}$.

§ 41

Theorem. Every function of the form $a \pm x$, in which x designates a free variable but a designates a measurable number completely independent of x, is continuous for every measurable value of x, and with respect to a positive as well as a negative increase. The same also holds of every function of the form ax or xa, and in the same way also of the function $\frac{x}{a}$ provided a is not zero. Finally every function of the form $\frac{a}{x}$ is also continuous for every value of x which is not zero.

Proof. Follows from §§ 27, 29, 36.

§ 42

Theorem. Every arbitrary power of a freely variable number x is also continuous for all measurable values of this variable with respect to a positive as well as a negative increase.

Proof. If the power is of the first degree, the proposition needs no proof because x^1 is identical with x itself. But if the power is of the second or higher degree, then if we represent this by x^n where n denotes a number >1, and the difference $\Delta Fx = \Delta(x^n) = (x+\Delta x)^n - x^n$ is, in its absolute value, $< n(x+\Delta x)^{n-1} \cdot \Delta x$ by §, if we put there $a = x + \Delta x$, $b = x$. This is an expression which obviously decreases indefinitely with Δx whatever value x may have, provided it is always measurable.

§ 43

Theorem. Every *integral* [*ganze*] *rational function* of a free variable is continuous for every measurable value of the latter with respect to a positive increase, as well as a negative increase.

Proof. We call a function of x rational and integral if it is equivalent for all values of x to an expression of the form $a + bx + cx^2 + dx^3 + \cdots + lx^m$, in which a, b, c, d, ..., l all designate measurable numbers but m designates an arbitrary actual number.[h] Now if

$$Fx = a + bx + cx^2 + dx^3 + \cdots + lx^m$$

then

$$\begin{aligned}\Delta Fx =& F(x+\Delta x) - Fx \\ =& [a + b(x+\Delta x) + c(x+\Delta x)^2 + d(x+\Delta x)^3 \\ & \quad + \cdots + l(x+\Delta x)^m] \\ & \quad - [a + bx + cx^2 + dx^3 + \cdots + lx^m] \\ =& b\Delta x + c[(x+\Delta x)^2 - x^2] + d[(x+\Delta x)^3 - x^3] \\ & \quad + \cdots + l[(x+\Delta x)^m - x^m].\end{aligned}$$

But from the previous § we know that each of the numbers enclosed in the square brackets decreases indefinitely with the indefinite decrease in Δx and can therefore be represented by $\Omega^1, \Omega^2, \ldots, \Omega^m$. Therefore the whole expression $\Delta Fx = b\Delta x + c\Omega^1 + d\Omega^2 + \cdots + l\Omega^m$. By §42 each of these terms decreases

[h] Here 'actual number' means whole number.

indefinitely and therefore (because their number is constant) so also does their algebraic sum, and therefore the value of ΔFx itself.

§ 44

Theorem. Every *fractional rational* [*gebrochene rationale*] function of a free variable is also continuous for every measurable value of the latter which does not make the denominator zero, and with respect to a positive as well as a negative increase.

Proof. We call a function Fx rational and fractional if for all values of x it is equivalent to a function of the form

$$\frac{a + bx + cx^2 + dx^3 + \cdots + lx^m}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots + \lambda x^\mu}.$$

As an abbreviation let us now denote the numerator of this latter expression by fx, and the denominator by ϕx, then fx and ϕx are a pair of rational and integral functions and therefore continuous for every value of x and in both directions. But

$$\begin{aligned}\Delta Fx &= f(x + \Delta x) - Fx^{\mathrm{i}}\\ &= \frac{f(x+\Delta x)}{\phi(x+\Delta x)} - \frac{fx}{\phi x}\\ &= \frac{fx + \Delta fx}{\phi x + \Delta\phi x} - \frac{fx}{\phi x},\end{aligned}$$

which, if neither ϕx nor $\phi(x + \Delta x) = \phi x + \Delta\phi x = 0$, can be put

$$\begin{aligned}&= \frac{\phi x\,.\,fx + \phi x\,.\,\Delta fx - fx\,.\,\phi x - fx\,.\,\Delta\phi x}{\phi x(\phi x + \Delta\phi x)}\\ &= \frac{\phi x\,.\,\Delta fx - fx\,.\,\Delta\phi x}{\phi x(\phi x + \Delta\phi x)}.\end{aligned}$$

Now, if ϕx, i.e. the denominator of the given fractional function is not $= 0$, then certainly also $\phi x + \Delta\phi x$ is not $= 0$, at least starting from a certain value of the difference $\Delta\phi x$, namely from such a value which in absolute value is $< \Delta x$. Therefore ΔFx can decrease indefinitely, providing the numerator in the latter expression, i.e. $\phi x\,.\,\Delta fx - fx\,.\,\Delta\phi x$, decreases indefinitely. But this happens for every value of x if Δx decreases indefinitely, because Δfx and $\Delta\phi x$ decrease indefinitely.

§ 45

Theorem. If it is true that a function Fx becomes *discontinuous* for a specific value of its variable x, and with respect to either a positive or negative increase of it, then one of the following three cases must occur: (i) either the expression Fx, or

[i] This should be $\Delta Fx = F(x + \Delta x) - Fx$.

the expression $F(x + \Delta x)$, denotes no number which is measurable, or remains measurable, if we put for Δx in the latter any arbitrary smaller value starting from a certain one, or (ii) the difference $F(x + \Delta x) - Fx$ in its absolute value always remains greater than a certain number, however small Δx may become, or (iii) this difference, for certain values of Δx, does become smaller than every given fraction $\frac{1}{N}$, but it does not remain so, instead for every number there is an even smaller Δx, for which $F(x + \Delta x) - Fx$ again becomes $\overline{\gtrless} \frac{1}{N}$.

Proof. It is self-evident that apart from these three cases, no other situation can occur; but that each of them can arise, under certain circumstances, we may show with the following examples.

1. The function $Fx = \frac{1}{1-x}$ is discontinuous for the value $x = 1$, because the value of Fx itself for $x = 1$ is not measurable, namely it is infinitely large. If the number y depends in such a way on x that we always have the equation $y^2 = 1 - x^2$ then for the value $x = 1$ and for a positive Δx, y is discontinuous because the expression $F(x + \Delta x)$ does not represent a measurable number at all. For this number would have to be of such a nature that its square $= 1 - (1 + \Delta x)^2 = -2\Delta x - \Delta x^2$, i.e. a *negative* number. But such a [number] is not known (§).
2. If we put, for $x = 1$ and all smaller values $Fx = 3x$, but for all larger values $Fx = 5x$, then for the value $x = 1$, and for a positive Δx, Fx is discontinuous, because the difference $F(x + \Delta x) - Fx = 5(1 + \Delta x) - 3 = 2 + 5\Delta x$ always remains >2.
3. If we put, for every value of x which is of the form $\frac{1}{2^n}$ where n can represent any arbitrary actual number, the value of $Fx = 1$, but for all other values of x, $Fx = 2x$, then Fx is discontinuous for the value $x = 0$ and a positive Δx for the third reason given in the theorem. That is, $Fx = F(0) = 0$ because the value $x = 0$ is not of the form $\frac{1}{2^n}$. But $F(x + \Delta x) = F(\Delta x)$ will be $= 1$ for every value of Δx which is of the form $\frac{1}{2^n}$, e.g. $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ etc., but for every other value it will be $= 2\Delta x$. Therefore the difference $F(x + \Delta x) - Fx$ is sometimes $= 1$, and sometimes $= 2\Delta x$. Therefore if Δx decreases indefinitely, [the difference] decreases also but not in such a way that there is not, for every Δx, a smaller one (namely one of the form $\frac{1}{2^n}$) for which this difference again becomes $= 1$.

§ 46

Theorem. The property of being continuous belongs to some functions only for a certain *isolated* value of their variables.

Proof. Consider a number $W = Fx$ which depends on the variable x according to a rule such that for all those values of x which are of the form $\frac{2m+1}{2^n}$, $W = 2x$, but for all other values $W = 3x$. I claim that this function of x is continuous for the value $x = 0$ but discontinuous for all other values. It is clear that W is to be called continuous for the value $x = 0$ because the difference $F(x + \Delta x) - Fx$, for this value of x, $= F(\Delta x) - F(0)$ decreases indefinitely with Δx. For $F(0) = 3.0 = 0$ because 0 is not of the form $\frac{2m+1}{2^n}$; but $F(\Delta x)$ is either $=2\Delta x$ or $=3\Delta x$ according to whether Δx is of the form $\frac{2m+1}{2^n}$ or not. But now it is obvious that not only $2\Delta x$ but also $3\Delta x$ decrease indefinitely with Δx itself. Therefore our function is continuous for $x = 0$. But for every other value it is discontinuous. The difference $F(x + \Delta x) - Fx$ can always only have one of the following four values, $3\Delta x$, $x + 3\Delta x$, $-x + 2\Delta x$, $2\Delta x$ according to whether neither $x + \Delta x$ nor x, or only $x + \Delta x$, or only x, or $x + \Delta x$ as well as x, are of the form $\frac{2m+1}{2^n}$. Providing x is not $= 0$, it may be seen that this difference decreases indefinitely with Δx only in the first and fourth cases, but does not do so in the second and third cases. But obviously there is no Δx so small that there would not be a smaller one which is of the form $\frac{2m+1}{2^n}$, since the fraction $\frac{2m+1}{2^n}$, if we indefinitely increase the exponent n for the same m, can become smaller than any given number. Therefore in each case, whether x is of the form $\frac{2m+1}{2^n}$ or not, the difference $F(x + \Delta x) - Fx$ does not decrease indefinitely with Δx, but for every Δx there is a smaller one for which this difference is, in the first case, $x + 3\Delta x$, in the second case $-x + 2\Delta x$.

§ 47

Theorem. A function can *observe*[j] the law of continuity as well for a certain isolated value, as also for a whole collection of values of its variables, as many as lie within certain limits a and b, or even generally for all values of its variables. Conversely, a function can also *break* the law of continuity, as well for a certain isolated value and with respect to one or both directions, as also for a whole collection of values of its variables, as many as lie within given limits a and b, or even generally for all values of its variables, and this both by jumps as well as gaps.

Proof. What the theorem states about the case in which a function can *observe* the law of continuity is already clear from above. Therefore it is only to be proved what it states about the breaking of this law.

1. If we now stipulate that the number W, for all values of x which are <1 has the value $4x$, but for $x = 1$ and all greater values, it has the value $5x$, then we obtain the example of a function which breaks the law of continuity for an isolated value, namely for $x = 1$, in respect of the negative direction, and indeed by a *jump*. If we had stipulated that W, for all values of x which are < 1 and for $x = 1$, has the value $4x$, but for all greater values has the value $5x$, then the function would be continuous for the value $x = 1$ only for a negative increase. Finally if we put $W = 4x$ for all values of $x < 1$, $W = 5x$ for all values of $x > 1$, but for $x = 1$, $W = 10$, then this function is continuous in neither direction for this last value, but for all other values it is continuous in both directions.
2. The function which we considered in §46 breaks the law of continuity for all values of its variables and does so by continual *jumps*.
3. If we stipulate that W has the value $W = ax$, for all values of x with the exception of all integer values of x, for which W does not even exist, or is not

[j] The German *beobachten* in the sense here of 'obey'.

measurable, then we have an example of a function which for certain isolated values of its variables, namely here for the values 1, 2, 3, 4, . . . , *in inf*. breaks the law of continuity by [having] a *gap*.
4. Finally, if we say that W does not even exist, or not be measurable, for all values of its variables lying within α and β, but for all other values it is $= ax$, then we have a function which has a *gap* for all values of its variables from $x = \alpha$ to $x = \beta$ (exclusive). And so on.

§ 48

Theorem. Merely from the circumstance that a function Fx is continuous for all values of its variables lying within a and b with respect to a *positive* (or negative) increase, it does not follow that it is also continuous with respect to a *negative* (or positive) increase.

Proof. If Fx is continuous for every value of x lying within a and b with respect to a positive Δx, then if we take ω positive, it must be that $F(x + \omega) - Fx = \Omega$, and if we take i so that $x - i$ also lies within a and b, then it must also be that $F(x - i + \omega) - F(x - i) = \overset{1}{\Omega}$. Now if we were allowed to put $i = \omega$ then the latter equation would certainly give $Fx - F(x - \omega) = \overset{1}{\Omega}$, i.e. the difference $F(x - \omega) - Fx$, in its absolute value, could also decrease indefinitely. From this we could be tempted to conclude that Fx is also continuous for a negative Δx. However, it is not to be forgotten that in the equation $F(x - i + \omega) - F(x - i) = \overset{1}{\Omega}$, the value of $\overset{1}{\Omega}$ depends not only on ω but also on $x - i$, and therefore also on i. Therefore it may not be argued immediately that $\overset{1}{\Omega}$ here denotes a number which decreases indefinitely with i, for it could be that for the value $i = \omega$, the difference $F(x - i) - Fx$ always remains *greater* than a certain measurable number, although the difference $F(x + \omega) - Fx$ decreases indefinitely, because the condition of continuity for the value $x - i$ and for a positive increase, only requires that $F(x - i + \omega) - F(x - i)$ decreases indefinitely *for the same* i with a certain ω. But this ω can be different, for different values of i, and specifically it could always be necessary that it becomes *smaller* than i in order to make the difference $F(x - i + \omega) - F(x - i) < \frac{1}{N}$. For example, if it were that $Fx = x^2$ for all values of $x < 2$, but $Fx = x^3$ for all values of $x \geqq 2$, then Fx would certainly be proved continuous for every positive increase. For even for the value $x = 2$, we would have

$$F(x + \Delta x) - Fx = (2 + \Delta x)^3 - 2^3 = 12\Delta x + 6\Delta x^2 + \Delta x^3.$$

But for a negative increase,

$$F(x - \Delta x) - Fx = (2 - \Delta x)^2 - 2^3 = -4 - 12\Delta x + 6\Delta x^2 - \Delta x^3.$$

Therefore here the function is discontinuous.

§ 49

Theorem. Merely from [the fact] that a function Fx is continuous for all values of its variable lying within a and b, it does not follow that for all values of x lying within these limits there must be one and the same number e, small enough to be able to claim that one need never make Δx, in its absolute value, $< e$, so that it turns out that the difference $F(x + \Delta x) - Fx < \frac{1}{N}$.

Proof. It is neither contradictory in itself, nor to the given concept of continuity, to assume that for every x, e.g. particularly for every x that approaches a certain limit c, there is necessarily another smaller Δx in order to satisfy the condition that $F(x+\Delta x) - Fx$ becomes and remains $<\frac{1}{N}$ so long as Δx is ever more reduced. We have an example in the function $Fx = \frac{1}{1-x}$ for such values of x which approach the value of 1 indefinitely. Namely, if we write for brevity $x = 1 - i$, then $F(x + \Delta x) - Fx = \frac{\Delta x}{i(i-\Delta x)}$. If this is to become $<\frac{1}{N}$, then it must be that $\Delta x < \frac{i^2}{N+i}$. Therefore the smaller i becomes, so much smaller must also Δx be made, and if i decreases indefinitely, i.e. if x approaches the limit 1 indefinitely, then Δx must become gradually smaller than any given number, just so that the difference $\Delta Fx < \frac{1}{N}$.

§ 50

Theorem. If we know about a number $W = Fx$ that it obeys the law of continuity either generally, or within given limits a and b of its variable x, and if it is known furthermore that for values of x which can approach as near as we please to a specific m lying within a and b, it takes values whose difference from the constant measurable number M decreases indefinitely, then we may conclude that for $x = m$ it changes into the value M, or that $Fm = M$.

Proof. Because of the continuity which the function $W = Fx$ displays for the value $x = m$, it must be that the difference $F(m + \omega) - Fm = \Omega$, if ω and Ω have the usual meaning. By the assumption, $F(m + \omega) - M = \overset{1}{\Omega}$, then by subtraction it must also be that $M - Fm = \overset{1}{\Omega} - \overset{2}{\Omega} = \overset{3}{\Omega}$, an equation from which, by *RZ* 7, §92, $M = Fm$ follows because M and Fm are constant.

§ 51

Corollary 1. In this way, with the assumption of continuity, we are in a position in some cases to determine the value of a function which would be undetermined without this assumption. For example, if we were given that a certain number W had to be $= a + x$ in all those cases in which another variable x had a *positive* value, then from this statement alone it is certainly not determined which value W has to take for the case of $x = 0$. But if we may assume, besides this, that W always varies only according to the law of continuity, or at least around the value $x = 0$, then by combining these facts it is now determined which value W has to take for $x = 0$. Namely, it may be nothing but the value $W = a$, because only with this can the difference $\Delta W = F(x + \Delta x) - Fx = (a + x + \Delta x) - (a + x) = \Delta x$ decrease indefinitely if Δx decreases indefinitely. For an even more curious example, *the value of the infinite number expression* $1 - 1 + 1 - 1 + \cdots$ *in inf.* which we were not able to determine (*RZ* 7, §29), can be decided as soon as we notice that the infinite series $1 - x + x^2 - x^3 + x^4 - \cdots$ *in inf.* for the value $x = 1$ changes into the infinite number expression $1 - 1 + 1 - 1 + \cdots$ *in inf.*, and we establish that this transition occurs according to the law of continuity. Because the infinite series $1 - x + x^2 - x^3 + x^4 - \cdots$ *in inf.* is measurable for every value of x which is <1, and it has the known value $\frac{1}{1+x}$, a value which approaches indefinitely the value $\frac{1}{2}$ if x approaches the value 1 indefinitely. Thus it may be seen that the law of continuity requires the value of the series $1 - 1 + 1 - 1 + \cdots$ *in inf.* to be $= \frac{1}{2}$.

§ 52

Corollary 2. It is not unusual, especially with expressions which have the form of a *fraction*, that for a certain value of a variable x occurring in them, they become the expression $\frac{0}{0}$, which according to § is in itself completely indeterminate, and can represent any arbitrary number. This happens, for example, with the expression $\frac{x^3-7x-6}{x^3-2x^2-9x+18}$ for the value $x = 3$. But if we may assume that the number which such an expression represents varies continuously, either in general or in the neighbourhood of that value of x which turns it into $\frac{0}{0}$, then it is often possible, by this fact, to determine completely the value which the number concerned takes in this case.

Thus, if we put instead of the value $x = \alpha$, which turns the given expression into $\frac{0}{0}$, the value $x = \alpha + \omega$, and we can find, by appropriate expansions, a number A which approaches the value of this expression as closely as we please as long as we decrease ω indefinitely, then we may conclude that A is the true value which our number takes for the value $x = \alpha$. In the above example we obtain, if we put for x the value $3 + \omega$,

$$\frac{(3+\omega)^3 - 7(3+\omega) - 6}{(3+\omega)^3 - 2(3+\omega)^2 - 9(3+\omega) + 18} = \frac{20\omega + 9\omega^2 + \omega^3}{6\omega + 7\omega^2 + \omega^3}$$

or if we divide the numerator and denominator by ω, $\frac{20+9\omega+\omega^2}{6+7\omega+\omega^2}$. It can now be seen from this expression that it approaches indefinitely the value $\frac{20}{6} = \frac{10}{3}$, with the indefinite decrease in ω. Therefore $\frac{10}{3}$ is the value which the given function, $\frac{x^3-7x-6}{x^3-2x^2-9x+18}$, must take for the value $x = 3$, as long as it obeys the law of continuity, either in general, or around this value. In a similar way it may be found that $\frac{a^3-x^3}{a-x}$ must take the value $3a^2$ for the value $x = a$, and $\frac{a^2-2ax+x^2}{a^2-x^2}$ must take the value 0 for $x = a$, if these functions are to be continuous for $x = a$.

§ 53

Theorem. If a function Fx, which is continuous for all values lying within a and b, takes one and the same value M for so many values of x that there are no two measurable numbers within a and b lying so close to one another that within them a value of x does not lie which makes Fx again $= M$, then I claim that Fx is really *not variable at all* within a and b but always remains $= M$.

Proof. Let x represent an arbitrary value of x lying between a and b, then, because Fx is continuous it must be that some i can be given small enough that for it, and all smaller values, $F(x+i) - Fx$ becomes and remains $< \frac{1}{N}$. If within a and b there are no two measurable numbers lying so close to one another that within them no value of x lies which makes $Fx = M$, then such a value must also lie within x and $x + i$. Let us denote this by $x + j$, then j is a number of the same sign as i and in its absolute value $< i$. Therefore also the relation stated above must hold for $x + j$, $F(x+j) - Fx < \frac{1}{N}$. But $F(x+j) = M$. Therefore $M - Fx$ in its absolute value must be $< \frac{1}{N}$. Now since M and Fx are a pair of measurable numbers completely independent of i, while the fraction $\frac{1}{N}$ can decrease indefinitely by diminishing i, then it is clear (*RZ* 7, §92) that it must be $M = Fx$. Therefore the function Fx has one and the same value for every value of x lying within a and b.

§ 54

Corollary. Therefore, conversely, if Fx denotes a number which is continuously *variable* within the limits a and b and dependent on x then it can indeed be the case that, within the limits a and b, Fx returns infinitely often to one and the same value M. Yet it must be possible, in every case, to give values of x lying within a and b which are so close to one another that no third value lies between them which makes $Fx = M$. In other words, among the values of x which make $Fx = M$, for every one [of them] there must be the *next* one to it, i.e. another lying so close to it that no third one can lie yet nearer.

§ 55

Theorem. If the infinitely many values which a single-valued function takes as its variable x approaches a certain measurable number c as close as we please, become greater, when taken in absolute value, than every measurable number, then this function is certainly not continuous for the value $x = c$.

Proof. If it were, then the value of Fc would have to be equal to some measurable number C and the difference $F(c \pm \omega) - Fc$ would have to decrease indefinitely merely by diminishing ω. However, by virtue of the assumption, $(Fc \pm \omega)$[k] becomes greater in its absolute value than every measurable number merely by

[k] This should be $F(c \pm \omega)$.

diminishing ω, and therefore $F(c \pm \omega) - Fc = F(c \pm \omega) - C$ certainly cannot decrease indefinitely.

Example. The function $\frac{a}{c-x}$ becomes greater than any given number if x approaches the value c indefinitely. Therefore this function is also not continuous for the value $x = c$.

§ 56

Theorem. If the infinitely many values which a function Fx takes, while its variable x takes all the values occurring from $x = a$ to $x = b$ inclusive, have the property that, for every measurable number, one of them can be found which exceeds this number, then this function is certainly not continuous for all values from $x = a$ to $x = b$ inclusive.

Proof. If we choose an infinite series of measurable numbers of which each successive one is greater than the previous ones and which reaches every arbitrary quantity, e.g. the numbers 1, 2, 3, 4, ... *in inf*., then there must also be a series of values of x, x^1, x^2, x^3, ... *in inf*. which all lie within a and b and have the property that the corresponding values of the function, Fx^1, Fx^2, Fx^3, Fx^4, ..., are each greater, taken in their absolute value, than the corresponding term in the first series, namely $Fx^1 > 1$, $Fx^2 > 2$, $Fx^3 > 3$, $Fx^4 > 4$, etc. Now we know from § that the infinitely many numbers x^1, x^2, x^3, x^4, ... either all of them, or a part of them which is so large that its multitude is itself infinite, can be enclosed in a pair of limits, p and q, which can approach one another as close as we please, and it follows from § that one of these limits could be represented by c, the other by $c \pm \omega$, if we denote by c a certain *constant* number lying not outside a and b, but by ω a number which can decrease indefinitely. But to an infinite multitude of numbers x^1, x^2, x^3, x^4, ... there also corresponds an infinite multitude of numbers Fx^1, Fx^2, Fx^3, Fx^4, ... and this latter [multitude] must certainly contain numbers which exceed any given measurable number. For because the first of these numbers or $Fx^1 > 1$, the second or $Fx^2 > 2$, the third or $Fx^3 > 3$, etc., so among n of these numbers, even if the smallest are chosen, is contained at least one which is $> n$. Now it follows from the previous theorem that our function certainly cannot be continuous for the value $x = c$, even if [it is] for every other [value], because within c and $c \pm \omega$ values of Fx appear which exceed any given measurable number. Now since c is either $= a$ or $= b$ or lies within a and b, what our theorem states is hereby proved.

§ 57

Corollary. Therefore if, on the contrary, a certain function Fx is continuous for all values from $x = a$ to $x = b$ inclusive, then there must be some measurable and constant number N, which is greater than every single value which this function takes from $x = a$ to $x = b$ inclusive, if we consider it in absolute value.

§ 58

Theorem. If a function Fx is continuous from $x = a$ to $x = b$ inclusive, and there is a constant measurable number C such that the infinitely many values of Fx, which appear if we successively give to its variable x an infinite number of values lying within a and b, approach the number C indefinitely, then there is also, among the values from $x = a$ to $x = b$ inclusive, at least one $= c$ for which $Fc = C$.

Proof. If we denote the infinitely many values of x which have the property that the values of Fx belonging to them approach the number C indefinitely, by x^{1}, x^{2}, x^{3}, x^{4}, ... *in inf.*, then it follows from § that even if not all these values, but an infinite multitude of them, can be enclosed between a pair of limits of the form c and $c \pm \omega$, if we denote by c a number lying not outside a and b. But hence it follows by §50, because our function is to be continuous for the value $x = c$, that it must be that $Fc = C$.

§ 59

Corollary. If we are allowed to leave out of account the two values Fa and Fb then it is not at all certain that for every constant number C, that the values of Fx approach indefinitely, there must be one among these values which is completely equal to it. Because we can only assume about the number c, for which we have to prove that $Fc = C$, that it may not lie outside a and b, but not at all that it must lie between a and b, it could sometimes happen that only the value which Fx becomes for $x = a$ or $x = b$ is equal to the given number C. Thus, for example, there is no value of the variable x lying within 0 and 4 for which the function $7 + 5x$ takes the value $C = 27$, although there is an infinite multitude of such values of x for which Fx approaches this value indefinitely. This is because that single value of x for which $7 + 5x$ becomes exactly $= 27$ is precisely one of those two limits, namely $x = 4$.

§ 60

Theorem. If a function Fx is continuous from $x = a$ to $x = b$ inclusive, then among all the values which it takes, if we imagine that x successively takes all the values from a to b inclusive, there is always a *greatest* in the sense that no other is greater than it, and there is also a *smallest* in the sense that no other is smaller than it.

Proof. The proposition holds even in the case when our function takes one and the same value from a to b inclusive, for then this constant value itself is its *greatest* as well as its *smallest* in the sense just defined. But if Fx takes different (i.e. unequal) values within the given limits a and b, then of course one and the same value cannot be called its greatest as well as its smallest. However, it will be enough if we only show that our function must have a *greatest* value, for everyone will see that the existence of a *smallest* value can be proved in a similar way. Now we already know from §57 that there must be a measurable and constant number

N which can be given and of which it can be said that all numbers $> N$ are also greater than any of the values which our function takes from $x = a$ to $x = b$ inclusive. Nevertheless it is clear that it cannot be asserted of every number, however small, that all numbers which are greater than it are also greater than all values of our function. For if we take a number D which is $< Fa$, then there is no doubt that of this it cannot be said that all numbers which are greater than D are also greater than all values of Fx, for Fa is a number which is $> D$ and yet not greater than every value of Fx, namely not [greater] than the value Fa itself. Therefore, by *RZ* 7, §109 there must be a constant measurable number M which is the *smallest* among those of which it can be said that all numbers which are greater than M must also be greater than all values which our function takes from $x = a$ to $x = b$ inclusive. Now I claim of this number M, that it is a value which our function itself takes for some value of its variable lying not outside a and b, and indeed a value such that it takes no greater value. If, among all the values Fx takes from $x = a$ to $x = b$ inclusive, there was not a single one which $= M$, then by the previous proposition there could be no values of Fx which approach the number M indefinitely. Accordingly there must be a number μ small enough to be able to claim that the difference $M - Fx$ remains $>\mu$ for all values from $x = a$ to $x = b$ inclusive. But from this it would follow that the number M is not the smallest of which it can be asserted that all greater numbers also have the property of being greater than all values of Fx. For $M - \mu$ would also be a number, and indeed a smaller one, of which the same could be asserted. Therefore the only remaining possibility is to grant that among the various values which our function takes from $x = a$ to $x = b$ inclusive there is at least one which $= M$. But there can be no greater value than this, because otherwise it would not be true that all numbers which are greater than M are also greater than all the values of our function from $x = a$ to $x = b$. It is therefore decided that among these values there occurs at least one which has no value greater than it.

§ 61

Corollary 1. The values Fa and Fb must necessarily be taken into account, for it can in no way be claimed that among all the values which a continuous function takes from $x = a$ to $x = b$ *exclusively* there must always occur a greatest and a smallest. For example, among all the values which the function $5x$ takes from $x = 1$ to $x = 10$ *exclusively* there is neither a greatest nor a smallest. No greatest, for to every value of x lying within 1 and 10, e.g. 9, for which this function takes the value 45, there is another one (one lying nearer to 10), e.g. $9\frac{1}{2}$, for which $5x$ has a greater value, namely $47\frac{1}{2}$. No smallest, for to every value of x lying within 1 and 10, e.g. 2, for which this function takes the value 10, there is another one (one lying nearer to 1), e.g. $1\frac{1}{2}$, for which its value $= 7\frac{1}{2} < 10$.

§ 62

Corollary 2. Therefore if, on the contrary, it appears that among all the values which a certain function Fx takes while its variable x is given all conceivable values from $x = a$ to $x = b$ with these included, there is either no greatest, or no smallest, in the sense that for every one there can be found a greater or a smaller, then there must be at least one among the values from $x = a$ to $x = b$ inclusive for which this function breaks the law of continuity. An example is the function $\frac{a}{c-x}$ for the value $x = c$.

§ 63

Theorem. There are functions Fx which vary according to such a law that for every measurable number M which can be specified as lying between two different values of Fx, Fa and Fb, which belong to the values of the variable $x = a$ and $x = b$, we are also in the position of finding a measurable number m lying between the numbers a and b for which the corresponding value of Fx, i.e. Fm, is equal to the given number M.

Proof. Such a function, for example, is cx. For if we first put a for x, then put b, we obtain the values $Fa = ca$ and $Fb = cb$. Now if M denotes a number lying between the values Fa and Fb, i.e. ca and cb, then it must be that $ca \lesseqgtr M \lesseqgtr cb$. Therefore also if we divide both sides by c, $a \lesseqgtr \frac{M}{c} \lesseqgtr b$, when c is positive, and $a \gtreqless \frac{M}{c} \gtreqless b$ when c is negative. But in each case $\frac{M}{c}$ is a number lying between a and b, and if we put this in the place of x in our function then we obtain $c \cdot \frac{M}{c} = M$. Therefore there is certainly a value of the variable, lying between a and b, for which the function cx takes the value lying between ca and cb.

§ 64

Definition. If a function Fx has the property just described, i.e. if to every number M which lies between two of its values Fa and Fb there is a value m for its variable x lying between the numbers a and b for which $Fm = M$, then to say it briefly, *the function Fx changes from no value Fa into another Fb without first having to take every M lying between them.* We are therefore indicating two things by this expression; *firstly*, that every number lying between the numbers Fa and Fb, i.e. every number which satisfies the relationships $M \lesseqgtr Fa$ and $M \gtreqless Fb$, is also one of the values which Fx takes some time, or perhaps even several times; and *secondly*, that this happens, at least once, if not more often, with a value $x = m$ which lies between the values a and b, i.e. which satisfies the relationships $m \lesseqgtr a$ and $m \gtreqless b$.

§ 65

Theorem. A function which is continuous either in general, or within given limits, does not change from one of its values into another different (i.e. unequal) value without first having to take all values lying between them at least once.

Proof. Let a and b be a pair of values of its variable lying within the limits for which the continuity of Fx applies, and for which the values Fa and Fb are unequal. Now let us denote the smaller of these by Fa, then if the number M represents an arbitrary number lying between Fa and Fb, $M - Fa$ is *positive*, and on account of the continuity of the function for $x = a$ there is a positive, as well as a negative number i, small enough that also $M - F(a + i)$ turns out positive. For this it is only necessary to take i, in its absolute value, so small that $F(a + i) - Fa$ becomes and remains $< (M - Fa)$, however much i is further decreased. For this value of i and for all smaller ones, it holds of the expression $M - F(a + i)$ that it is always positive. Now if we take i with the same sign which the difference $b - a$ has, and increase it gradually in absolute value until it reaches the absolute value of $b - a$, then $a + i = a + (b - a) = b$, and $M - F(a + i) = M - Fb$ is certainly no longer positive, but is already *negative*, because otherwise it could not be said that M lies between Fa and Fb. The property of making the expression $M - F(a + i)$ *positive* therefore belongs to the variable i for all values which are smaller than a certain one, but not for all values in general. Therefore by *RZ* 7, §109 there is, without doubt, a measurable number m which in its absolute value is the *greatest* of those of which it can be said that all values of i smaller in absolute value have this property. Let us now investigate the nature of this m, and which value the expression $M - F(a + i)$ takes if we give i the value m itself.

1. First of all it is clear from the way we determined m that it has the same sign as i, i.e. as $b - a$, but I add the remark that this m in its absolute value is $< b - a$. Because, as we have already remarked, $M - Fb$ is *negative*, so by the continuity of the function Fx for the value $x = b$ there must be a positive as well as a negative number ω small enough so that also $M - F(b - \omega)$ is negative. Therefore because it is permissible, let us stipulate that ω has the same sign as $b - a$, and therefore also the same sign as i and m. Now if m in its absolute value was not $< b - a$, then it would have to be either $= b - a$ or $> b - a$.

(a) In the first case, it would be that $b = a + m$, and so $M - F(b - \omega) = M - F(a + m - \omega)$ would be negative, i.e. there would be a value for $i = m - \omega$ which in its absolute value is $< m$, and yet makes the expression $M - F(a + i)$ negative. It would therefore not be true that *all* i which in their absolute value are $< m$, make $M - F(a + i)$ positive.
(b) In the second case, if m in its absolute value were $> b-a$, then $b-a$ itself would represent a value for i smaller than m which makes the expression $M - F(a+i) = M - Fb$ negative. This directly contradicts the assumption that all i which are smaller than m should make $M - F(a + i)$ positive. The only remaining possibility

is to grant that m, in its absolute value, is $< b - a$. But then by § $a + m$ is certainly a value which lies within the limits a and b.

2. The value which the expression $M - F(a+i)$ takes for $i = m$, i.e. $M - F(a+m)$, can neither be positive nor negative, for [the following reasons].

(a) If $M - F(a + m)$ were positive then because $a + m$ lies within a and b and so the function Fx is continuous for the value $x = a + m$, there would be an ω small enough that also $M - F(a + m \pm \omega)$ would be positive. But one of the values $m + \omega$ or $m - \omega$ is certainly in absolute value greater than m, therefore m would not be the greatest value of which it can be said that all i which are $<m$ make $M - F(a + i)$ positive.
(b) Just as little can $M - F(a + m)$ be negative, because otherwise there would be an ω small enough that also $M - F(a + m \pm \omega)$ would be negative. And since one of the values $m+\omega$ or $m-\omega$ is smaller than m, it would not be true that all i which are $<m$ make the expression $M - F(a + i)$ positive. Since also M is measurable and it follows from the continuity of the function Fx that the value $F(a + m)$ is likewise a measurable value then the difference $M - F(a+m)$ must also represent a measurable number. But now since it is neither positive nor negative there only remains that $M - F(a + m) = 0$ or $M = F(a + m)$. It is thus proved that there is a value of x, and indeed one lying within a and b, namely $a + m$, for which the function $Fx = F(a + m)$ becomes the required value M.

§ 66

Corollary. In this theorem it is only claimed, and proved, that for every value M lying between Fa and Fb there is *at least one* value of x lying between a and b which makes $Fx = M$ as long as the function Fx is continuous for all values of x from a to b inclusive. But we are in no way saying that there is only a single such value of x for which Fx becomes M. In fact there are cases where [the existence of] more than one such value can be proved. For example, the function $x^3 - 9x^2 + 26x + 1$ takes the the value 19 for $x = 1$ and the value 31 for $x = 5$, but the value 25 lying between 19 and 31 is taken for $x = 2$, as well as for $x = 3$ and even for $x = 4$. And not only several, but even *infinitely many* different values of the variable, which all lie within the limits a and b, can produce one and the same value M lying between Fa and Fb without Fx having to cease being a continuous function. If, for example, we stipulate that the function Fx for all $x < 5$ has the value $Fx = 4x$, but for all $x \geqq 5$ and $\leqq 10$ has the constant value $Fx = 20$, and finally for all x which are >10 has the value $7x - 50$, then one will easily be convinced that the way the function Fx varies is according to the law of continuity defined in §38, in that the difference $F(x + \Delta x) - Fx$ decreases indefinitely for every value of x if Δx decreases indefinitely. That is, for all $x < 5$, we have $F(x + \Delta x) - Fx = 4(x + \Delta x) - 4x = 4\Delta x$. For $x = 5$ and a negative Δx, we still have $F(x - \Delta x) - Fx = 4(5 - \Delta x) - 4.5 = -4\Delta x$. On the other hand, for a positive Δx, namely for all values of x which lie within 5 and 10, $F(x+\Delta x) - Fx$ is constant $= 20 - 20 = 0$, which does not contradict the requirement that ΔFx should decrease indefinitely because this only means that ΔFx has to become and remain $< \frac{1}{N}$, which in fact happens because, of course, $0 < \frac{1}{N}$. Also for the value $x = 10$ and a negative Δx we still have $F(x - \Delta x) - Fx = 20 - 20 = 0$. But for a positive Δx there results, $F(x+\Delta x) - Fx = 7(10+\Delta x) - 50 - 20 = 7.\Delta x$. The same happens for every x which is >10, $F(x+\Delta x) - Fx = 7(x+\Delta x) - 50 - 7x + 50 = 7\Delta x$. Therefore our function is without doubt continuous for all values of its variable. But for the value $x = 1$ it has the value 4, for $x = 11$ the value 27, and if we ask for which value of x lying within 1 and 11 it becomes the value 20 lying between 4 and 27, then the answer is that first of all it may be proved to take this value for $x = 5$, but then for an infinite multitude of values, namely for all from 5 to 10 inclusive remains [20]; only for values of x which are > 10 would other values be obtained.

§ 67

Theorem. Suppose a function Fy is continuous for the specific value of its variable which we just designate by y, either only in respect of a positive, or only in respect of a negative increase in y, or in both respects at once. We now consider this variable y itself as a function of another freely variable number x, where it is the case that this function $fx = y$ is likewise continuous for the specific value of x which makes $fx = y$, and this indeed in respect of such an increase in x that has the same sign as that belonging to the Δy in respect of which Fy is also continuous. Then I claim that the function of x, $F(fx)$ which arises if we put fx in place of y in Fy, is likewise continuous, and this in respect of the same positive or negative nature of the increase already mentioned.

Proof. Because the function Fy is to be continuous for that value of its variable which we denote just by y, then at least for a certain (positive or negative) sign of Δy the difference $F(y + \Delta y) - Fy$ decreases indefinitely as long as Δy decreases indefinitely. But if we put $y = fx$ then $y+\Delta y = f(x+\Delta x)$, and $\Delta y = f(x+\Delta x) - fx$. Now if fx is also continuous for that value of x which gives rise to the equation $fx = y$, and for that increase Δx (positive or negative), which produces a Δy with that sign as required for the continuity of Δy, then merely by reducing Δx, Δy can be reduced indefinitely. Therefore, if we put the value fx instead of y, and the value $f(x+\Delta x) - fx$ instead of Δy, then $F(y+\Delta y) - Fy = F(f(x+\Delta x)) - F(fx)$ can likewise decrease indefinitely, just as long as we let Δx decrease indefinitely.

§ 68

Note. The restriction made in this theorem concerning the positive or negative nature of the increase for which Fy is continuous is, indeed, not superfluous, although it is commonly omitted. For example, if the function $Fy = \sqrt{1-y}$ then it would only be continuous for the value $y = 1$ in respect of a negative increase.

But if we put $y = 4x^2 \pm \sqrt{x^2 - \frac{1}{4}}$ then this is a function which for that value of x which makes $fx = y = 1$, namely for $x = \frac{1}{2}$, is only continuous with respect to a positive Δx, which likewise gives rise only to a positive Δy. Therefore $F(fx)$ for the given value does in fact prove not to be continuous. On this assumption, $F(fx) = \sqrt{1 - 4x^2 \mp \sqrt{x^2 - \frac{1}{4}}}$ which becomes zero for $x = \frac{1}{2}$, but turns out imaginary for $x = \frac{1}{2} + \Delta x$, whether we take Δx positive or negative.

§ 69

Corollary. In the way that the continuity of the function $F(fx)$ can be deduced from the continuity of the function Fy for a certain value of y and from the continuity of fx for that value of x which makes $fx = y$ we cannot also immediately conclude, conversely, from the discontinuity of one of the functions Fy and fx for a certain value of its variable, the discontinuity of $F(fx)$ for this value. Whether in fact $F(fx)$ is discontinuous or not depends on particular circumstances. Thus $Fy = y + \frac{1}{1-y}$ is discontinuous for the value $y = 1$, and if we put $y = \frac{1}{x}$, then fx is a function which becomes discontinuous for $x = 0$, but now $F(fx)$ is discontinuous if $x = 0$ as well as for that value of x $(=1)$ which makes Fy discontinuous, i.e. $fx = y = 1$. On the other hand, if $Fy = \frac{1}{1-y}$ then for $y = fx = \frac{1}{x}$, $F(fx) = \frac{x}{x-1}$ is discontinuous only for the single value $x = 1$ which makes Fy discontinuous.

§ 70

Theorem. If a variable number depends on *several* free variables then it can be *continuous* with respect to one of them and with reference to a certain value of it, but be *discontinuous* with respect to others and in reference to another value.

Proof. For example, if $W = x + \frac{1}{10-y}$, this variable number is called continuous with respect to x, and indeed for every value of it (assuming that y is a measurable number different from 10), but in respect of y this number W is only called continuous for all those values which are $>$ or $<$ 10.

§ 71

Theorem. Simply from the fact that a certain function $F(x, y, z, \ldots)$ of *several* variables x, y, z, … is *continuous* or *discontinuous* with respect to one of them x, and with reference to certain values of it, as long as the other variables y, z, … are given the specific values y, z, …, it cannot be deduced that this function retains that continuity or discontinuity if a single one of the numbers y, z, … changes its specified value.

Proof. If the number $W = F(x, y, z, \ldots)$ is a function not only of x but also of $y, z, \ldots$, then every value of this number depends not only on the value of x but also on the values of y, z, …. And the law of this dependency can be such that the change in W, if x becomes $x \pm \Delta x$, for a certain value of $y, z, \ldots$ decreases indefinitely with Δx, where W is then called continuous, but with another value of $y, z, \ldots$ the change mentioned does not decrease indefinitely and then W is discontinuous, and of the opposite [kind]. Therefore simply from the fact that a function $W = F(x, y, z, \ldots)$ has the property that for certain values of $y, z, \ldots$ it follows the law of continuity in respect of the variable x, or on the contrary, does not obey it, we are not justified in concluding that it must also behave in just the same way with other values of $y, z, \ldots$. For example, the function $x^2 + \frac{1}{5+y}$ is continuous with respect to x if we take for y a value different from -5, but for the value $y = -5$ this function is discontinuous because $\frac{1}{5+y}$ becomes unmeasurable. The function $x^2 + \frac{1}{1-y} + \frac{1}{2-y} + \frac{1}{3-y} + \frac{1}{4-y} + \cdots$ *in inf*. is continuous for every value of x as long as y takes none of those values in the series of the natural numbers $1, 2, 3, 4, \ldots$. And so on.

§ 72

Theorem. Even if we know that the variable W depends only on the two variables x and y, and that it may be proved continuous for all values of x which lie within a and b while y maintains a definite value lying within c and d, and equally also proved continuous for all values of y which lie within c and d while x maintains a definite value lying within a and b, then we may nevertheless not conclude that W is also continuous even for a single value of x while y takes a value different from that previous one, and just as little, that W is continuous for a single value of y while x receives a value different from that previous one.

Proof. The law of dependency of W on the two variables x and y could even have the property that certainly for a specific value of y lying within c and d, the change in W with that of x, and just as much, for a specific value of x lying within a and b, the change in W with that of y decreases indefinitely, but that as soon as, in the first case, y and at the same time x, takes a different value, neither of these two things occurs any more. For what prevents us stipulating, for example, that for every value of x lying within a and b and while the specific value of $y = c + \gamma$ lying within c and d, W takes the measurable value $W = x^2$, but with every other value of y, W is not measurable, and furthermore, that for every value of y within c and d and while x has the specific value $x = a + \alpha$ lying within a and b, W takes the measurable value $W = y^2$, but is not measurable for every other value of y? Now in this case there obviously would not be a single value of y lying outside $c + \gamma$ and not a single value of x lying outside $a + \alpha$ of a kind that with these *two* values W would have a measurable value. Now if $W = F(x, y)$ is to be proved continuous for some value of x while y takes a value different from $c + \gamma$, then if we denote the value of x by x, but that of y by $c + \gamma + \Delta\gamma$, it would have to be that the difference $F(x + \Delta x, c + \gamma + \Delta\gamma) - F(x, c + \gamma + \Delta\gamma)$ decreases indefinitely with Δx. Therefore also the two expressions $F(x, c + \gamma + \Delta\gamma)$ and $F(x + \Delta x, c + \gamma + \Delta\gamma)$ must denote measurable numbers. However, by the assumption which we have just made at least one of these two expressions must represent a number which is not measurable, because either x, or $x + \Delta x$, has a value different from $a + \alpha$.

§ 73

Note. Experts know that even among that class of functions which follow a single identical law for all values of their variables there are very simple examples which could be quoted for the proof of our theorem. Thus the function $xy + (1 - x)\left[\sqrt{2-y} + \sqrt{y-2}\right]$ is continuous for every value of x if we give y the value 2, and for every value of y if we give x the value 1, because in both cases $(1 - x)\left[\sqrt{2-y} + \sqrt{y-2}\,\right]$ will be zero. But if we take for y any value different from 2, and for x any value different from 1, then in the first case no single value for x, in the second case no single value for y, can be found for which this function would be continuous.

§ 74

Definition. I say that a function of several variables $F(x, y, z, \ldots)$ has *continuity simultaneously for the value of x and for its positive or negative increase, also for the value y and its positive or negative increase, as also for the value z and its positive or negative increase, if the following circumstances occur*:

(a) if this function is continuous[1] for the value x and with the specified nature of its increase for every value of y, z, ... which do not lie outside the limits y and $y + \Delta y$, z and $z + \Delta z$, ... respectively, where Δy, Δz, ... have the specified signs;
(b) if this holds also of y, i.e. if the function is continuous for the value of y and the specified positive or negative nature of its increase, for every value of x, z, ... which does not lie outside the limits x and $x + \Delta x$, z and $z + \Delta z$, ... respectively, where Δx, Δz, ... have the specified signs;
(c) if this also holds of z, etc.

If the increases in x, y, z, ... can be positive as well as negative then I say simply that $F(x, y, z, \ldots)$ has *continuity* for the values x, y, z, ... And now it is easy to guess the meaning of the expressions, that $F(x, y, z, \ldots)$ has continuity for all values of x which lie within a and b, of y which lie within c and d, of z which lie within e and f, or for all values of x, y, z, ... with the exception of the former and the latter values, etc.

§ 75

Theorem. If a function of several variables $W = F(x, y, z, \ldots)$ is continuous with respect to all its variables for the values denoted by x, y, z, ... and in respect of their positive, as well as negative increase, then the difference

[1] The German *Stetigkeit äußert* here and in part (b) is literally 'expresses (or manifests) continuity'.

$\Delta W = F(x + \Delta x, y + \Delta y, z + \Delta z, \ldots) - F(x, y, z, \ldots)$ which occurs in W while *all those variables simultaneously* change by certain increases Δx, Δy, Δz, ... can become smaller, in its absolute value, than every given fraction $\frac{1}{N}$, and will also remain smaller, if the increases Δx, Δy, Δz, ... are taken small enough, and all the more so if they are reduced further.

Proof. According to §9 the increase referred to is

$$\begin{aligned}\Delta W &= F(x + \Delta x, y + \Delta y, z + \Delta z, \ldots) - F(x, y, z, \ldots)\\ &= \Delta_x F(x, y, z, \ldots) + \Delta_y F(x + \Delta x, y, z, \ldots)\\ &\qquad + \Delta_z F(x + \Delta x, y + \Delta y, z, \ldots) + \cdots .\end{aligned}$$

But because the given function $F(x, y, z, \ldots)$ is continuous for the value x for a positive as well as a negative increase, as long as y, z, ... retain their values, then $\Delta_x F(x, y, z, \ldots) = F(x + \Delta x, y, z, \ldots) - F(x, y, z, \ldots)$ decreases indefinitely with Δx. Furthermore, because the given function is also continuous for the value y as long as x, z, ... retain their values or change by a positive or negative increase which can be as small as we please, then also $\Delta_y F(x + \Delta x, y, z, \ldots) = F(x + \Delta x, y + \Delta y, z, \ldots) - F(x + \Delta x, y, z, \ldots)$ must decrease indefinitely with Δy. Likewise, because our function is to be continuous for the value z, as long as $x, y, \ldots$ retain their values, or change by a positive or negative increase which can be as small as we please, then also $\Delta_z F(x + \Delta x, y + \Delta y, z, \ldots) = F(x + \Delta x, y + \Delta y, z + \Delta z, \ldots) - F(x + \Delta x, y + \Delta y, z, \ldots)$ must be able to decrease indefinitely with Δz. Therefore there is no doubt that the algebraic sum of the three expressions just considered, i.e. ΔW itself, can decrease indefinitely if Δx, Δy, Δz, ... can also decrease indefinitely.

§ 76

Theorem. If a function of several variables $F(y, z, \ldots)$ is continuous for each of these variables, at least for the specific value of them which we denote by $y, z, \ldots,$ and with respect to a certain positive or negative increase, and now we consider these variables themselves as functions of a new freely variable number x, $y = fx$, $z = \phi x$ etc., for which it happens that these functions fx, ϕx, ... for one and the same value of x, namely that which produces the equations just mentioned, $fx = y$, $\phi x = z$, ..., and for one and the same positive or negative increase Δx, and for just those positive or negative increases Δy, Δz, ... are continuous then I claim that the function of x, $F(fx, \phi x, \ldots)$ which arises if we put fx, ϕx, ... in the place of $y, z, \ldots$ respectively is likewise continuous, and indeed with respect to the value of x already mentioned, which gives the equations $fx = y$, $\phi x = z$, ... and in respect of the positive or negative increase which produces the positive or negative increases Δy, Δz, ... mentioned.

Proof. If x becomes $x + \Delta x$, then $fx = y$ becomes $f(x + \Delta x) = y + \Delta y$, $\phi x = z$ becomes $\phi(x + \Delta x) = z + \Delta z$ etc. Therefore the change in $F(y, z, \ldots) = F(fx, \phi x, \ldots)$ if x becomes $x + \Delta x$, is

$$\begin{aligned} F(f(x + \Delta x), \phi(x + \Delta x), \ldots) - F(fx, \phi x, \ldots) \\ &= F(y + \Delta y, z + \Delta z, \ldots) - F(y, z, \ldots) \\ &= F(y + \Delta y, z, \ldots) - F(y, z, \ldots) \\ &\quad + F(y + \Delta y, z + \Delta z, \ldots) \\ &\quad - F(y + \Delta y, z, \ldots) + \cdots . \end{aligned}$$

But according to the assumption, for the stipulated value of x, and for a definite sign of Δx, fx, ϕx, ... are continuous functions, therefore the differences $f(x + \Delta x) - fx = \Delta y$, $\phi(x + \Delta x) - \phi x = \Delta z$, ... decrease indefinitely with Δx and since they also have that sign which is required if the function $F(y, z, \ldots)$ is to prove continuous, then it follows from the assumed continuity of this function for y that $F(y + \Delta y, z, \ldots) - F(y, z, \ldots)$ can decrease indefinitely with Δy or Δx. But in a similar way it follows from the assumption that the function $F(y, z, \ldots)$ is also continuous for the value z and for a value of y which lies not outside the limits y and $y + \Delta y$, therefore also for the value $y + \Delta y$ itself, that also the difference $F(y + \Delta y, z + \Delta z, \ldots) - F(y + \Delta y, z, \ldots)$ can decrease indefinitely if Δz decreases indefinitely, etc. Hence it clearly follows that the whole difference $F(f(x + \Delta x), \phi(x + \Delta x), \ldots) - F(fx, \phi x, \ldots)$ or $\Delta F(fx, \phi x, \ldots)$ can also decrease indefinitely with Δx, i.e. that $F(fx, \phi x, \ldots)$ is continuous for the specific value of x and the specific sign of Δx.

§ 77

Note. After what was said in §68 I need say no more about the necessity of the restriction introduced in this theorem concerning the positive or negative nature of the increases Δy, Δz, ...

§ 78

Corollary 1. The previous proposition only shows us that from the continuity of the function $F(y, z, \ldots)$ and $y = fx$, $z = \phi x$, we can infer, with due restrictions, the continuity of $F(fx, \phi x, \ldots)$. But it is already clear from §69 that we cannot also conclude, conversely, the discontinuity of $F(fx, \phi x, \ldots)$ from the discontinuity of the functions $F(y, z, \ldots)$, $y = fx$, $z = \phi x$, ... for a certain value of x.

§ 79

Corollary 2. Therefore every function which is an algebraic *sum*, likewise also a *product*, of a finite multitude of other functions of one and the same free variable x, is continuous for all values of this variable with at most the exception of those for which one or the other of these functions become discontinuous themselves. For a sum, and likewise also a product, of a finite multitude of freely variable numbers is continuous for all its values (§§ 27, 34).

§ 80

Corollary 3. A function which is a *quotient* of two other functions of one and the same free variable x, is continuous for all values of this variable with at most the exception of those values for which one or the other of these functions becomes discontinuous itself, or those values for which the divisor becomes the value zero.

§ 81

Theorem. Every *integral rational function* of a variable x, i.e. every function which is of the form $ax^n + bx^{n-1} + cx^{n-2} + \cdots + lx + m$, in which $a, b, c, \ldots, l, m$ denote measurable numbers, is continuous for every measurable value of its variable.

Proof. Every individual term of this algebraic sum, like ax^n, bx^{n-1}, $\ldots$, is continuous for every value of x (§42). Therefore also its whole sum is continuous.

§ 82

Theorem. Also every *fractional rational function* of a variable x, i.e. every function which can be represented by the form

$$\frac{ax^n + bx^{n-1} + cx^{n-2} + \cdots + lx + p}{\alpha x^m + \beta x^{m-1} + \gamma x^{m-2} + \cdots + \lambda x + \pi}$$

is continuous for every value of its variable x which does not make the denominator $\alpha x^m + \beta x^{m-1} + \gamma x^{m-2} + \cdots + \lambda x + \pi$ zero.

Proof. Follows from the earlier [results].

§ 83

Transition. Since we already know from §65 that all continuous functions have the property of not going from one of their values to another one without first having to pass through all the values lying in between at least once, this raises the question whether this property does not perhaps belong to continuous functions exclusively, in such a way that every function which has this property must also be continuous. The following provides the answer to this question.

§ 84

Theorem. Simply from the fact that a certain function does not go from one of its values to another without first having taken all values lying in between, once or several times, it by no means follows that its variation obeys the law of continuity.

Proof. The assumption that a certain number W only goes from each of its values to another so that it first takes all values lying in between, is consistent with the assumption that for every value of its variable x which is of the form $\frac{2m+1}{2^n}$, it $=ax$, but for every other value it has another arbitrary value. For if the value which W is to take for these other values of x is not stipulated then, because there is an *infinite multitude* of values of W between every two values which are determined by the previous assumption, i.e. between every two values of x which are of the form $\frac{2m+1}{2^n}$, then among these infinitely many values W can always take all those which lie between the two mentioned. But then W satisfies the condition of the theorem, but nevertheless as we know already from §46, it is continuous for no value of its variable.

§ 85

Theorem. There are functions for which it is the case that either in general or for all values of their variables lying within given limits, the *greater* the value of the latter, then also the *greater*, or on the contrary *the smaller*, is the value of the function.

Proof. For the function $Fx = ax$, as long as a is positive, to every greater x there belongs a greater value of W. Namely, if $x^2 > x^1$ then if a is positive, by *RZ* 7, §85, also $ax^2 > ax^1$, i.e. $Fx^2 > Fx^1$. But if on the contrary, a is negative, then to every greater x there always belongs a smaller Fx, for by § $ax^2 < ax^1$, i.e. $Fx^2 < Fx^1$.

§ 86

Definition. If a single-valued function of x, $=Fx$, has the property that it holds, either in general or for all values of its variable lying within given limits a and b, that to every greater value of this variable there also belongs an even greater value of [the function], then we say this function *increases*, or *rises*, either *always*, or *within the limits a and b*. But if, on the contrary, to every greater value of x there belongs a smaller value of the function, then we say that it *decreases*, or *falls*, either *always* or *within the limits a and b*.

§ 87

Corollary 1. Therefore if it is to be false that a function which has nothing but measurable values within a and b, always increases or decreases within these limits, then in the first case, there must be a pair of values of x lying within these limits, e.g. μ and ρ, which stand in the relationship $\mu < \rho$ while the corresponding values of the function stand in the relationship, $F\mu \geqq F\rho$. On the other hand, in the second case, the relationship $F\mu \leqq F\rho$ must occur. And whenever we find that $F\mu > F\rho$ then we can conclude that at least the function does not increase always, and if we find that $F\mu < F\rho$ that at least it does not *decrease* always, finally if $F\mu = F\rho$, that it neither always increases nor always decreases.

§ 88

Corollary 2. If a function Fx constantly [*beständig*] increases or constantly decreases within a and b, and x and $x + \Delta x$ are a pair of values lying within a and b, then in the first case $F(x + \Delta x) - Fx$ constantly has the same sign as Δx, but in the second case it constantly has the opposite sign from Δx.

§ 89

Theorem. If a function Fx always *increases* or *decreases* within a and b, then it takes each of its values within these limits only once.

Proof. For if μ and ρ were a pair of values of x lying within a and b for which $F\mu = F\rho$ then since μ and ρ are unequal to one another, and therefore one of them, we will take ρ, is the greater number, it would follow (from Corollary 1) that our function neither always increases nor always decreases within a and b.

§ 90

Theorem. If a function Fx *always increases* within a and b and $F\rho > F\mu$ then it must also be that $\rho > \mu$. And if, on the contrary, the function always decreases, and $F\mu > F\rho$, then it must be that $\mu < \rho$.

Proof. We need only prove the first part. Because $F\rho > F\mu$ then ρ and μ must be a pair of unequal numbers, and because both lie within a and b, they are therefore measurable, so it must be that either $\rho > \mu$ or $\mu > \rho$. However, the latter assumption $\mu > \rho$ would also give, because the function is to increase constantly within a and b, $F\mu > F\rho$. Since this is not so, it follows that $\rho > \mu$.

§ 91

Theorem. If a single-valued function Fx does not go from one of its values to another within a and b without first having taken all values lying in between, and we find three successive values of its variable x, ε, μ, ρ (i.e. three such values that, $\varepsilon < \mu < \rho$) for which the three corresponding measurable values of the function are neither in the relationship $F\varepsilon < F\mu < F\rho$, nor $F\varepsilon > F\mu > F\rho$, then there is no doubt that this function passes through at least one of its values within a and b twice.

Proof. If the three measurable numbers $F\varepsilon$, $F\mu$, $F\rho$ are neither in the relationship $F\varepsilon < F\mu < F\rho$, nor $F\varepsilon > F\mu > F\rho$, and we ignore the case where two, or even all three, are assumed *equal* (in which cases the truth of what our theorem states is self-evident) then it can only be that either $F\varepsilon < F\mu > F\rho$, or $F\varepsilon > F\mu < F\rho$.

1. If $F\varepsilon < F\mu > F\rho$ and it is not that $F\varepsilon = F\rho$, then there only remains either $F\varepsilon < F\rho$ or $F\varepsilon > F\rho$. In the first case $F\varepsilon < F\rho < F\mu$, therefore our function cannot go from the value $F\varepsilon$ to the value $F\mu$ without first having changed into the value $= F\rho$, i.e. there must be a measurable number λ between ε and μ for

which $F\lambda = F\rho$. Now since λ, lying between ε and μ, is certainly different from ρ, there are two numbers λ and ρ lying within a and b for which the function takes the same value. In the second case, if $F\varepsilon > F\rho$ we have $F\rho < F\varepsilon < F\mu$, therefore our function cannot go from the value $F\rho$ to the value $F\mu$ without first having taken a value $= F\varepsilon$, therefore there must be a value π between μ and ρ for which $F\pi = F\varepsilon$. Now since π, lying between μ and ρ, is certainly different from ε, there are two numbers ε and π lying within a and b for which our function takes the same value.

2. The proof is conducted in a similar way if the relationship $F\varepsilon > F\mu > F\rho$[m] is assumed.

§ 92

Corollary 1. Therefore if we know that a function Fx does not go from one of its values to another within a and b without first having taken all values lying in between, and it passes through each value only once, then the values of the function, corresponding to the three successive values ε, μ, ρ of its variable, are necessarily in one of the two relationships, either $F\varepsilon < F\mu < F\rho$ or $F\varepsilon > F\mu > F\rho$.

§ 93

Corollary 2. If $\varepsilon, \mu, \rho, \psi, \ldots$ denote more than three successive values of x which all lie within a and b, i.e. $\varepsilon < \mu < \rho < \psi < \cdots$ then either $F\varepsilon < F\mu < F\rho < F\psi < \cdots$ or $F\varepsilon > F\mu > F\rho > F\psi > \cdots$.

§ 94

Theorem. If a function does not go from one of its values to another within a and b without first having taken all values lying in between, and it passes through each one *only once*, and always remains measurable, then only one of the two [cases] occurs: either it always increases or always decreases within a and b.

Proof. Let μ and ρ be a pair of values of the variable x lying between a and b, and $\mu < \rho$. Now because the function Fx takes every value between a and b only once, $F\mu$ and $F\rho$ must be unequal, therefore, because they are both measurable, either $F\mu < F\rho$ or $F\mu > F\rho$. In the first case I claim the function increases, in the second, it decreases. It will be sufficient to prove only the first. This will be done by §91 if I show that for every two values of x lying between a and b, e.g. $\overset{1}{x}$ and $\overset{2}{x}$, which stand in the relationship $\overset{1}{x} < \overset{2}{x}$, also the relationship $F\overset{1}{x} < F\overset{2}{x}$ occurs. First of all let us distinguish the two cases: when one of the numbers $\overset{1}{x}, \overset{2}{x}$ is the same as one of μ, ρ, and when all four numbers are different

[m] This should be $F\varepsilon > F\mu < F\rho$.

from one another:

A. The assumption that one of the numbers $\overset{1}{x}$ or $\overset{2}{x}$ is the same as one of μ or ρ comprises the following four cases:

1. $\overset{1}{x} = \mu$. Then it must be that either $(\overset{1}{x} = \mu) < \rho < \overset{2}{x}$ or $(\overset{1}{x} = \mu) < \overset{2}{x} < \rho$. In the first case, it follows from the previous corollary that it must either be that $(F\overset{1}{x} = F\mu) < F\rho < F\overset{2}{x}$, or that $(F\overset{1}{x} = F\mu) > F\rho > F\overset{2}{x}$. Since the latter contradicts the condition that $F\mu < F\rho$, the former follows and therefore $F\overset{1}{x} < F\overset{2}{x}$. In the second case, it must be that either $(F\overset{1}{x} = F\mu) < F\overset{2}{x} < F\rho$, or $(F\overset{1}{x} = F\mu) > F\overset{2}{x} > F\rho$ and because the latter again contradicts the condition $F\mu < F\rho$ so the former holds, or $F\overset{1}{x} < F\overset{2}{x}$.
2. $\overset{1}{x} = \rho$. Then the relationship $\mu < (\rho = \overset{1}{x}) < \overset{2}{x}$ holds, from which it follows that only one of the two cases can occur, $F\mu < (F\rho = F\overset{1}{x}) < F\overset{2}{x}$ or $F\mu > (F\rho = F\overset{1}{x}) > F\overset{2}{x}$. But since the latter directly contradicts the condition $F\mu < F\rho$ then the former follows and therefore $F\overset{1}{x} < F\overset{2}{x}$.
3. $\overset{2}{x} = \mu$. Then the relationship $\overset{1}{x} < (\overset{2}{x} = \mu) < \rho$ holds, therefore it must be that either $F\overset{1}{x} < (F\overset{2}{x} = F\mu) < F\rho$ or $F\overset{1}{x} > (F\overset{2}{x} = F\mu) > F\rho$. The latter contradicts the condition that $F\mu < F\rho$. Therefore the former holds, and we have $F\overset{1}{x} < F\overset{2}{x}$.
4. $\overset{2}{x} = \rho$. Then it must be that either $\overset{1}{x} < \mu < (\rho = \overset{2}{x})$ or $\mu < \overset{1}{x} < (\rho = \overset{2}{x})$. From $\overset{1}{x} < \mu < (\rho = \overset{2}{x})$ follows one of the two cases: either $F\overset{1}{x} < F\mu < (F\rho = F\overset{2}{x})$ or $F\overset{1}{x} > F\mu > (F\rho = F\overset{2}{x})$. Now since the latter should not be so, because $F\mu < F\rho$, then the former follows or $F\overset{1}{x} < F\overset{2}{x}$. From $\mu < \overset{1}{x} < (\rho = \overset{2}{x})$ it follows that either $F\mu < F\overset{1}{x} < (F\rho = F\overset{2}{x})$ or $F\mu > F\overset{1}{x} > (F\rho = F\overset{2}{x})$. And because the latter contradicts the assumption $F\mu < F\rho$, then the former must occur, therefore it must be that $F\overset{1}{x} < F\overset{2}{x}$.

B. If all four numbers $\overset{1}{x}, \overset{2}{x}, \mu, \rho$ are different then only one of the three cases can occur: the numbers $\overset{1}{x}$ and $\overset{2}{x}$ both lie within μ and ρ, or both are outside, or only one is inside and the other is outside.

1. If $\overset{1}{x}$ and $\overset{2}{x}$ both lie within μ and ρ, then it can only be that $\mu < \overset{1}{x} < \overset{2}{x} < \rho$. And then it follows from §93 that either $F\mu < F\overset{1}{x} < F\overset{2}{x} < F\rho$ or $F\mu > F\overset{1}{x} > F\overset{2}{x} > F\rho$. Now since the latter contradicts the assumption $F\mu < F\rho$, so the former follows and therefore again $F\overset{1}{x} < F\overset{2}{x}$.
2. If $\overset{1}{x}$ and $\overset{2}{x}$ both lie outside μ and ρ then it can only be that $\overset{1}{x} < \mu < \rho < \overset{2}{x}$, or $\overset{1}{x} < \overset{2}{x} < \mu < \rho$, or $\mu < \rho < \overset{1}{x} < \overset{2}{x}$ and so it must be that either $F\overset{1}{x} < F\mu < F\rho < F\overset{2}{x}$, or $F\overset{1}{x} > F\mu > F\rho > F\overset{2}{x}$. Now since the latter is absurd, because $F\mu < F\rho$, then the former follows again, or $F\overset{1}{x} < F\overset{2}{x}$.
3. If one of the numbers $\overset{1}{x}$ and $\overset{2}{x}$ lies inside μ and ρ, and the other outside, then it can only be that either $\overset{1}{x} < \mu < \overset{2}{x} < \rho$, or $\mu < \overset{1}{x} < \rho < \overset{2}{x}$. The first, or the assumption $\overset{1}{x} < \mu < \overset{2}{x} < \rho$, can only give a choice between $F\overset{1}{x} < F\mu < F\overset{2}{x} < F\rho$,

or $F\overset{1}{x} > F\mu > F\overset{2}{x} > F\rho$, where on account of $F\mu < F\rho$, only the first can occur, therefore $F\overset{1}{x} < F\overset{2}{x}$. The second, or the assumption $\mu < \overset{1}{x} < \rho < \overset{2}{x}$, can only give the choice between $F\mu < F\overset{1}{x} < F\rho < F\overset{2}{x}$, or $F\mu > F\overset{1}{x} > F\rho > F\overset{2}{x}$, and because $F\mu < F\rho$ it follows that only the first, or that $F\overset{1}{x} < F\overset{2}{x}$, must hold. Thus in every case it may be proved that $F\overset{1}{x} < F\overset{2}{x}$, i.e. the function always increases.

§ 95

Theorem. If a function always increases or always decreases within a and b, and does not go from one of its values to another without first having passed through all values in between, then it is continuous within a and b.

Proof. Let x denote an arbitrary value of the variable lying between a and b, then the value of the function belonging to it must be $= Fx$, and if we take Δx so that $x + \Delta x$ also lies within a and b, $F(x + \Delta x)$, and hence also $F(x + \Delta x) - Fx$, represents a measurable number. If we put the latter $= D$, and denote by μ an absolute number which is < 1, then also μD, in absolute value, is $< D$ a nd hence $Fx + \mu D$ is a number which lies within Fx and $Fx + D = F(x + \Delta x)$, and is therefore one of the values which our function must take before it goes from the value Fx to the value $F(x + \Delta x)$. Now let us denote the value of the variable for which the function becomes the value $Fx + \mu D$, by $x + \pi \Delta x$, then $x + \pi \Delta x$ must also lie within x and $x + \Delta x$, therefore $\pi < 1$. But since μ can decrease indefinitely so also can $\mu.D$ decrease indefinitely and therefore become $< \frac{1}{N}$. Therefore to every fraction $\frac{1}{N}$, however small, there is a difference of $x = \pi \Delta x$, for which the difference $F(x + \pi \Delta x) - Fx < \frac{1}{N}$. And if we now take π ever smaller then, because our function always increases or decreases, the difference $F(x + \pi \Delta x) - Fx$ will also always become smaller, and so behaves exactly as the law of continuity requires.

§ 96

Theorem. If a, b, c denote three successive numbers (i.e. if $a < b < c$) then it is possible that a function always increases within a and b but always decreases within b and c, or conversely, it decreases within a and b but increases within b and c, and this happens even if the function is always continuous within a and c.

Proof. If b does not lie between a and c, but if either $a = c$ or b is smaller than the smaller of the two numbers a and c or is larger than the larger of them, then it would of course be absurd to require that one and the same function increases (or decreases) within a and b, but decreases (or increases) within b and c. For in the first case the limits a, b and c, b are not even different, but in the second case all numbers which the first (or second) encloses also lie within the second (or first). But if b lies between a and c then no value of x which lies within a and b also lies at the same time within b and c, and it is therefore possible that the function within

a and b obeys quite a different law than within b and c; this must be the case if we require that it increases there but decreases here, or conversely. The example of the function $ax - x^2$ should prove to us that there are functions which do this, even while they always follow the law of continuity from a to c. Of this function we know from §81 that as a rational, integral function it remains continuous for all values of x. Moreover if a is positive then I claim that it always increases from $x = 0$ to $x = \frac{a}{2}$ but always decreases from $x = \frac{a}{2}$ to $x = a$. That is, if x and $x + \Delta x$ are a pair of values lying within 0 and $\frac{a}{2}$ then the difference $F(x + \Delta x) - Fx = [a(x + \Delta x) - (x + \Delta x)^2] - [ax - x^2] = (a - 2x - \Delta x)\Delta x$ is obviously positive as long as Δx itself and the factor $a - 2x - \Delta x$ is positive, i.e. as long as x is positive and $<\frac{a}{2}$ and also $x + \Delta x < \frac{a}{2}$. Then also the sum $2x + \Delta x < a$. Therefore as long as $x + \Delta x$ is greater than x, and both lie within 0 and $\frac{a}{2}$, then $F(x + \Delta x)$ is greater than Fx and the function increases within 0 and $\frac{a}{2}$. But if $x > \frac{a}{2}$, and all the more if $x + \Delta x > \frac{a}{2}$, then the sum $2x + \Delta x > a$ therefore the difference $(a - 2x - \Delta x)\Delta x$ is negative, therefore $F(x + \Delta x) < Fx$, i.e. the function decreases. As an example of a function which within the same limits does exactly the opposite, decreases within 0 and $\frac{a}{2}$, but increases within $\frac{a}{2}$ and a, we have $x^2 - ax$ as long as a is positive. For here the difference $F(x + \Delta x) - Fx = (-a + 2x + \Delta x)\Delta x$ is therefore negative as long as x and $x + \Delta x$ are within 0 and $\frac{a}{2}$, and on the other hand, positive when $x > \frac{a}{2}$, and therefore also $x + \Delta x > \frac{a}{2}$. Whoever wants to illustrate more vividly for himself the rising and falling of these two functions may choose for a a definite number, e.g. 8, and calculate for different definite values of x the corresponding values of $ax - x^2$ and $x^2 - ax$, as, for instance, in the following table:

x	$8x - x^2$	$x^2 - 8x$
0	0	0
+1	+7	−7
+2	+12	−12
+3	+15	−15
+4	+16	−16
+5	+15	−15
+6	+12	−12
+7	+7	−7
+8	0	0

§ 97

Theorem. If a function Fx always increases within a and b and always decreases within b and c but is continuously variable around the value b, then the value Fb is *greater* than all values of the form $F(b \pm \omega)$ if we choose ω small enough and let it decrease indefinitely. But if, on the contrary, the function Fx decreases within a and b and increases within b and c, then the value Fb is *smaller* than all values of the form $F(b \pm \omega)$.

Proof. We need only prove the first part. If Fx always increases within a and b then the value of $F(b - \omega)$ always becomes greater while ω, starting from a certain value, decreases indefinitely, because $b - \omega$ increases. Furthermore, if Fx always decreases within b and c then also the value of $F(b + \omega)$ must always become larger if ω decreases because then also $b + \omega$ itself decreases. Now since, if our function is to be continuous around the value b, $F(b \pm \omega) - Fb$ must decrease indefinitely, it follows that the value Fb must be greater than every value which is of the form $F(b \pm \omega)$. For first of all, if Fb were to be *smaller* than $F(b - \omega)$ for some value of ω, then the difference $F(b - \omega) - Fb$ if ω now becomes ever smaller, cannot decrease indefinitely, because $F(b - \omega)$ increases. If Fb were to be *smaller* than $F(b + \omega)$ for some value of ω then just as little could the difference $F(b+\omega)-Fb$ decrease indefinitely if ω is made ever smaller, for $F(b+\omega)$ increases. Therefore there only remains that $Fb > F(b - \omega)$ as well as $Fb > F(b + \omega)$.

§ 98

Definition. If the value which a function Fx takes (regardless of whether it is continuous or discontinuous) for a certain value of its variable $x = b$ is greater or smaller than all those which can be represented by $F(b \pm \omega)$ if ω decreases indefinitely, starting from a certain value, then in the first case, we usually say that the value of Fb is a *greatest value*, or a *greatest*, a maximum; in the second case, on the other hand, that it is a *smallest value*, or a *smallest*, a minimum. There is a lack of one word which covers both values, assuming we do not want to use the word '*extreme*' (*Extrema*), which at least sounds better than the word suggested by *Busse*: '*eminences*' [*Eminenzien*]. In the special case that the relationship $F(b \pm \omega) \gtrless Fb$ holds only for a positive, or only for a negative ω, but the other relationship $F(b \mp \omega) \gtrless Fb$ does not hold, either because the function for $b + \omega$ has no value at all, or no measurable values, or because for a certain ω, and all smaller ones, the relationship $F(b \mp \omega) = Fb$ holds, I say that the value Fb is a half- or *one-sided* maximum or minimum, with respect to a positive or negative increase in x, according to whether the relationship $F(b + \omega) \gtrless Fb$ or $F(b - \omega) \lessgtr Fb$ holds. In contrast, I also sometimes call a maximum or minimum of the kind just described a *complete*, *whole* or *two-sided* maximum or minimum. But if I speak of simply a maximum or minimum, then I understand just the complete [kind].

§ 99

Theorem. If a certain value Fx,[n] of a function Fx is a *greatest* or a *smallest* in the sense just defined it does not also have to be, at the same time, what we called in §60 the *greatest* or *smallest* value of a function among all of its values occurring from $x = a$ to $x = b$ inclusive, even if it may be assumed that the number m is a value lying within a and b. Nevertheless, it is always possible to specify a pair of limits α and β lying as near one another [as desired] and enclosing the number m, in respect of which it holds that Fm is the greatest, or the smallest, of all values

[n] This should be Fm.

of the function which occur from $x = \alpha$ to $x = \beta$ inclusive. If, in the converse case, Fm is the *greatest* or *smallest* of all values which the function Fx takes from $x = a$ to $x = b$ inclusive, in the sense defined in §60, namely that no greater or no smaller value is there, then this same Fm is a *greatest* or a *smallest* in the sense of the previous §, for there would have to be a pair of numbers α and β between which m lies, with the property that Fx does not change its value at all within α and β.

Proof. 1. That a value Fm which is a greatest or a smallest in the sense just defined does not have to be the greatest or smallest of all [values] which the function Fx takes from $x = a$ to $x = b$ is self-evident. For the former it is only required that Fm be greater or smaller than every value of the form $F(m \pm \omega)$ if we let ω decrease indefinitely starting from a certain value. But this can occur even if certain other values of Fx, corresponding to a much greater or much smaller value of x, are in the first case larger, in the second case smaller, then Fm. For example, if we stipulate that the function Fx should take the value 10 for $x = a$ and the value 5 for $x = b$, but should be $=1$ for all other values of x, then $Fm = 5$ is without doubt a greatest in the sense of the previous § because $5 > 1$ but is in no way the *greatest value* which Fm takes from $x = a$ to $x = b$, for this is $= 10$.
2. But it can easily be seen that it is always possible to specify a pair of limits α and β lying as close to one another [as desired] and enclosing the number m in respect of which it holds that Fm is the greatest or smallest value of all those which the function Fx takes from $x = \alpha$ to $x = \beta$. For since there must be an ω small enough that for it, and for all smaller values, if Fm is a greatest the relationship $F(m - \omega) < Fm > F(m + \omega)$ holds, but if Fm is a smallest then the relationship $F(m - \omega) > Fm < F(m + \omega)$ holds, then those numbers $m - \omega$ and $m + \omega$ themselves form the required pair.
3. If Fm is the *greatest* or the *smallest* value of all those which the function Fx takes from $x = a$ to $x = b$ in the sense that at least no other of these values is greater or smaller, then of course it does not follow from this that Fm could also be called a *greatest* or a *smallest* in the sense of the previous §, because it could be that our function does not even change its value for all values lying within α and β to which also m belongs. This does not contradict the concept of a greatest or of a smallest value in the sense of §60, but it does [contradict] the [concept] of an extreme in the sense of the previous §.
4. But if this is not the case then there is no doubt that a value which deserves the name of the greatest, or the smallest, in the sense of §60 is also an extreme value. For providing we take ω small enough that $m \pm \omega$ still lies within a and b then if Fm is the greatest value of our function the relationship $F(m - \omega) < Fm > F(m + \omega)$ must necessarily hold, and if Fm is the smallest value the relationship $F(m - \omega) > Fm < F(m + \omega)$ must hold. For not to admit this would be claiming that there is some ω for which, in the first case either $F(m - \omega) > Fm$ or $F(m + \omega) < Fm$, but in the second case either $F(m - \omega) < Fm$ or $F(m + \omega) > Fm$. But in the first case Fm could obviously not be the greatest, and in the second case it obviously

could not be the smallest, of all values which Fx takes from $x = a$ to $x = b$. But if $F(m-\omega) < Fm > F(m+\omega)$ or $F(m-\omega) > Fm < F(m+\omega)$, and this relationship holds even if ω can be decreased indefinitely, then in the first case Fm is a maximum and in the second case a minimum in the sense of the previous §.

§ 100

Theorem. If a function is continuous from a to b inclusive and always increases or always decreases, then in the first case Fa is the *smallest value* and Fb is the *greatest value*, but in the second case Fa is the *greatest* and Fb is the *smallest* value of all those values which the function takes from $x = a$ to $x = b$ inclusive.

Proof. Let us again consider the first case. If we denote by x an arbitrary value of the variable lying between a and b then it is to be proved that $Fa < Fx < Fb$. But because $a+\omega$ and $b-\omega$ also denote a pair of values lying within a and b as long as we take $\omega < (b-a)$ then if we take ω small enough, so that x lies within $a+\omega$ and $b-\omega$, on account of the continual increasing of the function it must also be that $F(a+\omega) < Fx < F(b-\omega)$. Now since $F(a+\omega)$ and $F(b-\omega)$, with the indefinite decrease of ω, approach indefinitely the values Fa and Fb, it must also be that $Fa < Fx < Fb$.

§ 101

Theorem. Also, functions which for all values of their variable x lying within the given measurable limits a and b obey the law of continuity, can, merely by their variable taking the infinitely many values $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \overset{4}{x}, \ldots$, one after another, of which each successive value is greater (or smaller) than its immediate predecessor, but all lie within a and b, become alternately, now $\geqq$ than a certain constant number M, and then $\leqq$ a constant number m, different from the former. Likewise it can also alternately take a *positive* and then again a *negative* value, or (as we can say) *its sign changes infinitely many times*.

Proof. 1. Suppose we say that the number W, for all values of $x > 0$ and $\leqq \frac{1}{2}$ is to take the value $W = x$, but for all values of $x > \frac{1}{2}$ and $\leqq \frac{3}{4}$, the value $W = \frac{3-4x}{2}$, for all values of $x > \frac{3}{4}$ and $\leqq \frac{7}{8}$, the value $W = 4x - 3$, for all values of $x > \frac{7}{8}$ and $\leqq \frac{15}{16}$ the value $W = \frac{15-16x}{2}$. And generally for all values of $x > \frac{2^{2n-1}-1}{2^{2n-1}}$ and $\leqq \frac{2^{2n}-1}{2^{2n}}$ we suppose the value $W = \frac{2^{2n}-1-2^{2n}.x}{2}$, but for all values of $x > \frac{2^{2n}-1}{2^{2n}}$ and $\leqq \frac{2^{2n+1}-1}{2^{2n+1}}$ the value $W = 2^{2n}.x - 2^{2n} + 1$. Then it is easy to see that this function follows the law of continuity for all values of x lying within 0 and 1. For that it is continuous for those values of x which lie between 0 and $\frac{1}{2}$, then $\frac{1}{2}$ and $\frac{3}{4}$, $\frac{3}{4}$ and $\frac{7}{8}$, etc. follows because for these values W is everywhere of the form $a + bx$ (§81). But that W is also continuous for those values of x which appear in the infinite series,

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots, \frac{2^{2n-1}-1}{2^{2n-1}}, \frac{2^{2n}-1}{2^{2n}}, \ldots$$

is evident if we compare the values which any two successive definitions give for these cases while they approach one another indefinitely if we decrease indefinitely the difference in x. Thus for $x = \frac{2^{2n}-1}{2^{2n-1}}$, $W = \frac{1}{2}$ and for $x = \frac{2^{2n-1}-1}{2^{2n-1}} + \omega$, $W = \frac{1}{2} - 2^{2n-1}\omega$; for $x = \frac{2^{2n}-1}{2^{2n}}$, $W = 0$ and for $x = \frac{2^{2n}-1}{2^{2n}} + \omega$, $W = 2^{2n}\omega$. Hence we see at once that for this function those values of x which lie in the series

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots, \frac{2^{2n-1}-1}{2^{2n-1}}, \frac{2^{2n}-1}{2^{2n}}, \ldots$$

produce for W alternately the value $\frac{1}{2}$ and 0. Now since the values of x mentioned, of which the number is infinite, always remain within the limits 0 and 1, it may be seen that the number W in fact achieves what is stated in the first part of our theorem. That is, the numbers M and m are here $\frac{1}{2}$ and 0, with a and b being 0 and 1.

2. But hereby also the second part of the proposition is already proved. For if W represents a number which within the limits 0 and 1 alternately takes the values $\frac{1}{2}$ and 0 infinitely many times, then $W - \frac{1}{4}$ represents a number which within these same limits alternately becomes $\frac{1}{4}$ and $-\frac{1}{4}$, i.e. *positive* and then *negative*, infinitely many times.

§ 102

Note. It will be obvious to experts that also functions which are determinable by a single law, quite independent of the particular value of its variable, can have the property which this theorem describes. Thus, for example, $\sin\log(1-x)$ takes the values $+1$ and -1 infinitely many times within $x = 0$ and $x = 1$.

§ 103

Corollary. It is worth noting that in our theorem we assume the continuity of the relevant function Fx merely for all values of its variable lying *within* a and b. But we have left it undecided whether it also has to be continuous, or even whether it can be continuous, for these limit values a and b themselves.

§ 104

Theorem. If a function Fx follows the law of continuity for all values of its variable lying within a and b, and nevertheless it alternately becomes, infinitely many times, now $\overline{\gtrless}$ a certain constant number M, and then $\overline{\lessgtr}$ a constant number m (different from M), or its sign changes infinitely many times, then this can only happen in such a way that, in the first case, Fx takes every arbitrary value lying within M and m infinitely many times, and in the second case, the value *zero* infinitely many times. Indeed, [this happens] for values of x which have the property that each has a *next* [one], i.e. one lying so close that no third one can be alleged to lie even closer to it.

Proof. By the assumption of our theorem there is no doubt that a series of values of x can be specified of which each one is greater (or smaller) than its immediate predecessor, although they all lie within a and b, which have the further property that the values of the function Fx corresponding to them are, in the first case, alternately now $\overline{\gtrless} M$, then again $\overline{\lessgtr} m$, but in the second case, they are alternately positive or negative. But since, by §65, a continuous function cannot go from a value M to another value m without first having passed through, at least once, all values lying in between, there must be, if C denotes an arbitrary number lying between M and m, in the first case between every two terms of the series of values of x described before, at least one value of x for which Fx changes into the value C. Therefore if there is an infinite multitude of values of x for which Fx alternates between M and m then there must certainly be an infinite multitude of values of x lying in between them for which Fx changes into the value C. But that each of these values must have a next [one] to it follows from §54. The same also holds in the second case of our theorem, namely if Fx changes its sign infinitely many times, for between a value which is positive, and another which is negative, the number zero forms an intermediate value.

§ 105

Theorem. Suppose that when its variable x takes an infinite multitude of values, $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$ lying between the measurable limits a and b, a function Fx alternately becomes first $\overline{\gtrless}$ a certain constant number M, and then $\overline{\lessgtr}$ a certain constant number m different from M, and this alternation recurs infinitely many times with the infinite multitude of numbers $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$ Then I claim that this function certainly does *not* obey the law of continuity from a to b *inclusively*, but it is discontinuous either for $x = a$, or for $x = b$, *or* for some value of its variable lying within a and b, if not for several such values.

Proof. Because the infinitely many values of x, namely $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$ for which Fx alternately takes a value $\overline{\gtrless} M$ and then one $\overline{\lessgtr} m$, all lie within the measurable limits a and b, then it follows from § that it must be possible to enclose either all these values, or such a part of them that already contains an infinite multitude of them, within a pair of limits which can approach one another as close as we please. And these limits are either a and $a + \omega$, if we denote by ω an absolute number which can decrease indefinitely, or b and $b - \omega$, or finally a number c lying within a and b and, on the one hand, $c+\omega$ or on the other hand, $c-\omega$. Then I claim that our function is discontinuous, if the first case occurs, for the value $x = a$ and a positive increase, in the second case for the value $x = b$ and a negative increase, and finally in the third case for the value $x = c$. It will be enough if I only prove the first case. Now if the values $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$ for which Fx alternately becomes first $\overline{\gtrless} M$

and then $\leqq m$ accumulate so densely in the neighbourhood of the value $x = a$ that an infinite multitude of them can be enclosed by the two limits a and $a + \omega$, however small we may make ω, then it is obvious that the difference $F(a+\omega) - Fa$ with the indefinite decrease of ω does not behave as the law of continuity requires. That is, according to this [law] the difference mentioned should, in absolute value, become and remain $< \frac{1}{N}$, providing we take ω small enough, and all the more if we reduce it further. But here for every ω however small, there are two smaller [ones], for one of which $F(a + \omega) \geqq M$, and for the other $\leqq m$. Therefore if we denote the former by $\overset{1}{\omega}$, and the latter by $\overset{2}{\omega}$, then we have $F(a + \overset{1}{\omega}) \geqq M$ and $F(a + \overset{2}{\omega}) \leqq m$ whence $F(a + \overset{1}{\omega}) - F(a + \overset{2}{\omega}) \geqq (M - m)$. But according to the law of continuity it should also be that $F(a + \overset{1}{\omega}) - Fa < \frac{1}{N}$ and $F(a + \overset{2}{\omega}) - Fa < \frac{1}{N}$, therefore the difference $F(a+\overset{1}{\omega}) - F(a+\overset{2}{\omega})$ should always be $< \frac{2}{N}$ and so may not be $\geqq (M-m)$. Therefore Fx is certainly not continuous for the value $x = a$.

§ 106

Theorem. Also a function which follows the law of continuity for all values of its variable x from a inclusive to b inclusive can alternately rise and fall infinitely many times within these limits, yet it is required that neither its maxima nor its minima increase indefinitely in their absolute values, also that the difference between the greatest and smallest values which it alternately reaches while x continually increases or continually decreases, decreases indefinitely.

Proof. There is no difficulty in imagining a function continuous for each such value of its variable x for which a small enough i can be indicated to be able to claim that within x and $x+i$ or x and $x-i$ the function always increases or always decreases. However, if Fx alternately rises and falls infinitely many times while x either always increases or always decreases, but in both cases always remains within the limits a and b, then there must be an infinite multitude of values $\overset{1}{x}$, $\overset{2}{x}$, $\overset{3}{x}$, $\overset{4}{x}$, ... of the variable x, of which each successive one is greater (or smaller) than the previous one although they all lie within a and b, and for every two of these successive values it must be claimed that Fx increases or decreases within them, and for the following pair, the opposite, so that the values which Fx itself takes for the values $\overset{1}{x}$, $\overset{2}{x}$, $\overset{3}{x}$, $\overset{4}{x}$, ... are again maxima and minima. But from this follows, as already shown in the previous §, that either for the value $x = a$ or for $x = b$ or for one or more values of x lying within a and b, so many of the $\overset{1}{x}$, $\overset{2}{x}$, $\overset{3}{x}$, ... accumulate that every two limits, which can be formed from one such value and a different variable [one] which can approach each other as close [as we please], always enclose between them an infinite multitude of the $\overset{1}{x}$, $\overset{2}{x}$, $\overset{3}{x}$, ... If we therefore denote this one, or one of these several values of x, by c (where then c must either $= a$ or $= b$ or lie within a and b), then there is no i small enough to be able to claim that Fx always increases or always decreases within c and $c + i$ or within c and $c - i$. Now this fact certainly puts the continuity of the function Fx for the value $x = c$ in danger, and if either the maxima or the minima, in their absolute value, increase indefinitely then the function certainly could not be continuous because in contradiction to §60, among all the values of it from $x = a$ to $x = b$ inclusive, either no *greatest* or no *smallest* may be found in the sense defined there. Therefore it is necessary that those maxima, as well as those minima, do not exceed a certain measurable value. However, by the result of the previous § the continuity of the function would also be made impossible if those maxima and minima, which Fx alternately reaches infinitely many times, [are such that] the former all remain $\overline{\overline{>}}$ a certain constant number M, the latter all remain $\overline{\overline{<}}$ a certain constant number m. If, on the other hand, as is supposed in our present theorem, the difference between these maxima and minima decreases indefinitely, then our function can always satisfy for $x = c$ the condition which is required for continuity. For if, for every ω, there is a pair of smaller values $\overset{1}{\omega}$ and $\overset{2}{\omega}$ for which $F(c \pm \overset{1}{\omega})$ is a maximum and $F(c \pm \overset{2}{\omega})$ is a minimum, the former is therefore somewhat greater, the latter somewhat smaller, than the earlier value $F(c \pm \omega)$ then this does not prevent the difference $F(c \pm \omega) - Fc$ becoming and remaining $< \frac{1}{N}$ however much ω is further reduced. That is, if we have first taken ω only small enough that the difference between all maxima and minima which belong to smaller values of ω, remain $< \frac{1}{N}$ then if $Fc < F(c \pm \omega)$, the difference $F(c \pm \omega) - Fc$ will certainly not be able to turn out $> F(c \pm \overset{1}{\omega}) - Fc$, and if $Fc > F(c \pm \omega)$ the difference $F(c \pm \omega) - Fc$ will certainly not be able to turn out $> F(c \pm \overset{2}{\omega}) - Fc$ however much ω is further reduced. Therefore $F(c \pm \omega) - Fc$ always remains $< \frac{1}{N}$.

Example. Let the number W depend on x according to a law such that

for all values from $x = 0$ to $\frac{1}{2}$, $\quad W = x$

$$x = \frac{1}{2} \text{ to } \frac{3}{4}, \qquad W = 1 - x$$

$$x = \frac{3}{4} \text{ to } \frac{7}{8}, \qquad W = x - \frac{1}{2}$$

$$x = \frac{7}{8} \text{ to } \frac{15}{16}, \qquad W = \frac{5}{4} - x$$

$$x = \frac{15}{16} \text{ to } \frac{31}{32}, \qquad W = x - \frac{5}{8} \text{ etc.}$$

Generally, for all values of x of the form $\frac{2^{2n}-1}{2^{2n}}$ to $\frac{2^{2n+1}-1}{2^{2n+1}}$,

$$W = x - \frac{2^{2n} - 1}{3.2^{2n-1}},$$

and for all values of x of the form $\frac{2^{2n+1}-1}{2^{2n+1}}$ to $\frac{2^{2n+2}-1}{2^{2n+2}}$,

$$W = \frac{2^{2n+2} - 1}{3.2^{2n}} - x,$$

finally for $x = 1$, $\quad W = \frac{1}{3}$.

Then it is clear that this function rises and falls infinitely many times within the limits $x = 0$ and $x = 1$. That is, whenever the value of W is of the form $W = x - \frac{2^{2n}-1}{3.2^{2n-1}}$, then W increases if x increases. On the other hand, whenever the value of W is determined by the equation $W = \frac{2^{2n+2}-1}{3.2^{2n}} - x$, W decreases if x increases. The values of W which correspond to the values of the series,

$$0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots, \frac{2^{2n}-1}{2^{2n}}, \frac{2^{2n+1}-1}{2^{2n+1}}, \ldots$$

form the series,

$$\begin{array}{l} 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \frac{11}{32}, \frac{21}{64}, \ldots, \frac{2^{2n}-1}{3.2^{2n}}, \frac{2^{2n+1}+1}{3.2^{2n+1}}, \ldots \\ m, M, m, M, m, \; M, \; m, \ldots, \quad m, \quad M, \ldots, \end{array}$$

which alternate so that they represent first minima, then maxima, as they are indicated by the letters m and M put under them. But because the difference between every two successive values $= \frac{2^{2n+1}+1}{3.2^{2n+1}} - \frac{2^{2n}-1}{3.2^{2n}} = \frac{1}{2^{2n+1}}$ decreases indefinitely with the increase of n, this fact does not prevent our function being continuous from $x = 0$ to $x = 1$ inclusive. For $x = 0$, and for all values lying within 0 and 1, this continuity is self-evident. But that this function, if we give it the value $W = \frac{1}{3}$ for the value $x = 1$, is also continuous for $x = 1$ is then clear because the difference $F(1 - \omega) - \frac{1}{3}$ decreases indefinitely. Since for every value lying within 0 and 1, therefore also for the value $1 - \omega$, one of the two general equations holds, either $W = x - \frac{2^{2n}-1}{3.2^{2n-1}}$ or $W = \frac{2^{2n+2}-1}{3.2^{2n}} - x$, then $F(1 - \omega) - \frac{1}{3}$ is either of the form $1 - \omega - \frac{2^{2n}-1}{3.2^{2n-1}} - \frac{1}{3}$ or of the form $\frac{2^{2n+2}-1}{3.2^{2n}} - 1 + \omega - \frac{1}{3}$ if we increase n indefinitely. But under this condition both expressions decrease indefinitely, for the first is $= \frac{1}{3.2^{2n-1}} - \omega$, but the second $= -\frac{1}{3.2^{2n}} + \omega$.

§ 107

Note. If an example of [such] a function is required, which can be represented by a perfectly simple algebraic expression, then I cite

$$(1 - x)^2 \sin \log(1 - x)$$

which is continuous for all values from $x = 0$ to $x = 1$ inclusive, although there is an infinite multitude of greatest and smallest values within these limits, but which are not constant as in §101 but they themselves decrease indefinitely.

§ 108

Corollary 1. Therefore it does not contradict the continuity of a function that, while its variable x proceeds from a to b, it goes through an infinite series of values $F\overset{1}{x}, F\overset{2}{x}, F\overset{3}{x}, \ldots$ which alternate, first being greater then being smaller, and also that these values change their sign infinitely many times, i.e. at one time becoming positive, at another time negative. It is only required, in the first case, that the differences $F\overset{1}{x} - F\overset{2}{x}$, $F\overset{2}{x} - F\overset{3}{x}$, $F\overset{3}{x} - F\overset{4}{x}$, ..., in their absolute value, decrease indefinitely, and in the second case, that the values $F\overset{1}{x}$, $F\overset{2}{x}$, $F\overset{3}{x}$, ... themselves, taken absolutely, decrease indefinitely, because otherwise the difference between the greatest (positive) value and the smallest (negative) value of Fx could not decrease indefinitely as our theorem requires.

§ 109

Corollary 2. Whether that value which a continuous function, as the theorem describes it, takes for $x = c$, i.e. for the value of its variable in the neighbourhood of which there is an infinite multitude of its greatest and smallest values is itself a greatest, or smallest, or neither of these two, depends on particular circumstances. In the example that we appended to the theorem, $Fc = \frac{1}{3}$ and is therefore neither a greatest nor a smallest since every maximum $\frac{2^{2n+1}+1}{3.2^{2n+1}} > \frac{1}{3}$, but every minimum $\frac{2^{2n}-1}{3.2^{2n}} < \frac{1}{3}$. This happens because the maxima and minima in this function approach one another indefinitely in that the former become ever smaller and the latter become ever greater. But understandably this indefinite approach of the two can also come about if the successive maxima and minima both always become greater, or always become smaller, and then in the first case Fc will become a maximum, and in the second case a minimum. For example, if we imagine a function W which from $x = 0$ to $x = \frac{1}{2}$ always rises from the value 0 to the value $\frac{1}{2}$, but from $x = \frac{1}{2}$ to $x = \frac{3}{4}$ it always falls from the value $\frac{1}{2}$ to the value $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, from the value $x = \frac{3}{4}$ to $x = \frac{7}{8}$ it rises again from $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ to $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, from $x = \frac{7}{8}$ to $x = \frac{15}{16}$ it again falls from $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ to $\frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$, from $x = \frac{15}{16}$ to $x = \frac{31}{32}$ it rises again from $\frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$ to $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, from $x = \frac{31}{32}$ to $x = \frac{63}{64}$ it again falls from $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ to $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} - \frac{1}{16} = \frac{13}{16}$ and it proceeds indefinitely according to this gently decreasing law. Then it is clear that within the limits $x = 0$ and $x = 1$ this function has an infinite multitude of successive maxima and minima which both increase, and approach the value 1 indefinitely. If we therefore ascribe to the function W the value 1 for the value $x = 1$ it will be continuous from $x = 0$ to $x = 1$ inclusive, and for $x = 1$ it will again reach a maximum (in fact the greatest of all).

§ 110

Corollary 3. Therefore a continuous function (all the more certainly for one which does not even follow the law of continuity) can have an *extreme* value (a maximum or a minimum) for a certain value of its variable $x = c$ although there is no number ω however small, of which it could be said that this function continually increases (or decreases) within c and $c + \omega$ and continually decreases (or increases) within c and $c - \omega$.

§ 111

Theorem. It is possible that a function Fx follows the law of continuity from $x = a$ to $x = b$ inclusive although neither *at a nor at b nor at any value of x lying within a and b* can a small enough ω be found to be able to claim that within a and $a + \omega$, or within b and $b - \omega$, or within x and $x \pm \omega$, Fx does only one of two things: it always increases or it always decreases.

Proof.[0] 1. Firstof all, let us imagine a function of x, I shall call it $\overset{1}{y}$, which for $x = a$ takes the value A, for $x = b$ the value B which is different from, and perhaps greater than A, but within a and b it is *uniform*, i.e. it varies so that to equal increases in x correspond also equal (positive or negative) increases in $\overset{1}{y}$. Therefore for this function it must always be that $\overset{1}{y} = A + (x - a)\frac{B-A}{b-a}$.
2. Now let us imagine a second continuous function $\overset{2}{y}$ which follows a somewhat different procedure from the previous one, in that it does not always increase (or decrease) but rather displays a certain number [*Menge*], perhaps infinite, of alternations of rising and falling. For example, if we stipulate that $\overset{2}{y}$ gives the same value as $\overset{1}{y}$ for the two extreme values of x, a and b, and for the middle value $\frac{a+b}{2}$, that is the values A, B and $\frac{A+B}{2}$ respectively, but within a and $\frac{a+b}{2}$, and also within b and $\frac{a+b}{2}$ it first rises, then falls. In particular let this happen in such a way that the greatest value which $\overset{2}{y}$ reaches within a and $\frac{a+b}{2}$ occurs for $x = a + \frac{3}{8}(b - a)$ and $=A + \frac{5}{8}(B - A) = \frac{5}{8}(A + B)$, and is therefore not as large as B. Let us assume for greater simplicity that both the increasing as well as the decreasing always happen uniformly, so that for all values from $x = a$ to $x = a + \frac{3}{8}(b - a)$, $y^2 = A + \frac{5}{3}(b - a)\frac{B-A}{b-a}$ and for all values from $x = a + \frac{3}{8}(b - a)$ to $x = \frac{a+b}{2}$, $y^2 = \frac{A+B}{2} + \left(\frac{a+b}{2} - x\right)\frac{B-A}{b-a}$. From $x = \frac{a+b}{2}$ to $x = a + \frac{7}{8}(b - a)$, $\overset{2}{y}$ may rise again in the same way and be $=\frac{A+B}{2} + \frac{5}{3}\left(x - \frac{a+b}{2}\right)\frac{B-A}{b-a}$, but from $x = a + \frac{7}{8}(b - a)$ to b, $\overset{2}{y}$ should again fall and be $=B + (b - x)\frac{B-A}{b-a}$. So the greatest value which $\overset{2}{y}$ takes will be the one corresponding to $x = a + \frac{7}{8}(b - a)$, namely $B + \frac{B-A}{8}$. If on the contrary, $B < A$, then all that has been said so far would be valid as soon as we exchanged the words '*increasing*' and '*falling*', '*greatest*' and '*smallest value*'.
3. In a similar way to how we derived the function $\overset{2}{y}$ just from $\overset{1}{y}$ we can again derive a third function $\overset{3}{y}$ from $\overset{2}{y}$. That is, we undertake with each of the four pieces, in which the interval $b - a$ was divided by the previous procedure, what we did before with the whole interval, i.e. divide each of these pieces into four others within which $\overset{3}{y}$ one time rises, and the next time falls. As could be said of the function $\overset{2}{y}$, the greatest distance between the values of x within which it does only one of the two [things], either always rises or always falls is not

[0] The function defined in this proof, the 'Bolzano function', is introduced as an example of a function continuous on $[a, b]$ that is not monotonic for any subinterval. It is also an example of a continuous and nowhere differentiable function (cf. §135).

greater than $\frac{3}{8}(b-a)$, so it can be claimed of the function $\overset{3}{y}$ that this greatest distance is not greater than $\left(\frac{3}{8}\right)^2(b-a)$.

4. If we proceed also with $\overset{3}{y}$ just as with $\overset{2}{y}$ then we obtain a fourth continuous function $\overset{4}{y}$, for which the greatest distance within which it only rises or falls is $\left(\frac{3}{8}\right)^3(b-a)$. And so on.

5. Since these results could be continued indefinitely and the number $\left(\frac{3}{8}\right)^n(b-a)$ decreases indefinitely through the increase of n, we see that to every number ω, however small, a function $\overset{n}{y}$ could be found for which the greatest difference between the values of x within which the function always increases or decreases, is $<\omega$, although it follows the law of continuity without interruption. However, we should prove that not in different functions, but in one and the same function, such variety of alternations of rising and falling could occur that no ω, however small, can be given of which it can be said that this function within x and $x \pm \omega$ continually rises or falls.

6. I claim that we obtain a function with this property if we allow the function Fx to depend on the variable x according to a law such that every value of Fx, belonging to x, represents the limit to which the values of the function $\overset{n}{y}$, belonging to the same x, approach indefinitely with the indefinite increase of n, as long as both values are not completely equal to one another. I shall first have to show that such a function is in fact possible, then it will be easy to prove that it obeys the law of continuity and has the property which is stated in the theorem.

(a) The possibility of a function like the Fx I have just described, will be beyond doubt if we prove that to every value of x not lying outside a and b there is a definite measurable value for Fx. Now if $x=a$ then we have $Fa=A$, for $x=b$ we have $Fb=B$, for $x=\frac{a+b}{2}$, $F\left(\frac{a+b}{2}\right)=\frac{A+B}{2}$ and there is an infinite multitude of values of Fx which can be represented by a rational expression composed from A and B, because there are definite values in one of the functions $\overset{1}{y}, \overset{2}{y}, \overset{3}{y}, \ldots$ with which they coincide. Namely, these are all those values of Fx which belong to such values of x which arise sooner or later in the course of the division of the interval $(b-a)$. But that also to every arbitrary *other* value of x, which always lies within one of those limit values, there belongs a measurable value of Fx becomes clear as follows. If we suppose that $A<B$ then obviously among all values which the function $\overset{2}{y}$ takes the smallest $=A$ and the greatest $=\frac{9B-A}{8}$, but if on the contrary $A>B$, then A is the greatest and $\frac{9B-A}{8}$ the smallest. Therefore in both cases the difference between the greatest and the smallest value of the function $\overset{2}{y}$ is not greater than $\frac{9}{8}(B-A)$. Now since, in order to obtain the values of the function $\overset{3}{y}$ we do, with each of the four pieces into which the interval $(b-a)$ was divided, what was done with the whole interval $(b-a)$ in order to produce all of the values of $\overset{2}{y}$, it is clear that the greatest difference between those values of $\overset{3}{y}$ which belong to *one* of the four pieces mentioned is only $\frac{9}{8}\cdot\frac{5}{8}(B-A)$ because that interval which before was $(B-A)$ is here only $\frac{5}{8}(B-A)$. In a similar way the greatest difference occurring between the values which belong to one of the sixteen pieces in which the interval $(b-a)$ is divided for the formation of the function $\overset{3}{y}$ is $=\frac{9}{8}\left(\frac{5}{8}\right)^2(B-A)$. And thus, in general, the greatest difference between the values which belong to one of the (4^{n-2}) pieces in which the interval $(b-a)$ is divided for the formation of the function $\overset{n}{y}$, i.e. of which the corresponding x values have no greater difference than $\left(\frac{3}{8}\right)^{n-2}(b-a)$ from one another, is $=\frac{9}{8}.\left(\frac{5}{8}\right)^{n-2}(B-A)$. Therefore if we increase the number n, by r, by forming the functions $y^{n+1}, y^{n+2}, \ldots, y^{n+r}$ then the difference between those values of $\overset{n}{y}$ and $\overset{n+r}{y}$ of which the corresponding x do not differ from one another by more than $\left(\frac{3}{8}\right)^{n-2}(b-a)$ can in no case be greater than the sum which arises if we add the single differences just mentioned, that is,

$$\frac{9}{8}.\left(\frac{5}{8}\right)^{n-2}(B-A)+\frac{9}{8}.\left(\frac{5}{8}\right)^{n-1}(B-A)+\frac{9}{8}\left(\frac{5}{8}\right)^{n}(B-A)+\cdots \\ +\frac{9}{8}\left(\frac{5}{8}\right)^{n+r-2}(B-A).$$

Therefore this difference always remains smaller than the value of this series if we allow it to proceed indefinitely, i.e. smaller than $3\left(\frac{5}{8}\right)^{n-2}(B-A)$. Now if we understand by $\overset{n}{y}$ that value of the function $\overset{n}{y}$ which belongs to the same x as the definite value Fx, then in every case the difference between $\overset{n}{y}$ and Fx is $<3\left(\frac{5}{8}\right)^{n-2}(B-A)$. But since this difference decreases indefinitely with the indefinite increase in n it follows that the value of Fx can be determined as precisely as we please.

(b) It is also evident that this function follows the law of continuity, because $\overset{n}{y}$ follows the law of continuity, so the difference of the two values of $\overset{n}{y}$ which belong to x and $x+\Delta x$ decreases indefinitely with Δx. Therefore the difference of the values Fx and $F(x+\Delta x)$, which the former [the values of $\overset{n}{y}$] approach indefinitely closely, must also decrease indefinitely.

(c) But it is clear that in spite of its continuity, there is an infinite multitude of alternations of rising and falling for this function in such a way that for no value of x, providing it does not lie outside a and b, can there be a small enough ω given so that we can assert that Fx only always rises or only always falls within x and $x\pm\omega$, because to every value of ω however small, an n can be found so large that $\left(\frac{3}{8}\right)^{n-2}(b-a)<\omega$. With such a value of n, $\overset{n+1}{y}$ represents a function which, within every interval $=$ or $<\left(\frac{3}{8}\right)^{n-2}(b-a)$, rises at least twice and falls again twice. Therefore also $\overset{n+1}{y}$ rises and falls twice within x and $x\pm\omega$. Now since the highest and lowest values which the functions of the form $\overset{1}{y}, \overset{2}{y}, \overset{3}{y}, \ldots$ take with their alternating rising and falling, all appear in the function Fx, there is also in the latter, within x and $x\pm\omega$, four successive values, $\alpha, \beta, \gamma, \delta$ lying at as many successive values of x, for which $\beta>\alpha$ and $\gamma<\beta$, the function Fx neither always rises nor does it always fall within x and $x\pm\omega$.

§ 112

Corollary. Therefore for functions which are bound by no other condition than the law of continuity the condition specified in §106 that the maxima and minima alternate in succession is by no means obvious and does not necessarily hold. For if there is no ω, however small, such that it can be asserted that the function Fx always *rises* or always falls within x and $x \pm \omega$, then it may be seen that it is possible that between every two values x and $x \pm \omega$, of which the corresponding Fx and $F(x \pm \omega)$ are a pair of *extreme values*, a third value of x lies to which likewise an extreme value corresponds. But if this is so then we cannot speak *of a pair of extreme values following next to one another*, and therefore we cannot also claim that one of them must be a maximum and the other a minimum.

§ 113

Theorem. Suppose Fc is an *extreme value* of the function Fx and indeed an extreme value at least in respect of a *positive* (negative) increase of its variable x (§98). Furthermore, if $F(c + \gamma)$ is the *next* extreme value which there is in this function on the side of the *positive* (negative) increase, and finally suppose we know that this function obeys the law of continuity from $x = c$ to $x = c + \gamma$ inclusive. Then the extreme value which $F(c + \gamma)$ represents (if not on both sides) occurs at least with respect to a *negative* (positive) increase, and one of the two extremes Fc and $F(c + \gamma)$ must be found to be a maximum and the other a minimum.

Proof. 1. It will be sufficient if we only show that in the case when the value Fc is a maximum, and indeed with respect to a positive increase, the value $F(c + \gamma)$ must be a minimum, and indeed with respect to a negative increase. For the remaining cases the proof arises in a similar way. Now because the function Fx follows the law of continuity from $x = c$ to $x = c + \gamma$ inclusive, then by §60 there must be, among the whole collection of values which it takes from $x = c$ to $x = c + \gamma$, at least one which is the *greatest* and one which is the *smallest*, in the sense that the first has no *greater* next it, and the second has no *smaller* next it. Now if Fc were not such a greatest value there would have to be some value of x different from c, but not lying outside c and $c + \gamma$, which we shall represent by $c + \mu\gamma$, for which the corresponding value of Fx, or $F(c + \mu\gamma)$, is such a greatest value. By §99 this $F(c + \mu\gamma)$ would either have to be a maximum, or there would have to be a pair of numbers α and β enclosing the value $c + \mu\gamma$ such that our function for all values of x lying within α and β keeps the value $F(c + \mu\gamma)$ constant.
2. The first [hypothesis], or that $F(c + \mu\gamma)$ is a maximum, directly contradicts the condition of the theorem that there should be no maximum or minimum within c and $c + \gamma$.
3. Let us therefore only examine the second hypothesis. With this there must be some i small enough that for it, and for all smaller, the equation $F(c + \mu\gamma - i) = F(c + \mu\gamma)$ holds. In this equation i can be reduced unconditionally but not increased unconditionally. For if we take $i = \mu\gamma$ then $F(c + \mu\gamma - i) = Fc$ is

certainly not $= F(c + \mu\gamma)$. The property, of yielding an expression $F(c + \mu\gamma - i)$ which is $= F(c + \mu\gamma)$, therefore belongs to all values of i which are *smaller* than a certain value but *not to all values generally*. According to *RZ* 7, §109 there is therefore certainly a measurable number j which is the greatest of those of which it can be said that all values of i which are $< j$ have the property described here.

4. I now claim that this j must be $< \mu\gamma$, but the value $F(c + \mu\gamma - j)$ must be the last of those which are equal to the value $F(c + \mu\gamma)$. On the other hand, every value $F(c + \mu\gamma - j - \omega)$, which arises if we put instead of j, a number $j + \omega$, however small ω might be, must turn out $< F(c + \mu\gamma)$.

(α) It is clear that j must be $< \mu\gamma$ because we have already found before that Fc must be $< F(c + \mu\gamma)$.

(β) It follows from the fact that our function is continuous that $F(c + \mu\gamma - j)$ belongs to the values which equal the number $F(c + \mu\gamma)$. Because for all values of $i < j$ the equation $F(c + \mu\gamma - i) = F(c + \mu\gamma)$ holds, so it must also hold for $i = j$, i.e. $F(c + \mu\gamma - j)$ must also be $= F(c + \mu\gamma)$ (§53).

(γ) Finally, that $F(c+\mu\gamma-j-\omega)$ must be $< F(c+\mu\gamma)$ follows from the assumption that j is the greatest value of which it can be said that all smaller values satisfy the relationship $F(c + \mu\gamma - i) = F(c + \mu\gamma)$, and from the fact that all values of the function which are not equal to the greatest must be *smaller* than it.

5. From all this it clearly follows that the value $F(c + \mu\gamma - j)$ behaves altogether like a maximum (at least like a one-sided maximum). Now since it is assumed that there is no extreme value for our function within c and $c + \gamma$, we must also reject this second hypothesis. It therefore only remains to admit that the value Fc has no *greater* value above it among all the values of the function from $x = c$ to $x = c + \gamma$ inclusive.

6. But among these values there must also be one which has no *smaller* value below it, and if it is not acknowledged to me immediately that this is the value $F(c + \gamma)$, then this would have to be represented by an expression like $F(c + \mu\gamma)$ in which $\mu\gamma < \gamma$. But we know from §99 that every such *smallest* value of a function is either a minimum, or that there are two limits α and β enclosing the number $c + \mu\gamma$ which have the property that all values of Fx lying within them have the same value $F(c + \mu\gamma)$. Supposing that $F(c + \mu\gamma)$ is a minimum directly contradicts the condition that our function has no extreme value between c and $c + \gamma$. However, it is apparent on the second hypothesis, in a similar way as before, that it is necessary to admit some minimum, even only a one-sided minimum within c and $c + \gamma$. Since we cannot do this, there only remains to declare the value $F(c + \gamma)$ itself as a *smallest* value of the function. Then this value is certainly a minimum.

7. It is very easy to see that it must be so with respect to a negative increase in x (at least if it is not so in both respects). For the opposite assumption would have the consequence that there is an ω small enough to be able to assert that for this, and for all smaller values, $F(c + \gamma - \omega) = F(c + \gamma)$. And we already know how this

leads to the further consequence that there is a minimum of the form $F(c+\gamma-j)$, i.e. some extreme value between c and $c+\gamma$.

§ 114

Theorem. If a function Fx follows the law of continuity for all values of its variable x lying within a and b, but within these limits has several *extreme values* which may be one-sided or two-sided but at all events they are so distributed that among all the values of x which correspond to them each one has its *next*, then I claim that of every two extreme values which are next to one another, one must always be a maximum and the other must be a minimum. If it were that both extremes were merely *one-sided*, and that the function retained one and the same constant value for all values of its variable lying in between, then in this case, those two extreme values could be of the same kind, i.e. both maxima or both minima.

Proof. 1. If c is a value lying within a and b which corresponds to an extreme value Fc, in particular a *two-sided* maximum, then it follows from the previous proposition that the next extreme value bordering on Fc on the side of a positive, as well as a negative, increase in x (in case there is such) is a minimum. Conversely if Fc is a two-sided minimum it follows that the next extreme value bordering on it to both sides is a maximum.

2. But if Fc represents a merely *one-sided* maximum or minimum then by virtue of the concept of such an extreme value, there must be on one side an increase of $x=i$ which is small enough that for it and for all smaller [values], $F(c+i)=Fc$. But it can be proved, just as in the previous §, no. 3, that there must also be a greatest value of $i=j$ which satisfies the equation $F(c+j)=Fc$, so then $F(c+j+\omega)$ must already be $>$ or $<Fc$ from which it follows that $F(c+j)$ is again an extreme value, that is a one-sided extreme value.

3. But of which kind this extreme value is, is undetermined. With the same Fc it can be a maximum as well as a minimum, as we can easily be convinced by an example. For let us say that Fx for all values from $x=0$ to $x=c$ has the value ax, therefore for $x=c$ it becomes $=ac$, but then for all values from $x=c$ to $x=c+j$ it always remains $=ac$, then we have, if a and c are positive, the example of a one-sided maximum at the value ac. It does not prevent us arranging for the next one-sided extreme value, which occurs for $x=c+j$, that it may be a maximum, or also that it may be a minimum. The first happens if, for example, we establish that for all values of $x>c+j$, $Fx=ax-aj$, the second if we assume $Fx=2ac+aj-ax$.

§ 115

Theorem. Functions Fx can be imagined which, within certain limits a and b of their variable x, always increase or decrease, thereby also remaining always measurable, and yet they are not continuous even for a single value of x lying within a and b.

Proof. If Fx is not to be continuous for a single value of its variable x lying within a and b then (by §45) for each of these values one of the following three cases must occur:

(a) either the difference $F(x + \Delta x) - Fx$ must remain unmeasurable however small Δx may be taken, or
(b) this difference, although measurable, must be constantly $> \frac{1}{N}$, or finally
(c) it will indeed sometimes be $< \frac{1}{N}$ but it does not remain like this if we allow Δx to decrease indefinitely.

1. The first case certainly cannot occur with our function for any value of x lying within a and b. For such a [value], by the assumption in the theorem, the value of Fx is a measurable number, and if we take Δx small enough that $x + \Delta x$ lies within a and b, so also is the value of $F(x + \Delta x)$ and hence also the difference $F(x + \Delta x) - Fx$.
2. The third case can also never arise here, because it is assumed that Fx always *increases* or always *decreases* within a and b so the difference $F(x + \Delta x) - Fx$ taken in its absolute value, can always only decrease with the reduction in Δx. Thus, if for a certain value of Δx it has once become $< \frac{1}{N}$ then merely by our allowing Δx to decrease indefinitely it can never become $> \frac{1}{N}$ again.
3. It leads to no contradiction that the second case occurs for each value of x lying within a and b, as long as we do not assume that the number $\frac{1}{N}$, for which the relationship $F(x + \Delta x) - Fx > \frac{1}{N}$ should hold, remains one and the same for all values of x. Now it is certainly true that if Δx becomes an *infinitely small* number in the sense of the word defined in *RZ* 7, §22, the difference $F(x + \Delta x) - Fx$ even for this case must always remain, in its absolute value, greater than a given finite number. Therefore if we denote by x and $x + i$, a pair of arbitrary values of the free variable lying within a and b, and we imagine the difference i divided into an infinite multitude of infinitely small parts, perhaps as they would arise if we were to divide it by an arbitrary infinitely large number n, then we are indeed justified in claiming an infinite number of relationships hold like the following:

$$F\left(x + \frac{i}{n}\right) - Fx > \overset{1}{e}$$

$$F\left(x + \frac{2i}{n}\right) - F\left(x + \frac{i}{n}\right) > \overset{2}{e}$$

$$F\left(x + \frac{3i}{n}\right) - F\left(x + \frac{2i}{n}\right) > \overset{3}{e}$$

$$\cdots\cdots\cdots$$

$$F\left(x + \frac{ni}{n}\right) - F\left(x + \frac{n-1}{n}i\right) > \overset{n}{e}.$$

From these relationships it follows, by addition of the greater to the greater if we omit the equal parts which appear with opposite signs in the sum because they cancel each other out:

$$F(x+i) - Fx > \overset{1}{e} + \overset{2}{e} + \cdots + \overset{n}{e}.$$

Now on the right-hand side here there is surely a sum which consists of an infinite multitude of finite numbers of the same sign. However, we know from §, that such a sum does not need to be infinitely large. But if this sum is only finite then nothing prevents us from imagining the difference $F(x+i) - Fx$, which in its absolute value must always be greater than that sum, as only finitely large. Therefore the values of Fx can all remain measurable.

§ 116

Theorem. If a function Fx makes a *jump* (§38) for a certain value of its variable $x = c$, but it is continuous for others of the form $c + \omega$ or $c - \omega$, then a constant measurable number can always be specified with the property that the difference $F(c \pm \omega) - Fc$, in its absolute value, comes closer to it with the indefinite decrease in ω than to other number.

Proof. By the definition of a jump given in §38 the value Fc must be measurable, and because the function for all those x which are of the form $c + \omega$ or $c - \omega$ is to be continuous, then all those of its values which are either of the form $F(c+\omega)$ or $F(c-\omega)$ must be measurable. Hence the difference $F(c \pm \omega) - Fc$ is also measurable. Now if (because this is also possible) the value of $F(c \pm \omega)$, for all values of ω below a certain value remains constantly the same [*beständig einerlei*], e.g. $= A$, then $A - C$ is itself the number of which we speak in the theorem: a constant measurable number which the difference $F(c \pm \omega) - Fc$ approaches as near to as to no other. But if $F(c \pm \omega)$ is variable then it follows from the continuity that this function is to observe for all ω which are smaller than a certain [value], that there is a constant measurable number A which $F(c \pm \omega)$ approaches indefinitely. If we denote the different values which $F(c \pm \omega)$ becomes when we allow ω to decrease indefinitely by an arbitrary law, by $x^1, x^2, x^3, \ldots$, then it holds of these numbers that the difference $x^{n+m} - x^n$, considered in its absolute value, however large the number m may be taken, always remains smaller than a certain fraction $\frac{1}{N}$ which itself can be taken as small as we please if the number n has first been taken large enough initially. Now we know from *RZ* 7, §107 there is always one, and only one single measurable number A which the numbers $x^1, x^2, x^3, \ldots$ approach indefinitely. But if A is a number of the kind that $A - F(c \pm \omega)$ decreases indefinitely in its absolute value, i.e. becomes $= \Omega$, then $F(c \pm \omega) - Fc = A - \Omega - Fc = A - C - \Omega$. Therefore $A - C$ is that constant number of which this theorem asserts that the difference $F(c \pm \omega) - Fc$ can approach it as no other.

§ 117

Definition. I may be permitted to call the number which this theorem describes, i.e. that constant measurable number to which the difference $F(c \pm \omega) - Fc$ comes closer, with the indefinite decrease of ω, than to no other specifiable number, the *size of the jump* which the function Fx makes for the value $x = c$.

§ 118

Theorem. If a function does not break the law of continuity other than for certain isolated values of its variable, then nevertheless the multitude of jumps which it makes between two given limits of its variable can become infinitely large and [this can happen] even if within these limits only one of the two cases occurs: that it either always increases, or always decreases. However, in this case the sizes of its jumps, if we arranged them so that a greater one never followed a smaller, must form a measurable, and therefore convergent [*convergirend*], series. But if the condition that the function always increases or always decreases within the given limits is not assumed then the size of its jumps is not restricted by any other law than that each must be measurable.

Proof. 1. Let us imagine a function which for all values from $x = 0$ inclusive to $x = \frac{1}{2}$ exclusive takes the value $Fx = x$, for all values from $x = \frac{1}{2}$ inclusive to $x = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ exclusive, the value $Fx = \frac{1}{2} + x$, for all values from $x = \frac{3}{4}$ inclusive to $x = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ exclusive, the value $Fx = \frac{3}{4} + x$, for all values from $x = \frac{7}{8}$ inclusive to $x = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$ exclusive, the value $Fx = \frac{7}{8} + x$, and proceed indefinitely according to the law indicated here, so that in general all values from $x = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$ inclusive to $x = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n+1}}$ exclusive correspond to the value $Fx = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + x$. It is obvious that this function follows the law of continuity for all values lying within the limits 0 and $\frac{1}{2}$, then $\frac{1}{2}$ and $\frac{3}{4}$, then $\frac{3}{4}$ and $\frac{7}{8}$, etc. For all the values of the function for these values of its variable are of the simple form $a + x$, only from time to time a takes another value, but always a measurable value which is <1. But it is also obvious that this function departs from the law of continuity and makes a jump for the values $x = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$ etc. All these jumps, the multitude of which is infinite, lie within the limits $x = 0$ and $x = 1$. But the sizes of them are in exactly the order in which we enumerated them, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$, i.e. they form a convergent [*zusammenlaufende*] series. There is also no doubt that this function always increases within the limits 0 and 1 just mentioned. Therefore it is shown that a function which breaks the law of continuity only for certain isolated values of its variable can make an infinite multitude of jumps within given limits, as long as the sizes of these jumps decrease in such a way that they can be arranged into a convergent series.

2. It is also clear that under this last condition, an infinite multitude of jumps can occur with a function for which only one of the two cases holds, it either always increases or always decreases, because if a function always increases or decreases,

the difference $F(x + \Delta x) - Fx$ keeps the same sign for all values of x as long as we do not also change the sign of Δx. Therefore if we take two values of x of which one, α, approaches the limit a, the other, β, approaches the limit b, as close as we please, then it is clear that the difference $F\beta - F\alpha$, in its absolute value, can be considered as a sum, which together with several other summands, includes all those numbers which represent the size of the jumps occurring within α and β. Now since the number of these jumps can be increased as much as we please merely by moving the limits α and β ever closer to the limits a and b, it is decided that the difference $F\beta - F\alpha$, and hence at least one of the numbers $F\beta$ or $F\alpha$, must be infinitely large, unless the jumps mentioned were to become ever smaller so that when arranged in order of magnitude they form a series which, although it proceeds to infinity, nevertheless has a finite value.

3. But if the condition that our function only does one of the two things, either always increases or always decreases, is omitted, then it is not necessary to restrict the nature of its jumps in the way stated. For if the function sometimes increases, sometimes decreases, and perhaps sometimes does neither of the two, then the individual differences of the form $F(x + \Delta x) - Fx$, from which the difference $F\beta - F\alpha$ is composed, have different signs and hence their number, as well as their size, can be whatever we please without $F\beta - F\alpha$ needing to exceed a certain limit.

Second Section. Derived Functions

§ 119

Transition. In this main part we have proposed that we get to know the characteristic behaviour which a dependent number displays if we make clear the particular nature of the law of its dependency when the variable on which it depends assumes different values. But up until now we have paused only to observe whether the difference in the dependent number decreases indefinitely as long as the difference in its variable is reduced indefinitely, or whether this does not happen. It is easy to see, however, that there are many other characteristics of functions to be discovered if we compare those two differences with one another in a more exact way, in particular if we *divide* one by the other and examine the nature of the *quotient* that then arises. For while the *sum*, the *difference*, and the *product* of two numbers which both decrease indefinitely again produce a number which decreases indefinitely, the *quotient* of two such numbers has the peculiarity that it can take the greatest variety of values. That is, in certain circumstances such a quotient can be a constant number of any arbitrary value, in other circumstances it can yield a variable number, and this can sometimes increase indefinitely, at other times it may approach a given measurable number as closely as we please, and finally it may do none of these things. For example, if we consider the function $W = ax$, then $\Delta W = a\Delta x$ so the quotient $\frac{\Delta W}{\Delta x} = a$, i.e. a constant number completely independent of x and Δx. But if we have $W = ax^2$ then $\Delta W = (2ax + \Delta x)\Delta x$, therefore the quotient $\frac{\Delta W}{\Delta x} = 2ax + \Delta x$, a number which approaches the value $2ax$ indefinitely, a number which is not dependent on Δx, but certainly dependent on x. With the function $W = \frac{a}{x}$ then $\Delta W = -\frac{a\Delta x}{x(x+\Delta x)}$, therefore the quotient $\frac{\Delta W}{\Delta x} = -\frac{a}{x(x+\Delta x)}$, an expression which as long as x is not zero, approaches the value $-\frac{a}{x^2}$ indefinitely, but for $x = 0$ is unmeasurable. And so on.

§ 120

Definition. The case when a number can be specified which the quotient $\frac{\Delta Fx}{\Delta x}$ approaches indefinitely, with the indefinite decrease in Δx, is clearly especially noteworthy. We therefore say of such a number that it is the *derivative* of Fx, and thus understand by the *derivative* of a function Fx *for the value x and for a positive or negative* Δx, a measurable number M such that the difference $\frac{\Delta Fx}{\Delta x} - M$ for a definite value of x, and a definite positive or negative sign of Δx, becomes and remains smaller, in its absolute value, than any given number if Δx is taken small enough, and all the more so the further it is reduced. We say that the function Fx has a *derivative for the value x, and for the positive or negative increase* or in a positive or negative direction, if such a number M as we have just described can be specified for the particular value x, and for a positive or negative value of Δx. We say that Fx has a *two-sided* derivative, or a derivative in *two directions* or with respect to a positive as well as a negative increase, if such a number M can be specified for a positive as well as a negative value of Δx. If one and the same number M represents the derivative of Fx for the definite value of x *in both directions*, i.e. for a positive as well as a negative value of Δx, then we call it simply *the derivative of Fx for the value x*. Naturally a different number M is necessary for different values of x, and therefore we can speak of M generally being dependent on x. Thus we call a function of x which has the property that it represents, *for every value of x*, the derivative belonging to Fx, *the derived function* of Fx, and on the other hand, we call the Fx which belongs to this derived function the *primitive function* [*ursprüngliche Function*]. For example, we call $2ax$ the derived function of ax^2, because $2ax$ is that number, dependent merely on x, which the quotient $\frac{\Delta(ax^2)}{\Delta x} = \frac{a(x+\Delta x)^2 - ax^2}{\Delta x} = 2ax + a\Delta x$ approaches indefinitely with the indefinite decrease of Δx, and this is so for every value of x. In the special case when a function Fx has no derived function for a certain value of its variable x simply for the reason that the quotient $\frac{\Delta Fx}{\Delta x}$ is infinitely large for this particular value of x, or that it increases indefinitely with the indefinite decrease of Δx, then we say the derivative of Fx is *infinitely large*. For example, $\frac{1}{1-x}$ has no derivative for $x = 1$ because the quotient $\left(\frac{1}{1-x-\Delta x} - \frac{1}{1-x}\right) : \Delta x$ is infinitely large for the value $x = 1$. It is, therefore also said, (in a figurative sense) that the derived function of $\frac{1}{1-x}$ for $x = 1$ becomes infinitely large. If the derived number M of the function Fx is, in fact, dependent on x and it is apparent that it also has a derivative then we call this the *second* derivative in respect of the primitive function for M, i.e. in respect of Fx, and to make the difference clearer, the M described before is called the *first* derivative. Thus, for example, the function $2ax$ which is the derivative, i.e. the *first* derivative of the primitive ax^2, also has a derivative itself, namely $2a$, because $\frac{2a(x+\Delta x)-2ax}{\Delta x} = 2a$. We therefore call this number $2a$ the *second* derivative of the primitive ax^2. After this it will be self-evident what we call the *third, fourth,* ... and, generally, the *nth derivative*. The first derivative of a given function denoted by Fx is usually represented (following notation introduced by *Lagrange*) by $F'x$, the second by $F''x$, the third by $F'''x, \ldots$ the nth by $F^{(n)}x$. If the given function of x whose derivative we have to represent is composed of several symbols, each significant in themselves, like $ax^5 + bx^3$, it can be enclosed in brackets, so for example the third derivative of this function would be expressed by $(ax^5 + bx^3)'''$. But because one and the same expression is often composed of several numbers which can be considered as variable, e.g. $4x^3 - y^2x + xyz$ or $F(x, y, z, \ldots)$, and, naturally a completely different derivative arises according to which of its numbers are regarded as being variable. For example, $4x^3 - y^2x + xyz$ in *respect of x*, i.e. if we regard x as variable, has the derivative $12x^2 - y^2 + yz$, but with respect to y, it has the derivative $-2yx + xz$. We therefore often denote the derivative of a given function, like $F(x, y, z, \ldots)$, with respect to x by the notation $\frac{\mathrm{d}F(x,y,z,\ldots)}{\mathrm{d}x}$, and the derivative of this function with respect to y by $\frac{\mathrm{d}F(x,y,z,\ldots)}{\mathrm{d}y}$ etc. With this notation the symbols $\mathrm{d}x$, $\mathrm{d}y$, ... put underneath in the form of a divisor, may be thought of as merely indicating the variable in respect of which the derivative is to be taken, although the original meaning of these symbols is different. The second derivative of $F(x, y, z, \ldots)$ with respect to x, or the derivative of $\frac{dF(x,y,z,\ldots)}{dx}$ with respect to x, which should be represented in this method of notation by $\frac{d\frac{dF(x,y,z,\ldots)}{dx}}{dx}$, is represented more briefly by $\frac{d^2F(x,y,z,\ldots)}{dx^2}$, the third derivative by $\frac{d^3F(x,y,z,\ldots)}{dx^3}$ etc. If we want to indicate that the derivative of the function $F(x, y, z, \ldots)$ is firstly to be taken with respect to x, and then the derivative of latter is to be taken with respect to y, then this would be expressed by $\frac{d^2F(x,y,z,\ldots)}{dx\,dy}$. Therefore, for example, $\frac{d^2(x^5+y^3x^2)}{dx\,dy} = 6y^2x$. For the first derivative of $x^5 + y^3x^2$ with respect to x is $5x^4 + 2y^3x$, but the first derivative of $5x^4 + 2y^3x$ with respect to y is $6y^2x$. Conversely, in order to denote the *primitive* function of x which belongs to a given function, or number, M considered as *derivative*, we may make use of the signs $\int M\,dx$ or $\int [M]\,dx$ in which the symbol dx may be thought of as just being there to draw attention to which number (namely x) is to be considered here as the free variable. For example, a function which can be regarded as the primitive of $5x^4 + 2y^3x$, if x is to be considered as the variable in this expression, is represented by $\int [5x^4 + 2y^3x]\,dx$. And so on.

§ 121

Note. The concept of a *derivative* is also defined by *Lagrange*, *Lacroix*, *Cauchy* and most modern mathematicians essentially as we have done here. It is just worth remarking that the case where a function Fx has a derivative for a definite value of its variable x either only in respect of a *positive* increase, or only in respect of a *negative* increase, or if this derivative has a different value in each of these respects, is usually passed over in silence. But from the definition proposed here it follows that a function may have no derivative in certain cases where some people would not decline to grant it such [a derivative]. Thus many people would like to allow a derivative without hesitation to the function $Fx = (x^2 - 3x + 2)^{\frac{3}{2}} + (x^2 - 1)^{\frac{3}{2}}$, namely $\frac{3}{2}(x^2 - 3x + 2)^{\frac{1}{2}}.(2x - 3) + \frac{3}{1}(x^2 - 1)^{\frac{1}{2}}.x$, even for the value $x = 1$, and say that for this value it is $= 0$. Whereas holding strictly to the given definition I must say that this function has no derivative for the value mentioned. Since ΔFx here turns out to be imaginary for a positive as well as a negative increase, there is no number M, of which it can be said, that the quotient $\frac{\Delta Fx}{\Delta x}$ approaches it indefinitely with the indefinite decrease in Δx. Furthermore, since a derivative exists sometimes only for a positive Δx, sometimes only for a negative Δx, or the value of one is quite different from the value of the other, it would be desirable that we could express in the notation of a derivative (whenever necessary) in which respect it is to be taken: whether in respect of a positive increase, or of a negative increase. However, I shall be on my guard against adding yet another symbol to the already too many symbols which have been proposed—especially for the differential calculus.

§ 122

Theorem. There are functions which have for their derivative one and the same constant number for all values of their variable, and for a positive as well as a negative increase in the variable. And there are functions for which this is not the case, but the derivative of [the function] for different values of the variable is also different, and is thus itself a function of x. Some of these derived functions vary within certain limits according to the law of continuity, and even have a derivative again themselves, others do not.

Proof. If the given function is of the form ax, then

$$\frac{\Delta Fx}{\Delta x} = \frac{a(x + \Delta x) - ax}{\Delta x} = a.$$

Therefore the derivative $= a$ for all values of x, and for a positive as well as a negative Δx. But if we had, for example, $Fx = ax^2$, then

$$\frac{\Delta Fx}{\Delta x} = \frac{a(x + \Delta x)^2 - ax^2}{\Delta x} = 2ax + a\Delta x.$$

Therefore the derivative $= 2ax$ is different for different values of x. In this function we have, at the same time, an example of a function whose derivative is continuous, and which itself has another derivative, namely $2a$. If we put $Fx = x^2$ for all values of x, <2, and $Fx = x^3$ for $x = 2$ and for all greater values, then Fx for all values of $x < 2$ has the derivative $2x$, for $x = 2$ and all greater values, the derivative is $3x^2$. Therefore this derivative is itself a function of x which breaks the law of continuity with respect to a negative Δx for the value $x = 2$. For $\Delta F'x = F'(x + \Delta x) - F'x$ is for this value $= 2(2 - \Delta x) - 3.4 = -8 + 2\Delta x$. Etc.

§ 123

Theorem. A single-valued function Fx, for a definite (single) value of its variable x, and with respect to a definite sign of its increase, has a unique [*nur eine einzige*] derivative, if it has one at all.

Proof. For if Fx is single-valued, then by §11, Δx, and hence also $\frac{\Delta Fx}{\Delta x}$, is single-valued. Therefore by *RZ* 7, §92 there cannot be two different numbers, i.e. two unequal measurable numbers M and N, to which the value of the fraction $\frac{\Delta Fx}{\Delta x}$ comes as close as we please with the infinite decrease of Δx.

§ 124

Corollary. For the value $\Delta x = 0$, the quotient $\frac{\Delta Fx}{\Delta x}$ changes into the expression $\frac{0}{0}$ (§), which is always in itself indeterminate because, if $\Delta x = 0$, then also $\Delta Fx = Fx - Fx = 0$. But if the function Fx has a derivative for the definite value x and with respect to that sign which we are assuming for Δx, i.e. there is a certain measurable number M, independent of Δx, of a kind that with the indefinite decrease of Δx the difference $\frac{\Delta Fx}{\Delta x} - M$ becomes and remains $< \frac{1}{N}$, then there is always one and only one single value, namely M itself, which we must attribute to the quotient $\frac{\Delta Fx}{\Delta x}$ for $\Delta x = 0$, *if it is required that it should represent a number which obeys the law of continuity for the value x*. For if we designate by C the value which this quotient $\frac{\Delta Fx}{\Delta x}$ changes into for $\Delta x = 0$, then the condition of continuity requires that $\frac{\Delta Fx}{\Delta x} - C$ becomes and remains $< \frac{1}{N}$ if Δx decreases indefinitely. But since, with this indefinite decrease of Δx, also $\frac{\Delta Fx}{\Delta x} - M$ becomes and remains $< \frac{1}{N}$ then it is clear that it must be that $C = M$.

§ 125

Theorem. If a pair of functions fx and ϕx can be considered as the derivative of one and the same function Fx for all values of their variable lying within certain limits a and b, then for all values of x lying within a and b the equation $fx = \phi x$ has to hold.

Proof. If the equations

$$\frac{F(x+\Delta x) - Fx}{\Delta x} = fx + \overset{1}{\Omega} \quad \text{and} \quad \frac{F(x+\Delta x) - Fx}{\Delta x} = \phi x + \overset{2}{\Omega}$$

hold for all values of x lying within a and b, then for these same values the equation $fx + \overset{1}{\Omega} = \phi x + \overset{2}{\Omega}$ must also hold. But since fx and ϕx can be considered as constant numbers while $\overset{1}{\Omega}$ and $\overset{2}{\Omega}$ decrease indefinitely, this is only possible, by *RZ* 7, §92, if for all values of x lying within a and b, $fx = \phi x$.

§ 126

Corollary. For certain *isolated* values of x the equations

$$\frac{F(x+\Delta x) - Fx}{\Delta x} = fx + \overset{1}{\Omega} \quad \text{and} \quad \frac{F(x+\Delta x) - Fx}{\Delta x} = \phi x + \overset{2}{\Omega}$$

can hold without fx and ϕx having to be equivalent for every x. For example, if we put $\phi x = fx + (x-1)(x-2)(x-3)\ldots$ *in inf*. then fx and ϕx can in no way be regarded as a pair of equivalent functions. But if $\frac{F(x+\Delta x)-Fx}{\Delta x} = fx + \Omega$, then also the equation $\frac{F(x+\Delta x)-Fx}{\Delta x} = \phi x + \Omega$ holds for all values of x which are in the series $1, 2, 3, \ldots$, because the product $(x-1)(x-2)(x-3)\ldots$ *in inf*. vanishes for each of these values.

§ 127

Theorem. If a pair of functions are equivalent to one another for all values of their variables lying within certain limits a and b, then also their derivatives must be equivalent to one another within these limits. However, it need not be, conversely, that if the latter holds then the former holds.

Proof. 1. If we have $Fx = \Phi(x)$ for all values of x within a and b, then we also have, $F(x+\Delta x) = \Phi(x+\Delta x)$, providing we take $x + \Delta x$ between these limits, therefore also $F(x + \Delta x) - Fx = \Phi(x + \Delta x) - \Phi x$ and

$$\frac{F(x + \Delta x) - Fx}{\Delta x} = \frac{\Phi(x + \Delta x) - \Phi x}{\Delta x}.$$

Now if we denote the derivative of the function Fx by $F'x$, and the derivative of Φx by $\Phi' x$, then it should be that

$$\frac{F(x + \Delta x) - Fx}{\Delta x} = F'x + \overset{1}{\Omega} \quad \text{and} \quad \frac{\Phi(x + \Delta x) - \Phi x}{\Delta x} = \Phi' x + \overset{2}{\Omega},$$

and therefore also $F'x + \overset{1}{\Omega} = \Phi' x + \overset{2}{\Omega}$. Now since with the same x, $\overset{1}{\Omega}$ and $\overset{2}{\Omega}$ can decrease indefinitely it follows that it must be that $F'x = \Phi' x$ and this is so for all values of x which lie within a and b.

2. But conversely, merely because the equation $F'x = \Phi' x$ holds for all values of x lying within a and b, it is clear that it does not follow that also the equation $Fx = \Phi x$ must occur. Because even the two unequal functions $a + Fx$ and Fx have the same derivative if we designate by a an arbitrary number completely independent of x. For with this assumption, $\frac{\Delta(a+Fx)}{\Delta x} = \frac{\Delta Fx}{\Delta x}$, therefore the derivative of both functions is certainly the same.

§ 128

Corollary. Therefore if an equation holds for all values of a variable x appearing in it, or at least for all which lie within certain limits, then it will not be affected if we take the derivative with respect to this variable of both sides of it. For example, if in general $(1+x)^3 = 1+3x+3x^2+x^3$, then also in general $3(1+x)^2 = 3+6x+3x^2$. It would not be so if an equation only holds for one, or some isolated values of x, and we wanted to take the derivative. For example, $(x-3)(x-5)x^2 = (x-3)(x-5)x^4$ is a correct equation for $x = 0$ or $x = 3$ or $x = 5$, but in no way would the following be correct:

$$(x - 5)x^2 + (x - 3)x^2 + 2(x - 3)(x - 5)x = (x - 5)x^4 + (x - 3)x^4 + 4(x - 3)(x - 5)x^3$$

which arises if we take the derivatives of both sides of the previous [equation]. For this last equation still holds for $x = 0$, but not for $x = 3$ or $x = 5$.

§ 129

Theorem. If the number W is in fact independent of the number x which we are to regard as the variable, as happens if x appears in the expression itself either not at all, or only in such a way that the value of W does not change for every arbitrary value of x, as, for example, if $W = \frac{ax-bx}{cx}$, then the derivative of W with respect to x is $= 0$.

Proof. For now if we put $W = Fx$ then $F(x + \Delta x) = Fx$, therefore $\Delta Fx = 0$ and hence also $\frac{\Delta Fx}{\Delta x} = 0$. Therefore [there is] certainly no measurable number other than zero which the quotient $\frac{\Delta Fx}{\Delta x}$ approaches indefinitely.

§ 130

Theorem. If a function Fx has a derivative for the definite value x, with respect to a certain positive or negative increase, then it must also be *continuous* for this value of x, and with respect to the same increase. But it does not follow, conversely, from the continuity of a function for a definite value of its variable, and with respect to a certain sign, that it has a derivative in this respect.

Proof. 1. If Fx is to be continuous in the second degree[p] for the value x and with respect to that sign which we are assuming for Δx, then with the indefinite decrease of Δx the quotient $\frac{\Delta Fx}{\Delta x} = \frac{F(x+\Delta x)-Fx}{\Delta x}$ approaches indefinitely the measurable number M which is constant for the same x. From this it follows that the numerator of that fraction, or the difference $F(x + \Delta x) - Fx$, also represents a measurable number which decreases indefinitely with Δx itself. For in the opposite case, if this numerator did not even represent a measurable number, or always remained $> \frac{1}{N}$, or became so from time to time, while Δx is allowed to decrease indefinitely, then also the fraction $\frac{\Delta Fx}{\Delta x}$ would either not represent any measurable number or increase indefinitely. But if the difference $F(x + \Delta x) - Fx$ decreases indefinitely with Δx, then Fx is continuous in the first degree for the value x and for that sign of Δx which we established with $\frac{\Delta Fx}{\Delta x}$.
2. But we cannot infer, conversely, from the continuity of a function, or merely from the fact that $F(x + \Delta x) - Fx$ decreases indefinitely with Δx, the existence of a derivative, or the fact that the quotient $\frac{F(x+\Delta x)-Fx}{\Delta x}$ approaches indefinitely a constant and measurable number. We have learned this already from the example of the function $\frac{1}{1-x}$, for the value $x = 1$, which we considered in §120.

§ 131

Theorem. If it is to be true that a function Fx has no derivative for the value x with respect to a certain positive or negative increase, while it does have continuity in this respect, then only one of the following two cases can occur. Firstly, either the quotient $\frac{\Delta Fx}{\Delta x}$ increases indefinitely with the indefinite decrease of Δx, or secondly, there is indeed a certain measurable number M to which this difference can be brought as close as we please, but it does not remain at this closeness, instead to every Δx there is a smaller one for which the difference $\frac{\Delta Fx}{\Delta x} - M$ again becomes $> \frac{1}{N}$.

Proof. 1. There cannot be a third case because Δx and ΔFx should both always remain measurable. For if Δx and ΔFx are always measurable then also the value

[p] Bolzano uses the expression *stetig vom zweiten Grade* as a synonym for 'differentiable'. See §169 and the remark from 6 August 1833 in *MM* 2018 (*BGA* 2B13/1).

of the quotient $\frac{\Delta Fx}{\Delta x}$ is always measurable, and hence either there is a certain measurable number which is always greater than this quotient, in its absolute value, or it itself increases indefinitely. If there is a measurable number which is always $>\frac{\Delta Fx}{\Delta x}$ then according to *RZ* 7, §109 there is also an M which is the smallest of those of which it can be said that all greater numbers have the property of always remaining $>\frac{\Delta Fx}{\Delta x}$. This number M must either be exactly equal to the quotient $\frac{\Delta Fx}{\Delta x}$, for certain values of Δx, or come so close to it that the difference $M - \frac{\Delta Fx}{\Delta x}$ can become smaller than every given fraction $\frac{1}{N}$. For if this difference always remained $>\frac{1}{N}$ there would be a still smaller number than M of which, in the same way as it could be said of M itself, all greater numbers would always remain $>\frac{\Delta Fx}{\Delta x}$. However small the difference $M - \frac{\Delta Fx}{\Delta x}$ (or, what amounts to the same, the difference $\frac{\Delta Fx}{\Delta x} - M$, in its absolute value) becomes, for certain Δx, there must be other still smaller Δx for which $\frac{\Delta Fx}{\Delta x} - M$ again becomes $>\frac{1}{N}$. For if this were not to happen, then by the definition of §120, we would have to admit that Fx has a derivative and that M is this derivative.

2. We wish to prove, by a couple of examples, that each of the two cases stated in the theorem can occur in certain circumstances. If we assume that a certain number y depends on the number x by a law such that we always have the equation $y^2 = 1 - x^2$, where in addition it may be established that the value of W is always positive. On these assumptions, it is easy to prove that for all x which in their absolute value are not >1, W is a measurable and unique number. For if the symbols $a, b, c, d, \ldots$ denote arbitrary actual numbers, each >1, and we choose for the denominators $a, ab, abc, abcd, \ldots$, the numerators $\alpha, \beta, \gamma, \delta, \ldots$ according to a rule such that,

$$\left(\frac{\alpha}{a}\right)^2 \mathrel{\substack{=\\<}} 1 - x^2, \text{ but } \left(\frac{\alpha+1}{a}\right)^2 > 1 - x^2, \text{ furthermore}$$

$$\left(\frac{\alpha}{a} + \frac{\beta}{ab}\right)^2 \mathrel{\substack{=\\<}} 1 - x^2, \text{ but } \left(\frac{\alpha}{a} + \frac{\beta+1}{ab}\right)^2 > 1 - x^2, \text{ and also}$$

$$\left(\frac{\alpha}{a} + \frac{\beta}{ab} + \frac{\gamma}{abc}\right)^2 \mathrel{\substack{=\\<}} 1 - x^2, \text{ but } \left(\frac{\alpha}{a} + \frac{\beta}{ab} + \frac{\gamma+1}{abc}\right)^2 > 1 - x^2,$$

and in general,

$$\left(\frac{\alpha}{a} + \frac{\beta}{ab} + \cdots + \frac{\mu}{ab\ldots m}\right)^2 \mathrel{\substack{=\\<}} 1 - x^2, \text{ but}$$

$$\left(\frac{\alpha}{a} + \frac{\beta}{ab} + \cdots + \frac{\mu+1}{ab\ldots m}\right)^2 > 1 - x^2,$$

then it follows from *RZ* 7, §48 that the symbols $\alpha, \beta, \gamma, \ldots, \mu$ all denote actual numbers, or (some of them) zeros, but each has a unique value. Furthermore, we know from *RZ* 7, §§ 104, 105, that the infinite number expression $\frac{\alpha}{a} + \frac{\beta}{ab} + \frac{\gamma}{abc} + \cdots + \frac{\mu}{abc\ldots m} + \cdots$ *in inf*. represents a measurable number, and indeed a unique one, such that when put in the place of y it satisfies the equation $y^2 = 1 - x^2$. I now claim that the number y determined in this way is a function of x which has continuity for all values from $x = -1$ to $x = +1$ inclusive, but also has a derivative for all values from $x = -1$ to $x = +1$ exclusive. Because the equation $y^2 = 1 - x^2$ is to hold for all values of x, it must also hold if we put $x + \Delta x$ instead of x, and put $y + \Delta y$, instead of y, i.e. it must be that $(y+\Delta y)^2 = 1-(x+\Delta x)^2$ or $y^2+2y\Delta y+(\Delta y)^2 = 1-x^2-2x\Delta x-(\Delta x)^2$. Hence it follows by subtraction that $\Delta y(2y + \Delta y) = -\Delta x(2x + \Delta x)$ or $\Delta y = -\frac{(2x+\Delta x)\Delta x}{2y+\Delta y}$. Now first of all this equation proves the continuity of the function y, for every value of x from $x = -1$ to $x = +1$ inclusive, provided Δx is taken positive in the first case but negative in the last case. For if $x = -1$, $y = 0$, and $\Delta y = \frac{(2-\Delta x)\Delta x}{\Delta y}$ or $\Delta y^2 = (2 - \Delta x)\Delta x$ from which it is clear that it decreases indefinitely if Δx remains positive. It also gives for Δy an always measurable value which likewise decreases indefinitely. But the same equation, $\Delta y^2 = (2 - \Delta x)\Delta x$, also arises if we take $x = +1$ and Δx negative. Therefore it is proved that the function is continuous for $x = +1$, as well as for $x = -1$. But also [it is proved] for every value of x lying in between, since for each such value y has some measurable value different from zero, and the value of $\Delta y = -\frac{(2x+\Delta x)\Delta x}{2y+\Delta y}$ obviously decreases indefinitely with Δx. Indeed, for every value of x lying within $+1$ and -1, y also has a derivative. For the value $\frac{\Delta y}{\Delta x} = -\frac{2x+\Delta x}{2y+\Delta y}$ obviously approaches indefinitely the value $-\frac{x}{y}$ which is quite independent of Δx. y has no derivative for the two values $x = +1$ and -1, because for these values of x, $y = 0$ so $\frac{\Delta y}{\Delta x} = \frac{\mp 2-\Delta x}{\Delta y}$, and this increases indefinitely if Δx decreases indefinitely, because at the same time Δy also decreases indefinitely. We therefore have here the example of a function for which the quotient $\frac{\Delta Fx}{\Delta x}$ increases indefinitely with Δx. We obtain an example of the second case from the function y if it depends on x by such a rule that we have, for the following values of x, the values of y written next to them.

x		y
from 0	to $\frac{1}{2}$	x
$\frac{1}{2}$	$\frac{3}{4}$	$1 - x$
$\frac{3}{4}$	$\frac{7}{8}$	$x - \frac{1}{2}$
$\frac{7}{8}$	$\frac{15}{16}$	$\frac{5}{4} - x$
$\frac{15}{16}$	$\frac{31}{32}$	$x - \frac{5}{8}$
$\frac{31}{32}$	$\frac{63}{64}$	$\frac{21}{16} - x$
$\frac{63}{64}$	$\frac{127}{128}$	$x - \frac{31}{32}$
	etc.	

That is, in general, for every value of x from $\frac{2^{2n}-1}{2^{2n}}$ to $\frac{2^{2n+1}-1}{2^{2n+1}}$, $y = x - \frac{2^{2n}-1}{3.2^{2n-1}}$ and for every value of x from $\frac{2^{2n+1}-1}{2^{2n+1}}$ to $\frac{2^{2n+2}-1}{2^{2n+2}}$, $y = \frac{2^{2n+2}-1}{3.2^{2n}} - x$. It is self-evident on this assumption that y is continuous from $x = 0$ to $x = 1$ inclusive. But for this last value, by §106, it becomes $\frac{1}{3}$ because this is the limit which the expressions $x - \frac{2^{2n}-1}{3.2^{2n-1}}$ and $\frac{2^{2n+2}-1}{3.2^{2n}} - x$ approach indefinitely with the indefinite increase in n, if x is put $= 1$. But by no means does this function have a derivative for the value $x = 1$. Since for all values of x from $\frac{2^{2n}-1}{2^{2n}}$ to $\frac{2^{2n+1}-1}{2^{2n+1}}$, $\frac{\Delta y}{\Delta x} = +1$, and for all values of x from $\frac{2^{2n+1}-1}{2^{2n+1}}$ to $\frac{2^{2n+2}-1}{2^{2n+2}}$, $\frac{\Delta y}{\Delta x} = -1$, it may be seen that for the value $x = 1$ no negative Δx can be given small enough so that the quotient $\frac{\Delta y}{\Delta x}$ remains at one of these values $+1$ or -1 if Δx is made ever smaller.

§ 132

Note. The *second* case just considered, where a function has no derivative for a certain value of its variable simply because the quotient $\frac{F(x+\Delta x)-Fx}{\Delta x}$, although it always remains measurable, approaches no constant measurable number M indefinitely, so that there are not always smaller values of x for which the difference $\frac{\Delta Fx}{\Delta x} - M$ again becomes $> \frac{1}{N}$, also occurs with functions which follow a single identical law for all values of their variables. For the proof I just quote again the function considered several times already, $(c-x)\sin\log(c-x)$, which for $x = c$ and a negative increase in x without doubt possesses the continuity defined in §38, since the difference $F(x + \Delta x) - Fx$ for $x = 0$, is $= \omega \sin\log\omega$, and therefore certainly decreases indefinitely with ω. But there is no derivative to be found here, since the quotient $\frac{F(x+\Delta x)-Fx}{\Delta x} = \sin\log\omega$ oscillates indefinitely between the two extreme values $+1$ and -1. Consequently some propositions appearing in *Cauchy*'s *Cours d'Algèbre*,[q] particularly in Ch. 2, §3, are in need of a small correction. 'If the numerator and denominator of a fraction are infinitely small quantities whose values decrease indefinitely together with that of the variable x, then the value of this function for $x = 0$ is sometimes *finite*, sometimes *zero*, sometimes *infinite*.' [Correction is needed here]—at least in so far as the case where the value of that fraction remains *undetermined* because it continually oscillates between two limits, is not considered at all. *Cauchy* bases the proof of his proposition on the assumption, 'that every variable quantity which vanishes together with another [variable quantity] x, must be able to be represented in the form $K.x^n(1 \pm \varepsilon)$ in which K is a *constant* different from zero, or represents a variable quantity again vanishing with x itself'. However, it will certainly be acknowledged by that astute scholar that this assumption cannot be strictly proved, indeed, is not even generally true.

§ 133

Theorem. As the property of continuity of some functions belongs only to a certain isolated value of their variable (§46), so also the property of *having a derivative* can sometimes belong to a function only for an isolated value of its variable, and it is

[q] It should be *Cours d'Analyse*.

possible that a function also has a derivative for just the single value of its variable for which it is continuous.

Proof. Suppose that for every value of x which is of the form $\frac{2m+1}{2^n}$, Fx has the value $\left(1+\frac{1}{2^n}\right)x$, but for every other value of x, $Fx = x$, then it is apparent, as in §46, that $F(x+\Delta x) - Fx$ can always only have the following four values: $x+\Delta x$, $+\frac{x}{2^n} + \left(1+\frac{1}{2^n}\right)\Delta x$, $-\frac{x}{2^n} + \Delta x$, $\left(1+\frac{1}{2^n}\right)\Delta x$, according to whether neither x nor $x+\Delta x$ are of the form $\frac{2m+1}{2^n}$, or the first, or the second, or both, are of this form. Hence it is clear, as in §46, that this function is continuous only for the single value $x = 0$. But for just this value it also has a derivative. For $\frac{F(x+\Delta x)-Fx}{\Delta x}$ is either $= 1$, or $1+\frac{1}{2^n}$, and this latter value, where n increases indefinitely, approaches the value 1 indefinitely with the indefinite decrease of Δx. Therefore 1 may be regarded as the limit which the quotient $\frac{F(x+\Delta x)-Fx}{\Delta x}$ approaches indefinitely with the indefinite decrease of Δx.

§ 134

Theorem. As a function can have a derivative for both cases: for certain isolated values, as well as for a whole collection [*Inbegriff*] of values of its variable, as many as lie within certain limits a and b, or generally for all values of x, so also a function can *lack* a derivative for a certain isolated value as well as for a whole collection of values of its variable, as many as lie within certain limits a and b, and indeed generally for all its values.

Proof. Suppose that for all values of x which are <10, $Fx = x$, but for $x = 10$, $Fx = 20$, and finally for all values of x which are >10, $Fx = 1 + x$. Then for all values of x which are <10 the quotient

$$\frac{F(x+\Delta x) - Fx}{\Delta x} = \frac{(x+\Delta x) - x}{\Delta x} = 1,$$

therefore 1 itself is the derivative. In the same way for all values of x which are >10,

$$\frac{F(x+\Delta x) - Fx}{\Delta x} = \frac{(1+x+\Delta x) - (1+x)}{\Delta x} = 1,$$

i.e. also for all these values the derivative is 1. But for the value $x = 10$ the difference $F(x+\Delta x) - Fx$ for a positive Δx, $= 1 + 10 + \Delta x - 20 = -9 + \Delta x$, and for a negative Δx, $= (10 - \Delta x) - 20 = -10 - \Delta x$. Therefore this function for the value $x = 10$ is not even continuous, much less does it have a derivative (§130). Finally, it follows from this that there are also functions which fail to have a derivative for all values of their variables within certain limits, or for all values generally, because there are also functions which are not even continuous.

§ 135

Corollary. The function Fx considered in §111 with which the rising and falling alternates so frequently that for no value of x is there an ω small enough to be able to assert that Fx always increases or always decreases within x and $x \pm \omega$, gives us a proof that a function can even be continuous and yet have no derivative for so many values of its variable that between every two of them there is a third for which it can be proved that it again has no derivative.[r] For if x is one of those values for which the corresponding Fx coincides exactly with one of the values belonging to x which the functions $\overset{1}{y}, \overset{2}{y}, \overset{3}{y}, \ldots$[s] take, then it is easy to prove that there is no constant measurable number which the quotient $\frac{\Delta Fx}{\Delta x}$ approaches indefinitely with the indefinite decrease of Δx, but rather this quotient increases indefinitely, which amounts to saying that the derivative $F'x$ is infinitely great, i.e. does not exist at all (§120). That is, to every Δx however small, an n can be specified so large that $\left(\frac{3}{8}\right)^{n-2}(b-a)$ becomes $<\Delta x$. Now if we denote this $\left(\frac{3}{8}\right)^{n-2}(b-a)$ for brevity by α, but we denote the difference, by which the value of the function $\overset{n}{y}$ belonging to $x+\alpha$ is greater than the [value] Fx belonging to x, by β, then we know that to $x+\frac{3}{8}\alpha$ belongs an increase of $\overset{n}{y}=\frac{5}{8}\beta$, to $x+\left(\frac{3}{8}\right)^2\alpha$ belongs an increase of $\overset{n}{y}=\left(\frac{5}{8}\right)^2\beta$ and generally to $x+\left(\frac{3}{8}\right)^r\alpha$ belongs an increase of $\overset{n}{y}=\left(\frac{5}{8}\right)^r\beta$. Now since all the values belonging to the values of x just mentioned or increases of the function $\overset{n}{y}$ are at the same time values or increases of Fx, then we see that the ratio $\frac{\Delta Fx}{\Delta x}$, with the gradual reduction of Δx, can take all the values appearing in the following series:

$$\frac{5}{3}\frac{\beta}{\alpha}, \left(\frac{5}{3}\right)^2\frac{\beta}{\alpha}, \left(\frac{5}{3}\right)^3\frac{\beta}{\alpha}, \ldots, \left(\frac{5}{3}\right)^r\frac{\beta}{\alpha}.$$

Now since $\left(\frac{5}{3}\right)^r$ increases indefinitely, there is no doubt that also the ratio $\frac{\Delta Fx}{\Delta x}$ increases indefinitely.

§ 136

Note. The last part of the previous theorem contradicts to a certain extent what *Lagrange* and many others sometimes explicitly claim, and sometimes just tacitly assume: that every function, with at most the exception of some isolated [*isolirt*] values of its variable, but in all other cases, has a *derivative*. But it is worth remarking (as I already mentioned in §39) that these scholars take the word *function* in a much narrower sense, because they understand by it only such numbers, dependent on another number x which can be expressed by one of the seven signs: $a+x$, $a-x$, ax, $\frac{a}{x}$, x^n, a^x, $\log x$, or by a combination of several of these. Now what they claim certainly holds of such [functions] especially as with some of these signs it is already in the *meaning* of them, that they should denote numbers that vary only by the law of continuity, or always have a derivative. But

[r] Bolzano shows that the function is differentiable at no point of an everywhere dense set. It is in fact differentiable nowhere as first proved by Jarník in an article (in Czech) in 1922. An English version of this paper, 'On Bolzano's Function' appeared in Jarník (1981).

[s] Here, and in the remainder of the paragraph, Bolzano assumes the context of §111.

since I believe (§2) that a much wider concept must be associated with the word *function* then it will be necessary to allow of functions that they not only have no derivative, but they may even *break* the law of continuity not only for single values of their variables, but for all values lying within certain limits, and for all values *generally*. However, even some of those mathematicians who do take the concept of a *function* in a wider sense, and as I have done above, seem inclined to believe that every function has its derivative provided we exclude isolated values. In the *Annales de Mathématiques* by *J. D. Gergonne*, T. 21 (1830) there occurs on p. 182 an attempt by *Galais*[t] to prove this. Since the proof is very short, it may be quoted here as follows.[u]

> *Theorem.* Let Fx and fx be any two given functions; for some x and h we shall have
> $$\frac{F(x+h)-Fx}{f(x+h)-fx}=\phi(k)$$
> ϕ being a definite function and k a quantity intermediate between x and $x+h$.
>
> *Proof.* Let us put, in effect, $\frac{F(x+h)-Fx}{f(x+h)-fx}=P$. One will deduce $F(x+h)-P.f(x+h)=Fx-P.fx$, from which it can be seen that the function $Fx-P.fx$ does not change when x is changed to $x+h$ from which it follows, unless it remains constant between these limits which would only happen in some particular cases, that this function will have one or more *maxima* or *minima* between x and $x+h$. Let K be the value of x corresponding to one of them then one will evidently have $K=\psi(P)$, ψ being a definite function, then one must also have $P=\phi(K)$, ϕ being another equally definite function; this proves the theorem. It can be concluded from it as a corollary that the quantity, $\lim\frac{F(x+h)-Fx}{f(x+h)-fx}=\phi(x)$, for $h=0$, is necessarily a function of x which proves *a priori* the existence of derived functions.

This proof is not satisfactory to me. Without doubt the equation $\frac{F(x+h)-Fx}{f(x+h)-fx}=P$ requires that P is considered as a number which depends not only on x and h, but also on the nature of the functions which are being expressed by the symbols F and f. Now it is correct that the whole expression $Fx-P.fx$ does not change its value if x becomes $x+h$, from which it certainly follows (if the continuity of the functions Fx and fx is assumed) that that expression must have one or more maxima and minima within x and $x+h$. But it is not at all clear to me that if one denotes one of these by K, K must *evidently* be a function of the number P. That is, just as in the expression $Fx-P.fx$ not only does P appear, but also the symbols F and f, then it could well be, and in fact it is, that K does not depend merely on the value of P, but also on the nature of the functions which we denote by F and f. Perhaps someone may want to reply that the influence that the nature of the

[t] It appears thus in the *Annales*. It should be *Galois*.

[u] The following theorem and proof are copied by Bolzano in French. The translation here is given with grateful acknowledgement to Denise and David Fowler.

functions Fx and fx has on the determination of K is indeed undeniable, but that one could do without it for the determination of K, if K is allowed to be dependent only on P, because P itself already depends on F and f. Then I answer that this not a sound argument: 'If K and P both depend on one and the same function Fx (or on two functions Fx and fx) then it must also be that K can be determined through P (and P through K).' For example the length of a line s is determined by the abscissa x and the function for the ordinate $y = fx$; the same also holds of the area P which this line encloses with its co-ordinates. But could we really say that $s = \psi(P)$ or that $P = \phi(s)$?

§ 137

Theorem. There are functions which have a derivative for a certain value of their variable only in respect of a *positive* increase, or only in respect of a *negative* increase. There are also functions which for a certain value of their variable have one derivative for a positive sign of Δx, and another one for a negative sign of Δx. Finally, there are also functions and values of their variable for which one and the same derivative occurs in both respects (i.e. for a positive, as well as a negative increase).

Proof. The function y, already considered in §131, which is determined by the equation $y^2 = 1 - x^2$, provides us with an example of the first and second case. For with the value $x = -1$, y has a derivative only for a positive Δx, but for the value $x = +1$ only for a negative Δx. For an example of the third case we can now take y to have the value $3x$ for all values of x which are <4 to the value $x = 4$ inclusive, but for all greater values of x is to have the value $5x - 8$. For with this assumption, for the value $x = 4$ and for a positive Δx the quotient is $\frac{\Delta y}{\Delta x} = \frac{[5(4+\Delta x)-8]-3.4}{\Delta x} = 5$, but for a negative Δx it is $\frac{\Delta y}{\Delta x} = \frac{3(4-\Delta x)-3.4}{\Delta x} = 3$. Finally, that there are also functions with which the number M which the quotient $\frac{\Delta Fx}{\Delta x}$ approaches indefinitely keeps the same value whether we take Δx positive or negative scarcely needs a proof. Thus if $y = ax^2$ for every value of x and for a positive Δx, the quotient

$$\frac{\Delta y}{\Delta x} = \frac{a(x + \Delta x)^2 - ax^2}{\Delta x} = 2ax + a\Delta x,$$

and for a negative Δx,

$$\frac{\Delta y}{\Delta x} = \frac{a(x - \Delta x)^2 - ax^2}{-\Delta x} = 2ax - a\Delta x;$$

therefore $2ax$ is the number which the value of that quotient approaches indefinitely for both cases.

§ 138

Theorem. If a function has a derivative in both directions for all values of its variable lying within a and b which, moreover, follows the law of continuity for all the values of x mentioned then this derivative must be the same in both directions for every single value of x.

Proof. Let x denote an arbitrary value lying within a and b then there must be a positive Δx small enough so that the difference $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x$ turns out $<\frac{1}{N}$. Furthermore, because of the existence of a double-sided derivative of the function Fx (§130) and the explicit assumption that its derivative $F'x$ is itself also to be continuous for every x lying within a and b, there must be a positive i small enough that for it and for all smaller values the following three relationships hold:

$$F'(x-i) - F'x < \frac{1}{N}$$

$$F(x-i) - Fx < \frac{\Delta x}{N}$$

$$F(x+\Delta x - i) - F(x+\Delta x) < \frac{\Delta x}{N}$$

providing we take the differences in the three left-hand sides all in their absolute values. Therefore it is also the case that if we divide the last two relationships by the positive Δx and subtract [them] from one another (*RZ* 7, §§ 67, 127)

$$\frac{F(x+\Delta x-i)-F(x+\Delta x)}{\Delta x} - \frac{F(x-i)-Fx}{\Delta x} < \frac{2}{N},$$

which can also be written,

$$\frac{F(x+\Delta x-i)-F(x-i)}{\Delta x} - \frac{F(x+\Delta x)-Fx}{\Delta x} < \frac{2}{N}.$$

But since $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x < \frac{1}{N}$, it must also be that $\frac{F(x+\Delta x-i)-F(x-i)}{\Delta x} - F'x$, is and remains $<\frac{3}{N}$, if i decreases indefinitely.

1. Now if $i = \Delta x$ then

$$\frac{F(x+\Delta x-i)-F(x-i)}{\Delta x} = \frac{Fx - F(x-\Delta x)}{\Delta x} = \frac{F(x-\Delta x)-Fx}{-\Delta x}.$$

Therefore $\frac{F(x-\Delta x)-Fx}{-\Delta x} - F'x < \frac{3}{N}$. Now since $\frac{3}{N}$, just as much as $\frac{1}{N}$, denotes a number which can decrease indefinitely if Δx decreases indefinitely, there is no doubt that for this case $F'x$ behaves exactly as the derivative of Fx should with a *negative* Δx.

2. If $i > \Delta x$ then the relationship $\frac{F(x+\Delta x-i)-F(x-i)}{\Delta x} - F'x < \frac{3}{N}$ holds all the more certainly if we allow i to decrease and become $=\Delta x$. For the three relationships above, from which this last one comes, are not affected by any decrease in i. Therefore also in this case $\frac{F(x-\Delta x)-Fx}{-\Delta x} - F'x < \frac{3}{N}$, i.e. $F'x$ behaves here like a derivative of Fx in the negative direction.

3. Finally if $i < \Delta x$, then we need only remark that our function is to have a derivative for every value of x, therefore also for the value $x - i$. So an ω must be able to be given small enough that for it, and for all smaller values,

$\frac{F(x-i+\omega)-F(x-i)}{\omega} - F'(x-i)$ becomes and remains $<\frac{1}{N}$. But since we were to take, as already stipulated, i so small that $F'(x-i)-F'x < \frac{1}{N}$, then also $\frac{F(x-i+\omega)-F(x-i)}{\omega} - F'x < \frac{2}{N}$. Now since this relationship holds even if i and ω are allowed to decrease indefinitely then it must also hold if we allow the greater of these two numbers to become equal to the smaller. But then $\frac{Fx-F(x-\omega)}{\omega} - F'x < \frac{2}{N}$, i.e. $\frac{F(x-\omega)-Fx}{-\omega} - F'x < \frac{2}{N}$, i.e. the function Fx also has a derivative for a negative Δx.

§ 139

Theorem. If a function Fx has a derivative for all values of its variable lying within a and b, that breaks the law of continuity only for certain isolated values of x of which each has a next one on both sides, then this break consists in this, that if c represents such a value the derivative $F'c$ has one value M for a *positive* increase and another value R for a negative increase, while those derivatives which are of the form $F'(c+\omega)$ approach the value M indefinitely, but those derivatives which are of the form $F'(c-\omega)$ approach the value R indefinitely as ω decreases indefinitely.

Proof. Because each value of x for which the derivative $F'x$ breaks the law of continuity is to be an isolated one, and have a next one to it on both sides, let $c+i$ be this next one on the positive side and $c-j$ be the next one on the negative side. For all values of x which lie between c and $c+i$, as also for all lying between c and $c-j$, $F'x$ therefore follows the law of continuity. Therefore by the previous proposition for all these values the value of $F'x$ is the same in both directions. If this were also to be so for the value $x=c$ then no deviation from the law of continuity would be made known. Therefore if, on the contrary, a [deviation] is to occur then the two values of $F'c$ which we denoted by M and R in the theorem, must be unequal. But it may be proved as follows that the value of the derivative $F'(c+\omega)$ can approach the value M as closely as desired by indefinitely reducing ω. Because M represents the derivative of Fx for $x=c$ and a positive increase, then it must be that a positive ω can be given which is small enough that $\frac{F(c+2\omega)-Fc}{2\omega} - M$ becomes $<\frac{1}{N}$, and then all the more certainly $\frac{F(c+\omega)-Fc}{\omega} - M < \frac{1}{N}$. However

$$\frac{F(c+2\omega)-Fc}{2\omega} = \frac{F(c+2\omega)-F(c+\omega)}{2\omega} + \frac{F(c+\omega)-Fc}{2\omega}.$$

Therefore it must be that

$$\frac{F(c+2\omega)-F(c+\omega)}{\omega} + \frac{F(c+\omega)-Fc}{\omega} - 2M < \frac{2}{N},$$

and if we subtract $\frac{F(c+\omega)-Fc}{\omega} - M$,

$$\frac{F(c+2\omega)-F(c+\omega)}{\omega} - M < \frac{3}{N}.$$

But the quotient $\frac{F(c+2\omega)-F(c+\omega)}{\omega}$ approaches the value $F'(c+\omega)$ indefinitely. Therefore also it must be that $F'(c+\omega) - M < \frac{3}{N}$. In a similar way it may be proved that $F'(c-\omega) - R < \frac{3}{N}$.

§ 140

Theorem. Simply from the fact that a function Fx has a derivative with respect to a *positive* (or negative) Δx for all values of its variable lying within a and b, it does not follow that it must also have a derivative with respect to a negative (or positive) Δx.

Proof. [This can be shown] in a similar way as for the similar proposition §122, so that the same example can again be used here. If for all values of $x < 2$, $Fx = x^2$, but for $x = 2$ and all greater values, $Fx = x^3$, then for every $x < 2$, $\frac{\Delta Fx}{\Delta x} = 2x + \Delta x$, therefore $2x$ is the derivative with respect to a positive Δx. Similarly for every $x > 2$, $\frac{\Delta Fx}{\Delta x} = 3x^2 + 3x\Delta x + \Delta x^2$, therefore $3x^2$ is the required derivative with respect to a positive Δx. Finally, there is also a derivative for the value $x = 2$ with respect to a positive increase, for

$$\frac{\Delta Fx}{\Delta x} = \frac{(2+\Delta x)^3 - 2^3}{\Delta x} = 12 + 6\Delta x + \Delta x^2,$$

therefore this derivative is 12. Therefore our function has a derivative with respect to a positive increase for *all* values of x. It is not so for a difference with a negative sign for the value $x = 2$. Here

$$\frac{\Delta Fx}{\Delta x} = \frac{(2-\Delta x)^2 - 2^3}{\Delta x} = \frac{-4 - 4\Delta x + \Delta x^2}{\Delta x},$$

an expression which in its absolute value increases indefinitely if Δx decreases indefinitely.

§ 141

Theorem. Simply from the fact that a certain function Fx has a derivative for all values of its variable lying within a and b, it in no way follows that for all values of x lying within these limits there must be *one and the same number e* small enough to be able to assert that the difference $\frac{\Delta Fx}{\Delta x} - F'x$, considered in its absolute value (where $F'x$ represents the derivative with respect to x, and for the same sign as Δx), becomes $< \frac{1}{N}$, without it being necessary to take $\Delta x < e$.

Proof. If we put $Fx = \frac{1}{1-x}$ then what is claimed here occurs if the variable x approaches the value 1 indefinitely. That is, if for brevity we write $x = 1 - i$, then $\frac{\Delta Fx}{\Delta x} = \frac{1}{i(i-\Delta x)}$ therefore $\frac{1}{i^2}$ is the derivative of $\frac{1}{1-x}$ or $\frac{1}{i}$. Now if the difference

$$\frac{\Delta Fx}{\Delta x} - F'x = \frac{1}{i(i-\Delta x)} - \frac{1}{i^2} = \frac{\Delta x}{i^2(i-\Delta x)}$$

is to become $< \frac{1}{N}$ then Δx must be taken $< \frac{i^3}{N+i^2}$, a number which obviously becomes smaller than every given e as long as i decreases indefinitely, i.e. as long as x approaches the limit 1 indefinitely.

§ 142

Theorem. It is possible that a function Fx has a *derivative* with respect to a positive (or negative) increase for every value of its variable lying within a and b and yet does not follow the law of continuity in both directions for every value of its variable lying within a and b. But if the function has a derivative in both directions then certainly it must also always be continuous in both directions.

Proof. 1. If we put $Fx = x^2$ for all values of x which are < 4, and for $x = 4$ itself, but for every greater value $Fx = x^3$, then Fx has a derivative for all values of x at least in one direction. Namely for all values of $x < 4$, the derivative is $2x$, for all $x > 4$ the derivative is $3x^2$ in both directions, but for $x = 4$ [there is] only a derivative in the negative direction $= 2x = 8$, while for a positive increase

$$\frac{\Delta Fx}{\Delta x} = \frac{(4 + \Delta x)^3 - 4^2}{\Delta x} = \frac{48 + 48\Delta x + 12\Delta x^2 + \Delta x^3}{\Delta x}$$

increases indefinitely. For all these values this function is no longer continuous. For ΔFx here $= 48 + 48\Delta x + 12\Delta x^2 + \Delta x^3$ is always >48.
2. But if the function has a derivative in both directions for all values lying within a and b, then it must also be continuous in both directions for all these values. For if there was some value $x = c$ lying within a and b, for which the function is discontinuous for either a positive or negative ω, then $F(c+\omega) - Fc$ would have to be a difference which either never becomes $< \frac{1}{N}$, however much ω was decreased, or it does not always remain $< \frac{1}{N}$ if ω is allowed to decrease indefinitely. For because the function has a double-sided derivative for all values of x lying within a and b, the values Fc and $F(c + \omega)$ must both be measurable, therefore certainly (by §45) beyond the two cases mentioned no third case can occur. However, in neither of these two cases could the quotient $\frac{F(c+\omega)-Fc}{\omega}$ approach indefinitely a measurable number independent of ω, but on the contrary it would have to increase indefinitely with the decrease of ω. For if to every ω there is some smaller one for which $F(c+\omega) - Fc$ becomes $\overline{\overline{>}}\, \frac{1}{N}$ then $\frac{F(c+\omega)-Fc}{\omega} \overline{\overline{>}}\, \frac{1}{\omega N}$, which, providing ω is small enough, can become greater than every given number.

§ 143

Corollary. If we have simply been told that a function Fx has a derivative for every value of its variable lying within a and b without determining whether it is always in both directions, but it is established that this function is always continuous, we cannot conversely draw the conclusion that it has a derivative in both directions for every value lying within a and b. For if we put $Fx = x - \frac{2^{2n}-1}{3.2^{2n-1}}$, for all values of x from $\frac{2^{2n}-1}{2^{2n}}$ to $\frac{2^{2n+1}-1}{2^{2n+1}}$, but $Fx = \frac{2^{2n+2}-1}{3.2^{2n}} - x$ for all values of x from $\frac{2^{2n+1}-1}{2^{2n+1}}$

to $\frac{2^{2n+2}-1}{2^{2n+2}}$, and finally that for all $x \overline{\gtrless} 1$, $Fx = \frac{x^2}{3}$, then comparison with the last example in §131 shows us that this function is continuous with respect to a positive as well as a negative increase, for all values of x from 0 to 10 or to any arbitrarily large limit, and likewise that this function, for every value within these limits, also has a derivative in one, yet not in both, directions. That is, for $x = 1$ and for a positive Δx there is here the derivative $\frac{2x}{3} = \frac{2}{3}$, but for a negative Δx there is none.

§ 144

Theorem. If a function Fx has a derivative $F'x$ in both directions for all values of its variable lying within a and $a + h$, but for the value $x = a$ it at least has one in the direction of h, and for the value $x = a + h$ in the opposite direction, moreover if this derivative follows the law of *continuity* for all values of x just mentioned, then it must be possible to specify a number e small enough to be able to assert that the increase Δx need never be taken smaller than e so that the difference $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x$, in its absolute value, turns out smaller than a given fraction $\frac{1}{N}$ as long as both x and $(x + \Delta x)$ do not lie outside a and $a + h$.

Proof. If the opposite were [the case] and there was no number e small enough that what has just been said holds true of it, then the values of Δx which are necessary to maintain the relationship $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x < \frac{1}{N}$ for every value of x which does not lie outside a and $a + h$, must decrease indefinitely. To each Δx, however small it was taken, there would therefore be another one here which is yet smaller. Therefore, if we denote the values of x by $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \overset{4}{x}, \ldots$ which have the property that they each lie within a and $a + h$, and each successive Δx needs to be smaller than its predecessor in order to maintain the relationship $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x < \frac{1}{N}$ then these values would form a series which proceeds to infinity. But we know from § that there must be a certain measurable number c, not outside a and $a + h$ of a kind that an infinite multitude of terms of that series may be enclosed by the two limits c and $c \pm i$ where i in its absolute value may be taken as small as we please. Hence it follows directly that for values of x which approach c indefinitely, the value of Δx would have to decrease indefinitely in order to satisfy the condition $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x < \frac{1}{N}$. For if the limits c and $c \pm i$, however closely they approach one another, always enclose between them an infinite multitude of the terms $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$ *in inf*., then it is obvious there must be infinitely many of these values which are so close to c that the distance is smaller than every given number. Among these infinitely many values there must necessarily also be such as lie in the series $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$ *in inf*. so far from its beginning that their index is greater than every given [number], therefore the Δx belonging to the terms must become smaller than *every given fraction*[v] if the condition $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x < \frac{1}{N}$ is to be satisfied. But if Fx has a derivative in both directions for every value of x lying within a and

[v] Bolzano has added '*Quid hoc?* NB' at this point of the manuscript.

$a + h$, moreover for $x = a$ it has a derivative in the same direction as h, and for $x = a + h$ in the opposite direction, then for every value of c not lying outside a and $a + h$ there is a number e, of the same sign as i, and small enough that the difference $\frac{F(c+e)-Fc}{e} - F'c$ turns out $< \frac{1}{2.N}$. N.B. There can even be infinitely many such [values of] c and their corresponding [values of] e! Now I claim that no Δx need become smaller than this e which corresponds to some value of x bordering sufficiently closely on c, so that the relationship $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x < \frac{1}{N}$ occurs. That is, because of the assumption that the derivative $F'x$ is to be continuous for all values of x not lying outside a and $a + h$, and because also for all these values the function Fx itself, according to §130, must be continuous otherwise it could not possess derivatives in both directions for each of the values mentioned, there is no doubt that there must be some number j, of the same sign as i, so small that for it, and for all smaller numbers, the following three relationships all hold simultaneously:

$$F'(c + j) - F'c < \frac{1}{6N}$$

$$F(c + j) - Fc < \frac{e}{6N}$$

$$F(c + j + e) - F(c + e) < \frac{e}{6N}$$

if we take all the differences which form the left-hand sides in these relationships in their absolute values. But from these relationships it follows, by division of the last two by e and adding,

$$\frac{F(c + j + e) - F(c + e)}{e} + \frac{F(c + j) - Fc}{e} + F'(c + j) - F'c < \frac{1}{2N}.$$

Therefore also

$$\frac{F(c + j + e) - F(c + e)}{e} - \frac{F(c + j) - F(c)}{e} - (F'(c + j) - F'c) < \frac{1}{2N},$$

likewise

$$\frac{F(c + j + e) - F(c + j)}{e} - \frac{F(c + e) - Fc}{e} - (F'(c + j) - F'c) < \frac{1}{2N}.$$

Since also $\frac{F(c+e)-F(c)}{e} - F'c < \frac{1}{2N}$, it follows by addition that in every case $\frac{F(c+j+e)-F(c+j)}{e} - F'(c + j) < \frac{1}{N}$, where j may be taken as small as we please. But on this assumption $c + j$ represents every arbitrary value of x lying within c and $c + i$, and it is thus proved that for none of these values of x a value of Δx smaller than e is needed in order to maintain the relationship $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x < \frac{1}{N}$.

§ 145

Theorem. Suppose a function Fx has a derivative in both directions for all values of x lying within a and $a + h$, but for $x = a$ it has a derivative at least in the same direction as h, and for $x = a + h$ in the opposite direction. Finally, if this derivative is always continuous for all values of x just mentioned, then the equation

$$F(a+h) = Fa + \frac{h}{n}\left[F'a + F'\left(a+\frac{h}{n}\right) + F'\left(a+\frac{2h}{n}\right) + F'\left(a+\frac{3h}{n}\right) + \cdots + F'\left(a+\frac{(n-1)h}{n}\right)\right] + \Omega$$

always holds, in which n denotes an arbitrary actual number through the indefinite increase of which the number Ω can be decreased indefinitely.

Proof. If the function Fx has the property just stated then by the previous § there is a number e small enough to be able to claim that the increase Δx never needs to be taken smaller than e, so that the difference $\frac{F(x+\Delta x)-Fx}{\Delta x} - F'x$, in its absolute value, turns out smaller than the given fraction $\frac{1}{N}$ as long as x and $x + \Delta x$ do not lie outside the limits a and $a + h$. Certainly there is also a number n large enough that the quotient $\frac{h}{n}$, in its absolute value, becomes $\overline{\overline{<}}\, e$. Accordingly, if we write for brevity $\frac{h}{n} = \omega$, the following relationships will hold:

$$\frac{F(a+\omega) - F(a)}{\omega} - F'a < \frac{1}{N}$$

$$\frac{F(a+2\omega) - F(a+\omega)}{\omega} - F'(a+\omega) < \frac{1}{N}$$

$$\frac{F(a+3\omega) - F(a+2\omega)}{\omega} - F'(a+2\omega) < \frac{1}{N}$$

$$\cdots\cdots$$

$$\frac{F(a+n\omega) - F\left(a+\frac{n-1}{n}\omega\right)}{\omega} - F'\left(a+\frac{n-1}{n}\omega\right) < \frac{1}{N}$$

in which we will be permitted to take the left-hand sides in their actual, not merely absolute values, since if the former is different from the latter, it satisfies the relationship all the more certainly as a negative [value]. But on this assumption we obtain, by addition, if we omit the terms which cancel out as equal and opposite, and notice that the number of these relationships is n:

$$\frac{F(a+n\omega) - Fa}{\omega} - \left[F'a + F'(a+\omega) + F'(a+2\omega) + \cdots + F'\left(a+\frac{n-1}{n}\omega\right)\right] < \frac{n}{N}.$$

Or if we multiply by $\omega = \frac{h}{n}$ and substitute $n\omega = h$:

$$[F(a+h) - Fa] - \frac{h}{n}\left[F'a + F'\left(a + \frac{h}{n}\right) + F'\left(a + \frac{2h}{n}\right) + \cdots + F'\left(a + \frac{n-1}{n}h\right)\right] < \frac{h}{N}.$$

Now in this expression the number N can also be increased indefinitely with the same a and h by merely increasing n, because the increase of n allows Δx, and consequently also the fraction $\frac{1}{N}$, to decrease indefinitely. Then from this it must be that:

$$F(a+h) = Fa + \frac{h}{n}\left[F'a + F'\left(a + \frac{h}{n}\right) + F'\left(a + \frac{2h}{n}\right) + \cdots + F'\left(a + \frac{n-1}{n}h\right)\right] + \Omega.$$

Example. In this theorem the condition that the function Fx must belong to the class of those which are determined by an identical law for all values of their variable is neither explicitly nor implicitly assumed. So we could take for all x which are <2, $Fx = x^2$, but for $x = 2$ and all greater values, $Fx = x^3 - 8x + 12$. Furthermore, let $a = 1$ and $h = 3$, then all the conditions which the theorem requires are correct. The function Fx has a derivative for all values of x from $a = 1$ to $a + h = 4$ inclusive, and in both directions, and this derivative itself is continuous. Namely for all $x < 2$, $F'x = 2x$ with a positive as well as a negative Δx, for $x > 2$, $F'x = 3x^2 - 8$, likewise with a positive as well as a negative Δx. But for $x = 2$, for a positive Δx,

$$\frac{\Delta Fx}{\Delta x} = \frac{[(2+\Delta x)^3 - 8(2+\Delta x) + 12] - [2^3 - 8.2 + 12]}{\Delta x}$$
$$= 4 + 6\Delta x + \Delta x^3.$$

Therefore the derivative is $=4$. And similarly for a negative Δx,

$$\frac{\Delta Fx}{\Delta x} = \frac{(2-\Delta x)^2 - [2^3 - 8.2 + 12]}{\Delta x} = 4 + \Delta x.$$

Therefore the derivative is again $=4$. Now since also $2x$ and $3x^2 - 8$ for $x = 2$ become the value 4, then $F'x$ is obviously continuous. Therefore if we take $n = 10$, then we obtain

$$44 = 1 + \frac{3}{10}\left[2.\frac{10+13+16+19}{10} + 3.\frac{22^2+25^2+28^2+31^2+34^2+37^2}{100} - 6.8\right] \mp \Omega$$

$$= 47 \cdot 891 + \Omega.$$[w]

§ 146

Theorem. Suppose a function Fx has a derivative in both directions for all values of x within a and $a+h$, but for $x=a$ at least a derivative in the same direction as h, and for $x=a+h$ one in the opposite direction. Moreover, if we know that this derivative follows the law of continuity for all the values of x just mentioned, then there is always a number μ lying not outside 0 and 1, or (what amounts to the same thing) a number $a+\mu h$, lying not outside a and $a+h$, for which the equation $F(a+h) = Fa + h.F'(a+\mu h)$ applies.

Proof. By the previous §,

$$F(a+h) = Fa + \frac{h}{n}\left[F'a + F'\left(a+\frac{h}{n}\right) + F'\left(a+\frac{2h}{n}\right) + F'\left(a+\frac{3h}{n}\right) + \cdots + F'\left(a+\frac{n-1}{n}h\right)\right] + \Omega$$

in which Ω can decrease indefinitely merely by increasing n.

1. Now if (as this is not impossible) the numbers $F'a$, $F'\left(a+\frac{h}{n}\right)$, $F'\left(a+\frac{2h}{n}\right), \ldots, F'\left(a+\frac{n-1}{n}h\right)$ were all equal to one another and they remained so with every increase, however great, in the number n (perhaps because $F'x$ is not really dependent on x) then the truth of what our theorem assumes for this case would be beyond doubt. For we would have here,

$$F'a + F'\left(a+\frac{h}{n}\right) + F'\left(a+\frac{2h}{n}\right) + F'\left(a+\frac{3h}{n}\right) + \cdots + F'\left(a+\frac{n-1}{n}h\right) = nF'a.$$

Therefore $F(a+h) = Fa + h.F'a + \Omega$. Now since n does not even appear in this equation, it would have to be concluded that $\Omega = 0$. Therefore we may simply put $F(a+h) = Fa + h.F'a$, which coincides with the statement of our theorem because we may also take $\mu = 0$.

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[w] The correct answer is 38.491. Moreover, the example only states that $\Omega = 5.509$ for $n = 10$.

2. But if the numbers $F'a$, $F'\left(a+\frac{h}{n}\right)$, $F'\left(a+\frac{2h}{n}\right)$, ... are unequal, then because the function $F'x$ is to be continuous for all values of x lying not outside a and $a+h$, by §60 there must be a value $x = p$, likewise lying not outside a and $a + h$, for which $F'x$ becomes the greatest, and another value $= q$ for which $F'x$ becomes the smallest, in the sense that among all the values of $F'x$, from $F'a$ to $F'(a + h)$ inclusive there is none greater than $F'p$ and none smaller than $F'q$. On this assumption the double relationship certainly holds for every arbitrary value of n:

$$F'p > \frac{1}{n}\left[F'a + F'\left(a+\frac{h}{n}\right) + F'\left(a+\frac{2h}{n}\right) + F'\left(a+\frac{3h}{n}\right) + \cdots + F'\left(a+\frac{n-1}{n}h\right)\right] > F'q.$$

Therefore if h is positive then by multiplication by h and the addition of Fa and Ω, whereby the middle term of this double relationship becomes $F(a + h)$, we have (according to *RZ* 7, §§67, 85)

$$Fa + h.F'p + \Omega > F(a+h) > Fa + h.F'q + \Omega.$$

But if h is negative, then we have (*RZ* 5, §16)

$$Fa + h.F'p + \Omega < F(a+h) < Fa + h.F'q + \Omega.$$

Since however, Ω can decrease indefinitely, by *RZ* 7, §57, it must also be in the first case: $Fa + h.F'p > F(a + h) > Fa + h.F'q$, but in the second case: $Fa + h.F'p < F(a + h) < Fa + h.F'q$. Therefore in general

$$F'p \gtrless \frac{F(a+h) - Fa}{h} \lessgtr F'q.$$

From this it follows, by §65, that there must be some value of x lying within p and q, consequently also not outside a and $a+h$, which can therefore be reasonably represented by $a + \mu h$, for which the equation $F'(a + \mu h) = \frac{F(a+h)-Fa}{h}$ holds, or (what amounts to the same), the equation: $F(a + h) = Fa + h.F'(a + \mu h)$.

Example. If we take $Fx = x^2$ for all values of $x < 8$, but for $x = 8$ and all greater values, $Fx = 2x^2 - 16x + 64$, and furthermore, if we put $a = 6$, $h = 4$ then the condition of the theorem applies that Fx has a derivative in both directions for all values of x from $a = 6$ to $a+h = 10$ inclusive which, moreover, follows the law of continuity. Because for all values of $x < 8$, $F'x = 2x$, but for $x = 8$ and all greater values, $F'x = 4x - 16$, while for $x = 8$ both expressions change into one and the same value 16. There must therefore be a value for μ lying not outside 0 and 1, which satisfies the equation $F(10) = F(6) + 4.F'(6 + 4\mu)$. And so it is, since $F(10) = 104$, $F(6) = 36$, then $104 = 36 + 4.F'(6 + 4\mu)$, i.e. $F'(6 + 4\mu) = 17$. Now if we took $\mu < \frac{1}{2}$ then $F'(6 + 4\mu)$ would still be of the form $2x$, and there could of course be no value given for μ which would make $2(6 + 4\mu) = 17$. But if we take $\mu \overline{\gtrless} \frac{1}{2}$ then $F'(6 + 4\mu)$ must be of the form $4x - 16$, and we obtain for the determination of μ the equation $4(6+4\mu)-16=17$ which is satisfied by the value $\mu=\frac{9}{16}$.

§ 147

Corollary. If we omit one of the two conditions, either that the function Fx has a derivative in both directions for all values of x lying within a and b, and for the value $x=a$, at least a derivative in the same direction as h, but for $x=a+h$ one in the opposite direction, or the other that this derivative obeys the law of continuity for all the values mentioned, then the necessity of the statement ceases to apply. If, for example, we put $Fx=x^2$ for all values of $x<1$ and for $x=1$ itself, but for all greater values $Fx=4x-x^2+6$, then Fx would have a derivative for every value of x: for all <1, the double-sided $2x$, for all >1 the double-sided $4-2x$, but for $x=1$ only a derivative for a negative Δx, $=2x$. This derivative itself would follow the law of continuity for all values of x because for $x=2$ the value $2x$ coincides with that of $4-2x$, nevertheless the statement of the theorem does not hold here. For if, for example, we take $a=0$, $h=2$, there is no value for μ lying within 0 and 1 which satisfies the equation $F(2)=F(0)+2.F'(2\mu)$. That is, since $F(2)=10$, $F(0)=0$ then it must be that $F'(2\mu)=5$. Now if we take $\mu \overline{\overline{<}} \frac{1}{2}$ then $F(2\mu)$ is of the form $2x$ and the greatest value of it $=2.\frac{1}{2}=1$. But if we take $\mu>\frac{1}{2}$, then $F'(2\mu)$ is of the form $4-2x$ and the greatest value $=4$. Therefore in no case $F(2\mu)=5$. The second condition, namely the continuity of Fx, is just as essential. For if, for example, we take for all $x \overline{\overline{<}} 8$, $Fx=x^2$, but for all greater values $Fx=2x^2-64$, then Fx does have a derivative in both directions for all values of x, but this derivative itself does not vary according to the law of continuity, because for all x of the form $8-\omega$ it is <16, for all x of the form $8+\omega$ it is >32. Now if we were to take $a=6$, $h=4$ then in fact a value of μ would be found which would make $F(10)=F(6)+4.F'(6+4\mu)$. For it would have to be that $F'(6+4\mu)=25$. But if we take $\mu \overline{\overline{<}} \frac{1}{2}$ then $F'(6+4\mu)$ is of the form $2x$ therefore the greatest value $=16<25$, but if we take $\mu>\frac{1}{2}$ then $F'(6+4\mu)$ is of the form $4x$ therefore the smallest value is $32>25$.

§ 148

Theorem. If a function Fx has a derivative in both directions for all values of x lying within a and $a+h$ which also follows the law of continuity for the values of x mentioned, and furthermore, if the function Fx is also continuous for both the values $x=a$ and $x=a+h$, for the first one at least in the same sense as h, and for the second at least in the opposite sense, then the equation of the previous theorem still also holds for this case: $F(a+h)=Fa+h.F'(a+\mu h)$.

Proof. Let α and $\alpha+i$ be a pair of numbers lying within a and $a+h$, then Fx has a derivative, and indeed in both directions, not only for all values of x lying within α and $\alpha+i$ but also for $x=\alpha$ and for $x=\alpha+i$. Since the previous theorem certainly applies here, it must therefore be that $F(\alpha+i)=F\alpha+i.F'(\alpha+\mu i)$, and this remains so however close we move the value α towards a, and that of i towards h, providing we always do this so that α and $\alpha+i$ lie within a and $a+h$. If we let the difference $\alpha - a$ decrease in its absolute value indefinitely, then, because the given function Fx is continuous for the value $x = a$ and with respect to an increase of the same sign as h, the difference $F\alpha - Fa$ in its absolute value must also decrease indefinitely. The same must also hold of the difference $F'(\alpha + \mu i) - F'(a + \mu i)$. Therefore we also write $F(a+i) = Fa + i.F'(a+\mu i) + \Omega$. If the value $a+i$ is to lie within a and $a+h$, as we have assumed, then i, in its absolute value, must be $< h$. Therefore every number which can be represented by $\alpha + \mu i$, can all the more certainly be represented by $a + \mu h$ if μ is to denote nothing but a certain number lying not outside 0 and 1. Therefore without disturbing our equation we can also put $F'(a+\mu h)$, in place of $F'(a+\mu i)$, and hence obtain $F(a+i) = Fa + i.F'(a + \mu h) + \Omega$. But because Fx is also to be continuous for the value $x = a + h$, and indeed with respect to an increase of the opposite sign from h, if we move the number i indefinitely closer to the value h then the difference $F(a+h) - F(a+i)$ must, in its absolute value, also decrease indefinitely. It must therefore also be that, $F(a+h) = Fa + i.F'(a+\mu h) + \Omega^1$. Finally, since with the indefinite approach of i to h also the value of $i.F'(a+\mu h)$ approaches the value $h.F'(a+\mu h)$ indefinitely, we may certainly also write $F(a+h) = Fa + h.F'(a+\mu h) + \Omega^2$. Then if we determine the value of the number μ in such a way that the value of the expression $Fa = h.F'(a+\mu h)$ comes as close to the value $F(a+h)$ as possible, as must be permissible by the assumption of a value for μ which does not lie outside the limits 0 and 1, then all the expressions, $F(a+h)$, Fa and $h.F'(a+\mu h)$ are completely independent of i, and therefore Ω, since it is $= F(a+h) - Fa - h.F'(a+\mu h)$, must also be a value completely independent of i. Since in every case Ω can only have a value of such a kind that it becomes smaller than every given number with the indefinite approach of i towards h, it follows that here this value can be none other than zero. Accordingly $F(a+h) = Fa + h.F'(a+\mu h)$.

Example. If $Fx = y$ is such a function of x that we have (as in §131) the equation $y^2 = 1 - x^2$, then $\frac{\Delta y}{\Delta x} = -\frac{2x+\Delta x}{2y+\Delta y}$, which provided y is not zero approaches the value $-\frac{x}{y}$ indefinitely, which we can therefore regard as the derivative of Fx. But for $x = 1$, $y = 0$ and $-\frac{2x+\Delta x}{2y+\Delta y}$ increases indefinitely. Therefore for this value Fx has no derivative. Nevertheless the formula $F(a+h) = Fa + h.F'(a+\mu h)$ can also be applied to this case, for example, if we assume $a = \frac{3}{5}$ and $h = \frac{2}{5}$, therefore $a + h = 1$. Then here $y^2 = 1 - x^2 = 1 - \frac{9}{25} = \frac{16}{25}$, therefore $Fa = y = \frac{4}{5}$ and $F(a+h) = F(1) = 0$. Therefore it is sufficient, if a value for μ can be specified which makes

$$F'(a+\mu h) = -\frac{Fa}{h} = -\frac{4}{5} : \frac{2}{5} = \frac{4}{2} = 2,$$

or what amounts to the same,

$$(F'(a+\mu h))^2 = 4,$$

i.e.

$$\frac{x^2}{1-x^2} = \frac{\left(\frac{3}{5}+\frac{2\mu}{5}\right)^2}{1-\left(\frac{3}{5}+\frac{2\mu}{5}\right)^2} = 4$$

or

$$\frac{9+12\mu+4\mu^2}{16-12\mu-4\mu^2} = 4.$$

This is again possible. For $\mu = 0$, $\frac{9+12\mu+4\mu^2}{16-12\mu-4\mu^2}$ becomes < 4, but for $\mu = 1$ it becomes infinitely great. Therefore there is certainly a value for μ lying within 0 and 1 which satisfies the above equation.

§ 149

Theorem. Suppose a function Fx is continuous in both directions for all values of its variable lying within a and $a+h$, but for $x = a$ at least in the same sense as h, and for $x = a+h$ in the opposite direction. Furthermore, suppose that with at most the exception of certain *isolated* values of x, the multitude of which may be infinite, providing that to each of the values there is only one *next* to it, the function Fx also has a *derivative* $F'x$ in both directions, which is again continuous within every two of the values of x just mentioned. Then I claim that the equation $F(a+h) = Fa + h.M$ can be formed if one imagines M as a certain number which lies between the greatest and the smallest of the values which the derivative $F'x$ takes within a and $a+h$.

Proof. If we denote those values of x for which the derivative $F'x$ either does not exist at all, or breaks the law of continuity, in order from smaller to greater or conversely, by $a+h_1$, $a+h_1+h_2$, $a+h_1+h_2+h_3$, etc. then we must indeed assume the multitude of terms in this series can even be infinite. But we know that for all values of x which lie between two terms of the series which immediately follow one another, not only does a derivative $F'x$ exist in both directions but it is also continuous. Hence the following equations (the multitude of which can possibly be infinite) arise according to the previous §:

$$F(a+h_1) = Fa + h_1.F'(a+\mu_1 h_1)$$

$$F(a+h_1+h_2) = F(a+h_1) + h_2.F'(a+h_1+\mu_2 h_2)$$

$$F(a+h_1+h_2+h_3) = F(a+h_1+h_2) + h_3.F'(a+h_1+h_2+\mu_3 h_3)$$

etc. The rth of these equations is:

$$F(a+h_1+h_2+h_3+\cdots+h_r) = F(a+h_1+h_2+\cdots+h_{r-1}) + h_r.F'(a+h_1+h_2+\cdots+h_{r-1}+\mu_r h_r)$$

and the addition of all the equations gives, if we omit the equal terms from both sides,

$$\begin{aligned} F(a+h_1+h_2+h_3+\cdots+h_r) = Fa + h_1.F'(a+\mu_1 h_1) \\ + h_2.F'(a+h_1+\mu_2 h_2) \\ + h_3.F'(a+h_1+h_2+\mu_3 h_3) \\ +\cdots \\ + h_r.F'(a+h_1+h_2 \\ +\cdots+h_{r-1}+\mu_r h_r). \end{aligned}$$

But because the numbers $h_1, h_2, h_3, \ldots, h_r$ are all of the same sign, we know from §146 that the sum of the products which come after Fa can be put equal to a single product, one factor of which is the sum $(h_1+h_2+h_3+\cdots+h_r)$ the other is some intermediate value lying between the greatest and smallest values of $F(a+\mu_1 h_1)$, $F(a+h_1+\mu_2 h_2)$, etc. This intermediate value is also intermediate for all the values which $F'x$ takes from $x=a$ to $x=a+h$, therefore [it is] that which we have denoted in the theorem by M. Consequently, we may write the equation $F(a+h_1+h_2+h_3+\cdots+h_r) = Fa+(h_1+h_2+h_3+\cdots+h_r)M$. But with the indefinite increase of r the sum $h_1+h_2+h_3+\cdots+h_r$ changes into the value h. Accordingly $F(a+h)=Fa+h.M$.

Example. If we put the following [values] of Fx corresponding to the following values of x:

x		Fx
from 0	to 1	x
1	2	$2x-1$
2	3	$4x-5$
3	4	$8x-17$
4	5	$16x-49$
	etc.	

then the function Fx is continuous from $a=0$ to $a+h=5$ and satisfies all conditions which the theorem requires. But all the values which the derivative $F'x$ takes within a and $a+h$ are 1, 2, 4, 8, 16. Therefore the smallest value is 1 and the greatest is 16, and thus there must be a value for M lying between 1 and 16 which satisfies the equation $F(a+h)=Fa+h.M$, as there actually is, since $F(a+h)=31$ and $Fa=0$, but $h=5$ therefore $M=\frac{31}{5}=6\frac{1}{5}$.

§ 150

Theorem. If a function of two variables $F(x,y)$ has the property that for every value of the one variable x lying within a and b it becomes, in its absolute value, smaller than every given fraction *merely by the reduction of the other variable* y, and remains so if y is reduced ever further, and this function has, in respect of x (i.e. if x alone is considered as the variable), a derivative in both directions for every value of x lying within a and b, then I claim that also this derivative $\frac{dF(x,y)}{dx}$ possesses the property just described, of decreasing indefinitely for every value of x lying within a and b, simply by the indefinite decrease of y.

Proof. 1. First of all, the example of the following series shows us that there are functions of the kind which we are assuming here,

$$\frac{1}{1-x} - 1 - x - x^2 - x^3 - \cdots - x^r,$$

whose value $= \frac{x^{r+1}}{1-x}$ can conveniently be considered as a function of x and r, or also of x and $\frac{1}{r} = y$. Now as long as x is within the limits -1 and $+1$, then we know that the value of this series, simply by the increase of r, or what is the same thing, simply by the *reduction* of y, becomes smaller than every given fraction $\frac{1}{N}$ and remains so if we take r ever greater, i.e. y ever smaller.

2. Now if x is some value lying within a and b then the equation

$$\frac{F(x+\omega,y)-F(x,y)}{\omega} = \frac{dF(x,y)}{dx} + \Omega$$

holds in which *for the same x and y*, Ω decreases indefinitely with ω. Nevertheless in order to make $\Omega < \frac{1}{N}$ with a changed value of x or y, perhaps another and still smaller ω could be necessary. Therefore although with the same ω, i.e. with the same denominator of the fraction $\frac{F(x+\omega,y)-F(x,y)}{\omega}$, the numerator and hence also the whole value of the fraction, can be made as small as we please simply by the reduction of y, then we may only conclude from this that the algebraic sum $\frac{dF(x,y)}{dx} + \Omega$ decreases indefinitely, but not that the term $\frac{dF(x,y)}{dx}$ also decreases indefinitely on its own. Since it could even be that by the reduction of y, Ω increases and for different signs of both numbers $\frac{dF(x,y)}{dx}$ and Ω, only their *difference* decreases indefinitely. However, by §74, there must be an ω small enough to be able to claim that the function $\frac{dF(x,y)}{dx}$ obeys the law of continuity for all values from x to $x+\omega$. And from this it follows, by §146, that there would have to be a number μ lying not outside zero and 1, which produces the equation

$$F(x+\omega,y) = F(x,y) + \omega.\frac{dF(x+\mu\omega,y)}{dx}$$

or

$$\frac{F(x+\omega,y)-F(x,y)}{\omega} = \frac{dF(x+\mu\omega,y)}{dx}.$$

Now in this equation the value of μ may always be independent of the number y, which teaches us that the value of $\frac{dF(x+\mu\omega,y)}{dx}$, for the same x and ω, could decrease indefinitely simply by the reduction of y because also $F(x,y)$ and $F(x+\omega,y)$, therefore the numerator of the fraction $\frac{F(x+\omega,y)-F(x,y)}{\omega}$ with unchanged denominator, and thus the value of the whole fraction itself, can decrease indefinitely simply by the reduction of y. But since for every value of y or μ, it must be that

$$\frac{\mathrm{d}F(x+\mu\omega, y)}{\mathrm{d}x} = \frac{\mathrm{d}F(x, y)}{\mathrm{d}x} + \Omega,$$

because otherwise this function would not vary according to the law of continuity, then it must also hold of $\frac{\mathrm{d}F(x,y)}{\mathrm{d}x}$ that it can be reduced indefinitely by the mere reduction of y.

§ 151

Corollary 1. Also if the function $F(x, y)$ can be decreased indefinitely not by decreasing y, but by the unbounded *increase* of y, its derivative with respect to x, i.e. the function $\frac{\mathrm{d}F(x,y)}{\mathrm{d}x}$, must have this property in common with it. For if y increases indefinitely then, on the other hand, $\frac{1}{y}$ is a number which decreases indefinitely, and $F(x, y)$ is a function of x and $\frac{1}{y}$ to which the theorem can be applied.

§ 152

Corollary 2. Therefore if an equation of the form $F(x, y) = \Phi(x, y) + \Omega$ holds for all values of x lying within a and b in which by merely decreasing or increasing y indefinitely the value of Ω can be decreased indefinitely, and if both functions, $F(x, y)$ as well as $\Phi(x, y)$, have a derivative in both directions with respect to all values of x lying within a and b, then within these same limits for x the equation

$$\frac{\mathrm{d}F(x, y)}{\mathrm{d}x} = \frac{\mathrm{d}\Phi(x, y)}{\mathrm{d}x} + \overset{1}{\Omega}$$

holds, and indeed in such a way that merely by decreasing or increasing y indefinitely the value of $\overset{1}{\Omega}$ can be decreased indefinitely. For this assumption represents $F(x, y) - \Phi(x, y)$ as a function of x and y of which everything holds which is stipulated in our theorem. Therefore also the derivative of this function with respect to x, must be

$$\frac{\mathrm{d}F(x, y)}{\mathrm{d}x} - \frac{\mathrm{d}\Phi(x, y)}{\mathrm{d}x} + \overset{2}{\Omega}.$$

§ 153

Theorem. Every function of the form ax^n in which a denotes an arbitrary measurable number completely independent of x, but n denotes an actual number, has a derivative of the form nax^{n-1} for every measurable value of its variable x.[x]

Proof. If $n = 1$ then as we already know, we have $\frac{\Delta ax}{\Delta x} = a$. Therefore a itself is the required derivative which also coincides with the formula nax^{n-1}. But if $n > 1$ then we have,

$$\frac{\Delta ax^n}{\Delta x} = \frac{a(x+\Delta x)^n - ax^n}{\Delta x}$$

[x] The theorem is extended to negative integer exponents in F+ §11.

which by §42 lies between the limits nax^{n-1} and $na(x + \Delta x)^{n-1}$. But since the value of $na(x + \Delta x)^{n-1}$ indefinitely approaches the value nax^{n-1} if Δx decreases indefinitely (§42), there is no doubt that nax^{n-1} is the required derivative.

§ 154

Theorem. Every algebraic sum of two functions Fx and Φx of one and the same variable x which both have a derivative for the very same value of this variable, and both with respect to a positive increase of it or both with respect to a negative increase of it, also has a derivative itself for this value of the variable and with respect to just this (positive or negative) increase of it. In fact this derivative is the sum of the derivatives of the two addends, namely $F'x$ and $\Phi'x$.

Proof. If we put $Fx + \Phi x = W$ then

$$\frac{\Delta W}{\Delta x} = \frac{\Delta Fx}{\Delta x} + \frac{\Delta \Phi x}{\Delta x} = \frac{F(x + \Delta x) - Fx}{\Delta x} + \frac{\Phi(x + \Delta x) - \Phi x}{\Delta x}.$$

Now if Fx has a derivative for the value of x just assumed, and for the same sign that Δx has, then

$$\frac{F(x + \Delta x) - Fx}{\Delta x} = F'x + \overset{1}{\Omega},$$

and on a similar assumption about Φx, also

$$\frac{\Phi(x + \Delta x) - \Phi x}{\Delta x} = \Phi'x + \overset{2}{\Omega}.$$

Hence also

$$\frac{\Delta W}{\Delta x} = F'x + \overset{1}{\Omega} + \Phi'x + \overset{2}{\Omega} = F'x + \Phi'x + \overset{3}{\Omega},$$

from which it is clear that $F'x + \Phi'x$ is the required derivative of $W = Fx + \Phi x$.

§ 155

Corollary. It is self-evident that these arguments can be extended to any arbitrary, even infinite, multitude of summands, and therefore the derivative of $Fx + \Phi x + \Psi x + \cdots$ *in inf.* $= F'x + \Phi'x + \Psi'x + \cdots$ *in inf.*[y]

§ 156

Theorem. A product $a.Fx$ of two factors a and Fx of which the one is a and is a measurable number completely independent of x, the other is an arbitrary function of x which has a derivative for the fundamental value of x and with respect to a positive or negative increase, also has a derivative itself for just this value of x and with respect to the same (positive or negative) increase, in fact this derivative is $a.F'x$, a product of the constant factor a and the derivative of the other factor.

[y] See F+ §2.

Proof. If we write $a.Fx = W$, then

$$\frac{\Delta W}{\Delta x} = \frac{a.F(x+\Delta x) - a.Fx}{\Delta x}.$$

Therefore if the value of x assumed here is the same as that for which Fx has a derivative, and we take for Δx the same sign for which Fx has a derivative, then

$$\frac{F(x+\Delta x) - Fx}{\Delta x} = F'x + \Omega.$$

Therefore

$$\frac{\Delta W}{\Delta x} = a(F'x + \Omega) = a.F'x + \Omega^{1}.$$

Therefore $a.F'x$ is the derivative of W or $a.Fx$.

§ 157

Corollary. In a similar way also the derivative of a *quotient* $\frac{Fx}{a}$, whose divisor is a measurable number different from zero, is $= \frac{F'x}{a}$. For the previous a can, by the assumption just made, also denote the factor $\frac{1}{a}$.

§ 158

Theorem. Also a product of two variable factors $Fx.\Phi x$, each of which has a derivative for one and the same value of x, and with respect to the same positive or negative increase, likewise itself has a derivative in the same respect. Indeed this derivative is $Fx.\Phi' x + F'x.\Phi x$, i.e. we obtain it if we multiply each factor by the derivative of the other and add these products.

Proof. If we write $Fx.\Phi x = W$ then

$$\frac{\Delta W}{\Delta x} = \frac{F(x+\Delta x).\Phi(x+\Delta x) - Fx.\Phi x}{\Delta x}.$$

But if Fx has a derivative $= F'x$, and Φx a derivative $= \Phi' x$, then

$$\frac{F(x+\Delta x) - Fx}{\Delta x} = F'x + \overset{1}{\Omega} \quad \text{and} \quad \frac{\Phi(x+\Delta x) - \Phi x}{\Delta x} = \Phi' x + \overset{2}{\Omega},$$

therefore $F(x+\Delta x) = Fx + \Delta x.F'x + \Delta x.\overset{1}{\Omega}$, and $\Phi(x+\Delta x) = \Phi x + \Delta x.\Phi' x + \Delta x.\overset{2}{\Omega}$. Therefore by substitution,

$$\begin{aligned}\frac{\Delta W}{\Delta x} &= \frac{(Fx + \Delta x.F'x + \Delta x.\overset{1}{\Omega})(\Phi x + \Delta x.\Phi' x + \Delta x.\overset{2}{\Omega}) - Fx.\Phi x}{\Delta x} \\ &= F'x.\Phi x + Fx.\Phi' x + Fx.\overset{2}{\Omega} + \Phi x.\overset{1}{\Omega} \\ &\qquad + \Delta x[F'x.\Phi' x + F'x.\overset{2}{\Omega} + \Phi' x.\overset{1}{\Omega} + \overset{1}{\Omega}\overset{2}{\Omega}].\end{aligned}$$

Now since the terms which contain $\overset{1}{\Omega}$, $\overset{2}{\Omega}$ and Δx as factors decrease indefinitely if Δx decreases indefinitely, then it is clear that $\frac{\Delta W}{\Delta x} = Fx.\Phi'x + F'x.\Phi x + \overset{3}{\Omega}$ and thus $Fx.\Phi'x + F'x.\Phi x$ is the required derivative of W or $Fx.\Phi x$.

§ 159

Corollary. Therefore a product of 3, 4, ... and every arbitrary but always finite number of variable factors must also have a derivative for a certain value of the variable and with respect to a certain positive or negative increase if all these factors have their derivatives in this same direction. Thus, for example, the derivative of a product of three factors $Fx.\Phi x.\Psi x = F'x.\Phi x.\Psi x + Fx.\Phi'x.\Psi x + Fx.\Phi x.\Psi'x$. And so on.

§ 160

Theorem. A function $\frac{Fx}{\Phi x}$, which is a quotient of two others Fx and Φx, has a derivative for every value of x and with respect to every positive or negative increase if also both functions have their derivatives in this respect, and furthermore the value of Φx is not zero, actually this derivative $= \frac{F'x.\Phi x - Fx.\Phi'x}{(\Phi x)^2}$. That is, we obtain it if we multiply the derivative of the numerator by the denominator, and the derivative of the denominator by the numerator, subtract both products from one another and divide by the square of the denominator.

Proof. If we write $\frac{Fx}{\Phi x} = W$, then

$$\Delta W = \frac{F(x+\Delta x)}{\Phi(x+\Delta x)} - \frac{Fx}{\Phi x} = \frac{F(x+\Delta x).\Phi x - \Phi(x+\Delta x).Fx}{\Phi(x+\Delta x).\Phi x}.$$

Therefore

$$\frac{\Delta W}{\Delta x} = \frac{F(x+\Delta x).\Phi x - \Phi(x+\Delta x).Fx}{\Delta x.\Phi(x+\Delta x).\Phi x}.$$

If we put in here $F(x+\Delta x) = Fx + \Delta x.F'x + \Delta x.\overset{1}{\Omega}$, and $\Phi(x+\Delta x) = \Phi x + \Delta x.\Phi'x + \Delta x.\overset{2}{\Omega}$, and remove the common factor Δx in the numerator and the denominator, and notice that the terms which contain $\overset{1}{\Omega}$, $\overset{2}{\Omega}$ or Δx as factors decrease indefinitely, then as long as Φx is not exactly zero, we have

$$\frac{\Delta W}{\Delta x} = \frac{F'x.\Phi x - Fx.\Phi'x}{(\Phi x)^2} + \Omega.$$

Therefore $\frac{F'x.\Phi x - Fx.\Phi'x}{(\Phi x)^2}$ is the required derivative of the quotient $\frac{Fx}{\Phi x}$.

§ 161

Theorem. Every *integral* or *fractional, rational* function, for all values of its variable, with the exception with the latter of those for which its denominator becomes zero, is continuous in the second degree and its derivative is again only an integral or fractional, rational function which we can find according to the suggestion indicated in the previous theorem.

Proof. 1. Every integral, rational function is of the form $a + bx + cx^2 + dx^3 + ex^4 + \cdots + lx^m$ and thus by the results of §§ 153, 155, 156 its derivative for every arbitrary measurable value of x is $= b + 2cx + 3dx^2 + 4ex^3 + \cdots + mlx^{m-1}$ which is obviously again an integral, rational function.
2. Every fractional, rational function is of the form

$$\frac{a + bx + cx^2 + dx^3 + \cdots + lx^m}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots + \lambda x^\mu}.$$

Therefore its derivative for every value of x which does not make the denominator $\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots + \lambda x^\mu$ zero, is by §160,

$$\frac{\begin{array}{l}[(b + 2cx + 3dx^2 + \cdots + mlx^{m-1})(\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots) \\ \quad -(\beta + 2\gamma x + 3\delta x^2 + \cdots + \mu\lambda x^{\mu-1})(a + bx + cx^2 + \cdots + lx^m)]\end{array}}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots + \lambda x^\mu)^2}$$

which is again a rational function.

Example. 1. The first derived function of $8x^4 + 3x^3 - 5x$ is therefore $32x^3 + 9x^2 - 5$, the second [derivative], or the derivative of the latter, is $96x^2 + 18x$, the third [derivative], or the derivative of the one just found, $192x + 18$, the fourth 192, all subsequent ones $= 0$.
2. The derivative of the product $(x^3 + a)(3x^2 + 6)$ is, according to §158, $3x^2(3x^2 + 6) + (x^3 + a)6x$.
3. The derivative of the function $\frac{1-2x^2}{1+3x-5x^2}$, for every value of x which does not make the denominator $1 + 3x - 5x^2$ zero, is by §160

$$\frac{-4x(1 + 3x - 5x^2) - (1 - 2x^2)(3 - 10x)}{(1 + 3x - 5x^2)^2} = \frac{-3 + 12x - 32x^2 + 20x^3}{(1 + 3x - 5x^2)^2}.$$

The second derivative

$$[(1 + 3x - 5x^2)[12 - 64x + 60x^2] \\ = \frac{-2[-3 + 12x - 32x^2 + 20x^3][1 + 3x - 5x^2](3 - 10x)]}{(1 + 3x - 5x^2)^4}.$$

And so on.

§ 162

Corollary 1. Every integral, rational function of nth degree has a derivative which is an integral, rational function of degree $(n - 1)$. The second derivative is therefore only of degree $(n - 2)$. The $(n - 1)$th [derivative] is of the first degree, the nth [derivative] is already not a variable number but simply a constant number (sometimes perhaps even zero); the $(n + 1)$th [derivatives] and all successive ones are always and everywhere zero.

§ 163

Corollary 2. It is not like this with the *fractional* rational functions for which the number of their derivatives can go on indefinitely. Thus the function $\frac{1}{x}$ has for first derivative $-\frac{1}{x^2}$, for the second, $+\frac{2}{x^3}$, for the third $-\frac{2.3}{x^4}$, for the fourth $+\frac{2.3.4}{x^5}$ etc., from which the general form may soon be seen, namely the nth derivative is $\pm\frac{2.3...n}{x^{n+1}}$ where the sign $+$ holds for an even n, and the sign $-$ holds for an odd n.

§ 164

Theorem. If a function Fy has a derivative for the definite value of its variable which we denote just by y, either only with respect to a positive increase, or only with respect to a negative increase, or in both respects at once, and we now consider this variable y itself as a function of another freely variable number x for which it happens that this function $y = fx$, for the definite value of x which makes $fx = y$ also has a derivative, and does so at least with respect to such an increase of x with which Δy gets the same sign in respect of which also Fy has its derivative, then I claim that the function of x, $F(fx)$ which arises if we put fx in the place of y in Fy, likewise has a derivative for every value of x which makes $fx = y$, and with respect to the same positive or negative nature of the increase which has already been mentioned, and this derivative is $= F'(fx).f'x$. That is, we obtain the derivative of a function of the variable y, which we consider as itself a function of x, if we multiply the derivative of the first in respect of y by the derivative of y itself with respect to x.

Proof. Because Fy has a derivative with respect to y, then it must be that at least for a certain sign of Δy,

$$\frac{F(y+\Delta y) - Fy}{\Delta y} = F'y + \overset{1}{\Omega}.$$

If we put $y = fx$ then $y + \Delta y = f(x+\Delta x)$, and $\Delta y = f(x+\Delta x) - fx$, therefore

$$\frac{F(f(x+\Delta x)) - F(fx)}{f(x+\Delta x) - fx} = F'(fx) + \overset{1}{\Omega}$$

and $F(f(x+\Delta x)) - F(fx) = [f(x+\Delta x) - fx][F'(fx) + \overset{1}{\Omega}]$. Now if fx also has a derivative and indeed for the same sign of Δx which is needed to produce the Δy appearing above, then we have

$$\frac{f(x+\Delta x) - fx}{\Delta x} = f'x + \overset{2}{\Omega}.$$

This gives by substitution

$$\frac{F(f(x+\Delta x)) - F(fx)}{\Delta x} = [f'x + \overset{2}{\Omega}][F'(fx) + \overset{1}{\Omega}]$$
$$= F'(fx).f'x + \overset{3}{\Omega}$$

from which it is clear that $F'(fx).f'x$ is the derivative of $F(fx)$.

Example. The derivative of the function $(a+bx+cx^2+\cdots+lx^m)^n$ would therefore be, if we consider $a+bx+cx^2+\cdots+lx^m$ as y, $n(a+bx+cx^2+\cdots+lx^m)^{n-1}(b+2cx+\cdots+mlx^{m-1})$. For example, the derivative of $(4x^2+5x^4)^3 = 3(4x^2+5x^4)^2(8x+20x^3)$, etc.

§ 165

Note. The restriction made in the theorem concerning the positive or negative nature of the increase for which the derivatives of Fy and fx occur is easily shown by examples not to be superfluous, although it is usually omitted. Let us take Fy so that for a definite value of y it only has a derivative with respect to a negative Δy, if Fy were $= (1-y)^{\frac{5}{2}}$, [this is the case] for $y = 1$, furthermore, if we put y to be fx so that the value of x which makes $fx = y$, only has a derivative for a positive Δx and that from this only a positive Δy arises, as happens if $y = 4x^2 \pm \left(x^2 - \frac{1}{4}\right)^{\frac{3}{2}}$. Then $F(fx)$, for the value just determined which makes $fx = 1$, here $x = \frac{1}{2}$, will have no derivative at all. On this assumption $F(fx) = \left(1 - 4x^2 \mp \left(x^2 - \frac{1}{4}\right)^{\frac{1}{2}}\right)^{\frac{5}{2}}$ which becomes zero for $x = \frac{1}{2}$, but is imaginary for $x = \frac{1}{2} \pm \Delta x$, where the upper or lower sign for Δx can hold. Whoever is used to assuming the existence of a derivative everywhere where the usual rule of differentiation does not produce any imaginary or infinitely large expression would believe a derivative could be found here, for he would obtain $\frac{5}{2}\left(1 - 4x^2 \mp \left(x^2 - \frac{1}{4}\right)^{\frac{3}{2}}\right)^{\frac{5}{2}}\left(-8 \mp 3\left(x^2 - \frac{1}{4}\right)^{\frac{1}{2}}\right)x$ which for $x = \frac{1}{2}$ changes into the value 0.

§ 166

Corollary 1. Since we represent the derivative of a function Fy with respect to y according to the second method of notation given in §120 by $\frac{dFy}{dy}$, and the derivative of y with respect to x by $\frac{dy}{dx}$, then the derivative of $Fy = F(fx)$, if we consider x as freely variable, can be written $\frac{dFy}{dx}\left(=\frac{dF(fx)}{dx}\right) = \frac{dFy}{dy}.\frac{dy}{dx}$.

§ 167

Corollary 2. The theorem reminds us, by its similarity, of the one in §67 where we spoke of continuity of the first degree. As it was noted there (§69), from the continuity of both of the functions Fy and fx we may conclude the continuity of $F(fx)$, but conversely from the discontinuity of one of the functions Fy, fx for a certain value of its variable we may not immediately conclude the discontinuity of $F(fx)$ for this value; the same also holds for the continuity of the second degree. Thus for example, $Fy = \frac{y^2}{a-y}$ has no derivative for the value $y = a$, but if we put $y = fx = \frac{b}{x}$, a function which has no derivative for $x = 0$, then $F(fx) = \frac{b^2}{x^2\left(a-\frac{b}{x}\right)} = \frac{b^2}{ax^2-bx}$ has no derivative for two values of x, namely for the value $x = \frac{b}{a}$, which makes $y = a$ and for the value $x = 0$, which makes $fx = \frac{b}{x}$ discontinuous. On the other hand, if as in §69 $Fy = \frac{1}{1-y}$, and $y = fx = \frac{1}{x}$, then $F(fx) = \frac{x}{x-1}$ would have no derivative only for the single value $x = 1$.

§ 168

Theorem. If the equation $y = fx$ holds for all values of x lying within a and b, but for just these values the equation $x = \phi y$ also holds, and we find that fx has a derivative $f'x$ for some value of x lying within a and b at least with respect to a positive or negative increase of x, then also the function ϕy has a derivative for the value y and at least with respect to a (positive or negative) increase of y, as long as $f'x$ is not just $= 0$. In fact the equation $\phi' y = \frac{1}{f'x}$ holds.

Proof. If fx has a derivative for the value x and with respect to a positive increase then it must be that $\frac{f(x+\Delta x)-fx}{\Delta x} = f'x + \overset{1}{\Omega}$. Because the equation $fx = y$ holds for all values of x lying within a and b, then if we take x and $x + \Delta x$ only within these limits, $f(x + \Delta x) = y + \Delta y$ and $f(x + \Delta x) - fx = \Delta y$, and because x is to be $=\phi y$ it must also be that $x + \Delta x = \phi(y + \Delta y)$ and $\Delta x = \phi(y + \Delta y) - \phi y$. The combination of these equations gives

$$\frac{\phi(y + \Delta y) - \phi y}{\Delta y} = \frac{\Delta x}{f(x + \Delta x) - fx} = \frac{1}{f'x + \overset{1}{\Omega}},$$

which as long as $f'x$ is not $= 0$ can be put $= \frac{1}{f'x} + \overset{2}{\Omega}$ (§). But from this last equation it may be seen that also ϕy has a derivative at least with respect to such a sign as Δy has, and in fact the equation $\phi' y = \frac{1}{f'x}$ holds.

Example. If $y = fx = \frac{a}{x}$ then, on the other hand, $x = \phi y = \frac{a}{y}$ and because fx has a derivative for every value of x which is not zero, namely $f'x = \frac{-a}{x^2}$, then also ϕy must have a derivative, namely $\phi' y = \frac{-a}{y^2}$ for every value of y. And the equation holds,

$$-\frac{a}{y^2} = \frac{1}{(-a : x^2)} = -\frac{x^2}{a}.$$

§ 169

Definition. In exactly the same sense as in §74 continuity of the *first* degree was spoken about for functions of *several* variables I may use this way of speaking also about continuity of the *second* degree. Therefore I say, for example, that the function $F(x, y)$ is continuous in the second degree, or has a derivative for both of its variables x and y at least for this definite value of them, and with respect to a positive or negative increase, if $F(x, y)$ has a derivative with respect to x for the definite value of x, and for every value of y which does not lie outside the limits y and $y + \Delta y$, and also has a derivative with respect to y for the definite value of y and for every x which does not lie outside the limits x and $x + \Delta x$. And so on.

§ 170

Theorem. First of all suppose a function $F(y, z, \ldots)$ of *several* variables y, z, … has a derivative with respect to the variable y for the definite value y, then with respect to the variable z for the definite value z, etc. always at least with respect to a positive or negative increase. Now we consider these variables $y, z, \ldots$ themselves as again functions of a single free variable x, that is, $y = fx$, $z = \phi x$, etc. and these functions for that value of x which makes $fx = y$, $\phi x = z$ all have their derivatives with respect to x again at least for such an increase in x as required to produce increases in y and z with signs in respect of which the derivatives mentioned exist. Then I claim that also the function of x into which $F(y, z, \ldots)$ changes if we put for y, z, …, fx, ϕx, … respectively, has a derivative for every value of x which makes $fx = y$, $\phi x = z$, …, and in fact this derivative is,

$$\frac{\mathrm{d}F(x, y, \ldots)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}F(y, z, \ldots)}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}x} + \cdots ,$$ [z]

i.e. one finds the derivative of a function $F(y, z, \ldots)$ of several variables y, z, … which themselves are considered as functions of a single free variable x, if the derivatives of this function with respect to each of the variables $y, z, \ldots$ are taken as if they were independent of one another, then multiplied by the derivatives of these variables themselves with respect to x, and finally these products are combined into a sum.

Proof. It will be sufficient to prove the proposition only for the case of two variables y, z. Now here obviously,

$$\frac{\Delta F(y, z)}{\Delta x} = \frac{F(y + \Delta y, z + \Delta z) - F(y, z)}{\Delta x}$$

[z] The first factor of the first term on this line should be $\frac{\mathrm{d}F(y,z,\ldots)}{\mathrm{d}y}$.

and the last expression can also be written,

$$= \frac{F(y+\Delta y, z+\Delta z) - F(y+\Delta y, z) + F(y+\Delta y, z) - F(y,z)}{\Delta x}$$

$$= \frac{F(y+\Delta y, z) - F(y,z)}{\Delta x} + \frac{F(y+\Delta y, z+\Delta z) - F(y+\Delta y, z)}{\Delta x}$$

$$= \frac{F(y+\Delta y, z) - F(y,z)}{\Delta y}.\frac{\Delta y}{\Delta x}$$

$$+ \frac{F(y+\Delta y, z+\Delta z) - F(y+\Delta y, z)}{\Delta z}.\frac{\Delta z}{\Delta x}.$$

However, because the given function $F(y,z)$ has a derivative with respect to y,

$$\frac{F(y+\Delta y, z) - F(y,z)}{\Delta y} = \frac{\mathrm{d}F(y,z)}{\mathrm{d}y} + \overset{1}{\Omega},$$

and because y has a derivative with respect to x, $\frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x} + \overset{2}{\Omega}$. Therefore the product

$$\frac{F(y+\Delta y, z) - F(y,z)}{\Delta y}.\frac{\Delta y}{\Delta x} = \left[\frac{\mathrm{d}F(y,z)}{\mathrm{d}y} + \overset{1}{\Omega}\right]\left[\frac{\mathrm{d}y}{\mathrm{d}x} + \overset{2}{\Omega}\right]$$

$$= \frac{\mathrm{d}F(y,z)}{\mathrm{d}y}.\frac{\mathrm{d}y}{\mathrm{d}x} + \overset{3}{\Omega}.$$

Because the given function $F(y,z)$ is also to have a derivative with respect to z, then $\frac{F(y,z+\Delta z)-F(y,z)}{\Delta y}$ must approach (for the same y) a certain measurable number dependent merely on z, namely $\frac{\mathrm{d}F(y,z)}{\mathrm{d}z}$, as close as we please providing we take Δz small enough, and all the more the further we reduce it. But because the function $F(y,z)$ must also be continuous with respect to y it is possible to make the increase Δy so small (by reducing Δx, and thus also Δz) that $\frac{F(y+\Delta y,z+\Delta z)-F(y+\Delta y,z)}{\Delta z}$ approaches the value of $\frac{F(y,z+\Delta z)-F(y,z)}{\Delta z}$, and therefore also the value of $\frac{\mathrm{d}F(y,z)}{\mathrm{d}z}$ as close as we please. We may therefore write

$$\frac{F(y+\Delta y, z+\Delta z) - F(y+\Delta y, z)}{\Delta z} = \frac{\mathrm{d}F(y,z)}{\mathrm{d}z} + \overset{4}{\Omega}$$

and since also $\frac{\Delta z}{\Delta x} = \frac{\mathrm{d}z}{\mathrm{d}x} + \overset{5}{\Omega}$, we obtain

$$\frac{F(y+\Delta y, z+\Delta z) - F(y+\Delta y, z)}{\Delta z}.\frac{\Delta z}{\Delta x} = \left[\frac{\mathrm{d}F(y,z)}{\mathrm{d}z} + \overset{4}{\Omega}\right]\left[\frac{\mathrm{d}z}{\mathrm{d}x} + \overset{5}{\Omega}\right]$$

$$= \frac{\mathrm{d}F(y,z)}{\mathrm{d}z}.\frac{\mathrm{d}z}{\mathrm{d}x} + \overset{6}{\Omega}.$$

Therefore

$$\frac{\Delta F(y,z)}{\Delta x} = \frac{dF(y,z)}{dy}.\frac{dy}{dx} + \frac{dF(y,z)}{dz}.\frac{dz}{dx} + \overset{7}{\Omega}.$$

Thus $\frac{dF(y,z)}{dy}.\frac{dy}{dx} + \frac{dF(y,z)}{dz}.\frac{dz}{dx}$ is the derivative of $F(y,z)$.

Example. If it were that $F(y,z) = \frac{y+z}{y-z}$, then we would have $\frac{dF(y,z)}{dy} = \frac{-2z}{(y-z)^2}$, $\frac{dF(y,z)}{dz} = \frac{2y}{(y-z)^2}$ for every value of y and z, provided $y-z$ does not become $= 0$. Furthermore, if it were that $y = \frac{x^2-a^2}{x}$, $z = \frac{b^2 x}{x^2+a^2}$ then we would have $\frac{dy}{dx} = \frac{a^2+x^2}{x^2}$, $\frac{dz}{dx} = \frac{b^2(a^2-x^2)}{(x^2+a^2)^2}$ for every value of x with the exception of $x = 0$. Therefore with at most the exception of the value $x = 0$, and those which make $y - z = \frac{x^4-b^2x^2-a^4}{x(x^2+a^2)}$ zero, the derivative of $\frac{y+z}{y-z}$ with respect to x must be

$$\frac{-2z\frac{dy}{dx} + 2y\frac{dz}{dx}}{(y-z)^2} = \frac{-4b^2x(a^4+x^4)}{(x^4-b^2a^2-x^4)^2}.$$

§ 171

Note. The necessity of the restriction introduced, namely that of §164, is again clear here. For example, if we put $F(y,z) = y.z$ and $y = (x^2-1)^{\frac{3}{2}}$, $z = (x^2 - 3x + 2)^{\frac{3}{2}}$, then $F(y,z) = (x^2-1)^{\frac{3}{2}}(x^2-3x+2)^{\frac{3}{2}}$ has no derivative for the value $x = 1$, because the expression $F(y+\Delta y, z+\Delta z)$ becomes imaginary for a positive, as well as a negative Δx, namely $=(\pm 2\Delta x + \Delta x^2)^{\frac{3}{2}}.(\mp\Delta x + \Delta x^2)^{\frac{3}{2}}$. Therefore we would be wrong if we wished to conclude directly that, simply from the fact that $\frac{dF(y,z)}{dy}$, $\frac{dF(y,z)}{dz}$, $\frac{dy}{dx}$, $\frac{dz}{dx}$ are all real, also $\frac{dF(y,z)}{dx}$ must be real. Moreover, I am by no means claiming that the proposition can be reversed, and that for every value for which one of the derivatives just mentioned is missing, that also the derivative $\frac{dF(y,z)}{dx}$ must be missing. The example of the theorem itself proves the opposite. For although $y = \frac{x^2-a^2}{x}$ certainly has no derivative for the value $x = 0$, nevertheless

$$F(y,z) = \frac{x^4 + b^2x^2 - a^4}{x^4 - b^2x^2 - a^4}$$

has a derivative for the value $x = 0$, with respect to a positive as well as a negative Δx. For both cases, we have

$$\frac{F(y+\Delta y, z+\Delta z) - F(y,z)}{\Delta x} = \frac{2.b^2\Delta x}{\Delta x^4 - b^2\Delta x^2 - a^4},$$

therefore the derivative $= 0$.

§ 172

Corollary. If $F(x, y, z, \ldots)$ is a function of several variables among which we consider one, x, as independent but the others $y, z, \ldots$ as dependent on it, then we obtain the derivative of this function with respect to x if we first take the derivative of $F(x,y,z,\dots)$ with respect to x, as if x could vary without y, z, $\dots$, i.e. as if y, z, $\dots$ were constant numbers, then in the same way [take] the derivative of $F(x,y,z,\dots)$ with respect to y, to z, $\dots$, always as though each one could vary in itself alone, while the rest, together with x, all remain constant. Then each of these derivatives is multiplied by the derivative of its variable considered as a function of x, and all these are added up. If x, together with y, z, $\dots$ were to be dependent on another free variable u, then the theorem would apply and we would obtain the derivative of $F(x,y,z,\dots)$ if we looked for the derivative with respect to x, y, z, $\dots$ as was just described, and multiplied each one by the derivative of its variable with respect to u. But in the given case x is the same as u, therefore the derivative of x with respect to $u = 1$. Those derivatives which we obtain from $F(x,y,z,\dots)$ if we regard only x as variable, we need therefore only multiply by 1, i.e. not multiply at all.

Example. Suppose

$$F(x,y,z) = \frac{x^3y - y^2z^2 + az^3}{a^2 + x^2},$$

and y, z, $\dots$ are dependent on x, but x is freely variable, then the complete derivative of this expression would be

$$= \frac{3(a^2+x^2)x^2y - 2(x^3y - y^2z^2 + az^3)x}{(a^2+x^2)^2} + \frac{(x^3 - 2yz^2)}{a^2+x^2}\cdot\frac{dy}{dx} + \frac{(-2y^2z + 3az^2)}{a^2+z^2}\cdot\frac{dz}{dx}.$$ [a]

Now if $y = 6 + x$, $z = \frac{a^2-x^2}{x}$, then $\frac{dy}{dx} = 1$, $\frac{dz}{dx} = -1 - \frac{(a^2-x^2)}{x^2}$, therefore the complete derivative or

$$\text{d.}\frac{x^6(6+x) - x(6+x)^2(a^2-x^2)^2 + a(a^2-x^2)^3}{x^3(a^2+x^2)}$$

$$= \frac{3x^2(6+x)}{a^2+x^2} - 2\left[\frac{x^6(6+x) - (6+x^2)(a^2-x^2)^2 + a(a^2-x^2)^3}{x^2(a^2+x^2)^2} + \frac{x^5 - 2(6+x)(a^2-x^2)^2}{x^2(a^2+x^2)} - \frac{3a(a^2-x^2)^2 - 2x(6+x)^2(a^2-x^2)}{x^2(a^2+x^2)} - \frac{3a(a^2-x^2)^3 - 2x(6+x)^2(a^2-x^2)^2}{x^4(a^2+x^2)}\right].$$

[a] The denominator of the first factor of the last term should be $a^2 + x^2$.

§ 173

Theorem. Suppose a function $F(x,y)$ of two variables, independent of one another, has a derivative with respect to x as well as to y, $\frac{dF(x,y)}{dx}$, $\frac{dF(x,y)}{dy}$, both at least for those values of the variables which we denote by x and y, and in respect of a positive or negative increase. Suppose, furthermore, the derivative with respect to x or $\frac{dF(x,y)}{dx}$ again has a derivative with respect to y, namely $\frac{d\frac{dF(x,y)}{dy}}{dx}$, and indeed in respect of the same positive or negative increase which underlies $\frac{dF(x,y)}{dy}$.[b] Finally suppose, in the same way, that the derivative in respect of y, or $\frac{dF(x,y)}{dy}$ again has a derivative with respect to x, namely $\frac{d\frac{dF(x,y)}{dy}}{dx}$, and indeed in respect to the same positive or negative increase which underlies $\frac{dF(x,y)}{dx}$.[c] Then I claim that these last two derivatives, $\frac{d\frac{dF(x,y)}{dx}}{dy}$ and $\frac{d\frac{dF(x,y)}{dy}}{dx}$ are equal to one another.

Proof. Since $F(x,y)$ has a derivative with respect to x, at least in respect of a certain positive or negative increase, then if we denote this by Δx, it must be that

$$\frac{dF(x,y)}{dx} = \frac{F(x+\Delta x, y) - F(x,y)}{\Delta x} - \overset{1}{\Omega}.$$

But according to the definition given in §169 this derivative must also exist if we put $y + \Delta y$ in place of y. Therefore also,

$$\frac{dF(x,y+\Delta y)}{dx} = \frac{F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y)}{\Delta x} - \overset{2}{\Omega}.$$

In these equations $\overset{1}{\Omega}$, $\overset{2}{\Omega}$ denote a pair of numbers which decrease indefinitely with Δx. Therefore by subtraction and division by Δy,

$$\frac{\frac{dF(x,y+\Delta y)}{dx} - \frac{dF(x,y)}{dx}}{\Delta y}$$
$$= \frac{F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y) - F(x+\Delta x, y) + F(x,y)}{\Delta x . \Delta y} + \overset{3}{\Omega}$$

if we write $\overset{3}{\Omega}$ in place of $\frac{\overset{1}{\Omega}-\overset{2}{\Omega}}{\Delta y}$, since with the same Δy it can decrease indefinitely simply with the decrease of Δx. But the term on the left-hand side of this equation is nothing but the difference in the function $\frac{dF(x,y)}{dx}$ if y increases by Δy, divided by Δx, therefore by virtue of the assumption of the theorem, it $=\frac{d\frac{dF(x,y)}{dx}}{dy} + \overset{4}{\Omega}$. So

[b] The expression should be $\frac{dF(x,y)}{dx}$.

[c] The expression should be $\frac{dF(x,y)}{dy}$.

if we put $\overset{3}{\Omega} - \overset{4}{\Omega} = \overset{5}{\Omega}$, we obtain,

$$\frac{d\frac{dF(x,y)}{dx}}{dy} = \frac{F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y) - F(x+\Delta x, y) + F(x,y)}{\Delta x.\Delta y} + \overset{5}{\Omega}. \qquad (A)$$

In a similar way,

$$\frac{dF(x,y)}{dy} = \frac{F(x, y+\Delta y) - F(x,y)}{\Delta y} + \overset{6}{\Omega}$$

and

$$\frac{dF(x+\Delta x, y)}{dy} = \frac{F(x+\Delta x, y+\Delta y) - F(x+\Delta x, y)}{\Delta y} + \overset{7}{\Omega}$$

and by subtraction and division by Δx,

$$\frac{\frac{dF(x+\Delta x,y)}{dy} - \frac{dF(x,y)}{dy}}{\Delta x} = \frac{F(x+\Delta x, y+\Delta y) - F(x+\Delta x, y) - F(x, y+\Delta y) + F(x,y)}{\Delta x.\Delta y} + \overset{8}{\Omega}$$

if we write $\overset{8}{\Omega}$ in place of $\frac{\overset{6}{\Omega}-\overset{7}{\Omega}}{\Delta x}$, because with the same Δx, $\overset{6}{\Omega}$ and $\overset{7}{\Omega}$ can decrease indefinitely simply with the decrease of Δy. Furthermore, since the left-hand term is $=\frac{d\frac{dF(x,y)}{dy}}{dx} + \overset{9}{\Omega}$ we obtain

$$\frac{d\frac{dF(x,y)}{dy}}{dx} = \frac{F(x+\Delta x, y+\Delta y) - F(x+\Delta x, y) - F(x, y+\Delta y) + F(x,y)}{\Delta x.\Delta y} + \overset{10}{\Omega}. \qquad (B)$$

The comparison of the two equations (A) and (B) shows that

$$\frac{d\frac{dF(x,y)}{dx}}{dy} = \frac{d\frac{dF(x,y)}{dy}}{dx}$$

must hold, because the terms on the right-hand side differ only by an Ω.

§ 174

Note. One could be tempted to believe that from the existence of a derivative of $F(x,y)$ with respect to x, or $\frac{dF(x,y)}{dx}$ and again a derivative of the latter with respect to y, $\frac{d\frac{dF(x,y)}{dx}}{dy}$, then the existence of a derivative of $F(x,y)$ with respect to y or $\frac{dF(x,y)}{dy}$ could be immediately inferred. But it is not so. For example, if $F(x,y) = x^2y^2 + \sqrt{1-y}$, then for the value $y=1$ there is indeed a $\frac{dF(x,y)}{dx}$, namely,

$2xy^2$, likewise also there is a derivative of this derivative with respect to y, namely $\frac{\mathrm{d}\frac{\mathrm{d}F(x,y)}{\mathrm{d}x}}{\mathrm{d}y} = 4xy$. However, there is not a derivative of $F(x, y)$ with respect to y for the value $y = 1$, since $\frac{\mathrm{d}F(x,y)}{\mathrm{d}y} = 2x^2y - \frac{1}{2\sqrt{1-y}}$ becomes infinitely large for $y = 1$.

§ 175

Corollary. Since it can also be proved in a similar way that under restrictions similar to those stated in the theorem,

$$\frac{\mathrm{d}\frac{\mathrm{d}\frac{\mathrm{d}F(x,y,z)}{\mathrm{d}x}}{dy}}{\mathrm{d}z} = \frac{\mathrm{d}\frac{\mathrm{d}\frac{\mathrm{d}F(x,y,z)}{\mathrm{d}z}}{\mathrm{d}x}}{\mathrm{d}y}$$

then it is clear that it is generally unimportant which order the derivatives of one and the same function of several variables may be taken in. Therefore the equation holds $\frac{\mathrm{d}^3}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}F(x, y, z) = \frac{\mathrm{d}^3}{\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z}F(x, y, z) = \frac{\mathrm{d}^3}{\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}F(x, y, z)$ etc.

§ 176

Theorem. If a pair of functions have the same derivatives for all values of their variable lying within a and b, then their difference for some value of their variable lying within these limits consists at most of a constant number independent of x itself.

Proof. If Fx and Φx have the same derivative fx for all values of x lying within a and b, then by §145, if we take x and $h + x$ within a and b,

$$F(x+h) - Fx = \frac{h}{n}\left[fx + f\left(x + \frac{h}{n}\right) + f\left(x + \frac{2h}{n}\right) + \cdots + f\left(x + \frac{n-1}{n}h\right)\right] + \overset{1}{\Omega}$$

$$\Phi(x+h) - \Phi x = \frac{h}{n}\left[fx + f\left(x + \frac{h}{n}\right) + f\left(x + \frac{2h}{n}\right) + \cdots + f\left(x + \frac{n-1}{n}h\right)\right] + \overset{2}{\Omega}$$

where $\overset{1}{\Omega}$ and $\overset{2}{\Omega}$ can decrease indefinitely by the increase in n. Hence it arises by *RZ* 7, §92 that it must be that $F(x+h) - Fx = \Phi(x+h) - \Phi x$ or $F(x+h) - \Phi(x+h) = Fx - \Phi x$. Now if we put for x a constant value c lying within a and b, while we change h arbitrarily but always so that $c + h$ is a value lying within a and b, which I shall denote by x, then we see that $Fx - \Phi x = Fc - \Phi c$ is a constant number completely independent of x.

§ 177

Corollary. Therefore if we have first found a single function Fx which has the property that its derivative $F'x$ is equivalent to a given function fx or to a given constant number C completely independent of x, for all values of the variable x lying within certain limits a and b, then we also know *all functions* which can be regarded as primitive [functions] of the given derivative fx or C. At least this is in as much as we know that they can all differ by at most a single constant C, not dependent on x, which is connected with all the other, shared, parts by the sign $+$ or $-$, i.e. they must be of the form $Fx + C$. For if this were not the case, if we denote one of the previous [functions] by Φx, how could the difference $Fx - \Phi x$ be equal, for all values of x, to only one and the same constant C?

§ 178

Theorem. The primitive function of which the derivative is a constant number a, completely independent of x, must be of the form $ax + c$.

Proof. For ax is a function which, considered as a primitive function, gives the derivative a (§122).

§ 179

Theorem. If Fx, Φx, Ψx, ... are a finite or infinite multitude of functions which can be considered as the primitive functions of the derivatives fx, ϕx, ψx, ..., then every function which is to be regarded as a primitive of the algebraic sum $fx + \phi x + \psi x + \cdots$, must be of the form $Fx + \Phi x + \Psi x + \cdots + C$. That is, we find the form of the primitive function of an algebraic sum if we take the primitive functions of the individual summands and add to them an arbitrary constant.

§ 180

Theorem. If the given function which we are considering as a derivative, and of which we are to determine the corresponding primitive function, (as far as possible), is a product of two factors $a.F'x$ of which one, a, is completely independent of the variable x, the other $F'x$, on the other hand, has a primitive function Fx known to us, then the general form of the required primitive function is $a.Fx + C$.

Proof. Follows directly from §156.

§ 181

Theorem. If the given function of which we seek the primitive function is a *sum* of two products $Fx.\Phi'x + F'x.\Phi x$ in each of which we find only two factors, of which one is the derivative of one of the factors appearing in the other product (namely $\Phi'x$ is the derivative of Φx, and $F'x$ is the derivative of Fx) then the product of the two factors, which can be considered as primitive functions, namely $Fx.\Phi x$, forms the variable part of the primitive function to be found, which therefore must be of the form $C + Fx.\Phi x$.

Example. If we were given the following function as derivative: $2x(x^3 - b^2x) + (x^2 + a^2)(3x^2 - b^2)$ then we would only need to notice that $2x$ is the derivative of

$x^2 + a^2$, and on the other hand $3x^2 - b^2$ is the derivative of $x^3 - b^2x$, to be able to deduce immediately that the primitive function required has to be of the form $(x^2 + a^2)(x^3 - b^2x) + C$.

§ 182

Transition. Up until now it has not yet been shown that every function fx of a variable x can be considered to be a *derivative* with respect to this variable such that some other function, Fx, of this same variable x, must be specifiable which is related to the former as the *primitive* [function], or in respect of which the equation $\frac{F(x+\Delta x)-Fx}{\Delta x} = fx + \Omega$ holds. Now if in fact a certain function fx had the property that there is no specifiable function Fx of which it can be considered as the derivative, then the concept which the symbol $\int[fx]\,\mathrm{d}x$ expresses would be *empty* (*Einleitung*,[d] II, §4). In particular we may not be permitted to conclude directly from the fact that a certain *sum* of several functions $fx + \phi x + \cdots$ represents the *derivative* of a given function Fx, that also every single one of these functions fx, ϕx, ... can itself be considered as the derivative of a certain function. Hence it follows that we may not, necessarily, derive from the equation $\int[fx + \phi x + \cdots]\,\mathrm{d}x = Fx$, the equation $\int[fx]\,\mathrm{d}x + \int[\phi x]\,\mathrm{d}x + \cdots = Fx$. For example, by §181 the sum $Fx.\Phi'x + F'x.\Phi x$ is a function which can be regarded as the derivative of $Fx.\Phi x$. But we do not know whether also $Fx.\Phi'x$, and likewise $F'x.\Phi x$, are in themselves functions which can be considered as derivatives. Therefore while the notation $\int[Fx.\Phi'x + F'x.\Phi x]\,\mathrm{d}x$ expresses a non-empty idea, it could be that the idea which is expressed by the notation $\int[Fx.\Phi'x]\,\mathrm{d}x + \int[F'x.\Phi x]\,\mathrm{d}x$ is empty. Nevertheless it is easy to see that there would be great convenience if we were permitted generally to consider the notations such as $\int[fx + \phi x + \cdots]\,\mathrm{d}x$ and $\int[fx]\,\mathrm{d}x + \int[\phi x]\,\mathrm{d}x + \cdots$ as equivalent. We shall be able to do this if, by a certain broadening of the concept which we have previously connected with the notation $\int[fx]\,\mathrm{d}x$, we establish that in future this notation is only to have such a meaning that $\int[fx]\,\mathrm{d}x + \int[\phi x]\,\mathrm{d}x$ can always be put $= \int[fx + \phi x]\,\mathrm{d}x$.

§ 183

Definition. We therefore stipulate that the notation $\int[fx]\,\mathrm{d}x$ should in future denote a concept of such a kind that not only every function whose derivative $= fx$ is included in it, but that also $\int[fx]\,\mathrm{d}x + \int[\phi x]\,\mathrm{d}x = \int[fx + \phi x]\,\mathrm{d}x$ may be written, regardless whether a function whose derivative $= fx$, another whose derivative $= \phi x$, and finally a third whose derivative $= fx + \phi x$ can, in fact, be specified.

§ 184

Corollary. Therefore the equation holds generally: $\mathrm{d}\int[Fx]\,\mathrm{d}x = Fx$.

[d] This refers to Bolzano, *EG*.

§ 185

Theorem. If a given function which we are to consider as a derivative, is a product of two factors $Fx.\Phi'x$ of which the one, $\Phi'x$ represents the derivative of a known function Φx, then we shall be permitted to form the equation $\int[Fx.\Phi'x]\,\mathrm{d}x = Fx.\Phi x - \int[F'x.\Phi x]\,\mathrm{d}x + C$.

Proof. If we take the derivative of both sides of the equation then by §158, 184 we obtain, $Fx.\Phi'x = Fx.\Phi'x + F'x.\Phi x - F'x.\Phi x$, an identical equation. Moreover, since one side of this equation contains an arbitrary constant C, there is no doubt that if in fact there is a primitive function of the kind indicated by the symbol $\int[Fx.\Phi'x]\,\mathrm{d}x$, this must likewise be of the form of the other side of the equation. But in case there should be no such function specifiable then the correctness of this equation follows from the definition just established. For in consequence of this, $\int[Fx.\Phi'x]\,\mathrm{d}x + \int[F'x.\Phi x]\,\mathrm{d}x = \int[Fx.\Phi'x + F'x.\Phi x]\,\mathrm{d}x$, and by §181, this can be put $= Fx.\Phi x + C$. Therefore also $\int[Fx.\Phi'x]\,\mathrm{d}x = Fx.\Phi x - \int[F'x.\Phi x]\,\mathrm{d}x + C$.

Example. If we were to determine $\int(x^2 + a^2)(3x^2 - b^2)$ then, since we consider $3x^2 - b^2$ as the derivative of $x^3 - b^2x$, we would find

$$\int(x^2 + a^2)(3x^2 - b^2) = (x^2 + a^2)(x^3 - b^2x) - \int(x^3 - b^2x)2x + C$$

(as in the previous §185). Similarly, if we consider x^4 as the derivative of $\frac{x^5}{5}$, $\int x^4(x-1)^3$ would be found $=\frac{x^5}{5}(x-1)^3 - \frac{3}{5}\int x^5(x-1)^2$, and if we treat $\int x^5(x-1)^2$ in a similar way, by regarding x^5 as the derivative of $\frac{x^6}{6}$, then $\int x^5(x-1)^2 = \frac{x^6}{6}(x-1)^2 - \frac{2}{6}\int x^6(x-1)$. And $\int x^6(x-1) = \frac{x^7}{7}(x-1) - \frac{1}{7}\int x^7$. And $\int x^7 = \frac{x^8}{8}$. Therefore we obtain

$$\int x^4(x-1)^3 = \frac{x^5}{5}(x-1)^3 - \frac{3}{5.6}x^6(x-1)^2 + \frac{3.2}{5.6.7}x^7(x-1) - \frac{3.2.1}{5.6.7.8}x^8 + C.$$

On the other hand, $x^4(x-1)^3 = x^7 - 3x^6 + 3x^5 - x^4$, therefore by §179 $\int x^4(x-1)^3 = \frac{x^8}{8} - \frac{3}{7}x^7 + \frac{3x^6}{6} - \frac{x^5}{5} + C$ and expansion of the above formula shows that it is the same as this last one.

§ 186

Theorem. If the given function which we are to consider as a derivative, can be brought into the form $\frac{F'x.\Phi x - Fx.\Phi'x}{(\Phi x)^2}$ then the required primitive [function] is $\frac{Fx}{\Phi x} + C$.

Proof. Directly from §160.

Example. If we were given $\frac{(x^2+cx)(3x^2-a^2)-(x^3-a^2x)(2x+c)}{(x^2+cx)^2}$ then we could conclude that the primitive function must be $\frac{x^3-a^2x}{x^2+cx} + C$.

§ 187

Theorem. If the given function which we are to consider as a derivative can be brought to the form $F'(fx).f'x$, then the required primitive function is of the form $F(fx) + C$.

Proof. From §164.

Example. If we were given $15(x^3+a^3)^4x^2$ then we soon notice that this expression could also be factorized as $5(x^3 + a^3)^4.3x^2$, and now if we consider $x^3 + a^3$ as a single variable y, $5(x^3 + a^3)^4$ is the derivative of y^5, but $3x^2$ is the derivative of $x^3 + a^3$. Therefore $\int 15(x^3 + a^3)^4x^2 = (x^3 + a^3)^5 + C$.

§ 188

Theorem. Every integral, rational function can be considered as a derivative, and the primitive function corresponding to it can again only be an integral, rational function which is only one degree higher than the given function.

Proof. Every integral, rational function is of the form $a+bx+cx^2+dx^3+\cdots+lx^n$. Now let us put instead of a a function of which a can be regarded as the derivative (and such a function by the previous corollary is ax), furthermore, let us put instead of bx a function of which bx can be considered as the derivative (and such a function is $\frac{bx^2}{2}$, since the derivative of $\frac{bx^2}{2}$ is, by §153, $= \frac{2bx}{2} = bx$). Similarly, let us put in place of cx^2 a function of which cx^2 can be regarded as the derivative (and such a function is $\frac{cx^3}{3}$, since the derivative of $\frac{cx^3}{3}$, by §153, $= \frac{3cx^2}{3} = cx^2$) etc. Generally let us put in place of each term of the form lx^m a term of which lx^m can be regarded as the derived function (and such a term is $\frac{lx^{m+1}}{m+1}$ since the derivative of this expression is, by §153, $= \frac{(m+1)lx^m}{m+1} = lx^m$). Then the last corollary shows that the algebraic sum of all these terms, if we add a constant C which is completely independent of x but still entirely undetermined, must represent the general form of the primitive function of the given $a + bx + cx^2 + dx^3 + \cdots + lx^m$ which is to be found, i.e. the function $\int[a + bx + cx^2 + \cdots + lx^m]\mathrm{d}x$. This form is therefore $C + ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \frac{dx^4}{4} + \cdots + \frac{lx^{m+1}}{m+1}$. And since this is the form of an integral, rational function which rises only one degree higher than the given function then we have shown the truth of what our theorem states.

Example. If we are to consider $1 + 4x - 15x^2 + 8x^3$ as a derived function of which the primitive, i.e. $\int[1 + 4x - 15x^2 + 8x^3]\mathrm{d}x$ is sought, then we shall answer that this latter must be of the form $C + x + 2x^2 - 5x^3 + 2x^4$. Similarly for the given function $7 - x + 14x^2 - 3x^3 + 21x^4$, the form of the primitive function may be found $\int[7-x+14x^2-3x^3+21x^4]\mathrm{d}x = C+7x-\frac{x^2}{2}+\frac{14x^3}{3}-\frac{3x^4}{4}+\frac{21x^5}{5}$. And so on.

§ 189

Theorem. Every function whose nth derivative $= 0$, for every value of its variable lying within certain limits a and b, is, within these limits, a merely rational and integral function of degree $(n-1)$.

Proof. If the nth derivative $= 0$ then the $(n-1)$th derivative, which with respect to the former is a primitive function of it, must be a constant a completely independent of the variable x. The $(n-2)$th derivative, which is a primitive function with respect to the $(n-1)$th, must be of the form $ax+b$, where we denote by b an arbitrary constant completely independent of x. The $(n-3)$th derivative must have the form $\frac{ax^2}{2}+bx+c$, the $(n-4)$th must have the form $\frac{ax^3}{2.3}+\frac{bx^2}{2}+cx+d$, etc. Hence it is self-evident that the $n-(n-1)$th, i.e. the first derivative, must be of the form

$$\frac{ax^{n-2}}{2.3\ldots(n-2)}+\frac{bx^{n-3}}{2.3\ldots(n-3)}+\frac{cx^{n-4}}{2.3\ldots n-4}+\cdots+px+q$$

i.e. an integral, rational function of degree $(n-2)$.

§ 190

Theorem. If a function Fx always *increases* for all values of its variable lying within a and b, and thereby also always has a derivative $F'x$, then this must always be positive and can become zero for at most certain isolated values, (the multitude of which can however even be infinite). But if the function always *decreases* then its derivative is always *negative*, and can be zero at most for certain individual values. If, in the opposite case, the function Fx has a derivative $F'x$ which is always *positive* for all values of x lying within a and b, and is $= 0$ at most for individual values, then this function always increases from a to b as long as it is variable. And if the derivative is always negative or zero, then the function always decreases.

Proof. It will be enough to prove the proposition only for the *increasing* case.

1. Now if Fx always increases within a and b, then if Δx is positive and x and $x+\Delta x$ are both taken within a and b then by the definition of §§ 4, 86 it must be that $F(x+\Delta x) > Fx$. But because Fx has a derivative then it must be that $\frac{F(x+\Delta x)-Fx}{\Delta x} = F'x+\Omega$, if we can decrease Δx indefinitely. Now since the numerator as well as the denominator of the fraction $\frac{F(x+\Delta x)-Fx}{\Delta x}$ always remains positive, it is clear that that equation could not possibly hold if $F'x$ were negative. Therefore it must be either positive or zero. But the latter case can occur at most for certain isolated values. For in the opposite case, if there were a pair of numbers α and β specifiable and lying within a and b, of a kind that for all values from $x=\alpha$ to $x=\beta$, we had $F'x=0$, then by §177 for these same values of x lying within α and β, Fx would have to be a constant C completely independent of x, and thus it would not be true that our function always increases within a and b.

2. Conversely, if it is known that for all values of x lying within a and b the derivative $F'x$ is positive or is zero at most for certain isolated values, then if we denote by x and $x+i$ a pair of arbitrary values lying within a and b, by §145,

$$F(x+i) - Fx$$
$$= \frac{i\left[F'x + F'\left(x+\frac{i}{n}\right) + F'\left(x+\frac{2i}{n}\right) + \cdots + F'\left(x+\frac{n-1}{n}i\right)\right]}{n} + \Omega$$

and the expression contained in brackets consists of nothing but terms which are either positive or zero. But the latter cannot be the case, at least not for all terms of which this expression is composed if for the same x and i we increase the number n indefinitely. Because Ω decreases indefinitely with the indefinite increase of n, it would have to be that $F(x+i) - Fx = 0$, therefore, by §24, Fx itself would have to be constant for all values of x lying within a and b. Therefore if we only know that Fx is in fact variable then it follows from the above equation that the value $F(x+i) - Fx$ must always be positive, and by the definition of §86 this means that Fx always increases within a and b.

§ 191

Corollary. It is assumed in the condition of this theorem that the function referred to has a derivative for all values of its variable within a and b. But naturally a continual increase (and similarly a continual decrease) is possible even for functions which have no derivative and are not even continuous. In particular there is nothing to prevent a function which always increases (or decreases) from making a *jump* for certain isolated values of its variable; it will only be required that for continual increase the size of this jump is positive, and for continual decrease that this size is negative. Also the function which continually increases (or decreases) can obey continuity without having a derivative, and indeed this [can happen] so that for certain isolated values of its variable it has (as we say) an *infinitely large derivative*, i.e. the quotient $\frac{F(x+\Delta x)-Fx}{\Delta x}$ increases indefinitely with the decrease of Δx. It will only be be required that this quotient is positive for an uninterrupted increase, or that it is negative for an uninterrupted decrease. The reason follows directly from the way we proved the first part of our theorem. That is, that for Fx to continually increase it is required that $F(x+\Delta x)$ is always $>Fx$ and this is not consistent with the condition that $\frac{F(x+\Delta x)-Fx}{\Delta x}$ increases indefinitely if Δx decreases indefinitely. Finally it is easy to see that a continual increase, as well as a decrease, is possible if the derivative $F'x$ sometimes becomes zero, sometimes infinitely large, within the given limits a and b of the variable, not merely once or a few times, but even infinitely many times, providing every such value has one which is the next (§54).

§ 192

Theorem. If a function Fx has a derivative for all values of its variable from a to b inclusive which is constantly *positive* or, at most for certain isolated values, $=0$, then Fa is its *smallest* and Fb its *greatest* value in the sense that all others are greater than Fa and smaller than Fb. Conversely, if its derivative is always *negative*, or sometimes even zero, then Fa is the greatest value, and Fb the smallest value.

Proof. On this assumption Fx increases in the first case from a to b inclusive, and in the second case it decreases, from which the rest follows by §190.

§ 193

Theorem. If a function Fx has a derivative for a certain value of its variable $x = c$, with respect to a positive as well as a negative increase, but this derivative has a value different from zero on both sides, either the same value for a positive as well as a negative increase, or indeed a different (unequal) value but still with the same sign, then the value Fc is certainly not an *extreme value* in the sense of §98.

Proof. An extreme value in the sense of §98, i.e. a greatest or smallest value, only exists where there is an ω small enough that for it, and for all smaller values, either the relationship $F(c-\omega) < Fc > F(c+\omega)$ holds, or the relationship $F(c-\omega) > Fc < F(c+\omega)$ holds. But if the function Fx has a derivative for the value $x = c$ and this indeed with respect to a positive as well as a negative increase, then we have

$$\frac{F(c+\omega)-Fc}{\omega} = M + \overset{1}{\Omega}$$

and

$$\frac{F(c-\omega)-Fc}{-\omega} = \frac{Fc-F(c-\omega)}{\omega} = N + \overset{2}{\Omega}.$$

Now if M and N are both different from zero, and either equal to one another, or of different values but with the same signs, i.e. either both positive or both negative, then in the first case obviously $F(c-\omega) < Fc < F(c+\omega)$ but in the second case $F(c-\omega) > Fc > F(c+\omega)$. Therefore in neither of the two cases does the relationship occur which must hold for an extreme value.

§ 194

Corollary. Therefore, conversely, if the value Fc of a function Fx is to be an *extreme value*, then it must be that:

(a) either the function Fx *has no derivative* for $x = c$, either none only with respect to a negative increase, or none in both respects, or
(b) in one of these respects, or in both, the derivative must be equal to zero, or
(c) its value in neither of these two respects is zero, but it has a different sign in one respect from that in the other respect. Any other case is not conceivable.

§ 195

Theorem. In each of the cases just enumerated it is *possible*, but in none of them, apart from the last, is it *necessary* that the value Fc is an extreme value.

Proof. 1. If the function Fx has *no derivative* for $x = c$, either none with respect to a positive, or none with respect to a negative increase or none in both respects, then Fc can be a maximum, or minimum, but it is not necessary. For if we suppose that for all values from $x = 0$ to $x = c$ inclusive $Fx = ax^2$, but for all greater values of x, $Fx = ax^2 - b$, then Fx has a derivative for the value $x = c$ only with respect to a negative increase, but with respect to a positive increase no derivative exists. For

$$\frac{F(c-\omega) - Fc}{-\omega} = \frac{a(c-\omega)^2 - ac^2}{\omega} = -2ac + a\omega,$$

therefore $-2ac$ is the derivative for a negative ω. On the other hand,

$$\frac{F(c+\omega) - Fc}{\omega} = \frac{a(c+\omega)^2 - b - ac^2}{\omega} = \frac{-b + 2ac\omega + a\omega^2}{\omega},$$

an expression which increases indefinitely with the indefinite decrease of ω. Now if a and b are positive, then $Fc = ac^2$ is obviously greater than $F(c - \omega)$ and $F(c + \omega)$, for $F(c - \omega) = ac^2 - 2ac\omega + a\omega^2$ and $F(c + \omega) = ac^2 - b + 2ac\omega + a\omega^2$. Therefore Fc is a genuine maximum. But if b is negative, $F(c - \omega)$ is still always $<Fc$, but $F(c + \omega) > Fc$. Therefore there is neither a maximum nor a minimum here. It may easily be seen how by changing the sign of a the other cases which we mentioned above can be produced.

2. Also if the derivative $F'c = 0$ either only with respect to a positive increase, or only with respect to a negative increase, or even in both respects, Fc can be, but is not necessarily, an extreme value. For if we put $Fx = 2cx - x^2$, then for every positive as well as negative Δx, $F'x = 2c - 2x$. Therefore for $x = c$ the derivative $F'c = 0$. But the value of $Fc = c^2$ is a maximum. For $F(c - \omega)$ and $F(c + \omega)$ are both $= c^2 - \omega^2$, and therefore smaller than c^2. If we had put $Fx = x^2 - 2cx$, then we would have $F'x = 2x - 2c$, $F'c = 0$, $Fc = -c^2$, $F(c - \omega) = F(c + \omega) = \omega^2 - c^2 > -c^2$ therefore Fc would be a minimum. On the other hand, if we consider the function $Fx = (c - x)^3$, of which the derivative $F'x = -3(c - x)^2$ likewise becomes zero for $x = c$, then for this value of x, $Fc = 0$ and $F(c - \omega) = \omega^3$, $F(c + \omega) = -\omega^3$, therefore the one is greater and the other is smaller than Fc. Therefore Fc is neither a maximum nor a minimum. It is easy to invent examples, according to the foregoing, where the derivative $F'x$ becomes zero only with respect to a positive increase or negative increase: and sometimes the case of an extreme value occurs, sometimes it does not.

3. Finally if the derivative $F'c$, according to whether we consider it at one time with respect to a positive increase, and at another time with respect to a negative increase, takes two values which differ from zero and which take different signs, then the value of Fc is certainly an extreme value. If we denote the value of $F'c$ for

a positive increase by M, and for a negative increase by N, then

$$\frac{F(c+\omega)-Fc}{\omega}=M+\overset{1}{\Omega}$$

and

$$\frac{F(c-\omega)-Fc}{-\omega}=\frac{Fc-F(c-\omega)}{\omega}=N+\overset{2}{\Omega}$$

and because M and N denote a pair of numbers different from zero, then ω can be taken so small that the absolute values of $\overset{1}{\Omega}$ and $\overset{2}{\Omega}$ are smaller than the absolute values of M and N. For this, and for all smaller values, of ω the sign of $M+\overset{1}{\Omega}$ is the same as that of M, and the sign of $N+\overset{2}{\Omega}$ is the same as that of N, therefore one is $+$ and the other is $-$. From which it is self-evident that either we have $F(c-\omega) < Fc > F(c+\omega)$ or $F(c-\omega) > Fc < F(c+\omega)$, therefore Fc is always an extreme value. For example, if for all values of $x \overset{=}{<} c$ the function Fx were of the form ax^2, but for all greater values it were of the form $2ac^2 - ax^2$, then it would be that

$$\frac{F(c+\omega)-Fc}{\omega}=\frac{[2ac^2-a(c+\omega)^2]-ac^2}{\omega}=-2ac-a\omega$$

and

$$\frac{F(c-\omega)-Fc}{-\omega}=\frac{a(c-\omega)^2-ac^2}{-\omega}=+2ac-a\omega.$$

Therefore the value of $F'c$ for a positive increase $=-2ac$, but for a negative increase $=+\,2ac$. Therefore if both a and c have the same sign, i.e. if ac is positive, then we have $F(c-\omega) < Fc > F(c+\omega)$, i.e. Fc is a maximum, and in the opposite case, $F(c-\omega) > Fc < F(c+\omega)$, i.e. Fc is a minimum.

§ 196

Note. The small deviations from the usual theory which I am allowing myself in this presentation come from the fact that the cases where either a function itself, or only its derivative, become discontinuous, are usually not considered. *Cauchy*'s presentation, which is very observant of the first of these cases, for this reason corresponds much more closely with the one given here.

§ 197

Theorem. If a function Fx, for all values of its variable lying within the limits a and b, not only has a first derivative, but also a second, third, and generally all successive derivatives up to the nth inclusive, and this latter (in the case that it is variable) is also continuous, and furthermore, it is evident that for a certain value of x lying within a and b, the derivatives mentioned except the nth are all $=0$, then whenever n is an *odd* number, Fc is *not an extreme value*, but if n is even, then Fc is a maximum if $F^n c$ is negative, and Fc is a minimum if $F^n c$ is positive.

Proof. Because the given function Fx always has a derivative $F'x$ within a and b, which itself has a derivative, it is therefore continuous, and we have, since c is a value of x lying within a and b and therefore if we take ω small enough, $c+\omega$ must also represent such a value, by §146, $F(c+\omega) = Fc + \omega.F'(c+\overset{1}{\omega})$, as long as we denote by $\overset{1}{\omega}$ a certain number which in no case lies outside 0 and ω. But because the function $F'x$ also has a derivative, which itself again obeys the law of continuity, then we have, because $c+\overset{1}{\omega}$ also lies within a and b, by the same §146, that also $F'(c+\overset{1}{\omega}) = F'c + \overset{1}{\omega}.F''(c+\overset{2}{\omega})$, in which $\overset{2}{\omega}$ denotes a certain number lying not outside 0 and $\overset{1}{\omega}$. Now if $n > 2$ then also the function $F''x$ has a derivative which is itself continuous, and we thus obtain $F''(c+\overset{2}{\omega}) = F''c + \overset{2}{\omega}.F'''(c+\overset{3}{\omega})$, where $\overset{3}{\omega}$ denotes a number lying not outside 0 and $\overset{2}{\omega}$. It is self-evident how these arguments can be continued so that we finally obtain the equation $F^{n-1}(c+\overset{n-1}{\omega}) = F^{n-1}c + \overset{n-1}{\omega}.F^{n}(c+\overset{n}{\omega})$. But since the numbers $F'c$, $F''c$, ..., $F^{n-1}c$ are all zeros, then combining these equations gives:

$$F(c+\omega) = Fc + \omega.\overset{1}{\omega}.\overset{2}{\omega}\ldots\overset{n-1}{\omega}.F^{n}(c+\overset{n}{\omega}).$$

Now if we decrease ω indefinitely, therefore all the more certainly also $\overset{1}{\omega}$, $\overset{2}{\omega}$, ..., $\overset{n-1}{\omega}$, then it follows from the law of continuity which the function $F^{n}x$ observes, that it must be that $F^{n}(c+\overset{n}{\omega}) = F^{n}c + \overset{1}{\Omega}$. Therefore $F(c+\omega) = Fc + \omega.\overset{1}{\omega}.\overset{2}{\omega}\ldots\overset{n-1}{\omega}(F^{n}c + \overset{1}{\Omega})$. Now because $F^{n}c$ is not $=0$, then ω can always be considered such that in the absolute value of both numbers, $\overset{1}{\Omega} < F^{n}c$, and then $F^{n}c + \overset{1}{\Omega}$ is positive or negative according to whichever $F^{n}c$ is.

1. After this preliminary observation it is now easy to prove that the value of Fc is neither a maximum nor a minimum if the number n is *odd*. For everything we said before about the formula $F(c+\omega) = Fc + \omega.\overset{1}{\omega}.\overset{2}{\omega}\ldots\overset{n-1}{\omega}(F^{n}c+\overset{1}{\Omega})$ holds for a positive as well as a negative ω providing both are taken small enough. But if ω is positive then also all the numbers $\overset{1}{\omega}$, $\overset{2}{\omega}$, ..., $\overset{n-1}{\omega}$ are positive, and conversely if ω is negative, all these numbers are negative. Therefore if n is odd the number of factors in the product $\omega.\overset{1}{\omega}.\overset{2}{\omega}\ldots\overset{n-1}{\omega}$ is odd, therefore this product is positive or negative according to whichever ω is. Accordingly the term $\omega.\overset{1}{\omega}.\overset{2}{\omega}\ldots\overset{n-1}{\omega}(F^{n}c+\overset{1}{\Omega})$ changes its sign with ω. Therefore either $F(c+\omega) > Fc > F(c-\omega)$ or $F(c+\omega) < Fc < F(c-\omega)$, i.e. in no case is the value Fc here an extreme value.

2. But if n is *even* then the number of factors in the product

$$\omega.\overset{1}{\omega}.\overset{2}{\omega}\ldots\overset{n-1}{\omega}$$

is also even, therefore the sign remains the same, and therefore also the sign of the expression $\omega.\overset{1}{\omega}.\overset{2}{\omega}\ldots\overset{n-1}{\omega}(F^{n}c+\overset{1}{\Omega})$ is the same whether ω is taken positive or negative. Consequently if $F^{n}c$ is itself *positive*, then $F(c+\omega)$ as well as

$F(c-\omega) > Fc$, therefore Fc is a *minimum*. But if $F^n c$ is *negative*, then $F(c+\omega)$ as well as $F(c-\omega) < Fc$, therefore Fc is a *maximum*.

Example. 1. The function $Fx = a^4+b^3x-c^2x^2$ for every value of x has a derivative $F'x = b^3 - 2c^2x$ which becomes zero for the value $x = \frac{b^3}{2c^2}$. But the second derivative of this function F^2c is $= -2c^2$, and is therefore negative for every value of x. Hence it follows that the value $x = \frac{b^3}{2c^2}$ takes the function $a^4 + b^3x - c^2x^2$ to a maximum.
2. The function $Fx = \frac{x^2-2ax+a^2}{x}$ gives the derivative $F'x = 1 - \frac{a^2}{x^2}$, which becomes zero for $x = +a$ as well as for $x = -a$. The second derivative $F''x = \frac{2a^2}{x^3}$, therefore for $x = +a$ it becomes positive $= \frac{2}{a}$, for $x = -a$ it becomes negative $= -\frac{2}{a}$. Therefore $x = +a$ gives a minimum, namely $Fa = 0$, but $x = -a$ gives a maximum, namely $F(-a) = -4a$. In order to be convinced that these values are in fact a minimum, and a maximum in the sense of §98, we need only put $a \pm \omega$ and $-a \pm \omega$ in the place of x, whereby we obtain $\frac{\omega^2}{a\pm\omega}$ and $-4a - \frac{\omega^2}{a\mp\omega}$ respectively.
3. The function $(a+x)^4$ has, for its first derivative $4(a+x)^3$, for its second derivative $12(a+x)^2$, for its third derivative $24(a+x)$, and for its fourth derivative the constant number 24. If we therefore put $x = -a$ then all these derivatives up to the last become $= 0$. Therefore this function has a minimum for $x = -a$, because 4 is even and 24 positive.
4. On the other hand, the function $(a+x)^5$ has the first derivative $5(a+x)^4$, the second derivative $20(a+x)^3$, the third derivative $60(a+x)^2$, the fourth derivative $120(a+x)$ and the fifth derivative $= 120$. Now since all these derivatives up to the last vanish for $x = -a$, but $n = 5$ is odd, then this function has neither a maximum nor a minimum for the value $x = -a$.

§ 198

Theorem. Suppose the function Fx follows the law of continuity for all values of its variable lying within a and b, and also has a *derivative* which becomes *infinitely great* at most for certain isolated values, each of which has one next to it. Moreover, suppose the [derivative] observes the law of continuity again with at most the exception of certain isolated values each of which has one next to it, then there can be a possibly infinite number of values of x lying within a and b for which the value of the function becomes an *extreme value*. But to each such value of x there is also a next one, and for every two extreme values of the function which belong to two values of x next to one another, one is always a maximum and the other a minimum, unless the function had certain limits lying within a and b within which it did not change its value at all, where if it took the same value from $x = \alpha$ exclusively to $x = \beta$ inclusively, $F\alpha$ and $F\beta$ are a pair of *one-sided* extreme values, they could both be maxima as well as minima.

Proof. 1. Some of the examples already considered show us that a function, as described by the theorem could possess an infinite number of values of its variable lying within a and b, for which its own value becomes an extreme value. In particular there is the example of §106, in which for every value from $x = \frac{2^{2n}-1}{2^{2n}}$ to $x = \frac{2^{2n+1}-1}{2^{2n+1}}$ it is stipulated that the function $Fx = x - \frac{2^{2n}-1}{3.2^{2n-1}}$, and for every value from $x = \frac{2^{2n+1}-1}{2^{2n+1}}$ to $x = \frac{2^{2n+1}-1}{2^{2n+2}}$, $Fx = \frac{2^{2n+2}-1}{3.2^{2n}} - x$. Then for all values of x which are of the form $\frac{2^n-1}{2^n}$, such extreme values arise.

2. It is clear, as follows, that for a function of this kind, for every value of x which has such an extreme value there must be a second one specifiable which is next to it, and indeed on the side of the negative increases, as well as the positive increases in x. A function which is continuous, and has a derivative with at most the exception of certain isolated values, can have an *extreme value*, by §193, only for such values of its variable for which its derivative is infinitely great, or $=0$, or its sign for its two directions changes. Now the values of x which make $F'x$ *infinitely great* should, according to the explicit assumption, only be isolated and each one has one next to it. But concerning those values of x which *make $F'x$ zero*, it follows from the assumed continuity of $F'x$, by §53, in case there were certain limits α and β lying not outside a and b within which $F'x$ became zero so often that between every two values of x which made $F'x = 0$ could be specified a third, which did the same, then the function Fx itself would have to be constant between those limits, where for this reason there would be at most a one-sided maximum or minimum for α and β. Finally, concerning the values of x at which the *sign of $F'x$ changes*, on account of the continuity with which $F'x$ is to change, between every two values for which $F'x = 0$ there must be a third such value. Therefore it is proved that this value, and therefore all values for which Fx can take an extreme value, can only be isolated and such that for every one there is a next one to it.

3. The remaining things in this theorem follow directly from §114.

§ 199

Theorem. If a function Fx for all values of x lying within a and $a+h$ not only has a *first* derivative, but also a *second*, third, and every successive derivative up to the nth inclusive and in both directions, if also this last derivative follows the law of continuity within the values stated, and if, finally, the function Fx is also continuous for the two values $x = a$ and $x = a+h$, for the first at least in the same sense as h, and for the second at least in the opposite direction, then the equation always holds,

$$F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \cdots + \frac{h^n}{2.3\ldots n}.F^n(a+\mu h)$$

in which μ denotes a certain number lying not outside 0 and 1.

Proof. 1. We shall prove this theorem first for the case where a *first*, *second*, ... and nth derivative do not merely exist for all values lying within a and $a+h$, but also for the values $x = a$ and $x = a+h$ themselves, and likewise its continuity extends up to $a+h$ inclusive. Moreover, if $n = 1$, then the proposition merely states that $F(a+h) = Fa + h.F'(a+\mu h)$ which was already proved in §146.

2. For $n = 2$ it will be asserted that $F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''(a+\mu h)$. Now if we denote an arbitrary variable number not outside 0 and h by y, and we note that if $F'x$ is considered as a primitive function, $F''x$ represents its first derivative, then the conditions already stated justify, by the proposition just mentioned, the equation $F'(a+y) = F'a + y.F''(a+\mu y)$, where μy denotes a certain number lying not outside 0 and y. But because the function $F''x$ is to be continuous from $x = a$ to $x = a+h$ inclusive then among all its values from $x = a$ to $x = a+h$ there is (by §60) a smallest, as well as a greatest, in the sense that the former has none smaller below it, and the latter has none greater above it. If we denote the smallest by $F''p$, the greatest by $F''q$, then p and q are a pair of numbers lying not outside a and $a+h$ which are completely independent of y for the same constant a and h. Thus for each of the values of y within 0 and h the relationships hold: $F'(a+y) - F'a \gtreqless y.F''p$ and $\lesseqgtr y.F''q$. Accordingly $F'(a+y) - F'a - y.F''p$ and $y.F''q - F'(a+y) + F'a$ are a pair of expressions which always remain zero or positive for all values of the variable y lying not outside 0 and h. Now as long as a is regarded as constant, but y is regarded as variable, by §166, $F'(a+y)$ can be considered as the derivative of $F(a+y)$, because the derivative of $a+y = 1$. On this assumption $F'a$ is the derivative of $y.F'a$, and $y.F''p$ is the derivative of $\frac{y^2}{2}.F''p$, because $F'a$ and $F''p$, by this assumption, denote constant numbers. Thus the whole expression $F'(a+y) - F'a - y.F''p$ can be considered as the derivative of a function which is of the form $C + F(a+y) - y.F'a - \frac{y^2}{2}.F''p$. And in a completely similar way the expression $y.F''q - F'(a+y) + F'a$ can be regarded as the derivative of $D + \frac{y^2}{2}.F''q - F(a+y) + y.F'a$, if we denote by C and D a pair of arbitrary constants. Now let us determine these constants in such a way that these functions become zero for $y = 0$. For this purpose it is only necessary to make $C = -Fa$ and $D = +Fa$. Now therefore $-Fa + F(a+y) - y.F'a - \frac{y^2}{2}.F''p$ and $Fa + \frac{y^2}{2}.F''q - F(a+y) + y.F'a$ represent two functions of y which vanish for $y = 0$, but whose derivatives, $F'(a+y) - F'a - y.F''p$ and $y.F''q - F'(a+y) + F'a$ are always positive or zero for all values from $y = 0$ to $y = h$. From §190 we know that such functions, if q is positive, must themselves be positive or zero. Therefore $-Fa + F(a+y) - y.F'a - \frac{y^2}{2}.F''p$, as well as $Fa + \frac{y^2}{2}.F''q - F(a+y) + y.F'a$, is either positive or zero. Therefore $F(a+y) - Fa - y.F'a \gtreqless \frac{y^2}{2}F''p$ and $\lesseqgtr \frac{y^2}{2}F''q$. Therefore also, if we take $y = h$, $F(a+h) - Fa - h.F'a \gtreqless \frac{h^2}{2}.F''p$ and $\lesseqgtr \frac{h^2}{2}.F''q$. Now since $F''x$ is to be continuous, at least in the first degree, then by §65 there is a value of x lying between p and q, therefore also between a and $a+h$, which can therefore be represented by $a + \mu h$ of a kind that we obtain the equation $F(a+h) - Fa - h.F'a = \frac{h^2}{2}.F''(a+\mu h)$ or $F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''(a+\mu h)$. In a completely similar way, we obtain $F(a-h) = Fa - h.F'a + \frac{h^2}{2}.F''(a-\mu h)$, from which we see that that formula holds generally for a positive value, as well as a negative value of h.

3. For $n = 3$ the theorem asserts that $F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''(a+\mu h)$. If we denote again by y an arbitrary variable number lying not outside 0 and

h, $F''(a+y) = F''a + y.F'''(a+\mu y)$ and if we denote by p and q those values of x for which $F'''x$ takes the smallest and greatest values among all those from $x = a$ to $x = a+h$, then it must be that $F''(a+y) - F''a \gtreqless y.F'''p$ and $\lesseqgtr y.F'''q$. Accordingly $F''(a+y) - F''a - y.F'''p$ and $y.F'''q - F''(a+y) + F''a$ denote a pair of expressions which for all values lying not outside 0 and h are either positive or zero. Therefore if h is positive, the primitive functions of which these can be regarded as derivatives, namely $C + F'(a+y) - y.F''a - \frac{y^2}{2}.F'''p$ and $D + \frac{y^2}{2}.F'''q - F'(a+y) + y.F''a$ must always be positive or zero, i.e. if we determine C and D so that they vanish for $y = 0$, i.e. if we take $C = -F'a$ and $D = +F'a$. Therefore $-F'a + F'(a+y) - yF''a - \frac{y^2}{2}.F'''p$ and $F'a + \frac{y^2}{2}.F'''q - F'(a+y) + y.F''a$ are again a pair of functions of y which are always positive or zero. But $F'(a+y)$ can be considered as the derivative of $C + F(a+y)$, $F'a$ as the derivative of $y.F'a$, $y.F''a$ as the derivative of $\frac{y^2}{2}.F''a$, and $\frac{y^2}{2}.F'''p$ as the derivative of $\frac{y^3}{2.3}.F'''p$, and similarly $\frac{y^2}{2}.F'''q$ as the derivative of $\frac{y^3}{2.3}.F'''q$. Therefore by §190, $C - y.F'a + F(a+y) - \frac{y^2}{2}.F''a - \frac{y^3}{2.3}.F'''p$ and $D + y.F'a + \frac{y^3}{2.3}.F'''q - F(a+y) - \frac{y^2}{2}.F''a$ are two primitive functions of y which must always remain positive or zero provided we determine their constants C and D so that both functions vanish for $y = 0$, i.e. if we put $C = -Fa$ and $D = +Fa$. Therefore we know that the two expressions $F(a+y) - Fa - y.F'a - \frac{y^2}{2}.F''a - \frac{y^3}{2.3}.F'''p$ and $-F(a+y) + Fa + y.F'a + \frac{y^2}{2}.F''a + \frac{y^3}{2.3}.F'''q$ are always positive or zero, i.e. that $F(a+y) - Fa - y.F'a - \frac{y^2}{2}.F''a \gtreqless \frac{y^3}{2.3}.F'''p$ and $\lesseqgtr \frac{y^3}{2.3}.F'''q$. Therefore also if we put $y = h$, $F(a+h) - Fa - h.F'a - \frac{h^2}{2}.F''a \gtreqless \frac{h^3}{2.3}.F'''p$ and $\lesseqgtr \frac{h^3}{2.3}.F'''q$. There must therefore be a value $a+\mu h$ lying not outside p and q, and therefore also not outside a and $a+h$, which gives rise to the equation $F(a+h) - Fa - h.F'a - \frac{h^2}{2}.F''a = \frac{h^3}{2.3}.F'''(a+\mu h)$ or $F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''(a+\mu h)$. In a similar way the equation for $-h$ follows, $F(a-h) = Fa - h.F'a + \frac{h^2}{2}.F''a - \frac{h^3}{2.3}.F'''(a-\mu h)$.

4. Now it is easy to prove that our formula holds completely generally for every value of n. For assuming that it holds for a definite value of n, then it can be shown directly that it also holds for the next larger value $n+1$, assuming that the given function has an $(n+1)$th derivative which is continuous for all values from $x = a$ to $x = a+h$ inclusive. That is, because the formula holds for the value n, but $F^{n+1}x$ is not the $(n+1)$th but only the nth derivative of $F'x$, then we obtain $F'(a+h) = F'a + h.F''a + \frac{h^2}{2}.F'''a + \frac{h^3}{2.3}.F^{\mathrm{IV}}a + \cdots + \frac{h^n}{2.3\ldots n}.F^{n+1}(a+\mu h)$, or also for every y which does not lie outside 0 and h, $F'(a+y) = F'a + y.F''a + \frac{y^2}{2}.F'''a + \frac{y^3}{2.3}.F^{\mathrm{IV}}a + \cdots + \frac{y^n}{2.3\ldots n}.F^{n+1}(a+\mu y)$. Therefore if we denote the smallest value which the continuous function $F^{n+1}x$ takes among all values of x from a to $a+h$, by $F^{n+1}p$, and the greatest by $F^{n+1}q$, then it must be that, $F'(a+y) - F'a - y.F''a - \frac{y^2}{2}.F'''a - \frac{y^3}{2.3}.F^{\mathrm{IV}}a - \cdots \gtreqless \frac{y^n}{2.3\ldots n}.F^{n+1}p$ and $\lesseqgtr \frac{y^n}{2.3\ldots n}.F^{n+1}q$. Accordingly $F'(a+y) - F'a - y.F''a - \frac{y^2}{2}.F'''a - \frac{y^3}{2.3}.F^{\mathrm{IV}}a - \cdots - \frac{y^n}{2.3\ldots n}.F^{n+1}p$ and $-F'(a+y) + F'a + y.F''a + \frac{y^2}{2}.F'''a + \frac{y^3}{2.3}.F^{\mathrm{IV}}a + \cdots + \frac{y^n}{2.3\ldots n}.F^{n+1}q$ are a pair of functions

of y which always remain positive or zero. But $F'(a+y)$ can be considered as the derivative of $F(a+y)$, $F'a$ as the derivative of $y.F'a$, $y.F''a$ as the derivative of $\frac{y^2}{2}.F''a$, $\frac{y^2}{2}.F'''a$ as the derivative of $\frac{y^3}{2.3}.F'''a$, $\frac{y^3}{2.3}.F^{\mathrm{IV}}a$ as the derivative of $\frac{y^4}{2.3.4}.F^{\mathrm{IV}}a$, etc. Finally $\frac{y^n}{2.3\ldots n}.F^{n+1}p$ can be considered as the derivative of $\frac{y^{n+1}}{2.3\ldots n+1}.F^{n+1}p$, and similarly $\frac{y^n}{2.3\ldots n}.F^{n+1}q$ as the derivative of $\frac{y^{n+1}}{2.3\ldots n+1}.F^{n+1}q$. Therefore for the two complete expressions, the first can be considered as the derivative of $C+F(a+y)-y.F'a-\frac{y^2}{2}.F''a-\frac{y^3}{2.3}.F'''a-\frac{y^4}{2.3.4}.F^{\mathrm{IV}}a-\cdots-\frac{y^{n+1}}{2.3\ldots n+1}.F^{n+1}p$ and the second as the derivative of $D-F(a+y)+y.F'a+\frac{y^2}{2}.F''a+\frac{y^3}{2.3}.F'''a+\frac{y^4}{2.3.4}.F^{\mathrm{IV}}a+\cdots+\frac{y^{n+1}}{2.3\ldots n+1}.F^{n+1}q$. If we therefore determine the constants C and D so that both expressions vanish for $y=0$, i.e. if we put $C=-Fa$ and $D=+Fa$, then it follows from §190 that the two functions $F(a+y)-Fa-y.F'a-\frac{y^2}{2}.F''a-\frac{y^3}{2.3}.F'''a-\frac{y^4}{2.3.4}.F^{\mathrm{IV}}a-\cdots-\frac{y^{n+1}}{2.3\ldots n+1}.F^{n+1}p$ and $-F(a+y)+Fa+y.F'a+\frac{y^2}{2}.F''a+\frac{y^3}{2.3}.F'''a+\frac{y^4}{2.3.4}.F^{\mathrm{IV}}a+\cdots+\frac{y^{n+1}}{2.3\ldots n+1}.F^{n+1}q$ are always positive or zero. Therefore also if we take $y=h$, $F(a+h)-Fa-h.F'a-\frac{h^2}{2}.F''a-\frac{h^3}{2.3}.F'''a-\frac{h^4}{2.3.4}.F^{\mathrm{IV}}a-\cdots-\frac{h^n}{2.3\ldots n}.F^na \geqq \frac{h^{n+1}}{2.3\ldots n+1}.F^{n+1}p$ and $\leqq \frac{h^{n+1}}{2.3\ldots n+1}.F^{n+1}q$. Thus there is a value $a+\mu h$ lying not outside p and q, therefore also not outside a and $a+h$, for which the equation holds $F(a+h)=Fa+h.F'a+\frac{h^2}{2}.F''a+\frac{h^3}{2.3}.F'''a+\frac{h^4}{2.3.4}.F^{\mathrm{IV}}a+\cdots+\frac{h^{n+1}}{2.3\ldots n+1}.F^{n+1}(a+\mu h)$. And in a similar way as this has been proved for a positive value of h, it may also be proved for a negative value.

5. Finally, if the function Fx has a first, second, and nth derivative for all values of x lying within a and $a+h$, but not for these [values] themselves, then provided we take α and $\alpha+i$ within a and $a+h$, by what was proved previously, we may form the equation, $F(\alpha+i)=F\alpha+i.F'\alpha+\frac{i^2}{2}.F''\alpha+\frac{i^3}{2.3}.F'''\alpha+\cdots+\frac{i^n}{2.3\ldots n}.F^n(\alpha+\mu i)$. From this, by similar arguments to those in §148, we may derive, $F(a+h)=Fa+h.F'a+\frac{h^2}{2}.F''a+\frac{h^3}{2.3}.F'''a+\cdots+\frac{h^n}{2.3\ldots n}.F^n(a+\mu h)$. For if we decrease the difference $\alpha-a$ indefinitely then, because Fx is to be continuous for the value $x=a$, and with respect to an increase of the same sign as h, the difference $F\alpha-Fa$, in its absolute value, must also decrease indefinitely. The same must also hold of the differences $F'\alpha-F'a$, $F''\alpha-F''a,\ldots$, $F^n(\alpha+\mu i)-F^n(a+\mu i)$ for the same reason. We may therefore also write $F(a+i)=Fa+i.F'a+\frac{i^2}{2}.F''a+\frac{i^3}{2.3}.F'''a+\cdots+\frac{i^n}{2.3\ldots n}.F^n(a+\mu i)+\Omega$ and likewise, because every number which can be represented by $a+\mu i$ can also be represented by $a+\mu h$, since $h>i$, $F(a+i)=Fa+i.F'a+\frac{i^2}{2}.F''a+\frac{i^3}{2.3}.F'''a+\cdots+\frac{i^n}{2.3\ldots n}.F^n(a+\mu h)+\Omega$. But because Fx is also to be continuous for the value $x=a+h$, and indeed with respect to an increase of the opposite sign from h, then, if we move i indefinitely close to the value h, then the difference $F(a+h)-F(a+i)$, in its absolute value, must also decrease indefinitely. Therefore it must also be that $F(a+h)=Fa+i.F'a+\frac{i^2}{2}.F''a+\frac{i^3}{2.3}.F'''a+\cdots+\frac{i^n}{2.3\ldots n}.F^n(a+\mu h)+\overset{1}{\Omega}$. Then since with the indefinite approach of i to h, every term of the series, $i.F'a$, $\frac{i^2}{2}.F''a$,

$\frac{i^3}{2.3}.F'''a, \ldots, \frac{i^n}{2.3\ldots n}.F^n(a+\mu h)$, also approaches indefinitely the corresponding one of the following [series], $h.F'a$, $\frac{h^2}{2}.F''a$, $\frac{h^3}{2.3}.F'''a, \ldots$, $\frac{h^n}{2.3\ldots n}.F^n(a+\mu h)$, while the number of them remains unchanged, then we may certainly also write (§) $F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \cdots + \frac{h^n}{2.3\ldots n}.F^n(a+\mu h) + \overset{2}{\Omega}$. Finally if we determine the number μ so that the value of the expression $Fa + h.F'a + \frac{h^2}{2}.F''a + \cdots + \frac{h^n}{2.3\ldots n}.F^n(a+\mu h)$ approaches the value of $F(a+h)$ as closely as possible, then all the expressions up to Ω appearing in the equation just given are completely independent of i, and thus Ω itself must also have a value completely independent of i, from which it then obviously follows that this could only be the value zero.

Example. The tacit condition that the function Fx has to belong to the class of those functions which are determined by an identical rule for all values of their variable, applies as little to this theorem as it does to that of §146 of which this is only a further development. For example, if Fx were a function of such a kind that for all values of $x \overline{\overline{<}} 1$, $Fx = \frac{4x^4-4x^3+12x^2-4x+1}{24}$ but for all higher [values] $Fx = \frac{3x^5+5x^4+10x^3+30x^2-5x-2}{120}$, then Fx would have a derivative for all values of x, not only a first, but also a second, third and all successive ones indefinitely. That is we would have the following:

For $x \overline{\overline{<}} 1$	For $x > 1$
$F'x = \frac{4x^3-3x^2+6x-1}{6}$	$= \frac{3x^4+4x^3+6x^2+12x-1}{24}$
$F''x = 2x^2 - x + 1$	$= \frac{x^3+x^2+x+1}{2}$
$F'''x = 4x - 1$	$= \frac{3x^2+2x+1}{2}$
$F^{\mathrm{IV}}x = 4$	$= 3x + 1$
$F^{\mathrm{V}}x = 0$	$= 3$

And since these formulae give the same value for $x = 1$ up to $F^{\mathrm{IV}}x$, the fourth derivative still varies continuously. Therefore the formula $F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \frac{h^4}{2.3.4}.F^{\mathrm{IV}}(a+\mu h)$ must be applicable for every value of a and h. If, in fact, we put $a = 0$ and $h = 10$ then we have $F(10) = \frac{90737}{30} = \frac{1}{24} - \frac{10}{6} + \frac{100}{2} - \frac{1000}{6} + \frac{10000}{24}F^{\mathrm{IV}}(10\mu)$. Therefore $F^{\mathrm{IV}}(10\mu) = \frac{377142}{50000}$. Now if $\mu \overline{\overline{<}} \frac{1}{10}$ then $F^{\mathrm{IV}}(10\mu)$ would be $=4$, which gives an incorrect result. But if we take $\mu > \frac{1}{10}$, then $F^{\mathrm{IV}}(10\mu)$ is of the form $3x + 1 = 30\mu + 1$, then we have $\mu = \frac{327143}{1500000}$, therefore certainly between $\frac{1}{10}$ and 1. But if we were to take $a = 1$, $h = 9$ then we would have $\frac{90737}{30} = \frac{3}{8} + 9 + \frac{9^2}{2}.2 + \frac{9^3.3}{2.3} + \frac{9^4}{2.3.4}F^{\mathrm{IV}}(1+9\mu)$. Now for $\mu = 0$ it would be that $F^{\mathrm{IV}}(1+9\mu) = 4$, again therefore an impossible result. Therefore it must be that $\mu > 0 < 1$, and we have $\mu = \frac{177263}{885735}$ etc.

§ 200

Corollary 1. By means of this important theorem which is usually called *Taylor's theorem* after its inventor, the value of the increase $F(a+h) - Fa$ in a function when its variable goes from a definite value $x = a$ to another arbitrary value denoted by $a + h$, can be calculated as long as the values of the derived functions $F'x$, $F''x$, $F'''x$, ... up to the nth, $F^n x$, for the value $x = a$, are known, and furthermore that we know that this nth derivative is continuous for all values from $x = a$ to $x = a + h$, and finally, that we know a means of determining the last term $\frac{h^n}{2.3...n}.F^n(a + \mu h)$.

§ 201

Corollary 2. If Fx is an *integral, rational function* of the mth degree then the mth derivative of it is constant, and the $(m + 1)$th derivative and all successive ones are $=0$ for every value of x (§162). Therefore the condition holds which is required for the application of Taylor's theorem, for every such function, and for all values of a and h, because the term $\frac{i^n}{2.3...n}F^n(a+\mu h)$ and all successive ones vanish as soon as $n > m+1$, so the increase in such a function, if its variable goes from the value a to the arbitrary value $a + h$, can be represented by a series of increasing powers of the increase h. For example, if $Fx = 4x^2 - x^3$, then $F'x = 8x - 3x^2$, $F''x = 8 - 6x$, $F'''x = -6$ and all successive derivatives $= 0$ for every value of x. Therefore we would have, $4(a+h)^2 - (a+h)^3 = 4a^2 - a^3 + h(8a - 3a^2) + \frac{h^2}{2}(8 - 6a) + \frac{h^3}{2.3}(-6)$ as may also be confirmed by expanding each side of the equation.

§ 202

Corollary 3. As an especially noteworthy example let us consider here the formula for the increase of a power whose root is variable, namely x^n. We already know all the derivatives which such a power has, namely starting from the first they are, nx^{n-1}, $n.n - 1.x^{n-2}$, $n.n - 1.n - 2.x^{n-3}$, ..., so that the $(n - 1)$th, $= n.n - 1.n - 2 \ldots 2x$, and the nth, $= n.n - 1.n - 2 \ldots 2.1$ is constant, and the $(n + 1)$th and all following ones are $= 0$. Therefore

$$\begin{aligned}(a + h)^n = a^n + na^{n-1}h + \frac{n.n - 1}{1.2}a^{n-2}.h^2 + \frac{n.n - 1.n - 2}{1.2.3}a^{n-3}h^3 \\ + \cdots + \frac{n.n - 1.n - 2 \ldots n - m}{1.2.3 \ldots m + 1}a^{n-m-1}h^{m+1} \\ + \cdots + \frac{n.n - 1 \ldots 1}{1.2.3 \ldots n}h^n\end{aligned}$$

a formula which is usually called the *binomial formula*. The coefficients appearing in the individual terms of it: 1, n, $\frac{n.n-1}{1.2}$, $\frac{n.n-1.n-2}{1.2.3}$, $\frac{n.n-1.n-2...n-m}{1.2.3...(m+1)}$, ... are called the binomial coefficients.

Example. By this formula we have $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$, $(a+h)^4 = a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4$. Etc. If we put $a = 1$ and $h = 1$ then for every integer value of n,

$$2^n = 1 + n + \frac{n.n-1}{1.2} + \frac{n.n-1.n-2}{1.2.3} + \frac{n.n-1.n-2.n-3}{1.2.3.4} + \cdots + 1.$$

But if we put $a = 1$ and $h = -1$, then because $1 - 1 = 0$ and $0^n = 0$ for every integer value of n it must be that

$$1 - n + \frac{n.n-1}{1.2} - \frac{n.n-1.n-2}{1.2.3} + \frac{n.n-1.n-2.n-3}{1.2.3.4} - \cdots \pm 1 = 0.$$

And so on.

§ 203

Corollary 4. The example of the function $\frac{1}{4-x}$ can prove to us how essential in this theorem is the condition that even the last derivative $F^n x$ obeys the law of continuity for all values of x lying within a and $a+h$ (for the others it is obvious). The multitude of derivatives here goes on indefinitely; they form the series

$$\frac{1}{(4-x)^2}, \quad \frac{2}{(4-x)^3}, \quad \frac{2.3}{(4-x)^4}, \quad \frac{2.3.4}{(4-x)^5}, \quad \ldots, \quad \frac{2.3\ldots n}{(4-x)^{n+1}},^{\text{e}} \quad \text{etc.}$$

But for the value $x = 4$ these derivatives all become discontinuous. Therefore if we take, for example, $a = 3$, but $h = 2$, then we can never assert of the function $F^n x$, however n is determined, that it remains continuous for all values from $x = a$ to $x = a + h$ (i.e. from 3 to 5). Now since $F(a+h) = \frac{1}{4-5} = -1$, $Fa = 1$, $F''a = 2$, $F'''a = 2.3$, $F^{\mathrm{IV}}a = 2.3.4, \ldots$ and generally $F^r a = 2.3.4\ldots r$, if the formula of the theorem were also to hold for this case, it should be that $-1 = 1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + \frac{2^n}{(1-2\mu)^{n+1}}$. But now for every *even* value of n, which therefore makes $n+1$ an odd value, there is actually a value for μ, which satisfies this equation, namely the greatest value which μ can take $= 1$, this gives $\frac{2^n}{(1-2\mu)^{n+1}} = -2^n$, and it is certainly true that $-1 = 1+2+2^2+2^3+\cdots+2^{n-1}-2^n$ or $2^n - 1 = 1+2+2^2+2^3+\cdots+2^{n-1}$ (§). However, for an *odd* n there is no value at all for μ which makes that equation true because $\frac{2^n}{(1-2\mu)^{n+1}}$ always remains positive if n is odd, and therefore $n+1$ is even.

§ 204

Corollary 5. If it can be proved that the series

$$\frac{h^n}{2.3\ldots n}.F^n a + \frac{h^{n+1}}{2.3\ldots n+1}.F^{n+1}a + \ldots + \frac{h^{n+r}}{2.3\ldots(n+r)}.F^{n+r}(a+\mu h)^{\text{f}}$$

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[e] The continuation dots in the final numerator are omitted in F(2).

[f] The continuation dots and plus sign immediately following them in the series are omitted in F(2).

for every value which μ can take, becomes and remains smaller than every given fraction $\frac{1}{N}$ merely by increasing n, however large r is subsequently made, then it will also follow from RZ7, §107 that we may put $F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \cdots$ *in inf*. For example the rth derivative of $\frac{1}{1+x} = \pm\frac{2.3\ldots r}{(1+x)^{r+1}}$. Therefore if we put $Fx = \frac{1}{1+x}$, $a = 0$, $h > 0$ and <1, then in each case

$$\frac{h^r}{2.3\ldots r}.F^r(a+\mu h) \gtreqless \frac{h^r}{(1+h)^{r+1}}$$

and the series

$$\frac{h^n}{2.3\ldots n}.F^n a + \frac{h^{n+1}}{2.3\ldots n+1}.F^{n+1}a + \cdots + \frac{h^{n+r}}{2.3\ldots(n+r)}.F^{n+r}(a+\mu h)$$

decreases indefinitely if we increase n indefinitely, because it $= \pm\left(h^n + h^{n+1} + h^{n+2} + \cdots + \frac{h^{n+r}}{(1+h)^{n+r+1}}\right)$. Therefore we may write $\frac{1}{1+h} = 1 - h + h^2 - h^3 + h^4 - h^5 + \cdots$ *in inf*.

§ 205

Theorem. If a function Fx has the property that not only the first n of its derived functions (as was assumed in the foregoing theorem) but also all following ones, *are measurable numbers* for all values of x lying within a and $a+h$, but for the definite value $x = a$ they take values of such a kind that $F^r x$, for every x lying within a and $a+h$, in its absolute value, always remains smaller than a given number M, however large we wish r to become, then we always have the equation,

$$F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \frac{h^4}{2.3.4}.F^{\mathrm{IV}}a + \cdots + \frac{h^r}{2.3.4\ldots r}.F^r a + \Omega$$

or also

$$F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \frac{h^4}{2.3.4}.F^{\mathrm{IV}}a + \cdots + \frac{h^r}{2.3\ldots r}.F^r a + \cdots \textit{ in inf}.$$

Proof. By virtue of the previous theorem and with the assumption of the present theorem, that $F^r x$ for every value of r represents a measurable number, then as long as we take x within a and $a+h$ we have the equation,

$$F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \cdots + \frac{h^r}{2.3\ldots r}.F^r a + \frac{h^{r+1}}{2.3\ldots(r+1)}.F^{r+1}(a+\mu h)$$

and if we increase r by s,

$$F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \cdots$$
$$+ \frac{h^r}{2.3\ldots r}.F^r a + \frac{h^{r+1}}{2.3\ldots(r+1)}.F^{r+1}a + \cdots$$
$$+ \frac{h^{r+s+1}}{2.3\ldots(r+s+1)}.F^{r+s+1}(a+\mu h)$$

in which the symbol μ denotes a certain number (not the same in both expressions) which will never lie outside the limits 0 and 1. Now if there is a number M large enough to be able to assert that $F^r x$, in its absolute value, always remains $< M$, however large r may be taken, providing x is always within a and $a+h$, then every term of the series

$$\frac{h^{r+1}}{2.3\ldots(r+1)}.F^{r+1}a + \frac{h^{r+2}}{2.3\ldots(r+2)}.F^{r+2}a + \cdots$$
$$+ \frac{h^{r+s+1}}{2.3\ldots(r+s+1)}.F^{r+s+1}(a+\mu h)$$

which appears in the second equation in place of the term $\frac{h^{r+1}}{2.3\ldots(r+1)}.F^{r+1}(a+\mu h)$ of the first one is, in its absolute value, smaller than the corresponding term of the series

$$\frac{h^{r+1}}{2.3\ldots(r+1)}.M + \frac{h^{r+2}}{2.3\ldots(r+2)}.M + \cdots + \frac{h^{r+s+1}}{2.3\ldots(r+s+1)}.M.$$

Consequently the value of that first series is in every case smaller than the value of the last series, even in the case when we allow all terms to take one and the same sign, i.e. h positive. But it can be proved that the value of this last series can be made smaller than every given number, providing we take r large enough, after which s may then increase indefinitely. For this series is

$$\frac{h^{r+1}M}{2.3\ldots(r+1)}\left[1 + \frac{h}{r+2} + \frac{h^2}{(r+2)(r+3)} + \cdots \right.$$
$$\left. + \frac{h^s}{(r+2)(r+3)\ldots(r+s+1)}\right],$$

which decreases more rapidly than the following series,

$$\frac{h^{r+1}M}{2.3\ldots(r+1)}\left[1 + \frac{h}{r+2} + \left(\frac{h}{r+2}\right)^2 + \cdots + \left(\frac{h}{r+2}\right)^s\right].$$

Therefore if we first take r so large that $r+2$ is greater than $2h$ for example, then $1 + \frac{h}{r+2} + \left(\frac{h}{r+2}\right)^2 + \cdots$ *in inf.*, by §, remains < 2. Therefore the series just given, for this and every greater value of r, is always $< \frac{h^{r+1}.M}{3.4\ldots(r+1)}$. But the value of this last expression will obviously become all the smaller the greater we take r, and can be decreased indefinitely. For whenever we increase r only by one, we multiply this expression by a new factor of the form $\frac{h}{r+2}$ which is therefore $< \frac{1}{2}$. But from § we know that by an arbitrary number of repetitions of such a procedure, i.e. by continually multiplying by fractions which are $=$ or $< \frac{1}{2}$, every given measurable number can be decreased indefinitely. Therefore all the more certainly can the series

$$\frac{h^{r+1}}{2.3\ldots(r+1)}.F^{r+1}a + \frac{h^{r+2}}{2.3\ldots(r+2)}.F^{r+2}a + \cdots + \frac{h^{r+s+1}}{2.3\ldots(r+s+1)}.F^{r+s+1}(a+\mu h)$$

be decreased indefinitely by merely increasing r. But from this it follows by *RZ* 7, §107,

$$F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \cdots + \frac{h^r}{2.3\ldots r}.F^r a + \Omega$$

and

$$F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \frac{h^3}{2.3}.F'''a + \cdots + \frac{h^r}{2.3\ldots r}.F^r a + \cdots \textit{in inf}.$$

§ 206

Corollary 1. If it happens for a particular value of a, or even generally for every value of a, that the $(n+1)$th derivative $F^{n+1}a$ and all the ones *following* it are merely zeros, then we have $F(a+h) = Fa + h.F'a + \frac{h^2}{2}.F''a + \cdots + \frac{h^n}{2.3\ldots n}.F^n a$. From §201 we know that this case occurs for every value of a, as long as the function Fx is simply a rational and integral function, but this phenomenon also occurs for some functions of another kind for individual values of a.

§ 207

Corollary 2. If we put $h = z - a$, and thus $a + h = z$, which will be permitted if all derivatives of our function, for all values of the variable within a and z are measurable numbers, and moreover if there is some constant number M specifiable [which is] large enough to be able to assert that the values of all these derivatives, within the limits of their variable just mentioned, always remain $<M$, then we obtain

$$Fz = Fa + (z-a).F'a + \frac{(z-a)^2}{2}.F''a + \frac{(z-a)^3}{2.3}.F'''a + \frac{(z-a)^4}{2.3.4}.F^{\mathrm{IV}}a + \cdots + \frac{(z-a)^r}{2.3\ldots r}.F^r a + \Omega$$

or

$$Fz = Fa + (z - a).F'a + \frac{(z-a)^2}{2}.F''a + \frac{(z-a)^3}{2.3}.F'''a + \cdots in\,inf.$$

This formula, especially for the particular case, when $a = 0$, is called *Maclaurin*'s formula.

§ 208

Note. Without going into a detailed examination here of the various kinds of proofs which have been given before of *Taylor*'s and *Maclaurin*'s theorems, I only remind [the reader] that the incorrectness of most of them shows itself through the fact that (as the saying goes) they *prove too much*. That is, arguments are permitted from which, if they were allowed, it would have to follow that this formula would also hold in certain cases in which it decidedly does not hold. The one given by *Lagrange* (in *Théorie des fonctions analytiques* and in the *Leçons sur le calcul des fonctions*), is regarded as one of the strictest proofs which also many others have essentially still followed, for example, *Kästner*, *Tempelhof*, *Pfaff*, *Bohnenberger*, *Mayer*, *Prasse*, *Pasquich*, *Eytelwein*, *Brosius*, *Ohm*, *Young* (*Elements of the differential calculus*, London, 1831). The weakness of this proof (as also *Grunert* remarks in the continuation of *Klügels Wörterbuch*, B. 5, S. 8), consists in the assumption that $F(x+i)-Fx$ can be represented in the form $Ai^{\alpha}+Bi^{\beta}+\cdots$ which, since $F(x+i) - Fx$ can denote every arbitrary function of i, is fundamentally no different from the assumption that every function of a variable x must be of the form $Ax^{\alpha} + Bx^{\beta} + \cdots$. But this assumption is not only still unproven in such generality but is decidedly false. It is well known, for example, that the function $\log x$ cannot be represented by such a series for any real number taken as the base. One may admit this, but say that the case where such an expansion does not apply is exceptional, and only occurs for certain values of the variable. Therefore as long as x can be *in its full generality* (as they say), the assumption $F(x+i) - Fx = Ax^{\alpha} + Bx^{\beta} + \cdots$ may be allowed. I do not want to criticize here the very improper way of speaking that one takes x in its full generality if it is taken so that it cannot represent *every arbitrary number*. This does mean exactly that an expression is *not* taken in its full generality. Therefore we should rather say that the equation $F(x+i) - Fx = Ax^{\alpha} + Bi^{\beta} + \cdots$ does not hold for every value of x, not even for every value of i, but always for *certain* values, even (if we wish) infinitely many values. But it is a question of the proof of this assertion. Those mathematicians who, like *Lagrange* himself, or more recently and most emphatically like *Ohm* (in his *System der Mathematik*, Theil 3, S. 58), understand by a function nothing but an expression in which one or more letters denoting a variable number, and possibly also certain numerical expressions, are combined by means of one or more operation signs, do indeed claim something here which can be completely justified by their concept. Some of them, particularly *Ohm*, have provided proofs for their assertion in which essentially nothing can be criticized. However, we may not forget that the proposition which they prove in such a way under the name of *Taylor* 's theorem, is a truth of very limited scope and use, with the statement of which we cannot be very satisfied from the present scientific perspective. That is to say, with that concept of a function the proposition would only be applicable to such variable and dependent numbers and quantities of which we already know in advance that their relationship of dependency can be represented by one or more operation signs. But there is a much more general truth, namely the one that every number which depends on another according to such a law that a derivative can be specified, must also obey the *Taylor* formula given in §199. Everyone sees that this is a much more general assertion. Or else how could it be that only by the use of *Taylor*'s theorem can it be proved that every number which depends on another according to the law just mentioned must be representable by an expression which includes nothing but those operations [mentioned], since it is often necessary that we apply [this result] to an infinite multitude and even here exceptions occur for individual values?

§ 209

Theorem. If for all values of its variable lying within a and b a function Fx not only has a first derivative, but also a second, third, ... and nth derivative, and if also this last derivative is continuous at least for all values of x lying within a and b, then for every value of x lying within a and b there is a positive, as well as a negative i, so small that we may not only form the equation

$$F(x \pm i) = Fx \pm i.F'x + \frac{i^2}{2}.F''x \pm \frac{i^3}{2.3}.F'''x \pm \cdots \pm \frac{i^n}{2.3\ldots n}.F^n(x \pm \mu i)$$

but that also every term in the series on the right-hand side which is not itself $= 0$, is, in its absolute value, greater than the algebraic sum of all those following it.

Proof. 1. If x is a value lying within a and b and we take i small enough that as well as being $< x - a$ it is also $< x - b$, then $x + i$ as well as $x - i$ lies within a and b. The function Fx therefore has the first n derivatives for all values of its variable from $x - i$ to $x + i$, and the last of them is still continuous. Therefore it follows from §199 that we have the equation:

$$F(x \pm i) = Fx \pm i.F'x + \frac{i^2}{2}.F''x \pm \frac{i^3}{2.3}.F'''x + \cdots \pm \frac{i^n}{2.3\ldots n}.F^n(x \pm \mu i),$$

for which μ denotes a certain number lying not outside 0 and 1. But if this equation holds first for a definite value of i, then it holds all the more for every smaller value.

2. If it is only required to find a value for i so that the last but one term in the series, namely $\frac{i^{n-1}}{2.3\ldots n-1}.F^{n-1}x$, becomes, in its absolute value, greater than the last term $\frac{i^n}{2.3\ldots n}.F^n(x + \mu i)$, then it is easy to satisfy this demand providing $F^{n-1}x$ is not $= 0$ for the given value of x, as we have already required in the theorem. That is, because $F^n x$ is continuous for all values of x lying within a and b, then among all its values there is a *greatest* in the sense of §60. If we denote this by $F^n q$, or else

another arbitrary greater number Q, then there is no doubt that $F^n(a+\mu i)$ in its absolute value is $\overline{\gtrless} Q$. Therefore if we take an i so small that as well as the two conditions $i < x - a$ and $i < b - x$, the third condition $i < \frac{n.F^{n-1}x}{Q}$ is also satisfied, then certainly also the relationship $i < \frac{n.F^{n-1}x}{F^n(x+\mu i)}$ holds, and consequently also, if we multiply both sides by $\frac{i^{n-1}}{2.3...n}$,

$$\frac{i^{n-1}.i}{2.3\ldots n} < \frac{i^{n-1}.n.F^{n-1}\,x}{2.3\ldots n.F^n(x+\mu i)},$$

that is

$$\frac{i^n}{2.3\ldots n}.F^n(x+\mu i) < \frac{i^{n-1}.F^{n-1}x}{2.3\ldots(n-1)}.$$

3. If it is not just the last but one term, but some earlier term, of which it is required, that in its absolute value it becomes greater than the algebraic sum of all the terms following it, then we can immediately reduce this problem to the one just considered, since we can always shorten the series on the right-hand side so that the given term becomes the last but one in it. For if the nth derivative $F^n x$ has the property of being continuous then all earlier ones also have this property because they themselves have a derivative. Therefore if, for example, it is the third term, or $\frac{i^2}{2}.F''x$, that we are to make greater than the algebraic sum of all following ones, then we just take μ such that $F(x+i) = Fx+i.F'x+\frac{i^2}{2}.F''x+\frac{i^3}{2.3}.F'''(x+\mu i)$ where the term $\frac{i^3}{2.3}.F'''(x+\mu i)$ = the sum of all terms following $\frac{i^2}{2}.F''x$, or $=\frac{i^3}{2.3}.F'''x + \frac{i^4}{2.3.4}.F^{\mathrm{IV}}x+\cdots+\frac{i^n}{2.3...n}.F^n(x+\mu i)$. Now if we choose i so small that as well as the two relationships $i < x-a$ and $i < b-x$, the third relationship $i < \frac{3.F''x}{F'''(x+\mu i)}$, also holds which is always possible if $F''x$ is not $= 0$, then $\frac{i^2}{2}.F''x > \frac{i^3}{2.3}.F'''(x+\mu i)$, therefore also $>\frac{i^3}{2.3}.F'''x+\frac{i^4}{2.3.4}.F^{\mathrm{IV}}x+\cdots+\frac{i^n}{2.3...n}.F^n(x+\mu i)$. Likewise if the first term Fx is to become greater than all following ones, or if $F(x+i) = Fx+iF(x+\mu i)$, it is only necessary for us to take $i < \frac{Fx}{F'(x+\mu i)}$.

Example. The function $\frac{1}{x}$ has all derivatives for all values of its variable lying within 0 and every other number, there is therefore always the equation,

$$\frac{1}{x+i} = \frac{1}{x} - \frac{1}{x^2} + \frac{i^2}{x^3} - \frac{i^3}{x^4} + \frac{i^4}{x^5} - \cdots \pm \frac{i^{n-1}}{(x+\mu i)^n}.$$

If we put here $x = \frac{1}{2}$, for example, then

$$\frac{1}{\frac{1}{2}+i} = 2 - 4i + 8i^2 - 16i^3 + 32i^4 - \cdots \pm \frac{i^{n-1}}{\left(\frac{1}{2}+\mu i\right)^n}.$$

Now if we want to find a value for i for which the $(n-1)$th term of this series, i.e. $\frac{i^{n-2}}{x^{n-1}}$, in its absolute value, becomes greater than the nth term, then for μ we need only choose that value lying not outside 0 and 1 for which the absolute value of $(F^{n-2}(x+\mu i)) = \frac{(n-2)!}{\left(\frac{1}{2}+\mu i\right)^{n-1}}$ becomes greatest. This obviously happens for $\mu = 0$, when this term changes into $(n-2)!2^{n-1}$. Therefore so as to obtain $i < \frac{(n-1).F^{n-2}x}{F^{n-1}q}$, we must choose $i < \frac{(n-1)(n-2)!2^{n-1}}{(n-1)!2^n} = \frac{1}{2}$. And in fact the value $i = \frac{1}{2}$ is already enough for our purpose, for the term $\frac{i^{n-2}}{x^{n-1}}$, with this value, becomes $= 2$, while for $\mu = \sqrt[n]{2} - 1$ the last term $\frac{i^{n-1}}{(x+\mu i)^n} = 1$. It also happens for this value, quite correctly, that,

$$\frac{1}{\frac{1}{2} + \frac{1}{2}} = 1 = 2 - 2 + 2 - 2 + 2 - 2 + \cdots \pm 2 \mp 1.$$

§ 210

Theorem. Suppose a function of x, for all values of its variable x from a inclusive to $a + h$, allows the following equation:

$$Fx = A + B(x-a) + C(x-a)^2 + D(x-a)^3 + \cdots + R(x-a)^r + \Omega$$

in which the symbols $A, B, C, D, \ldots, R$ denote certain measurable numbers *independent* of x, but Ω decreases indefinitely by simply increasing r. Moreover, suppose it is known that the function Fx, at least for the value a and for an increase which has the same sign as h, not only has a first derivative but also a second, third and all following derivatives indefinitely, then it must be that:

$$A = Fa, \quad B = F'a, \quad C = \frac{F''a}{2}, \quad D = \frac{F'''a}{2.3}, \ldots, \quad R = \frac{F^r a}{2.3 \ldots r}.$$

Proof. That we must have $Fa = A$ follows from the fact that the given equation is to hold for the value $x = a$, and here it changes into $Fa = A$. (*RZ* 7, §57) Furthermore, since not only the term on the left-hand side, i.e. Fx, but also all the terms appearing on the right-hand side of the equality sign, A, $B(x-a)$, $C(x-a)^2$, $D(x-a)^3, \ldots, R(x-a)^r$, up to Ω, have a first, second, third and all successive derivatives, because all these terms are simply rational functions of x, then if we take the first derivative of both sides of the equation, by the previous §,

$$F'x = B + 2C(x-a) + 3D(x-a)^2 + \cdots + rR(x-a)^{r-1} + \overset{\prime}{\Omega} \qquad (1)$$

and this equation must hold for the same values as the given one. Therefore if we take $x = a$, we have $F'a = B + \overset{\prime}{\Omega}$, from which by RZ 7, §92, $B = F'a$ follows. Similiarly, if we again take the derivative of both sides of equation (1),

$$F''x = 2C + 2.3D(x-a) + \cdots + (r-1)rR(x-a)^{r-2} + \overset{\prime\prime}{\Omega} \qquad (2)$$

from which, for $x = a$, there follows $F''a = 2C + \overset{\prime\prime}{\Omega}$ and $C = \frac{F''a}{2}$. In the same way $F'''x = 2.3D + \cdots + (r-2)(r-1)rR(x-a)^{r-3} + \overset{\prime\prime\prime}{\Omega}$ and from this for $x = a$, $F'''a = 2.3D + \overset{\prime\prime\prime}{\Omega}$, or $D = \frac{F'''a}{2.3}$. It is self-evident how these arguments can always be continued and, in general, give $R = \frac{F^r a}{2.3 \ldots r}$.

§ 211

Theorem. Suppose for a certain value of x lying within a and b a pair of functions Fx and Φx are equal in value to one another, and this also holds of their first derivatives, second derivatives, up to the nth derivatives, so that the values of the $(n+1)$th derivatives, $F^{n+1}x$ and $\Phi^{n+1}x$ are the first that differ for the specific value of x concerned. Moreover, suppose those derivatives are also continuous, then I claim that the difference $F(x+i) - \Phi(x+i)$, in its absolute value, simply by decreasing i, can be made smaller than the difference between one of these functions and a third function fx, for which not all the following $(n+1)$ equations hold, $Fx = fx$, $F'x = f'x$, $F''x = f''x$, $\ldots$, $F^n x = f^n x$, as they do between Fx and Φx.

Proof. Because the two functions Fx and Φx have at least $(n+1)$ derivatives, and these are also continuous, then the two equations hold:

$$F(x+i) = Fx + i.F'x + \frac{i^2}{2}.F''x + \cdots + \frac{i^n}{2.3\ldots n}.F^n x + \frac{i^{n+1}}{2.3\ldots(n+1)}.F^{n+1}(x+\mu i),$$

$$\Phi(x+i) = \Phi x + i.\Phi'x + \frac{i^2}{2}.\Phi''x + \cdots + \frac{i^n}{2.3\ldots n}.\Phi^n x + \frac{i^{n+1}}{2.3\ldots(n+1)}.\Phi^{n+1}(x+\nu i)$$

where μ and ν denote a pair of numbers lying not outside 0 and 1. Because also for the definite value of x it should be that: $Fx = \Phi x$, $F'x = \Phi'x$, $F''x = \Phi''x$, $\ldots$, $F^n x = \Phi^n x$, then we obtain,

$$F(x+i) - \Phi(x+i) = \frac{i^{n+1}}{2.3\ldots(n+1)}.[F^{n+1}(x+\mu i) - \Phi^{n+1}(x+\nu i)].$$

The difference $F^{n+1}(x+\mu i) - \Phi^{n+1}(x+\nu i)$ varies with i, but, if we can decrease i indefinitely starting from a certain value j, it can never become greater than it becomes when we put for $F^{n+1}(x+\mu i)$ the greatest value, and for $\Phi^{n+1}(x+\nu i)$ the smallest value, which the two functions $F^{n+1}(x+i)$ and $\Phi^{n+1}(x+i)$ take within x and $x+j$. There must be such greatest and smallest values of them by virtue of their assumed continuity. If we denote this greatest difference by Q, then in its absolute value,

$$F(x+i) - \Phi(x+i) \mathrel{\substack{=\\<}} \frac{i^{n+1}.Q}{2.3\ldots(n+1)}.$$

Now if, in the relationship between the two functions Fx and fx not even the first equation mentioned in the theorem holds, namely $Fx = fx$, then however much the difference $F(x+i) - f(x+i)$ may be decreased with i, it certainly does not decrease indefinitely because otherwise from the assumed continuity of the functions Fx and fx it would have to follow that $Fx = fx$. There is therefore a number P, which is small enough so as to remain always smaller than this difference, in its absolute value. Therefore $P < F(x+i) - f(x+i)$. If we then take i so small that $i^{n+1} < 2.3\ldots(n+1)\frac{P}{Q}$, then also $\frac{i^{n+1}Q}{2.3\ldots(n+1)} < P$, and all the more certainly $F(x+i) - \Phi(x+i)$ in its absolute value. But if $Fx = fx$, and several of the equations $F'x = f'x$ also hold, then let $F^m x = f^m x$ be the highest derivatives which equal one another, and so $m < n$. But we have,

$$F(x+i) = Fx + i.F'x + \frac{i^2}{2}.F''x + \cdots + \frac{i^m}{2.3\ldots m}.F^m x + \frac{i^{m+1}}{2.3\ldots(m+1)}.F^{m+1}(x+\mu i),$$

$$f(x+i) = fx + i.f'x + \frac{i^2}{2}.f''x + \cdots + \frac{i^m}{2.3\ldots m}.f^m x + \frac{i^{m+1}}{2.3\ldots(m+1)}.f^{m+1}(x+\nu i)$$

where μ and ν have not the same, but similar, meanings to those in the previous equations. Therefore,

$$F(x+i) - f(x+i) = \frac{i^{m+1}}{2.3\ldots(m+1)}.[F^{m+1}(x+\mu i) - f^{m+1}(x+\nu i)].$$

But because $F^{m+1}x \overset{n}{=} f^{m+1}x$,[g] then $F^{m+1}(x+\mu i)$ and $f^{m+1}(x+\nu i)$ do not approach one another indefinitely with the indefinite decrease of i. Therefore if we denote by P a number so small that the difference $[F^{m+1}(x+\mu i) - f^{m+1}(x+\nu i)]$, in its absolute value, always remains greater than it, then we have,

$$F(x+i) - f(x+i) > \frac{i^{m+1}}{2.3\ldots(m+1)}.P.$$

If we therefore choose a value for i small enough that $i^{n-m} < (m+2)(m+3)\ldots(n+1)\frac{P}{Q}$, then also

$$\frac{i^{n+1}}{2.3\ldots(n+1)}.Q < \frac{i^{m+1}.P}{2.3\ldots(m+1)}.$$

All the more certainly, $F(x+i) - \Phi(x+i) < F(x+i) - f(x+i)$.

Example. Let $Fx = \frac{3x^4-2x^3+6x+5}{12}$, $\Phi x = x^3 - 2x^2 + 2x, fx = x^3 + 2x^2 - 6x + 4$, then $F'x = \frac{2x^3-x^2+1}{2}$, $\Phi' x = 3x^2 - 4x + 2$, $f'x = 3x^2 + 4x - 6$; $F''x = 3x^2 - x$, $\Phi'' x = 6x - 4$, $f''x = 6x + 4$; $F'''x = 6x - 1$, $\Phi''' x = 6$, $f'''x = 6$; $F^{\mathrm{IV}}x = 6$, $\Phi^{\mathrm{IV}}x = 0$, $f^{\mathrm{IV}}x = 0$. Now if we take for x the particular value 1, then $Fx = \Phi x = fx = 1$, $F'x = \Phi' x = f'x = 1$, $F''x = \Phi'' x = 2$, but

[g] The notation $\overset{n}{=}$ is evidently being used for 'not equal to'.

$f''x = 10$, $F'''x = 5$, $\Phi'''x = f'''x = 6$, $F^{IV}x = 6$, $\Phi^{IV}x = f^{IV}x = 0$. Consequently, in accordance with our theorem the two functions $F(x+i)$ and $\Phi(x+i)$ come closer to one another for the smallest value of i than $F(x+i)$ and $f(x+i)$, or $\Phi(x+i)$ and $f(x+i)$. In fact, for the value $x = 1$,

$$F(1+i) = 1 + i + i^2 + \frac{5i^3}{6} + \frac{i^4}{4}$$

$$\Phi(1+i) = 1 + i + i^2 + i^3$$

$$f(1+i) = 1 + i + 5i^2 + i^3.$$

$$F(1+i) - \Phi(1+i) = -\frac{i^3}{6} + \frac{i^4}{4}$$

Therefore the difference $F(1+i) - f(1+i) = -4i^2 - \frac{i^3}{6} + \frac{i^4}{4}$ where the second difference, in its absolute value, is obviously greater than the first; for example, if we just put $i = 1$.

§ 212

Theorem. Suppose a function $F(x, y)$ of two variables x, y independent of one another, has a first, second, ... and mth derivative with respect to x, for every value of x lying within x and $x + \Delta x$, and it has a first, second, ... and nth derivative with respect to y for every value of y lying within y and $y + \Delta y$. Furthermore, suppose also that $\frac{dF(x,y)}{dx}$ has a first, second, ..., pth derivative with respect to y, that $\frac{d^2F(x,y)}{dx^2}$ has a first, second, ..., qth derivative with respect to y, etc., and that $\frac{d^mF(x,y)}{dx^m}$ has a first, second, ..., tth derivative with respect to y. Finally, suppose that these last derivatives are continuous within the limits x and $x + \Delta x$, and y and $y + \Delta y$, and the function $F(x, y)$ itself is also continuous for the values x and $x + \Delta x$, y and $y + \Delta y$, then the equation can always be formed:

$$\begin{aligned} F(x+\Delta x, y + \Delta y) &\\ = F(x, y) + \Delta x.\frac{dF(x,y)}{dx} + \frac{\Delta x^2}{2}.\frac{d^2F(x,y)}{dx^2} + \cdots &\\ + \frac{\Delta x^m}{2.3\ldots m}.\frac{d^mF(x+\mu\Delta x, y)}{dx^m} + \Delta y.\frac{dF(x,y)}{dy} &\\ + \frac{\Delta y^2}{2}.\frac{d^2F(x,y)}{dy^2} + \cdots + \frac{\Delta y^n}{2.3\ldots n}.\frac{d^nF(x, y+\nu\Delta y)}{dy^n} & \end{aligned}$$

$$+\Delta x.\Delta y.\frac{d^2F(x,y)}{dx\,dy}+\Delta x.\frac{\Delta y^2}{2}.\frac{d^2F(x,y)}{dx\,dy^2}{}^{h}+\cdots$$

$$+\Delta x.\frac{\Delta y^p}{2.3\ldots p}.\frac{d^{p+1}F(x,y+\pi\Delta y)}{dx\,dy^p}$$

$$+\frac{\Delta x^2}{2}.\frac{d^3F(x,y)}{dx^2dy}{}^{i}+\frac{\Delta x^2}{2}.\frac{\Delta y^2}{2}.\frac{d^4F(x,y)}{dx^2dy^2}+\cdots$$

$$+\frac{\Delta x^2}{2}.\frac{\Delta y^q}{2.3\ldots q}.\frac{d^{q+2}F(x,y+\chi\Delta y)}{dx^2dy^q}$$

$$+\frac{\Delta x^m}{2.3\ldots m}.\Delta y.\frac{d^{m+1}F(x+\mu\Delta x,y)}{dx^m dy}$$

$$+\frac{\Delta x^m.\Delta y^2}{2.3\ldots m.2}.\frac{d^{m+2}F(x+\mu\Delta x,y)}{dx^m dy^2}+\cdots$$

$$+\frac{\Delta x^m.\Delta y^t}{2.3\ldots m.2.3\ldots t}.\frac{d^{m+t}F(x+\mu\Delta x,y+\tau\Delta y)}{dx^m dy^t}.$$

Proof. From the conditions assumed, it follows by §199 that if we let x increase by Δx,

$$F(x+\Delta x,y)=F(x,y)+\Delta x.\frac{dF(x,y)}{dx}+\frac{\Delta x^2}{2}.\frac{d^2F(x,y)}{dx^2}+\cdots$$
$$+\frac{\Delta x^m}{2.3\ldots m}.\frac{d^mF(x+\mu\Delta x,y)}{dx^m}. \quad (1)$$

But if y also increases by Δy, then it must be that

$$F(x+\Delta x,y+\Delta y)=F(x,y+\Delta y)+\Delta x.\frac{dF(x,y+\Delta y)}{dx}$$
$$+\frac{\Delta x^2}{2}.\frac{d^2F(x,y+\Delta y)}{dx^2}+\cdots$$
$$+\frac{\Delta x^m}{2.3\ldots m}.\frac{d^mF(x+\mu\Delta x,y+\Delta y)}{dx^m}. \quad (2)$$

However the term $F(x,y+\Delta y)$ can again be expanded in the following way.

$$F(x,y+\Delta y)=F(x,y)+\Delta y.\frac{dF(x,y)}{dy}+\frac{\Delta y^2}{2}.\frac{d^2F(x,y)}{dy^2}+\cdots$$
$$+\frac{\Delta y^n}{2.3\ldots n}.\frac{d^nF(x,y+\nu\Delta y)}{dy^n}. \quad (3)$$

[h] The final factor of this term should, presumably, be $\frac{d^3F(x,y)}{dxdy^2}$.

[i] There should, presumably, be a factor Δy in this term.

Furthermore, the term $\Delta x.\frac{\mathrm{d}F(x,y+\Delta y)}{\mathrm{d}x}$ can also be expanded if we consider $\frac{\mathrm{d}F(x,y)}{\mathrm{d}x}$ as a primitive function, of which the first, second, ..., pth derivatives with respect to y can be represented by $\frac{\mathrm{d}^2F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$, $\frac{\mathrm{d}^3F(x,y)}{\mathrm{d}x\,\mathrm{d}y^2}$, ..., $\frac{\mathrm{d}^{p+1}F(x,y)}{\mathrm{d}x\,\mathrm{d}y^p}$. We therefore obtain,

$$\begin{aligned}\frac{\Delta x.\mathrm{d}F(x,y+\Delta y)}{\mathrm{d}x} = \Delta x.\frac{\mathrm{d}F(x,y)}{\mathrm{d}x} + \Delta x.\Delta y.\frac{\mathrm{d}^2F(x,y)}{\mathrm{d}x\mathrm{d}y} \\ + \Delta x.\frac{\Delta y^2}{2}.\frac{\mathrm{d}^3F(x,y)}{\mathrm{d}x\mathrm{d}y^2} + \cdots \\ + \frac{\Delta x.\Delta y^p}{2.3\ldots p}.\frac{\mathrm{d}^{p+1}F(x,y+\pi\Delta y)}{\mathrm{d}x\,\mathrm{d}y^p}.\end{aligned} \tag{4}$$

Similarly the term $\frac{\Delta x^2}{2}.\frac{\mathrm{d}^2F(x,y+\Delta y)}{\mathrm{d}x^2}$ can be expanded further if we consider $\frac{\mathrm{d}^2F(x,y)}{\mathrm{d}x^2}$ as a primitive function of which the first, second, ..., qth derivatives with respect to y can be represented by $\frac{\mathrm{d}^3F(x,y)}{\mathrm{d}x^2\mathrm{d}y}$, $\frac{\mathrm{d}^4F(x,y)}{\mathrm{d}x^2\mathrm{d}y^2}$, ..., $\frac{\mathrm{d}^{q+2}F(x,y)}{\mathrm{d}x^2\mathrm{d}y^q}$. Therefore

$$\begin{aligned}\frac{\Delta x^2}{2}.\frac{\mathrm{d}^2F(x,y+\Delta y)}{\mathrm{d}x^2} = \frac{\Delta x^2}{2}.\frac{\mathrm{d}^2F(x,y)}{\mathrm{d}x^2} + \frac{\Delta x^2}{2}.\Delta y.\frac{\mathrm{d}^3F(x,y)}{\mathrm{d}x^2\mathrm{d}y} \\ + \frac{\Delta x^2}{2}.\frac{\Delta y^2}{2}.\frac{\mathrm{d}^4F(x,y)}{\mathrm{d}x^2\mathrm{d}y^2} + \cdots \\ + \frac{\Delta x^2}{2}.\frac{\Delta y^q}{2.3\ldots q}.\frac{\mathrm{d}^{q+2}F(x,y+K\Delta y)}{\mathrm{d}x^2\mathrm{d}y^q}.\end{aligned} \tag{5}$$

And so on. Finally, the term $\frac{\Delta x^m}{2.3\ldots m}.\frac{\mathrm{d}^mF(x+\mu\Delta x,y+\Delta y)}{\mathrm{d}x^m}$ can also be expanded if we consider $\frac{\mathrm{d}^mF(x+\mu\Delta x,y)}{\mathrm{d}x^m}$ as a primitive function of which the first, second, ..., tth derivatives with respect to y can be represented by $\frac{\mathrm{d}^{m+1}F(x+\mu\Delta x,y)}{\mathrm{d}x^m\mathrm{d}y}$, $\frac{\mathrm{d}^{m+2}F(x+\mu\Delta x,y)}{\mathrm{d}x^m\mathrm{d}y^2}$, ..., $\frac{\mathrm{d}^{m+t}F(x+\mu\Delta x,y)}{\mathrm{d}x^m\mathrm{d}y^t}$. We therefore obtain

$$\begin{aligned}\frac{\Delta x^m}{2.3\ldots m}.\frac{\mathrm{d}^mF(x+\mu\Delta x,y+\Delta y)}{\mathrm{d}x^m} \\ = \frac{\Delta x^m}{2.3\ldots m}.\frac{\mathrm{d}^mF(x+\mu\Delta x,y)}{\mathrm{d}x^m} + \frac{\Delta x^m}{2.3\ldots m}.\Delta y.\frac{\mathrm{d}^{m+1}F(x+\mu\Delta x,y)}{\mathrm{d}x^m\mathrm{d}y} \\ + \frac{\Delta x^m}{2.3\ldots m}.\frac{\Delta y^2}{2}.\frac{\mathrm{d}^{m+2}F(x+\mu\Delta x,y)}{\mathrm{d}x^m\mathrm{d}y^2} + \cdots \\ + \frac{\Delta x^m.\Delta y^t}{2.3\ldots m.1.2\ldots t}.\frac{\mathrm{d}^{m+t}F(x+\mu\Delta x,y+\tau\Delta y)}{\mathrm{d}x^m\mathrm{d}y^t}.\end{aligned} \tag{6}$$ [j]

The substitution of the values (3), (4), (5), (6) into (2) gives the equation of the theorem.

[j] The continuation dots in 1, 2, ..., t in this term are missing in $F(2)$.

§ 213

Corollary 1. It is self-evident that similar equations can be given for functions of three or more variables.

§ 214

Corollary 2. If $F(x, y, z, \ldots)$ is *an integral, rational* function of several free variables x, y, z, ... then all the conditions hold which are required for the application of the formula in the previous theorem for every value of x, y, z, ... and Δx, Δy, $\Delta z, \ldots$. In fact, if the function is of the mth degree, all derivatives which have been obtained as derivatives higher than m are $=0$. The change in such a function if its variables increase by arbitrary amounts Δx, Δy, Δz, ..., i.e. $F(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) - F(x, y, z, \ldots)$, can therefore always be represented by a series of increasing powers of these increases. Thus, for example, if we had: $F(x, y) = x^3 + 2x^2y - 2xy^2 + y^3$, $\frac{dF(x,y)}{dx} = 3x^2 + 4xy - 2y^2$, $\frac{d^2F(x,y)}{dx^2} = 6x + 4y$, $\frac{d^3F(x,y)}{dx^3} = 6$, $\frac{dF(x,y)}{dy} = 2x^2 - 4xy + 3y^2$, $\frac{d^2F(x,y)}{dy^2} = -4x + 6y$, $\frac{d^3F(x,y)}{dy^3} = 6$, $\frac{d^2F(x,y)}{dx\,dy} = 4x - 4y$, $\frac{d^3F(x,y)}{dx\,dy^2} = -4$, $\frac{d^3F(x,y)}{dx^2dy} = 4$, and all following ones $=0$. Therefore

$$
\begin{aligned}
&(x + \Delta x)^3 + 2(x + \Delta x)^2(y + \Delta y) \\
&\quad -2(x + \Delta x)(y + \Delta y)^2 + (y + \Delta y)^3 \\
&\qquad = x^3 + 2x^2y - 2xy^2 + y^3 + (3x^2 + 4xy - 2y^2)\Delta x \\
&\qquad\quad + (6x + 4y)\frac{\Delta x^2}{2}^{\text{k}} + 6.\frac{\Delta x^3}{2.3} + (2x^2 - 4xy + 3y^2)\Delta y, \\
&\qquad\quad + (-4x + 6y)\frac{\Delta y^2}{2} + 6.\frac{\Delta y^3}{2.3} + (4x - 4y)\Delta x.\Delta y \\
&\qquad\quad - 4\Delta x.\frac{\Delta y^2}{2} + 4.\frac{\Delta x^2}{2}.\Delta y
\end{aligned}
$$

as the expansion of both sides of the equation also confirms.

§ 215

Corollary 3. If not[k] only the first m or n derivatives of $F(x, y)$ with respect to each variable x, y, ..., but also all following derivatives are measurable numbers (including the case where these derivatives from a certain one onwards are zero), then the series of which the right-hand side of the equation is composed can be continued as far as we please, and if it turns out that the terms converge then it can be regarded, by §204, as proceeding indefinitely.

---

[k] The denominator of this term in $F(2)$ is x, but we have assumed it should be 2.

§ 216

Corollary 4. In particular this last result always holds if all derivatives of $F(x, y)$ always remain smaller than a certain constant number M, which may be proved in a similar way to that of §205.

§ 217

Corollary 5. It can also be proved in a similar way to that of §209 that in all cases where the formula of the theorem can be applied it is also possible, if the increases Δx, Δy are all put equal to one another, and the series is arranged in powers of Δx, to take this Δx so small that every term which is multiplied with one and the same power of Δx, providing its coefficient is not $= 0$, is *greater* than the sum of all successive terms.

Improvements and Additions to the Theory of Functions

Theory of Functions

Improvements to the section on the differential calculus

§1 [...?...] on 21.[a] First of all, it is well to check whether the theorem put forward here: that the derivative of a sum $Fx + \phi x = F'x + \phi' x$, was not already needed earlier. But in any case what must be called the *corollary* is to change not to an *infinite*, but to a *finite* number [of terms]. But then the theorem follows:

§2 *Theorem.* If the algebraic sum of the infinitely many functions $f_1 x + f_2 x + f_3 x + \cdots f_r x + \cdots$ *in inf*. forms a *converging* series for all values of the variable x which lie within certain limits a and b, and each of these functions $f_r x$ has its derivative $f_r' x$, then the sum which appears if we take the derivatives of each summand, $f_1' x + f_2' x + f_3' x + \cdots$ *in inf*. also converges, and this sum is to be considered as the derivative of the former.

Proof. 1. Because $f_1 x + f_2 x + f_3 x + \cdots$ *in inf*. converges, then $f_r x + f_{r+1} x + \cdots$ *in inf*. represents a function of x and r which, with every value of x lying within a and b, can be reduced indefinitely merely by the increase of r (§150). Also the series consisting of s terms, $f_{r+1} x + f_{r+2} x + \cdots + f_{r+s} x$ represents such a function however s may be taken. However, since this latter series consists merely of a finite number of terms, certainly has a derivative which is $f_{r+1}' x + f_{r+2}' x + \cdots + f_{r+s}' x$. Therefore according to the theorem of §150 this derivative must also have the property of decreasing indefinitely merely through the increase in r, for every value of x lying within a and b, while s remains completely arbitrary. However if $f_{r+1}' x + f_{r+2}' x + \cdots + f_{r+s}' x$ can be made as small as desired for every value of s, as long as r is taken large enough, then it follows from the § that the series $f_1' x + f_2' x + f_3' x + \cdots + f_r' x +$ *in inf*. is a *convergent* one.

[a] The notation [...?...] is used in *BGA* 2A10/1 for something illegible in the mss. The number 21 refers to a pagination of the manuscript of *F*. The sheet concerns *F* §§ 154,155.

2. It is clear that this series may be considered as the derivative of $f_1x+f_2x+f_3x+\cdots+f_rx+$ *in inf*. as follows. If we write $Fx = f_1x+f_2x+\cdots+f_rx+\Omega(x,r)$,[b] and $F(x+\Delta x) = f_1(x+\Delta x)+f_2(x+\Delta x)+\cdots+f_r(x+\Delta x)+\Omega(x+\Delta x,r)$, then $\Omega(x,r)$ and $\Omega(x+\Delta x,r)$ denote a pair of numbers which can decrease indefinitely merely by increasing r, as long as x and $x+\Delta x$ lie within a and b. Therefore also

$$\frac{F(x+\Delta x)-Fx}{\Delta x} = \frac{f_1(x+\Delta x)-f_1x}{\Delta x}+\frac{f_2(x+\Delta x)-f_2x}{\Delta x}+\cdots + \frac{f_r(x+\Delta x)-f_rx}{\Delta x}+\frac{\Omega_\Delta-\Omega}{\Delta x}$$

or if we put in place of each term of the form $\frac{f_r(x+\Delta x)-f_rx}{\Delta x} = f'_rx+\overset{r}{\omega}$, we put the latter, where $\overset{1}{\omega}, \overset{2}{\omega}, \ldots, \overset{r}{\omega}$ will denote numbers which can decrease indefinitely through diminishing Δx,

$$\frac{F(x+\Delta x)-Fx}{\Delta x} = [f'_1x+f'_2x+f'_3x\cdots f'_rx]+\overset{1}{\omega}+\overset{2}{\omega}+\cdots + \overset{r}{\omega}+\frac{\Omega(x+\Delta x,r)-\Omega(x,r)}{\Delta x}$$

Now if we increase r by s then the terms $\frac{f_{r+1}(x+\Delta x)-f_{r+1}x}{\Delta x}, \frac{f_{r+2}(x+\Delta x)-f_{r+2}x}{\Delta x}, \ldots, \frac{f_{r+s}(x+\Delta x)-f_{r+s}x}{\Delta x}$ are added in to the above equations, therefore we obtain,

$$\begin{aligned}&\frac{F(x+\Delta x)-Fx}{\Delta x}-[f'_1x+f'_2x+\cdots+f'_rx]\\ &=\overset{1}{\omega}+\overset{2}{\omega}+\cdots+\overset{r}{\omega}\\ &\quad+\frac{[f_{r+1}(x+\Delta x)+\cdots+f_{r+s}(x+\Delta x)]-[f_{r+1}x+\cdots+f_{r+s}x]}{\Delta x}\\ &\quad+\frac{\Omega(x+\Delta x,r+s)-\Omega(x,r+s)}{\Delta x}.\end{aligned}$$

Now r can always be taken so great that

$$\frac{f_{r+1}(x+\Delta x)-f_{r+1}x}{\Delta x}+\frac{f_{r+2}(x+\Delta x)-f_{r+2}x}{\Delta x}+\cdots + \frac{f_{r+s}(x+\Delta x)-f_{r+s}x}{\Delta x}$$

will become, and will remain, $<\frac{1}{3N}$, however small Δx may be taken and however large s may become; because by diminishing Δx the individual quotients $\frac{f_{r+s}(x+\Delta x)-f_{r+s}x}{\Delta x}$ always get closer to a definite value, namely $f'_{r+s}x$, which itself can become as small as desired if r is taken large enough. Therefore if Δx is

[b] Bolzano wrote the term $\Omega(x,r)$ with x inside the Ω and r as a centred superscript. Thus $\Omega(x+\Delta x, r+s)$ required an especially large Ω symbol. We thought it unnecessary to imitate this here.

taken so small that also $\overset{1}{\omega} + \cdots + \overset{r}{\omega} < \frac{1}{3N}$, and finally s is taken so large that $\frac{\Omega(x+\Delta x, r+s) - \Omega(x, r+s)}{\Delta x} < \frac{1}{3N}$, then the difference $\frac{F(x+\Delta x) - Fx}{\Delta x} - [f_1'x + \cdots + f_r'x] < \frac{1}{N}$, where by diminishing Δx and increasing r, N can become as large as desired. Therefore $\frac{F(x+\Delta x)-Fx}{\Delta x} = f_1'x + f_2'x + \cdots + \textit{in inf.} + \Omega$, where Ω decreases indefinitely through the diminishing of Δx. Therefore $f_1'x + \cdots$ *in inf.* is the derivative of Fx.

§3 *Theorem.* If a pair of infinite series $A + Bx + Cx^2 + Dx^3 + \cdots$ in inf. and $A + Bx + Cx^2 + Dx^3 + \cdots$ *in inf.* of which one converges and the other is continually equivalent [to it] for all values of x which are < than a certain one, then it must be that $A = A$, $B = B$, $C = C, \cdots$

Proof. If both series converge and $A + Bx + Cx^2 + Dx^3 + \cdots = A + Bx + Cx^2 + Dx^3 + \cdots$ *in inf.* then it must be possible to give a value of x with $A - A + (B - B)x + (C - C)x^2 + \cdots = 0$. Therefore $A = A$. Both functions must have a *derivative* $B - B + 2(C - C)x + \cdots = 0$. Therefore $B = B$, etc.

§4 Here among others also the propositions: If two series converge then their sum also converges; their product if the terms all have the same signs etc. (from Cauchy's *Algèbre*).

§5 For the differential calculus. The theorem that every continuous function, among all its values from $x = a$ to $x = b$ inclusive, must have a greatest and smallest value,[c] (and indeed some other similar propositions) must also be extended to functions of two or more variables. Then for a function of two variables I use this proposition with the theory of equations where I claim that the modulus of an expression which is a polynomial in $\alpha + \beta\sqrt{-1}$ has a minimum.

§6 *Theorem.* If a function Fx is continuous for all values of the variable x from $x = a$ to $x = b$ inclusive, then there is a certain number e sufficiently small that for all values of x which do not lie outside a and b, the increase Δx does not need to become $<e$ so that the difference $F(x + \Delta x) - Fx$ turns out < than a given number $\frac{1}{N}$.[d]

Proof. Assume that for the following values of x which do not lie outside a and b, $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$ an ever smaller Δx must be taken so that the condition $F(x+\Delta x) - Fx < \frac{1}{N}$ is satisfied. Now if the number of terms of this series is only finite, then there is no doubt that there is an $\overset{n}{x}$ among them for which the Δx belonging to it is the smallest (i.e. has no smaller one after it), and the theorem is already proved. But if the series goes to infinity then the suspicion arises that the Δx becomes smaller than every given fraction. But we know from § that there is in every case a number c, lying not outside a and b, with the property that within c and $c+j$ or $c-j$ there are an infinite number of those terms of the series $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$, included, and indeed this [is true] even if j is decreased indefinitely. Now if the function is to be

[c] This is the theorem in *F* §60.

[d] It is apparent from the proof that the concept of uniform continuity is intended.

continuous also for $x = c$, then there must be a number e small enough that $F(c+e) - F(c) < \frac{1}{2N}$. I claim about this number e that no Δx needs to be smaller than it so that the relationship $F(x + \Delta x) - Fx < \frac{1}{N}$ is produced. There will be no doubt that Δx need never become smaller than e if I show that Δx need not become $<e$ even for such an x' which lies within the limits c and $c + j$ if j decreases indefinitely, i.e. generally for a value of $x = c \pm j$, where j can decrease indefinitely. Now if j can decrease indefinitely then one can surely find a value of it that is so small that for it and for all smaller ones, the following two relationships hold at the same time

$$F(c - j) - Fc < \frac{1}{4N} \qquad \text{and}$$

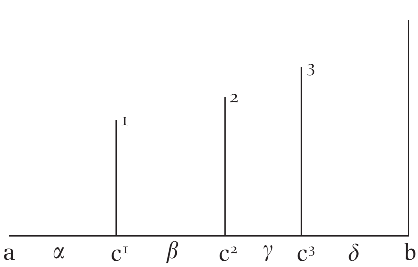
$$F(c + e - j) - F(c + e) < \frac{1}{4N}$$

But then by subtraction

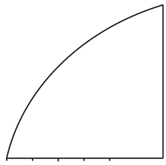
$$F(c + e - j) - F(c - j) - (F(c + e) - Fc) < \frac{1}{2N}$$

and because $F(c + e) - Fc < \frac{1}{2N}$, also $F(c - j + e) - F(c - j) < \frac{1}{N}$. Therefore all the more certainly does the relationship hold for every other value of x that $F(x + e) - Fx < \frac{1}{N}$.

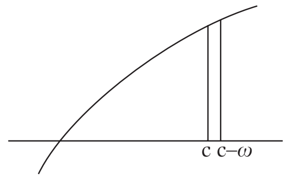
Objection. There could be an ∞ multitude of such c of which each one has an ∞ multitude of the $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}$, near to it, and consequently an infinite multitude of $\overset{1}{e}, \overset{2}{e}, \overset{3}{e}, \ldots$. If the values of Δx which belong to the above values of x so that the relationship $F(x + \Delta x) - Fx < \frac{1}{N}$ is satisfied, are not all equal to one another, then we can think of them as a decreasing series of the kind $\Delta\overset{1}{x}, \Delta\overset{2}{x}, \Delta\overset{3}{x}$—all the others are among these terms, so that this series has ∞ many terms if there is no smallest Δx. If there is only a single value of $x = c$ in whose neighbourhood the smallest terms of the series group? Not necessarily. There can be several, even infinitely many $\overset{1}{c}, \overset{2}{c}, \overset{3}{c}, \ldots$ which approach indefinitely the $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots$ belonging to $\Delta\overset{1}{x}, \Delta\overset{2}{x}, \Delta\overset{3}{x}, \ldots$. If there are ∞ many c for which Δx must decrease indefinitely so that $f(x + \Delta x) - fx < \frac{1}{N}$ therefore must be $<e$, then the difference $fb - fa$ consists of infinitely many $\frac{1}{N}$. Let us add to the above condition the further condition of being continually increasing. Therefore if no number n is large enough to give a quotient $\frac{i}{n}$ small enough that $f(x+\frac{i}{n}) - fx < \frac{1}{N}$ for all values from $x = a$ to $x = b$ then on the contrary $f(x + \frac{i}{n}) - fx > \frac{1}{N}$.



Therefore if we divide $[ab]$ into the parts $\alpha + \beta + \gamma + \delta + \cdots$ whose multitude may be ∞ then the difference $fb - fa \geqq [f(a+\alpha) - fa] + [f(a+\alpha+\beta) - f(a+\alpha)] + \cdots > \frac{1}{N} + \frac{1}{N} + \cdots$ i.e. infinitely large.



$$f(a+\alpha) - fa \geq \tfrac{1}{N}$$
$$f(a+\alpha+\beta) - f(a+\alpha) > \tfrac{1}{N}$$



If the differences $\overset{1}{e}, \overset{2}{e}, \overset{3}{e}, \ldots$ become ever smaller so the values of x, $a+\overset{1}{\alpha}$, $a+\overset{1}{\alpha}+\overset{2}{\alpha}, a+\overset{2}{\alpha}+\overset{3}{\alpha}+\ \ldots$, approach the value $a+i$ then I say there must be some value of x for which fx is continuous. That is, there must be some value c in whose neighbourhood the differences $\overset{1}{e}, \overset{2}{e}, \ldots$ become smaller than any given $\bar{e}$. For this c, fc is continuous at least in a negative respect. For $f(c-\omega) - fc$ must be able to become smaller than any given number if ω is made small enough. Suppose it remained that $f(c-\omega)-fc > u$, however small ω was made. Suppose the function fx increases continuously from $x = a$ to $x = b$ in such a way that the increases in fx for $a, a+\overset{1}{\alpha}, a+\overset{1}{\alpha}+\overset{2}{\alpha}, a+\overset{1}{\alpha}+\overset{2}{\alpha}+\overset{3}{\alpha}, \ldots$ etc., namely

$$f(a+\overset{1}{\alpha}) - fa > \overset{1}{e}$$

$$f(a+\overset{1}{\alpha}+\overset{2}{\alpha}) - f(a+\overset{1}{\alpha}) > \overset{2}{e}$$

$$f(a+\overset{1}{\alpha}+\overset{2}{\alpha}+\overset{3}{\alpha}) - f(a+\overset{1}{\alpha}+\overset{2}{\alpha}) > \overset{3}{e}$$

...

but $\overset{1}{e}, \overset{2}{e}, \overset{3}{e}, \ldots$ themselves become ever smaller.

If a function fx increases continually from $x = a$ to $x = b$ but remains finite and there are values of x, $a+\overset{1}{\alpha}, a+\overset{1}{\alpha}+\overset{2}{\alpha}$, which lie within a and b then the following differences must all be positive and finite

$$f(a+\overset{1}{\alpha}) - fa = \overset{1}{e}$$

$$f(a+\overset{1}{\alpha}+\overset{2}{\alpha}) - f(a+\overset{1}{\alpha}) = \overset{2}{e}$$

$$f(a+\overset{1}{\alpha}+\overset{2}{\alpha}+\overset{3}{\alpha}) - f(a+\overset{1}{\alpha}+\overset{2}{\alpha}) = \overset{3}{e}$$

Let $\overset{1}{\alpha}, \overset{2}{\alpha}, \overset{3}{\alpha}, \ldots$ decrease indefinitely, then also the values $\overset{1}{e}, \overset{2}{e}, \ldots$ must decrease indefinitely. For if the latter all remain $> \frac{1}{N}$ then their sum would be ∞ large. But if the values of $\overset{1}{e}, \overset{2}{e}, \ldots$ decrease indefinitely as $\overset{1}{\alpha}, \overset{2}{\alpha}, \ldots$ decrease indefinitely then for a certain value of x, $= c$ to which $a+\overset{1}{\alpha}+\overset{2}{\alpha}+\ldots$ approaches indefinitely, fc must be continuous at least in a negative respect.

If $fc - f(c-\omega) < \frac{1}{N}$ then it is continuous at least at b or at other values lying before b.

Therefore it is proved that even if it makes some, perhaps ∞ many jumps, it may not continually jump throughout any interval, however small. Therefore it is clear that it is continuous with the exception of certain isolated values (of finite or infinite multitude).

§7 *Theorem.* If the equation $Fx = f_1x + f_2x + f_3x + \cdots$ *in inf*. holds for every value of x within certain limits a and b, and each of these functions has a primitive which we denote by $\int Fx, \int fx, \ldots$ then the equation $\int Fx = \int f_1x + \int f_2x + \int f_3x + \cdots$ *in inf*. $+ C$ also holds, providing the series $\int f_1x + \int f_2x + \int f_3x + \cdots$ *in inf*. converges.

Proof. If this were not so it would certainly have to be that $\int Fx = \int f_1x + \int f_2x + \int f_3x + \cdots + \phi x$, where ϕx represents some function of x. But then differentiation would give $Fx = f_1x + f_2x + f_3x + \cdots + \frac{d\phi x}{dx}$. Therefore it must be that $\frac{d\phi x}{dx} = 0$, and therefore $\phi x = C$.

§8 *Assumption.* If $f_1x + f_2x + f_3x +$ *in inf*., converges (for all values of x within a and b) then also the series of primitives $\int f_1x + \int f_2x + \cdots$ *in inf*. converges.

Proof. Since $f_1x + \cdots f_{r+1}x + \cdots +$ *in inf*. converges then it must be that $f_{r+1}x + f_{r+2}x + \cdots$ *in inf*. $= \Omega$ provided r is taken sufficiently large. Now suppose $\int f_{r+1}x + \int f_{r+2}x + \cdots = \phi(x, r)$ then by differentiation it must be that $f_{r+1}x + f_{r+2}x + \cdots$ *in inf*. $= \frac{d\phi(x,r)}{dx} = \Omega$. Therefore [this is] merely by decrease (equally increase) in r, therefore [it is] not dependent on x. Therefore it is really $= \Phi(r)$ and consequently $\Phi(x, r) = x\Phi(r)$. Where with ∞ increase in r our Φr must decrease indefinitely. Therefore $\int f_{r+1} + \cdots$ *in inf*. $= \Omega$.

§9 On the binomial series. *Cauchy* also proves in his *Exercices* 1. Heft that it converges for $x = 1$, and $n = -1$ to $n = \infty$.

Additions to the section on the differential calculus

§10 *Taylor's* theorem must be presented more generally than *Slivka* has proved it (*Analecta Mathematica* 16 Heft p. 1369, or in the *Beylage*).

Nevertheless it must first be specifically proved that $\phi(x, y)$ is a function which has a derivative. Also the appropriate consideration must be given to powers with negative exponents or zero exponent, and the proof I leave to the reader.

§11 *Theorem.* The function x^{-n} has a derivative for every measurable value of x for which $-nx^{-n-1}$ is measurable, and this derivative is then $-nx^{-n-1}$.

Proof. If we write $y = x^{-n}$, then $y.x^n = 1$, $(y + \Delta y)(x + \Delta x)^n = 1$ or $(y + \Delta y)(x^n + nx^{n-1}\Delta x + \frac{n.n-1}{1.2}x^{n-2}\Delta x^2 + \cdots) = 1$. Therefore $y.x^n + y.x^{n-1}\Delta x + y.\frac{n.n-1}{1.2}.x^{n-2}\Delta x^2 + \cdots + \Delta y.x^n + \Delta y.n.x^{n-1}\Delta x + \cdots = 1$, $\frac{\Delta y}{\Delta x} =$ etc. First $(a+b)^0 = a^0 + 0.a^{-1}b + \cdots$

§12 *Theorem.* Therefore $(1 + x)^{-n} = 1 - nx - \frac{n.-n-1}{1.2}x^2 + \cdots$ whenever the series converges.

Proof. From *Taylor's* theorem, since $fx + if'x + \cdots$, is $1 - nx + \cdots$, for $x = 1$, $i = x, fx = x^{-n}$.

§13 Where the theorem of the differentiation of powers appears, the corollary gives that $a+bx^{\beta}+cx^{\gamma}+dx^{\delta}+\cdots$, if $\alpha, \beta, \gamma, \ldots$ denote whole positive or negative numbers, and the number of terms is arbitrary, has the derivative $\beta bx^{\beta-1}+\gamma cx^{\gamma-1}+\delta dx^{\delta-1}+\cdots$.

With integration the theorem §188 is (probably) conceived in this way: $x^{\pm m}$ can be considered as a derivative, if $\frac{x^{\pm m+1}}{\pm m+1}$ represents a measurable number. But if not, then $x^{\pm m}$ need not always have a primitive.

But even for x^{-1} there is a *primitive*. Namely $C+x-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots$ *in inf*. For this differentiated gives $1-(x-1)+(x-1)^2-(x-1)^3+\cdots=\frac{1}{1+(x-1)}=\frac{1}{x}$. For the special value $C=-1$ the series $C+x-\frac{(x-1)^2}{2}+\cdots$ vanishes for $x=1$. It is called the lgnt x. d lgnt $x^n=\frac{1}{x^n}.nx^{n-1}=\frac{n}{x}=n.$d lgnt x. Integrated lgnt $x^n=C+n$lgnt x for $x=1$, $0=C=n.0$, therefore $C=0$.

$$\text{lgnt}\, x^n = n\,\text{lgnt}\, x \qquad 1=(\mathrm{e}-1)-\frac{(\mathrm{e}-1)^2}{2}+\cdots$$

$$\text{lgnt}\, \mathrm{e}^n = n \qquad \text{lgnt}\, \mathrm{e} = 1$$

§14 Also to this section belongs the theorem in which case the series

$$1+mx+\frac{m.m-1}{1.2}x^2+\frac{m.m-1.m-2}{1.2.3}x^3+\cdots$$

and also

$$1-mx+\frac{m.m-1}{1.2}x^2+\frac{m.m-1.m-2}{1.2.3}x^3+\cdots$$

converges or diverges if m denotes an arbitrary measurable number. Following *Ettingshausen*, or *Ohm*, *Aufsätze aus dem Gebiete der höheren Mathematik*.

§15 *Theorem.* If the quotient of two consecutive terms of a series U_{n+1} and U_n, $\frac{U_{n+1}}{U_n}$, can be brought into the form

$$\frac{n^{\mu}+An^{\mu-1}+Bn^{\mu-2}+\cdots+L}{n^{\mu}+an^{\mu-1}+bn^{\mu-2}+\cdots+l}=\frac{P}{p}$$

and we find by comparison of the coefficients that H and h are the first coefficients (i.e. belonging to the highest powers) which are not equal, then the terms of the series, at least from a certain initial term, increase if the difference $H-h$ is positive, and decrease if it is negative.

Proof. Very easy, since $P-p$ will be positive in the first case, negative in the second, if n is sufficiently great, therefore the quotient $\frac{P}{p}$ is >1 or <1.

§16 *Corollary.* It is not asserted that the terms of the series U, $\frac{U^2}{U^1}$, ... can become *greater* in the first case, *smaller* in the second case, than every given number.

§17 *Theorem.* If the coefficients which differ from one another in magnitude are those of the second term, namely of the power $n^{\mu-1}$, and this difference $A-a$

is positive, then the terms of the series will be > than any given number, if it is negative, < than any given number, and if it is = 0, the terms of the series approach a constant number.

Proof. 1. If $A - a$ is positive, then there will always be a positive number π of such a kind that $\frac{(U_n)^\pi}{n} = V_n$ continually increases. That is, the value following that of $\frac{(U_n)^\pi}{n}$ is $\frac{(U_{n+1})^\pi}{n+1}$, therefore the quotient $\frac{n(U_{n+1})^\pi}{(n+1)(U_n)^\pi} = \frac{n.P^\pi}{(n+1).p^\pi} = \frac{n}{n+1}\frac{(n^{\pi\mu}+\pi An^{\pi\mu-1}+\cdots)}{n^{\pi\mu}+\pi an^{\pi\mu-1}+\cdots} = \frac{n^{\pi\mu+1}+\pi An^{\pi\mu}+\cdots}{n^{\pi\mu+1}+(\pi a+1)n^{\pi\mu}}$. By a suitable assumption about π, also πA can be made $>\pi a + 1$ if $A > a$, and consequently the terms of the series $\frac{(U_1)^\pi}{1}, \frac{(U_2)^\pi}{2}, \frac{(U_3)^\pi}{3}, \ldots$ increase. Therefore $(U_n)^\pi$ must become greater than any given number. Therefore also U_n.

2. If $A - a$ is negative then there will be a number π great enough that $n(U_n)^\pi$, with constant n, will become smaller [than] the next consecutive value, namely $(n+1)(U_{n+1})^\pi$. Therefore the quotient $\frac{n+1}{n}\frac{(U_{n+1})^\pi}{(U_n)^\pi} = \frac{n^{\mu+1}+(\pi A+1)n^{\pi\mu}+\cdots}{n^{\pi\mu+1}+\pi a.n^\pi+\cdots}$. Therefore there is always a π large enough that $\pi a > (\pi A + 1)$ if $a > A$. Then the terms of the series $1(U_1)^\pi, 2(U_2)^\pi, 3(U_3)^\pi, \ldots$ decrease continually. Therefore $\left(\frac{U_n}{n}\right)^\pi$ must be able to become smaller than any given number merely by increasing n. Therefore also U_n. That is, the terms of the given series become smaller than any given number.

3. But if $A = a$, then let us distinguish the two cases, whether the series is increasing or decreasing.

(a) If it is increasing then we consider the number $\left(\frac{n}{n-1}\right)^\pi .U_n$. This forms a series which continually decreases starting from a certain value. For the subsequent value is $\left(\frac{n+1}{n}\right)^\pi .U_{n+1}$. Therefore the quotient

$$\left(\frac{n^2-1}{n^2}\right)^\pi \frac{n^\mu + An^{\mu-1} + B.n^{\mu-2} + \cdots}{n^\mu + an^{\mu-1} + bn^{\mu-2} + \cdots}$$
$$= \frac{n^{\mu+2\pi} + An^{\mu+2\pi-1} + (B-\pi)^{\mu+2\pi-2} + \cdots}{n^{\mu+2\pi} + an^{\mu+2\pi-1} + bn^{\mu+2\pi-2}}$$

where because $A = a$, at the most $B - \pi$ and b can be made different and with suitable increase in π they can certainly always be made so different that $B - \pi$ becomes smaller than b. Therefore the series decreases. Thus while the series

$$\text{(I)} \qquad U_1, U_2, U_3, U_4, \ldots U_n, U_{n+1}, \ldots$$

increases, the series

$$\left(\frac{1}{0}\right)^\pi .U_1, \left(\frac{2}{1}\right)^\pi .U_2, \left(\frac{3}{2}\right)^\pi .U_3, \ldots \left(\frac{n}{n-1}\right)^\pi .U_n, \left(\frac{n+1}{n}\right)^\pi .U_{n+1}$$

decreases, and nevertheless $\left(\frac{n}{n-1}\right)^\pi U_n$ always remains $>U_n$. Therefore the series (I) must always stay within certain limits.

(b) In the second case if the series *decreases*, we consider the number $\left(\frac{n-1}{n}\right)^\pi .U_n$ etc.

§18 *Corollary.* Therefore the series

$$1, \frac{\alpha.\beta}{1.\gamma}, \frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma.(\gamma+1)}, \frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}, \text{ etc.}$$

(See *von Ettingshausen.*)

§19 *Corollary* 2. (Binomial coefficients)
(How the proof of the proposition that as $Fx = A + Bx + Cx^2 + Dx^3 + \cdots$ *in inf.* for all values of x, also $A = F0$, $B = F'0$, $C = F^2 0, \ldots$, the proposition arises that if for all x, $A + Bx + Cx^2 + \cdots$ *in inf.* $= 0$, then it must be that $A = 0$, $B = 0$, $C = 0, \ldots$. Because $Fx = 0$ for all values of x.)

§20 *Proof of the proposition which appears in an example*, that if x is not of the form $\frac{2m+1}{2^n}$ then there is a $\Delta x < \frac{1}{N}$ specifiable which gives an $x + \Delta x$ of the form $\frac{2m+1}{2^n}$.

First n is taken so large that $2^n \gtreqqless 2N$, and then if the following series of numbers is formed

$$\frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \frac{4}{2^n}, \ldots \textit{ in inf.}$$

then x, must necessarily either be equal to a term of this series or lie in between two consecutive terms.

(1) If it is the former, and $x = \frac{p}{2^n}$, then p must be even otherwise x would be of the form $\frac{2m+1}{2^n}$. But if p is even then $p+1$ is odd, therefore if we put $\Delta x = \frac{1}{2^n}$, $x + \Delta x = \frac{p+1}{2^n}$, which is of the required form $\frac{2m+1}{2^n}$.

(2) But if x is in between two terms of the series, e.g. $\frac{p}{2^n}$ and $\frac{p+1}{2^n}$ so that $x > \frac{p}{2^n}$ and $< \frac{p+1}{2^n}$, then also $\frac{p+1}{2^n} - x < \frac{1}{2^n}$ and also $\frac{p+2}{2^n} - x < \frac{2}{2^n} < \frac{2}{2N}$. Therefore if we take $\Delta x = \frac{p+1}{2^n} - x$ or $= \frac{p+2}{2^n} - x$, then in the first case $x + \Delta x = \frac{p+1}{2^n}$, in the second case $\frac{p+2}{2^n}$, and the the latter or the former (according to whether p is even or odd) is of the form $\frac{2m+1}{2^n}$.

For the Main Part on Continuity

§21 *Theorem.* If a function Fx increases (or decreases) and does so for all values of its variable from $x = a$ to $x = b$ inclusive, and thereby always remains measurable, then with at most the exception of certain isolated values of x (whose multitude, incidentally, can either be finite or infinite), it must always be continuous.

Proof. If we divide the difference $b - a = i$ into an infinite set of parts $\overset{1}{\alpha}, \overset{2}{\alpha}, \overset{3}{\alpha}, \ldots$ which become ever smaller, so that we have $i = \overset{1}{\alpha} + \overset{2}{\alpha} + \overset{3}{\alpha} + \cdots$ *in inf.* and we set the differences

$$f(a + \overset{1}{\alpha}) - fa = \overset{1}{e}$$

$$f(a + \overset{1}{\alpha} + \overset{2}{\alpha}) - f(a + \overset{1}{\alpha}) = \overset{2}{e}$$

$$f(a + \overset{1}{\alpha} + \overset{2}{\alpha} + \overset{3}{\alpha}) - f(a + \overset{1}{\alpha} + \overset{2}{\alpha}) = \overset{3}{e}$$

And so on. Then the increases $\overset{1}{e}, \overset{2}{e}, \overset{3}{e}$ must generally designate numbers which are measurable and positive or negative, and it must be that

$$\overset{1}{e} + \overset{2}{e} + \overset{3}{e} + \cdots \textit{in inf.} = Fb - Fa.$$

Therefore, with the exception of a certain finite number of these numbers, it must necessarily hold of the infinitely many others that they are each individually smaller than a certain given number $\frac{1}{N}$, however small this itself may be taken. For if the opposite were to be the case and only a finite number of these numbers, or even none of them, were $< \frac{1}{N}$, then there would have to be an infinite number of them $=$ or $> \frac{1}{N}$ and then the sum $\overset{1}{e} + \overset{2}{e} + \cdots$ would certainly be infinitely large. If we divide the distance i into an arbitrary number of equal parts, e.g. 2, and we again divide these, and so on continually, and ask where the x belonging to this infinite multitude of the $\overset{1}{e}, \overset{2}{e}, \overset{3}{e}, \ldots$ lies, then the answer to this question must be that at least one part of this set, and in fact such a part which contains infinitely many of them, must be able to be bound by a pair of limits c and $c \pm i$, where i can be taken as small as desired, while c always remains the same. Now for this value of c I claim that the function fx is certainly continuous, at least with respect to a Δx with a sign opposite to that of Δi. For if we take $a + \overset{1}{\alpha} + \overset{2}{\alpha} + \cdots + \overset{n}{\alpha}$ such that it is $> c - \omega$, and $a + \overset{1}{\alpha} + \overset{2}{\alpha} + \cdots + \overset{N}{\alpha} + \overset{m+1}{\alpha} < c$ which may always be allowed, because the parts $\overset{1}{\alpha}, \overset{2}{\alpha}, \ldots$ are completely arbitrary, then certainly $F(a + \overset{1}{\alpha} + \cdots + \overset{n+1}{\alpha}) - F(a + \overset{1}{\alpha} + \cdots + \overset{n}{\alpha})$ can become $< \frac{1}{N}$. All the more certainly must $F(c - \omega) - Fc$ (or actually $Fc - F(c - \omega)$) be $< \frac{1}{N}$. Therefore the function is continuous for $x = c$. Now since c denotes some value lying not outside a and b, and what we have just proved of a and b also holds of every two limits which are arbitrarily close to one another and which are not outside a and b, it follows that there can be no distance, however small, within which the function does not have at least one value for which it is continuous. Therefore the values for which it is discontinuous are isolated.

§22 *Note.* Someone perhaps thinks that the following proposition could be established. If a function fx is continuous for $x = a$ to $x = b$ inclusive, then there must be some one number e sufficiently small that the increase Δx, for all values from $x = a$ to $x = b$ need only become $< e$ so that the difference $F(x + \Delta x) - Fx$ turns out smaller than a given fraction $\frac{1}{N}$. But this proposition is in fact incorrect.

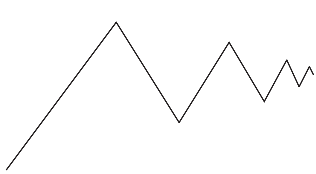


Fig. 1.

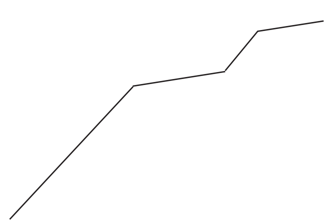


Fig. 2.

For example, if we let fx proceed according to the law of Fig. 1, so that the pieces always become smaller as in Fig. 2, the slope of the pieces to the horizontal is certainly always climbing, but climbing ever more gently (the angle is always becoming smaller), so that Δx must decrease indefinitely if the line approaches its final point to which it attains, continuously.

Relationships between continuous functions

§23[e] *Theorem.* If fx, ϕx are a pair of functions which are continuous in the neighbourhood of the value x, then also $fx.\phi x$, and likewise $\frac{fx}{\phi x}$, or $\frac{\phi x}{fx}$, are continuous in the neighbourhood of the value x.

Proof. If we suppose that $Fx = fx.\phi x$, $F(x + \Delta x) - Fx = f(x + \Delta x)\phi(x + \Delta x) - fx.\phi x = (fx + \omega)(\phi x + \omega') - fx.\phi x = \Omega$, where ω, ω', Ω denote quantities which can become as small as desired merely by decreasing Δx. And so on.

§24 *Theorem.* If fx is a continuous function of x, then also $(fx)^{\pm\frac{m}{n}}$ is continuous in the neighbourhood of the same value of x.

Proof. Well-known application of the concept of continuity to the determination of the value of different functions in particular cases.

§25 *Theorem.* If we only know about a variable quantity y that its changes obey the law of continuity, and that in all cases, where the variable x is actually present, $y = a + x$, then in the case when x is not present or $= 0$, the value of $y = a$.

Proof. Let the value of y for $x = 0$ be M, then if we consider a as variable and x as the increase in a, $F(a + x) - Fa$ can become as small as desired if x is taken small enough. But $Fa = M$, and $F(a + x) = y = a + x$. Therefore $a + x - M$ must be able to become as small as desired, if x is taken small enough. Therefore it must be that $a = M$.

§26 *Theorem.* If everything is as before, only that we have $y = a - x$, then for $x = 0$, $y = a$, but for $x = a$, [y] does not exist at all but = zero.

Proof. (Similar)

§27 *Theorem.* If $y = ax^n + bx^{n-1} + \cdots + l$ and the exponents $n, \ldots,$ are all positive, then for $x = 0$, $y = \pm l$.

§28 *Theorem.* If $y = \frac{a+bx+\cdots}{\alpha+\beta x+\cdots}$ then for $x = 0$, $y = \frac{a}{\alpha}$.

§29 *Theorem.* If $y = \frac{fx.(a-x)}{\phi x.(a-x)}$, and fa, ϕa are not $= 0$, then for $x = a$, $y = \frac{fa}{\phi a}$.

[e] The *BGA* edition has '13' here.

§30 *Note.* Sometimes the matter is presented as though the theorems just mentioned were independent of the assumption that the functions referred to obey the law of continuity, but as I believe I have shown, it is not so.

§31 *Problem.* To specify a way in which the value of the fraction may be determined in some cases if the numerator and denominator of a fraction become zero for a certain value of x, if both (numerator and denominator) are polynomials, and assuming that the value of the fraction itself varies according to the law of continuity.

Solution. It must be that $x - a$ is a factor of the numerator and denominator. Therefore we remove it etc.

For example, to determine $\frac{3x^3-3x^2+12x-36}{x^2-5x+6}$ for $x=2$. It turns out that $\frac{(x^2+x+6)(x-2)}{(x-3)(x-2)} = -12$.

§32 *Transition.* [Note. More suitable in the section *On the discovery of functions*, following later.] Since we have already seen above that an expression of the form p^n can denote a real number not only if n is an integer,[f] but also if n is a whole-numbered or fractional, positive or negative, quantity. This leads us to the question whether such an expression might not represent a quantity, even in other cases, namely even if n denotes an irrational quantity. But it is understood the expression p^n has no meaning in such a case unless we extend the concept of an exponent, which we have already extended several times, even more. For if n is an irrational quantity, then it cannot be represented by any fraction whose numerator and denominator are just integers [*Zahlen*], and it is only if the numerator and denominator are just integers that the symbol $p^{\frac{m}{n}}$ has a meaning according to the previous determinations of the concept, since it denotes a quantity which is so composed that a product of n factors equal to it is equal to a product of m factors equal to the quantity p. Therefore merely because we have defined that the expression p^n should denote a quantity which has the value just described in all those cases where n is a whole-numbered or fractional, positive or negative, quantity, would not determine at all what this expression means for the case of an irrational value of n. But just as we have already extended several expressions, which are undetermined in a certain case, to expressions of a determinate quantity, by adding to their concept the condition of continuity, so we can also attempt now [to see] whether this condition is not sufficient to give a determinate meaning to the expression p^x. However if wished to go ahead for this purpose immediately from the concept that p^n or p^x should denote a quantity which in all those cases where x is an integral or fractional quantity has the value which we connected with such a symbol previously but in the remaining cases it varies according to the law of continuity, then it might be possible that this concept would contain a kind of redundancy since some of the properties which we would attribute by virtue

[f] It is clear from what follows that Bolzano means here by *Zahl* an integer.

of this definition of the function a^x may perhaps already be a consequence of the others. In order to assure ourselves about this it will be advisable to attach the condition of continuity first of all only to one of the properties stated above and see whether the concept of a function which we obtain in this way is already narrow enough to let us use it. If it is too wide then we can add a second property, etc. The method in which we use those properties here will be most appropriate if it can be understood most easily why we compose a concept of just these, and no other, characteristics. Now the property that $p^m.p^n = p^{m+n}$ is the first which comes to mind with the concept of power. No one should therefore be surprised if we first investigate what has to be the property of a continuous function of the quantities p and x if it is to have the property that $p^x.p^y = p^{x+y}$, whatever x and y may denote.

§33 *Problem.* To investigate which properties a continuous function of the quantities x and y must have, if for every value of x and y the equation $p^x.p^y = p^{x+y}$ or $f(p, x).f(p, y) = f(p, x + y)$ is to hold.

Solution. Even without using the law of continuity, it arises from the condition $p^x.p^y = p^{x+y}$ that at least one of the values of $\frac{p^x}{p^y}$ must be $= p^{x-y}$, whatever x and y are. For if we put $\frac{p^x}{p^y} = M$, then $M.p^y = p^x$. But if $M = p^{x-y}$, then we have $p^{x-y}.p^y = p^{x-y+y} = p^x$. Furthermore, if we use the help of the law of continuity then it follows that it must be that $p^{x+\omega} = p^x + \Omega$, if ω, Ω denote a pair of quantities which can become smaller than every given quantity. Furthermore since $p^{x+\omega} = p^x.p^\omega$ it follows that $p^x.p^\omega = p^x + \Omega$, therefore $p^x(p^\omega - 1) = \Omega$, therefore $p^\omega = 1 + \Omega$, and hence it must be that $p^0 = 1$. But what p^1 or $p^2, p^3, \ldots$ denote remains completely undetermined. For even if we allowed, for example, $p^1 = p.p$, the condition of continuity and that $p^x.p^y = p^{x+y}$, will be satisfied.

§34 *Transition.* This concept is therefore still too wide to yield a really useful function. We shall therefore add a new condition and what presents itself most naturally is the determination that it should be the case that $p^1 = p$.

§35 *Problem.* To investigate the properties of a continuous function of p and x which satisfies the double condition that $p^x.p^y = p^{x+y}$ and $p^1 = p$.

Note. It should be noticed that p may not be a quantity capable of taking opposite values (i.e. not always positive values), if p^x is to be a continuous function of x, since if $x = \frac{m}{n}$ of one kind [of value], $(-p)^{\frac{m}{n}}$ changes its sign ∞ many times if x increases by Δx.

Solution. Here we have

1. for every whole number n, $p^n = p.p \ldots$, a product of n factors $= p$.
2. $p^{-n} = \frac{1}{p^n}$ if n is a whole number.
3. $(p^n)^m = p^{mn}$, if n, m are whole numbers.
4. $\frac{p^n}{p^m} = p^{n-m}$, whatever kind of whole numbers n, m may be.
5. $(p^x)^n = p^{nx}$, if n is a whole number and whatever x is.

6. $(p^x)^{-n} = p^{-nx}$, if n is a whole number.
7. One of the values of $(p^x)^{\frac{1}{n}}$, if n is a whole number, $=$ one of the values which $p^{\frac{x}{n}}$ has. For $[(p^x)^{\frac{1}{n}}]^n = p^x$ and $[p^{\frac{x}{n}}]^n = p^{\frac{nx}{n}} = p^x$.
8. One of the values of $(p^x)^{\frac{m}{n}} =$ one of the values of $p^{\frac{mx}{n}}$.
9. One of the values of $(a^x)^i$, whatever sort of i even irrational, $=$ one of the values of a^{ix}. For $(a^x)^i = (a^x)^{\frac{n}{m}+\omega}$, where n, m denote integers and ω a quantity which can be as small as desired. Therefore according to the law of continuity $(a^x)^i = a^{\frac{n.x}{m}} + \Omega$. Now if it were the case that $(a^x)^i = a^{x.i} + A$, then we would have one of the values of $a^{xi} = a^{xi} + A = a^{\frac{mx}{n}} + \Omega$, $a^{xi} - a^{\frac{mx}{n}} = \Omega - A = \Omega$. Therefore $A = 0$.
10. Whatever x and y are, then one of the values of $(a^x)^{\frac{1}{y}} =$ one of the values of $a^{\frac{x}{y}}$.
11. If m is an integral positive or negative quantity then p^m always has one and only one real value. If m is an integral odd positive or negative quantity then $p^{\frac{1}{m}}$ always has one and only one real value, but if m is even then $p^{\frac{1}{m}}$, if p is positive, has two real, equal and opposite values; if p is negative it has no values.
12. Also if p is positive, p^x always has at least two real equal and opposite values, whatever x may be.

§36 *Corollary.* Therefore there is certainly no doubt that by assuming this meaning of the expression p^x we obtain a useful extension of the concept of a power; for all earlier rules hold here with some restrictions which can easily be imagined.

§37 *Definition.* Therefore this concept of a power will in fact be adopted and now a quantity p^x is called the xth power of p, if p^x is a quantity of such a kind which obeys the law of continuity and is determined by the condition that $p^x.p^y = p^{x+y}$ and $p^1 = p$. On this the theory of the binomial theorem with irrational exponents—and likewise the theory of logarithms and exponential quantities. (From the *Abhandlung*.[g])

Seventh Section. On the determinable functions

§38 *Transition.* The mere assumption, that a certain quantity depending on another one x, so that $y = fx$, varies according to the law of continuity in the sense previously given, does not, as we have seen, determine whether or not, within certain values of x however close to one another they lie, the difference $f(x + \Delta x) - fx$ has the same sign. Indeed, not only this difference, but also the value of the quantity fx itself may sometimes be positive, and again sometimes negative. Much less does the latter assumption of the continuity of a function determine the relationship between the increase in the argument and the quantity y, or the quotient $\frac{\Delta y}{\Delta x}$. For example, whether this quotient approaches as close

[g] Referring to *BL* §§ 41, 64, and 66.

as desired, for the same x, a constant quantity independent of Δx, provided Δx is taken sufficiently small, or whether this is not the case. According to the difference in these circumstances functions can be divided into two very significant classes and the properties of one of these classes, which I call the *determinable* functions, are to be considered in this section.

§39 *Theorem.* If a function of x, fx, however else it is composed, always remains finite within certain limits of x, or if at least only $f(x + \Delta x) - fx$ always remains finite within these limits, then it can be asserted of every value of x lying within these limits that with at most the exception of certain isolated cases, that also the quotient $\frac{f(x+\Delta x)-fx}{\Delta x}$, which we obtain if we divide the increase in the function fx by the increase in the argument x, remains within certain finite limits however small we take the increase Δx.

Proof. 1. Suppose, on the contrary, this were not so, and that therefore the quotient $\frac{f(x+\Delta x)-fx}{\Delta x}$, not merely for certain isolated values of x but for *all* values lying within certain limits a and b, could either increase indefinitely or decrease indefinitely merely by diminishing Δx. Then the former and the latter would contradict the assumption. For suppose first of all that $\frac{f(x+\Delta x)-fx}{\Delta x}$ could become smaller than every given quantity merely by diminishing Δx, and this in fact happens for all values of x which lie within the limits a and b. Then if x and $x + i$ are taken within these limits, but n denotes some integer which may be taken as large as desired, we would have,

$$\frac{f(x+\frac{i}{n})-fx}{\frac{i}{n}} < d$$

$$\frac{f(x+\frac{2i}{n})-f(x+\frac{i}{n})}{\frac{i}{n}} < d$$

$$\frac{f(x+\frac{3i}{n})-f(x+\frac{2i}{n})}{\frac{i}{n}} < d$$

......

$$\frac{f(x+i)-f(x+\frac{n-1}{n}i)}{\frac{i}{n}} < d.$$

Therefore by addition

$$\frac{f(x+i)-fx}{\frac{i}{n}} < nd$$

or $f(x+i) - fx < id$. In this expression d could be taken as small as desired. Therefore it must be that $f(x+i) - fx = 0$ for every value of x and $x + i$ which lies within a and b, and therefore fx would really not be dependent on x.

2. If on the other hand the quotient $\frac{f(x+\Delta x)-fx}{\Delta x}$ could always be made > a certain given quantity D merely by decreasing Δx then by similar arguments as previously we get that $f(x+i)-fx > nD$ and therefore the difference $f(x+i)-fx$ must be infinitely great.

3. Therefore it is at most for certain isolated values of x that $\frac{f(x+\Delta x)-fx}{\Delta x}$ can become smaller or greater than every given quantity merely by decreasing Δx. For other values, and therefore for all values lying within given limits, $\frac{f(x+\Delta x)-fx}{\Delta x}$ must remain a finite quantity lying within certain limits d and D, however small Δx may be taken.

§40 *Corollary.* Therefore every function which remains finite for all values of its variable lying within certain limits a and b is, with at most the exception of certain isolated values, continuously variable at all the other points.

Proof. Since $\frac{f(x+\Delta x)-fx}{\Delta x}$ remains finite, then we may take Δx arbitrarily and $f(x+\Delta x)-fx$ must become as small as desired—and remain—

§41 *Corollary.* Therefore if fx is a continuous function of x within the limits a and b then within these limits $\frac{f(x+\Delta x)-fx}{\Delta x}$ always remains finite with at most the exception of certain isolated values, however small Δx may be made.

§42 *Note.* A further property, namely that the quotient $\frac{f(x+\Delta x)-fx}{\Delta x}$, merely by decreasing Δx, cannot increase or decrease indefinitely, even for certain isolated values of x, or that in the cases where it remains finite it can be brought as close as desired to a certain finite quantity dependent only on x, cannot, as far as I see, be proved.

For example, $2ax - x^2$ is a continuously variable quantity for every value of x within $x = 0$ and $x = 2a$, and yet the quotient

$$\frac{f(x+\Delta x)-fx}{\Delta x} = \frac{2a(x+\Delta x)-(x+\Delta x^2)-2ax+x^2}{\Delta x} = 2a-2x-\Delta x$$

for $x = a$, becomes smaller than every given quantity. Similarly if we take $y = fx = (a^2-x^2)^{\frac{1}{3}}$, then fx is continuously variable for all values of x, and nevertheless $\frac{\Delta y}{\Delta x} = -\frac{2x}{3(a^2-x^2)^{\frac{2}{3}}} + \Omega$ increases indefinitely if x approaches the value a. Finally the case is also not inconceivable, that although a quantity $y = fx$ is a continuous variable, i.e. although $f(x+\Delta x)-fx$ may become smaller than every given quantity merely by decreasing Δx, but the quotient $\frac{f(x+\Delta x)-fx}{\Delta x}$ for certain values of x, even perhaps for all values within certain given limits, always remains finite but approaches no constant quantity (independent of x), but on the contrary it oscillates to and fro indefinitely between certain given limits. In fact we read in *Cauchy*'s *Lehrbuch der algebraischen Analysis* §§2, 3 the theorem, 'If the numerator and denominator of a fraction are infinitely small quantities whose values decrease indefinitely at the same time with the variable α, then the value of this function which corresponds to $\alpha = 0$, is sometimes finite, sometimes zero and sometimes infinite.' Now if this were correct it would also hold of functions which were continuous, then we can indeed assert that the quotient $\frac{f(x+\Delta x)-fx}{\Delta x}$ for every value of x, merely by decreasing Δx, becomes greater than every given quantity, or can be brought as near as desired to zero, or to a finite quantity. And the case that such an expression oscillates indefinitely between two finite limits would never be able to occur. However if I consider the way in which Cauchy proves this proposition then we shall hardly be satisfied with this proof for our present purpose. It is based simply on the assumption that every variable quantity which vanishes at the same time as another one α can be expressed in the form $\kappa\alpha^n(1 \pm \varepsilon)$, in which κ is a *constant* finite quantity, but ε represents a variable quantity which always vanishes at the same time as α. This assumption, which *Cauchy* already makes in §1 of this chapter, is not proved by anything, and I do not believe that it is self-evident that it is even consistent with the truth. For even if κ is not *constant* but a *variable*, nevertheless represents a quantity which remains within certain limits, then $\kappa\alpha^n(1 \pm \varepsilon)$ will vanish at the same time as α. But if κ is variable then it can no longer be asserted that the ratio of the two quantities of the form $\kappa\alpha^n(1 \pm \varepsilon)$ may become either $= 0$, or $= \infty$, or can be brought as near as desired to a certain finite quantity if only α is taken small enough. For the limit of this ratio is then $= \frac{\kappa'}{\kappa}$, and can therefore itself be variable (and dependent on α). And in fact, even among those functions which we can express with our usual symbols, there are some which contradict the claim of *Cauchy*.

Dr. Bernard Bolzano's

Paradoxien des Unendlichen

herausgegeben

aus dem schriftlichen Nachlasse des Verfassers

von

Dr. Fr. Přihonsky.

Je suis tellement pour l'infini actuel, qu'au lieu d'admettre, que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte par-tout, pour mieux marquer les perfections de son Auteur. *(Leibniz, Opera omnia studio Ludov. Dutens. Tom. II. part 1. p. 243.)*

Leipzig,
bei C. H. Reclam sen.
1854.

Dr. Bernard Bolzano's

Paradoxes of the Infinite

edited from the writings of the author

by

Dr. Fr. Příhonský

Je suis tellement pour l'infini actuel, qu'au lieu d'admettre, que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte par-tout, pour mieux marquer les perfections de son Auteur.

— Leibniz, *Opera omnia studio Ludov. Dutens.*, Tom. II, part x, p. 243

Leipzig
C. H. Reclam
1851

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Title page quotation:

I stand for actual infinity so much that instead of admitting that Nature abhors it, as it is commonly said, I hold that [Nature] assumes it everywhere, in order to signal better the perfections of its Author.

Editor's Preface[a]

The author of this remarkable work on the *Paradoxes of the Infinite* began writing it in 1847 while staying in the company of the editor at the charming villa in Liboch, near Melnik. But he was interrupted by other tasks and did not complete it until the summer months of the following year, the last of his life. The work showed that in spite of his advancing age and the visible decline of his physical powers (he was in his sixty-seventh year at that time) his mental powers had lost nothing of their vigour and alertness. He also showed the learned world his unusual insight into the most abstract depths of mathematics, natural science and metaphysics. Indeed, if Bolzano had written, and left us, nothing else but this work then our firm belief is that on account of this alone he would have to be accounted one of the most distinguished minds of our century. He understands how to solve, with remarkable ease, the most interesting and complex questions which have for long engaged those studying the *a priori* sciences. He can sort them out, in front of the reader with such clarity that anyone who is not a complete stranger to the area, even if he has understood very little before, can follow the author's exposition and grasp at least the majority of his propositions. Furthermore, the experts, provided they give some attention to the work (and surely this is not too much to expect?) are sure to notice how important are the ideas which Bolzano puts forward here, and in other works (especially his *Logic* and *Athanasia*). And they should notice that with these views he aims at nothing less than a complete transformation of all previous scientific presentations.

The editor received this work in manuscript form from the heirs of the author and undertook to have it printed as soon as possible. This obligation was welcome as it coincided with his innermost feelings. Bolzano was his unforgettable teacher and friend. He would gladly have done this earlier if significant obstacles had not been in the way which he was only able to overcome in the course of this year. Now he is at last in the position of being able to improve the manuscript which is not always very readable, and even in places incorrect. He could also facilitate the use of the book with a detailed list of contents and he could find a suitable place for publication. He chose Leipzig because he expects this will offer a greater distribution of the work and also because this famous city of books will be honoured (he is by birth a Bohemian). He is confident that once Bolzano's great genius is generally recognized it will not be the least title to fame for Leipzig to have contributed to the appearance of these *Paradoxes*.

Budissin,
10th July 1850

[a] By Dr Příhonský, who also compiled the following list of *Contents*.

Contents

[b] The German *Gegenständlichkeit* means of a concept there *are* objects associated with that concept. This is in contrast to a concept or idea being 'empty' [*gegenstandlos*]. Thus 'objectivity' is being used here in a specific and unusual sense.

§24. Two sums of quantities that are equal to one another pairwise, may, if their multitude is infinite, not be immediately put equal, but only if both multitudes have equal determining grounds [*Bestimmungsgründe*].

§25. There is an infinite entity in the domain of the actual.

§26. The principle of the 'universal determinateness of all actual existence' does not contradict this assertion.

§27. But those mathematicians surely go wrong who speak of infinitely great intervals of time that are nevertheless bounded on both sides, or as happens more often, they speak of infinitely small parts of time. Similarly they speak of infinitely large and infinitely small distances. Also physicists, and metaphysicians, go wrong if they assume or claim that there are forces in the universe that are infinitely greater or smaller than others.

§28. The chief paradoxes of the infinite in the domain of mathematics. First of all, in the general theory of quantity, and especially in the theory of numbers. How the paradox of a calculation of the infinite can be resolved.

§29. A calculation with the infinitely great does in fact exist.

§30. Similarly a calculation with the infinitely small [exists].

§§31–32. The falsity of some concepts that even mathematicians cherish about the infinitely small and the infinitely great.

§33. The caution to be observed with calculation with the infinite, in order to avoid going astray.

§34. A more precise determination of the concept of zero. Zero may never be used as a divisor in an equation that is to be more than a mere identity.

§35. Contradictions that arise from the occasionally encountered assertion that infinitely small quantities, when combined with certain others through addition or subtraction become zero or vanish.

§36. These contradictions are not avoided by the assumption of some mathematicians that infinitely small quantities are mere zeros and infinitely large ones are quotients resulting from the division of a finite quantity by a mere zero.

§37. How the author conceives the method of calculation with the infinite in order to free it from all contradiction.

§38. Paradoxes of the infinite in the applied part of the theory of quantity, and indeed in the theory of time and space. The concept of the continuum or of continuous extension already contains apparent contradictions. How these may be resolved.

§39. Paradoxes in the concept of time.

§40. Paradoxes in the concept of space.

§41. How most of the paradoxes in the theory of space are explained by the author's concept of space.

§§42–43. How a false conception of the theory of the infinitely great has led some mathematicians to false ideas.

§44. J. Schultz's calculation of the magnitude of infinite space, and wherein the mistake in this calculation really lies.

§45. Also the theory of the infinitely small has been the cause of so many absurd assertions.

§46. What we are to think of Galilei's proposition that the circumference of a circle is as large as its centre.

§47. An explanation of the theorem that the common cycloid has an infinite curvature at the point where it meets its base line.

§48. How it comes about that many spatial extensions are spread through an infinite space yet have only a finite magnitude; in contrast other figures that are bounded in a finite space are infinite in magnitude, and others again remain finite despite an infinity of revolutions around a fixed point.

§49. Some other paradoxical relationships of spatial extensions that possess an infinite magnitude.

§50. Paradoxes of the infinite in the domain of physics and metaphysics. What truths must be acknowledged in order rightly to evaluate these paradoxes. A proof that there are no two altogether equal things, and hence there are no two altogether equal atoms (simple substances) in the universe. Further, there are necessarily simple substances and that these substances are variable.

§51. Some prejudices that must be abandoned before the paradoxes here can be correctly evaluated. There is no dead, purely inert matter.

§52. It is a prejudice of the schools that the hypothesis of a direct action of substances is not permitted.

§53. It is likewise a prejudice to believe that a direct action at a distance is impossible.

§54. The interpenetration of substances must be unconditionally denied.

§55. The prejudice that mental entities[c] are completely non-spatial, in so far that they cannot occupy the place of a single point. There are no differences between created substances apart from those of degree.

[c] The German here is *geistiger Wesen* which Steele renders, perfectly soundly, as 'spiritual beings'. Both ideas are in the German phrase.

§56. The great paradox of the connection between mental and material substance is automatically resolved on this view.

§57. The erroneous idea of a construction of the universe from mere forces without substances.

§58. In God's creation there is neither a highest nor a lowest level of being.

§59. Variation in the density of bodies is quite compatible with the continuous filling of infinite space with substances, and it is not necessary to assume that substances interpenetrate each other.

§60. Every substance in the world is continually interacting with every other substance.

§61. Among [substances] there are 'dominant' substances, but none of them possesses forces infinitely greater than those of the 'dominated' ones.

§62. Whether every arbitrary collection of substances must contain one dominating substance.

§63. Together with the dominant substances there is yet another world-stuff [*Weltstoff*], the ether, which fills all the remaining spaces outside distinguished substances, and connects all bodies of the world. An attraction and repulsion take place between the substances, and how the author conceives of them. How it comes about that material entities that differ in their forces, that is, in the degree of their mutual attractions, are nevertheless all equal in weight, or that their weights are proportional to their masses.

§64. How the dominance of certain substances or atoms over others is manifested, and what is the consequence of it.

§65. No distinguished substance experiences such a great change that it would be freed thereby from its immediate environment.

§66. Where one body ends and another begins, that is, the question of the boundaries of a body.

§67. Whether, and if, bodies can be in direct contact with one another.

§68. The possible kinds of movements occurring in the universe.

§69. Whether an atom in the universe at any time describes a perfectly straight or a perfectly curved line. Whether the author's views on the infinitude of the universe allow for any movement of the entire [universe] in a given direction, or even a turning motion of it about a world axis, or about a world centre?

§70. Two paradoxes made famous by Euler.

§ 1

Certainly most of the *paradoxical* assertions which we meet with in the area of mathematics, though not all of them as *Kästner* suggests, are propositions that either contain the concept of *infinity* directly, or depend on it in some way for their attempted proof. It is even more indisputable that precisely those mathematical paradoxes which deserve our greatest attention are of this kind. This is because decisions on very important questions in many another subject, such as metaphysics and physics, depend on a satisfactory resolution of their apparent contradictions.

This is the reason why in the present work I am dealing exclusively with the consideration of the paradoxes of the infinite. But it is self-evident that it would not be possible to recognize the appearance of contradiction which is attached to these mathematical paradoxes for what it is, a mere appearance, if we did not make abundantly clear what concept we actually associate with the infinite. Therefore we do this first.

§ 2

The word already indicates that the *infinite* is contrasted [*entgegensetze*] with everything that is merely *finite*. And the fact that the *name* of the former is derived from that of the latter, shows that we think of the *concept* of the infinite as one which arises from that of the finite only through the addition of a new component (such indeed is the mere concept of *negation*). Finally, that both concepts are applied to *multitudes* [*Mengen*], or more specifically to *pluralities* [*Vielheiten*] (i.e. to multitudes of units),[d] therefore also to *quantities*, cannot be denied because it is precisely *mathematics*, i.e. the theory of quantity, where we speak most frequently of the infinite. Since here finite as well as infinite pluralities, and besides *finite* quantities not only *infinitely large* but even *infinitely small* quantities arise as objects of our consideration—and even calculation. Without assuming that both those concepts (namely of the finite and of the infinite) can always only be applied to objects to which *magnitude* and *plurality* can be referred in some respect, we may hope that a more precise investigation of the question of the circumstances in which we define a multitude as finite or as infinite, will also give us information about the *infinite in general*.

§ 3

For this purpose we must nevertheless go back to one of the simplest concepts of our understanding so as to agree first of all on the word we wish to use for its designation. It is the concept which underlies the conjunction '*and*' which I believe, if it is to stand out as clearly as required in countless cases for the purpose of mathematics as well as philosophy, can be expressed most suitably by

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[d] On the translation of *Menge* and *Vielheit* see §4 and the *Note on the Translations*.

the words: *a collection* [*Inbegriff*] *of certain things* or *a whole* [*Ganze*] *consisting of certain parts*. That is, if it is agreed that we wish to interpret these words in such a wide sense that they may be asserted in all propositions where the conjunction '*and*' is usually used, e.g. in the following: '*The sun, the earth and the moon* have a mutual effect on one another', '*the rose and the concept of a rose* are a pair of very different things', 'the names *Socrates and son of Sophroniskus* designate one and the same person'—the object which is spoken about in these propositions is *a certain collection of things*, a *whole* consisting of certain *parts*. Namely, in the first one, it is that whole which the sun, earth and moon form together of which it is stated that it is a whole whose parts have mutual effect on one another. In the second proposition it is the collection which the two objects 'the rose and the concept of a rose' jointly make up of which it is judged that they are two very different things etc. These few examples should already be enough for agreement about the concept spoken of here, at least if we add that any arbitrary object A can be combined with all the other arbitrary objects B, C, D, ... into a collection or (to speak more correctly) already forms a collection in itself. About this collection several more or less important truths can be stated provided each of the ideas A, B, C, D, ... does in fact represent *another* object, or provided none of the propositions 'A is the same as B', 'A is the same as E', 'B is the same as C' etc. is true. For if, for example, A is the same thing as B, then it is of course absurd to speak of a collection of the things A and B.

§ 4

There are collections, which, although containing the same parts $A, B, C, D, \ldots$, nevertheless present themselves as *different* (we call it essentially different) according to the viewpoint (concept) under which we interpret them. For example, a complete glass and a glass broken into pieces considered as a drinking vessel. We call the basis [*Grund*] for this difference in such collections, the *mode of combination* or *arrangement* of its parts. A collection which we put under a concept so that the arrangement of its parts is unimportant (in which therefore nothing essential changes for us if we merely change this arrangement) I call a *multitude* [*Menge*]. And a multitude whose parts are all considered as *units* of a certain kind A, i.e. as objects which come under the concept A, is called a *plurality* [*Vielheit*] of A.

§ 5

It is well known that there are also collections whose parts themselves are compound, i.e. are again collections. Among these are also such as we consider from a viewpoint for which nothing essential changes in them if we conceive the parts of the parts as parts of the whole itself. I call them, with a word borrowed from mathematicians, *sums* [*Summen*]. For it is just the concept of a sum that it must be that $A + (B + C) = A + B + C$.

§ 6

If we consider an object as belonging to a kind [*Gattung*] of thing of which every two, M and N, can have no other relationship to one another than that they are either equal to one another, or that one of them presents itself as a sum which includes a part equal to the other one. That is, that either $M = N$ or $M = N + \nu$ or $N = M + \mu$, where the same must again hold of the parts ν and μ, namely that they are either equal to one another, or one is to be viewed as a part contained in the other, then we consider this object as a *quantity* [*Größe*].

§ 7

If a given collection of things $\ldots, A, B, C, D, E, F, \ldots, L, M, N, \ldots$ has the property that for every part M some one, and only one, other part N can be identified of a kind that we can *determine* by the *same rule* for all parts of the collection either N by its relationship to M, or M by its relationship to N, then I call this collection a *series* [*Reihe*] and its parts the *terms* of this series. I call that rule by which either N is determinable through its relationship to M, or M is determinable through its relationship to N, the *rule of formation* [*Bildungsgesetz*] of the series. One of these terms, whichever one wants, I call (without wishing to designate by this name the concept of an actual sequence in time or space) the *previous* or *preceding* term, the other the *following* or *succeeding* term. Every term M which has a previous term, as well as a following term, i.e. which is not only itself derivable from another but from which also again another term is derivable according to the rule of formation holding for the series, I call an *interior* term of the series, from which it is self-evident which terms, if they exist, I call *exterior*, the *first* or the *last* term.*

§ 8

Let us imagine a *series* of which the *first* term is a *unit* of the kind A, but every succeeding term is derived from its predecessor by our taking an object equal to it and combining it with a new unity of kind A into a sum. Then obviously all the terms appearing in this series—with the exception of the first which is a *mere unit* of the kind A—are *pluralities of the kind A* and in fact these are such as I call *finite* or *countable pluralities*, indeed I call them straightforwardly (and even including the first term) *numbers* [*Zahlen*], and more definitely, *whole* numbers.

* More precise discussions about this, as also about some of the concepts put forward in the previous paragraphs, are found in the *Wissenschaftslehre*.[e]

[e] Explanations of the terms for collections are given in *WL* §§ 82–86.

§ 9

According to the different nature of the concept designated here by *A* there may sometimes be a greater and sometimes a smaller multitude of objects which it comprehends, i.e. the units of the kind *A*. And therefore there is sometimes a greater and sometimes a smaller multitude of terms in the series being discussed. In particular there can even be so many of them that this series, to the extent that it is to exhaust *all* these units (taken in themselves), may have absolutely *no last term*. We shall prove this in more detail in what follows. Therefore assuming this for the time being I shall call a plurality which is greater than every finite one, i.e. a plurality which has the property that every finite multitude represents only a part of it, an *infinite plurality*.

§ 10

I hope it will be granted that the definition put forward here of both the concepts of a *finite* and of an *infinite* plurality truly determine the difference between them as intended by those who have used these expressions in a strict sense. It will also be granted that there is no hidden circularity in these definitions. Therefore it only a question of whether through a mere definition of what is called an infinite *plurality* we are in a position to determine what is [the nature of] the *infinite* in general. This would be the case if it should prove that, strictly speaking, there is nothing other than pluralities to which the concept of infinity may be applied in its true meaning, i.e. if it should prove that infinity is really only a property of a plurality or that everything which we have defined as *infinite* is only called so because, and in so far as, we discover a property in it which can be regarded as an infinite plurality. Now it seems to me that is really the case. The mathematician obviously never uses this word in any other sense. For generally it is nearly always quantities with whose determination he is occupied and for which he makes use of the assumption of one of those of the same kind for the *unit*, and then of the concept of a number. If he finds a quantity greater than every number of the unit taken, then he calls it *infinitely large*; if he finds one so small that every multiple of it is smaller than the unit, then he calls it *infinitely small*. Outside these two classes of infinities and the kinds further derived from them of infinitely greater and infinitely smaller *quantities of higher order*, which all proceed from the same concepts, there is no other infinity for him.

§ 11

Now some philosophers, particularly of more recent times, like *Hegel* and his followers, are not satisfied with this infinity so well known to mathematicians. They call it contemptuously ‘the bad infinity’ and claim to know a much higher one, the true, the *qualitative infinity* which they find especially in *God* and generally only in the *absolute*. If they, like *Hegel*, *Erdmann* and others, imagine the mathematical infinity only as a quantity which is *variable* and has no limit to its growth (which is, of course, as we shall soon see, what some mathematicians have put forward as the definition of their concept), then I would agree with them in their criticism of this concept of a quantity itself never *reaching* but only *growing* into infinity. A *truly infinite* quantity, e.g. the length of the whole straight line unbounded in both directions (i.e. the magnitude of that spatial thing which contains all points which are determined by their merely conceptual relationship to two given points), needs precisely not to be variable, as it is in fact not in the example mentioned. A quantity which can always be taken greater than it has already been taken, and may become greater than every given (finite) quantity can nevertheless always remain a merely finite quantity, as holds in particular of every number quantity [*Zahlgröße*] 1, 2, 3, 4, ... What I do not concede is merely that the philosopher may know an object on which he is justified in conferring the predicate of being infinite without first having identified in some respect an infinite magnitude or plurality in this object. If I can prove that even in God as that being which we consider as the most perfect unity, viewpoints can be identified from which we see in him an infinite plurality, and that it is only from these viewpoints that we attribute infinity to him, then it will hardly be necessary to demonstrate further that similar considerations underlie all other cases where the concept of infinity is well justified. Now I say we call God infinite because we concede to him powers of more than one kind that have an infinite magnitude. Thus we must attribute to him a power of knowledge that is true omniscience, that therefore comprehends an infinite multitude of truths because all truths in general etc. And what would be the concept that anyone would want to press upon us in place of the concept of true infinity put forward here? It should be the universe [*das All*], which comprehends every possible thing, the absolute universe, apart from which there is nothing. According to this statement there would be an infinity which included, according to our definition, infinitely many things. It would be a collection of not only all actual things, but also all those things which have no reality, the propositions and truths in themselves. And thus even with all the other errors in mind which are mixed up in this theory of the universe there should be no basis for abandoning our concept of the infinite so as to adopt that other one.

§ 12

I cannot also help rejecting as incorrect many other definitions of infinity, which have been proposed even by mathematicians in the opinion that they present only the components of this one and the same concept.

1. In fact, as I have just mentioned earlier, some mathematicians, among them even *Cauchy* (in his *Cours d'Analyse* and many other writings), and the author of the article '*Unendlich*' in *Klügel*'s *Wörterbuch*, have believed infinity to be defined if they describe it as a variable quantity whose value increases *without bound* and which can be proved to become greater than *every given quantity however large*. The *limit* of this unbounded increase is the *infinitely large quantity*. Thus the tangent of a right angle, thought of as a continuous quantity, is unbounded, without end,

and in the *proper sense infinite*. The mistakenness of this definition is clear from the fact that what mathematicians call a *variable quantity* is not really a quantity but is the mere concept, the mere *idea* of a quantity, and in fact such an idea that is concerned not with a single quantity but an infinite multitude of quantities differing from one another in value, i.e. quantities distinguishable by their *magnitude* [*Großheit*]. What is called infinite are indeed not those *different* values which, in the example mentioned here, are represented by the expression tang. ϕ for different values of ϕ, but only that single value which is imagined (although wrongly in this case) that that expression takes for the value $\phi = \frac{\pi}{2}$. Also it is certainly a contradiction to speak of the limit of an unbounded increase, and equally for the definition of the infinitely small, of the limit of an unbounded decrease. And if the infinitely large is defined by the former, then by analogy the infinitely small should be defined by the latter, i.e. the mere *zero* (a nothing). But this is certainly incorrect and neither *Cauchy* nor *Grunert* allow themselves to say it.

2. If the definition just considered was too wide, in contrast that adopted by Spinoza and many other philosophers as well as mathematicians, that *only that is infinite which is capable of no further increase*, or to which nothing more can be attached (added), is much too narrow. The mathematician is allowed to add to every quantity, even infinitely large ones, other quantities, and not only finite ones but even other quantities which are already infinite. Indeed he may even multiply the infinitely large infinitely many times etc. And if some dispute whether this procedure is even a legitimate one, which mathematicians, providing they do not reject everything infinite, will not have to admit that the length of a straight line which is bounded on only one side while on the other side it continues indefinitely, is infinitely large and nevertheless can be increased by additions on the first side?

3. No more satisfactory is the definition of those who adhere precisely to the components of the word and say infinite is *what has no end*. If they think thereby only of an end in time, a cessation, then they could only call things which are in time, finite or infinite. However we also ask about things which are not in time, e.g. lines or quantities in general, whether they are finite or infinite. But if they take the word in a wider sense, roughly equivalent to *limits* in general, then I point out *firstly* that there many objects for which one cannot reasonably show that a limit exists for them without attributing to the word a highly unreliable, confused meaning, and which nevertheless nobody counts as infinite. Thus every simple part of time or space (a point in time or space) has no limit, but is instead usually considered itself only as a limit (of a time interval or line), indeed most of them are directly defined so that this belongs to their nature. But it occurs to nobody (unless they are Hegel) to wish to see an infinity in a mere point. Just as little does the mathematician regard the circumference of a circle and many other lines and surfaces which turn back on themselves as a limit and consider them only as finite things. (It would have to be that he may come to speak of the infinite multitude of the points contained in them, and in that respect he must also recognize in every bounded line something infinite.) *Secondly*, I remark that there are many objects which are undeniably bounded, but are regarded as quantities belonging to the

infinite. It is so not only with the straight line already mentioned earlier, which only extends into infinity on one side, but also with the surface area which a pair of infinite parallel lines encloses between them, or the two indefinitely extended arms of an angle drawn in the plane, and several others. Thus also in rational psychology we shall call an intellect infinitely large if, even without being omniscient, it is just capable of surveying some infinite multitude of truths, e.g. just the complete infinite series of decimal places which the single quantity $\sqrt{2}$ contains.

4. Most commonly what is called infinitely large is what is greater than every quantity that *could be given* [*angebliche Größe*]. Here we need most of all a more exact determination of what is in mind with the words '*could be given*'. Should it only mean that something is *possible*, i.e. *can* have reality, or only that it is *nothing contradictory*? In the first case, the concept of *finite thing* is limited solely to that kind of thing which has *real existence* [*Wirklichkeit*], either they are real at all times, or have been or will be real at certain times, or at least *could* become real at some time. In fact it is in this sense which *Fries* (*Naturphilosophie*, §47) seems to have taken the infinite when he calls it the *incompletable*. But usage applies the concept of finite and also that of infinite both to objects which have real existence, like God, and also to others which cannot be spoken of as having any existence at all such as the pure propositions and truths in themselves, together with their components the ideas in themselves, since we assume finite as well as infinite multitudes of them. But if by 'what could be given' is understood everything which is just *not contradictory*, then one already puts into the definition of the concept that there may be no infinity, for a quantity which is to be greater than every [quantity] which is not contradictory, would also have to be greater than itself, which is, of course, absurd. However there is still a third meaning in which the words '*could be given*' could be taken, if one understood by them only such a thing as *can only be given to us*, i.e. can become an object of *our experience*. But I ask everyone whether—if a beneficial use is to be made of it in science—he does not in any case take the words '*finite*' and '*infinite*' in a sense, and he must necessarily adopt only such a sense, that they refer to a certain *internal* property of the object which we call thus, but in no way do they refer to a mere relationship of it to our *perception*, even to our *sense awareness* (whether we may be able, or not, to have *experiences* of it). Thus the question of whether something is finite or infinite can certainly not depend upon whether the object in question possesses a quantity which we are able to perceive (for example, to look at, or not).

§ 13

If we have now come to agreement on which concept we shall associate with the word '*infinite*' and if we have also made clear the components from which we compose this concept, then the next question is whether it also has *objectivity*,[f] i.e. whether there are also things to which it can be applied, multitudes, which we

[f] See the footnote on p. 596.

may call infinite in the sense defined? And I venture to *affirm* this categorically. In the *realm of those things which make no claim to reality but only to possibility*, there are indisputably multitudes which are infinite. *The multitude of propositions and truths in themselves* is, as may very easily be seen, infinite. For if we consider some truth, perhaps the proposition that there are actually truths, or otherwise any arbitrary truth, which I shall designate by *A*, then we find that the proposition that the words '*A is true*' express is different from *A* itself, for the latter obviously has a completely different subject from the former. Namely, its subject is the whole proposition *A* itself. However, by the rule by which we derived from the proposition *A*, this different one, which I shall call *B*, we can again derive from *B* a third proposition *C*, and continue in this way without end. The collection of all these propositions in which each successive one stands in the relationship just given to the one immediately before it, in that it makes it its subject and states of it that it is a true proposition, this collection—I say—comprises a multitude of parts (propositions) which are greater than every finite multitude. For without my reminder the reader may notice the similarity between the series of these propositions formed by the rule just given, and the *series of numbers* considered in §8. This is a similarity consisting in this, that to every term of the latter there is a term of the former corresponding to it, that therefore for every number, however large, there is also a number of distinct propositions equal to it, and that we can always form new propositions, or to say it better, that there are such propositions in themselves regardless of whether we form them or not. Whence it follows that the collection of all these propositions has a plurality which is greater than every number, i.e. is infinite.

§ 14

Nevertheless, simple and clear as the proof just given is, there are a considerable number of scholarly and intelligent men who declare the proposition which I believe I have proved here, to be not only paradoxical but downright false. They deny that there *is any infinity*. According to their claim, not only among things which have reality but also among the others, there is no single thing, not even a collection of several things, for which an infinite multitude of parts could in any respect be assumed. We shall consider later the arguments which they raise against infinity in the realm of reality because we shall also bring forward later the reasons for the existence of such an infinity. Therefore let us examine here the arguments through which it is to be proved that there may never be something infinite, not even among the things which make no claim to being real.

1. They say, 'There can never be an infinite multitude, just for this reason because an infinite multitude *can never be united into a whole, can never be gathered together in thought*.' I must immediately call this assertion an error which is produced by the false view that in order to think of a whole consisting of certain objects *a*, *b*, *c*, *d*, . . . one would first have to have formed *ideas* which represent each one of

these objects individually (individual ideas of them). It is definitely not so; I can imagine the multitude, or the collection if preferred, the *whole* [*Ganze*] of the inhabitants of Prague or of Beijing without imagining each of these inhabitants individually, i.e. through an idea corresponding exclusively to each one. I am actually doing this just now, since I am speaking of this very multitude, and, for example, make the judgement, that their number in Prague lies between the two numbers 100 000 and 120 000. That is, as soon as we possess an idea A which represents each of the objects $a, b, c, d, \ldots,$ but nothing else, it is extremely easy to reach an idea which represents the *collection* which all these objects make up together. In fact nothing extra is needed other than the concept which the word '*collection*' denotes, connected with the idea A in such a way as indicated by the words: *the collection of all A*. By this single remark, whose correctness I believe must be clear to everyone, all difficulty which may be found with the concept of a multitude if it consists of infinitely many parts, is removed. As soon as a category [*Gattungsbegriff*] for each of these parts exists, but which covers nothing else, as is the case with the concept: '*The multitude of all propositions or truths in themselves*', where the required category is already: 'a proposition or truth in itself'. However, I cannot leave uncriticized a *second* error which is revealed in that objection.

It is the opinion, 'that a multitude would not exist unless first somebody were to exist who *conceives* it'. Whoever asserts this, in order to be as consistent as one can actually be with an error, should not only assert that there may be no *infinite* multitude of propositions and truths in themselves, but he should assert that actually there may *not be any* propositions and truths in themselves *at all*. For if we have brought about a clear awareness in ourselves of the concept of propositions and truths in themselves and do not in fact doubt the objectivity of them, then we could hardly make assertions like the one just mentioned, and could certainly not persist with them. In order to show this in a way clear to everybody, I permit myself to put the question whether there do not exist at the poles of the earth fluid as well as solid bodies, air, water, rocks and such like, whether these bodies do not act upon another according to certain laws, e.g. that the speeds which they impart to one another on impact are inversely proportional to their masses and such like, and whether all these things occur even if no person, or any other thinking being, is there to observe it? If one agrees with this (and who would not have to agree?) then there are also propositions and truths in themselves which express all these proceedings without anyone knowing and thinking them. And in these propositions there is frequent reference to wholes and multitudes, for every body is a whole and produces many of its effects only through the multitude of parts of which it consists. Therefore there are multitudes and wholes without the presence of a being which conceives them. And if this were not so, if these multitudes were not there themselves, how could the judgements which we make about them be true? Or rather, what would be the meaning of these judgements if they should only become true if somebody is there who perceives these proceedings? If I say, 'This boulder broke off from that cliff in front of my eyes, cut through the air, and crashed down below,' this would have to have roughly the following meaning: While I thought of certain simple entities together up there, a combination of them arose which I call a boulder, this combination withdrew from certain others, which, while I think of them together, united into a whole which I call a cliff, etc.

2. However, one might say, 'for all this it remains true that it is only *our act* [*Werk*], and in fact a largely very arbitrary act, whether we want to think of certain simple objects together in a collection or not, and only if we do this first do relationships arise between them. The central particle in this button on my coat and the central particle in the top of that tower there have nothing to do with one another and have no connection with one another at all, only through my present thinking of them together does any kind of connection between them originate.' Even this I must contradict. The two particles were, even before the thinking being put together their ideas, in mutual effect on one another, e.g. through the force of attraction and such like, and if, on the other hand, that thinking being does not, by virtue of his thoughts, also adopt actions which produce a change in the relationships between the two particles, then it is absolutely untrue that it is only through that thinking of them together that relationships arise among them, which apart from this would not be there. If I should judge truly that the former [particle] is lower, and the latter is higher, and that therefore the latter may be pulled up by the former by some small amount in height etc, then all this would have to be the case even if I had not thought about it, etc.

3. Other people say, 'It is not the case that for a collection to exist it is necessary for it to have *actually been thought* by a thinking being, but rather it is necessary that it *could* be thought. Now because no being is possible that can imagine each one of an infinite multitude of things individually and then connect these ideas, then also no collection which comprises an infinite multitude of things as parts in itself is possible.'

We have already seen in no.1 how much in error is the assumption that is repeated here, that for the thinking of a collection the thinking of all its parts individually, i.e. the thinking of each individual part by means of a single idea representing it, is required. Also we do not need to refer at the outset to the omniscient being as such a being for which the conception of an infinite multitude of things, each one individually, causes no trouble. However, we may not even grant the first assumption, namely that the existence of a collection of things rests on the condition that such a collection *can be thought of*. For the '*capacity of a thing to be thought of*' can never include the basis of its possibility, instead it is exactly conversely that the possibility of a thing is firstly the basis on which a reasonable being, providing it is not mistaken, and the thing is *possible*, *can think* it, or as we say (but improperly) finds it *thinkable*. One will be even more convinced of the complete correctness of this remark and the fact that the admittedly very widespread view which I am attacking here is completely untenable, if one tries to clarify the components of which the highly important concept of possibility consists. That one calls that which is *possible*, what *can* be, is obviously not an analysis of this concept, for the concept of possibility is altogether involved in the word '*can*'. But it would be still more incorrect to wish to set up the definition that that is possible *which can be thought*. We can even think, in the true sense of the word where it concerns mere *representation*, of the impossible, and we actually think it whenever we judge about it and e.g. explain something as impossible, as when we say that there is, and can be, no quantity which represented by 0 or $\sqrt{-1}$. But even if one understands by thinking here not a mere representing but an actual *asserting* [*Fürwahrhalten*], it is false that everything is possible which we can assert as true. By mistake we sometimes even hold the impossible, e.g. that we had found the square of the circle, as true. Therefore it would have to be said (as I already adopted in modified form above) that is possible about which a thinking being, if it judged the truth appropriately, expresses the judgement that it can be, i.e. that it is possible. A definition which contains an obvious circularity! We are therefore required to drop completely the reference to a thinking being for the definition of the possible and look for another characteristic. One sometimes hears people say that 'possible' is '*what does not contradict itself*'. Of course, everything which already contains a contradiction within itself, e.g. that a sphere is not a sphere, is impossible. But not everything impossible is of such a kind that the contradiction which is already in the components from which we have composed the idea of it, is found. It is impossible that a solid which is enclosed by seven plane polygonal surfaces, may be enclosed by equal polygonal surfaces. But the contradictory nature does not lie open to view in the words which are connected here. We must therefore extend our definition further. But if we want to say that the impossible is what stands in contradiction to some truth, then we would by this be defining everything which is not, as impossible, because the proposition that it is, would contradict the truth, that it is not. We would therefore admit no difference between the possible and the actual, and even the necessary, which nevertheless we all do distinguish. Accordingly we see the domain of truths which the impossible contradicts must be limited only to a certain class, and now we can hardly fail to notice which class of truths this is. They are the pure conceptual truths [*Begriffswahrheiten*]. Whatever some pure conceptual truth contradicts is called the *impossible*. Therefore the possible is what stands in contradiction with no pure conceptual truth. Whoever has once realized that this is the correct concept of possibility, to them it can hardly occur to make the assertion that something is only possible if it is thought, i.e. is viewed as possible by a thinking being which does not err in its judgement. For this is to say: 'A proposition contradicts no pure conceptual truth if it contradicts no pure conceptual truth that there is a thinking being which judges of this proposition the truth that it contradicts no pure conceptual truth.' Who does not see how irrelevant here is this addition of a thinking being? But if it is decided that the *thinking* does not make the possibility, where is there some reason for concluding from the supposed circumstance that an infinite multitude of things cannot be *thought together* that such multitudes cannot exist?

§ 15

I consider it now as sufficiently proved and defended that there are infinite multitudes, at least among the things which have no reality, in particular that the multitude of all *truths in themselves* is an infinite multitude. It will also be admitted, that in a similar way as it is derived in §13, the multitude of *all numbers* (the so-called natural, or whole numbers, whose concept we defined in §8) is infinite. But this proposition does sound *paradoxical* and we might actually consider it as the *first paradox* appearing in the area of mathematics, because the one considered before properly belongs to a more general science than the theory of quantity.

It might be said, 'If every number, as a concept, is a merely finite multitude, how can the multitude of *all* numbers be an infinite multitude? If we consider the series of natural numbers:

$$1, 2, 3, 4, 5, 6, \ldots$$

then we notice that the multitude of numbers which this series contains, starting from the first (the unit) up to some other one, e.g. the number 6, is always represented by this latter one itself. Thus the multitude of *all* numbers must be as large as the *last* of them and thus itself be a number and therefore not infinite.'

The deceptiveness of this argument disappears as soon as it is remembered that in the multitude of all numbers, in their natural series, there is *no last one*. Therefore the concept of a last (highest) number is an empty one because it is a self-contradictory concept. For according to the *rule of formation* given in the definition of such a series (§8) every one of its terms has a *succeeding one*. This paradox may therefore be considered as solved by this single remark.

§ 16

If the multitude of *numbers* (namely the so-called *whole* numbers) is infinite, then it is all the more certain that the multitude of quantities (according to the definition appearing in §6 and *Wissenschaftslehre*, §87) is *infinite*. For as a consequence of that definition not only are all numbers also quantities, but there are even more quantities than numbers, because also the fractions $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \ldots$, and the so-called *irrational* expressions $\sqrt{2}, \sqrt[3]{2}, \ldots, \pi, e, \ldots$ designate quantities. In consequence of this definition it is no contradiction to speak of quantities which are *infinitely large*, and of others which are *infinitely small*, as long as one understands by an *infinitely large* quantity only one which, once given a basic unit, appears as a whole for which every finite multitude of these units is only a part; and one understands by an *infinitely small* quantity one for which the unit itself appears as a whole of which every finite plurality of this quantity constitutes only a part. The multitude of all numbers appears immediately as an indisputable example of an infinitely large *quantity*. I say as a quantity, but not, of course, as an example of an infinitely large *number*, for this infinitely large plurality is of course not to be called a number, as we have just remarked in the previous paragraph. On the other hand, if we now make the quantity which appeared *infinitely large* with respect to another one taken as the unit, now itself as the unit, and we measure the one previously considered as the unit with it, then this latter one will now be represented as *infinitely small*.

§ 17

A most important class of infinitely large quantities which nevertheless do not belong in the domain of the actual, although they can be *determinations* of the actual, are *time* and *space*. Neither time nor space is something actual, for they are neither *substances* nor *properties* of substances, but they occur merely as determinations of all incomplete (bounded, finite or—what amounts to the same thing—dependent, created) *substances*. This is because each of the latter must always be in a certain time and also in a certain space such that every simple substance must stay at every *time instant*, i.e. in every simple part of time, and in some simple part of space, i.e. in some point of [space]. Now in time, as well as in space, the multitude of simple parts or points, of which the former and latter consist is *infinite*. Not only is the multitude of simple parts from which the whole of time and the whole of space is composed, i.e. the multitude of moments and points[g] that there are in general, infinitely large, but also the multitude of moments which lie between every two moments α and β, however close to one another, and in the same way the multitude of points which lie between every two points *a* and *b* however close to one another, is infinite. I need hardly go into a justification of these propositions since there is scarcely any mathematician, who, supposing he does not deny every infinity in general, would not concede them to us. But the opponents of *all* infinity, in order not to have to admit the infinity presented here so clearly escape behind the pretext, 'that we can of course always *think of adding* to the points in time and space, more than we have already thought of, but that the multitude of those which there are in *reality*, always remains only a finite multitude.' But I reply to this that neither time nor space, therefore also neither the simple parts of time nor those of space, are something real, so that it is absurd to speak of a finite multitude of them which exists in *reality*. And it is even more absurd to imagine that these parts only obtain their reality through our *thinking*. For it would follow from this that the properties of time, as well as those of space, depend on our thinking [them] or accepting [them] as true, and that therefore the ratio of the diameter to the circumference of a circle was rational as long as we mistakenly regarded it as if it were rational, and that space would have all those properties which we would get to know subsequently, would also then be accepted! But if the opponents rectify the above expression [by saying] that only thinking which is in accordance with truth may determine the true properties of time and space, then they say something completely tautological, namely that what is true, is true. From this then there is certainly not the slightest thing that can be concluded against the infinity of time and space as we have claimed.

[g] *Zeitpuncte* and *Raumpuncte* have been translated as 'moments' and 'points', respectively.

It is, in any case, inept to say that time and space contain only as many points as we imagine.

§ 18

Although every quantity, and generally every object, which counts for us as infinite in some respect, must be able to be considered in this respect as a whole consisting of an infinite multitude of parts, it does not hold conversely that every quantity which we consider as the sum of an infinite multitude of other quantities, which are all finite, must itself be an infinite quantity. For example, it is generally recognized, that irrational quantities, like $\sqrt{2}$, are finite quantities with respect to their underlying unit, although they can be viewed as composed from an infinite multitude of fractions of the form

$$\frac{14}{10} + \frac{1}{100} + \frac{4}{1\,000} + \frac{2}{10\,000} + \cdots,$$

of which the numerator and denominator are whole numbers. Equally the sum of the infinite series of summands of the form: $a + ae + ae^2 + \cdots$ *in inf*. is equal to the finite quantity $\frac{a}{1-e}$ as long as $e < 1$.* Therefore there is certainly nothing

* Since the usual proof for the summation of this series does not seem to be completely strict, it may be permitted to sketch out the following on this occasion. If we take $a = 1$ and e positive (because the application to other cases follows directly), and if we put as a symbolic equation

$$S = 1 + e + e^2 + \cdots \textit{ in inf.} \tag{1}$$

then at least it is certain that S designates a positive quantity, no matter whether it is finite or infinitely large. But also for every arbitrary whole numbered value of n,

$$S = 1 + e + e^2 + \cdots + e^{n-1} + e^n + e^{n+1} + \cdots \textit{ in inf.}$$

or also

$$S = \frac{1 - e^n}{1 - e} + e^n + e^{n+1} + \cdots \textit{ in inf.} \tag{2}$$

for which we can also write

$$S = \frac{1 - e^n}{1 - e} + \overset{1}{P} \tag{3}$$

if we designate the value of the infinite series $e^n + e^{n+1} + \cdots$ *in inf*. by $\overset{1}{P}$, for which we certainly know at least this, that $\overset{1}{P}$ designates a quantity, measurable or unmeasurable, but at any rate positive, which is dependent on e and n. But we can represent the same infinite series in the following way:

$$e^n + e^{n+1} + \cdots \textit{ in inf.} = e^n[1 + e + \cdots \textit{ in inf.}].$$

Now here the sum consisting of infinitely many terms in the brackets on the right-hand side of the equation, namely

$$[1 + e + e^2 + \cdots \textit{ in inf.}]$$

has completely the appearance of the series put forward in the symbolic equation (1) $= S$, but nevertheless it is not to be regarded as identical with it, since the *multitude* of summands here and in (1), although in both cases infinite, is not the same, rather here it is indisputably *n terms less* than in (1).

contradictory in the assertion that a sum of infinitely many finite quantities may itself be only a finite quantity, because otherwise it could not be proved to be true. But the paradox that might be perceived in this, is only produced because it is forgotten how the terms being added here become ever smaller and smaller. For that a sum of summands [*Addenden*], each successive one of which takes, for example, half the value of its predecessor, can never become more than double the first term, cannot really upset anybody because for each of the terms of this series, however much later, the series is always short of that double value by exactly as much as this last term.

§ 19

Even with the examples of the infinite considered so far it could not escape our notice that not all infinite multitudes are to be regarded as *equal to one another in respect of their plurality*, but that some of them are *greater* (or *smaller*) than others, i.e. another multitude is contained as a part in one multitude (or on the contrary one multitude occurs in another as a mere part). This also is a claim which sounds to many *paradoxical*. And of course everyone who defines infinity as something such that it is capable of no further increase, must find it not only paradoxical but directly *contradictory*, that one infinity may be greater than another one. However, we have already found above, that this view rests on a concept of infinity which does not coincide at all with the normal use of the word. After our definition, which corresponds not only to usage but also to the purpose of science, no one

Therefore with complete confidence we can only put the equation $[1 + e + e^2 + \cdots \textit{in inf.}] = S - \overset{2}{P}$ in which we may assume that $\overset{2}{P}$ designates a quantity which is dependent on n and always positive. Accordingly we obtain

$$S = \frac{1 - e^n}{1 - e} + e^n \left[S - \overset{2}{P} \right] \tag{4}$$

or

$$S[1 - e^n] = \frac{1 - e^n}{1 - e} - e^n \overset{2}{P},$$

or finally

$$S = \frac{1}{1 - e} - \frac{e^n}{1 - e^n} . \overset{2}{P}. \tag{5}$$

Combining the two equations (3) and (5) gives

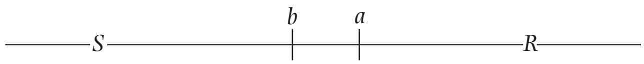
$$\frac{-e^n}{1 - e} + \overset{1}{P} = \frac{-e^n}{1 - e^n} . \overset{2}{P}$$

or

$$\overset{1}{P} + \frac{e^n}{1 - e^n} . \overset{2}{P} = + \frac{e^n}{1 - e}$$

from which we see that if we take n arbitrarily great, and thereby the value of $\frac{e^n}{1-e}$ is brought down below every arbitrary quantity $\frac{1}{N}$ however small, then also each of the quantities $\overset{1}{P}$ and $\frac{e^n}{1-e^n} . \overset{2}{P}$ must itself fall below every arbitrary value. But if this is so then each of the equations (3) and (5) shows that, because S has only an unchanging value for the same value of e and therefore cannot depend on n, $S = \frac{1}{1-e}$.

can find anything controversial or even noteworthy in the idea that one infinite multitude should be greater than another one. For example, to whom must it not be clear, that the length of the straight line



continuing without limit in the direction *aR* is an infinite length? But that the straight line *bR* going in the same direction from the point *b* may be called greater than *aR*, by the piece *ba*? And that the straight line continuing without limit on both sides *aR* and *aS* may be called greater by a quantity which is itself infinite? And so on.

§ 20

Let us now turn to the consideration of a highly remarkable peculiarity which can occur, indeed actually always occurs, in the relationship of two multitudes *if they are both infinite*, but which previously has been overlooked to the detriment of knowledge of some important truths in metaphysics, as well as physics and mathematics. Even now, as I am stating it, it will be found paradoxical to such a degree that it might be very necessary to dwell on its consideration somewhat longer. I claim that two multitudes, that are both infinite, can stand in such a relationship to each other that, *on the one hand*, it is possible to combine each thing belonging to one multitude, with a thing of the other multitude, into a pair, with the result that no single thing in both multitudes remains without connection to a pair, and no single thing appears in two or more pairs, and also, *on the other hand* it is possible that one of these multitudes contains the other in itself as a mere *part*, so that the pluralities which they represent if we consider the members of them all as equal, i.e. as units, have the *most varied relationships* to one another.

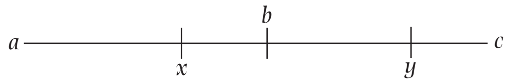
I shall offer the proof of this claim through two examples, in which what has been said indisputably occurs.

1. If we take two arbitrary (abstract) quantities, e.g. 5 and 12, then it is clear that the multitude of quantities which there are between zero and 5 (or which are smaller than 5) is infinite, likewise also the multitude of quantities which are smaller than 12 is infinite. And equally certainly the latter multitude is greater since the former is indisputably only a part of it. If we put any other quantity in the place of the quantities 5 and 12, we cannot avoid the judgement that those two multitudes do not always have the same relationship to one another but rather the most varied kinds of relationships occur. However, no less true than all these things is the following: if x denotes any quantity lying between zero and 5, and we determine the relationship between x and y by the equation

$$5y = 12x,$$

then also the value of y is a quantity lying between zero and 12, and conversely whenever y lies between zero and 12, then x lies between zero and 5. It also follows from that equation that to every value of x there belongs only one value of y, and conversely. From these two things it is clear that to every quantity $=x$, in the multitude of quantities lying between zero and 5 there is one quantity, $=y$, in the multitude of quantities lying between zero and 12, which can be combined with the former into a pair with the result that no single one of the things of which these two multitudes consist, remains without combination into a pair, and also no single one occurs in two or more combinations.

2. The second example will be taken from a spatial object. Whoever already knows that the properties of space are based on those of time, and the properties of time are based on those of abstract numbers and quantities did not of course need to learn from an example that there are such infinite multitudes as we have found generally among quantities also in time and space. Yet it is on account of the correct application of our proposition which we have to make subsequently that it is necessary to consider individually at least one case where there exist such multitudes. Therefore let a, b, c be three arbitrary points in a straight line, and let the ratio of the distances $ab : ac$ also be completely arbitrary, suppose ac denotes the greater of the two. Then although the multitude of points which lie in ab and ac are both infinite,



nevertheless the multitude of points which lie in ac exceeds that of the points in ab, because in ac as well as all the points of ab there also lie all those of bc, which do not occur in ab. We cannot even help admitting that if the ratio of the distances $ab : ac$ is altered arbitrarily then the ratio of these two multitudes will become very different. Nevertheless the same holds for these two multitudes which was proved before for the two multitudes of quantities which lie between 0 and 5 and between 0 and 12 in respect of the pairs which can be formed from each of the things from one multitude and each of the things from the other multitude. For let x be some point in ab, then if we take the point y in the direction ax, so that the ratio

$$ab : ac = ax : ay$$

holds, then y will also be a point in ac. And conversely if y is a point in ac, and if we determine ax from ay by the same equation, then x will be a point of ab. And every other x will determine another y, and conversely every other y will determine another x. But these two truths again show that for every point of ab a point of ac can be chosen, and for every point of ac, a point of ab can be chosen, with the result that of the pairs which we form from every two such points, it can be asserted that there is no single point in the multitude of points of ab, or in the multitude of points of ac, which does not appear in one of these pairs, and also none which appears in two or more pairs.

§ 21

Therefore merely for the reason that two multitudes A and B stand in such a relation to one another that to every part[h] a occurring in one of them A, we can seek out according to a certain rule, a part b occurring in B, with the result that all the pairs $(a + b)$ which we form in this way contain everything which occurs in A or B and contains each thing only once—merely from this circumstance we can—as we see—in no way conclude *that these two multitudes are equal to one another if they are infinite* with respect to the plurality of their parts (i.e. if we disregard all differences between them). But rather they are able, in spite of that relationship between them that is the same for both of them, to have a relationship of inequality in their plurality, so that one of them can be presented as a whole, of which the other is a part. An equality of these multiplicities may only be concluded if some other reason is added, such as that both multitudes have exactly the same determining grounds[i] [*Bestimmungsgründe*], e.g. they have exactly the same way of being formed [*Entstehungsweise*].

§ 22

The paradox which, as I do not deny at all, is attached to these assertions, arises solely from the circumstance that that mutual relationship which we find with the two multitudes being compared with one another, consisting in [the fact] that we can put together the parts of them in pairs in the way already mentioned several times, is indeed sufficient to define them as completely equal in respect of the *plurality* of their parts in every case where these multitudes are *finite*. Namely, if two finite multitudes are of such a nature that to every thing a of one of them we can find a b of the other one and join them into a pair so that in neither of the two multitudes does there remain a thing for which there is nothing corresponding in the other one, and there is also nothing which appears in two or more pairs, are always equal to one another in their plurality. It therefore appears that this should also be the case if these multitudes, instead of being finite, are infinite.

I say, it appears, because more exact consideration shows that in no way does it need to be so, since the reason why it happens for all finite multitudes lies precisely in their finiteness, and is therefore lacking with the infinite multitudes. Namely if both multitudes A and B are finite, or (for this is also sufficient) we know only of one of them A, that is it finite, and we disregard all differences between the things of which they consist, in order to consider now both multitudes only in respect of their plurality, then, while we designate some arbitrary thing in the multitude A by 1, some other arbitrary thing by 2 etc., in such a way that for every successive thing we always give for its designation the number of the things which we have considered so far (including this one itself), we must sometime arrive at

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[h] Sometimes, as in §5, ‘part’ [*Teil*] is used in the sense of ‘subset’, here and in §23 it is used in the sense of ‘element’.

[i] Steele, *PI* uses ‘mode of specification’ for this interesting concept.

a thing in A after the designation of which nothing more remains which is still undesignated. This is a direct consequence of the concept of a finite or countable plurality. Now let this last [thing] just spoken of in A get the number n for its designation, then the number of things in $A = n$. Now because to every thing in A there should be one found in B, that can be combined with it in a pair, then if we designate each of the things from B with exactly the symbol which that thing from A has with which it is paired, it must happen that there are also n things in B which we have used up in this way, since every one of [that multitude] gets a symbol which shows how many we have used so far. Therefore it is clear that of the things in B there are certainly not fewer than n, for this number corresponds to an actual [thing] (the one which we used last). But there are also no more of them, for if there was a single one beyond those used up so far, then there would be for this one nothing in A with which it could be combined in a pair, which contradicts the assumption. Accordingly the number of things in B is neither smaller nor greater than n, therefore $= n$. Therefore both multitudes have one and the same plurality, or as one can also say, *equal* plurality. Obviously this conclusion becomes void as soon as the multitude of things in A is an *infinite* multitude, for now not only do we never reach, by *counting*, the last thing in A, but rather, by virtue of the definition of an infinite multitude, in itself there is no such *last* thing in A, i.e. however many have already been designated, there are always others to designate. Therefore, in spite of the fact that likewise there never lack things in the multitude B which can be joined with those in A into new pairs, any reason to conclude that the two multitudes have one and the same plurality does not apply.

§ 23

What has now been said does show that the *reason*, which produces the necessary equality of finite multitudes as soon as the much discussed relationship holds between them, *becomes void with infinite multitudes*. But it does not show us, how and why it is that with the latter an inequality may often arise. This will only become clear from consideration of the examples mentioned. These show us, in fact, that the parts a and b taken from the two multitudes being compared, and which we combine into a pair $(a + b)$, *do not appear in their multitudes in completely the same way*. For if the parts a' and b' form a second pair and we compare the relationships in which a and a' appear in the multitude A and in which b and b' appear in the multitude B, then it is immediately evident that they are different. Let us take (in the first example) two quantities[j] quite arbitrarily from the multitude of quantities which lie between 0 and 5, say 3 and 4, then the [parts] in B belonging

[j] The word ‘quantity’ is used in this section to reflect the use of *Größe*, although ‘number’ might appear more natural.

to them (forming a pair with them) are obviously

$$\frac{12}{5}.3 \quad \text{and} \quad \frac{12}{5}.4, \qquad \text{i.e.} \qquad 7\frac{1}{5} \quad \text{and} \quad 9\frac{3}{5}.$$

Now if we understand (as we should) by the *relationship* between two things the collection into a whole of *all* known properties, then we may take into account for the relationship in which the parts 3 and 4 stand to one another in the one multitude, and $7\frac{1}{5}$ and $9\frac{3}{5}$ in the other multitude, not merely that ratio which is usually called *geometric*, but rather looking at everything which belongs here, therefore also in particular at the *arithmetic* difference between the quantities 3 and 4, which is quite different from between the quantities $7\frac{1}{5}$ and $9\frac{3}{5}$, since the former = 1 and the latter = $2\frac{2}{5}$. Therefore although every quantity in *A* or *B* can be joined with one and only one unique [quantity] in *B* or *A* into a pair nevertheless the multitude of quantities in *B* is different (greater) than in *A* because the *distance* which every two such quantities in *B* have from one another is different (greater) than the *distance* which separates the two corresponding quantities in *A* from one another. Hence it follows naturally that every two of these quantities in *B* have *between them* a different (greater) multitude of such quantities than is the case in *A*, and therefore it is no surprise that the *whole* multitude of quantities in *B* is different (greater) than in *A*. It is completely similar in the two examples, therefore we wish to say no more about these than that the points in *ab* that are joined in thought with the points of *ac* in pairs, are all standing *nearer* to one another than the corresponding ones in *ac*, since the distance of every two there to the distance of every two here is always in the ratio of *ab* : *ac*.

§ 24

If we may now regard the proposition of §20 as sufficiently proved and clarified by the foregoing, then it follows as the next consequence of it *that we may not immediately put equal to one another, two sums of quantities which are equal to one another pair-wise* (i.e. every one from one with every one from the other), *if their multitude is infinite*, unless we have convinced ourselves that the infinite plurality of these quantities in both sums is the same. That the summands determine their sums, and that therefore equal summands also give equal sums, is indeed completely indisputable, and holds not only if the multitude of these summands is finite but also if it is infinite. But because there are different infinite multitudes, in the latter case it must also be proved that the infinite multitude of these summands in the one sum is exactly the same as in the other. But by our proposition it is in no way sufficient, to be able to conclude this, if in some way one can discover for every term occurring on one sum, another equal to it in the other sum. Instead this can only be concluded with certainty if *both multitudes have the same basis for their determination*. If this is overlooked we shall see subsequently, from some examples, what absurdities may be involved in calculation with infinity.

§ 25

I now come to the claim that there is an infinity not merely among the things which have no reality, but *also in the area of reality itself*. Whoever has arrived through a series of arguments from purely conceptual truths, or in some other way, to the highly important conviction *that there is a God*, a being which has the ground of his being in nothing else, and just for this reason is an *altogether perfect* being, i.e. all perfections and powers which can be present together, and each of them in the highest degree in which they can be together, are combined in him, who therefore takes on the existence of a being which has infinitude in more than one respect, in his *knowing*, his *willing*, his *external effect* (his power). He *knows infinitely many things* (namely the universe of truths), he *wills infinitely many things* (namely the sum of all possible good things), and *everything, which he wants, he puts into reality* through his power to produce external effect. From this last property of God arises the further consequence that there are beings outside of him, namely *created* beings which we call, in contrast to him, *finite beings*, of which nevertheless some infinite things can be proved. For already the *multitude* of these beings must be an infinite one, likewise the multitude of the *circumstances* which each single one of these beings experiences during however short a time, must be infinitely great (because each such time contains infinitely many moments) etc. Therefore we also meet with infinity everywhere in the area of reality.

§ 26

Nevertheless several of those scholars who realize they cannot deny infinity with those things which have no reality (like the mere propositions and truths in themselves) refuse to admit this. For to admit an infinity also in the area of reality, would, they think, be forbidden by the ancient principle *that all reality must have a general definiteness* [*durchgängige Bestimmtheit*]. However I believe I have already shown in the *Wissenschaftslehre* (Bd. I, §45) that this principle also holds of the unreal things in exactly the sense in which it holds of all *real* things. Namely, it holds generally simply in the sense that for every two *contradictory* properties, one must belong to each object (each arbitrary thing), and the other must be denied of it. Therefore if it were established that we violate this principle by the acceptance of an infinity of things which have reality, then we might also not speak of any infinity of the unreal objects of our thought, therefore we might not even admit an infinite multitude of truths in themselves or of mere numbers. But we do not violate the principle referred to at all when we declare something as infinite. We are only saying that in a certain respect there are in this object a plurality of parts which is greater than every arbitrary number, therefore indeed a multiplicity *which cannot be determined by a mere number*. But from this it does not follow at all that this plurality is *something which cannot be determined in any way*; it certainly does not follow that there is even a single pair of contradictory properties *b* and not-*b* of which both of them would have to be denied. What has no colour, e.g. a proposition, that may of course not be determined by the statement of its colour, whatever has no sound, cannot be determined by the statement of its sound etc. But on that account such things are certainly not incapable of being determined and do not make an exception to the principle that of the two predicates *b* or not-*b* (blue or not-blue, harmonious or unharmonious etc.), if we interpret them thus, as we must, so that they remain contradictory, one of them belongs to each thing. In just the same way as not being blue, or not being fragrant is a determination of Pythagoras' theorem (of course, only a very wide one), also the mere statement that the multitude of points between *m* and *n* is infinite is a determination of this multitude. And it may often not need many statements in order to determine such an infinite multitude of things *completely*, i.e. so that *all* its properties follow merely from the few that have been stated. Thus we have the infinite multitude of points just mentioned between *m* and *n* determined in the most complete way as soon as we only determine the two points *m* and *n* themselves (say by an intuition referring to them). For then it is decided precisely for every other point, merely by those few words, whether it belongs to this multitude or not.

§ 27

If in the foregoing I have been allowed to defend many assumptions of infinity against incorrect denials of them, I must now acknowledge with equal candour that many scholars especially among *mathematicians*, have gone too far in the opposite direction. They have adopted sometimes an *infinitely large*, and sometimes an *infinitely small*, where according to my own conviction there is none.

1. I have no objection to the assumption of an *infinitely large time interval*, if one understands by it a time interval which has either no start or no end or even neither the one nor the other (the whole of time or the collection of all moments of time in general is such). But I find it necessary to think of the *ratio* which one time interval, or distance between two moments, has to every other time interval or distance between two moments, as a merely finite ratio completely determined by mere concepts, therefore never to assume a time interval bounded by beginning and end as infinitely greater or smaller than another such time interval. But it is well known that many mathematicians do exactly this since they speak not only of infinitely large amounts of time, which nevertheless are to be bounded on both sides, but even more often of *infinitely small parts of time*, in comparison with which every *finite* time interval, e.g. a second, would have to be regarded as infinitely large.

2. A similar thing holds of the *distances between every two points in space*, which in my view can always stand in a merely finite relationship (completely determinable by pure concepts) to one another while nothing is more usual with our mathematicians than to speak of *infinitely large* and *infinitely small distances*.

3. Finally also with the *forces* in the universe which are assumed in metaphysics as well as physics, none of which we must suppose to be infinitely greater or smaller than another one but all are in a relationship to every other that is completely

determinable by mere concepts however often one allows oneself to do the opposite. The reasons for which I claim all these things I will not of course be able to make completely clear to anyone here who does not even know the concepts which I connect with the words *intuition* and *concept*, *derivability* of a proposition from others, *objective consequence* of a truth from other truths, and many others, finally also the *definitions* of time and space. Nevertheless whoever has at least read the two works, *Versuch einer objektiven Begründung der Lehre von der Zusammensetzung der Kräfte*,* and *Versuch einer objektiven Begründung der Lehre von der drei Dimensionen des Raumes*,** should find the following proof not entirely unintelligible.

From the definitions of time and space it follows directly that all *dependent* (i.e. created) substances always have a mutual effect on one another. Also, it may be allowed that of every two moments α and β, however near or far they may be from one another, the state of the world in the earlier one α may be considered as a *cause*, and the state of the world in the later one β as an *effect* (at least indirectly), as long as the direct actions of God which occurred in the intervening time $\alpha\beta$ are counted into the cause. Hence it follows further that from the statement of the two moments α and β, from the statement of all the *forces* which the created substances have in the moment α, from the statement of the *places* where each of them is, and finally a statement of the divine influences which one or other of those substances experienced within $\alpha\beta$—then as well as the *forces* which these substances experience at the moment β, also the *places* which belong to them are derivable in the same way as an *effect* must be derivable (equally whether directly or indirectly) from its complete cause. Now this further requires that all properties of the effect can be derived from the properties of its cause, by means of a principle [*Obersatz*], composed from nothing but pure concepts, of the form: Every cause with the property $u, u', u'', \ldots$ has an effect with the property $w, w', w'', \ldots$. An easy consequence from this which we require now for our purpose, is: Every circumstance of the cause which *does not hold equally* for the effect, i.e. which is of such a nature that the effect does not remain the same however the circumstance changes, must *be completely determined* through mere concepts for which at most some intuitions which are also required for the determination of the effect are taken as their basis.

Now after these preliminaries, our assertions made above are easily established. For if there were:

1. even only two moments α and β whose distance from one another was infinitely many times greater or smaller than the distance of two others γ and δ, then the absurdity would follow from this that the state of the world which is to occur at the moment β can absolutely not be determined from that state which occurs at the moment α together with the divine actions occurring in the time interval and also the size of the time interval $\alpha\beta$. Also for the determination of the state in which the created being exists, indeed only the *magnitudes of its forces* in a single

* Prague, 1842, published by Kronberger & Řiwnač.

** Prague, 1843, published by Kronberger & Řiwnač.

moment α, the basis of a proper time unit is necessary. For because these forces are merely *forces of change* then their magnitude cannot be judged other than with respect to a given time interval within which they bring about a given effect. Therefore if we take the time interval $\gamma\delta$ as this time unit (which we must be allowed to do), then even in the most favourable case, if with this time unit, all forces of the created substances, as they are at the moment α can be determined precisely, and if everything else which belongs to the complete cause of the state of the world ocurring at the moment β can be determined precisely, nevertheless the distance at which this moment itself stands from α cannot be determined by that time unit in that it proves to be infinitely large or infinitely small. Therefore *conversely* if it is to be allowed that every arbitrary state of the world (under the conditions already mentioned several times) should be considered as cause of every arbitrary later [state], then there may not be two moments α and β whose distance from one another compared with the distance in which another pair γ and δ stand proves to be infinitely great or small.

2. If there were even only two points in space *a* and *b*, whose distance from one another in comparison with the distance of another pair *c* and *d* was infinitely large or small, then for the determination of the state of the world belonging to some moment α would belong, among other things, the determination of the magnitude of the force (perhaps of attraction or of repulsion) which the substance *A*, occurring at that moment in the place *a*, exerts on *B* occurring in the place *b*. But if we adopt (as is always permitted) the distance *cd* as the unit of length, then this would, even in the most favourable case, when we were successful with all other forces, prove for this one to be a force which is impossible. For if the force of attraction or repulsion which substance *A* exerts on a substance completely similar to *B* at the distance (= *cd*) taken for the unit of length, were to have a completely determinate magnitude, then directly from the fact that this magnitude is determinate, the magnitude of the attraction or repulsion with which *A* acts on *B* is indeterminate if the ratio of the distances $ab : cd$ on which it depends were to be infinite and therefore indeterminate.

3. Finally if there were even a single force *k* which appeared to be infinitely large or small in comparison with another one *l*, then if we denote the moment when this ratio holds by α, for this moment even in the most favourable case where all other forces had been shown to be finite for the units of time and space chosen for their measurement, and where also *l* was finite, the quantity *k* would, just for this reason, turn out to be an infinitely large or small quantity, i.e. as indeterminate. But thereby the whole state of the world at the moment α would appear indeterminate, therefore the derivation of some later state of the world as an effect produced by it would be impossible.

§ 28

Now I believe in the foregoing I have established the basic rules according to which all strange-sounding theories which we have to set out in the following, can be judged. It must be decided whether they should be renounced as errors or must be retained as propositions, which in spite of their appearance of contradiction, are nevertheless truths. The order in which we set out these paradoxes may determine the scientific area to which they belong, and their true importance, greater or lesser.

The first and most comprehensive science in whose domain we meet with paradoxes of the infinite is—as some examples have already shown—*the general theory of quantity* where such paradoxes are not missing even in *number theory*. Therefore it is with these that we shall begin.

Even the *concept* of a *calculation of the infinite* has, I admit, the appearance of being self-contradictory. To want to *calculate* something means to attempt a *determination of something* through numbers. But how can one determine the infinite through numbers—that infinite which according our own definition must always be something which we can consider as a multitude consisting of infinitely many parts, i.e. as a multitude which is greater than every number, which therefore cannot possibly be determined by the statement of a mere number? But this doubtfulness disappears if we take into account that a calculation of the infinite done correctly does not aim at a calculation of that which is determinable through no number, namely not a calculation of the infinite plurality in itself, but only a determination of the *relationship* of one infinity to another. This is a matter which is feasible, in certain cases at any rate, as we shall show by several examples.

§ 29

Whoever admits that there are infinite pluralities and therefore also infinite quantities generally, cannot also deny that there are infinite quantities which differ from one another according to their quantity (magnitude) in various ways. For example if we denote the series of natural numbers by

$$1, 2, 3, 4, \ldots, n, n+1, \ldots \textit{in inf}.$$

then the expression

$$1 + 2 + 3 + 4 + \cdots + n + (n+1) + \cdots \textit{in inf}.$$

will be the *sum* of these natural numbers, and the following expression

$$1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 + (n+1)^0 + \cdots \textit{in inf}.$$

in which the single summands, 1^0, 2^0, 3^0, ... all represent mere units, represents just the *number* [*Menge*] of all natural numbers. If we designate this by $\overset{0}{N}$ and therefore form the merely symbolic equation

$$1^0 + 2^0 + 3^0 + \cdots + n^0 + (n+1)^0 + \cdots \textit{in inf}. = \overset{0}{N} \qquad (1)$$

and in the same way we designate the number of natural numbers from $(n+1)$ by $\overset{n}{N}$, and therefore form the equation

$$(n+1)^0 + (n+2)^0 + (n+3)^0 + \cdots \textit{in inf.} = \overset{n}{N}. \qquad (2)$$

Then we obtain by subtraction the certain and quite unobjectionable equation

$$1^0 + 2^0 + 3^0 + \cdots + n^0 = n = \overset{0}{N} - \overset{n}{N} \qquad (3)$$

from which we therefore see how two infinite quantities $\overset{0}{N}$ and $\overset{n}{N}$ sometimes have a completely definite finite difference.

On the other hand if we designate the quantity which represents the *sum* of all natural numbers by $\overset{0}{S}$, or assert the merely symbolic equation

$$1 + 2 + 3 + \cdots + n + (n+1) + \cdots \textit{in inf.} = \overset{0}{S} \qquad (4)$$

then we will certainly realize that $\overset{0}{S}$ must be far greater than $\overset{0}{N}$. But it is not so easy to determine precisely the difference between these two infinite quantities or even their (geometrical) *ratio* to one another. For if, as some people have done, we wanted to form the equation

$$\overset{0}{S} = \frac{\overset{0}{N} \cdot (\overset{0}{N} + 1)}{2}$$

then we could hardly justify it on any other ground than that for every finite multitude of terms the equation

$$1 + 2 + 3 + \cdots + n = \frac{n.(n+1)}{2}$$

holds, from which it appears to follow that for the complete infinite multitude of numbers n just becomes $\overset{0}{N}$. However it is in fact not so, because with an infinite series it is absurd to speak of a last term which has the value $\overset{0}{N}$.

The purely symbolic equation (4) underlying all this will surely allow the derivation, through successive multiplication of both sides by $\overset{0}{N}$, of the following equations:

$$1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} + \cdots \textit{in inf.} = (\overset{0}{N})^2$$

$$1^0.(\overset{0}{N})^2 + 2^0.(\overset{0}{N})^2 + 3^0.(\overset{0}{N})^2 + \cdots \textit{in inf.} = (\overset{0}{N})^3 \quad \text{etc.}$$

from which we are convinced that there also infinite quantities of so-called *higher orders*, of which one exceeds the other infinitely many times. But it also certainly follows from this there are infinite quantities which have every arbitrary rational, as well as irrational, ratio $\alpha : \beta$ to one another, because, as long as $\overset{0}{N}$ denotes some infinite quantity which always remains the same, $\alpha.\overset{\circ}{N}$ and $\beta.\overset{\circ}{N}$ are likewise a pair of infinite quantities which are in the ratio $\alpha : \beta$.

It is no less clear that it will be found that the whole *multitude* (plurality) of quantities which lie between two given quantities, e.g. 7 and 8, although it is equal to an *infinite* [multitude] and therefore cannot be determined by any number however great, depends solely on the magnitude of the distance of those two boundary quantities from one another, i.e. on the quantity $8 - 7$, and therefore must be an equal [multitude] whenever this distance is equal. Assuming this, if we designate the multitude of all quantities lying between a and b by

$$\text{Mult.}\ (b - a)$$

there will be innumerable equations of the following form:

$$\text{Mult.}\ (8 - 7) = \text{Mult.}\ (13 - 12)$$

and also of the form

$$\text{Mult.}\ (b - a) : \text{Mult.}\ (d - c) = b - a : d - c$$

against the correctness of which no valid objection can be made.

§ 30

Now as these few examples are sufficient to show that a *calculation with the infinitely large* may exist, so one with the *infinitely small* may also exist. For if $\overset{\circ}{N}$ is infinitely large, then indeed

$$\frac{1}{\overset{\circ}{N}}$$

necessarily represents a quantity which is infinitely small, and at least in the *general* theory of quantity, we shall have no reason to describe such an idea as altogether empty. In order to give a single example, if the question is raised of what is the probability if someone who shoots a bullet at random, shoots it in such a way that its centre goes precisely through the centre of that apple hanging on this tree, then everyone must admit that the multitude of all possible cases of an equal or still smaller probability is infinite whence it follows that the degree of that probability has a magnitude $=$ or $< \frac{1}{\infty}$. But with this it is proved that we have infinitely many of the infinitely small quantities, of which they have every arbitrary ratio one to another. In particular, it can even be infinitely greater. Therefore there also exist infinitely many orders among the infinitely large, as also among the infinitely small quantities, and by following certain rules it will indeed be possible to find very often correct equations between quantities of this kind.

For example, if it is first decided that the value of variable quantity y depends on another x in such a way that the equation,

$$y = x^4 + ax^3 + bx^2 + cx + d$$

always holds between them, and it is compatible with the nature of that special kind of quantity which x and y designate here, that they can also become infinitely small and therefore can take an infinitely small increment, then if we can increase x by an infinitely small part designated by $\mathrm{d}x$, and the change which y undergoes is designated by $\mathrm{d}y$, then also the following equation must necessarily hold,

$$y + \mathrm{d}y = (x + \mathrm{d}x)^4 + a(x + \mathrm{d}x)^3 + b(x + \mathrm{d}x)^2 + c(x + \mathrm{d}x) + d,$$

from which also follows without contradiction,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (4x^3 + 3ax^2 + 2bx + c) + (6x^2 + 3ax + b)\mathrm{d}x + (4x + a)\mathrm{d}x^2 + \mathrm{d}x^3$$

which represents the ratio of the two infinitely small quantities as a quantity dependent not only on a, b, c and x but also on the value of the variable $\mathrm{d}x$ itself.

§ 31

However most mathematicians who ventured to calculate with the infinite have gone much further than is allowed by the principles established here. Not only did they permit the assumption, without thinking, sometimes of an infinitely large and sometimes of an infinitely small among quantities which in their nature are incapable of such (of which examples are to be mentioned subsequently) but they even presumed sometimes to make quantities which arise from the summation of infinite series equal to one another, sometimes to set one as greater or smaller than the other, merely because in both of them corresponding terms which stand in such relationships of equality or inequality can be found although their multitudes were obviously unequal. They ventured to state that not only does every infinitely small quantity, or also one of a *higher* order next to one of a *lower* order, *vanish like a mere zero* in the summation with a finite [quantity], but also every infinitely *great* quantity of lower order in the summation next to one of a *higher* order *vanishes like a mere zero*. In order to justify to some extent their method of calculation based on this proposition, they think of the claim that it is permissible to consider a mere zero as divisor and that the quotient

$$\frac{1}{0}$$

basically denotes nothing but an *infinitely large quantity*, but the quotient $\frac{0}{0}$ denotes a completely *indeterminate quantity*. We must show how false and misleading these concepts are because even these days they are still more or less fashionable.

§ 32

It was only in 1830, in *Gergonne*'s *Annales de Mathématique* (Vol. 20, No. 12), someone with the signature *M. R. S.* attempted to prove that the well-known infinite series

$$a - a + a - a + a - a + \cdots \textit{ in inf}.$$

has the value $\frac{a}{2}$, since he believed, having put this value $= x$, he may conclude that

$$x = a - a + a - a + \cdots \textit{in inf.} = a - (a - a + a - a + \cdots \textit{in inf.})$$

and the series enclosed in the brackets is identical with the one to be calculated therefore may again be put $= x$ which gives,

$$x = a - x$$

and therefore

$$x = \frac{a}{2}.$$

The false inference here is not deeply hidden. The series in the brackets obviously does not have the same multitude of terms as the one put $= x$ at first, rather it is lacking the first a. Therefore its value, supposing it could actually be stated, would have to be denoted by $x - a$. But this would have given the identical equation

$$x = a + x - a.$$

'But,' it might be said, 'there is something paradoxical here in that this series which is certainly not infinitely large, should have no exactly determinable, measurable value, the more so since it may arise through an indefinitely continued division by $2 = 1 + 1$ into a: an origin which speaks entirely for the correctness of the assumption that its true value is $\frac{a}{2}$.'

I may draw attention to the fact, which is not in itself incomprehensible, that there may be *quantity expressions* which designate *no actual quantity*, as we generally accept, and must accept, zero itself as one such expression.

In particular a *series*, if we want to consider it only as a quantity, namely only as the *sum* of its terms must, by virtue of the *concept* of a sum (which belongs to multitudes, i.e. to those totalities for which no attention is paid to the *order* of their parts) have such a nature that it undergoes no change in value when we make a change in the order of its terms. With quantities especially it must be that:

$$(A + B) + C = A + (B + C) = (A + C) + B.$$

This property now offers us a clear proof that the expression [*Zeichnung*] under discussion:

$$a - a + a - a + a - a + \cdots \textit{in inf.}$$

is not an expression of an actual quantity. For we should surely change nothing in the quantity represented here, supposing one was represented, if we altered that expression thus:

$$(a - a) + (a - a) + (a - a) + \cdots \textit{in inf.} \tag{1}$$

because here nothing else has happened than that every two adjacent terms in a partial sum have been combined. This certainly must be possible, because the given series should actually have no *last* term. But then we obtain

$$0 + 0 + 0 + \cdots \textit{in inf}.$$

which is obviously only $= 0$.

Nevertheless, just as little can anything be altered in the quantity which that expression represents, supposing it does actually represent one, if we re-arrange it thus:

$$a + (-a + a) + (-a + a) + (-a + a) + \cdots \textit{in inf}. \tag{2}$$

where, with the omission of the first [term], we combine every two successive terms in a partial sum, or also:

$$-a + (a - a) + (a - a) + (a - a) + \cdots \textit{in inf}. \tag{3}$$

which is obtained from (1) if the terms in each pair are transposed and in the expression obtained the same change is made as that by which (2) arises from (1). Therefore if the given quantity expression were *not empty* then the expressions (1), (2) and (3) would all have to denote the same quantity, because it is clear that the idea of a sum of one and the same multitude of quantities cannot represent several quantities *different from one another*, as is the case for example with the ideas $\sqrt{+1}$, arcsin $= \frac{1}{2}$ etc. However, the quantity idea [*Größenvorstellung*] under consideration here:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots \textit{in inf}.$$

if it is not to be altogether empty, with the same justification with which we wanted to put it equal to zero (which is usually called a quantity albeit in a figurative sense), it must also be put $= +a$, and also $= -a$. This is altogether absurd and therefore justifies the conclusion that we have here an absolutely empty idea.

It is true that the series under discussion is produced by an indefinitely continued division of $2 = 1 + 1$ into a, but all series which are produced in such a way can, of course, just because that division always leaves a remainder (here alternately $-a$ and $+a$), only give the true value of the quotient (here $\frac{a}{2}$), if the remainders arising from further division become smaller than every quantity however small. This occurs in the case of the series, $a + ae + ae^2 + \cdots$ *in inf.*, considered in §18, which is produced by the division of $1 - e$ into a provided $e < 1$. But if, as in the previous case, $e = 1$, or even if $e > 1$, where therefore the remainder rises ever higher the further the division is continued, nothing could be more understandable than that the value of the series cannot become equal to the quotient $\frac{a}{1-e}$. Or how should, for example, the series with alternating signs:

$$1 - 10 + 100 - 1\,000 + 10\,000 - 100\,000 + \cdots \textit{in inf}.$$

which arises through the indefinitely continued division of $1 + 10$ into 1, be able to become $= \frac{1}{11}$? Whoever really wanted to put the series

$$1 + 10 + 100 + 1\,000 + \cdots \textit{in inf}.$$

composed of purely positive terms, equal to the negative value $-\frac{1}{9}$, merely because $\frac{1}{1-10}$ expands into this series? Nevertheless the person *M. R. S.* mentioned before still defends such summations and wants to prove, for example, the correctness of the equation

$$1 - 2 + 4 - 8 + 16 - 32 + 64 - 128 + \cdots \textit{in inf.} = \frac{1}{3}$$

only for the reason that

$$\begin{aligned} x &= 1 - 2 + 4 - 8 + 16 - 32 + 64 - \cdots \\ &= 1 - 2(1 - 2 + 4 - 8 + 16 - 32 + \cdots) \\ &= 1 - 2x. \end{aligned}$$

Here it is again overlooked that the series contained in the brackets is not the same one as taken originally because it no longer has the same multitude of terms. It is also clear that this number expression is empty in a similar way as with the one considered earlier, because it leads to contradictory results. For on the one hand, it would have to be that:

$$\begin{aligned} &1 - 2 + 4 - 8 + 16 - 32 + 64 - \cdots \\ &\quad = 1 + (-2 + 4) + (-8 + 16) + (-32 + 64) + \cdots \\ &\quad = 1 + 2 + 8 + 32 + 64 + \cdots \end{aligned}$$

on the other hand, equally certainly:

$$\begin{aligned} &= (1 - 2) + (4 - 8) + (16 - 32) + (64 - 128) + \cdots \\ &= -1 - 4 - 16 - 64 - \cdots \end{aligned}$$

so that therefore, by a doubly justified procedure, the same expression results in one time an infinitely large positive value and another time an infinitely large negative value.

§ 33

Therefore if we wish to avoid getting onto the wrong track in our calculations with the infinite then we may never allow ourselves to declare two infinitely large quantities, which originated from the summation of the terms of two infinite series, as equal, or one to be greater or smaller than the other, because every term in the one is either equal to one in the other series, or greater or smaller than it. We may, just as little, declare such a sum as the greater just because it includes all the terms of the other and in addition many, even infinitely many, terms (which are all positive), which are absent in the other. For even in spite of that it can be smaller, even infinitely smaller, than the latter. An example is supplied by the very well-known sum of the *squares* of all natural numbers compared with the sum of the *first powers* of these numbers. Certainly no one can deny that every term of the series of all *squares*

$$\left.\begin{array}{l} 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + \cdots \textit{in inf.} = \\ 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + \cdots \textit{in inf.} \quad = \end{array}\right\} \overset{2}{S}$$

because it is also a natural number, also appears in the series of first powers of the natural numbers

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+\cdots \textit{in inf.} = \overset{1}{S}$$

and likewise in the latter series $\overset{1}{S}$, together with all the terms of $\overset{2}{S}$ there appear many (even infinitely many) terms which are missing from $\overset{2}{S}$ because they are not square numbers. Nevertheless $\overset{2}{S}$, the sum of all square numbers, is not smaller but is indisputably greater than $\overset{1}{S}$, the sum of the first powers of all numbers. For first of all, in spite of all appearance to the contrary, the *multitude of terms* [*Gliedermenge*] in both series (not considered as sums, and therefore not divisible into arbitrary multitudes of parts) is certainly the same. By the fact that we raise every single term of the series $\overset{1}{S}$ to the square into the series $\overset{2}{S}$, we alter merely the nature (the magnitude) of these terms not their plurality. But if the multitude of terms in $\overset{1}{S}$ and $\overset{2}{S}$ is the same, then it is clear that $\overset{2}{S}$ must be much greater than $\overset{1}{S}$, since, with the exception of the *first* term, each of the remaining terms in $\overset{2}{S}$ is definitely greater than the corresponding one in $\overset{1}{S}$. So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms as $\overset{1}{S}$, namely:

$$0, 2, 6, 12, 20, 30, 42, 56, \ldots, n(n-1), \ldots \textit{in inf.},$$

in which, with the exception of the first *two* terms, all succeeding terms are greater than the corresponding terms in $\overset{1}{S}$, so that the sum of the whole series is again indisputably greater than $\overset{1}{S}$. If we therefore subtract from this remainder the series $\overset{1}{S}$ for the second time, then we obtain as the *second* remainder a series of the same number of terms

$$-1, 0, 3, 8, 15, 24, 35, 48, \ldots, n(n-2), \ldots \textit{in inf.}$$

in which, with the exception of the first *three* terms, all the following terms are greater than the corresponding ones in $\overset{1}{S}$, so that also this third remainder is without contradiction greater than $\overset{1}{S}$. Now since these arguments can be continued without end it is clear that the sum $\overset{2}{S}$ is infinitely greater than the sum $\overset{1}{S}$, while in general we have

$$\overset{2}{S} - m\overset{1}{S} = (1 - m) + (2^2 - 2m) + (3^2 - 3m) + (4^2 - 4m) + \cdots + (m^2 - m^2) + \cdots + n(n - m) + \cdots \textit{in inf}.^{\text{k}}$$

In this series only a finite multitude of terms, namely the first $m - 1$ are negative and the mth is 0, but all succeeding ones are positive and increase indefinitely.

§ 34

Before we can put the incorrectness of the other assertions mentioned in §31 in a proper light we must determine [*bestimmen*] the concept of zero rather more precisely than is usually done.*

All mathematicians indisputably wish to know that only such a concept is connected with the symbol 0 that the two equations

$$A - A = 0 \qquad \text{I}$$

$$A \pm 0 = A \qquad \text{II}$$

may always be written, whatever kind of quantity expression A is, regardless whether it corresponds to an actual quantity or is quite empty. Now here everyone will admit that this can only be permitted if we consider the symbol 0 itself not as the idea of an actual quantity, but rather as the mere absence of a quantity and the notation $A \pm 0$ as a demand for the possible quantity, which A denotes, if in truth *we wish neither to add nor subtract something*. But it would be wrong to believe that the mere explanation that zero is an empty quantity idea is sufficient for the complete determination of the concept which mathematicians associate with this symbol. For obviously there are other notations for quantities in mathematics, as in particular the sign $\sqrt{-1}$ which has become so very important in analysis, which are likewise empty, which nevertheless we may not view and deal with as equivalent to 0. But if we determine the meaning of the symbol 0 more precisely by the definition: it is to be understood in such a way that the two equations I and II hold generally, then we establish a concept which on the one hand is quite wide enough for what is required by previous usage and the interests of science and yet on the other hand is also sufficiently narrow to prevent any misuse of it.

But, on further consideration, it is not merely the concept of zero which is determined in a special way by stipulating the general validity of the two equations I and II, but also the concepts of *addition* and *subtraction*, which appear here

* I very gladly grant to *Herr M. Ohm* the merit, in his very valuable *Versuche eines vollkommen konsequenten Systems der Mathematik* (2. Aufl., Berlin, 1828), to have been the first to have drawn the attention of the mathematical public to the difficulties in the concept of zero.

[k] The German first edition has only $\overset{2}{S} - \overset{1}{S}$ on the left-hand side of this equation.

with the symbols $+$ and $-$, undergo a particular extension which is very much to the advantage of science.

Furthermore, the same advantage of science requires that the concept of *multiplication* may be understood so broadly that whatever A is (whether finite, or an infinitely large or infinitely small quantity, or even a merely empty quantity idea like $\sqrt{-1}$ or 0) the equation:

$$0 \times A = A \times 0 = 0 \qquad \text{III}$$

can be formed.

Finally, also in the interests of science, we must require that the concept of *division* be conceived as generally as possible so as not to contradict one of the three equations already established, therefore also in the equation:

$$B \times \left(\frac{A}{B}\right) = \left(\frac{A}{B}\right) \times B = A \qquad \text{IV}$$

to give the symbol B such a wide range as those three equations allow in the generality already belonging to them. Now all these permit that B may designate any arbitrary finite, as well as infinitely large or infinitely small actual quantity, as well as the imaginary $\sqrt{-1}$, but absolutely not that B may become put $= 0$, i.e. that at any time we do not use zero, or some expression equivalent to zero, as a *divisor*. For since by III it must be that $0(A) = 0$, whatever A is, then if we put $B = 0$ in IV it would also have to be that $B(\frac{A}{B}) = 0$ which would agree with the equation $B\left(\frac{A}{B}\right) = A$ required in IV only in the single case when $A = 0$. Therefore in order not to fall into contradiction we must establish the rule *that zero or an expression equivalent to zero may never be used as a divisor in an equation which is to be anything other than a mere identity*, as perhaps

$$\frac{A}{0} = \frac{A}{0}.$$

That the observation of this rule is absolutely necessary is proved, apart from what has just been said, from the highly absurd consequences which arise from completely correct premisses as soon as we allow ourselves divisions by zero.

Let a be any kind of real quantity, then if division by an expression equivalent to zero, e.g. $1 - 1$, is to be permitted, then by the well-known and certainly quite correct method of division, the following equation arises:

$$\frac{a}{1-1} = a + a + \cdots + a + \frac{a}{1-1}$$

where arbitrarily many of the summands of the form a can appear. Now if we subtract from both sides the same quantity expression $\frac{a}{1-1}$ the highly absurd equation arises:

$$a + a + \cdots + a = 0.$$

If a and b are a pair of different quantities then the two identical equations hold:

$$a - b = a - b$$
$$b - a = b - a$$

Therefore also by addition $a - a = b - b$

or $a(1 - 1) = b(1 - 1)$.

Now if it is permitted to divide the two sides of an equation by a factor equivalent to zero, then we obtain the absurd result $a = b$, whatever a and b may be. Nevertheless it is generally known that an incorrect result may be reached much too easily with larger calculations if a common factor is cancelled from both sides of an equation without first being convinced that it is not zero.

§ 35

It will now be easy to show how wrong is the assertion put forward by so many that not only does an infinitely small quantity of higher order *vanish like a mere zero* in combination by addition or subtraction with another of lower order or with a finite quantity, but also every finite quantity, and even every infinitely large quantity of each arbitrarily high order in combination by addition or subtraction with another infinitely large quantity of higher order *vanishes like a mere zero*. Now if this is to be understood—and in the usual expositions which read rather carelessly, like the expressions just used, one is not warned against such misinterpretation—if this, I say, is to be explained in such a way that from the combination [*Komplexe*] of the two quantities $M \pm m$, of which the first is infinitely greater than the second, the latter may be dropped altogether, even if in the course of the calculation the quantity M itself may disappear (maybe by subtraction of a quantity equal to it), then I hardly need to prove the error of this rule.

Nevertheless it will be said that this is not what is meant. If the quantities M and $M \pm m$ are said to be equal, then it is not meant that they yield an equal result if in further calculation they enter into new combinations by additions or subtractions, but rather their equality only consists in this, that in the process of *measuring*, namely by a quantity N which has equal status with them and stands in a finite (therefore completely determinable) ratio to one of them, e.g. to M, they give equal results. This would in fact be the least that one is justified to require in a definition of a pair of quantities being *equally great*. But do M and $M \pm m$ achieve even this much? If one of them, e.g. M, stands in an irrational ratio to the measure N, then it can certainly happen that, with the most usual method of measuring, which seeks, for every arbitrary number q however large, another number p with the property that

$$\frac{M}{N} > \frac{p}{q} < \frac{p+1}{q}\text{[1]}$$

[1] It is printed thus in the first edition. It clearly means what we would now write as $\frac{p}{q} < \frac{M}{N} < \frac{p+1}{q}$. Similarly for the equation two lines later.

and it can happen that $\frac{M \pm m}{N}$ always remains within the same limits, i.e. that also

$$\frac{M \pm m}{N} > \frac{p}{q} < \frac{p+1}{q}.$$

But if the ratio $\frac{M}{N}$ is rational then there is a q for which

$$\frac{M}{N} = \frac{p}{q}$$

and on the other hand $\frac{M \pm m}{N}$ is either $>$ or $< \frac{p}{q}$, where therefore there is a difference made known between these quantities even in comparison with mere *numbers* (finite quantities). Therefore how can we call them equal to one another?

§ 36

In order to avoid such contradictions several mathematicians have taken refuge, following *Euler*'s procedure, in the explanation that infinitely small quantities were in fact *mere zeros* but that the infinitely large quantities were the quotients which arise from a finite quantity from division by a mere zero. With this statement the vanishing or dispensing of an infinitely small quantity in combination by addition with a finite quantity was more than justified. But it was all the more difficult to make comprehensible the existence of infinitely large quantities, likewise the possibility of the emergence of a finite quantity from the division of two infinitely small or large quantities, and the existence of infinitely small and infinitely large quantities of higher order. For on this view the infinitely large quantities arose from a division by zero or a quantity expression equivalent to zero (which is actually an empty idea), therefore in a way forbidden by the laws of calculation. But to all those finite or even infinite quantities which could arise by division of an infinite quantity into another infinite quantity there cling the many blemishes of illegitimate birth.

What seems to support best the correctness of this calculation with zeros is surely the way in which the value of a quantity y, that is dependent on the variable x and is to be determined by the equation

$$y = \frac{Fx}{\Phi x}$$

may be calculated in the special cases when a certain value of $x = a$ makes either the denominator alone of this fraction equal to zero, or the denominator and numerator together equal to zero. In the first case, if $\Phi a = 0$ but Fa remains a finite quantity it is concluded that y has become *infinitely large*. On the other hand, in the second case, when $Fa = 0$ as well as $\Phi a = 0$, then it is concluded that the two expressions Φx and Fx contain a factor of the form $(x - a)$ once or several times and therefore must be of the form

$$\Phi x = (x-a)^m . \phi x; \quad Fx = (x-a)^n . fx$$

where ϕx or fx can possibly also represent constants. Now if $m > n$ then it is concluded that after removing the common factors in the denominator and the numerator (which does not change the value of the fraction $\frac{Fx}{\Phi x}$), the former still becomes zero for $x = a$, and therefore the assertion still holds that the value $x = a$ gives an infinitely large y. But if $m = n$ then, since it must be that $\frac{Fx}{\Phi x} = \frac{fx}{\phi x}$, the finite quantity which $\frac{fa}{\phi a}$ expresses, is viewed as the correct value of y. And finally if $m < n$ then it is concluded that because now

$$\frac{Fx}{\Phi x} = \frac{(x-a)^{n-m}.fx}{\phi x}$$

becomes zero for $x = a$, that the value $x = a$ makes the quantity y zero.

My opinion of this procedure is as follows. If the value of y belonging to $x = a$ in the specified cases is declared to be infinitely large, then that can obviously only happen to be true if the quantity y is of a kind which *can* become infinitely large. In the first place, it remains true that this result does not arise from the given expression, which here calls for a division by zero. Merely from the circumstance that is stated, the value of y is always the one which the given expression $\frac{Fx}{\Phi x}$ specifies, we can only argue for the nature of the quantity y for all those values of x which represent a real quantity, but not for those with which this expression is *empty*, as is the case if its denominator becomes zero, or even only its numerator is zero, and certainly if both are zero at once. It could well be said that the quantity y, in the case mentioned first where only $\Phi x = 0$, may become *greater* than every given quantity, and in the second case where only $Fx = 0$, it may become *smaller* than every given quantity. Finally in the third case where $\frac{Fx}{\Phi x}$ contains an equal number of factors of the form $(x - a)$ in the denominator and numerator, [the quantity y] can approach as close as desired to the value $\frac{fa}{\phi a}$ while x approaches as close to the value a as desired. However nothing follows from all this about the nature of this value when the expression $\frac{Fx}{\Phi x}$ is *empty*, i.e. represents no value at all, because it either takes the value 0 itself, or the form $\frac{c}{0}$, or indeed the form $\frac{0}{0}$. For the proposition about the equality of the value of two fractions of which one differs from the other only by the removal of a common factor in the denominator and numerator holds indeed in all cases except in the case where this factor is a *zero*. Because otherwise with the same justification with which we claim to maintain that $\frac{2.0}{3.0} = \frac{2}{3}$, it might also be maintained that any arbitrary quantity, e.g. $1000 = \frac{2}{3}$. For it is certain that $3000.0 = 0$ as well as $2.0 = 0$. Therefore if $\frac{2.0}{3.0}$ may be put $= \frac{2}{3}$, then also

$$\frac{2 \times (3000.0)}{3 \times (2.0)} = \frac{(2.3000).0}{(3.2).0} = \frac{2.3000}{3.2} = 1000.$$

The fallacy which is obvious here, attracted less attention above because the division with a factor $(x - a)$ equivalent to zero is done in a form which disguises this zero value. And because the removal of this is allowed in every other case, it is assumed all the more confidently that it may also be allowed in this case, because the resulting value for y is just as one is entitled to expect, that is, if it is a *finite* value, exactly as the law of continuity requires it, zero, if the neighbouring values decrease towards zero, and infinitely large if the neighbouring values increase indefinitely. But it is being forgotten here that the law of continuity may not be followed by all variable quantities. So a quantity which becomes as small as desired while x is brought as near as desired to the value a does not on this account have to become zero for $x = a$, and just as little, if it grows indefinitely as x approaches the value a, may it actually become infinite for $x = a$. Particularly in geometry there are numerous quantities which do not follow any law of continuity, for example the magnitudes of lines and angles which serve for the determination of the circumscribing lines and surfaces of polygons and polyhedra etc.

§ 37

Although we can reproach, not unjustly as I believe, previous presentations of the *theory of the infinite* with many important defects, it is nevertheless well known that *mostly quite correct results* are obtained if the rules which are generally established for calculation with the infinite are followed with suitable care. Such results could never have arisen if there were not a way of understanding and using these methods of calculation which is actually perfectly correct. I am happy to think that it might have been fundamentally this that the clever discoverers of that method had in mind although they were not immediately in a position to explain their ideas on it perfectly clearly, a matter which is generally only achieved in difficult cases after repeated attempts.

Let me outline briefly here how I believe this method of calculating has to be understood so that it may be completely justified. It will be sufficient to speak of the procedure which is followed with the so-called *differential* and *integral calculus*, for the method of calculating with the *infinitely large* arises easily through mere contrast, especially after all that *Cauchy* has achieved on this already.

Therefore I definitely do not need here the narrow assumption, which normally would be regarded as necessary, that the quantities used in calculation can become *infinitely small*, a restriction whereby all limited [*begrenzte*] temporal and spatial quantities, also all forces of finite matter, therefore basically all quantities whose determination we are mostly concerned with are excluded in advance from the scope of this method of calculation. I require nothing other than that these quantities in case they are *variable*, and yet not freely variable but *dependent* on one or more other quantities, have their *derivatives* (*une fonction derivée* according to the definition of Lagrange) if not for all values of their *determining parts* at least for all those for which the calculation is to be validly applied. In other words, if x is a freely variable quantity and $y = fx$ designates a quantity dependent on it, then if our calculation is to give a correct result for all values of x lying between $x = a$ and $x = b$, y must depend on x in such a way that for all values of x lying between a and b the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - fx}{\Delta x}$$

which arises when we divide the increase in y by the increase in x belonging to it, approaches a quantity $f'x$, which is either constant or depends on x alone, as close as desired provided Δx is taken small enough and then it remains as close, or approaches even closer, if Δx is made still smaller.*

If an equation between x and y is given then it is usually a very easy and well-known business to find this derivative of y. For example, if it were that

$$y^3 = ax^2 + a^3 \tag{1}$$

then one would have here for every Δx, which is not zero,

$$(y + \Delta y)^3 = a(x + \Delta x)^2 + a^3 \tag{2}$$

which gives by well-known rules

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{2ax + a\Delta x}{3y^2 + 3y\Delta y + \Delta y^2} \\ &= \frac{2ax}{3y^2} + \frac{3ay^2\Delta x - 6axy\Delta y - 2ax\Delta y^2}{9y^4 + 9y^3\Delta y + 3y^2\Delta y^2}. \end{aligned} \tag{3}$$

And the required *derived function* of y or (in *Lagrange*'s notation) y' would be

$$\frac{2ax}{3y^2},$$

a function which arises from the expression of

$$\frac{\Delta y}{\Delta x}$$

if, after its proper development, namely one in which we separate the terms in the numerator and the denominator which are multiplied by Δx or by Δy from the others, therefore in the expression

$$\frac{2ax + a\Delta x}{3y^2 + 3y\Delta y + \Delta y^2},$$

we put both Δx as well as $\Delta y = 0$.

* It can be shown that all *dependent variable quantities*, provided they are generally *determinable*, must be bound by this law in the sense that exceptions to it, if in an infinite multitude, may always only occur for *isolated values* of its *free variables*.[m]

[m] This claim that determinable functions are differentiable with possibly the exception of infinitely many isolated values may appear to be in conflict with the function Bolzano defined in F § 111. The function is intuitively determinable but it is proved in F § 135 that it is not differentiable on a dense set of values. It depends, as van Rootselaar points out in his detailed note (*PU*(5), p. 142), on exactly what Bolzano meant by 'determinable' and by 'isolated'. By drawing on material from the diaries he shows it to be 'plausible' that Bolzano meant by a determinable function one that is

I do not need to speak of the many uses there are for the finding of this *derivative*, of the ways in which for every finite increase in x the corresponding finite increase in y can be calculated by means of such derivatives, and how, if conversely only the derivative $f'x$ is given, also the original function fx can be determined up to a constant.

But because, as would be noticed just now, we obtain the derived function of a dependent quantity y with respect to its variable x, supposing it was first developed so that neither Δx nor Δy appeared anywhere as divisors, as soon as put in the expression

$$\frac{\Delta y}{\Delta x}$$

the Δx as well as the $\Delta y = 0$, then it might not be inappropriate to represent the derivative by a notation as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

providing we explain here *on the one hand* that all the Δx, Δy, or perhaps the $\mathrm{d}x$, $\mathrm{d}y$ written in their place, which appear in the development of $\frac{\Delta y}{\Delta x}$ are to be treated and viewed as *mere zeros*. But *on the other hand* the notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not to be viewed as a *quotient* of $\mathrm{d}y$ by $\mathrm{d}x$, but only as a *symbol* of the derivative of y by x.

It is clear that in no way could the objection be made to such a procedure that it assumes ratios between quantities which do not even exist (zero to zero), for that notation is known to be regarded as nothing but a *mere sign*.

Furthermore it will be just as perfectly correct if the *second* derived function of y by x, i.e. that quantity dependent merely on x (or perhaps also completely constant) which the quotient

$$\frac{\Delta^2 y}{\Delta x^2}$$

approaches as closely as desired as long as Δx may also be taken as small as desired, is denoted by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

and this is interpreted so that the quantities Δx, $\Delta^2 y$ appearing in the development of $\frac{\Delta^2 y}{\Delta x^2}$ are treated and considered as mere zeros, and the notation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ must not be regarded as a division of zero into zero but only the *symbol* of the function into which the development of $\frac{\Delta^2 y}{\Delta x^2}$ proceeds following the required change just described.

piecewise monotonic. The Bolzano function does not have this property, thus resolving the conflict. However, there remains much of interest, requiring further investigation, about Bolzano's notion of a determinable function. See also on this *F*+ § 38 on p. 586. For the history of this problem see Berg (1962) p. 26.

Once these meanings of the symbols $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... are assumed we can prove strictly that every variable quantity dependent on another free variable x in a determinable way

$$y = fx,$$

is governed, with at most the exception of certain isolated values of x and Δx, by the equation

$$f(x + \Delta x) = fx + \Delta x.\frac{dfx}{dx} + \frac{\Delta x^2}{1.2}.\frac{d^2fx}{dx^2} + \frac{\Delta x^3}{1.2.3}.\frac{d^3fx}{dx^3} + \cdots + \frac{\Delta x^n}{1.2\ldots n}.\frac{d^nf(x + \mu\Delta x)}{dx^n}$$

in which $\mu < 1$.*

No one is unaware of how many important truths of the general theory of quantity [*Größenlehre*] (especially in the so-called higher analysis) can be established through this single equation. But also in applied mathematics, in the theory of space (geometry) and the theory of forces (statics, mechanics etc.) this equation paves the way for the solution of the most difficult problems e.g. the rectification of lines, the complanation of surfaces, the cubature of solids without needing some contradictory assumption of the infinitely small, as well as another alleged axiom such as the well-known Archimedean axiom and several others.

But if it is permitted to put forward equations in the previously defined sense of such a kind as, for example, the formula for the rectification of curves with a rectangular co-ordinate system

$$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]}$$

then it will also be possible, without danger of error, to write down equations of the following kind

$$d(a + bx + cx^2 + dx^3 + \cdots) = b\,dx + 2cx\,dx + 3dx^2dx + \cdots;$$

$$ds^2 = dx^2 + dy^2 + dz^2;$$

or if r denotes the radius of the circle of curvature of a line of simple curvature,

$$r = -\frac{ds^3}{d^2y.dx}$$

and many others, in which we consider the signs dx, dy, dz, ds, d^2y etc. not as signs of actual quantities but rather we consider them as equivalent to zero, and we see in the whole equation nothing other *than a complex sign which is so*

* The proof of this theorem for every kind of dependency of y on x, no matter if known to us and representable by the symbols used so far, or not, has already been written down by the author for a long time, and will perhaps soon be published.

constructed that if we make only genuine changes in it which algebra allows with all signs for actual quantities (therefore here also a division with dx etc.)—*an incorrect result is never produced if we eventually manage to see the signs* dx, dy *etc. disappear on both sides of the equation.*

It is easy to comprehend that this is so and must be so. For if, for example, the equation

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]}$$

is perfectly correct, how could it be that the equation

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$$

is not also perfectly correct since the former can also be derived from the latter directly by the kind of procedure just mentioned?

Finally, it is easy to think that also no error could be produced if, in some equation which contains the signs dx, dy, . . . , then for the abbreviation of all those summands for which we know for certain, in advance, that they will be omitted at the conclusion of the calculation as equivalent to zero, we omit them directly at the outset. For example, if we come across in some calculation the equation (arising from (1) and (2)),

$$3y^2.\Delta y + 3y\Delta y^2 + \Delta y^3 = 2ax\Delta x + a\Delta x^2$$

which, with the transition of the symbols equivalent to zero, takes the form

$$3y^2.\mathrm{d}y + 3y.\mathrm{d}y^2 + \mathrm{d}y^3 = 2ax\,\mathrm{d}x + a.\mathrm{d}x^2.$$

We can immediately see that the summands which contain the higher powers $\mathrm{d}y^2$, $\mathrm{d}y^3$, $\mathrm{d}x^2$ will, at any rate eventually, be omitted and therefore we can immediately put

$$3y^2\mathrm{d}y = 2ax\,\mathrm{d}x$$

from which then the required derivative of y with respect to x arises directly

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2ax}{3y^2}.$$

This whole procedure, to say it finally in one word, rests on quite similar principles to those on which calculation with the so-called *imaginary quantities* (which are mere notations just like our dx, dy, . . .) rests, or also the abbreviated methods of division discovered in recent times and other similar calculation abbreviations. It is sufficient here, just as it is there, to prove the justification of the procedure, that we give to the signs introduced

$$\left(\mathrm{d}x, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \ldots, \sqrt{-1}, \left(\sqrt{-1}\right)^3, \frac{\sqrt{-1}}{-\sqrt{-1}}\text{etc.}\right)$$

only such meanings, and we allow ourselves to make only such changes to them that *it is always the case in the end that if finite signs appear instead of the empty* *signs so that they signify actual quantities, then both sides of the equation are actually equivalent to one another.*

§ 38

If we turn to the applied part of mathematics we encounter the first paradoxes in the field of the *theory of time* in the concept of *time itself*, especially in so far as it is to be a *continuous extension*. But it rests on the *apparent contradictions* famous from ancient times which are believed to be found in the concept of a *continuous extension* of a continuum, in the same way in the temporal, as in the spatial, even in the material. Therefore we shall consider them all together.

It is very well known that everything extended, according to its concept, must be composed of parts; it is further recognized that the being [*Dasein*] of what is extended can be explained without circularity from the composition of parts which are themselves extended. Nonetheless some also claim to find a contradiction in the assumption that extension arises from parts which have no extension but are absolutely simple (points in time, or space, atoms, i.e. simple substances in the universe in the realm of reality).

If it would be asked what is found objectionable in this last explanation then they would sometimes say that a property which is lacking in all the parts cannot belong to the whole, and sometimes that every two points, in time as well as in space, and likewise also every two substances, always have a distance from one another, therefore can never form a *continuum*.

But it really does not need much reflection to see the absurdity in these objections. An attribute which is lacking in all the parts should also not belong to the whole? Precisely the converse! Every whole has, and must have, some properties which the parts lack. An automaton has the attribute of imitating almost perfectly certain movements of a living human being, but the individual parts, the springs, small wheels etc. lack this property. That every two instants in time are separated by an infinite multitude of instants in between, that likewise between every two points in space there is an infinite multitude of points lying in between them, that even in the realm of reality between every two substances there is an infinite multitude of others—is of course conceded, but what follows from this which contains a contradiction? Only this much follows, that with two points alone, or even with three, four or any merely *finite* multitude of them, no extension is produced. We admit all this, indeed we admit that even an infinite multitude of points is not always sufficient to produce a continuum, e.g. a line however short, if these points do not also have the proper *arrangement*. If we attempt to gain a clear awareness [*Bewußtsein*] of the concept which we designate by the expression a *continuous extension* or a *continuum* then we cannot help defining a continuum to exist where, and only where, a collection of simple objects (of points in time or space or even of substances) occurs which are so arranged that every single one of them has at least one neighbour in this collection at every distance however small. If this is not the case, for example, if among a given collection of points in space even only a single one occurs which is not so thickly surrounded by neighbours that for every distance, provided it is taken small enough, a neighbour can be indicated, then we say that this point stands *single* (isolated) and that that collection accordingly does not present a perfect continuum. On the other hand, if there is not a single point that is isolated in this sense in the collection of points under consideration, therefore each of them has at least one neighbour for every distance however small, then there remains nothing which could justify us denying to this collection the name of continuum. For what more would we require?

'This, *that every point has one which it touches directly!*' comes the reply. Here however, something is required which is an obvious impossibility, which ends in a contradiction in itself. For when do you want to say that a pair of points touch one another? Perhaps if the boundary of one of them (say the right-hand side of it) coincides with the boundary of the other one (say the left-hand side of it)? But surely points are *simple* parts of space, they therefore have no boundaries, no right and left sides. If one had only a part in common with the other then it would be absolutely the same as it, and if it is to have something different from it, then both must lie completely outside one another and there must therefore be space for another point lying between them. Indeed because the same holds of these intermediate points in comparison with those two, [there is space] for an infinite multitude of points.

'*But that is all incomprehensible!*' they say. Certainly, it cannot be grasped in one's fingers, or perceived with one's eyes, but it will surely be known by the understanding, and known as something which can be necessarily so and not otherwise, so that a contradiction is then only assumed if it is presented as other [than it is], if it is presented incorrectly.

However, it is continued: 'How incomprehensible it is, to imagine in the smallest line an accumulation of infinitely many points, even an infinite multitude of such accumulations of points, as must be done in the usual theory! For even the smallest line can be divided into an infinite multitude of other lines, since it may first be divided into two halves, then these again may be divided into two halves and so on without end!' I find nothing wrong and nothing strange in this whole chain of ideas up to the single expression of *a smallest line* which many people could only miss from lack of attention because there is no such thing and cannot be such, and of this very thing being considered it is immediately explained that it can be divided into smaller ones. Every infinite multitude, not only the points in a line, can be divided into parts which themselves contain infinite multitudes, indeed into infinitely many such parts. For if ∞ denotes an infinite multitude, then also $\frac{\infty}{2}, \frac{\infty}{4}, \frac{\infty}{8}, \ldots$ are *infinite multitudes*. Thus it is with the concept of the infinite.

In case the previous discussions, after further consideration, were to turn out satisfactorily, the following might eventually be said. 'But how should we interpret the claim of those mathematicians who explain that extension cannot be produced by any accumulation of points however large, and that by division into a multitude of parts, however large, extension can never be resolved into simple points?' Strictly speaking, it should of course be said *on the one hand*, that a finite multitude can never provide an extension and an infinite multitude only does so if the condition, already mentioned several times, that every point has certain neighbours at every sufficiently small distance. *On the other hand*, it should be admitted that not *every division* of a given spatial thing into parts, namely no division into such parts whose number is only finite, not even every one such as goes on indefinitely (e.g. by continued bisection), as we saw before, reaches the simple parts. Nevertheless, one must insist on the fact that ultimately every continuum can arise from nothing else but from points and only points. And provided they are correctly understood, both things are perfectly compatible.

§ 39

It can be anticipated that the properties of that particular continuous extension, *time*, may give rise to particular problems. Especially for those philosophers who, like the sceptics, purposely aim at confusing and finding apparent contradictions everywhere instead of elucidating human concepts, the theory of time must offer welcome material. But we shall only mention here the most important things, not everything which would arise here concerns the concept of infinity.

The question may be raised whether time is something *actual*, and if so, whether it is substance or attribute, and in the first case whether it is created or uncreated? 'If the former,' one might think, 'it must have had a beginning, also at some future time it must have an end, therefore it must change, accordingly there needs to be another time in which it changes. It would be even more absurd to define it as *God himself*, or as an *attribute* belonging to him. Certainly *time* may be compared here *to eternity*—what is it? How is it possible that an infinite multitude not only of moments but *of complete time intervals* may be contained in a single small period however short, e.g. in a single glance of the eyes [*Blick mit dem Auge*], of which every simple instant is called a *moment* [*Augenblick*]? But in fact (it is said, in the end) *time does not exist at all!* For the time that is passed, just because is has passed, is obviously there no more, and the future, because it is future, is not here now. Finally, what is present is nothing other than a *mere moment* in the strictest sense of the word which has no *duration*, therefore no claims to the *name* of time.'

As a consequence of my ideas time is certainly *nothing actual* in the proper sense of the word where we attribute being actual only to *substances* and their *forces*. I therefore regard it neither as God himself, nor as a created substance, nor even as an *attribute* either of God, or of some other created substance, or of a collection of several substances. It is also certainly nothing *variable*, but rather it is that *in which* all variation takes place. If the opposite is said, as in the proverb, '*times change*', then it was already noticed long ago that by time here is understood only the things and circumstances which occur in it. Now to state this more precisely time itself, is that *determination* occurring in every (variable or what is equivalent) dependent substance whose idea we must add to the idea of this substance in order to be able, for every two *contradictory attributes b* and not-*b*, to attribute one of them to it truly and to disallow the other. More precisely the *determination* mentioned here is *a single simple part* of time, an instant or moment, in which we have to imagine the substance *x*, to which we want to attribute with certainty one of two contradictory attributes *b* and not-*b*, in such a way that our decision must therefore actually state that *x* at time instant *t* has either attribute *b* or not-*b*. If it is admitted that this is a correct definition of the concept of a moment then I can also state clearly what time itself, and indeed *the whole time*, or *eternity* is, namely that whole [*Ganze*] to which all moments belong as parts. And every *finite* time, i.e. every *time interval* or *time period* contained within two given moments I define as the collection of all the moments which lie between those two boundary moments. As a consequence of these definitions there is therefore no difference between time and eternity if the former is understood (as often happens) as not restricted to a finite time but rather to the whole of time (endless in both directions). But there is indeed a great difference in the way that God, and the variable or created beings, occur in this time. Namely, the latter are in time *in that they themselves vary in it*, but God is for all time completely constant. This has given rise to him alone being called *eternal*, but the other beings, his creatures, being called *temporal beings*. To represent the fact that every small period, however short, like a glance, already contains an infinite multitude of complete time intervals in a form accessible to the senses may be a difficult problem for *our imagination*. It is sufficient if our *mind* can grasp it and recognize it as something which cannot be otherwise. The objective reason for this can be seen from the concept of time which we have indicated here, but the analysis of it here would be too lengthy. It would be absurd if we had claimed that the same multitude of moments may be put into a short time as into a longer time, or that the infinitely many time intervals into which the former can be divided were of an equal length as for some longer time.

Finally, the fallacy which wishes to destroy completely the reality of the concept of time is so obvious that it scarcely needs one word for its refutation. We admit indeed that time generally is nothing existing and that neither past nor future time exists, even the present has no existence, but how should it follow from this that time is *nothing*? For are not propositions and truths in themselves also something although it may not occur to anyone to claim that they were something existing—providing they were not being confused with their conception in the consciousness of a thinking being, therefore with actual *ideas or judgements*?

§ 40

It is well known with regard to the paradoxes in the *theory of space* that is not even known how to define space. It is frequently held to be something existing, it is sometimes confused with the substances which occur in it, and sometimes it is even identified with God or at least with an attribute of the Deity. Even the great *Newton* thought of defining space as 'the sensorium of the Deity' [*das Sensorium der Gottheit*]. Not only do the substances occurring in space often *move*, but space itself can change, i.e. the locations of their positions. It used to be believed (since *Descartes*) that not all substances, but only the so-called material substances, may occur in space, until finally *Kant* got the unfortunate idea, still repeated by many today, of considering space as well as time not to be something objective, but to be a mere (subjective) *form of our intuition*. Since that time the question has been raised whether other beings may not have another space, e.g. with two or four dimensions. Finally *Herbart* has moreover presented us with a double space, both *rigid and continuous*, and a similarly double [concept of] time. I have already explained myself about all these things in other places.

For me space, in a similar way as for time, is *not an attribute* of substances but only a *determination* of them. Indeed, I call those determinations of created substances which state the reason why, with their attributes, they produce exactly these changes in one another at a certain time, the *places* at which they occur. The collection of all places I call *space*, the whole of space. This definition puts me in the position of deriving the theory of space from that of time *objectively*, and therefore showing, for example, both that, and why, space has three dimensions, and several other things.

Therefore the paradoxes which have been found already in the *concept* of space, in that *objectivity* which befits it despite the fact that it is nothing actual, in the infinite multitude of its parts and in the continuous whole which they form among one another despite the fact that no two of these simple parts (points) touch one another directly—these apparent contradictions I believe I should not discuss further but may consider them as dealt with.

The first thing which still requires further illumination might well be the concept of the *magnitude* [*Größe*] of a spatial extension. There is no dispute that *magnitude* applies to all extension. And it is also agreed that the magnitudes appearing (in the one temporal dimension[n] or the *three* spatial dimensions) can be determined only by their ratio to the one which has been adopted arbitrarily as the *unit of measurement*. Also [it is agreed] that this extension adopted for the unit must be of just the same kind as that which is being measured by it, therefore a *line* for *lines*, a surface for surfaces, a solid for solids.* But if we now ask in what this really consists, what we call the magnitude of a spatial extension, then one would surely like [to say], especially since such an extension consists of nothing

* Perhaps it is not unwelcome to some people to read here the definition of these three kinds of spatial extension. If the definition given in §38 of *extension in general* is admitted as correct (and it has the merit that it can easily be extended to those quantities of the *general theory of quantity*, which are called *continuously variable*) then I say a spatial extension is extended *simply*, or is a line, if every point, for every sufficiently small distance, has one or several, but never so many neighbours that their collection *in itself alone* forms an extension; I say further that a spatial extension is *doubly* extended, or a *surface*, if every point, for every sufficiently small distance, has a whole line of points for its neighbours. Finally I say that a spatial extension is *triply* extended, or a *solid*, if every point for every sufficiently small distance, has a whole surface full of points for its neighbours.

[n] German *Ausdehnung* also means 'extension' which is how the word has usually been rendered.

but points arranged according to a certain rule, that with a magnitude we should never look at the order but only at the multitude of points. We should be very inclined to conclude that it is just this *multitude of points* that we think of by the magnitude of any spatial thing. *The name itself* also seems to confirm this when we call the magnitude of a surface or a solid, the *content* [*Inhalt*] of this spatial thing. Nevertheless a closer consideration shows that this is not so. Or how otherwise could we assume, as we do generally and unhesitatingly, that the magnitude of a spatial thing, e.g. a cube, does not change in the least whether we include in the calculation of the content, the boundary of it, here therefore the surface area of the cube (which itself has a magnitude), or not? And thus we proceed without any question if we find the magnitude of a cube of side 2 to be eight times as large as a cube whose side = 1, in spite of the fact that the former one has a surface area of size which is 12 square units less than the latter since by their composition into a single cube of 24 such squares, half of them—those in the interior of the larger cube—are lost. Hence it follows that by the magnitude of a spatial extension, whether it is a line, surface or solid, we really think of nothing but a quantity which is derived from an extension assumed as unit of the same kind as that to be measured by such a rule that if, proceeding by this rule we derive from the piece M the quantity m and from the piece N the quantity n, and following the same rule we obtain from the piece produced by the combination of the pieces M and N, the quantity $m + n$, equally whether we take the boundaries of M and N and the whole consisting of the two, $M + N$, into account or not. That the most general formulae which the science of space [*Raumwissenschaft*] has for the rectification, complanation and cubature, can in fact be derived from this concept without needing any other assumption, especially not the misnamed 'axioms' of Archimedes, has already been shown in the work mentioned in §37.

§ 41

Relying on the definitions given so far we may now, without fear of being charged with a contradiction, put forward propositions like the following, paradoxical as some of them may appear to the usual mode of thinking.

1. The collection of all points which lie between the two points *a* and *b*, represents an extension of simple kind, or a line. This is so just as much when we include the points *a* and *b*, when it is a *bounded* straight line, as when one or the other or both of the boundary points is not included, when it is therefore *unbounded*. But in each case the length of it is always as before. Every such unbounded straight line has, on the side where its boundary point is missing, for this reason, no *extreme* (furthest) point, but beyond each one there is a further one, although its distance always remains finite.
2. The perimeter of a triangle, *abc* can be composed, (1) from the straight line *ab* bounded on both sides, (2) from *ac* bounded only on one side, at *c*, and (3) from *bc* unbounded on both sides; but its length is equal to the sum of the three lengths of *ab*, *bc*, and *ca*.

3. Let us imagine that the straight line *az* is bisected by the point *b*, the piece *bz* is again bisected by the point *c*, that *cz* is again bisected by the point *d*, and this is continued without end. If we assume that these infinitely many points of bisection *b*, *c*, *d*, . . . and the point *z* are to be thought of as omitted from the collection of points lying between *a* and *z*, then the collection of all the remaining ones still deserves the name of a *line*, and its magnitude is to be the same as before. But if we include *z* in the collection then the whole is no longer a continuous extension, for the point *z* is isolated, because for it there is no distance, however small, of which it could be said that for this distance and for all smaller distances it has neighbours in this collection of points. Namely for all distances of the form $\frac{az}{2^n}$ a neighbour is missing for *z*.
4. If the distance of the points *a* and *b* equals the distance of the points α and β, then the multitude of points between *a* and *b* must be assumed equal to the multitude of points between α and β.
5. Extensions which have an equal multitude of points are also of equal magnitude, but not conversely, *two* extensions, which are of equal magnitude, need not have equally many points.
6. For a pair of spatial things which are perfectly similar to one another, the multitudes of their points must be related exactly as their magnitudes.
7. Therefore if the ratio of magnitudes between two perfectly similar spatial things is irrational, then the ratio between the multitudes of their points is irrational. Therefore there are *numbers* (indeed only infinite ones) *whose ratio, in every way that is chosen, is irrational.*

§ 42

Among these propositions, whose number (as may be seen) could easily be increased, to my knowledge only the sixth has been given any attention in the writings of mathematicians up till now. But only in the sense that the [following] proposition has been put forward contradicting it: *similar lines, however different in their magnitude, have an equal number of points. Dr J. K. Fischer* asserts such a thing (*Grundriß der gesamten höheren Mathematik*, Leipzig, 1809, Bd. II, §51, Anm.), particularly of similar and concentric circular arcs, for the additional reason that through every point of one a radius may be drawn which meets one point of the other. But it is well known that *Aristotle* has already considered this paradox. *Fischer*'s method of argument obviously reveals the opinion that a pair of multitudes, if they are also infinite, must be equal to one another as soon as every part of one can be joined together into a pair with one part of the other. After discovering this error no further refutation of that theory is needed. Moreover it cannot even be understood in this regard why, if it were correct, we would have to limit this assertion of the equal numbers of points only to circular arcs which are concentric and similar, since the same reason could also be given for all *straight* lines and for the most different kinds, no less than for similar curves.

§ 43

There is hardly any truth in the theory of space that teachers of this subject have sinned against more often than this one: *that every distance lying between two points in space, as also every straight line bounded on both sides, is only finite*, i.e. it stands to every other in a relationship which is precisely determinable by mere concepts. For there will scarcely be a geometer who has not at times spoken of *infinitely large* distances and a straight line, which is to have its boundary points on both sides [but which] under certain circumstances would have become *infinitely large*. As an example it is sufficient here to refer to that well-known pair of lines which are called (understood in a geometrical sense of the words) the tangent and secant of an angle or arc. According to the explicit definition these should be a pair of straight lines which are bounded on both sides, and yet how little there is which teaches us to doubt that for a right angle the tangent, as well as the secant, would become *infinitely large*. Nevertheless one is punished immediately for this mistaken theory by the embarrassment into which one lands as soon as one is to state whether these two infinitely large quantities are to be viewed as *positive* or as *negative*? For obviously the same reason holds, which could be quoted for the one, as for the other; because as is well known a straight line drawn through the centre of a circle parallel to a tangent line of it, has a completely equal relationship to both sides of this touching, therefore it intersects it on one side as little as it does on the other side. Also in the quantity expression for these two lines which $= \frac{1}{0}$, since zero can be viewed as neither positive nor negative, lies not the least reason to define this supposedly infinite quantity as positive or as negative. Therefore it is not merely paradoxical, but quite false, to assume the existence of an infinitely large tangent of a right angle, as also of all angles of the form

$$\pm n\pi + \frac{\pi}{2}.$$

That there is, strictly speaking, also for the angle $= 0$ or for the angle $= \pm n.\pi$ neither sine nor tangent, is only mentioned occasionally. The difference in these two assumptions is merely that with the latter no false result arises if one considers the products as not existing at all in cases where these number expressions appear as factors, but where they occur as divisors, one concludes that the calculation requires something unlawful.

§ 44

It was an equally unjustified procedure, which nevertheless has fortunately found few imitators, when *Joh. Schultz* wanted to calculate *the magnitude of a whole infinite space*, from the circumstance that from every given point *a*, on all sides outwards, i.e. in every direction which there is, straight lines can be imagined drawn indefinitely, and from the further circumstance that every conceivable point *m* of the universe must lie on one and only one of these lines. The conclusion drawn that the whole of infinite space may be viewed as a *sphere* which is described from the arbitrarily chosen point *a* with a radius of magnitude $= \infty$, whence it immediately arises that the whole infinite space has precisely the magnitude $\frac{4}{3}\pi\infty^3$.

Without doubt it would be one of the most important theorems of the science of space if this could be justified as true. And hardly anything well-founded can be said against the two premisses (which nevertheless I did not present here precisely according to the exposition of *Schultz* which I do not have to hand). For if someone wished to say that the second premiss must be wrong because from it a very unequal distribution of points in space would follow, namely a much denser accumulation around the arbitrarily chosen centre point *a*, he should recognize that he has not overcome the prejudice we attacked in §21 [and following sections]. *Schultz* is mistaken, and quite obviously mistaken, just in this, that he assumes that the straight lines which have to be drawn out in all directions from the point *a* into the unbounded, if every point of space is to lie on some one of them, are nevertheless *radii*, therefore lines bounded on both sides. For only on this assumption does the spherical form of space follow and the calculation of its magnitude as $= \frac{4}{3}\pi\infty^3$. But from this error the absurdity also follows that—because to every sphere there is a circumscribing cylinder, or even a circumscribing cube, indeed many other spatial things, e.g. there must be infinitely many other spheres of equal diameter enclosing it—the alleged whole space is not the whole, but a mere part, which again has infinitely many other spaces outside it.

The single remark that a line drawn out to infinity only on one side is, for this reason, not a line bounded on this side and that therefore we can speak of a boundary point of it just as little as of the point of a sphere, or the curve of a straight line or a single point, or the point of intersection of two parallel lines—this single remark, I say, is sufficient to show the invalidity of most of the paradoxes which *Boscovich* brought forward in his *Diss. de transformatione locorum geometricorum* (appended to his *Elem. univ. Matheseos*, T. III, Romae, 1754).

§ 45

Infinitely small distances and lines in space have also been assumed as often as the infinitely large, particularly if there is an apparent need to treat lines or surfaces of which no part (which is itself extended) is straight or flat, nevertheless as though they are straight or flat, for example, in order to be able to determine more easily their length, the magnitude of their curvature, or even perhaps some properties of them important for mechanics. Indeed one is allowed to speak in such cases even of distances which are supposed to be measured by infinitely small quantities of the second, third and other higher orders.

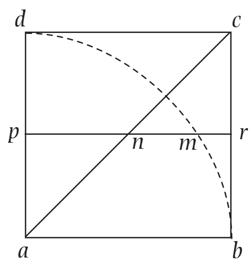
For the fact that one only rarely reaches a false result by this procedure, particularly in geometry, one must be grateful for the circumstance mentioned in §37 that the variable quantities which refer to spatial extensions which are *determinable*, must be of such a nature that they have, with at most the exception of single *isolated values*, a first, second and every successive *derived* function. For where such [things] exist, what is asserted of the so-called infinitely small lines, surfaces and solids, holds in common with all lines, surfaces and solids, which—although they always remain finite—can be taken as small as desired, i.e. can decrease (as they say) indefinitely. Therefore these variable quantities were really these for which, what had been stated falsely of the infinitely small distances, was valid.

But it is obvious that with such a presentation of the subject many paradoxes, and even completely wrong things, have had to be brought out and apparently be proved. How objectionable it sounded for example, when it was asserted of every curved line and surface that it is nothing other than an object composed from infinitely many straight lines and plane surfaces, which would have to be assumed infinitely small, especially if in addition infinitely small lines and surfaces would be acknowledged by them which are nevertheless curved. How strange it would be if it was asserted of lines which have no curvature at all at one of their points, but have, for example, a turning point, that its curvature in this point is infinitely small, and that its radius of curvature is therefore infinitely large; or of lines which, at one of their points taper into a cusp, that its curvature here was infinitely large, and that its radius of curvature was infinitely small, and similar things.

§ 46

As a really striking, and at the same time very simple, example of the subject and cause of the absurdities that the assumption of such infinitely small distances presents, let me quote here a proposition from the report of *Kästner* (*Anfangsgründe der höheren Analysis*, Bd. II, Vorr.) which had already been stated by *Galileo* in his *Discorsi e dimostrazioni matematiche* etc., surely only with a view to arousing some thinking, namely that the *circumference of a circle is as large as its centre*.

In order to gain an idea of the way that one might try and prove this, the reader may think of a square *abcd*, in which from *a* as the centre, with the radius $ab = a$, the quadrant *bd* is described, then



the straight line *pr* parallel to *ab* is drawn which cuts both sides of the square *ad* and *bc* in *p* and *r*, the diagonal *ac* in *n*, and the quadrant in *m*: in short, the well-known figure, by which one usually proves that a circle with the radius *pn* is equal to the annulus which is left behind after removing the circle on *pm* from that on *pr*, or that

$$\pi . pn^2 = \pi . pr^2 - \pi . pm^2.$$

If *pr* continually draws nearer to *ab*, obviously the circle on *pn* becomes continually smaller, and the annulus between the circles on *pm* and *pr* becomes ever thinner. Therefore geometers who did not find infinitely small distances objectionable, extended this relationship also to the case when *pr* drew infinitely close to *ab*, therefore for example, the distance *ap* becomes = d*x*, when the equation

$$\pi . \mathrm{d}x^2 = \pi . a^2 - \pi(a^2 - \mathrm{d}x^2)$$

should hold, which is also in fact justified as a mere identity. But in this case their idea of the circle on *pn* has become an infinitely small thing of the second order; the annulus on the other hand which remains after the removal of the circle on *pm* from that on *pr*, now only has the width

$$mr = \frac{1}{2} . \frac{\mathrm{d}x^2}{a} + \frac{1}{2.4} . \frac{\mathrm{d}x^4}{a^3} + \cdots \text{ [o]}$$

which itself was an infinitely small thing of the second order. Now if it were assumed especially that *pr* goes into *ab* completely, then the infinitely small circle on *pn* contracts into the single point *a*, and the infinitely thin annulus of width *mr* changes into the mere circumference of the circle with diameter *ab*. Therefore one seems to be justified in concluding that the mere centre *a* of every arbitrary circle on *ab* would be as large as its whole circumference.

The deceptiveness of this proof is chiefly produced by the introduction of the infinitely small. Through this the reader is led to a series of thoughts which lets him overlook much more easily how absurd are the assertions that, of the circle on *pn*, if instead of the point *p* the point *a* is finally considered, and no radius like *pn* exists any more, yet *the centre a still remains*, and that the annulus originating from the removal of the circle with the smaller radius *pm* from the circle with the larger radius *pr*, if both radii and therefore also the circles become equal to one another, becomes the circumference of the previously greater one. For indeed with infinitely small quantities one is used to the same quantities sometimes being considered as equal to one another, then one, as an infinitely small quantity of a higher order, being greater or smaller than the other, then also as being considered as being completely equal to zero. If we wish to proceed logically then we may conclude, from the correctly stated equation

$$\pi . pn^2 = \pi . pr^2 - \pi . pm^2$$

which compares the mere quantities (surface areas) of the circles concerned, nothing but that for the case when *pr* and *pm* become equal to one another, the circle on *pn* has no magnitude, therefore it does not exist at all.

[o] The first edition has long dashes to the left of *mr* and in place of each of the two '+' signs.

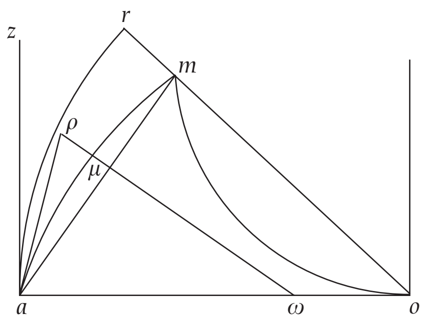
It is indeed true (and I have set out the premisses leading to this truth at §41) that there are circles with and without circumferences, and that this alters nothing in their magnitudes, which depend solely on the magnitude of their radii. And from this someone could perhaps want to derive a new apparent proof for *Galileo*'s proposition since he may start from the demand, surely allowable, that one should imagine the circle on *pm* as without circumference, but the circle on *pr* as together with its circumference. Then of course, after taking away the circle on *pm* from the one on *pr*, if we go from *pr* to *ab* in fact only the circumference of the circle on *ab* is left remaining. But also now no circle around *a* can be spoken of which has contracted into a single point, and still less would it be allowed to call on the above equation in order to deduce from it that the point *a* and that circumference were equally great since the equation stated only deals with the magnitudes of the three circles—they may be considered with or without circumferences.

§ 47

The example just discussed was, as already mentioned, not put forward by its discoverer in order to be wondered at as a truth. But it is taught as a serious truth about the common cycloid, that at the point where it meets its base line it has an infinitely large curvature or (what amounts to the same) an infinitely small radius of curvature, and it stands here in a vertical direction. This is also completely correct if it is understood that the radius of curvature decreases indefinitely while the cycloidal arc approaches the base line indefinitely. And also it should be understood that its direction at the point of starting is itself a vertical direction. What was said of the radius of curvature having become infinitely small or zero, consists (when expressed correctly) merely in this, that (because the curve, as is well known, continues over its baseline on both sides indefinitely, and therefore has *no boundary points*) two pieces of arc meet one another in this point, and indeed in such a way that, because they are both vertical to the baseline, they form a *cusp* with one another, and indeed one such that both have one and the same direction, or (as already said no less correctly), their directions here form an *angle* of zero.

However, one can be convinced by calculation that all this is the case and yet not understand how it comes about or even how it is possible. In order to make clear by what means the paradox is to be resolved, we must understand first of all why the direction in which the common cycloid rises above its baseline is a vertical one.

From the way the common cycloid can be constructed, namely that from every point *o* of the basis one describes an arc of a circle touching the latter with the radius of the generating circle, and cutting off from this a piece *om* equal in length to the distance of the point *o* from the starting point *a*, *m* is considered as a point of the cycloid. It follows immediately that the angle *mao* comes ever closer to a right angle, the closer the point *o* is moved towards *a*, since the angle *moa*, whose size is the half arc *om*, gets ever smaller and the relationship of the two sides *oa* and *om*



in the triangle *moa* approaches ever more to the relationship of equality. Therefore the angle on the third side *am* differs less and less from a right angle. The actual calculation shows this quite clearly. Hence, moreover, it follows that the cycloidal arc *am* lies wholly on the same side of the chord *am*, namely between it and the vertical *at* raised from *a*, therefore that the latter denotes the direction of the curve at the point *a*. If further we describe from *o* as centre an arc of a circle proceeding from *a*, on *oa*, then it is obvious that this cuts the chord *om* first in a point *r* of its extension, because it must be that $or = oa > om$. Now if μ is some point of the curve lying nearer to *a* then there is for it an ω lying still nearer to *a* in *ao* of such a kind that the same holds of the chord $\omega\mu$ which has just been asserted of *om*, namely that an arc of a circle described from ω as centre with the radius ωa meets the extension of $\omega\mu$ somewhere beyond μ in ρ. But because $\omega a < oa$ the arc $a\rho$ lies within the arc *ar*, and therefore between the cycloid arc $a\mu$ and the circular arc *ar*. Therefore we see that to every circular arc *ar* described with the radius *oa*, which touches the cycloid *am* in *a*, there is another $a\rho$ which comes even closer to it in this region. In other words, there is no circle so small that it could be viewed as a measure for the curvature occurring at *a*, in the case that there is a curvature here. Therefore in truth there is no curvature here, but the curve, which does not end at this point, has here, as we already know, a cusp.

§ 48

It has frequently been found paradoxical that some spatial extensions, *which extend through an infinite space* (i.e. have points whose distances from one another exceed any given distance) nevertheless possess only a *finite magnitude*, and again others which are *limited to a quite finite space*, (i.e. whose complete multitude of points are situated so that their distances from one another do not exceed a given distance), yet they possess an *infinite magnitude*. Finally many a spatial extension has a finite magnitude although it *makes infinitely many rotations around one point*.

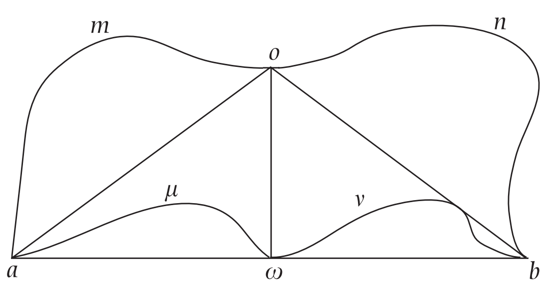
1. Before anything else we must distinguish here whether by the spatial extension of which we speak here is to be understood a whole consisting of several *separated*

parts (e.g. such is the hyperbola with four branches), or only an *absolutely connected whole*, i.e. only such an extension which has no individual part which itself represents another extension in which there was not at least *one point* which, in relation to the other parts, again forms an extension with them.

No one who thinks of the fact that an infinite series of quantities, if they decrease in geometric ratio, have a merely finite sum, will find it objectionable that an extension consisting of separate parts could spread itself out over an infinite space without thereby becoming infinitely large. In this sense therefore also a *line* can extend indefinitely and yet be only finite, like that which arises, if we draw a bounded straight line *ab* from a given point *a* in the given direction *aR*, then a straight line *cd* which remains always at an equal distance, and is only half so long as the previous one, and continue with the same rule indefinitely.

But if we speak only of such spatial extensions which provide a *connected whole*—and that should now always be the case in what follows—then clearly among *extensions of the lowest kind*, i.e. *lines*, none could be found which stretch indefinitely, without at the same time having an *infinite magnitude* (length). For this already necessarily follows from the well-known truth that the shortest absolutely connected line which is to join two given points to one another is just the straight line between them.*

* Because the proof of this truth is so short, I shall include it in this note. If the line *amonb* is not straight, then there must be some point *o* in it which lies outside the straight line *ab*, and if we drop from *o* the

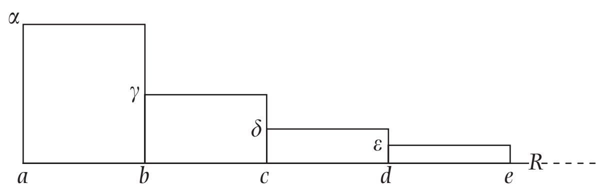


perpendicular *oω* onto *ab*, then the distances are [related by],

$$a\omega < ao, \quad b\omega < bo.$$

But since all systems of two points are similar to one another, there is a line *aμω* between the points *a* and *ω* similar to the piece *amo* of the given *amonb* which lies between the point *a* and *o*, and likewise a line *bνω* between the points *b* and *o*, similar to the piece *bno* of the given *bnoma* lying between the points *b* and *o*. But this *similarity* also requires that the length of the straight line *aω* is related to the length of *aμω* as the length of the straight line *ao* is to the length of the piece *amo*, and the length of the straight line *bω* is to the length *bνω* as the length of the straight line *bo* is to the length of the piece *bno*. Now because $a\omega < ao$, then also it must be that $a\mu\omega < amo$ and because $b\omega < bo$, then it

It is different from the case of lines, with *surfaces*, which, with the same length can be made as small as desired merely by reducing their width, and with *solids*, which, for the same length and width, can become as small as desired merely by reducing their height. Hence it is understandable why surfaces with an infinite *length*, and *solids* with an infinite *length* and an infinite *width*, sometimes nevertheless only possess a finite magnitude. An example, which even the most inexperienced will find understandable, is provided if we require that he imagines drawn indefinitely on the indefinitely extended straight line *aR* the equal pieces $ab = 1 = bc = cd =$ etc.,



then on the first piece *ab* he will imagine the square $b\alpha$, on the second *bc* the rectangle $c\gamma$ that has only half the height of *bc*, and so for each subsequent one, a rectangle half as high as the directly preceding one, when he will certainly very soon recognize that the connected surface which is presented to him here extends indefinitely and yet is not bigger than 2. It will not be more difficult to him to imagine a cube whose side = 1, and to think of a second solid whose base area is a square of side 2, therefore four times as big as the base area of the previous cube, but the height comes to only $\frac{1}{8}$. Then to put after this one a third one whose base area is again four times as big as the directly preceding one but whose height comes to $\frac{1}{8}$ of the solid before, and to imagine that this is continued according to the same law indefinitely. He will understand that the length and width of the solids which are produced in succession increase indefinitely although their volume [*körperlicher Inhalt*] becomes ever smaller, indeed so that every successive one is half of the one directly preceding. Therefore the quantities make a whole in the shape of a pyramid that evidently never exceeds the volume = 2 in spite of its infinite basis.

2. As the case considered before, where an extension had something infinite in it (an infinite length, or even width), and yet was of only finite magnitude, occurred only with the two higher kinds of extension, *surfaces* and *solids*, but not with lines, then the opposite occurs in the case which we now come to discuss, where an extension which appears finite because it is restricted to a completely finite space, nevertheless in fact has an infinite magnitude. This case can only occur with the

must also be that $b\nu\omega < bno$. Consequently also the whole $a\mu\omega\nu b <$ the whole *amonb*. The curved line *amonb* is therefore not the shortest between *a* and *b*, but $a\mu\omega\nu b$ is shorter.

two lower kinds of extension, *lines* and *surfaces*, but in no way can it happen with solids. A solid, in which there are no points whose distances from one another exceed any given quantity can certainly not be infinitely great. This follows directly from the well-known truth that among all solids with points whose distances from one another do not exceed a given distance ε, the greatest is a sphere of diameter ε. For this contains all those points, and its magnitude is only $\frac{\pi}{6}.\varepsilon^3$, every other solid not exceeding this space, must therefore necessarily be *smaller* than $\frac{\pi}{6}.\varepsilon^3$. On the other hand, there are infinitely many *lines* which can be drawn in the space of a single surface however small, e.g. a *square foot*, and to each of them we can attribute a magnitude which is at least finite, e.g. the length of a foot, also by the addition of one, or even infinitely many, connecting lines [we can] combine them all into a single absolutely connected line whose length then must certainly be infinite. And in completely the same way there are infinitely many *surfaces* which can be drawn in the space of a single *solid*, however small, e.g. a *cubic foot*, for each of which we can attribute a magnitude, e.g. a square foot, and by the addition of one or even infinitely many connecting surfaces we can combine all these surfaces into a single one whose magnitude will then undoubtedly be infinite. All this can surprise nobody who does not forget that it is not the same unit with which we measure lines, surfaces and solids, and that although the number of points in every line, however small, is infinite, in a surface this number must nevertheless be assumed infinitely many times greater than in a line, and finally in a solid, with equal certainty, it must be infinitely many times greater than in a surface.

3. The third paradox mentioned at the beginning of this paragraph is that there are also extensions which make an infinite number of revolutions [*Umläufen*] around a certain point, and nevertheless have a finite magnitude. If such an extension is to be *linear* then this can only happen, as we saw in no. 1, if the whole line is situated in a finite space. But on this condition there is absolutely nothing incomprehensible in the phenomenon that it has a finite length although it makes infinitely many revolutions around a given point, providing the further condition is satisfied that these revolutions beginning from a finite magnitude decrease in the appropriate way indefinitely. [This is] a requirement which is possible through the circumstance that it is a *mere point*, around which those revolutions are to occur. For if this is allowed, that the distances of the individual points of such a revolution have from this centre, and therefore also have from one another, can be decreased indefinitely, then the circle itself shows us that the length of this revolution can be reduced indefinitely. The *logarithmic spiral*, if merely that piece of it is to be observed, that, starting from a given point continually approaches the centre without actually reaching it, will have given our readers an example of a line like the one discussed here.

But if the spatial extension which makes infinitely many revolutions around a given point, is to be a *surface* or a *solid*, then the restriction is not even needed that none of the points of the spatial thing are further than a particular distance from its centre. For in order to make myself understandable in the shortest way, let the reader just imagine the spiral mentioned as a kind of abscissae-line from each point of which ordinates emanate at right angles to it and to its plane. The collection of all these ordinates then obviously forms a surface (of a cylindrical kind) which on the one side approaches the centre in infinitely many windings, without ever reaching it, but on the other side it is indefinitely far away. How large this surface is will depend on the rule according to which we allow the ordinates to increase or decrease. But the part going to the centre will always remain finite as long as we cannot increase the ordinates on this side indefinitely (i.e. over the finite branch of the abscissa) because every surface in which neither the abscissa nor the ordinate grows indefinitely, is finite. But also the part of the surface which stands above the other branch of the spiral extending indefinitely will remain finite as long as the ordinates decrease at a faster rate than the abscissae (i.e. the arclengths of the spiral) increase. Therefore if we choose for the abscissae-line the *natural* spiral, where the branch moving towards the centre with radius $= 1$ has length $\sqrt{2}$, and take for the boundary of the surface the arc of a hyperbola of higher kind for which the equation is $yx^2 = a^3$, then that part of this surface which belongs [to values of x] from $x = a$ to all higher values of x only has the magnitude a^2, while the other part, belonging to all smaller values of x, grows indefinitely. But if we take $a > \sqrt{2}$ and transfer the endpoint of the abscissa $x = a$ to the point of the spiral which has the radius 1 then its centre coincides with the endpoint of the abscissa $x = a - \sqrt{2}$, therefore it has a finite ordinate and the part of it which lies over this branch of the spiral is not greater than

$$a^3\left(\frac{1}{a} - \frac{1}{x}\right) = a^2 - \frac{a^3}{a - \sqrt{2}} = -\left(\frac{a^3}{a - \sqrt{2}} - a^2\right).$$

The whole surface covering the spiral on both sides (which we must obtain by adding both magnitudes in their positive value) is therefore

$$= a^2 + \left(\frac{a^3}{a - \sqrt{2}} - a^2\right) = \frac{a^3}{a - \sqrt{2}}.$$

Therefore, for instance, for $a = 2$ the whole surface comes to only $4(2 + \sqrt{2})$.

It is a very similar situation also with solid extensions. Only it is to be observed that here if one wants to allow the part of the solid moving towards the centre to increase its extension in breadth [*Breite*] and thickness it would encroach on the space of its own adjacent revolutions (to the right and the left). If one wanted to avoid this and have a solid all of whose parts are separate then among other ways to achieve the aim is that one adds, to a surface of such a kind as the one just considered which always increased its breadth on approach to the centre, a third dimension, a thickness, which diminishes towards the centre in such a ratio that it always amounts to less than half of the distance between two adjacent turnings of the spiral.

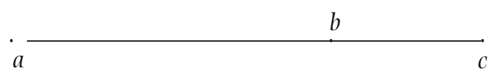
§ 49

Spatial extensions which have an infinite magnitude, precisely with regard to this magnitude itself, are of so many kinds and often have such paradoxical relationships that we must at least give some of them special consideration.

There is the fact that a spatial thing which contains an infinite multitude of points does not, on that account, have to be a continuous extension, as also the fact that with a continuous extension we do not even determine the number of points through its magnitude. Of two extensions which we regard as equally great, one can contain even infinitely many points more or less than the other. Indeed, a surface can contain infinitely many lines, and a solid infinitely many surfaces, more or less, than an extension of the same kind which is considered as equally great. We can consider all these things as sufficiently explained by what has been said already.

1. The first thing, to which we want to direct the readers' attention, is that the multitude of points which a single line *az*, however short, contains, is a multitude which must be considered as *infinitely greater* than that infinite multitude which we select from the former if, starting from one of its boundary points *a*, we extract a second *b* at a suitable distance, after this at a smaller distance a third *c*, and continue without end reducing those distances according to a rule such that the infinite multitude of them has a sum equal to, or smaller than, the distance *az*. For since also the infinitely many pieces *ab*, *bc*, *cd*, . . . , into which *az* is divided are all finite lines, then we can do with each of them what we have just required of *az*, that is we can prove that in each of them again there is such an infinite multitude of points as in *az*, and all these points are at the same time in *az*. Therefore the whole *az* must contain such an infinite multitude of points infinitely many times.

2. Every straight line, indeed every spatial extension in general, which is not only similar to another one but also (geometrically) *equal* (i.e. coincides with it in all *characteristics which are conceptually representable* [*begrifflich darstellbaren*] through comparison with a given distance) must also have an equal multitude of points belonging to it providing we also assume the two have the same kind of *boundary*, i.e. in two straight lines the endpoints may be included or not included. For the opposite could only occur if there were distances which, although equal, permitted an unequal multitude of points between the two points which are at these distances. But that contradicts the concept which we associate with the word *geometrically equal*, for we only call a distance *ac* unequal with another *ab*, indeed



greater than the latter, if, in the case that b and c both lie in the same direction, the point b is between a and c, and therefore all points lying between a and b also lie between a and c, but not conversely that all between a and c are also between a and b.

3. If we designate the multitude of points that lie between a and b, including a and b, by E, and take the straight line ab as the unit of all *lengths* then the multitude of points in the straight line ac, which has the length n (by which we now understand only a whole number) if its endpoints a and c are to be counted in, is $= nE - (n - 1)$.

4. The multitude of points in a *square area* whose side is $= 1$, (the usual unit for areas), will be, if we include the perimeter, $= E^2$.

5. The multitude of points in every *rectangle*, of which one side has the length m and the other side has the length n, will, with the inclusion of the perimeter, be

$$= mnE^2 - [n(m-1) + m(n-1)]E + (m-1)(n-1).$$

6. The multitude of points in a *cube* whose side $= 1$ (the usual unit for a volume), will be, if we count in the points of the surface, $= E^3$.

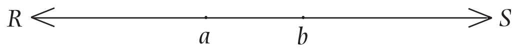
7. The multitude of points in a *parallelepiped* whose sides have the lengths m, n, r, with the inclusion of the surface, will be:

$$\begin{aligned} mnr.E^3 &- [nr(m-1) + mr(n-1) + mn(r-1)]E^2 \\ &+ [m(n-1)(r-1) + n(m-1)(r-1) + r(m-1)(n-1)]E \\ &- (m-1)(n-1)(r-1). \end{aligned}$$

8. We must ascribe to a straight line which extends indefinitely on both sides an infinite length and a multitude of points which is infinitely many times as great as the multitude of points $= E$ in the straight line taken as the unit. We must also grant to *all* such straight lines, equal lengths and *equal* multitudes of points, because the determining pieces, the two points by which any such pair of straight lines can be determined, through which they go, if we assume the distance between these points to be equally great, are not only similar to one another, but also (geometrically) equal.

9. The position of a point chosen arbitrarily in such a straight line is completely similar on both sides of the line, and clearly presents only such characteristics as are conceptually intelligible [*begrifflich erfassbaren*], as does the position of every other point of the kind. Nevertheless, it cannot be said that such a point divides the line into two *equally long* pieces, for if we said that of a point a, then we would have to assert it also of every other point b for the same reason, which is

self-contradictory since, if $aR = aS$ then also it could not be bR $(= ba + aR) = bS$ $(= aS - ab)$.



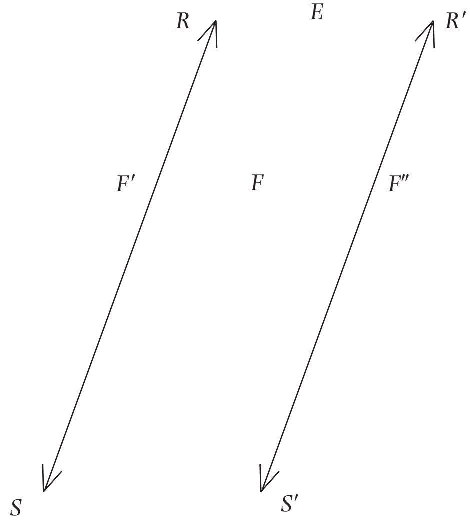
Therefore we must instead claim that a line, unbounded on both sides, *does not have a midpoint*, i.e. has no point which could be determined by a relationship to this line that is merely conceptually intelligible.

10. We must grant to the plane surfaces, which two parallel lines unbounded on each side *enclose* between them, (i.e. to the collection of all those points contained in the perpendiculars from every point of one of these parallels to the other), an *infinitely great* surface area, and a number of points which is infinitely greater than the number in the square assumed as unit area $= E^2$. We must also attribute to all such parallel strips, if they have the same *width* (length of the perpendicular), an equal magnitude and number of points. For they can be determined in such a manner that the determining pieces are not only similar to one another but also geometrically equal, e.g. if we determine them through the statement of an isosceles right-angled triangle of equal side for which we establish that one of these parallels goes through the base line and the other goes through the apex of the triangle.

11. The position of a perpendicular chosen arbitrarily from such parallel strips, is similar on both sides of the surface, and presents no other characteristics that are conceptually intelligible than those the position of every other such perpendicular presents. Nevertheless it cannot be said that such a perpendicular divides the surface into two *geometrically equal* pieces. For this assumption would involve us immediately in a completely similar contradiction as no.9, and this proves its falsity.

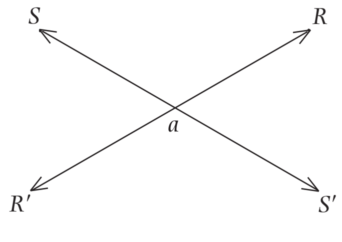
12. To a plane which extends in all directions indefinitely, we must allow an infinitely great surface area and a multitude of points which is infinitely greater than the multitude of points which are in one parallel strip. But as we allow all such parallel strips to be of equal width with one another, so we must allow all such boundless planes the same infinite multitude of points. For it also holds of them that they can be determined in a not merely similar, but also (geometrically) equal manner; as, e.g. if we determine each one through three points lying in it which form a similar and equal triangle.

13. The position of an unbounded line chosen arbitrarily in such an unbounded plane is completely similar on both sides of the plane. Moreover, it presents the same characteristics that are conceptually representable as the position of every other straight line of the kind.

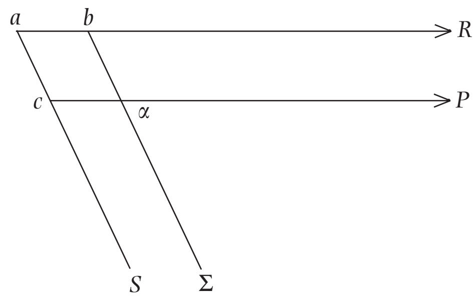


But this is not to say that such a straight line divides the plane into two *geometrically equally large* pieces. For if we were to assert that of one straight line RS, then we would have to admit it also of every other $R'S'$, which leads to an obvious contradiction as soon as we take these lines parallel to one another.

14. Two unbounded straight lines which lie in the same plane but do not run parallel to one another, therefore intersect somewhere and form four angles (equal in pairs), divide the whole surface of the unbounded plane into four pieces of which each two spanned by the *equal* (similar) angles $RaS = R'aS'$, $RaS' = R'aS$ are *similar* to one another.



Each of these four *angle spaces* [*Winkelräume*] contains an infinite multitude of *parallel strips* stretching indefinitely on one side, such as we considered in no. 11, of any arbitrary width. And after we remove, in thought, every finite multitude of them there still remains an angle space, spanned by an *equal angle* as at the beginning. However, after no. 9 and no. 11, we are as little justified in calling the arms of this angle, or even the parallel strips, which we can demonstrate as parts of its surface area, equal to one another, as (indeed for the same reasons as there) we are justified in calling these infinite angle spaces, even with equal (similar) angles, equal to one another, i.e. equally great. Thus it is obvious of the two angle



surfaces RaS and $P\alpha\Sigma$ that the first is greater than the second, although the angles themselves are equal to one another, if $b\,\Sigma$ # aS, cP # aR.[p]

15. We must define the solid space, which two unbounded, parallel planes enclose (i.e. the collection of all those points, which all the perpendiculars falling from every point of one plane to the other, contain), this *unbounded solid layer* (as it could be called) as *infinitely large*, however the *width* of it (the length of one such perpendicular) may be. But with equal widths we may define these magnitudes, indeed also the multitude of points in two solid layers, as equal, according to the same arguments which we have already applied several times (no.s 8, 10, 12).

16. The position which a *parallel strip* perpendicular to its planes chosen arbitrarily in an unbounded *solid layer* [occupies] is completely similar on its two sides in that solid layer, and also the position which another parallel strip of this kind has in the same layer, or even in every other arbitrary unbounded solid layer, is similar. Nevertheless it cannot be said that those two parts into which the solid layer is divided by such a parallel strip would have to be of *equal magnitude*.

17. Two unbounded planes which *intersect* one another divide the whole infinite space into four infinitely large parts of which each opposite two are undoubtedly *similar*, but may not immediately be held to be of equal magnitude.

18. Just as little may the solid spaces, which enclose two *similar* or (as is commonly said) equal *solid corners* between their indefinitely extended side surfaces [*Seitenflächen*], be given as being equally great.

19. Also the two parts into which a *single infinite plane* divides the whole space, are, although similar, not to be considered as *geometrically equal*, i.e. as of equal magnitude, much less as consisting of equal multitudes of points.

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p Evidently Bolzano is using the symbol # to mean 'is parallel to'.

§ 50

Now we still need to have a short discussion of those paradoxes which we meet with in the area of *metaphysics* and *physics*.

In these sciences I put forward the propositions: '*there are no two absolutely equal things, therefore also no two absolutely equal atoms*, or simple substances [*Substanzen*], in the universe; but such simple substances must necessarily be assumed as soon as *composite bodies* are accepted; finally one must also assume that all these simple substances are *variable* and change continually.' I assert all these because it seems to me they are truths which can be proved as strictly and clearly as some theorem of mathematics. Nevertheless I must be apprehensive that most physicists will only be shaking their heads while they listen to these propositions. They pride themselves on only putting forward truths which *experience* teaches them; but experience does not indicate any difference between the smallest particles of a solid, especially of the same kind, e.g. between the smallest particles of gold that we have obtained from this or from that mine. Furthermore, experience surely teaches us that every body is composite, but nobody has perceived atoms which would be absolutely simple and therefore also without any extension; finally, experience shows that the different elements, e.g. oxygen, hydrogen, etc., sometimes enter this combination with each other, sometimes the other, and as well as this, they display sometimes these, sometimes those, effects—but that they themselves would become altered in their inner nature, and that, for example, oxygen would gradually transform itself into another element, that would simply be invented.

1. In my view it is an error that *experience* teaches what is claimed here. Experience, merely, direct experience or perception, without being combined with certain purely conceptual truths, teaches us nothing but that we are having these, and those, intuitions or ideas. Where these ideas come to us from, whether through the effect of some object different from us, indeed whether they even need a cause at all, what properties these may have, about these things direct perception teaches us nothing at all, but we infer them only from certain conceptual truths which we have to supply through reason, and for the most part the inference is made according to a mere rule of probability, e.g. that this red which we are seeing just now, is produced by a diseased condition of our eyes, but that perfume is produced by a flower being nearby. On the other hand, in order to see that between every two things there must be some difference it does not need any inferences of the mere probability drawn from experience, but with a little thought we can recognize it with complete certainty. If A and B are two things then just for this reason the truth holds that the thing A is not the thing B, a truth which assumes that there are two ideas A and B, of which one represents only the thing A and not B, the other, only the thing B and not A. And in this fact, there is already a difference (and indeed an intrinsic one) between the things A and B. If we see in this way that every two things necessarily have certain differences, how can we believe ourselves justified in doubting such a difference merely because

sometimes we do not perceive it? Since a special sharpness of the senses, and many other circumstances, are needed for this perception.

2. It is correct that only experience teaches us that of the things which affect us, there are several, particularly all those which intuitions mediate to us, which are *composite*. Yet experience only teaches this under the assumption of certain purely conceptual truths: such as that different effects can only be produced by different causes etc. But no less certain are the conceptual truths, that every cause has to be something real, and everything real is either a substance or a collection of several substances, or an attribute of one or more substances, likewise, that attributes, that are something real, cannot exist without the existence of a substance in which they occur, and collections of substances [cannot exist] without simple [substances] which form the parts of these collections. But from this the existence of simple substances follows with strict necessity, and it will be ridiculous to wish not to accept the latter because they cannot be seen. And [it would be] all the more absurd when further reflection teaches that every body, which is to be perceivable by our senses, must be composite and composed from an infinite multitude of simple parts.

3. It is a similar fallacy from the non-perceivable to the non-existent, if one wishes not to admit that all finite substances are liable to an unceasing *change*. In our own *soul* we know well enough the changeability of its states, ideas, properties and powers; we are brought to infer a similar thing also about the souls of animals and about plants, by mere analogy. But that all those substances which show no noticeable change over a period of a century, do in fact change, we will assume justified only on the grounds of reason. Whoever disputes this, wants to deny it, at least with reference to the so-called *inanimate material* and in respect of its *simple parts* or atoms, may see himself driven to the claim that all changes which appear to us in this part of the creation, for example, if a piece of ice which a short time ago was solid, has now melted and in the next hour evaporates in vapour, that (I say) all these changes are nothing but mere changes in the spatial relationships of the smaller or larger particles of this solid, nothing thereby changes in the inner nature of those particles themselves. But how could one fail to notice that with this explanation one runs into a contradiction? For if nothing could change in the simple substances themselves (in their inner nature), then how could changes in their spatial relationships among one another be produced, and what consequences should these merely external changes have, what purpose should they serve, and how could it even be known? All these questions can only be answered rationally if we allow to the simple substances—those which are not altogether perfect, therefore can assume more forces [*Kräfte*][q] than they already have—just because of this, the capability of *change* through mutual effect on one another. And let us consider their *positions* as those determinations of

[q] The German *Kräfte* has occurred before but becomes more significant in the more metaphysical context of this, and succeeding sections. It can mean both 'forces' as here and 'powers' (as in the next section) though often neither is a very satisfactory translation.

them which contain the reason why, with exactly this size of forces in a given period, one produces exactly this change on the other, and not a larger or smaller change. Only under this assumption, which is so clear to common sense, does every contradiction in the theory of the universe disappear, and we only needed to rise above some almost obsolete *scholastic opinions*, in order to find everything in harmony.

§ 51

1. The first of these *scholastic opinions* which we must give up is the *dead* or merely *inert material* invented by the older physicists, whose simple parts, if it has such, should all be equal to one another and eternally unchangeable, and have no forces of their own at all, except the so-called *force of inertia*. Whatever is *actual*, must indeed also *act*, and therefore have *powers* to act. But a *limited* substance which for that reason is also variable, has no power which, by its nature, allows a change in its acting, therefore especially no power of creation, but it must have mere *powers of change*, which moreover can either be *inherent* like the power of *sensation*, or *transient*, like the power of motion.

At any rate, in order gradually to learn to estimate with sufficient precision the result of a certain combination of several bodies, it may still be permitted to imagine the case first of all far simpler and, instead of the infinite multitude of forces which are in fact acting together here, to assume only the presence of a few of them, indeed to imagine bodies and their properties which in reality are not even present, in order to determine what these would produce. Only we may not assume, without having first expressly taken the matter into account, that the result which would have to appear in this fictitious case, would coincide to a certain degree with what would occur in reality. The neglect of this precaution has been the cause of many famous paradoxes, as we shall see.

§ 52

2. It is another prejudice of the scholastics that *any assumption of a direct action of one substance on another in science is not permitted*. It is only true that we may never assume, without first having to prove it, that a certain action results directly. It is true that all scientific study would cease if with every phenomenon that we experienced we wanted to explain it by saying that it was produced directly. However, we obviously go too far and fall into a new, likewise very detrimental, error if we explain every action which one substance exerts on another as merely *indirect*, and so want to allow no direct action anywhere. For how could there be an indirect action if there is no direct action? Since this is clear enough we will not delay with this any longer, but it is enough to say how remarkable it is that such a great and cautious thinker as *Leibniz*, just from this motive—because no *means* was known to him whereby substances which are simple should be able to influence one another—came to that unfortunate *hypothesis of the pre-established harmony* which spoils his otherwise beautiful system of cosmology.

§ 53

3. Intimately related to this prejudice, and therefore already disproved, is that even older one, that there is no effect possible (no direct effect that is) of one substance on another situated *at a distance* from it. In the most blatant contradiction of this idea, I claim instead, that every effect of one substance (situated in space, therefore limited) on another is an *actio in distans*, for the perfectly simple reason that every two different substances in each moment also occupy two different simple places, therefore must have a *distance* between them. I have already discussed above the apparent contradiction which lies between this and one of our other claims, that space is to be continuously filled.

§ 54

4. But of course with this we also reject another, more recent, scholastic prejudice which wants to perceive a *penetration* of substances, especially in every *chemical* combination. I absolutely deny any possibility of such a penetration, because, as far as I see, it is already in the *concept of a simple place* (or point), that it is a place which can only accommodate a *single* (simple) substance. Where there are two atoms, there are also two places. From our *definition* of space, which has been repeated already several times, it also follows directly that only the magnitude of the distance between two atoms acting on one another determines the *magnitude* of the change which one produces on the other during a given time interval. If two or more substances could be in one and the same place for however short a time then the magnitude of their mutual influence would be absolutely indeterminable, and if it were even only a single moment, it would not determine its state in that moment.

§ 55

5. Since *Descartes* a new prejudice has arisen in the schools. Since he believed (from a very commendable motive) that he could not sufficiently state the difference between *thinking* and *non-thinking* substances (*mind* and *matter* as he called them) he came to that almost inconceivable claim, so striking to common sense, that a mental entity may be viewed not only not as something which is *extended*, i.e. as consisting of parts, but not even as something occurring in space, therefore not even a being occupying, by its presence, a single *point* in space. Now since at a later time, *Kant* went so far as to define space (no less than time) as a pair of pure forms of our sensibility [*Sinnlichkeit*] to which no object in itself corresponds; and since he directly contrasts *two worlds*, one *intelligible world* of the *mind*, and one *world of the senses* then it is not surprising if the prejudice of the non-spatial nature of mental entities is established so deeply, in Germany at least, that it still exists to the present day in our schools. With regard to the grounds on which I believe I have attacked this prejudice I must refer to other writings, chiefly to the *Wissenschaftslehre* and *Athanasia*. Everyone will have to admit this much, that the

view I put forward, as a consequence of which all created substances for the same reason must be located in space as well as in time, and any difference in their powers is merely a difference of degree, already recommends itself through its simplicity before any others known up until now.

§ 56

6. With this view is also removed the great paradox that has always been found previously in the *connection between the mental and the material substances*. How the material could influence the mind, and how the latter could influence the former, if both were of such different kinds, has been declared a mystery not open to human investigation. But from the above views it follows that this mutual influence must be, at least partly, a *direct* influence, to that extent therefore it could not be something mysterious and secret in itself. Yet with this we certainly do not wish to have said that there is not very much that is worthy of knowing and investigating in that part of this influence which is *mediated* in some way, especially *through organisms*.

§ 57

7. If, in times past, *substances without forces* were imagined, then conversely people in more recent times want to construct the universe from mere *forces without substances*. It was without doubt the fact, that every substance does not make its existence known other than through its effects, therefore through its *forces*, that caused the erroneous definition of the concept of substance that it was a collection of *mere* forces. And the sense-laden image to which the etymology of the words: *substance*, *substratum*, *subject*, *bearer*, and such like, points, appears to offer a clear proof that the generally prevailing theory, that for the existence of a substance it needs a proper *thing*, to which those forces belong as its *properties*, is a mere illusion of the sense faculty, for there is certainly no need here for a *bearer*, a *support* in the proper sense of the word. But must we then stay with this sense-based [*sinnlich*] interpretation? Every arbitrary something, even the mere *concept* of *nothing*, we must consider as an object to which belongs not merely one, but a whole collection of infinitely many properties. Do we therefore think of every arbitrary thing as a bearer in the proper sense? Certainly not! But if we think of a thing with the determination that it is an *actual thing* and such an actual thing that is no property of another *actual thing*, then we conceive it under the concept of a *substance* according to the correct definition of the word. And of such substances, apart from the one uncreated one, there is an infinite multitude of created ones. According to the prevailing usage, we call forces, all those properties of these substances which we must assume as most immediate (i.e. direct) ground of something else inside or outside the substance causing it. A force which is not found to be in any substance as an attribute of it, would, for this very reason, not be called a mere force, but a *substance* existing in itself. Because as a cause it

would have to be something actual, accordingly an actual thing which occurs in *no* other actual thing.

§ 58

That no *degree of existence is the highest*, and none is the *lowest* in God's creation, further that to every degree, however high, and at every time, however early, there have been creatures which by their rapid progress have already risen to this degree, but also that to every degree, however low, and at every time, however late, there will be creatures which, in spite of their continual progress occur now only at this degree—these paradoxes need no further justification after all that we said about similar relationships (§ 38 ff.) with time and space.

§ 59

However, more objectionable is the *paradox*: 'it could be, notwithstanding that the *entire infinite space of the whole universe* is filled at all times with substances in such a way that no single point or even a moment is without a substance inherent in it and that no single point accommodates two or more [substances]—that nevertheless there is an infinite multitude of different *degrees of density* with which different parts of space are filled at different times, in such a way that the same multitude of substances that at this moment, for example, occupies this cubic foot, at another times may be spread out through a space a million times bigger, and again at another time may become forced into a space a thousand times smaller, without any point standing empty with the expansion into a bigger space, nor with the compression into a smaller space any point needing to take two or more atoms.'

I know perfectly well that with this I claim something which seems to most physicists until now as an absurdity. For because they suppose that the fact of the *unequal density of solids* cannot be combined with the assumption of a continuously filled space, they assume a kind of *porosity* as a general property of all bodies, even those with which (as with gases and the aether) not the slightest observation speaks for it. And in these *pores*, of which the larger are generally to be filled with gases, therefore really only in the never-seen pores of the fluidities, physicists still assume up till now their so-called *vacuum dispertitum*, i.e. certain empty spaces in such multitudes and extension, that scarcely the billionth part of a space filled with mere aether contains true matter. Nevertheless I hope that if all that was said in §20 ff. is taken into consideration, it will become clear enough that it contains absolutely nothing impossible that the same (infinite) multitude of atoms may at one time be spread through a greater space and at another time drawn into a smaller space, without it causing in the first case even a single point in that space to be solitary, and without a single point in the second case having to take two atoms.

§ 60

And now one might scarcely take much offence at an assertion (which all the same has already been put forward in the older metaphysics in the theory *de nexu cosmico*) that every substance in the world stands in a state of continuous exchange with every other [substance], but so that the change which one produces in the other becomes all the smaller, the greater the distance lying between them; and that the *total result of the influence of all of them on every individual one* is a change which—apart from the case in which there is a direct act of God—proceeds according to the well-known *law of continuity*, because a deviation from this latter requires a force which in comparison with a continuous one would have to be *infinitely great*.

§ 61

The theory already put forward in the first edition of *Athanasia* (1829) of *dominant substances* can easily be derived from mere concepts. Equally easily people will discover paradoxes in it. Therefore it is necessary to say a few words about them here.

I start in the work mentioned above from the idea that, because it is well known that between every two substances in the universe at every time there must be some difference of finite magnitude, there are also substances at every time which are already so advanced in their powers that they exert a kind of superiority over all substances lying around them in an area, however small it may be. It would be an error, which immediately brought this assumption into the suspicion of an inner contradiction, if someone wanted to imagine that such a dominant substance would have to possess powers which exceeded those of the ones *dominated* by an infinite amount. But this is not at all the case. For if we suppose, in a space of finite magnitude, e.g. in that of a sphere, there occurs a substance (say at the centre of it) which in its forces surpasses each of the others in a finite ratio, as it would be for example, if each of the latter were to be only half as strong as it was. Now although there can be no doubt at all that the total effect of these infinitely many weaker substances there, where they happen *to combine* in their activity (as, for example, we shall soon hear, what tends to happen with the striving for the approach to a central body) surpasses infinitely many times the activity of the one stronger one, there can and must be other cases where those forces are not seeking the same goal, particularly this must be if we wish to keep in mind now merely that effect which every substance occurring in space exerts for itself alone on every other and experiences mutually from it—it can be stated as a rule, that this mutual effect, on the side of the stronger substance, is all the stronger in the same proportion to its strength. Therefore in this example, the substance which we assume as at least twice as strong as each of its neighbouring ones, will act on each of them at least twice as strongly as they act back on it. And this is all that we imagine when we say that it *dominates* the others.

§ 62

However, someone may perhaps say, if things behave in this way, then one must find not merely in some, but in every space, however small, indeed in every arbitrary collection of atoms, a dominant one. For there must be a strongest as well as a weakest atom in every collection of several atoms. Nevertheless I hope none of my readers needs the explanation that this holds at most of finite multitudes, but where an infinite multitude occurs, every member can have an even greater one above it (or an even smaller one below it), without it being that any one of them exceeds a given finite quantity (or sinks below it).

§ 63

These *dominant* substances which, according to their concept, appear in each finite space only in finite numbers, but each one surrounded with a shell [*Hülle*], sometimes greater, sometimes smaller, of merely inferior substances, are what, when combined into clusters of finite magnitude, form what we call the manifold *bodies* appearing in the world (gaseous, as well as liquid, solid, organic etc.). In contrast with them, I call all the other material of the world, which, without having distinguished [*ausgezeichnete*] atoms, fills all the spaces existing elsewhere, and therefore combines all bodies of the world, the *aether*. Here is not the place to discuss how many phenomena so far explained only imperfectly, or even not at all, from the previous assumptions (even if they were admitted as assumptions), may be explained with great facility. I must allow myself, in accord with the purpose of this writing, just a few indications by which apparent contradictions will be elucidated.

If all created substances are distinguished from one another only through the *degree* of their powers, therefore each of them must be allowed some, however little, degree of *sensitivity*, and all [of them] act on all [others], then nothing is more understandable than that for every two, however formed, and all the more certainly for every two distinguished substances, *not every distance* appears *equally acceptable to them* (equally good for them), because the magnitudes of the effects which they exert as well as those they experience, depend on the magnitude of the distance. If the distance at which they are situated is greater than that which is acceptable to one of them, then there will be a striving in it to shorten this distance, therefore a so-called *attraction*, but in the opposite case a *repulsion*. Neither the former nor the latter must we think of as always mutual, much less always as a result accompanying an actual change of position. But we may indeed assume with certainty that for every two substances in the universe there is a distance great enough such that for this one and for all greater, there is mutual attraction, and equally also a distance small enough that for it and for all smaller there is a mutual repulsion. But how much the magnitudes of the two distances spoken of here which are the boundaries of the attraction and the repulsion for the two given substances, are, may vary with the time, not only according to the nature of these substances themselves, but also according to the nature of the neighbouring substances lying in the whole region. It is indisputable that all the influence which two substances exert on one another in otherwise similar circumstances must decrease with the increase of their distance from one another, if only for the reason that the number of those which could take effect at the same distance and had a claim to the same effect, increases as the square of that distance. Further, as the superiority that every distinguished substance has over a merely inferior one, rises always only as a finite quantity, against which the number of the latter exceeds that of the former infinitely many times, so it is understandable that the magnitude of the attraction which all the substances in a given space exerts on one atom lying outside it, if the distance of it is sufficiently large, is close to the one which there would be if that space contained no distinguished substances but only an equally large number of common atoms. This combined with the earlier [assertions], leads to the important conclusion: *that a force of attraction exists between all bodies, providing their distance from one another is sufficiently great, which varies directly as the sum of their masses* (i.e. the multitude of their atoms), *and inversely as the square of their distance*. No physicist or astronomer in our day denies that this law may be observed in the universe. But it seems to have been rarely considered how difficult it is to square this with the usual view of the nature of the elementary parts of the different bodies. Namely if things actually behave as they have usually been represented up to now, that those 55 or more *simple elements* which our chemists have got to know from the earth, the masses of all the bodies occurring here would be formed in the way that each of them would be a mere collection of atoms of one, or another, or some of these elements put together. Thus, for example, gold would be a mere collection of nothing but gold atoms, sulphur would be a collection of nothing but sulphur atoms, etc. Then explain to me, whoever can, how it comes about that elements which are so different in their powers, especially in the degree of their mutual attraction, are nevertheless absolutely the same as one another in their *weight*, i.e. that their weights are proportional to their masses. For that this occurs proves directly the well-known experience that spheres of any arbitrary element, if they are of equal weights, on collision with one another behave exactly as bodies of equal mass would have to do, e.g. with equal speed (as long as the effect of the elasticity is mutual or taken into account) they bring one another to rest. But if we assume that all bodies actually consist of nothing other than an infinite multitude of aether [atoms] in which there occurs a quite negligible number of distinguished atoms whose forces surpass those of an aether atom only finitely many times, then it is understandable that the force of attraction which these bodies experience from the side of the whole globe can in no case be noticeably increased by the small number of those distinguished atoms, their weight must therefore only be proportional to their mass. Nevertheless there is now no lack of physicists who consider *caloric* [*Wärmestoffe*] (therefore fundamentally the same element which I identify with the aether) as a fluid, which occurs in all bodies and can never be completely expelled from them. Therefore if they had not unfortunately conceived the idea that this caloric was *imponderable* and if they had taken the view that the multitude of atoms which are present in every particular body compared with the caloric is insignificant (and how near to this they were when they sometimes postulated that the former separated by distances which, compared with their diameters, are infinitely large), then it would soon have become perfectly clear to them that it may be just its caloric which determines the weight of all bodies.

§ 64

It is easy to think that that domination which a distinguished substance exerts over its immediate neighbourhood, consists, if in nothing else, at least in a certain stronger attraction of its neighbouring atoms, as a consequence of which these are forced together and *denser* around it than they would otherwise be; and for this very reason they have a tendency to move themselves again, on a given opportunity, somewhat further from this point of attraction as well as from one another, therefore *to repel* one another. This is a matter to which much experience points, but for the explanation of it an *original* force of repulsion between the particles of the *aether* has quite unnecessarily been assumed.

§ 65

From this fact an easy proof follows of the proposition which I have put forward already in the *Athanasia*, *that no distinguished substance experiences in its shell such a change* that it may not keep a certain part (however small it may be) of its nearest surroundings. Certainly no one will be concerned that a distinguished substance *a* should be deprived of the aether atoms immediately surrounding it, if, among all the neighbours lying immediately round about it, of distinguished rank *b*, *c*, *d*, *e*, . . . , none changes its distance from *a*. But what may cause some concern is something like [what happens] if some of them, or even all of them, disappear. Even if this does happen only a part of the aether particles surrounding *a* can follow the escaping substances *b*, *c*, *d*, *e*, . . . , but a part, and indeed of those which are nearest to *a*, must always remain, although we do not only admit, but even assert it as necessary, that it will expand into a larger space. In fact according to the detailed circumstances, aether atoms from certain distant regions could flow into it, and press into those spaces which are filled with a comparatively more diffuse aether on account of the far too great distances into which the substances *a*, *b*, *c*, *d*, *e*, . . . have just dispersed. But there is no reason to suppose that this aether coming from a distance should push away and displace the [atoms] surrounding substance *a* at the moment. Instead of completely driving away the aether surrounding the substance *a* the aether flowing in must rather hinder its spreading further and force it together compactly until its density balances the attractive forces of all the surrounding atoms.

§ 66

Next some questions can be answered in a manner which could have been found paradoxical if it were not for the explanation provided by the foregoing material. Of such a kind is the question about the *boundaries of a body*: where exactly does one body end and another begin? I understand by the *boundary* of a body the collection of those *outermost* aether atoms that still belong to it, i.e. those that are attracted by the distinguished atoms of the body more strongly than they are by other dominant atoms in the neighbourhood. This happens in such a way that in so far as the body changes its position in relation to its neighbourhood (i.e. moves away from it), it will *pull them along* with it, perhaps not with the same speed, but so that no separation takes place and no extraneous atoms come in between. Assuming this concept of a boundary it is obvious that the boundary of a body is something very variable, indeed that it is changing almost continually. Any change whatsoever proceeds partly in the boundary itself, and partly in the neighbouring bodies because understandably all such changes can produce many a change in the magnitude, as well as in the direction, of the attraction which acts on the atoms of a body, not only the subsidiary [*dienenden*] ones but even its dominant ones. Thus for example, several particles of this quill which shortly before were being attracted by those of the rest of the quill more strongly than by the surrounding air, therefore belonged to it, but now are attracted by my fingers more strongly than by the material of the quill, and are themselves therefore torn away. More exact consideration shows that some bodies at certain places do not even have any boundary atoms, i.e. no atoms which are the *outermost* among those which still belong to it and would still be drawn along with it if its position changes. For in fact, whenever one of two neighbouring bodies has an outermost atom at some place, pulled along with it, for this very reason the other has no such outermost [atom] because all aether atoms occurring behind the former already belong to the latter.

§ 67

Here also the question may be answered of whether and when bodies are standing in direct *contact* with one another, or are separated by a space in between? If I may be allowed (as seems most expedient to me) the definition that a pair of bodies touch one another where the outermost atoms, which according to the definition of the previous paragraph, belong to one of them, form a continuous extension with certain atoms of the other one, then it will certainly not be denied that there are many bodies which touch each other mutually. This is not only so if one, or indeed both, are fluid, but also if they are solid, in so far as the air attaching to them, in the usual conditions on earth, is removed between them through strong pressure or in some other way. If a pair of bodies are not touching one another, then, because there is no completely empty space, the space in between must be filled with some other body or at least by mere aether. Therefore it can be asserted that really every body stands on all sides in contact with some other bodies, or for lack of them, with mere aether.

§ 68

In respect of the different kinds of motion occurring in the universe, it could be believed that in the circumstance (in our view) that no part of space is empty there is never any other motion possible than one in which the whole mass is moved simultaneously and forms a single extension where every part of the mass always only occupies a place which directly before another part of the mass had occupied. But whoever has kept in mind what was said in §59 about the different degrees of density with which space can be filled, will understand that many other motions can and must take place. Particularly one motion, the *vibrating motion*, must occur not only with all aether atoms, but also with nearly all distinguished atoms, almost unceasingly, for a reason which is so clear that I do not need to give it. After this, *rotating* motion must also be very common to these [distinguished atoms] especially with *solid* bodies. How this is to be imagined, how it may be explained, that, if the axis of rotation is a material line (which as a consequence of our views it must always be), the same atoms which now occur on this side of it, after a half turn get to the opposite side, without tearing off? [These problems] can only mislead those who forget that in a continuum, just as much as outside of it, every atom is at a certain distance from every other, and therefore can circle the latter without tearing off or having to turn around with it. The latter, the rotation around itself, with a simple spatial object, would be something self-contradictory.

§ 69

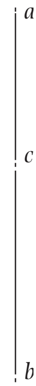
Without wanting to assert that even a single dominant or common atom in the universe at some time may describe a perfectly straight or perfectly circular path (which indeed, with the infinitely many perturbations which every atom has due to the effect of all the others, would have an infinitely great improbability), we may nevertheless not declare such motions as something which would be impossible in itself. But we may surely assert that the describing of, for example, a *broken line* could then only come about if the speed of the atom gradually reduced towards the end of the piece *ab*, so that at the point *b* it became zero, whereupon if the motion is not interrupted by a finite period of rest, in each of the moments following the arrival in *b* there must again be a speed (increasing from zero).

It is not so with certain other lines, as for example, with the logarithmic spiral. Even apart from all the perturbations from outside, it is contradictory that only that branch of this line which, starting from any point of it towards the centre, may be described by the motion of an atom in a finite time. And it is still more absurd to require that the moving atom finally arrive at the centre of the spiral. In order to prove this for the case where the atom proceeds in its path with uniform speed, imagine first of all that it moves alone. Then it is soon apparent that its progress in the spiral can be considered as if it were composed of two motions: one uniformly in the line directed towards the centre, and the other an angular turning around this centre whose speed, increasing uniformly, must become greater than every finite quantity if the atom is to get as near to the centre as desired. Therefore it is certain that there is no force in nature which could give it this speed, much less is there a force which could impart such a speed to a complete mass of atoms spread through three dimensions, as is required if that atom in it is to traverse in a finite time all the infinitely many turns of the spiral up to the centre. But even if it had this, could it really be said of it that it may reach the centre? I at least do not think so. For although it may be said that this centre forms a continuum with the points of the spiral (which belong to it quite undeniably) because a neighbour may be found among them for however small a distance, this linear extension lacks a second property which every one must have if it is to be able to be described through the motion of an atom: namely that at each of its points, it has one or more definite directions. This is well known not to be the case at the centre point.

Finally, this is also the place for the teasing question of whether, with our views of the infinity of the universe, there could also be a progression of the universe in some given direction, or even a rotation of it around a given world axis, or world centre? We reply, that neither the one, nor the other motion has been declared impossible because there cannot be found places for every atom to move to, but it surely must be declared impossible because of the lack of causes (forces) which should produce such a motion. For neither a *physical reason* or state of affairs [*Einrichtung*], which is absolutely necessary (i.e. which is a mere consequence of purely theoretical conceptual truths), nor a *moral reason* or state of affairs which is only *conditionally* necessary (i.e. which we only meet with in the world because God causes every beneficial event for the well-being of his creation)—can be conceived on the basis of which such a motion could occur in the world.

§ 70

Let us conclude these considerations with two paradoxes made especially famous by *Euler*. *Boscovich* has already drawn attention to the fact that to one and the same question, namely how an atom *a* moves if it is attracted by a force at *c* which is inversely proportional to the square of the distance, one obtains a different answer according to the case considered as one in which the elliptic motion changes gradually if its speed of projection decreases to zero, or if the matter is judged merely in itself quite apart from this fiction. If the atom *a* had obtained, through projection (or in some other way), a lateral speed at the beginning of its motion perpendicular to *ac*, then (apart from any resistance in the medium) it must describe an ellipse of which one focus is at *c*. If this lateral speed decreases indefinitely then also the smaller axis of this ellipse decreases indefinitely, on account of which *Euler* argued that in the case where the atom has no speed at all at the point *a*, an oscillation of it between the points *a* and *c* must occur, this is the only motion into which that elliptical motion may pass without breaking the law of continuity. On the other hand others, chiefly *Busse*, found it absurd,



that the atom, whose speed in the direction *ac* should increase indefinitely when approaching *c*, should here, be locked in its course and pushed back in the opposite direction without any reason being given (for the presence of an atom preventing the passage through this place, like an atom fixed and impenetrable, would certainly not be assumed). They therefore claimed that it must instead continue its motion in the direction *ac* beyond *c* but now with diminishing speed until it reaches the end of $cb = ca$, and then it returns in a similar way from *b* to *a* again, and so on without end. In my view nothing could be decided here through *Euler*'s appeal to the law of continuity. For the phenomenon which is disputed here violates the kind of continuity, which provably in fact dominates in the changes in the universe (in the growth or decline of the forces of individual substances), just as little if the oscillation of the atom occurred within the limits *a* and *b*, as if it occurred within *a* and *c*. But this rule is violated, in a way which is absolutely not justified, when forces are assumed here, namely a force of attraction, which grows indefinitely, and therefore it should not be a surprise if from contradictory premisses, contradictory conclusions can be derived. Hence one may nevertheless see that not only *Euler*'s but also *Busse*'s answer to the question is incorrect, because it assumes something which is in itself impossible, namely the infinitely great speed at the point *c*. If this mistake is corrected, so it is assumed that the speed with which the atom progresses changes according to such a rule that it always remains finite, and finally if it is considered that one can never speak of the motion of a single atom without a medium in which it moves, and assuming a greater or smaller multitude of atoms moving with it at the same time, then a quite different result comes to light, with whose more exact description we do not need to concern ourselves here.

The *second* paradox that we want to say a few words about, concerns *pendulum motion* and is as follows. The half period of a simple pendulum whose length $= r$, is well known to be $= \frac{\pi}{2}\sqrt{\frac{r}{g}}$, calculated through an infinitely small arc, while the time of descent over the chord of this arc, which is usually considered as of equal length with it, gives it as $= \sqrt{2}.\sqrt{\frac{r}{g}}$. That *Euler* saw a paradox in this surely rests solely on his incorrect idea of the *infinitely small* which he imagined as equivalent to zero. But in fact there are no infinitely small arcs any more than there are infinitely small chords, but what mathematicians assert of their so-called infinitely small arcs and chords are really only proved by them of arcs and chords which can become as small as desired. The above two equations understood correctly can have no other meaning than: the half period of a pendulum approaches the quantity $\frac{\pi}{2}\sqrt{\frac{r}{g}}$ as much as desired if the arc through which it can swing is taken as small as desired. But the time of descent on the chord of this arc, under the same circumstances, approaches as precisely as desired, the quantity $\sqrt{2}.\sqrt{\frac{r}{g}}$. Now that these two quantities are different, that therefore the arc and its chord differ in respect of the time of descent mentioned, however small they are taken, is something no more strange than many other differences between them whose vanishing no one expects as long as both just exist. For example, the arc always has a curvature, and indeed one whose magnitude we could measure with $\frac{1}{r}$, while the chord always remains straight, i.e. has no curvature.

Selected Works of Bernard Bolzano

This section consists of a small selected and annotated list of Bolzano's works. It contains the works translated in this volume (I), together with the other works by Bolzano (II) to which reference has been made in this volume. There is an extensive Bolzano bibliography in *BGA* E2/1 (1972) together with Supplement I (1981) and Supplement II (1987). A comprehensive bibliography up to 1999 is forthcoming in *BGA* E2/3 edited by Jan Berg and Edgar Morscher. The more compact bibliographies contained in Berg (1962) and Sebestik (1990) are still very useful. There is some deliberate overlap of this section and the Bibliography: where a version of a work by Bolzano has been subject to major editing, selection or translation, it may also appear in the Bibliography under the name of the editor.

I Works translated in this volume

BG (1) *Betrachtungen über einige Gegenstände der Elementargeometrie* Prague: Karl Barth 1804

(2) 2nd edition of (1) in *Geometrische Arbeiten*, edited and with notes by Jan Vojtěch. Royal Bohemian Society of Sciences, Prague, 1948. pp. 9–49. [This work is Vol.5 of the series *SBB*.]

(3) Facsimile edition of (1) contained in *EMW* pp.1–80.

(4) *Preface* of (1) appears in an earlier translation by Russ in Ewald (1996) pp. 172–174.

BD (1) *Beyträge zu einer begründeteren Darstellung der Mathematik Erste Lieferung* Prague: Caspar Widtmann 1810

(2) *Philosophie der Mathematik oder Beiträge zu einer begründeteren Darstellung der Mathematik. Neu herausgegeben mit Einleitung und Anmerkungen von Dr Heinrich Fels.* Paderborn: Ferdinand Schöningh 1926 [2nd edition of (1) with introduction and notes by Heinrich Fels.]

(3) Facsimile edition of (1) with an introduction to the reprint by Hans Wußing. Darmstadt: Wissenschaftliche Buchgesellschaft 1974

(4) Facsimile edition of (1) contained in *EMW* pp. 83–250.

(5) French translation of the *Appendix* of (1) in Laz (1993).

(6) *Contributions to a better-grounded presentation of mathematics.* An earlier translation by Russ in Ewald (1996) pp. 174–224.

BL (1) *Der binomische Lehrsatz, und aus Folgerung aus ihm der polynomische, und die Reihen, die zur Berechnung des Logarithmen und Exponentialgrößen dienen, genauer als bisher erwiesen*. Prague: C.W. Enders 1816

(2) Facsimile edition of (1) contained in *EMW* pp. 213–415.

RB (1) *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegensetztes Resultat gewähren, wenigstens eine reele Wurzel der Gleichung liege* Prague: Gottliebe Haase 1817

(2) The same printing as (1) in Proceedings of the Royal Bohemian Society of Sciences [Abhandlungen der königlichen böhmischen Gesellschaft der Wissenschaften] Vol.5. Prague 1818

(3) 2nd edition of (1) edited by Philip E.B. Jourdain. Leipzig: Wilhelm Engelmann. 1905. In *Ostwald's Klassiker der exakten Wissenschaften* 153.

(4) *Démonstration purement analytique du théorème: entre deux valeurs quelconques qui donnent deux résultats de signes opposés se trouve au moins une racine réelle de l'équation*. Revue d'histoire des sciences et de leurs applications 17 1964 pp. 136–164 Tr. Jan Sebestik.

(5) The first English translation of (1) is in Russ (1980*b*).

(6) Facsimile edition of (1) in *EMW* pp. 417–476.

(7) *Purely analytic proof of the theorem that between any two values which give results of opposite sign there lies at least one real root of the equation*. An improved translation by Russ in Ewald (1996) pp. 225–248.

DP (1) *Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahmen des Archimedes, und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst: zugleich als Probe einer gänzlichen Umstaltung der Raumwissenschaft, allen Mathematikern zur Prüfung vorgelegt*. Leipzig: Paul Gotthelf Kummer 1817

(2) 2nd edition of (1) in *Geometrische Arbeiten*, edited and with notes by Jan Vojtěch. Royal Bohemian Society of Sciences, Prague, 1948. pp. 67–138. [This work is Vol.5 of the series *SBB*.]

(3) Facsimile edition of (1) in *EMW* pp. 479–583.

RZ (1) *Theorie der reellen Zahlen im Bolzanos handschriftlichen Nachlasse* Ed. Karel Rychlík Prague: Verlag der Tschechoslowakischen Akademie der Wissenschaften 1962.

[A partial edition of the seventh section of *Reine Zahlenlehre* headed *Unendliche Größenbegriffe*. The manuscript sources for this are Series Nova 3467–3469 held in the Austrian National Library [Österreichischen Nationalbibliothek].]

(2) *Reine Zahlenlehre*

Siebenter Abschnitt. Unendliche Größenbegriffe.

BGA 2A8 Ed. Jan Berg. Stuttgart: Frommann-Holzboog 1976 pp. 100–168

[This is a very detailed transcription made from the manuscripts cited in (1) with useful editorial footnotes.]

F (1) *Functionenlehre. Herausgegeben und mit Anmerkungen versehen von Dr Karel Rychlík*. Prague: Königliche böhmische Gesellschaft der Wissenschaften. 1930.

[This is Vol.1 of *SBB*. It is a partial edition of the manuscript sources Series Nova 3471a and 3471b in the Austrian National Library.]

(2) *Functionenlehre*

BGA 2A10/1 Ed. Bob van Rootselaar Stuttgart: Frommann-Holzboog 2000 pp. 23–165.

[This is a very detailed transcription of the manuscripts cited in (1) with useful editorial footnotes.]

F+ (1) *Verbesserungen und Zusätze zur Functionenlehre*

BGA 2A10/1 Ed. Bob van Rootselaar Stuttgart: Frommann-Holzboog 2000 pp. 167–190.

[A very detailed transcription of manuscripts Series Nova 3472 7,8 in the Austrian National Library. It represents a new and improved edition of the corrections to *Functionenlehre* first reported in Rootselaar (1969).]

PU (1) *Dr Bolzano's Paradoxien des Unendlichen herausgegeben aus dem schriftlichen Nachlasse des verfassers von Dr. Fr. Příhonský* Leipzig: Reclam 1851

(2) 2nd unaltered edition Berlin: Mayer & Müller, 1889

(3) Reprint of (1) edited by Alois Höfler and with notes by Hans Hahn. Leipzig: Felix Meiner. 1921

(4) *Paradoxes of the Infinite by Dr. Bernard Bolzano*. Translated from the German of the posthumous edition by Dr.Příhonský and furnished with a historical introduction by Donald A Steele. London: Routledge & Kegan Paul. 1950

(5) A further edition of (3) with introductions and notes by Bob van Rootselaar. Hamburg: Felix Meiner. 1975

(6) *Les paradoxes de l'infini. Introduction, traduction de l'allemand et notes* Hourya Sinaceur. Paris. 1993

[The introduction summarises some of the material from *RZ* on infinite quantity concepts. The footnotes are extensive and useful.]

II Other works by Bolzano

BGA *Bernard Bolzano Gesamtausgabe* Ed. Eduard Winter, Jan Berg, Friederich Kambartel, Jaromír Loužil, Edgar Morscher and Bob van Rootselaar. Stuttgart: Frommann-Holzboog. 1969–

[Complete Edition. A magnificent edition and the definitive resource for Bolzano scholars, there are projected to be about 120 volumes over 60 of which are already available at the time of writing. For further details of all volumes and latest availability see www.frommann-holzboog.de.]

SBB *Spisy Bernada Bolzana/Oevres de Bernard Bolzano/Bernard Bolzano's Schriften*. Prague: Royal Bohemian Society of Sciences. 1930–1948 5 vols.

WL (1) *Dr B Bolzanos Wissenschaftslehre. Versuch einer ausführlichen und größentheils neuen Darstellung der Logik mit steter Rücksicht auf deren bisherige Bearbeiter*. Sulzbach: Seidel. 1837 4 vols.

[The title page declares it was edited by several of Bolzano's friends and has a Preface by Dr J.Ch.A. Heinroth.]

(2) *Dr B. Bolzano's gesammelte Schriften. Neue Ausgabe in zwölf Bänden*. Vienna: Braumüller. 1882

[Unaltered version of (1) appears here as Vols.7—10.]

(3) *Werke Bernard Bolzanos*. Ed. Alois Höfler. Leipzig: Felix Meiner. 1914, 1915 2 vols.

[Consists of facsimile edition of vols.1 and 2 of (1).]

(4) *Bernard Bolzanos Wissenschaftslehre in vier Bänden*. Ed. Wolfgang Schultz. Leipzig: Felix Meiner. 1929–31 4 vols.

[Selected paragraphs from vols.1 and 2 of (1).]

(6) *Theory of Science. Attempt at a detailed and in the main novel exposition of logic with constant attention to earlier authors*. Ed. and tr. with *Introduction* by Rolf George. Oxford: Oxford University Press. 1972

[This contains slections from all main parts of (1) and offers good coverage by means of summaries as well as complete translations.]

(7) *Theory of Science*. Ed.with *Introduction* by Jan Berg. Tr. Burnham Terrell. Dordrecht: Reidel. 1973

[This volume contains a substantial technical introduction from a modern point of view as well as some extracts from Bolzano's correspondence.]

(8) *BGA* edition of (1) in 12 volumes from *BGA* 1,11/1 to *BGA* 1,14/3 1985–2000

EG (1) *Einleitung zur „Größenlehre"* in *BGA* 2A7 pp. 23–216.

[The inverted commas indicate a title of a project rather than a single volume. See p.345. This work represents an important bridging work between *WL* and the later mathematical works.]

(2) *On the Mathematical Method and the Correspondence with Exner*. Paul Rusnock and Rolf George Amsterdam: Rodopi 2004.

[The first part of this work is a translation of a substantial part of (1). It is a highly condensed version of the logic in *WL* but oriented towards mathematics. Exner is a philosopher whose misunderstandings provoke Bolzano to clarify his fundamental logical concepts carefully and instructively.]

EMW *Early Mathematical Works* Acta historiae rerum naturalium necnon technicarum Special Issue 12 Prague 1981.

[A facsimile edition of the originals of the five early mathematical works including a useful Introduction by Luboš Nový and Jaroslav Folta, translated into English by Jaroslav Tauer.]

DDR *Versuch einer objectiven Begründung der Lehre von den drei Dimensionen des Raumes* Prague: Kronberger & Řiwnač 1843.

[*Translation:* Attempt at an objective grounding of the theory of the three dimensions of space. This work, and the following one, were composed much earlier than they were published, probably in the 1810's. They are both mentioned in *PU* §27, both have versions in the proceedings of the Royal Bohemian Society of Sciences and they both appear in *BGA* 1,18.]

ZK *Versuch einer objectiven Begründung der Lehre von der Zusammensetzung der Kräfte* Prague: Kronberger & Řiwnač 1842

[*Translation:* Attempt at an objective grounding of the theory of the composition of forces. See note on the previous work.]

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Square brackets indicate some of the more common variants in spelling.

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